CONTEMPORARY CALCULUS



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Contemporary Calculus

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TABLE OF CONTENTS

Licensing

0: Welcome to Calculus

- 0.1: A Preview of Calculus
- 0.2: Lines in the Plane
- 0.3: Functions and Their Graphs
- 0.4: Combinations of Functions
- 0.5: Mathematical Language
- 0.A: Answers

1: Limits and Continuity

- 1.0: Tangent Lines, Velocities and Graphs
- 1.1: The Limit of a Function
- 1.2: Properties of Limits
- 1.3: Continuous Functions
- 1.4: Definition of Limit
- 1.A: Answers

2: The Derivative

- 2.0: Introduction to Derivatives
- 2.1: The Definition of Derivative
- 2.2: Derivatives: Properties and Formulas
- 2.3: More Differentiation Patterns
- 2.4: The Chain Rule
- 2.5: Applications of the Chain Rule
- 2.6: Related Rates
- 2.7: Newton's Method
- 2.8: Linear Approximation and Differentials
- 2.9: Implicit and Logarithmic Differentiation
- 2.A: Answers

3: Derivatives and Graphs

- 3.1: Finding Maximums and Minimums
- 3.2: Mean Value Theorem
- 3.3: The First Derivative and the Shape of f
- 3.4: The Second Derivative and the Shape of f
- 3.5: Applied Maximum and Minimum Problems
- 3.6: Asymptotic Behavior of Functions
- 3.7: L'Hôpital's Rule
- 3.A: Answers

4: The Integral

- 4.0: Area
- 4.1: Sigma Notation and Riemann Sums

- 4.2: The Definite Integral
- 4.3: Properties of the Definite Integral
- 4.4: Areas, Integrals and Antiderivatives
- 4.5: The Fundamental Theorem of Calculus
- 4.6: Finding Antiderivatives
- 4.7: First Applications of Definite Integrals
- 4.8: Using Tables [and Technology] to Find Antiderivatives
- 4.9: Approximating Definite Integrals
- 4.A: Answers

5: Applications of Definite Integrals

- 5.1: Volumes by Slicing
- 5.2: Disks and Washers
- 5.3: Arclength and Surface Area
- 5.4: More Work
- 5.5: Volumes: Tubes
- 5.6: Moments and Centers of Mass
- 5.7: Improper Integrals
- 5.8: Additional Applications
- 5.A: Answers

6: Differential Equations

- 6.1: Introduction to Differential Equations
- 6.2: The Differential Equation y = f'(x)
- 6.3: Separable Equations
- 6.4: Exponential Growth and Decay
- 6.5: Heating, Cooling and Mixing
- 6.A: Answers

7: Transcendental Functions

- 7.1: One-to-One Functions
- 7.2: Inverse Functions
- 7.3: Inverse Trigonometric Functions
- 7.4: Derivatives of Inverse Trigonometric Functions
- 7.5: Integrals Involving Inverse Trigonometric Functions
- 7.A: Answers

8: Integration Techniques

- 8.1: Finding Antiderivatives: A Review
- 8.2: Integration by Parts
- 8.3: Partial Fraction Decomposition
- 8.4: Trigonometric Substitution

Index

Glossary

Detailed Licensing

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

0: Welcome to Calculus

Calculus was first developed more than 300 years ago by Sir Isaac Newton and Gottfried Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to a wide variety of problems besides planetary motion. They have used calculus to help understand physical, biological, economic and social phenomena and to describe and solve related problems.

The discovery, development and application of calculus is a great intellectual achievement—and now you have the opportunity to share in that achievement. You should feel exhilarated. You may also be somewhat concerned (a common reaction among students just beginning to study calculus). You need to be concerned enough to work to master calculus, yet confident enough to keep going when you (at first) don't understand something.

Part of the beauty of calculus is that it relies upon a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe and solve problems in a variety of fields. This book tries to emphasize both the beauty and the power.

In Section 0.1 (Preview) we will look at the main ideas that will continue throughout the book: the problems of finding tangent lines and computing areas. We will also consider a process that underlies both of those problems: the limiting process of approximating a solution and then getting better and better approximations until we finally get an exact solution.

Sections 0.2 (Lines), 0.3 (Functions) and 0.4 (Combinations of Functions) contain review material. These sections emphasize concepts and skills you will need in order to succeed in calculus. You should have worked with most of these concepts in previous courses, but the emphasis and use of the material here may be different than in those earlier classes. Section 0.5 (Mathematical Language) discusses a few key mathematical phrases. It considers their use and meaning and some of their equivalent forms. It will be difficult to understand the meaning and subtleties of calculus if you don't understand how these phrases are used and what they mean.

0.1: A Preview of Calculus
0.2: Lines in the Plane
0.3: Functions and Their Graphs
0.4: Combinations of Functions
0.5: Mathematical Language
0.A: Answers

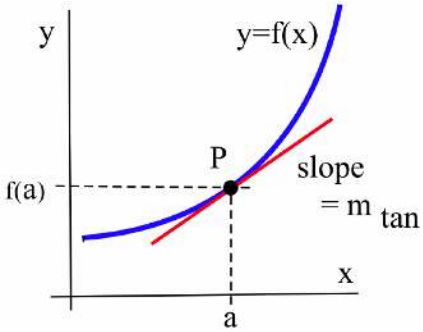
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0.1: A Preview of Calculus

Calculus can be viewed as an attempt—a historically successful attempt—to solve two fundamental problems. In this section we begin to examine geometric forms of those two problems and some fairly simple attempts to solve them. At first, the problems themselves may not appear very interesting or useful—and the methods for solving them may seem crude—but these simple problems and methods have led to one of the most beautiful, powerful and useful creations in mathematics: Calculus.

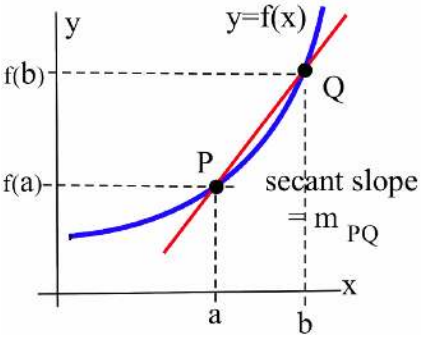
Finding the Slope of a Tangent Line

Suppose we have the graph of a function $y = f(x)$ and we want to find an equation of a line tangent to the graph at a particular point P on the graph.



(We will offer a precise definition of "tangent" in Section 1.0; for now, think of the tangent line as a line that touches the curve at P and stays close to the graph of $y = f(x)$ near P.) We know that P is on the tangent line, so if its x-coordinate is $x = a$, then the y-coordinate of P must be $y = f(a)$: $P = (a, f(a))$. The only other information we need to find an equation of the tangent line is its slope, $m_{\tan}$, but that is where the difficulty arises.

In algebra, we needed two points in order to determine a slope. So far, we only have the point P. Let's simply pick a second point, call it Q, on the graph of $y = f(x)$. If the x-coordinate of Q is b, then the y-coordinate is $f(b)$: $Q = (b, f(b))$.



So the slope of the line through P and Q is:

$$m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{f(b) - f(a)}{b - a}$$

If we drew the graph of $y = f(x)$ on a wall, put nails at the points P and Q, and laid a straightedge on the nails, then the straightedge would have slope m_{PQ}. But the slope m_{PQ} can be very different from the value we want (the slope $m_{\tan}$ of the tangent line). The key idea is that when the point Q is close to the point P, then the slope m_{PQ} should be close to the slope we want, $m_{\tan}$. Physically, if we slide the nail at Q along the graph toward the fixed point P, then the slope, $m_{PQ} = \frac{f(b)-f(a)}{b-a}$, of the straightedge gets closer and closer to the slope, $m_{\tan}$, of the tangent line. If the value of b is very close to a, then the point Q is very close to P, and the value of m_{PQ} is very close to the value of $m_{\tan}$.

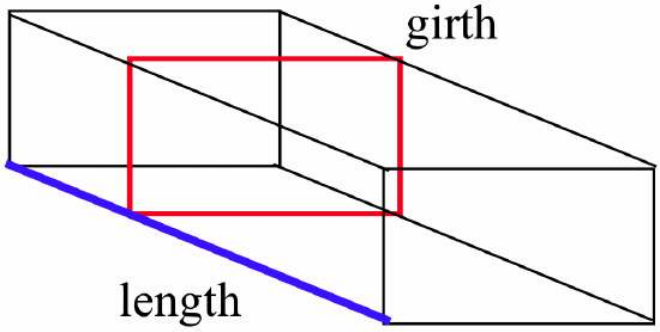
Rather than defacing walls with graphs and nails, we can instead calculate $m_{PQ} = \frac{f(b)-f(a)}{b-a}$ and examine the values of m_{PQ} as b gets closer and closer to a. We say that $m_{\tan}$ is the **limiting value** of m_{PQ} as b gets very close to a, and we write:

$$m_{\tan} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{0.1.1}$$

Eventually we will call the slope $m_{\tan}$ of the tangent line the **derivative** of the function $f(x)$ at the point P, and call this part of calculus **differential calculus**. Chapters 2 and 3 begin the study of differential calculus.

The slope of the tangent line to the graph of a function will tell us important information about the function and will allow us to solve problems such as:

- The U.S. Postal Service requires that the length plus the girth of a package not exceed 84 inches.

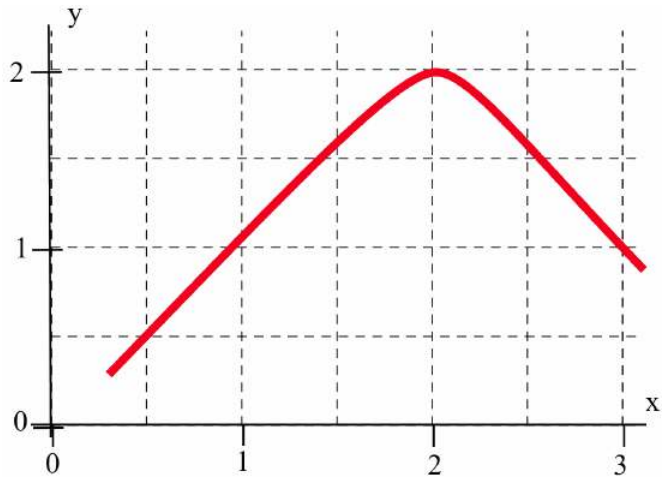


  What is the largest volume that can be mailed in a rectangular box?
- An oil tanker was leaking oil and a 4-inch-thick oil slick had formed. When first measured, the slick had a radius of 200 feet, and the radius was increasing at a rate of 3 feet per hour. At that time, how fast was the oil leaking from the tanker.

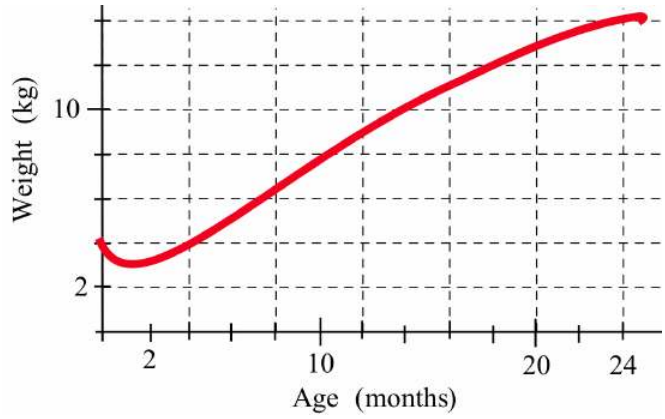
Derivatives will even help us find solutions to equations such as $x^2 = \sin(x)$ and $x^9 + 5x^5 + x^3 + 3 = 0$.

Problems

1. Sketch the lines tangent to the curve shown below at $x = 1$, 2 and 3. Estimate the slope of each of the tangent lines you drew.



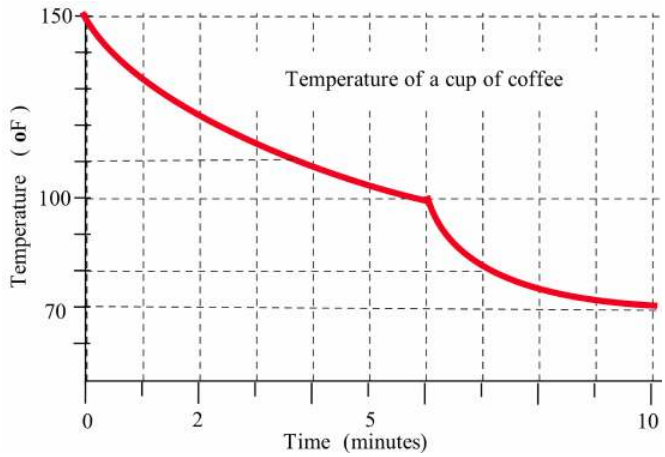
2. A graph of the weight of a “typical” child from age 0 to age 24 months appears below.



(Your answers should have the units "kg per month.")

a. What was the average weight gain from month 0 to month 24?
b. What was the average weight gain from month 9 to month 12? From month 12 to month 15?
c. Approximately how fast was the child gaining weight at age 12 months? At age 3 months?

3. The graph below shows the temperature of a cup of coffee during a 10-minute period.



(Each of your answers in (a)–(c) should have the units "degrees per minute.")

a. What was the average rate of cooling from minute 0 to minute 10?
b. What was the average rate of cooling from minute 7 to minute 8? From minute 8 to minute 9?
c. What was the rate of cooling at minute 8? At minute 2?
d. When was cold milk added to the coffee?

4. Describe a method for determining the slope of a steep hill at a point midway up the hill by:

a. using a ruler, a long piece of string, a glass of water and a loaf of bread.
b. using a protractor, a piece of string and a helium.

Finding the Area of a Shape

Suppose we need to find the area of a leaf as part of a study of how much energy a plant gets from sunlight. One method for finding the area would be to trace the shape of the leaf onto a piece of paper and then divide the region into "easy" shapes such as rectangles and triangles (whose areas we can easily calculate). We could add all of these "easy" areas together to approximate the area of the leaf.



Figure 0.1.1

A modification of this method would be to trace the shape onto a piece of graph paper and then count the number of squares completely inside the edge of the leaf to get a lower estimate of the area, and count the number of squares that touch the leaf to get an upper estimate of the area. If we repeat this process with smaller and smaller squares, we will have to do more counting and adding, but our estimates should be closer together—and closer to the actual area of the leaf.

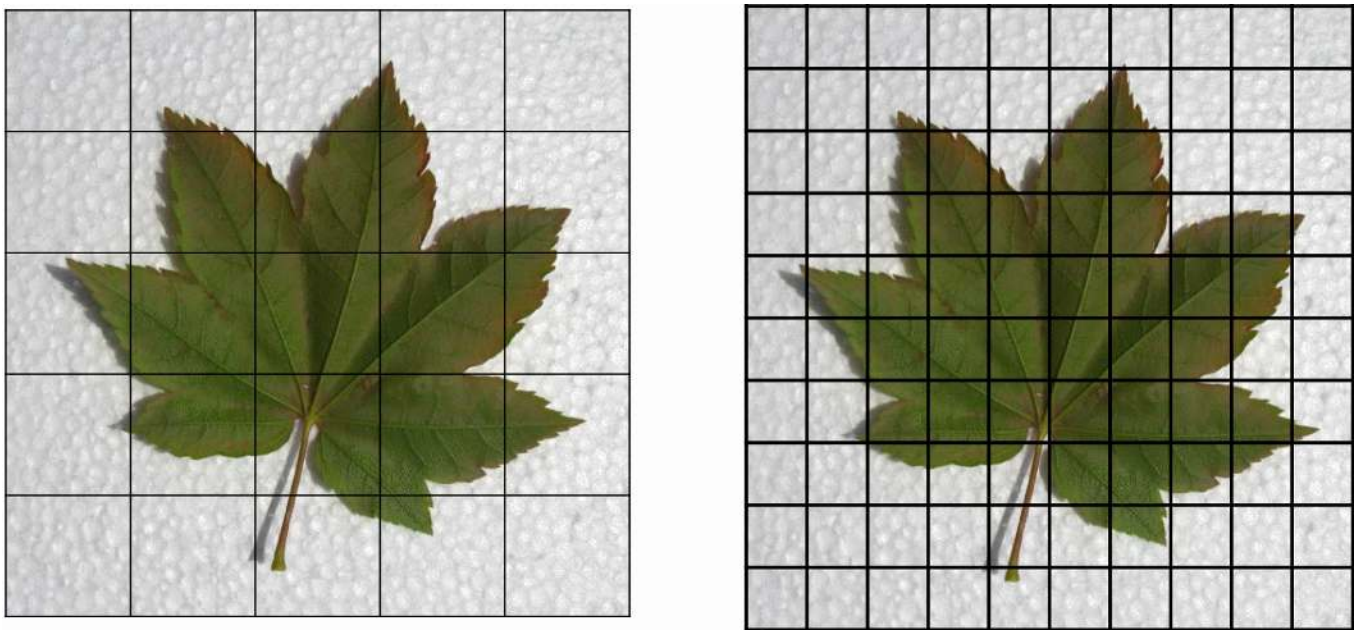
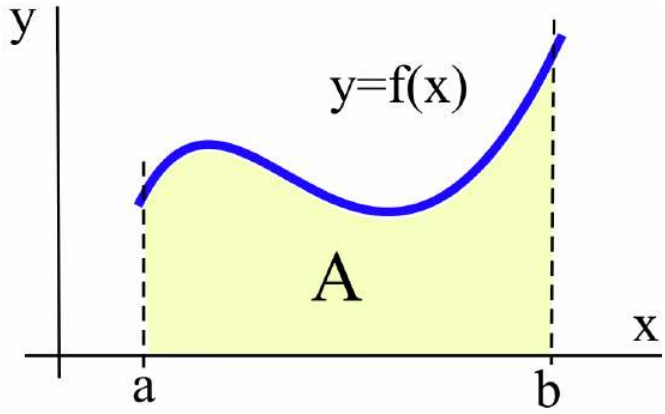


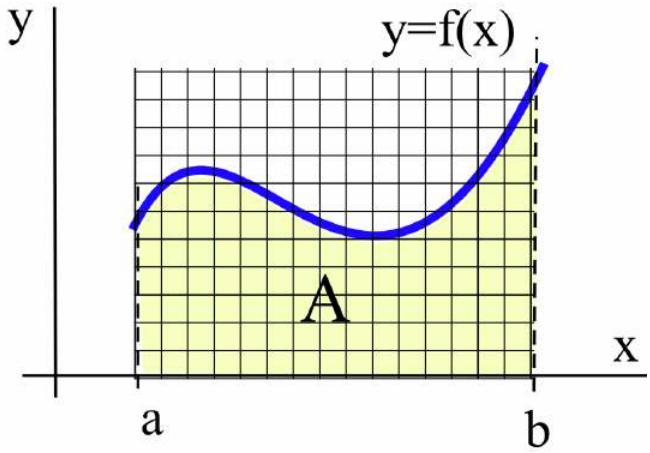
Figure 0.1.2

(We could also approximate the area of the leaf using a sheet of paper, scissors and an accurate scale. How?)

We can calculate the area A between the graph of a function $y = f(x)$ and the x-axis by using similar methods.



We can divide the area into strips of width w and determine the lower and upper values of $y = f(x)$ on each strip. Then we can approximate the area of each rectangle and add all of the little areas together to get A_w, an approximation of the exact area.



The key idea is that if w is small, then the rectangles are narrow, and the approximate area A_w should be very close to the actual area A. If we take narrower and narrower rectangles, the approximate areas get closer and closer to the actual area:

$$A = \lim_{w \to 0} A_w \tag{0.1.2}$$

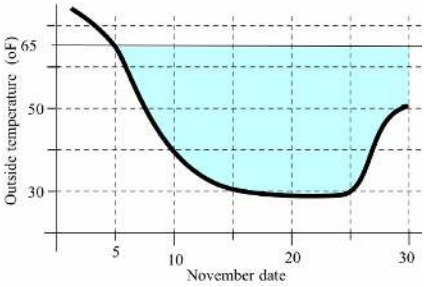
The process described above is the basis for a technique called **integration**, and this part of calculus is called **integral calculus**. Integral calculus and integration will begin in Chapter 4. The process of taking the limit of a sum of "little" quantities will give us important information about a function and will also allow us to solve problems such as:

- Find the length of the graph of $y = sin(x)$ over one period (from $x = 0$ to $x = 2\pi$).
- Find the volume of a torus ("doughnut") of radius 1 inch that has a hole of radius 2 inches.
- A car starts at rest and has an acceleration of $5 + 3\sin(t)$ feet per second per second in the northerly direction at time t seconds. Where will the car be, relative to its starting position, after 100 seconds?

Problems

5. Approximate the area of the leaf in Figure 0.1.2 using (a) the grid on the left. (b) the grid on the right.

6. A graph showing temperatures during the month of November appears below.



   a. Approximate the shaded area between the temperature curve and the 65° line from Nov. 15 to Nov. 25.
   b. The area of the "rectangle" is (base)(height) so what are the units of your answer in part (a)?
   c. Approximate the shaded area between the temperature curve and the 65° line from Nov. 5 to Nov. 30.
   d. Who might use or care about these results?

7. Describe a method for determining the volume of a compact fluorescent light bulb using a ruler, a large can, a scale and a jug of water.

A Unifying Process: Limits

We used similar processes to "solve" both the tangent line problem and the area problem. First, we found a way to get an approximate solution, and then we found a way to improve our approximation. Finally, we asked what would happen if we continued improving our approximations "forever": that is, we "took a limit."

For the tangent line problem, we let the point Q get closer and closer and closer to P, the limit as b approached a. In the area problem, we let the widths of the rectangles get smaller and smaller, the limit as w approached 0. Limiting processes underlie derivatives, integrals and several other fundamental topics in calculus, and we will examine limits and their properties in detail in Chapter 1.

Two Sides of the Same Coin: Differentiation and Integration

Just as the set-up of each of the two basic problems involved a limiting process, the solutions to the two problems are also related. The process of differentiation used to solve the tangent line problem and the process of integration used to solve the area problem turn out to be "opposites" of each other: each process undoes the effect of the other process. The Fundamental Theorem of Calculus in Chapter 4 will show how this "opposite" effect works.

Extensions of the Main Problems

The first five chapters present the two key ideas of calculus, show "easy" ways to calculate derivatives and integrals, and examine some of their applications. And there is more. Through the ensuing chapters, we will examine new functions and find ways to calculate their derivatives and integrals. We will extend the approximation ideas to use "easy" functions, such as polynomials, to approximate the values of "hard" functions, such as $sin(x)$ and e^x . In later chapters, we will extend the notions of "tangent lines" and "areas" to 3-dimensional space as "tangent planes" and "volumes."

> Success in calculus will require time and effort on your part, but such a beautiful and powerful field is worth that time and effort.

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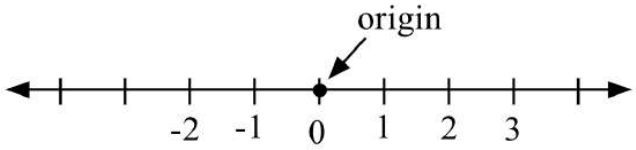
0.2: Lines in the Plane

The first graphs and functions you encountered in algebra were straight lines and their equations. These lines were easy to graph, and the equations were easy to evaluate and to solve. They described a variety of physical, biological and financial phenomena such as $d = rt$ relating the distance d traveled to the rate r and time t spent traveling, and $C = \frac{5}{9}(F - 32)$ for converting the temperature in degrees Fahrenheit (F) to degrees Celsius (C).

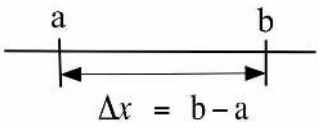
The first part of calculus—differential calculus—will deal with ideas, techniques and applications of tangent lines to the graphs of functions, so it is important that you understand the graphs, properties and equations of straight lines.

The Real Number Line

The real numbers (consisting of all integers, fractions, rational and irrational numbers) can be represented as a line, called the **real number line**. Once we have selected a starting location, called the **origin**, a positive direction (usually up or to the right), and unit of length, then every number can be located as a point on the number line.



If we move from a point $x = a$ to a point $x = b$ on the line, then we will have moved an **increment** of $b - a$. We denote this increment with the symbol Δx (read "delta x").



Note

The capital Greek letter delta (Δ) appears often in calculus to represent the "change" in something.

Caution

Δx does not mean Δ *times* x, but rather the *difference* between two x-coordinates.

- If b is larger than a, then we will have moved in the positive direction, and $\Delta x = b - a$ will be positive.
- If b is smaller than a, then $\Delta x = b - a$ will be negative and we will have moved in the negative direction.
- Finally, if $\Delta x = b - a = 0$, then $a = b$ and we did not move at all.

We can also use the Δ notation and absolute values to express the distance that we have moved. On the number line, the distance from $x = a$ to $x = b$ is

$$\text{dist}(a, b) = \begin{cases} b - a & \text{if } b \geq a \\ a - b & \text{if } b < a \end{cases}$$

or:

$$\text{dist}(a, b) = |b - a| = |\Delta x| = \sqrt{(\Delta x)^2}$$

The **midpoint** of the interval from $x = a$ to $x = b$ is the point M such that $\text{dist}(a, M) = \text{dist}(M, b)$, or $|M - a| = |b - M|$. If $a < M < b$,

$$M - a = b - M \Rightarrow 2M = a + b \Rightarrow M = \frac{a + b}{2}$$

It's not difficult to check that this formula also works when $b < M < a$.

Example 0.2.1

Find the length and midpoint of the interval from $x = -3$ to $x = 6$.

Solution

$\text{dist}(-3, 6) = |6 - (-3)| = |9| = 9$; $M = \frac{(-3)+6}{2} = \frac{3}{2}$.

Practice 0.2.1

Find the length and midpoint of the interval from $x = -7$ to $x = -2$.

Answer

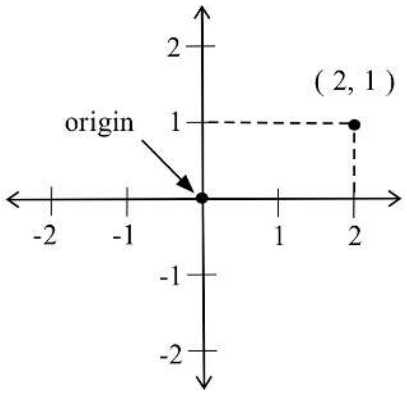
Length $= \text{dist}(-7, -2) = |(-7) - (-2)| = |-5| = 5$. The midpoint is at $\frac{(-7)+(-2)}{2} = \frac{-9}{2} = -4.5$.

Note

You should work through each of the Practice problems as you encounter them when reading this book. You can click on "Answer" above to check your answer and see a solution.

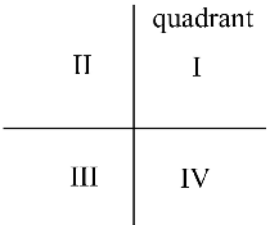
The Cartesian Plane

Two perpendicular number lines, called **coordinate axes**, determine a **real number plane**. The axes intersect at a point called the **origin**. Each point P in the plane can be described by an **ordered pair** (x, y) of numbers that specify how far, and in which directions, we must move from the origin to reach the point P. We can locate the point $P = (x, y)$ in the plane by starting at the origin and moving x units horizontally and then y units vertically. Similarly, we can label each point in the plane with the ordered pair (x, y), which directs us how to reach that point from the origin.



In this book, a point in the plane will be labeled either with a name, say P, or with an ordered pair, say (x, y), or with both: $P = (x, y)$.}

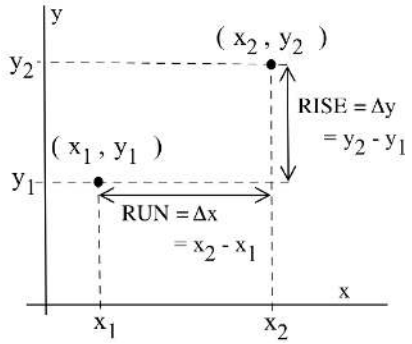
This coordinate system is called the **rectangular coordinate system** or the **Cartesian** coordinate system (after René Descartes), and the resulting plane the **Cartesian** plane. The coordinate axes divide the plane into four **quadrants**, labeled quadrants I, II, III and IV moving counterclockwise from the upper-right quadrant. We will often call the horizontal axis the **x-axis** and the vertical axis the **y-axis** and then refer to the plane as the **xy-plane**. This choice of (x\) and y as labels for the axes is a common choice, but we will sometimes prefer to use different labels—and even different units of measurement—on the two axes.



Increments and Distance Between Points in the Plane

If we move from a point $P = (x_1, y_1)$ in the plane to another point $Q = (x_2, y_2)$, then we will need to consider two **increments** or changes.

- The increment in the x (horizontal) direction is $x_2 - x_1$, denoted by $\Delta x = x_2 - x_1$.
- The increment in the y (vertical) direction is $y_2 - y_1$, denoted by $\Delta y = y_2 - y_1$.



Computing the **distance** between the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ involves a simple application of the Pythagorean Theorem:

$$\text{dist}(P, Q) = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

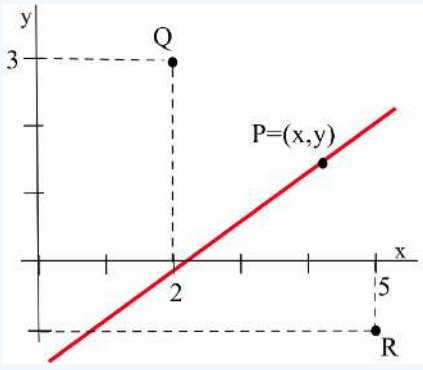
The **midpoint** M of the line segment joining P and Q is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where we have just used the one-dimension midpoint formula for each coordinate.

Example 0.2.2

Find an equation describing all the points $P = (x, y)$ equidistant from $Q = (2, 3)$ and $R = (5, -1)$.



Solution

The points $P = (x, y)$ must satisfy $\text{dist}(P, Q) = \text{dist}(P, R)$ so:

$$\sqrt{(x-2)^2+(y-3)^2} = \sqrt{(x-5)^2+(y-(-1))^2}$$

By squaring each side we get:

$$(x-2)^2+(y-3)^2 = (x-5)^2+(y+1)^2$$

Expanding we get:

$$x^2-4x+4+y^2-6y+9 = x^2-10x+25+y^2+2y+1$$

and canceling like terms yields:

$$-4x-6y+13 = -10x+2y+26$$

so $y = 0.75x - 1.625$, the equation of a line. Every point on the line $y = 0.75x - 1.625$ is equally distant from both Q and R.

? Practice 0.2.2

Find an equation describing all points $P = (x, y)$ equidistant from $Q = (1, -4)$ and $R = (0, -3)$.

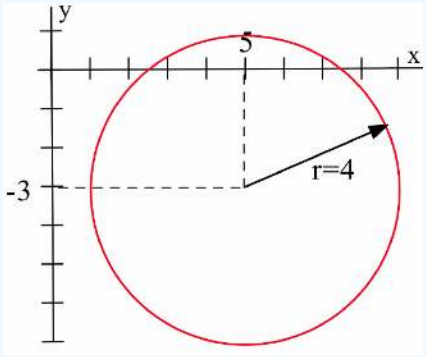
Answer

$\text{dist}(P, Q) = \text{dist}(P, R) \Rightarrow (x-1)^2 + (y+4)^2 = (x-0)^2 + (y+3)^2$; squaring each side and simplifying eventually yields $y = x - 4$.

A circle with radius r and center at the point $C = (a, b)$ consists of all points $P = (x, y)$ at a distance of r from the center C: the points P that satisfy $\text{dist}(P, C) = r$.

✓ Example 0.2.3

Find an equation of a circle with radius $r = 4$ and center$C = (5, -3)$.



Solution

A circle consists of the set of points $P = (x, y)$ at a fixed distance r from the center point C, so this circle will be the set of points$P = (x, y)$ at a distance of4 units from the point$C = (5, -3)$; P will be on this circle if $\text{dist}(P, C) = 4$.

Using the distance formula and rewriting:

$$\sqrt{(x-5)^2+(y+3)^2} = 4 \quad \Rightarrow \quad (x-5)^2+(y+3)^2 = 16$$

which we can also express as $x^2 - 10x + 25 + y^2 + 6y + 9 = 16$.

? Practice 0.2.3

Find an equation of a circle with radius $r = 5$ and center$C = (-2, 6)$.

Answer

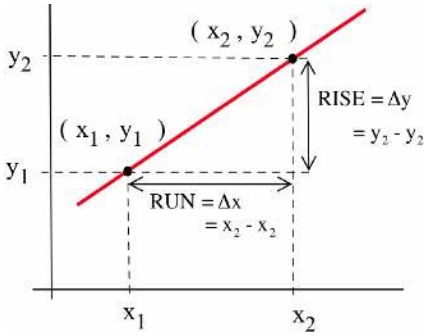
The point $P=(x,y)$ is on the circle when it is 5 units from the center $C=(-2,6)$, so $\text{dist}(P,C)=5$. Then $\text{dist}\,((x,y),(-2,6))=5$, so $\sqrt{(x+2)^2+(y-6)^2}=5 \Rightarrow (x+2)^2+(y-6)^2=25$.

The Slope Between Points in the Plane

In one dimension (on the number line), our only choice was to move in the positive direction (so the x-values were increasing) or in the negative direction. In two dimensions (in the plane), we can move in infinitely many directions, so we need a precise way to describe direction.

The **slope** of the line segment joining $P=(x_1,y_1)$ to $Q=(x_2,y_2)$ is

$$m=\text{slope from } P \text{ to } Q=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x1}=\frac{\Delta y}{\Delta x}$$



The slope of a line measures how fast we rise or fall as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. Most of our work will occur in two dimensions, and slope will be a very useful concept that will appear often.

If P and Q have the same x-coordinate, then $x_1=x_2 \Rightarrow x=0$. The line from P to Q is thus **vertical** and the slope $m=\frac{\Delta y}{\Delta x}$ is **undefined** because $\Delta x=0$. If P and Q have the same y-coordinate, then $y_1=y_2 \Rightarrow \Delta y=0$, so the line is **horizontal** and the slope is $m=\frac{\Delta y}{\Delta x}=\frac{0}{\Delta x}=0$ (assuming $\Delta x \neq 0$).

? Practice 0.2.4

For $P=(-3,2)$ and $Q=(5,-14)$, find Δx, Δy, and the slope of the line segment from P to Q.

Answer

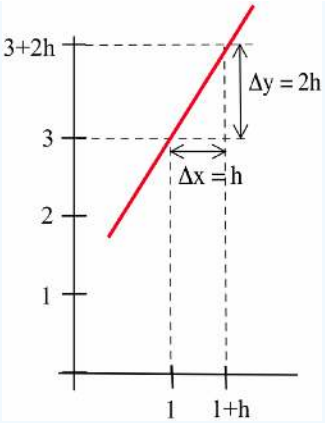
$\Delta x=5-(-3)=8$ and $\Delta y=-14-2=-16$, so: $\text{slope}=\frac{\Delta y}{\Delta x}=\frac{-16}{8}=-2$

If the coordinates of P or Q contain variables, then the slope m is still given by $m=\frac{\Delta y}{\Delta x}$, but we will need to use algebra to evaluate and simplify m.

✓ Example 0.2.4

Find the slope of the line segment from $P=(1,3)$ to $Q=(1+h,3+2h)$.

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Solution

$y_1 = 3$ and $y_2 = 3 + 2h$, so $\Delta y = (3 + 2h) - (3) = 2h$; $x_1 = 1$ and $x_2 = 1 + h$, so $\Delta x = (1 + h) - (1) = h$. The slope is:

$$m = \frac{\Delta y}{\Delta x} = \frac{2h}{h} = 2$$

In this example, the value of m is constant (2) and does not depend on the value of h.

? Practice 0.2.5

Find the slope and midpoint of the line segment from $P = (2, -3)$ to $Q = (2 + h, -3 + 5h)$.

Answer

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(3a^2 + 5a) - (3x^2 + 5x)}{a - x} = \frac{3(a^2 - x^2) + 5(a - x)}{a - x} = \frac{3(a + x)(a - x) + 5(a - x)}{a - x} = 3(a + x) + 5$$

✓ Example 0.2.5

Find the slope between the points $P = (x, x^2 + x)$ and $Q = (a, a^2 + a)$ for $a \neq x$.

Solution

$y_1 = x^2 + x$ and $y_2 = a^2 + a \Rightarrow \Delta y = (a^2 + a) - (x^2 + x)$; $x_1 = x$ and $x_2 = a$, so $\Delta x = a - x$ and the slope is:

$$\begin{aligned} m = \frac{\Delta y}{\Delta x} &= \frac{(a^2 + a) - (x^2 + x)}{a - x} \\ &= \frac{a^2 - x^2 + a - x}{a - x} = \frac{(a - x)(a + x) + (a - x)}{a - x} \\ &= \frac{(a - x)\left((a + x) + 1\right)}{a - x} = (a + x) + 1 \end{aligned}$$

Here the value of m depends on the values of both a and x.

? Practice 0.2.6

Find the slope between the points $P = (x, 3x^2 + 5x)$ and $Q = (a, 3a^2 + 5a)$ for $a \neq x$.

Answer

$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(-3 + 5h) - (-3)}{(2 + h) - 2} = \frac{5h}{h} = 5$. The midpoint is at $\left(\frac{(2) + (2 + h)}{2}, \frac{(-3 + 5h) + (-3)}{2}\right) = \left(2 + \frac{h}{2}, -3 + \frac{5h}{2}\right)$

In application problems, it is important to read the information and the questions very carefully—including the units of measurement of the variables can help you avoid "silly" answers.

Example 0.2.6

In 1970, the population of Houston was 1,233,535 and in 1980 it was 1,595,138. Find the slope of the line through the points $(1970, 1233535)$ and $(1980, 1595138)$.

Solution

$m = \frac{\Delta y}{\Delta x} = \frac{1595138 - 1233535}{1980 - 1970} = \frac{361603}{10} = 36,160.3$ but $36,160.3$ is just a number that may or may not have any meaning to you.

If we include the units of measurement along with the numbers we will get a more meaningful result:

$$\frac{1595138 \text{ people} - 1233535 \text{ people}}{\text{year } 1980 - \text{year } 1970} = \frac{361603 \text{ people}}{10 \text{ years}} = 36,160.3 \frac{\text{people}}{\text{year}}$$

which says that during the decade from 1970 to 1980 the population of Houston grew at an average rate of $36,160$ people per year.

If the x-unit is time (in hours) and the y-unit is distance (in kilometers), then

$$m = \frac{\Delta y \text{ km}}{\Delta x \text{ hours}}$$

so the units for m are $\frac{\text{km}}{\text{hour}}$ (``kilometers per hour''), a measure of velocity, the rate of change of distance with respect to time.

If the x-unit is the number of employees at a bicycle factory and the y-unit is the number of bicycles manufactured, then

$$m = \frac{\Delta y \text{ bicycles}}{\Delta x \text{ employees}}$$

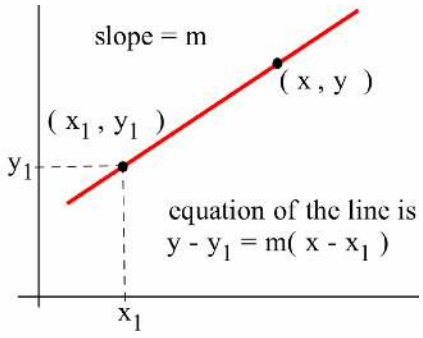
and the units for m are $\frac{\text{bicycles}}{\text{employee}}$ (``bicycles per employee''), a measure of the rate of production per employee.

Equations of Lines

Every (non-vertical) line has the property that the slope of the segment between any two points on the line is the same, and this constant slope property of straight lines leads to ways of finding equations to represent non-vertical lines.

Point-Slope Form

In calculus, we will usually know a point on a line and the slope of that line, so the point-slope form will be the easiest to apply. Other forms of equations for lines can be derived from the point-slope form.

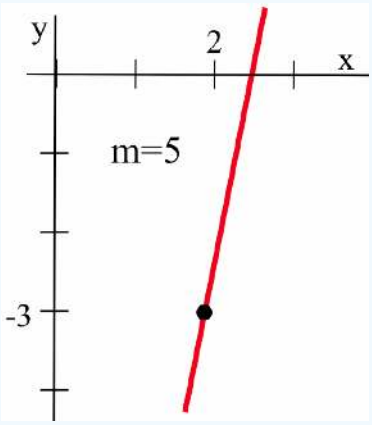


If L is a non-vertical line through a known point $P = (x_1, y_1)$ with a known slope m, then the equation of the line L is:

Point-Slope: $y - y_1 = m(x - x_1)$

Example 0.2.7

Find an equation of the line through $(2, -3)$ with slope 5.



Solution

We can simply use the point-slope formula: $m = 5$, $y_1 = -3$ and $x_1 = 2$, so $y - (-3) = 5(x - 2)$, which simplifies to $y = 5x - 13$.

An equation for a **vertical line** through a point $P = (a, b)$ is $x = a$. All points $Q = (x, y)$ on the vertical line through the point P have the same (x\)-coordinate as P.

Two-Point Form

If two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are on the line L, then we can calculate the slope between them and use the first point and the point-slope equation to find an equation for L:

Two-Point: $y - y_1 = m(x - x_1)$ where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Once we have the slope, m, it does not matter whether we use P or Q as the point. Either choice will result in the same equation for the line once we simplify it.

Slope-Intercept Form

It is common practice to rewrite an equation of a line into the form $y = mx + b$, the **slope-intercept form** of the line. The line $y = mx + b$ has slope m and crosses the y-axis at the point $(0, b)$.

Practice 0.2.7

Use the $\frac{\Delta y}{\Delta x}$ definition of slope to calculate the slope of the line $y = mx + b$.

Answer

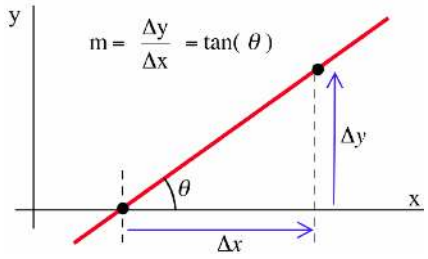
Let $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$. Then:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m \tag{0.2.1}$$

The point-slope and the two-point forms are usually more useful for *finding* an equation of a line, but the slope-intercept form is usually the most useful form for an *answer* because it allows us to easily picture the graph of the line and to quickly calculate y-values given x-values.

Angles Between Lines

The **angle of inclination** of a line with the x-axis is the smallest angle $\$\theta$ that the line makes with the positive (x\)-axis as measured from the x-axis counterclockwise to the line.

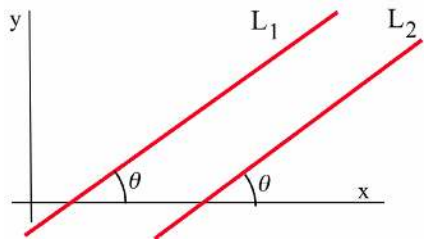


Because the slope $m = \frac{\Delta y}{\Delta x}$ and because $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$ in a right triangle, $m = \tan(\theta)$.

> The slope of a line is the tangent of its angle of inclination.

Parallel Lines

Two parallel lines L_1 and L_2 make equal angles with the x-axis, so their angles of inclination will be equal and hence so will their slopes.



Similarly, if the slopes, m_1 and m_2, of two lines are equal, then the equations of the lines (in slope-intercept form) will always differ by a constant:

$$y_1 - y_2 = (m_1x + b_1) - (m_2x + b_2) = (m_1 - m_2)x + (b_1 - b_2) = b_1 - b_2$$

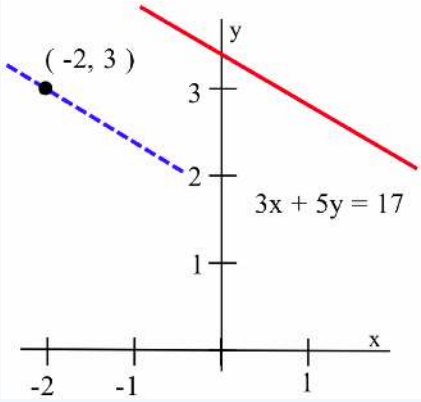
which is a constant, so the lines will be parallel.

The two preceding ideas can be combined into a single statement:

> Two non-vertical lines L_1 and L_2 with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$.

? Practice 0.2.8

Find an equation of the line that contains the point $(-2, 3)$ and is parallel to the line $3x + 5y = 17$.



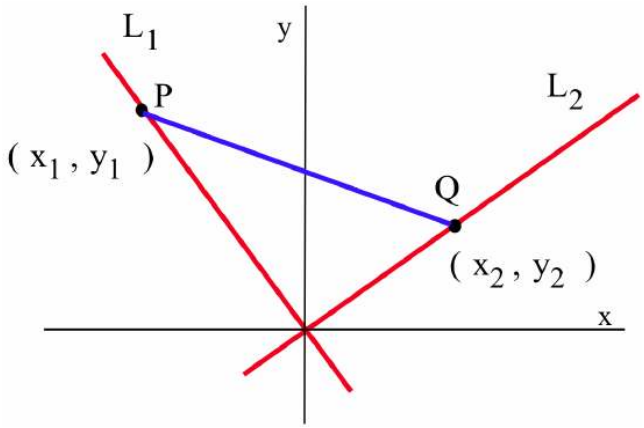
Answer

The line $3x + 5y = 17$ has slope $-\frac{3}{5}$, so the slope of the parallel line is $m = -\frac{3}{5}$. Using the form $y = -\frac{3}{5}x + b$ and the point $(-2, 3)$ on the line, we have $3 = -\frac{3}{5}(-2) + b \Rightarrow b = \frac{9}{5} \Rightarrow y = -\frac{3}{5}x + \frac{9}{5}$,
or $5y + 3x = 9$.

Perpendicular Lines

If two lines are perpendicular, the situation is a bit more complicated.

Assume L_1 and L_2 are two non-vertical lines that intersect at the origin (for simplicity), with $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ points away from the origin on L_1 and L_2, respectively.



Then the slopes of L_1 and L_2 will be $m_1 = \frac{y_1}{x_1}$ and $m_2 = \frac{y_2}{x_2}$. The line connecting P and Q forms the third side of triangle OPQ, which will be a right triangle if and only if L_1 and L_2 are perpendicular.

In particular, L_1 and L_2 are perpendicular if and only if the triangle OPQ satisfies the Pythagorean Theorem:

$$(\text{dist}(O,P))^2 + (\text{dist}(O,Q))^2 = (\text{dist}(P,Q))^2$$

or:

$$(x_1 - 0)^2 + (y_1 - 0)^2 + (x_2 - 0)^2 + (y_2 - 0)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Squaring and simplifying, this reduces to $0 = -2x_1x_2 - 2y_1y_2$, so:

$$\frac{y_2}{x_2} = -\frac{x_1}{y_1} \Rightarrow m_2 = \frac{y_2}{x_2} = -\frac{x_1}{y_1} = -\frac{1}{\frac{y_1}{x_1}} = -\frac{1}{m_1}$$

We have just proved the following result:

Theorem

Two non-vertical lines L_1 and L_2 with slopes m_1 and m_2 are perpendicular if and only if their slopes are negative reciprocals of each other: $m_2 = -\frac{1}{m_1}$.

Practice 0.2.9

Find an equation of the line that goes through the point $(2, -5)$ and is perpendicular to the line $3y - 7x = 2$.

Answer

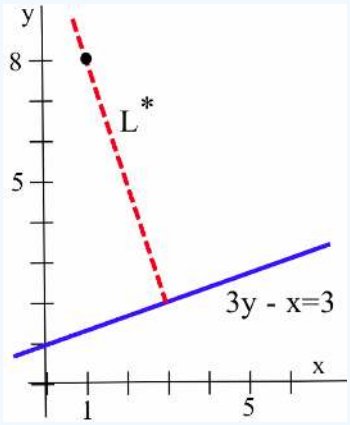
The line $3y - 7x = 2$ has slope $\frac{7}{3}$, so the slope of the perpendicular line is $m = -\frac{3}{7}$. Using the form $y = -\frac{3}{7}x + b$ and the point $(2, -5)$ on the line, we have $-5 = -\frac{3}{7}(2) + b \Rightarrow b = -\frac{29}{7} \Rightarrow y = -\frac{3}{7}x - \frac{29}{7}$, or $7y + 3x = -29$.

Example 0.2.8

Find the distance (that is, the shortest distance) from the point $(1, 8)$ to the line $L : 3y - x = 3$.

Solution

This is a sophisticated problem that requires several steps to solve. First we need a picture of the problem:



We will find an equation for the line L^* through the point $(1, 8)$ and perpendicular to L. Then we will find the point P where L and L^* intersect. Finally, we will find the distance from P to $(1, 8)$.

Step 1: L has slope $\frac{1}{3}$ so L^* has slope $m = -\frac{1}{\frac{1}{3}} = -3$, and L^* has equation $y - 8 = -3(x - 1)$, which simplifies to $y = -3x + 11$.

Step 2: We can find the point where L\$intersects\(L^* by replacing the y-value in the equation for L with the y-value from our equation for L^*:\[3(-3x + 11) - x = 3 \Rightarrow x = 3 \Rightarrow y = -3x + 11 = -3(3) + 11 = 2\nonumber

\]which tells us that L and L^* intersect at $P = (3, 2)$.

Step 3: Finally, the distance from L to $(1, 8)$ is just the distance from the point $(1, 8)$ to the point $P = (3, 2)$, which is

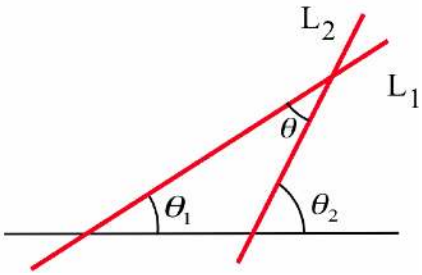
$$\sqrt{(1-3)^2 + (8-2)^2} = \sqrt{40} \approx 6.325$$

The distance is (exactly) $\sqrt{40}$, or (approximately) 6.325.

Angle Formed by Intersecting Lines

If two lines that are not perpendicular intersect at a point (and neither line is vertical), then we can use some geometry and trigonometry to determine the angles formed by the intersection of those lines.

Because θ_2 in the figure below is an exterior angle of the triangle ABC, θ_2 is equal to the sum of the two opposite interior angles, so $\theta_2 = \theta_1 + \theta \Rightarrow \theta = \theta_2 - \theta_1$:



From trigonometry, we then know that:

$$\tan(\theta) = \tan(\theta_2 - \theta_1) = \frac{\tan(\theta_2) - \tan(\theta_1)}{1 + \tan(\theta_2)\tan(\theta_1)} = \frac{m_2 - m_1}{1 + m_2 m_1}$$

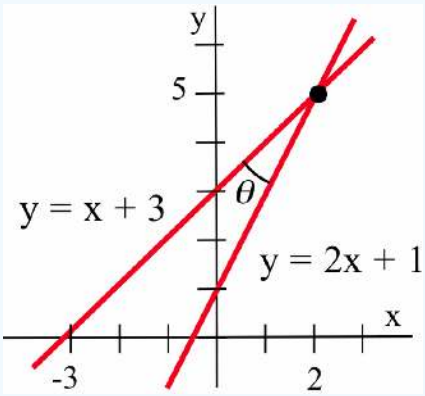
The range of the arctan function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so $\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right)$ always gives the smaller of the angles. The larger angle is $\pi - \theta$ (or $180° - \theta°$ if we measure the angles in degrees).

The smaller angle θ formed by two non-perpendicular lines with slopes m_1 and m_2 is:

$$\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right)$$

Example 0.2.9

Find the point of intersection and the angle between the lines $y = x + 3$ and $y = 2x + 1$.



Solution

Solving the first equation for y and then substituting into the second equation:

$$(x + 3) = 2x + 1 \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad y = 2 + 3 = 5$$

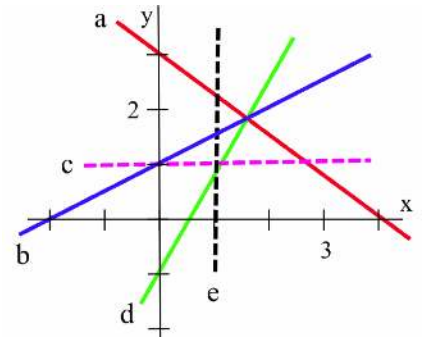
The point of intersection is $(2, 5)$. Because both lines are in slope-intercept form, it is easy to see that $m_1 = 1$ and $m_2 = 2$:

$$\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right) = \arctan\left(\frac{2 - 1}{1 + 2 \cdot 1}\right)$$
$$= \arctan\left(\frac{1}{3}\right) \approx 0.322 \text{ radians} = 18.43°$$

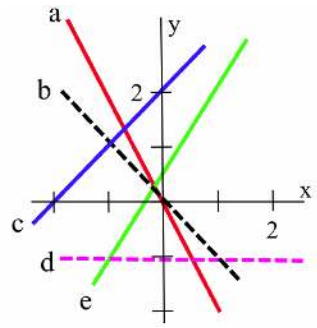
The lines intersect at an angle of (approximately) $18.43°$.

Problems

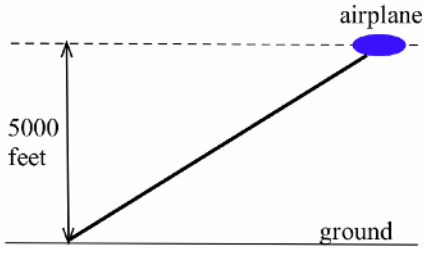
1. Estimate the slope of each line shown below.



2. Estimate the slope of each line shown below.



3. Compute the slope of the line that passes through:
   a. $(2,4)$ and$(5,8)$
   b. $(-2,4)$ and$(3,-5)$
   c. $(2,4)$ and(x,x^2)
   d. $(2,5)$ and$(2+h,1+(2+h)^2)$
   e. (x,x^2+3) and(a,a^2+3)
4. Compute the slope of the line that passes through:
   a. $(5,-2)$ and $(3,8)$
   b. $(-2,-4)$ and $(5,-3)$
   c. $(x,3x+5)$ and $(a,3a+5)$
   d. $(4,5)$ and $(4+h,5-3h)$
   e. $(1,2)$ and $(x,1+x^2)$
   f. $(2,-3)$ and $(2+h,1-(1+h)^2)$
   g. (x,x^2) and $(x+h,x^2+2xh+h^2)$
   h. (x,x^2) and $(x-h,x^2-2xh+h^2)$
5. A small airplane at an altitude of 5,000 feet is flying east at 300 feet per second (a bit over 200 miles per hour), and you are watching it with a small telescope as it passes directly overhead.



   a. What is the **slope** of the telescope 5, 10 and 20 seconds after the plane passes overhead?
   b. What is the **slope** of the telescope t seconds after the plane passes overhead?

c. After the plane passes overhead, is the slope of the telescope increasing, decreasing or staying the same?

6. You are at the origin, $(0,0)$, and are watching a small bug at the point $(t, 1+t^2)$ at time t seconds.
   a. What is the **slope** of your line of vision when $t = 5$, 10 and 15 seconds?
   b. What is the **slope** of your line of vision at an arbitrary time t?
7. The blocks in a city are all perfect squares. A friend gives you directions to a good restaurant: "Go north 3 blocks, turn east and go 5 blocks, turn south and go 7 blocks, turn west and go 3 blocks." How far away (straight-line distance) is it?
8. At the restaurant (see previous problem), a fellow diner gives you directions to a hotel: "Go north 5 blocks, turn right and go 6 blocks, turn right and go 3 blocks, turn left and go 2 blocks." How far away is the hotel from the restaurant?
9. The bottom of a 20-foot ladder is 4 feet from the base of a wall.
   a. How far up the wall does the ladder reach?
   b. What is the slope of the ladder?
   c. What angle does it make with the ground?
10. Let $P = (1,-2)$ and $Q = (5,4)$. Find:
    a. the midpoint R of the line segment PQ.
    b. the point T that is $\frac{1}{3}$ of the way from P to Q:
       $\text{dist}(P,T) = \frac{1}{3}\text{dist}(P,Q)$
    c. the point S that is $\frac{2}{5}$ of the way from P to Q.
11. If $P = (2,3)$, $Q = (8,11)$ and $R = (x,y)$, where:
    $$x = 2a + 8(1-a),\ y = 3a + 11(1-a),\ 0 \le a \le 1$$
    a. Verify that R is on the line segment PQ.
    b. Verify that $\text{dist}(P,R) = (1-a)\cdot\text{dist}(P,Q)$.
12. A rectangular box is 24 inches long, 18 inches wide and 12 inches high.
    a. Find the length of the longest (straight) stick that will fit into the box.
    b. What angle (in degrees) does that stick make with the base of the box?
13. The lines $y = x$ and $y = 4 - x$ intersect at $(2,2)$.
    a. Show that the lines are perpendicular.
    b. Graph the lines together on your calculator using the "window" $[-10,10]\times[-10,10]$.
    c. Why do the lines not appear to be perpendicular on the calculator display?
    d. Find a suitable window so that the lines do appear perpendicular.
14. Two lines both go through the point $(1,2)$, one with slope 3 and one with slope $-\frac{1}{3}$.
    a. Find equations for the lines.
    b. Choose a suitable window so that the lines will appear perpendicular, and then graph them together on your calculator
15. Sketch the line with slope m that goes through the point P, then find an equation for the line.
    a. $m = 3$, $P = (2,5)$
    b. $m = -2$, $P = (3,2)$
    c. $m = -\frac{1}{2}$, $P = (1,4)$
16. Sketch the line with slope m that goes through the point P, then find an equation for the line.
    a. $m = 5$, $P = (2,1)$
    b. $m = -\frac{2}{3}$, $P = (1,3)$
    c. $m = \pi$, $P = (1,-3)$
17. Find an equation for each line.
    a. L_1 goes through the point $(2,5)$ and is parallel to $3x - 2y = 9$.
    b. L_2 goes through the point $(-1,2)$ and is perpendicular to $2x = 7 - 3y$.
    c. L_3 goes through the point $(3,-2)$ and is perpendicular to $y = 1$
18. Find a value for the constant A, B or D so that:
    a. the line $y = 2x + A$ goes through $(3,10)$.

b. the line $y = Bx + 2$ goes through $(3, 10)$.
c. the line $y = Dx + 7$ crosses the y-axis at the point $(0, 4)$.
d. the line $Ay = Bx + 1$ goes through the points $(1, 3)$ and $(5, 13)$

19. Find the shortest distance between the circles with centers $C_1 = (1, 2)$ and $C_2 = (7, 10)$ with radii r_1 and r_2 when:

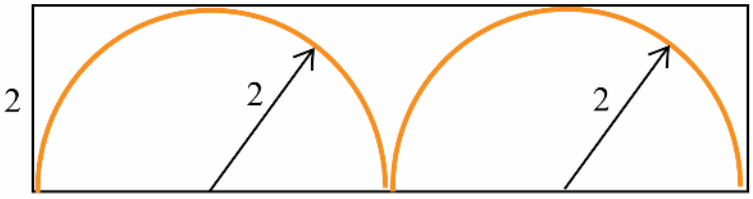
a. $r_1 = 2$ and $r_2 = 4$
b. $r_1 = 2$ and $r_2 = 7$
c. $r_1 = 5$ and $r_2 = 8$
d. $r_1 = 3$ and $r_2 = 15$
e. $r_1 = 12$ and $r_2 = 1$

20. Find an equation of the circle with center C and radius r when:

a. $C = (2, 7)$ and $r = 4$
b. $C = (3, -2)$ and $r = 1$
c. $C = (-5, 1)$ and $r = 7$
d. $C = (-3, -1)$ and $r = 4$

21. Explain how to show, without graphing, whether a point $P = (x, y)$ is inside, on, or outside the circle with center $C = (h, k)$ and radius r.

22. A box with a base of dimensions 2 cm and 8 cm is definitely big enough to hold two semicircular rods with radii of 2 cm, as shown in this figure:



a. Will these same two rods fit in a box 2 cm high and 7.6 cm wide?
b. Will they fit in a box 2 cm high and 7.2 cm wide? (Suggestion: Turn one of the rods over.)

23. Show that an equation of the circle with center $C = (h, k)$ and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

24. Find an equation of the line tangent to the circle $x^2 + y^2 = 25$ at the point P when:

a. $P = (3, 4)$
b. $P = (-4, 3)$
c. $P = (0, 5)$
d. $P = (-5, 0)$

25. Find an equation of the line tangent to the circle with center $C = (3, 1)$ at the point P when:

a. $P = (8, 13)$
b. $P = (-10, 1)$
c. $P = (-9, 6)$
d. $P = (3, 14)$

26. Find the center $C = (h, k)$ and the radius r of the circle that goes through the three points:

a. $(0, 1)$, $(1, 0)$ and $(0, 5)$
b. $(1, 4)$, $(2, 2)$ and $(8, 2)$
c. $(1, 3)$, $(4, 12)$ and $(8, 4)$

27. How close does:

a. the line $3x - 2y = 4$ come to the point $(2, 5)$?
b. the line $y = 5 - 2x$ come to the point $(1, -2)$?
c. the circle with radius 3 and center at $(2, 3)$ come to the point $(8, 3)$

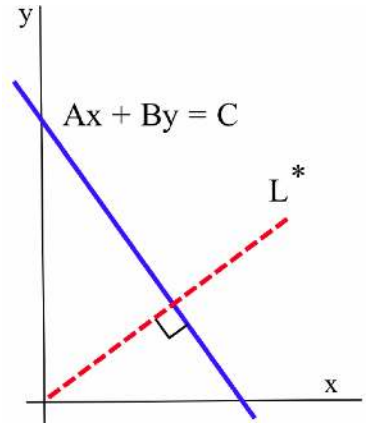
28. How close does:

a. the line $2x - 5y = 4$ come to the point $(1, 5)$?

b. the line $y = 3 - 2x$ come to the point $(5, -2)$?

c. the circle with radius 4 and center at $(4, 3)$ come to the point $(10, 3)$

29. Follow the steps below (and refer to the figure below) to find a formula for the distance from the origin to the line $Ax + By = C$.



a. Show that the line L given by $Ax + By = C$ has slope $m = -\frac{A}{B}$.

b. Find the equation of the line L^* that goes through $(0, 0)$ and is perpendicular to L.

c. Show that L and L^* intersect at the point: $(x, y) = \left(\frac{AC}{A^2 + B^2}, \frac{BC}{A^2 + B^2}\right)$

d. Show that the distance from the origin to the point (x, y) is: $\frac{|C|}{\sqrt{A^2 + B^2}}$

30. Show that a formula for the distance from the point (p, q) to the line $Ax + By = C$ is: $\frac{|Ap + Bq - C|}{\sqrt{A^2 + B^2}}$

(The steps will be similar to those in the previous problem, but the algebra will be more complicated.)

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0.3: Functions and Their Graphs

When you prepared for calculus, you learned to manipulate functions by adding, subtracting, multiplying and dividing them, as well as evaluating functions of functions (composition). In calculus, we will continue to work with functions and their applications. We will create new functions by operating on old ones. We will gather information from the graphs of functions and from derived functions. We will find ways to describe the point-by-point behavior of functions as well as their behavior "close to" certain points and also over entire intervals. We will find tangent lines to graphs of functions and areas between graphs of functions. And, of course, we will see how these ideas can be used in a variety of fields.

This section and the next one review information and procedures you should already know about functions before we begin calculus.

What Is a Function?

Let's begin with a (very) general definition of a function:

Definition

A **function** from a set X to a set Y is a rule for assigning to each element of the set X a single element of the set Y. A function assigns a unique (exactly one) output element from the set Y to each input element from the set X.

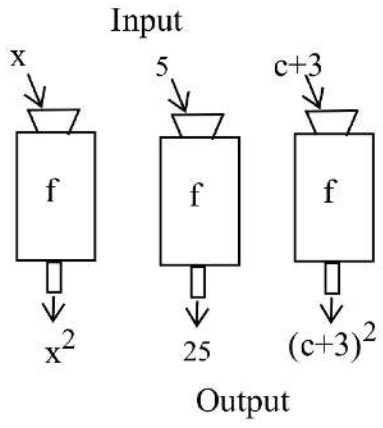
The rule that defines a function is often given in the form of an equation, but it could also be given in words or graphically or by a table of values. In practice, functions are given in all of these ways, and we will use all of them in this book.

In the definition of a function, the set X of all inputs is called the **domain** of the function. The set Y of all outputs produced from these inputs is called the **range** of the function. Two different inputs (elements in the domain) can be assigned to the same output (an element in the range) but one input cannot lead to two different outputs.

Most of the time we will work with functions whose domains and ranges are real numbers, but there are other types of functions all around us. Final grades for this course provide an example of a function. For each student, the instructor will assign a final grade based on some rule for evaluating that student's performance. The domain of this function consists of all students registered for the course, and the range consists of grades (either letters like A, B, C, D, F, or numbers like 4.0, 3.7, 2.0, 1.7, 0.0). Two students can receive the same final grade, but only one grade can be assigned to each student.

Function Machines

Functions are abstract structures, but sometimes it is easier to think of them in a more concrete way. One such way is to imagine that a function is a special-purpose computer, a machine that accepts inputs, does something to those inputs according to a defining rule, and produces an output. The output is the value of the function associated with the given input value.



If the defining rule for a function f is "multiply the input by itself," $f(\text{input}) = (\text{input})(\text{input})$, then the figure and table in the margin show the results of putting the inputs x, 5, 2.5, a, $c+3$ and $x+h$ into the machine f.

Input (x)	Output (x^2)
x	x^2
a	a^2
2.5	6.25
$x+h$	$(x+h)^2$

? Practice 0.3.1

If we have a function machine g whose rule is "divide 3 by the input and add 1," $g(x) = \frac{3}{x} + 1$, what outputs do we get from the inputs x, 5, a, $c+3$ and $x+h$? What happens if the input is 0?

Answer

Create an input-output table using the function rule:

input	output
x	$\frac{3}{x} + 1$
5	$\frac{3}{5} + 1 = 1.6$
a	$\frac{3}{a} + 1$
$c+3$	$\frac{3}{c+3} + 1$
$x+h$	$\frac{3}{x+h} + 1$

If $x = 0$, then $g(0) = \frac{3}{0} + 1$ is not defined (because of division by 0).

You expect your calculator to behave as a function: each time you press the same sequence of keys (input) you expect to see the same display (output). In fact, if your calculator did not produce the same output each time, you would need a new calculator. (On many calculators there is a feature that does not produce the same output each time you use it. What is it?)

Functions Defined by Equations

If the domain of a function consists of a collection of real numbers (perhaps all real numbers) and the range is also a collection of real numbers, then the function is called a **numerical function**. We can give the rule for a numerical function in several ways, but usually write it as a formula. If the rule for a numerical function, f, is "the output is the input number multiplied by itself," then we could write the rule as

$$f(x) = x \cdot x = x^2 \tag{0.3.1}$$

The use of an "x" to represent the input is simply a matter of convenience and custom. We could also represent the same function by:

- $f(a) = a^2$
- $f(\#) = \#^2$ or
- $f(\text{input}) = (\text{input})^2$.

For the function f defined by $f(x) = x^2 - x$, we can see that:

- $f(3) = 3^2 - 3 = 6$
- $f(0.5) = (0.5)^2 - (0.5) = -0.25$
- $f(-2) = (-2)^2 - (-2) = 6$

Notice that the two different inputs 3 and -2 both lead to the output of 6. That is allowable for a function.

We can also evaluate f if the input contains variables. If we replace the "x" with something else in the notation "$f(x)$," then we must replace the "x" with the same thing everywhere in the formula:

- $f(c) = c^2 - c$
- $f(a+1) = (a+1)^2 - (a+1) = (a^2 + 2a + 1) - (a+1) = a^2 + a$
- $f(x+h) = (x+h)^2 - (x+h) = (x^2 + 2xh + h^2) - (x+h)$

and, in general: $f(\text{input}) = (\text{input})^2 - (\text{input})$

For more complicated expressions, we can just proceed step-by-step:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{((x+h)^2-(x+h))-(x^2-x)}{h} \\ &= \frac{((x^2+2xh+h^2)-(x+h))-(x^2-x)}{h} \\ &= \frac{2xh+h^2-h}{h} = \frac{h(2x+h-1)}{h} = 2x+h-1\end{aligned}$$

? Practice 0.3.2

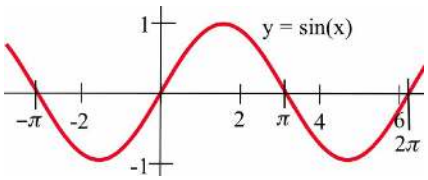
For the function g defined by $g(t) = t^2 - 5t$, evaluate $g(1)$, $g(-2)$, $g(w+3)$, $g(x+h)$, $g(x+h) - g(x)$ and $\frac{g(x+h)-g(x)}{h}$.

Answer

$g(t) = t^2 - 5t$
$g(1) = 1^2 - 5(1) = -4$
$g(-2) = (-2)^2 - 5(-2) = 14$
$g(w+3) = (w+3)^2 - 5(w+3) = w^2 + 6w + 9 - 5w - 15 = w^2 + w - 6$
$g(x+h) = (x+h)^2 - 5(x+h) = x^2 + 2xh + h^2 - 5x - 5h$
$g(x+h) - g(x) = (x^2 + 2xh + h^2 - 5x - 5h) - (x^2 - 5x) = 2xh + h^2 - 5h$
$\frac{g(x+h)-g(x)}{h} = \frac{2xh+h^2-5h}{h} = 2x + h - 5$

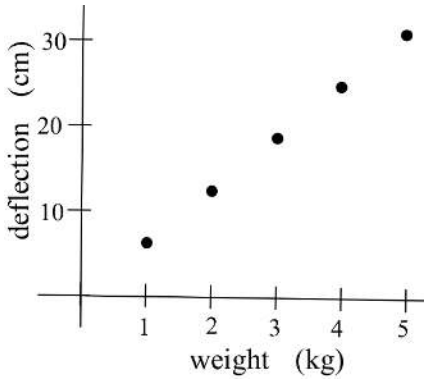
Functions Defined by Graphs and Tables of Values

The **graph** of a numerical function f consists of a plot of ordered pairs (x, y) where x is in the domain of f and $y = f(x)$, such as the graph of $f(x) = \sin(x)$ for $-4 \le x \le 7$ below:

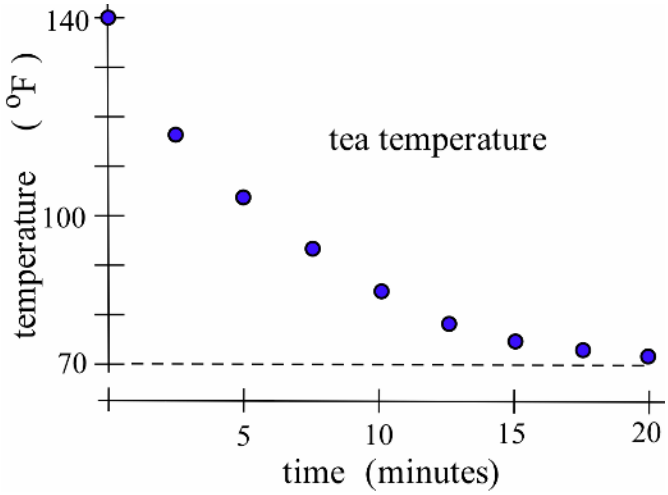


A **table** of values of a numerical function consists of a list of (some of) the ordered pairs (x, y) where $y = f(x)$.

A function can be defined by a graph or by a table of values, and these types of definitions are common in applied fields. The outcome of an experiment will depend on the input, but the experimenter may not know the “rule” for predicting the outcome. In that case, the experimenter usually represents the function of interest as a table of measured outcome values versus input values, or as a graph of the outcomes versus the inputs. The table below shows the deflections obtained when weights were loaded at the end of a wooden stick:



The next graph shows the temperature of a hot cup of tea as a function of the time as it sits in a 68°F room. In these experiments, the “rule” for the function is that $f(\text{input}) =$ actual outcome of the experiment:



Tables have the advantage of presenting the data explicitly, but it is often difficult to detect patterns simply from lists of numbers.

Graphs usually obscure some of the precision of the data, but more easily allow us to detect patterns visually—we can actually see what is happening with the numbers

Creating Graphs of Functions

Most people understand and can interpret pictures more quickly than tables of data or equations, so if we have a function defined by a table of values or by an equation, it is often useful (and necessary) to create a picture of the function: a graph.

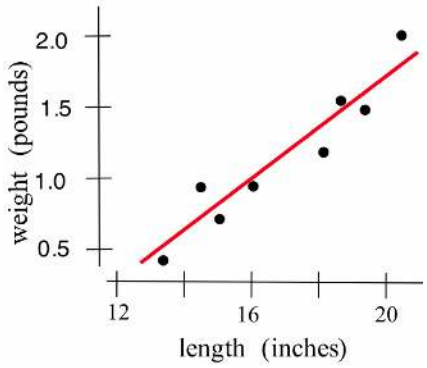
A Graph from a Table of Values

If we have a table of values for a function, perhaps consisting of measurements obtained from an experiment, then we can plot the ordered pairs in the xy-plane to get a graph consisting of a collection of points.

The table below shows the lengths and weights of trout caught (and released) during several days of fishing.

length (in.)	weight (lbs.)
13.5	0.4
14.5	0.9
15.0	0.7
16.0	0.9
18.0	1.2
18.5	1.6
19.5	1.5
20.5	1.7
20.5	2.1

The graph below plots those values along with a line that comes “close” to the plotted points. From the graph, you could estimate that a 17-inch trout should weigh slightly more than one pound.



A Graph from an Equation

Creating the graph of a function given by an equation is similar to creating one from a table of values: we need to plot enough points (x, y) where $y = f(x)$ so we can be confident of the shape and location of the graph of the entire function. We can find a point (x, y) that satisfies $y = f(x)$ by picking a value for x and then calculating the value for y by evaluating $f(x)$. Then we can enter the (x, y) value in a table or simply plot the point (x, y).

If you recognize the form of the equation and know something about the shape of graphs of that form, you may not have to plot many points. If you do not recognize the form of the equation, then you will need to plot more points, maybe 10 or 20 or 234: it depends on how complicated the graph appears and on how important it is to you (or your boss) to have an accurate graph. Evaluating $y = f(x)$ at many different values for x and then plotting the points (x, y) is usually not very difficult, but it can be very time-consuming. Fortunately, calculators and computers can often do the evaluation and plotting for you.

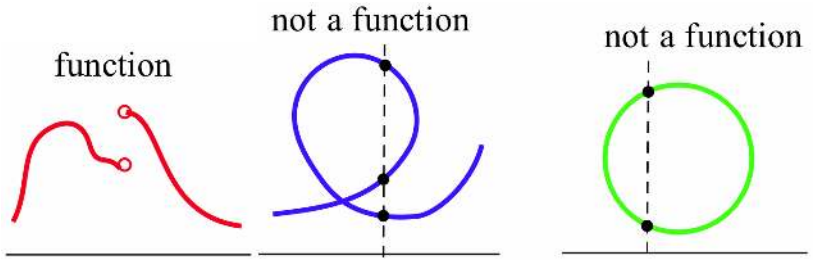
Is Every Graph the Graph of a Function?

The definition of “function” requires that each element of the domain (each input value) be sent by the function to exactly one element of the range (to exactly one output value), so for each input x-value there will be exactly one output y-value, $y = f(x)$. The points (x, y_1) and (x, y_2) cannot both be on the graph of f unless $y_1 = y_2$. The graphical interpretation of this result is called the {\bf Vertical Line Test}.

Vertical Line Test for a Function

A graph is the graph of a function if and only if a vertical line drawn through any point in the domain intersects the graph at exactly one point.

The figure below shows the graph of a function, followed by two graphs that are not graphs of functions, along with vertical lines that intersect those graphs at more than one point. Non-functions are not "bad," and are often necessary to describe complicated phenomena.

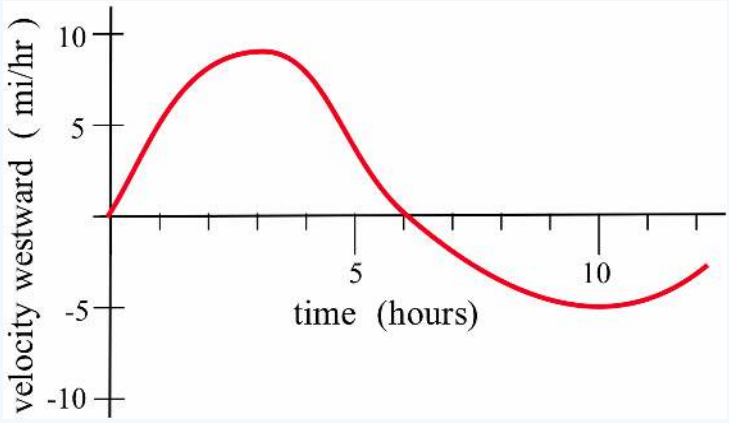


Reading Graphs Carefully

Calculators and computers can help students, reporters, financial analysts and scientists create graphs quickly and easily. Because of this, graphs are being used more often than ever to present information and justify arguments. This text takes a distinctly graphical approach to the ideas and meaning of calculus. Calculators and computers can help us create graphs, but we need to be able to read them carefully. The next examples illustrate some types of information that can be obtained by carefully reading and understanding graphs.

Example 0.3.1

A boat starts from St. Thomas and sails due west with the velocity shown in the figure below:



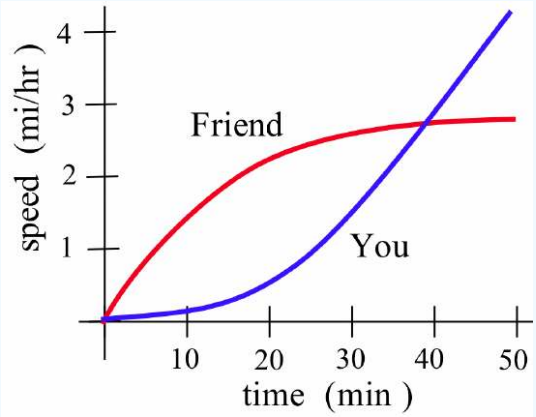
a. When is the boat traveling the fastest?
b. What does a negative velocity away from St. Thomas mean?
c. When is the boat the farthest away from St. Thomas?

Solution

a. The greatest speed is 10 mph at $t = 3$ hours.
b. It means that the boat is heading back toward St. Thomas.
c. The boat is farthest from St. Thomas at $t = 6$ hours. For $t < 6$ the boat's velocity is positive, and the distance from the boat to St. Thomas is increasing. For $t > 6$ the boat's velocity is negative, and the distance from the boat to St. Thomas is decreasing.

Practice 0.3.3

You and a friend start out together and hike along the same trail but walk at different speeds, as shown in the figure:



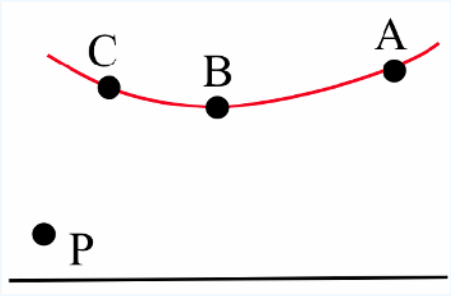
a. Who is walking faster at $t = 20$?
b. Who is ahead at $t = 20$?
c. When are you and your friend farthest apart?
d. Who is ahead when $t = 50$?

Answer

a. Friend
b. Friend
c. At $t = 40$. Before that, your friend is walking faster and increasing the distance between you. Then you start to walk faster than your friend and start to catch up.
d. Friend. You are walking faster than your friend at $t = 50$, but you still have not caught up.

Example 0.3.2

Which has the largest slope in the figure below: the line through the points A and P, the line through B and P, or the line through C and P?

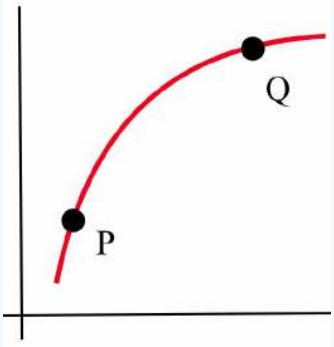


Solution

The line through C and P: $m_{PC} > m_{PB} > m_{PA}$.

Pratice 0.3.4

In the figure below, the point Q on the curve is fixed, and the point P is moving to the right along the curve toward the point Q. As P moves toward Q, is the indicated value increasing, decreasing, remaining constant, or doing something else?



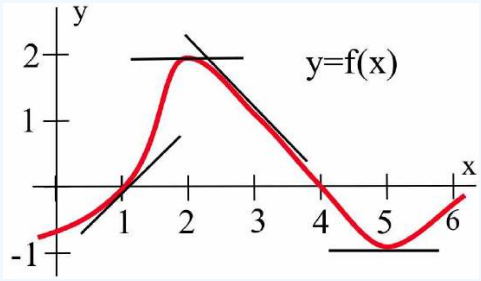
a. x-coordinate of P
b. x-increment from P to Q
c. slope from P to Q

Answer

a. The x-coordinate is increasing.
b. The x-increment Δx is decreasing.
c. The slope of the line through P and Q is decreasing.

Example 0.3.3

The graph of $y = f(x)$ appears below:



Let $g(x)$ be the **slope** of the line tangent to the graph of $f(x)$ at the point $(x, f(x))$.

a. Estimate the values $g(1)$, $g(2)$ and $g(3)$.
b. For what value(s) of x is $g(x) = 0$?
c. At what value(s) of x is $g(x)$ largest?
d. Sketch the graph of $y = g(x)$.

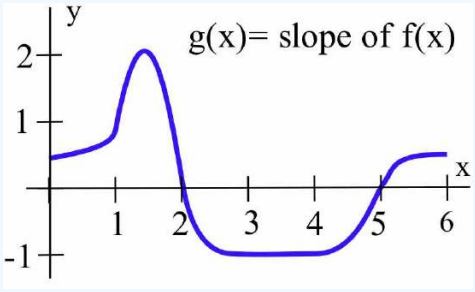
Solution

a. The figure shows the graph of $y = f(x)$ with several tangent lines to the graph of f. From this graph, we can estimate that $g(1)$ (the slope of the line tangent to the graph of f at $(1, 0)$) is approximately equal to 1. Similarly, $g(2) \approx 0$ and $g(3) \approx -1$.
b. The slope of the tangent line appears to be horizontal (slope $= 0$) at $x = 2$ and at $x = 5$.
c. The tangent line to the graph of f appears to have greatest slope (be steepest) near $x = 1.5$.
d. We can build a table of values of $g(x)$:

x	$f(x)$	$g(x)$
0	-1.0	0.5
1	0.0	1.0
2	2.0	0.0
3	1.0	-1.0
4	0.0	-1.0
5	-1.0	0.0
6	-0.5	0.5

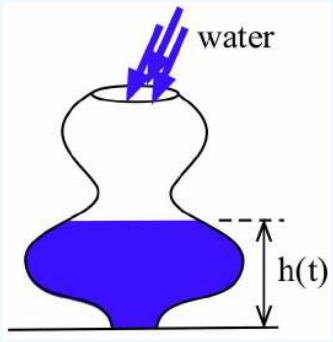
and then sketch the graph of these values. A graph of $y = g(x)$ appears below:

e.



Practice 0.3.5

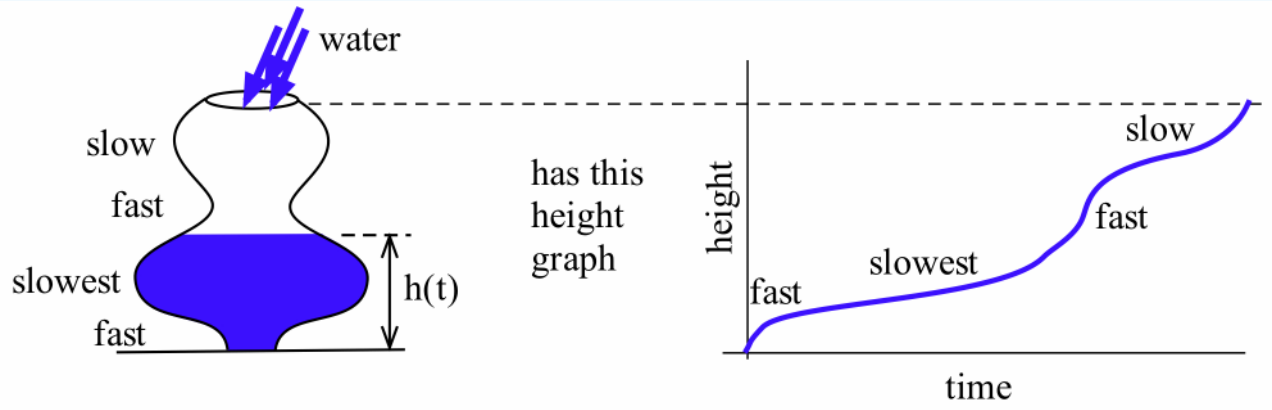
Water flows into a container at a constant rate of 3 gallons per minute into the container shown below:



Starting with an empty container, sketch the graph of the height of the water in the container as a function of time.

Answer

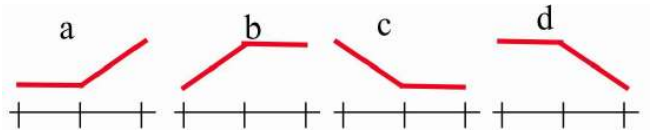
See the figure:



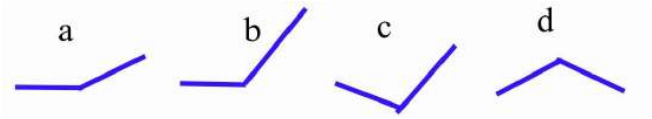
Problems

In Problems 1–4, match the numerical triples to the graphs. For example, in Problem 1, A: 3, 3, 6 is "over and up" so it matches graph (a).

1. A: 3, 3, 6; B: 12, 6, 6; C: 7, 7, 3 D: 2, 4, 4



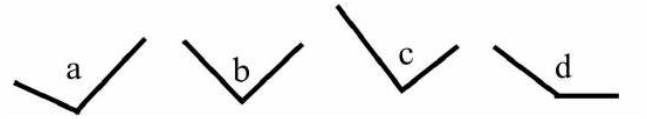
2. A: 7, 10, 7; B: 17, 17, 25; C: 4, 4, 8 D: 12, 8, 16



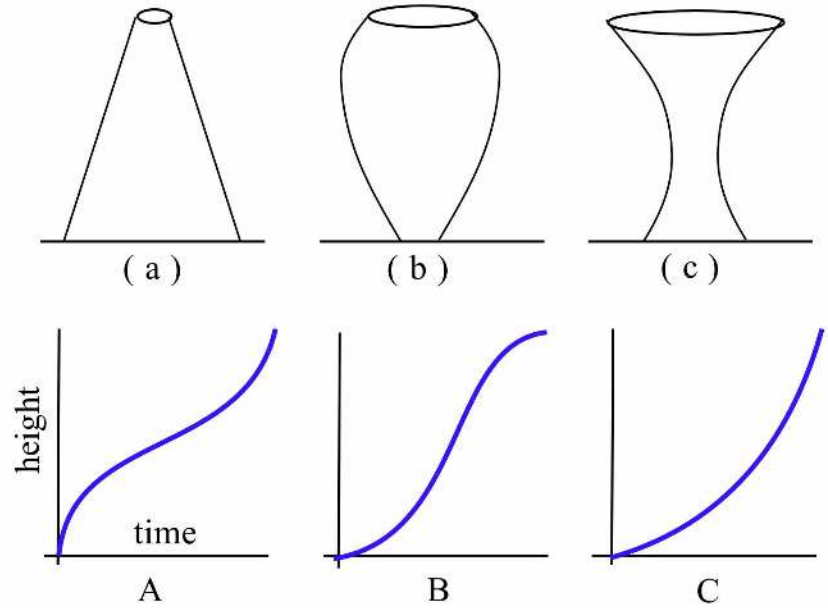
3. A: 7, 14, 10; B: 23, 45, 22; C: 8, 12, 8 D: 6, 9, 3



4. A: 6, 3, 9; B: 8, 1, 1; C: 12, 6, 9 D: 3.7, 1.9, 3.6

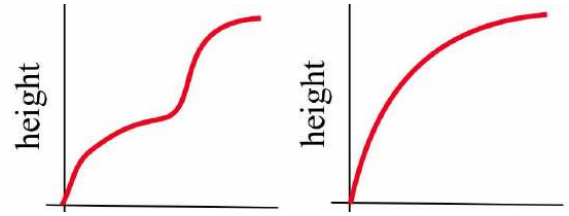


5. Water is flowing at a steady rate into each of the bottles shown below:



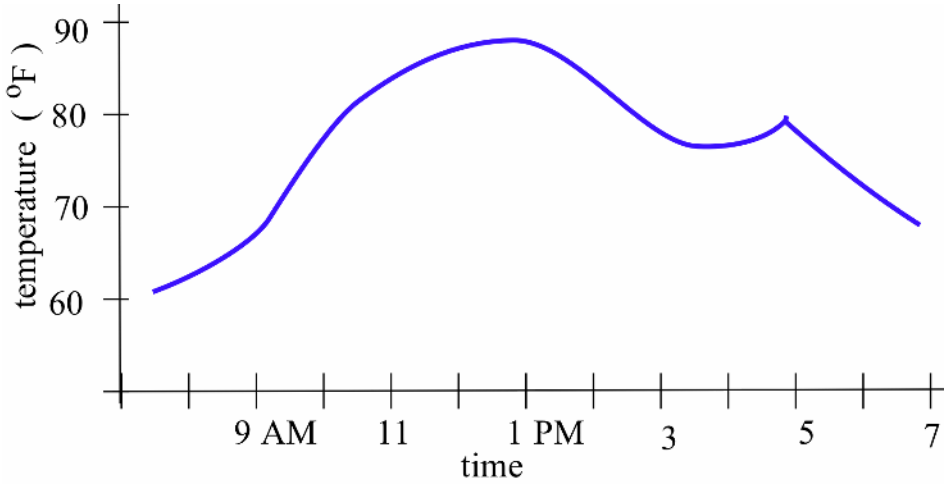
Match each bottle shape with the graph of the height of the water as a function of time

6. Sketch shapes of bottles that will have the water height versus time graphs shown below:



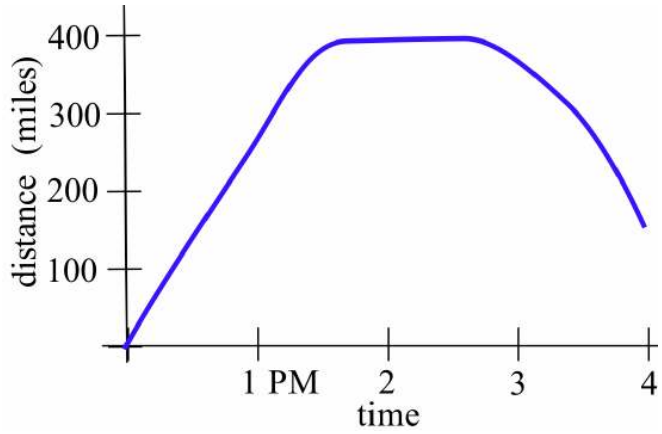
7. If $f(x) = x^2 + 3$, $g(x) = \sqrt{x-5}$ and $H(x) = \dfrac{x}{x-2}$:
   a. evaulate $f(1)$, $g(1)$ and $H(1)$.
   b. graph $f(x)$, $g(x)$ and $H(x)$ for $-5 \le x \le 10$.
   c. evaluate $f(3x)$, $g(3x)$ and $H(3x)$.
   d. evaluate $f(x+h)$, $g(x+h)$ and $H(x+h)$.
8. Find the slope of the line through the points P and Q when:
   a. $P = (1,3)$, $Q = (2,7)$
   b. $P = (x, x^2+2)$, $Q = (x+h, (x+h)^2+2)$
   c. $P = (1,3)$, $Q = (x, x^2+2)$
   d. P, Q as in (b) with $x = 2$, $x = 1.1$, $x = 1.002$
9. Find the slope of the line through the points P and Q when:
   a. $P = (1,5)$, $Q = (2,7)$
   b. $P = (x, x^2+3x-1)$, $Q = (x+h, (x+h)^2+3(x+h)-1)$
   c. P, Q as in (b) with $x = 1.3$, $x = 1.1$, $x = 1.002$
10. If $f(x) = x^2 + x$ and $g(x) = \frac{3}{x}$, evaluate and simplify $\dfrac{f(a+h)-f(a)}{h}$ and $\dfrac{g(a+h)-g(a)}{h}$ when $a = 1$, $a = 2$, $a = -1$, $a = x$.
11. If $f(x) = x^2 - 2x$ and $g(x) = \sqrt{x}$, evaluate and simplify $\dfrac{f(a+h)-f(a)}{h}$ and $\dfrac{g(a+h)-g(a)}{h}$ when $a = 1$, $a = 2$, $a = 3$, $a = x$.

12. The temperatures shown below were recorded during a 12-hour period in Chicago:



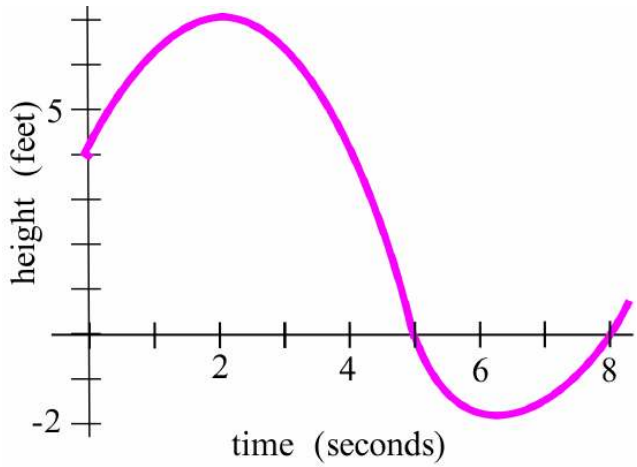
a. At what time was the temperature the highest? Lowest?
b. How fast was the temperature rising at 10 a.m.? At 1 p.m.?
c. What could have caused the drop in temperature between 1 p.m. and 3 p.m.?

13. The graph below shows the distance of an airplane from an airport during a long flight:



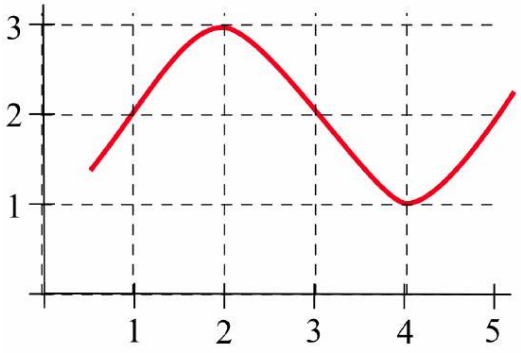
a. How far was the airplane from the airport at 1 p.m.? At 2 p.m.?
b. How fast was the distance changing at 1 p.m.?
c. How could the distance from the plane to the airport remain unchanged from 1:45 p.m. until 2:30 p.m. without the airplane falling?

14. The graph below shows the height of a diver above the water level at time t seconds:



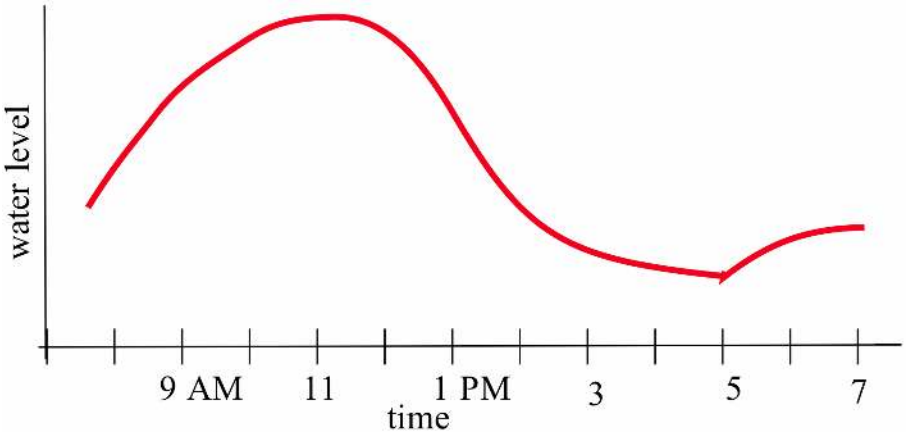
a. What was the height of the diving board?
b. When did the diver hit the water?
c. How deep did the diver get?
d. When did the diver return to the surface?

15. Refer to the curve shown below:



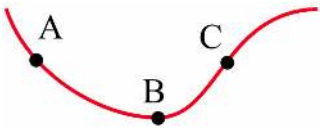
a. Sketch the lines tangent to the curve at $x = 1, 2, 3, 4$ and 5.
b. For what value(s) of x is the value of the function largest? Smallest?
c. For what value(s) of x is the slope of the tangent line largest? Smallest?

16. The figure below shows the height of the water (above and below mean sea level) at a Maine beach:



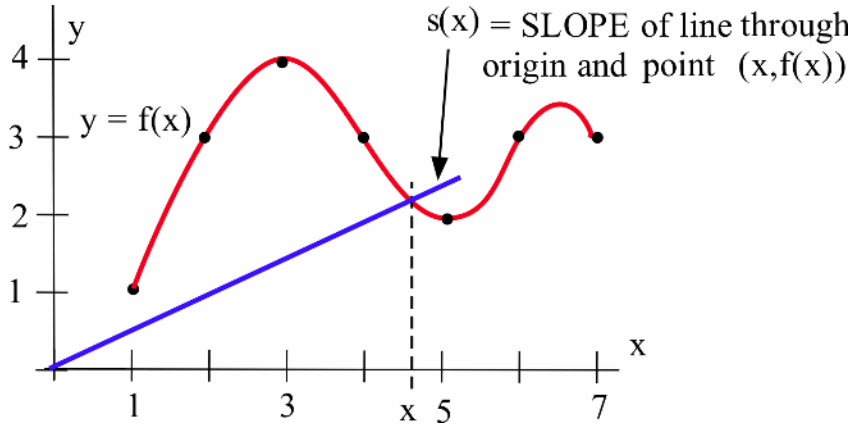
a. At which time(s) was the most beach exposed? The least?
b. At which time(s) was the current the strongest?

17. Imagine that you are ice skating, **from left to right**, along the path shown below:



Sketch the path you will follow if you fall at points A, B and C.

18. Define $s(x)$ to be the {\bf slope} of the line through the points $(0,0)$ and $(x, f(x))$ where $f(x)$ is the function graphed below:

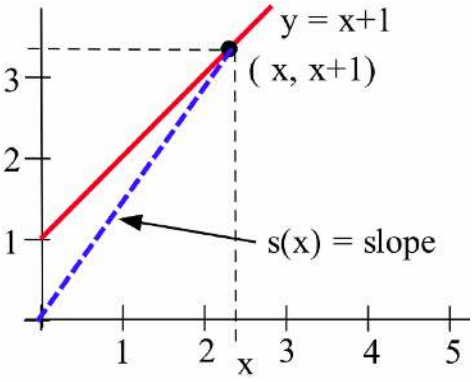


For example, $s(3) =$ {\bf slope} of the line through $(0,0)$ and $(3, f(3))$ = $\frac{4}{3}$.

a. Evaluate $s(1)$, $s(2)$ and $s(4)$.

b. For which integer value of x between 1 and 7 is $s(x)$ smallest?

19. Let $f(x) = x + 1$ and define $s(x)$ to be the {\bf slope} of the line through the points $(0,0)$ and $(x, f(x))$, as shown below:

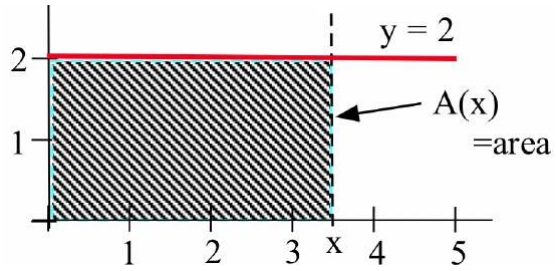


For example, $s(2) =$ slope of the line through $(0,0)$ and $(2,3)$ = $\frac{3}{2}$.

a. Evaluate $s(1)$, $s(3)$ and $s(4)$.

b. Find a formula for $s(x)$.

20. Define $A(x)$ to be the area of the rectangle bounded by the coordinate axes, the line $y = 2$ and a vertical line at x, as shown below:

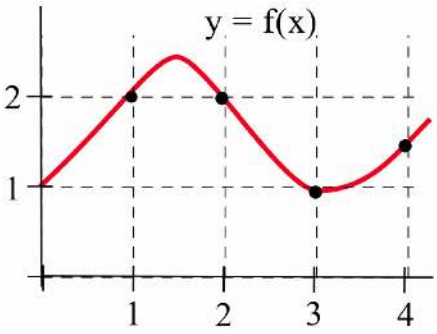


For example, $A(3) =$ area of a 2×3 rectangle $= 6$.

a. Evaluate $A(1)$, $A(2)$ and $A(5)$.

b. Find a formula for $A(x)$.

21. Using the graph of $y = f(x)$ below:

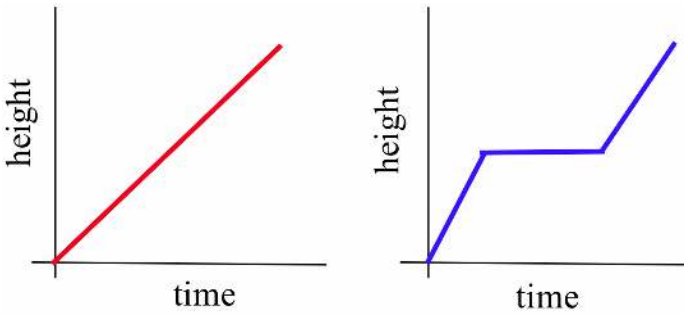


let $g(x)$ be the slope of the line tangent to the graph of $f(x)$ at the point $(x, f(x))$. Complete the table, estimating values of the slopes as best you can.

x	$f(x)$	$g(x)$
0	1	1
1		
2		
3		
4		

22. Sketch the graphs of water height versus time for water pouring into a bottle shaped like:
    a. a milk carton
    b. a spherical glass vase
    c. an oil drum (cylinder) lying on its side
    d. a giraffe \item you

23. Design bottles whose graphs of (highest) water height versus time will look like those shown below:



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0.4: Combinations of Functions

Sometimes a physical or economic situation behaves differently depending on various circumstances. In these situations, a more complicated formula may be needed to describe the situation.

Multiline Definitions of Functions: Putting Pieces Together

Sales Tax: Some states have different rates of sales tax depending on the type of item purchased. As an example, for many years food purchased at restaurants in Seattle was taxed at a rate of 10%, while most other items were taxed at a rate of 9.5% and food purchased at grocery stores had no tax assessed. We can describe this situation by using a **multiline** function: a function whose defining rule consists of several pieces. Which piece of the rule we need to use will depend on what we buy. In this example, we could define the tax T on an item that costs x to be:

$$T(x) = \begin{cases} 0 & \text{if } x \text{ if is the cost of a food at a grocery store} \\ 0.10x & \text{if } x \text{ is the cost of food at a restaurant} \\ 0.095x & \text{if } x \text{ is the cost of any other item} \end{cases}$$

To find the tax on a \$2 can of stew, we would use the first piece of the rule and find that the tax is \$0. To find the tax on a \$30 restaurant bill, we would use the second piece of the rule and find that the tax is \$3.00. The tax on a \$150 textbook requires using the third rule: the tax would be \$14.25.

Wind Chill Index: The rate at which a person's body loses heat depends on the temperature of the surrounding air and on the speed of the air. You lose heat more quickly on a windy day than you do on a day with little or no wind. Scientists have experimentally determined this rate of heat loss as a function of temperature and wind speed, and the resulting function is called the Wind Chill Index, WCI. The WCI is the temperature on a still day (no wind) at which your body would lose heat at the same rate as on the windy day. For example, the WCI value for $30°$F air moving at 15 miles per hour is $9°$F: your body loses heat as quickly on a $30°$F day with a 15 mph wind as it does on a $9°$F day with no wind.

If T is the Fahrenheit temperature of the air and v is the speed of the wind in miles per hour, then the WCI can be expressed as a multiline function of the wind speed v (and of the temperature T):

$$\text{WCI} = \begin{cases} T & \text{if } 0 \le v \le 4 \\ 91.4 - \frac{10.45 + 6.69\sqrt{v} - 0.447v}{22}(91.5 - T) & \text{if } 4 < v \le 45 \\ 1.60T - 55 & \text{if } v > 45 \end{cases} \tag{0.4.1}$$

The WCI value for a still day ($0 \le v \le 4$ mph) is just the air temperature. The WCI for wind speeds above 45 mph are the same as the WCI for a wind speed of 45 mph. The WCI for wind speeds between 4 mph and 45 mph decrease as the wind speeds increase. This WCI function depends on two variables: the temperature and the wind speed; but if the temperature is constant, then the resulting formula for WCI only depends on the wind speed. If the air temperature is $30°$F ($T = 30$), then the formula for the Wind Chill Index is:

$$\text{WCI}_{30} = \begin{cases} 30° & \text{if } 0 \le v \le 4 \text{ mph} \\ 62.19 - 18.70\sqrt{v} + 1.25v & \text{if } 4 \text{ mph} < v \le 45 \text{ mph} \\ -7° & \text{if } v > 45 \text{ mph} \end{cases} \tag{0.4.2}$$

The WCI graphs for temperatures of $40°$F, $30°$F and $20°$F appear below:

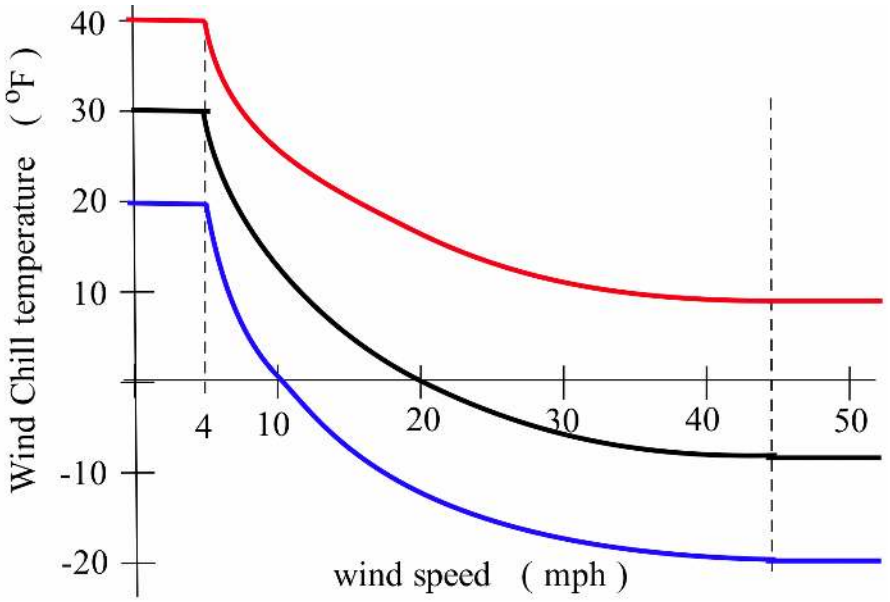


Figure 0.4.*i*: From UMAP Module 658, "Windchill," by William Bosch and L.G.\ Cobb, 1984.

? Practice 0.4.1

A Hawaiian condo rents for $380 per night during the tourist season (from December 15 through April 30), and for $295 per night otherwise. Define a multiline function that describes these rates.

Answer

$C(x)$ is the cost for one night on date x: $C(x) = \begin{cases} \$380 & \text{if } x \text{ is between December 15 and April 30} \\ \$295 & \text{if } x \text{ is any other date} \end{cases}$

✓ Example 0.4.1

Define $f(x)$ by:

$$f(x) = \begin{cases} 2 & \text{if } x < 0 \\ 2x & \text{if } 0 \le x < 2 \\ 1 & \text{if } 2 < x \end{cases}$$

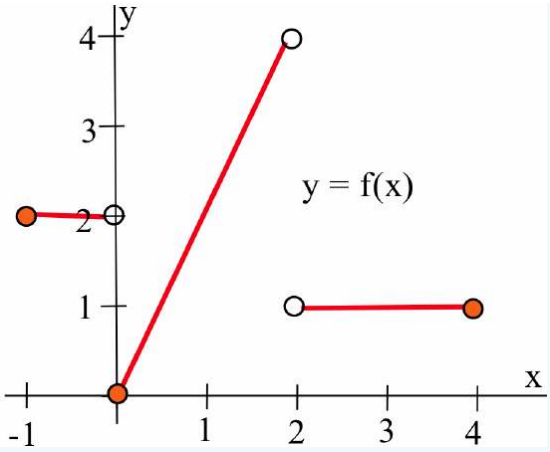
Evaluate $f(-3), f(0), f(1), f(4)$ and $f(2)$. Graph $y = f(x)$ on the interval $-1 \le x \le 4$.

Solution

To evaluate the function at different values of x, we must first decide which line of the rule applies. If $x = -3 < 0$, then we need to use the first line, so $f(-3) = 2$.

When $x = 0$ or $x = 1$, we need the second line of the function definition, so $f(0) = 2(0) = 0$ and $f(1) = 2(1) = 2$.

At $x = 4$ we need the third line, so $f(4) = 1$. Finally, at $x = 2$, none of the lines apply: the second line requires $x < 2$ and the third line requires $2 < x$, so $f(2)$ is undefined. The graph of $f(x)$ appears below:



Note the "hole" above $x = 2$, which indicates $f(2)$ is not defined by the rule for f.

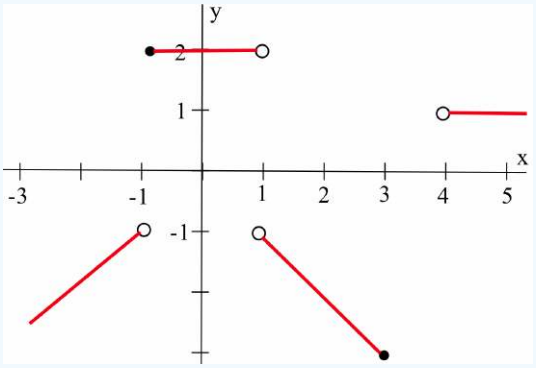
Practice 0.4.2

Define $g(x)$ by: $g(x) = \begin{cases} x & \text{if } x < -1 \\ 2 & \text{if } -1 \le x < 1 \\ -x & \text{if } 1 < x \end{cases}$

Graph $y = g(x)$ for $-3 \le x \le 6$) and evaluate $g(-3)$, $g(-1)$, $g(0)$, $g(\frac{1}{2})$, $g(1)$, $g(\frac{\pi}{3})$, $g(2)$, $g(3)$, $g(4)$ and $g(5)$.

Answer

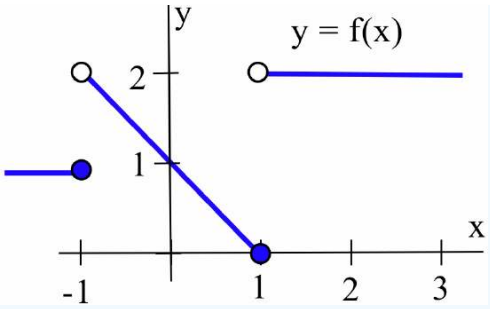
A graph and table appear below:



x	$g(x)$		x	$g(x)$
-3	-3		$\frac{\pi}{3}$	$-\frac{\pi}{3}$
-1	2		2	-2
0	2		3	-3
$\frac{1}{2}$	2		4	undefined
1	undefined		5	1

Exercise 0.4.3

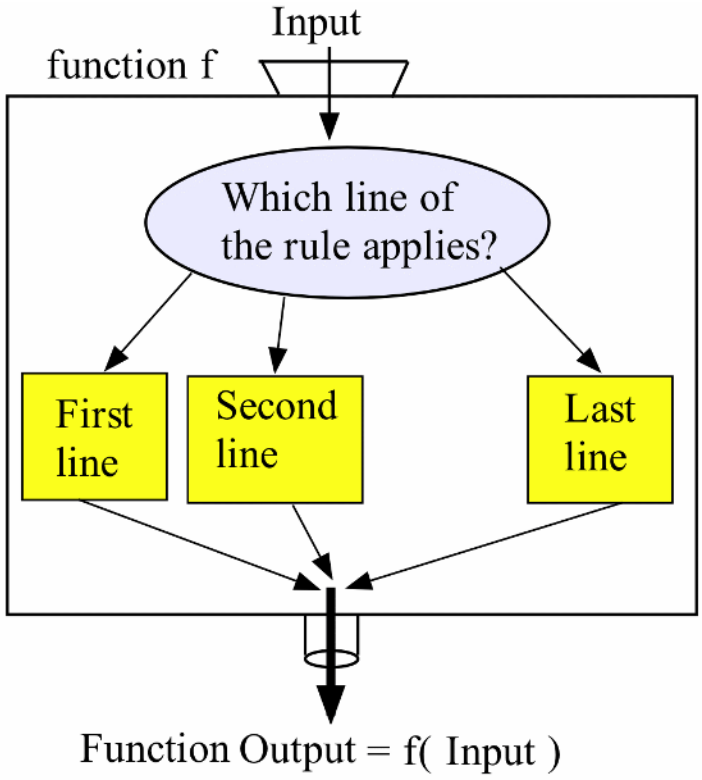
Write a multiline definition for the function whose graph appears below:



Answer

Define $f(x)$ as: $f(x) = \begin{cases} 1 & \text{if } x \leq -1 \\ 1-x & \text{if } -1 < x \leq 1 \\ 2 & \text{if } 1 < x \end{cases}$

We can think of a multiline function as a machine that first examines the input value to decide which line of the function rule to apply:



Composition of Functions: Functions of Functions

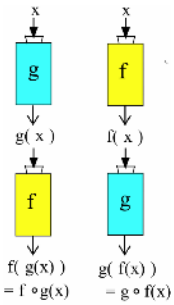
Basic functions are often combined with each other to describe more complicated situations. Here we will consider the composition of functions—functions of functions.

The **composite** of two functions f and g, written $f \circ g$, is: $f \circ g(x) = f(g(x))$

The domain of the composite function $f \circ g(x) = f(g(x))$ consists of those x-values for which $g(x)$ and $f(g(x))$ are both defined: we can evaluate the composition of two functions at a point x only if each step in the composition is defined.

If we think of our functions as machines, then composition is simply a new machine consisting of an arrangement of the original machines. The composition $f \circ g$ of the function machines f and g shown below left is an arrangement of the machines so that the

original input x goes into machine g, the output from machine g becomes the input into machine f, and the output from machine f is our final output. The composition of the function machines $f \circ g(x) = f(g(x))$ is only valid if x is an allowable input into g (that is, x is in the domain of g) and if $g(x)$ is then an allowable input into f.



The composition $g \circ f$ involves arranging the machines so the original input goes into f, and the output from f then becomes the input for g (shown above right).

Example 0.4.2

For $f(x) = \sqrt{x-2}$, $g(x) = x^2$ and $h(x) = \begin{cases} 3x & \text{if } x < 2 \\ x-1 & \text{if } 2 \le x \end{cases}$ evaluate $f \circ g(3)$, $g \circ f(6)$, $f \circ h(2)$ and $h \circ g(-3)$.

Find the formulas and domains of $f \circ g(x)$ and $g \circ f(x)$.

Solution

$f \circ g(3) = f(g(3)) = f(3^2) = f(9) = \sqrt{9-2} = \sqrt{7} \approx 2.646$; $g \circ f(6) = g(f(6)) = g(\sqrt{6-2}) = g(\sqrt{4}) = g(2) = 2^2 = 4$; $f \circ h(2) = f(h(2)) = f(2-1) = f(1) = \sqrt{1-2} = \sqrt{-1}$, which is undefined; $h \circ g(-3) = h(g(-3)) = h(9) = 9 - 1 = 8$; $f \circ g(x) = f(g(x)) = f(x^2) = \sqrt{x^2-2}$, and the domain of $f \circ g$ consists of those x-values for which $x^2 - 2 \ge 0$, so the domain of $f \circ g$ is all x such that $x \ge \sqrt{2}$ or $x \le -\sqrt{2}$; $g \circ f(x) = g(f(x)) = g(\sqrt{x-2}) = \left(\sqrt{x-2}\right)^2 = x - 2$, but this last equality is true only when $x - 2 \ge 0 \Rightarrow x \ge 2$, so the domain of $g \circ f$ is all $x \ge 2$.

Practice 0.4.4

For $f(x) = \frac{x}{x-3}$, $g(x) = \sqrt{1+x}$ and $h(x) = \begin{cases} 2x & \text{if } x \le 1 \\ 5-x & \text{if } 1 < x \end{cases}$ evaluate $f \circ g(3)$, $f \circ g(8)$, $g \circ f(4)$, $f \circ h(1)$, $f \circ h(3)$, $f \circ h(2)$ and $h \circ g(-1)$.

Find formulas for $f \circ g(x)$ and $g \circ f(x)$.

Answer

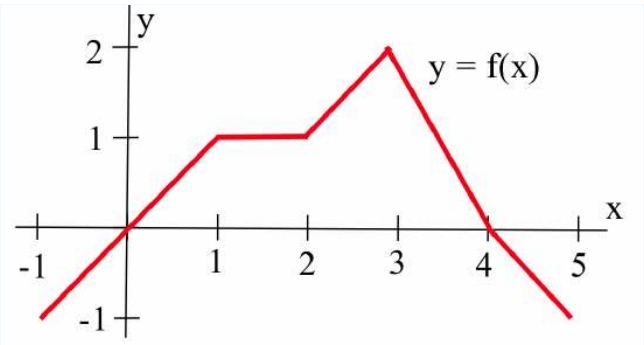
$f \circ g(3) = f(2) = \frac{2}{-1} = -2$; $f \circ g(8) = f(3)$ is undefined; $g \circ f(4) = g(4) = 5$; $f \circ h(1) = f(2) = \frac{2}{-1} = -2$; $f \circ h(3) = f(2) = -2$; $f \circ h(2) = f(3)$ is undefined; $h \circ g(-1) = h(0) = 0$; $f \circ g(x) = f(\sqrt{1+x}) = \frac{1+x}{\sqrt{1+x}-3}$, $g \circ f(x) = g\left(\frac{x}{x-3}\right) = \sqrt{1+\frac{x}{x-3}}$

Shifting and Stretching Graphs

Some common compositions are fairly straightforward; you should recognize the effect of these compositions on graphs of the functions.

Example 0.4.3

The figure below shows the graph of $y = f(x)$:

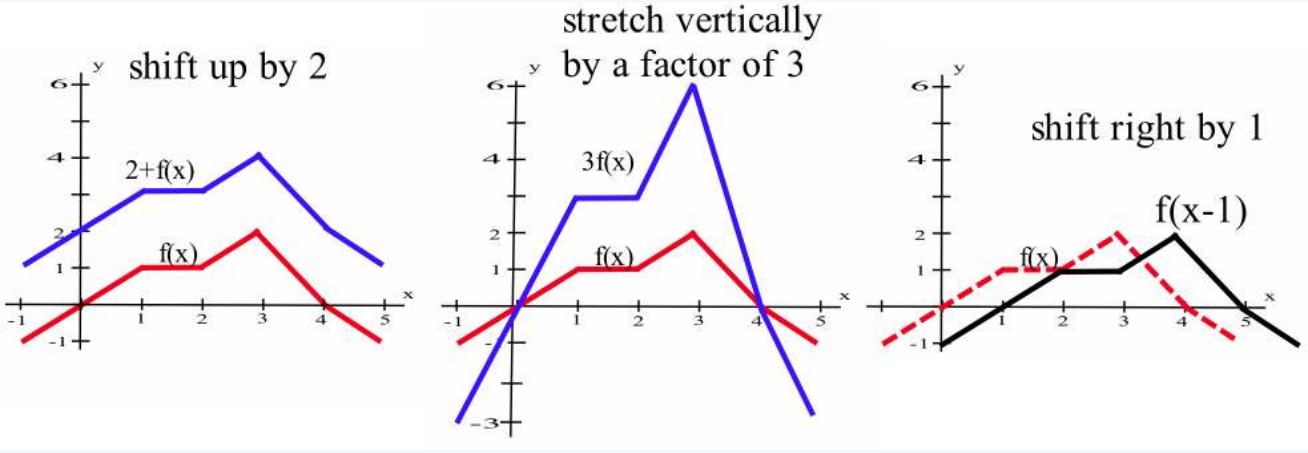


Graph:

a. $2 + f(x)$
b. $3 \cdot f(x)$
c. $f(x-1)$

Solution

All of the new graphs appear below:



a. Adding 2 to all of the values of $f(x)$ rigidly shifts the graph of $f(x)$ upward 2 units.
b. Multiplying all of the values of $f(x)$ by 3 leaves all of the roots (zeros) of f fixed: if x is a root of f then $f(x) = 0 \Rightarrow 3 \cdot f(x) = 3(0) = 0$ so x is also a root of $3 \cdot f(x)$. If x is not a root of f, then the graph of $3f(x)$ looks like the graph of $f(x)$ stretched vertically by a factor of \(3).
c. The graph of $f(x-1)$ is the graph of $f(x)$ rigidly shifted 1 unit to the right.

We could also get these results by examining the graph of $y = f(x)$, creating a table of values for $f(x)$ and the new functions:

x	$f(x)$	$2 + f(x)$	$3f(x)$	$x - 1$	$f(x-1)$
-1	-1	1	-3	-2	$f(-2)$ not defined
0	0	2	0	-1	$f(0-1) = -1$
1	1	3	3	0	$f(1-1) = 0$
2	1	3	3	1	$f(2-1) = 1$
3	2	4	6	2	$f(3-1) = 1$
4	0	2	0	3	$f(4-1) = 2$
5	-1	1	-3	4	$f(5-1) = 0$

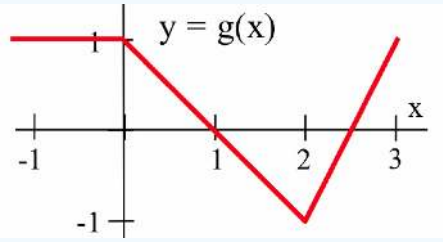
and then graphing those new functions.

If k is a positive constant, then:

- the graph of $y = k + f(x)$ will be the graph of $y = f(x)$ rigidly shifted up by k units
- the graph of $y = kf(x)$ will have the same roots as the graph of $f(x)$ and will be the graph of $y = f(x)$ vertically stretched by a factor of k
- the graph of $y = f(x - k)$ will be the graph of $y = f(x)$ rigidly shifted right by k units
- the graph of $y = f(x + k)$ will be the graph of $y = f(x)$ rigidly shifted left by k units

Practice 0.4.5

The figure below shows the graph of $y = g(x)$:

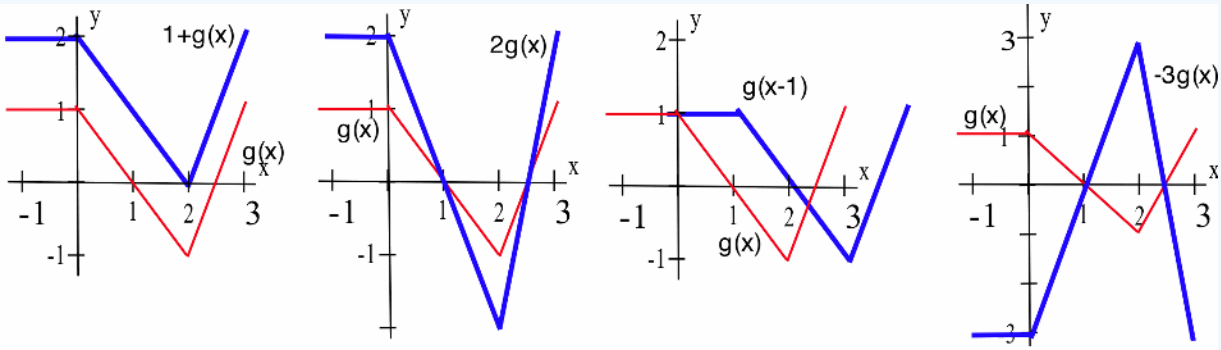


Graph:

a. $1 + g(x)$
b. $2 \cdot g(x)$
c. $g(x - 1)$
d. $-3 \cdot g(x)$

Answer

See the figure below:



Iteration of Functions

Certain applications feed the output from a function machine back into the same machine as the new input. Each time through the machine is called an **iteration** of the function.

Example 0.4.4

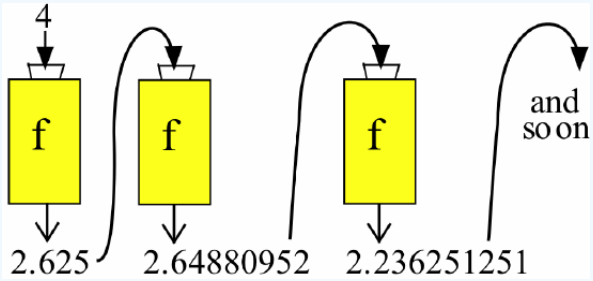
Suppose $f(x) = \frac{\frac{5}{x} + x}{2}$ and we start with the input $x = 4$ and repeatedly feed the output from f back into f. What happens?

Solution

Creating a table:

iteration	input	output
1	4	$f(4) = \frac{\frac{5}{4}+4}{2} = 2.625000000$
2	2.625000000	$f(f(4)) = \frac{\frac{5}{2.625}+2.625}{2} = 2.264880952$
3	2.264880952	$f(f(f(4))) = 2.236251251$
4	2.236251251	2.236067985
5	2.236067985	2.236067977
6	2.236067977	2.236067977

Once we have obtained the output **2.236067977**, we will just keep getting the same output (to 9 decimal places). You might recognize this output value as an approximation of $\sqrt{5}$.



This algorithm always finds $\pm\sqrt{5}$. If we start with any positive input, the values will eventually get as close to $\sqrt{5}$ as we want. Starting with any negative value for the input will eventually get us close to $-\sqrt{5}$. We cannot start with $x = 0$, as $\frac{5}{0}$ is undefined.

? Practice 0.4.6

What happens if we start with the input value $x = 1$ and iterate the function $f(x) = \frac{\frac{9}{x}+x}{2}$ several times? Do you recognize the resulting number? What do you think will happen to the iterates of $g(x) = \frac{\frac{A}{x}+x}{2}$? (Try several positive values of A.)

Answer

Using $f(x) = \frac{\frac{9}{x}+x}{2}$, $f(1) = \frac{\frac{9}{1}+1}{2} = 5$, $f(5) = \frac{\frac{9}{5}+5}{2} = 3.4$, $f(3.4) \approx 3.023529412$ and $f(3.023529412) \approx 3.000091554$. The next iteration gives $f(3.000091554) \approx 3.000000001$: these values are approaching 3, the square root of 9.

With $A = 6$, $f(x) = \frac{\frac{6}{x}+x}{2}$, so $f(1) = \frac{\frac{6}{1}+1}{2} = 3.5$, $f(3.5) = \frac{\frac{6}{3.5}+3.5}{2} = 2.607142857$, and the next iteration gives $f(2.607142857) \approx 2.45425636$. Then $f(2.45425636) \approx 2.449494372$, $f(2.449494372) \approx 2.449489743$ and $f(2.449489743) \approx 2.449489743$ (the output is the same as the input to 9 decimal places): these values are approaching 2.449489743, an approximation of $\sqrt{6}$.

For any positive value A, the iterates of $f(x) = \frac{\frac{A}{x}+x}{2}$ (starting with any positive x) will approach $\sqrt{A}$.

Two Useful Functions: Absolute Value and Greatest Integer

Two functions (one of which should be familiar to you, the other perhaps not) possess useful properties that let us describe situations in which an object abruptly changes direction or jumps from one value to another value. Their graphs will have corners and breaks, respectively.

The **absolute value function** evaluated at a number x, $y = f(x) = |x|$, is the distance between the number x and 0.

Note

Some calculators and computer programming languages represent the absolute value function by $\mathbf{abs}(x)$ or $\mathbf{ABS}(x)$.

If x is greater than or equal to 0, then $|x|$ is simply $x - 0 = x$. If x is negative, then $|x|$ is $0 - x = -x = -1 \cdot x$, which is positive because:

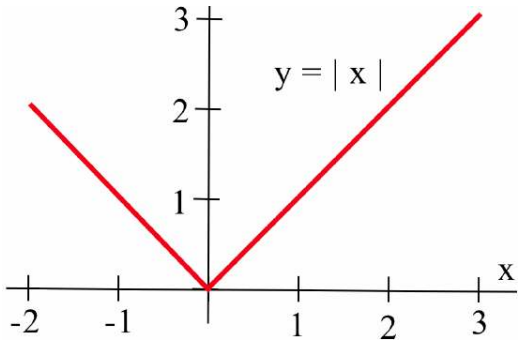
$$-1 \cdot (\text{negative number}) = \text{a positive number} \tag{0.4.3}$$

Definition: $|x|$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We can also write: $|x| = \sqrt{x^2}$.

The domain of $y = f(x) = |x|$ consists of all real numbers. The range of $f(x) = |x|$ consists of all numbers larger than or equal to zero (all non-negative numbers). The graph of $y = f(x) = |x|$ has no holes or breaks, but it does have a sharp corner at $x = 0$:



The absolute value will be useful for describing phenomena such as reflected light and bouncing balls that change direction abruptly or whose graphs have corners. The absolute value function has a number of properties we will use later.

Properties of $|x|$

For all real numbers a and b:

- $|a| = 0 \cdot |a| = 0$ if and only if $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a + b| \leq |a| + |b|$

This last property is widely known as the **triangle inequality**.

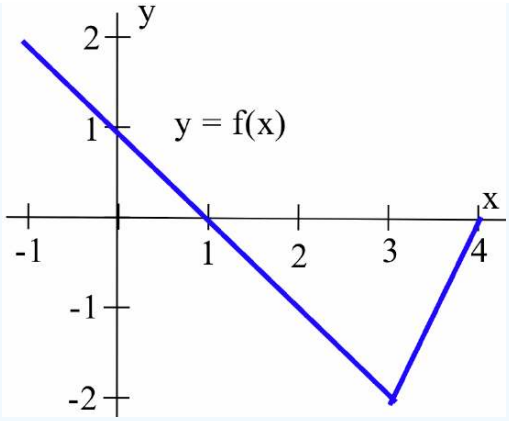
Taking the absolute value of a function has an interesting effect on the graph of the function: for any function $f(x)$, we have:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

In other words, if $f(x) \geq 0$, then $|f(x)| = f(x)$, so the graph of $|f(x)|$ is the same as the graph of $f(x)$. If $f(x) < 0$, then $|f(x)| = -f(x)$, so the graph of $|f(x)|$ is just the graph of $f(x)$ "flipped" about the x-axis, and it lies above the x-axis. The graph of $|f(x)|$ will always be on or above the x-axis.

Example 0.4.5

The figure below shows the graph of $f(x)$:

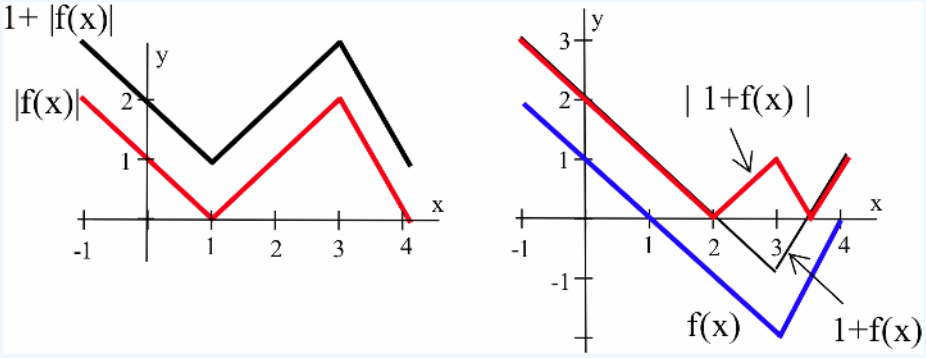


Graph:

a. $|f(x)|$
b. $|1+f(x)|$
c. $1+|f(x)|$

Solution

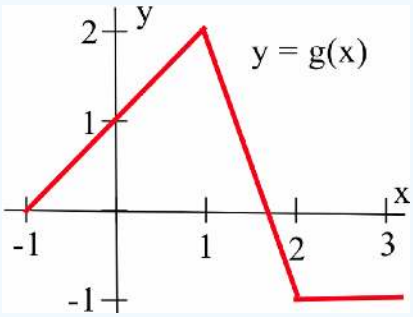
The graphs appear below:



In (b), we rigidly shift the graph of f up 1 unit before taking the absolute value. In (c), we take the absolute value before rigidly shifting the graph up 1 unit.

? Practice 0.4.7

The figure below shows the graph of $g(x)$:



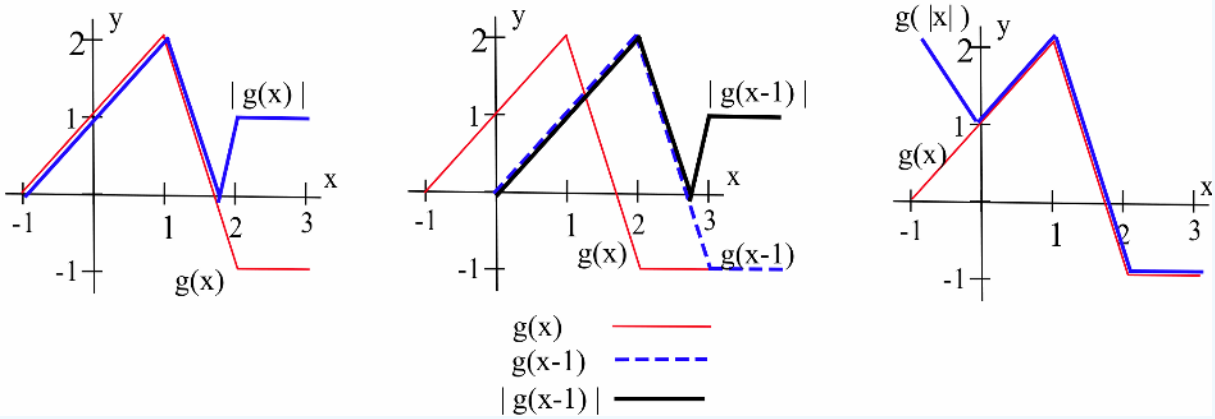
Graph:

a. $|g(x)|$
b. $|g(x-1)|$
c. $g(|x|)$

https://math.libretexts.org/@go/page/209838

Answer

The figure below shows some intermediate steps and final graphs:



The **greatest integer function** (or **floor function**) evaluated at a number x, $y = f(x) = \lfloor x \rfloor$, is the largest integer less than or equal to x.

The value of $\lfloor x \rfloor$ is always an integer and $\lfloor x \rfloor$ is always less than or equal to x. For example, $\lfloor 3.2 \rfloor = 3$, $\lfloor 3.9 \rfloor = 3$ and $\lfloor 3 \rfloor = 3$. If x is positive, then $\lfloor x \rfloor$ **truncates** x (drops the fractional part of x). If x is negative, the situation is different: $\lfloor -4.2 \rfloor \neq -4$ because -4 is not less than or equal to -4.2: $\lfloor -4.2 \rfloor = -5$, $\lfloor -4.7 \rfloor = -5$ and $\lfloor -4 \rfloor = -4$.

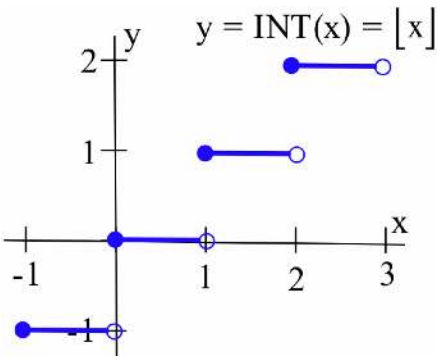
Note

Historically, many textbooks have used the square brackets [ ] to represent the greatest integer function, while calculators and many programming languages use $\mathbf{INT}(x)$ or $\mathbf{floor}(x)$.

Definition: $\lfloor x \rfloor$

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{largest integer strictly less than } x & \text{if } x \text{ is not an integer} \end{cases}$$

The domain of $f(x) = \lfloor x \rfloor$ is all real numbers. The range of $f(x) = \lfloor x \rfloor$ is only the integers. The graph of $y = f(x) = \lfloor x \rfloor$ appears below.



It has a jump break{\thinspace}---{\thinspace}a “step”—at each integer value of x, so $f(x) = \lfloor x \rfloor$ is called a {\bf step function}. Between any two consecutive integers, the graph is horizontal with no breaks or holes.

The greatest integer function is useful for describing phenomena that change values abruptly, such as postage rates as a function of weight. As of January 26, 2014, the cost to mail a first-class retail “flat” (such as a manila envelope) was \$0.98 for the first ounce and another \$0.21 for each additional ounce.

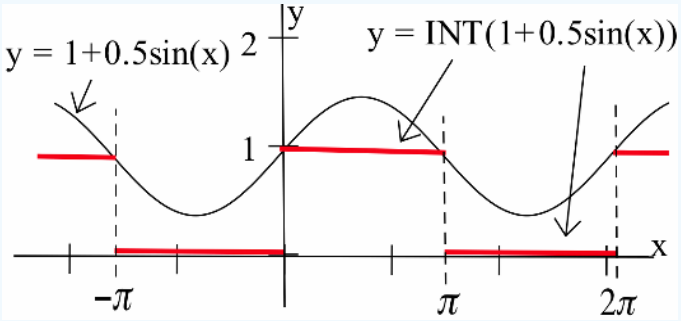
The $\lfloor x \rfloor$ function can also be used for functions whose graphs are “square waves,” such as the on and off of a flashing light.

Example 0.4.6

Graph $y = \lfloor 1 + 0.5\sin(x) \rfloor$.

Solution

One way to create this graph is to first graph $y = 1 + 0.5\sin(x)$, the thin curve in figure below:



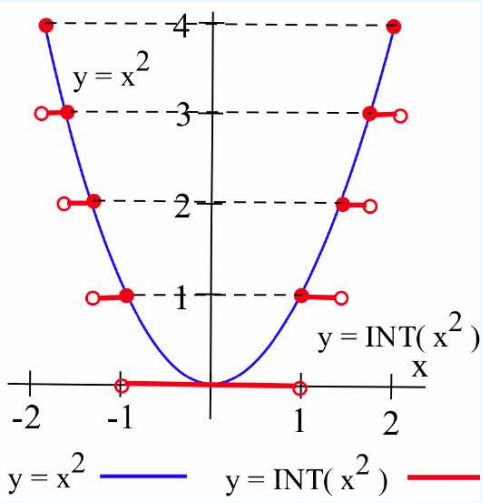
and then apply the greatest integer function to y to get the thicker "square wave" pattern.

Practice 0.4.8

Sketch the graph of $y = \lfloor x^2 \rfloor$ for $-2 \le x \le 2$.

Answer

The figure below shows the graph of $y = x^2$ and the (thicker) graph of $y = \lfloor x^2 \rfloor$:

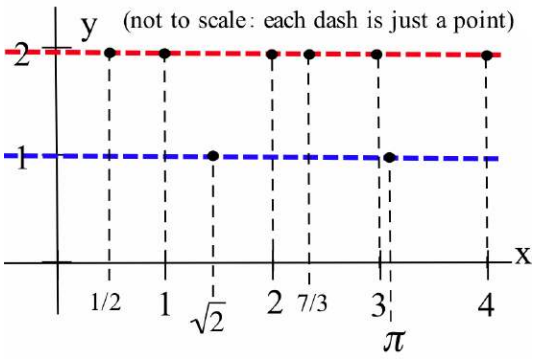


A Really "Holey" Function

The graph of $\lfloor x \rfloor$ has a break or jump at each integer value, but how many breaks can a function have? The next function illustrates just how broken or "holey" the graph of a function can be.

Define a function $h(x)$ as: $h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$

Then $h(3) = 2$, $h\left(\frac{5}{3}\right) = 2$ and $h\left(-\frac{2}{5}\right) = 2$, because 3, $\frac{5}{3}$ and $-\frac{2}{5}$ are all rational numbers. Meanwhile, $h(\pi) = 1$, $h(\sqrt{7}) = 1$ and $h(\sqrt{2}) = 1$, because π, $\sqrt{7}$ and $\sqrt{2}$ are all irrational numbers. These and some other points are plotted in the figure below:



In order to analyze the behavior of $h(x)$ the following fact about rational and irrational numbers is useful.

Fact

Every interval contains both rational and irrational numbers.

Equivalently: If a and b are real numbers and $a < b$, then there is:

- a rational number R between a and b ($a < R < b$)
- an irrational number I between a and b ($a < I < b$

The above fact tells us that between any two places where $y = h(x) = 1$ (because x is rational) there is a place where $y = h(x)$ is 2, because there is an irrational number between any two distinct rational numbers. Similarly, between any two places where $y = h(x) = 2$ (because x is irrational) there is a place where $y = h(x) = 1$, because there is a rational number between any two distinct irrational numbers.

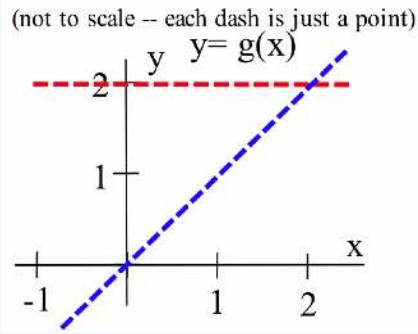
The graph of $y = h(x)$ is impossible to actually draw, because every two points on the graph are separated by a hole. This is also an example of a function that your computer or calculator cannot graph, because in general it can not determine whether an input value of x is irrational.

Example 0.4.7

Sketch the graph of $g(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ x & \text{if } x \text{ is an irrational number} \end{cases}$

Solution

A sketch of the graph of $y = g(x)$ appears below:



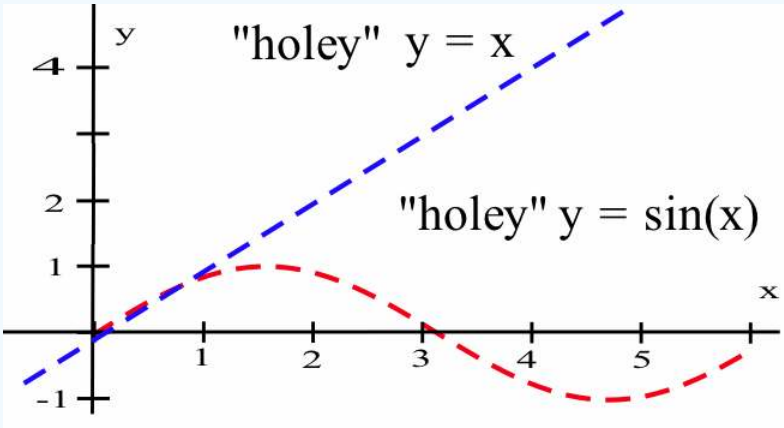
When x is rational, the graph of $y = g(x)$ looks like the "holey" horizontal line $y = 2$. When x is irrational, the graph of $y = g(x)$ looks like the "holey" line $y = x$.

Practice 0.4.8

Sketch the graph of $h(x) = \begin{cases} \sin(x) & \text{if } x \text{ is a rational number} \\ x & \text{if } x \text{ is an irrational number} \end{cases}$

Answer

The figure below shows the "holey" graph of $y = x$ with a hole at each rational value of x and the "holey" graph of $y = \sin(x)$ with a hole at each irrational value of x. Together they form the graph of $r(x)$.



(This is a very crude image, since we can't really see the individual holes, which have zero width.)

Problems

1. If T is the Celsius temperature of the air and v is the speed of the wind in kilometers per hour, then:

$$\text{WCI} = \begin{cases} T & \text{if } 0 \le v \le 6.5 \\ 33 - \frac{10.45 + 5.29\sqrt{v} - 0.279v}{22}(33 - T) & \text{if } 6.5 < v \le 72 \\ 1.6T - 19.8 & \text{if } v > 72 \end{cases}$$

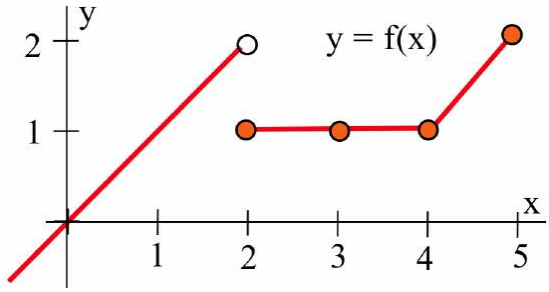
   a. Determine the Wind Chill Index for:

      i. a temperature of $0°$C and a wind speed of 49 km/hr

      ii. a temperature of $11°$C and a wind speed of 80 km/hr.

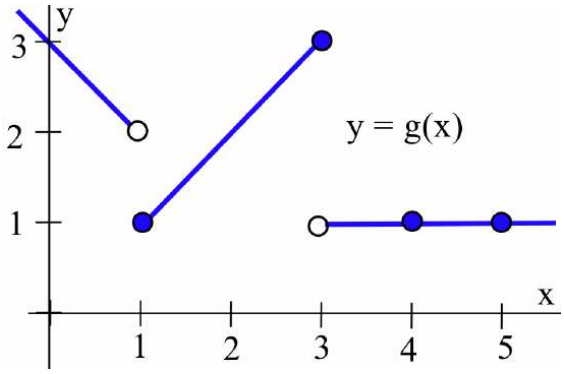
   b. Write a multiline function definition for the WCI if the temperature is $11°$C.

2. Use the graph of $y = f(x)$ below to evaluate $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$:



   Write a multiline function definition for f.

3. Use the graph of $y = g(x)$ below to evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$ and $g(5)$:

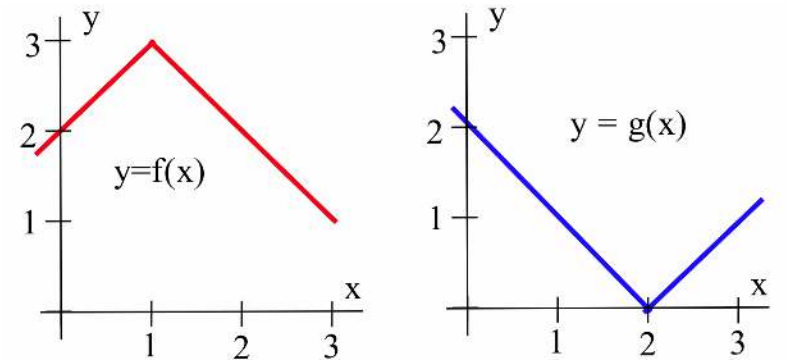


Write a multiline function definition for g.

4. Use the values given in the table below, along with $h(x) = 2x + 1$, to determine the missing values of $f \circ g$, $g \circ f$ and $h \circ g$.

x	$f(x)$	$g(x)$	$f \circ g(x)$	$g \circ f(x)$	$h \circ g(x)$
-1	2	0			
0	1	2			
1	-1	1			
2	0	2			

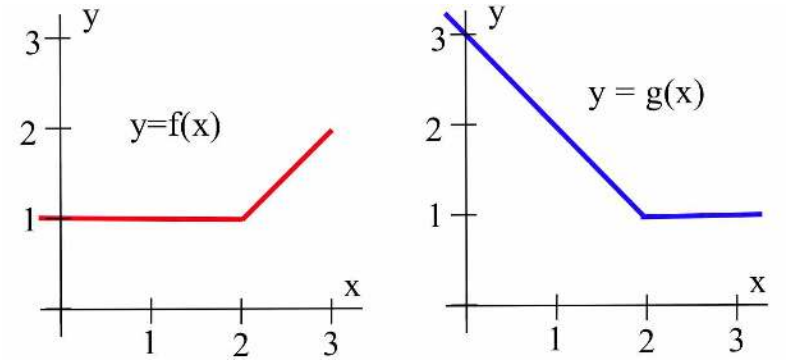
5. Use the graphs shown below:



and the function $h(x) = x - 2$ to determine the values of:

a. $f(f(1))$, $f(g(2))$, $f(g(0))$, $f(g(1))$
b. $g(f(2))$, $g(f(3))$, $g(g(0))$, $g(f(0))$
c. $f(h(3))$, $f(h(4))$, $h(g(0))$, $h(g(1))$

6. Use the graphs shown below:



and the function $h(x) = 5 - 2x$ to determine the values of:

a. $h(f(0))$, $f(h(1))$, $f(g(2))$, $f(f(3))$
b. $g(f(0))$, $g(f(1))$, $g(h(2))$, $h(f(3))$
c. $f(g(0))$, $f(g(1))$, $f(h(2))$, $h(g(3))$

7. Defining $h(x) = x - 2$, $f(x)$ as:

$$f(x) = \begin{cases} 3 & \text{if } x < 1 \\ x - 2 & \text{if } 1 \le x < 3 \\ 1 & \text{if } 3 \le x \end{cases}$$

and $g(x)$ as:

$$g(x) = \begin{cases} x^2 - 3 & \text{if } x < 0 \\ \lfloor x \rfloor & \text{if } 0 \le x \end{cases}$$

a. evaluate $f(x)$, $g(x)$ and $h(x)$ for $x = -1, 0, 1, 2, 3$ and 4.
b. evaluate $f(g(1))$, $f(h(1))$, $h(f(1))$, $f(f(2))$, $g(g(3.5))$.
c. graph $f(x)$, $g(x)$ and $h(x)$ for $-5 \le x \le 5$.

8. Defining $h(x) = 3$, $f(x)$ as:

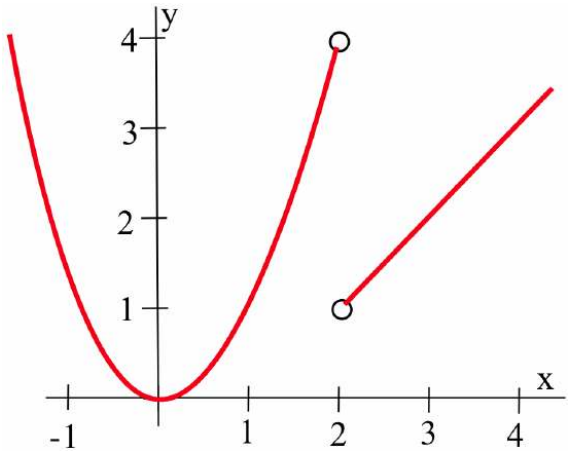
$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ 1 & \text{if } 1 \le x < 3 \\ 2 - x & \text{if } 3 \le x \end{cases}$$

and $g(x)$ as:

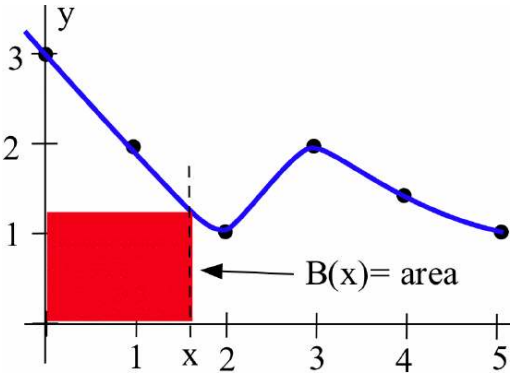
$$g(x) = \begin{cases} |x + 1| & \text{if } x < 0 \\ 2x & \text{if } 0 \le x \end{cases}$$

a. evaluate $f(x)$, $g(x)$ and $h(x)$ for $x = -1, 0, 1, 2, 3$ and 4.
b. evaluate $f(g(1))$, $f(h(1))$, $h(f(1))$, $f(f(2))$, $g(g(3.5))$.
c. graph $f(x)$, $g(x)$ and $h(x)$ for $-5 \le x \le 5$.

9. You are planning to take a one-week vacation in Europe, and the tour brochure says that Monday and Tuesday will be spent in England, Wednesday in France, Thursday and Friday in Germany, and Saturday and Sunday in Italy. Let $L(d)$ be the location of the tour group on day d and write a multiline function definition for $L(d)$.
10. A state has just adopted the following state income tax system: no tax on the first $10,000 earned, 1% of the next $10,000 earned, 2% of the next $20,000 earned, and 3% of all additional earnings. Write a multiline function definition for $T(x)$, the state income tax due on earnings of x dollars.
11. Write a multiline function definition for the curve $y = f(x)$ shown below:

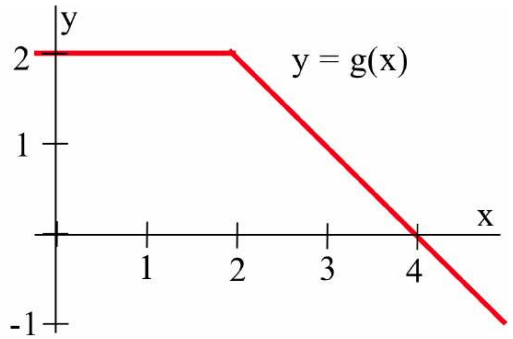


12. Define $B(x)$ to be the **area** of the rectangle whose lower left corner is at the origin and whose upper right corner is at the point $(x, f(x))$ for the function f shown below:



For example, $B(3) = 6$. Evaluate $B(1)$, $B(2)$, $B(4)$ and $B(5)$.

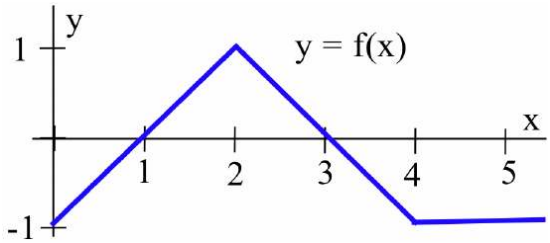
13. Define $B(x)$ to be the **area** of the rectangle whose lower left corner is at the origin and whose upper right corner is at the point $(x, \frac{1}{x})$.
    a. Evaluate $B(1)$, $B(2)$ and $B(3)$.
    b. Show that $B(x) = 1$ for all $x > 0$.
14. For $f(x) = |9 - x|$ and $g(x) = \sqrt{x-1}$:
    a. evaluate $f \circ g(1)$, $f \circ g(3)$, $f \circ g(5)$, $f \circ g(7)$, $f \circ g(0)$.
    b. evaluate $f \circ f(2)$, $f \circ f(5)$, $f \circ f(-2)$.
    c. Does $f \circ f(x) = |x|$ for all values of x?
15. The function $g(x)$ is graphed below:



Graph:

    a. $g(x) - 1$
    b. $g(x - 1)$
    c. $|g(x)|$
    d. $\lfloor g(x) \rfloor$

16. The function $f(x)$ is graphed below:



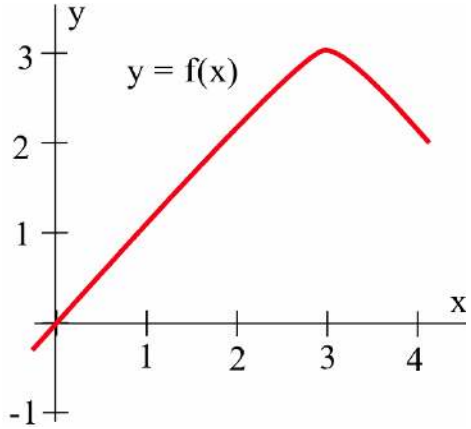
Graph:

    a. $f(x) - 2$
    b. $f(x - 2)$
    c. $|f(x)|$
    d. $\lfloor f(x) \rfloor$

17. Find A and B so that $f(g(x)) = g(f(x))$ when:
    a. $f(x) = 3x + 2$ and $g(x) = 2x + A$
    b. $f(x) = 3x + 2$ and $g(x) = Bx - 1$
18. Find C and D so that $f(g(x)) = g(f(x))$ when:
    a. $f(x) = Cx + 3$ and $g(x) = Cx - 1$
    b. $f(x) = 2x + D$ and $g(x) = 3x + D$
19. Graph $y = f(x) = x - \lfloor x \rfloor$ for $-1 \le x \le 3$. This function is called the "fractional part of x" and its graph an example of a "sawtooth" graph.
20. The function $f(x) = \lfloor x + 0.5 \rfloor$ rounds off x to the **nearest** integer, while $g(x) = \dfrac{\lfloor 10x + 0.5 \rfloor}{10}$ rounds off x to the nearest tenth (the first decimal place). What function will round off x to:
    a. the nearest hundredth (two decimal places)?
    b. the nearest thousandth (three decimal places)?
21. Modify the function in Example 6 to produce a "square wave" graph with a "long on, short off, long on, short off" pattern.
22. Many computer languages contain a "signum" or "sign" function defined by:

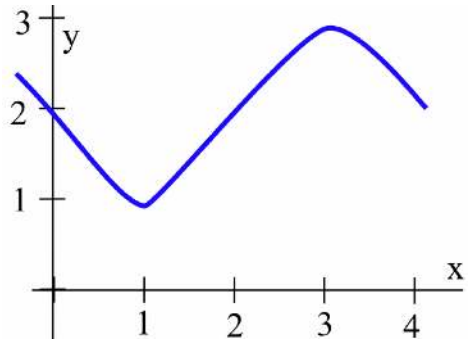
$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

    a. Graph $\operatorname{sgn}(x)$.
    b. Graph $\operatorname{sgn}(x - 2)$.
    c. Graph $\operatorname{sgn}(x - 4)$.
    d. Graph $\operatorname{sgn}(x - 2) \cdot \operatorname{sgn}(x - 4)$.
    e. Graph $1 - \operatorname{sgn}(x - 2) \cdot \operatorname{sgn}(x - 4)$.
    f. For real numbers a and b with $a < b$, describe the graph of $1 - \operatorname{sgn}(x - a) \cdot \operatorname{sgn}(x - b)$.
23. Define $g(x)$ to be the **slope** of the line tangent to the graph of $y = f(x)$ at (x, y), with $y = f(x)$ shown here:



    a. Estimate $g(1)$, $g(2)$, $g(3)$ and $g(4)$.
    b. Graph $y = g(x)$ for $0 \le x \le 4$.

24. Define $h(x)$ to be the **slope** of the line tangent to the graph of $y = f(x)$ at (x, y), with $y = f(x)$ shown here:



a. Estimate $h(1)$, $h(2)$, $h(3)$ and $h(4)$.
b. Graph $y = h(x)$ for $0 \le x \le 4$.

25. Using the **cos** (cosine) button on your calculator several times produces iterates of $f(x) = \cos(x)$. What number will the iterates approach if you use the **cos** button 20 or 30 times starting with:

a. $x = 1$?
b. $x = 2$?
c. $x = 10$?

(Be sure your calculator is in radian mode.)

26. Let $f(x) = 1 + \sin(x)$.

a. What happens if you start with $x = 1$ and repeatedly feed the output from f back into f?
b. What happens if you start with $x = 2$ and examine the iterates of f?

(Be sure your calculator is in radian mode.)

27. Starting with $x = 1$, do the iterates of $f(x) = \frac{x^2+1}{2x}$ approach a number? What happens if you start with $x = 0.5$ or $x = 4$?
28. Let $f(x) = \frac{x}{2} + 3$.

a. What are the iterates of f if you start with $x = 2$? $x = 4$? $x = 6$?
b. Find a number c so that $f(c) = c$. This value of c is called a **fixed point** of f.
c. Find a fixed point of $g(x) = \frac{x}{2} + A$.

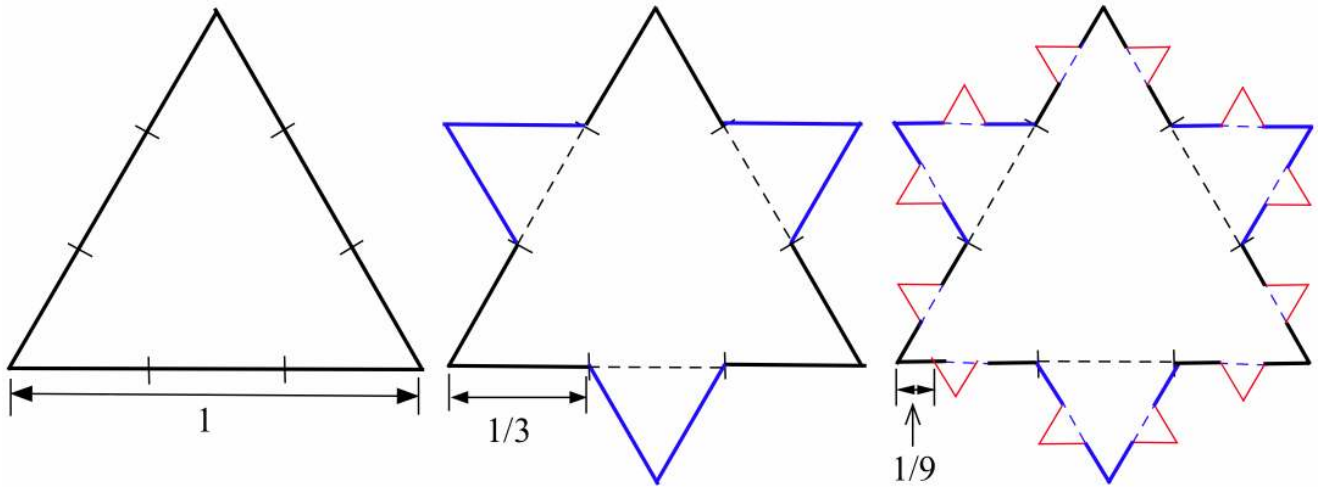
29. Let $f(x) = \frac{x}{3} + 4$.

a. What are the iterates of f if you start with $x = 2$? $x = 4$? $x = 6$?
b. Find a number c so that $f(c) = c$.
c. Find a fixed point of $g(x) = \frac{x}{3} + A$.

30. Some iterative procedures are geometric rather than numerical. Start with an equilateral triangle with sides of length 1, as shown at left in the figure below.

- Remove the middle third of each line segment.
- Replace the removed portion with two segments with the same length as the removed segment.

The first two iterations of this procedure are shown at center and right in the figure below:



Repeat these steps several more times, each time removing the middle third of each line segment and replacing it with two new segments. What happens to the length of the shape with each iteration? (The result of iterating over and over with this procedure is called **Koch's Snowflake**, named for Helga von Koch.)

31. Sketch the graph of: $p(x) = \begin{cases} 3-x & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$
32. Sketch the graph of: $q(x) = \begin{cases} x^2 & \text{if } x \text{ is a rational number} \\ x+11 & \text{if } x \text{ is an irrational number} \end{cases}$

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0.5: Mathematical Language

The calculus concepts we will explore in this book are simple and powerful, but sometimes subtle. To succeed in calculus you will need to master some techniques, but (more importantly) you will need to understand ideas and be able to work with the ideas in words and pictures—very clear words and pictures.

You also need to understand some of the common linguistic constructions used in mathematics. In this section, we will discuss a few of the most common mathematical phrases, the meanings of these phrases, and some of their equivalent forms.

Your calculus instructor is going to use these types of statements, and it is very important that you understand exactly what your instructor means. You have reached a level in mathematics where the precise use of language is important.

Equivalent Statements

Two statements are **equivalent** if they always have the same logical value (a logical value is either "true" or "false"): that is, they are both true or are both false.

The statements "$x = 3$" and "$x + 2 = 5$" are equivalent statements, because if one of them is true then so is the other—and if one of them is false then so is the other.

The statements "$x = 3$" and "$x^2 - 4x + 3 = 0$" are not equivalent, because $x = 1$ makes the second statement true but the first one false.

AND and OR

In everyday language, we use the words "and" and "or" all the time, but in mathematics we must use them very carefully.

> The compound statement "A **and** B are true" is equivalent to "both of A and B are true."

If A or if B or if both are false, then the statement "A and B are true" is false. The statement "$x^2 = 4$ and $x > 0$)$"istruewhen\(x = 2$ and is false for every other value of x.

> The compound statement "A **or** B is true" is equivalent to "at least one of A or B is true."

If both A and B are false, then the statement "A or B is true" is false.

The statement "$x^2 = 4$ or $x > 0$" is true if $x = -2$ or x is any positive number. The statement is false when $x = -3$ (and for lots of other values of x).

? Practice 0.5.1

Which values of x make each statement true?

a. $x < 5$
b. $x + 2 = 6$
c. $x^2 - 10x + 24 = 0$
d. "(a) and (b)"
e. "(a) or (c)"

Answer

a. All values of x less than 5.
b. $x = 4$
c. Both $x = 4$ and $x = 6$.
d. $x = 4$
e. $x = 6$ and all x less than 5.

Negation of a Statement

For some simple statements, we can construct the negation just by adding the word "not."

statement	negation of the statement
x is equal to 3 ($x = 3$)	x is **not** equal to 3 ($x \neq 3$)
x is less than 5 ($x < 5$)	x is **not** less than 5 ($x \not< 5$) x is greater than or equal to 5 ($x \geq 5$)

When the statement contains words such as "all," "no" or "some," then the negation becomes more complicated.

statement	negation of the statement
All x satisfy A. Every x satisfies A.	At least one x does not satisfy A. There is an x that does not satisfy A. Some x does not satisfy A.
No x satisfies A. Every x does not satisfy A.	At least one x satisfies A. Some x satisfies A.
There is an x that satisfies A. At least one x satisfies A. Some x satisfies A.	No x satisfies A. Every x does not satisfy A.

We can also negate compound statements containing "and" and "or."

statement	negation of the statement
A and B are both true.	At least one of A or B is not true.
A and B and C are all true.	At least one of A or B or C is not true.
A or B is true.	Both A and B are not true.

? Practice 0.5.2

Write the negation of each statement.

a. $x + 5 = 3$
b. All prime numbers are odd.
c. $x^2 < 4$
d. x divides 2 and x divides 3.
e. No mathematician can sing well.

Answer

a. $x + 5 < 3$
b. At least one prime number is even. (There is an even prime number.)
c. $x^2 = 4$
d. x does not divide 2 or x does not divide 3.
e. At least one mathematician can sing well. (There is a mathematician who can sing well.)

If…Then…: A Very Common Structure in Mathematics

The most common and basic structure used in mathematical language is the "If {some hypothesis} then {some conclusion}" sentence. Almost every result in mathematics can be stated using one or more "If…then…" sentences.

> "**If A then B**" means that when the hypothesis A is true, the conclusion B must also be true.

When the hypothesis is false, the "If…then…" sentence makes no claim about the truth or falsity of the conclusion—the conclusion may be either true or false.

Even in everyday life you have probably encountered "If…then…" statements for a long time. A parent might try to encourage a child with a statement like "If you clean your room then I will buy you an ice cream cone."

To show that an "If…then…" statement is not valid (not true), all we need to do is find a single example where the hypothesis is true and the conclusion is false. Such an example with a true hypothesis and false conclusion is called a counterexample for the "If…then…" statement. A valid "If…then…" statement has no counterexample.

> A **counterexample** to the statement "If A then B" is an example in which A is true and B is false.

The only way for the statement "If you clean your room then I will buy you an ice cream cone" to be false is if the child cleaned the room and the parent did not buy the ice cream cone. If the child did not clean the room but the parent bought the ice cream cone anyway, we would say that the statement was true.

The statement "If n is a positive integer, then $n^2 + 5n + 5$ is a prime number" has hypothesis "n is a positive integer" and conclusion "n^2 + 5n + 5\) is a prime number." This "If…then…" statement is false, because replacing n with the number 5 will make the hypothesis true and the conclusion false. The number 5 is a counterexample for the statement.

Every invalid "If…then…" statement has at least one counterexample, and the most convincing way to show that a statement is not valid is to find a counterexample to the statement.

Several other language structures are equivalent to the "If…then…" form. The statements below all mean the same as "If A then B":

- "All A are B."
- "Every A is B."
- "Each A is B."
- "Whenever A, then B."
- "B whenever A."
- "A only if B."
- "A implies B."
- "$A \Rightarrow B$" (the symbol "⇒" means "implies")

? Practice 0.5.3

Restate "If {a shape is A square} then {the shape is A rectangle}" as many ways as you can.

Answer

Here are several ways:

- All squares are rectangles.
- Every square is a rectangle.
- Each square is a rectangle.
- Whenever a shape is a square, then it is a rectangle.
- A shape is a rectangle whenever it is a square.
- A shape is a square only if it is a rectangle.
- A shape is a square implies that it is a rectangle.
- Being a square implies being a rectangle.

"If…then…" statements occur hundreds of times in every mathematics book, including this one. It is important that you are able to recognize the various forms of "If…then…" statements and that you are able to distinguish the hypotheses from the conclusions.

Contrapositive Form of an "If…Then…" Statement

The statement "If A then B" means that if the hypothesis A is true, then the conclusion B is guaranteed to be true.

Suppose we know that in a certain town the statement "If {a building is a church} then {the building is green}" is a true statement. What can we validly conclude about a red building? Based on the information we have, we can validly conclude that the red building is "not a church" because every church is green. We can also conclude that a blue building is not a church. In fact, we can conclude that every "not green" building is "not a church." That is, if the conclusion of a valid "If…then…" statement is false, then the hypothesis must also be false.

The **contrapositive** form of "If A then B" is "If {negation of B} then {negation of A}" or "If {B is false} then {A is false}."

The statement "If A then B" and its contrapositive "If {not B} then {not A}" are equivalent.

What about a green building in the aforementioned town? The green building may or may not be a church—perhaps every post office is also painted green. Or perhaps every building in town is green, in which case the statement "If {a building is a church} then {the building is green}" is certainly true.

? Practice 0.5.4

Write the contrapositive form of each statement.

a. If a function is differentiable then it is continuous.
b. All men are mortal.
c. If $x = 3$ then $x^2 - 5x + 6 = 0$
d. If {2 divides x and 3 divides x} then {6 divides x}.

Answer

a. If a function is not continuous then it is not differentiable.
b. All immortals are not men.
c. $x^2 - 5x + 6 \neq 0 \Rightarrow x \neq 3$
d. If $\{6$ does not divide $x\}$ then $\{2$ does not divide x or 3 does not divide $x\}$.

Converse of an "If…then…" Statement

If we switch the hypothesis and the conclusion of an "If A then B" statement, we get the **converse** "If B then A." For example, the converse of "If {a building is a church} then {the building is green}" is "If {a building is green} then {the building is a church}."

The converse of an "If…then…" statement is *not equivalent* to the original "If…then…" statement. For example, the statement "If $x = 2$ then $x^2 = 4$" is true, but the converse statement "If $x^2 = 4$ then $x = 2$" is not true because $x = -2$ makes the hypothesis of the converse true and the conclusion false

The **converse** of "If A then B" is "If B then A."

The statement "If A then B" and its converse "If B then A" are **not equivalent**.

Wrap-up

The precise use of language by mathematicians (and mathematics books) is an attempt to clearly communicate ideas from one person to another, but this requires that both people understand the use and rules of the language. If you don't understand this usage, the communication of the ideas will almost certainly fail.

Problems

In Problems 1–2, define the sets A, B and C as $A = \{1, 2, 3, 4, 5\}$, $B = \{0, 2, 4, 6\}$ and $C = \{-2, -1, 0, 1, 2, 3\}$. List all values of x that satisfy each statement.

1. a. x is in A **and** x is in B
   b. x is in A **or** x is in C
   c. x is not in B **and** x is in C
2. a. x is not in B or C
   b. x is in B and C but not in A
   c. x is not in A but is in B or C

In Problems 3–5, list or describe all the values of x that make each statement true.

3. a. $x^2 + 3 > 1$
   b. $x^3 + 3 > 1$
   c. $\lfloor x \rfloor \leq |x|$
4. a. $\frac{x^2+3x}{x} = x + 3$
   b. $x > 4$ and $x < 9$
   c. $|x| = 3$ and $x < 0$
5. a. $x + 5 = 3$ or $x^2 = 9$
   b. $x + 5 = 3$ and $x^2 = 9$
   c. $|x + 3| = |x| + 3$

In Problems 6–8, write the **contrapositive** of each statement. If false, give a **counterexample**.

6. a. If $x > 3$ then $x^2 > 9$.
   b. Every solution of $x^2 - 6x + 8 = 0$ is even.
7. a. If $x^2 + x - 6 = 0$ then $x = 2$ or $x = -3$.
   b. All triangles have 3 sides.
8. a. Every polynomial has at least one zero.
   b. If I exercise and eat right then I will be healthy.

In Problems 9–11, write the **contrapositive** of each statement.

9. a. If your car is properly tuned, it will get at least 24 miles per gallon.
   b. You can have pie if you eat your vegetables.
10. a. A well-prepared student will miss less than 15 points on the final exam.
    b. I feel good when I jog.
11. a. If you love your country, you will vote for me.
    b. If guns are outlawed then only outlaws will have guns.

In 12–15, write the **negation** of each statement.

12. a. It is raining.
    b. Some equations have solutions.
    c. $f(x)$ and $g(x)$ are polynomials.
13. a. $f(x)$ or $g(x)$ is positive.
    b. x is positive.
    c. 8 is a prime number.

14. a. Some months have six Mondays.
    b. All quadratic equations have solutions.
    c. The absolute value of a number is positive.
15. a. For all numbers a and b, $|a+b| = |a| + |b|$.
    b. All snakes are poisonous.
    c. No dog can climb trees.
16. Write an "If…then…" statement that is true but whose converse is false.
17. Write an "If… then…" statement that is true and whose converse is true.
18. Write an "If… then…" statement that is false and whose converse is false.

In 19–22, determine whether each statement is true or false. If false, give a counterexample.

19. a. If a and b are real numbers then:
    $$(a+b)^2 = a^2 + b^2$$
    b. If $a > b$ then $a^2 > b^2$.
    c. If $a > b$ then $a^3 > b^3$.
20. a. For all real numbers a and b, $|a+b| = |a| + |b|$
    b. For all real numbers a and b, $\lfloor a \rfloor + \lfloor b \rfloor = \lfloor a+b \rfloor$.
    c. If $f(x)$ and $g(x)$ are linear functions, then $f(g(x))$ is a linear function.
21. a. If $f(x)$ and $g(x)$ are linear functions then $f(x) + g(x)$ is a linear function.
    b. If $f(x)$ and $g(x)$ are linear functions then $f(x) \cdot g(x)$ is a linear function.
    c. If x divides 6 then x divides 30.
22. a. If x divides 50 then x divides 10.
    b. If x divides yz then x divides y or z.
    c. If x divides a^2 then x divides a.

In 23–26, rewrite each statement in the form of an "If… then…" statement and determine whether it is true or false. If the statement is false, give a counterexample.

23. a. The sum of two prime numbers is a prime.
    b. The sum of two prime numbers is never a prime number.
    c. Every prime number is odd.
    d. Every prime number is even.
24. a. Every square has 4 sides.
    b. All 4-sided polygons are squares.
    c. Every triangle has 2 equal sides.
    d. Every 4-sided polygon with equal sides is a square.
25. a. Every solution of $x + 5 = 9$ is odd.
    b. Every 3-sided polygon with equal sides is a triangle.
    c. Every calculus student studies hard.
    d. All (real) solutions of $x^2 - 5x + 6 = 0$ are even.
26. a. Every line in the plane intersects the x-axis.
    b. Every (real) solution of $x^2 + 3 = 0$ is even.
    c. All birds can fly.
    d. No mammal can fly.

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0.A: Answers

Important Note about Precision of Answers: In many of the problems in this book you are required to read information from a graph and to calculate with that information. You should take reasonable care to read the graphs as accurately as you can (a small straightedge is helpful), but even skilled and careful people make slightly different readings of the same graph. That is simply one of the drawbacks of graphical information. When answers are given to graphical problems, the answers should be viewed as the best approximations we could make, and they usually include the word "approximately" or the symbol "≈" meaning "approximately equal to." Your answers should be close to the given answers, but you should not be concerned if they differ a little. (Yes those are vague terms, but it is all we can say when dealing with graphical information.)

Section 0.1

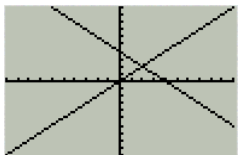
1. approx.\ 1, 0, -1
3. a. $\approx \dfrac{70^\circ - 150^\circ}{10 \text{ min } - 0 \text{ min}} = -8 \dfrac{^\circ}{\text{min}}$
   b. $\approx 6 \dfrac{^\circ}{\text{min}}$ cooling; $5 \dfrac{^\circ}{\text{min}}$ cooling
   c. $\approx 5.5 \dfrac{^\circ}{\text{min}}$ cooling; $10 \dfrac{^\circ}{\text{min}}$ cooling
   d. When $t = 6$ min.
5. We estimate that the area is approximately (very approximately) 9 cm^2.
7. **Method 1**: Measure the diameter of the can, fill it half full of water, measure the height of the water and calculate the volume. Submerge the bulb, measure the height of the water again, and calculate the new volume. The volume of the bulb is the difference of the two calculated volumes. **Method 2**: Fill the can with water and weigh it. Submerge the bulb (displacing a volume of water equal to the volume of the bulb), remove the bulb, and weigh the can again. By subtracting, find the weight of the displaced water and use the fact that the density of water is 1 gram per 1 cubic centimeter to determine the volume of the bulb.

Section 0.2

1. a. $-\frac{3}{4}$
   b. $\frac{1}{2}$
   c. 0 \item 2
   d. undefined
3. a. $\frac{4}{3}$
   b. $-\frac{9}{5}$
   c. $x + 2$ (if $x \neq 2$)
   d. $4 + h$ (if $h \neq 0$)
   e. $a + x$ (if $a \neq x$)
5. a. $t = 5$: $\frac{5000}{1500} = \frac{10}{3}$; $t = 10$: $\frac{5000}{3000} = \frac{5}{3}$; $t = 20$: $\frac{5000}{6000} = \frac{5}{6}$
   b. any $t > 0$: $\frac{5000}{300t} = \frac{50}{3t}$
   c. Decreasing, since the numerator remains constant at 5000 while the denominator increases.
7. The restaurant is 4 blocks south and 2 blocks east. The distance is $\sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47$ blocks.
9. a. $\sqrt{20^2 - 4^2} = \sqrt{384} \approx 19.6$ feet
   b. $\frac{\sqrt{384}}{4} \approx 4.9$
   c. $\tan(\theta) = \frac{\sqrt{384}}{4} \approx 4.9$ so $\theta \approx 1.37$ ($\approx 78.5^\circ$)
11. a. The equation of the line through $P = (2, 3)$ and $Q = (8, 11)$ is $y - 3 = \frac{8}{6}(x - 2) \Rightarrow 6y - 8x = 2$. Substituting $x = 2a + 8(1 - a) = 8 - 6a$ and $y = 3a + 11(1 - a) = 11 - 8a$ into the equation for the line, we get: $6(11 - 8a) - 8(8 - 6a) = 66 - 48a - 64 + 48a = 2$ for all a, so the point with $x = 2a + 8(1 - a)$ and $y = 3a + 11(1 - a)$ is on the line through P and Q for any a. Furthermore, $2 \leq 8 - 6a \leq 8$ for $0 \leq a \leq 1$, so the point in question must be on the line *segment* PQ.
    b. $\text{dist}(P, Q) = \sqrt{6^2 + 8^2} = 10$, while:

$$\begin{aligned}\text{dist}(P,R) &= \sqrt{(8-6a-2)^2+(11-8a-3)^2}\\ &= \sqrt{(6-6a)^2+(8-8a)^2}\\ &= \sqrt{6^2(1-a)^2+8^2(1-a)^2}\\ &= \sqrt{100(1-a)^2} = 10\cdot|1-a|\\ &= |1-a|\cdot\text{dist}(P,Q)\end{aligned}$$

13. a. $m_1 \cdot m_2 = (1)(-1) = -1$

b. 

c. Because 20 units of x-values are physically wider on the screen than 20 units of y-values.

d. Set the window so that: $\text{xmax} - \text{xmin}) \approx 1.7(\text{ymax} - \text{ymin})$

15. a. $y-5 = 3(x-2)$ or $y = 3x-1$

b. $y-2 = -2(x-3)$ or $y = 8-2x$

c. $y-4 = -\frac{1}{2}(x-1)$ or $y = -\frac{1}{2}x+\frac{9}{2}$

17. a. $y-5 = \frac{3}{2}(x-2)$ or $y = \frac{3}{2}x+2$

b. $y-2 = \frac{3}{2}(x+1)$ or $y = \frac{3}{2}x+\frac{7}{2}$

c. $x = 3$

19. Distance between the centers $= \sqrt{6^2+8^2} = 10$.

a. $10-2-4 = 4$

b. $10-2-7 = 1$

c. 0 (they intersect)

d. $15-10-3 = 2$

e. $12-10-1 = 1$

21. Find $\text{dist}(P,C) = \sqrt{(x-h)^2+(y-k)^2}$ and compare the value to r:

$$P \text{ is } \begin{cases} \text{inside the circle} & \text{if} \quad \text{dist}(P,C) < r \\ \text{on the circle} & \text{if} \quad \text{dist}(P,C) = r \\ \text{outside the circle} & \text{if} \quad \text{dist}(P,C) > r \end{cases}$$

23. A point $P = (x,y)$ lies on the circle if and only if its distance from $C = (h,k)$ is r: $\text{dist}(P,C) = r$. So P is on the circle if and only if:

$$\sqrt{(x-h)^2+(y-k)^2} = r$$

if and only if:

$$(x-h)^2+(y-k)^2 = r^2$$

25. a. $-\frac{5}{12}$

b. undefined (vertical line)

c. $\frac{12}{5}$

d. 0 (horizontal line)

27. a. ≈ 2.22

b. ≈ 2.24

c. (by inspection) 3 units, which occurs at the point $(5,3)$

29. a. If $B \neq 0$, we can solve for y: $y = -\frac{A}{B}x + \frac{C}{B}$, so the slope is $m = -\frac{A}{B}$.

b. The required slope is $\frac{B}{A}$ (the negative reciprocal of $-\frac{A}{B}$) and the y-intercept is 0, so the equation is $y = \frac{B}{A}x$ or $Bx - Ay = 0$.

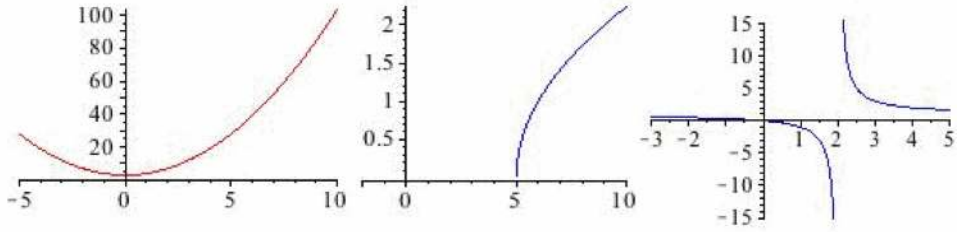
c. Solve the equations $Ax+By = C$ and $Bx-Ay = 0$ simultaneously to get: $x = \dfrac{AC}{A^2+B^2}$ and $y = \dfrac{BC}{A^2+B^2}$

d. The distance from this point to the origin is: $\sqrt{\left(\frac{AC}{A^2+B^2}\right)^2+\left(\frac{BC}{A^2+B^2}\right)^2}$

$$=\sqrt{\frac{A^2C^2}{(A^2+B^2)^2}+\frac{B^2C^2}{(A^2+B^2)^2}}$$
$$=\sqrt{\frac{(A^2+B^2)C^2}{(A^2+B^2)^2}}$$
$$=\sqrt{\frac{C^2}{A^2+B^2}}=\frac{|C|}{\sqrt{A^2+B^2}}$$

Section 0.3

1. A: a, B: c, C: d, D:b
3. A: b, B: c, C: d, D: a
5. a. C
   b. A
   c. B
7. a. $f(1)=4$, $g(1)$ is undefined, $H(1)=-1$



   b.
   c. $f(3x)=(3x)^2+3=9x^2+3$, $g(3x)=\sqrt{3x-5}$ (for $x\geq\frac{5}{3}$), $H(3x)=\frac{3x}{3x-2}$
   d. $f(x+h)=(x+h)^2+3=x^2+2xh+h^2+3$, $g(x+h)=\sqrt{x+h-5}$, $H(x+h)=\frac{x+h}{x+h-2}$
9. a. $m=2$
   b. $m=2x+3+h$
   c. If $x=1.3$, then $m=5.6+h$; if $x=1.1$, then $m=5.2+h$; if $x=1.002$, then $m=5.004+h$
11. a. $\frac{f(a+h)-f(a)}{h}=2a+h-2$ (if $h\neq 0$). If $a=1$: h. If $a=2$: $2+h$. If $a=3$: $4+h$. If $a=x$: $2x+h-2$. $\frac{g(a+h)-g(a)}{h}=\frac{1}{\sqrt{a+h}+\sqrt{a}}$ $=\frac{\sqrt{a+h}-\sqrt{a}}{h}$. If $a=1$: $\frac{\sqrt{1+h}-1}{h}$. If $a=2$: $\frac{\sqrt{2+h}-2}{h}$. If $a=3$: $\frac{\sqrt{3+h}-3}{h}$. If $a=x$: $\frac{\sqrt{x+h}-x}{h}$.
13. a. Approx.\ 250 miles, 375 miles. \item Approx.\ 200 miles/hour.
    b. By flying along a circular arc about 375 miles from the airport (or by landing at another airport).
15. a. FIGURE ans003_15.png
    b. Largest: $x=2$; smallest: $x=4$.
    c. Largest: at $x=5$; smallest at $x=3$.

17. The path of the slide is a straight line tangent to the graph of the path at the point of fall: FIGURE ans003_17.png}

19. a. $s(1)=2$, $s(3)=\frac{4}{3}$, $s(4)=\frac{5}{4}$
    b. $s(x)=\frac{x+1}{x}$

21. Approximate values:

x	$f(x)$	$g(x)$
0	1.0	1.0
1	2.0	1.0
2	2.0	−1.0
3	1.0	0.0
4	1.5	0.5

23. On your own.

Section 0.4

1. a. ≈ -18, ≈ -2.2

   b. If $T = 11°\text{C}$, $\text{WCI}_{11} = \begin{cases} 11 & \text{if } 0 \le v \le 6.5 \\ 22.55 - 5.29\sqrt{v} + 0.279v & \text{if } 6.5 < v \le 72 \\ -2.2 & \text{if } v > 72 \end{cases}$

3. $g(0) = 3$, $g(1) = 1$, $g(2) = 2$, $g(3) = 3$, $g(4) = 1$, $g(5) = 1$. $g(x) = \begin{cases} 3 - x & \text{if } x < 1 \\ x & \text{if } 1 \le x <\le 3 \\ 1 & \text{if } x > 3 \end{cases}$

5. a. $f(f(1)) = 1$, $f(g(2)) = 2$,\ $f(g(0)) = 2$, $f(g(1)) = 3$

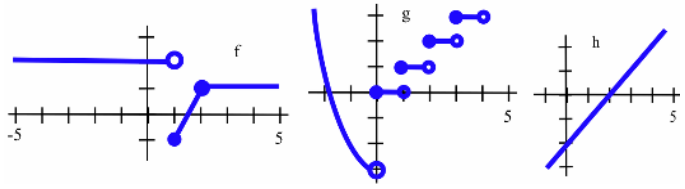
   b. $g(f(2)) = 0$, $g(f(3)) = 1$,\ $g(g(0)) = 0$, $g(f(0)) = 0$

   c. $f(h(3)) = 3$, $f(h(4)) = 2$,\ $h(g(0)) = 0$, $h(g(1)) = -1$

7. a.

x	−1	0	1	2	3	4
$f(x)$	3	3	−1	0	1	1
$g(x)$	−2	0	1	2	3	4
$h(x)$	−3	−2	−1	0	1	2

   b. $f(g(1)) = -1$, $f(h(1)) = 3$, $h(f(1)) = -3$,\ $f(f(2)) = 3$, $g(g(3.5)) = 3$

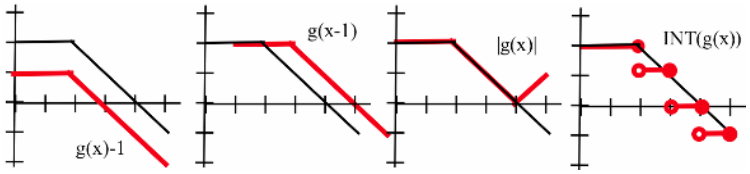
   c. 

9. If $L(d)$ represents location on day d: $L(d) = \begin{cases} \text{England} & \text{if } d = \text{Monday or Tuesday} \\ \text{France} & \text{if } d = \text{Wednesday} \\ \text{Germany} & \text{if } d = \text{Thursday or Friday} \\ \text{Italy} & \text{if } d = \text{Saturday} \end{cases}$

11. Assuming the left portion is part of a parabola: $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$

13. a. $B(1) = 1 \cdot f(1) = 1 \cdot \frac{1}{1} = 1$, $B(2) = 2 \cdot \frac{1}{2} = 1$, $B(3) = 3 \cdot \frac{1}{3} = 1$.

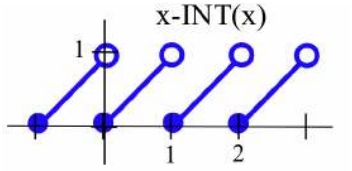
    b. $B(x) = x \cdot f(x) = x \cdot \frac{1}{x} = 1$ (if $x > 0$)

15. 

17. a. $f(g(x)) = 6x + 2 + 3A$, $g(f(x)) = g(3x + 2) = 6x + 4 + A$. If $f(g(x)) = g(f(x))$, then $A = 1$.

    b. $f(g(x)) = 3Bx - 1$, $g(f(x)) = 3Bx + 2B - 1$. If $f(g(x)) = g(f(x))$, then $B = 0$.

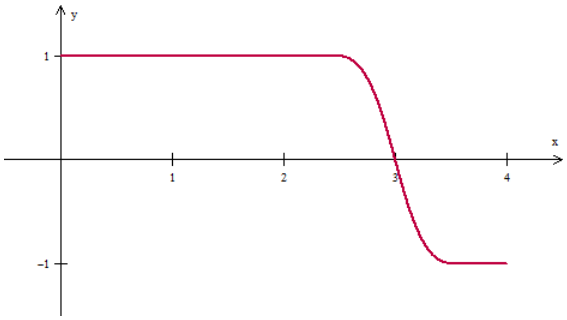
19. Graph of $f(x) = x - \lfloor x \rfloor$:



21. $f(x) = \lfloor 1.3 + 0.5\sin(x) \rfloor$ works. The value of A in $f(x) = \lfloor A + 0.5\sin(x) \rfloor$ determines the relative lengths of the long and short parts of the pattern.

23. a. $g(1) = 1$, $g(2) = 1$, $g(3) = 0$, $g(4) = -1$.

b.



25. ≈ 0.739 starting with $x = 1, 2, 10$ or any value

27. $f(1) = \frac{2}{2} = 1$. $f(0.5) = 1.25$, $f(1.25) = 1.025$, $f(1.025) \approx 1.0003049$, $f(1.0003049) \approx 1.000000046, \ldots f(4) = 2.125$, $f(2.125) \approx 1.297794$, $f(1.297794) \approx 1.034166$, $f(1.034166) \approx 1.000564, \ldots$

29. a. $f(2) = \frac{14}{3} \approx 4.7$, $f(\frac{14}{3}) = \frac{50}{9} \approx 5.6$, $f(\frac{50}{9}) = \frac{158}{27} \approx 5.85$, $f(\frac{158}{27}) = \frac{482}{81} \approx 5.95 \ldots f(4) = \frac{16}{3} \approx 5.3$, $f(\frac{16}{3}) = \frac{52}{9} \approx 5.8$, $f(\frac{52}{9}) = \frac{160}{27} \approx 5.93$, $f(\frac{160}{27}) = \frac{484}{81} \approx 5.975$ \ldots $f(6) = 6$.

b. $c = 6$

c. Solve $c = g(c) = \frac{c}{3} + A$ to get $3c = c + 3A \Rightarrow 2c = 3A \Rightarrow c = \frac{3A}{2}$ is a fixed point of g.

31. On your own.

Section 0.5

1. a. $x = 2, 4$

   b. $x = -2, -1, 0, 1, 2, 3, 4, 5$

   c. $x = -2, -1, 1, 3$

3. a. all x (all real numbers)

   b. $x > \sqrt[3]{-2}$

   c. all x

5. a. $x = -2, -3, 3$

   b. no values of x

   c. $x \geq 0$

7. a. If $x \neq 2$ and $x \neq -3$, then $x^2 + x - 6 \neq 0$. True.

   b. If an object does not have 3 sides, then it is not a triangle. True.

9. a. If your car does not get at least 24 miles per gallon, then it is not tuned properly.

   b. If you cannot have dessert, then you did not eat your vegetables.

11. a. If you will not vote for me, then you do not love your country.

    b. If not only outlaws have guns, then guns are not outlawed. (poor English) If someone legally has a gun, then guns are not illegal.

13. a. Both $f(x)$ and $g(x)$ are not positive.

    b. x is not positive. (\)x \leq 0\))

    c. 8 is not a prime number.

15. a. For some numbers a and b, $|a + b| \neq |a| + |b|$.

https://math.libretexts.org/@go/page/210054

b. Some snake is not poisonous.
c. Some dog can climb trees.

17. If x is an integer, then $2x$ is an even integer. True. Converse: If $2x$ is an even integer, then x is an integer. True. (It is not likely that these were the statements you thought of; there are lots of other examples.)

19. a. False. If $a = 3$, $b = 4$, then $(a+b)^2 = 7^2 = 49$, but $a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25$.
b. False. If $a = -2$, $b = -3$, then $a > b$, but $a^2 = 4 < 9 = b^2$.
c. True.

21. a. True.
b. False. If $f(x) = x + 1$ and $g(x) = x + 2$, Then $f(x) \cdot g(x) = x^2 + 3x + 2$ is not a linear function.
c. True.

23. a. If a and b are prime numbers, then $a + b$ is prime. False: take $a = 3$ and $b = 5$.
b. If a and b are prime numbers, then $a + b$ is not prime. False: take $a = 2$ and $b = 3$.
c. If x is a prime number, then x is odd. False: take $x = 2$. (This is the only counterexample.)
d. If x is a prime number, then x is even. False: take $x = 3$ (or 5 or 7 or…)

25. a. If x is a solution of $x + 5 = 9$, then x is odd. False: take $x = 4$.
b. If a 3-sided polygon has equal sides, then it is a triangle. True. (We also have non-equilateral triangles.)
c. If a person is a calculus student, then that person studies hard. False (unfortunately), but we won't mention names.
d. If x is a (real number) solution of $x^2 - 5x + 6 = 0$, then x is even. False: take $x = 3$.

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CHAPTER OVERVIEW

1: Limits and Continuity

1.0: Tangent Lines, Velocities and Graphs

1.1: The Limit of a Function

1.2: Properties of Limits

1.3: Continuous Functions

1.4: Definition of Limit

1.A: Answers

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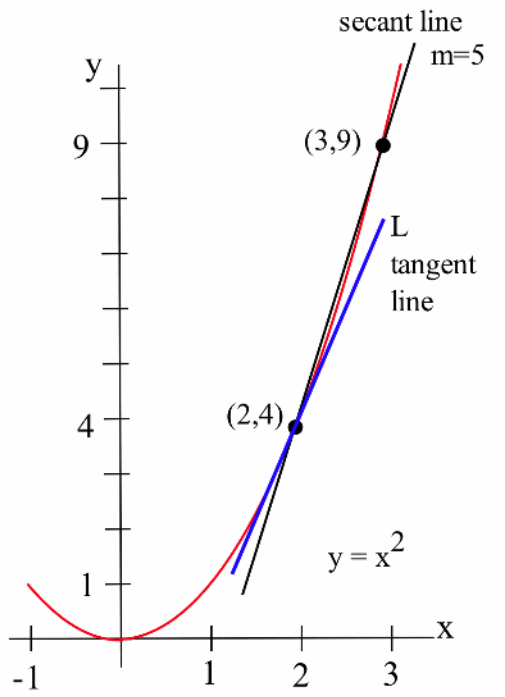
1.0: Tangent Lines, Velocities and Graphs

In Section 0.2, we estimated the slope of a line tangent to the graph of a function *at a point*. At the end of Section 0.3, we constructed a new function that gave the slope of the line tangent to the graph of a given function *at each point*. In both cases, before we could calculate a slope, we had to *estimate* the tangent line from the graph of the given function, a method that required an accurate graph and good estimating. In this section we will begin to look at a more precise method of finding the slope of a tangent line that does not require a graph or any estimation by us. We will start with a non-applied problem and then look at two applications of the same idea.

The Slope of a Line Tangent to a Function at a Point

Our goal is to find a way of exactly determining the slope of the line that is tangent to a function (to the graph of the function) at a point in a way that does not require us to actually have the graph of the function

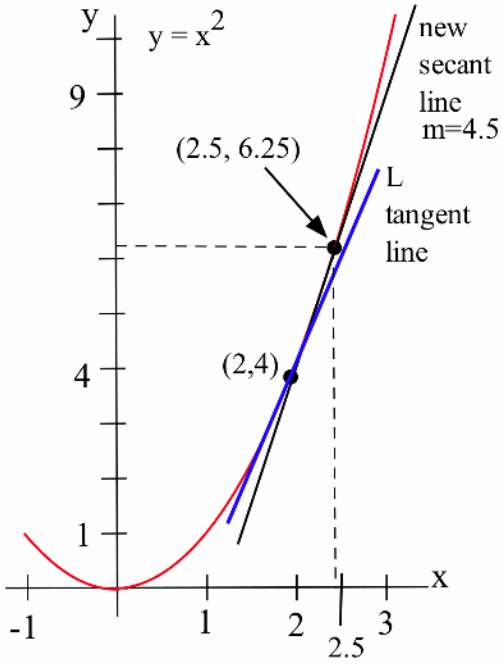
Let's start with the problem of finding the slope of the line L, which is tangent to $f(x) = x^2$ at the point $(2, 4)$:



We could estimate the slope of L from the graph above, but we won't. Instead, we can see that the line through $(2, 4)$ and $(3, 9)$ on the graph of f is an approximation of the slope of the tangent line, and we can calculate that slope exactly:

$$m = \frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5 \tag{1.0.1}$$

But $m = 5$ is only an *estimate* of the slope of the tangent line—and not a very good estimate. It's too big. We can get a better estimate by picking a second point on the graph of f closer to $(2, 4)$—the point $(2, 4)$ is fixed and it must be one of the two points we use:



From the figure above we can see that the slope of the line through the points $(2,4)$ and $(2.5,6.25)$ is a better approximation of the slope of the tangent line at $(2,4)$:

$$m = \frac{\Delta y}{\Delta x} = \frac{6.25-4}{2.5-2} = \frac{2.25}{0.5} = 4.5 \tag{1.0.2}$$

This is a better estimate, but still an approximation.

We can continue picking points closer and closer to $(2,4)$ on the graph of f, and then calculating the slopes of the lines through each of these points (x,y) and the point $(2,4)$:

points to the left of $(2,4)$			points to the right of $(2,4)$		
x	$y=x^2$	slope	x	$y=x^2$	slope
1.5	2.25	3.5	3	9	5
1.9	3.61	3.9	2.5	6.25	4.5
1.99	3.9601	3.99	2.01	4.0401	4.01

The only thing special about the x-values we picked is that they are numbers close—and very close—to $x=2$. Someone else might have picked other nearby values for x. As the points we pick get closer and closer to the point $(2,4)$ on the graph of $y=x^2$, the slopes of the lines through the points and $(2,4)$ are better approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4.

? Practice 1.0.1

What is the slope of the line through $(2,4)$ and (x,y) for $y=x^2$ and $x=1.994$? For $x=2.0003$?

Answer

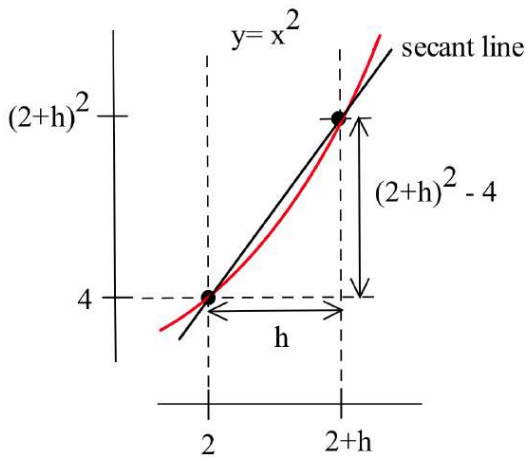
If $x=1.994$, then $y=3.976036$, so the slope between $(2,4)$ and (x,y) is:

$$\frac{4-y}{2-x} = \frac{4-3.976036}{2-1.994} = \frac{0.023964}{0.006} \approx 3.994$$

If $x=2.0003$, then $y \approx 4.0012$, so the slope between $(2,4)$ and (x,y) is:

$$\frac{4-y}{2-x} = \frac{4-4.0012}{2-2.0003} = \frac{-0.0012}{-0.0003} \approx 4.0003$$

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near $(2,4)$. Define $x = 2 + h$ so h is the increment from 2 to x:



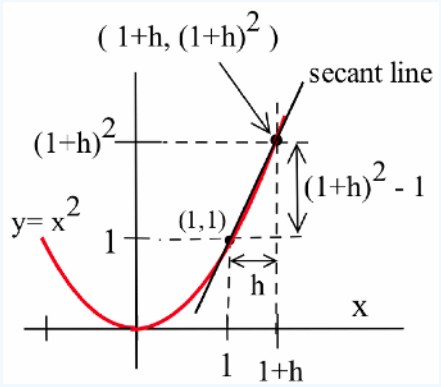
If h is small, then $x = 2 + h$ is close to 2 and the point $(2 + h, f(2 + h)) = (2 + h, (2 + h)^2)$ is close to $(2, 4)$. The slope m of the line through the points $(2, 4)$ and $(2 + h, (2 + h)^2)$ is a good approximation of the slope of the tangent line at the point $(2, 4)$:

$$m = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{(4 + 4h + h^2) - 4}{h}$$
$$= \frac{4h + h^2}{h} = \frac{h(4+h)}{h} = 4 + h$$

If h is very small, then $m = 4 + h$ is a very good approximation to the slope of the tangent line, and $m = 4 + h$ also happens to be very close to the value 4. The value $m = 4 + h$ is called the slope of the **secant line** through the two points $(2, 4)$ and $(2 + h, (2 + h)^2)$. The limiting value 4 of $m = 4 + h$ as h gets smaller and smaller is called the slope of the **tangent line** to the graph of f at $(2, 4)$.

Example 1.0.1

Find the slope of the line tangent to $f(x) = x^2$ at the point $(1, 1)$ by evaluating the slope of the secant line through $(1, 1)$ and $(1 + h, f(1 + h))$ and then determining what happens as h gets very small.



Solution

The slope of the secant line through the points $(1, 1)$ and $(1 + h, f(1 + h))$ is:

$$m = \frac{f(1+h)-1}{(1+h)-1} = \frac{(1+h)^2-1}{h} = \frac{(1+2h+h^2)-1}{h}$$
$$= \frac{2h+h^2}{h} = \frac{h(2+h)}{h} = 2+h$$

As h gets very small, the value of m approaches the value 2, the slope of tangent line at the point $(1,1)$.

? Practice 1.0.2

Find the slope of the line tangent to the graph of $y = f(x) = x^2$ at the point $(-1,1)$ by finding the slope of the secant line, m_{sec}, through the points $(-1,1)$ and $(-1+h, f(-1+h))$ and then determining what happens to m_{sec} as h gets very small.

Answer

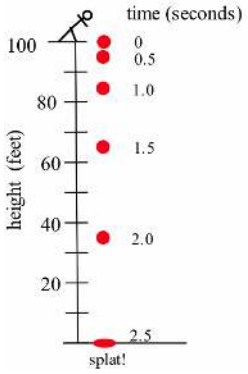
Computing m_{sec}:

$$\frac{f(-1+h)-(1)}{(-1+h)-(-1)} = \frac{(-1+h)^2-1}{h} = \frac{1-2h+h^2-1}{h} = \frac{h(-2+h)}{h} = -2+h$$

As $h \to 0$, $m_{\text{sec}} = -2+h \to -2$.

Falling Tomato

Suppose we drop a tomato from the top of a 100-foot building:



and record its position at various times during its fall:

time (sec)	height (ft)
0.0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

Some questions about this situation are easy to answer directly from the table:

a. How long did it take for the tomato to drop 100 feet? (2.5 seconds)
b. How far did the tomato fall during the first second? (\)100 - 84 = 16\) feet)
c. How far did the tomato fall during the last second? (\)64 - 0 = 64\) feet)
d. How far did the tomato fall between $t = 0.5$ and $t = 1$? (96 - 84 = 12\) feet)

Other questions require a little calculation:

e. What was the average velocity of the tomato during its fall?

$$\text{average velocity} = \frac{\text{distance fallen}}{\text{total time}} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \frac{\text{ft}}{\text{sec}} \tag{1.0.3}$$

f. What was the average velocity between $t = 1$ and $t = 2$ seconds?

$$\text{average velocity} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \frac{\text{ft}}{\text{sec}} \tag{1.0.4}$$

Some questions are more difficult.

g. How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want the {\bf instantaneous velocity}, the velocity at an instant in time. Unfortunately, the tomato is not equipped with a speedometer, so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, $-40 \frac{\text{ft}}{\text{sec}}$. But the tomato fell slowly at the beginning and rapidly near the end, so this estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at $t = 1$ by calculating the average velocities over a short time interval near $t = 1$. The average velocity between $t = 0.5$ and $t = 1$ is:

$$\frac{-12 \text{ feet}}{0.5 \text{ sec}} = -24 \frac{\text{ft}}{\text{sec}} \tag{1.0.5}$$

and the average velocity between $t = 1$ and $t = 1.5$ is

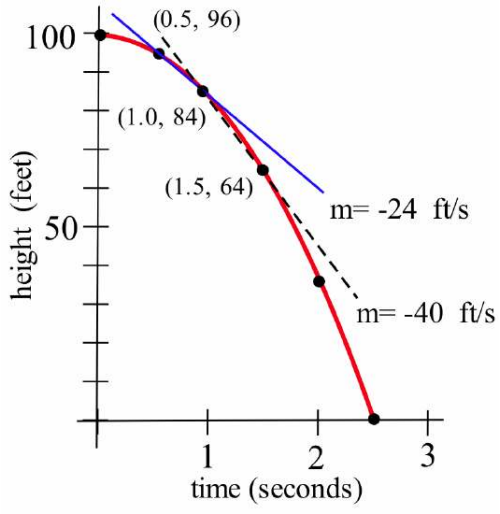
$$\frac{-20 \text{ feet}}{0.5 \text{ sec}} = -40 \frac{\text{ft}}{\text{sec}} \tag{1.0.6}$$

so we can be reasonably sure that the instantaneous velocity is between $-24 \frac{\text{ft}}{\text{sec}}$ and $-40 \frac{\text{ft}}{\text{sec}}$.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is:

$$\frac{\Delta\text{position}}{\Delta\text{time}} \tag{1.0.7}$$

which is the slope of the secant line through two points on the graph of height versus time:



$$\begin{aligned}\text{average velocity} &= \frac{\Delta\text{position}}{\Delta\text{time}} \\ &= \text{slope of the secant line through two points}\end{aligned}$$

The instantaneous velocity at a particular time and height is the slope of the tangent line to the graph at the point given by that time and height.

instantaneous velocity = slope of the line tangent to the graph

? Practice 1.0.3

Estimate the instantaneous velocity of the tomato 2 seconds after it was dropped.

Answer

The average velocity between $t = 1.5$ and $t = 2.0$ is:

$$\frac{36 - 64 \text{ feet}}{2.0 - 1.5 \text{ sec}} = -56 \frac{\text{feet}}{\text{sec}}$$

The average velocity between $t = 2.0$ and $t = 2.5$ is:

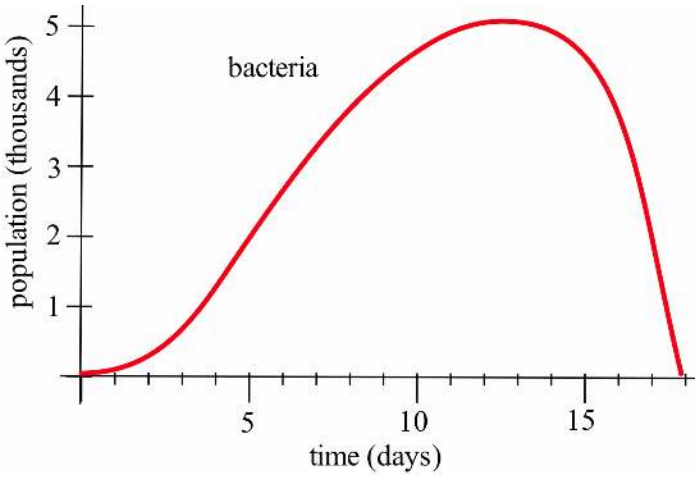
$$\frac{0 - 36 \text{ feet}}{2.5 - 2.0 \text{ sec}} = -72 \frac{\text{feet}}{\text{sec}}$$

The velocity at $t = 2.0$ is somewhere between $-56 \frac{\text{feet}}{\text{sec}}$ and $-72 \frac{\text{feet}}{\text{sec}}$, probably around the middle of this interval:

$$\frac{(-56) + (-72)}{2} = -64 \frac{\text{feet}}{\text{sec}}$$

Growing Bacteria

Suppose we set up a machine to count the number of bacteria growing on a Petri plate:



At first there are few bacteria, so the population grows slowly. Then there are more bacteria to divide, so the population grows more quickly. Later, there are more bacteria and less room and nutrients available for the expanding population, so the population grows slowly again. Finally, the bacteria have used up most of the nutrients and the population declines as bacteria die.

The population graph can be used to answer a number of questions:

a. What is the bacteria population at time $t = 3$ days? (about 500 bacteria)
b. What is the population increment from $t = 3$ to $t = 10$ days? (about $4,000$ bacteria)
c. What is the {\bf rate} of population growth from $t = 3$ to $t = 10$ days?

To answer this last question, we compute the average change in population during that time:

$$\text{average change in population} = \frac{\text{change in population}}{\text{change in time}}$$
$$= \frac{\Delta\text{population}}{\Delta\text{time}} = \frac{4000 \text{ bacteria}}{7 \text{ days}} \approx 570 \frac{\text{bacteria}}{\text{day}}$$

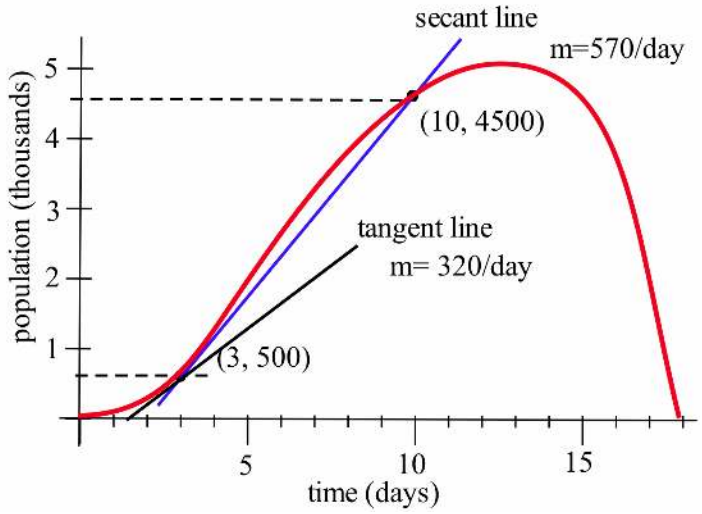
This is the slope of the secant line through $(3, 500)$ and $(10, 4500)$.

> average population growth rate = $\frac{\Delta\text{population}}{\Delta\text{time}}$ = slope of the secant line through two points

Now for a more difficult question:

d. What is the rate of population growth on the third day, at $t = 3$?

This question asks for the instantaneous rate of population change, the slope of the line tangent to the population curve at $(3, 500)$. If we sketch a line approximately tangent to the curve at $(3, 500)$ and pick two points near the ends of the tangent line segment:



we can estimate that the instantaneous rate of population growth is approximately $320 \frac{\text{bacteria}}{\text{day}}$.

> instantaneous population growth rate = slope of the line tangent to the graph

? Practice 1.0.4

Find approximate values for:

a. the average change in population between $t = 9$ and $t = 13$.
b. the rate of population growth at $t = 9$ days.

Answer

a. When $t = 9$ days, the population is approximately $P = 4,200$ bacteria. When $t = 13$, $P \approx 5,000$. The average change in population is approximately:

$$\frac{5000 - 4200 \text{ bacteria}}{13 - 9 \text{ days}} = \frac{800 \text{ bacteria}}{4 \text{ days}} = 200 \frac{\text{bacteria}}{\text{day}}$$

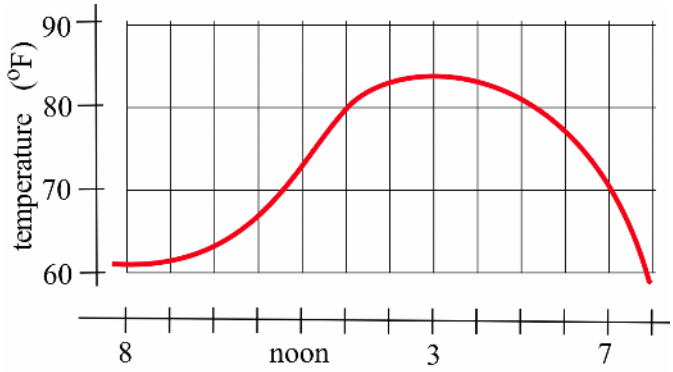
b. To find the rate of population growth at $t = 9$ days, sketch the line tangent to the population curve at the point $(9, 4200)$ and then use $(9, 4200)$ and another point on the tangent line to calculate the slope of the line. Using the approximate values $(5, 2800)$ and $(9, 4200)$, the slope of the tangent line at the point $(9, 4200)$ is approximately:

$$\frac{4200-2800 \text{ bacteria}}{9-5 \text{ days}} = \frac{1400 \text{ bacteria}}{4 \text{ days}} \approx 350 \frac{\text{bacteria}}{\text{day}}$$

The tangent line problem, the instantaneous velocity problem and the instantaneous growth rate problem are all similar. In each problem we wanted to know how rapidly something was **changing at an instant in time**, and each problem turned out to involve finding the **slope of a tangent line**. The approach in each problem was also the same: find an approximate solution and then examine what happens to the approximate solution over shorter and shorter intervals. We will often use this approach of finding a limiting value, but before we can use it effectively we need to describe the concept of a limit with more precision.

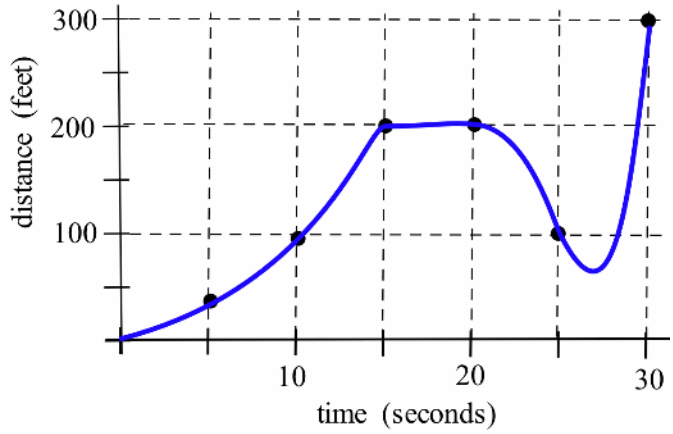
Problems

1. a. What is the slope of the line through $(3,9)$ and (x,y) for $y=x^2$ when:
   i. $x=2.97$?
   ii. $x=3.001$?
   iii. $x=3+h$?
   b. What happens to this last slope when h is very small (close to 0)?
   c. Sketch the graph of $y=x^2$ for x near 3.
2. a. What is the slope of the line through $(-2,4)$ and (x,y) for $y=x^2$ when:
   i. $x=-1.98$?
   ii. $x=-2.03$?
   iii. $x=-2+h$?
   b. What happens to this last slope when h is very small (close to 0)?
   c. Sketch the graph of $y=x^2$ for x near -2.
3. a. What is the slope of the line through $(2,4)$ and (x,y) for $y=x^2+x-2$ when:
   i. $x=1.99$?
   ii. $x=2.004$?
   iii. $x=2+h$?
   b. What happens to this last slope when h is very small (close to 0)?
   c. Sketch the graph of $y=x^2+x-2$ for x near 2.
4. a. What is the slope of the line through $(-1,-2)$ and (x,y) for $y=x^2+x-2$ when:
   i. $x=-0.98$?
   ii. $x=-1.03$?
   iii. $x=-1+h$?
   b. What happens to this last slope when h is very small (close to 0)?
   c. Sketch the graph of $y=x^2+x-2$ for x near -1.
5. The figure below shows the temperature during a day in Ames:



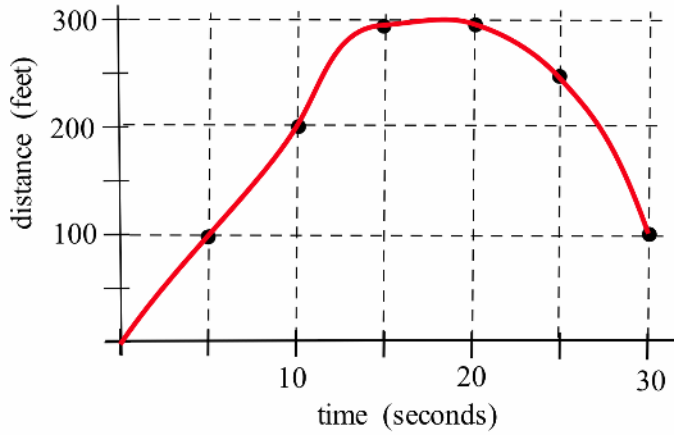
   a. What was the average change in temperature from 9 a.m.\ to 1 p.m.?
   b. Estimate how fast the temperature was rising *at* 10 a.m.\ and *at* 7 p.m.

6. The figure below shows the distance of a car from a measuring position located on the edge of a straight road:



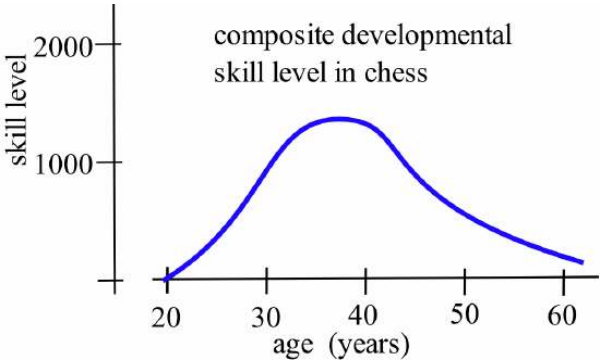
a. What was the average velocity of the car from $t = 0$ to $t = 30$ seconds?
b. What was the average velocity from $t = 10$ to $t = 30$ seconds?
c. About how fast was the car traveling *at* $t = 10$ seconds? At $t = 20$? At $t = 30$?
d. What does the horizontal part of the graph between $t = 15$ and $t = 20$ seconds tell you?
e. What does the negative velocity at $t = 25$ represent?

7. The figure below shows the distance of a car from a measuring position located on the edge of a straight road::



a. What was the average velocity of the car from $t = 0$ to $t = 20$ seconds?
b. What was the average velocity from $t = 10$ to $t = 30$ seconds?
c. About how fast was the car traveling *at* $t = 10$ seconds? At $t = 20$? At $t = 30$?

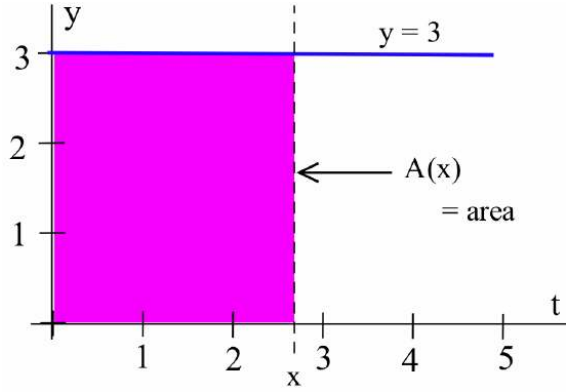
8. The figure below shows the composite developmental skill level of chessmasters at different ages as determined by their performance against other chessmasters.



(From ``Rating Systems for Human Abilities," by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)

a. At what age is the “typical” chessmaster playing the best chess?
b. At approximately what age is the chessmaster’s skill level increasing most rapidly?
c. Describe the development of the ``typical" chessmaster's skill in words.
d. Sketch graphs that you {\bf think} would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician and a professional in your major field.

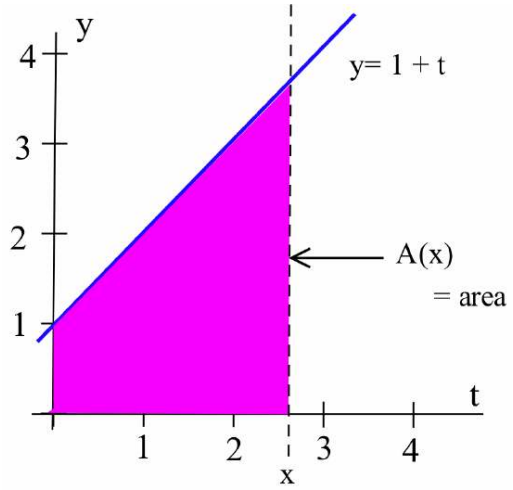
9. Define $A(x)$ to be the **area** bounded by the t- (horizontal) and y-axes, the horizontal line $y = 3$, and the vertical line at x:



For example, $A(4) = 12$ is the area of the 4×3 rectangle.

a. Evaluate $A(0)$, $A(1)$, $A(2)$, $A(2.5)$ and $A(3)$.
b. What area would $A(4) - A(1)$ represent?
c. Graph $y = A(x)$ for $0 \le x \le 4$.

10. Define $A(x)$ to be the **area** bounded by the t- (horizontal) and y-axes, the line $y = t + 1$, and the vertical line at x:



For example, $A(4) = 12$.

a. Evaluate $A(0)$, $A(1)$, $A(2)$, $A(2.5)$ and $A(3)$.
b. What area would $A(3) - A(1)$ represent in the figure?
c. Graph $y = A(x)$ for $0 \le x \le 4$.

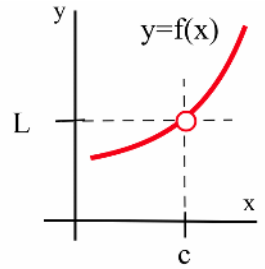
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1.1: The Limit of a Function

Calculus has been called the study of continuous change, and the limit is the basic concept that allows us to describe and analyze such change. An understanding of limits is necessary to understand derivatives, integrals and other fundamental topics of calculus.

The Idea (Informally)

The limit of a function at a point helps describe the behavior of the function when the input variable is near—**but does not equal**—a specified number:



If the values of $f(x)$ are all "very close"—as close as we want—to one number L as we restrict values of x to be "very close" to (but not equal to) a number c, then we say:

> "the limit of $f(x)$, as x approaches c, is L"

and we write:

> $\lim_{x \to c} f(x) = L$

Note

The symbol $\to$ means "approaches" or "gets very close to."

It is very important to note that:

> $f(c)$ is a single number that describes the behavior (value) of f *at* the point $x = c$

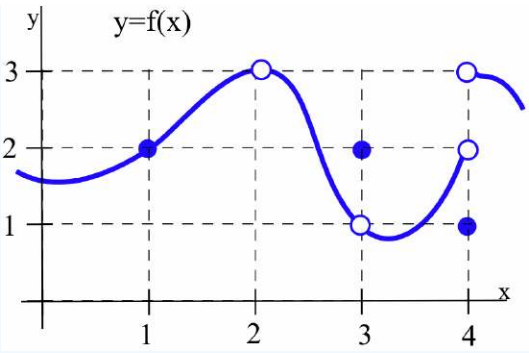
while:

> $\lim_{x \to c} f(x)$ is a single number that describes the behavior of f *near* (but not *at*) the point $x = c$

If we have a graph of the function $f(x)$ near $x = c$, then it is often easy to estimate $\lim_{x \to c} f(x)$.

Example 1.1.1

Use the graph of $y = f(x)$ given here:



to estimate the following limits.

a. $\lim_{x\to 1} f(x)$
b. $\lim_{x\to 2} f(x)$
c. $\lim_{x\to 3} f(x)$
d. $\lim_{x\to 4} f(x)$

Solution

Each of these limits involves a different issue, as you may be able to tell from the graph.

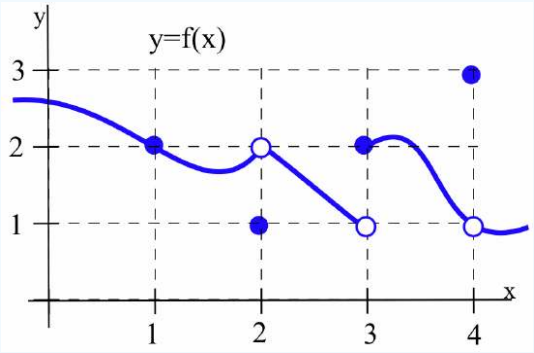
a. $\lim_{x\to 1} f(x) = 2$: When x is very close to 1, the values of $f(x)$ appear to be very close to $y = 2$. In this example, it happens that $f(1) = 2$, but that is irrelevant for the limit. The only thing that matters is what happens for x {\em close to} 1 but with $x \neq 1$.
b. $f(2)$ is undefined, but we only care about the behavior of $f(x)$ for x {\em close to} 2, where the values of $f(x)$ appear to be close to 3. If we restrict x close enough to 2, the values of $f(x)$ will be as close to 3 as we want, so $\lim_{x\to 2} f(x) = 3$.
c. When x is close to 3, the values of $f(x)$ appear to be close to 1, so $\lim_{x\to 3} f(x) = 1$. For this limit it is completely irrelevant that $f(3) = 2$: we only care about what happens to $f(x)$ for x close to (but not equal to) 3.
d. This one is harder and we need to be careful. When x is close to 4 and slightly {\em less than} 4 (\)x\) is just to the left of 4 on the x-axis) then the values of $f(x)$ are close to 2. But if x is close to 4 and slightly *larger than* 4 then the values of $f(x)$ are close to 3.

   If we know only that x is very close to 4, then we cannot say whether $y = f(x)$ will be close to 2 or close to 3—it depends on whether x is on the right or the left side of 4. In this situation, the $f(x)$ values are not all close to a single number when x is close to 4, so we say that $\lim_{x\to 4} f(x)$ **does not exist**.

In (d), it is irrelevant that $f(4) = 1$. The limit, as x approaches 4, would still cease to exist if $f(4)$ was 3 or 2 or anything else.

? Practice 1.1.1

Use the graph of $y = f(x)$ here:



to estimate the following limits:

a. $\lim\limits_{x\to 1} f(x)$
b. $\lim\limits_{t\to 2} f(t)$
c. $\lim\limits_{x\to 3} f(x)$
d. $\lim\limits_{w\to 4} f(w)$

Answer

a. 2
b. 2
c. does not exist (no limit)
d. 1

Example 1.1.2

Determine the value of $\lim\limits_{x\to 3} \dfrac{2x^2-x-1}{x-1}$.

Solution

We need to investigate the values of $f(x)=\frac{2x^2-x-1}{x-1}$ when x is close to 3. If the $f(x)$ values are all arbitrarily close to — or even equal to —some number L, then L will be the limit.

One way to keep track of both the x and the $f(x)$ values is to set up a table and to pick several x values that get closer and closer (but not equal) to 3.

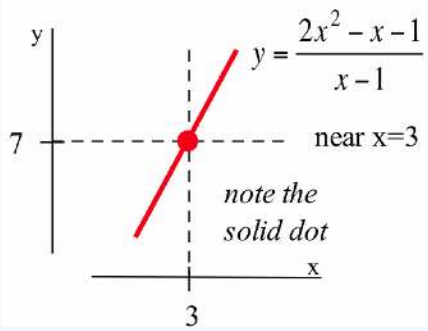
We can pick some values of x that approach 3 from the left, say $x=2.91$, 2.9997, 2.999993 and 2.9999999, and some values of x that approach 3 from the right, say $x=3.1$, 3.004, 3.0001 and 3.000002. The only thing important about these particular values for x is that they get closer and closer to 3 without actually equaling 3. You should try some other values "close to 3" to see what happens. Our table of values is:

x	$f(x)$	x	$f(x)$
2.9	6.82	3.1	7.2
2.9997	6.9994	3.004	7.008
2.999993	6.999986	3.0001	7.0002
2.9999999	6.9999998	3.000002	7.000004
↓	↓	↓	↓
3	7	3	7

As the x values get closer and closer to 3, the $f(x)$ values are all close to 7. In fact, we can get $f(x)$ as close to 7 as we want ("arbitrarily close") by taking the values of x very close ("sufficiently close") to 3. We write:

$$\lim_{x\to 3} \frac{2x^2-x-1}{x-1}=7$$

Instead of using a table of values, we could have graphed $y=f(x)$ for x close to 3:



and used the graph to answer the limit question. This graphical approach is easier, particularly if you have a calculator or computer do the graphing work for you, but it is really very similar to the "table of values" method: in each method you need to evaluate $y = f(x)$ at many values of x near 3.

In the previous example, you might have noticed that if we just evaluate $f(3)$, then we get the correct answer, 7. That works for this particular problem, but it often fails. The next example (identical to the previous one, except $x \to 1$) illustrates one such difficulty.

Example 1.1.3

Find $\lim\limits_{x \to 1} \dfrac{2x^2 - x - 1}{x - 1}$.

Solution

You might try to evaluate $f(x) = \frac{2x^2 - x - 1}{x - 1}$ at $x = 1$, but $f(1) = \frac{0}{0}$, so f is not defined at $x = 1$.

It is tempting—*but wrong*—to conclude that this function does not have a limit as x approaches 1.

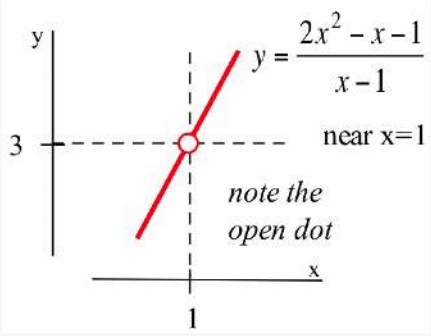
Table Method: Trying some "test" values for x that get closer and closer to 1 from both the left and the right, we get:

x	$f(x)$	x	$f(x)$
0.9	2.82	1.1	3.2
0.9998	2.9996	1.003	3.006
0.999994	2.999988	1.0001	3.0002
0.9999999	2.9999998	1.000007	3.000014
↓	↓	↓	↓
1	3	1	3

The function f is not defined at $x = 1$, but when x gets close to 1, the values of $f(x)$ are very close to 3. We can ensure that $f(x)$ is as close to 3 as we want by restricting x to be very close to 1, so:

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3$$

Graph Method: We can graph $y = f(x) = \frac{2x^2 - x - 1}{x - 1}$ for x close to 1:



and notice that whenever x is close to 1, the values of $y = f(x)$ are close to 3; f is not defined at $x = 1$, so the graph has a hole above $x = 1$, but we only care about what $f(x)$ is doing for x *close to* but *not equal to* 1.

Algebra Method: We could have found the same result by noting:

$$f(x) = \frac{2x^2 - x - 1}{x - 1} = \frac{(2x + 1)(x - 1)}{x - 1} = 2x + 1$$

as long as $x \neq 1$. The “x\rightarrow 1\)” part of the limit means that x is *close to* 1 but *not equal to* 1, so our division step is valid and:

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x \to 1} [2x + 1] = 3$$

which is the same answer we obtained using the first two methods.

Three Methods for Evaluating Limits

The previous example utilized three different methods, each of which led us to the same answer for the limit.

The Algebra Method

The algebra method involves algebraically simplifying the function before trying to evaluate its limit. Often, this simplification just means factoring and dividing, but sometimes more complicated algebraic or even trigonometric steps are needed.

The Table Method

To evaluate a limit of a function $f(x)$ as x approaches c, the table method involves calculating the values of $f(x)$ for “enough” values of x very close to c so that we can “confidently” determine a limiting value of $f(x)$. If $f(x)$ is well behaved, we may not need to use very many values for x. However, this method is usually used with complicated functions, and then we need to evaluate $f(x)$ for lots of values of x.

A computer or calculator can often make the function evaluations easier, but their calculations are subject to “round off” errors. The result of any computer calculation that involves both large and small numbers should be viewed with some suspicion. For example, the function

$$f(x) = \frac{((0.1)^x + 1) - 1}{(0.1)^x} = \frac{(0.1)^x}{(0.1)^x} = 1$$

for every value of x, and my calculator gives the correct answer for some values of x: $f(3) = 1$, and $f(8)$ and $f(9)$ both equal 1.

But my calculator says $((0.1)^{10} + 1) - 1 = 0$, so it evaluates $f(10)$ to be 0, definitely an incorrect value.

Your calculator may evaluate $f(10)$ correctly, but try $f(35)$ or $f(107)$.

> Calculators are too handy to be ignored, but they are too prone to these types of errors to be believed uncritically. Be careful

The Graph Method

The graph method is closely related to the table method, but we create a graph of the function instead of a table of values, and then we use the graph to determine the limiting value of $f(x)$ (if there is one).

Which Method Should You Use?

In general, the algebraic method is preferred because it is precise and does not depend on which values of x we chose or the accuracy of our graph or precision of our calculator. **If you can evaluate a limit algebraically, you should do so.** Sometimes, however, it will be very difficult to evaluate a limit algebraically, and the table or graph methods offer worthwhile alternatives. Even when you can algebraically evaluate the limit of a function, it is still a good idea to graph the function or evaluate it at a few points just to check that your algebraic answer is reasonable.

The table and graph methods have the same advantages and disadvantages. Both can be used on complicated functions that are difficult to handle algebraically or whose algebraic properties you don't know.

Often both methods can be easily programmed on a calculator or computer. However, these two methods are very time-consuming by hand and are prone to round-off errors on computers. You need to know how to use these methods when you can't figure out how to use the algebraic method, but you need to use these two methods warily.

Example 1.1.4

Evaluate each limit.

a. $\lim\limits_{x\to 0} \dfrac{x^2+5x+6}{x^2+3x+2}$

b. $\lim\limits_{x\to -2} \dfrac{x^2+5x+6}{x^2+3x+2}$

Solution

The function in each limit is the same but x is approaching a different number in each of them.

a. Because $x \to 0$, we know that x is getting closer and closer to 0, so the values of the x^2, $5x$ and $3x$ terms get as close to 0 as we want. The numerator approaches 6 and the denominator approaches 2, so the values of the whole function get arbitrarily close to $\frac{6}{2} = 3$, the limit.

b. As x approaches -2, the numerator and denominator approach 0, and a small number divided by a small number can be almost anything—the ratio depends on the size of the top compared to the size of the bottom. More investigation is needed.

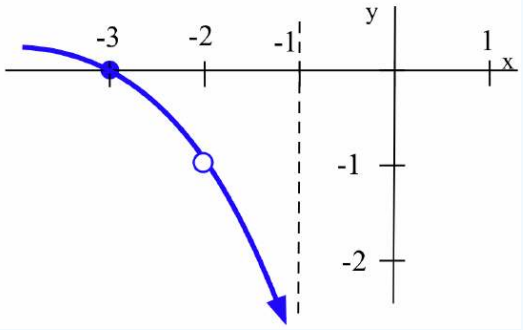
Table Method: If we pick some values of x close to (but not equal to) -2, we get the table:

x	x^2+5x+6	x^2+3x+2	$\frac{x^2+5x+6}{x^2+3x+2}$
-1.97	0.0309	-0.0291	-1.061856
-2.005	-0.004975	0.005025	-0.990050
-1.9998	0.00020004	-0.00019996	-1.00040008
-2.00003	-0.00002999	0.0000300009	-0.9996666
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
-2	0	0	-1

Even though the numerator and denominator are each getting closer and closer to 0, their ratio is getting arbitrarily close to -1, which appears to be the limit.

Graph Method: A graph of $y = f(x) = \frac{x^2+5x+6}{x^2+3x+2}$:

$$y = \frac{x^2+5x+6}{x^2+3x+2}$$
$$= \frac{(x+2)(x+3)}{(x+2)(x+1)}$$



appears to show that the values of $f(x)$ are very close to -1 when the x-values are close to -2.

Algebra Method: Factoring the numerator and denominator:

$$f(x) = \frac{x^2+5x+6}{x^2+3x+2} = \frac{(x+2)(x+3)}{(x+2)(x+1)}$$

We know $x \to -2$ so $x \neq -2$ and we can divide the top and bottom by $(x+2)$. Then

$$f(x) = \frac{(x+3)}{(x+1)} \to \frac{1}{-1} = -1$$

You should remember the technique used in the previous example:

If $\lim\limits_{x\to c} \dfrac{\text{polynomial}}{\text{another polynomial}} = \dfrac{0}{0}$, try dividing the top and bottom by $x-c$.

Practice 1.1.2

Evaluate each limit.

a. $\lim\limits_{x\to 2} \dfrac{x^2-x-2}{x-2}$
b. $\lim\limits_{t\to 0} \dfrac{t\cdot\sin(t)}{t^2+3t}$
c. $\lim\limits_{w\to 2} \dfrac{w-2}{\ln(\frac{w}{2})}$

Answer

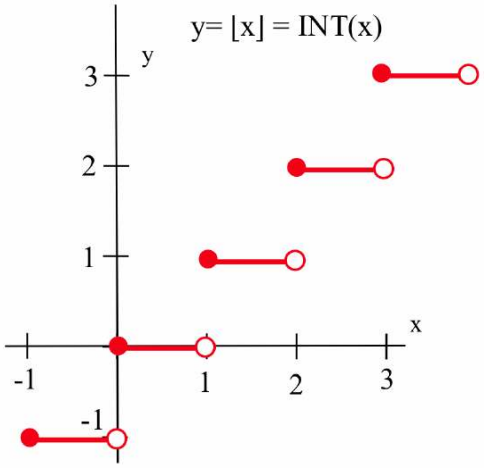
a. $\lim\limits_{x\to 2} \dfrac{(x+1)(x-2)}{x-2} = \lim\limits_{x\to 2}(x+1) = 3$
b. $\lim\limits_{t\to 0} \dfrac{t\sin(t)}{t(t+3)} = \lim\limits_{t\to 0} \dfrac{\sin(t)}{t+3} = \dfrac{0}{3} = 0$
c. $\lim\limits_{w\to 2} \dfrac{w-2}{\ln(\frac{w}{2})} = 2$ To see this, make a graph or a table:

w	$\frac{w-2}{\ln(\frac{w}{2})}$	w	$\frac{w-2}{\ln(\frac{w}{2})}$
2.2	2.098411737	1.9	1.949572575
2.01	2.004995844	1.99	1.994995823
2.003	2.001499625	1.9992	1.999599973
2.0001	2.00005	1.9999	1.99995
↓	↓	↓	↓
2	2	2	2

One-Sided Limits

Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point.

If we let x approach 3 from the left (\)x\) is close to 3 and $x < 3$) then the values of $\lfloor x \rfloor = \text{INT}(x)$ equal 2:



If we let x approach 3 from the right (\)x\) is close to 3 and $x > 3$) then the values of $\lfloor x \rfloor = \text{INT}(x)$ equal 3.

On the number line we can approach a point from the left or the right, and that leads to **one-sided limits**.

Definition: Left and Right Limits

The **left limit** as x approaches c of $f(x)$ is L if the values of $f(x)$ are as close to L as we want when x is very close to but left of c (\)x < c\)):

$$\lim_{x \to c^-} f(x) = L$$

The **right limit**, $\lim_{x \to c^+} f(x)$, requires that x lie to the right of c (\)x > c\)).

Example 1.1.5

Evaluate $\lim_{x \to 2^-} x - \lfloor x \rfloor$ and $\lim_{x \to 2^+} x - \lfloor x \rfloor$.

Solution

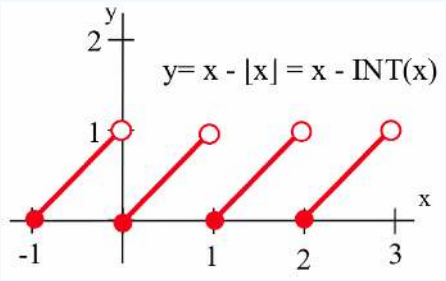
The left-limit notation $x \to 2^-$ requires that x be close to 2 and that x be to the left of 2, so $x < 2$. If $1 < x < 2$, then $\lfloor x \rfloor = 1$ and:

$$\lim_{x \to 2^-} x - \lfloor x \rfloor = \lim_{x \to 2^-} x - 1 = 2 - 1 = 1$$

If x is close to 2 and is to the right of 2, then $2 < x < 3$, so $\lfloor x \rfloor = 2$ and:

$$\lim_{x \to 2^+} x - \lfloor x \rfloor = \lim_{x \to 2^+} x - 2 = 2 - 2 = 0$$

A graph of $f(x) = x - \lfloor x \rfloor$ appears here:



If the left and right limits of $f(x)$ have the same value at $x = c$:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

then the value of $f(x)$ is close to L whenever x is close to c, and it does not matter whether x is left or right of c, so

$$\lim_{x \to c} f(x) = L$$

Similarly, if:

$$\lim_{x \to c} f(x) = L$$

then $f(x)$ is close to L whenever x is close to c and less than c, and whenever x is close to c and greater than c, so:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

We can combine these two statements into a single theorem.

One-Sided Limit Theorem

$\lim_{x \to c} f(x) = L$ if and only if \(\displaystyle \lim_{x \to c^{-}} \, f(x) = \lim_{x \to c^{+}} \, f(x) = L\

This theorem has an important corollary.

Corollary

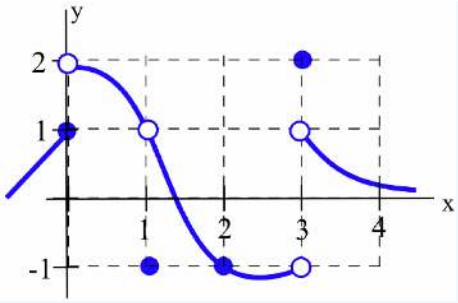
If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then $\lim_{x \to c} f(x)$ does not exist.

One-sided limits are particularly useful for describing the behavior of functions that have steps or jumps.

To determine the limit of a function involving the greatest integer or absolute value or a multiline definition, definitely consider both the left and right limits.

? Practice 1.1.3

Use this graph:



to evaluate the one- and two-sided limits of f at $x = 0$, 1, 2 and 3.

Answer

$\lim\limits_{x\to 0^-} f(x) = 1$, $\lim\limits_{x\to 0^+} f(x) = 2$, $\lim\limits_{x\to 0} f(x)$ DNE
$\lim\limits_{x\to 1^-} f(x) = 1$, $\lim\limits_{x\to 1^+} f(x) = 1$, $\lim\limits_{x\to 1} f(x) = 1$
$\lim\limits_{x\to 2^-} f(x) = -1$, $\lim\limits_{x\to 2^+} f(x) = -1$, $\lim\limits_{x\to 2} f(x) = -1$
$\lim\limits_{x\to 3^-} f(x) = -1$, $\lim\limits_{x\to 3^+} f(x) = 1$, $\lim\limits_{x\to 3} f(x)$ DNE

? Practice 1.1.4

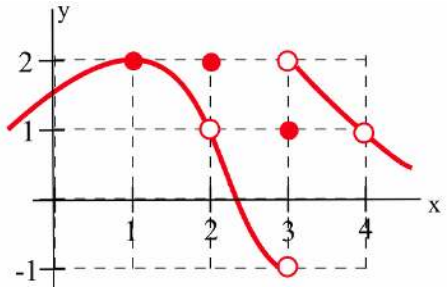
Defining $f(x)$ as: $f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x < 3 \\ 2 & \text{if } 3 < x \end{cases}$ find the one- and two-sided limits of f at 1 and 3.

Answer

$\lim\limits_{x\to 1^-} f(x) = 1$, $\lim\limits_{x\to 1^+} f(x) = 1$, $\lim\limits_{x\to 1} f(x) = 1$
$\lim\limits_{x\to 3^-} f(x) = 3$, $\lim\limits_{x\to 3^+} f(x) = 2$, $\lim\limits_{x\to 3} f(x)$ DNE

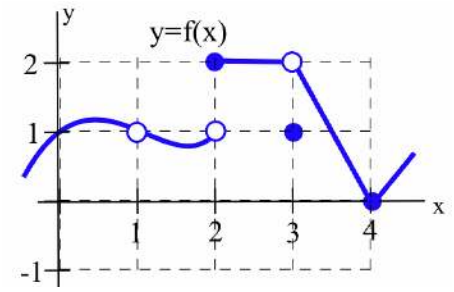
Problems

1. Use the graph below to estimate the limits:



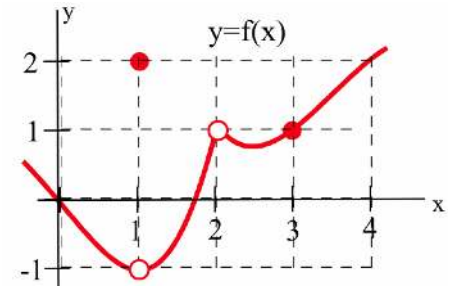
a. $\lim\limits_{x\to 1} f(x)$
b. $\lim\limits_{x\to 2} f(x)$
c. $\lim\limits_{x\to 3} f(x)$
d. $\lim\limits_{x\to 4} f(x)$

2. Use the graph below to estimate the limits.



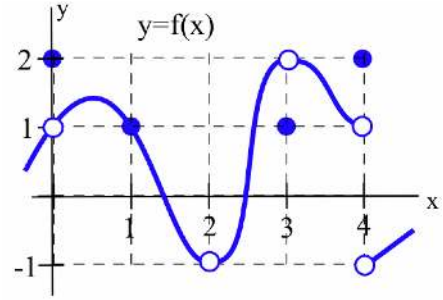
a. $\lim_{x\to 1} f(x)$
b. $\lim_{x\to 2} f(x)$
c. $\lim_{x\to 3} f(x)$
d. $\lim_{x\to 4} f(x)$

3. Use the graph below to estimate the limits.



a. $\lim_{x\to 1} f(2x)$
b. $\lim_{x\to 2} f(x-1)$
c. $\lim_{x\to 3} f(2x-5)$
d. $\lim_{x\to 0} f(4+x)$

4. Use the graph below to estimate the limits.



a. $\lim_{x\to 1} f(3x)$
b. $\lim_{x\to 2} f(x+1)$
c. $\lim_{x\to 3} f(2x-4)$
d. $\lim_{x\to 0} |f(4+x)|$

In Problems 5–11, evaluate (or estimate) each limit.

5. a. $\lim_{x\to 1} \dfrac{x^2+3x+3}{x-2}$
   b. $\lim_{x\to 2} \dfrac{x^2+3x+3}{x-2}$
6. a. $\lim_{x\to 0} \dfrac{x+7}{x^2+9x+14}$

b. $\lim_{x\to 3} \dfrac{x+7}{x^2+9x+14}$

c. $\lim_{x\to -4} \dfrac{x+7}{x^2+9x+14}$

d. $\lim_{x\to -7} \dfrac{x+7}{x^2+9x+14}$

7. a. $\lim_{x\to 1} \dfrac{\cos(x)}{x}$

b. $\lim_{x\to \pi} \dfrac{\cos(x)}{x}$

c. $\lim_{x\to -1} \dfrac{\cos(x)}{x}$

8. a. $\lim_{x\to 7} \sqrt{x-3}$

b. $\lim_{x\to 9} \sqrt{x}-3$

c. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

9. a. $\lim_{x\to 0^-} |x|$

b. $\lim_{x\to 0^+} |x|$

c. $\lim_{x\to 0} |x|$

10. a. $\lim_{x\to 0^-} \dfrac{|x|}{x}$

b. $\lim_{x\to 0^+} \dfrac{|x|}{x}$

c. $\lim_{x\to 0} \dfrac{|x|}{x}$

11. a. $\lim_{x\to 5} |x-5|$

b. $\lim_{x\to 3} \dfrac{|x-5|}{x-5}$

c. $\lim_{x\to 5} \dfrac{|x-5|}{x-5}$

12. Find the one- and two-sided limits as $x \to 0$, 1, and 2 of: $f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } 0 < x \le 2 \\ 1 & \text{if } 2 < x \end{cases}$

13. Find the one- and two-sided limits as $x \to 1$, 2, 4 and 5 of $g(x) = \begin{cases} 1 & \text{if } x \le 2 \\ \frac{8}{x} & \text{if } 2 < x < 4 \\ 6-x & \text{if } 4 < x \end{cases}$

In 14–17, use a calculator or computer to get approximate answers accurate to 2 decimal places.

14. a. $\lim_{x\to 0} \dfrac{2^x-1}{x}$

b. $\lim_{x\to 1} \dfrac{\log_{10}(x)}{x-1}$

15. a. $\lim_{x\to 0} \dfrac{3^x-1}{x}$

b. $\lim_{x\to 1} \dfrac{\ln(x)}{x-1}$

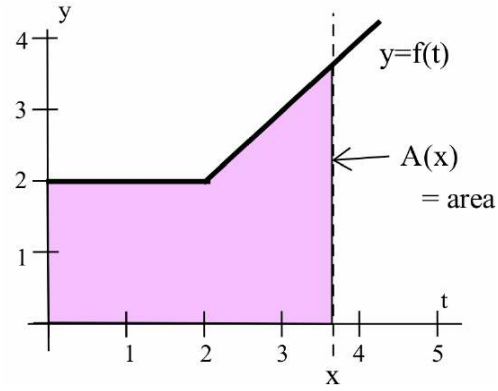
16. a. $\lim_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}$

b. $\lim_{x\to 0} \dfrac{\sin(3x)}{5x}$

17. a. $\lim_{x\to 16} \dfrac{\sqrt{x}-4}{x-16}$

b. $\lim_{x\to 0} \dfrac{\sin(7x)}{2x}$

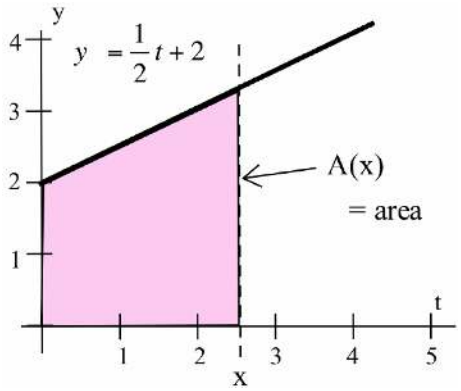
18. Define $A(x)$ to be the **area** bounded by the t- and y-axes, the "bent line" in the figure below:



and the vertical line $t = x$. For example, $A(4) = 10$.

a. Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.
b. Graph $y = A(x)$ for $0 \le x \le 4$.
c. What area does $A(3) - A(1)$ represent?

19. Define $A(x)$ to be the **area** bounded by the t- and y-axes, the line $y = \frac{1}{2}t + 2$ and the vertical line $t = x$:



For example, \)A(4) = 12\).

a. Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.
b. Graph $y = A(x)$ for $0 \le x \le 4$.
c. What area does $A(3) - A(1)$ represent?

20. Sketch the graph of $f(t) = \sqrt{4t - t^2}$ for $0 \le t \le 4$ (you should get a semicircle). Define $A(x)$ to be the area bounded below by the t-axis, above by the graph $y = f(t)$ and on the right by the vertical line at $t = x$.

a. Evaluate $A(0)$, $A(2)$ and $A(4)$.
b. Sketch a graph $y = A(x)$ for $0 \le x \le 4$.
c. What area does $A(3) - A(1)$ represent?

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1.2: Properties of Limits

This section presents results that make it easier to calculate limits of combinations of functions or to show that a limit does not exist. The main result says we can determine the limit of "elementary combinations" of functions by calculating the limit of each function separately and recombining these results to get our final answer.

Main Limit Theorem

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then:

a. $\lim_{x\to a} [f(x) + g(x)] = L + M$
b. $\lim_{x\to a} [f(x) - g(x)] = L - M$
c. $\lim_{x\to a} k \cdot f(x) = k \cdot L$
d. $\lim_{x\to a} f(x) \cdot g(x) = L \cdot M$
e. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ (if $M \neq 0$)
f. $\lim_{x\to a} [f(x)]^n = L^n$
g. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

When n is an even integer in part (g), we need $L \geq 0$ and $f(x) \geq 0$ for x near a.

The Main Limit Theorem says we get the same result if we first perform the algebra and then take the limit or if we take the limits first and then perform the algebra: for example, (a) says that the limit of the sum equals the sum of the limits.

A proof of the Main Limit Theorem is not inherently difficult, but it requires a more precise definition of the limit concept than we have at the moment, and it then involves a number of technical difficulties.

Practice 1.2.1

For $f(x) = x^2 - x - 6$ and $g(x) = x^2 - 2x - 3$, evaluate:

a. $\lim_{x\to 1} [f(x) + g(x)]$
b. $\lim_{x\to 1} f(x) \cdot g(x)$
c. $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$
d. $\lim_{x\to 3} [f(x) + g(x)]$
e. $\lim_{x\to 3} f(x) \cdot g(x)$
f. $\lim_{x\to 3} \dfrac{f(x)}{g(x)}$
g. $\lim_{x\to 2} [f(x)]^3$
h. $\lim_{x\to 2} \sqrt{1 - g(x)}$

Answer

a. -10
b. 24
c. $\frac{3}{2}$
d. 0
e. 0
f. $\frac{5}{4}$
g. -64
h. 2

Limits of Some Very Nice Functions: Substitution

As you may have noticed in the previous example, for some functions $f(x)$ it is possible to calculate the limit as x approaches a simply by substituting $x = a$ into the function and then evaluating $f(a)$, but sometimes this method does not work. The following results help to (partially) answer the question about when such a substitution is valid.

Two Easy Limits

$\lim_{x \to a} k = k$ and $\lim_{x \to a} x = a$

We can use the preceding Two Easy Limits and the Main Limit Theorem to prove the following Substitution Theorem.

Substitution Theorem For Polynomial and Rational Functions

If: $P(x)$ and $Q(x)$ are polynomials and a is any number

then: $\lim_{x \to a} P(x) = P(a)$ and $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ (as long as $Q(a) \neq 0$).

The Substitution Theorem says that we can calculate the limits of polynomials and rational functions by substituting (as long as the substitution does not result in a division by 0).

Practice 1.2.2

Evaluate each limit.

a. $\lim_{x \to 2} \left[5x^3 - x^2 + 3\right]$

b. $\lim_{x \to 2} \frac{x^3 - 7x}{x^2 + 3x}$

c. $\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - x - 2}$

Answer

a. 39

b. $-\frac{3}{5}$

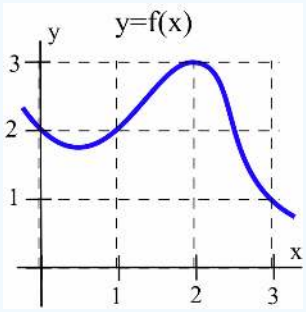
c. $\frac{2}{3}$

Limits of Other Combinations of Functions

So far we have concentrated on limits of single functions and elementary combinations of functions. If we are working with limits of other combinations or compositions of functions, the situation becomes slightly more difficult, but sometimes these more complicated limits have useful geometric interpretations.

Example 1.2.1

Use the graph below to estimate each limit.



a. $\lim_{x\to 1}\,[3+f(x)]$
b. $\lim_{x\to 1}\,f(2+x)$
c. $\lim_{x\to 0}\,f(3-x)$
d. $\lim_{x\to 2}\,[f(x+1)-f(x)]$

Solution

a. $\lim_{x\to 1}\,[3+f(x)]$ requires a straightforward application of part (a) of the Main Limit Theorem:

$$\lim_{x\to 1}\,[3+f(x)]=\lim_{x\to 1}\,3+\lim_{x\to 1}\,f(x)=3+2=5$$

b. We first need to examine what happens to the quantity $2+x$ as $x\to 1$ before we can consider the limit of $f(2+x)$. When x is very close to 1, the value of $2+x$ is very close to 3, so the limit of $f(2+x)$ as $x\to 1$ is equivalent to the limit of $f(w)$ as $w\to 3$ (where $w=2+x$) and it is clear from the graph that $\lim_{w\to 3}\,f(w)=1$, so:

$$\lim_{x\to 1}\,f(2+x)=\lim_{w\to 3}\,f(w)=1$$

In most situations it is not necessary to formally substitute a new variable w for the quantity $2+x$, but it is still necessary to think about what happens to the quantity $2+x$ as $x\to 1$.

c. As $x\to 0$ the quantity $3-x$ will approach 3, so we want to know what happens to the values of f when the input variable is approaching 3:

$$\lim_{x\to 0}\,f(3-x)=1$$

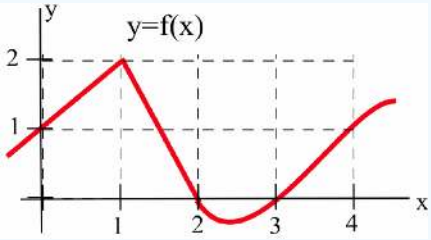
d. Using part (b) of the Main Limit Theorem:

$$\begin{aligned}\lim_{x\to 2}\,[f(x+1)-f(x)]&=\lim_{x\to 2}\,f(x+1)-\lim_{x\to 2}\,f(x)\\&=\lim_{w\to 3}\,f(w)-\lim_{x\to 2}\,f(x)=1-3=-2\end{aligned}$$

(Notice the use of the substitution $w=x+1$ above.)

? Practice 1.2.3

Use the graph below to estimate each limit.



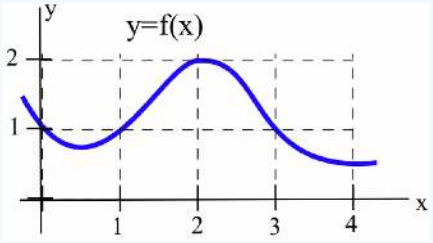
a. $\lim_{x\to 1}\,f(2x)$
b. $\lim_{x\to 2}\,f(x-1)$
c. $\lim_{x\to 0}\,3\cdot f(4+x)$
d. $\lim_{x\to 2}\,f(3x-2)$

Answer

a. 0
b. 2
c. 3
d. 1

Example 1.2.2

Use the graph below to estimate each limit.



a. $\lim_{h\to 0} f(3+h)$
b. $\lim_{h\to 0} f(3)$
c. $\lim_{h\to 0} [f(3+h)-f(3)]$
d. $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$

The last limit is a special type of limit we will encounter often in this book, while the first three parts are the steps we need to evaluate it.

Solution

a. As $h \to 0$, the quantity $w = 3+h$ will approach 3, so

$$\lim_{h\to 0} f(3+h) = \lim_{w\to 3} f(w) = 1$$

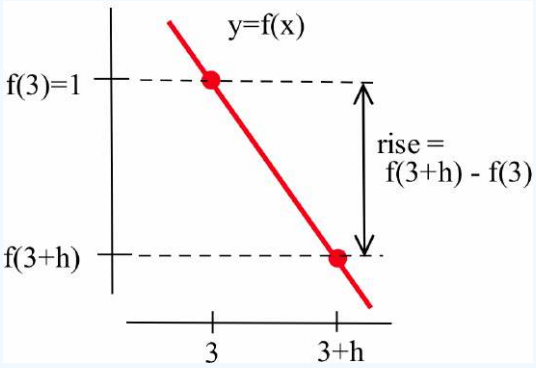
b. $f(3)$ is a constant (equal to 1) and does not depend on h in any way, so:

$$\lim_{h\to 0} f(3) = f(3) = 1$$

c. This limit is just an algebraic combination of the first two limits:

$$\lim_{h\to 0} [f(3+h)-f(3)] = \lim_{h\to 0} f(3+h) - \lim_{h\to 0} f(3) = 1-1 = 0$$

The quantity $f(3+h)-f(3)$ also has a geometric interpretation: it is the change in the y-coordinates, the Δy, between the points $(3, f(3))$ and $(3+h, f(3+h))$:



d. As $h \to 0$, the numerator and denominator of $\dfrac{f(3+h)-f(3)}{h}$ both approach 0, so we cannot immediately determine the value of the limit. But if we recognize that $f(3+h)-f(3) = \Delta y$ for the two points $(3, f(3))$ and $(3+h, f(3+h))$ and that $h = \Delta x$ for the same two points, then we can interpret $\dfrac{f(3+h)-f(3)}{h}$ as $\frac{\Delta y}{\Delta x}$, which is the slope of the secant line through the two points:

$$\begin{aligned}\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} &= \lim_{\Delta x\to 0} [\text{slope of the secant line}] \\ &= \text{slope of the tangent line at } (3, f(3)) \\ &\approx -1\end{aligned}$$

This last limit represents the slope of line tangent to the graph of f at the point $(3, f(3))$. (It is a pattern we will encounter often.)

Tangent Lines as Limits

If we have two points on the graph of the function $y = f(x)$:

$$(x, f(x)) \text{ and } (x+h, f(x+h))$$

then $\Delta y = f(x+h) - f(x)$ and $\Delta x = (x+h) - (x) = h$, so the slope of the secant line through those points is:

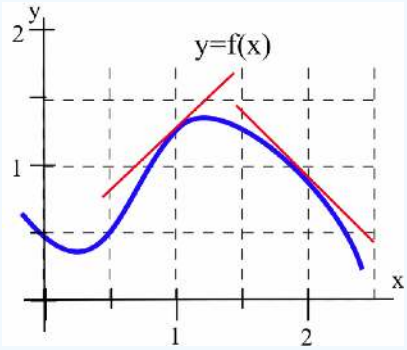
$$m_{\text{sec}} = \frac{\Delta y}{\Delta x}$$

and the slope of the line tangent to the graph of f at the point $(x, f(x))$ is, by definition,

$$m_{\text{tan}} = \lim_{\Delta x \to 0} [\text{slope of the secant line}] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1.2.3

Give a geometric interpretation for the following limits and **estimate** their values for the function whose graph appears below.



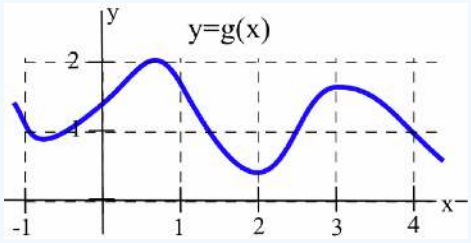
a. $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$
b. $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

Solution

a. The limit represents the slope of the line tangent to the graph of $f(x)$ at the point $(1, f(1))$, so $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \approx 1$.
b. The limit represents the slope of the line tangent to the graph of $f(x)$ at the point $(2, f(2))$, so $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \approx -1$.

Practice 1.2.4

Give a geometric interpretation for the following limits and estimate their values for the function whose graph appears below.



a. $\lim_{h \to 0} \frac{g(1+h) - g(1)}{h}$
b. $\lim_{h \to 0} \frac{g(3+h) - g(3)}{h}$

c. $\lim_{h \to 0} \dfrac{g(h) - g(0)}{h}$

Answer

a. slope of the line tangent to the graph of g at the point $(1, g(1))$; estimated slope ≈ -2
b. slope of the line tangent to the graph of g at the point $(3, g(3))$; estimated slope ≈ 0
c. slope of the line tangent to the graph of g at the point $(0, g(0))$; estimated slope ≈ 1

Comparing the Limits of Functions

Sometimes it is difficult to work directly with a function. However, if we can compare our complicated function with simpler ones, then we can use information about the simpler functions to draw conclusions about the complicated one. If the complicated function is always between two functions whose limits are equal, then we know the limit of the complicated function.

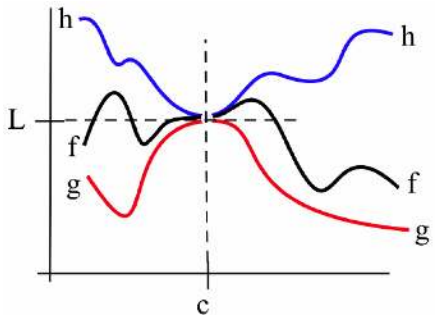
Squeezing Theorem

If: $g(x) \leq f(x) \leq h(x)$ for all x near (but not equal to) c

and: $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

then: $\lim_{x \to c} f(x) = L$.

The figure below shows the idea behind the proof of this theorem:



The function $f(x)$ gets "squeezed" between the smaller function $g(x)$ and the bigger function $h(x)$. Because $g(x)$ and $h(x)$ converge to the same limit, L, so must $f(x)$.

We can use the Squeezing Theorem to evaluate some "hard" limits by squeezing a "complicated" function in between two "simpler" functions with "easier" limits.

Example 1.2.4

Use the inequality $-|x| \leq \sin(x) \leq |x|$ to determine:

a. $\lim_{x \to 0} \sin(x)$
b. $\lim_{x \to 0} \cos(x)$

Solution

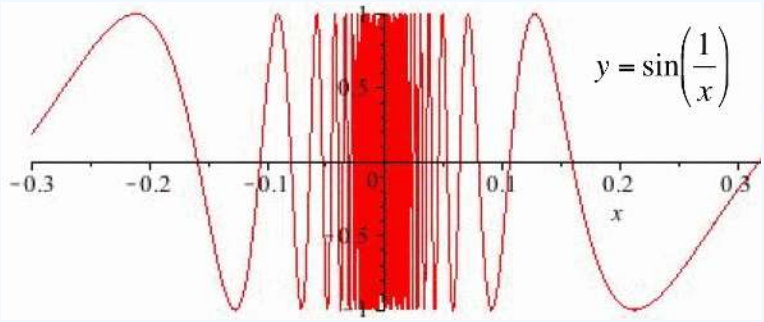
a. $\lim_{x \to 0} |x| = 0$ and $\lim_{x \to 0} -|x| = 0$ so, by the Squeezing Theorem, $\lim_{x \to 0} \sin(x) = 0$.
b. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\cos(x) = \sqrt{1 - \sin^2(x)}$, so $\lim_{x \to 0} \cos(x) = \lim_{x \to 0} \sqrt{1 - \sin^2(x)} = \sqrt{1 - 0^2} = 1$.

Example 1.2.5

Evaluate $\lim_{x \to 0} x \cdot \sin\left(\dfrac{1}{x}\right)$.

Solution

In the graph of $\sin\left(\frac{1}{x}\right)$:



the y-values change very rapidly for values of x near 0, but they all lie between -1 and 1:

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

so, if $x > 0$, multiplying this inequality by x we get:

$$-x \le x \cdot \sin\left(\frac{1}{x}\right) \le x$$

which we can rewrite as:

$$-|x| \le x \cdot \sin\left(\frac{1}{x}\right) \le |x|$$

because $|x| = x$ when $x > 0$.

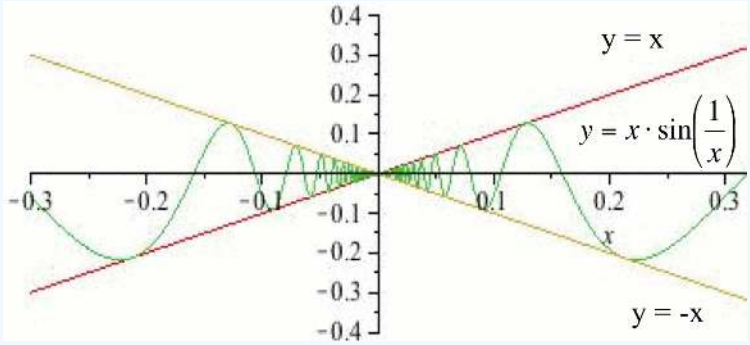
If $x < 0$, when we multiply the original inequality by x we get:

$$-x \ge x \cdot \sin\left(\frac{1}{x}\right) \ge x \quad \Rightarrow \quad |x| \ge x \cdot \sin\left(\frac{1}{x}\right) \ge -|x|$$

because $|x| = -x$ when $x < 0$. Either way we have:

$$-|x| \le x \cdot \sin\left(\frac{1}{x}\right) \le |x|$$

for all $x \neq 0$, and in particular for x near 0.



Both "simple" functions ($-|x|$ and $|x|$) approach 0 as $x \to 0$, so

$$\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

by the Squeezing Theorem.

? Practice 1.2.5

If $f(x)$ is always between x^2+2 and $2x+1$, what can you say about $\lim_{x\to 1} f(x)$?

Answer

$\lim_{x\to 1}\left[x^2+2\right]=3$ and $\lim_{x\to 1}\left[2x+1\right]=3$ so $\lim_{x\to 1} f(x)=3$

? Practice 1.2.6

Use the relation $\cos(x)\le\frac{\sin(x)}{x}\le 1$ (Problem 27 guides you through the steps to prove this relation) to show that:

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

Answer

$\lim_{x\to 0}\cos(x)=1$ and $\lim_{x\to 0}1=1$ so $\lim_{x\to 0}\frac{\sin(x)}{x}=1$

List Method for Showing that a Limit Does Not Exist

If the limit of $f(x)$, as x approaches c, exists and equals L, then we can guarantee that the values of $f(x)$ are as close to L as we want by restricting the values of x to be very, very close to c. To show that a limit, as x approaches c, does **not** exist, we need to show that no matter how closely we restrict the values of x to c, the values of $f(x)$ are not **all** close to a single, finite value L.

One way to demonstrate that $\lim_{x\to c} f(x)$ does not exist is to show that the left and right limits exist but are not equal.

Another method of showing that $\lim_{x\to c} f(x)$ does not exist uses two (infinite) lists of numbers, $\{a_1,a_2,a_3,a_4,\ldots\}$ and $\{b_1,b_2,b_3,b_4,\ldots\}$, with entries that become arbitrarily close to the value c as the subscripts get larger, but for which the corresponding lists of function values, $\{f(a_1),f(a_2),f(a_3),f(a_4),\ldots\}$ and $\{f(b_1),f(b_2),f(b_3),f(b_4),\ldots\}$ approach two different numbers as the subscripts get larger.

✓ Example 1.2.6

For $f(x)$ defined as:

$$f(x)=\begin{cases}1 & \text{if } x<1\\ x & \text{if } 1<x<3\\ 2 & \text{if } 3<x\end{cases}$$

show that $\lim_{x\to 3} f(x)$ does not exist.

Solution

We could use one-sided limits to show that this limit does not exist, but instead we will use the list method.

One way to define values of $\{a_1,a_2,a_3,a_4,\ldots\}$ that approach 3 from the right is to define $a_1=3+1$, $a_2=3+\frac{1}{2}$, $a_3=3+\frac{1}{3}$, $a_4=3+\frac{1}{4}$ and, in general, $a_n=3+\frac{1}{n}$. Then $a_n>3$ so $f(a_n)=2$ for all subscripts n, and the values in the list $\{f(a_1),f(a_2),f(a_3),f(a_4),\ldots\}$ are all close to 2—in fact, all of the $f(a_n)$ values *equal* 2.

We can define values of $\{b_1,b_2,b_3,b_4,\ldots\}$ that approach 3 from the left by $b_1=3-1$, $b_2=3-\frac{1}{2}$, $b_3=3-\frac{1}{3}$, $b_4=3-\frac{1}{4}$, and, in general, $b_n=3-\frac{1}{n}$. Then $b_n<3$ so $f(b_n)=b_n=3-\frac{1}{n}$ for each subscript n, and the values in the list $\{f(b_1),f(b_2),f(b_3),f(b_4),\ldots\}=\{2,2.5,2\frac{2}{3},2\frac{3}{4},2\frac{4}{5},\ldots,3-\frac{1}{n},\ldots\}$ are all close to 3 for large values of n.

Because the values in the lists $\{f(a_1),f(a_2),f(a_3),f(a_4),\ldots\}$ and $\{f(b_1),f(b_2),f(b_3),f(b_4),\ldots\}$ have two different limiting values, we can conclude that $\lim_{x\to 3} f(x)$ does not exist.

Example 1.2.7

Define $h(x)$ as:

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

(the "holey" function introduced in Section 0.4). Use the list method to show that $\lim_{x \to 3} h(x)$ does not exist.

Solution

Let $\{a_1, a_2, a_3, a_4, ...\}$ be a list of rational numbers that approach 3: for example, $a_1 = 3 + 1$, $a_2 = 3 + \frac{1}{2}$, $a_3 = 3 + \frac{1}{3}, \ldots, a_n = 3 + \frac{1}{n}$. Then $f(a_n) = 2$ for all n, so:

$$\{f(a_1), f(a_2), f(a_3), f(a_4), ...\} = \{2, 2, 2, 2, ...\}$$

and the $f(a_n)$ values are all "close to" (in fact, equal) 2.

If $\{b_1, b_2, b_3, b_4, ...\}$ is a list of irrational numbers that approach 3 (for example, $b_1 = 3 + \pi$, $b_2 = 3 + \frac{\pi}{2}, \ldots, b_n = 3 + \frac{\pi}{n}$) then:

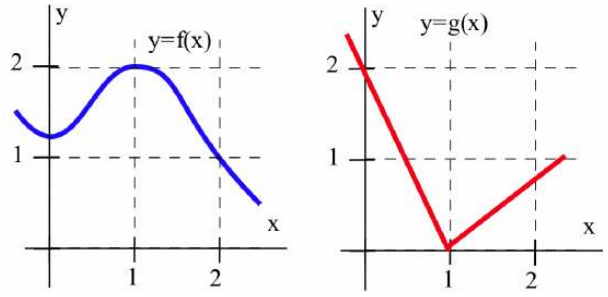
$$\{f(b_1), f(b_2), f(b_3), f(b_4), ...\} = \{1, 1, 1, 1, ...\}$$

and the $f(b_n)$ values are all close to 1 for large values of n.

Because the $f(a_n)$ and $f(b_n)$ values become close to two different numbers, the limit of $f(x)$ as $x \to 3$ does not exist. A similar argument will work as x approaches any number c, so for every c we can show that $\lim_{x \to c}(x)$ does not exist. The "holey" function does not have a limit as x approaches *any* value c.

Problems

1. Use the functions f and g defined by the graphs below to determine the following limits.

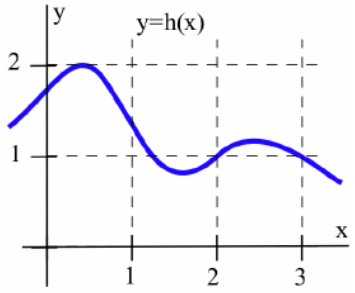


   a. $\lim_{x \to 1} [f(x) + g(x)]$
   b. $\lim_{x \to 1} f(x) \cdot g(x)$
   c. $\lim_{x \to 1} \frac{f(x)}{g(x)}$
   d. $\lim_{x \to 1} f(g(x))$

2. Use the functions f and g defined by the graphs in Problem 1 to determine the following limits.

   a. $\lim_{x \to 2} [f(x) + g(x)]$
   b. $\lim_{x \to 2} f(x) \cdot g(x)$
   c. $\lim_{x \to 2} \frac{f(x)}{g(x)}$
   d. $\lim_{x \to 2} f(g(x))$

3. Use the function h defined by the graph below to determine the following limits.

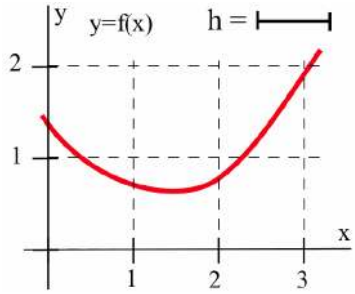


a. $\lim_{x \to 2} h(2x-2)$
b. $\lim_{x \to 2} [x + h(x)]$
c. $\lim_{x \to 2} h(1+x)$
d. $\lim_{x \to 3} h\left(\frac{x}{2}\right)$

4. Use the function h defined by the graph in Problem 3 to determine the following limits.

a. $\lim_{x \to 2} h(5-x)$
b. $\lim_{x \to 0} [h(3+x) - h(3)]$
c. $\lim_{x \to 2} x \cdot h(x-1)$
d. $\lim_{x \to 0} \frac{h(3+x) - h(3)}{x}$

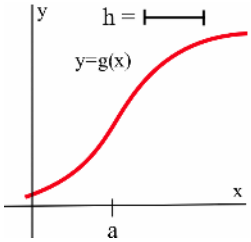
5. Label the parts of the graph of f (below):



that are described by:

a. $2+h$
b. $f(2)$
c. $f(2+h)$
d. $f(2+h) - f(2)$
e. $\frac{f(2+h) - f(2)}{(2+h) - 2}$
f. $\frac{f(2-h) - f(2)}{(2-h) - 2}$

6. Label the parts of the graph of g (below):

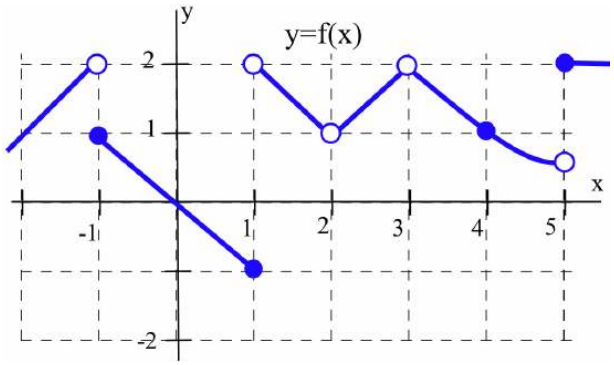


that are described by:

a. $a+h$
b. $g(a)$

c. $g(a+h)$
d. $g(a+h)-g(a)$
e. $\dfrac{g(a+h)-g(a)}{(a+h)-a}$
f. $\dfrac{g(a-h)-g(a)}{(a-h)-a}$

7. Use the graph below to estimate each limit:



a. $\lim\limits_{x\to 1^+} f(x)$
b. $\lim\limits_{x\to 1^-} f(x)$
c. $\lim\limits_{x\to 1} f(x)$
d. $\lim\limits_{x\to 3^+} f(x)$
e. $\lim\limits_{x\to 3^-} f(x)$
f. $\lim\limits_{x\to 3} f(x)$
g. $\lim\limits_{x\to -1^+} f(x)$
h. $\lim\limits_{x\to -1^-} f(x)$
i. $\lim\limits_{x\to -1} f(x)$

8. Use the graph from Problem 7 to estimate:

a. $\lim\limits_{x\to 2^+} f(x)$
b. $\lim\limits_{x\to 2^-} f(x)$
c. $\lim\limits_{x\to 2} f(x)$
d. $\lim\limits_{x\to 4^+} f(x)$
e. $\lim\limits_{x\to 4^-} f(x)$
f. $\lim\limits_{x\to 4} f(x)$
g. $\lim\limits_{x\to -2^+} f(x)$
h. $\lim\limits_{x\to -2^-} f(x)$
i. $\lim\limits_{x\to -2} f(x)$

9. The Lorentz contraction formula in relativity theory says the length L of an object moving at v miles per second with respect to an observer is:

$$L = A \cdot \sqrt{1-\frac{v^2}{c^2}}$$

where c is the speed of light (a constant).

a. Determine the object's "rest length" ($v = 0$).
b. Determine: $\lim\limits_{v\to c^-} L$

10. Evaluate each limit.

a. $\lim\limits_{x\to 2^+} \lfloor x \rfloor$

b. $\lim_{x \to 2^-} \lfloor x \rfloor$
c. $\lim_{x \to -2^+} \lfloor x \rfloor$
d. $\lim_{x \to -2^-} \lfloor x \rfloor$
e. $\lim_{x \to -2.3} \lfloor x \rfloor$
f. $\lim_{x \to 3} \left\lfloor \frac{x}{2} \right\rfloor$
g. $\lim_{x \to 3} \frac{\lfloor x \rfloor}{2}$
h. $\lim_{x \to 0^+} \frac{\lfloor 2 + x \rfloor - \lfloor 2 \rfloor}{x}$

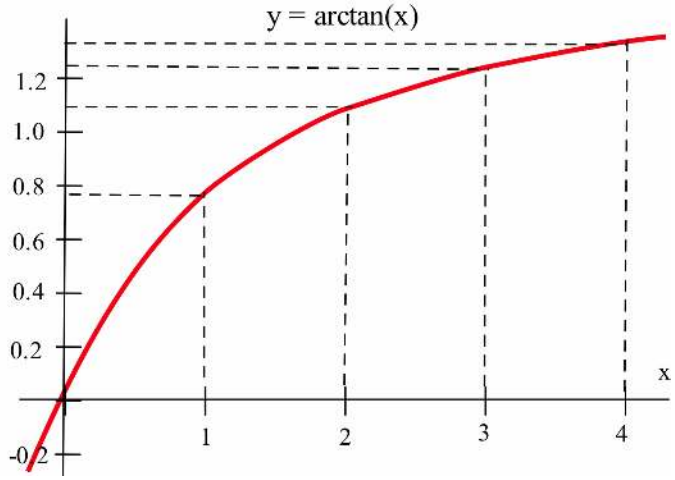
11. For $f(x)$ and $g(x)$ defined as:\[
12. For $f(x)$ and $g(x)$ defined as:

$$f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x \end{cases} \qquad g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

determine the following limits:

a. $\lim_{x \to 2} [f(x) + g(x)]$
b. $\lim_{x \to 2} \frac{f(x)}{g(x)}$
c. $\lim_{x \to 2} f(g(x))$
d. $\lim_{x \to 0} \frac{g(x)}{f(x)}$
e. $\lim_{x \to 1} \frac{f(x)}{g(x)}$
f. $\lim_{x \to 1} g(f(x))$

13. Give geometric interpretations for each limit and use a calculator to estimate its value.



a. $\lim_{h \to 0} \frac{\arctan(0 + h) - \arctan(0)}{h}$
b. $\lim_{h \to 0} \frac{\arctan(1 + h) - \arctan(1)}{h}$
c. $\lim_{h \to 0} \frac{\arctan(2 + h) - \arctan(2)}{h}$

14. a. What does $\lim_{h \to 0} \frac{\cos(h) - 1}{h}$ represent in relation to the graph of $y = \cos(x)$? It may help to recognize that:

$$\frac{\cos(h) - 1}{h} = \frac{\cos(0 + h) - \cos(0)}{h}$$

b. Graphically and using your calculator, estimate $\lim_{h \to 0} \frac{\cos(h) - 1}{h}$.

15. a. What does the ratio $\frac{\ln(1 + h)}{h}$ represent in relation to the graph of $y = \ln(x)$? It may help to recognize that:

$$\frac{\ln(1+h)}{h} = \frac{\ln(1+h) - \ln(1)}{h}$$

b. Graphically and using your calculator, determine $\lim_{h\to 0} \frac{\ln(1+h)}{h}$.

16. Use your calculator (to generate a table of values) to help you estimate the value of each limit.

a. $\lim_{h\to 0} \frac{e^h - 1}{h}$

b. $\lim_{c\to 0} \frac{\tan(1+c) - \tan(1)}{c}$

c. $\lim_{t\to 0} \frac{g(2+t) - g(2)}{t}$ when $g(t) = t^2 - 5$.

17. a. For $h > 0$, find the slope of the line through the points $(h, |h|)$ and $(0, 0)$.

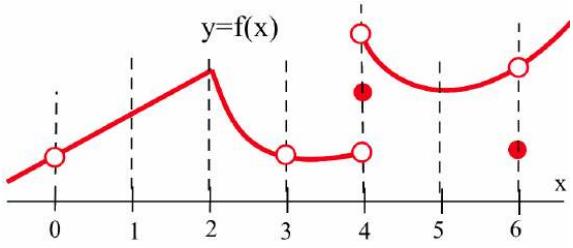
b. For $h < 0$, find the slope of the line through the points $(h, |h|)$ and $(0, 0)$.

c. Evaluate $\lim_{h\to 0^-} \frac{|h|}{h}$, $\lim_{h\to 0^+} \frac{|h|}{h}$ and $\lim_{h\to 0} \frac{|h|}{h}$.

In 17–18, describe the behavior at each integer of the function $y = f(x)$ in the figure provided, using one of these phrases:

- "connected and smooth"
- "connected with a corner"
- "not connected because of a simple hole that could be plugged by adding or moving one point"
- "not connected because of a vertical jump that could not be plugged by moving one point"

17. A piecewise blue function, y = f(x), defined over the range x = -1 to x = 5. The function shows multiple discontinuities. From x = -1 to x = 1, it increases linearly to a solid point at (0, 1.5) and then is constant until an open circle at (1, 1.5). From x = 1 to x = 2, it curves downward, ending at an open circle near (2, 0.5), with an open circle near (1.5, 1) and a solid point near (3, 2). From x = 2 to x = 3, it increases linearly to an open circle at (3, 1.5). For x 3, the function smoothly decreases to a minimum near x = 4, then increases smoothly. There is a single solid, isolated point at (3, 2.5)." class="internal default" style="width: 470px; height: 210px;" width="470px" height="210px" src="/@api/deki/files/139907/fig102_16.png">

18. 

19. Use the list method to show that $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

20. Show that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist. (Suggestion: Let $f(x) = \sin\left(\frac{1}{x}\right)$ and let $a_n = \frac{1}{n\pi}$ so that $f(a_n) = \sin\left(\frac{1}{a_n}\right) = \sin(n\pi) = 0$ for every n. Then pick $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ so that $f(b_n) = \sin\left(\frac{1}{b_n}\right) = \sin(2n\pi + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$ for all n.)

In Problems 21–26, use the Squeezing Theorem to help evaluate each limit.

26. $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

27. $\lim_{x\to 0} \sqrt[3]{x} \sin\left(\frac{1}{x^3}\right)$

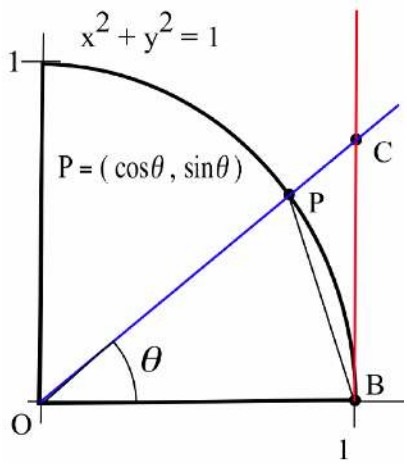
28. $\lim_{x\to 0} 3 + x^2 \sin\left(\frac{1}{x}\right)$

29. $\lim_{x\to 1^-} \sqrt{1-x^2} \cos\left(\frac{1}{x-1}\right)$

30. $\lim_{x\to 0} x^2 \cdot \left\lfloor \frac{1}{x^2} \right\rfloor$

31. $\lim_{x\to 0} (-1)^{\lfloor \frac{1}{x} \rfloor} (1 - \cos(x))$

32. This problem outlines the steps of a proof that $\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$. Refer to the figure below, assume that $0 < \theta < \frac{\pi}{2}$, and justify why each statement must be true.



a. Area of $\triangle OPB = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\sin(\theta)$

b. $\dfrac{\text{area of the sector (the pie shaped region) } OPB}{\text{area of the whole circle}} = \dfrac{\theta}{2\pi}$

c. area of the sector $OPB = \pi \cdot \dfrac{\theta}{2\pi} = \dfrac{\theta}{2}$

d. The line L through the points $(0,0)$ and $P = (\cos(\theta), \sin(\theta))$ has slope $m = \dfrac{\sin(\theta)}{\cos(\theta)}$, so $C = \left(1, \dfrac{\sin(\theta)}{\cos(\theta)}\right)$

e. area of $\triangle OCB = \dfrac{1}{2}(\text{base})(\text{height}) = \dfrac{1}{2}(1)\dfrac{\sin(\theta)}{\cos(\theta)}$

f. area of $\triangle OPB$ < area of sector OPB < area of $\triangle OCB$

g. $\dfrac{1}{2}\sin(\theta) < \dfrac{\theta}{2} < \dfrac{1}{2}(1)\dfrac{\sin(\theta)}{\cos(\theta)} \Rightarrow \sin(\theta) < \theta < \dfrac{\sin(\theta)}{\cos(\theta)}$

h. $1 < \dfrac{\theta}{\sin(\theta)} < \dfrac{1}{\cos(\theta)} \Rightarrow 1 > \dfrac{\sin(\theta)}{\theta} > \cos(\theta)$

i. $\lim\limits_{\theta \to 0^+} 1 = 1$ and $\lim\limits_{\theta \to 0^+} \cos(\theta) = 1$.

j. $\lim\limits_{\theta \to 0^+} \dfrac{\sin(\theta)}{\theta} = 1$

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1.3: Continuous Functions

In Section 1.2 we saw a few "nice" functions whose limits as $x \to a$ simply involved substituting a into the function: $\lim_{x \to a} f(x) = f(a)$. Functions whose limits have this substitution property are called **continuous functions** and such functions possess a number of other useful properties.

In this section we will examine what it means graphically for a function to be continuous (or not continuous), state some properties of continuous functions, and look at a few applications of these properties—including a way to solve horrible equations such as $\sin(x) = \dfrac{2x+1}{x-2}$.

Definition of a Continuous Function

We begin by formally stating the definition of this new concept.

Definition: Continuity at a Point

A function f is **continuous** at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

The graph below illustrates some of the different ways a function can behave at and near a point:

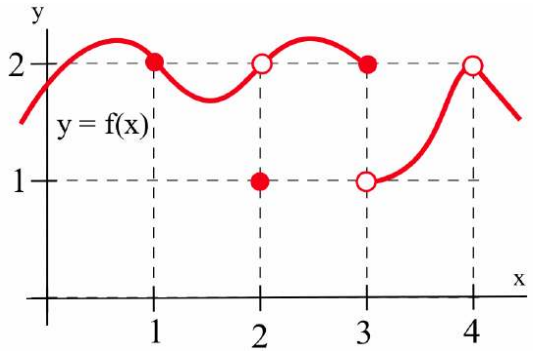


Figure 1.3.1

and this corresponding table contains some numerical information about the example function f and its behavior:

a	$f(a)$	$\lim_{x \to a} f(x)$
1	2	2
2	1	2
3	2	DNE
4	undefined	2

We can conclude from the information in the table that f is continuous at 1 because $\lim_{x \to 1} f(x) = 2 = f(1)$.

We can also conclude that f is not continuous at 2 or 3 or 4, because $\lim_{x \to 2} f(x) \neq f(2)$, $\lim_{x \to 3} f(x) \neq f(3)$ and $\lim_{x \to 4} f(x) \neq f(4)$.

Graphical Meaning of Continuity

When x is close to 1, the values of $f(x)$ are close to the value $f(1)$, and the graph of f does not have a hole or break at $x = 1$. The graph of f is "connected" at $x = 1$ and can be drawn without lifting your pencil. At $x = 2$ and $x = 4$ the graph of f has "holes," and at $x = 3$ the graph has a "break." The function f is also continuous at 1.7 (why?) and at every point shown *except* at 2, 3 and 4.

Informally, we can say:

- A function is **continuous** at a point if the graph of the function is **connected** there.

- A function is **not continuous** at a point if its graph has a **hole** or **break** at that point.

Sometimes the definition of "continuous" (the substitution condition for limits) is easier to use if we chop it into several smaller pieces and then check whether or not our function satisfies each piece.

Theorem

f is continuous at a if and only if:

i. f is defined at a
ii. the limit of $f(x)$, as $x \to a$, exists\ (so the left limit and right limits exist and are equal)
iii. the value of f at a equals the value of the limit as $x \to a$:

$$\lim_{x \to a} f(x) = f(a)$$

If f satisfies conditions (i), (ii) and (iii), then f is continuous at a. If f does not satisfy one or more of the three conditions at a, then f is not continuous at a.

For $f(x)$ in Figure 1.3.1, all three conditions are satisfied for $a = 1$, so f is continuous at 1. For $a = 2$, conditions (i) and (ii) are satisfied but not (iii), so f is not continuous at 2. For $a = 3$, condition (i) is satisfied but (ii) is violated, so f is not continuous at 3. For $a = 4$, condition (i) is violated, so f is not continuous at 4.

A function is **continuous on an interval** if it is continuous at every point in the interval.

A function f is **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$ and is **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.

Example 1.3.1

Is the function $f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 2 & \text{if } 1 < x \le 2 \\ \frac{1}{x-3} & \text{if } x > 2 \end{cases}$ continuous at $x = 1$? At $x = 2$? At $x = 3$?

Solution

We could answer these questions by examining a graph of $f(x)$, but let's try them without the benefit of a graph. At $x = 1$, $f(1) = 2$ and the left and right limits are equal:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} [x+1] = 2 = \lim_{x \to 1^+} 2 = \lim_{x \to 1^+} f(x)$$

and their common limit matches the value of the function at $x = 1$:

$$\lim_{x \to 1} f(x) = 2 = f(1)$$

so f is continuous at 1.

At $x = 2$, $f(2) = 2$, but the left and right limits are not equal:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 1^-} 2 = 2 \neq -1 = \lim_{x \to 2^+} \frac{1}{x-3} = \lim_{x \to 2^+} f(x)$$

so f fails condition (ii), hence is not continuous at 2. We can, however, say that f is continuous from the left (but not from the right) at 2.

At $x = 3$, $f(3) = \frac{1}{0}$, which is undefined, so f is not continuous at 3 because it fails condition (i).

Example 1.3.2

Where is $f(x) = 3x^2 - 2x$ continuous?

Solution

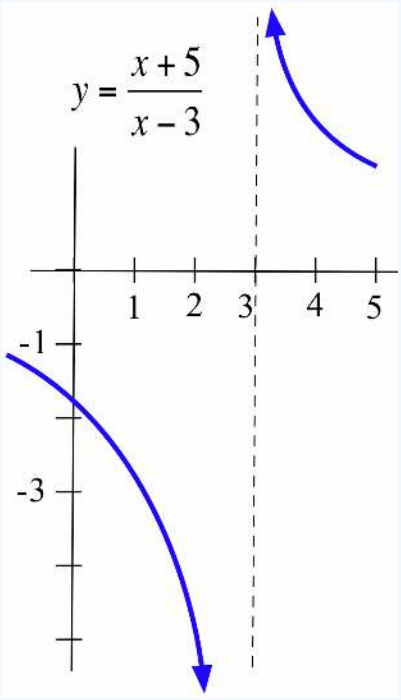
By the Substitution Theorem for Polynomial and Rational Functions, $\lim_{x \to a} P(x) = P(a)$ for any polynomial $P(x)$ at any point a, so every polynomial is continuous everywhere. In particular, $f(x) = 3x^2 - 2x$ is continuous everywhere.

Example 1.3.3

Where is the function $g(x) = \dfrac{x+5}{x-3}$ continuous? Where is $h(x) = \dfrac{x^2+4x-5}{x^2-4x+3}$ continuous?

Solution

Because $g(x)$ is a rational function, the Substitution Theorem for Polynomial and Rational Functions says it is continuous everywhere except where its denominator is 0: g is continuous everywhere except at $x = 3$. The graph of g:

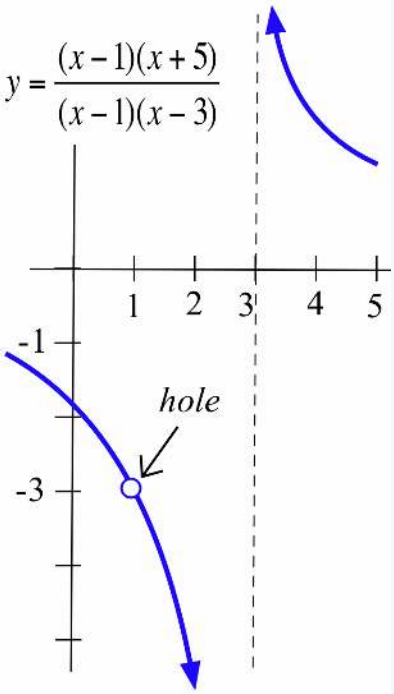


is "connected" everywhere except at $x = 3$, where it has a vertical asymptote.

We can rewrite the rational function $h(x)$ as:

$$h(x) = \frac{(x-1)(x+5)}{(x-1)(x-3)}$$

and note that its denominator is 0 at $x = 1$ and $x = 3$, so h is continuous everywhere except 3 and 1. The graph of h:



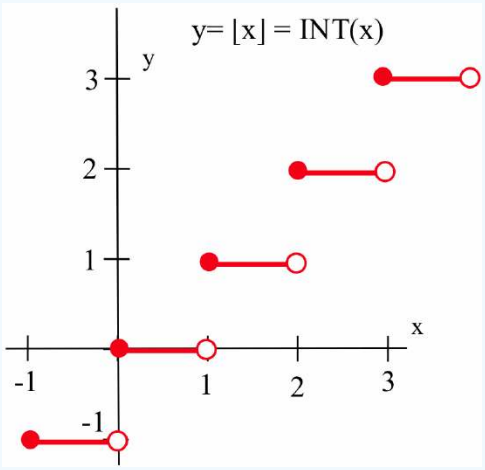
is "connected" everywhere except at **3**, where it has a vertical asymptote, and **1**, where it has a hole: $f(1) = \frac{0}{0}$ is undefined.

Example 1.3.4

Where is $f(x) = \lfloor x \rfloor$ continuous?

Solution

The graph of $y = \lfloor x \rfloor$ seems to be "connected" except at each integer, where there is a "jump":



If a is an integer, then $\lim_{x \to a^-} \lfloor x \rfloor = a - 1$ and $\lim_{x \to a^+} \lfloor x \rfloor = a$ so $\lim_{x \to a} \lfloor x \rfloor$ is undefined, and $\lfloor x \rfloor$ is not continuous at $x = a$.

If a is *not* an integer, then the left and right limits of $\lfloor x \rfloor$, as $x \to a$, both equal $\lfloor a \rfloor$ so:

$$\lim_{x \to a} \lfloor x \rfloor = \lfloor a \rfloor$$

hence $\lfloor x \rfloor$ is continuous at $x = a$.

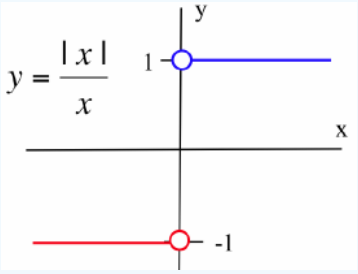
Summarizing: $\lfloor x \rfloor$ is continuous everywhere except at the integers. In fact, $f(x) = \lfloor x \rfloor$ is continuous from the right everywhere and is continuous from the left everywhere except at the integers.

? Practice 1.3.1

Where is $f(x) = \frac{|x|}{x}$ continuous?

Answer

$f(x) = \frac{|x|}{x}$ is continuous everywhere **except** at $x = 0$, where this function is not defined:



If $a > 0$, then $\lim_{x \to a} \frac{|x|}{x} = 1 = f(a)$ so f is continuous at a.

If $a < 0$, then $\lim_{x \to a} \frac{|x|}{x} = -1 = f(a)$ so f is continuous at a.

But $f(0)$ is not defined and

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \neq 1 = \lim_{x \to 0^+} \frac{|x|}{x}$$

so $\lim_{x \to a} \frac{|x|}{x}$ does not exist.

Why Do We Care Whether a Function Is Continuous?

There are several reasons for us to examine continuous functions and their properties:

- Many applications in engineering, the sciences and business are continuous or are modeled by continuous functions or by pieces of continuous functions.
- Continuous functions share a number of useful properties that do not necessarily hold true if the function is not continuous. If a result is true of all continuous functions and we have a continuous function, then the result is true for our function. This can save us from having to show, one by one, that each result is true for each particular function we use. Some of these properties are given in the remainder of this section.
- Differential calculus has been called the study of **continuous** change, and many of the results of calculus are guaranteed to be true only for continuous functions. If you look ahead into Chapters 2 and 3, you will see that many of the theorems have the form "If f is continuous and (some additional hypothesis), then (some conclusion)."

Combinations of Continuous Functions

Not only are most of the basic functions we will encounter continuous at most points, so are basic combinations of those functions.

Theorem

If: $f(x)$ and $g(x)$ are continuous at a
and: k is any constant
then: the elementary combinations of f and g

- $k \cdot f(x)$
- $f(x) + g(x)$
- $f(x) - g(x)$
- $f(x) \cdot g(x)$

- $\frac{f(x)}{g(x)}$ (as long as $g(a) \neq 0$)

are continuous at a.

The continuity of a function is defined using limits, and all of these results about simple combinations of continuous functions follow from the results about combinations of limits in the Main Limit Theorem.

Our hypothesis is that f and g are both continuous at a, so we can assume that

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a)$$

and then use the appropriate part of the Main Limit Theorem.

For example,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a)$$

so $f + g$ is continuous at a.

? Practice 1.3.2

Prove: If f and g are continuous at a, then $k \cdot f$ and $f - g$ are continuous at a (where k a constant).

Answer

a. To prove that $k \cdot f$ is continuous at a, we need to prove that $k \cdot f$ satisfies the definition of continuity at a: $\lim_{x \to a} k \cdot f(x) = k \cdot f(a)$. Using results about limits, we know

$$\lim_{x \to a} k \cdot f(x) = k \cdot \lim_{x \to a} f(x) = k \cdot f(a)$$

(because f is continuous at a) so $k \cdot f$ is continuous at a.

b. To prove that $f - g$ is continuous at a, we need to prove that $f - g$ satisfies the definition of continuity at a: $\lim_{x \to a} [f(x) - g(x)] = f(a) - g(a)$. Again using information about limits:

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = f(a) - g(a)$$

(because f and g are both continuous at a) so $f - g$ is continuous at a.

Composition of Continuous Functions

If: $g(x)$ is continuous at a
and: $f(x)$ is continuous at $g(a)$
then: $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(g(a))$
so: $f \circ g(x) = f(g(x))$ is continuous at a.

The proof of this result involves some technical details, but just formalizes the following line of reasoning:

The hypothesis that "g is continuous at a" means that if x is close to a then $g(x)$ will be close to $g(a)$. Similarly, "\)f\) is continuous at $g(a)$" means that if $g(x)$ is close to $g(a)$ then $f(g(x)) = f \circ g(x)$ will be close to $f(g(a)) = f \circ g(a)$. Finally, we can conclude that if x is close to a, then $g(x)$ is close to $g(a)$ so $f \circ g(x)$ is close to $f \circ g(a)$ and therefore $f \circ g$ is continuous at $x = a$.

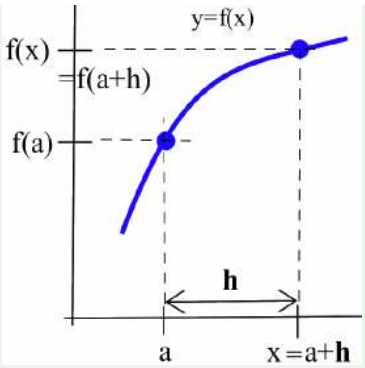
The next theorem presents an alternate version of the limit condition for continuity, which we will use occasionally in the future.

Theorem

$\lim_{x \to a} f(x) = f(a)$ if and only if $\lim_{h \to 0} f(a + h) = f(a)$

Proof

Let's define a new variable h by $h = x - a$ so that $x = a + h$:



Then $x \to a$ if and only if $h = x - a \to 0$, so $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a+h)$ and therefore $\lim\limits_{x \to a} f(x) = f(a)$ if and only if $\lim\limits_{h \to 0} f(a+h) = f(a)$.

We can restate the result of this theorem as:

A function f is continuous at a if and only if $\lim\limits_{h \to 0} f(a+h) = f(a)$.

Which Functions Are Continuous?

Fortunately, the functions we encounter most often are either continuous everywhere or continuous everywhere except at a few places.

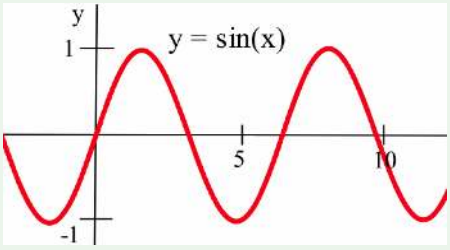
Theorem 1.3.1

The following functions are continuous everywhere

a. polynomials
b. $\sin(x)$ and $\cos(x)$
c. $|x|$

Proof

a. This follows from the Substitution Theorem for Polynomial and Rational Functions and the definition of continuity.
b. The graph of $y = \sin(x)$:

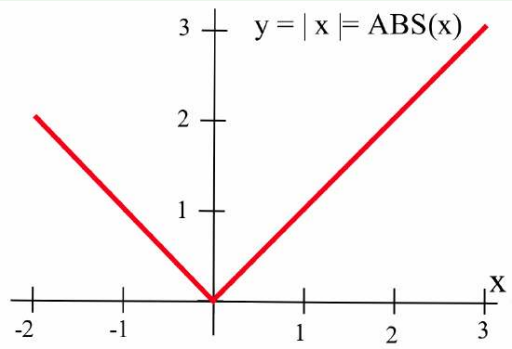


clearly has no holes or breaks, so it is reasonable to think that $\sin(x)$ is continuous everywhere. Justifying this algebraically, recall the angle addition formula for $\sin(\theta)$ and the results from Section 1.2 that $\lim\limits_{h \to 0} \cos(h) = 1$ and $\lim\limits_{h \to 0} \sin(h) = 0$. For every real number a:

$$\begin{aligned} \lim_{h \to 0} \sin(a+h) &= \lim_{h \to 0} [\sin(a)\cos(h) + \cos(a)\sin(h)] \\ &= \lim_{h \to 0} \sin(a) \cdot \lim_{h \to 0} \cos(h) + \lim_{h \to 0} \cos(a) \cdot \lim_{h \to 0} \sin(h) \\ &= \sin(a) \cdot 1 + \cos(a) \cdot 0 = \sin(a) \end{aligned}$$

so $f(x) = \sin(x)$ is continuous at every point. The justification for $f(x) = \cos(x)$ is similar.

c. For $f(x) = |x|$, when $x > 0$, then $|x| = x$ and its graph:



is a straight line and is continuous because x is a polynomial. When $x < 0$, then $|x| = -x$ and it is also continuous. The only questionable point is the "corner" on the graph when $x = 0$, but the graph there is only bent, not broken:

$$\lim_{h \to 0^+} |0 + h| = \lim_{h \to 0^+} h = 0$$

and:

$$\lim_{h \to 0^-} |0 + h| = \lim_{h \to 0^-} -h = 0$$

so:

$$\lim_{h \to 0} |0 + h| = 0 = |0|$$

and $f(x) = |x|$ is also continuous at 0.

A continuous function can have corners but not holes or breaks

Even functions that fail to be continuous at some points are often continuous most places:

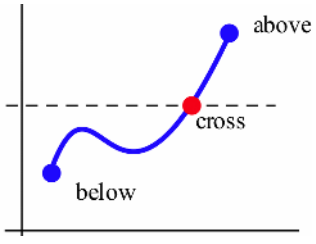
- A rational function is continuous *except* where the denominator is 0.
- The trig functions $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$ are continuous *except* where they are undefined.
- The greatest integer function $\lfloor x \rfloor$ is continuous *except* at each integer.
- But the "holey" function

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

is **discontinuous everywhere**.

Intermediate Value Property of Continuous Functions

Because the graph of a continuous function is connected and does not have any holes or breaks in it, the values of the function can not "skip" or "jump over" a horizontal line:



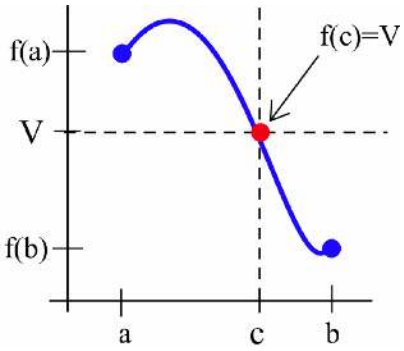
If one value of the continuous function is below the line and another value of the function is above the line, then **somewhere** the graph will cross the line. The next theorem makes this statement more precise. The result seems obvious, but its proof is technically difficult and is not given here.

Intermediate Value Theorem for Continuous Functions

If: f is continuous on the interval $[a, b]$
and: V is any value between $f(a)$ and $f(b)$
then: there is a number c between a and b so that $f(c) = V$.

(That is, f actually takes on each intermediate value between $f(a)$ and $f(b)$.)

If the graph of f connects the points $(a, f(a))$ and $(b, f(b))$ and V is any number between $f(a)$ and $f(b)$, then the graph of f must cross the horizontal line $y = V$ somewhere between $x = a$ and $x = b$.



Since f is continuous, its graph cannot "hop" over the line $y = V$.

We often take this theorem for granted in some common situations:

- If a child's temperature rose from $98.6°$F to $101.3°$F, then there was an instant when the child's temperature was exactly $100°$F. (In fact, every temperature between $98.6°$F and $101.3°$F occurred at some instant.)
- If you dove to pick up a shell 25 feet below the surface of a lagoon, then at some instant in time you were 17 feet below the surface. (Actually, you want to be at 17 feet twice. Why?)
- If you started driving from a stop (velocity $= 0$) and accelerated to a velocity of 30 kilometers per hour, then there was an instant when your velocity was exactly 10 kilometers per hour.

But we cannot apply the Intermediate Value Theorem if the function is not continuous:

- In 1987 it cost 22¢ to mail a first-class letter inside the United States, and in 1990 it cost 25¢ to mail the same letter, but we cannot conclude that there was a time when it cost 23¢ or 24¢ or ¢ to send the letter. (Postal rates did not increase in a continuous fashion. They jumped directly from 22¢ to 25¢.)
- Prices, taxes and rates of pay change in jumps—discrete steps—without taking on the intermediate values.

The Intermediate Value Theorem (IVT) is an example of an "existence theorem": it concludes that something exists (a number c so that $f(c) = V$). But like many existence theorems, it does not tell us how to find the the thing that exists (the value of c) and is of no use in actually finding those numbers or objects.

Bisection Algorithm for Approximating Roots

The IVT can help us finds roots of functions and solve equations. If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs (one is positive and one is negative), then 0 is an intermediate value between $f(a)$ and $f(b)$ so f will have a root c between $x = a$ and $x = b$ where $f(c) = 0$.

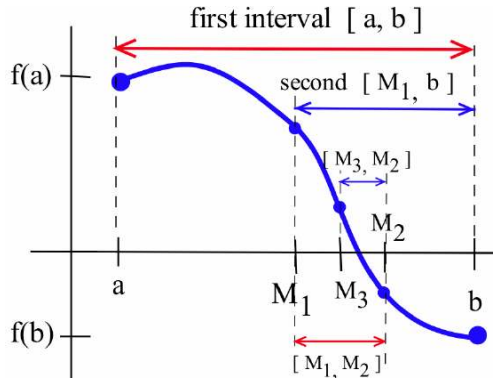
While the IVT does not tell us how to find c, it lays the groundwork for a method commonly used to approximate the roots of continuous functions.

Bisection Algorithm for Finding a Root of $f(x)$

1. Find two values of x (call them a and b) so that $f(a)$ and $f(b)$ have opposite signs. (The IVT will then guarantee that $f(x)$ has a root between a and b.)
2. Calculate the midpoint (or **bisection point**) of the interval $[a, b]$, using the formula $m = \frac{a+b}{2}$, and evaluate $f(m)$.

3. a. If $f(m) = 0$, then m is a root of f and we are done.
   b. If $f(m) \neq 0$, then $f(m)$ has the sign opposite $f(a)$ or $f(b)$:
      i. if $f(a)$ and $f(m)$ have opposite signs, then f has a root in $[a, m]$ so put $b = m$
      ii. if $f(b)$ and $f(m)$ have opposite signs, then f has a root in $[m, b]$ so put $a = m$
4. Repeat steps 2 and 3 until a root is found exactly or is approximated closely enough.

The length of the interval known to contain a root is cut in half each time through steps 2 and 3, so the Bisection Algorithm quickly "squeezes" in on a root (see margin figure).



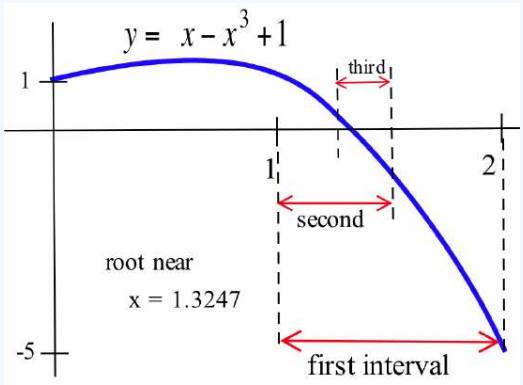
The steps of the Bisection Algorithm can be done "by hand," but it is tedious to do very many of them that way. Computers are very good with this type of tedious repetition, and the algorithm is simple to program.

Example 1.3.5

Find a root of $f(x) = -x^3 + x + 1$.

Solution

$f(0) = 1$ and $f(1) = 1$ so we cannot conclude that f has a root between 0 and 1. $f(1) = 1$ and $f(2) = -5$ have opposite signs, so by the IVT (this function is a polynomial, so it is continuous everywhere and the IVT applies) we know that there is a number c between 1 and 2 such that $f(c) = 0$:



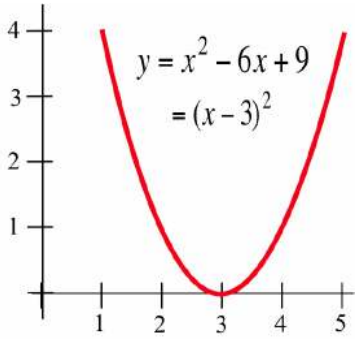
The midpoint of the interval $[1, 2]$ is $m = \frac{1+2}{2} = \frac{3}{2} = 1.5$ and $f(\frac{3}{2}) = -\frac{7}{8}$ so f changes sign between 1 and 1.5 and we can be sure that there is a root between 1 and 1.5. If we repeat the operation for the interval $[1, 1.5]$, the midpoint is $m = \frac{1+1.5}{2} = 1.25$, and $f(1.25) = \frac{19}{64} > 0$ so f changes sign between 1.25 and 1.5 and we know f has a root between 1.25 and 1.5.

Repeating this procedure a few more times, we get:

a	b	$m = \frac{b+a}{2}$	$f(a)$	$f(b)$	$f(m)$	root between	
1	2		1	−5		\(1	2
1	2	1.5	1	−5	−0.875	\(1	1.5
1	1.5	1.25	1	−0.875	0.2969	\(1.25	1.5
1.25	1.5	1.375	0.2969	−0.875	−0.2246	\(1.25	1.375
1.25	1.375	1.3125	0.2969	−0.2246	0.0515	\(1.3125	1.375
1.3125	1.375	1.34375					

If we continue the table, the interval containing the root will squeeze around the value 1.324718.

The Bisection Algorithm has one major drawback: there are some roots it does not find. The algorithm requires that the function take on both positive and negative values near the root so that the graph actually crosses the x-axis. The function $f(x) = x^2 - 6x + 9 = (x-3)^2$ has the root $x = 3$ but is never negative:



We cannot find two starting points a and b so that $f(a)$ and $f(b)$ have opposite signs, so we cannot use the Bisection Algorithm to find the root $x = 3$. In Chapter 2 we will see another method—Newton's Method—that does find roots of this type.

The Bisection Algorithm requires that we supply two starting x-values, a and b, at which the function has opposite signs. These values can often be found with a little "trial and error," or we can examine the graph of the function to help pick the two values.

Finally, the Bisection Algorithm can also be used to solve equations, because the solution of any equation can always be transformed into an equivalent problem of finding roots by moving everything to one side of the equal sign. For example, the problem of solving the equation $x^3 = x + 1$ can be transformed into the equivalent problem of solving $x^3 - x - 1 = 0$ or of finding the roots of $f(x) = x^3 - x - 1$, which is equivalent to the problem we solved in the previous example.

Example 1.3.6

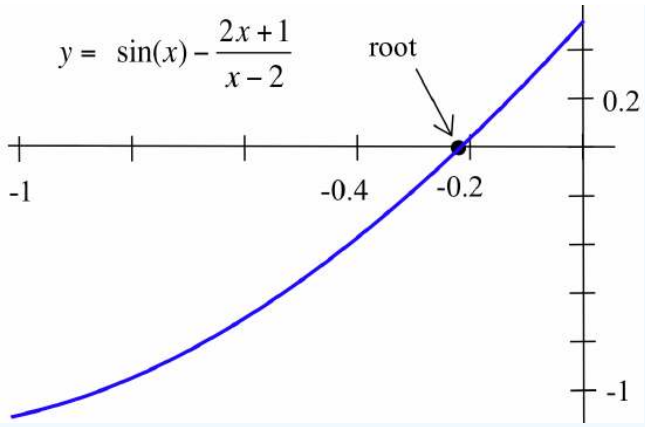
Find all solutions of $\sin(x) = \dfrac{2x+1}{x-2}$ (with x in radians.)

Solution

We can convert this problem of solving an equation to the problem of finding the roots of

$$f(x) = \sin(x) - \frac{2x+1}{x-2} = 0$$

The function $f(x)$ is continuous everywhere except at $x = 2$, and the graph of $f(x)$:

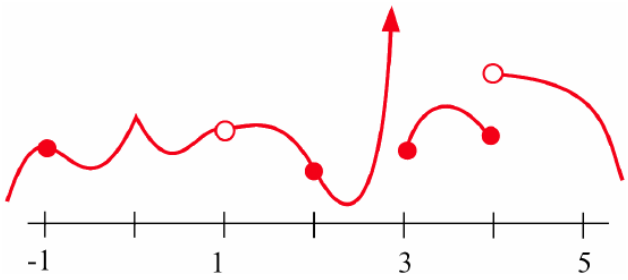


can help us find two starting values for the Bisection Algorithm. The graph shows that $f(-1)$ is negative and $f(0)$ is positive, and we know $f(x)$ is continuous on the interval $[-1, 0]$. Using the algorithm with the starting interval $[-1, 0]$, we know that a root is contained in the shrinking intervals $[-0.5, 0]$, $[-0.25, 0]$, $[-0.25, -0.125], \ldots$, $[-0.238281, -0.236328], \ldots$, $[-0.237176, -0.237177]$ so the root is approximately -0.237177.

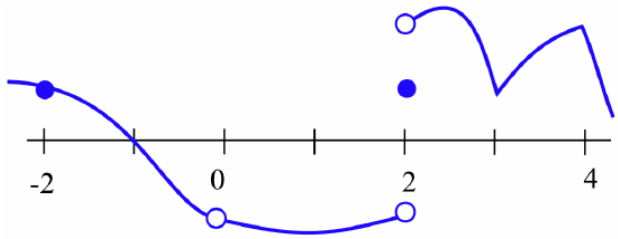
We might notice that $f(0) = 0.5 > 0$ while $f(\pi) = 0 - \frac{2\pi+1}{\pi-2} \approx -6.38 < 0$. Why is it wrong to conclude that $f(x)$ has another root between $x = 0$ and $x = \pi$?

Problems

1. At which points is the function in the graph below discontinuous?



2. At which points is the function in the graph below discontinuous?



3. Find at least one point at which each function is not continuous and state which of the three conditions in the definition of continuity is violated at that point.
   a. $\frac{x+5}{x-3}$
   b. $\frac{x^2+x-6}{x-2}$
   c. $\sqrt{\cos(x)}$
   d. $\frac{\lfloor x^2 \rfloor}{x}$
   e. $\frac{x}{\sin(x)}$
   f. $\frac{x}{x}$
   g. $\ln(x^2)$
   h. $\frac{\pi}{x^2-6x+9}$

i. $\tan(x)$

4. Which *two* of the following functions are not continuous? Use appropriate theorems from this section to justify that each of the other functions is continuous.
   a. $\dfrac{7}{\sqrt{2+\sin(x)}}$
   b. $\cos^2(x^5-7x+\pi)$
   c. $\dfrac{x^2-5}{1+\cos^2(x)}$
   d. $\dfrac{x^2-5}{1+\cos(x)}$
   e. $\lfloor 3+0.5\sin(x)\rfloor$
   f. $\lfloor 0.3\sin(x)+1.5\rfloor$
   g. $\sqrt{\cos(\sin(x))}$
   h. $\sqrt{x^2-6x+10}$
   i. $\sqrt[3]{\cos(x)}$
   j. $2^{\sin(x)}$
   k. $1-3^{-x}$
   l. $\arctan(1-x^2)$

5. A continuous function f has the values:

x	0	1	2	3	4	5
$f(x)$	5	3	-2	-1	3	-2

   a. f has at least roots between 0 and 5.
   b. $f(x)=4$ in at least places between $x=0$ and $x=5$.
   c. $f(x)=2$ in at least places between $x=0$ and $x=5$.
   d. $f(x)=3$ in at least places between $x=0$ and $x=5$.
   e. Is it possible for $f(x)$ to equal 7 for some x-value(s) between 0 and 5?

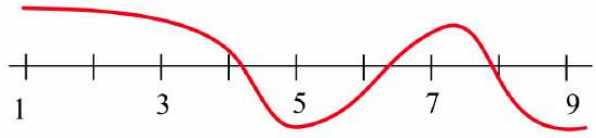
6. A continuous function g has the values:

x	1	2	3	4	5	6	7
$g(x)$	-3	1	4	-1	3	-2	-1

   a. g has at least roots between 1 and 5.
   b. $g(x)=3.2$ in at least places between $x=1$ and $x=7$.
   c. $g(x)=-0.7$ in at least places between $x=3$ and $x=7$.
   d. $g(x)=1.3$ in at least places between $x=2$ and $x=6$.
   e. Is it possible for $g(x)$ to equal π for some x-value(s) between 5 and 6?

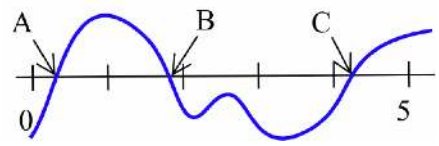
7. This problem asks you to verify that the Intermediate Value Theorem is true for some particular functions, intervals and intermediate values. In each problem you are given a function f, an interval $[a,b]$ and a value V. Verify that V is between $f(a)$ and $f(b)$ and find a value of c in the given interval so that $f(c)=V$.
   a. $f(x)=x^2$ on $[0,3]$, $V=2$
   b. $f(x)=x^2$ on $[-1,2]$, $V=3$
   c. $f(x)=\sin(x)$ on $[0,\frac{\pi}{2}]$, $V=\frac{1}{2}$
   d. $f(x)=x$ on $[0,1]$, $V=\frac{1}{3}$
   e. $f(x)=x^2-x$ on $[2,5]$, $V=4$
   f. $f(x)=\ln(x)$ on $[1,10]$, $V=2$

8. Two students claim that they both started with the points $x=1$ and $x=9$ and applied the Bisection Algorithm to the function graphed below:



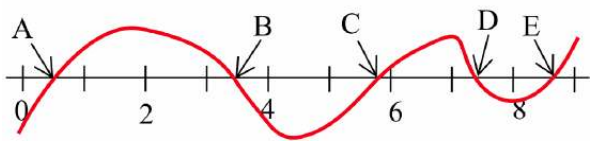
The first student says that the algorithm converged to the root near $x = 8$, but the second claims that the algorithm will converge to the root near $x = 4$. Who is correct?

9. Two students claim that they both started with the points $x = 0$ and $x = 5$ and applied the Bisection Algorithm to the function graphed below:



The first student says that the algorithm converged to the root labeled A, but the second claims that the algorithm will converge to the root labeled B. Who is correct?

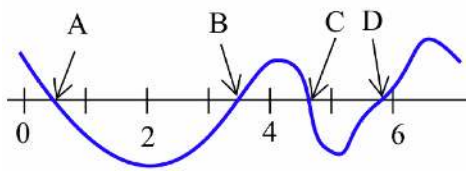
10. If you apply the Bisection Algorithm to the function graphed below:



which root does the algorithm find if you use:

a. starting points 0 and 9?
b. starting points 1 and 5?
c. starting points 3 and 5?

11. If you apply the Bisection Algorithm to the function graphed below:



which root does the algorithm find if you use:

a. starting points 3 and 7?
b. starting points 5 and 6?
c. starting points 1 and 6?

In Problems 12–17, use the IVT to verify each function has a root in the given interval(s). Then use the Bisection Algorithm to narrow the location of that root to an interval of length less than or equal to 0.1.

12. $f(x) = x^2 - 2$ on $[0, 3]$
13. $g(x) = x^3 - 3x^2 + 3$ on $[-1, 0]$, $[1, 2]$, $[2, 4]$
14. $h(t) = t^5 - 3t + 1$ on $[1, 3]$
15. $r(x) = 5 - 2^x$ on $[1, 3]$
16. $s(x) = \sin(2x) - \cos(x)$ on $[0, \pi]$
17. $p(t) = t^3 + 3t + 1$ on $[-1, 1]$
18. Explain what is wrong with this reasoning: If $f(x) = \frac{1}{x}$ then

$$f(-1) = -1 < 0 \quad \text{and} \quad f(1) = 1 > 0$$

so f must have a root between $x = -1$ and $x = 1$.

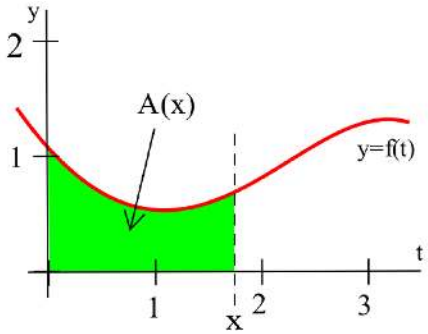
19. Each of the following statements is false for some functions. For each statement, sketch the graph of a counterexample.

a. If $f(3) = 5$ and $f(7) = -3$, then f has a root between $x = 3$ and $x = 7$.

b. If f has a root between $x = 2$ and $x = 5$, then $f(2)$ and $f(5)$ have opposite signs.

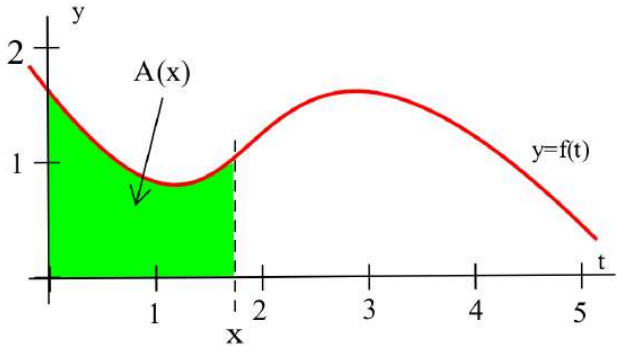
c. If the graph of a function has a sharp corner, then the function is not continuous there.

20. Define $A(x)$ to be the **area** bounded by the t- and y-axes, the curve $y = f(t)$, and the vertical line $t = x$ as shown in the figure:



It is clear that $A(1) < 2$ and $A(3) > 2$. Do you think there is avalue of x between 1 and 3 so that $A(x) = 2$? If so, justify your conclusion and estimate the location of the value of x that makes $A(x) = 2$. If not, justify your conclusion.

21. Define $A(x)$ to be the area bounded by the t- and y-axes, the curve $y = f(t)$, and the vertical line $t = x$ as shown in the figure:



a. Shade the part of the graph represented by $A(2.1) - A(2)$ and estimate the value of $\frac{A(2.1) - A(2)}{0.1}$.

b. Shade the part of the graph represented by $A(4.1) - A(4)$ and estimate the value of $\frac{A(4.1) - A(4)}{0.1}$.

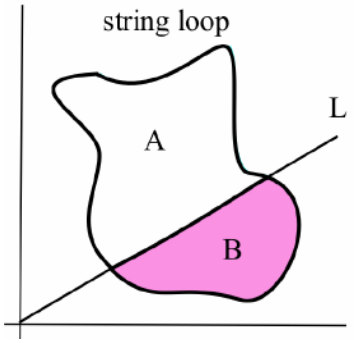
22. a. A square sheet of paper has a straight line drawn on it from the lower-left corner to the upper-right corner. Is it possible for you to start on the left edge of the sheet and draw a "connected" line to the right edge that does not cross the diagonal line?

b. Prove: If f is continuous on the interval $[0, 1]$ and $0 \le f(x) \le 1$ for all x, then there is a number c with $0 \le c \le 1$ such that $f(c) = c$. (The number c is called a "fixed point" of f because the image of c is the same as c: f does not "move" c.) Hint: Define a new function $g(x) = f(x) - x$ and start by considering the values $g(0)$ and $g(1)$.

c. What does part (b) have to do with part (a)?

d. Is the theorem in part (b) true if we replace the closed interval $[0, 1]$ with the open interval $(0, 1)$?

23. A piece of string is tied in a loop and tossed onto quadrant I enclosing a single region:



a. Is it always possible to find a line L passing through the origin so that L divides the region into two equal areas? (Justify your answer.)

b. Is it always possible to find a line L parallel to the x-axis so that L divides the region into two equal areas? (Justify your answer.)

c. Is it always possible to find two lines, L parallel to the x-axis and M parallel to the y-axis, so that L and M divide the region into four equal areas? (Justify your answer.)

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1.4: Definition of Limit

It may seem strange that we have been using and calculating the values of limits for quite a while without having a precise definition of "limit," but the history of mathematics shows that many concepts—including limits—were successfully used before they were precisely defined or even fully understood. We have chosen to follow the historical sequence, emphasizing the intuitive and graphical meaning of limit because most students find these ideas and calculations easier than the definition. (This intuitive and graphical understanding of limit was sufficient for the first 100-plus years of the development of calculus—from Newton and Leibniz in the late 1600s to Cauchy in the early 1800s—and it is sufficient for using and understanding the results in beginning calculus.}

Mathematics, however, is more than a collection of useful tools, and part of its power and beauty comes from the fact that in mathematics terms are precisely defined and results are rigorously proved. Mathematical tastes (what is mathematically beautiful, interesting, useful) change over time, but because of careful definitions and proofs, the results remain true—everywhere and forever. Textbooks seldom give all of the definitions and proofs, but it is important to mathematics that such definitions and proofs exist.

The goal of this section is to provide a precise definition of the limit of a function. The definition will not help you calculate the values of limits, but it provides a precise statement of what a limit is. The definition of limit is then used to verify the limits of some functions and prove some general results.

The Intuitive Approach

The precise ("formal") definition of limit carefully states the ideas that we have already been using graphically and intuitively. The following side-by-side columns show some of the phrases we have been using to describe limits, and those phrases—particularly the last ones—provide the basis on which to build the definition of limit.

A Particular Limit	General Limit
$\lim_{x\to 3} 2x-1=5$	$\lim_{x\to a} f(x)=L$
"as the values of x approach 3, the values of $2x-1$ approach (are arbitrarily close to) 5"	"as the values of x approach a, the values of $f(x)$ approach (are arbitrarily close to) L"
"when x is close to 3 (but not equal to 3), the value of $2x-1$ is close to 5"	"when x is close to a (but not equal to a), the value of $f(x)$ is close to L"
"we can guarantee that the values of $2x-1$ are as close to 5 as we want by restricting the values of x to be sufficiently close to 3 (but not equal to 3)"	"we can guarantee that the values of $f(x)$ are as close to L as we want by restricting the values of x to be sufficiently close to a(but not equal to a)"

Let's examine what the last phrase ("we can…") means for the Particular Limit in the previous discussion.

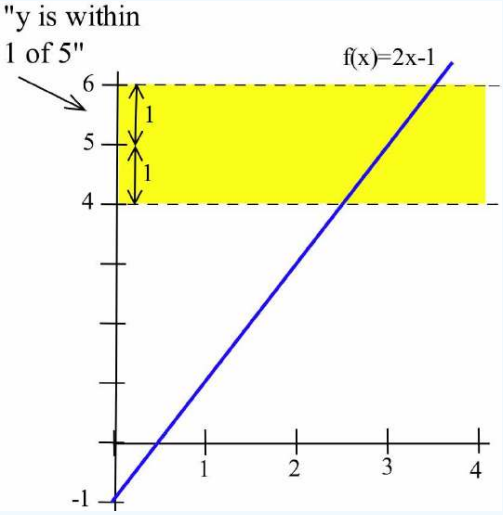
Example 1.4.1

We know $\lim_{x\to 3} 2x-1=5$ and need to show that we can guarantee that the values of $f(x)=2x-1$ are as close to 5 as we want by restricting the values of x to be sufficiently close to 3. What values of x guarantee that $f(x)=2x-1$ is within:

a. 1 unit of 5?
b. 0.2 units of 5?
c. E units of 5?

Solution

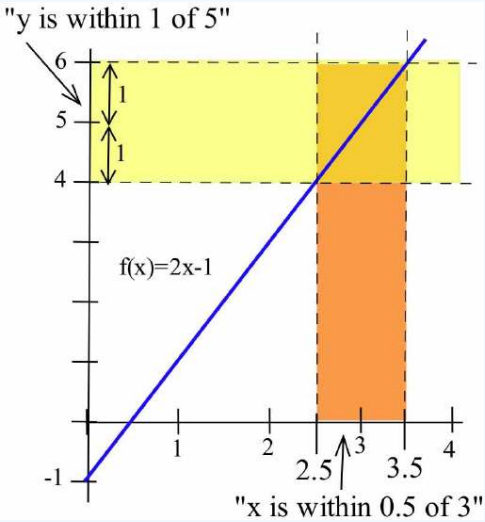
a. "Within 1 unit of 5" means between $5-1=4$ and $5+1=6$, so the question can be rephrased as "for what values of x is $y=2x-1$ between 4 and 6: $4<2x-1<6$?" We want to know which values of x ensure the values of $y=2x-1$ are in the the shaded band:



The algebraic process is straightforward:

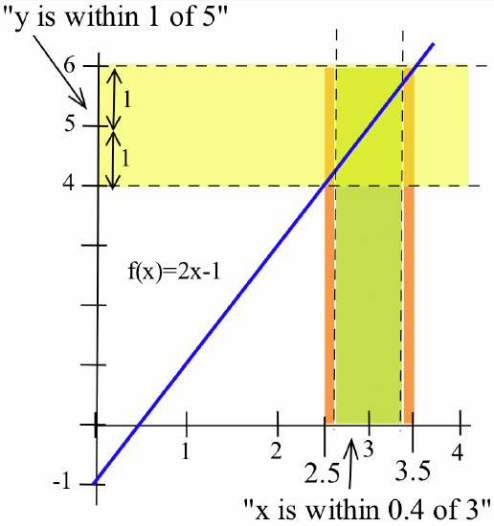
$$4 < 2x - 1 < 6 \quad \Rightarrow \quad 5 < 2x < 7 \quad \Rightarrow \quad 2.5 < x < 3.5$$

We can restate this result as follows: "If x is within 0.5 units of 3, then $y = 2x - 1$ is within 1 unit of 5."

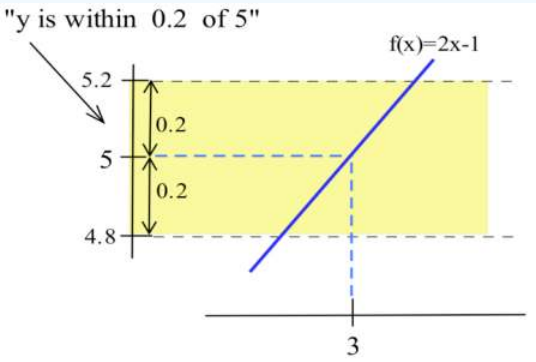


Any smaller distance also satisfies the guarantee: for example, "If x is within 0.4 units of 3, then $y = 2x - 1$ is within 1 unit

of 5."



b. "Within 0.2 units of 5" means between $5-0.2=4.8$ and $5+0.2=5.2$, so the question can be rephrased as "for which values of x is $y=2x-1$ between 4.8 and 5.2: $4.8<2x-1<5.2$?" Solving for x, we get $5.8<2x<6.2$ and $2.9<x<3.1$. "If x is within 0.1 units of 3, then $y=2x-1$ is within 0.2 units of 5."



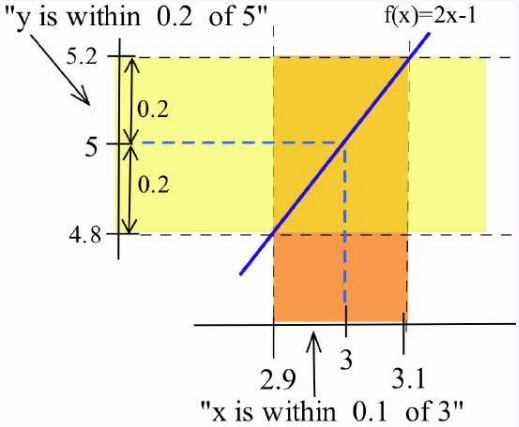
Any smaller distance also satisfies the guarantee.

Rather than redoing these calculations for every possible distance from 5, we can do the work once, generally:

c. "Within E unit of 5" means between $5-E$ and $5+E$, so the question becomes, "For what values of x is $y=2x-1$ between $5-E$ and $5+E$: $5-E<2x-1<5+E$?" Solving $5-E<2x-1<5+E$ for x, we get:

$$6-E<2x<6+E \quad \Rightarrow \quad 3-\frac{E}{2}<x<3+\frac{E}{2}$$

"If x is within $\frac{E}{2}$ units of 3, then $y = 2x - 1$ is within E units of 5."



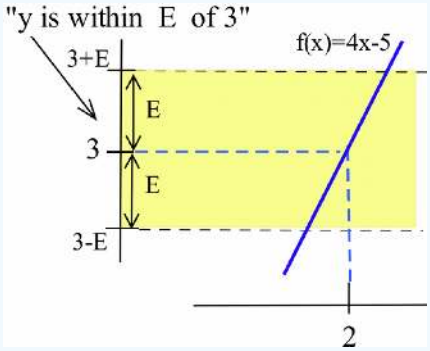
Any smaller distance also works.

Part (c) of Example 1 illustrates the power of general solutions in mathematics. Rather than redoing similar calculations every time someone demands that $f(x) = 2x - 1$ be within some given distance of 5, we did the calculations once. And then we can quickly respond for any given distance. For the question "What values of x guarantee that $f(x) = 2x - 1$ is within 0.4, 0.1 or 0.006 units of 5?" we can answer, "If x is within 0.2 (\)= \frac{0.4}{2}\)), 0.05 (\)=\frac{0.1}{2}\)) or 0.003 ($= \frac{0.006}{2}$) units of 3."

Practice 1.4.1

Knowing that $\lim_{x \to 2} 4x - 5 = 3$, determine which values of x guarantee that $f(x) = 4x - 5$ is within

a. 1 unit of 3.
b. 0.08 units of 3.
c. E units of 3.



Answer

a. $3 - 1 < 4x - 5 < 3 + 1 \Rightarrow 7 < 4x < 9 \Rightarrow 1.75 < x < 2.25$:\ "$x$ within $\frac{1}{4}$ unit of 2."
b. $3 - 0.08 < 4x - 5 < 3 + 0.08 \Rightarrow 7.92 < 4x < 8.08 \Rightarrow 1.98 < x < 2.02$: "$x$ within 0.02 units of 2."
c. $3 - E < 4x - 5 < 3 + E \Rightarrow 8 - E < 4x < 8 + E \Rightarrow 2 - \frac{E}{4} < x < 2 + \frac{E}{4}$: "$x$ within $\frac{E}{4}$ units of 2."

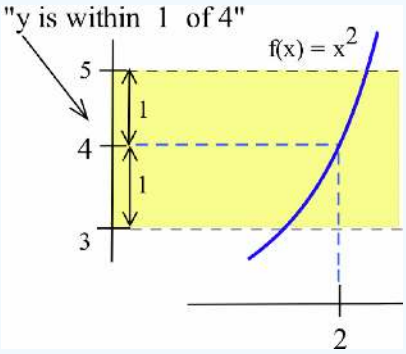
The same ideas work even if the graphs of the functions are not straight lines, but the calculations become more complicated.

Example 1.4.2

Knowing that $\lim_{x \to 2} x^2 = 4$, determine which values of x guaranteethat $f(x) = x^2$ is within:

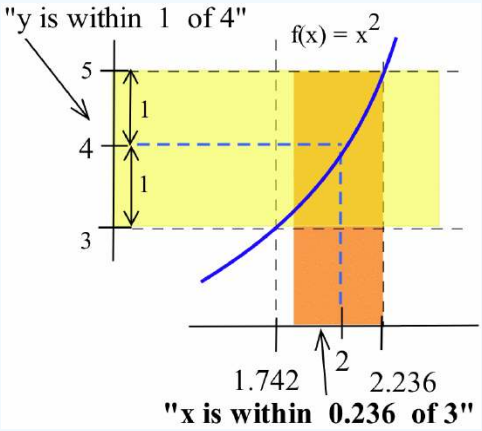
a. 1 unit of 4.
b. 0.2 units of 4.

State each answer in the form: "If x is within units of 2, then $f(x)$ is within units of 4."



Solution

a. If x^2 is within 1 unit of 4 (and x is near 2, hence positive) then $3 < x^2 < 5$ so $\sqrt{3} < x < \sqrt{5}$ or $1.732 < x < 2.236$. The interval containing these x values extends from $2 - \sqrt{3} \approx 0.268$ units to the left of 2 to $\sqrt{5} - 2 \approx 0.236$ units to the right of 2.



Because we want to specify a single distance on each side of 2, we can pick the {\em smaller} of the two distances, 0.236, and say: "If x is within 0.236 units of 2, then $f(x)$ is within 1 unit of 4."

b. Similarly, if x^2 is within 0.2 units of 4 (and x is near 2, so $x > 0$) then $3.8 < x^2 < 4.2$ so $\sqrt{3.8} < x < \sqrt{4.2}$ or $1.949 < x < 2.049$. The interval containing these x values extends from $2 - \sqrt{3.8} \approx 0.051$ units to the left of 2 to $\sqrt{4.2} - 2 \approx 0.049$ units to the right of 2. Again picking the smaller of the two distances, we can say: "If x is within 0.049 units of 2, then $f(x)$ is within 1 unit of 4."

The situation in Example $1.4.2$—with different distances on the left and right sides—is very common, and we *always* pick our single distance to be the *smaller* of the distances to the left and right. By using the smaller distance, we can be certain that if x is within that smaller distance on either side, then the value of $f(x)$ is within the specified distance of the value of the limit.

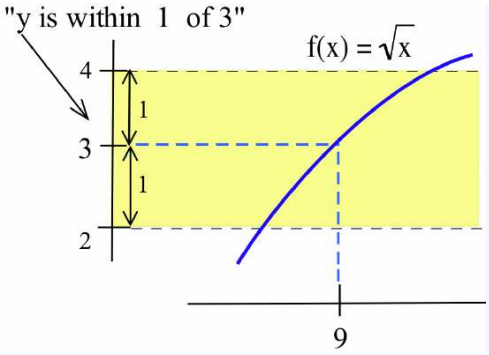
? Practice 1.4.2

Knowing that $\lim_{x \to 9} \sqrt{x} = 3$, determine which values of x guarantee that $f(x) = \sqrt{x}$ is within:

a. 1 unit of 3.

b. 0.2 units of 3.

State each answer in the form: "If x is within units of 9, then $f(x)$ is within units of 3."



Answer

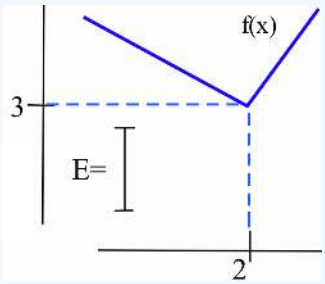
a. "Within 1 unit of 3": If $2 < \sqrt{x} < 4$, then $4 < x < 16$, which extends from 5 units to the left of 9 to 7 units to right of 9. Using the smaller of these two distances from 9: "If x is within 5 units of 9, then $\sqrt{x}$ is within 1 unit of 3."

b. "Within 0.2 units of 3": If $2.8 < \sqrt{x} < 3.2$, then $7.84 < x < 10.24$, which extends from 1.16 units to the left of 9 to 1.24 units to the right of 9. "If x is within 1.16 units of 9, then x is within 0.2 units of 3.

The same ideas can also be used when the function and the specified distance are given graphically, and in that case we can give the answer graphically.

Example 1.4.3

In the figure below, $\lim\limits_{x \to 2} f(x) = 3$:

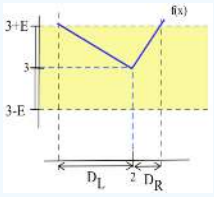


Which values of x guarantee that $y = f(x)$ is within E units (given graphically) of 3? State your answer in the form: "If x is within ({\em show a distance D graphically}) of 2, then $f(x)$ is within E units of 3."

Solution

The solution process requires several steps:

i. Use the given distance E to find the values $3 - E$ and $3 + E$ on the y-axis:

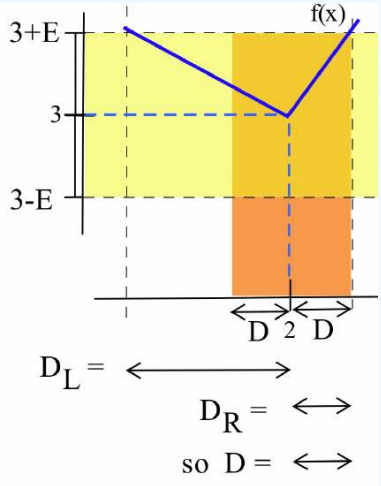


ii. Sketch the horizontal band with lower edge at $y = 3 - E$ and upper edge at $y = 3 + E$.

iii. Find the first locations to the right and left of $x = 2$ where the graph of $y = f(x)$ crosses the lines $y = 3 - E$ and $y = 3 + E$, and at these locations draw vertical line segments extending to the x-axis.

iv. On the x-axis, graphically determine the distance from 2 to the vertical line on the left (labeled D_L) and from 2 to the vertical line on the right (labeled D_R).

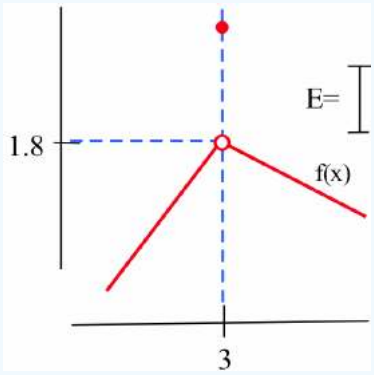
v. Let the length D be the smaller of the lengths D_L and D_R.



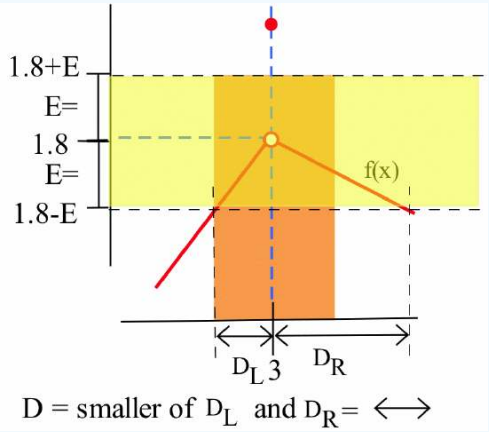
If x is within D units of 2, then $f(x)$ is within E units of 3.

Practice 1.4.3

In figure below, $\lim_{x \to 3} f(x) = 1.8$. Which values of x guarantee that $y = f(x)$ is within E units (given graphically) of 1.8?



Answer



The Formal Definition of Limit

The ideas from the previous Examples and Practice problems, restated for general functions and limits, provide the basis for the definition of limit given below. The use of the lowercase Greek letters ϵ (epsilon) and δ (delta) in the definition is standard, and this

definition is sometimes called the “epsilon-delta” definition of a limit.

Definition of $\lim_{x\to a} f(x) = L$

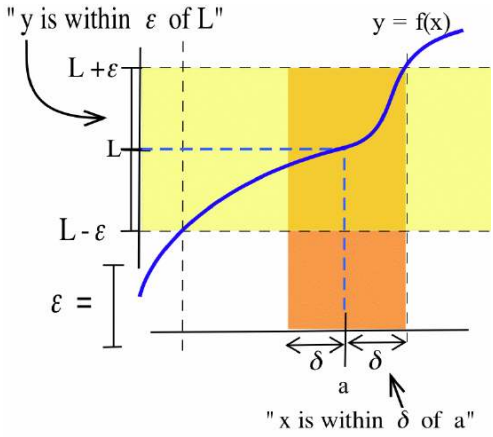
For every given number $\epsilon > 0$ there is a number $\delta > 0$ so that:

if: x is within δ units of a (and $x \neq a$)

then: $f(x)$ is within ϵ units of L

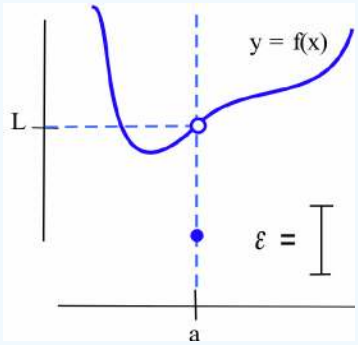
Equivalently: $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

In this definition, ϵ represents the given distance on either side of the limiting value $y = L$, and δ is the distance on each side of the point $x = a$ on the x-axis that we have been finding in the previous examples. This definition has the form of a “challenge and response”: for any positive challenge ϵ (make $f(x)$ within ϵ of L), there is a positive response δ (start with x within δ of a and $x \neq a$).



Example 1.4.4

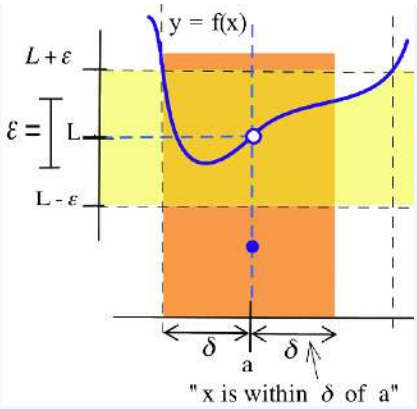
As seen in the figure below, $\lim_{x\to a} f(x) = L$, with a value for ϵ given graphically as a length:



Find a length for δ that satisfies the definition of limit (so “if x is within δ of a, and $x \neq a$, then $f(x)$ is within ϵ of L”).

Solution

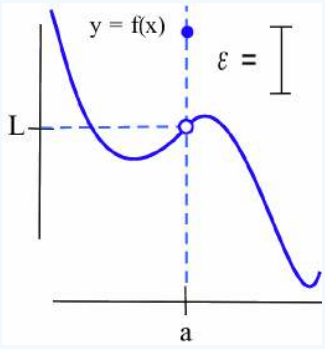
Follow the steps outlined in Example 1.4.3. The length for δ is shown here:



and any shorter length for δ also satisfies the definition.

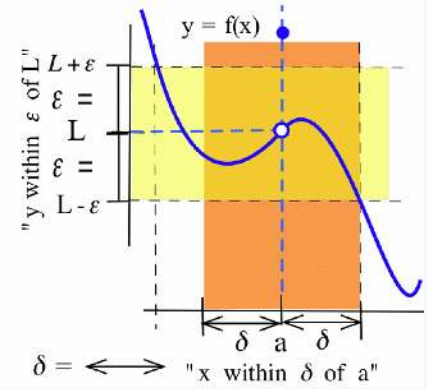
Practice 1.4.4

In figure below, $\lim_{x \to a} f(x) = L$, with a value for ϵ given graphically:



Find a length for δ that satisfies the definition of limit.

Answer



Example 1.4.5

Prove that $\lim_{x \to 3} 4x - 5 = 7$.

Solution

We need to show that "for every given $\epsilon > 0$ there is a $\delta > 0$ so that if x is within δ units of 3 (and $x \neq 3$) then $4x - 5$ is within ϵ units of 7."

Actually, there are two things we need to do. First, we need to find a value for δ (typically depending on ϵ) and, second, we need to show that our δ really does satisfy the "if…then…" part of the definition.

Finding δ is similar to part (c) in Example 1 and Practice 1: Assume $4x-5$ is within ϵ units of 7 and solve for x. If $7-\epsilon<4x-5<7+\epsilon$ then $12-\epsilon<4x<12+\epsilon \Rightarrow 3-\frac{\epsilon}{4}<x<3+\frac{\epsilon}{4}$ so x is within $\frac{\epsilon}{4}$ units of 3. Put $\delta=\frac{\epsilon}{4}$.

To show that $\delta=\frac{\epsilon}{4}$ satisfies the definition, we merely reverse the order of the steps in the previous paragraph. Assume that x is within δ units of 3. Then $3-\delta<x<3+\delta$, so:

$$\begin{aligned} 3-\frac{\epsilon}{4}<x<3+\frac{\epsilon}{4} \quad &\Rightarrow \quad 12-\epsilon<4x<12+\epsilon \\ &\Rightarrow \quad 7-\epsilon<4x-5<7+\epsilon \end{aligned}$$

so we can conclude that $f(x)=4x-5$ is within ϵ units of 7. This formally verifies that $\lim\limits_{x\to 3} 4x-5=7$.

Practice 1.4.5

Prove that $\lim\limits_{x\to 4} 5x+3=23$.

Answer

Given any $\epsilon>0$, take $\delta=\frac{\epsilon}{5}$. If x is within $\delta=\frac{\epsilon}{5}$ of 4, then $4-\frac{\epsilon}{5}<x<4+\frac{\epsilon}{5}$ so:

$$-\frac{\epsilon}{5}<x-4<\frac{\epsilon}{5} \Rightarrow -\epsilon<5x-20<\epsilon \Rightarrow -\epsilon<(5x+3)-23<\epsilon$$

so, finally, $f(x)=5x+3$ is within ϵ of $L=23$.

We have shown that "for any $\epsilon>0$, there is a $\delta>0$ (namely $\delta=\frac{\epsilon}{5}$)" so that the rest of the definition is satisfied.

The method used to prove the values of the limits for these particular linear functions can also be used to prove the following general result about the limits of linear functions.

Theorem

$\lim\limits_{x\to a} mx+b=ma+b$

Proof

Let $f(x)=mx+b$.

Case 1: $m=0$. Then $f(x)=0x+b=b$ is simply a constant function, and any value for $\delta>0$ satisfies the definition. Given any value of $\epsilon>0$, let $\delta=1$ (any positive value for δ works). If x is is within 1 unit of a, then $f(x)-f(a)=b-b=0<\epsilon$, so we have shown that for any $\epsilon>0$ there is a $\delta>0$ that satisfies the limit definition.

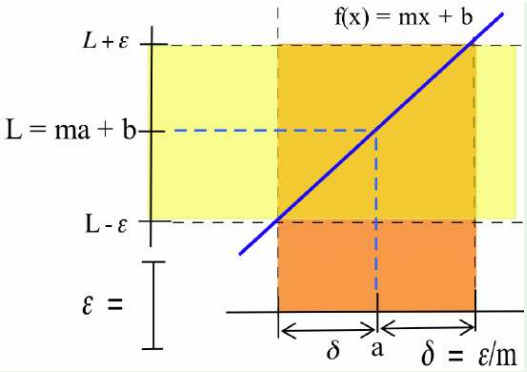
Case 2: $m\neq 0$. For any $\epsilon>0$, put $\delta=\frac{\epsilon}{|m|}>0$. If x is within $\delta=\frac{\epsilon}{|m|}$ of a then

$$a-\frac{\epsilon}{|m|}<x<a+\frac{\epsilon}{|m|} \Rightarrow \frac{\epsilon}{|m|}<x-a<\frac{\epsilon}{|m|} \Rightarrow |x-a|<\frac{\epsilon}{|m|}$$

Then the distance between $f(x)$ and $L=ma+b$ is:

$$\begin{aligned} |f(x)-L| &= |(mx+b)-(ma+b)| = |mx-ma| \\ &= |m|\cdot|x-a|<|m|\frac{\epsilon}{|m|}=\epsilon \end{aligned}$$

so $f(x)$ is within ϵ of $L=ma+b$.

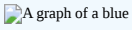


In each case, we have shown that "given any $\epsilon > 0$, there is a $\delta > 0$" that satisfies the rest of the limit definition.

If there is even a single value of ϵ for which there is no δ, then we say that the limit "{\bf does not exist}."

Example 1.4.6

With $f(x)$ defined as: $f(x) = \begin{cases} 2 & \text{if } x < 1 \\ 4 & \text{if } x > 1 \end{cases}$ use the limit definition to prove that $\lim_{x \to 1} f(x)$ does not exist.


A graph of a blue piecewise graph with f(x) = 2 for x<1, f(x) = 4 for x1, and open dots at (1,2) and (1,4)." class="internal default" style="width: 255px; height: 380px;" width="255px" height="380px" src="/@api/deki/files/140114/fig104_14.png">

Solution

One common proof technique in mathematics is called "proof by contradiction" and that is the method we use here:

- We assume that the limit does exist and equals some number L.
- We show that this assumption leads to a contradiction
- We conclude that the assumption must have been false.

We therefore conclude that the limit does not exist.

First, assume that the limit exists: $\lim_{x \to 1} f(x) = L$ for some value for L. Let $\epsilon = \frac{1}{2}$. Then, because we are assuming that the limit exists, there is a $\delta > 0$ so that if x is within δ of 1 then $f(x)$ is within ϵ of L. (The definition says "for every ϵ" so we can certainly pick $\frac{1}{2}$ as our ϵ value; why we chose this particular value for ϵ shows up later in the proof.)

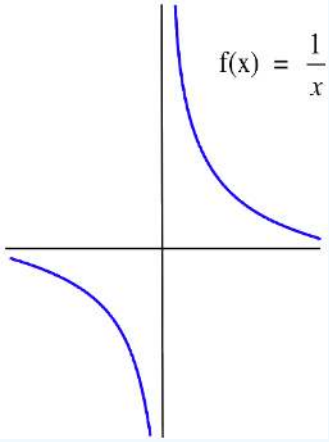
Next, let x_1 be between 1 and $1 + \delta$. Then $x_1 > 1$ so $f(x_1) = 4$. Also, x_1 is within δ of 1 so $f(x_1) = 4$ is within $\frac{1}{2}$ of L, which means that L is between 3.5 and 4.5: $3.5 < L < 4.5$.

Let x_2 be between 1 and $1 - \delta$. Then $x_2 < 1$, so $f(x_2) = 2$. Also, x_2 is within δ of 1 so $f(x_2) = 2$ is within $\frac{1}{2}$ of L, which means that L is between 1.5 and 2.5: $1.5 < L < 2.5$.

These inequalities provide the contradiction we hoped to find. There is no value L that satisfies both $3.5 < L < 4.5$ {\bf and} $1.5 < L < 2.5$, so our assumption must be false: $f(x)$ does not have a limit as $x \to 1$.

Practice 1.4.6

Use the limit definition to prove that $\lim_{x \to 0} \frac{1}{x}$ does not exist.



Answer

Using "proof by contradiction" as in the solution to Example **1.4.6** (this is a much more sophisticated [= harder] problem):

- Assume that the limit exists: $\lim_{x\to 0} \frac{1}{x} = L$ for some value of L. Let $\epsilon = 1$. Since we're assuming that the limit exists, there is a $\delta > 0$ so that if x is within δ of 0 then $f(x) = \frac{1}{x}$ is within $\epsilon = 1$ of L. (The definition says "for every ϵ" so we can pick $\epsilon = 1$. For this particular limit, the definition fails for every $\epsilon > 0$.) FIGURE
- Let x_1 be between 0 and $0 + \delta$ and also require that $x_1 < \frac{1}{2}$. Then $0 < x_1 < \frac{1}{2}$ so $f(x_1) = \frac{1}{x_1} > 2$. Because x_1 is within δ of 0, $f(x_1) > 2$ is within $\epsilon = 1$ of L, so $L > 2 - \epsilon = 1$: that is, $1 < L$. Let x_2 be between 0 and $0 - \delta$ and also require $x_2 > -\frac{1}{2}$. Then $0 > x_2 > \frac{1}{2}$ so $f(x_2) = \frac{1}{x_2} < -2$. Since x_2 is within δ of 0, $f(x_2) < -2$ is within $\epsilon = 1$ of L, so $L < -2 + \epsilon = -1 \Rightarrow -1 > L$.
- The two inequalities derived above provide the contradiction we were hoping to find. There is no value L that satisfies {\bf both} $1 < L$ and $L < -1$, so we can conclude that our assumption was false and that $f(x) = \frac{1}{x}$ does not have a limit as $x \to 0$.

Proofs of Two Limit Theorems

We conclude with proofs of two parts of the Main Limit Theorem so you can see how such proofs proceed—you have already used these theorems to evaluate limits. (There are rigorous proofs of all of the other limit properties in the Main Limit Theorem, but they are somewhat more complicated than the proofs given here.)

Theorem

If: $\lim_{x\to a} f(x) = L$

then: $\lim_{x\to a} k \cdot f(x) = kL$

Proof

Case $k = 0$: The theorem is true but not very interesting:

$$\lim_{x\to a} k \cdot f(x) = \lim_{x\to a} 0 \cdot f(x) = \lim_{x\to a} 0 = 0 = 0 \cdot L = kL$$

Case $k \neq 0$: Because $\lim_{x\to a} f(x) = L$, then, by the definition, for every $\epsilon > 0$ there is a $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$. For any $\epsilon > 0$, we know $\frac{\epsilon}{|k|} > 0$, so pick a value of δ that satisfies $|f(x) - L| < \frac{\epsilon}{|k|}$ whenever $|x - a| < \delta$.

When $|x - a| < \delta$ ("\)x\) is within δ of a") then $|f(x) - L| < \frac{\epsilon}{|k|}$ ("\)f(x)\) is within $\frac{\epsilon}{|k|}$ of L") so $|k| \cdot |f(x) - L| < \epsilon \Rightarrow |k \cdot f(x) - k \cdot L| < \epsilon$ (that is, $k \cdot f(x)$ is within ϵ of $k \cdot L$).

Theorem

If: $\lim_{x\to a} f(x) = L$ \quad and \quad $\lim_{x\to a} g(x) = M$

then: $\lim_{x\to a} [f(x) + g(x)] = L + M$.

Proof

Given any $\epsilon > 0$, we know $\frac{\epsilon}{2} > 0$, so there is a number $\delta_f > 0$ such that when $|x - a| < \delta_f$ then $|f(x) - L| < \frac{\epsilon}{2}$ ("if x is within δ_f of a, then $f(x)$ is within $\frac{\epsilon}{2}$ of L").

Likewise, there is a number $\delta_g > 0$ such that when $|x - a| < \delta_g$ then $|g(x) - M| < \frac{\epsilon}{2}$ ("if x is within δ_g of a, then $g(x)$ is within $\frac{\epsilon}{2}$ of M").

Let δ be the smaller of δ_f and δ_g. If $|x - a| < \delta$ then $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$ so:

$$\begin{aligned} |(f(x) + g(x)) - (L + M))| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

so $f(x) + g(x)$ is within ϵ of $L + M$ whenever x is within δ of a.

Here we use the "triangle inequality":

$$|a + b| \leq |a| + |b|$$

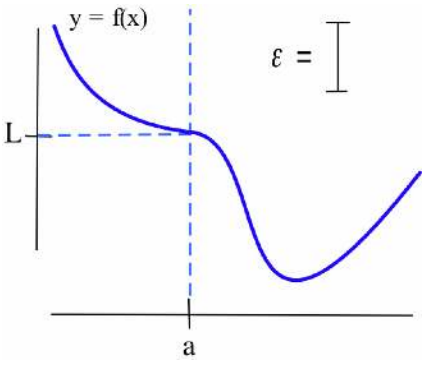
Problems

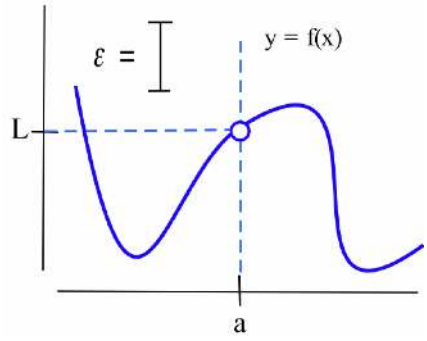
In Problems 1–4, state each answer in the form "If x is within units of…"

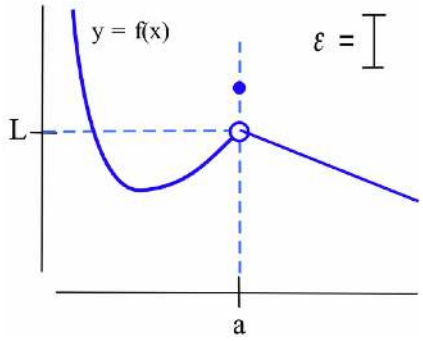
1. Knowing that $\lim_{x\to 3} 2x + 1 = 7$, what values of x guarantee that $f(x) = 2x + 1$ is within:
   a. 1 unit of 7?
   b. 0.6 units of 7?
   c. 0.04 units of 7?
   d. ϵ units of 7?
2. Knowing that $\lim_{x\to 1} 3x + 2 = 5$, what values of x guarantee that $f(x) = 3x + 2$ is within:
   a. 1 unit of 5?
   b. 0.6 units of 5?
   c. 0.09 units of 5?
   d. ϵ units of 5?
3. Knowing that $\lim_{x\to 2} 4x - 3 = 5$, what values of x guarantee that $f(x) = 4x - 3$ is within:
   a. 1 unit of 5?
   b. 0.4 units of 5?
   c. 0.08 units of 5?
   d. ϵ units of 5?
4. Knowing that $\lim_{x\to 1} 5x - 3 = 2$, what values of x guarantee that $f(x) = 5x - 3$ is within:
   a. 1 unit of 2?
   b. 0.5 units of 2?
   c. 0.01 units of 2?
   d. ϵ units of 2?
5. For Problems 1–4, list the slope of each function f and the δ (as a function of ϵ). For these linear functions f, how is δ related to the slope?
6. You have been asked to cut two boards (exactly the same length after the cut) and place them end to end. If the combined length must be within 0.06 inches of 30 inches, then each board must be within how many inches of 15?

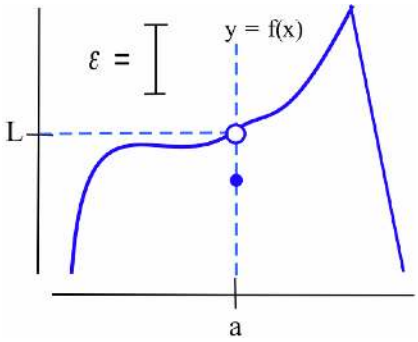
7. You have been asked to cut three boards (exactly the same length after the cut) and place them end to end. If the combined length must be within 0.06 inches of 30 inches, then each board must be within how many inches of 10?
8. Knowing that $\lim\limits_{x\to 3} x^2 = 9$, what values of x guarantee that $f(x) = x^2$ is within:
   a. 1 unit of 9?
   b. 0.2 units of 9?
9. Knowing that $\lim\limits_{x\to 2} x^3 = 8$, what values of x guarantee that $f(x) = x^3$ is within:
   a. 0.5 units of 8?
   b. 0.05 units of 8?
10. Knowing that $\lim\limits_{x\to 16} \sqrt{x} = 4$, what values of x guarantee that $f(x) = \sqrt{x}$ is within:
    a. 1 unit of 4?
    b. 0.1 units of 4?
11. Knowing that $\lim\limits_{x\to 3} \sqrt{1+x} = 2$, what values of x guarantee that $f(x) = \sqrt{1+x}$ is within:
    a. 1 unit of 2?
    b. 0.0002 units of 2?
12. You must cut four pieces of wire (all the same length) and form them into a square. If the area of the square must be within 0.06 in^2 of 100 in^2, then each piece of wire must be within how many inches of 10 in?
13. You need to cut four pieces of wire (all the same length) and form them into a square. If the area of the square must be within 0.06 in^2 of 25 in^2, then each piece of wire must be within how many inches of 5 in?

Problems 14–17 give $\lim\limits_{x\to a} f(x) = L$, the function f and a value for ϵ graphically. Find a length for δ that satisfies the limit definition for the given function and value of ϵ.

14.


15.


16. 

17. 

18. Redo each of Problems 14–17 taking a new value of ϵ to be half the value of ϵ given in the problem.

In Problems 19–22, use the limit definition to prove that the given limit does not exist. (Find a value for $\epsilon > 0$ for which there is no δ that satisfies the definition.)

19. With $f(x)$ defined as: $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 3 & \text{if } x > 2 \end{cases}$ show that $\lim\limits_{x \to 2} f(x)$ does not exist.

20. Show that $\lim\limits_{x \to 3} \lfloor x \rfloor$ does not exist.

21. With $f(x)$ defined as: $f(x) = \begin{cases} x & \text{if } x < 2 \\ 6 - x & \text{if } x > 2 \end{cases}$ show that $\lim\limits_{x \to 2} f(x)$ does not exist.

22. With $f(x)$ defined as: $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$ show that $\lim\limits_{x \to 2} f(x)$ does not exist.

23. Prove: If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ then $\lim\limits_{x \to a} [f(x) - g(x)] = L - M$.

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1.A: Answers

Important Note about Precision of Answers: In many of the problems in this book you are required to read information from a graph and to calculate with that information. You should take reasonable care to read the graphs as accurately as you can (a small straightedge is helpful), but even skilled and careful people make slightly different readings of the same graph. That is simply one of the drawbacks of graphical information. When answers are given to graphical problems, the answers should be viewed as the best approximations we could make, and they usually include the word "approximately" or the symbol "≈" meaning "approximately equal to." Your answers should be close to the given answers, but you should not be concerned if they differ a little. (Yes those are vague terms, but it is all we can say when dealing with graphical information.)

Section 1.0

1. a. $m = \frac{y-9}{x-3}$. If $x = 2.97$, $m = \frac{-0.1791}{-0.03} = 5.97$. If $x = 3.001$, $m = \frac{0.006001}{0.001} = 6.001$. If $x = 3 + h$, $m = \frac{(3+h)^2 - 9}{(3+h) - 3} = \frac{9 + 6h + h^2 - 9}{h} = 6 + h$
   b. When h is close to 0, $6 + h$ is close to 6.

3. a. $m = \frac{y-4}{x-2}$. If $x = 1.99$, $m = \frac{-0.0499}{-0.01} = 4.99$. If $x = 2.004$, $m = \frac{0.020016}{0.004} = 5.004$. If $x = 2 + h$, $m = \frac{[(2+h)^2 + (2+h) - 2] - 4}{(2+h) - 2} = 5 + h$
   b. When h is very small, $5 + h$ is very close to 5.

5. All of these answers are **approximate**. Your answers should be close to these numbers.
   a. average rate of temperature change $\approx \frac{80° - 64°}{1 \text{ p.m.} - 9 \text{ a.m.}} = \frac{16°}{4 \text{ hours}} = 4 \frac{°}{\text{hour}}$
   b. At 10 a.m., temperature was rising about $5°$ per hour; at 7 p.m., its was rising about $-10°$/hr (**falling** about $10°$/hr).

7. All of these answers are **approximate**. Your answers should be close to these numbers.
   a. average velocity $\approx \frac{300 \text{ ft} - 0 \text{ ft}}{20 \text{ sec} - 0 \text{ sec}} = 15 \frac{\text{ft}}{\text{sec}}$
   b. average velocity $\approx \frac{100 \text{ ft} - 200 \text{ ft}}{30 \text{ sec} - 10 \text{ sec}} = -5 \frac{\text{ft}}{\text{sec}}$
   c. At $t = 10$ seconds, velocity ≈ 30 feet per second (between 20 and 35 ft/sec); at $t = 20$ seconds, velocity ≈ -1 feet per second; at $t = 30$ seconds, velocity ≈ -40 feet per second.

9. a. $A(0) = 0$, $A(1) = 3$, $A(2) = 6$, $A(2.5) = 7.5$, $A(3) = 9$
   b. The area of the rectangle bounded below by the t-axis, above by the line $y = 3$, on the left by the vertical line $t = 1$ and on the right by the vertical line $t = 4$.
   c. Graph of $y = 3x$.

Section 1.1

1. a. 2
   b. 1
   c. DNE (does not exist)
   d. 1

3. a. 1
   b. -1
   c. -1
   d. 2

5. a. -7
   b. $\frac{13}{0}$ (DNE)

7. a. 0.54 (radian mode!)
   b. -0.318
   c. -0.54

9. a. 0
   b. 0

c. **0**

11. a. **0**
    b. **−1**
    c. DNE

13. The one- and two-sided limits agree at $x = 1$, $x = 4$ and $x = 5$, but not at $x = 2$:

$$\lim_{x\to 1^-} g(x) = 1 \qquad \lim_{x\to 1^+} g(x) = 1 \qquad \lim_{x\to 1} g(x) = 1$$

$$\lim_{x\to 2^-} g(x) = 1 \qquad \lim_{x\to 2^+} g(x) = 4 \qquad \lim_{x\to 2} g(x) = \text{DNE}$$

$$\lim_{x\to 4^-} g(x) = 2 \qquad \lim_{x\to 4^+} g(x) = 2 \qquad \lim_{x\to 4} g(x) = 2$$

$$\lim_{x\to 5^-} g(x) = 1 \qquad \lim_{x\to 5^+} g(x) = 1 \qquad \lim_{x\to 5} g(x) = 1$$

15. a. **1.0986**
    b. **1**

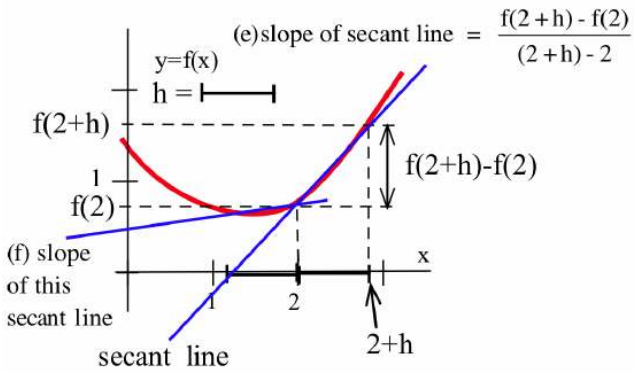
17. a. **0.125**
    b. **3.5**

19. a. $A(0) = 0$, $A(1) = 2.25$, $A(2) = 5$, $A(3) = 8.25$
    b. $A(x) = 2x + \frac{1}{4}x^2$
    c. The area of the trapezoid bounded below by the t-axis, above by the line $y = \frac{1}{2}t + 2$, on the left by the vertical line $t = 1$ and on the right by the vertical line $t = 3$.

Section 1.2

1. a. **2**
   b. **0**
   c. DNE
   d. **1.5**

3. a. **1**
   b. **3**
   c. **1**
   d. $\approx$ **0.8**

5. 

7. a. **2**
   b. **−1**
   c. DNE
   d. **2**
   e. **2**
   f. **2**
   g. **1**

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h. 2
i. DNE

9. a. When $v = 0$, $L = A$.
b. 0

11. a. 4
b. 1
c. 2
d. 0
e. 1
f. 1

13. a. Slope of the line tangent to the graph of $y = \cos(x)$ at the point $(0, 1)$.
b. slope $= 0$

15. a. ≈ 1
b. ≈ 3.43
c. ≈ 4

17. At $x = -1$: "connected and smooth"; at $x = 0$: "connected with a corner"; at $x = 1$: "simple hole"; at $x = 2$: "vertical jump"; at $x = 3$: "simple hole"; at $x = 4$: "corner"; at $x = 5$: "smooth"

19. Many lists will work. Here is one example: Put $a_n = 2 + \frac{1}{n}$ so a_n approaches 2 and $\frac{|a_n - 2|}{a_n - 2} = 1$ for all n. Put $b_n = 2 - \frac{1}{n}$ so $b_n \to 2$ and $\frac{|b_n - 2|}{b_n - 2} = -1$ for all n.

21. $-x^2 \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq x^2$ so limit is 0

23. $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ so limit is 3

25. $\frac{1}{x^2} - 1 < \left\lfloor \frac{1}{x^2} \right\rfloor \leq \frac{1}{x^2}$ so limit is 1

Section 1.3

1. Discontinuous at 1, 3 and 4.

3. a. Discontinuous at $x = 3$: fails condition (i) there.
b. At $x = 2$: fails (i).
c. Where $\cos(x) < 0$ (for example, at $x = \pi$): fails (i).
d. Where x^2 is an integer (for example, at $x = 1$ or $x = 2$): fails (ii).
e. Where $\sin(x) = 0$ (for example, at $x = 0$, $x = \pm\pi$, $\pm 2\pi$ ...): fails (i).
f. At $x = 0$: fails (i).
g. At $x = 0$: fails (i).
h. At $x = 3$: fails (i).
i. At $x = \frac{\pi}{2}$: fails (i).

5. a. $f(x) = 0$ for at least 3 values of x in the interval $0 \leq x \leq 5$.
b. 1
c. 3
d. 2
e. Yes. (It does not have to happen, but it is possible.)

7. a. $f(0) = 0$, $f(3) = 9$ and $0 \leq 2 \leq 9$; $c = \sqrt{2} \approx 1.414$
b. $f(-1) = 1$, $f(2) = 4$ and $1 \leq 3 \leq 4$; $c = \sqrt{3} \approx 1.732$
c. $f(0) = 0$, $f(\frac{\pi}{2} = 1$ and $0 \leq \frac{1}{2} \leq 1$; $c = \arcsin(\frac{1}{2}) \approx 0.524$
d. $f(0) = 0$, $f(1) = 1$ and $0 \leq \frac{1}{3} \leq 1$; $c = \frac{1}{3}$
e. $f(2) = 2$, $f(5) = 20$ and $2 \leq 4 \leq 20$; $c = \frac{1+\sqrt{17}}{2} \approx 2.561$
f. $f(1) = 0$, $f(10) \approx 2.30$ and $0 \leq 2 \leq 2.30$; $c = e^2 \approx 7.389$

9. Neither student is correct. The bisection algorithm converges to the root labeled C.

11. a. D

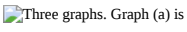
b. D

c. hits B

13. $[-0.9375, -0.875]$, root ≈ -0.879; $[1.3125, 1.375]$, root ≈ 1.347; \)\left[2.5, 2.5625\right]\), root ≈ 2.532

15. $[2.3125, 2.375]$, root ≈ 2.32

17. $[-0.375, -0.3125]$, root ≈ -0.32

19.  Three graphs. Graph (a) is of a red function f that is positive for x<4.5 and negative for x4.5. The graph has open dots on both pieces at x = 4.5. A caption reads: f(3)=5 and f(7)=-3 but f does not have a root between x = 3 and x = 7. Graph (b) is of a red function that begins postive, becomes negative and then becomes positive again, along with a blue graph f that is always positive except at 4.5, where it just touches the x-axis. A caption reads: both functions have a root between x=2 and x=5 but f(2) and f(5) are both positive. Graph (c) is a red function f that consists of a line segment with small positive slope, then bends at x=2 and becomes a line segment with steeper negative slope. A caption reads: f has a corner at x=2 but f is continuous a x-2." class="internal default" style="width: 769px; height: 257px;" width="769px" height="257px" src="/@api/deki/files/140140/ans103_19.png">

21. a. $A(2.1) - A(2)$ is the area of the region bounded below by the t-axis, above by the graph of $y = f(t)$, on the left by the vertical line $t = 2$, and on the right by the vertical line $t = 2.1$. $\frac{A(2.1)-A(2)}{0.1} \approx f(2)$ or $f(2.1) \quad \Rightarrow \quad \frac{A(2.1)-A(2)}{0.1} \approx 1$

b. $A(4.1) - A(4)$ is the area of the region bounded below by the t-axis, above by the graph of $y = f(t)$, on the left by the vertical line $t = 4$, and on the right by the vertical line $t = 4.1$. $\frac{A(4.1)-A(4)}{0.1} \approx f(4) \approx 2$

23. a. Yes (you justify).

b. Yes.

c. Try it.

Section 1.4

1. a. If x is within $\frac{1}{2}$ unit of 3 then $2x + 1$ is within 1 unit of 7.

b. 0.3

c. 0.02

d. $\frac{\epsilon}{2}$

3. a. If x is within $\frac{1}{4}$ unit of 2 then $4x - 3$ is within 1 unit of 5.

b. 0.1

c. 0.02

d. $\frac{\epsilon}{4}$\end{enumerate*}

5. In 1, $m = 2$, $\delta = \frac{\epsilon}{2}$; in 2, $m = 3$, $\delta = \frac{\epsilon}{3}$; in 3, $m = 4$, $\delta = \frac{\epsilon}{4}$; in 4, $m = 5$, $\delta = \frac{\epsilon}{5}$. In general: $\delta = \frac{\epsilon}{|m|}$

7. 0.02 inches

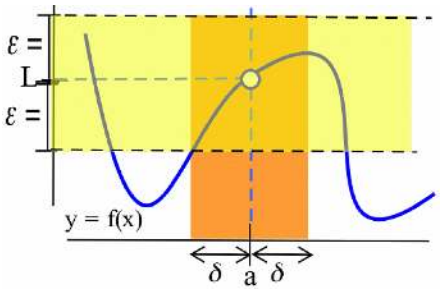
9. a. Any value of x between $\sqrt[3]{7.5} \approx 1.957$ and $\sqrt[3]{8.5} \approx 2.043$: If x is within 0.043 units of 2 then x^3 will be within 0.5 units of 8.

b. Any x between $\sqrt[3]{7.95} \approx 1.9958$ and $\sqrt[3]{8.05} \approx 2.0042$: If x is within 0.0042 units of 2 then x^3 will be within 0.05 units of 8.

11. a. Any value of x between 0 and 8: If x is within 3 units of 3 then $\sqrt{1+x}$ will be within 1 unit of 2.

b. Any x between 2.99920004 and 3.00080004: If x is within 0.00079996 units of 3 then $\sqrt{1+x}$ will be within 0.0002 units of 2.

13. 0.0059964 inches

15. 

17. On your own.

19. Assume $\lim_{x \to 2} f(x)$ exists. Let $\epsilon = 0.1$, so there must be a δ so that:

$$2 - \delta < x < 2 + \delta \Rightarrow L - 0.1 < f(x) < L + 0.1 \qquad (1.A.1)$$

Now, $f(x) = 4$ for any x with $2 - \delta < x < 2$, so it must be true that:

$$\begin{aligned} 2 - \delta < x < 2 &\Rightarrow L - 0.1 < 4 < L + 0.1 \\ &\Rightarrow L > 3.9 \end{aligned}$$

Similarly, $f(x) = 3$ for any x with $2 < x < 2 + \delta$, so it must be true that:

$$\begin{aligned} 2 < x < 2 + \delta &\Rightarrow L - 0.1 < 3 < L + 0.1 \\ &\Rightarrow L < 3.1 \end{aligned}$$

But no value of L can simultaneously satisfy $L > 3.9$ and $L < 3.1$, so we have reached a contradiction and our assumption must be false: $\lim\limits_{x \to 2} f(x)$ does not exist.

21. Assume $\lim\limits_{x \to 2} f(x)$ exists. Let $\epsilon = 0.1$, so there must be a δ so that:

$$2 - \delta < x < 2 + \delta \Rightarrow L - 0.1 < f(x) < L + 0.1 \tag{1.A.2}$$

We can assume that $\delta \leq 1$ (if not, we can replace our initial δ with $\delta = 1$ because any smaller value will also work). Now, $f(x) = x$ for any x with $2 - \delta < x < 2$, so it must be true that:

$$\begin{aligned} 2 - \delta < x < 2 &\Rightarrow L - 0.1 < x < L + 0.1 \\ &\Rightarrow L < 0.1 + x < 2.1 \end{aligned}$$

On the other hand, $f(x) = 6 - x$ for any x with $2 < x < 2 + \delta \leq 3$, so it must be true that:

$$\begin{aligned} 2 < x < 2 + \delta &\Rightarrow L - 0.1 < 6 - x < L + 0.1 \\ &\Rightarrow L > 5.9 - x \geq 2.9 \end{aligned}$$

(because $x \leq 3$). But no value of L can simultaneously satisfy $L < 2.1$ and $L > 2.9$, so we have reached a contradiction and our assumption must be false: $\lim\limits_{x \to 2} f(x)$ does not exist.

23. Given any $\epsilon > 0$, we know $\frac{\epsilon}{2} > 0$, so there is a number $\delta_f > 0$ such that $|x - a| < \delta_f \Rightarrow |f(x) - L| < \frac{\epsilon}{2}$. Likewise, there is a number $\delta_g > 0$ such that $|x - a| < \delta_g \Rightarrow |g(x) - L| < \frac{\epsilon}{2}$. Let δ be the smaller of δ_f and δ_g. If $|x - a| < \delta$ then $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$ so:

$$\begin{aligned} &|(f(x) - g(x)) - (L - M))| \\ &\quad = |(f(x) - L) - (g(x) - M)| \\ &\quad \leq |f(x) - L| + |g(x) - M| \\ &\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

so $f(x) - g(x)$ is within ϵ of $L - M$ whenever x is within δ of a.

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CHAPTER OVERVIEW

2: The Derivative

The two previous chapters have laid the foundation for the study of calculus. They provided a review of some material you will need and started to emphasize the various ways we will view and use functions: functions given by graphs, equations and tables of values.

Chapter 2 will focus on the idea of tangent lines. We will develop a definition for the derivative of a function and calculate derivatives of some functions using this definition. Then we will examine some of the properties of derivatives, see some relatively easy ways to calculate the derivatives, and begin to look at some ways we can use them.

2.0: Introduction to Derivatives
2.1: The Definition of Derivative
2.2: Derivatives: Properties and Formulas
2.3: More Differentiation Patterns
2.4: The Chain Rule
2.5: Applications of the Chain Rule
2.6: Related Rates
2.7: Newton's Method
2.8: Linear Approximation and Differentials
2.9: Implicit and Logarithmic Differentiation
2.A: Answers

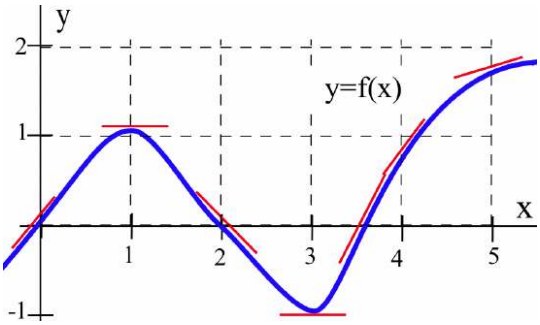
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2.0: Introduction to Derivatives

This section begins with a very graphical approach to slopes of tangent lines. It then examines the problem of finding the slopes of the tangent lines for a single function, $y = x^2$, in some detail—and illustrates how these slopes can help us solve fairly sophisticated problems.

Slopes of Tangent Lines: Graphically

The figure below shows the graph of a function $y = f(x)$.

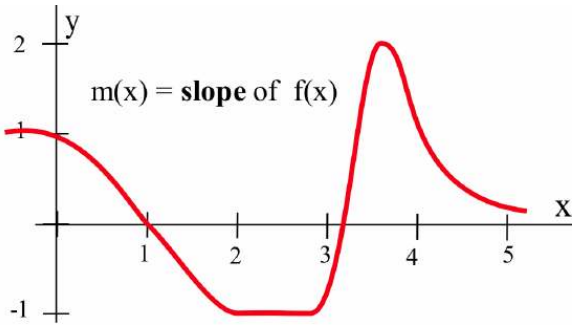


We can use the information in the graph to fill in the table:

x	$y = f(x)$	$m(x)$
0	0	1
1	1	0
2	0	-1
3	-1	0
4	1	1
5	2	$\frac{1}{2}$

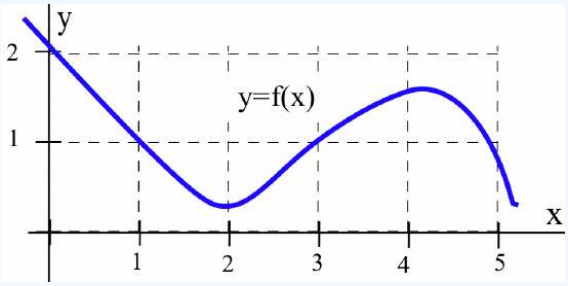
where $m(x)$ is the (estimated) **slope** of the line tangent to the graph of $y = f(x)$ at the point (x, y). We can estimate the values of $m(x)$ at some non-integer values of x as well: $m(0.5) \approx 0.5$ and $m(1.3) \approx -0.3$, for example. We can even say something about the behavior of $m(x)$ over entire intervals: if $0 < x < 1$, then $m(x)$ is positive, for example.

The values of $m(x)$ definitely depend on the values of x (the slope varies as x varies, and there is at most one slope associated with each value of x) so $m(x)$ is a function of x. We can use the results in the table to help sketch a graph of the function $m(x)$:



Practice 2.0.1

A graph of $y = f(x)$ appears below. Set up a table of (estimated) values for x and $m(x)$, the slope of the line tangent to the graph of $y = f(x)$ at the point (x, y), and then sketch a graph of the function $m(x)$.

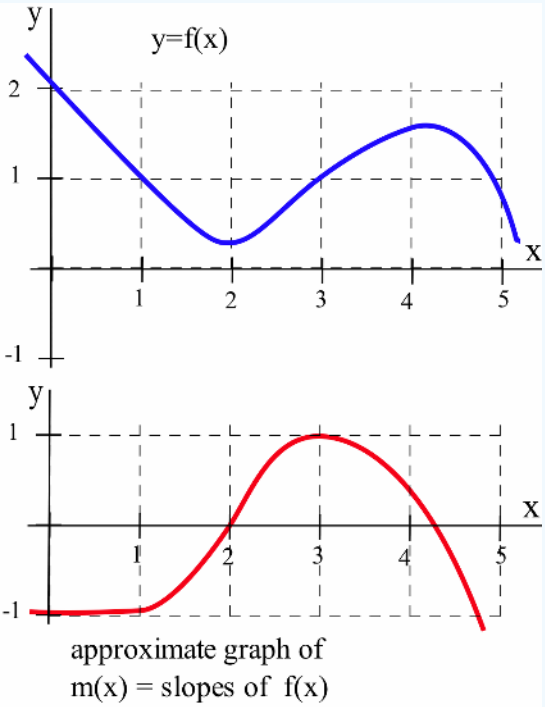


Answer

Approximate values of $m(x)$ appear in the table:

x	$f(x)$	$m(x)$
0	2	-1
1	1	-1
2	$\frac{1}{3}$	0
3	1	1
4	$\frac{3}{2}$	$\frac{1}{2}$
5	1	-2

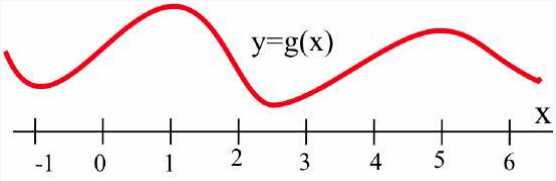
The figure shows a graph of $m(x)$ below a graph of $y = f(x)$:



In some applications, we need to know where the graph of a function $f(x)$ has horizontal tangent lines (that is, where the slope of the tangent line equals 0). The slopes of the lines tangent to graph of $y = f(x)$ in Practice $2.0.1$ are 0 when $x = 2$ or $x \approx 4.25$.

? Practice 2.0.2

At what values of x does the graph of $y = g(x)$ (below) have horizontal tangent lines?

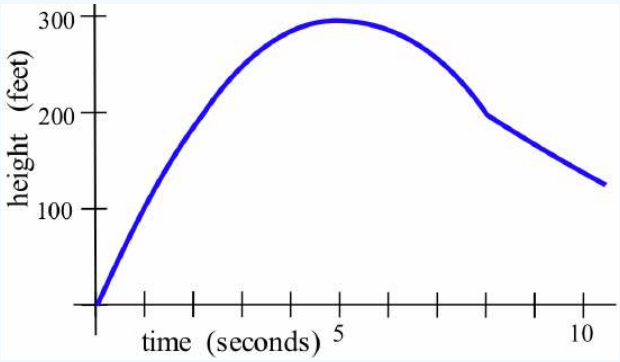


Answer

The tangent lines to the graph of g are horizontal (slope $= 0$) when $x \approx -1$, 1, 2.5 and 5.

✓ Example 2.0.1

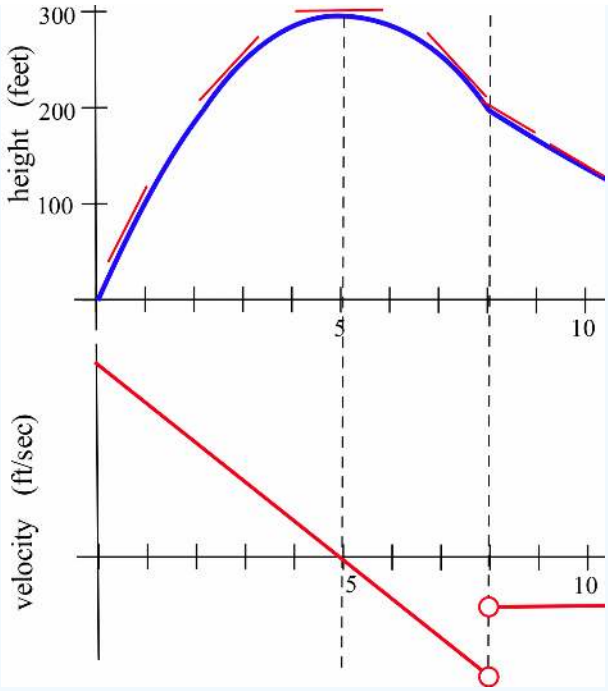
The graph of the height of a rocket at time t appears below:



Sketch a graph of the **velocity** of the rocket at time t. (Remember that instantaneous velocity corresponds to the **slope of the line tangent** to the graph of position or height function.)

Solution

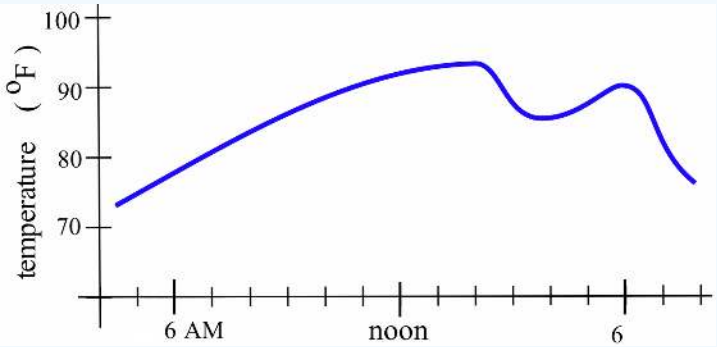
The top figure below shows some sample tangent line segments:



while the bottom figure shows the velocity of the rocket. (What so you think happened at time $t = 8$?)

? Practice 2.0.3

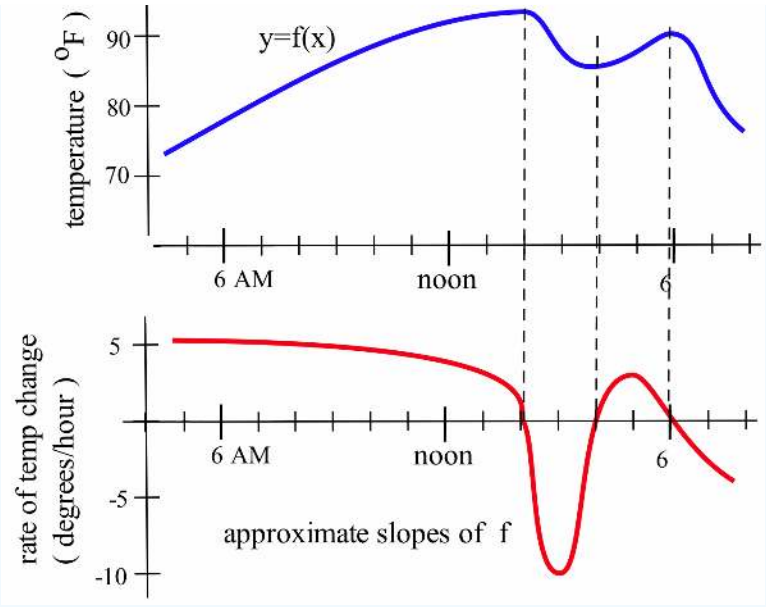
The graph below shows the temperature during a summer day in Chicago.



Sketch a graph of the **rate** at which the temperature is changing at each moment in time. (As with instantaneous velocity, the instantaneous rate of change for the temperature corresponds to the slope of the line tangent to the temperature graph.

Answer

The figure below shows a graph of the approximate rate of temperature change (slope) below a graph of the temperature.

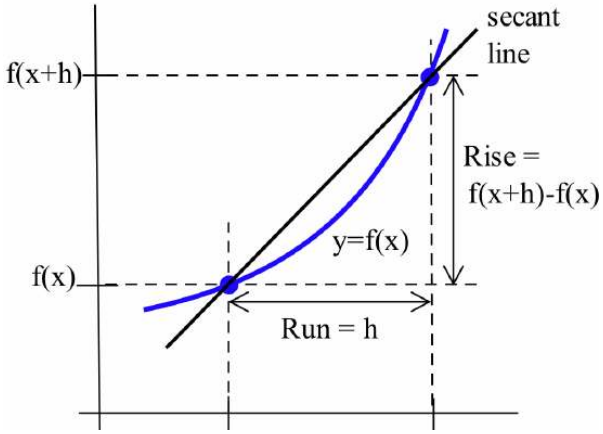


The function $m(x)$, the slope of the line tangent to the graph of $y = f(x)$ at $(x, f(x))$, is called the **derivative** of $f(x)$.

We used the idea of the slope of the tangent line all throughout Chapter 1. In Section 2.1, we will formally define the derivative of a function and begin to examine some of its properties, but first let's see what we can do when we have a formula for $f(x)$.

Tangents to $y = x^2$

When we have a formula for a function, we can determine the slope of the tangent line at a point $(x, f(x))$:



by calculating the slope of the secant line through the points $(x, f(x))$ and $(x + h, f(x + h))$:

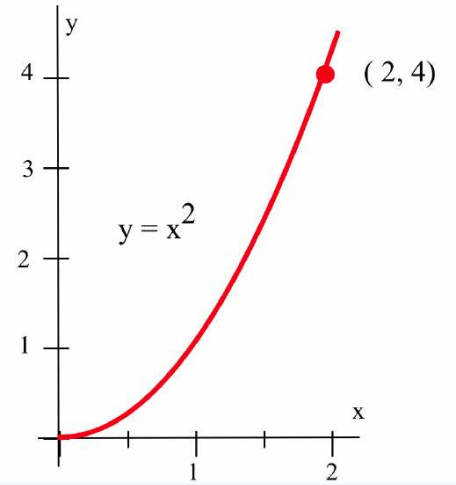
$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - (x)} \tag{2.0.1}$$

and then taking the limit of m_{sec} as h approaches 0:

$$m_{\text{tan}} = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - (x)} \tag{2.0.2}$$

Example 2.0.2

Find the slope of the line tangent to the graph of the function $y = f(x) = x^2$ at the point $(2, 4)$.



Solution

In this example, $x = 2$, so $x + h = 2 + h$ and $f(x+h) = f(2+h) = (2+h)^2$. The slope of the tangent line at $(2,4)$ is

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{(2+h)-(2)} \\ &= \lim_{h\to 0} \frac{(2+h)^2-2^2}{h} = \lim_{h\to 0} \frac{4+4h+h^2-4}{h} \\ &= \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} [4+h] = 4 \end{aligned}$$

The line tangent to $y = x^2$ at the point $(2,4)$ has slope 4.

We can use the point-slope formula for a line to find an equation of this tangent line:

$$y - y_0 = m(x - x_0) \;\Rightarrow\; y - 4 = 4(x-2) \;\Rightarrow\; y = 4x - 4$$

? Practice 2.0.4

Use the method of Example 2.0.2 to show that the slope of the line tangent to the graph of $y = f(x) = x^2$ at the point $(1,1)$ is $m_{\tan} = 2$. Also find the values of $m_{\tan}$ at $(0,0)$ and $(-1,1)$.

Answer

At $(1,1)$, the slope of the tangent line is:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{(1+h)-(1)} \\ &= \lim_{h\to 0} \frac{(1+h)^2-1^2}{h} = \lim_{h\to 0} \frac{1+2h+h^2-1}{h} \\ &= \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} [2+h] = 2 \end{aligned}$$

so the line tangent to $y = x^2$ at the point $(1,1)$ has slope 2. At $(0,0)$:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(0+h)-f(1)}{(0+h)-(0)} \\ &= \lim_{h\to 0} \frac{(0+h)^2-0^2}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0 \end{aligned}$$

so the line tangent to $y = x^2$ at $(0,0)$ has slope 0. At $(-1,1)$:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{(-1+h)-(-1)} \\ &= \lim_{h\to 0} \frac{(-1+h)^2-(-1)^2}{h} = \lim_{h\to 0} \frac{1-2h+h^2-1}{h} \\ &= \lim_{h\to 0} \frac{-2h+h^2}{h} = \lim_{h\to 0} [-2+h] = -2 \end{aligned}$$

so the line tangent to $y = x^2$ at the point $(-1, 1)$ has slope -2.

It is possible to compute the slopes of the tangent lines one point at a time, as we have been doing, but that is not very efficient. You should have noticed in Practice 4 that the algebra for each point was very similar, so let"s do all the work just once, for an arbitrary point $(x, f(x)) = (x, x^2)$ and then use the general result to find the slopes at the particular points we're interested in.

The slope of the line tangent to the graph of $y = f(x) = x^2$ at the arbitrary point (x, x^2) is:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-(x)} \\ &= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} [2x+h] = 2x \end{aligned}$$

The slope of the line tangent to the graph of $y = f(x) = x^2$ at the point (x, x^2) is $m_{\tan} = 2x$. We can use this general result at any value of x without going through all of the calculations again. The slope of the line tangent to $y = f(x) = x^2$ at the point $(4, 16)$ is $m_{\tan} = 2(4) = 8$ and the slope at (p, p^2) is $m_{\tan} = 2(p) = 2p$. The value of x determines the location of our point on the curve, (x, x^2), as well as the slope of the line tangent to the curve at that point, $m_{\tan} = 2x$. The slope $m_{\tan} = 2x$ is a **function** of x and is called the **derivative** of $y = x^2$.

Simply knowing that the slope of the line tangent to the graph of $y = x^2$ is $m_{\tan} = 2x$ at a point (x, y) can help us quickly find an equation of the line tangent to the graph of $y = x^2$ at any point and answer a number of difficult-sounding questions.

✔ Example 2.0.3

Find equations of the lines tangent to $y = x^2$ at the points $(3, 9)$ and (p, p^2).

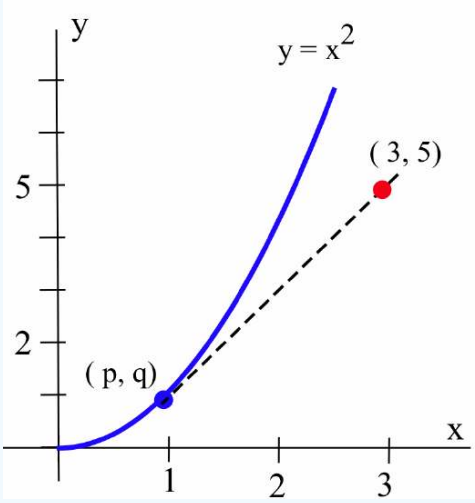
Solution

At $(3, 9)$, the slope of the tangent line is $2x = 2(3) = 6$, and the equation of the line is $y - 9 = 6(x - 3) \Rightarrow y = 6x - 9$.

At (p, p^2), the slope of the tangent line is $2x = 2(p) = 2p$, and the equation of the line is $y - p^2 = 2p(x - p) \Rightarrow y = 2px - p^2$.

✔ Example 2.0.4

A rocket has been programmed to follow the path $y = x^2$ in space (from left to right along the curve, as seen in the figure below), but an emergency has arisen and the crew must return to their base, which is located at coordinates $(3, 5)$.



At what point on the path $y = x^2$ should the captain turn off the engines so that the ship will coast along a path tangent to the curve to return to the base?

Solution

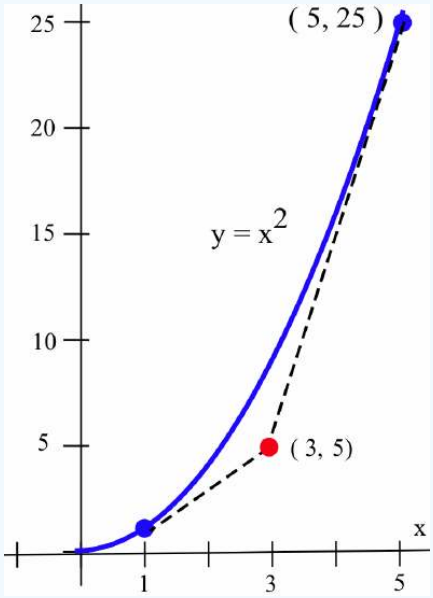
You might spend a few minutes trying to solve this problem without using the relation $m_{\tan} = 2x$, but the problem is much easier if we do use that result.

Let's assume that the captain turns off the engine at the point (p, q) on the curve $y = x^2$ and then try to determine what values p and q must have so that the resulting tangent line to the curve will go through the point $(3, 5)$. The point (p, q) is on the curve $y = x^2$, so $q = p^2$ and the equation of the tangent line, found in Example 2.0.3, must then be $y = 2px - p^2$.

To find the value of p so that the tangent line will go through the point $(3, 5)$, we can substitute the values $x = 3$ and $y = 5$ into the equation of the tangent line and solve for p:

$$\begin{aligned} y = 2px - p^2 \;\Rightarrow\; 5 = 2p(3) - p^2 \;&\Rightarrow\; p^2 - 6p + 5 = 0 \\ &\Rightarrow\; (p-1)(p-5) = 0 \end{aligned}$$

The only solutions are $p = 1$ and $p = 5$, so the only possible points are $(1, 1)$ and $(5, 25)$.



You can verify that the tangent lines to $y = x^2$ at $(1, 1)$ and $(5, 25)$ both go through the point $(3, 5)$. Because the ship is moving from left to right along the curve, the captain should turn off the engines at the point $(1, 1)$. (Why not at $(5, 25)$?)

? Practice 2.0.5

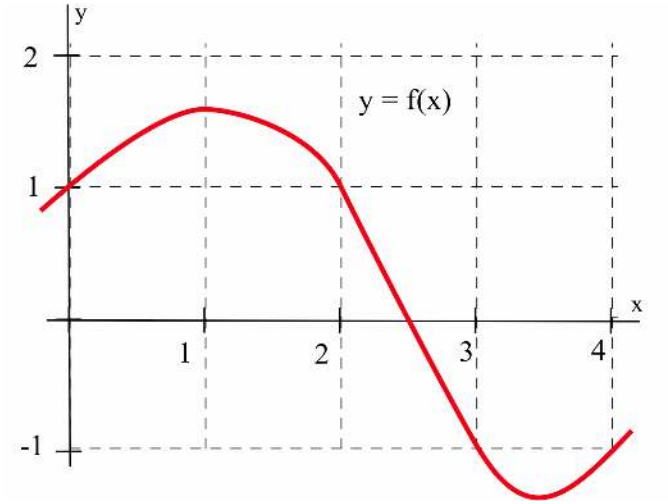
Verify that if the rocket engines in Example 4 are shut off at $(2,4)$, then the rocket will go through the point $(3,8)$.

Answer

From Example 2.0.4 we know the slope of the tangent line is $m_{\tan} = 2x$, so the slope of the tangent line at $(2,4)$ is $m_{\tan} = 2x = 2(2) = 4$. The tangent line has slope 4 and goes through the point $(2,4)$, so an equation of the tangent line (using $y - y_0 = m(x - x_0)$) is $y - 4 = 4(x - 2)$ or $y = 4x - 4$. The point $(3,8)$ satisfies the equation $y = 4x - 4$, so the point $(3,8)$ lies on the tangent line.

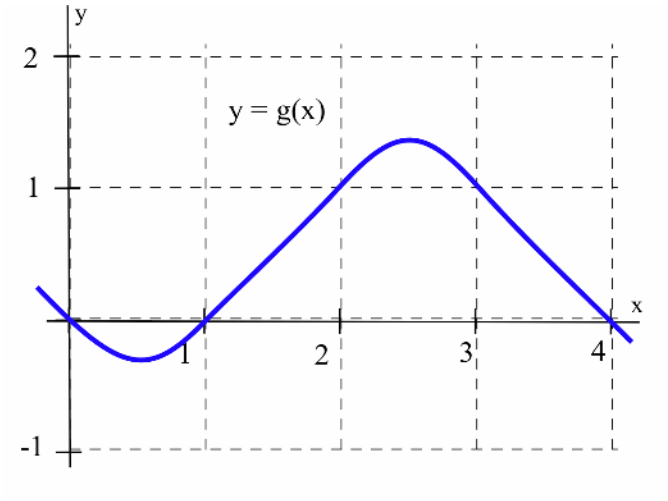
Problems

1. Use the function $f(x)$ graphed below to fill in the table and then graph $m(x)$, the estimated slope of the tangent line to $y = f(x)$ at the point (x, y).



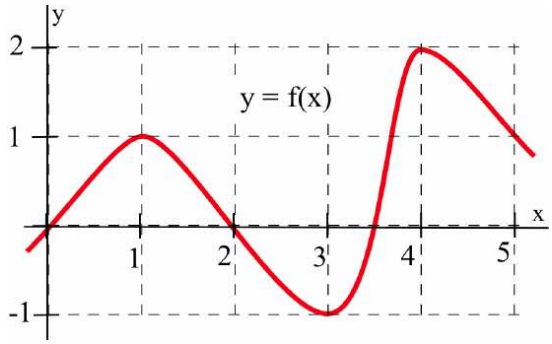
x	$f(x)$	$m(x)$	x	$f(x)$	$m(x)$
0.0			2.5		
0.5			3.0		
1.0			3.5		
1.5			4.0		
2.0					

2. Use the function $g(x)$ graphed below to fill in the table and then graph $m(x)$, the estimated slope of the tangent line to $y = g(x)$ at the point (x, y).



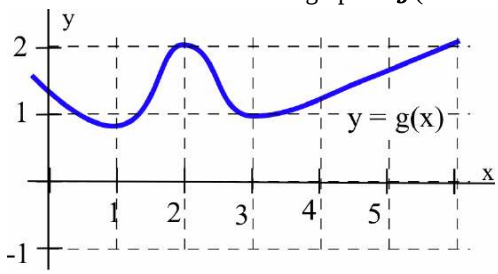
x	$g(x)$	$m(x)$	x	$g(x)$	$m(x)$
0.0			2.5		
0.5			3.0		
1.0			3.5		
1.5			4.0		
2.0					

3. a. At what values of x does the graph of f (shown below) have a horizontal tangent line?



   b. At what value(s) of x is the value of f the largest? Smallest?
   c. Sketch a graph of $m(x)$, the slope of the line tangent to the graph of f at the point $(x, f(x))$.

4. a. At what values of x does the graph of g (shown below) have a horizontal tangent line?

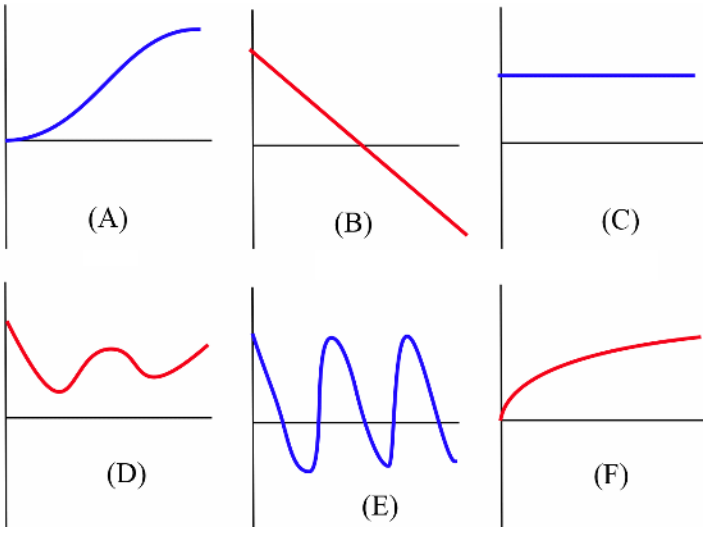


   b. At what value(s) of x is the value of g the largest? Smallest?
   c. Sketch a graph of $m(x)$, the slope of the line tangent to the graph of g at the point $(x, g(x))$.

5. a. Sketch the graph of $f(x) = \sin(x)$ on the interval $-3 \leq x \leq 10$.
   b. Sketch a graph of $m(x)$, the slope of the line tangent to the graph of $\sin(x)$ at the point $(x, \sin(x))$.

c. Your graph in part (b) should look familiar. What function is it?

6. Match the situation descriptions with the corresponding time-velocity graphs shown below.



a. A car quickly leaving from a stop sign.
b. A car sedately leaving from a stop sign.
c. A student bouncing on a trampoline.
d. A ball thrown straight up.
e. A student confidently striding across campus to take a calculus test.
f. An unprepared student walking across campus to take a calculus test.

Problems 7–10 assume that a rocket is following the path $y = x^2$, from left to right.

7. At what point should the engine be turned off in order to coast along the tangent line to a base at $(5, 16)$?
8. At $(3, -7)$?
9. At $(1, 3)$?
10. Which points in the plane can not be reached by the rocket? Why not?

In Problems 11–16, perform these steps:

a. Calculate and simplify: $m_{\text{sec}} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)}$
b. Determine $m_{\text{tan}} = \lim\limits_{h \to 0} m_{\text{sec}}$.
c. Evaluate m_{tan} at $x = 2$.
d. Find an equation of the line tangent to the graph of f at $(2, f(2))$.

11. $f(x) = 3x - 7$
12. $f(x) = 2 - 7x$
13. $f(x) = ax + b$ where a and b are constants
14. $f(x) = x^2 + 3x$
15. $f(x) = 8 - 3x^2$
16. $f(x) = ax^2 + bx + c$ where a, b and c are constants

In Problems 17–18, use the result: $f(x) = ax^2 + bx + c \;\Rightarrow\; m_{\text{tan}} = 2ax + b$

17. Given $f(x) = x^2 + 2x$, at which point(s) $(p, f(p))$ does the line tangent to the graph at that point also go through the point $(3, 6)$?
18. a. If $a \neq 0$, then what is the shape of the graph of $y = f(x) = ax^2 + bx + c$?
    b. At what value(s) of x is the line tangent to the graph of $f(x)$ horizontal?

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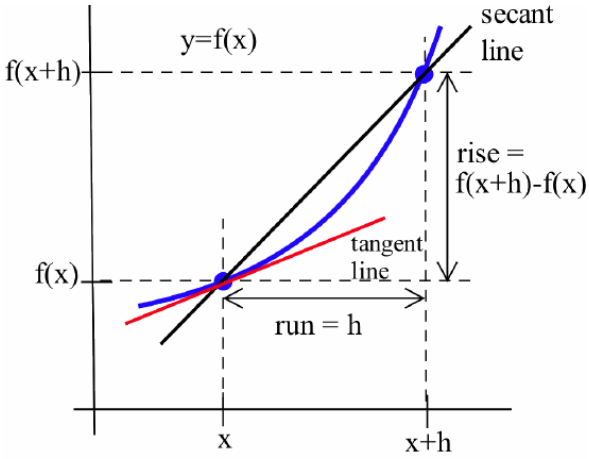
2.1: The Definition of Derivative

The graphical idea of a **slope of a tangent line** is very useful, but for some purposes we need a more algebraic definition of the **derivative of a function**. We will use this definition to calculate the derivatives of several functions and see that these results agree with our graphical understanding. We will also look at several different interpretations for the derivative, and obtain a theorem that will allow us to easily and quickly determine the derivative of any fixed power of x.

In the previous section we found the slope of the tangent line to the graph of the function $f(x) = x^2$ at an arbitrary point $(x, f(x))$ by calculating the slope of the secant line through the points $(x, f(x))$ and $(x + h, f(x + h))$:

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

and then taking the limit of m_{sec} as h approached 0:



That approach to calculating slopes of tangent lines motivates the definition of the derivative of a function.

Definition of the Derivative

The derivative of a function f is a new function, f' (pronounced "eff prime"), whose value at x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists and is finite.

This is **the** definition of differential calculus, and you must know it and understand what it says. The rest of this chapter and all of Chapter 3 are built on this definition, as is much of what appears in later chapters. It is remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative f') will lead to so many important ideas and applications.

Notation

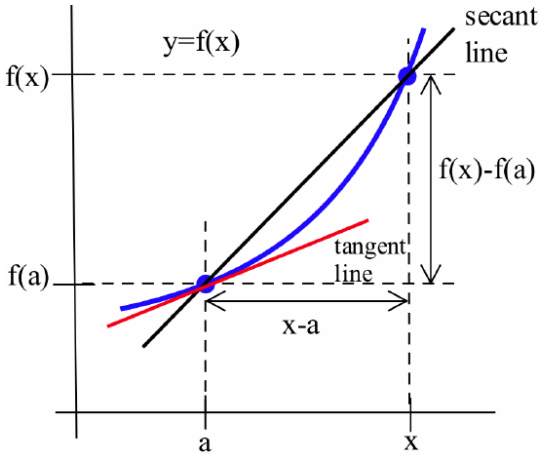
There are three commonly used notations for the derivative of $y = f(x)$:

- $f'(x)$ emphasizes that the derivative is a function related to f
- $\mathbf{D}(f)$ emphasizes that we perform an operation on f to get f'
- $\frac{df}{dx}$ emphasizes that the derivative is the limit of $\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$

We will use all three notations so that you can become accustomed to working with each of them.

The function $f'(x)$ gives the slope of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$, or the instantaneous rate of change of the function f at the point $(x, f(x))$.

If, in the figure below, we let x be the point $a + h$, then $h = x - a$:



As $h \to 0$, we see that $x \to a$ and:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We will use whichever of these two forms is more convenient algebraically in a particular situation.

Calculating Some Derivatives Using the Definition

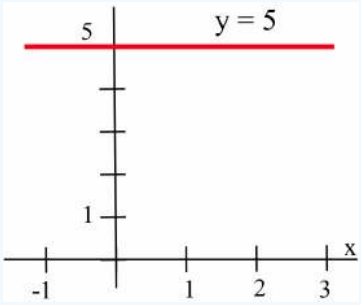
Fortunately, we will soon have some quick and easy ways to calculate most derivatives, but first we will need to use the definition to determine the derivatives of a few basic functions. In Section 2.2, we will use those results and some properties of derivatives to calculate derivatives of combinations of the basic functions. Let's begin by using the graphs and then the definition to find a few derivatives.

Example 2.1.1

Graph $y = f(x) = 5$ and estimate the {\bf slope} of the tangent line at each point on the graph. Then use the definition of the derivative to calculate the exact slope of the tangent line at each point. Your graphical estimate and the exact result from the definition should agree.

Solution

The graph of $y = f(x) = 5$ is a horizontal line, which has slope 0, so we should expect that its tangent line will also have slope 0.



Using the definition: With $f(x) = 5$, then $f(x+h) = 5$ no matter what h is, so:

$$\mathrm{D}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{5-5}{h} = \lim_{h \to 0} 0 = 0$$

and this agrees with our graphical estimate of the derivative.

Using similar steps, it is easy to show that the derivative of {\em any} constant function is 0.

Theorem

If $f(x) = k$, then $f'(x) = 0$.

Practice 2.1.1

Graph $y = f(x) = 7x$ and estimate the slope of the tangent line at each point on the graph. Then use the definition of the derivative to calculate the exact slope of the tangent line at each point.

Answer

The graph of $f(x) = 7x$ is a line through the origin. The slope of the line is 7. For all x:

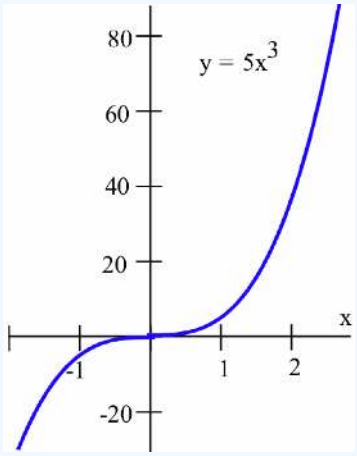
$$m_{\tan} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{7(x+h)-7x}{h} = \lim_{h\to 0} \frac{7h}{h} = \lim_{h\to 0} 7 = 7$$

Example 2.1.2

Describe the derivative of $y = f(x) = 5x^3$ graphically and compute it using the definition. Find an equation of the line tangent to $y = 5x^3$ at the point $(1, 5)$.

Solution

It appears from the graph of $y = f(x) = 5x^3$ (see margin) that $f(x)$ is increasing, so the slopes of the tangent lines are positive except perhaps at $x = 0$, where the graph seems to flatten out.



With $f(x) = 5x^3$ we have:

$$f(x+h) = 5(x+h)^3 = 5(x^3 + 3x^2h + 3xh^2 + h^3)$$

and using this last expression in the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5x^3}{h}$$
$$= \lim_{h\to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h} = \lim_{h\to 0} (15x^2 + 15xh + 5h^2) = 15x^2$$

so $\mathrm{D}(5x^3) = 15x^2$, which is positive except when $x = 0$ (as we predicted from the graph).

The function $f'(x) = 15x^2$ gives the slope of the line tangent to the graph of $f(x) = 5x^3$ at the point $(x, f(x))$. At the point $(1, 5)$, the slope of the tangent line is $f'(1) = 15(1)^2 = 15$. From the point-slope formula, an equation of the tangent line to f at that point is $y - 5 = 15(x - 1)$ or $y = 15x - 10$.

Practice 2.1.2

Use the definition to show that the derivative of $y = x^3$ is $\frac{dy}{dx} = 3x^2$. Find an equation of the line tangent to the graph of $y = x^3$ at the point $(2, 8)$.

Answer

$f(x) = x^3 \Rightarrow f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ so:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

At the point $(2, 8)$, the slope of the tangent line is $3(2)^2 = 12$ so an equation of the tangent line is $y - 8 = 12(x - 2)$ or $y = 12x - 16$.

If f has a derivative at x, we say that f is **differentiable** at x. If we have a point on the graph of a differentiable function and a slope (the derivative evaluated at the point), it is easy to write an equation of the tangent line.

Tangent Line Formula

If: $f(x)$ is differentiable at $x = a$

then: an equation of the line tangent to f at $(a, f(a))$ is:

$$y = f(a) + f'(a)(x - a)$$

Proof

The tangent line goes through the point $(a, f(a))$ with slope $f'(a)$ so, using the point-slope formula, $y - f(a) = f'(a)(x - a)$ or $y = f(a) + f'(a)(x - a)$.

Practice 2.1.3

The derivatives $\mathrm{D}(x) = 1$, $\mathrm{D}(x^2) = 2x$, $\mathrm{D}(x^3) = 3x^2$ exhibit the start of a pattern. Without using the definition of the derivative, what do you think the following derivatives will be? $\mathrm{D}(x^4)$, $\mathrm{D}(x^5)$, $\mathrm{D}(x^{43})$, $\mathrm{D}(\sqrt{x}) = \mathrm{D}(x^{\frac{1}{2}})$ and $\mathrm{D}(x^\pi)$. (Just make an intelligent "guess" based on the pattern of the previous examples.)

Answer

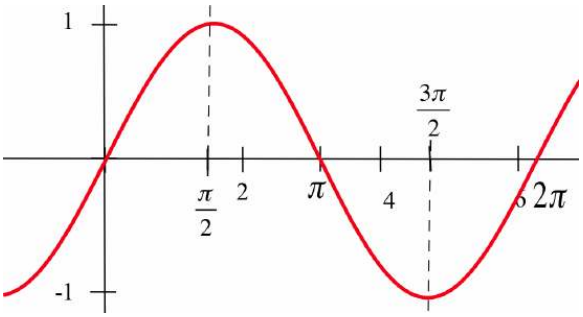
$\mathrm{D}(x^4) = 4x^3$, $\mathrm{D}(x^5) = 5x^4$, $\mathrm{D}(x^{43}) = 43x^{42}$, $\mathrm{D}(\sqrt{x}) = \mathrm{D}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, $\mathrm{D}(x^\pi) = \pi x^{\pi-1}$

Before further investigating the "pattern" for the derivatives of powers of x and general properties of derivatives, let's compute the derivatives of two functions that are not powers of x: $\sin(x)$ and $|x|$.

Theorem

$\mathrm{D}(\sin(x)) = \cos(x)$

The graph of $y = f(x) = \sin(x)$ should be very familiar to you:



The graph has horizontal tangent lines (slope $= 0$) when $x = \pm\frac{\pi}{2}$ and $x = \pm\frac{3\pi}{2}$ and so on. If $0 < x < \frac{\pi}{2}$, then the slopes of the tangent lines to the graph of $y = \sin(x)$ are positive. Similarly, if $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then the slopes of the tangent lines are negative. Finally, because the graph of $y = \sin(x)$ is periodic, we expect that the derivative of $y = \sin(x)$ will also be periodic. Note that the function $\cos(x)$ possesses all of those desired properties for the slope function.

Proof

With $f(x) = \sin(x)$, apply an angle addition formula to get:

$$f(x+h) = \sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

and use this formula in the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(\sin(x)\cos(h) + \cos(x)\sin(h)) - \sin(x)}{h} \end{aligned}$$

This limit looks formidable, but just collect the terms containing $\sin(x)$:

$$\lim_{h\to 0} \frac{(\sin(x)\cos(h) - \sin(x)) + \cos(x)\sin(h)}{h}$$

so you can factor out $\sin(x)$ from the first two terms, rewriting as:

$$\lim_{h\to 0} \left[\sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h}\right]$$

Now calculate the limits separately:

$$\lim_{h\to 0} \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \lim_{h\to 0} \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

The first and third limits do not depend on h, and we calculated the second and fourth limits in Section 1.2:

$$\sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)$$

So $\mathrm{D}(\sin(x)) = \cos(x)$ and the various properties we expected of the derivative of $y = \sin(x)$ by examining its graph are true of $\cos(x)$.

? Practice 2.1.4

Show that $\mathrm{D}(\cos(x)) = -\sin(x)$ using the definition.

You will need the angle addition formula for cosine to rewrite $\cos(x+h)$ as:

$$\cos(x) \cdot \cos(h) - \sin(x) \cdot \sin(h)$$

Answer

Proceeding as we did to find the derivative to $\sin(x)$:

$$\mathrm{D}(\cos(x)) = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} = \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-cos(x)}{h}$$
$$= \lim_{h\to 0}\left[\cos(x)\cdot\frac{\cos(h)-1}{h} - \sin(x)\frac{\sin(h)}{h}\right] = \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x)$$

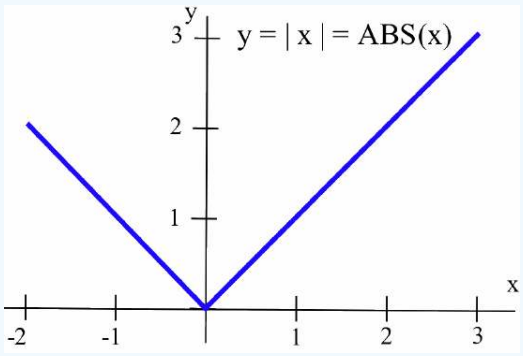
The derivative of $\cos(x)$ resembles the situation for $\sin(x)$ but differs by an important negative sign. You should memorize both of these important derivatives.

Example 2.1.3

For $y = |x|$, find $\frac{dy}{dx}$.

Solution

The graph of $y = f(x) = |x|$ is a "V" shape with its vertex at the origin:



When $x > 0$, the graph is just $y = |x| = x$, which is part of a line with slope $+1$, so we should expect the derivative of $|x|$ to be $+1$. When $x < 0$, the graph is $y = |x| = -x$, which is part of a line with slope -1, so we expect the derivative of $|x|$ to be -1. When $x = 0$, the graph has a corner, and we should expect the derivative of $|x|$ to be undefined at $x = 0$, as there is no single candidate for a line tangent to the graph there.

Using the definition, consider the same three cases discussed previously: $x > 0$, $x < 0$ and $x = 0$.

If $x > 0$, then, for small values of h, $x + h > 0$, so:

$$\mathrm{D}(f(x)) = \lim_{h\to 0}\frac{|x+h|-|x|}{h} = \lim_{h\to 0}\frac{x+h-x}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

If $x < 0$, then, for small values of h, $x + h < 0$, so:

$$\mathrm{D}(f(x)) = \lim_{h\to 0}\frac{|x+h|-|x|}{h} = \lim_{h\to 0}\frac{-(x+h)-(-x)}{h} = \lim_{h\to 0}\frac{-h}{h} = -1$$

When $x = 0$, the situation is a bit more complicated:

$$f'(0) = \lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

This is undefined, as $\lim_{h\to 0^+}\frac{|h|}{h} = +1$ and $\lim_{h\to 0^-}\frac{|h|}{h} = -1$, so:

$$\mathrm{D}(|x|) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

or, equivalently, $\mathrm{D}(|x|) = \frac{|x|}{x}$.

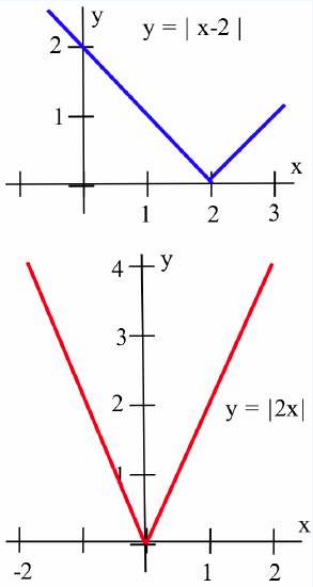
The derivative of $|x|$ agrees with the function $\mathrm{sgn}(x)$ defined in Chapter 0, except at $x = 0$: $\mathrm{D}(|x|)$ is undefined at $x = 0$ but $\mathrm{sgn}(0) = 0$.

? Practice 2.1.5

Graph $y = |x-2|$ and $y = |2x|$ and use the **graphs** to determine $\mathbf{D}(|x-2|)$ and $\mathbf{D}(|2x|)$.

Answer

Here are the graphs of $y = |x-2|$ $y = |2x|$:



$$\mathbf{D}(|x-2|) = \begin{cases} 1 & \text{if } x > 2 \\ \text{undefined} & \text{if } x = 2 \\ -1 & \text{if } x < 2 \end{cases} = \frac{|x-2|}{x-2}$$

$$\mathbf{D}(|2x|) = \begin{cases} 2 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -2 & \text{if } x < 0 \end{cases} = \frac{2|x|}{x}$$

So far we have emphasized the derivative as the slope of the line tangent to a graph. That very visual interpretation is very useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are employed in a wide variety of fields and applications, and some of these fields use other interpretations. A few commonly used interpretations of the derivative follow.

Interpretations of the Derivative

General

Rate of Change: The function $f'(x)$ is the rate of change of the function at x. If the units for x are years and the units for $f(x)$ are people, then the units for $\frac{df}{dx}$ are $\frac{\text{people}}{\text{year}}$, a rate of change in population.

Graphical

Slope: $f'(x)$ is the slope of the line tangent to the graph of f at $(x, f(x))$.

Physical

Velocity: If $f(x)$ is the position of an object at time x, then $f'(x)$ is the velocity of the object at time x. If the units for x are hours and $f(x)$ is distance, measured in miles, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{miles}}{\text{hour}}$, miles per hour, which is a measure of velocity.

Acceleration: If $f(x)$ is the velocity of an object at time x, then $f'(x)$ is the acceleration of the object at time x. If the units for x are hours and $f(x)$ has the units $\frac{\text{miles}}{\text{hour}}$, then the units for the acceleration $f'(x) = \frac{df}{dx}$ are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}^2}$, "miles per hour per hour."

Magnification: $f'(x)$ is the magnification factor of the function f for points close to x. If a and b are two points very close to x, then the distance between $f(a)$ and $f(b)$ will be close to $f'(x)$ times the original distance between a and b: $f(b) - f(a) \approx f'(x)(b-a)$.

Business

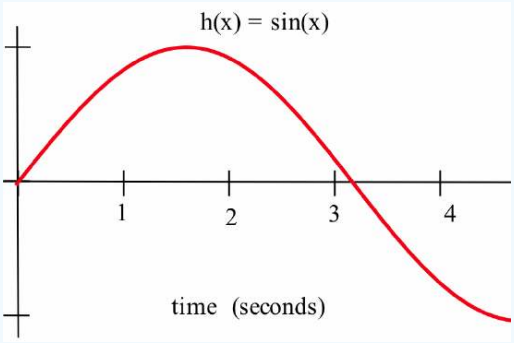
Marginal Cost: If $f(x)$ is the total cost of producing x objects, then $f'(x)$ is the marginal cost, at a production level of x: (approximately) the additional cost of making one more object once we have already made x objects. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the cost per bicycle.

Marginal Profit: If $f(x)$ is the total profit from producing and selling x objects, then $f'(x)$ is the marginal profit: the profit to be made from producing and selling one more object. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the profit per bicycle.

In financial contexts, the word "marginal" usually refers to the derivative or rate of change of some quantity. One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus remain useful.

Example 2.1.4

A small cork is bobbing up and down, and at time t seconds it is $h(t) = \sin(t)$ feet above the mean water level:



Find the height, velocity and acceleration of the cork when $t = 2$ seconds. (Include the proper units for each answer.)

Solution

$h(t) = \sin(t)$ represents the height of the cork at any time t, so the height of the cork when $t = 2$ is $h(2) = \sin(2) \approx 0.91$ feet above the mean water level.

The velocity is the derivative of the position, so $v(t) = \frac{d}{dt}h(t) = \frac{d}{dt}\sin(t) = \cos(t)$. The derivative of position is the limit of $\frac{\Delta h}{\Delta t}$, so the units are $\frac{\text{feet}}{\text{seconds}}$. After 2 seconds, the velocity is $v(2) = \cos(2) \approx -0.42$ feet per second.

The acceleration is the derivative of the velocity, so $a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\cos(t) = -\sin(t)$. The derivative of velocity is the limit of $\frac{\Delta v}{\Delta t}$, so the units are $\frac{\text{feet/second}}{\text{seconds}}$ or $\frac{\text{feet}}{\text{second}^2}$. After 2 seconds the acceleration is $a(2) = -\sin(2) \approx -0.91 \frac{\text{ft}}{\text{sec}^2}$.

Practice 2.1.6

Find the height, velocity and acceleration of the cork in the previous example after 1 second.

Answer

$h(t) = \sin(t)$ so $h(1) = \sin(1) \approx 0.84$ ft

$v(t) = \cos(t)$ so $v(1) = \cos(1) \approx 0.54$ ft/sec

$a(t) = -\sin(t)$ so $a(1) = -\sin(1) \approx -0.84$ ft/sec^2

A Most Useful Formula: $\mathbf{D}(x^n)$

Functions that include powers of x are very common (every polynomial is a sum of terms that include powers of x) and, fortunately, it is easy to calculate the derivatives of such powers. The "pattern" emerging from the first few examples in this section is, in fact, true for all powers of x. We will only state and prove the "pattern" here for positive integer powers of x, but it is also true for other powers (as we will prove later).

Theorem

If n is a positive integer, then: $\mathbf{D}(x^n) = n \cdot x^{n-1}$

This theorem is an example of the power of {\em generality} and {\em proof} in mathematics. Rather than resorting to the definition when we encounter a new exponent p in the form x^p (imagine using the definition to calculate the derivative of x^{307}), we can justify the pattern for all positive integer exponents n, and then simply apply the result for whatever exponent we have. We know, from the first examples in this section, that the theorem is true for $n = 1$, 2 and 3, but no number of {\em examples} would guarantee that the pattern is true for all exponents. We need a proof that what we {\em think} is true really {\em is} true.

Proof

With $f(x) = x^n$, $f(x+h) = (x+h)^n$, and in order to simplify $f(x+h) - f(x) = (x+h)^n - x^n$, we will need to expand $(x+h)^n$. However, we really only need to know the first two terms of the expansion and to know that all of the other terms of the expansion contain a power of h of at least 2.

The Binomial Theorem from algebra says (for $n > 3$) that:

$$(x+h)^n = x^n + n \cdot x^{n-1}h + a \cdot x^{n-2}h^2 + b \cdot x^{n-3}h^3 + \cdots + h^n$$

where a and b represent numerical coefficients. (Expand $(x+h)^n$ for a few different values of n to convince yourself of this result.) Then:

$$\mathbf{D}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Now expand $(x+h)^n$ to get:

$$\lim_{h \to 0} \frac{x^n + n \cdot x^{n-1}h + a \cdot x^{n-2}h^2 + b \cdot x^{n-3}h^3 + \cdots + h^n - x^n}{h}$$

Eliminating $x^n - x^n$ we get:

$$\lim_{h \to 0} \frac{n \cdot x^{n-1}h + a \cdot x^{n-2}h^2 + b \cdot x^{n-3}h^3 + \cdots + h^n}{h}$$

and we can then factor h out of the numerator:

$$\lim_{h \to 0} \frac{h(n \cdot x^{n-1} + a \cdot x^{n-2}h + b \cdot x^{n-3}h^2 + \cdots + h^{n-1})}{h}$$

and divide top and bottom by the factor h:

$$\lim_{h \to 0} \left[n \cdot x^{n-1} + a \cdot x^{n-2}h + b \cdot x^{n-3}h^2 + \cdots + h^{n-1}\right]$$

We are left with a polynomial in h and can now compute the limit by simply evaluating the polynomial at $h = 0$ to get $\mathbf{D}(x^n) = n \cdot x^{n-1}$.

Note

You may also be familiar with Pascal's triangle:

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Among many beautiful and amazing properties, the numbers in row n of the triangle (counting the first row as row 0) give the coefficients in the expansion of $(A+B)^n$. Notice that each entry in the interior of the triangle is the sum of the two numbers immediately above it.

Practice 2.1.7

Calculate $\mathrm{D}(x^5)$, $\frac{d}{dx}(x^2)$, $\mathrm{D}(x^{100})$, $\frac{d}{dt}(t^{31})$ and $\mathrm{D}(x^0)$.

Answer

$\mathrm{D}(x^5) = 5x^4$, $\frac{d}{dx}(x^2) = 2x^1 = 2x$, $\mathrm{D}(x^{100}) = 100x^{99}$, $\frac{d}{dt}(t^{31}) = 31t^{30}$ and $\mathrm{D}(x^0) = 0x^{-1} = 0$ or $\mathrm{D}(x^0) = \mathrm{D}(1) = 0$

We will occasionally use the result of the theorem for the derivatives of **all** constant powers of x even though it has only been proven for positive integer powers, so far. A proof of a more general result (for all rational powers of x) appears in Section 2.9

Example 2.1.5

Find $\mathrm{D}\left(\frac{1}{x}\right)$ and $\frac{d}{dx}(\sqrt{x})$.

Solution

Rewriting the fraction using a negative exponent:

$$\mathrm{D}\left(\frac{1}{x}\right) = \mathrm{D}(x^{-1}) = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Rewriting the square root using a fractional exponent:

$$\frac{d}{dx}(\sqrt{x}) = \mathrm{D}(x^{\frac{1}{2}}) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

These results can also be obtained by using the definition of the derivative, but the algebra involved is slightly awkward.\end{solution}

Practice 2.1.8

Find $\mathrm{D}(x^{\frac{3}{2}})$, $\frac{d}{dx}(x^{\frac{1}{3}})$, $\mathrm{D}\left(\frac{1}{\sqrt{x}}\right)$ and $\frac{d}{dt}(t^{\pi})$.

Answer

$\mathrm{D}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$, $\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$, $\mathrm{D}(\frac{1}{\sqrt{x}}) = \mathrm{D}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}}$, $\frac{d}{dt}(t^{\pi}) = \pi t^{\pi-1}$.

Example 2.1.6

It costs $\sqrt{x}$ hundred dollars to run a training program for x employees.

a. How much does it cost to train 100 employees? 101 employees? If you already need to train 100 employees, how much additional money will it cost to add 1 more employee to those being trained?

b. For $f(x) = \sqrt{x}$, calculate $f'(x)$ and evaluate f' at $x = 100$. How does $f'(100)$ compare with the last answer in part (a)?

Solution

a. Put $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ hundred dollars, the cost to train x employees. Then $f(100) = \$1000$ and $f(101) = \$1004.99$, so it costs \$4.99 additional to train the 101st employee.

b. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ so $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ hundred dollars $= 5.00$. Clearly $f'(100)$ is very close to the actual additional cost of training the 101st employee.

Important Information and Results

This section contains a great deal of important information that we will continue to use throughout the rest of the course. (So it is worthwhile to collect here some of those important ideas.)

Definition of Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (Valid if the limit exists and is finite.)

Notations for the Derivative: $f'(x)$, $\mathrm{D}(f(x))$, $\frac{df}{dx}$

Tangent Line Equation: $y = f(a) + f'(a) \cdot (x - a)$ (An equation of the line tangent to the graph of f at $(a, f(a))$.)

Formulas:

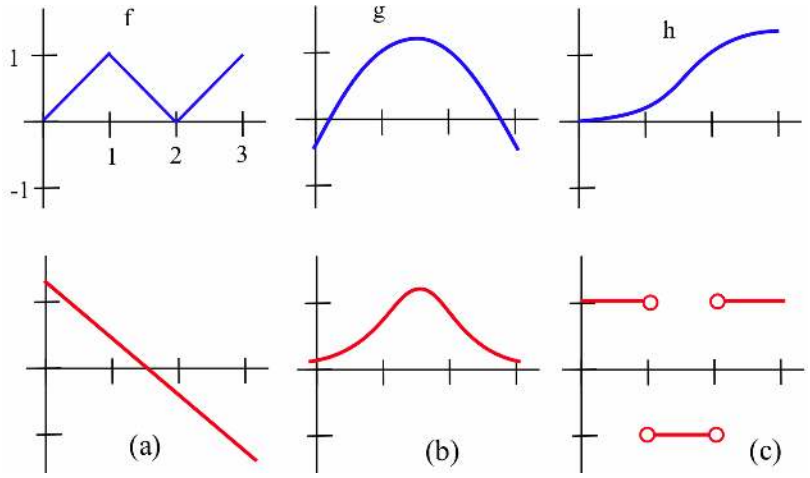
- $\mathrm{D}(\text{constant}) = 0$
- $\mathrm{D}(x^n) = n \cdot x^{n-1}$ (Proved for n = positive integer, but true for all constants n.
- $\mathrm{D}(\sin(x)) = \cos(x)$ and $\mathrm{D}(\cos(x)) = -\sin(x)$
- $\mathrm{D}(|x|) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} = \frac{|x|}{x}$

Interpretations of $f'(x)$:

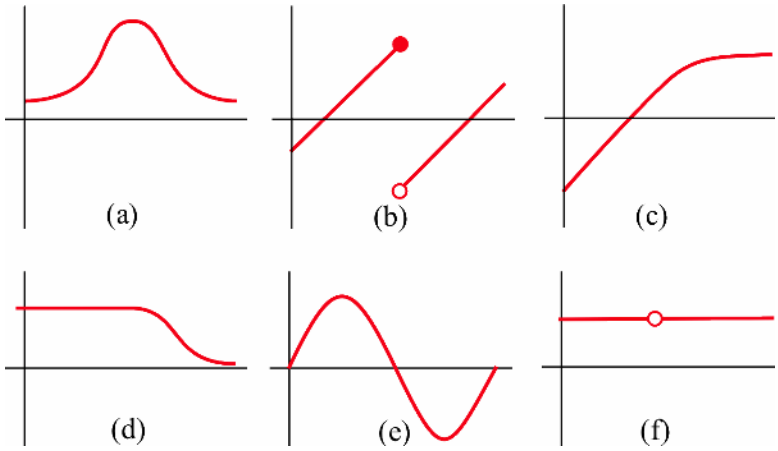
- Slope of a line tangent to a graph
- Instantaneous rate of change of a function at a point
- Velocity or acceleration
- Magnification factor
- Marginal change

Problems

1. Match the functions f, g and h (shown in the top row of the figure below) with the graphs of their derivatives (shown in the bottom row).



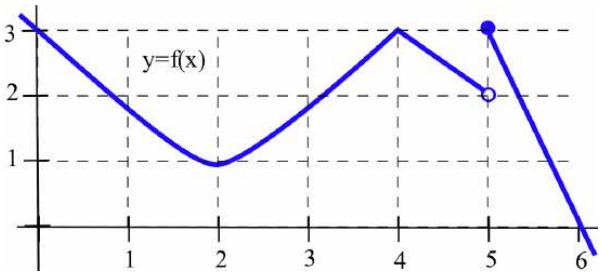
2. The figure below shows six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.



In Problems 3–6, find the slope m_{sec} of the secant line through the two given points and then calculate $m_{\text{tan}} = \lim_{h \to 0} m_{\text{sec}}$.

3. $f(x) = x^2$
   a. $(-2, 4)$, $(-2+h, (-2+h)^2)$
   b. $(0.5, 0.25)$, $(0.5+h, (0.5+h)^2)$
4. $f(x) = 3 + x^2$
   a. $(1, 4)$, $(1+h, 3+(1+h)^2)$
   b. $(x, 3+x^2)$, $(x+h, 3+(x+h)^2)$
5. $f(x) = 7x - x^2$
   a. $(1, 6)$, $(1+h, 7(1+h)-(1+h)^2)$
   b. $(x, 7x-x^2)$, $(x+h, 7(x+h)-(x+h)^2)$
6. $f(x) = x^3 + 4x$
   a. $(1, 5)$, $(1+h, (1+h)^3+4(1+h))$
   b. (x, x^3+4x), $(x+h, (x+h)^3+4(x+h))$

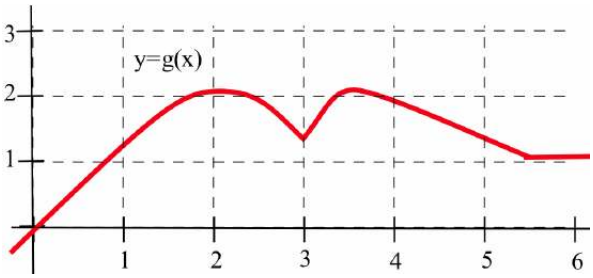
7. Use the graph below to estimate the values of these limits.



(It helps to recognize what the limit represents.)

a. $\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$
b. $\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$
c. $\lim_{w\to 0} \dfrac{f(2+w)-1}{w}$
d. $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$
e. $\lim_{h\to 0} \dfrac{f(4+h)-f(4)}{h}$
f. $\lim_{s\to 0} \dfrac{f(5+s)-f(5)}{s}$

8. Use the graph below to estimate the values of these limits.



a. $\lim_{h\to 0} \dfrac{g(0+h)-g(0)}{h}$
b. $\lim_{h\to 0} \dfrac{g(1+h)-g(1)}{h}$
c. $\lim_{w\to 0} \dfrac{g(2+w)-2}{w}$
d. $\lim_{h\to 0} \dfrac{g(3+h)-g(3)}{h}$
e. $\lim_{h\to 0} \dfrac{g(4+h)-g(4)}{h}$
f. $\lim_{s\to 0} \dfrac{g(5+s)-g(5)}{s}$

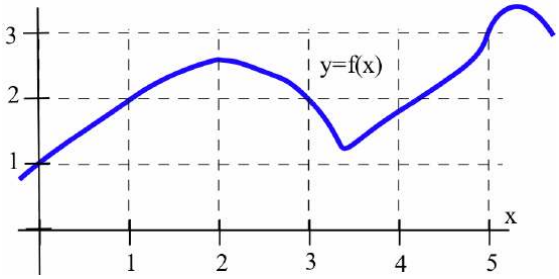
In Problems 9–12, use the definition of the derivative to calculate $f'(x)$ and then evaluate $f'(3)$.

9. $f(x)=x^2+8$
10. $f(x)=5x^2-2x$
11. $f(x)=2x^3-5x$
12. $f(x)=7x^3+x$
13. Graph $f(x)=x^2$, $g(x)=x^2+3$ and $h(x)=x^2-5$. Calculate the derivatives of f, g and h.
14. Graph $f(x)=5x$, $g(x)=5x+2$ and $h(x)=5x-7$. Calculate the derivatives of f, g and h.

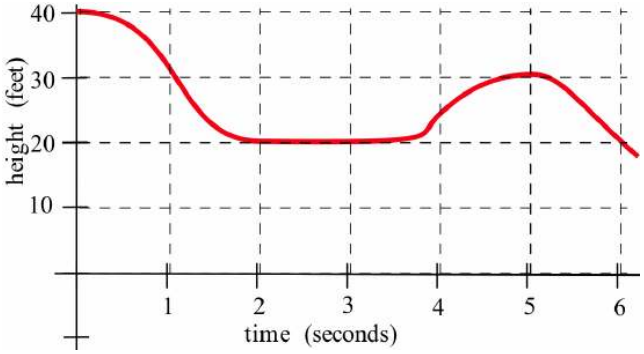
In Problems 15–18, find the slopes and equations of the lines tangent to $y=f(x)$ at the given points.

15. $f(x)=x^2+8$ at $(1,9)$ and $(-2,12)$.
16. $f(x)=5x^2-2x$ at $(2,16)$ and $(0,0)$.
17. $f(x)=\sin(x)$ at $(\pi,0)$ and $(\frac{\pi}{2},1)$.
18. $f(x)=|x+3|$ at $(0,3)$ and $(-3,0)$.

19. a. Find an equation of the line tangent to the graph of $y = x^2 + 1$ at the point $(2, 5)$.
    b. Find an equation of the line perpendicular to the graph of $y = x^2 + 1$ at $(2, 5)$.
    c. Where is the line tangent to the graph of $y = x^2 + 1$ horizontal?
    d. Find an equation of the line tangent to the graph of $y = x^2 + 1$ at the point (p, q).
    e. Find the point(s) (p, q) on the graph of $y = x^2 + 1$ so the tangent line to the curve at (p, q) goes through the point $(1, -7)$.
20. a. Find an equation of the line tangent to the graph of $y = x^3$ at the point $(2, 8)$.
    b. Where, if ever, is the line tangent to the graph of $y = x^3$ horizontal?
    c. Find an equation of the line tangent to the graph of $y = x^3$ at the point (p, q).
    d. Find the point(s) (p, q) on the graph of $y = x^3$ so the tangent line to the curve at (p, q) goes through the point $(16, 0)$.
21. a. Find the angle that the line tangent to $y = x^2$ at $(1, 1)$ makes with the x-axis.
    b. Find the angle that the line tangent to $y = x^3$ at $(1, 1)$ makes with the x-axis.
    c. The curves $y = x^2$ and $y = x^3$ intersect at the point $(1, 1)$. Find the angle of intersection of the two curves (actually the angle between their tangent lines) at the point $(1, 1)$.
22. The figure below shows the graph of $y = f(x)$. Sketch a graph of $y = f'(x)$.



23. The figure below shows the graph of the height of an object at time t.



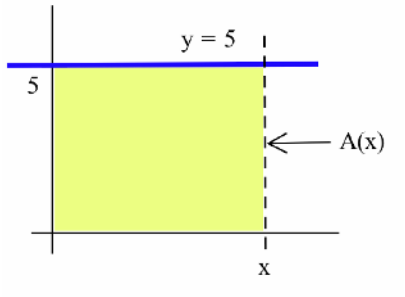
    Sketch a graph of the object's upward velocity. What are the units for each axis on the velocity graph?
24. Fill in the table with units for $f'(x)$.

units for x	units for f(x)	units for f'(x)
hours	miles	
people	automobiles	
dollars	pancakes	
days	trout	
seconds	miles per second	
seconds	gallons	
study hours	test points	

25. A rock dropped into a deep hole will drop $d(x) = 16x^2$ feet in x seconds.

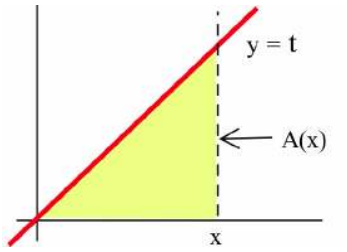
a. How far into the hole will the rock be after 4 seconds? After 5 seconds?
b. How fast will it be falling at exactly 4 seconds? After 5 seconds? After x seconds?

26. It takes $T(x) = x^2$ hours to weave x small rugs. What is the marginal production time to weave a rug? (Be sure to include the units with your answer.)
27. It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls. What is the marginal production cost to make a golf ball? What is the marginal production cost when $x = 25$? When $x = 100$? (Include units.)
28. Define $A(x)$ to be the {\bf area} bounded by the t- and y-axes, the line $y = 5$ and a vertical line at $t = x$:



a. Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.
b. Find a formula for $A(x)$ valid for $x \geq 0$.
c. Determine $A'(x)$.
d. What does $A'(x)$ represent?

29. Define $A(x)$ to be the {\bf area} bounded by the t-axis, the line $y = t$, and a vertical line at $t = x$:



a. Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.
b. Find a formula for $A(x)$ valid for $x \geq 0$.
c. Determine $A'(x)$.
d. What does $A'(x)$ represent?

30. Compute each derivative.
a. $\mathbf{D}(x^{12})$
b. $\frac{d}{dx}(\sqrt[7]{x})$
c. $\mathbf{D}\left(\frac{1}{x^3}\right)$
d. $\frac{d}{dx}(x^c)$
e. $\mathbf{D}(|x-2|)$

31. Compute each derivative.
a. $\mathbf{D}(x^9)$
b. $\frac{d}{dx}(x^{\frac{2}{3}})$
c. $\mathbf{D}\left(\frac{1}{x^4}\right)$
d. $\frac{d}{dx}(x^\pi)$
e. $\mathbf{D}(|x+5|)$

In Problems 32–37, find a function f that has the given derivative. (Each problem has several correct answers, just find one of them.)

32. $f'(x) = 4x + 3$
33. $f'(x) = 3x^2 + 8x$
34. $\mathrm{D}(f(x)) = 12x^2 - 7$
35. $f'(t) = 5\cos(t)$
36. $\frac{d}{dx} f(x) = 2x - \sin(x)$
37. $\mathrm{D}(f(x)) = x + x^2$

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2.2: Derivatives: Properties and Formulas

This section begins with a look at which functions have derivatives. Then we'll examine how to calculate derivatives of elementary combinations of basic functions. By knowing the derivatives of some basic functions and just a few differentiation patterns, you will be able to calculate the derivatives of a tremendous variety of functions. This section contains most—but not quite all—of the general differentiation patterns you will ever need.

Which Functions Have Derivatives?

A function must be continuous in order to be differentiable.

Theorem

If: a function is differentiable at a point

then: it is continuous at that point.

It is vital to understand what this theorem tells us and what it does **not** tell us: If a function is differentiable at a point, then the function is automatically continuous there. If the function is continuous at a point, then the function may or may not be differentiable there.

Proof

Assume that the hypothesis (f is differentiable at the point c) is true. Then $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ must exist and be equal to $f'(c)$. We want to show that f must necessarily be continuous at c, so we need to show that $\lim_{h\to 0} f(c+h)=f(c)$.

It's not yet obvious why we want to do so, but we can write:

$$f(c+h)=f(c)+\frac{f(c+h)-f(c)}{h}\cdot h$$

and then compute the limit of both sides of this expression:

$$\begin{aligned}\lim_{h\to 0} f(c+h) &= \lim_{h\to 0}\left(f(c)+\frac{f(c+h)-f(c)}{h}\cdot h\right)\\ &= \lim_{h\to 0} f(c)+\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\cdot h\right)\\ &= \lim_{h\to 0} f(c)+\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right)\cdot\lim_{h\to 0} h\\ &= f(c)+f'(c)\cdot 0=f(c)\end{aligned}$$

Therefore f is continuous at c.

We often use the contrapositive form of this theorem, which tells us about some functions that do **not** have derivatives.

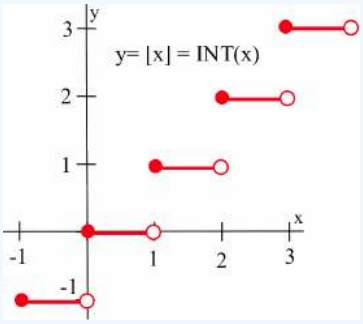
Contrapositive form of the Theorem

If: f is not continuous at a point

then: f is not differentiable at that point.

Example 2.2.1

Show that $f(x)=\lfloor x\rfloor$ is not continuous and not differentiable at $x=2$.



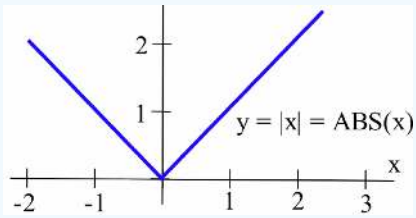
Solution

The one-sided limits $\lim_{x\to 2^+}\lfloor x\rfloor=2$ and $\lim_{x\to 2^-}\lfloor x\rfloor=1$ have different values, so $\lim_{x\to 2}\lfloor x\rfloor$ does not exist. Therefore $f(x)=\lfloor x\rfloor$ is not continuous at 2, and as a result it is not differentiable at 2.

Lack of continuity implies lack of differentiability, but the next examples show that continuity is **not** enough to guarantee differentiability.

Example 2.2.2

Show that $f(x) = |x|$ is continuous but **not** differentiable at $x = 0$.



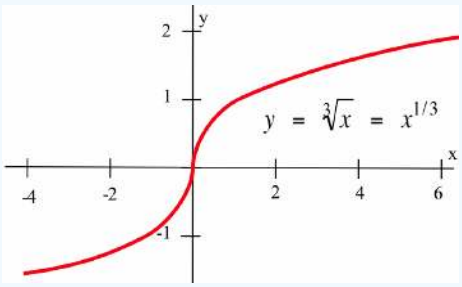
Solution

We know that $\lim_{x \to 0} |x| = 0 = |0|$, so f is continuous at 0, but in Section 2.1 we saw that $|x|$ was not differentiable at $x = 0$.

A function is not differentiable at a cusp or a "corner."

Example 2.2.3

Show that $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ is continuous but **not** differentiable at $x = 0$.



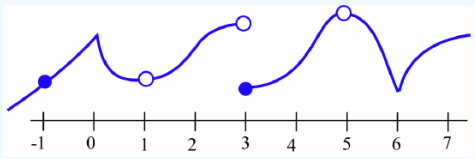
Solution

We can verify that $\lim_{x \to 0^+} \sqrt[3]{x} = \lim_{x \to 0^-} \sqrt[3]{x} = 0$, so $\lim_{x \to 0} \sqrt[3]{x} = 0 = \sqrt[3]{0}$ so f is continuous at 0. But $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$, which is undefined at $x = 0$, so f is not differentiable at 0.

A function is not differentiable where its tangent line is vertical.

Practice 2.2.1

At which integer values of x is the graph of f in the figure below continuous? Differentiable?



Answer

f is continuous at $x = -1$, 0, 2, 4, 6 and 7. f is differentiable at $x = -1$, 2, 4, and 7.

Graphically, a function is **continuous** if and only if its graph is "connected" and does not have any holes or breaks. Graphically, a function is **differentiable** if and only if it is continuous and its graph is "smooth" with no corners or vertical tangent lines.

Derivatives of Elementary Combinations of Functions

We now begin to compute derivatives of more complicated functions built from combinations of simpler functions.

Example 2.2.4

The derivative of $f(x) = x$ is $D(f(x)) = 1$ and the derivative of $g(x) = 5$ is $D(g(x)) = 0$. What are the derivatives of the elementary combinations: $3 \cdot f$, $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$?

Solution

The first three derivatives follow "nice" patterns:

$$\mathrm{D}(3\cdot f(x)) = \mathrm{D}(3x) = 3 = 3\cdot 1 = 3\cdot \mathrm{D}(f(x))$$
$$\mathrm{D}(f(x)+g(x)) = \mathrm{D}(x+5) = 1 = 1+0 = \mathrm{D}(f(x)) + \mathrm{D}(g(x))$$
$$\mathrm{D}(f(x)-g(x)) = \mathrm{D}(x-5) = 1 = 1-0 = \mathrm{D}(f(x)) - \mathrm{D}(g(x))$$

yet the other two derivatives fail to follow the same "nice" patterns: $\mathrm{D}(f(x)\cdot g(x)) = \mathrm{D}(5x) = 5$ but $\mathrm{D}(f(x))\cdot \mathrm{D}(g(x)) = 1\cdot 0 = 0$, and $\mathrm{D}\left(\frac{f(x)}{g(x)}\right) = \mathrm{D}\left(\frac{x}{5}\right) = \frac{1}{5}$ but $\frac{\mathrm{D}(f(x))}{\mathrm{D}(g(x))} = \frac{1}{0}$ is undefined.

The two very simple functions in the previous example show that, in general, $\mathrm{D}(f\cdot g) \neq \mathrm{D}(f)\cdot \mathrm{D}(g)$ and $\mathrm{D}\left(\frac{f}{g}\right) \neq \frac{\mathrm{D}(f)}{\mathrm{D}(g)}$.

? Practice 2.2.2

For $f(x) = 6x + 8$ and $g(x) = 2$, compute the derivatives of $3\cdot f$, $f+g$, $f-g$, $f\cdot g$ and $\frac{f}{g}$.

Answer

$f(x) = 6x+8$ and $g(x) = 2$ so $\mathrm{D}(f(x)) = 6$ and $\mathrm{D}(g(x)) = 0$

$\mathrm{D}(3\cdot f(x)) = 3\cdot \mathrm{D}(f(x)) = 3(6) = 18$

$\mathrm{D}(f(x)+g(x)) = \mathrm{D}(f(x)) + \mathrm{D}(g(x)) = 6+0 = 6$

$\mathrm{D}(f(x)-g(x)) = \mathrm{D}(f(x)) - \mathrm{D}(g(x)) = 6-0 = 6$

$\mathrm{D}(f(x)\cdot g(x)) = f(x)g'(x) + g(x)f'(x) = (6x+8)(0) + (2)(6) = 12$

$\mathrm{D}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{(2)(6)-(6x+8)(0)}{2^2} = \frac{12}{4} = 3$

Main Differentiation Theorem

If: f and g are differentiable at x then:

a. **Constant Multiple Rule**: $\mathrm{D}(k\cdot f(x)) = k\cdot \mathrm{D}(f(x))$
b. **Sum Rule**: $\mathrm{D}(f(x)+g(x)) = \mathrm{D}(f(x)) + \mathrm{D}(g(x))$
c. **Difference Rule**: $\mathrm{D}(f(x)-g(x)) = \mathrm{D}(f(x)) - \mathrm{D}(g(x))$
d. **Product Rule**: $\mathrm{D}(f(x)\cdot g(x)) = f(x)\cdot \mathrm{D}(g(x)) + g(x)\cdot \mathrm{D}(f(x))$
e. **Quotient Rule**: $\mathrm{D}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot \mathrm{D}(f(x)) - f(x)\cdot \mathrm{D}(g(x))}{[g(x)]^2}$

Part (e) requires that \(g(x) \neq 0\.

This theorem says that the simple patterns in the previous example for constant multiples of functions and sums and differences of functions are true for all differentiable functions. It also includes the correct patterns for derivatives of products and quotients of differentiable functions.

The proofs of parts (a), (b) and (c) of this theorem are straightforward, but parts (d) and (e) require some clever algebraic manipulations. Let’s look at some examples before tackling the proof.

✓ Example 2.2.5

Recall that $\mathrm{D}(x^2) = 2x$ and $\mathrm{D}(\sin(x)) = \cos(x)$. Find $\mathrm{D}(3\sin(x))$ and $\frac{d}{dx}(5x^2 - 7\sin(x))$.

Solution

Computing $\mathrm{D}(3\sin(x))$ requires part (a) of the theorem with $k = 3$ and $f(x) = \sin(x)$ so $\mathrm{D}(3\cdot \sin(x)) = 3\cdot \mathrm{D}(\sin(x)) = 3\cos(x)$, while $\frac{d}{dx}(5x^2 - 7\sin(x))$ uses part (c) of the theorem with $f(x) = 5x^2$ and $g(x) = 7\sin(x)$ so:

$$\begin{aligned}\frac{d}{dx}(5x^2 - 7\sin(x)) &= \frac{d}{dx}(5x^2) - \frac{d}{dx}(7\sin(x)) \\ &= 5\cdot \frac{d}{dx}(x^2) - 7\cdot \frac{d}{dx}(\sin(x)) \\ &= 5(2x) - 7(\cos(x))\end{aligned}$$

which simplifies to $10x - 7\cos(x)$.

? Practice 2.2.3

Find $\mathrm{D}(x^3 - 5\sin(x))$ and $\frac{d}{dx}(\sin(x) - 4x^3)$.

Answer

$\mathrm{D}(x^3 - 5\sin(x)) = \mathrm{D}(x^3) - 5\cdot \mathrm{D}(\sin(x)) = 3x^2 - 5\cos(x)$

$\frac{d}{dx}(\sin(x) - 4x^3) = \frac{d}{dx}\sin(x) - 4\cdot \frac{d}{dx}x^3 = \cos(x) - 12x^2$

? Practice 2.2.4

The table below gives the values of functions f and g, as well as their derivatives, at various points. Fill in the missing values for $\mathrm{D}(3\cdot f(x))$, $\mathrm{D}(2\cdot f(x) + g(x))$ and $\mathrm{D}(3\cdot g(x) - f(x))$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\mathrm{D}(3f(x))$	$\mathrm{D}(2f(x)+g(x))$	$\mathrm{D}(3g(x)-f(x))$
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Answer

$\mathrm{D}(3f(x))$	$\mathrm{D}(2f(x)+g(x))$	$\mathrm{D}(3g(x)-f(x))$
−6	−1	11
−3	−2	1
6	5	1

? Practice 2.2.5

Use the Main Differentiation Theorem to complete the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\mathrm{D}(f(x)\cdot g(x))$	$\mathrm{D}\left(\frac{f(x)}{g(x)}\right)$	$\mathrm{D}\left(\frac{g(x)}{f(x)}\right)$
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Answer

$\mathrm{D}(f(x)\cdot g(x))$	$\mathrm{D}\left(\frac{f(x)}{g(x)}\right)$	$\mathrm{D}\left(\frac{g(x)}{f(x)}\right)$
$3\cdot 3+(-4)(-2)=17$	$\frac{-4(-2)-(3)(3)}{(-4)^2}=-\frac{1}{16}$	$\frac{(3)(3)-(-4)(-2)}{3^2}=\frac{1}{9}$
$2\cdot 0+1(-1)=-1$	$\frac{1(-1)-(2)(0)}{1^2}=-1$	$\frac{2(0)-1(-1)}{2^2}=\frac{1}{4}$
$4\cdot 1+3\cdot 2=10$	$\frac{3(2)-(4)(1)}{3^2}=\frac{2}{9}$	$\frac{4(1)-3(2)}{4^2}=-\frac{1}{8}$

Note

Many calculus students find it easier to remember the Product Rule in words: “the first function times the derivative of the second plus the second function times the derivative of the first.”

✓ Example 2.2.6

Determine $\mathrm{D}(x^2\cdot\sin(x))$ and $\frac{d}{dx}\left(\frac{x^3}{\sin(x)}\right)$.

Solution

Use the Product Rule with $f(x)=x^2$ and $g(x)=\sin(x)$:

$$\begin{aligned}\mathrm{D}(x^2\cdot\sin(x)) &= \mathrm{D}(f(x)\cdot g(x)) = f(x)\cdot\mathrm{D}(g(x))+g(x)\cdot\mathrm{D}(f(x))\\ &= x^2\cdot\mathrm{D}(\sin(x))+\sin(x)\cdot\mathrm{D}(x^2)\\ &= x^2\cdot\cos(x)+\sin(x)\cdot 2x = x^2\cos(x)+2x\sin(x)\end{aligned}$$

Use the Quotient Rule with $f(x)=x^3$ and $g(x)=\sin(x)$:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^3}{\sin(x)}\right) &= \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\\ &= \frac{g(x)\cdot\mathrm{D}(f(x))-f(x)\cdot\mathrm{D}(g(x))}{[g(x)]^2}\\ &= \frac{\sin(x)\cdot\mathrm{D}(x^3)-x^3\cdot\mathrm{D}(\sin(x))}{[\sin(x)]^2}\\ &= \frac{\sin(x)\cdot 3x^2-x^3\cdot\cos(x)}{\sin^2(x)}\\ &= \frac{3x^2\sin(x)-x^3\cdot\cos(x)}{\sin^2(x)}\end{aligned}$$

which could also be rewritten in terms of $\csc(x)$ and $\cot(x)$.

Note

The Quotient Rule in words: "the bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared."

Practice 2.2.6

Find $\mathrm{D}((x^2+1)(7x-3))$, $\frac{d}{dt}\left(\frac{3t-2}{5t+1}\right)$ and $\mathrm{D}\left(\frac{\cos(x)}{x}\right)$.

Answer

$\mathrm{D}((x^2+1)(7x-3)) = (x^2+1)\mathrm{D}(7x-3) + (7x-3)\mathrm{D}(x^2+1) = (x^2+1)(7) + (7x-3)(2x) = 21x^2 - 6x + 7$

or: $\mathrm{D}((x^2+1)(7x-3)) = \mathrm{D}(7x^3 - 3x^2 + 7x) = 21x^2 - 6x + 7$

$$\frac{d}{dt}\left(\frac{3t-2}{5t+1}\right) = \frac{(5t+1)\mathrm{D}(3t-2) - (3t-2)\mathrm{D}(5t+1)}{(5t+1)^2} = \frac{(5t+1)(3) - (3t-2)(5)}{(5t+1)^2} = \frac{13}{(5t+1)^2}$$

$$\mathrm{D}\left(\frac{\cos(x)}{x}\right) = \frac{x\mathrm{D}(\cos(x)) - \cos(x)\mathrm{D}(x)}{x^2} = \frac{x(-\sin(x)) - \cos(x)(1)}{x^2} = \frac{-x\cdot\sin(x) - \cos(x)}{x^2}$$

Now that we've seen how to use the theorem, let's prove it.

The only general fact we have about derivatives is the definition as a limit, so our proofs here will need to recast derivatives as limits and then use some results about limits. The proofs involve applications of the definition of the derivative and results about limits.

Main Differentiation Theorem

Proof

a. Using the derivative definition and the limit laws:

$$\begin{aligned}\mathrm{D}(k\cdot f(x)) &= \lim_{h\to 0}\frac{k\cdot f(x+h) - k\cdot f(x)}{h}\\ &= \lim_{h\to 0} k\cdot\frac{f(x+h)-f(x)}{h}\\ &= k\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = k\cdot\mathrm{D}(f(x))\end{aligned}$$

b. You try it (see Practice problem that follows).

c. Once again using the derivative definition and the limit laws:

$$\begin{aligned}\mathrm{D}(f(x)-g(x)) &= \lim_{h\to 0}\frac{[f(x+h)-g(x+h)]-[f(x)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]-[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \mathrm{D}(f(x)) - \mathrm{D}(g(x))\end{aligned}$$

The proofs of parts (d) and (e) of the theorem are more complicated but only involve elementary techniques, used in just the right way. Sometimes we will omit such computational proofs, but the Product and Quotient Rules are fundamental techniques you will need hundreds of times.

e. By the hypothesis, f and g are differentiable, so:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

and:

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x)$$

Also, both f and g are continuous (why?) so $\lim_{h\to 0} f(x+h) = f(x)$ and $\lim_{h\to 0} g(x+h) = g(x)$. Let $P(x) = f(x)\cdot g(x)$. Then $P(x+h) = f(x+h)\cdot g(x+h)$ and:

$$\begin{aligned}\mathrm{D}(f(x)\cdot g(x)) = \mathrm{D}(P(x)) &= \lim_{h\to 0}\frac{P(x+h)-P(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}\end{aligned}$$

At this stage we need to use some cleverness to add and subtract $f(x)\cdot g(x+h)$ from the numerator (you'll see why shortly):

$$\lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) + [-f(x)\cdot g(x+h) + f(x)\cdot g(x+h)] - f(x)g(x)}{h}$$

We can then split this giant fraction into two more manageable limits:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h) - f(x)g(x)}{h}$$

and then factor out a common factor from each numerator:

$$\lim_{h\to 0} g(x+h)\cdot\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot\frac{g(x+h)-g(x)}{h}$$

Taking limits of each piece (and using the continuity of $g(x)$) we get:

$$\mathrm{D}(f(x)\cdot g(x)) = g(x)\cdot f'(x) + f(x)\cdot g'(x) = g\cdot\mathrm{D}(f) + f\cdot\mathrm{D}(g)$$

The steps for a proof of the Quotient Rule appear in Problem 69.

Practice 2.2.7

Prove the Sum Rule: $\mathbf{D}(f(x)+g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x))$. (Refer to the proof of part (c) for guidance.)

Answer

Mimicking the proof of the Difference Rule:

$$\begin{aligned}\mathbf{D}(f(x)+g(x)) &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \mathbf{D}(f(x)) + \mathbf{D}(g(x))\end{aligned}$$

Using the Differentiation Rules

You definitely need to memorize the differentiation rules, but it is vitally important that you also know **how** to use them. Sometimes it is clear that the function we want to differentiate is a sum or product of two obvious functions, but we commonly need to differentiate functions that involve several operations and functions. Memorizing the differentiation rules is only the first step in learning to use them.

Example 2.2.7

Calculate $\mathbf{D}(x^5 + x\cdot\sin(x))$.

Solution

This function is more difficult because it involves both an addition and a multiplication. Which rule(s) should we use — or, more importantly, which rule should we use **first**?

First apply the Sum Rule to trade one derivative for two easier ones:

$$\begin{aligned}\mathbf{D}(x^5 + x\cdot\sin(x)) &= \mathbf{D}(x^5) + \mathbf{D}(x\cdot\sin(x))\\ &= 5x^4 + [x\cdot\mathbf{D}(\sin(x)) + \sin(x)\cdot\mathbf{D}(x)]\\ &= 5x^4 + x\cdot\cos(x) + \sin(x)\end{aligned}$$

This last expression involves no more derivatives, so we are done.

If instead of computing the derivative you were evaluating the function $x^5 + x\sin(x)$ for some particular value of x, you would:

- raise x to the 5th power
- calculate $\sin(x)$
- multiply $\sin(x)$ by x and, finally,
- **add** (sum) the values of x^5 and $x\sin(x)$

Notice that the **final** step of your **evaluation** of f indicates the **first** rule to use to calculate the **derivative** of f.

Practice 2.2.8

Which differentiation rule should you apply **first** for each of the following?

a. $x\cdot\cos(x) - x^3\cdot\sin(x)$
b. $(2x-3)\cos(x)$
c. $2\cos(x) - 7x^2$
d. $\dfrac{\cos(x)+3x}{\sqrt{x}}$

Answer

a. difference rule
b. product rule
c. difference rule
d. quotient rule

Practice 2.2.9

Calculate $\mathbf{D}\left(\dfrac{x^2-5}{\sin(x)}\right)$ and $\dfrac{d}{dt}\left(\dfrac{t^2-5}{t\cdot\sin(t)}\right)$.

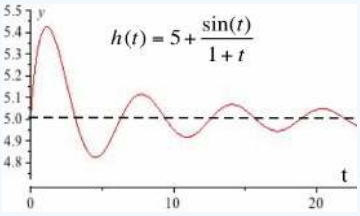
Answer

$$\mathbf{D}\left(x^2 - 5\sin(x)\right) = \frac{\sin(x)\mathbf{D}(x^2-5) - (x^2-5)\mathbf{D}(\sin(x))}{(\sin(x))^2} = \frac{\sin(x)(2x) - (x^2-5)\cos(x)}{\sin^2(x)}$$

$$\frac{d}{dt}\left(t^2 - 5t\cdot\sin(t)\right) = \frac{t\cdot\sin(t)\mathbf{D}(t^2-5) - (t^2-5)\mathbf{D}(t\cdot\sin(t))}{(t\cdot\sin(t))^2} = \frac{t\cdot\sin(t)(2t) - (t^2-5)\left[t\cos(t)+\sin(t)\right]}{t^2\cdot\sin^2(t)}$$

Example 2.2.8

A mass attached to a spring oscillates up and down but the motion becomes “damped” due to friction and air resistance. The height of the mass after t seconds is given by $h(t) = 5 + \frac{\sin(t)}{1+t}$ (in feet).



Find the height and velocity of the mass after 2 seconds.

Solution

The height is $h(2) = 5 + \frac{\sin(2)}{1+2} \approx 5 + \frac{0.909}{3} = 5.303$ feet above the ground. The velocity is $h'(2)$, so we must first compute $h'(t)$ and then evaluate the derivative at time $t = 2$:

$$h'(t) = \frac{(1+t)\cdot\cos(t) - \sin(t)\cdot 1}{(1+t)^2}$$

so $h'(2) = \frac{3\cos(2) - \sin(2)}{9} \approx \frac{-2.158}{9} \approx -0.24$ feet per second.

Practice 2.2.10

What are the height and velocity of the weight in Example $2.2.8$ after 5 seconds? What are the height and velocity of the weight be after a “long time” has passed?

Answer

$h(5) = 5 + \frac{\sin(5)}{1+5} \approx 4.84$ ft.; $v(5) = h'(5) = \frac{(1+5)\cos(5) - \sin(5)}{(1+5)^2} \approx 0.074$ ft/sec

“long time”: $h(t) = 5 + \frac{\sin(t)}{1+t} \approx 5$ feet when t is very large

$h'(t) = \frac{(1+t)\cos(t) - \sin(t)}{(1+t)^2} = \frac{\cos(t)}{1+t} - \frac{\sin(t)}{(1+t)^2} \approx 0$ ft/sec when t is very large

Example 2.2.9

Calculate $\mathrm{D}(x\cdot\sin(x)\cdot\cos(x))$.

Solution

Clearly we need to use the Product Rule, because the only operation in this function is multiplication. But the Product Rule deals with a product of **two** functions and here we have the product of three: x and $\sin(x)$ and $\cos(x)$. If, however, we think of our two functions as $f(x) = x\cdot\sin(x)$ and $g(x) = \cos(x)$, then we do have the product of two functions and:

$$\begin{aligned}\mathrm{D}(x\cdot\sin(x)\cdot\cos(x)) &= \mathrm{D}(f(x)\cdot g(x))\\ &= f(x)\cdot\mathrm{D}(g(x)) + g(x)\cdot\mathrm{D}(f(x))\\ &= x\sin(x)\cdot\mathrm{D}(\cos(x)) + \cos(x)\cdot\mathrm{D}(x\sin(x))\end{aligned}$$

We are not done, but we have traded one hard derivative for two easier ones. We know that $\mathrm{D}(\cos(x)) = -\sin(x)$ and we can use the Product Rule (again) to calculate $\mathrm{D}(x\sin(x))$. Then the
last line of our calculation above becomes:

$$x\sin(x)\cdot[-\sin(x)] + \cos(x)\cdot[x\mathrm{D}(\sin(x)) + \sin(x)\mathrm{D}(x)]$$

and then:

$$-x\sin^2(x) + \cos(x)\,[x\cos(x) + \sin(x)(1)]$$

which simplifies to $-x\sin^2(x) + x\cos^2(x) + \cos(x)\sin(x)$.

Evaluating a Derivative at a Point

The derivative of a function $f(x)$ is a new **function** $f'(x)$ that tells us the slope of the line tangent to the graph of f at each point x. To find the slope of the tangent line at a particular point $(c, f(c))$ on the graph of f, we should *first* calculate the derivative $f'(x)$ and *then* evaluate the function $f'(x)$ at the point $x = c$ to get the **number** $f'(c)$. If you mistakenly evaluate f first, you get a number $f(c)$, and the derivative of a constant is always equal to 0.

Example 2.2.10

Determine the slope of the line tangent to the graph of $f(x) = 3x + \sin(x)$ at $(0, f(0))$ and $(1, f(1))$.

Solution

$f'(x) = \mathrm{D}(3x + \sin(x)) = \mathrm{D}(3x) + \mathrm{D}(\sin(x)) = 3 + \cos(x)$. When $x = 0$, the graph of $y = 3x + \sin(x)$ goes through the point $(0, 3(0) + \sin(0)) = (0, 0)$ with slope $f'(0) = 3 + \cos(0) = 4$. When $x = 1$, the graph goes through the point $(1, 3(1) + \sin(1)) \approx (1, 3.84)$ with slope $f'(1) = 3 + \cos(1) \approx 3.54$.

? Practice 2.2.11

Where do $f(x) = x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have horizontal tangent lines?

Answer

$f'(x) = 2x - 10$ so $f'(x) = 0 \Rightarrow 2x - 10 = 0 \Rightarrow x = 5$

$g'(x) = 3x^2 - 12$ so $g'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Important Information and Results

This section, like the last one, contains a great deal of important information that we will continue to use throughout the rest of the course, so we collect here some of those important results.

Differentiability and Continuity: If a function is differentiable then it must be continuous. If a function is not continuous then it cannot be differentiable. A function may be continuous at a point and not differentiable there.

Graphically: *Continuous* means "connected"; *differentiable* means "continuous, smooth and not vertical."

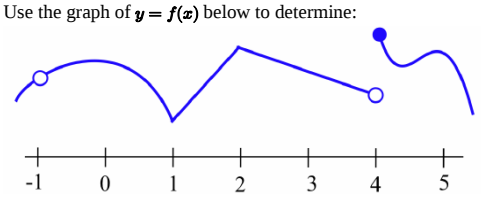
Differentiation Patterns:

- $[k \cdot f(x)]' = k \cdot f'(x)$
- $[f(x) + g(x)]' = f'(x) + g'(x)$
- $[f(x) - g(x)]' = f'(x) - g'(x)$
- $[f(x) \cdot g(x)]' = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
- $\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- The *final step* used to evaluate a function f indicates the *first rule* used to differentiate f.

Evaluating a derivative at a point: First differentiate and *then* evaluate.

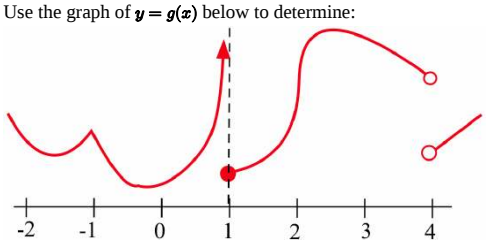
Problems

1. Use the graph of $y = f(x)$ below to determine:



   a. at which integers f is continuous.
   b. at which integers f is differentiable.

2. Use the graph of $y = g(x)$ below to determine:



   a. at which integers g is continuous.
   b. at which integers g is differentiable.

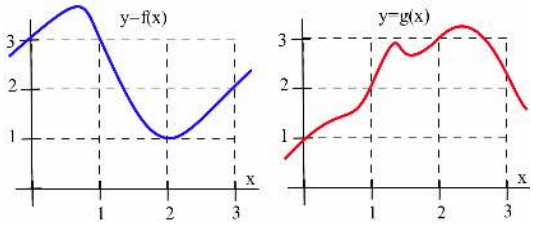
3. Use the values given in the table to determine the values of $f \cdot g$, $\mathrm{D}(f \cdot g)$, $\frac{f}{g}$ and $\mathrm{D}\left(\frac{f}{g}\right)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$f(x) \cdot g(x)$	$\mathrm{D}(f(x) \cdot g(x))$	$\frac{f(x)}{g(x)}$	$\mathrm{D}\left(\frac{f(x)}{g(x)}\right)$
0	2	3	1	5				
1	−3	2	5	−2				
2	0	−3	2	4				
3	1	−1	0	3				

4. Use the values given in the table to determine the values of $f \cdot g$, $\mathrm{D}(f \cdot g)$, $\frac{f}{g}$ and $\mathrm{D}\left(\frac{f}{g}\right)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$f(x)\cdot g(x)$	$\mathrm{D}(f(x)\cdot g(x))$	$\frac{f(x)}{g(x)}$	$\mathrm{D}\left(\frac{f(x)}{g(x)}\right)$
0	4	2	3	−3				
1	0	3	2	1				
2	−2	5	0	−1				
3	−1	−2	−3	4				

5. Use the information in the figure below left to plot the values of the functions $f+g$, $f\cdot g$ and $\frac{f}{g}$ and their derivatives at $x=1$, 2 and 3.



6. Use the information in the figure above right to plot the values of the functions $2f$, $f-g$ and $\frac{g}{f}$ and their derivatives at $x=1$, 2 and 3.
7. Calculate $\mathrm{D}((x-5)(3x+7))$ by:
   a. using the Product Rule.
   b. expanding and then differentiating.

   Verify that both methods give the same result.
8. Calculate $\mathrm{D}\left(\frac{x^3-3x+2}{\sqrt{x}}\right)$ by:
   a. using the Product Rule.
   b. expanding and then differentiating.

   Verify that both methods give the same result.

In Problems 9–26, compute each derivative.

9. $\frac{d}{dx}\left(19x^3-7\right)$
10. $\frac{d}{dt}\left(5\cos(t)+\frac{\pi}{2}\right)$
11. $\mathrm{D}(\sin(x)+\cos(x))$
12. $\mathrm{D}(7\sin(x)-3\cos(x))$
13. $\mathrm{D}(x^2\cdot\cos(x))$
14. $\mathrm{D}(\sqrt{x}\cdot\sin(x))$
15. $\mathrm{D}(\sin^2(x))$
16. $\mathrm{D}(\cos^2(x))$
17. $\frac{d}{dx}\left(\frac{\cos(x)}{x^2}\right)$
18. $\frac{d}{dt}\left(\frac{\sin(t)}{t^3}\right)$
19. $\frac{d}{dx}\left(\frac{1}{1+x^2}\right)$
20. $\frac{d}{dt}\left(\frac{t}{1+t^3}\right)$
21. $\frac{d}{d\theta}\left(\frac{1}{\cos(\theta)}\right)$
22. $\frac{d}{d\theta}\left(\frac{1}{\sin(\theta)}\right)$
23. $\frac{d}{d\theta}\left(\frac{\sin(\theta)}{\cos(\theta)}\right)$
24. $\frac{d}{d\theta}\left(\frac{\cos(\theta)}{\sin(\theta)}\right)$
25. $\mathrm{D}\left(8x^5-3x^4+2x^3+7x^2-12x+147\right)$
26. a. $\mathrm{D}(\sin(x))$
    b. $\mathrm{D}\left(\sin(x)+7\right)$
    c. $\mathrm{D}(\sin(x)-8000)$
    d. $\mathrm{D}(\sin(x)+k)$
27. Find values for the constants a, b and c so that the parabola $f(x)=ax^2+bx+c$ has $f(0)=0$, $f'(0)=0$ and $f'(10)=30$.
28. If f is a differentiable function, how are the:
    a. graphs of $y=f(x)$ and $y=f(x)+k$ related?
    b. derivatives of $f(x)$ and $f(x)+k$ related?
29. If f and g are differentiable functions that always differ by a constant (\)f(x) - g(x) = k\) for all x) then what can you conclude about their graphs? Their derivatives?
30. If f and g are differentiable functions whose sum is a constant (\)f(x) + g(x) = k\) for all x) then what can you conclude about their graphs? Their derivatives?
31. If the product of f and g is a constant (that is, $f(x)\cdot g(x)=k$ for all x) then how are $\frac{\mathrm{D}(f(x))}{f(x)}$ and $\frac{\mathrm{D}(g(x))}{g(x)}$ related?
32. If the quotient of f and g is a constant ($\frac{f(x)}{g(x)}=k$ for all x) then how are $g\cdot f'$ and $f\cdot g'$ related?

In Problems 33–40:

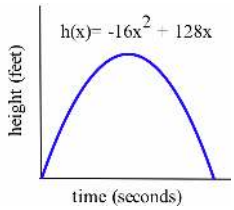
a. calculate $f'(1)$

b. determine where $f'(x) = 0$.

33. $f(x) = x^2 - 5x + 13$
34. $f(x) = 5x^2 - 40x + 73$
35. $f(x) = 3x - 2\cos(x)$
36. $f(x) = |x + 2|$
37. $f(x) = x^3 + 9x^2 + 6$
38. $f(x) = x^3 + 3x^2 + 3x - 1$
39. $f(x) = x^3 + 2x^2 + 2x - 1$
40. $f(x) = \dfrac{7x}{x^2 + 4}$
41. $f(x) = x \cdot \sin(x)$ and $0 \le x \le 5$. (You may need to use the Bisection Algorithm or the "trace" option on a calculator to approximate where $f'(x) = 0$.)
42. $f(x) = Ax^2 + Bx + C$, where B and C are constants and $A \neq 0$ is constant.
43. $f(x) = x^3 + Ax^2 + Bx + C$ with constants A, B and C. Can you find conditions on the constants A, B and C that will guarantee that the graph of $y = f(x)$ has two distinct "turning points"? (Here a "turning point" means a place where the curve changes from increasing to decreasing or from decreasing to increasing, like the vertex of a parabola.)

In Problems 44–51, where are the functions differentiable?

44. $f(x) = |x|\cos(x)$
45. $f(x) = \tan(x)$
46. $f(x) = \dfrac{x - 5}{x + 3}$
47. $f(x) = \dfrac{x^2 + x}{x^2 - 3x}$
48. $f(x) = |x^2 - 4|$
49. $f(x) = |x^3 - 1|$
50. $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$
51. $f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$
52. For what value(s) of A is $f(x) = \begin{cases} Ax - 4 & \text{if } x < 2 \\ x^2 + x & \text{if } x \ge 2 \end{cases}$ differentiable at $x = 2$?
53. For what values of A and B is $f(x) = \begin{cases} Ax + B & \text{if } x < 1 \\ x^2 + x & \text{if } x \ge 1 \end{cases}$ differentiable at $x = 1$?
54. An arrow shot straight up from ground level (get out of the way!) with an initial velocity of 128 feet per second will be at height $h(x) = -16x^2 + 128x$ feet after x seconds (see figure below).



a. Determine the velocity of the arrow when $x = 0$, 1 and 2 seconds.
b. What is the velocity of the arrow, $v(x)$, at any time x?
c. At what time x will the velocity of the arrow be 0?
d. What is the greatest height the arrow reaches?
e. How long will the arrow be aloft?
f. Use the answer for the velocity in part (b) to determine the acceleration, $a(x) = v'(x)$, at any time x.

55. If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet per second, its height will be $h(x) = -2.65x^2 + 128x$ feet after x seconds. Redo parts (a)–(e) of problem 40 using this new formula for $h(x)$.
56. In general, if an arrow is shot straight upward with an initial velocity of 128 feet per second from ground level on a planet with a constant gravitational acceleration of g feet per second\)^2\) then its height will be $h(x) = -\frac{g}{2}x^2 + 128x$ feet after x seconds. Answer the questions in problem 40 for arrows shot on Mars and Jupiter.

object	g (ft/sec^2)	g (cm/sec^2)
Mercury	11.8	358
Venus	20.1	887
Earth	32.2	981
moon	5.3	162
Mars	12.3	374
Jupiter	85.3	2601
Saturn	36.6	1117
Uranus	34.4	1049
Neptune	43.5	1325

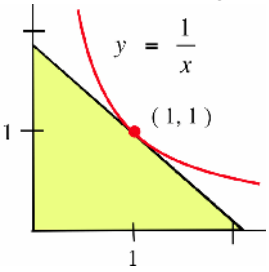
Source: CRC Handbook of Chemistry and Physics

57. If an object on Earth is propelled upward from ground level with an initial velocity of v_0 feet per second, then its height after x seconds will be $h(x) = -16x^2 + v_0x$.
    a. Find the object's velocity after x seconds.
    b. Find the greatest height the object will reach.
    c. How long will the object remain aloft?

58. In order for a 6-foot-tall basketball player to dunk the ball, the player must achieve a vertical jump of about 3 feet. Use the information in the previous problems to answer the following questions.
    a. What is the smallest initial vertical velocity the player can have and still dunk the ball?
    b. With the initial velocity achieved in part (a), how high would the player jump on the moon?

59. The best high jumpers in the world manage to lift their centers of mass approximately 3.75 feet.
    a. What is the initial vertical velocity these high jumpers attain?
    b. How long are these high jumpers in the air?
    c. How high would they lift their centers of mass on the moon?

60. a. Find an equation for the line L that is tangent to the curve $y = \frac{1}{x}$ at the point $(1, 1)$.
    b. Determine where L intersects the x-axis and the y-axis.
    c. Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis (see figure below).



61. a. Find an equation for the line L that is tangent to the curve $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.
    b. Graph $y = \frac{1}{x}$ and L and determine where L intersects the x-axis and the y-axis.
    c. Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis.

62. a. Find an equation for the line L that is tangent to the curve $y = \frac{1}{x}$ at the point $(p, \frac{1}{p})$ (assuming $p \neq 0$).
    b. Determine where L intersects the x-axis and the y-axis.
    c. Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis.
    d. How does the area of the triangle in part (c) depend on the initial point $(p, \frac{1}{p})$?

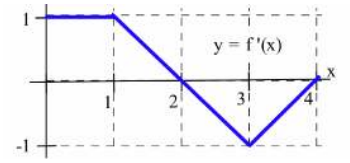
63. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point $(1, 4)$ and is tangent to the line $y = 9x - 13$ at the point $(3, 14)$.

64. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point $(0, 1)$ and is also tangent to the line $y = 3x - 2$ at the point $(2, 4)$.

65. a. Find a function f so that $\mathrm{D}(f(x)) = 3x^2$.
    b. Find another function g with $\mathrm{D}(g(x)) = 3x^2$.
    c. Can you find more functions whose derivatives are $3x^2$?

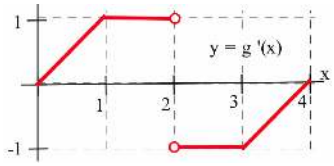
66. a. Find a function f so that $f'(x) = 6x + \cos(x)$.
    b. Find another function g with $g'(x) = f'(x)$.

67. The graph of $y = f'(x)$ appears below.



    a. Assume $f(0) = 0$ and sketch a graph of $y = f(x)$.
    b. Assume $f(0) = 1$ and graph $y = f(x)$.

68. The graph of $y = g'(x)$ appears below. Assume that g is continuous.



    a. Assume $g(0) = 0$ and sketch a graph of $y = g(x)$.
    b. Assume $g(0) = 1$ and graph $y = g(x)$.

69. Assume that f and g are differentiable functions and that $g(x) \neq 0$. State why each step in the following proof of the Quotient Rule is valid.

Proof of the Quotient Rule

$$
\begin{aligned}
\mathrm{D}\left(\frac{f(x)}{g(x)}\right) &= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)g(x)-g(x+h)f(x)}{g(x+h)g(x)}\right] \\
&= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left[\frac{f(x+h)g(x)+(-f(x)g(x)+f(x)g(x))-g(x+h)f(x)}{h}\right] \\
&= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left[g(x)\frac{f(x+h)-f(x)}{h}+f(x)\frac{g(x)-g(x+h)}{h}\right] \\
&= \frac{1}{[g(x)]^2}\left[g(x)\cdot f'(x)-f(x)\cdot g'(x)\right] \\
&= \frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{[g(x)]^2}
\end{aligned}
$$

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2.3: More Differentiation Patterns

Polynomials are very useful, but they are not the only functions we need. This section uses the ideas of the two previous sections to develop techniques for differentiating **powers of functions**, and to determine the derivatives of some particular functions that occur often in applications: the **trigonometric** and **exponential** functions.

As you focus on learning how to differentiate different types and combinations of functions, it is important to remember what derivatives are and what they measure. Calculators and computers are available to calculate derivatives. Part of your job as a professional will be to decide which functions need to be differentiated and how to use the resulting derivatives. You can succeed at that only if you understand what a derivative is and what it measures.

A Power Rule for Functions: $\mathbf{D}(f^n(x))$

If we apply the Product Rule to the product of a function with itself, a pattern emerges.

$$\begin{aligned}
\mathbf{D}_{(f^2)} &= \mathbf{D}_{(f\cdot f)} = \mathbf{D}_{(f)}\cdot f + f\cdot \mathbf{D}_{(f)} = 2f\cdot \mathbf{D}(f)\\
\mathbf{D}_{(f^3)} &= \mathbf{D}_{(f^2\cdot f)} = \mathbf{D}_{(f)}\cdot f^2 + f\cdot \mathbf{D}_{(f^2)} = f^2\cdot \mathbf{D}(f) + f\cdot 2f\cdot \mathbf{D}(f) = 3f^2\cdot \mathbf{D}(f)\\
\mathbf{D}_{(f^4)} &= \mathbf{D}_{(f^3\cdot f)} = \mathbf{D}_{(f)}\cdot f^3 + f\cdot \mathbf{D}_{(f^3)} = f^3\cdot \mathbf{D}(f) + f\cdot 3f^2\cdot \mathbf{D}(f) = 4f^3\cdot \mathbf{D}(f)
\end{aligned}$$

? Practice 2.3.1

What is the pattern here? What do you think the results will be for $\mathbf{D}(f^5)$ and $\mathbf{D}(f^{13})$?

Answer

The pattern is $\mathbf{D}(f^n(x)) = n\cdot f^{n-1}(x)\cdot \mathbf{D}(f(x))$: $\mathbf{D}(f^5(x)) = 5f^4(x)\cdot \mathbf{D}(f(x))$ and $\mathbf{D}(f^{13}(x)) = 13f^{12}(x)\cdot \mathbf{D}(f(x))$

We could keep differentiating higher and higher powers of $f(x)$ by writing them as products of lower powers of $f(x)$ and using the Product Rule, but the Power Rule for Functions guarantees that the pattern we just saw for the small integer powers also works for all constant powers of functions.

Power Rule for Functions

If: p is any constant

then: $\mathbf{D}(f^p(x)) = p\cdot f^{p-1}(x)\cdot \mathbf{D}(f(x))$

The Power Rule for Functions is a special case of a more general theorem, the Chain Rule, which we will examine in Section 2.4, so we will wait until then to prove the Power Rule for Functions.

✓ Example 2.3.1

Use the Power Rule for Functions to find:

a. $\mathbf{D}((x^3-5)^2)$
b. $\frac{d}{dx}\left(\sqrt{2x+3x^5}\right)$
c. $\mathbf{D}(\sin^2(x))$ (Remember: $\sin^2(x) = [\sin(x)]^2$)

Solution

a. To match the pattern of the Power Rule for $\mathbf{D}((x^3-5)^2)$, let $f(x) = x^3-5$ and $p = 2$. Then:

$$\begin{aligned}
\mathbf{D}((x^3-5)^2) &= \mathbf{D}(f^p(x)) = p\cdot f^{p-1}(x)\cdot \mathbf{D}(f(x))\\
&= 2(x^3-5)^1\mathbf{D}(x^3-5) = 2(x^3-5)(3x^2) = 6x^2(x^3-5)
\end{aligned}$$

(Check that you get the same answer by first expanding $(x^3-5)^2$ and then taking the derivative.)

b. To match the pattern for $\frac{d}{dx}\left(\sqrt{2x+3x^5}\right)=\frac{d}{dx}\left((2x+3x^5)^{\frac{1}{2}}\right)$, let $f(x)=2x+3x^5$ and take p = $\frac{1}{2}$. Then:

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{2x+3x^5}\right)&=\frac{d}{dx}\left(f^p(x)\right)=p\cdot f^{p-1}(x)\cdot\frac{d}{dx}(f(x))\\&=\frac{1}{2}(2x+3x^5)^{-\frac{1}{2}}\frac{d}{dx}(2x+3x^5)\\&=\frac{1}{2}(2x+3x^5)^{-\frac{1}{2}}(2+15x^4)=\frac{2+15x^4}{2\sqrt{2x+3x^5}}\end{aligned}$$

c. To match the pattern for $\mathrm{D}(\sin^2(x))$, let $f(x)=\sin(x)$ and $p=2$:

$$\begin{aligned}\mathrm{D}(\sin^2(x))&=\mathrm{D}(f^p(x))=p\cdot f^{p-1}(x)\cdot\mathrm{D}(f(x))\\&=2\sin^1(x)\mathrm{D}(\sin(x))=2\sin(x)\cos(x)\end{aligned}$$

(We could also rewrite this last expression as $\sin(2x)$.)

Practice 2.3.2

Use the Power Rule for Functions to find:

a. $\frac{d}{dx}\left((2x^5-\pi)^2\right)$
b. $\mathrm{D}\left(\sqrt{x+7x^2}\right)$
c. $\mathrm{D}(\cos^4(x))$

Answer

$\frac{d}{dx}(2x^5-\pi)^2=2(2x^5-\pi)^1\mathrm{D}(2x^5-\pi)=2(2x^5-\pi)(10x^4)=40x^9-20\pi x^4$

$\mathrm{D}\left((x+7x^2)^{\frac{1}{2}}\right)=\frac{1}{2}(x+7x^2)^{-\frac{1}{2}}\mathrm{D}(x+7x^2)=\frac{1+14x}{2\sqrt{x+7x^2}}$

$\mathrm{D}\left((\cos(x))^4\right)=4(\cos(x))^3\mathrm{D}(\cos(x))=4(\cos(x))^3(-\sin(x))=-4\cos^3(x)\sin(x)$

Example 2.3.2

Use calculus to show that the line tangent to the circle $x^2+y^2=25$ at the point $(3,4)$ has slope $-\frac{3}{4}$.

Solution

The top half of the circle is the graph of $f(x)=\sqrt{25-x^2}$ so:

$$f'(x)=\mathrm{D}\left((25-x^2)^{\frac{1}{2}}\right)=\frac{1}{2}(25-x^2)^{-\frac{1}{2}}\cdot\mathrm{D}(25-x^2)=\frac{-x}{\sqrt{25-x^2}}$$

and $f'(3)=\frac{-3}{\sqrt{25-3^2}}=-\frac{3}{4}$. As a check, you can verify that the slope of the radial line through the center of the circle $(0,0)$ and the point $(3,4)$ has slope $\frac{4}{3}$ and is perpendicular to the tangent line that has a slope of $-\frac{3}{4}$.

Derivatives of Trigonometric Functions

We have some general rules that apply to any elementary combination of differentiable functions, but in order to use the rules we still need to know the derivatives of some basic functions. Here we will begin to add to the list of functions whose derivatives we know.

We already know the derivatives of the sine and cosine functions, and each of the other four trigonometric functions is just a ratio involving sines or cosines. Using the Quotient Rule, we can easily differentiate the rest of the trigonometric functions.

Theorem

- $\mathrm{D}(\tan(x))=\sec^2(x)$
- $\mathrm{D}(\sec(x))=\sec(x)\cdot\tan(x)$
- $\mathrm{D}(\cot(x))=-\csc^2(x)$
- $\mathrm{D}(\csc(x))=-\csc(x)\cdot\cot(x)$

Proof

From trigonometry, we know $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$ and $\csc(x) = \dfrac{1}{\sin(x)}$.

From calculus, we already know $\mathrm{D}(\sin(x)) = \cos(x)$ and $\mathrm{D}(\cos(x)) = -\sin(x)$. So:

$$\begin{aligned}\mathrm{D}(\tan(x)) &= \mathrm{D}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot \mathrm{D}(\sin(x)) - \sin(x)\cdot \mathrm{D}(\cos(x))}{(\cos(x))^2} \\ &= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)\end{aligned}$$

Similarly:

$$\begin{aligned}\mathrm{D}(\sec(x)) &= \mathrm{D}\left(\frac{1}{\cos(x)}\right) = \frac{\cos(x)\cdot \mathrm{D}(1) - 1\cdot \mathrm{D}(\cos(x))}{(\cos(x))^2} \\ &= \frac{\cos(x)\cdot 0 - (-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\cdot\tan(x)\end{aligned}$$

Instead of the Quotient Rule, we could have used the Power Rule to calculate $\mathrm{D}(\sec(x)) = \mathrm{D}((\cos(x))^{-1})$.

Practice 2.3.3

Use the Quotient Rule on $f(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$ to prove that $f'(x) = -\csc^2(x)$.

Answer

Mimicking the proof for the derivative of $\tan(x)$:

$$\begin{aligned}\mathrm{D}\left(\frac{\cos(x)}{\sin(x)}\right) &= \frac{\sin(x)\cdot \mathrm{D}(\cos(x)) - \cos(x)\cdot \mathrm{D}(\sin(x))}{(\sin(x))^2} \\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)(\cos(x))}{\sin^2(x)} \\ &= \frac{-(\sin^2(x)+\cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)\end{aligned}$$

Practice 2.3.4

Prove that $\mathrm{D}(\csc(x)) = -\csc(x)\cdot\cot(x)$. The justification of this result is very similar to the justification for $\mathrm{D}(\sec(x))$.

Answer

Mimicking the proof for the derivative of $\sec(x)$:

$$\begin{aligned}\mathrm{D}(\csc(x)) &= \mathrm{D}\left(\frac{1}{\sin(x)}\right) = \frac{\sin(x)\cdot \mathrm{D}(1) - 1\cdot \mathrm{D}(\sin(x)}{\sin^2(x)} \\ &= \frac{\sin(x)\cdot 0 - \cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} = -\cot(x)\csc(x)\end{aligned}$$

Practice 2.3.5

Find:

a. $\mathrm{D}(x^5\tan(x))$
b. $\dfrac{d}{dt}\left(\dfrac{\sec(t)}{t}\right)$
c. $\mathrm{D}\left(\sqrt{\cot(x)-x}\right)$

Answer

$$D(x^5 \cdot \tan(x)) = x^5 D(\tan(x)) + \tan(x) D(x^5) = x^5 \sec^2(x) + \tan(x)(5x^4)$$

$$\frac{d}{dt}\left(\frac{\sec(t)}{t}\right) = \frac{tD(\sec(t)) - \sec(t)D(t)}{t^2} = \frac{t\sec(t)\tan(t) - \sec(t)}{t^2}$$

$$D\left((\cot(x) - x)^{\frac{1}{2}}\right) = \frac{1}{2}(\cot(x) - x)^{-\frac{1}{2}} D(\cot(x) - x)$$
$$= \frac{1}{2}(\cot(x) - x)^{-\frac{1}{2}}(-\csc^2(x) - 1) = \frac{-\csc^2(x) - 1}{2\sqrt{\cot(x) - x}}$$

Derivatives of Exponential Functions

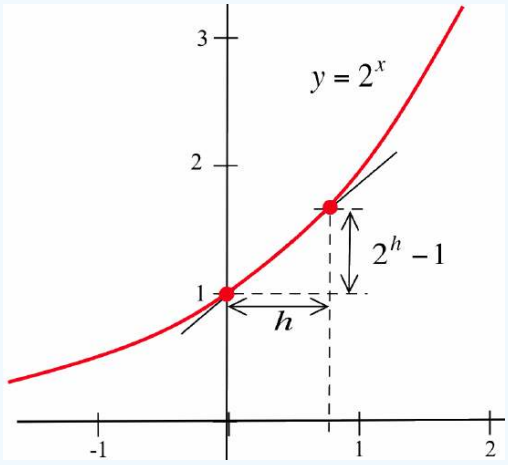
We can estimate the value of a derivative of an exponential function (a function of the form $f(x) = a^x$ where $a > 0$) by estimating the slope of the line tangent to the graph of such a function, or we can numerically approximate those slopes.

Example 2.3.3

Estimate the value of the derivative of $f(x) = 2^x$ at the point $(0, 2^0) = (0, 1)$ by approximating the slope of the line tangent to $f(x) = 2^x$ at that point.

Solution

We can get estimates from the graph of $f(x) = 2^x$ by carefully graphing $y = 2^x$ for small values of x (so that x is near 0), sketching secant lines, and then measuring the slopes of the secant lines:



We can also estimate the slope numerically by using the definition of the derivative:

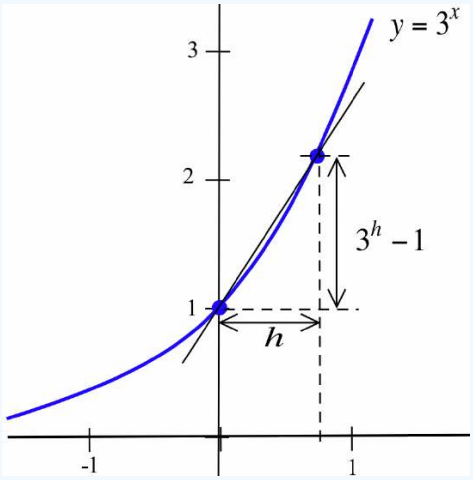
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{2^{0+h} - 2^0}{h} = \lim_{h \to 0} \frac{2^h - 1}{h}$$

and evaluating $\frac{2^h - 1}{h}$ for some very small values of h. From the table below we can see that $f'(0) \approx 0.693$.

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$	$\frac{e^h-1}{h}$
$+0.1$	0.717734625		
-0.1	0.669670084		
$+0.01$	0.695555006		
-0.01	0.690750451		
$+0.001$	0.693387463		
-0.001	0.692907009		
↓	↓	↓	↓
0	≈ 0.693	≈ 1.099	1

? Practice 2.3.6

Fill in the table above for $\frac{3^h-1}{h}$ and show that the slope of the line tangent to $g(x)=3^x$ at $(0,1)$ is approximately 1.099.



Answer

Filling in values for both 3^x and e^x:

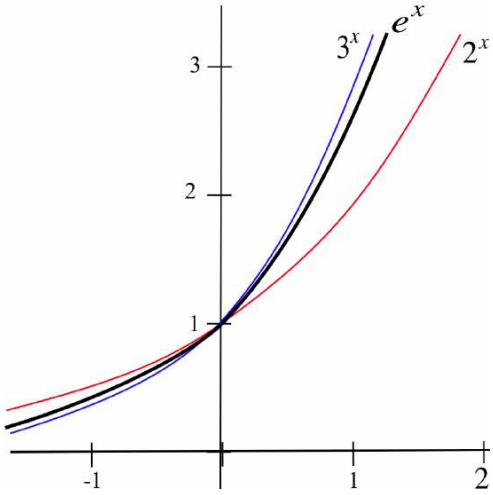
h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$	$\frac{e^h-1}{h}$
$+0.1$	0.717734625	1.161231740	1.0517091808
-0.1	0.669670084	1.040415402	0.9516258196
$+0.01$	0.695555006	1.104669194	1.0050167084
-0.01	0.690750451	1.092599583	0.9950166251
$+0.001$	0.693387463	1.099215984	1.0005001667
-0.001	0.692907009	1.098009035	0.9995001666
↓	↓	↓	↓
0	≈ 0.693	≈ 1.0986	1

At $(0,1)$, the slope of the tangent to $y=2^x$ is less than 1 and the slope of the tangent to $y=3^x$ is slightly greater than 1. You might expect that there is a number b between 2 and 3 so that the slope of the tangent to $y=b^x$ is exactly 1. Indeed, there is such a number, e

≈ 2.71828182845904, with

$$\lim_{h\to 0}\frac{e^h-1}{h}=1$$

The number e is irrational and plays a very important role in calculus and applications. (In fact, e is a "transcendental" number, which means that it is not the root of any polynomial equation with rational coefficients.)



We have not proved that this number e with the desired limit property actually exists, but if we assume it does, then it becomes relatively straightforward to calculate $\mathrm{D}(e^x)$. (Don't worry—we'll tie up some of these loose ends in Chapter 7.)

Theorem

$\mathrm{D}(e^x)=e^x$

Proof

Using the definition of the derivative:

$$\begin{aligned}\mathrm{D}(e^x)&=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h}\\&=\lim_{h\to 0}e^x\cdot\frac{e^h-1}{h}=\lim_{h\to 0}e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}\\&=e^x\cdot 1=e^x\end{aligned}$$

The function $f(x)=e^x$ is its own derivative: $f'(x)=f(x)$.

Graphically: the **height** of $f(x)=e^x$ at any point and the **slope** of the tangent to $f(x)=e^x$ at that point are the same: as the graph gets higher, its slope gets steeper. (Notice that the limit property of e that we assumed was true actually says that for $f(x)=e^x$, $f'(0)=1$. So knowing the derivative of $f(x)=e^x$ at a single point ($x=0$) allows us to determine its derivative at every other point.)

Example 2.3.4

Find:

a. $\frac{d}{dt}\left(t\cdot e^t\right)$
b. $\mathrm{D}\left(\frac{e^x}{\sin(x)}\right)$
c. $\mathrm{D}(e^{5x})$

Solution

a. Using the Product Rule with $f(t)=t$ and $g(t)=e^t$:

$$\frac{d}{dt}\left(t\cdot e^t\right)=t\cdot\mathrm{D}(e^t)+e^t\cdot\mathrm{D}(t)=t\cdot e^t+e^t\cdot 1=(t+1)e^t$$

b. Using the Quotient Rule with $f(x) = e^x$ and $g(x) = \sin(x)$:

$$\begin{aligned} \mathrm{D}\left(\frac{e^x}{\sin(x)}\right) &= \frac{\sin(x)\cdot \mathrm{D}(e^x) - e^x \cdot \mathrm{D}(\sin(x))}{[\sin(x)]^2} \\ &= \frac{\sin(x)\cdot e^x - e^x(\cos(x))}{\sin^2(x)} \end{aligned}$$

c. Using the Power Rule for Functions with $f(x) = e^x$ and $p = 5$:

$$\mathrm{D}((e^x)^5) = 5(e^x)^4 \cdot \mathrm{D}(e^x) = 5e^{4x}\cdot e^x = 5e^{5x}$$

where we have rewritten e^{5x} as $(e^x)^5$.

? Practice 2.3.7

Find:

a. $\mathrm{D}(x^3e^x)$
b. $\mathrm{D}((e^x)^3)$.

Answer

$\mathrm{D}(x^3e^x) = x^3\mathrm{D}(e^x) + e^x\mathrm{D}(x^3) = x^3e^x + e^x\cdot 3x^2 = x^2e^x(x+3)$

$\mathrm{D}\left((e^x)^3\right) = 3\left(e^x\right)^2\mathrm{D}(e^x) = 3e^{2x}\cdot e^x = 3e^{3x}$

Higher Derivatives: Derivatives of Derivatives

The derivative of a function f is a new function f' and if this new function is differentiable we can calculate the derivative of this new function to get the derivative of the derivative of f, denoted by f'' and called the {\bf second derivative} of f.

For example, if $f(x) = x^5$ then $f'(x) = 5x^4$ and $f''(x) = (f'(x))' = (5x^4)' = 20x^3$.

Definitions

Given a differentiable function f:

- the first derivative is $f'(x)$, the rate of change of f.
- the second derivative is $f''(x) = (f'(x))'$, the rate of change of f'.
- the third derivative is $f'''(x) = (f''(x))'$, the rate of change of f''.

For $y = f(x)$, we write $f'(x) = \frac{dy}{dx}$, so we can extend that notation to write $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, $f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ and so on.

? Practice 2.3.8

Find f', f'' and f''' for $f(x) = 3x^7$, $f(x) = \sin(x)$ and $f(x) = x\cdot\cos(x)$.

Answer

$f(x) = 3x^7 \Rightarrow f'(x) = 21x^6 \Rightarrow f''(x) = 126x^5 \Rightarrow f'''(x) = 630x^4$

$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x) \Rightarrow f'''(x) = -\cos(x)$

$f(x) = x\cdot\cos(x) \Rightarrow f'(x) = -x\sin(x) + \cos(x) \Rightarrow f''(x) = -x\cos(x) - 2\sin(x) \Rightarrow f'''(x) = x\sin(x) - 3\cos(x)$

If $f(x)$ represents the position of a particle at time x, then $v(x) = f'(x)$ will represent the velocity (rate of change of the position) of the particle and $a(x) = v'(x) = f''(x)$ will represent the acceleration (the rate of change of the velocity) of the particle.

✓ Example 2.3.5

The height (in feet) of a particle at time t seconds is given by $t^3 - 4t^2 + 8t$. Find the height, velocity and acceleration of the particle when $t = 0$, 1 and 2 seconds.

Solution

$f(t) = t^3 - 4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet and $f(2) = 8$ feet. The velocity is given by $v(t) = f'(t) = 3t^2 - 8t + 8$ so $v(0) = 8$ ft/sec, $v(1) = 3$ ft/sec and $v(2) = 4$ ft/sec. At each of these times the velocity is positive and the particle is moving upward (increasing in height). The acceleration is $a(t) = 6t - 8$ so $a(0) = -8$ ft/sec\)^2\), $a(1) = -2$ ft/sec\)^2\) and $a(2) = 4$ ft/sec\)^2\).

We will examine the geometric (graphical) meaning of the second derivative in the next chapter.

A Really “Bent” Function

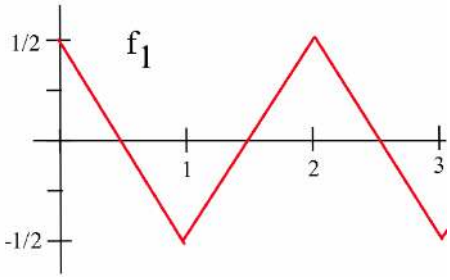
In Section 1.2 we saw that the “holey” function

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

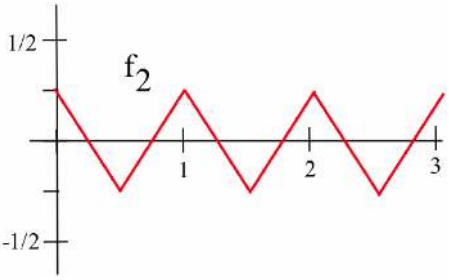
is discontinuous at every value of x, so $h(x)$ is not differentiable anywhere. We can create graphs of continuous functions that are not differentiable at several places just by putting corners at those places, but how many corners can a continuous function have? How badly can a continuous function fail to be differentiable?

In the mid-1800s, the German mathematician Karl Weierstrass surprised and even shocked the mathematical world by creating a function that was {\bf continuous everywhere but differentiable nowhere}—a function whose graph was everywhere connected and everywhere bent! He used techniques we have not investigated yet, but we can begin to see how such a function could be built.

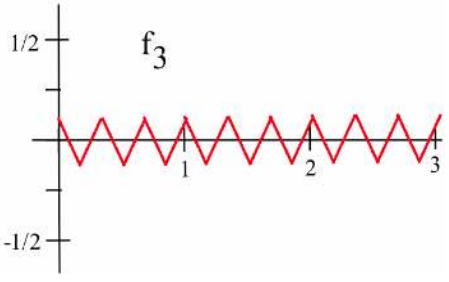
Start with a function f_1 that zigzags between the values $\frac{1}{2}$ and $-\frac{1}{2}$ and has a “corner” at each integer:



This starting function f_1 is continuous everywhere and is differentiable everywhere except at the integers. Next create a list of functions f_2, f_3, f_4,…, each of which is “shorter” than the previous one but with many more “corners” than the previous one. For example, we might make f_2 zigzag between the values $\frac{1}{4}$ and $-\frac{1}{4}$ and have “corners” at $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, etc.:



f_3 zigzag between $\frac{1}{9}$ and $-\frac{1}{9}$ and have “corners” at $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{3}{3} = \pm 1$, etc.:



and so on. If we add f_1 and f_2, we get a continuous function (because the sum of two continuous functions is continuous) with corners at 0, $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$,…. If we then add f_3 to the previous sum, we get a new continuous function with even more corners. If we continue adding the

functions in our list “indefinitely,” the final result will be a continuous function that is differentiable nowhere.

We haven’t developed enough mathematics here to precisely describe what it means to add an infinite number of functions together or to verify that the resulting function is nowhere differentiable—but we will. You can at least start to imagine what a strange, totally “bent” function it must be. Until Weierstrass created his “everywhere continuous, nowhere differentiable” function, most mathematicians thought a continuous function could only be “bad” in a few places. Weierstrass’ function was (and is) considered “pathological,” a great example of how bad something can be. The mathematician Charles Hermite expressed a reaction shared by many when they first encounter the Weierstrass function: “I turn away with fright and horror from this lamentable evil of functions which do not have derivatives.”

Important Results

Power Rule for Functions: $\mathrm{D}(f^p(x)) = p \cdot f^{p-1}(x) \cdot \mathrm{D}(f(x))$

Derivatives of the Trigonometric Functions:

$\mathrm{D}(\sin(x)) = \cos(x)$, $\mathrm{D}(\cos(x)) = -\sin(x)$

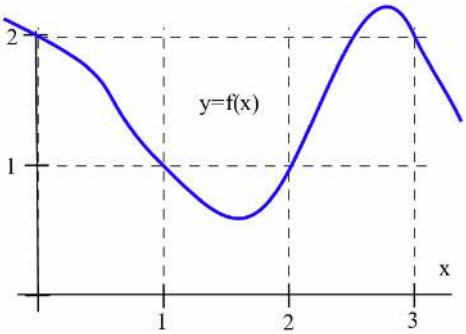
$\mathrm{D}(\tan(x)) = \sec^2(x)$, $\mathrm{D}(\cot(x)) = -\csc^2(x)$

$\mathrm{D}(\sec(x)) = \sec(x)\tan(x)$, $\mathrm{D}(\csc(x)) = -\csc(x)\cot(x)$

Derivative of the Exponential Function: $\mathrm{D}(e^x) = e^x$

Problems

1. Let $f(1) = 2$ and $f'(1) = 3$. Find the values of each of the following derivatives at $x = 1$.
   a. $\mathrm{D}(f^2(x))$
   b. $\mathrm{D}(f^5(x))$
   c. $\mathrm{D}(\sqrt{f(x)})$
2. Let $f(2) = -2$ and $f'(2) = 5$. Find the values of each of the following derivatives at $x = 2$.
   a. $\mathrm{D}(f^2(x))$
   b. $\mathrm{D}(f^{-3}(x))$
   c. $\mathrm{D}(\sqrt{f(x)})$
3. For $x = 1$ and $x = 3$ estimate the values of $f(x)$, whose graph appears below:



   as well as $f'(x)$ and
   a. $\frac{d}{dx}\left(f^2(x)\right)$
   b. $\mathrm{D}\left(f^3(x)\right)$
   c. $\mathrm{D}\left(f^5(x)\right)$
4. For $x = 0$ and $x = 2$ estimate the values of $f(x)$ (from the previous Problem), $f'(x)$ and
   a. $\mathrm{D}\left(f^2(x)\right)$
   b. $\frac{d}{dx}\left(f^3(x)\right)$
   c. $\frac{d}{dx}\left(f^5(x)\right)$

In Problems 5–10, find $f'(x)$.

10. $f(x) = (2x - 8)^5$
11. $f(x) = (6x - x^2)^{10}$
12. $f(x) = x \cdot (3x + 7)^5$

13. $f(x) = (2x+3)^6 \cdot (x-2)^4$
14. $f(x) = \sqrt{x^2+6x-1}$
15. $f(x) = \dfrac{x-5}{(x+3)^4}$
16. A mass attached to the end of a spring is at a height of $h(t) = 3 - 2\sin(t)$ feet above the floor t seconds after it is released.
    a. Graph $h(t)$.
    b. At what height is the mass when it is released?
    c. How high does above the floor and how close to the floor does the mass ever get?
    d. Determine the height, velocity and acceleration at time t. (Be sure to include the correct units.)
    e. Why is this an unrealistic model of the motion of a mass attached to a real spring?
17. A mass attached to a spring is at a height of $h(t) = 3 - \dfrac{2\sin(t)}{1+0.1t^2}$ feet above the floor t seconds after it is released.
    a. Graph $h(t)$.
    b. At what height is the mass when it is released?
    c. Determine the velocity of the mass at time t.
    d. What happens to the height and the velocity of the mass a "long time" after it is released?
18. The kinetic energy K of an object of mass m and velocity v is $\frac{1}{2}mv^2$.
    a. Find the kinetic energy of an object with mass m and height $h(t) = 5t$ feet at $t = 1$ and $t = 2$ seconds.
    b. Find the kinetic energy of an object with mass m and height $h(t) = t^2$ feet at $t = 1$ and $t = 2$ seconds.
19. An object of mass m is attached to a spring and has height $h(t) = 3 + \sin(t)$ feet at time t seconds.
    a. Find the height and kinetic energy of the object when $t = 1$, 2 and 3 seconds.
    b. Find the rate of change in the kinetic energy of the object when $t = 1$, 2 and 3 seconds.
    c. Can K ever be negative? Can $\dfrac{dK}{dt}$ ever be negative? Why?

In Problems 15–20, compute $f'(x)$.

15. $f(x) = x \cdot \sin(x)$
16. $f(x) = \sin^5(x)$
17. $f(x) = e^x - \sec(x)$
18. $f(x) = \sqrt{\cos(x)+1}$
19. $f(x) = e^{-x} + \sin(x)$
20. $f(x) = \sqrt{x^2-4x+3}$

In Problems 21–26, find an equation for the line tangent to the graph of $y = f(x)$ at the given point.

21. $f(x) = (x-5)^7$ at $(4,-1)$
22. $f(x) = e^x$ at $(0,1)$
23. $f(x) = \sqrt{25-x^2}$ at $(3,4)$
24. $f(x) = \sin^3(x)$ at $(\pi,0)$
25. $f(x) = (x-a)^5$ at $(a,0)$
26. $f(x) = x \cdot \cos^5(x)$ at $(0,0)$
27. a. Find an equation for the line tangent to $f(x) = e^x$ at the point $(3,e^3)$.
    b. Where will this tangent line intersect the x-axis?
    c. Where will the tangent line to $f(x) = e^x$ at the point (p,e^p) intersect the x-axis?

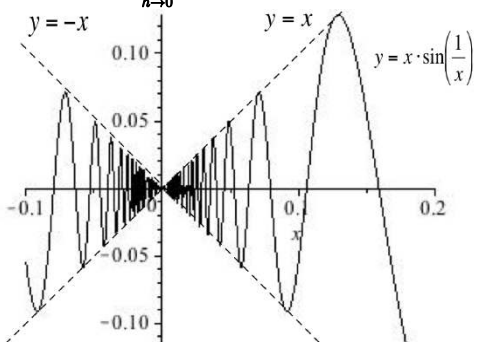
In Problems 28–33, calculate f' and f''.

28. $f(x) = 7x^2 + 5x - 3$
29. $f(x) = \cos(x)$
30. $f(x) = \sin(x)$
31. $f(x) = x^2 \cdot \sin(x)$
32. $f(x) = x \cdot \sin(x)$
33. $f(x) = e^x \cdot \cos(x)$
34. Calculate the first 8 derivatives of $f(x) = \sin(x)$. What is the pattern? What is the 208th derivative of $\sin(x)$?
35. What will the second derivative of a quadratic polynomial be? The third derivative? The fourth derivative?
36. What will the third derivative of a cubic polynomial be? The fourth derivative?

37. What can you say about the n-th and $(n+1)$-st derivatives of a polynomial of degree n?

In Problems 38–42, you are given f'. Find a function f with the given derivative.

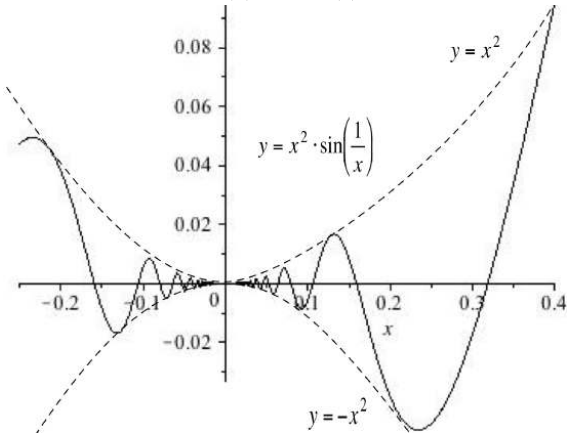
38. $f'(x) = 4x + 2$
39. $f'(x) = 5e^x$
40. $f'(x) = 3 \cdot \sin^2(x) \cdot \cos(x)$
41. $f'(x) = 5(1 + e^x)^4 \cdot e^x$
42. $f'(x) = e^x + \sin(x)$
43. The function $f(x)$ defined as $f(x) = \begin{cases} x \cdot \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ shown below is continuous at 0 because we can show (using the Squeezing Theorem) that $\lim_{h \to 0} f(x) = 0 = f(0)$.



Is f differentiable at 0? To answer this question, use the definition of $f'(0)$ and consider

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

44. The function $f(x)$ defined as $f(x) = \begin{cases} x^2 \cdot \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ shown below is continuous at 0 because we can show (using the Squeezing Theorem) that $\lim_{h \to 0} f(x) = 0 = f(0)$.



Is f differentiable at 0? To answer this question, use the definition of $f'(0)$ and consider

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

The number e appears in a variety of unusual situations. Problems 45–48 illustrate a few of these.

45. Use your calculator to examine the values of $f(x) = \left(1 + \frac{1}{x}\right)^x$ when x is relatively large (for example, $x = 100$, 1000 and $10,000$. Try some other large values for x. If x is large, the value of $f(x)$ is close to what number?
46. If you put \)1 into a bank account that pays 1\% interest per year and compounds the interest x times a year, then after one year you will have $\left(1 + \frac{0.01}{x}\right)^x$ dollars in the account.
    a. How much money will you have after one year if the bank calculates the interest once a year?
    b. How much money will you have after one year if the bank calculates the interest twice a year?
    c. How much money will you have after one year if the bank calculates the interest 365 times a year?

d. How does your answer to part (c) compare with $e^{0.01}$?

47. Define $n!$ to be the product of all positive integers from 1 through n. For example, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$ and $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

a. Calculate the value of the sums:

$$s_1 = 1 + \frac{1}{1!}$$
$$s_2 = 1 + \frac{1}{1!} + \frac{1}{2!}$$
$$s_3 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$$
$$s_4 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$$
$$s_5 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$
$$s_6 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$

b. What value do the sums in part (a) seem to be approaching?

c. Calculate s_7 and s_8.

48. If it is late at night and you are tired of studying calculus, try the following experiment with a friend. Take the 2 through 10 of hearts from a regular deck of cards and shuffle these nine cards well. Have your friend do the same with the 2 through 10 of spades. Now compare your cards one at a time. If there is a match, for example you both play a 5, then the game is over and you win. If you make it through the entire nine cards with no match, then your friend wins. If you play the game {\bf many times}, then the ratio:

$$\frac{\textbf{total number of games played}}{\textbf{number of times your friend wins}}$$

will be approximately equal to e.

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2.4: The Chain Rule

The Chain Rule is the **most important and most often used** of the differentiation patterns. It enables us to differentiate **composites** of functions such as $y = \sin(x^2)$. It is a powerful tool for determining the derivatives of some **new functions** such as logarithms and inverse trigonometric functions. And it leads to important **applications** in a variety of fields. You will need the Chain Rule hundreds of times in this course. Practice with it now will save you time—and points—later. Fortunately, with practice, the Chain Rule is also easy to use. We already know how to differentiate the composition of some functions.

Example 2.4.1

For $f(x) = 5x - 4$ and $g(x) = 2x + 1$, find $f \circ g(x)$ and $\mathbf{D}(f \circ g(x))$.

Solution

Writing $f \circ g(x) = f(g(x)) = 5(2x + 1) - 4 = 10x + 1$, we can compute that $\mathbf{D}(f \circ g(x)) = \mathbf{D}(10x + 1) = 10$.

Practice 2.4.1

For $f(x) = 5x - 4$ and $g(x) = x^2$, find $f \circ g(x)$, $\mathbf{D}(f \circ g(x))$, $g \circ f(x)$ and $\mathbf{D}(g \circ f(x))$.

Answer

$f(x) = 5x - 4$ and $g(x) = x^2 \Rightarrow f'(x) = 5$ and $g'(x) = 2x$, so $f \circ g(x) = f(g(x)) = f(x^2)) = 5x^2 - 4$ and $\mathbf{D}(5x^2 - 4) = 10x$ or:

$$\mathbf{D}(f \circ g(x)) = f'(g(x)) \cdot g'(x) = 5 \cdot 2x = 10x \tag{2.4.1}$$

\)g\circ f(x) = g(f(x)) = g(5x - 4) = (5x - 4)^2 = 25x^2 - 40x + 16\) and $\mathbf{D}(25x^2 - 40x + 16) = 50x - 40$ or:

$$\mathbf{D}(g \circ f(x)) = g'(f(x)) \cdot f'(x) = 2(5x - 4) \cdot 5 = 50x - 40 \tag{2.4.2}$$

Some compositions, however, are still very difficult to differentiate. We know the derivatives of $g(x) = x^2$ and $h(x) = \sin(x)$, and we know how to differentiate certain combinations of these functions, such as $x^2 + \sin(x)$, $x^2 \cdot \sin(x)$ and even $\sin^2(x) = (\sin(x))^2$. But the derivative of the simple composition $f(x) = h \circ g(x) = \sin(x^2)$ is hard—until we know the Chain Rule. (To see just how difficult, try using the definition of derivative on it.)

Example 2.4.2

a. Suppose amplifier Y doubles the strength of the output signal from amplifier U, and U triples the strength of the original signal x. How does the final signal out of Y compare with the original signal x?

b. Suppose y changes twice as fast as u, and u changes three times as fast as x. How does the rate of change of y compare with the rate of change of x?

Solution

In each case we are comparing the result of a composition, and the answer to each question is **6**, the product of the two amplifications or rates of change. In part (a), we have that:

$$\frac{\text{signal out of } Y}{\text{signal } x} = \frac{\text{signal out of } Y}{\text{signal out of } U} \cdot \frac{\text{signal out of } U}{\text{signal } x} = 2 \cdot 3 = 6$$

In part (b):

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = 2 \cdot 3 = 6$$

These examples are simple cases of the Chain Rule for differentiating a composition of functions.

The Chain Rule

We can express the chain rule using more than one type of notation. Each will be useful in various situations.

Chain Rule (Leibniz notation form)

If: y is a differentiable function of u

and: u is a differentiable function of x

then: y is a differentiable function of x

and: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Idea for a proof. If $\Delta u \neq 0$ then:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$
$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right) = \frac{dy}{du} \cdot \frac{du}{dx}$$

The key step here is to argue that $\Delta x \to 0$ implies $\Delta u \to 0$, which follows from the continuity of u as as function of x.

Although this nice short argument gets to the heart of why the Chain Rule works, it is not quite valid. If $\frac{du}{dx} \neq 0$, then it is possible to show that $\Delta u \neq 0$ for all "very small" values of Δx, and the "idea for a proof" becomes a real proof. There are, however, functions for which $\Delta u = 0$ for infinitely many small values of Δx (no matter how close to 0 we restrict Δx) and this creates problems with the simple argument outlined above. (A justification that holds true for {\bf all} cases is more complicated and provides no new conceptual insight. Problem 84 at the end of this section guides you through a rigorous proof of the Chain Rule.)

The symbol $\frac{dy}{du}$ is a single symbol, as is $\frac{du}{dx}$, so we cannot eliminate du from the product $\frac{dy}{du}\frac{du}{dx}$ in the Chain Rule by "cancelling" du as we can with Δu in the fractions $\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$. It is, however, perfectly fine to use the {\em idea} of cancelling du to help you remember the proper statement of the Chain Rule.

Example 2.4.3

Write $y = \cos(x^2 + 3)$ as $y = \cos(u)$ with $u = x^2 + 3$ and find $\frac{dy}{dx}$.

Solution

$y = \cos(u) \Rightarrow \frac{dy}{du} = -\sin(u)$ and $u = x^2 + 3 \Rightarrow \frac{du}{dx} = 2x$. Using the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot 2x = -2x \cdot \sin(x^2 + 3)$$

Notice that in the last step we have eliminated the intermediate variable u to express the derivative only in terms of x.

Practice 2.4.2

Find $\frac{dy}{dx}$ for $y = \sin(4x + e^x)$.

Answer

$$\frac{d}{dx}\left(\sin(4x + e^x)\right) = \cos(4x + e^x) \cdot \mathrm{D}(4x + e^x) = \cos(4x + e^x) \cdot (4 + e^x)$$

We can also state the Chain Rule in terms of composition of functions. The notation is different, but the meaning is precisely the same.

Chain Rule (composition form)

If: g is differentiable at x

and: f is differentiable at $g(x)$

then: the composite $f \circ g$ is differentiable at x

and: $(f \circ g)'(x) = \mathrm{D}(f(g(x))) = f'(g(x)) \cdot g'(x)$.

You may find it easier to think of the result of the composition form of the Chain Rule in words: "the derivative of the outside function (evaluated at the original inside function) times the derivative of the inside function" where f is the outside function and g is the inside function.

Example 2.4.4

Differentiate $\sin(x^2)$.

Solution

We can write the function $\sin(x^2)$ as the composition $f \circ g$ of two simple functions: $f(x) = \sin(x)$ and $g(x) = x^2$: $f \circ g(x) = f(g(x)) = f(x^2) = \sin(x^2)$. Both f and g are differentiable functions with derivatives $f'(x) = \cos(x)$ and $g'(x) = 2x$, so the Chain Rule says:

$$\begin{aligned} \mathrm{D}(\sin(x^2)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x) = \cos(g(x)) \cdot 2x \\ = \cos(x^2) \cdot 2x = 2x\cos(x^2) \end{aligned}$$

Check that you get the same answer using the Leibniz notation. (If you tried using the definition of derivative to calculate the derivative of this function at the beginning of this section, you can really appreciate the power of the Chain Rule for differentiating compositions of functions, even simple ones like these.)

Example 2.4.5

The table below gives values for f, f', g and g' at various points. Use these values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and $x = 0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
-1	2	3	1	0		
0	-1	1	3	2		
1	1	0	-1	3		
2	3	-1	0	1		
3	0	2	2	-1		

Solution

$(f \circ g)(-1) = f(g(-1)) = f(3) = 0$, $(f \circ g)(0) = f(g(0)) = f(1) = 1$, $(f \circ g)'(-1) = f'(g(-1)) \cdot g'(-1) = f'(3) \cdot 0 = 2 \cdot 0 = 0$ and $(f \circ g)'(0) = f'(g(0)) \cdot g'(0) = f'(1) \cdot 2 = (-1)(2) = -2$.

Practice 2.4.3

Fill in the table in Example 2.4.5 for $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1$, 2 and 3.

Answer

To fill in the last column, compute:

$$\begin{aligned} f'(g(1)) \cdot g'(1) = f'(0) \cdot 3 = (3)(3) = 9 \\ f'(g(2)) \cdot g'(2) = f'(-1) \cdot 1 = (1)(1) = 1 \\ f'(g(3) \cdot g'(3) = f'(2) \cdot (-1) = (0)(-1) = 0 \end{aligned}$$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
1	1	0	-1	3	-1	9
2	3	-1	0	1	2	3
3	0	2	2	-1	3	0

Neither form of the Chain Rule is inherently superior to the other—use the one you prefer or the one that appears most useful in a particular situation. The Chain Rule will be used hundreds of times in the rest of this book, and it is important that you master its usage. The time you spend now mastering and understanding how to use the Chain Rule will be paid back tenfold over the next several chapters.

Example 2.4.6

Determine $\mathrm{D}\left(e^{\cos(x)}\right)$ using each form of the Chain Rule.

Solution

Using the Leibniz notation: $y = e^u$ and $u = \cos(x)$ so we have $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = -\sin(x)$. Applying the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (-\sin(x)) = -\sin(x) \cdot e^{\cos(x)}$$

We can also write the function $e^{\cos(x)}$ as the composition of $f(x) = e^x$ with $g(x) = \cos(x)$, so the Chain Rule says:

$$\mathrm{D}(e^{\cos(x)}) = f'(g(x)) \cdot g'(x) = e^{g(x)} \cdot (-\sin(x)) = -\sin(x) \cdot e^{\cos(x)}$$

because $\mathrm{D}(e^x) = e^x$ and $\mathrm{D}\left(\cos(x)\right) = -\sin(x)$.

Practice 2.4.4

Calculate $\mathrm{D}\left(\sin(7x-1)\right)$, $\frac{d}{dx}\left(\sin(ax+b)\right)$ and $\frac{d}{dt}\left(e^{3t}\right)$.

Answer

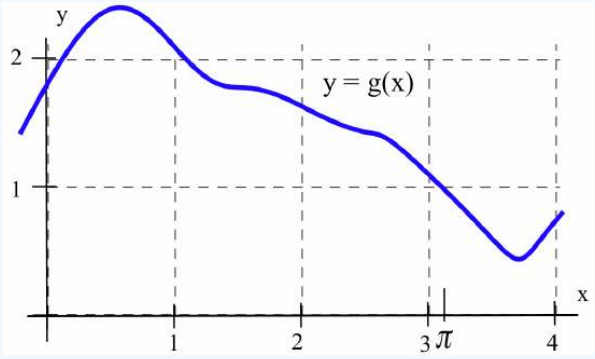
$\mathrm{D}\left(\sin(7x-1)\right) = \cos(7x-1) \cdot \mathrm{D}(7x-1) = 7 \cdot \cos(7x-1)$

$\frac{d}{dx}\left(\sin(ax+b)\right) = \cos(ax+b) \cdot \mathrm{D}(ax+b) = a \cdot \cos(ax+b)$

$\frac{d}{dt}\left(e^{3t}\right) = e^{3t} \cdot \frac{d}{dt}\left(3t\right) = 3 \cdot e^{3t}$

Practice 2.4.5

Use the graph of g given below along with the Chain Rule to estimate $\mathrm{D}\left(\sin(g(x))\right)$ and $\mathrm{D}\left(g(\sin(x))\right)$ at $x = \pi$.



Answer

$\mathrm{D}\left(\sin(g(x))\right) = \cos(g(x)) \cdot g'(x)$. At $x = \pi$, $\cos(g(\pi)) \cdot g'(\pi) \approx \cos(0.86) \cdot (-1) \approx -0.65$. $\mathrm{D}\left(g(\sin(x))\right) = g'(\sin(x)) \cdot \cos(x)$. At $x = \pi$, $g'(\sin(\pi)) \cdot \cos(\pi) = g'(0) \cdot (-1) \approx -2$

The Chain Rule is a general differentiation pattern that can be used along with other general patterns like the Product and Quotient Rules.

Example 2.4.7

Determine $\mathbf{D}\left(e^{3x}\cdot\sin(5x+7)\right)$ and $\dfrac{d}{dx}\left(\cos(x\cdot e^x)\right)$.

Solution

The function $e^{3x}\sin(5x+7)$ is a product of two functions so we need the Product Rule first:

$$\begin{aligned}\mathbf{D}(e^{3x}\cdot\sin(5x+7)) &= e^{3x}\cdot\mathbf{D}(\sin(5x+7))+\sin(5x+7)\cdot\mathbf{D}(e^{3x})\\ &= e^{3x}\cdot\cos(5x+7)\cdot 5+\sin(5x+7)\cdot e^{3x}\cdot 3\\ &= 5e^{3x}\cos(5x+7)+3e^{3x}\sin(5x+7)\end{aligned}$$

The function $\cos(x\cdot e^x)$ is a composition of cosine with a product so we need the Chain Rule first:

$$\begin{aligned}\frac{d}{dx}\left(\cos(x\cdot e^x)\right) &= -\sin(x\cdot e^x)\cdot\frac{d}{dx}(x\cdot e^x)\\ &= -\sin(xe^x)\cdot\left(x\cdot\frac{d}{dx}\left(e^x\right)+e^x\cdot\frac{d}{dx}\left(x\right)\right)\\ &= -\sin(xe^x)\cdot(xe^x+e^x)\end{aligned}$$

We could also write this last answer as $-(x+1)e^x\sin(e^x)$.

Sometimes we want to differentiate a composition of more than two functions. We can do so if we proceed in a careful, step-by-step way.

Example 2.4.8

Find $\mathbf{D}(\sin(\sqrt{x^3+1}))$.

Solution

The function $\sin(\sqrt{x^3+1})$ can be viewed as a composition $f\circ g$ of $f(x)=\sin(x)$ and $g(x)=\sqrt{x^3+1}$. Then:

$$\begin{aligned}(\sin(\sqrt{x^3+1}))' &= f'(g(x))\cdot g'(x)=\cos(g(x))\cdot g'(x)\\ &= \cos(\sqrt{x^3+1})\cdot\mathbf{D}(\sqrt{x^3+1})\end{aligned}$$

For the derivative of $\sqrt{x^3+1}$, we can use the Chain Rule again or its special case, the Power Rule:

$$\begin{aligned}\mathbf{D}(\sqrt{x^3+1}) &= \mathbf{D}((x^3+1)^{\frac{1}{2}})=\frac{1}{2}(x^3+1)^{-\frac{1}{2}}\cdot\mathbf{D}(x^3+1)\\ &= \frac{1}{2}(x^3+1)^{-\frac{1}{2}}\cdot 3x^2\end{aligned}$$

Finally, $\mathbf{D}\left(\sin(\sqrt{x^3+1})\right)=\cos(\sqrt{x^3+1})\cdot\frac{1}{2}(x^3+1)^{-\frac{1}{2}}\cdot 3x^2$, which can be rewritten as $\dfrac{3x^2\cos(\sqrt{x^3+1})}{2\sqrt{x^3+1}}$.

This example was more complicated than the earlier ones, but it is just a matter of applying the Chain Rule twice, to a composition of a composition. If you proceed step by step and don't get lost in the details of the problem, these multiple applications of the Chain Rule are relatively straightforward.

We can also use the Leibniz form of the Chain Rule for a composition of more than two functions. If $y=\sin(\sqrt{x^3+1})$, then $y=\sin(u)$ with $u=\sqrt{w}$ and $w=x^3+1$. The Leibniz form of the Chain Rule says:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dw}\cdot\frac{dw}{dx}=\cos(u)\cdot\frac{1}{2\sqrt{w}}\cdot 3x^2\\ &= \cos(\sqrt{x^3+1})\cdot\frac{1}{2\sqrt{x^3+1}}\cdot 3x^2\end{aligned}$$

which agrees with our previous answer.

Practice 2.4.6

a. Find $\mathbf{D}(\sin(\cos(5x)))$.
b. For $y = e^{\cos(3x)}$, find $\frac{dy}{dx}$.

Answer

$$\mathbf{D}\left(\sin(\cos(5x))\right) = \cos(\cos(5x)) \cdot \mathbf{D}(\cos(5x)) = \cos(\cos(5x)) \cdot (-\sin(5x)) \cdot \mathbf{D}(5x) = -5 \cdot \sin(5x) \cdot \cos(\cos(5x))$$

$$\frac{d}{dx}\left(e^{\cos(3x)}\right) = e^{\cos(3x)} \cdot \mathbf{D}(\cos(3x)) = e^{\cos(3x)}(-\sin(3x))\mathbf{D}(3x) = -3 \cdot \sin(3x) \cdot e^{\cos(3x)}$$

The Chain Rule and Tables of Derivatives

With the Chain Rule, the derivatives of all sorts of strange and wonderful functions become available. If we know f' and g', then we also know the derivatives of their composition: $(f(g(x))' = f'(g(x)) \cdot g'(x)$.

We have begun to build a list of derivatives of "basic" functions, such as x^n, $\sin(x)$ and e^x. We will continue to add to that list later in the course, but if we peek ahead at the rest of that list—spoiler alert!—to (for example) see that $\mathbf{D}(\arctan(x)) = \frac{1}{1+x^2}$, then we can use the Chain Rule to compute derivatives of compositions of those functions.

Example 2.4.9

Given that $\mathbf{D}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$, compute the derivatives $\mathbf{D}(\arcsin(5x))$ and $\frac{d}{dx}(\arcsin(e^x))$.

Solution

Write $\arcsin(5x)$ as the composition of $f(x) = \arcsin(x)$ with $g(x) = 5x$. We know $g'(x) = 5$ and $f'(x) = \frac{1}{\sqrt{1-x^2}}$, so we have $f'(g(x)) = \frac{1}{\sqrt{1-(g(x))^2}} = \frac{1}{\sqrt{1-25x^2}}$. Then:

$$\mathbf{D}(\arcsin(5x)) = f'(g(x)) \cdot g'(x) = \frac{1}{\sqrt{1-(5x)^2}} \cdot 5 = \frac{5}{\sqrt{1-25x^2}}$$

We can write $y = \arcsin(e^x)$ as $y = \arcsin(u)$ with $u = e^x$, and we know that $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = e^x$ so:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

We can generalize this result to say that $\mathbf{D}(\arcsin(f(x))) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$ or, in Leibniz notation, $\frac{d}{du}(\arcsin(u)) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

Practice 2.4.7

Given that $\mathbf{D}(\arctan(x)) = \frac{1}{1+x^2}$, compute the derivatives $\mathbf{D}(\arctan(x^3))$ and $\frac{d}{dx}(\arctan(e^x))$.

Answer

$$\mathbf{D}\left(\arctan(x^3)\right) = \frac{1}{1+(x^3)^2} \cdot \mathbf{D}(x^3) = \frac{3x^2}{1+x^6}$$

$$\frac{d}{dx}(\arctan(e^x)) = \frac{1}{1+(e^x)^2} \cdot \mathbf{D}(e^x) = \frac{e^x}{1+e^{2x}}$$

Appendix D at the end of this book shows the derivative patterns for a variety of functions. You may not know much about some of the functions, but with the given differentiation patterns and the Chain Rule you should be able to calculate derivatives of compositions that involve these new functions. It is just a matter of following the pattern.

Practice 2.4.8

Use the patterns $\mathbf{D}(\sinh(x)) = \cosh(x)$ and $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to determine:

a. $\mathbf{D}(\sinh(5x-7))$

b. $\frac{d}{dx}\left(\ln(3+e^{2x})\right)$
c. $\mathrm{D}(\arcsin(1+3x))$

Answer

$$\mathrm{D}(\sinh(5x-7)) = \cosh(5x-7)\cdot \mathrm{D}(5x-7) = 5\cdot\cosh(5x-7)$$

$$\frac{d}{dx}\left(\ln(3+e^{2x}\right) = \frac{1}{3+e^{2x}}\cdot \mathrm{D}(3+e^{2x}) = \frac{2e^{2x}}{3+e^{2x}}$$

$$\mathrm{D}(\arcsin(1+3x)) = \frac{1}{\sqrt{1-(1+3x)^2}}\cdot \mathrm{D}(1+3x) = \frac{3}{\sqrt{1-(1+3x)^2}}$$

Example 2.4.10

If $\mathrm{D}(F(x)) = e^x\cdot\sin(x)$, find $\mathrm{D}(F(5x))$ and $\frac{d}{dt}\left(F(t^3)\right)$.

Solution

$\mathrm{D}(F(5x)) = \mathrm{D}(F(g(x))$ with $g(x) = 5x$ and we know that $F'(x) = e^x\cdot\sin(x)$ so:

$$\mathrm{D}(F(5x)) = F'(g(x))\cdot g'(x) = e^{g(x)}\cdot\sin(g(x))\cdot 5 = e^{5x}\cdot\sin(5x)\cdot 5$$

With $y = F(u)$ and $u = t^3$ we know $\frac{dy}{du} = e^u\cdot\sin(u)$ and $\frac{du}{dt} = 3t^2$ so:

$$\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dt} = e^u\cdot\sin(u)\cdot 3t^2 = e^{t^3}\cdot\sin(t^3)\cdot 3t^2$$

Notice that we have eliminated the intermediate variable u (which didn’t appear in the original problem) from the final answer.

Proof of the Power Rule For Functions

We started using the Power Rule For Functions in Section 2.3. Now we can easily prove it.

Power Rule For Functions

If: p is any constant

then: $\mathrm{D}(f^p(x)) = p\cdot f^{p-1}(x)\cdot\mathrm{D}(f(x))$.

Proof

Write $y = f^p(x)$ as $y = u^p$ with $u = f(x)$. Then $\frac{dy}{du} = p\cdot u^{p-1}$ and $\frac{du}{dx} = f'(x)$ so:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = p\cdot u^{p-1}\cdot f'(x) = p\cdot f^{p-1}(x)\cdot f'(x)$$

by the Chain Rule.

Problems

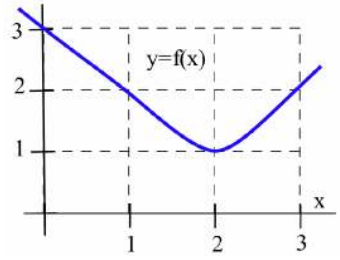
In Problems 1–6 , find two functions f and g so that the given function is the composition of f and g.

1. $y = (x^3-7x)^5$
2. $y = \sin^4(3x-8)$
3. $y = \sqrt{(2+\sin(x))^5}$
4. $y = \frac{1}{\sqrt{x^2+9}}$
5. $y = |x^2-4|$
6. $y = \tan(\sqrt{x})$
7. For each function in Problems 1–6, write y as a function of u for some u that is a function of x.

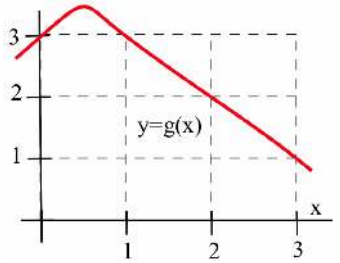
For Problems 8–9, use the values given in this table to determine the indicated quantities:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f\circ g)(x)$	$(f\circ g)'(x)$
-2	2	-1	1	1		
-1	1	2	0	2		
0	-2	1	2	-1		
1	0	-2	-1	2		
2	1	0	1	-1		

8. $(f\circ g)(x)$ and $(f\circ g)'(x)$ at $x=1$ and $x=2$.
9. $(f\circ g)(x)$ and $(f\circ g)'(x)$ at $x=-2$, -1 and 0.
10. Using the figure below, estimate the values of $g(x)$, $g'(x)$, $(f\circ g)(x)$, $f'(g(x))$ and $(f\circ g)'(x)$ at $x=1$.



11. Using the figure below, estimate the values of $g(x)$, $g'(x)$, $(f\circ g)(x)$, $f'(g(x))$ and $(f\circ g)'(x)$ at $x=2$.



In Problems 12–22, compute the derivative.

12. $\mathrm{D}\left((x^2+2x+3)^{87}\right)$
13. $\mathrm{D}\left(\left(1-\frac{3}{x}\right)^4\right)$
14. $\frac{d}{dx}\left(x+\frac{1}{x}\right)^5$
15. $\mathrm{D}\left(\frac{5}{\sqrt{2+\sin(x)}}\right)$
16. $\frac{d}{dt}\left(t\cdot\sin(3t+2)\right)$
17. $\frac{d}{dx}\left(x^2\cdot\sin(x^2+3)\right)$
18. $\frac{d}{dx}\left(\sin(2x)\cdot\cos(5x+1)\right)$
19. $\mathrm{D}\left(\frac{7}{\cos(x^3-x)}\right)$
20. $\frac{d}{dt}\left(\frac{5}{3+e^t}\right)$
21. $\mathrm{D}\left(e^x+e^{-x}\right)$
22. $\mathrm{D}\left(e^x-e^{-x}\right)$
23. An object attached to a spring is at a height of $h(t)=3-\cos(2t)$ feet above the floor t seconds after it is released.
    a. At what height was it released?
    b. Determine its height, velocity and acceleration at any time t.
    c. If the object has mass m, determine its kinetic energy $K=\frac{1}{2}mv^2$ and $\frac{dK}{dt}$ at any time t.
24. An employee with d days of production experience will be able to produce approximately $P(d)=3+15(1-e^{-0.2d})$ items per day.
    a. Graph $P(d)$.

b. Approximately how many items will a beginning employee be able to produce each day?
c. How many items will a very experienced employee be able to produce each day?
d. What is the marginal production rate of an employee with 5 days of experience? (Include units for your answer. What does this mean?)

25. The air pressure $P(h)$, in pounds per square inch, at an altitude of h feet above sea level is approximately $P(h) = 14.7e^{-0.0000385h}$.
    a. What is the air pressure at sea level?
    b. What is the air pressure at 30,000 feet?
    c. At what altitude is the air pressure 10 pounds per square inch?
    d. If you are in a balloon that is 2,000 feet above the Pacific Ocean and is rising at 500 feet per minute, how fast is the air pressure on the balloon changing?
    e. If the temperature of the gas in the balloon remained constant during this ascent, what would happen to the volume of the balloon?

Find the indicated derivatives in Problems 26--33.

26. $\mathrm{D}\left(\dfrac{(2x+3)^2}{(5x-7)^3}\right)$
27. $\dfrac{d}{dz}\sqrt{1+\cos^2(z)}$
28. $\mathrm{D}\left(\sin(3x+5)\right)$
29. $\dfrac{d}{dx}\tan(3x+5)$
30. $\dfrac{d}{dt}\cos(7t^2)$
31. $\mathrm{D}\left(\sin(\sqrt{x+1})\right)$
32. $\mathrm{D}\left(\sec(\sqrt{x+1})\right)$
33. $\dfrac{d}{dx}\left(e^{\sin(x)}\right)$

In Problems 34–37 , calculate $f'(x)\cdot x'(t)$ when $t=3$ and use these values to determine the value of $\dfrac{d}{dt}\left(f(x(t))\right)$ when $t=3$.

34. $f(x)=\cos(x)$, $x=t^2-t+5$
35. $f(x)=\sqrt{x}$, $x=2+\dfrac{21}{t}$
36. $f(x)=e^x$, $x=\sin(t)$
37. $f(x)=\tan^3(x)$, $x=8$

In Problems 38–43, find a function that has the given function as its derivative. (You are given a function $f'(x)$ and are asked to find a corresponding function $f(x)$.)

38. $f'(x)=(3x+1)^4$
39. $f'(x)=(7x-13)^{10}$
40. $f'(x)=\sqrt{3x-4}$
41. $f'(x)=\sin(2x-3)$
42. $f'(x)=6e^{3x}$
43. $f'(x)=\cos(x)e^{\sin(x)}$

If two functions are equal, then their derivatives are also equal. In Problems 44–47 , differentiate each side of the trigonometric identity to get a new identity.

44. $\sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$
45. $\cos(2x)=\cos^2(x)-\sin^2(x)$
46. $\sin(2x)=2\sin(x)\cdot\cos(x)$
47. $\sin(3x)=3\sin(x)-4\sin^3(x)$

Derivatives of Families of Functions

So far we have emphasized derivatives of particular functions, but sometimes we want to investigate the derivatives of a whole family of functions all at once. In 48--71, A, B, C and D represent constants and the given formulas describe families of functions.

For Problems 48–65, calculate $y'=\frac{dy}{dx}$.

48. $y=Ax^3-B$
49. $y=Ax^3+Bx^2+C$
50. $y=\sin(Ax+B)$

51. $y = \sin(Ax^2 + B)$
52. $y = Ax^3 + \cos(Bx)$
53. $y = \sqrt{A + Bx^2}$
54. $y = \sqrt{A - Bx^2}$
55. $y = A - \cos(Bx)$
56. $y = \cos(Ax + B)$
57. $y = \cos(Ax^2 + B)$
58. $y = A \cdot e^{Bx}$
59. $y = x \cdot e^{Bx}$
60. $y = e^{Ax} + e^{-Ax}$
61. $y = e^{Ax} - e^{-Ax}$
62. $y = \dfrac{\sin(Ax)}{x}$
63. $y = \dfrac{Ax}{\sin(Bx)}$
64. $y = \dfrac{1}{Ax + B}$
65. $y = \dfrac{Ax + B}{Cx + D}$

In Problems 66–71:

a. find y'
b. find the value(s) of x so that $y' = 0$ and
c. find y''.

Typically your answer in part (b) will contain A's, B's and (sometimes) C's.

66. $y = Ax^2 + Bx + C$
67. $y = Ax(B - x) = ABx - Ax^2$
68. $y = Ax(B - x^2) = ABx - Ax^3$
69. $y = Ax^2(B - x) = ABx^2 - Ax^3$
70. $y = Ax^2 + Bx$
71. $y = Ax^3 + Bx^2 + C$

In Problems 72–83, use the differentiation patterns $\mathbf{D}(\arctan(x)) = \dfrac{1}{1 + x^2}$, $\mathbf{D}(\arcsin(x)) = \dfrac{1}{\sqrt{1 - x^2}}$ and $\mathbf{D}(\ln(x)) = \dfrac{1}{x}$. We have not derived the derivatives for these functions (yet), but if you are handed the derivative pattern then you should be able to use that pattern to compute derivatives of associated composite functions.

72. $\mathbf{D}\left(\arctan(7x)\right)$
73. $\mathbf{D}\left(\arctan(x^2)\right)$
74. $\dfrac{d}{dt}\left(\arctan(\ln(t))\right)$
75. $\dfrac{d}{dx}\left(\arctan(e^x)\right)$
76. $\dfrac{d}{dw}\left(\arcsin(4w)\right)$
77. $\dfrac{d}{dx}\left(\arcsin(x^3)\right)$
78. $\mathbf{D}\left(\arcsin(\ln(x))\right)$
79. $\mathbf{D}\left(\arcsin(e^t)\right)$
80. $\mathbf{D}\left(\ln(3x + 1)\right)$
81. $\mathbf{D}\left(\ln(\sin(x))\right)$
82. $\dfrac{d}{dx}\left(\ln(\arctan(x))\right)$
83. $\dfrac{d}{ds}\left(\ln(e^s)\right)$
84. To prove the Chain Rule, assume $g(x)$ is differentiable at $x = a$ and $f(x)$ is differentiable at $x = g(a)$. We need to show that

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

exists and is equal to $f'(g(a)) \cdot g'(a)$. To do this, define a new function F as:

$$F(y) = \begin{cases} \frac{f(y)-f(g(a))}{y-g(a)} & \text{if } y \neq g(a) \\ f'(g(a)) & \text{if } y = g(a) \end{cases}$$

and justify each of the following statements.

a. $F(y)$ is continuous at $y = g(a)$ because:

$$\lim_{y \to g(a)} F(y) = \lim_{y \to g(a)} \frac{f(y) - f(g(a))}{y - g(a)} = F(g(a))$$

b. By considering separately the cases $g(x) = g(a)$ and $g(x) \neq g(a)$:

$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x)) \cdot \frac{g(x) - g(a)}{x - a}$$

for all $x \neq a$.

c. $\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \to a} F(g(x)) \cdot \frac{g(x) - g(a)}{x - a}$

d. $\lim_{x \to a} F(g(x)) \cdot \frac{g(x) - g(a)}{x - a} = F(g(a)) \cdot g'(a)$

e. $\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = f'(g(a)) \cdot g'(a)$

f. $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$

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2.5: Applications of the Chain Rule

The Chain Rule can help us determine the derivatives of logarithmic functions like $f(x) = \ln(x)$ and general exponential functions like $f(x) = a^x$. We will also use it to answer some applied questions and to find slopes of graphs given by parametric equations.

Derivatives of Logarithms

You know from precalculus that the natural logarithm $\ln(x)$ is defined as the inverse of the exponential function e^x: $e^{\ln(x)} = x$ for $x > 0$. We can use this identity along with the Chain Rule to determine the derivative of the natural logarithm.

Theorem

$\mathrm{D}(\ln(x)) = \dfrac{1}{x}$ and $\mathrm{D}\left(\ln(g(x))\right) = \dfrac{g'(x)}{g(x)}$

Proof

We know that $\mathrm{D}(e^u) = e^u$, so using the Chain Rule we have $\mathrm{D}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$. Differentiating each side of the identity $e^{\ln(x)} = x$, we get:

$$\mathrm{D}\left(e^{\ln(x)}\right) = \mathrm{D}(x) \Rightarrow e^{\ln(x)} \cdot \mathrm{D}(\ln(x)) = 1$$
$$\Rightarrow x \cdot \mathrm{D}(\ln(x)) = 1 \Rightarrow \mathrm{D}(\ln(x)) = \frac{1}{x}$$

The function $\ln(g(x))$ is the composition of $f(x) = \ln(x)$ with $g(x)$ so the Chain Rule says:

$$\mathrm{D}\left(\ln(g(x)\right) = \mathrm{D}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

Graph $f(x) = \ln(x)$ along with $f'(x) = \dfrac{1}{x}$ and compare the behavior of the function at various points with the values of its derivative at those points. Does $y = \dfrac{1}{x}$ possess the properties you would expect to see from the derivative of $f(x) = \ln(x)$?

You can remember the differentiation pattern for the the natural logarithm in words as: "one over the inside times the the derivative of the inside."

Example 2.5.1

Find $\mathrm{D}(\ln(\sin(x)))$ and $\mathrm{D}(\ln(x^2 + 3))$.

Solution

Using the pattern $\mathrm{D}(\ln(g(x)) = \dfrac{g'(x)}{g(x)}$ with $g(x) = \sin(x)$:

$$\mathrm{D}(\ln(\sin(x))) = \frac{g'(x)}{g(x)} = \frac{\mathrm{D}(\sin(x))}{\sin(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

With $g(x) = x^2 + 3$, $\mathrm{D}(\ln(x^2 + 3)) = \dfrac{g'(x)}{g(x)} = \dfrac{2x}{x^2 + 3}$.

We can use the Change of Base Formula from precalculus to rewrite any logarithm as a natural logarithm, and then we can differentiate the resulting natural logarithm.

Change of Base Formula for Logarithms

$\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all positive a, b and x.

Note

Your calculator likely has two logarithm buttons: **ln** for the natural logarithm (base e) and **log** for the common logarithm (base 10). Be careful, however, as more advanced mathematics texts (as well as the Web site Wolfram|Alpha) use **log** for the (base e) natural logarithm.}

Example 2.5.2

Use the Change of Base formula and your calculator to find $\log_\pi(7)$ and $\log_2(8)$.

Solution

$\log_\pi(7) = \dfrac{\ln(7)}{\ln(\pi)} \approx \dfrac{1.946}{1.145} \approx 1.700$. (Check that $\pi^{1.7} \approx 7$.) Likewise, $\log_2(8) = \dfrac{\ln(8)}{\ln(2)} = 3$.

Practice 2.5.1

Find the values of $\log_9 20$, $\log_3 20$ and $\log_\pi e$.

Answer

$$\log_9(20) = \frac{\log(20)}{\log(9)} \approx 1.3634165 \approx \frac{\ln(20)}{\ln(9)}$$

$$\log_3(20) = \frac{\log(20)}{\log(3)} \approx 2.726833 \approx \frac{\ln(20)}{\ln(3)}$$

$$\log_\pi(e) = \frac{\log(e)}{\log(\pi)} \approx 0.8735685 \approx \frac{\ln(e)}{\ln(\pi)} = \frac{1}{\ln(\pi)}$$

Putting $b = e$ in the Change of Base Formula, $\log_a(x) = \dfrac{\log_e(x)}{\log_e(a)} = \dfrac{\ln(x)}{\ln(a)}$, so any logarithm can be written as a natural logarithm divided by a constant. This makes any logarithmic function easy to differentiate.

Theorem

$\mathrm{D}\left(\log_a(x)\right) = \dfrac{1}{x\ln(a)}$ and $\mathrm{D}\left(\log_a(f(x))\right) = \dfrac{f'(x)}{f(x)} \cdot \dfrac{1}{\ln(a)}$

Proof

$\mathrm{D}\left(\log_a(x)\right) = \mathrm{D}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln(a)} \cdot \mathrm{D}(\ln x) = \dfrac{1}{\ln(a)} \cdot \dfrac{1}{x} = \dfrac{1}{x\ln(a)}$. The second differentiation formula follows from the Chain Rule.

Practice 2.5.2

Calculate $\mathrm{D}\left(\log_{10}(\sin(x))\right)$ and $\mathrm{D}\left(\log_\pi(e^x)\right)$.

Answer

$$\mathrm{D}\left(\log_{10}(\sin(x))\right) = \frac{1}{\sin(x)\cdot\ln(10)}\mathrm{D}(\sin(x)) = \frac{\cos(x)}{\sin(x)\cdot\ln(10)}$$

$$\mathrm{D}\left(\log_\pi(e^x)\right) = \frac{1}{e^x\cdot\ln(\pi)}\mathrm{D}(e^x) = \frac{e^x}{e^x\cdot\ln(\pi)} = \frac{1}{\ln(\pi)}$$

The number e might seem like an "unnatural" base for a natural logarithm, but of all the possible bases, the logarithm with base e has the nicest and easiest derivative. The natural logarithm is even related to the distribution of prime numbers. In 1896, the mathematicians Hadamard and Vallèe-Poussin proved the following conjecture of Gauss (the Prime Number Theorem): For large values of N,

$$\text{number of primes less than } N \approx \frac{N}{\ln(N)}$$

Derivative of a^x

Once we know the derivative of e^x and the Chain Rule, it is relatively easy to determine the derivative of a^x for any $a > 0$.

Theorem

$\mathrm{D}(a^x) = a^x \cdot \ln(a)$ \ for $a > 0$.

Proof

If $a > 0$, then $a^x > 0$ and $a^x = e^{\ln(a^x)} = e^{x \cdot \ln(a)}$, so we have: $\mathrm{D}(a^x) = \mathrm{D}\left(e^{\ln(a^x)}\right) = \mathrm{D}\left(e^{x \cdot \ln(a)}\right) = e^{x \cdot \ln(a)} \cdot \mathrm{D}(x \cdot \ln(a)) = a^x \cdot \ln(a)$.

Example 2.5.3

Calculate $\mathrm{D}(7^x)$ and $\frac{d}{dt}\left(2^{\sin(t)}\right)$.

Solution

$\mathrm{D}(7^x) = 7^x \cdot \ln(7) \approx (1.95)7^x$. We can write $y = 2^{\sin(t)}$ as $y = 2^u$ with $u = \sin(t)$. Using the Chain Rule: $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = 2^u \cdot \ln(2)\cos(t) = 2^{\sin(t)} \cdot \ln(2) \cdot \cos(t)$.

Practice 2.5.3

Calculate $\mathrm{D}\left(\sin(2^x)\right)$ and $\frac{d}{dt}\left(3^{t^2}\right)$.

Answer

$\mathrm{D}\left(\sin(2^x)\right) = \cos(2^x)\mathrm{D}\left(2^x\right) = \cos(2^x) \cdot 2^x \cdot \ln(2)$

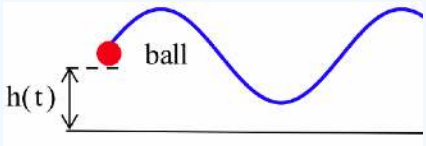
$\frac{d}{dt}\left(3^{t^2}\right) = 3^{t^2}\ln(3)\mathrm{D}(t^2) = 3^{t^2}\ln(3) \cdot 2t$

Some Applied Problems

Let's examine some applications involving more complicated functions.

Example 2.5.4

A ball at the end of a rubber band is oscillating up and down, and its height (in feet) above the floor at time t seconds is $h(t) = 5 + 2\sin\left(\frac{t}{2}\right)$ (with t in radians).



a. How fast is the ball traveling after 2 seconds? After 4 seconds? After 60 seconds?
b. Is the ball moving up or down after 2 seconds? After 4 seconds? After 60 seconds?
c. Is the vertical velocity of the ball ever 0?

Solution

a. $v(t) = h'(t) = \mathrm{D}\left(5 + 2\sin\left(\frac{t}{2}\right)\right) = 2\cos\left(\frac{t}{2}\right) \cdot \frac{1}{2}$ so $v(t) = \cos\left(\frac{t}{2}\right)$ feet/second: $v(2) = \cos\left(\frac{2}{2}\right) \approx 0.540$ ft/s, $v(4) = \cos\left(\frac{4}{2}\right) \approx -0.416$ ft/s, and $v(60) = \cos\left(\frac{60}{2}\right) \approx 0.154$ ft/s.
b. The ball is moving up at $t = 2$ and $t = 60$, down when $t = 4$.

c. $v(t) = \cos\left(\frac{t}{2}\right) = 0$ when $\frac{t}{2} = \frac{\pi}{2} \pm k \cdot \pi \Rightarrow t = \pi \pm 2\pi k$ for integer k.

Example 2.5.5

If 2,400 people now have a disease, and the number of people with the disease appears to double every 3 years, then the number of people expected to have the disease in t years is $y = 2400 \cdot 2^{\frac{t}{3}}$.

a. How many people are expected to have the disease in 2 years?
b. When are 50,000 people expected to have the disease?
c. How fast is the number of people with the disease growing now? How fast is it expected to be growing 2 years from now?

Solution

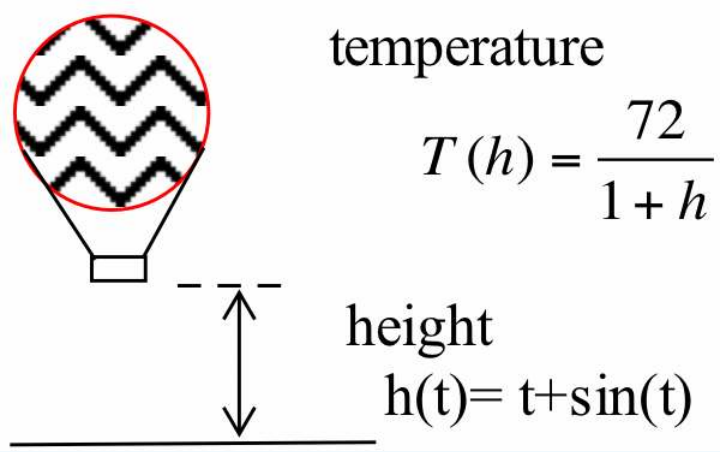
a. In 2 years, $y = 2400 \cdot 2^{\frac{2}{3}} \approx 3{,}810$ people.
b. We know $y = 50000$ and need to solve $50000 = 2400 \cdot 2^{\frac{t}{3}}$ for t. Taking logarithms of each side of the equation: $\ln(50000) = \ln\left(2400 \cdot 2^{\frac{2}{3}}\right) = \ln(2400) + \frac{t}{3} \cdot \ln(2)$ so $10.819 \approx 7.783 + 0.231t$ and $t \approx 13.14$ years. We expect 50,000 people to have the disease about 13 years from now.
c. This question asks for $\frac{dy}{dt}$ when $t = 0$ and $t = 2$.

$$\frac{dy}{dt} = \frac{d}{dt}\left(2400 \cdot 2^{\frac{t}{3}}\right) = 2400 \cdot 2^{\frac{t}{3}} \cdot \ln(2) \cdot \frac{1}{3} \approx 554.5 \cdot 2^{\frac{t}{3}}$$

Now, at $t = 0$, the rate of growth of the disease is approximately \hspace*{1.5em}\)554.5\cdot 2^0 \approx 554.5\) people/year. In 2 years, the rate of growth will be approximately $554.5 \cdot 2^{\frac{2}{3}} \approx 880$ people/year.

Example 2.5.6

You are riding in a balloon, and at time t (in minutes) you are $h(t) = t + \sin(t)$ thousand feet above sea level. If the temperature at an elevation h is $T(h) = \frac{72}{1+h}$ degrees Fahrenheit, then how fast is the temperature changing when $t = 5$ minutes?



Solution

As t changes, your elevation will change. And, as your elevation changes, so will the temperature. It is not difficult to write the temperature as a function of time, and then we could calculate $\frac{dT}{dt} = T'(t)$ and evaluate $T'(5)$. Or we could use the Chain Rule:

$$\frac{dT}{dt} = \frac{dT}{dh} \cdot \frac{dh}{dt} = -\frac{72}{(1+h)^2} \cdot (1 + \cos(t))$$

At $t = 5$, $h(5) = 5 + \sin(5) \approx 4.04$ so $T'(5) \approx -\frac{72}{(1+4.04)^2} \cdot (1 + 0.284) \approx -3.64$ °/minute.

? Practice 2.5.4

Write the temperature T in the previous example as a function of the variable t alone and then differentiate T to determine the value of $\frac{dT}{dt}$ when $t = 5$ minutes.

Answer

$$T = \frac{72}{1+h} = \frac{72}{1+t+\sin(t)} \Rightarrow$$

$$\frac{dT}{dt} = \frac{(1+t+\sin(t))\cdot 0 - 72\cdot \mathrm{D}(1+t+\sin(t))}{(1+t+\sin(t))^2} = \frac{-72(1+\cos(t))}{(1+t+\sin(t))^2}$$

When $t = 5$, $\frac{dT}{dt} = \frac{-72(1+\cos(5))}{(1+5+\sin(5))^2} \approx -3.63695$.

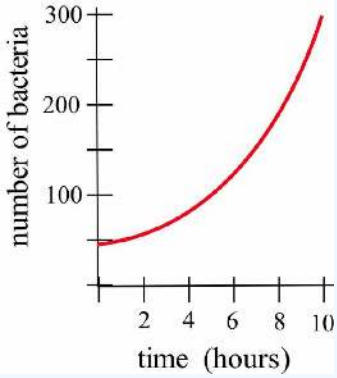
✓ Example 2.5.7

A scientist has determined that, under optimum conditions, an initial population of 40 bacteria will grow "exponentially" to $f(t) = 40\cdot e^{\frac{t}{5}}$ bacteria after t hours.

a. Graph $y = f(t)$ for $0 \le t \le 15$. Calculate $f(0)$, $f(5)$ and $f(10)$.
b. How fast is the population increasing at time t? (Find $f'(t)$.)
c. Show that the rate of population increase, $f'(t)$, is proportional to the population, $f(t)$, at any time t. (Show $f'(t) = K\cdot f(t)$ for some constant K.)

Solution

a. The graph of $y = f(t)$ appears below:



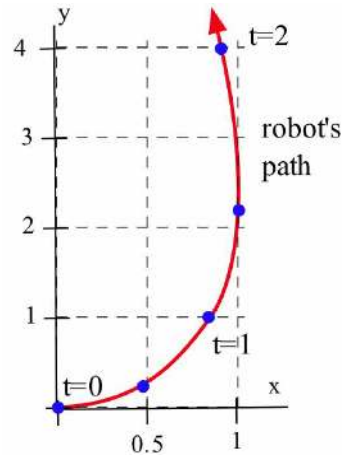
$f(0) = 40\cdot e^{\frac{0}{5}} = 40$ bacteria, $f(5) = 40\cdot e^{\frac{5}{5}} = 40e \approx 109$ bacteria and $f(10) = 40\cdot e^{\frac{10}{5}} \approx 296$ bacteria.

b. $f'(t) = \frac{d}{dt}(f(t)) = \frac{d}{dt}\left(40\cdot e^{\frac{t}{5}}\right) = 40\cdot e^{\frac{t}{5}}\cdot \frac{d}{dt}\left(\frac{t}{5}\right) = 40\cdot e^{\frac{t}{5}}\cdot \frac{1}{5} = 8\cdot e^{\frac{t}{5}}$ bacteria/hour.

c. $f'(t) = 8\cdot e^{\frac{t}{5}} = \frac{1}{5}\cdot 40e^{\frac{t}{5}} = \frac{1}{5}f(t)$ so $f'(t) = K\cdot f(t)$ with $K = \frac{1}{5}$. The rate of change of the population is proportional to its size.

Parametric Equations

Suppose a robot has been programmed to move in the xy-plane so at time t its x-coordinate will be $\sin(t)$ and its y-coordinate will be t^2. Both x and y are functions of the independent parameter t: $x(t) = \sin(t)$ and $y(t) = t^2$. The path of the robot can be found by plotting $(x, y) = (x(t), y(t))$ for lots of values of t:



t	$x(t)=\sin(t)$	$y(t)=t^2$	point
0	0	0	$(0,0)$
0.5	0.48	0.25	$(0.48,0.25)$
1.0	0.84	1	$(0.84,1)$
1.5	1.00	2.25	$(1,2.25)$
2.0	0.91	4	$(0.91,4)$

Typically we know $x(t)$ and $y(t)$ and need to find $\frac{dy}{dx}$, the slope of the tangent line to the graph of $(x(t),y(t))$. The Chain Rule says:

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

so , algebraically solving for $\frac{dy}{dx}$, we get:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

If we can calculate $\frac{dy}{dt}$ and $\frac{dx}{dt}$, the derivatives of y and x with respect to the parameter t, then we can determine $\frac{dy}{dx}$, the rate of change of y with respect to x.

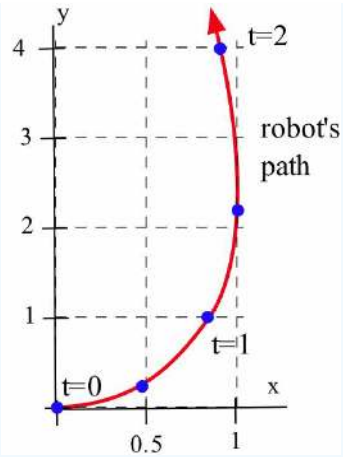
Theorem

If : $x=x(t)$ and $y=y(t)$ are differentiable with respect to t and $\frac{dx}{dt}\neq 0$

then: $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Example 2.5.8

Find the slope of the tangent line to the graph of $(x,y)=(\sin(t),t^2)$ when $t=2$.



Solution

$\frac{dx}{dt} = \cos(t)$ and $\frac{dy}{dt} = 2t$. When $t = 2$, the object is at the point $(\sin(2), 2^2) \approx (0.91, 4)$ and the slope of the tangent line is:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\cos(t)} = \frac{2 \cdot 2}{\cos(2)} \approx \frac{4}{-0.42} \approx -9.61$$

Notice in the figure that the slope of the tangent line to the curve at $(0.91, 4)$ is negative and very steep.

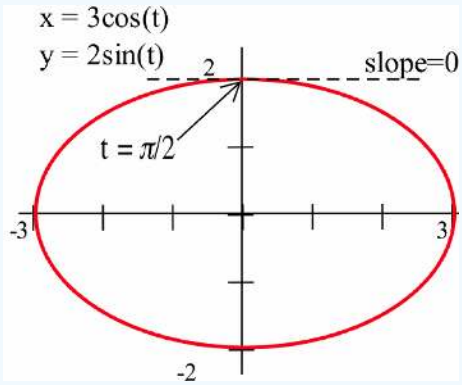
Practice 2.5.5

Graph $(x, y) = (3\cos(t), 2\sin(t))$ and find the slope of the tangent line when $t = \frac{\pi}{2}$.

Answer

$x(t) = 3\cos(t) \Rightarrow \frac{dx}{dt} = -3\sin(t)$, $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} = 2\cos(t)$:

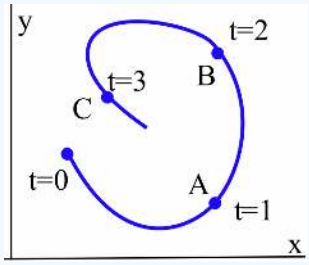
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos(t)}{-3\sin(t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{2\cos(\frac{\pi}{2})}{-3\sin(\frac{\pi}{2})} = \frac{2 \cdot 0}{-3 \cdot 1} = 0$$



When we calculated $\frac{dy}{dx}$, the slope of the tangent line to the graph of $(x(t), y(t))$, we used the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Each of these also has a geometric meaning: $\frac{dx}{dt}$ measures the rate of change of $x(t)$ with respect to t: it tells us whether the x-coordinate is increasing or decreasing as the t-variable increases (and how fast it is changing), while $\frac{dy}{dt}$ measures the rate of change of $y(t)$ with respect to t.

Example 2.5.9

For the parametric graph below:



determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive or negative when $t = 2$.

Solution

As we move through the point B (where $t = 2$) in the direction of increasing values of t, we are moving to the left, so $x(t)$ is decreasing and $\frac{dx}{dt} < 0$. The values of $y(t)$ are increasing, so $\frac{dy}{dt} > 0$. Finally, the slope of the tangent line, $\frac{dy}{dx}$, is negative.

As a check on the sign of $\frac{dy}{dx}$ in the previous example:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\text{positive}}{\text{negative}} = \text{negative}$$

Practice 2.5.6

For the parametric graph in Example 2.5.9, tell whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive or negative at $t = 1$ and $t = 3$.

Answer

$x = 1$: positive, positive, positive. $x = 3$: positive, negative, negative.

Speed

If we know the position of an object at any time, then we can determine its speed. The formula for speed comes from the distance formula and looks a lot like it, but involves derivatives.

Theorem

If: $x = x(t)$ and $y = y(t)$ give the location of an object at time t and both are differentiable functions of t

then: the speed of the object is

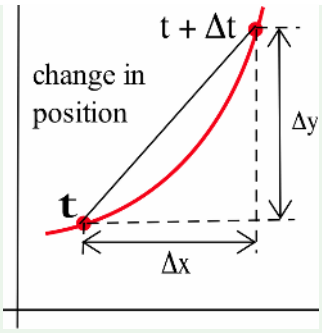
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Proof

The speed of an object is the limit, as $\Delta t \to 0$, of:

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2}}$$
$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \to \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

as $\Delta t \to 0$.



Example 2.5.10

Find the speed of the object whose location at time t is $(x, y) = (\sin(t), t^2)$ when $t = 0$ and $t = 1$.

Solution

$\frac{dx}{dt} = \cos(t)$ and $\frac{dy}{dt} = 2t$ so:\[\mbox{speed } = \sqrt{\left(\cos(t)\right)^2 + (2t)^2} = \sqrt{\cos^2(t) + 4t^2}]When $t = 0$, speed $= \sqrt{\cos^2(0) + 4(0)^2} = \sqrt{1 + 0} = 1$. When $t = 1$, speed $= \sqrt{\cos^2(1) + 4(1)^2} \approx \sqrt{0.29 + 4} \approx 2.07$.

Practice 2.5.7

Show that an object located at $(x, y) = (3\sin(t), 3\cos(t))$ at time t has a constant speed. (This object is moving on a circular path.)

Answer

$x(t) = 3\sin(t) \Rightarrow \frac{dx}{dt} = 3\cos(t)$ and $y(t) = 3\cos(t) \Rightarrow \frac{dy}{dt} = -3\sin(t)$. So:

$$\begin{aligned} \text{speed} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3\cos(t))^2 + (-3\sin(t))^2} \\ &= \sqrt{9 \cdot \cos^2(t) + 9 \cdot \sin^2(t)} = \sqrt{9} = 3 \quad \text{(a constant)} \end{aligned}$$

Practice 2.5.8

Is the object at $(x, y) = (3\cos(t), 2\sin(t))$ at time t traveling faster at the top of the ellipse (\)t = \frac{\pi}{2}\)) or at the right edge ($t = 0$)?

Answer

$x(t) = 3\cos(t) \Rightarrow \frac{dx}{dt} = -3\sin(t)$ and $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} = 2\cos(t)$ so:

$$\begin{aligned} \text{speed} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-3\sin(t))^2 + (2\cos(t))^2} \\ &= \sqrt{9 \cdot \sin^2(t) + 4 \cdot \cos^2(t)} \end{aligned}$$

When $t = 0$, the speed is $\sqrt{9 \cdot 0^2 + 4 \cdot 1^2} = 2$.

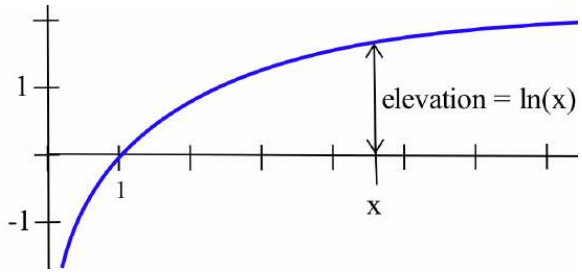
When $t = \frac{\pi}{2}$, the speed is $\sqrt{9 \cdot 1^2 + 4 \cdot 0^2} = 3$ (faster).

Problems

In Problems 1–27, differentiate the given function.

1. $\ln(5x)$

2. $\ln(x^2)$
3. $\ln(x^k)$
4. $\ln(x^x) = x \cdot \ln(x)$
5. $\ln(\cos(x))$
6. $\cos(\ln(x))$
7. $\log_2(5x)$
8. $\log_2(kx)$
9. $\ln(\sin(x))$
10. $\ln(kx)$
11. $\log_2(\sin(x))$
12. $\ln(e^x)$
13. $\log_5(5^x)$
14. $\ln\left(e^{f(x)}\right)$
15. $x \cdot \ln(3x)$
16. $e^x \cdot \ln(x)$
17. $\dfrac{\ln(x)}{x}$
18. $\sqrt{x + \ln(3x)}$
19. $\ln\left(\sqrt{5x-3}\right)$
20. $\ln(\cos(t))$
21. $\cos(\ln(w))$
22. $\ln(ax+b)$
23. $\ln\left(\sqrt{t+1}\right)$
24. 3^x
25. $5^{\sin(x)}$
26. $x \cdot \ln(x) - x$
27. $\ln(\sec(x) + \tan(x))$
28. Find the slope of the line tangent to $f(x) = \ln(x)$ at the point $(e, 1)$. Find the slope of the line tangent to $g(x) = e^x$ at the point $(1, e)$. How are the slopes of f and g at these points related?
29. Find a point P on the graph of $f(x) = \ln(x)$ so the tangent line to f at P goes through the origin.
30. You are moving from left to right along the graph of $y = \ln(x)$:

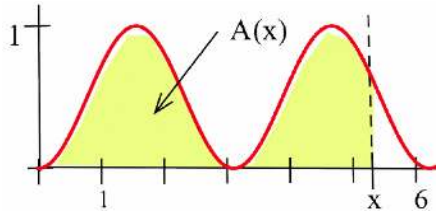


   a. If the x-coordinate of your location at time t seconds is $x(t) = 3t + 2$, then how fast is your elevation increasing?
   b. If the x-coordinate of your location at time t seconds is $x(t) = e^t$, then how fast is your elevation increasing?
31. The percent of a population, $p(t)$, who have heard a rumor by time t is often modeled by $p(t) = \dfrac{100}{1 + Ae^{-t}} = 100\,(1+Ae^{-t})^{-1}$ for some positive constant A. Calculate $p'(t)$, the rate at which the rumor is spreading.
32. If we start with A atoms of a radioactive material that has a "half-life" (the time it takes for half of the material to decay) of 500 years, then the number of radioactive atoms left after t years is $r(t) = A \cdot e^{-Kt}$ where $K = \dfrac{\ln(2)}{500}$. Calculate $r'(t)$ and show that $r'(t)$ is proportional to $r(t)$ (that is, $r'(t) = b \cdot r(t)$ for some constant b).

In Problems 33–41, find a function with the given derivative.

33. $f'(x) = \dfrac{8}{x}$

34. $h'(x) = \dfrac{3}{3x+5}$
35. $f'(x) = \dfrac{\cos(x)}{3+\sin(x)}$
36. $g'(x) = \dfrac{x}{1+x^2}$
37. $g'(x) = 3e^{5x}$
38. $h'(x) = e^2$
39. $f'(x) = 2x \cdot e^{x^2}$
40. \mbox{$g'(x) = \cos(x)e^{\sin(x)}$}
41. $h'(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$
42. Define $A(x)$ to be the **area** bounded between the t-axis, the graph of $y = f(t)$ and a vertical line at $t = x$:

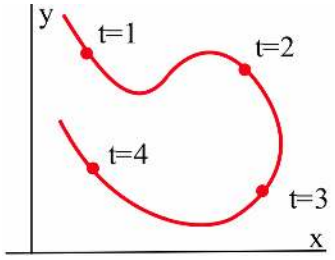


The area under each "hump" of f is 2 square inches.

a. Graph $A(x)$ for $0 \le x \le 9$.
b. Graph $A'(x)$ for $0 \le x \le 9$.

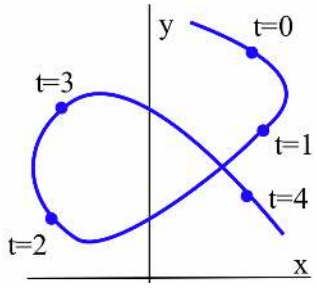
Problems 43--48 involve parametric equations.

43. At time t minutes, robot A is at $(t, 2t+1)$ and robot B is at $(t^2, 2t^2+1)$.
    a. Where is each robot when $t = 0$ and $t = 1$?
    b. Sketch the path each robot follows during the first minute.
    c. Find the slope of the tangent line, $\dfrac{dy}{dx}$, to the path of each robot at $t = 1$ minute.
    d. Find the speed of each robot at $t = 1$ minute.
    e. Discuss the motion of a robot that follows the path $(\sin(t), 2\sin(t)+1)$ for 20 minutes.
44. Let $x(t) = t+1$ and $y(t) = t^2$.
    a. Graph $(x(t), y(t))$ for $-1 \le t \le 4$.
    b. Find $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, the tangent slope $\dfrac{dy}{dx}$, and speed when $t = 1$ and $t = 4$.
45. For the parametric graph shown below:



determine whether $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$ and $\dfrac{dy}{dx}$ are positive, negative or 0 when $t = 1$ and $t = 3$.

46. For the parametric graph shown below:



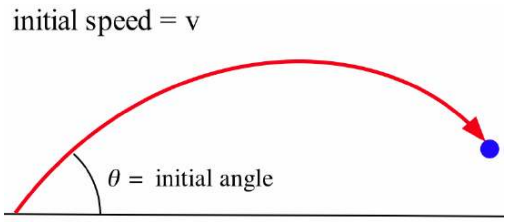
determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive, negative or 0 when $t = 1$ and $t = 3$.

47. The parametric graph $(x(t), y(t))$ defined by $x(t) = R \cdot (t - \sin(t))$ and $y(t) = R \cdot (1 - \cos(t))$ is called a **cycloid**, the path of a light attached to the edge of a rolling wheel with radius R.
    a. Graph $(x(t), y(t))$ for $0 \le t \le 4\pi$.
    b. Find $\frac{dx}{dt}$, $\frac{dy}{dt}$, the tangent slope $\frac{dy}{dx}$, and speed when $t = \frac{\pi}{2}$ and $t = \pi$.

48. Describe the motion of particles whose locations at time t are $(\cos(t), \sin(t))$ and $(\cos(t), -\sin(t))$.

49. a. Describe the path of a robot whose location at time t is $(3 \cdot \cos(t), 5 \cdot \sin(t))$.
    b. Describe the path of a robot whose location at time t is $(A \cdot \cos(t), B \cdot \sin(t))$.
    c. Give parametric equations so the robot will move along the same path as in part (a) but in the opposite direction.

50. After t seconds, a projectile hurled with initial velocity v and angle θ will be at $x(t) = v \cdot \cos(\theta) \cdot t$ feet and $y(t) = v \cdot \sin(\theta) \cdot t - 16t^2$ feet:



(This formula neglects air resistance.)
    a. For an initial velocity of 80 feet/second and an angle of $\frac{\pi}{4}$, find $T > 0$ so that $y(T) = 0$. What does this value for t represent physically? Evaluate $x(T)$.
    b. For v and θ in part (a), calculate $\frac{dy}{dx}$. Find T so that $\frac{dy}{dx} = 0$ at $t = T$, and evaluate $x(T)$. What does $x(T)$ represent physically?
    c. What initial velocity is needed so a ball hit at an angle of $\frac{\pi}{4} \approx 0.7854$ will go over a 40-foot-high fence 350 feet away?
    d. What initial velocity is needed so a ball hit at an angle of 0.7 radians will go over a 40-foot-high fence 350 feet away?

51. Use the method from the proof that $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to compute the derivative $\mathbf{D}(\arctan(x))$:
    a. Rewrite $y = \arctan(x)$ as $\tan(y) = x$.
    b. Differentiate both sides using the Chain Rule and solve for y'.
    c. Use the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ and the fact that $\tan(y) = x$ to show that $y' = \frac{1}{1 + x^2}$.

52. Use the method from the proof that $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to compute the derivative $\mathbf{D}(\arcsin(x))$:
    a. Rewrite $y = \arcsin(x)$ as $\sin(y) = x$.
    b. Differentiate both sides using the Chain Rule and solve for y'.
    c. Use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ and the fact that $\sin(y) = x$ to show that $y' = \frac{1}{\sqrt{1 - x^2}}$.

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2.6: Related Rates

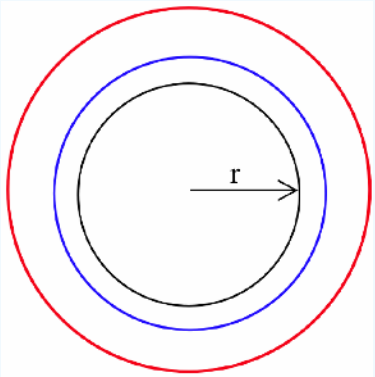
Throughout the next several sections we'll look at a variety of applications of derivatives. Probably no single application will be of interest or use to everyone, but at least some of them should be useful to you. Applications also reinforce what you have been practicing: they require that you recall what a derivative means and require you to use the differentiation techniques covered in the last several sections. Most people gain a deeper understanding and appreciation of a tool as they use it, and differentiation is both a powerful concept and a useful tool.

The Derivative as a Rate of Change

In Section 2.1, we discussed several interpretations of the derivative of a function. Here we will examine the "rate of change of a function" interpretation. If several variables or quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other variables must be changing.

Example 2.6.1

The radius of a circle is increasing at a rate of 10 feet each second:



and we want to know how fast the **area** of the circle is increasing when the radius is 5 feet. What can we do?

Solution

We could get an *approximate* answer by calculating the area of the circle when the radius is 5 feet:

$$A = \pi r^2 = \pi(5 \text{ feet})^2 \approx 78.6 \text{ feet}^2 \tag{2.6.1}$$

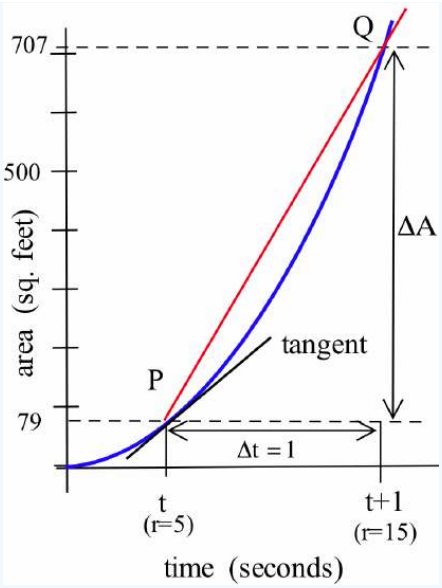
and the area 1 second later when the radius is 10 feet larger than before:

$$A = \pi r^2 = \pi(15 \text{ feet})^2 \approx 706.9 \text{ feet}^2 \tag{2.6.2}$$

and then computing:

$$\frac{\Delta \text{area}}{\Delta \text{time}} = \frac{706.9 \text{ feet}^2 - 78.6 \text{ feet}^2}{1 \text{ second}} = 628.3 \frac{\text{ft}^2}{\text{sec}} \tag{2.6.3}$$

This approximate answer represents the average change in area during the 1-second period when the radius increased from 5 feet to 15 feet. It is also the slope of the secant line through the points P and Q in the figure below:

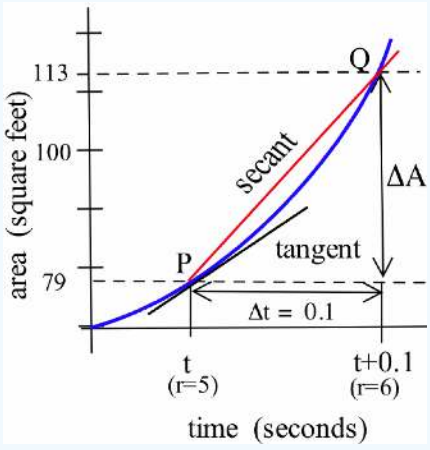


and it is clearly not a very good approximation of the instantaneous rate of change of the area, the slope of the tangent line at the point P.

We could get a *better approximation* by calculating $\frac{\Delta A}{\Delta t}$ over a shorter time interval, say $\Delta t = 0.1$ seconds. In this scenario, the original area is still 78.6 ft^2 but the new area (after $t = 0.1$ seconds has passed) is $A = \pi(6 \text{ feet})^2 \approx 113.1$ ft^2 (why is the new radius 6 feet?) so:

$$\frac{\Delta A}{\Delta t} = \frac{113.1 \text{ feet}^2 - 78.6 \text{ feet}^2}{0.1 \text{ second}} = 345 \frac{\text{ft}^2}{\text{sec}} \tag{2.6.4}$$

This is the slope of the secant line through the points P and Q in the figure below:



which represents a much better approximation of the slope of the tangent line at P—but it is still only an approximation. Using derivatives, we can get an **exact** answer without doing very much work at all.

We know that the two variables in this problem, the radius r and the area A, are related to each other by the formula $A = \pi r^2$. We also know that both r and A are changing over time, so each of them is a function of an additional variable t (time, in seconds): $r(t)$ and $A(t)$.

We want to know the rate of change of the area "when the radius is 5 feet" so if $t = 0$ corresponds to the particular moment in time when the radius is 5 feet, we can write $r(0) = 5$.

The statement that "the radius is increasing at a rate of 10 feet each second" can be translated into a mathematical statement about the rate of change, the derivative of r (radius) with respect to t (time): if $t = 0$ corresponds to the moment when the

radius is 5 feet, then $r'(0) = \frac{dr}{dt} = 10$ ft/sec.

The question about the rate of change of the area is a question about $A'(t) = \frac{dA}{dt}$.

Collecting all of this information…

- **variables**: $r(t)$ = radius at time t, $A(t)$ = area at time t
- **we know**: $r(0) = 5$ feet and $r'(0) = 10$ ft/sec
- **we want to know**: $A'(0)$, the rate of change of area with respect to time at the moment when $r = 5$ feet
- **connecting equation**: $A = \pi r^2$ or $A(t) = \pi [r(t)]^2$

To find $A'(0)$ we must first find $A'(t)$ and then evaluate this derivative at $t = 0$. Differentiating both sides of the connecting equation, we get:

$$A(t) = \pi [r(t)]^2 \Rightarrow A'(t) = 2\pi [r(t)]^1 \cdot r'(t) \Rightarrow A'(t) = 2\pi \cdot r(t) \cdot r'(t) \tag{2.6.5}$$

(Notice that we have used the Power Rule for Functions—or, more generally, the Chain Rule—because the area is a function of the radius, which is a function of time.)

Now we can plug in $t = 0$ and use the information we know:

$$A'(0) = 2\pi \cdot r(0) \cdot r'(0) = 2\pi \cdot 5 \cdot 10 = 100\pi \tag{2.6.6}$$

When the radius is 5 feet, the area is increasing at 100π ft^2/sec ≈ 314.2 square feet per second.

Before considering other examples, let's review the solution to the previous example. The statement "the radius is increasing at a rate of 10 feet each second" implies that this rate of change is the same at $t = 0$ (the moment in time we were interested in) as at any other time during this process, say $t = 1.5$ or $t = 98$: $r'(0) = r'(1.5) = r'(98) = 10$. But we only used the fact that $r'(0) = 10$ in our solution. (We should take care in future problems to consider whether the information we are given about rates of change holds true all the time or just at a particular moment in time. That didn't matter in our first example, but it might in other situations.)

Next, notice that we let $t = 0$ correspond to the particular moment in time the question asked about (the moment when $r = 5$). But this choice was arbitrary: we could have let this moment correspond to $t = 2.8$ or $t = 7\pi$ and the eventual answer would have been the same.

Finally, notice that we explicitly wrote each variable (and their derivatives) as a function of the time variable, t: $A(t)$, $r(t)$, $A'(t)$ and $r'(t)$. Consequently, we used the composition form of the Chain Rule:

$$(A \circ r)'(t) = A'(r(t)) \cdot r'(t) \tag{2.6.7}$$

Let's redo the previous example using the Leibniz form of the Chain Rule, keeping the above observations in mind.

✔ Example 2.6.1

Solution

We know that the two variables in this problem, the radius r and the area A, are related to each other by the formula $A = \pi r^2$. We also know that both r and A are changing over time, so each of them is a function of an additional variable t (time, in seconds).

We want to know the rate of change of the area "when the radius is 5 feet," which translates to evaluating $\frac{dA}{dt}$ at the moment when $r = 5$. We write this in Leibniz notation as:

$$\left.\frac{dA}{dt}\right|_{r=5} \tag{2.6.8}$$

The statement that "the radius is increasing at a rate of 10 feet each second" translates into $\frac{dr}{dt} = 10$. From the connecting equation $A = \pi r^2$ we know that $\frac{dA}{dr} = 2\pi r$. Furthermore,the Chain Rule tells us that:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} \tag{2.6.9}$$

We know that $\frac{dA}{dr} = 2\pi r$ and $\frac{dr}{dt} = 10$ are *always* true, so we can rewrite the Chain Rule statement above as:

$$\frac{dA}{dt} = 2\pi r \cdot 10 = 20\pi r \tag{2.6.10}$$

Finally, we evaluate both sides at the moment in time we are interested in (the moment when $r = 5$):

$$\left.\frac{dA}{dt}\right|_{r=5} = \left.20\pi r\right|_{r=5} = 20\pi \cdot 5 = 100\pi \approx 314.2 \tag{2.6.11}$$

which is the same answer we found in the original solution.

The key steps in finding the rate of change of the area of the circle were:

- write the known information in a mathematical form, expressing rates of change as derivatives: $\frac{dr}{dt} = 10$ ft/sec
- write the question in a mathematical form: $\frac{dA}{dt} = ?$
- find an equation connecting or relating the variables: $A = \pi r^2$
- differentiate both sides of the connecting equation using the Chain Rule (and other differentiation patterns as necessary): $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$
- substitute all of the known values that are *always* true into the equation resulting from the previous step and (if necessary) solve for the desired quantity in the resulting equation: $\frac{dA}{dt} = 2\pi r \cdot 10$
- substitute all of the known values that are true at the particular moment in time the question asks about into the equation resulting from the previous step: $\left.\frac{dA}{dt}\right|_{r=5} = \left.2\pi r \cdot 10\right|_{r=5} = 100\pi$

Example 2.6.2

Divers' lives depend on understanding situations involving related rates. In water, the pressure at a depth of x feet is approximately $P(x) = 15\left(1 + \frac{x}{33}\right)$ pounds per square inch (compared to approximately $P(0) = 15$ pounds per square inch at sea level). Volume is inversely proportional to the pressure, $V = \frac{k}{P}$, so doubling the pressure will result in half the original volume. Remember that volume is a function of the pressure: $V = V(P)$.

a. Suppose a diver's lungs, at a depth of 66 feet, contained 1 cubic foot of air and the diver ascended to the surface without releasing any air. What would happen?
b. If a diver started at a depth of 66 feet and ascended at a rate of 2 feet per second, how fast would the pressure be changing?

(Dives deeper than 50 feet also involve a risk of the "bends," or decompression sickness, if the ascent is too rapid. Tables are available that show the safe rates of ascent from different depths.)

Solution

a. The diver would risk rupturing his or her lungs. The 1 cubic foot of air at a depth of 66 feet would be at a pressure of $P(66) = 15\left(1 + \frac{66}{33}\right) = 45$ pounds per square inch (psi). Because the pressure at sea level, $P(0) = 15$ psi, is only $\frac{1}{3}$ as great, each cubic foot of air would expand to 3 cubic feet, and the diver's lungs would be in danger. Divers are taught to release air as they ascend to avoid this danger.

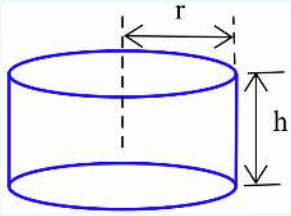
b. The diver is ascending at a rate of 2 feet/second so the rate of change of the diver's depth with respect to time is $\frac{dx}{dt} = -2$ ft/s. (Why is this rate of change negative?) The pressure is $P = 15\left(1 + \frac{x}{33}\right) = 15 + \frac{15}{33}x$, a function of x, so using the Chain Rule:

$$\frac{dP}{dt} = \frac{dP}{dx} \cdot \frac{dx}{dt} = \frac{15}{33}\frac{\text{psi}}{\text{ft}} \cdot \left(-2\frac{\text{ft}}{\text{sec}}\right) = -\frac{30}{33}\frac{\text{psi}}{\text{sec}} \approx -0.91\frac{\text{psi}}{\text{sec}} \tag{2.6.12}$$

The rates of change in this problem are constant (they hold true at any moment in time during the ascent) so we are done.

Example 2.6.3

The height of a cylinder is increasing at 7 meters per second and the radius is increasing at 3 meters per second. How fast is the volume changing when the cylinder is 5 meters high and has a radius of 6 meters?



Solution

First we need to translate our known information into a mathematical format. The height and radius are given: at the particular moment in time the question asks about, $h =$ height $= 5$ m and $r =$ radius $= 6$ m. We are also told how fast h and r are changing at this moment in time: $\frac{dh}{dt} = 7$ m/sec and $\frac{dr}{dt} = 3$ m/sec. Finally, we are asked to find $\frac{dV}{dt}$, and we should expect the units of $\frac{dV}{dt}$ to be the same as $\frac{\Delta V}{\Delta t}$, which are m^3/sec.

- **variables**: $h(t) =$ height at time t seconds, $r(t) =$ radius at time t, $V(t) =$ volume at time t.
- **we know**: at a particular moment in time, $h = 5$ m, $\frac{dh}{dt} = 7$ m/sec, $r = 6$ m and $\frac{dr}{dt} = 3$ m/sec
- **we want to know**: $\frac{dV}{dt}$ at this particular moment in time

We also need an equation that relates the variables h, r and V (all of which are functions of time t) to each other:

- **connecting equation**: $V = \pi r^2 h$

Differentiating each side of this equation with respect to t (remembering that h, r and V are functions of t), we have:

$$\begin{aligned}\frac{dV}{dt} = \frac{d}{dt}\left(\pi r^2 h\right) &= \pi r^2 \cdot \frac{dh}{dt} + h \cdot \frac{d}{dt}\left(\pi r^2\right) \\ &= \pi r^2 \cdot \frac{dh}{dt} + h \cdot 2\pi r \cdot \frac{dr}{dt}\end{aligned}$$

using the Product Rule (on the product $\pi r^2 \cdot h$) and the Power Rule for Functions (on πr^2, remembering that r is actually a function of t).

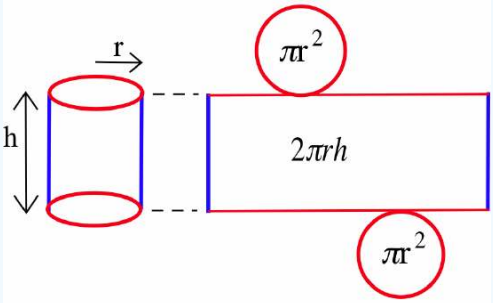
The rest of the solution just involves substituting values and doing some arithmetic. At the particular moment in time we're interested in:

$$\begin{aligned}\frac{dV}{dt} &= \pi \cdot 6^2\,\text{m}^2 \cdot 7\,\frac{\text{m}}{\text{sec}} + 5\,\text{m} \cdot 2\pi \cdot 6\,\text{m} \cdot 3\,\frac{\text{m}}{\text{sec}} \\ &= 432\pi\,\frac{\text{m}^3}{\text{sec}} \approx 1357.2\,\frac{\text{m}^3}{\text{sec}}\end{aligned}$$

The volume of the cylinder is increasing at a rate of 1,357.2 cubic meters per second. (It is always encouraging when the units of our answer are the ones we expect.)

Practice 2.6.1

How fast is the **surface area** of the cylinder changing in Example 2.6.3?



Assume that h, r, $\frac{dh}{dt}$ and $\frac{dr}{dt}$ have the same values as in the example and use the figure above to help you determine an equation relating the surface area of the cylinder to the variables h and r. The cylinder includes a top and bottom.

Answer

The surface area is $S = 2\pi rh + 2\pi r^2$. From the Example, we know that $\frac{dh}{dt} = 7$ m/sec and $\frac{dr}{dt} = 3$ m/sec, and we want to know how fast the surface area is changing
when $h = 5$ m and $r = 6$ m.

$$\begin{aligned}\frac{dS}{dt} &= 2\pi r \cdot \frac{dh}{dt} + 2\pi \frac{dr}{dt} \cdot h + 2\pi \cdot 2r \cdot \frac{dr}{dt} \\ &= 2\pi(6\,\text{m})\left(7\,\frac{\text{m}}{\text{sec}}\right) + 2\pi\left(3\frac{\text{m}}{\text{sec}}\right)(5m) + 2\pi\,(2 \cdot 6\,\text{m})\left(3\,\frac{\text{m}}{\text{sec}}\right) \\ &= 186\pi\,\frac{\text{m}^2}{\text{sec}} \approx 584.34\,\frac{\text{m}^2}{\text{sec}}\end{aligned}$$

(Note that the units represent a rate of change of area.)

? Practice 2.6.2

How fast is the **volume** of the cylinder in Practice **2.6.1** changing if the radius is decreasing at a rate of 3 meters per second? (The height, radius and rate of change of the height are the same as in Example **2.6.3**: 5 m, 6 m and 7 m/sec respectively.)

Answer

The volume is $V =$. We know that $\frac{dr}{dt} = -3$ m/sec and that $h = 5$ m, $r = 6$ m and $\frac{dh}{dt} = 7$ m/sec.

$$\begin{aligned}\frac{dV}{dt} &= \pi r^2 \cdot \frac{dh}{dt} + \pi \cdot 2r \cdot \frac{dr}{dt} \\ &= \pi(6\,\text{m})^2\left(7\,\frac{\text{m}}{\text{sec}}\right) + \pi(2 \cdot 6\,\text{m})\left(-3\,\frac{\text{m}}{\text{sec}}\right) \\ &= 72\pi\,\frac{\text{m}^3}{\text{sec}} \approx 226.19\,\frac{\text{m}^3}{\text{sec}}\end{aligned}$$

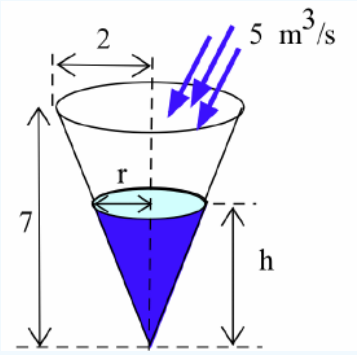
(Note that the units represent a rate of change of volume.)

Usually, the most difficult part of Related Rates problems is to find an equation that relates or connects all of the variables. In the previous problems, the relating equations required a knowledge of geometry and formulas for areas and volumes (or knowing where to look them up). Other Related Rates problems may require information about similar triangles, the Pythagorean Theorem or trigonometric identities: the information required varies from problem to problem.

It is a good idea—a very good idea—to draw a picture of the physical situation whenever possible. It is also a good idea, particularly if the problem is very important (your next raise depends on getting the right answer), to calculate at least one **approximate** answer as a check of your exact answer.

Example 2.6.4

Water is flowing into a conical tank at a rate of 5 m^3/sec. If the radius of the top of the cone is 2 m, the height is 7 m, and the depth of the water is 4 m, then how fast is the water level rising?



Solution

Let's define our variables to be $h =$ height (or depth) of the water in the cone and $V =$ the volume of the water in the cone. Both h and V are changing, and both of them are functions of time t. We are told in the problem that $h = 4$ m and $\frac{dV}{dt} = 5$ m^3/sec, and we are asked to find $\frac{dh}{dt}$. We expect that the units of $\frac{dh}{dt}$ will be the same as $\frac{\Delta h}{\Delta t}$, which are meters/second.

- **variables**: $h(t) =$ height at time t seconds, $r(t) =$ radius of the top surface of the water at time t, $V(t)$ = volume of water at time t
- **we know**: $\frac{dV}{dt} = 5$ m^3/sec (always true) and $h = 4$ m (at a particular moment)
- **we want to know**: $\frac{dh}{dt}$ at this particular moment

Unfortunately, the equation for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, also involves an additional variable r, the radius of the cone at the top of the water. This is a situation in which a picture can be a great help by suggesting that we have a pair of similar triangles:

$$\frac{r}{h} = \frac{\text{top radius}}{\text{total height}} = \frac{2\text{ m}}{7\text{ m}} = \frac{2}{7} \quad \Rightarrow \quad r = \frac{2}{7}h \tag{2.6.13}$$

Knowing this, we can rewrite the volume of the water contained in the cone, $V = \frac{1}{3}\pi r^2 h$, as a function of the single variable h:

- **connecting equation**: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{7}h\right)^2 h = \frac{4}{147}\pi h^3$

The rest of the solution is reasonably straightforward.

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \frac{d}{dh}\left(\frac{4}{147}\pi h^3\right)\cdot\frac{dh}{dt} \tag{2.6.14}$$

We know $\frac{dV}{dt} = 5$ always holds, and the derivative is easy to compute:

$$5 = \frac{4}{49}\pi h^2 \cdot \frac{dh}{dt} \tag{2.6.15}$$

At the particular moment in time we want to know about (when $h = 4$):

$$5 = \frac{4}{49}\pi h^2\Big|_{h=4}\cdot\frac{dh}{dt}\Big|_{h=4} \quad \Rightarrow \quad 5 = \frac{64\pi}{49}\cdot\frac{dh}{dt}\Big|_{h=4} \tag{2.6.16}$$

and we can now solve for the quantity of interest:

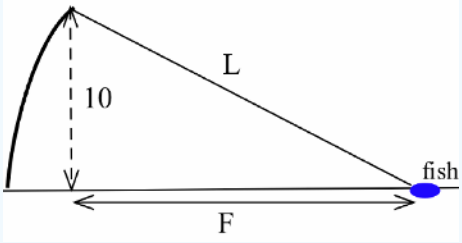
$$\frac{dh}{dt}\Big|_{h=4} = \frac{5}{\frac{64\pi}{49}} = \frac{245}{64\pi} \approx 1.22\,\frac{\text{m}}{\text{sec}} \tag{2.6.17}$$

This example was a bit more challenging because we needed to use similar triangles to get an equation relating V to h and because we eventually needed to do some arithmetic to solve for $\frac{dh}{dt}$.

Practice 2.6.3

A rainbow trout has taken the fly at the end of a 60-foot fishing line, and the line is being reeled in at a rate of 30 feet per minute. If the tip of the rod is 10 feet above the water and the trout is at the surface of the water, how fast is the trout being pulled toward the angler? (Hint: Draw a picture and use the Pythagorean Theorem.)

Answer



We know $\frac{dL}{dt} = -30\ \frac{\text{ft}}{\text{min}}$ (always true); F represents the distance from the fish to a point directly below the tip of the rod, and the distance from that point to the angler remains constant, so $\frac{dF}{dt}$ will equal the rate at which the fish is moving toward the angler. We want to know $\frac{dF}{dt}\big|_{L=60}$. The Pythagorean Theorem connects F and L: $F^2 + 10^2 = L^2$. Differentiating with respect to t and using the Power Rule for Functions:

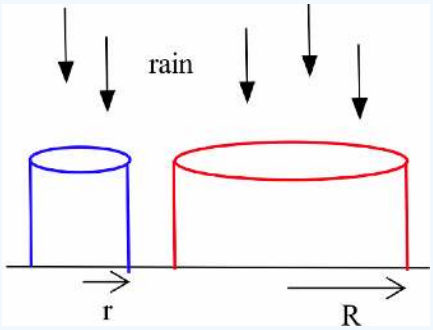
$$2F \cdot \frac{dF}{dt} + 0 = 2L \cdot \frac{dL}{dt} \quad \Rightarrow \quad \frac{dF}{dt} = \frac{L}{F} \cdot \frac{dL}{dt} \tag{2.6.18}$$

At a particular moment in time, $L = 60 \Rightarrow F^2 + 10^2 = 60^2 \Rightarrow F = \sqrt{3600 - 100} = \sqrt{3500} = 10\sqrt{35}$ so:

$$\frac{dF}{dt}\Big|_{L=60} = -30 \cdot \frac{60}{10\sqrt{35}} = -\frac{180}{\sqrt{35}} \approx -30.43\ \frac{\text{ft}}{\text{min}} \tag{2.6.19}$$

Example 2.6.5

When rain is falling vertically, the amount (volume) of rain collected in a cylinder is proportional to the area of the **opening** of the cylinder.



If you place a narrow cylindrical glass and a wide cylindrical glass out in the rain:

a. which glass will collect water faster?
b. in which glass will the water level rise faster?

Solution

Let's assume that the smaller glass has a radius of r and the larger glass has a radius of R, so that $R > r$. The areas of their openings are πr^2 and πR^2, respectively. Call the volume of water collected in each glass v (for the smaller glass) and V (for the larger glass).

FIGURE

a. The smaller glass will collect water at the rate $\frac{dv}{dt} = K \cdot \pi r^2$ and the larger at the rate $\frac{dV}{dt} = K \cdot \pi R^2$ so $\frac{dV}{dt} > \frac{dv}{dt}$ and the larger glass will collect water faster than the smaller glass.

b. The volume of water in each glass is a function of the radius of the glass and the height of the water in the glass: $v = \pi r^2 h$ and $V = \pi R^2 H$ where h and H are the heights of the water levels in the smaller and larger glasses, respectively. The heights h and H vary with t (in other words, they are each functions of t) while the radii (r and R) remain constant, so:

$$\frac{dv}{dt} = \frac{d}{dt}\left(\pi r^2 h\right) = \pi r^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{\frac{dv}{dt}}{\pi r^2} = \frac{K\pi r^2}{\pi r^2} = K \tag{2.6.20}$$

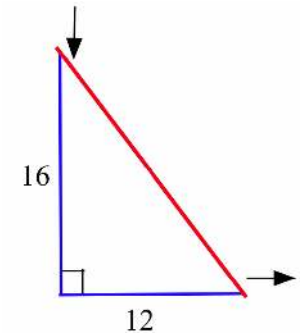
Similarly:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\pi R^2 H\right) = \pi R^2 \frac{dH}{dt} \Rightarrow \frac{dH}{dt} = \frac{\frac{dV}{dt}}{\pi R^2} = \frac{K\pi R^2}{\pi R^2} = K \tag{2.6.21}$$

So $\frac{dh}{dt} = K = \frac{dH}{dt}$, which tells us the water level in each glass is rising at the same rate. In a one-minute period, the larger glass will collect more rain, but the larger glass also requires more rain to raise its water level by a fixed amount. How do you think the volumes and water levels would change if we placed a small glass and a large plastic (rectangular) box side by side in the rain?

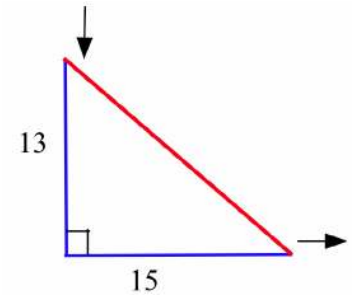
Problems

1. An expandable sphere is being filled with liquid at a constant rate from a tap (imagine a water balloon connected to a faucet). When the radius of the sphere is 3 inches, the radius is increasing at 2 inches per minute. How fast is the liquid coming out of the tap? ($V = \frac{4}{3}\pi r^3$)
2. The 12-inch base of a right triangle is growing at 3 inches per hour, and the 16-inch height of the triangle is shrinking at 3 inches per hour:



   a. Is the area increasing or decreasing?
   b. Is the perimeter increasing or decreasing?
   c. Is the hypotenuse increasing or decreasing?

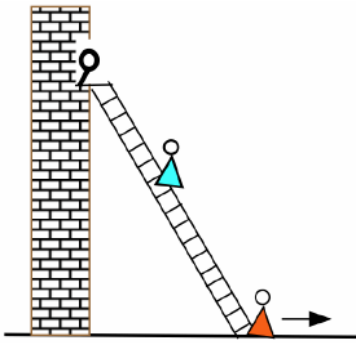
3. One hour later the right triangle in the previous problem is 15 inches long and 13 inches high:



   and the base and height are changing at the same rate as in Problem 2.

   a. Is the area increasing or decreasing now?
   b. Is the hypotenuse increasing or decreasing?
   c. Is the perimeter increasing or decreasing?

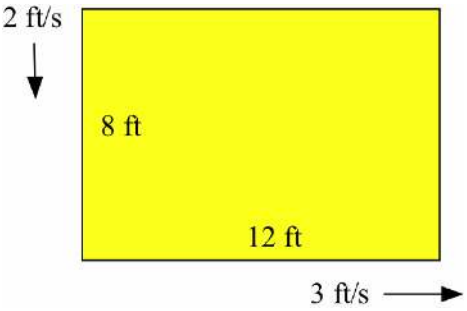
4. A young woman and her boyfriend plan to elope, but she must rescue him from his mother, who has locked him in his room. The young woman has placed a 20-foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second:



How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is:

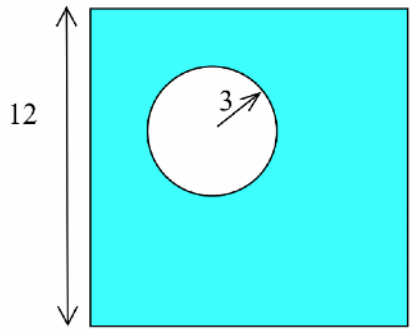
   a. 12 feet from the bottom of the wall?
   b. 16 feet from the bottom of the wall?
   c. 19 feet from the bottom of the wall?

5. The length of a 12-foot by 8-foot rectangle is increasing at a rate of 3 feet per second and the width is decreasing at 2 feet per second:



   a. How fast is the perimeter changing?
   b. How fast is the area changing?

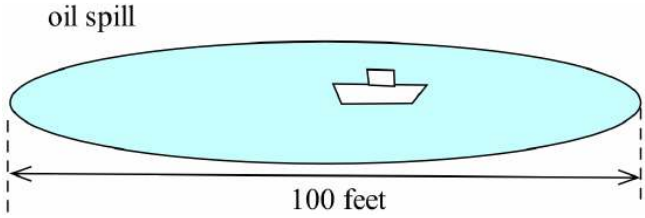
6. A circle of radius 3 inches is inside a square with 12-inch sides:



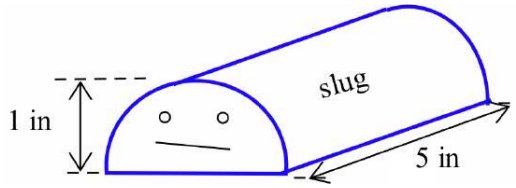
How fast is the area between the circle and square changing if the radius is increasing at 4 inches per minute and the sides are increasing at 2 inches per minute?

7. An oil tanker in Puget Sound has sprung a leak, and a circular oil slick is forming. The oil slick is 4 inches thick everywhere, is 100 feet in diameter, and the diameter is increasing at 12 feet per hour. Your job, as the Coast Guard commander or the tanker's

captain, is to determine how fast the oil is leaking from the tanker.



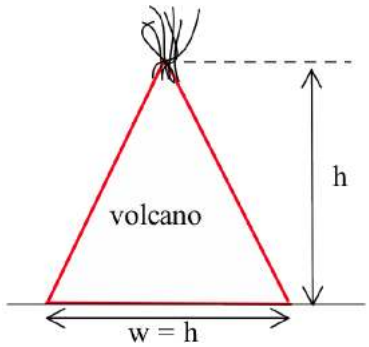
8. A mathematical species of slug has a semicircular cross section and is always 5 times as long as it is high:



When the slug is 5 inches long, it is growing at 0.2 inches per week.

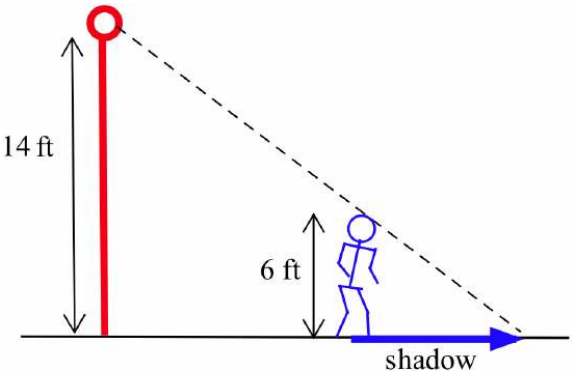
a. How fast is its volume increasing?
b. How fast is the area of its “foot” (the part of the slug in contact with the ground) increasing?

9. Lava flowing from a hole at the top of a hill is forming a conical mountain whose height is always the same as the width of its base:



If the mountain is increasing in height at 2 feet per hour when it is 500 feet high, how fast is the lava flowing (that is, how fast is the volume of the mountain increasing)? ($V = \frac{1}{3}\pi r^2 h$)

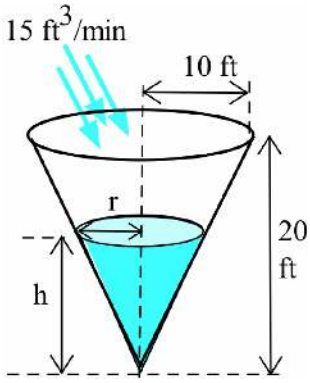
10. A 6-foot-tall person is walking away from a 14-foot lamp post at 3 feet per second.



When the person is 10 feet away from the lamp post:

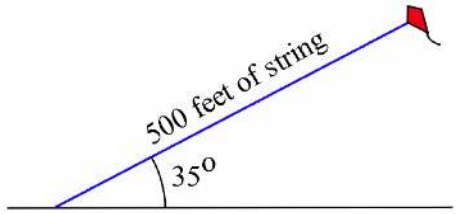
a. how fast is the length of the shadow changing?
b. how fast is the tip of the shadow moving away from the lamp post?

11. Redo the previous problem if the person is 20 feet from the lamp post.
12. Water is being poured at a rate of 15 cubic feet per minute into a conical reservoir that is 20 feet deep and has a top radius of 10 feet:



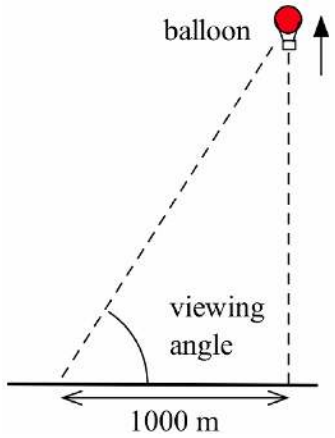
a. How long will it take to fill the empty reservoir?
b. How fast is the water level rising when the water is 4 feet deep?
c. How fast is the water level rising when the water is 16 feet deep?

13. The string of a kite is perfectly taut and always makes an angle of $35°$ above horizontal.



a. If the kite flyer has let out 500 feet of string, how high is the kite?
b. If the string is let out at a rate of 10 feet per second, how fast is the kite's height increasing?

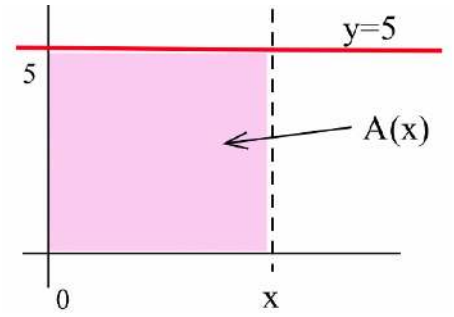
14. A small tracking telescope is viewing a hot-air balloon rise from a point 1,000 meters away from a point directly under the balloon.



a. When the viewing angle is $20°$, it is increasing at a rate of $3°$ per minute. How high is the balloon, and how fast is it rising?
b. When the viewing angle is $80°$, it is increasing at a rate of $2°$ per minute. How high is the balloon, and how fast is it rising?

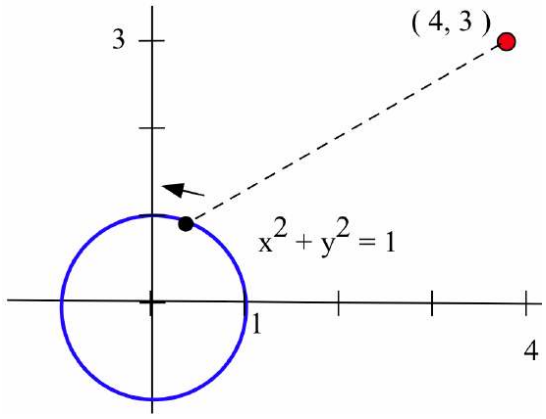
15. The 8-foot diameter of a spherical gas bubble is increasing at 2 feet per hour, and the 12-foot-long edges of a cube containing the bubble are increasing at 3 feet per hour. Is the volume contained between the spherical bubble and the cube increasing or decreasing? At what rate?
16. In general, the strength S of an animal is proportional to the cross-sectional area of its muscles, and this area is proportional to the square of its height H, so the strength $S = aH^2$. Similarly, the weight W of the animal is proportional to the cube of its height, so $W = bH^3$. Finally, the relative strength R of an animal is the ratio of its strength to its weight. As the animal grows, show that its strength and weight increase, but that the relative strength decreases.
17. The snow in a hemispherical pile melts at a rate proportional to its exposed surface area (the surface area of the hemisphere). Show that the height of the snow pile is decreasing at a constant rate.

18. If the rate at which water vapor condenses onto a spherical raindrop is proportional to the surface area of the raindrop, show that the radius of the raindrop will increase at a constant rate.
19. Define $A(x)$ to be the area bounded by the t- and y-axes, and the lines $y = 5$ and $t = x$.



a. Find a formula for A as a function of x.
b. Determine $A'(x)$ when $x = 1$, 2, 4 and 9.
c. If x is a function of time, $x(t) = t^2$, find a formula for A as a function of t.
d. Determine $A'(t)$ when $t = 1$, 2 and 3.
e. Suppose instead $x(t) = 2 + \sin(t)$. Find a formula for $A(t)$ and determine $A'(t)$.

20. The point P is going around the circle $x^2 + y^2 = 1$ twice a minute.

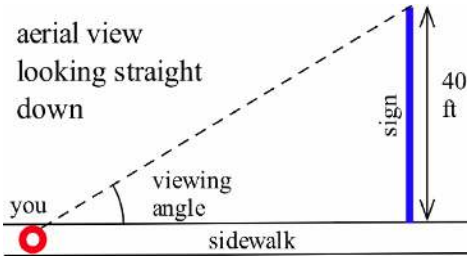


How fast is the distance between the point P and the point $(4, 3)$ changing:

a. when $P = (1, 0)$?
b. when $P = (0, 1)$?
c. when $P = (0.8, 0.6)$?

(Suggestion: Write x and y as parametric functions of time t.)

21. You are walking along a sidewalk toward a 40-foot-wide sign adjacent to the sidewalk and perpendicular to it.



If your viewing angle θ is $10°$:

a. how far are you from the corner of the sign?
b. how fast is your viewing angle changing if you are walking at 25 feet per minute?
c. how fast are you walking if the angle is increasing at $2°$ per minute?

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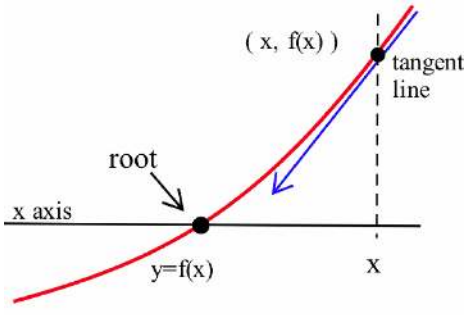
2.7: Newton's Method

Newton's method is a process that can find roots of functions whose graphs cross or just "kiss" the x-axis. Although this method is a bit harder to apply than the Bisection Algorithm, it often finds roots that the Bisection Algorithm misses, and it usually finds them faster.

Off on a Tangent

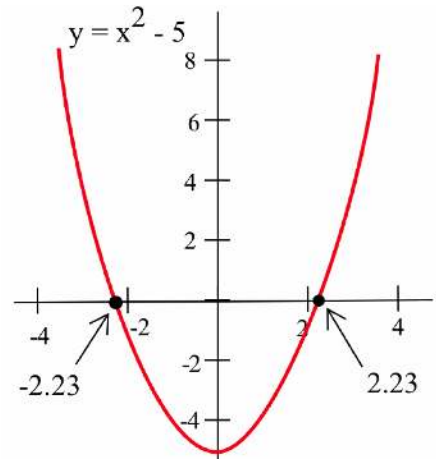
The basic idea of Newton's Method is remarkably simple and graphical: at a point $(x, f(x))$ on the graph of f, the tangent line to the graph "points toward" a root of f, a place where the graph touches the x-axis.

To find a root of f, we just pick a starting value x_0, go to the point $(x_0, f(x_0))$ on the graph of f, build a tangent line there, and follow the tangent line to where it crosses the x-axis, say at x_1.



If x_1 is a root of f, we are done. If x_1 is not a root of f, then x_1 is usually closer to the root than x_0 was, and we can repeat the process, using x_1 as our new starting point. Newton's method is an **iterative** procedure—that is, the output from one application of the method becomes the starting point for the next application.

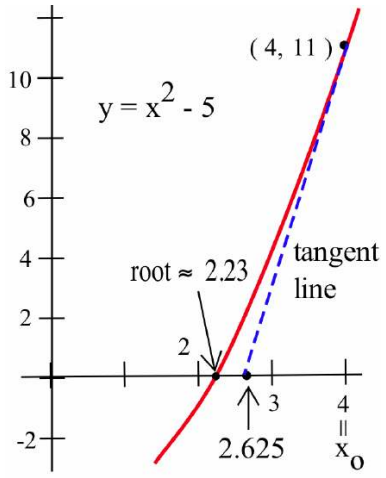
Let's begin with the function $f(x) = x^2 - 5$, whose roots we already know ($x = \pm\sqrt{5} \approx \pm 2.236067977$), to illustrate Newton's method.



First, pick some value for x_0, say $x_0 = 4$, and move to the point $(x_0, f(x_0)) = (4, 11)$ on the graph of f. The tangent line to the graph of f at $(4, 11)$ "points to" a location on the x-axis that is closer to the root of f than the point we started with. We calculate this location on the x-axis by finding an equation of the line tangent to the graph of f at $(4, 11)$ and then finding where this line intersects the x-axis.

At $(4, 11)$, the line tangent to f has slope $f'(4) = 2(4) = 8$, so an equation of the tangent line is $y - 11 = 8(x - 4)$. Setting $y = 0$, we can find where this line crosses the x-axis:

$$0 - 11 = 8(x - 4) \Rightarrow x = 4 - \frac{11}{8} = \frac{21}{8} = 2.625$$



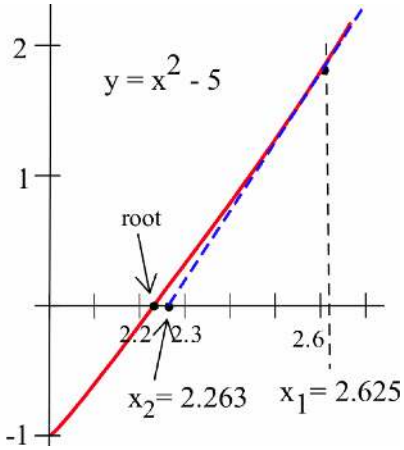
Call this new value x_1: The point $x_1 = 2.625$ is closer to the actual root $\sqrt{5}$, but it certainly does not equal the actual root. So we can use this new x-value, $x_1 = 2.625$, to repeat the procedure:

- move to the point $(x_1, f(x_1)) = (2.625, 1.890625)$
- find an equation of the tangent line at $(x_1, f(x_1))$:

$$y - 1.890625 = 5.25(x - 2.625)$$

- find x_2, the x-value where this new line intersects the x-axis:

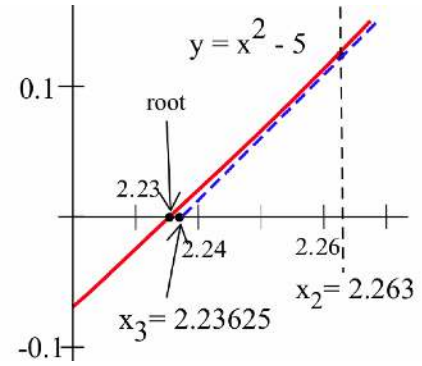
$$y - 1.890625 = 5.25(x - 2.625) \Rightarrow 0 - 1.890625 = 5.25(x_2 - 2.625)$$
$$\Rightarrow x_2 = 2.264880952$$



Repeating this process, each new estimate for the root of $f(x) = x^2 - 5$ becomes the starting point to calculate the next estimate. We get:

Iteration and Convergence	
$x_0 = 4$	(0 correct digits)
$x_1 = 2.625$	(1 correct digit)
$x_2 = 2.262880952$	(2 correct digits)
$x_3 = 2.236251252$	(4 correct digits)
$x_4 = 2.236067985$	(8 correct digits)

It only took 4 iterations to get an approximation within 0.000000008 of the exact value of $\sqrt{5}$.



One more iteration gives an approximation x_5 that has 16 correct digits. If we start with $x_0 = -2$ (or any negative number), then the values of x_n approach $-\sqrt{5} \approx -2.23606$.

? Practice 2.7.1

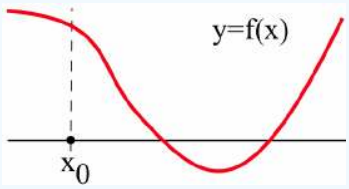
Find where the tangent line to $f(x) = x^3 + 3x - 1$ at $(1, 3)$ intersects the x-axis.

Answer

$f'(x) = 3x^2 + 3$, so the slope of the tangent line at $(1, 3)$ is $f'(1) = 6$ and an equation of the tangent line is $y - 3 = 6(x - 1)$ or $y = 6x - 3$. The y-coordinate of a point on the x-axis is 0 so putting $y = 0$ in this equation: $0 = 6x - 3 \Rightarrow x = \frac{1}{2}$. The line tangent to the graph of $f(x) = x^3 + 3x + 1$ at the point $(1, 3)$ intersects the x-axis at the point $(\frac{1}{2}, 0)$.

? Practice 2.7.2

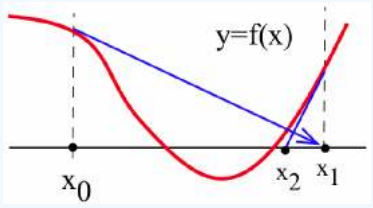
A starting point and a graph of f appear below:



Label the approximate locations of the next two points on the x-axis that will be found by Newton's method.

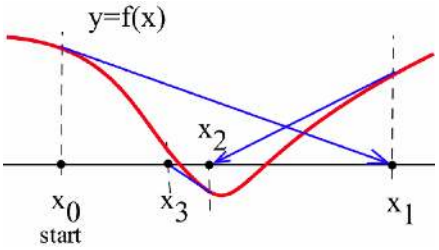
Answer

The approximate locations of x_1 and x_2 appear in the figure below:



The Algorithm for Newton's Method

Rather than deal with each particular function and starting point, let's find a pattern for a general function f.



For the starting point x_0, the slope of the tangent line at the point $(x_0, f(x_0))$ is $f'(x_0)$ so the equation of the tangent line is $y - f(x_0) = f'(x_0) \cdot (x - x_0)$. This line intersects the x-axis at a point $(x_1, 0)$, so:

$$0 - f(x_0) = f'(x_0) \cdot (x_1 - x_0) \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Starting with x_1 and repeating this process we get $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ and so on. In general, starting with x_n, the line tangent to the graph of f at $(x_n, f(x_n))$ intersects the x-axis at $(x_{n+1}, 0)$ with $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, our new estimate for the root of f.

Algorithm for Newton's Method

1. Pick a starting value x_0 (preferably close to a root of $f(x)$).
2. For each x_n, calculate a new estimate $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
3. Repeat step 2 until the estimates are "close enough" to a root or until the method "fails."

When we use Newton's method with $f(x) = x^2 - 5$, the function in our first example, we have $f'(x) = 2x$ so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^2 - 5}{2x_n} = \frac{2{x_n}^2 - {x_n}^2 + 5}{2x_n}$$
$$= \frac{{x_n}^2 + 5}{2x_n} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$$

The new approximation, x_{n+1}, is the average of the previous approximation, x_n, and 5 divided by the previous approximation, $\frac{5}{x_n}$. (Problem 16 helps you show this pattern—called Heron's method—approximates the square root of any positive number: just replace 5 with the number whose square root you want to find.)

Example 2.7.1

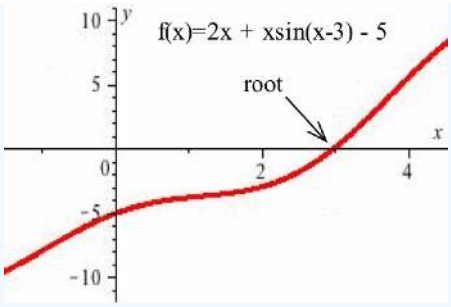
Use Newton's method to approximate the root(s) of $f(x) = 2x + x \cdot \sin(x + 3) - 5$.

Solution

$f'(x) = 2 + x\cos(x + 3) + \sin(x + 3)$ so:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n + x_n \cdot \sin(x_n + 3) - 5}{2 + x_n \cdot \cos(x_n + 3) + \sin(x_n + 3)}$$

The graph of $f(x)$ indicates only one root of f, which is near $x = 3$:



so pick $x_0 = 3$. Then Newton's method yields the values $x_0 = 3$, $x_1 = 2.96484457$, $x_2 = 2.96446277$, $x_3 = 2.96446273$ (the underlined digits agree with the exact answer).

If we had picked $x_0 = 4$ in the previous example, Newton's method would have required 4 iterations to get 9 digits of accuracy. For $x_0 = 5$, 7 iterations are needed to get 9 digits of accuracy. If we pick $x_0 = 5.1$, then the values of x_n are not close to the actual root after even 100 iterations: $x_{100} \approx -49.183$. Picking a "good" value for x_0 can result in values of x_n that get close to the root quickly. Picking a "poor" value for x_0 can result in x_n values that take many more iterations to get close to the root—or that don't approach the root at all.

The graph of the function can help you pick a "good" x_0.

Practice 2.7.3

Put $x_0 = 3$ and use Newton's method to find the first two iterates, x_1 and x_2, for the function $f(x) = x^3 - 3x^2 + x - 1$.

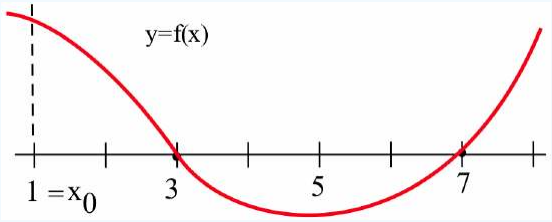
Answer

Using $f'(x) = 3x^2 + 3$ and $x_0 = 3$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{2}{10} = 2.8$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8 - \frac{f(2.8)}{f'(2.8)} = 2.8 - \frac{0.232}{7.72} \approx 2.769948187$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.769292663$$

Example 2.7.2

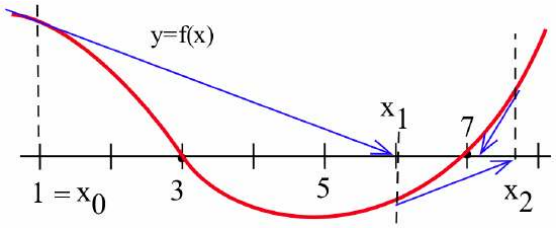
The function graphed below has roots at $x = 3$ and $x = 7$:



If we pick $x_0 = 1$ and apply Newton's method, which root do the iterates (the values of x_n) approach?

Solution

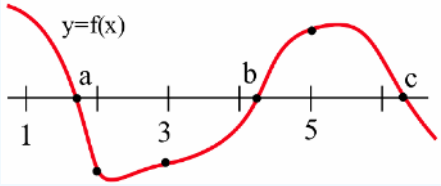
The iterates of $x_0 = 1$ are labeled in the graph below:



They are approaching the root at **7**.

? Practice 2.7.4

For the function graphed below:

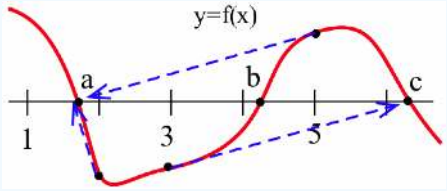


which root do the iterates of Newton's method approach if:

a. $x_0 = 2$?
b. $x_0 = 3$?
c. $x_0 = 5$?

Answer

The figure below shows the first iteration of Newton's Method for $x_0 = 2$, 3 and 5:



If $x_0 = 2$, the iterates approach the root at a; if $x_0 = 3$, they approach the root at c; and if $x_0 = 5$, they approach the root at a.

Iteration

We have been emphasizing the geometric nature of Newton's method, but Newton's method is also an example of **iterating a function**. If $N(x) = x - \dfrac{f(x)}{f'(x)}$, the "pattern" in the algorithm, then:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = N(x_0)$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = N(x_1) = N(N(x_0)) = N \circ N(x_0)$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = N(x_2) = N(N(N(x_0))) = N \circ N \circ N(x_0)$$

and, in general:

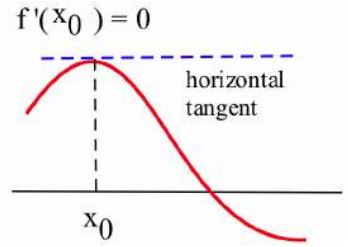
$$x_n = N(x_{n-1}) = n\text{th iteration of } N \text{ starting with } x_0$$

At each step, we use the output from N as the next input into N.

What Can Go Wrong?

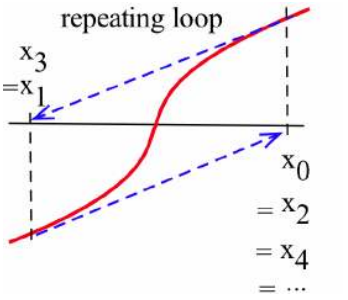
When Newton's method works, it usually works very well and the values of x_n approach a root of f very quickly, often doubling the number of correct digits with each iteration. There are, however, several things that can go wrong.

An obvious problem with Newton's method is that $f'(x_n)$ can be 0:

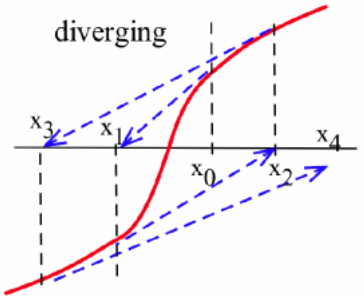


Then the algorithm tells us to divide by 0 and x_{n+1} is undefined. Geometrically, if $f'(x_n) = 0$, the tangent line to the graph of f at x_n is horizontal and does not intersect the x-axis at any point. If $f'(x_n) = 0$, just pick another starting value x_0 and begin again. In practice, a second or third choice of x_0 usually succeeds.

There are two other less obvious difficulties that are not as easy to overcome—the values of the iterates x_n may become locked into an infinitely repeating loop:



or they may actually move farther away from a root:



Example 2.7.3

Put $x_0 = 1$ and use Newton's method to find the first two iterates, x_1 and x_2, for the function $f(x) = x^3 - 3x^2 + x - 1$.

Solution

This is the function from the previous Practice Problem, but with a different starting value for x_0: $f'(x) = 3x^2 - 6x + 1$ so,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{-2} = 0$$

$$\text{and } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{1} = 1$$

which is the same as x_0, so $x_3 = x_1 = 0$ and $x_4 = x_2 = 1$. The values of x_n alternate between 1 and 0 and do not approach a root.

Newton's method behaves badly at only a few starting points for this particular function—for most starting points, Newton's method converges to the root of this function. There are some functions, however, that defeat Newton's method for **almost every** starting point.

? Practice 2.7.5

For $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ and $x_0 = 1$, verify that $x_1 = -2$, $x_2 = 4$ and $x_3 = -8$. Also try $x_0 = -3$ and verify that the same pattern holds: $x_{n+1} = -2x_n$. Graph f and explain why the Newton's method iterates get farther and farther away from the root at 0.

Answer

$f(x) = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. If $x_0 = 1$, then:

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{\frac{1}{3}} = 1 - 3 = -2$$
$$x_2 = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{(-2)^{\frac{1}{3}}}{\frac{1}{3}(-2)^{-\frac{2}{3}}} = -2 - \frac{-2}{\frac{1}{3}} = 4$$
$$x_3 = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{(4)^{\frac{1}{3}}}{\frac{1}{3}(4)^{-\frac{2}{3}}} = 4 - \frac{4}{\frac{1}{3}} = -8$$

and so on. If $x_0 = -3$, then:

$$x_1 = -3 - \frac{f(-3)}{f'(-3)} = -3 - \frac{(-3)^{\frac{1}{3}}}{\frac{1}{3}(-3)^{-\frac{2}{3}}} = -3 + 9 = 6$$
$$x_2 = 6 - \frac{f(6)}{f'(6)} = 6 - \frac{(6)^{\frac{1}{3}}}{\frac{1}{3}(6)^{-\frac{2}{3}}} = 6 - \frac{6}{\frac{1}{3}} = -12$$

The graph of $f(x) = \sqrt[3]{x}$ has a shape similar to the figure below: FIGURE and the behavior of the iterates is similar to the pattern shown in that figure. Unless $x_0 = 0$ (the only root of f) the iterates alternate in sign and double in magnitude with each iteration: they get progressively farther from the root with each iteration.

Newton's method is powerful and quick and very easy to program on a calculator or computer. It usually works so well that many people routinely use it as the first method they apply. If Newton's method fails for their particular function, they simply try some other method.

Chaotic Behavior and Newton's Method

An algorithm leads to **chaotic behavior** if two starting points that are close together generate iterates that are sometimes far apart and sometimes close together: $|a_0 - b_0|$ is small but $|a_n - b_n|$ is large for lots (infinitely many) of values of n and $|a_n - b_n|$ is small for lots of values of n. The iterates of the next simple algorithm exhibit chaotic behavior.

A Simple Chaotic Algorithm: Starting with any number between 0 and 1, double the number and keep the fractional part of the result: x_1 is the fractional part of $2x_0$, x_2 is the fractional part of $2x_1$, and in general, $x_{n+1} = 2x_n - \lfloor 2x_n \rfloor$.

If $x_0 = 0.33$, then the iterates of this algorithm are 0.66, $0.32 =$ fractional part of $2 \cdot 0.66$, 0.64, 0.28, 0.56, … The iterates for two other starting values close to 0.33 are given below as well as the iterates of 0.470 and 0.471:

x_0	0.32	0.33	0.34	0.470	0.471
x_1	0.64	0.66	0.68	0.940	0.942
x_2	0.28	0.32	0.36	0.880	0.884
x_3	0.56	0.64	0.72	0.760	0.768
x_4	0.12	0.28	0.44	0.520	0.536
x_5	0.24	0.56	0.88	0.040	0.072
x_6	0.48	0.12	0.76	0.080	0.144
x_7	0.96	0.24	0.56	0.160	0.288
x_8	0.92	0.48	0.12	0.320	0.576
x_9	0.84	0.96	0.24	0.640	0.152

There are starting values as close together as we want whose iterates are far apart infinitely often.

Many physical, biological and financial phenomena exhibit chaotic behavior. Atoms can start out within inches of each other and several weeks later be hundreds of miles apart. The idea that small initial differences can lead to dramatically diverse outcomes is sometimes called the "butterfly effect" from the title of a talk ("Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?") given by Edward Lorenz, one of the first people to investigate chaos. The "butterfly effect" has important implications about the possibility—or rather the impossibility—of accurate long-range weather forecasting. Chaotic behavior is also an important aspect of studying turbulent air and water flows, the incidence and spread of diseases, and even the fluctuating behavior of the stock market.

Newton's method often exhibits chaotic behavior and—because it is relatively easy to study—is often used as a model to investigate the properties of chaotic behavior. If we use Newton's method to approximate the roots of $f(x) = x^3 - x$ (with roots 0, $+1$ and -1), then starting points that are very close together can have iterates that converge to different roots. The iterates of 0.4472 and 0.4473 converge to the roots 0 and $+1$, respectively. The iterates of the median value 0.44725 converge to the root -1, and the iterates of another nearby point, $\frac{1}{\sqrt{5}} \approx 0.44721$, simply cycle between $-\frac{1}{\sqrt{5}}$ and $+\frac{1}{\sqrt{5}}$ and do not converge at all.

? Practice 2.7.6

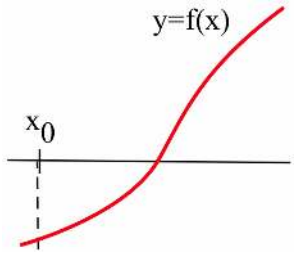
Find the first four Newton's method iterates of $x_0 = 0.997$ and $x_0 = 1.02$ for $f(x) = x^2 + 1$. Try two other starting values very close to 1 (but not equal to 1) and find their first four iterates. Use the graph of $f(x) = x^2 + 1$ to explain how starting points so close together can quickly have iterates so far apart.

Answer

If $x_0 = 0.997$, then $x_1 \approx -0.003$, $x_2 \approx 166.4$, $x_3 \approx 83.2$, $x_4 \approx 41.6$. If $x_0 = 1.02$, then $x_1 \approx 0.0198$, $x_2 \approx -25.2376$, $x_3 \approx -12.6$ and $x_4 \approx -6.26$.

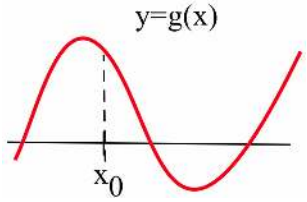
Problems

1. The graph of $y = f(x)$ appears below:



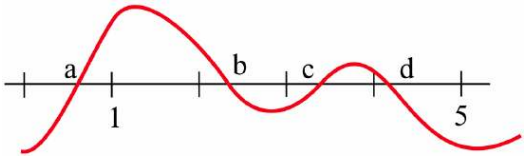
Estimate the locations of x_1 and x_2 when you apply Newton's method with the given starting value x_0.

2. The graph of $y = g(x)$ appears below:



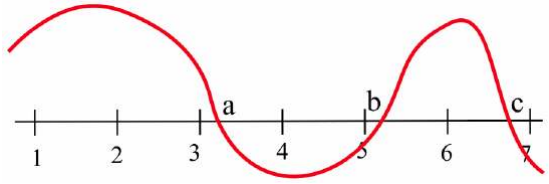
Estimate the locations of x_1 and x_2 when you apply Newton's method starting value with the value x_0 shown in the graph.

3. The function graphed below has several roots:



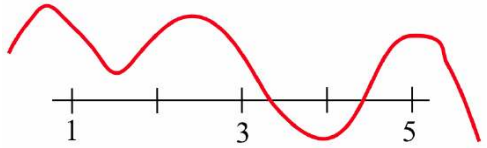
Which root do the iterates of Newton's method converge to if we start with $x_0 = 1$? With $x_0 = 5$?

4. The function graphed below has several roots:



Which root do the iterates of Newton's method converge to if we start with $x_0 = 2$? With $x_0 = 6$?

5. What happens to the iterates if we apply Newton's method to the function graphed below:



and start with $x_0 = 1$? With $x_0 = 5$?

6. What happens if we apply Newton's method to a function f and start with $x_0 =$ a root of f?
7. What happens if we apply Newton's method to a function f and start with $x_0 =$ a maximum of f?

In Problems 8–9, a function and a value for x_0 are given. Apply Newton's method to find x_1 and x_2.

8. $f(x) = x^3 + x - 1$ and $x_0 = 1$
9. $f(x) = x^4 - x^3 - 5$ and $x_0 = 2$

In Problems 10–11, use Newton's method to find a root, accurate to 2 decimal places, of the given functions using the given starting points.

10. $f(x) = x^3 - 7$ and $x_0 = 2$
11. $f(x) = x - \cos(x)$ and $x_0 = 0.7$

In Problems 12–15, use Newton's method to find all roots or solutions, accurate to 2 decimal places, of the given equation. It is helpful to examine a graph to determine a "good" starting value x_0.

12. $2 + x = e^x$
13. $\dfrac{x}{x+3} = x^2 - 2$
14. $x = \sin(x)$
15. $x = \sqrt[5]{3}$
16. Show that if you apply Newton's method to $f(x) = x^2 - A$ to approximate $\sqrt{A}$, then:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$$

so the new estimate of the square root is the average of the previous estimate and A divided by the previous estimate. This method of approximating square roots is called Heron's method.

17. Use Newton's method to devise an algorithm for approximating the cube root of a number A.
18. Use Newton's method to devise an algorithm for approximating the n-th root of a number A.

Problems 19–22 involve chaotic behavior.

19. The iterates of numbers using the Simple Chaotic Algorithm have some interesting properties.
    a. Verify that the iterates starting with $x_0 = 0$ are all equal to 0.
    b. Verify that if $x_0 = \frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ and, in general, $\frac{1}{2^n}$, then the n-th iterate of x_0 is 0 (and so are all iterates beyond the n-th iterate.)
20. When Newton's method is applied to the function $f(x) = x^2 + 1$, most starting values for x_0 lead to chaotic behavior for x_n. Find a value for x_0 so that the iterates alternate: $x_1 = -x_0$ and $x_2 = -x_1 = x_0$.
21. The function $f(x)$ defined as:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

    is called a "stretch and fold" function.
    a. Describe what f does to the points in the interval $[0, 1]$.
    b. Examine and describe the behavior of the iterates of $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$ and $\frac{2}{9}$.
    c. Examine and describe the behavior of the iterates of 0.10, 0.105 and 0.11.
    d. Do the iterates of f lead to chaotic behavior?
22. a. After many iterations (50 is fine) what happens when you apply Newton's method starting with $x_0 = 0.5$ to:
        i. $f(x) = 2x(1 - x)$
        ii. $f(x) = 3.3x(1 - x)$
        iii. $f(x) = 3.83x(1 - x)$
    b. What do you think happens to the iterates of $f(x) = 3.7x(1 - x)$? What actually happens?
    c. Repeat parts (a)–(b) with some other starting values x_0 between 0 and 1 $(0 < x_0 < 1)$.

    (The behavior of the iterates of f depends in a strange way on the numerical value of the leading coefficient. The behavior exhibited in part (b) is an example of "chaos.")

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2.8: Linear Approximation and Differentials

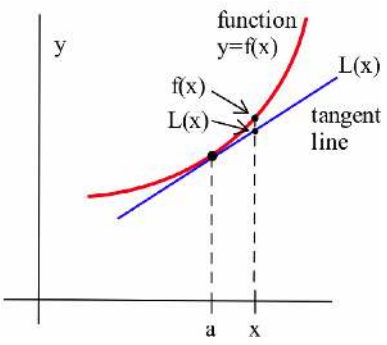
Newton's method used tangent lines to "point toward" a root of a function. In this section we examine and use another geometric characteristic of tangent lines:

If: f is differentiable at a, c is close to a

and: $y = L(x)$ is the line tangent to $f(x)$ at $x = a$

then: $L(c)$ is close to $f(c)$.

We can use this idea to approximate the values of some commonly used functions and to predict the "error" or uncertainty in a computation if we know the "error" or uncertainty in our original data. At the end of this section, we will define a related concept called the **differential** of a function.



Linear Approximation

Because this section uses tangent lines extensively, it is worthwhile to recall how we find the equation of the line tangent to $f(x)$ where $x = a$: the tangent line goes through the point $(a, f(a))$ and has slope $f'(a)$ so, using the point-slope form $y - y_0 = m(x - x_0)$ for linear equations, we have $y - f(a) = f'(a) \cdot (x - a) \Rightarrow y = f(a) + f'(a) \cdot (x - a)$.

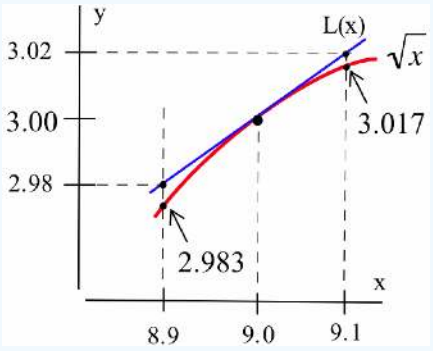
Theorem

If: f is differentiable at $x = a$

then: an equation of the line L tangent to f at $x = a$ is $L(x) = f(a) + f'(a) \cdot (x - a)$

Example 2.8.1

Find a formula for $L(x)$, the linear function tangent to the graph of $f(x) = \sqrt{x}$ at the point $(9, 3)$.



Evaluate $L(9.1)$ and $L(8.88)$ to approximate $\sqrt{9.1}$ and $\sqrt{8.88}$.

Solution

$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ so $f(9) = 3$ and $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$. Thus:

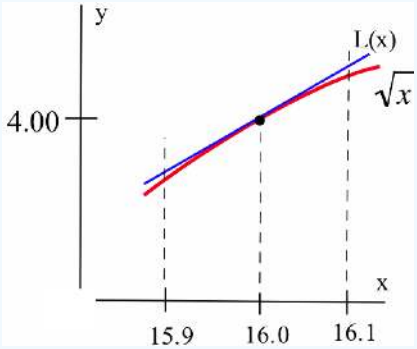
$$L(x) = f(9) + f'(9) \cdot (x - 9) = 3 + \frac{1}{6}(x - 9)$$

If x is close to 9, then the value of $L(x)$ should be a good approximation of the value of x. The number 9.1 is close to 9 so $\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.016666$. Similarly, $\sqrt{8.88} = f(8.88) \approx L(8.88) = 3 + \frac{1}{6}(8.88 - 9) = 2.98$. In fact, $\sqrt{9.1} \approx 3.016621$, so our estimate using $L(9.1)$ is within 0.000045 of the exact answer; $\sqrt{8.88} \approx 2.979933$ (accurate to 6 decimal places) and our estimate is within 0.00007 of the exact answer.

In each case in the previous example, we got a good estimate of a square root with very little work. The graph in the margin indicates the graph of the tangent line $y = L(x)$ lies slightly above the graph of $y = f(x)$; consequently (as we observed), each estimate is slightly larger than the exact value.

? Practice 2.8.1

Find a formula for $L(x)$, the linear function tangent to the graph of $f(x) = \sqrt{x}$ at the point $(16, 4)$. Evaluate $L(16.1)$ and $L(15.92)$ to approximate $\sqrt{16.1}$ and $\sqrt{15.92}$.



Are your approximations using L larger or smaller than the exact values of the square roots?

Answer

$f(x) = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$. At the point $(16, 4)$ on the graph of f, the slope of the tangent line is $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$. An equation of the tangent line is $y - 4 = \frac{1}{8}(x - 16)$ or $y = \frac{1}{8}x + 2$: $L(x) = \frac{1}{8}x + 2$. So:

$$\sqrt{16.1} \approx L(16.1) = \frac{1}{8}(16.1) + 2 = 4.0125$$
$$\sqrt{15.92} \approx L(15.92) = \frac{1}{8}(15.92) + 2 = 3.99$$

? Practice 2.8.2

FInd a formula for $L(x)$, the linear function tangent to the graph of $f(x) = x^3$ at the point $(1, 1)$ and use $L(x)$ to approximate $(1.02)^3$ and $(0.97)^3$. Do you think your approximations using L are larger or smaller than the exact values?

Answer

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$. At $(1, 1)$, the slope of the tangent line is $f'(1) = 3$. An equation of the tangent line is $y - 1 = 3(x - 1)$ or $y = 3x - 2$: $L(x) = 3x - 2$. So:

$$(1.02)^3 \approx L(1.02) = 3(1.02) - 2 = 1.06$$
$$(0.97)^3 \approx L(0.97) = 3(0.97) - 2 = 0.91$$

The process we have used to approximate square roots and cubics can be used to approximate values of any differentiable function, and the main result about the linear approximation follows from the two statements in the boxes. Putting these two statements together, we have the process for Linear Approximation.

Linear Approximation Process

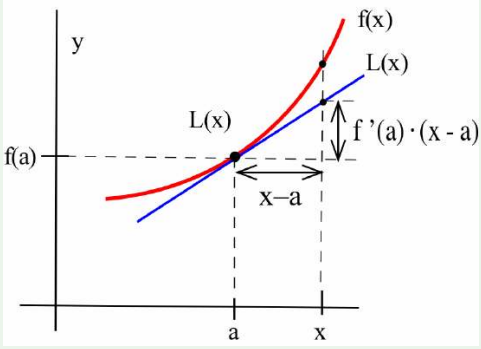
If: f is differentiable at a and $L(x) = f(a) + f'(a) \cdot (x - a)$

then: (geometrically) the graph of $L(x)$ is close to the graph of $f(x)$ when x is close to a

and: (algebraically) the values of the $L(x)$ approximate the values of $f(x)$ when x is close to a:

$$f(x) \approx L(x) = f(a) + f'(a) \cdot (x - a)$$

Sometimes we replace "$x - a$" with "Δx" in the last equation, and the statement becomes $f(x) \approx f(a) + f'(a) \cdot \Delta x$.



Example 2.8.2

Use the linear approximation process to approximate the value of $e^{0.1}$.

Solution

$f(x) = e^x \Rightarrow f'(x) = e^x$ so we need to pick a value a near $x = 0.1$ for which we know the exact value of $f(a) = e^a$ and $f'(a) = e^a$: $a = 0$ is an obvious choice. Then:

$$\begin{aligned} e^{0.1} = f(0.1) \approx L(0.1) &= f(0) + f'(0) \cdot (0.1 - 0) \\ &= e^0 + e^0 \cdot (0.1) = 1 + 1 \cdot (0.1) = 1.1 \end{aligned}$$

You can use your calculator to verify that this approximation is within 0.0052 of the exact value of $e^{0.1}$.

Practice 2.8.3

Approximate the value of $(1.06)^4$, the amount \$1 becomes after 4 years in a bank account paying 6% interest compounded annually. (Take $f(x) = x^4$ and $a = 1$.)

Answer

$f(x) = x^4 \Rightarrow f'(x) = 4x^3$. Taking $a = 1$ and $\Delta x = 0.06$:

$$\begin{aligned} (1.06)^4 = f(1.06) \approx L(1.06) &= f(1) + f'(1) \cdot (0.06) \\ &= 1^4 + 4(1^3)(0.06) = 1.24 \end{aligned}$$

Practice 2.8.4

Use the linear approximation process and the values in the table below to estimate the value of f when $x = 1.1$, 1.23 and 1.38.

x	$f(x)$	$f'(x)$
1.0	0.7854	0.5
1.2	0.8761	0.4098
1.4	0.9505	0.3378

Answer

Using values given in the table:

$$\begin{aligned} f(1.1) &\approx f(1) + f'(1) \cdot (0.1) \\ &= 0.7854 + (0.5)(0.1) = 0.8354 \\ f(1.23) &\approx f(1.2) + f'(1.2) \cdot (0.03) \\ &= 0.8761 + (0.4098)(0.03) = 0.888394 \\ f(1.38) &\approx f(1.4) + f'(1.4) \cdot (-0.02) \\ &= 0.9505 + (0.3378)(-0.02) = 0.943744 \end{aligned}$$

We can approximate **functions** as well as numbers (specific values of those functions).

Example 2.8.3

Find a linear approximation formula $L(x)$ for $\sqrt{1+x}$ when x is small. Use your result to approximate $\sqrt{1.1}$ and $\sqrt{0.96}$.

Solution

$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}$, so because "$x$ is small," we know that x is close to 0 and we can pick $a = 0$. Then $f(a) = f(0) = 1$ and $f'(a) = f'(0) = \frac{1}{2}$ so

$$\sqrt{1+x} \approx L(x) = f(0) + f'(0) \cdot (x - 0) = 1 + \frac{1}{2}x = 1 + \frac{x}{2}$$

Taking $x = 0.1$, $\sqrt{1.1} = \sqrt{1+0.1} \approx 1 + \frac{0.1}{2} = 1.05$; taking $x = -0.04$, $\sqrt{0.96} = \sqrt{1+(-.04)} \approx 1 + \frac{-0.04}{2} = 0.98$. Use your calculator to determine by how much each estimate differs from the true value.

Applications of Linear Approximation to Measurement "Error"

Most scientific experiments use instruments to take measurements, but these instruments are not perfect, and the measurements we get from them are only accurate up to a certain level of precision. If we know this level of accuracy of our instruments and measurements, we can use the idea of linear approximation to estimate the level of accuracy of results we calculate from our measurements.

If we measure the side x of a square to be 8 inches, then we would of course calculate its area to be $8^2 = 64$ square inches. Suppose, as would reasonable with a real measurement, that our measuring instrument could only measure or be read to the nearest 0.05 inches. Then our measurement of 8 inches would really mean some number between $8 - 0.05 = 7.95$ inches and $8 + 0.05 = 8.05$ inches, so the true area of the square would be between $7.95^2 = 63.2025$ and $8.05^2 = 64.8025$ square inches. Our possible "error" or "uncertainty," because of the limitations of the instrument, could be as much as $64.8025 - 64 = 0.8025$ square inches, so we could report the area of the square to be 64 ± 0.8025 square inches. We can also use the linear approximation method to estimate the "error" or uncertainty of the area. (For a function as simple as the area of a square, this linear approximation method really isn't needed, but it serves as a useful and easily understood illustration of the technique.)

For a square with side x, the area is $A(x) = x^2$ and $A'(x) = 2x$. If Δx represents the "error" or uncertainty of our measurement of the side then, using the linear approximation technique for $A(x)$, $A(x) \approx A(a) + A'(a) \cdot \Delta x$ so the uncertainty of our calculated area is $A(x) - A(a) \approx A'(a) \cdot \Delta x$. In this example, $a = 8$ inches and $\Delta x = 0.05$ inches, so $A(8.05) \approx A(8) + A'(8) \cdot (0.05) = 64 + 2(8) \cdot (0.05) = 64.8$ square inches, and the uncertainty in our calculated area is approximately $A(8 + 0.05) - A(8) \approx A'(8) \cdot \Delta x = 2(8 \text{ inches})(0.05 \text{ inches}) = 0.8$ square inches. (Compare this approximation of the biggest possible error with the exact answer of 0.8025 square inches computed previously.) This process can be summarized as:

Linear Approximation Error

If: the value of the x-variable is measured to be $x = a$ with a maximum "error" of Δx units

then: Δf, the "error" in estimating $f(x)$, is:

$$\Delta f = f(x) - f(a) \approx f'(a) \cdot \Delta x$$

Practice 2.8.5

If we measure the side of a cube to be 4 cm with an uncertainty of 0.1 cm, what is the volume of the cube and the uncertainty of our calculation of the volume? (Use linear approximation.)

Answer

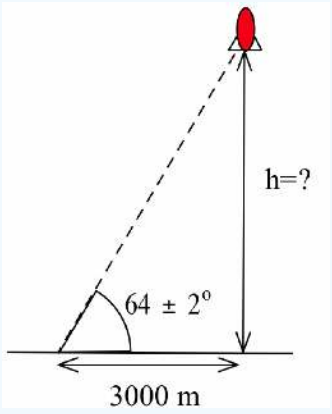
$f(x) = x^3 \Rightarrow f'(x) = 3x^2$ so $f(4) = 4^3 = 64$ cm^3 and the "error" is:

$$\Delta f \approx f'(x) \cdot \Delta x = 3x^2 \cdot \Delta x$$

When $x = 4$ and $\Delta x = 0.1$, $\Delta f \approx 3(4)^2(0.1) = 4.8$ cm^3.

Example 2.8.4

We are using a tracking telescope to follow a small rocket. Suppose we are 3,000 meters from the launch point of the rocket, and, 2 seconds after the launch we measure the angle of the inclination of the rocket to be 64° with a possible "error" of 2°.



How high is the rocket and what is the possible error in this calculated height?

Solution

Our measured angle is $x = 1.1170$ radians with $\Delta x = 0.0349$ radians (all trigonometric work should be in radians), and the height of the rocket at an angle x is $f(x) = 3000\tan(x)$ so $f(1.1170) \approx 6151$ m. Our uncertainty in the height is $\Delta f \approx f'(x) \cdot \Delta x \approx 3000 \cdot \sec^2(x) \cdot \Delta x = 3000\sec^2(1.1170) \cdot 0.0349 \approx 545$ m. If our measured angle of 64° can be in error by as much as 2°, then our calculated height of 6,151 m can be in error by as much as 545 m. The height is 6151 ± 545 meters

Practice 2.8.6

Suppose we measured the angle of inclination in the previous Example to be $43° \pm 1°$. Estimate the height of the rocket in the form "height $\pm$ error."

Answer

$43° \pm 1°$ is equivalent to 0.75049 ± 0.01745 radians, so with $f(x) = 3000\tan(x)$ we have $f(0.75049) = 3000\tan(0.75049) \approx 2797.5$ m and $f'(x) = 3000\sec^2(x)$. So:

$$\Delta f \approx f'(x)\cdot\Delta x = 3000\sec^2(x)\cdot\Delta x$$
$$= 3000\sec^2(0.75049)\cdot(0.01745) = 97.9 \text{ m}$$

The height of the rocket is 2797.5 ± 97.9 m.

In some scientific and engineering applications, the calculated **result** must be within some given specification. You might need to determine how accurate the initial measurements must be in order to guarantee the final calculation is within that specification. Added precision usually costs time and money, so it is important to choose a measuring instrument good enough for the job but which is not too expensive.

Example 2.8.5

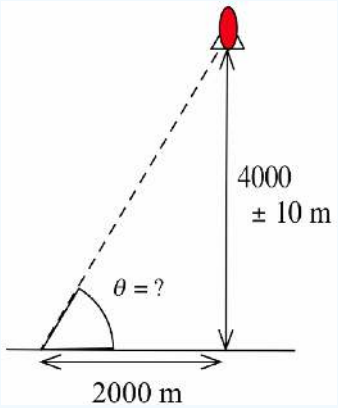
Your company produces ball bearings (small metal spheres) with a volume of 10 cm^3 and the volume must be accurate to within 0.1 cm^3. What radius should the bearings have—and what error can you tolerate in the radius measurement to meet the accuracy specification for the volume?

Solution

We want $V = 10$ and we know that the volume of a sphere is $V = \frac{4}{3}\pi r^3$, so solve $10 = \frac{4}{3}\pi r^3$ for r to get $r = 1.3365$ cm. $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = 4\pi r^2$ so $\Delta V \approx V'(r)\cdot\Delta r$. In this case we know that $\Delta V = 0.1$ cm^3 and we have calculated $r = 1.3365$ cm, so $0.1 \text{ cm}^3 = V'(1.3365 \text{ cm})\cdot\Delta r = (22.45 \text{ cm}^2)\cdot\Delta r$. Solving for Δr, we get $\Delta r \approx 0.0045$ cm. To meet the specification for allowable error in volume, we must allow the radius to vary no more than 0.0045 cm. If we instead measure the diameter of the sphere, then we want the diameter to be $d = 2r = 2(1.3365 \pm 0.0045) = 2.673 \pm 0.009$ cm.

Practice 2.8.7

You want to determine the height of a rocket to within 10 meters when it is 4,000 meters high:



How accurate must your angle of measurement be? (Do your calculations in radians.)

Answer

$f(\theta) = 2000\tan(\theta) \Rightarrow f'(\theta) = 2000\sec^2(\theta)$ and we know $f(\theta) = 4000$, so:

$$4000 = 2000\tan(\theta) \Rightarrow \tan(\theta) = 2 \Rightarrow \theta \approx 1.10715 \text{ (radians)}$$

and thus $f'(1.10715) = 2000\sec^2(1.10715) \approx 10000$. Finally, the "error" is given by $\Delta f \approx f'(\theta)\cdot\Delta\theta$ so:

$$10 \approx 10000\cdot\Delta\theta \;\Rightarrow\; \Delta\theta \approx \frac{10}{10000} = 0.001 \text{ (radians)} \;\approx 0.057^\circ$$

Relative Error and Percentage Error

The “error” we’ve been examining is called the **absolute error** to distinguish it from two other commonly used terms, the **relative error** and the **percentage error**, which compare the absolute error with the magnitude of the number being measured. An “error” of 6 inches in measuring the Earth's circumference would be extremely small, but a 6-inch error in measuring your head for a hat would result in a very bad fit.

Definition

The **Relative** Error of f is: $\frac{\text{error of } f}{\text{value of } f} = \frac{\Delta f}{f}$\ \vspace{0.25cm}

The **Percentage** Error of f is: $\frac{\Delta f}{f} \cdot 100\%$.

Example 2.8.6

If the relative error in the calculation of the area of a circle must be less than 0.4, then what relative error can we tolerate in the measurement of the radius?

Solution

$A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$ and $\Delta A \approx A'(r) \cdot \Delta r = 2\pi r \Delta r$. The Relative Error of A is:

$$\frac{\Delta A}{A} \approx \frac{2\pi r \Delta r}{\pi r^2} = 2\frac{\Delta r}{r}$$

We can guarantee that the Relative Error of A, $\frac{\Delta A}{A}$, is less than 0.4 if the Relative Error of r, $\frac{\Delta r}{r} = \frac{1}{2}\frac{\Delta A}{A}$, is less than $\frac{1}{2}(0.4) = 0.2$.

Practice 2.8.8

If you can measure the side of a cube with a percentage error less than 3%, then what will the percentage error for your calculation of the surface area of the cube be?

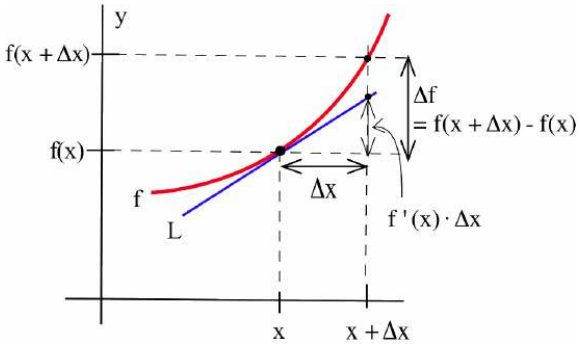
Answer

$A(r) = 6r^2 \Rightarrow A'(r) = 12r \Rightarrow \Delta A \approx A'(r) \cdot \Delta r = 12r \cdot \Delta r$ and we also know that $\frac{\Delta r}{r} < 0.03$, so the percentage error is:

$$\frac{\Delta A}{A} \cdot 100\% = \frac{12r \cdot \Delta r}{6r^2} \cdot 100\% = \frac{2\Delta r}{r} \cdot 100\% < 200(0.03)\% = 6\%$$

The Differential of f

As shown in the figure below, the change in value of the function f near the point $(x, f(x))$ is $\Delta f = f(x + \Delta x) - f(x)$ and the change along the tangent line is $f'(x) \cdot \Delta x$:



If Δx is small, then we have used the approximation that $\Delta f \approx f'(x) \cdot \Delta x$. This leads to the definition of a new quantity, df, called the **differential** of f.

Definition

The differential of f is $df = f'(x) \cdot dx$ where dx is any real number.

The differential of f represents the change in f, as x changes from x to $x + dx$, along the tangent line to the graph of f at the point $(x, f(x))$. If we take dx to be the number Δx, then the differential is an approximation of Δf: $\Delta f \approx f'(x) \cdot \Delta x = f'(x) \cdot dx = df$.

Example 2.8.7

Determine the differential for the functions $f(x) = x^3 - 7x$, $g(x) = \sin(x)$ and $h(r) = \pi r^2$.

Solution

$df = f'(x) \cdot dx = (3x^2 - 7)\,dx$, $dg = g'(x) \cdot dx = \cos(x)\,dx$, and $dh = h'(r)\,dr = 2\pi r\,dr$.

Practice 2.8.9

Determine the differentials of $f(x) = \ln(x)$, $u = \sqrt{1 - 3x}$ and $r = 3\cos(\theta)$. (While we will do very little with differentials for a while, we will use them extensively in integral calculus.)

Answer

Computing differentials:

$$df = f'(x) \cdot dx = \frac{1}{x}\,dx$$
$$du = \frac{du}{dx} \cdot dx = \frac{-3}{2\sqrt{1 - 3x}}\,dx$$
$$dr = \frac{dr}{d\theta}\,d\theta = -3\sin(\theta)\,d\theta$$

The Linear Approximation "Error" $|f(x) - L(x)|$

An approximation is most valuable if we also have have some measure of the size of the "error," the distance between the approximate value and the value being approximated. Typically, we will not know the exact value of the error (why not?), but it is useful to know an upper bound for the error. For example, if one scale gives the weight of a gold pendant as 10.64 grams with an error less than 0.3 grams (10.64 ± 0.3 grams) and another scale gives the weight of the same pendant as 10.53 grams with an error less than 0.02 grams (10.53 ± 0.02 grams), then we can have more faith in the second approximate weight because of the smaller "error" guarantee. Before finding a guarantee on the size of the error of the linear approximation process, we will check how well the linear approximation process approximates values of some functions we can compute exactly. Then we will prove one bound on the possible error and state a somewhat stronger bound.

Example 2.8.8

Given the function $f(x) = x^2$, evaluate the expressions $f(2 + \Delta x)$, $L(2 + \Delta x)$ and $|f(2 + \Delta x) - L(2 + \Delta x)|$ for $\Delta x = 0.1$, 0.05, 0.01, 0.001 and for a general value of Δx.

Solution

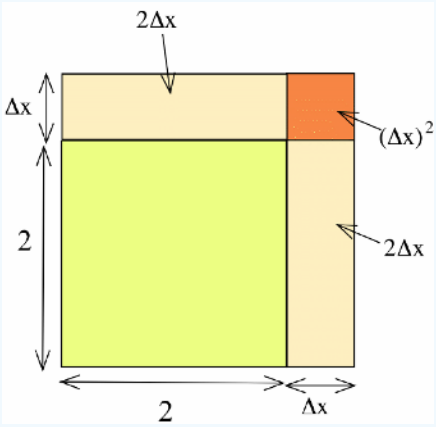
$f(2 + \Delta x) = (2 + \Delta x)^2 = 2^2 + 4\Delta x + (\Delta x)^2$ and $L(2 + \Delta x) = f(2) + f'(2) \cdot \Delta x = 2^2 + 4 \cdot \Delta x$. Then:

Δx	$f(2+\Delta x)$	$L(2+\Delta x)$	$\lvert f(2+\Delta x)-L(2+\Delta x)\rvert$
0.1	4.41	4.4	0.01
0.05	4.2025	4.2	0.0025
0.01	4.0401	4.04	0.0001
0.001	4.004001	4.004	0.000001

Cutting the value of Δx in half makes the error one fourth as large. Cutting Δx to $\frac{1}{10}$ as large makes the error $\frac{1}{100}$ as large. In general:

$$\begin{aligned} |f(2+\Delta x)-L(2+\Delta x)| &= \left|\left(2^2+4\cdot\Delta x+(\Delta x)^2\right)-\left(2^2+4\cdot\Delta x\right)\right| \\ &= (\Delta x)^2 \end{aligned}$$

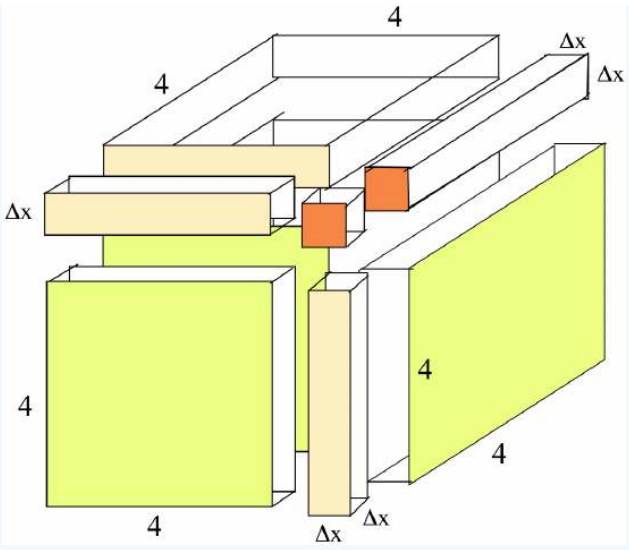
This function and error also have a nice geometric interpretation:



$f(x)=x^2$ is the area of a square of side x so $f(2+\Delta x)$ is the area of a square of side $2+\Delta x$, and that area is the sum of the pieces with areas 2^2, $2\cdot\Delta x$, $2\cdot\Delta x$ and $(\Delta x)^2$. The linear approximation $L(2+\Delta x)=2^2+4\cdot\Delta x$ to the area of the square includes the three largest pieces, 2^2, $2\cdot\Delta x$ and $2\cdot\Delta x$, but omits the small square with area $(\Delta x)^2$ so the approximation is in error by the amount $(\Delta x)^2$.

? Practice 2.8.10

Given $f(x)=x^3$, evaluate $f(4+\Delta x)$, $L(4+\Delta x)$ and $\lvert f(4+\Delta x)-L(4+\Delta x)\rvert$ for $\Delta x=0.1,\ 0.05,\ 0.01,\ 0.001$ and for a general value of Δx. Use the figure below:



to give a geometric interpretation of $f(4+\Delta x)$, $L(4+\Delta x)$ and $|f(4+\Delta x)-L(4+\Delta x)|$.

Answer

$f(x)=x^3 \Rightarrow f'(x)=3x^2$ so:

$$L(4+\Delta x)=f(4)+f'(4)\Delta x=4^3+3(4)^2\Delta x=64+48\Delta x$$

Evaluating the various quantities at the indicated points:

Δx	$f(4+\Delta x)$	$L(4+\Delta x)$	$\lvert f(4+\Delta x)-L(4+\Delta x)\rvert$
0.1	68.921	68.8	0.121
0.05	66.430125	66.4	0.030125
0.01	64.481201	64.48	0.001201
0.001	64.048012	64.048	0.000012

$f(4+\Delta x)$ is the actual volume of the cube with side length $4+\Delta x$.

$L(4+\Delta x)$ is the volume of the cube with side length 4 ($V=64$) plus the volume of the three "slabs" ($V=3\cdot 4^2\cdot\Delta x$).

$|f(4+\Delta x)-L(4+\Delta x)|$ is the volume of the "leftover" pieces from L: the three "rods" ($V=3\cdot 4\cdot(\Delta x)^2$) and the tiny cube ($V=(\Delta x)^3$).

In the previous Example and previous Practice problem, the error $|f(a+\Delta x)-L(a+\Delta x)|$ was very small, proportional to $(\Delta x)^2$, when Δx was small. In general, this error approaches 0 as $\Delta x\to 0$.

Theorem

If: $f(x)$ is differentiable at a

and: $L(a+\Delta x)=f(a)+f'(a)\cdot\Delta x$

then: $\lim\limits_{\Delta x\to 0}|f(a+\Delta x)-L(a+\Delta x)|=0$

and: $\lim\limits_{\Delta x\to 0}\dfrac{|f(a+\Delta x)-L(a+\Delta x)|}{\Delta x}=0.\backslash$

Proof

First rewrite the quantity inside the absolute value as:

$$f(a+\Delta x) - L(a+\Delta x) = f(a+\Delta x) - f(a) - f'(a)\cdot\Delta x$$
$$= \left[\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right]\cdot\Delta x$$

But f is differentiable at $x=a$ so $\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x} = f'(a)$, which we can rewrite as $\lim_{\Delta x\to 0}\left[\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right] = 0$. Thus:

$$\lim_{\Delta x\to 0}[f(a+\Delta x) - L(a+\Delta x)] = \lim_{\Delta x\to 0}\left[\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right]\cdot\lim_{\Delta x\to 0}\Delta x = 0\cdot 0 = 0$$

Not only does the difference $f(a+\Delta x) - L(a+\Delta x)$ approach 0, but this difference approaches 0 so fast that we can divide it by Δx, another quantity approaching 0, and the quotient still approaches 0.

In the next chapter we will be able to prove that the error of the linear approximation process is in fact proportional to $(\Delta x)^2$. For now, we just state the result.

Theorem

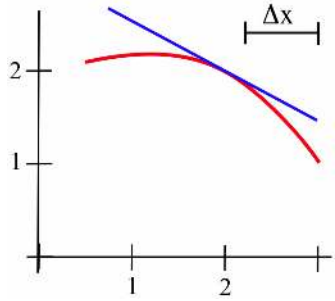
If: f is differentiable at a

and: $|f''(x)| \le M$ for all x between a and $a+\Delta x$

then: $|\text{“error”}| = |f(a+\Delta x) - L(a+\Delta x)| \le \frac{1}{2}M\cdot(\Delta x)^2$.

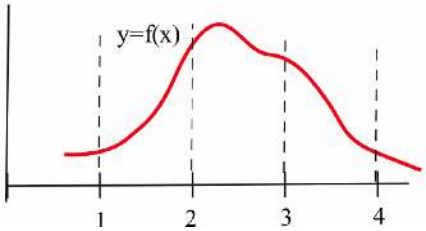
Problems

1. The figure below shows the tangent line to a function g at the point $(2,2)$ and a line segment Δx units long:



   a. On the figure, label the locations of
      i. $2+\Delta x$ on the x-axis
      ii. the point $(2+\Delta x, g(2+\Delta x))$
      iii. the point $(2+\Delta x, g(2)+g'(2)\cdot\Delta x)$
   b. How large is the “error,” $(g(2)+g'(2)\cdot\Delta x) - (g(2+\Delta x))$?

2. In the figure below, is the linear approximation $L(a+\Delta x)$ larger or smaller than the value of $f(a+\Delta x)$ when:



   a. $a=1$ and $\Delta x = 0.2$?
   b. $a=2$ and $\Delta x = -0.1$?
   c. $a=3$ and $\Delta x = 0.1$?
   d. $a=4$ and $\Delta x = 0.2$?

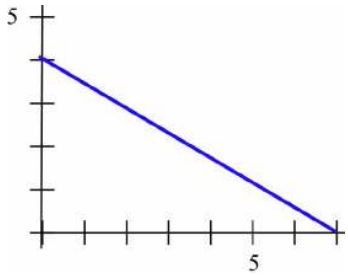
e. $a = 4$ and $\Delta x = -0.2$?

In Problems 3–4, find a formula for the linear function $L(x)$ tangent to the given function f at the given point $(a, f(a))$. Use the value $L(a + \Delta x)$ to approximate the value of $f(a + \Delta x)$.

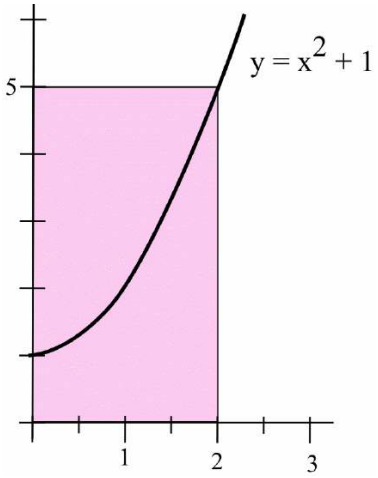
4. a. $f(x) = \sqrt{x}$, $a = 4$, $\Delta x = 0.2$
   b. $f(x) = \sqrt{x}$, $a = 81$, $\Delta x = -1$
   c. $f(x) = \sin(x)$, $a = 0$, $\Delta x = 0.3$
5. a. $f(x) = \ln(x)$, $a = 1$, $\Delta x = 0.3$
   b. $f(x) = e^x$, $a = 0$, $\Delta x = 0.1$
   c. $f(x) = x^5$, $a = 1$, $\Delta x = 0.03$
6. Show that $(1 + x)^n \approx 1 + nx$ if x is "close to" 0. (Suggestion: Put $f(x) = (1 + x)^n$ and $a = 0$ and then replace Δx with x.)

In Problems 6–7, use the linear approximation process to obtain each formula for x "close to" 0.

6. a. $(1 - x)^n \approx 1 - nx$
   b. $\sin(x) \approx x$
   c. $e^x \approx 1 + x$
7. a. $\ln(1 + x) \approx x$
   b. $\cos(x) \approx 1$
   c. $\tan(x) \approx x$
   d. $\sin\left(\frac{\pi}{2} + x\right) \approx 1$
8. The height of a triangle is exactly 4 inches, and the base is measured to be 7 ± 0.5 inches:



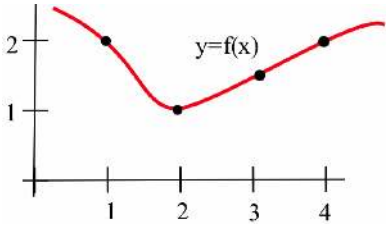
   Shade a part of the figure that represents the "error" in the calculation of the area of the triangle.
9. A rectangle has one side on the x-axis, one side on the y-axis and a corner on the graph of $y = x^2 + 1$:



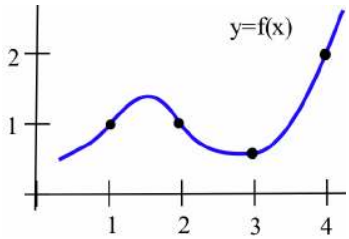
   a. Use Linear Approximation of the area formula to estimate the increase in the area of the rectangle if the base grows from 2 to 2.3 inches.
   b. Calculate exactly the increase in the area of the rectangle as the base grows from 2 to 2.3 inches.
10. You know that you can measure the diameter of a circle to within 0.3 cm of the exact value.
    a. How large is the "error" in the calculated area of a circle with a measured diameter of 7.4 cm?

b. How large is the "error" with a measured diameter of 13.6 cm?
c. How large is the percentage error in the calculated area with a measured diameter of d?

11. You are minting gold coins that must have a volume of 47.3 ± 0.1 cm^3. If you can manufacture the coins to be exactly 2 cm high, how much variation can you allow for the radius?
12. If F is the fraction of carbon-14 remaining in a plant sample Y years after it died, then $Y = 5700 \ln(0.5) \cdot \ln(F)$.
    a. Estimate the age of a plant sample in which $83 \pm 2\%$ (0.83 ± 0.02) of the carbon-14 remains.
    b. Estimate the age of a plant sample in which $13 \pm 2\%$ (0.13 ± 0.02) of the carbon-14 remains.
13. Your company is making dice (cubes) and specifications require that their volume be 87 ± 2 cm^3. How long should each side be and how much variation can be allowed in order to meet the specifications?
14. If the specifications require a cube with a surface area of 43 ± 0.2 cm^2, how long should each side be and how much variation can be allowed in order to meet the specifications?
15. The period P, in seconds, for a pendulum to make one complete swing and return to the release point is $P = 2\pi\sqrt{\frac{L}{g}}$ where L is the length of the pendulum in feet and g is 32 feet/sec^2.
    a. If $L = 2$ feet, what is the period?
    b. If $P = 1$ second, how long is the pendulum?
    c. Estimate the change in P if L increases from 2 feet to 2.1 feet.
    d. The length of a 24-foot pendulum is increasing 2 inches per hour. Is the period getting longer or shorter? How fast is the period changing?
16. A ball thrown at an angle θ (with the horizontal) with an initial velocity v will land $\frac{v^2}{g} \cdot \sin(2\theta)$ feet from the thrower.
    a. How far away will the ball land if $\theta = \frac{\pi}{4}$ and $v = 80$ feet/second?
    b. Which will result in a greater change in the distance: a 5% error in the angle θ or a 5% error in the initial velocity v?
17. For the function graphed below:



estimate the value of df when
    a. $x = 2$ and $dx = 1$
    b. $x = 4$ and $dx = -1$
    c. $x = 3$ and $dx = 2$
18. For the function graphed below:



estimate the value of df when
    a. $x = 1$ and $dx = 2$
    b. $x = 2$ and $dx = -1$
    c. $x = 3$ and $dx = 1$
19. Calculate the differentials df for the following functions:
    a. $f(x) = x^2 - 3x$

b. $f(x) = e^x$
c. $f(x) = \sin(5x)$
d. $f(x) = x^3 + 2x$ with $x = 1$ and $dx = 0.2$
e. $f(x) = \ln(x)$ with $x = e$ and $dx = -0.1$
f. $f(x) = \sqrt{2x + 5}$ with $x = 22$ and $dx = 3$.

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2.9: Implicit and Logarithmic Differentiation

This short section presents two more differentiation techniques, both more specialized than the ones we have already seen—and consequently used on a smaller class of functions. For some functions, however, one of these techniques may be the only method that works. The idea of each method is straightforward, but actually using each of them requires that you proceed carefully and practice.

Implicit Differentiation

In our work up until now, the functions we needed to differentiate were either given **explicitly** as a function of x, such as $y = f(x) = x^2 + \sin(x)$, or it was fairly straightforward to find an explicit formula, such as solving $y^3 - 3x^2 = 5$ to get $y = \sqrt[3]{5 + 3x^2}$. Sometimes, however, we will have an equation relating x and y that is either difficult or impossible to solve explicitly for y, such as $y^2 + 2y = \sin(x) + 4$ (difficult) or $y + \sin(y) = x^3 - x$ (impossible). In each case, we can still find $y' = f'(x)$ by using **implicit differentiation**.

The key idea behind implicit differentiation is to *assume* that y is a function of x even if we cannot explicitly solve for y. This assumption does not require any work, but we need to be very careful to treat y as a function when we differentiate and to use the Chain Rule or the Power Rule for Functions.

Example 2.9.1

Assume y is a function of x and compute each derivative:

a. $\mathrm{D}(y^3)$
b. $\frac{d}{dx}\left(x^3y^2\right)$
c. $(\sin(y))'$

Solution

a. We need the Power Rule for Functions because y is a function of x:

$$\mathrm{D}(y^3) = 3y^2 \cdot \mathrm{D}(y) = 3y^2 \cdot y'$$

b. We need to use the Product Rule and the Chain Rule:

$$\frac{d}{dx}\left(x^3y^2\right) = x^3 \cdot \frac{d}{dx}\left(y^2\right) + y^2 \cdot \frac{d}{dx}\left(x^3\right) = x^3 \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 3x^2$$

c. We just need to remember that $\mathrm{D}(\sin(u)) = \cos(u)$ and then use the Chain Rule: $(\sin(y))' = \cos(y) \cdot y'$.

Practice 2.9.1

Assume that y is a function of x. Calculate:

a. $\mathrm{D}\left(x^2 + y^2\right)$
b. $\frac{d}{dx}\left(\sin(2 + 3y)\right)$.

Answer

$\mathrm{D}(x^2 + y^2) = 2x + 2y \cdot y'$

$\frac{d}{dx}\left(\sin(2 + 3y)\right) = \cos(2 + 3y) \cdot \mathrm{D}(2 + 3y) = \cos(2 + 3y) \cdot 3y'$

Implicit Differentiation

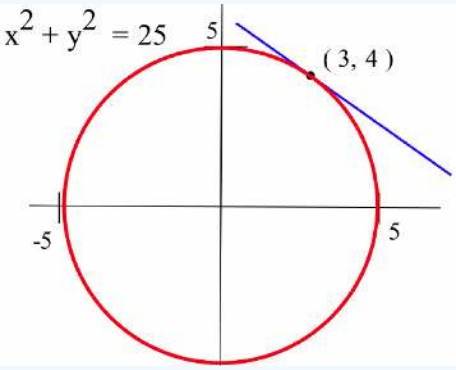
To determine y', differentiate each side of the defining equation, treating y as a function of x, and then algebraically solve for y'.

Example 2.9.2

Find the slope of the tangent line to the circle $x^2+y^2=25$ at the point $(3,4)$ with and without implicit differentiation.

Solution

Explicitly: We can solve $x^2+y^2=25$ for y: $y=\pm\sqrt{25-x^2}$ but because the point $(3,4)$ is on the top half of the circle:



we just need $y=\sqrt{25-x^2}$ so:

$$\mathrm{D}(y)=\mathrm{D}\left(\sqrt{25-x^2}\right)=\frac{1}{2}\left(25-x^2\right)^{-\frac{1}{2}}\cdot(-2x)=\frac{-x}{\sqrt{25-x^2}}$$

Replacing x with 3, we have $y'=\frac{-3}{\sqrt{25-3^2}}=-\frac{3}{4}$.

Implicitly: We differentiate each side of the equation $x^2+y^2=25$ treating y as a function of x and then solve for y':

$$\mathrm{D}\left(x^2+y^2\right)=\mathrm{D}(25)\ \Rightarrow\ 2x+2y\cdot y'=0\ \Rightarrow y'=\frac{-2x}{2y}=-\frac{x}{y}$$

so at the point $(3,4)$, $y'=-\frac{3}{4}$, the same answer we found explicitly.

Practice 2.9.2

Find the slope of the tangent line to $y^3-3x^2=15$ at the point $(2,3)$ with and without implicit differentiation.

Answer

Explicitly: $y'=\frac{1}{3}\left(3x^2+15\right)^{-\frac{2}{3}}\mathrm{D}\left(3x^2+15\right)=\frac{1}{3}\left(3x^2+15\right)^{-\frac{2}{3}}(6x)$. When $(x,y)=(2,3)$, $y'=\frac{1}{3}\left(3(2)^2+15\right)^{\frac{2}{3}}(6\cdot 2)=4\,(27)^{-\frac{2}{3}}=\frac{4}{9}$.

Implicitly: $\mathrm{D}\left(y^3-3x^2\right)=\mathrm{D}(15)\Rightarrow 3y^2\cdot y'-6x=0$ so $y'=\frac{2x}{y^2}$.
When $(x,y)=(2,3)$, $y'=\dfrac{2\cdot 2}{3^2}=\dfrac{4}{9}$.

In the previous Example and Practice problem, it was easy to explicitly solve for y, and then we could differentiate y to get y'. Because we could explicitly solve for y, we had a choice of methods for calculating y'. Sometimes, however, we cannot explicitly solve for y and the only way to determine y' is with implicit differentiation.

Example 2.9.3

Determine y' at $(0,2)$ for $y^2+2y=\sin(x)+8$.

Solution

Assuming that y is a function of x and differentiating each side of the equation, we get:

$$\mathrm{D}\left(y^2+2y\right)=\mathrm{D}\left(\sin(x)+8\right) \;\Rightarrow\; 2y\cdot y'+2y'=\cos(x)$$
$$\Rightarrow\; (2y+2)y'=\cos(x) \;\Rightarrow\; y'=\frac{\cos(x)}{2y+2}$$

so, at the point $(0,2)$, $y'=\dfrac{\cos(0)}{2(2)+2}=\dfrac{1}{6}$. (We could have first solved the equation explicitly for y using the quadratic formula. Do you see how? Would that make the problem easier or harder than using implicit differentiation?)

? Practice 2.9.3

Determine y' at $(1,0)$ for $y+\sin(y)=x^3-x$.

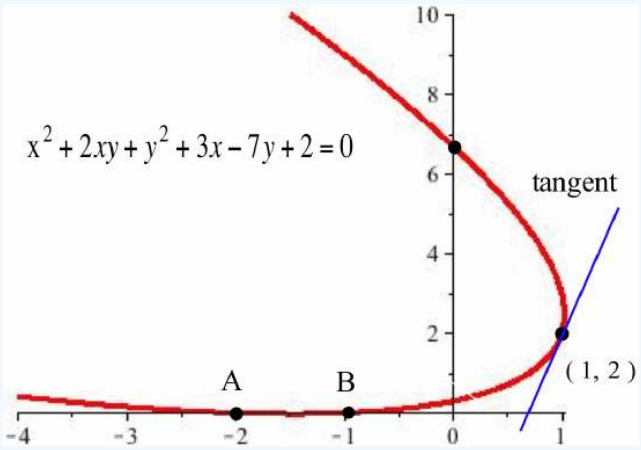
Answer

$y+\sin(y)=x^3-x \Rightarrow \mathrm{D}\left(y+\sin(y)\right)=\mathrm{D}\left(x^3-x\right) \Rightarrow y'+\cos(y)\cdot y'=3x^2-1 \Rightarrow y'\cdot(1+\cos(y))=3x^2-1$, so we have $y'=\frac{3x^2-1}{1+\cos(y)}$. When $(x,y)=(1,0)$, $y'=\dfrac{3(1)^2-1}{1+\cos(0)}=1$.

In practice, the equations may be rather complicated, but if you proceed carefully and step by step, implicit differentiation is not difficult. Just remember that **y must be treated as a function** so every time you differentiate a term containing a y you should use the Chain Rule and get something that has a y'. The algebra needed to solve for y' is always easy—if you differentiated correctly, the resulting equation will be a linear equation in the variable y'.

✓ Example 2.9.4

Find an equation of the tangent line L to the "tilted" $x^2+2xy+y^2+3x-7y+2=0$ (graphed below) at the point $(1,2)$.



Solution

The line goes through the point $(1,2)$ so we need to find the slope there. Differentiating each side of the equation, we get:

$$\mathrm{D}\left(x^2+2xy+y^2+3x-7y+2\right)=\mathrm{D}(0)$$

which yields:

$$2x+2x\cdot y'+2y+2y\cdot y'+3-7y'=0$$
$$\Rightarrow\; (2x+2y-7)y'=-2x-2y-3$$
$$\Rightarrow\; y'=\frac{-2x-2y-3}{2x+2y-7}$$

so the slope at $(1,2)$ is $m=y'=\dfrac{-2-4-3}{2+4-7}=9$. Finally, an equation for the line is $y-2=9(x-1)$ so $y=9x-7$.

Practice 2.9.4

Find the points where the parabola graphed above crosses the y-axis, and find the slopes of the tangent lines at those points.

Answer

To find where the parabola crosses the y-axis, we can set $x = 0$ and solve for the values of y: $y^2 - 7y + 2 = 0 \Rightarrow y = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(2)}}{2(1)} = \frac{7 \pm \sqrt{41}}{2} \approx 0.3$ and 6.7. The parabola crosses the y-axis (approximately) at the points $(0, 0.3)$ and $(0, 6.7)$. From Example 4, we know that $y' = \dfrac{-2x - 2y - 3}{2x + 2y - 7}$, so at the point $(0, 0.3)$, the slope is approximately $\dfrac{0 - 0.6 - 3}{0 + 0.6 - 7} \approx 0.56$, and at the point $(0, 6.7)$, the slope is approximately $\dfrac{0 - 13.4 - 3}{0 + 13.4 - 7} \approx -2.56$.

Implicit differentiation provides an alternate method for differentiating equations that can be solved explicitly for the function we want, and it is the *only* method for finding the derivative of a function that we cannot describe explicitly.

Logarithmic Differentiation

In Section 2.5 we saw that $D\left(\ln(f(x))\right) = \dfrac{f'(x)}{f(x)}$. If we simply multiply each side by $f(x)$, we have: $f'(x) = f(x) \cdot \mathrm{D}\left(\ln(f(x))\right)$. When the logarithm of a function is simpler than the function itself, it is often easier to differentiate the logarithm of f than to differentiate f itself.

Logarithmic Differentiation

$$f'(x) = f(x) \cdot \mathrm{D}\left(\ln(f(x))\right)$$

In words: The derivative of f is f times the derivative of the natural logarithm of f. Usually it is easiest to proceed in three steps:

- Calculate $\ln\left(f(x)\right)$ and simplify.
- Calculate $\mathrm{D}\left(\ln(f(x))\right)$ and simplify
- Multiply the result in the previous step by $f(x)$.

Let's examine what happens when we use this process on an "easy" function, $f(x) = x^2$, and a "hard" one, $f(x) = 2^x$. Certainly we don't need to use logarithmic differentiation to find the derivative of $f(x) = x^2$, but sometimes it is instructive to try a new algorithm on a familiar function.

$f(x) = x^2$:

- $\ln\left(f(x)\right) = \ln(x^2) = 2 \cdot \ln(x)$
- $\mathrm{D}\left(\ln\left(f(x)\right)\right) = \mathrm{D}\left(2 \cdot \ln(x)\right) = \frac{2}{x}$
- $f'(x) = f(x) \cdot \mathrm{D}\left(\ln\left(f(x)\right)\right) = x^2 \cdot \frac{2}{x} = 2x$

$f(x) = 2^x$:

- $\ln\left(f(x)\right) = \ln(2^x) = x \cdot \ln(2)$
- $\mathrm{D}\left(\ln\left(f(x)\right)\right) = \mathrm{D}\left(x \cdot \ln(2)\right) = \ln(2)$
- $f'(x) = f(x) \cdot \mathrm{D}\left(\ln\left(f(x)\right)\right) = 2^x \cdot \ln(2)$

Logarithmic differentiation is the easiest way to find the derivative of $f(x) = 2^x$ (if we don't remember the pattern for differentiating a^x from Section 2.5).

Example 2.9.5

Use the pattern $f'(x) = f(x) \cdot \mathrm{D}\left(\ln(f(x))\right)$ to find the derivative of $f(x) = (3x + 7)^5 \sin(2x)$.

Solution

Apply the natural logarithm to both sides and rewrite:

$$\ln\left(f(x)\right) = \ln\left((3x + 7)^5 \cdot \sin(2x)\right) = 5\ln(3x + 7) + \ln\left(\sin(2x)\right)$$

so:

$$\begin{aligned} \mathrm{D}\left(\ln(f(x))\right) &= \mathrm{D}\left(5\ln(3x+7)+\ln\left(\sin(2x)\right)\right) \\ &= 5\cdot\frac{3}{3x+7}+2\cdot\frac{\cos(2x)}{\sin(2x)} \end{aligned}$$

Then:

$$\begin{aligned} f'(x) &= f(x)\cdot\mathrm{D}\left(\ln(f(x))\right) \\ &= (3x+7)^5\sin(2x)\left(\frac{15}{3x+7}+2\cdot\frac{\cos(2x)}{\sin(2x)}\right) \\ &= 15(3x+7)^4\sin(2x)+2(3x+7)^5\cos(2x) \end{aligned}$$

the same result we would obtain using the Product Rule.

? Practice 2.9.5

Use logarithmic differentiation to find the derivative of $f(x)=(2x+1)^3(3x^2-4)^7(x+7)^4$.

Answer

Applying the formula $f'(x)=f(x)\cdot\mathrm{D}\left(\ln\left(f(x)\right)\right)$ to the function $f(x)=(2x+1)^3(3x^2-4)^7(x+7)^4$, we have:

$$\ln\left(f(x)\right)=3\cdot\ln(2x+1)+7\cdot\ln(3x^2-4)+4\cdot\ln(x+7)$$

so:

$$\mathrm{D}\left(\ln\left(f(x)\right)\right)=\frac{3}{2x+1}(2)+\frac{7}{3x^2-4}(6x)+\frac{4}{x+7}(1)$$

and thus:

$$f'(x)=f(x)\cdot\mathrm{D}\left(\ln\left(f(x)\right)\right)=(2x+1)^3(3x^2-4)^7(x+7)^4\cdot\left[\frac{6}{2x+1}+\frac{42x}{3x^2-4}+\frac{4}{x+7}\right]$$

We could have differentiated the functions in the previous Example and Practice problem without logarithmic differentiation. There are, however, functions for which logarithmic differentiation is the **only** method we can use. We know how to differentiate x raised to a constant power, $\mathrm{D}\left(x^p\right)=p\cdot x^{p-1}$, and a constant to a variable power, $\mathrm{D}\left(b^x\right)=b^x\ln(b)$, but the function $f(x)=x^x$ has both a variable base and a variable power, so neither differentiation rule applies. We need to use logarithmic differentiation.

✓ Example 2.9.6

Find $\mathrm{D}\left(x^x\right)$, assuming that $x>0$.

Solution

Apply the natural logarithm to both sides and rewrite:

$$\ln\left(f(x)\right)=\ln\left(x^x\right)=x\cdot\ln(x)$$

so:

$$\begin{aligned} \mathrm{D}\left(\ln\left(f(x)\right)\right) &= \mathrm{D}\left(x\cdot\ln(x)\right)=x\cdot\mathrm{D}\left(\ln(x)\right)+\ln(x)\cdot\mathrm{D}(x) \\ &= x\cdot\frac{1}{x}+\ln(x)\cdot 1=1+\ln(x) \end{aligned}$$

Then $\mathrm{D}\left(x^x\right)=f'(x)=f(x)\mathrm{D}\left(\ln\left(f(x)\right)\right)=x^x\left(1+\ln(x)\right)$.

? Practice 2.9.6

Find $\mathrm{D}\left(x^{\sin(x)}\right)$ assuming that $x > 0$.

Answer

Using $f'(x) = f(x) \cdot \mathrm{D}\left(\ln\left(f(x)\right)\right)$ with $f(x) = x^{\sin(x)}$:

$$\ln\left(f(x)\right) = \ln\left(x^{\sin(x)}\right) = \sin(x) \cdot \ln(x)$$

so:

$$D\left(\ln\left(f(x)\right)\right) = \mathrm{D}\left(\sin(x) \cdot \ln(x)\right) = \sin(x) \cdot \mathrm{D}\left(\ln(x)\right) + \ln(x) \cdot \mathrm{D}(\sin(x)) = \sin(x) \cdot \frac{1}{x} + \ln(x) \cdot \cos(x)$$

and thus:

$$f'(x) = f(x) \cdot \mathrm{D}\left(\ln\left(f(x)\right)\right) = x^{\sin(x)} \cdot \left[\frac{\sin(x)}{x} + \ln(x) \cdot \cos(x)\right]$$

Logarithmic differentiation is an alternate method for differentiating some functions such as products and quotients, and it is the only method we've seen for differentiating some other functions such as variable bases to variable exponents.

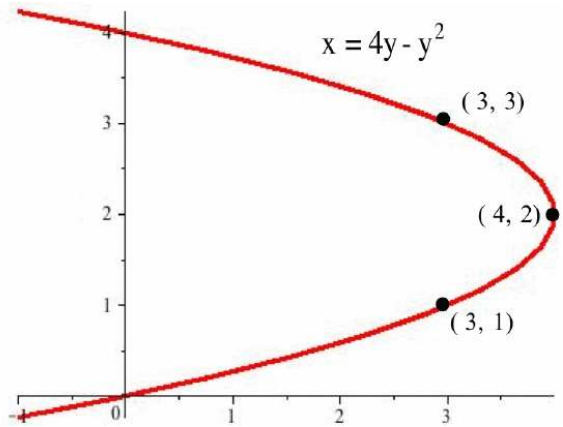
Problems

In Problems 1–10 find $\frac{dy}{dx}$ in two ways:

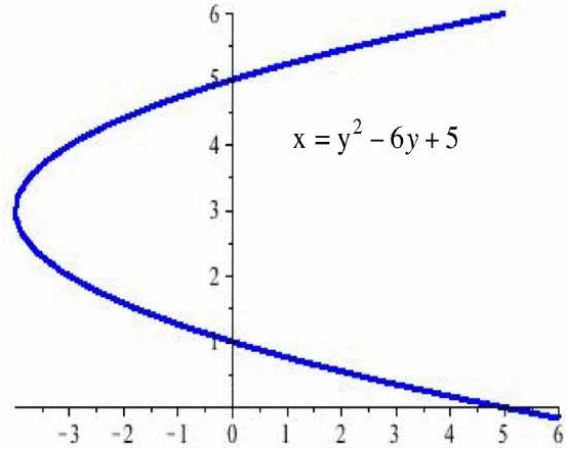
a. by differentiating implicitly and
b. by explicitly solving for y and then differentiating.

Then find the value of $\frac{dy}{dx}$ at the given point using your results from both the implicit and the explicit differentiation.

1. $x^2 + y^2 = 100$, point: $(6, 8)$
2. $x^2 + 5y^2 = 45$, point: $(5, 2)$
3. $x^2 - 3xy + 7y = 5$, point: $(2, 1)$
4. $\sqrt{x} + \sqrt{y} = 5$, point: $(4, 9)$
5. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point: $(0, 4)$
6. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point: $(3, 0)$
7. $\ln(y) + 3x - 7 = 0$, point: $(2, e)$
8. $x^2 - y^2 = 16$, point: $(5, 3)$
9. $x^2 - y^2 = 16$, point: $(5, -3)$
10. $y^2 + 7x^3 - 3x = 8$, point: $(1, 2)$
11. Find the slopes of the lines tangent to the graph of $x = 4y - y^2$ (see figure below) at the points $(3, 1)$, $(3, 3)$ and $(4, 2)$.



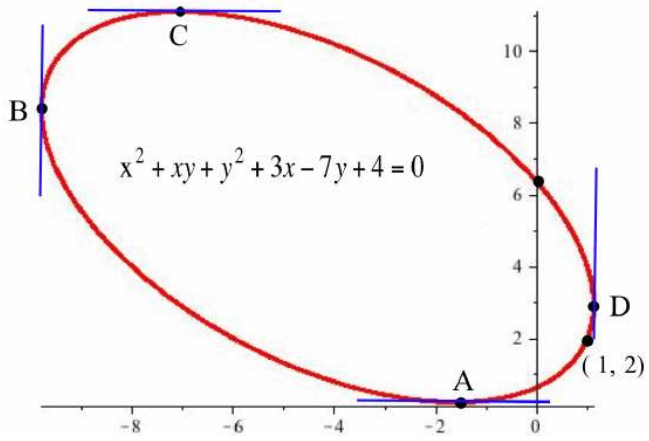
12. Find the slopes of the lines tangent to the graph of $x = 4y - y^2$ (see figure above) at the points where the graph crosses the y-axis.
13. Find the slopes of the lines tangent to the graph of $x = y^2 - 6y + 5$ (see figure below) at the points $(5, 0)$, $(5, 6)$ and $(-4, 3)$.



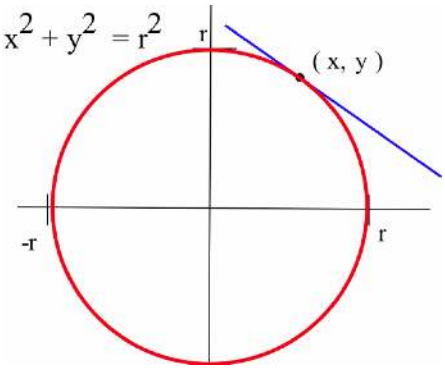
14. Find the slopes of the lines tangent to the graph of $x = y^2 - 6y + 5$ (see figure above) at the points where the graph crosses the y-axis.

In Problems 15–22, find $\frac{dy}{dx}$ using implicit differentiation and then find the slope of the line tangent to the graph of the equation at the given point.

15. $y^3 - 5y = 5x^2 + 7$, point: $(1, 3)$
16. $y^2 - 5xy + x^2 + 21 = 0$, point: $(2, 5)$
17. $y^2 + \sin(y) = 2x - 6$, point: $(3, 0)$
18. $y + 2x^2y^3 = 4x + 7$, point: $(3, 1)$
19. $e^y + \sin(y) = x^2 - 3$, point: $(2, 0)$
20. $\left(x^2+y^2+1\right)^2 - 4x^2 = 81$, point: $(0, 2\sqrt{2})$
21. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$, point: $(8, 1)$
22. $x + \cos(xy) = y + 3$, point: $(2, 0)$
23. Find the slope of the line tangent to the ellipse $x^2 + xy + y^2 + 3x - 7y + 4 = 0$ (shown in the figure below) at the point $(1, 2)$.



24. Find the slopes of the tangent lines at the points where the ellipse $x^2 + xy + y^2 + 3x - 7y + 4 = 0$ (shown in the figure above) crosses the y-axis.
25. Find y' for $y = Ax^2 + Bx + C$ and for the equation $x = Ay^2 + By + C$.
26. Find y' for $y = Ax^3 + B$ and for $x = Ay^3 + B$.
27. Find y' for $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
28. In Chapter 1 we assumed that the tangent line to a circle at a point was perpendicular to the radial line passing through that point and the center of the circle. Use implicit differentiation to prove that the line tangent to the circle $x^2 + y^2 = r^2$:



at an arbitrary point (x, y) is perpendicular to the line passing through $(0, 0)$ and (x, y).

Problems 29–31 use the figure from Problems 23–24.

29. Find the coordinates of point A where the tangent line to the ellipse is horizontal.
30. Find the coordinates of point B where the tangent line to the ellipse is vertical.
31. Find the coordinates of points C and D.

In 32–40, find $\frac{dy}{dx}$ in two ways:

a. by using the "usual" differentiation patterns and
b. by using logarithmic differentiation.

32. $y = x \cdot \sin(3x)$
33. $y = (x^2 + 5)^7 (x^3 - 1)^4$
34. $y = \dfrac{\sin(3x - 1)}{x + 7}$
35. $y = x^5 \cdot (3x + 2)^4$
36. $y = 7^x$
37. $y = e^{\sin(x)}$
38. $y = \cos^7(2x + 5)$
39. $y = \sqrt{25 - x^2}$
40. $y = \dfrac{x \cdot \cos(x)}{x^2 + 1}$

In Problems 41–46, use logarithmic differentiation to find $\frac{dy}{dx}$.

41. $y = x^{\cos(x)}$
42. $y = (\cos(x))^x$
43. $y = x^4 \cdot (x - 2)^7 \cdot \sin(3x)$
44. $y = \dfrac{\sqrt{x + 10}}{(2x + 3)^3 \cdot (5x - 1)^7}$
45. $y = (3 + \sin(x))^x$
46. $y = \sqrt{\dfrac{x^2 + 1}{x^2 - 1}}$

In Problems 47–50, use the values in each table to calculate the values of the derivative in the last column.

47.

x	$f(x)$	$\ln(f(x))$	$D(\ln(f(x)))$	$f'(x)$
1	1	0.0	1.2	
2	9	2.2	1.8	
3	64	4.2	2.1	

48.

x	$g(x)$	$\ln(g(x))$	$\mathrm{D}(\ln(g(x)))$	$g'(x)$
1	5	1.6	0.6	
2	10	2.3	0.7	
3	20	3.0	0.8	

49.

x	$f(x)$	$\ln(f(x))$	$\mathrm{D}(\ln(f(x)))$	$f'(x)$
1	5	1.6	-1	
2	2	0.7	0	
3	7	1.9	2	

50.

x	$g(x)$	$\ln(g(x))$	$\mathrm{D}(\ln(g(x)))$	$g'(x)$
2	1.4	0.3	1.2	
3	3.3	1.2	0.6	
7	13.6	2.6	0.2	

Problems 51–55 illustrate how logarithmic differentiation can be used to verify some differentiation patterns we already know (51–52, 54) and to derive some new patterns (53, 55). Assume that all of the functions are differentiable and that the function combinations are defined.

51. Use logarithmic differentiation on $f \cdot g$ to re-derive the Product Rule: $\mathrm{D}(f \cdot g) = f \cdot g' + g \cdot f'$.
52. Use logarithmic differentiation on $\frac{f}{g}$ to re-derive the quotient rule: $\mathrm{D}\left(\frac{f}{g}\right) = \frac{g \cdot f' - f \cdot g'}{g^2}$.
53. Use logarithmic differentiation to obtain a product rule for three functions: $\mathrm{D}(f \cdot g \cdot h) =$?.
54. Use logarithmic differentiation on the exponential function a^x (with $a > 0$) to show that its derivative is $a^x \ln(a)$.
55. Use logarithmic differentiation to determine a pattern for the derivative of f^g: $\mathrm{D}(f^g) =$?.
56. In Section 2.1 we proved the Power Rule $\mathrm{D}(x^n) = n \cdot x^{n-1}$ for any positive integer n.
    a. Why does this formula hold for $n = 0$?
    b. Use the Quotient Rule to prove that $\mathrm{D}(x^{-m}) = -m \cdot x^{-m-1}$ for any positive integer m and conclude that the Power Rule holds for all integers.
    c. Now let $y = x^{\frac{p}{q}}$ where p and q are integers so that $y^q = x^p$. Use implicit differentiation to show that the Power Rule holds for all rational exponents. (We still have not considered the case where $y = x^a$ with a an irrational number, because we haven't actually *defined* what x^a means for a irrational. We will take care of that—and the extension of the Power Rule to all real exponents—in Chapter 7.)

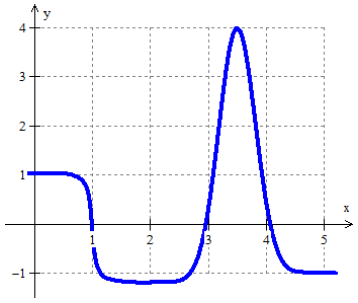
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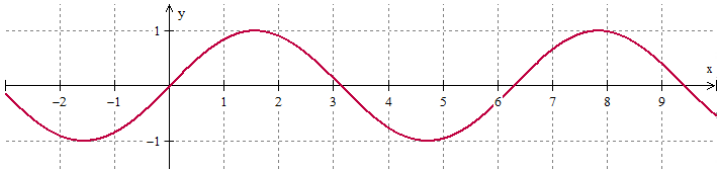
2.A: Answers

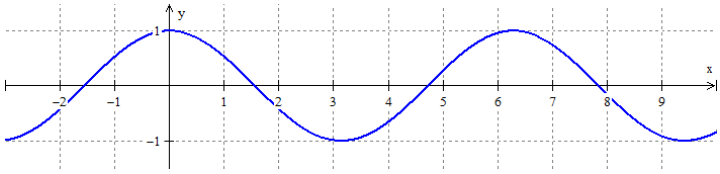
Section 2.0

1. Values are approximate; your answers may vary.

x	$f(x)$	$m(x)$	x	$f(x)$	$m(x)$
0.0	1.0	1	2.5	0.0	-2
0.5	1.4	$\frac{1}{2}$	3.0	-1.0	-2
1.0	1.6	0	3.5	-1.3	0
1.5	1.4	$-\frac{1}{2}$	4.0	-1.0	1
2.0	1.0	-2			

3. a. At $x = 1$, 3 and 4.
   b. Largest at $x = 4$; smallest at $x = 3$.
   c.


5. a.


   b.


   c. $m(x) = \cos(x)$

7. Assume we turn off the engine at the point (p, q) on the curve $y = x^2$ and then find values of p and q so the tangent line to $y = x^2$ at the point (p, q) goes through the given point $(5, 16)$: (p, q) is on $y = x^2$ so $q = p^2$ and %an equation of the tangent line to $y = x^2$ at (p, p^2) is $y = 2px - p^2$ so, substituting $x = 5$ and $y = 16$, we have $16 = 2p(5) - p^2 \Rightarrow p^2 - 10p + 16 = 0 \Rightarrow p = 2$ or $p = 8$. The solution we want (moving left to right along the curve) is $p = 2$, $q = p^2 = 4$ ($p = 8$, $q = 64$ would be the right-to-left solution).

9. Impossible: $(1, 3)$ sits "inside" the parabola.

11. a. $m_{\text{sec}} = \dfrac{f(x+h) - f(x)}{(x+h) - x} = \dfrac{[3(x+h) - 7] - [3x - 7]}{(x+h) - x} = \dfrac{3h}{h} = 3$
    b. $m_{\text{tan}} = \lim\limits_{h \to 0} m_{\text{sec}} = \lim\limits_{h \to 0} 3 = 3$.
    c. At $x = 2$, $m_{\text{tan}} = 3$.
    d. $f(2) = -1$ so the tangent line is $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

11. a. $m_{\text{sec}} = \dfrac{[a(x+h) + b] - [ax + b]}{(x+h) - x} = \dfrac{ah}{h} = a$
    b. $m_{\text{tan}} = \lim\limits_{h \to 0} m_{\text{sec}} = \lim\limits_{h \to 0} a = a$.
    c. At $x = 2$, $m_{\text{tan}} = a$.
    d. $f(2) = 2a + b$ so the tangent line is $y - (2a + b) = a(x - 2)$ or $y = ax + b$.

13. a. $m_{\text{sec}} = \frac{[8 - 3(x+h)^2] - [8 - 3x^2]}{(x+h) - x} = -6x - 3h$
    b. $m_{\text{tan}} = \lim\limits_{h \to 0} m_{\text{sec}} = \lim\limits_{h \to 0} [-6x - 3h] = -6x$.
    c. At $x = 2$, $m_{\text{tan}} = -6(2) = -12$
    d. $f(2) = -4$ so the tangent line is $y - (-4) = -12(x - 2)$ or $y = -12x + 20$.

15. $a=1, b=2, c=0 \Rightarrow m_{\tan}=(2)(1)(x)+2=2x+2$ so we need p to satisfy $6-(p^2+2p)=(2p+2)(3-p) \Rightarrow p^2-6p=0 \Rightarrow p=0$ or $p=6$ so the points are $(0,0)$ and $(6,48)$.

Section 2.1

1. derivative
   a. of g
   b. of h
   c. of f
3. a. $m_{\text{sec}}=h-4 \Rightarrow m_{\tan}=\lim\limits_{h\to 0} m_{\text{sec}}=-4$
   b. $m_{\text{sec}}=h+1 \Rightarrow m_{\tan}=\lim\limits_{h\to 0} m_{\text{sec}}=1$
5. a. $m_{\text{sec}}=5-h \Rightarrow m_{\tan}=\lim\limits_{h\to 0} m_{\text{sec}}=5$
   b. $m_{\text{sec}}=7-2x-h \Rightarrow m_{\tan}=7-2x$
7. a. -1
   b. -1
   c. 0
   d. $+1$
   e. DNE
   f. DNE
9. Using the definition:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[(x+h)^2+8]-[x^2+8]}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} [2x+h] = 2x \end{aligned}$$

   so $f'(3)=2\cdot 3=6$
11. Using the definition:

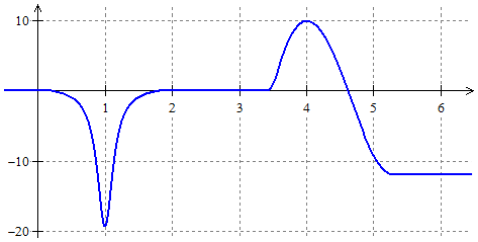
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[2(x+h)^3-5(x+h)]-[2x^3-5x]}{h} \\ &= \lim_{h\to 0} \frac{6x^2h+6xh^2+2h^3-5h}{h} \\ &= \lim_{h\to 0} \left[6x^2+6xh+2h^2-5\right] = 6x^2-5 \end{aligned}$$

   so $f'(3)=6\cdot 3^2-5=49$
13. For any constant C, if $f(x)=x^2+C$, then:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2+C]-[x^2+C]}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} [2x+h] = 2x \end{aligned}$$

   The graphs of $f(x)=x^2$, $g(x)=x^2+3$ and $h(x)=x^2-5$ are "parallel" parabolas: g is f shifted up 3 units; h is f shifted down 5 units.
15. $f'(x)=2x \Rightarrow f'(1)=2$ so an equation of the tangent line at $(1,9)$ is $y-9=2(x-1)$ or $y=2x+7$; $f'(-2)=-4$ so an equation of the tangent line at $(-2,12)$ is $y-12=-4(x+2)$ or $y=-4x+4$.
17. $f'(x)=\cos(x) \Rightarrow f'(\pi)=\cos(\pi)=-1$ so an equation of the tangent line at $(\pi,0)$ is $y-0=-1(x-\pi)$ or $y=-x+\pi$; $f'\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0$ so an equation of the tangent line at $\left(\frac{\pi}{2},1\right)$ is $y-1=0\left(x-\frac{\pi}{2}\right)$ or $y=1$.
19. a. $y-5=4(x-2)$ or $y=4x-3$
    b. $x+4y=22$ or $y=-0.25x+5.5$
    c. $f'(x)=2x$, so the tangent line is horizontal only if $x=0$: at the point $(0,1)$.
    d. $f'(p)=2p$ (the slope of the tangent line) so $y-q=2p(x-p)$ or $y=2px+(q-2p^2)$. Since $q=p^2+1$, the tangent line is $y=2px+(p^2+1-2p^2)=2px-p^2+1$.
    e. We need p such that $-7=2p(1)-p^2+1$, so $p^2-2p-8=0 \Rightarrow p=-2$ or $p=4$. There are two such points: $(-2,5)$ and $(4,17)$.
21. a. $y'(x)=2x$, so $y'(1)=2$ and the angle is $\arctan(2)\approx 1.107$ radians $\approx 63^\circ$
    b. $y'(x)=3x^2$ so $y'(1)=3$ and the angle is $\arctan(3)\approx 1.249$ radians $\approx 72^\circ$
    c. $1.249-1.107$ radians = 0.142 radians (or $71.57^\circ-63.43^\circ\approx 8.1^\circ$)

23. Units on the horizontal axis are “seconds”; units on the vertical axis are “feet per second.”



25. a. $d(4) = 256$ ft; $d(5) = 400$ ft
    b. $d'(x) = 32x \Rightarrow d'(4) = 128$ ft/sec and $d'(5) = 160$ ft/sec.
27. Marginal production cost is $C'(x) = \frac{1}{2\sqrt{x}}$ dollars per golf ball, so $C'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10} = 0.10$ per ball and $C'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05$ per ball.
29. a. $A(0) = 0$, $A(1) = \frac{1}{2}$, $A(2) = 2$ and $A(3) = \frac{9}{2}$.
    b. $A(x) = \dfrac{x^2}{2}$ $(x \geq 0)$
    c. $A'(x) = x$
    d. $A'(x)$ represents the rate at which $A(x)$ is increasing (the rate at which area is accumulating).
31. a. $9x^8$
    b. $\dfrac{2}{3x^{\frac{1}{3}}}$
    c. $\dfrac{-4}{x^5}$
    d. $\pi x^{\pi-1}$
    e. 1 if $x > -5$ and -1 if $x < -5$
33. $f(x) = x^3 + 4x^2$ (plus any constant)
35. $f(t) = 5 \cdot \sin(t)$
37. $f(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3$

Section 2.2

1. a. $0, 1, 2, 3, 5$
   b. $0, 3, 5$
3. In the table below, **und.** means “undefined”:

x	$f(x) \cdot g(x)$	$(f(x) \cdot g(x))'$	$\frac{f(x)}{g(x)}$	$\left(\frac{f(x)}{g(x)}\right)'$
0	2	13	2	-7
1	-15	16	$-\frac{3}{5}$	$\frac{4}{25}$
2	0	-6	0	$-\frac{3}{2}$
3	0	3	und.	und.

5.

x	$f+g$	$(f+g)'$	$f \cdot g$	$(f \cdot g)'$	$\frac{f}{g}$	$\left(\frac{f}{g}\right)'$
1	5	0	6	2	$\frac{3}{2}$	$-\frac{10}{4}$
2	4	$\frac{1}{2}$	3	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{18}$
3	4	0	4	0	1	1

7. a. $\mathrm{D}\left((x-5)(3x+7)\right) = (x-5) \cdot 3 + (3x+7) \cdot 1$
   b. $\mathrm{D}\left(3x^2 - 8x - 35\right) = 6x - 8$ (the same result)
9. $19 \cdot 3x^2 = 57x^2$
10. $\cos(x) - \sin(x)$
13. $x^2 \cdot [-\sin(x)] + \cos(x) \cdot 2x$
15. Applying the Product Rule:

$$\begin{aligned} \mathrm{D}\left(\sin^2(x)\right) &= \sin(x) \cdot \cos(x) + \sin(x) \cdot \cos(x) \\ &= 2\sin(x)\cos(x) = \sin(2x) \end{aligned}$$

17. $\dfrac{d}{dx}\left(\dfrac{\cos(x)}{x^2}\right) = \dfrac{x^2(-\sin(x)) - (\cos(x))(2x)}{(x^2)^2} = \dfrac{-x\left[x\sin(x) + 2\cos(x)\right]}{x^4} = -\dfrac{x\sin(x) + 2\cos(x)}{x^3}$
19. $\dfrac{(1+x^2)\cdot 0 - 1 \cdot 2x}{(1+x^2)^2} = -\dfrac{2x}{(1+x^2)^2}$

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21. $\dfrac{\cos(\theta)\cdot 0 - 1\cdot(-\sin(\theta))}{(\cos(\theta))^2} = \dfrac{\sin(\theta)}{\cos^2(\theta)}$
23. $\dfrac{\cos(\theta)\cdot\cos(\theta) - \sin(\theta)\cdot(-\sin(\theta))}{(\cos(\theta))^2} = \dfrac{1}{\cos^2(\theta)}$
25. $40x^4 - 12x^3 + 6x^2 + 14x - 12$
27. $f(x) = ax^2 + bx + c$ so $f(0) = c$, hence $f(0) = 0 \Rightarrow c = 0$. $f'(x) = 2ax + b$ so $f'(0) = b$, hence $f'(0) = 0 \Rightarrow b = 0$. Finally, $f'(10) = 20a + b = 20a$ so $f'(10) = 30 \Rightarrow 20a = 30 \Rightarrow a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2}x^2 + 0x + 0 = \frac{3}{2}x^2$.
29. Their graphs are vertical shifts of each other, and their derivatives are equal.
31. $f(x)\cdot g(x) = k \Rightarrow \mathrm{D}\left(f(x)\cdot g(x)\right) = \mathrm{D}(k) = 0$ so $f(x)\cdot g'(x) + g(x)\cdot f'(x) = 0$. If $f(x) \neq 0$ and $g(x) \neq 0$, then $\frac{f'(x)}{f(x)} = -\frac{g'(x)}{g(x)}$.
33. a. $f'(x) = 2x - 5$, so $f'(1) = -3$
    b. $f'(x) = 0$ only if $x = \frac{5}{2}$.
35. a. $f'(x) = 3 + 2\sin(x)$: $f'(1) = 3 + 2\sin(1) \approx 4.7$
    b. $f'(x) > 0$ because $\sin(x) > -\frac{3}{2}$ for all x.
37. a. $f'(x) = 3x^2 + 18x = 3x(x + 6)$, so $f'(1) = 21$
    b. $f'(x) = 0$ only if $x = 0$ or $x = -6$.
39. a. $f'(x) = 3x^2 + 4x + 2$, so $f'(1) = 9$
    b. $f'(x) > 0$ (the discriminant $4^2 - 4(3)(2) < 0$).
41. a. $f'(x) = x\cos(x) + \sin(x) \Rightarrow f'(1) = \cos(1) + \sin(1) \approx 1.38$
    b. The graph of $f'(x)$ crosses the x-axis infinitely often. The root of f' at $x = 0$ is easy to see; approximate others, such as those near $x = 2.03$ and 4.91 and -2.03, using technology.
43. $f'(x) = 3x^2 + 2Ax + B$: the graph of $y = f(x)$ has two distinct "vertices" if $f'(x) = 0$ for two distinct values of x. This occurs if the discriminant of $3x^2 + 2Ax + B > 0$: $(2A)^2 - 4(3)(B) > 0$.
45. Everywhere except at $x = (2k+1)\frac{\pi}{2}$.
47. Everywhere except at $x = 0$ and $x = 3$.
49. Everywhere except at $x = 1$.
51. Everywhere. The only possible difficulty is at $x = 0$: the definition of the derivative gives $f'(0) = 1$ and the derivatives of the "two pieces" of f match (and equal 1) at $x = 0$.
53. Continuity of f at $x = 1$ requires $A + B = 2$. The "left derivative" of f at $x = 1$ is $\mathrm{D}(Ax + B) = A$ and the "right derivative" of f at $x = 1$ is 3 (if $x > 1$ then $\mathrm{D}(x^2 + x) = 2x + 1$), so to achieve differentiability we need $A = 3$ and $B = 2 - A = -1$.
55. a. $h(x) = 128x - 2.65x^2 \Rightarrow h'(x) = 128 - 5.3x$ so $h'(0) = 128$ ft/sec, $h'(1) = 122.7$ ft/sec and $h'(2) = 117.4$ ft/sec
    b. $v(x) = h'(x) = 128 - 5.3x$ ft/sec
    c. $v(x) = 0$ when $x = \frac{128}{5.3} \approx 24.15$ sec
    d. $h(24.15) \approx 1545.66$ ft
    e. about 48.3 seconds: 24.15 up and 24.15 down
57. a. $h(x) = v_o x - 16x^2 \Rightarrow h'(x) = v_o - 32x$ ft/sec
    b. Max height when $x = \frac{v_o}{32}$: max height $= h\left(\frac{v_o}{32}\right) = v_o\left(\frac{v_o}{32}\right) - 16\left(\frac{v_o}{32}\right)^2 = \frac{(vo)^2}{64}$ ft
    c. Time aloft $= 2\left(\frac{v_o}{32}\right) = \frac{v_o}{16}$ sec
59. a. $\frac{(v_o)^2}{64} = 3.75 \Rightarrow v_o = 8\sqrt{3.75} \approx 15.5$ ft/sec
    b. $2\left(\frac{v_o}{32}\right) = \frac{8\sqrt{3.75}}{16} \approx 0.97$ sec
    c. Max lift $= \frac{(v_o)^2}{2g} = \frac{(8\sqrt{3.75})^2}{2(5.3)} = \frac{240}{10.6} \approx 22.64$ ft
61. a. $y' = -\frac{1}{x^2}$ so $y'(2) = -\frac{1}{4}$ and the tangent line is $y - 1/2 = (-\frac{1}{4})(x - 2)$ or $y = -\frac{1}{4}x + 1$
    b. x-intercept at $(4, 0)$; y-intercept at $(0, 1)$
    c. $A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(4)(1) = 2$
63. Because $(1, 4)$ and $(3, 14)$ are on the parabola, we need $a + b + c = 4$ and $9a + 3b + c = 14$. Subtracting the first equation from the second, $8a + 2b = 10$; $f'(x) = 2ax + b$ so $f'(3) = 6a + b = 9$, the slope of $y = 9x - 13$. Now solve the system $8a + 2b = 10$ and $6a + b = 9$ to get $a = 2$ and $b = -3$. Then use $a + b + c = 4$ to get $c = 5$: $a = 2$, $b = -3$, $c = 5$.
65. a. $f(x) = x^3$
    b. $g(x) = x^3 + 1$
    c. If $h(x) = x^3 + C$ for any constant C, then $\mathrm{D}\left(h(x)\right) = 3x^2$.
67. a. For $0 \le x \le 1$, $f'(x) = 1$ so $f(x) = x + C$; because $f(0) = 0$, we know $C = 0$ and $f(x) = x$. For $1 \le x \le 3$, $f'(x) = 2 - x$ so $f(x) = 2x - \frac{1}{2}x^2 + K$; because $f(1) = 1$, we know $K = -\frac{1}{2}$ and $f(x) = 2x - \frac{1}{2}x^2 - \frac{1}{2}$. For $3 \le x \le 4$, $f'(x) = x - 4$ so $f(x) = \frac{1}{2}x^2 - 4x + L$; because $f(3) = 1$, we know $L = \frac{17}{2}$ and $f(x) = \frac{1}{2}x^2 - 4x + \frac{17}{2}$.
    b. A vertical shift, up 1 unit, of the graph in (a).

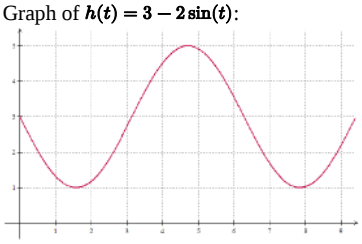
Section 2.3

1. $\mathrm{D}(f^2(x)) = 2\cdot f^1(x)\cdot f'(x)$; at $x = 1$, $\mathrm{D}(f^2(x)) = 2(2)(3) = 12$. $\mathrm{D}(f^5(x)) = 5\cdot f^4(x)\cdot f'(x)$; at $x = 1$, $\mathrm{D}(f^5(x)) = 5(2^4)(3) = 240$. $\mathrm{D}(f^{\frac{1}{2}}(x)) = \frac{1}{2}\cdot f^{-\frac{1}{2}}(x)\cdot f'(x)$; at $x = 1$, $\mathrm{D}(f^{\frac{1}{2}}(x)) = (\frac{1}{2})(2^{-\frac{1}{2}})(3) = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$.

3.

x	$f(x)$	$f'(x)$	$D(f^2)$	$D(f^3)$	$D(f^5)$
1	1	-1	-2	-3	-5
3	2	-3	-12	-36	-240

5. $f'(x) = 5 \cdot (2x-8)^4 \cdot (2) = 10(2x-8)^4$
7. $f'(x) = x \cdot 5 \cdot (3x+7)^4 \cdot 3 + 1 \cdot (3x+7)^5 = (3x+7)^4[15x + (3x+7)] = (3x+7)^4(18x+7)$
9. $f'(x) = \frac{1}{2}(x^2+6x-1)^{-\frac{1}{2}} \cdot (2x+6)$, which we can rewrite as $\frac{x+3}{\sqrt{x^2+6x-1}}$
11. a. Graph of $h(t) = 3 - 2\sin(t)$:



b. When $t = 0$, $h(0) = 3$ feet.
c. highest: 5 feet above the floor; lowest: 1 foot above the floor.
d. $v(t) = h'(t) = -2\cos(t)$ ft/sec, $a(t) = v'(t) = 2\sin(t)$ ft/sec^2.
e. This spring oscillates forever. The motion of a real spring would "damp out" due to friction.

13. a. If $h(t) = 5t$, then $v(t) = h'(t) = 5$ and $K(1) = K(2) = \frac{1}{2}m(5^2) = 12.5m$.
b. If $h(t) = t^2$, then $v(t) = h'(t) = 2t$ so $v(1) = 2$ and $v(2) = 4$, hence $K(1) = \frac{1}{2}m(2^2) = 2m$ and $K(2) = \frac{1}{2}m(4^2) = 8m$.
15. $x \cdot (\sin(x))' + \sin(x) \cdot (x)' = x\cos(x) + \sin(x)$
17. $f'(x) = e^x - \sec(x) \cdot \tan(x)$
19. $f'(x) = -e^{-x} + \cos(x)$
21. $f'(x) = 7(x-5)^6(1)$ so $f'(4) = 7(-1)^6(1) = 7$. Tangent line: $y-(-1) = 7(x-4)$ or $y = 7x - 29$.
23. $f'(x) = \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{25-x^2}}$ so $f'(3) = \frac{-3}{\sqrt{25-9}} = -\frac{3}{4}$. Tangent line: $y-4 = -\frac{3}{4}(x-3)$ or $3x+4y = 25$.
25. $f'(x) = 5(x-a)^4(1)$ so $f'(a) = 5(a-a)^4(1) = 0$. Tangent line: $y-0 = 0(x-a)$ or $y = 0$.
27. a. $f'(x) = e^x$ so $f'(3) = e^3$. Tangent line: $y - e^3 = e^3(x-3)$ or $y = e^3x - 2e^3$.
b. $0 - e^3 = e^3(x-3) \Rightarrow -1 = x-3 \Rightarrow x = 2$
c. $f'(p) = e^p$ so tangent line at (p, e^p) is $y - e^p = e^p(x-p)$; x-intercept: $0 - e^p = e^p(x-p) \Rightarrow -1 = x - p \Rightarrow x = p-1$
29. $f'(x) = -\sin(x) \Rightarrow f''(x) = -\cos(x)$
31. $f'(x) = x^2\cos(x) + 2x\sin(x) \Rightarrow f''(x) = -x^2\sin(x) + 2x\cos(x) + 2x\cos(x) + 2\sin(x) = -x^2\sin(x) + 4x\cos(x) + 2\sin(x)$
33. $f'(x) = e^x\cos(x) - e^x\sin(x) \Rightarrow f''(x) = -2e^x\sin(x)$
35. q' = linear, q'' = constant, $q''' = q^{(4)} = \cdots = 0$
37. $p^{(n)}$ = constant $\Rightarrow p^{(n+1)} = 0$
39. $f(x) = 5e^x$
41. $f(x) = (1+e^x)^5$
43. No. Using the definition of the derivative:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(0+h)\cdot\sin\left(\frac{1}{0+h}\right)-0}{h}$$

which simplifies to $\lim_{h\to 0}\sin\left(\frac{1}{h}\right)$; to see that this last limit does not exist, graph $\sin\left(\frac{1}{h}\right)$ or evaluate $\sin\left(\frac{1}{h}\right)$ for small values of h.
45. $\left(1+\frac{1}{x}\right)^x \approx 2.718\ldots = e$ when x is large.
47. a. $s_2 = 2.5$, $s_3 \approx 2.67$, $s_4 \approx 2.708$, $s_5 \approx 2.716$, $s_6 \approx 2.718$, $s_7 \approx 2.71825$, $s8 \approx 2.718178$
b. They are approaching e.

Section 2.4

1. $f(x) = x^5$, $g(x) = x^3 - 7x \Rightarrow f \circ g(x) = (x^3-7x)^5$
3. Setting $f(x) = x^{\frac{5}{2}}$ and $g(x) = 2 + \sin(x)$ yields $f \circ g(x) = (2+\sin(x))^{\frac{5}{2}}$. (The functions $f(x) = \sqrt{x}$ and $g(x) = (2+\sin(x))^5$ also work.)
5. $f(x) = |x|$, $g(x) = x^2 - 4 \Rightarrow f \circ g(x) = |x^2-4|$
7. $y = u^5$, $u = x^3 - 7x$; $y = u^4$, $u = \sin(3x-8)$; $y = u^{\frac{5}{2}}$, $u = 2 + \sin(x)$; $y = \frac{1}{\sqrt{u}}$, $u = x^2 + 9$; $y = |u|$, $u = x^2 - 4$; $y = \tan(u)$, $u = \sqrt{x}$

9.

x	$f\circ g(x)$	$(f\circ g)'(x)$
-2	1	0
-1	1	2
0	0	1
1	2	2
2	-2	-2

11. $g(2)\approx 2$, $g'(2)\approx -1$, $(f\circ g)(2)=f(g(2))\approx f(2)\approx 1$, $f'(g(2))\approx f'(2)\approx 0$, $(f\circ g)'(2)=f'(g(2))\cdot g'(2)\approx 0$
13. $\mathrm{D}\left(\left(1-\frac{3}{x}\right)^4\right)=4\left(1-\frac{3}{x}\right)^3\cdot\frac{3}{x^2}$
15. $5\left(-\frac{1}{2}\right)(2+\sin(x))^{-\frac{3}{2}}\cdot\cos(x)=\frac{-5\cos(x)}{2(2+\sin(x))^{\frac{3}{2}}}$
17. $x^2\cdot[\cos(x^2+3)\cdot 2x]+\sin(x^2+3)\cdot 2x=2x\left[x^2\cos(x^2+3)+\sin(x^2+3)\right]$
19. $\frac{7}{\cos(x^3-x)}=7\sec(x^3-x)$ so $\mathrm{D}(7\sec(x^3-x))=7(3x^2-1)\cdot\sec(x^3-x).\tan(x^3-x)$
21. $\mathrm{D}\left(e^x+e^{-x}\right)=e^x-e^{-x}$
23. a. $h(0)=2$ feet above the floor.
    b. $h(t)=3-\cos(2t)$ ft, $v(t)=h'(t)=2\sin(2t)$ ft/sec, $a(t)=v'(t)=4\cos(2t)$ ft/sec^2
    c. $K=\frac{1}{2}mv^2=\frac{1}{2}m\left(2\sin(2t)\right)^2=2m\sin^2(2t)$, $\frac{dK}{dt}=8m\cdot\sin(2t)\cdot\cos(2t)$
25. a. $P(0)=14.7$ psi (pounds per square inch), $P(30000)\approx 4.63$ psi
    b. $10=14.7e^{-0.0000385h}\Rightarrow\frac{10}{14.7}=e^{-0.0000385h}\Rightarrow h=\frac{1}{-0.0000385}\ln\left(\frac{10}{14.7}\right)\approx$ 10,007 ft
    c. $\frac{dP}{dh}=14.7(-0.0000385)e^{-0.0000385h}$ psi/ft. At $h=$ 2,000 ft, $\frac{dP}{dh}=14.7(-0.0000385)e^{-0.0000385(2000)}$ psi/ft ≈ -0.000524 psi/ft. Finally, $\frac{dP}{dt}=500(-0.000524)\approx -0.262$ psi/min.
    d. If temperature is constant, then (from physics!) we know (pressure)(volume) is constant, so decreasing pressure means increasing volume.
27. $\frac{2\cos(z)\left[-\sin(z)\right]}{2\sqrt{1+\cos^2(z)}}=\frac{-\sin(2z)}{2\sqrt{1+\cos^2(z)}}$
29. $\frac{d}{dx}\tan(3x+5)=3\cdot\sec^2(3x+5)$
31. $\mathrm{D}\left(\sin\left(\sqrt{x+1}\right)\right)=\cos\left(\sqrt{x+1}\right)\cdot\frac{1}{2\sqrt{x+1}}$
33. $\frac{d}{dx}\left(e^{\sin(x)}\right)=e^{\sin(x)}\cdot\cos(x)$
35. $f(x)=\sqrt{x}\Rightarrow f'(x)=\frac{1}{2\sqrt{x}}$; $x(t)=2+\frac{21}{t}\Rightarrow x'(t)=-\frac{21}{t^2}$. At $t=3$, $x=9$ and $x'(t)=-\frac{21}{9}=-\frac{7}{3}$ so $\frac{d}{dt}\left(f(x(t))\right)=\left(\frac{1}{2\sqrt{9}}\right)\left(-\frac{7}{3}\right)=-\frac{7}{18}$.
37. $f(x)=\tan^3(x)\Rightarrow f'(x)=3\cdot\tan^2(x)\cdot\sec^2(x)$; $x(t)=8\Rightarrow x'(t)=0$. When $t=3$, $x=8$ and $x'(t)=0$ so $\frac{d}{dt}\left(f(x(t))\right)=0$.
39. $f(x)=\frac{1}{77}(7x-13)^{11}$
41. $f(x)=-\frac{1}{2}\cos(2x-3)$
43. $f(x)=e^{\sin(x)}$
45. $-2\sin(2x)=2\cos(x)\left[-\sin(x)\right]-2\sin(x)\cdot\cos(x)$ or $\sin(2x)=2\sin(x)\cdot\cos(x)$
47. $3\cos(3x)=3\cos(x)-12\sin^2(x)\cdot\cos(x)$ so $\cos(3x)=\cos(x)\left[1-4\sin^2(x)\right]=\cos(x)\left[1-4+4\cos^2(x)\right]=4\cos^3(x)-3\cos(x)$
49. $y'=3Ax^2+2Bx$
51. $y'=2Ax\cdot\cos\left(Ax^2+B\right)$
53. $y'=\dfrac{Bx}{\sqrt{A+Bx^2}}$
55. $y'=B\cdot\sin(Bx)$
57. $y'=-2Ax\cdot\sin(Ax^2+B)$
59. $y'=x\left(B\cdot e^{Bx}\right)+e^{Bx}=(Bx+1)\cdot e^{Bx}$
61. $y'=A\cdot e^{Ax}+A\cdot e^{-Ax}$
63. $y'=\dfrac{A\cdot\sin(Bx)-Ax\cdot B\cdot\cos(Bx)}{\sin^2(Bx)}$
65. $y'=\dfrac{(Cx+D)A-(Ax+B)C}{(Cx+D)^2}=\dfrac{AD-BC}{(Cx+D)^2}$
67. a. $y'=AB-2Ax$
    b. $x=\frac{AB}{2A}=\frac{B}{2}$
    c. $y''=-2A$
69. a. $y'=2ABx-3Ax^2=Ax(2B-3x)$
    b. $x=0$, $\frac{2B}{3}$
    c. $y''=2AB-6Ax$
71. a. $y'=3Ax^2+2Bx=x(3Ax+2B)$
    b. $x=0$, $-\frac{2B}{3A}$
    c. $y''=6Ax+2B$
73. $\dfrac{d}{dx}\left(\arctan(x^2)\right)=\dfrac{2x}{1+x^4}$

75. $\mathrm{D}\left(\arctan(e^x)\right) = \dfrac{1}{1+(e^x)^2}\cdot e^x = \dfrac{e^x}{1+e^{2x}}$

77. $\mathrm{D}(\arcsin(x^3)) = \dfrac{3x^2}{\sqrt{1-x^6}}$

79. $\dfrac{d}{dt}\left(\arcsin(e^t)\right) = \dfrac{1}{\sqrt{1-(e^t)^2}}\cdot e^t = \dfrac{e^t}{\sqrt{1-e^{2t}}}$

81. $\dfrac{d}{dx}\left(\ln(\sin(x))\right) = \dfrac{1}{\sin(x)}\cdot\cos(x) = \cot(x)$

83. $\frac{d}{ds}\left(\ln(e^s)\right) = \frac{1}{e^s}\cdot e^s = 1$ or $\frac{d}{ds}\left(\ln(e^s)\right) = \frac{d}{ds}(s) = 1$

Section 2.5

1. $\mathrm{D}(\ln(5x)) = \dfrac{1}{5x}\cdot 5 = \dfrac{1}{x}$

3. $\mathrm{D}(\ln(x^k)) = \dfrac{1}{x^k}\cdot kx^{k-1} = \dfrac{k}{x}$

5. $\mathrm{D}(\ln(\cos(x))) = \dfrac{1}{\cos(x)}\cdot(-\sin(x)) = -\tan(x)$

7. $\mathrm{D}(\log_2(5x)) = \dfrac{1}{5x}\cdot\dfrac{1}{\ln(2)}\cdot(5) = \dfrac{1}{x\ln(2)}$

9. $\mathrm{D}(\ln(\sin(x))) = \dfrac{1}{\sin(x)}\cdot\cos(x) = \cot(x)$

11. $\mathrm{D}(\log_2(\sin(x))) = \dfrac{1}{\sin(x)}\cdot\dfrac{1}{\ln(2)}\cdot\cos(x) = \dfrac{\cot(x)}{\ln(2)}$

13. $\mathrm{D}(\log_5(5^x)) = \mathrm{D}(x) = 1$

15. $\mathrm{D}(x\cdot\ln(3x)) = x\cdot\dfrac{1}{3x}\cdot 3 + \ln(3x)\cdot 1 = 1+\ln(3x)$

17. $\mathrm{D}\left(\dfrac{\ln(x)}{x}\right) = \dfrac{x\cdot\frac{1}{x}-\ln(x)\cdot 1}{x^2} = \dfrac{1-\ln(x)}{x^2}$

19. $\mathrm{D}\left(\ln\left((5x-3)^{\frac{1}{2}}\right)\right) = \dfrac{1}{(5x-3)^{\frac{1}{2}}}\cdot\mathrm{D}\left((5x-3)^{\frac{1}{2}}\right) = \dfrac{1}{(5x-3)^{\frac{1}{2}}}\cdot\dfrac{1}{2}(5x-3)^{-\frac{1}{2}}\cdot\mathrm{D}(5x-3) = \dfrac{5}{2}\cdot\dfrac{1}{5x-3}$

21. $\mathrm{D}\left(\cos(\ln(w))\right) = -\sin(\ln(w))\cdot\dfrac{1}{w} = -\dfrac{\sin(\ln(w))}{w}$

23. $\dfrac{d}{dt}\left(\sqrt{\ln(t+1)}\right) = \dfrac{1}{2(t+1)}$

25. $\mathrm{D}\left(5^{\sin(x)}\right) = 5^{\sin(x)}\cdot\ln(5)\cdot\cos(x)$

27. $\frac{1}{\sec(x)+\tan(x)}\cdot(\sec(x)\tan(x)+\sec^2(x)) = \sec(x)$

29. $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$. Let $P = (p, \ln(p))$. So we need $y - \ln(p) = \frac{1}{p}(x-p)$ with $x = 0$ and $y = 0$: $-\ln(p) = -1 \Rightarrow p = e$, hence $P = (e, 1)$.

31. $100(-1)(1+Ae^{-t})^{-2}\cdot(Ae^{-t})(-1) = \dfrac{100Ae^{-t}}{(1+Ae^{-t})^2}$

33. $f(x) = 8\ln(x) +$ any constant

35. $f(x) = \ln(3+\sin(x)) +$ any constant

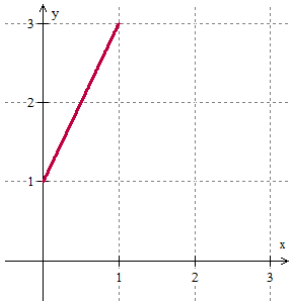
37. $g(x) = \frac{3}{5}e^{5x} +$ any constant

39. $f(x) = e^{x^2} +$ any constant

41. $h(x) = \ln(\sin(x)) +$ any constant

43. a. When $t = 0$, A is at $(0,1)$ and B is at $(0,1)$. When $t = 1$, A is at $(1,3)$ and B is at $(1,3)$.

    b. Both robots traverse the same path (the line segment $y = 2x+1$ for $0 \le x \le 1$):



    c. $\frac{dy}{dx} = 2$ for each, because $y = 2x+1$.

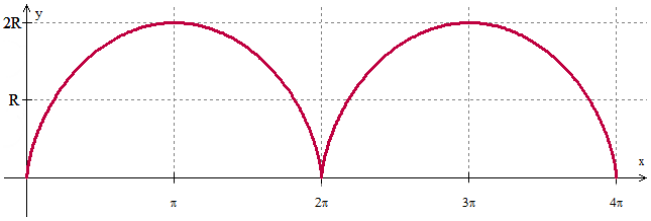
    d. A: $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2$, speed $= \sqrt{1^2+2^2} = \sqrt{5}$

    B: $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 4t$, speed $= \sqrt{(2t)^2+(4t)^2} = 2\sqrt{5}t$. At $t = 1$, B's speed is $2\sqrt{5}$.

    e. Moves along the same path $y = 2x+1$, but to the right and up for about 1.57 minutes, reverses direction and returns to its starting point, then continues left and down for 1.57 minutes, reverses, and continues to oscillate.

45. When $t = 1$: $\frac{dx}{dt} = +$, $\frac{dy}{dt} = -$, $\frac{dy}{dx} = -$. When $t = 3$, $\frac{dx}{dt} = -$, $\frac{dy}{dt} = -$, $\frac{dy}{dx} = +$.

47. a. $x(t) = R(t - \sin(t))$, $y(t) = R(1 - \cos(t))$:



b. $\frac{dx}{dt} = R(1 - \cos(t))$, $\frac{dy}{dt} = R\sin(t)$, so $\frac{dy}{dx} = \frac{\sin(t)}{1-\cos(t)}$. When $t = \frac{\pi}{2}$, then $\frac{dx}{dt} = R$ and $\frac{dy}{dt} = R$ so $\frac{dy}{dx} = 1$ and speed $\sqrt{R^2 + R^2} = R\sqrt{2}$. When $t = \pi$, $\frac{dx}{dt} = 2R$ and $\frac{dy}{dt} = 0$ so $\frac{dy}{dx} = 0$ and speed = $\sqrt{(2R)^2 + 0^2} = 2R$.

49. a. The ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$.

b. The ellipse $\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 = 1$ if $A \neq 0$ and $B \neq 0$. If $A = 0$, the motion is oscillatory along the x-axis; if $B = 0$, the motion is oscillatory along the y-axis.

c. $(3 \cdot \cos(t), -5 \cdot \sin(t))$ works.

Section 2.6

1. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$. We know $\frac{dr}{dt}\big|_{r=3} = 2$ in/min, so $\frac{dV}{dt}\big|_{r=3} = 4\pi(3\text{ in})^2(2\text{ in/min}) = 72\pi$ in^3/min ≈ 226.19 in^3/min.

3. a. $A = \frac{1}{2}bh \Rightarrow \frac{dA}{dt} = \frac{1}{2}\left[b\frac{dh}{dt} + h\frac{db}{dt}\right]$ so with $b = 15$ in, $h = 13$ in, $\frac{db}{dt} = 3\frac{\text{in}}{\text{hr}}$, $\frac{dh}{dt} = -3\frac{\text{in}}{\text{hr}}$:

$$\frac{dA}{dt} = \frac{1}{2}\left[(15\text{ in})\left(-3\frac{\text{in}}{\text{hr}}\right) + (13\text{ in})\left(3\frac{\text{in}}{\text{hr}}\right)\right]$$

which is < 0, so A is decreasing.

b. Hypotenuse $C = \sqrt{b^2 + h^2}$ so:

$$\frac{dC}{dt} = \frac{b\frac{db}{dt} + h\frac{dh}{dt}}{\sqrt{b^2 + h^2}} = \frac{15(3) + 13(-3)}{\sqrt{15^2 + 13^2}}$$

which is > 0 so C is increasing.

c. Perimeter $P = b + h + C$ so:

$$\frac{dP}{dt} = \frac{db}{dt} + \frac{dh}{dt} + \frac{dC}{dt} = 3 + (-3) + \frac{6}{\sqrt{394}}$$

which is > 0 so P is increasing.

5. a. Perimeter $P = 2x + 2y \Rightarrow \frac{dP}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt} = 2(3\text{ ft/sec}) + 2(-2\text{ ft/sec}) = 2$ ft/sec.

b. Area $A = xy \Rightarrow \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} \Rightarrow \frac{dA}{dt} = (12\text{ ft})\left(-2\frac{\text{ft}}{\text{sec}}\right) + (8\text{ ft})\left(3\frac{\text{ft}}{\text{sec}}\right) = 0$ ft^2/sec.

7. Volume $V = \pi r^2 h = \pi r^2\left(\frac{1}{3}\right) \Rightarrow \frac{dV}{dt} = \frac{2\pi}{3}r \cdot \frac{dr}{dt}$. So when $r = 50$ ft and $\frac{dr}{dt} = 6\frac{\text{ft}}{\text{hr}}$, we have $\frac{dV}{dt} = \frac{2\pi}{3}(50\text{ ft})\left(6\frac{\text{ft}}{\text{hr}}\right) = 200\pi\frac{\text{ft}^3}{\text{hr}} \approx 628.32\frac{\text{ft}^3}{\text{hr}}$.

9. $w(t) = h(t)$ for all t so $\frac{dw}{dt} = \frac{dh}{dt}$. $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{w}{2} = \frac{h}{2}$ so $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$. When $h = 500$ ft and $\frac{dh}{dt} = 2\frac{\text{ft}}{\text{hr}}$, then $\frac{dV}{dt}$ $= \frac{\pi}{4}(500)^2(2) = 125000\pi\frac{\text{ft}^3}{\text{hr}}$.

11. Let x be the distance from the lamp post to the person and L be the length of the shadow (both in feet). By similar triangles, $\frac{L}{6} = \frac{x}{8} \Rightarrow L = \frac{3}{4}x$. We also know that $\frac{dx}{dt} = 3\frac{\text{ft}}{\text{sec}}$.

a. $\frac{dL}{dt} = \frac{3}{4}\frac{dx}{dt} = \frac{3}{4}\left(3\frac{\text{ft}}{\text{sec}}\right) = 2.25\frac{\text{ft}}{\text{sec}}$

b. $\frac{d}{dt}(x + L) = \frac{dx}{dt} + \frac{dL}{dt} = 5.25\frac{\text{ft}}{\text{sec}}$

(We don't actually need the value of x.)

13. a. $\sin(35°) = \frac{h}{500} \Rightarrow h = 500\sin(35°) \approx 287$ ft

b. L = length of the string so $h = L\sin(35°)$ and $\frac{dh}{dt} = \sin(35°)\frac{dL}{dt} \approx (0.57)\left(10\frac{\text{ft}}{\text{sec}}\right) = 5.7\frac{\text{ft}}{\text{sec}}$

15. $V = s^3 - \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 3s^2\frac{ds}{dt} - 4\pi r^2\frac{dr}{dt}$ so when $r = 4$ ft, $\frac{dr}{dt} = 1\frac{\text{ft}}{\text{hr}}$, $s = 12$ ft and $\frac{ds}{dt} = 3\frac{\text{ft}}{\text{hr}}$: $\frac{dV}{dt} = 3(12\,\text{ft})^2\left(3\frac{\text{ft}}{\text{hr}}\right) - 4\pi(4\,\text{ft})^2\left(1\frac{\text{ft}}{\text{hr}}\right) \approx 1094.94\frac{\text{ft}^3}{\text{hr}}$. The volume is increasing at about 1,095 $\frac{\text{ft}^3}{\text{hr}}$.

17. Given: $\frac{dV}{dt} = k \cdot 2\pi r^2$ with k constant. We also have $V = \frac{2}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 2\pi r^2\frac{dr}{dt}$ so $k \cdot 2\pi r^2 = 2\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = k$. The radius r is changing at a constant rate.

19. a. $A = 5x$

b. $\frac{dA}{dx} = 5$ for all $x > 0$.

c. $A = 5t^2$

d. $\frac{dA}{dt} = 10t$. When $t = 1$, $\frac{dA}{dt} = 10$; when $t = 2$, $\frac{dA}{dt} = 20$; when $t = 3$, $\frac{dA}{dt} = 30$.

e. $A = 10 + 5 \cdot \sin(t) \Rightarrow \frac{dA}{dt} = 5 \cdot \cos(t)$.

21. In (b) and (c) we must use radians because our formulas for derivatives of trigonometric functions assume angles are measured in radians.

a. $\tan(10°) = \frac{40}{x} \Rightarrow x = \frac{40}{\tan(10°)} \approx 226.9$ ft.

b. $x = \frac{40}{\tan(\theta)} = 40\cot(\theta) \Rightarrow \frac{dx}{dt} = -40\csc^2(\theta)\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{\sin^2(\theta)}{40}\frac{dx}{dt}$ so when $\theta = 10° \approx 0.1745$ radians and $\frac{dx}{dt} = -25\frac{\text{ft}}{\text{min}}$:

$$\frac{d\theta}{dt} = -\frac{\sin^2(0.1745)}{40}(-25) \approx \frac{(0.1736)^2(25)}{40}$$

$\approx 0.0188 \frac{\text{rad}}{\text{min}} \approx 1.079 \frac{^\circ}{\text{min}}$.

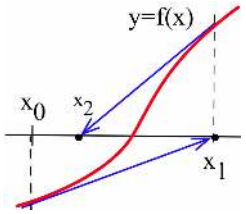
c. $\frac{dx}{dt} = -40\csc^2(\theta)\frac{d\theta}{dt} = \frac{-40}{\sin^2(\theta)}\frac{d\theta}{dt}$ so when $\theta = 10^\circ \approx 0.1745$ rad and $\frac{d\theta}{dt} = 2\frac{^\circ}{\text{min}} \approx 0.0349 \frac{\text{rad}}{\text{min}}$:

$$\frac{d\theta}{dt} \approx \frac{-40}{(0.1736)^2}(0.0349) \approx -46.3 \frac{\text{ft}}{\text{min}}$$

(The "-" indicates the distance to the sign is decreasing: you are approaching the sign.) Your speed is $46.3 \frac{\text{ft}}{\text{min}}$.

Section 2.7

1. The locations of x_1 and x_2 appear below:



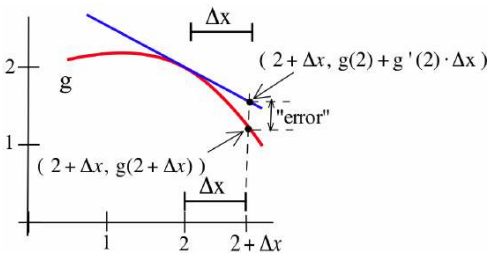
3. $x_0 = 1$: a; $x_0 = 5$: b
5. $x_0 = 1$: 1, 2, 1, 2, 1,…
   $x_0 = 5$: x_1 is undefined because $f'(5) = 0$
7. If f is differentiable, then $f'(x_0) = 0$ and $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is undefined.
9. $f(x) = x^4 - x^3 - 5 \Rightarrow f'(x) = 4x^3 - 3x^2$ so $x_0 = 2 \Rightarrow x_1 = 2 - \frac{3}{20} = \frac{37}{20} = 1.85 \Rightarrow x_2 = 1.85 - \frac{1.85^4 - 1.85^3 - 5}{4(1.85)^3 - 3(1.85)^2} \approx 1.824641$
11. $f(x) = x - \cos(x) \Rightarrow f'(x) = 1 + \sin(x)$ so $x_0 = 0.7 \Rightarrow x_1 = 0.7394364978 \Rightarrow x2 = 0.7390851605$, so root ≈ 0.74.
13. To solve $\frac{x}{x+3} = x^2 - 2$ we search for roots of $f(x) = x^2 - 2 - \frac{x}{x+3}$. If $x_0 = -4$, then the iterates $x_n \to -3.3615$; if $x_0 = -2$, then $x_n \to -1.1674$; if $x_0 = 2$, then the iterates $x_n \to 1.5289$.
15. For $x^5 - 3 = 0$ and $x_0 = 1$, $x_n \to 1.2457$.
17. $f(x) = x^3 - A \Rightarrow f'(x) = 3x^2$ so:

$$x_{n+1} = x_n - \frac{x_n^3 - A}{3x_n^2} = \frac{1}{3}\left[2x_n + \frac{A}{x_n^2}\right]$$

19. a. $2(0) - \lfloor 2(0) \rfloor = 0 - 0 = 0$
    b. $2(\frac{1}{2}) - \lfloor 2(\frac{1}{2}) \rfloor = 1 - 1 = 0$
    $2(\frac{1}{4}) - \lfloor 2(\frac{1}{4}) \rfloor = \frac{1}{2} - 0 = \frac{1}{2} \to 0$
    $2(\frac{1}{8}) - \lfloor 2(\frac{1}{8}) \rfloor = \frac{1}{4} - 0 = \frac{1}{4} \to \frac{1}{2} \to 0$
    $2(\frac{1}{2^n}) - \lfloor 2(\frac{1}{2^n}) \rfloor = \frac{1}{2^{n-1}} - 0 = \frac{1}{2^{n-1}}$
    $\to \frac{1}{2^{n-2}} \to \cdots \to \frac{1}{4} \to \frac{1}{2} \to 0$
21. a. If $0 \le x \le \frac{1}{2}$, then f stretches x to twice its value, $2x$. If $\frac{1}{2} < x \le 1$, then f stretches x to twice its value ($2x$) and "folds" the part above the value 1 ($2x - 1$) to below 1: $1 - (2x - 1) = 2 - 2x$.
    b. $f(\frac{2}{3}) = \frac{2}{3}$
    $f(\frac{2}{5}) = \frac{4}{5}$, $f(\frac{4}{5}) = \frac{2}{5}$, and the values continue to cycle.
    $f(\frac{2}{7}) = \frac{4}{7}$, $f(\frac{4}{7}) = \frac{6}{7}$, $f(\frac{6}{7}) = \frac{2}{7}$, and the values continue to cycle.
    $f(\frac{2}{9}) = \frac{4}{9}$, $f(\frac{4}{9}) = \frac{8}{9}$, $f(\frac{8}{9}) = \frac{2}{9}$, and the values continue to cycle.
    c. 0.1, 0.2, 0.4, 0.8, 0.4, 0.8, and the pair of values 0.4 and 0.8 continue to cycle.
    0.105, 0.210, 0.42, 0.84, **0.32**, 0.64, 0.72, 0.56, 0.88, 0.24, 0.48, 0.96, 0.08, 0.16, **0.32**,…0.11, 0.22, 0.44, **0.88**, 0.24, 0.48, 0.96, 0.08, 0.16, 0.32, 0.64, 0.72, 0.56, **0.88**,…
    d. Probably so.

Section 2.8

1. a. The desired points appear below:



   b. The "error" appears in the figure above.
3. a. $f(81) = 9$ and $f'(81) = \frac{1}{18}$ so tangent line is $y - 9 = \frac{1}{18}(x - 81)$ or $y = \frac{1}{18}(x - 81) + 9$ and $\sqrt{80} = f(80) \approx \frac{1}{18}(80 - 81) + 9 = \frac{161}{18} \approx 8.944$
   b. $f(0) = 0$ and $f'(0) = 1$ so tangent line is $y = x$ and $\sin(0.3) = f(0.3) \approx 0.3$

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4. a. $f(1)=0$ and $f'(1)=1$ so tangent line is $y-0=1(x-1)$ or $y=x-1$ and $\ln(1.3)=f(1.3)\approx 1.3-1=0.3$
   b. $f(0)=1$ and $f'(0)=1$ so tangent line is $y-1=1(x-0)$ or $y=1+x$ and $e^{0.1}=f(0.1)\approx 1.1$
   c. $f(1)=1$ and $f'(1)=5$ so tangent line is $y-1=5(x-1)$ or $y=5x-4$ and $(1.03)^5\approx 5(0.03)-4=1.15$
5. $f(x)=(1+x)^n\Rightarrow f'(x)=n(1+x)^{n-1}$ so $f(0)=1$ and $f'(0)=n$, hence tangent line is $y-1=n(x-0)$ or $y=1+nx$ and $(1+x)^n\approx 1+nx$ (when x is close to 0)
6. a. $f(x)=(1-x)^n\Rightarrow f'(x)=-n(1-x)^{n-1}$ so $f(0)=1$ and $f'(0)=-n$, hence tangent line is $y-1=-n(x-0)$ or $y=1-nx$ and $(1-x)^n\approx 1-nx$ (when x is close to 0)
   b. $f(x)=\sin(x)\Rightarrow f'(x)=\cos(x)$ so $f(0)=0$ and $f'(0)=1$, hence tangent line is $y-0=1(x-0)$ or $y=x$ and $\sin(x)\approx x$ (for x close to 0)
   c. $f(x)=e^x\Rightarrow f'(x)=e^x$ so $f(0)=1$ and $f'(0)=1$, hence tangent line is $y-1=1(x-0)$ or $y=x+1$ and $e^x\approx x+1$ (for x near 0)
7. a. $f(x)=\ln(1+x)\to f'(x)=\frac{1}{1+x}$ so $f(0)=0$ and $f'(0)=1$, hence tangent line is $y=x$ and $\ln(1+x)\approx x$
   b. $f(x)=\cos(x)\Rightarrow f'(x)=-\sin(x)$ so $f(0)=1$ and $f'(0)=0$, hence tangent line is $y=1$ and $\cos(x)\approx 1$
   c. $f(x)=\tan(x)\Rightarrow f'(x)=\sec^2(x)$, so $f(0)=0$ and $f'(0)=1$, hence tangent line is $y=x$ and $\tan(x)\approx x$
   d. $f(x)=\sin\left(\frac{\pi}{2}+x\right)\Rightarrow f'(x)=\cos\left(\frac{\pi}{2}+x\right)$ so $f(0)=1$ and $f'(0)=0$, hence tangent line is $y=1$ and $\sin\left(\frac{\pi}{2}+x\right)\approx 1$.
9. a. Area $A(x)=(\text{base})(\text{height})=x(x^2+1)=x^3+x\Rightarrow A'(x)=3x^2+1\Rightarrow A'(2)=13$ so $\Delta A\approx A'(2)\cdot\Delta x=(13)(2.3-2)=3.9$
   b. $(\text{base})(\text{height})=(2.3)((2.3)^2+1)=14.467$ so actual $\Delta A=14.467-(2)(2^2+1)=4.467$
11. $V=\pi r^2h=2\pi r^2$ and $\Delta V=2\pi\cdot 2r\Delta r=4\pi r\Delta r$. Because $V=2\pi r^2=47.3$, we have $r=\sqrt{\frac{47.3}{2\pi}}\approx 2.7437$ cm. We know $|\Delta V|\le 0.1$ so, using $\Delta V=4\pi r\Delta r$, we have $0.1\ge 4\pi(2.7437)\,|\Delta r|$ and $|\Delta r|\le\frac{0.1}{4\pi(2.7437)}\approx 0.0029$ cm. The required tolerance is ± 0.0029 cm. (Reality check: A coin 2 cm high is quite unusual; 47.3 cm^3 of gold weighs around 2 pounds!)
13. $V=x^3\Rightarrow\Delta V\approx 3x^2\Delta x$, $V=87\Rightarrow x=\sqrt[3]{87}\approx 4.431$ cm so $\Delta x\approx\frac{\Delta V}{3x^2}\approx\frac{2}{3(4.431)^2}\approx 0.034$ cm
15. With $P=2\pi\sqrt{\frac{L}{g}}$ and $g=32\,\frac{\text{ft}}{\text{sec}^2}$:
    a. $P=2\pi\sqrt{\frac{2}{32}}=\frac{\pi}{2}\approx 1.57$ sec
    b. $1=2\pi\sqrt{\frac{L}{32}}\Rightarrow L=\frac{8}{\pi^2}\approx 0.81$ ft
    c. $dP=\frac{2\pi}{\sqrt{32}}\cdot\frac{1}{2\sqrt{2}}\,dL=\frac{2\pi}{16}(0.1)\approx 0.039$ sec
    d. $2\,\frac{\text{in}}{\text{hr}}=\frac{1}{6}\,\frac{\text{ft}}{\text{hr}}=\frac{1}{21600}\,\frac{\text{ft}}{\text{sec}}$ so $\frac{dP}{dt}=\frac{2\pi}{4\sqrt{2}}\cdot\frac{1}{2\sqrt{24}}\cdot\frac{1}{21600}\approx 5.25\times 10^{-6}>0$ (increasing)
17. a. $df=f'(2)\,dx\approx(0)(1)=0$
    b. $df=f'(4)\,dx\approx(0.3)(-1)=-0.3$
    c. $df=f'(3)\,dx\approx(0.5)(2)=1$
19. a. $f'(x)=2x-3\Rightarrow df=(2x-3)\,dx$
    b. $f'(x)=e^x\Rightarrow df=e^x\,dx$
    c. $f'(x)=5\cos(5x)\Rightarrow df=5\cos(5x)\,dx$
    d. $f'(x)=3x^2+2\Rightarrow df=(3x^2+2)\,dx$ so when $x=1$ and $dx=0.2$, $df=(3\cdot 1^2+2)(0.2)=1$
    e. $f'(x)=\frac{1}{x}\Rightarrow df=\frac{1}{x}\,dx$ so when $x=e$ and $dx=-0.1$, $df=\frac{1}{e}(-0.1)=-\frac{1}{10e}$
    f. $f'(x)=\frac{1}{\sqrt{2x+5}}\Rightarrow df=\frac{1}{\sqrt{2x+5}}\,dx$ so when $x=22$ and $dx=3$, $df=\frac{1}{\sqrt{49}}(3)=\frac{3}{7}$

Section 2.9

1. a. $x^2+y^2=100\Rightarrow 2x+2y\cdot y'=0\Rightarrow y'=-\frac{x}{y}$ so at $(6,8)$, $y'=-\frac{6}{8}=-\frac{3}{4}$
   b. $y=\sqrt{100-x^2}\Rightarrow y'=\frac{-x}{\sqrt{100-x^2}}$ so at $(6,8)$, $y'=-\frac{6}{\sqrt{100-36}}=-\frac{6}{8}=-\frac{3}{4}$
3. a. $x^2-3xy+7y=5\Rightarrow 2x-3(y+xy')+7y'=0\Rightarrow y'=\frac{3y-2x}{7-3x}\Rightarrow y'|_{(2,1)}=\frac{3-4}{7-6}=-1$
   b. $y=\frac{5-x^2}{7-3x}\Rightarrow y'=\frac{(7-3x)(-2x)-(5-x^2)(-3)}{(7-3x)^2}$ so at $(2,1)$, $y'=\frac{(1)(-4)-(1)(-3)}{(1)^2}=-1$
5. a. $\frac{x^2}{9}+\frac{y^2}{16}=1\Rightarrow\frac{2x}{9}+\frac{2y}{16}y'=0\Rightarrow y'=-\frac{16}{9}\cdot\frac{x}{y}\Rightarrow y'|_{(0,4)}=0$
   b. $y=4\sqrt{1-\frac{x^2}{9}}=\frac{4}{3}\sqrt{9-x^2}\Rightarrow y'=\frac{4}{3}\cdot\frac{-x}{\sqrt{9-x^2}}\Rightarrow y'|_{(0,4)}=0$
7. a. $\ln(y)+3x-7=0\Rightarrow\frac{1}{y}y'+3=0\Rightarrow y'=-3y\Rightarrow y'|_{(2,e)}=-3e$
   b. $y=e^{7-3x}\Rightarrow y'=-3e^{7-3x}\Rightarrow y'|_{(2,e)}=-3e^{7-6}=-3e$
9. a. $x^2-y^2=16\Rightarrow 2x-2yy'=0\Rightarrow y'=\frac{x}{y}\Rightarrow y'|_{(5,-3)}=-\frac{5}{3}$
   b. The point $(5,-3)$ is on the bottom half of the hyperbola so $y=-\sqrt{x^2-16}\Rightarrow y'=-\frac{x}{\sqrt{x^2-16}}\Rightarrow y'|_{(5,-3)}=-\frac{5}{\sqrt{25-16}}=-\frac{5}{3}$
11. $x=4y-y^2\Rightarrow 1=4y'-2y\cdot y'\Rightarrow y'=\frac{1}{4-2y}$. At $(3,1)$, $y'=\frac{1}{4-2(1)}=\frac{1}{2}$; at $(3,3)$, $y'=\frac{1}{4-2(3)}=-\frac{1}{2}$; at $(4,2)$, $y'=\frac{1}{4-2(2)}$ is undefined (the tangent line is vertical).
13. $x=y^2-6y+5\Rightarrow 1=2y\cdot y'-6y'\Rightarrow y'=\frac{1}{2y-6}$. At $(5,0)$, $y'=\frac{1}{2(0)-6}=-\frac{1}{6}$; at $(5,6)$, $y'=\frac{1}{2(6)-6}=\frac{1}{6}$; at $(-4,3)$, $y'=\frac{1}{2(3)-6}$ is undefined (vertical tangent line).
15. $3y^2\cdot y'-5y'=10x\Rightarrow y'=\frac{10x}{3y^2-5}\Rightarrow m=y'|_{(1,3)}=\frac{10}{22}=\frac{5}{11}$

17. $y^2+\sin(y)=2x-6 \Rightarrow 2y\cdot y'+\cos(y)\cdot y'=2 \Rightarrow y'=\dfrac{2}{2y+\cos(y)} \Rightarrow m=y'|_{(3,0)}=\dfrac{2}{0+1}=2$
19. $e^y+\sin(y)=x^2-3 \Rightarrow e^y\cdot y'+\cos(y)\cdot y'=2x \Rightarrow y'=\dfrac{2x}{e^y+\cos(y)} \Rightarrow m=y'|_{(2,0)}=$
21. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=5 \Rightarrow \dfrac{2}{3}x^{-\frac{1}{3}}+\dfrac{2}{3}y^{-\frac{1}{3}}y'=0 \Rightarrow y'=$
23. $2x+x\cdot y'+y+2\cdot y'+3-7y'=0 \Rightarrow y'=\dfrac{-2x-y-3}{x+2y-7} \Rightarrow y'|_{(1,2)}=\dfrac{-2(1)-(2)-3}{(1)+2(2)-7}=\dfrac{7}{2}$
25. Explicitly: $y=Ax^2+Bx+C \Rightarrow y'=2Ax+B$
    Implicitly: $x=Ay^2+By+C \Rightarrow 1=2Ayy'+By' \Rightarrow y'=\frac{1}{2Ay+B}$
27. $Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \Rightarrow 2Ax+Bx\cdot y'+By+2Cy\cdot y'+D+Ey'=0 \Rightarrow y'=\frac{-2Ax-By-D}{Bx+2Cy+E}$
28. $x^2+y^2=r^2 \Rightarrow 2x+2y\cdot y'=0 \Rightarrow y'=-\dfrac{x}{y}$ so the slope of the tangent line is $-\frac{x}{y}$. The slope of the line through the points $(0,0)$ and (x,y) is $\frac{y}{x}$, so the slopes of the lines are negative reciprocals of each other and the lines are perpendicular.
29. From the solution to problem 23, we know that $y'=\frac{-2x-y-3}{x+2y-7}$ so $y'=0$ when $-2x-y-3=0 \Rightarrow y=-2x-3$. Substituting $y=-2x-3$ into the original equation, we have:

$$\begin{aligned} x^2+x(-2x-3)+(-2x-3)^2 \\ +3x-7(-2x-3)+4=0 \end{aligned}$$

    $\Rightarrow 3x^2+26x+34=0 \Rightarrow x=\frac{-26\pm\sqrt{26^2-4(3)(34)}}{2(3)}=\frac{-13\pm\sqrt{67}}{3}\approx -1.605$ or -7.062.
    If $x\approx -1.605$ (point A), then $y=-2x-3\approx -2(-1.605)-3=0.21$ so coordinates of A are $(-1.605, 0.21)$; if $x\approx -7.062$ (point C), then $y=-2x-3\approx -2(-7.062)-3=11.124$ so C is $(-7.062, 11.124)$.
31. From the solution to 29, point C is $(-7.062, 11.124)$. At D, $y'=\frac{-2x-y-3}{x+2y-7}$ is undefined so $x+2y-7=0 \Rightarrow x=7-2y$. Substituting $x=7-2y$ into the original equation:

$$(7-2y)^2+(7-2y)y+y^2+3(7-2y)-7y+4=0$$

    so $3y^2-34y+74=0 \Rightarrow y=\frac{34\pm\sqrt{34^2-4(3)(74)}}{2(3)}=\frac{17\pm\sqrt{67}}{3}\approx 8.395$ or 2.938.
    If $y\approx 2.938$ (point D), then $x=7-2y\approx 7-2(2.938)=1.124$ so point D is $(1.124, 2.938)$. Similarly, point B is $(-9.79, 8.395)$.
33. a. Using the Product Rule: $y=(x^2+5)^7(x^3-1)^4$ so:

$$\begin{aligned} y' &= (x^2+5)^7\cdot 4(x^3-1)^3\cdot 3x^2 \\ &\quad +7(x^2+5)^6\cdot 2x\cdot(x^3-1)^4 \\ &= (x^2+5)^6(x^3-1)^3(2x)\left[6x^3+30x+7x^3-7\right] \\ &= (x^2+5)^6(x^3-1)^3(2x)\left[13x^3+30x-7\right] \end{aligned}$$

    b. $\ln(y)=7\ln(x^2+5)+4\ln(x^3-1)$ so:

$$\frac{y'}{y}=\frac{14x}{x^2+5}+\frac{12x^2}{x^3-1}$$

    and solving for y' yields:

$$\begin{aligned} y' &= y\left[\frac{14x}{x^2+5}+\frac{12x^2}{x^3-1}\right] \\ &= (x^2+5)^7(x^3-1)^4\left[\frac{14x}{x^2+5}+\frac{12x^2}{x^3-1}\right] \end{aligned}$$

    which is the same as part (a). (Really! It is!)
35. a. $y=x^5(3x+2)^4$ so:

$$\begin{aligned} y' &= x^5\mathrm{D}\left((3x+2)^4\right)+(3x+2)^4\mathrm{D}\left(x^5\right) \\ &= x^5\cdot 4(3x+2)^3(3)+(3x+2)^4\cdot 5x^4 \end{aligned}$$

    b. $\ln(y)=5\ln(x)+4\ln(3x+2)$ so:

$$\frac{y'}{y}=\frac{5}{x}+\frac{12}{3x+2}$$

    and solving for y' yields:

$$\begin{aligned} y' &= y\left[\frac{5}{x}+\frac{12}{3x+2}\right] \\ &= \left[x^5(3x+2)^4\right]\cdot\left[\frac{5}{x}+\frac{12}{3x+2}\right] \end{aligned}$$

    which is the same as in part (a).
37. a. $y=e^{\sin(x)} \Rightarrow y'=e^{\sin(x)}\cdot\cos(x)$
    b. $\ln(y)=\sin(x) \Rightarrow \dfrac{y'}{y}=\cos(x) \Rightarrow y'=y\cdot\cos(x)=e^{\sin(x)}\cdot\cos(x)$

39. a. $y = \sqrt{25 - x^2} \Rightarrow y' = \dfrac{-x}{\sqrt{25 - x^2}}$

b. $\ln(y) = \frac{1}{2}\ln(25 - x^2)$ so:

$$\frac{y'}{y} = \frac{1}{2} \cdot \frac{-2x}{25 - x^2} = \frac{-x}{25 - x^2}$$

and solving for y' yields:

$$y' = \sqrt{25 - x^2} \cdot \frac{-x}{25 - x^2} = \frac{-x}{\sqrt{25 - x^2}}$$

41. $y = x^{\cos(x)} \Rightarrow \ln(y) = \cos(x) \cdot \ln(x) \Rightarrow \dfrac{y'}{y} = \cos(x) \cdot \dfrac{1}{x} - \ln(x) \cdot \sin(x)$ so:

$$y' = y\left[\frac{\cos(x)}{x} - \ln(x) \cdot \sin(x)\right] = x^{\cos(x)}\left[\frac{\cos(x)}{x} - \ln(x) \cdot \sin(x)\right]$$

43. $\ln(y) = 4\ln(x) + 7\ln(x - 2) + \ln(\sin(3x)) \Rightarrow \dfrac{y'}{y} = \dfrac{4}{x} + \dfrac{7}{x - 2} + \dfrac{3\cos(3x)}{\sin(3x)}$ so:

$$y' = y\left[\frac{4}{x} + \frac{7}{x - 2} + 3\cot(3x)\right] = x^4(x - 2)^7 \sin(3x)\left[\frac{4}{x} + \frac{7}{x - 2} + 3\cot(3x)\right]$$

45. $\ln(y) = x \cdot \ln(3 + \sin(x)) \Rightarrow \dfrac{y'}{y} = x \cdot \dfrac{\cos(x)}{3 + \sin(x)} + \ln(3 + \sin(x))$ so:

$$y' = (3 + \sin(x))^x\left[\frac{x\cos(x)}{3 + \sin(x)} + \ln(3 + \sin(x))\right]$$

47. $f'(1) = 1(1.2) = 1.2$; $f'(2) = 9(1.8) = 16.2$

$f'(3) = 64(2.1) = 134.4$

49. $f'(1) = 5(-1) = -5$; $f'(2) = 2(0) = 0$

$f'(3) = 7(2) = 14$

51. $\ln(f \cdot g) = \ln(f) + \ln(g) \Rightarrow \dfrac{\mathrm{D}(f \cdot g)}{f \cdot g} = \dfrac{f'}{f} + \dfrac{g'}{g}$ so $\mathrm{D}(f \cdot g) = (f \cdot g)\left[\frac{f'}{f} + \frac{g'}{g}\right] = f' \cdot g + g' \cdot f$

52. $\ln\left(\dfrac{f}{g}\right) = \ln(f) - \ln(g) \Rightarrow \dfrac{\mathrm{D}\left(\frac{f}{g}\right)}{\left(\frac{f}{g}\right)} = \dfrac{f'}{f} - \dfrac{g'}{g}$ so:

$$\begin{aligned}\mathrm{D}\left(\frac{f}{g}\right) &= \left(\frac{f}{g}\right)\left[\frac{f'}{f} - \frac{g'}{g}\right] = \frac{f'}{g} - \frac{f \cdot g'}{g^2} \\ &= \frac{f \cdot g' - g \cdot f'}{g^2}\end{aligned}$$

53. $\ln(f \cdot g \cdot h) = \ln(f) + \ln(g) + \ln(h) \Rightarrow \dfrac{\mathrm{D}(f \cdot g \cdot h)}{f \cdot g \cdot h} = \dfrac{f'}{f} + \dfrac{g'}{g} + \dfrac{h'}{h}$ so:

$$\begin{aligned}\mathrm{D}(f \cdot g \cdot h) &= (f \cdot g \cdot h)\left[\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right] \\ &= f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'\end{aligned}$$

54. $\ln(a^x) = x\ln(a) \Rightarrow \dfrac{\mathrm{D}(a^x)}{a^x} = \ln(a) \Rightarrow \mathrm{D}(a^x) = a^x \ln(a)$

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CHAPTER OVERVIEW

3: Derivatives and Graphs

In this chapter, we explore what the first and second derivatives of a function tell us about the graph of that function and apply this graphical knowledge to locate the extreme values of a function.

3.1: Finding Maximums and Minimums
3.2: Mean Value Theorem
3.3: The First Derivative and the Shape of f
3.4: The Second Derivative and the Shape of f
3.5: Applied Maximum and Minimum Problems
3.6: Asymptotic Behavior of Functions
3.7: L'Hôpital's Rule
3.A: Answers

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3.1: Finding Maximums and Minimums

In theory and applications, we often want to maximize or minimize some quantity. An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance. A manufacturer may want to maximize profits and market share or minimize waste. A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Many natural objects follow minimum or maximum principles, so if we want to model natural phenomena we may need to maximize or minimize. A light ray travels along a "minimum time" path. The shape and surface texture of some animals tend to minimize or maximize heat loss. Systems reach equilibrium when their potential energy is minimized. A basic tenet of evolution is that a genetic characteristic that maximizes the reproductive success of an individual will become more common in a species.

Calculus provides tools for analyzing functions and their behavior and for finding maximums and minimums.

Methods for Finding Maximums and Minimums

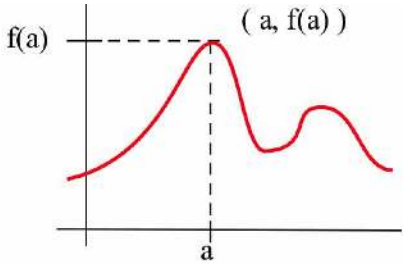
We can try to find where a function f is largest or smallest by evaluating f at lots of values of x, a method that is not very efficient and may not find the exact place where f achieves its extreme value. If we try hundreds or thousands of values for x, however, then we can often find a value of f that is close to the maximum or minimum. In general, this type of exhaustive search is only practical if you have a computer do the work.

The graph of a function provides a visual way of examining lots of values of f, and it is a good method, particularly if you have a computer to do the work for you. It is still inefficient, however, as you (or a computer) still need to evaluate the function at hundreds or thousands of inputs in order to create the graph—and we still may not find the exact location of the maximum or minimum.

Calculus provides ways to drastically narrow the number of points we need to examine to find the exact locations of maximums and minimums. Instead of examining f at thousands of values of x, calculus can often guarantee that the maximum or minimum must occur at one of three or four values of x, a substantial improvement in efficiency.

A Little Terminology

Before we examine how calculus can help us find maximums and minimums, we need to carefully define these concepts.



Definitions

- f has a **maximum** or **global maximum** at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of f.
- The **maximum value** of f is then $f(a)$ and this maximum value of f **occurs at** a.
- The **maximum point** on the graph of f is $(a, f(a))$.

The previous definition involves the overall biggest value a function attains on its entire domain. We are sometimes interested in how a function behaves locally rather than globally.

Definition

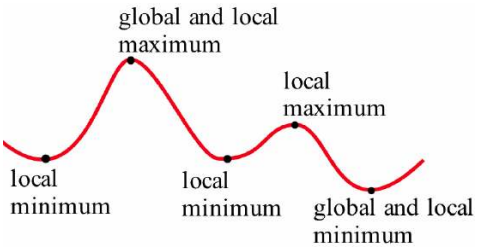
f has a **local** or **relative maximum** at $x = a$ if $f(a) \geq f(x)$ for all x "near" a, (that is, in some open interval that contains a).

Global and local **minimums** are defined similarly by replacing the $\geq$ symbol with $\leq$ in the previous definitions.

Definition

f has a **global extreme** at $x = a$ if $f(a)$ is a global maximum or minimum.

The figure below shows graphical examples of local and global extremes of a function:



You should notice that every global extreme is also a local extreme, but there are local extremes that are not global extremes. If $h(x)$ is the height of the Earth above sea level at location x, then the global maximum of h is $h(\text{summit of Mt. Everest}) = 29{,}028$ feet. The local maximum of h for the United States is $h(\text{summit of Mt. McKinley}) = 20{,}320$ feet. The local minimum of h for the United States is $h(\text{Death Valley}) = -282$ feet.

Finding Maximums and Minimums of a Function

One way to narrow our search for a maximum value of a function f is to eliminate those values of x that, for some reason, cannot possibly make f maximum.

Theorem

If: $f'(a) > 0$ or $f'(a) < 0$

then: $f(a)$ is not a local maximum or minimum.

Proof

Assume that $f'(a) > 0$.

A red increasing curve with the point on the curve where the input equals a marked by a black dot. A caption reads f'(a)0." class="internal default" style="width: 155px; height: 187px;" width="155px" height="187px" src="/@api/deki/files/140328/fig301_3a.png">

By definition:

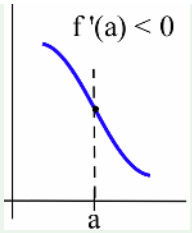
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

so $f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} > 0$. This means that the right and left limits are both positive: $f'(a) = \lim_{\Delta x \to 0^+} \frac{f(a+\Delta x) - f(a)}{\Delta x} > 0$ and $f'(a) = \lim_{\Delta x \to 0^-} \frac{f(a+\Delta x) - f(a)}{\Delta x} > 0$.

Considering the right limit, we know that if we restrict $\Delta x > 0$ to be sufficiently small, we can guarantee that $\frac{f(a+\Delta x) - f(a)}{\Delta x} > 0$ so, multiplying each side of this last inequality by the positive number Δx, we have $f(a+\Delta x) - f(a) > 0 \Rightarrow f(a+\Delta x) > f(a)$ for all sufficiently small values of $\Delta x > 0$, so any open interval containing $x = a$ will also contain values of x with $f(x) > f(a)$. This tell us that $f(a)$ is not a maximum.

Considering the left limit, we know that if we restrict $\Delta x < 0$ to be sufficiently small, we can guarantee that $\frac{f(a+\Delta x) - f(a)}{\Delta x} > 0$ so, multiplying each side of this last inequality by the negative number Δx, we have $f(a+\Delta x) - f(a) < 0 \Rightarrow f(a+\Delta x) < f(a)$ for all sufficiently small values of $\Delta x < 0$, so any open interval containing $x = a$ will also contain values of x with $f(x) < f(a)$. This tell us that $f(a)$ is not a minimum.

The argument for the "$f'(a) < 0$" case is similar.



When we evaluate the derivative of a function f at a point $x = a$, there are only four possible outcomes: $f'(a) > 0$, $f'(a) < 0$, $f'(a) = 0$ or $f'(a)$ is undefined. If we are looking for extreme values of f, then we can eliminate those points at which f' is positive or negative, and only two possibilities remain: $f'(a) = 0$ or $f'(a)$ is undefined.

Theorem

If: f is defined on an open interval

and: $f(a)$ is a local extreme of f

then: either $f'(a) = 0$ or f is not differentiable at a.

Example 3.1.1

Find the local extremes of $f(x) = x^3 - 6x^2 + 9x + 2$.

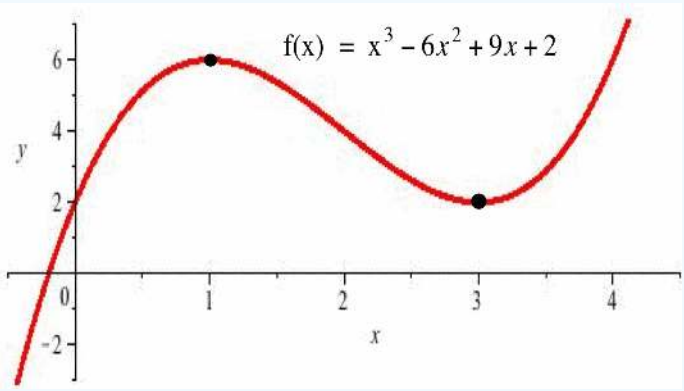
Solution

An extreme value of f can occur only where $f'(x) = 0$ or where f is not differentiable; $f(x)$ is a polynomial, so it is differentiable for all values of x, and we can restrict our attention to points where $f'(x) = 0$.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$$

so $f'(x) = 0$ only at $x = 1$ and $x = 3$.

The only possible locations of local extremes of f are at $x = 1$ and $x = 3$. We don't know yet whether $f(1)$ or $f(3)$ is a local extreme of f, but we can be certain that no other point is a local extreme. The graph of f:



shows that $(1, f(1)) = (1, 6)$ appears to be a local maximum and $(3, f(3)) = (3, 2)$ appears to be a local minimum. This function does not have a global maximum or minimum.

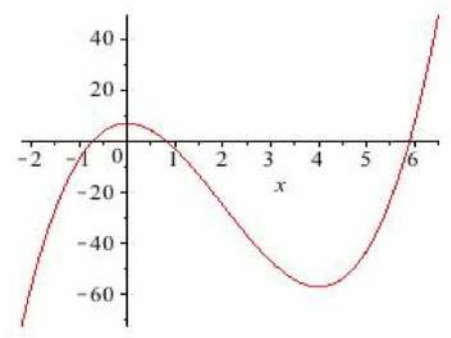
Practice 3.1.1

Find the local extremes of $f(x) = x^2 + 4x - 5$ and of $g(x) = 2x^3 - 12x^2 + 7$.

Answer

$f(x) = x^2 + 4x - 5$ is a polynomial so f is differentiable for all x and $f'(x) = 2x + 4$; $f'(x) = 0$ when $x = -2$ so the only candidate for a local extreme is $x = -2$. Because the graph of f is a parabola opening up, the point $(-2, f(-2)) = (-2, -9)$ is a local minimum.

$g(x) = 2x^3 - 12x^2 + 7$ is a polynomial so g is differentiable for all x and $g'(x) = 6x^2 - 24x = 6x(x - 4)$ so $g'(x) = 0$ when $x = 0$ or 4, so the only candidates for a local extreme are $x = 0$ and $x = 4$. The graph of g:

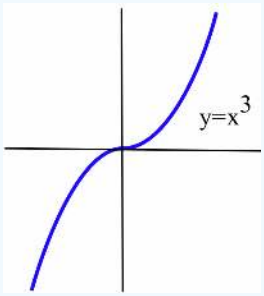


indicates that g has a local maximum at $(0, 7)$ and a local minimum at $(4, -57)$.

It is important to recognize that the two conditions "$f'(a) = 0$" or "f not differentiable at a" do not guarantee that $f(a)$ is a local maximum or minimum. They only say that $f(a)$ **might** be a local extreme or that $f(a)$ is a **candidate** for being a local extreme.

Example 3.1.2

Find all local extremes of $f(x) = x^3$.



Solution

$f(x) = x^3$ is differentiable for all x, and $f'(x) = 3x^2$ equals 0 only at $x = 0$, so the only candidate is the point $(0, 0)$. But if $x > 0$ then $f(x) = x^3 > 0 = f(0)$, so $f(0)$ is not a local maximum. Similarly, if $x < 0$ then $f(x) = x^3 < 0 = f(0)$ so $f(0)$ is not a local minimum. The point $(0, 0)$ is the only candidate to be a local extreme of f, but this candidate did not turn out to be a local extreme of f. The function $f(x) = x^3$ does not have any local extremes.

Theorem

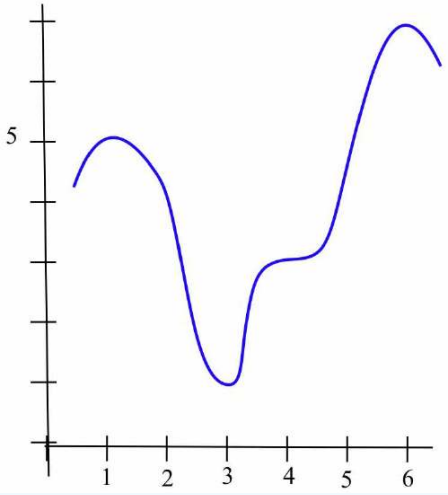
If: $f'(a) = 0$ or f is not differentiable at a

then: the point $(a, f(a))$ is a candidate to be a local extreme but may not actually be a local extreme.

Practice 3.1.2

Sketch the graph of a differentiable function f that satisfies the conditions: $f(1) = 5$, $f(3) = 1$, $f(4) = 3$ and $f(6) = 7$; $f'(1) = 0$, $f'(3) = 0$, $f'(4) = 0$ and $f'(6) = 0$; the only local maximums of f are at $(1, 5)$ and $(6, 7)$; and the only local minimum is at $(3, 1)$.

Answer

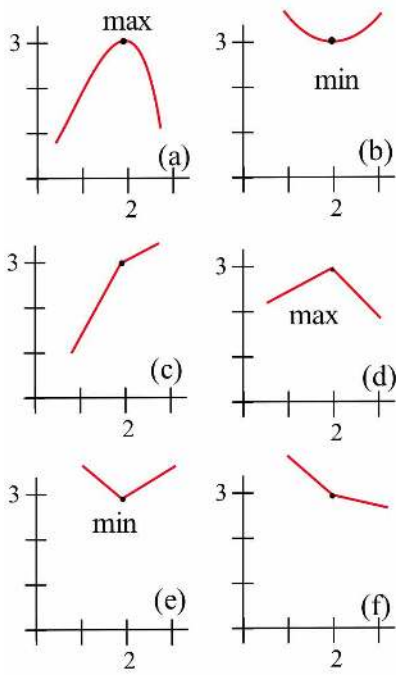


x	$f(x)$	$f'(x)$	max/min
1	5	0	local max
3	1	0	local min
4	3	0	neither
6	7	0	local max

Is $f(a)$ a Maximum or Minimum or Neither?

Once we have found the candidates $(a, f(a))$ for extreme points of f, we still have the problem of determining whether the point is a maximum, a minimum or neither.

One method involves graphing (or letting a calculator graph) the function near a, and then drawing a conclusion from the graph. All of the graphs below:

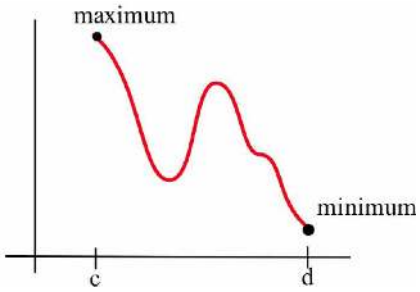


have $f(2) = 3$, and on each of the graphs $f'(2)$ either equals 0 or is undefined. It is clear from the graphs below:that the point $(2, 3)$ is: a local maximum in (a) and (d); a local minimum in (b) and (e); and not a local extreme in (c) and (f).

In Sections 3.3 and 3.4, we will investigate how information about the first and **second** derivatives of f can help determine whether the candidate $(a, f(a))$ is a maximum, a minimum or neither.

Endpoint Extremes

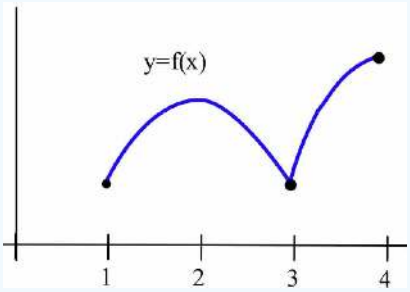
So far we have discussed finding extreme values of functions over the entire real number line or on an open interval, but in practice we may need to find the extreme of a function over some closed interval $[c, d]$. If an extreme value of f occurs at $x = a$ between c and d $(c < a < d)$ then the previous reasoning and results still apply: either $f'(a) = 0$ or f is not differentiable at a. On a closed interval, however, there is one more possibility: an extreme can occur at an **endpoint** of the closed interval, at $x = c$ or $x = d$:



We can extend our definition of a local extreme at $x = a$ (which requires $f(a) \geq f(x)$ [or $f(a) \leq f(x)$] for all x in some *open* interval containing a) to include $x = a$ being the endpoint of a closed interval: $f(a) \geq f(x)$ [or $f(a) \leq f(x)$] for all x in an interval of the form $[a, a + h)$ (for left endpoints) or $(a - h, a]$ (for right endpoints), where $h > 0$ is a number small enough to guarantee the "half-open" interval is in the domain of $f(x)$. Using this extended definition, the function in the margin has a local maximum (which is also a global maximum) at $x = c$ and a local minimum (also a global minimum) at $x = d$.

? Practice 3.1.3

List all of the extremes $(a, f(a))$ of the function in the figure below:



on the interval $[1, 4]$ and state whether $f'(a) = 0$, f is not differentiable at a, or a is an endpoint.

Answer

$(1, f(1))$ is a global minimum; $x = 1$ is an endpoint

$(2, f(2))$ is a local maximum; $f'(2) = 0$

$(3, f(3))$ is a local/global minimum; f is not differentiable at $x = 3$

$(4, f(4))$ is a global maximum; $x = 4$ is an endpoint

✓ Example 3.1.3

Find the extreme values of $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \leq x \leq 6$.

Solution

We need to find investigate points where $f'(x) = 0$, points where f is not differentiable, and the endpoints:

- $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$, so $f'(x) = 0$ only at $x = -1$ and $x = 3$.
- f is a polynomial, so it is differentiable everywhere.
- The endpoints of the interval are $x = -2$ and $x = 6$.

Altogether we have four points in the interval to examine, and any extreme values of f can only occur when x is one of those four points: $f(-2) = 3$, $f(-1) = 10$, $f(3) = -22$ and $f(6) = 59$. The (global) minimum of f on $[-2, 6]$ is -22 when $x = 3$, and the (global) maximum of f on $[-2, 6]$ is 59 when $x = 6$.

Sometimes the function we need to maximize or minimize is more complicated, but the same methods work.

Example 3.1.4

Find the extreme values of $f(x) = \frac{1}{3}\sqrt{64 + x^2} + \frac{1}{5}(10 - x)$ for $0 \le x \le 10$.

Solution

This function comes from an application we will examine in section 3.5. The only possible locations of extremes are where $f'(x) = 0$ or $f'(x)$ is undefined or where x is an endpoint of the interval $[0, 10]$.

$$\begin{aligned} f'(x) &= \mathrm{D}\left(\frac{1}{3}(64+x^2)^{\frac{1}{2}} + \frac{1}{5}(10 - x)\right) \\ &= \frac{1}{3} \cdot \frac{1}{2}(64 + x^2)^{-\frac{1}{2}} \cdot 2x - \frac{1}{5} \\ &= \frac{x}{3\sqrt{64 + x^2}} - \frac{1}{5} \end{aligned}$$

To find where $f'(x) = 0$, set the derivative equal to 0 and solve for x:

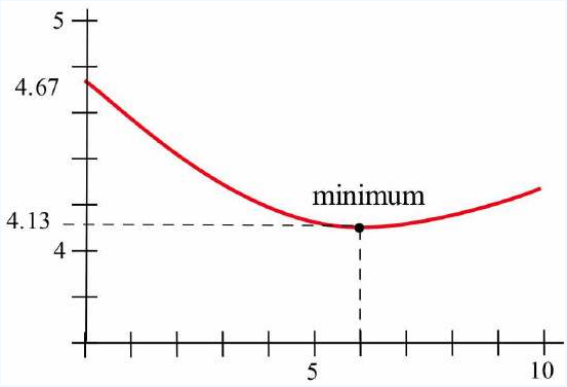
$$\begin{aligned} \frac{x}{3\sqrt{64 + x^2}} - \frac{1}{5} = 0 \;\Rightarrow\; \frac{x}{3\sqrt{64 + x^2}} = \frac{1}{5} \;\Rightarrow\; \frac{x^2}{576 + 9x^2} = \frac{1}{25} \\ \Rightarrow\; 16x^2 = 576 \;\Rightarrow x = \pm 6 \end{aligned}$$

but only $x = 6$ is in the interval $[0, 10]$. Evaluating f at this point gives $f(6) \approx 4.13$.

We can evaluate the formula for $f'(x)$ for any value of x, so the derivative is always defined.

Finally, the interval $[0, 10]$ has two endpoints, $x = 0$ and $x = 10$, and $f(0) \approx 4.67$ while $f(10) \approx 4.27$.

The maximum of f on $[0, 10]$ must occur at one of the points $(0, 4.67)$, $(6, 4.13)$ and $(10, 4.27)$, and the minimum must occur at one of these three points as well.



The maximum value of f is 4.67 at $x = 0$, and the minimum value of f is 4.13 at $x = 6$.

? Practice 3.1.4

Rework the previous Example to find the extreme values of $f(x) = \frac{1}{3}\sqrt{64+x^2} + \frac{1}{5}(10-x)$ for $0 \le x \le 5$.

Answer

This is the same function used in Example 4, but now the interval is $[0,5]$ instead of $[0,10]$. See the Example for the calculations.

Critical points:

- endpoints: $x = 0$ and $x = 5$
- f is differentiable for all $0 < x < 5$: none
- $f'(x) = 0$: none in $[0,5]$

$f(0) \approx 4.67$ is the maximum of f on $[0,5]$ $f(5) \approx 4.14$ is the minimum of f on $[0,5]$.

Critical Numbers

The points at which a function **might** have an extreme value are called **critical numbers**.

Definition

A **critical number** for a function f is a value $x = a$ in the domain of f so that:

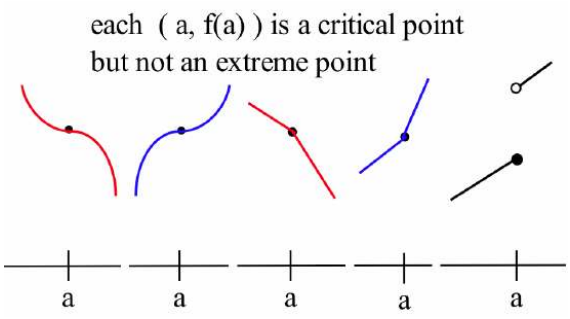
- $f'(a) = 0$ or
- f is not differentiable at a or
- a is an **endpoint** of a closed interval to which f is restricted.

If we are trying to find the extreme values of f on an **open** interval $c < x < d$ or on the entire number line, then the set of inputs to which f is restricted will not include any endpoints, so we will not need to worry about any endpoint critical numbers.

We can now give a very succinct description of where to look for extreme values of a function.

> An extreme value of f can only occur at a critical number.

The critical numbers only give possible locations of extremes; some critical numbers are not locations of extremes. In other words, critical numbers are the candidates for the locations of maximums and minimums. (Section 3.5 is devoted entirely to translating and solving maximum and minimum problems.)



Which Functions Have Extremes?

Some functions don't have extreme values: Example 2 showed that $f(x) = x^3$ (defined on the entire number line) did not have a maximum or minimum.

Example 3.1.5

Find the extreme values of $f(x) = x$.

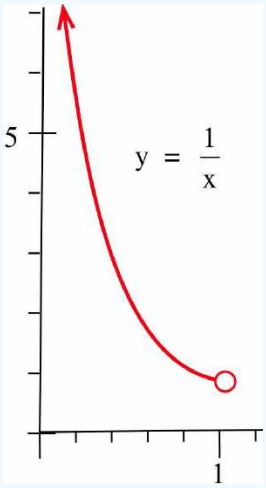
Solution

Because $f'(x) = 1 > 0$ for all x, the first theorem in this section guarantees that f has no extreme values. The function $f(x) = x$ does not have a maximum or minimum on the real number line.

With the previous function, the domain was so large that we could always make the function output larger or smaller than any given value by choosing an appropriate input x. The next example shows that we can encounter the same difficulty even on a "small" interval.

Example 3.1.6

Show that $f(x) = \frac{1}{x}$ does not have a maximum or minimum on the interval $(0, 1)$.



Solution

f is continuous for all $x \neq 0$ so f is continuous on the interval $(0, 1)$. For $0 < x < 1$, $f(x) = \frac{1}{x} > 0$ and for any number a strictly between 0 and 1, we can show that $f(a)$ is neither a maximum nor a minimum of f on $(0, 1)$, as follows.

Pick b to be any number between 0 and a: $0 < b < a$. Then $f(b) = \frac{1}{b} > \frac{1}{a} = f(a)$, so $f(a)$ is not a maximum. Similarly, pick c to be any number between a and 1: $a < c < 1$. Then $f(a) = \frac{1}{a} > \frac{1}{c} = f(c)$, so $f(a)$ is not a minimum. The interval $(0, 1)$ is not "large," yet f does not attain an extreme value anywhere in $(0, 1)$.

How would the situation change if we changed the interval in this example to $(0, 1]$? To $[1, 2]$?

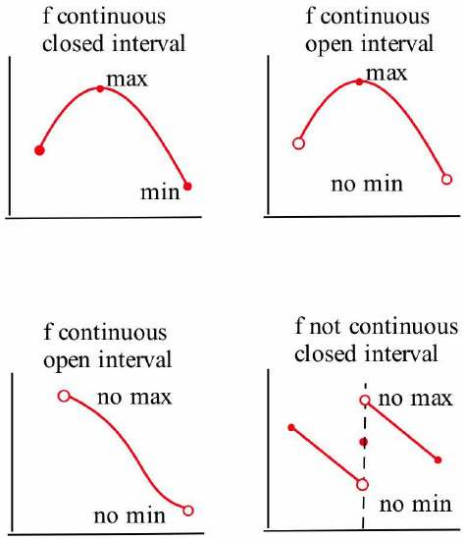
The Extreme Value Theorem provides conditions that guarantee a function to have a maximum and a minimum.

Extreme Value Theorem

If: f is continuous on a **closed** interval $[a, b]$

then: f attains both a maximum and minimum on $[a, b]$.

The proof of this theorem is difficult, so we omit it. The figure below:

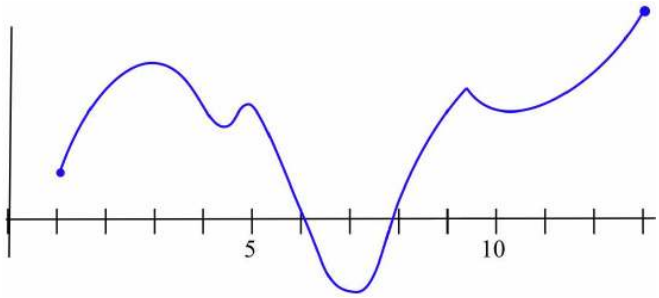


illustrates some of the possibilities for continuous and discontinuous functions on open and closed intervals.

The Extreme Value Theorem guarantees that certain functions (continuous ones) on certain intervals (closed ones) must have maximums and minimums. Other functions on other intervals **may** or **may not** have maximums and minimums.

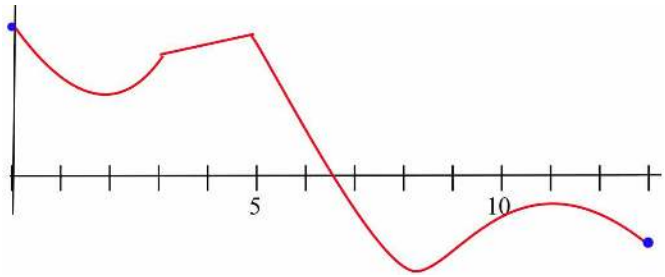
Problems

1. Label all of the local maximums and minimums of the function in the figure below:



   Also label all of the critical points.

2. Label the local extremes and critical points of the function graphed below:



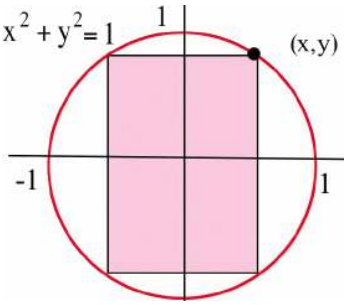
In Problems 3–22, find all of the critical points and local maximums and minimums of each function.

3. $f(x) = x^2 + 8x + 7$
4. $f(x) = 2x^2 - 12x + 7$
5. $f(x) = \sin(x)$
6. $f(x) = x^3 - 6x^2 + 5$
7. $f(x) = \sqrt[3]{x}$
8. $f(x) = 5x - 2$
9. $f(x) = xe^{5x}$

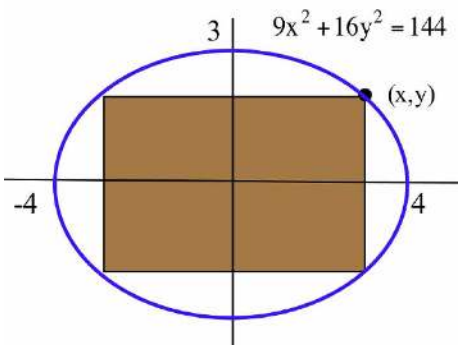
10. $f(x) = \sqrt[3]{1+x^2}$
11. $f(x) = (x-1)^2(x-3)$
12. $f(x) = \ln(x^2 - 6x + 11)$
13. $f(x) = 2x^3 - 96x + 42$
14. $f(x) = 5x + \cos(2x+1)$
15. $f(x) = e^{-(x-2)^2}$
16. $f(x) = |x+5|$
17. $f(x) = \dfrac{x}{1+x^2}$
18. $f(x) = \dfrac{x^3}{1+x^4}$
19. $f(x) = (x-2)^{\frac{2}{3}}$
20. $f(x) = (x^2-1)^{\frac{2}{3}}$
21. $f(x) = \sqrt[3]{x^2-4}$
22. $f(x) = \sqrt[3]{x-2}$
23. Sketch the graph of a continuous function f with:
    a. $f(1) = 3$, $f'(1) = 0$ and the point $(1,3)$ a relative maximum of f.
    b. $f(2) = 1$, $f'(2) = 0$ and the point $(2,1)$ a relative minimum of f.
    c. $f(3) = 5$, f is not differentiable at $x = 3$, and the point $(3,5)$ a relative maximum of f.
    d. $f(4) = 7$, f is not differentiable at $x = 4$, and the point $(4,7)$ a relative minimum of f.
    e. $f(5) = 4$, $f'(5) = 0$ and the point $(5,4)$ not a relative minimum or maximum of f.
    f. $f(6) = 3$, f not differentiable at 6, and $(6,3)$ not a relative minimum or maximum of f.

In Problems 24–37, find all critical points and local extremes of each function on the given intervals.

24. $f(x) = x^2 - 6x + 5$ on the entire real number line
25. $f(x) = x^2 - 6x + 5$ on $[-2,5]$
26. $f(x) = 2 - x^3$ on the entire real number line
27. $f(x) = 2 - x^3$ on $[-2,1]$
28. $f(x) = x^3 - 3x + 5$ on the entire real number line
29. $f(x) = x^3 - 3x + 5$ on $[-2,1]$
30. $f(x) = x^5 - 5x^4 + 5x^3 + 7$ on $(-\infty, \infty)$
31. $f(x) = x^5 - 5x^4 + 5x^3 + 7$ on $[0,2]$
32. $f(x) = \dfrac{1}{x^2+1}$ on $(-\infty, \infty)$
33. $f(x) = \dfrac{1}{x^2+1}$ on $[1,3]$
34. $f(x) = 3\sqrt{x^2+4} - x$ on $(-\infty, \infty)$
35. $f(x) = 3\sqrt{x^2+4} - x$ on $[0,2]$
36. $f(x) = xe^{-5x}$ on $(-\infty, \infty)$
37. $f(x) = x^3 - \ln(x)$ on $\left[\frac{1}{2}, 2\right]$
38. a. Find two numbers whose sum is 22 and whose product is as large as possible. (Suggestion: call the numbers x and $22 - x$).
    b. Find two numbers whose sum is $A > 0$ and whose product is as large as possible.
39. Find the coordinates of the point in the first quadrant on the circle $x^2 + y^2 = 1$ so that the rectangle in the figure below has the largest possible area.



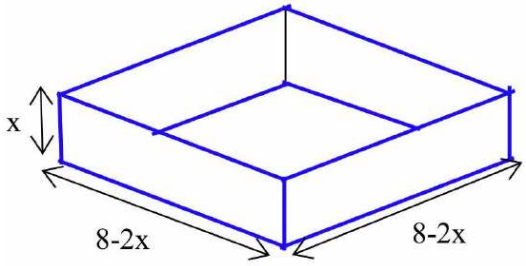
40. Find the coordinates of the point in the first quadrant on the ellipse $9x^2 + 16y^2 = 144$ so that the rectangle in the figure below



has:

a. the largest possible area.
b. The smallest possible area.

41. Find the value for x so the box shown below



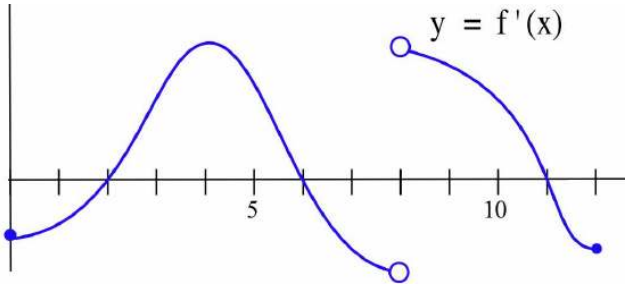
has:

a. the largest possible volume.
b. The smallest possible volume.

42. Find the radius and height of the cylinder that has the largest volume ($V = \pi r^2 h$) if the sum of the radius and height is 9.
43. Suppose you are working with a polynomial of degree 3 on a closed interval.
    a. What is the largest number of critical points the function can have on the interval?
    b. What is the smallest number of critical points it can have?
    c. What are the patterns for the most and fewest critical points a polynomial of degree n on a closed interval can have?
44. Suppose you have a polynomial of degree 3 divided by a polynomial of degree 2 on a closed interval.
    a. What is the largest number of critical points the function can have on the interval?
    b. What is the smallest number of critical points it can have?
45. Suppose $f(1) = 5$ and $f'(1) = 0$. What can we conclude about the point $(1, 5)$ if:
    a. $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$?
    b. $f'(x) < 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$?
    c. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$?
    d. $f'(x) > 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$?

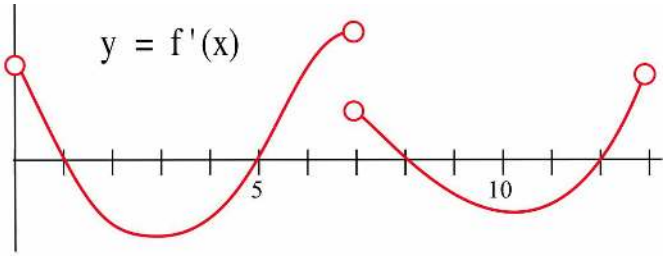
46. Suppose $f(2) = 3$ and f is continuous but not differentiable at $x = 2$. What can we conclude about the point $(2, 3)$ if:
    a. $f'(x) < 0$ for $x < 2$ and $f'(x) > 0$ for $x > 2$?
    b. $f'(x) < 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$?
    c. $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$?
    d. $f'(x) > 0$ for $x < 2$ and $f'(x) > 0$ for $x > 2$?

47. The figure below shows the graph of $f'(x)$, which is continuous on $(0, 12)$ except at $x = 8$.



    a. Which values of x are critical points of $f(x)$?
    b. At which values of x does f attain a local maximum?
    c. At which values of x does f attain a local minimum?

48. The figure below shows the graph of $f'(x)$, which is continuous on $(0, 13)$ except at $x = 7$.



    a. Which values of x are critical points?
    b. At which values of x does f attain a local maximum?
    c. At which values of x does f attain a local minimum?

49. State the contrapositive form of the Extreme Value Theorem.

50. Imagine the graph of $f(x) = 1 - x$. Does f have a **maximum** value for x in the given interval?
    a. $[0, 2]$
    b. $[0, 2)$
    c. $(0, 2]$
    d. $(0, 2)$
    e. $(1, \pi]$

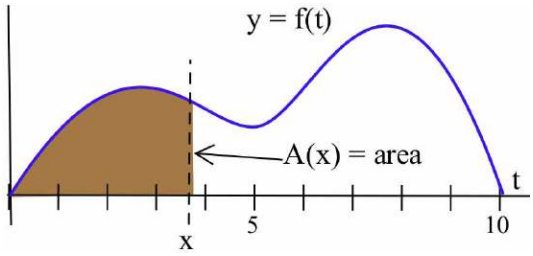
51. Imagine the graph of $f(x) = 1 - x$. Does f have a **minimum** value for x in the given interval?
    a. $[0, 2]$
    b. $[0, 2)$
    c. $(0, 2]$
    d. $(0, 2)$
    e. $(1, \pi]$

52. Imagine the graph of $f(x) = x^2$. Does f have a **maximum** value for x in the given interval?
    a. $[-2, 3]$
    b. $[-2, 3)$
    c. $(-2, 3]$
    d. $[-2, 1)$
    e. $(-2, 1]$

53. Imagine the graph of $f(x) = x^2$. Does f have a **minimum** value for x in the interval I?

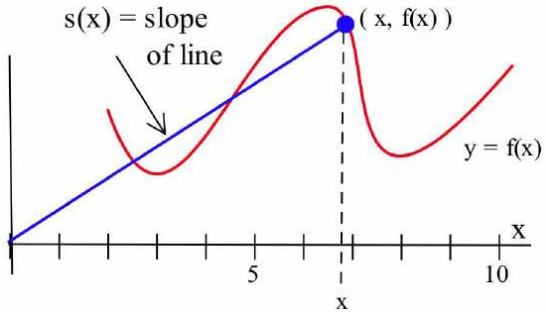
a. $[-2, 3]$
b. $[-2, 3)$
c. $(-2, 3]$
d. $[-2, 1)$
e. $(-2, 1]$

54. Define $A(x)$ to be the **area** bounded between the t-axis, the graph of $y = f(t)$ and a vertical line at $t = x$:



a. At what value of x is $A(x)$ minimum?
b. At what value of x is $A(x)$ maximum?

55. Define $S(x)$ to be the **slope** of the line through the points $(0, 0)$ and $(x, f(x))$ in the figure below:



a. At what value of x is $S(x)$ minimum?
b. At what value of x is $S(x)$ maximum?

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3.2: Mean Value Theorem

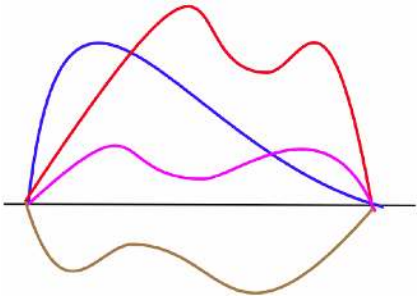
If you averaged 30 miles per hour during a trip, then at some instant during the trip you were traveling exactly 30 miles per hour.

That relatively obvious statement is the Mean Value Theorem as it applies to a particular trip. It may seem strange that such a simple statement would be important or useful to anyone, but the Mean Value Theorem is important and some of its consequences are very useful in a variety of areas. Many of the results in the rest of this chapter depend on the Mean Value Theorem, and one of the corollaries of the Mean Value Theorem will be used every time we calculate an "integral" in later chapters. A truly delightful aspect of mathematics is that an idea as simple and obvious as the Mean Value Theorem can be so powerful.

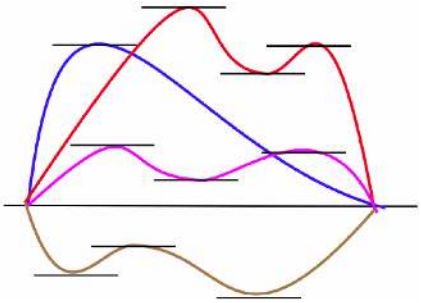
Before we state and prove the Mean Value Theorem and examine some of its consequences, we will consider a simplified version called Rolle's Theorem.

Rolle's Theorem

Pick any two points on the x-axis and think about all of the differentiable functions that pass through those two points.



Because our functions are differentiable, they must be continuous and their graphs cannot have any holes or breaks. Also, since these functions are differentiable, their derivatives are defined everywhere between our two points and their graphs can not have any "corners" or vertical tangents. The graphs of the functions in the margin figure can still have all sorts of shapes, and it may seem unlikely that they have any common properties other than the ones we have stated, but Michel Rolle (1652–1719) found one. He noticed that every one of these functions has one or more points where the tangent line is horizontal:



and this result is named after him.

Rolle's Theorem

If: $f(a) = f(b)$

and: $f(x)$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$

then: there is at least one number c between a and b so that $f'(c) = 0$.

Proof

We consider three cases: when $f(x) = f(a)$ for all x in (a, b), when $f(x) > f(a)$ for some x in (a, b), and when $f(x) < f(a)$ for some x in (a, b).

Case I: If $f(x) = f(a)$ for all x between a and b, then the graph of f is a horizontal line segment and $f'(c) = 0$ for *all* values of c strictly between a and b.

Case II: Suppose $f(x) > f(a)$ for some x in (a, b). Because f is continuous on the closed interval $[a, b]$, we know from the Extreme Value Theorem that f must attain a maximum value on the closed interval $[a, b]$. Because $f(x) > f(a)$ for some value of x in $[a, b]$, then the maximum of f must occur at some value c strictly between a and b: $a < c < b$. (Why can't the maximum be at a or b?) Because $f(c)$ is a local maximum of f, c is a critical number of f, meaning $f'(c) = 0$ or $f'(c)$ is undefined. But f is differentiable at all x between a and b, so the only possibility is that $f'(c) = 0$. (Notice that Rolle's Theorem tells us that [at least one] number c with the required properties exists, but does not tell us how to find c.)

Case III: Suppose $f(x) < f(a)$ for some x in (a, b). Then, arguing as we did in Case II, f attains a minimum at some value $x = c$ strictly between a and b, and so $f'(c) = 0$.

In each case, there is *at least one* value of c between a and b so that $f'(c) = 0$.

Example 3.2.1

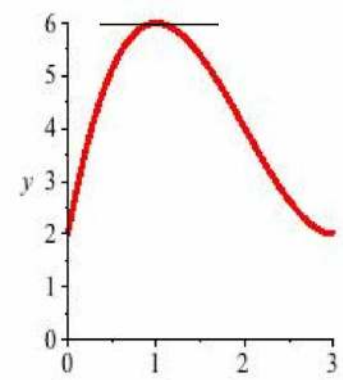
Show that $f(x) = x^3 - 6x^2 + 9x + 2$ satisfies the hypotheses of Rolle's Theorem on the interval $[0, 3]$ and find a value of c that the theorem tells you must exist.

Solution

Because f is a polynomial, it is continuous and differentiable everywhere. Furthermore, $f(0) = 2 = f(3)$, so Rolle's Theorem applies. Differentiating:

$$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$$

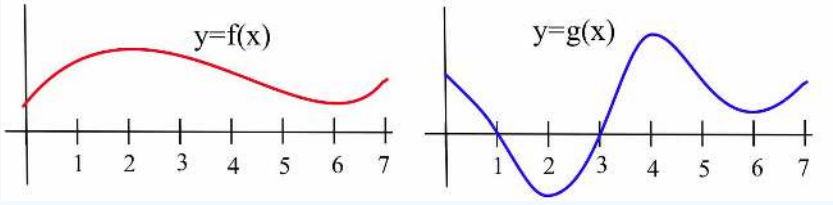
so $f'(x) = 0$ when $x = 1$ and when $x = 3$. The value $c = 1$ is between 0 and 3. The figure below:



shows a graph of f.

Practice 3.2.1

Find the value(s) of c for Rolle's Theorem for the functions graphed below.



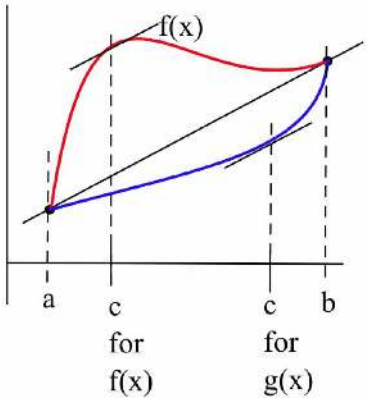
Answer

$f'(x) = 0$ when $x = 2$ and 6, so $c = 2$ and $c = 6$.

$g'(x) = 0$ when $x = 2$, 4 and 6, so $c = 2$, $c = 4$ and $c = 6$.

The Mean Value Theorem

Geometrically, the Mean Value Theorem is a "tilted" version of Rolle's Theorem:



In each theorem we conclude that there is a number c so that the slope of the tangent line to f at $x = c$ is the same as the slope of the line connecting the two ends of the graph of f on the interval $[a, b]$. In Rolle's Theorem, the two ends of the graph of f are at the same height, $f(a) = f(b)$, so the slope of the line connecting the ends is zero. In the Mean Value Theorem, the two ends of the graph of f do not have to be at the same height, so the line through the two ends does not have to have a slope of zero.

Mean Value Theorem

If: $f(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$

then: there is at least one number c between a and b so the line tangent to the graph of f at $x = c$ is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$:

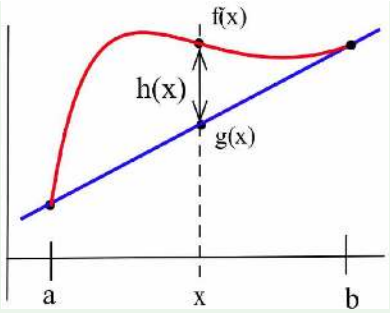
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

The proof of the Mean Value Theorem uses a tactic common in mathematics: introduce a new function that satisfies the hypotheses of some theorem we already know and then use the conclusion of that previously proven theorem. For the Mean Value Theorem we introduce a new function, $h(x)$, which satisfies the hypotheses of Rolle's Theorem. Then we can be certain that the conclusion of Rolle's Theorem is true for $h(x)$ and the Mean Value Theorem for f will follow from the conclusion of Rolle's Theorem for h.

First, let $g(x)$ be the linear function passing through the points $(a, f(a))$ and $(b, f(b))$ of the graph of f. The function g goes through the point $(a, f(a))$ so $g(a) = f(a)$. Similarly, $g(b) = f(b)$. The slope of the linear function $g(x)$ is $\frac{f(b) - f(a)}{b - a}$ so $g'(x) = \frac{f(b) - f(a)}{b - a}$ for all x between a and b, and g is continuous and differentiable. (The formula for g is $g(x) = f(a) + m(x - a)$ with $m = \frac{f(b) - f(a)}{b - a}$.)

Define $h(x) = f(x) - g(x)$ for $a \le x \le b$:



The function h satisfies the hypotheses of Rolle's theorem:

- $h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$
- $h(x)$ is continuous for $a \le x \le b$ because both f and g are continuous there
- $h(x)$ is differentiable for $a < x < b$ because both f and g are differentiable there

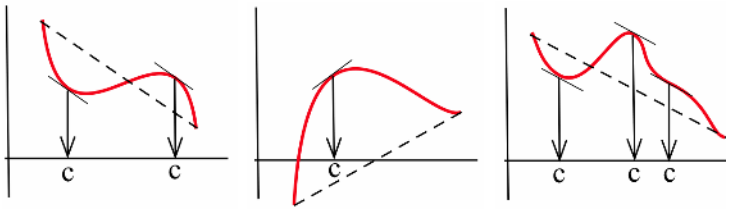
so the conclusion of Rolle's Theorem applies to h: there is a c between a and b so that $h'(c) = 0$.

The derivative of $h(x) = f(x) - g(x)$ is $h'(x) = f'(x) - g'(x)$ so we know that there is a number c between a and b with $h'(c) = 0$. But:

$$0 = h'(c) = f'(c) - g'(c) \Rightarrow f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}$$

which is exactly what we needed to prove.

Graphically, the Mean Value Theorem says that there is at least one point c where the slope of the tangent line, $f'(c)$, equals the slope of the line through the end points of the graph segment, $(a, f(a))$ and $(b, f(b))$. The figure below shows the locations of the parallel tangent lines for several functions and intervals:



The Mean Value Theorem also has a very natural interpretation if $f(x)$ represents the position of an object at time x: $f'(x)$ represents the velocity of the object at the **instant** x and $\frac{f(b) - f(a)}{b - a} = \frac{\text{change in position}}{\text{change in time}}$ represents the **average** (mean) velocity of the object during the time interval from time a to time b. The Mean Value Theorem says that there is a time c (between a and b) when the **instantaneous** velocity, $f'(c)$, is equal to the **average** velocity for the entire trip, $\frac{f(b) - f(a)}{b - a}$. If your average velocity during a trip is 30 miles per hour, then at some instant during the trip you were traveling exactly 30 miles per hour.

? Practice 3.2.2

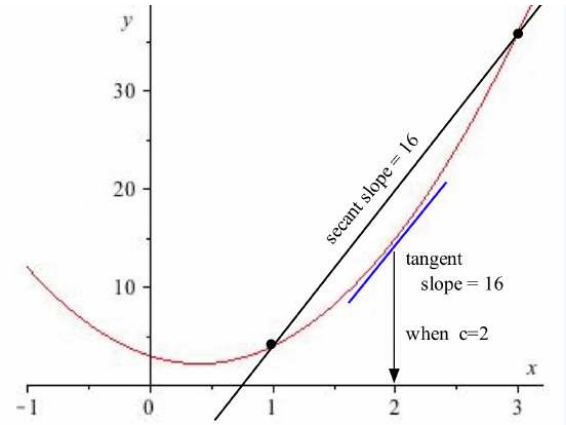
For $f(x) = 5x^2 - 4x + 3$ on the interval $[1, 3]$, calculate $m = \frac{f(b) - f(a)}{b - a}$ and find the value(s) of c so that $f'(c) = m$.

Answer

With $f(x) = 5x^2 - 4x + 3$ on $[1, 3]$, $f(1) = 4$ and $f(3) = 36$ so:

$$m = \frac{f(b) - f(a)}{b - a} = \frac{36 - 4}{3 - 1} = 16$$

\f'(x) = 10x - 4\) so $f'(c) = 10c - 4 = 16 \Rightarrow 10c = 20 \Rightarrow c = 2$. The graph of f showing the location of c appears below:



Some Consequences of the Mean Value Theorem

If the Mean Value Theorem was just an isolated result about the existence of a particular point c, it would not be very important or useful. However, the Mean Value Theorem is the basis of several results about the behavior of functions over entire intervals, and it is these consequences that give it an important place in calculus for both theoretical and applied uses.

The next two corollaries are just the first of many results that follow from the Mean Value Theorem.

We already know, from the Main Differentiation Theorem, that the derivative of a constant function $f(x) = K$ is always 0, but can a non-constant function have a derivative that is always 0? The first corollary says no.

Corollary 3.2.1

If: $f'(x) = 0$ for all x in an interval I

then: $f(x) = K$, a constant, for all x in I.

Proof

Assume $f'(x) = 0$ for all x in an interval I. Pick any two points a and b (with $a \neq b$) in the interval. Then, by the Mean Value Theorem, there is a number c between a and b so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. By our assumption, $f'(x) = 0$ for all x in I, so we know that $0 = f'(c) = \dfrac{f(b) - f(a)}{b - a}$ and thus $f(b) - f(a) = 0 \Rightarrow f(b) = f(a)$. But a and b were two arbitrary points in I, so the value of $f(x)$ is the same for any two values of x in I, and f is a constant function on the interval I.

We already know that if two functions are “parallel” (differ by a constant), then their derivatives are equal, but can two non-parallel functions have the same derivative? The second corollary says no.

Corollary 3.2.2

If: $f'(x) = g'(x)$ for all x in an interval I

then: $f(x) - g(x) = K$, a constant, for all x in I, so the graphs of f and g are “parallel” on the interval I

Proof

This corollary involves two functions instead of just one, but we can imitate the proof of the Mean Value Theorem and introduce a new function $h(x) = f(x) - g(x)$. The function h is differentiable and $h'(x) = f'(x) - g'(x) = 0$ for all x in I so, by Corollary 1, $h(x)$ is a constant function and $K = h(x) = f(x) - g(x)$ for all x in the interval. Thus $f(x) = g(x) + K$.

We will use Corollary 3.2.2 hundreds of times in Chapters 4 and 5 when we work with “integrals.” Typically you will be given the derivative of a function, $f'(x)$, and be asked to find **all** functions f that have that derivative. Corollary 2 tells us that if we can find **one** function f that has the derivative we want, then the only other functions that have the same derivative are of the form $f(x) + K$ where K is a constant: once you find one function with the right derivative, you have essentially found all of them.

Example 3.2.2

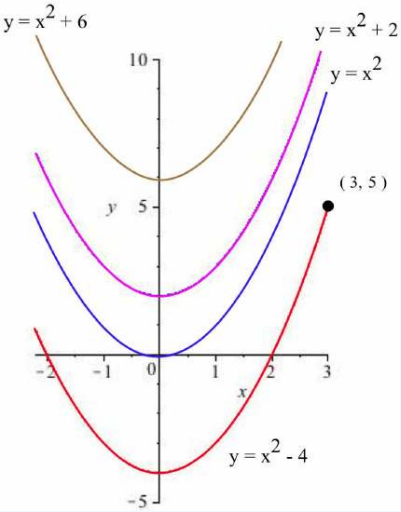
a. Find **all** functions whose derivatives equal $2x$.

b. Find a function $g(x)$ with $g'(x) = 2x$ and $g(3) = 5$.

Solution

a. Observe that $f(x) = x^2 \Rightarrow f'(x) = 2x$, so one function with the derivative we want is $f(x) = x^2$. Corollary 2 guarantees that every function g whose derivative is $2x$ has the form $g(x) = f(x) + K = x^2 + K$.

b. Because $g'(x) = 2x$, we know that g must have the form $g(x) = x^2 + K$, but this gives a whole "family" of functions:



and we want to find one member of that family. We also know that $g(3) = 5$ so we want to find the member of the family that passes through the point $(3, 5)$. Replacing $g(x)$ with 5 and x with 3 in the formula $g(x) = x^2 + K$, we can solve for the value of K: $5 = g(3) = (3)^2 + K \Rightarrow K = -4$. The function we want is $g(x) = x^2 - 4$.

Practice 3.2.3

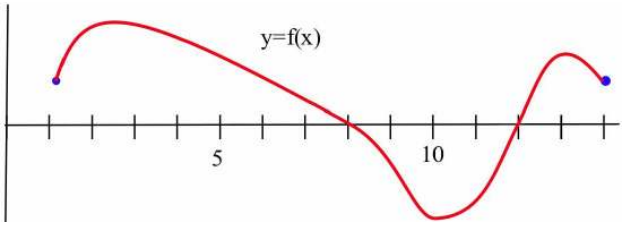
Restate Corollary **3.2.2** as a statement about the positions and velocities of two cars.

Answer

If two cars have the same velocities during an interval of time (so that $f'(t) = g'(t)$ for t in I) then the cars are always a constant distance apart during that time interval. (Note: "Same velocity" means **same speed** and **same direction**. If two cars are traveling at the same speed but in different directions, then the distance between them changes and is not constant.)

Problems

1. In the figure below, find the number(s) "c" that Rolle's Theorem promises (guarantees).

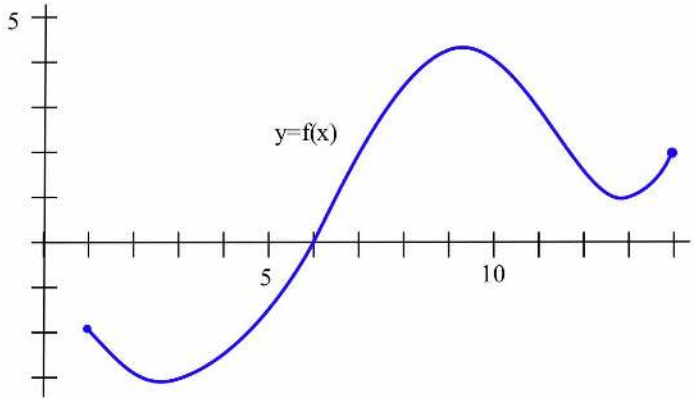


For Problems 2–4, verify that the hypotheses of Rolle's Theorem are satisfied for each of the functions on the given intervals, and find the value of the number(s) "c" that Rolle's Theorem promises.

2. a. $f(x) = x^2$ on $[-2, 2]$
   b. $f(x) = x^2 - 5x + 8$ on $[0, 5]$
3. a. $f(x) = \sin(x)$ on $[0, \pi]$

b. $f(x) = \sin(x)$ on $[\pi, 5\pi]$

4. a. $f(x) = x^3 - x + 3$ on $[-1, 1]$
   b. $f(x) = x \cdot \cos(x)$ on $[0, \frac{\pi}{2}]$
5. Suppose you toss a ball straight up and catch it when it comes down. If $h(t)$ is the height of the ball t seconds after you toss it, what does Rolle's Theorem say about the velocity of the ball? Why is it easier to catch a ball that someone on the ground tosses up to you on a balcony, than for you to be on the ground and catch a ball that someone on a balcony tosses down to you?
6. If $f(x) = \frac{1}{x^2}$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x) = -\frac{2}{x^3}$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?
7. If $f(x) = |x|$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x)$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?
8. If $f(x) = x^2$, then $f'(x) = 2x$ is never 0 on the interval $[1, 3]$. Why doesn't this function violate Rolle's Theorem?
9. If I take off in an airplane, fly around for a while and land at the same place I took off from, then my starting and stopping heights are the same but the airplane is always moving. Why doesn't this violate Rolle's theorem, which says there is an instant when my velocity is 0?
10. Prove the following corollary of Rolle's Theorem: If $P(x)$ is a polynomial, then between any two roots of P there is a root of P'.
11. Use the corollary in Problem 10 to justify the conclusion that the only root of $f(x) = x^3 + 5x - 18$ is 2. (Suggestion: What could you conclude about f' if f had another root?)
12. In the figure below, find the location(s) of the "c" that the Mean Value Theorem promises:

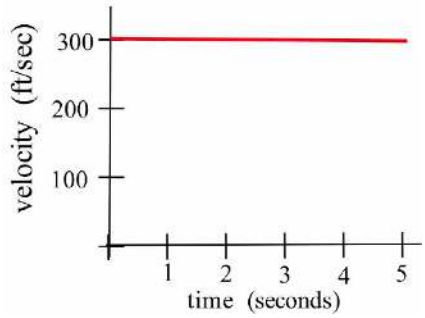


In Problems 13–15, verify that the hypotheses of the Mean Value Theorem are satisfied for each of the functions on the given intervals, and find the number(s) "c" that the Mean Value Theorem guarantees.

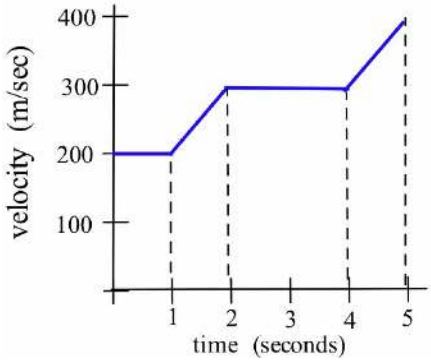
13. a. $f(x) = x^2$ on $[0, 2]$
    b. $f(x) = x^2 - 5x + 8$ on $[1, 5]$
14. a. $f(x) = \sin(x)$ on $[0, \frac{\pi}{2}]$
    b. $f(x) = x^3$ on $[-1, 3]$
15. a. $f(x) = 5 - \sqrt{x}$ on $[1, 9]$
    b. $f(x) = 2x + 1$ on $[1, 7]$
16. For the quadratic functions in parts (a) and (b) of Problem 13, the number c turned out to be the midpoint of the interval: $c = \frac{a+b}{2}$.
    a. For $f(x) = 3x^2 + x - 7$ on $[1, 3]$, show that $f'(2) = \dfrac{f(3) - f(1)}{3 - 1}$.
    b. For $f(x) = x^2 - 5x + 3$ on $[2, 5]$, show that $f'\left(\frac{7}{2}\right) = \dfrac{f(5) - f(2)}{5 - 2}$.
    c. For $f(x) = Ax^2 + Bx + C$ on $[a, b]$, show that $f'\left(\frac{a+b}{2}\right) = \dfrac{f(b) - f(a)}{b - a}$.
17. If $f(x) = |x|$, then $f(-1) = 1$ and $f(3) = 3$ but $f'(x)$ is never equal to $\dfrac{f(3) - f(-1)}{3 - (-1)} = \dfrac{1}{2}$. Why doesn't this violate the Mean Value Theorem?

In Problems 18–19, you are a traffic-court judge. In each case, a driver has challenged a speeding ticket and you need to decide if the ticket is appropriate.

18. A tolltaker says, “Your Honor, based on the elapsed time from when the car entered the toll road until the car stopped at my booth, I know the average speed of the car was 83 miles per hour. I did not actually see the car speeding, but I know it was and I gave the driver a ticket.”
19. The driver in the next case heard the tolltaker and says, “Your Honor, my average velocity on that portion of the toll road was only 17 miles per hour, so I could not have been speeding.”
20. Find three different functions (f, g and h) so that $f'(x) = g'(x) = h'(x) = \cos(x)$.
21. Find a function f so that $f'(x) = 3x^2 + 2x + 5$ and $f(1) = 10$.
22. Find $g(x)$ so that $g'(x) = x^2 + 3$ and $g(0) = 2$.
23. Find values for A and B so that the graph of the parabola $f(x) = Ax^2 + B$ is:
    a. tangent to $y = 4x + 5$ at the point $(1, 9)$.
    b. tangent to $y = 7 - 2x$ at the point $(2, 3)$.
    c. tangent to $y = x^2 + 3x - 2$ at the point $(0, 2)$.
24. Sketch the graphs of several members of the “family” of functions whose derivatives always equal 3. Give a formula that defines every function in this family.
25. Sketch the graphs of several members of the “family” of functions whose derivatives always equal $3x^2$. Give a formula that defines every function in this family.
26. At t seconds after takeoff, the upward velocity of a helicopter was $v(t) = 3t^2 + 2t$ feet/second. Two seconds after takeoff, the helicopter was 80 feet above sea level. Find a formula for the height of the helicopter at every time t.
27. Assume that a rocket is fired from the ground and has the upward velocity shown in the figure below:

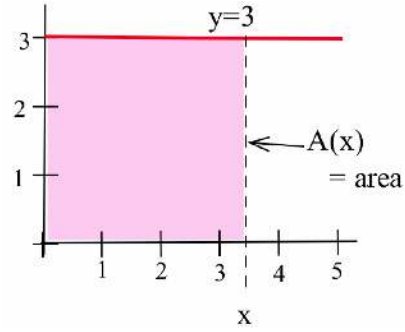


    Estimate the height of the rocket when $t = 1$, 2 and 5 seconds.
28. The figure below shows the upward velocity of a rocket:



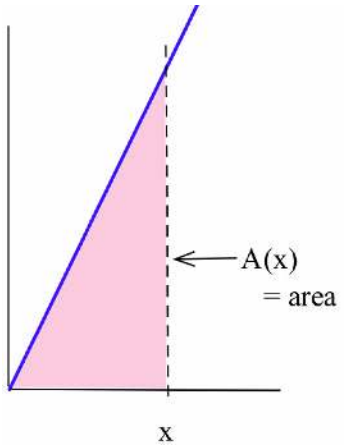
    Use the information in the graph to estimate the height of the rocket when $t = 1$, 2 and 5 seconds.
29. Determine a formula for $f(x)$ if you know: $f''(x) = 6$, $f'(0) = 4$ and $f(0) = -5$.
30. Determine a formula for $g(x)$ if you know: $g''(x) = 12x$, $g'(1) = 9$ and $g(2) = 30$.

31. Define $A(x)$ to be the **area** bounded by the t-axis, the line $y = 3$ and a vertical line at $t = x$:



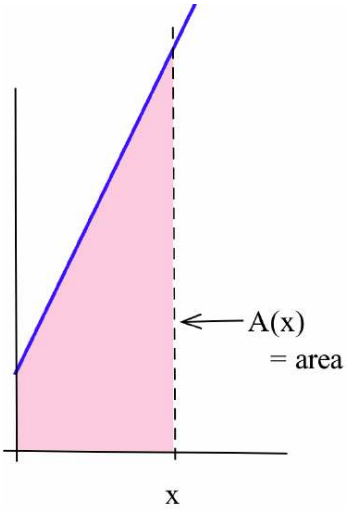
a. Find a formula for $A(x)$.
b. Determine $A'(x)$.

32. Define $A(x)$ to be the **area** bounded by the t-axis, the line $y = 2t$ and a vertical line at $t = x$:



a. Find a formula for $A(x)$.
b. Determine $A'(x)$.

33. Define $A(x)$ to be the **area** bounded by the t-axis, the line $y = 2t + 1$ and a vertical line at $t = x$:



a. Find a formula for $A(x)$.
b. Determine $A'(x)$.

In Problems 34–36, given a list of numbers $a_1, a_2, a_3, a_4, \ldots$, the **consecutive differences** between numbers in the list are: $a_2 - a_1$, $a_3 - a_2, a_4 - a_3, \ldots$

34. If $a_1 = 5$ and the consecutive difference is always 0, what can you conclude about the numbers in the list?
35. If $a_1 = 5$ and the consecutive difference is always 3, find a formula for a_n.
36. Suppose the "a" list starts with 3, 4, 7, 8, 6, 10, $13, \ldots$, and there is a "b" list that has the same consecutive differences as the "a" list.
    a. If $b_1 = 5$, find the next six numbers in the "b" list. How is b_n related to a_n?
    b. If $b_1 = 2$, find the next six numbers in the "b" list. How is b_n related to a_n?
    c. If $b_1 = B$, find the next six numbers in the "b" list. How is b_n related to a_n?

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3.3: The First Derivative and the Shape of f

This section examines some of the interplay between the shape of the graph of a function f and the behavior of its derivative, f'. If we have a graph of f, we will investigate what we can conclude about the values of f'. And if we know values of f', we will investigate what we can conclude about the graph of f.

Definitions

Given any interval I, a function f is…

- **increasing** on I if, for all x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- **decreasing** on I if, for all x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
- **monotonic** on I if f is increasing or if f is or decreasing on I

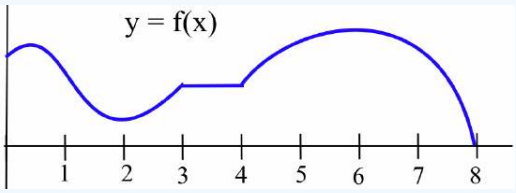
In this definition, I can be of the form (a, b), $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$ or $(-\infty, \infty)$, where $a < b$.

Graphically, f is increasing (decreasing) if, as we move from left to right along the graph of f, the height of the graph increases (decreases).

These same ideas make sense if we consider $h(t)$ to be the height (in feet) of a rocket at time t seconds. We naturally say that the rocket is rising or that its height is increasing if the height $h(t)$ increases over a period of time, as t increases.

Example 3.3.1

List the intervals on which the function graphed below is increasing or decreasing.

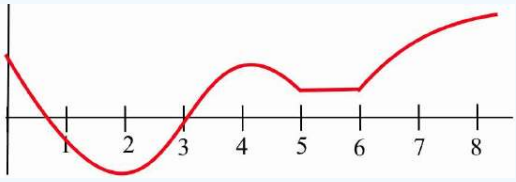


Solution

f is increasing on the intervals $[0, 0.3]$ (approximately), $[2, 3]$ and $[4, 6]$. f is decreasing on (approximately) $[0.3, 2]$ and $[6, 8]$. On the interval $[3, 4]$ the function is not increasing or decreasing—it is **constant**. It is also valid to say that f is increasing on the intervals $[0.5, 0.8]$ and $(0.5, 0.8)$ as well as many others, but we usually talk about the longest intervals on which f is monotonic.

Practice 3.3.1

List the intervals on which the function graphed below is increasing or decreasing.



Answer

g is increasing on $[2, 4]$ and $[6, 8]$; g is decreasing on $[0, 2]$ and $[4, 5]$; g is constant on $[5, 6]$.

If we have an accurate graph of a function, then it is relatively easy to determine where f is monotonic, but if the function is defined by a formula, then a little more work is required. The next two theorems relate the values of the derivative of f to the monotonicity of f. The first theorem says that if we know where f is monotonic, then we also know something about the values of f'. The second theorem says that if we know about the values of f' then we can draw conclusions about where f is monotonic.

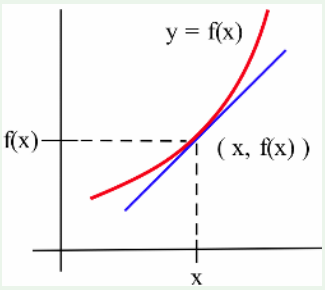
First Shape Theorem

For a function f that is differentiable on an interval (a, b):

- if f is increasing on (a, b) then $f'(x) \geq 0$ for all x in (a, b)
- if f is decreasing on (a, b) then $f'(x) \leq 0$ for all x in (a, b)
- if f is constant on (a, b), then $f'(x) = 0$ for all x in (a, b)

Proof

Most people find a picture such as the one below:



to be a convincing justification of this theorem: if the graph of f increases near a point $(x, f(x))$, then the tangent line is also increasing, and the slope of the tangent line is positive (or perhaps zero at a few places). A more precise proof, however, requires that we use the definitions of the derivative of f and of "increasing" (given above).

Case I: Assume that f is increasing on (a, b). We know that f is differentiable, so if x is any number in the interval (a, b) then

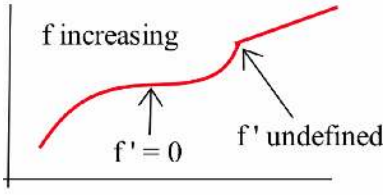
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3.3.1}$$

and this limit exists and is a finite value. If h is any small enough **positive** number so that $x + h$ is also in the interval (a, b), then $x < x + h \Rightarrow f(x) < f(x + h)$ (by the definition of "increasing"). We know that the numerator, $f(x + h) - f(x)$, and the denominator, h, are both positive, so the limiting value, $f'(x)$, must be positive or zero: $f'(x) \geq 0$.

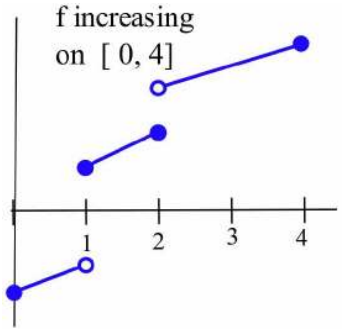
Case II: (The proof of this part is very similar to the "increasing" proof.) Assume that f is decreasing on (a, b). If $x < x + h$, then $f(x) > f(x + h)$ (by the definition of "decreasing"). So the numerator of the limit, $f(x + h) - f(x)$, will be negative but the denominator, h, will still be positive, so the limiting value, $f'(x)$, must be negative or zero: $f'(x) \leq 0$.

Case III: The derivative of a constant is 0, so if f is constant on (a, b) then $f'(x) = 0$ for all x in (a, b).

The previous theorem is easy to understand, but you need to pay attention to exactly what it says and what it does **not** say. It is possible for a differentiable function that is increasing on an interval to have horizontal tangent lines at some places in the interval:



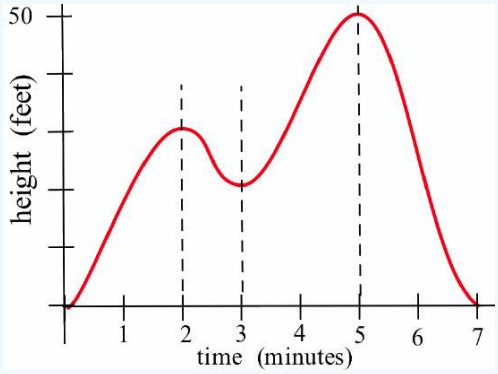
It is also possible for a continuous function that is increasing on an interval to have an undefined derivative at some places in the interval. Finally, it is possible for a function that is increasing on an interval to fail to be continuous at some places in the interval:



The First Shape Theorem has a natural interpretation in terms of the height $h(t)$ and upward velocity $h'(t)$ of a helicopter at time t. If the height of the helicopter is increasing ($h(t)$ is an increasing function), then the helicopter has a positive or zero upward velocity: $h'(t) \geq 0$. If the height of the helicopter is not changing, then its upward velocity is 0: $h'(t) = 0$.

Example 3.3.2

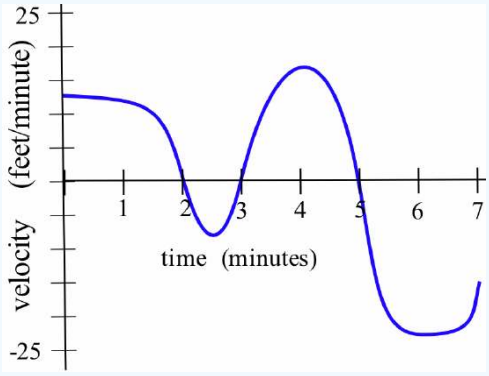
The figure below shows the height of a helicopter during a period of time:



Sketch the graph of the upward velocity of the helicopter, $\frac{dh}{dt}$.

Solution

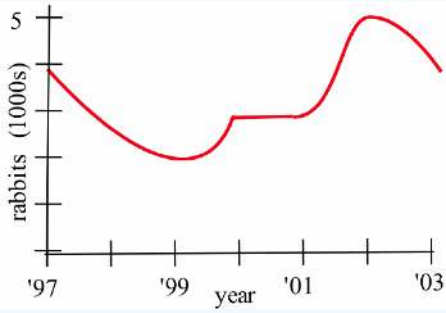
The graph of $v(t) = \frac{dh}{dt}$ appears below:



Notice that $h(t)$ has a local maximum when $t = 2$ and $t = 5$, and that $v(2) = 0$ and $v(5) = 0$. Similarly, $h(t)$ has a local minimum when $t = 3$, and $v(3) = 0$. When h is increasing, v is positive. When h is decreasing, v is negative.

Practice 3.3.2

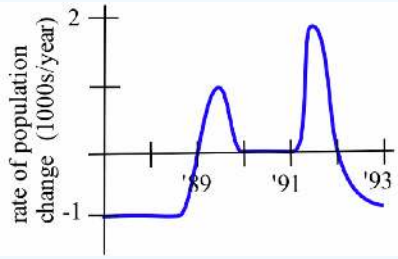
The figure below shows the population of rabbits on an island during a 6-year period:



Sketch the graph of the rate of population change, $\frac{dR}{dt}$, during those years.

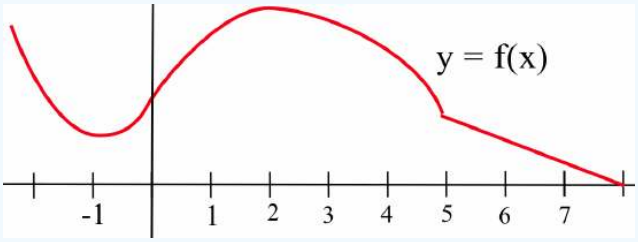
Answer

The graph below shows the rate of population change, $\frac{dR}{dt}$:



Example 3.3.3

A graph of f appears below:



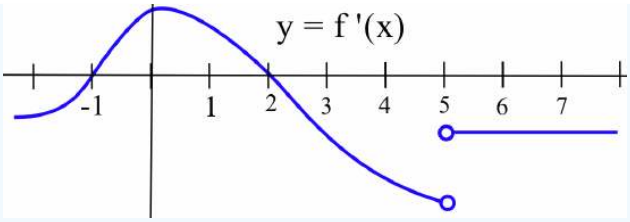
Sketch a graph of f'.

Solution

It is a good idea to look first for the points where $f'(x) = 0$ or where f is not differentiable (the critical points of f). These locations are usually easy to spot, and they naturally break the problem into several smaller pieces. The only numbers at which $f'(x) = 0$ are $x = -1$ and $x = 2$, so the only places the graph of $f'(x)$ will cross the x-axis are at $x = -1$ and $x = 2$: we can therefore plot the points $(-1, 0)$ and $(2, 0)$ on the graph of f'. The only place where f is not differentiable is at the "corner" above $x = 5$, so the graph of f' will not be defined for $x = 5$. The rest of the graph of f is relatively easy to sketch:

- if $x < -1$ then $f(x)$ is decreasing so $f'(x)$ is negative
- if $-1 < x < 2$ then $f(x)$ is increasing so $f'(x)$ is positive
- if $2 < x < 5$ then $f(x)$ is decreasing so $f'(x)$ is negative
- if $5 < x$ then $f(x)$ is decreasing so $f'(x)$ is negative

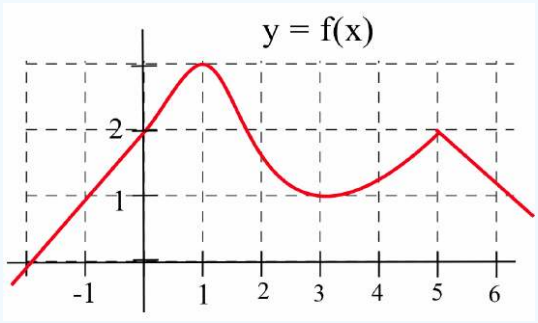
A graph of f' appears below:



$f(x)$ is continuous at $x = 5$, but not differentiable at $x = 5$ (indicated by the "hole").

Practice 3.3.3

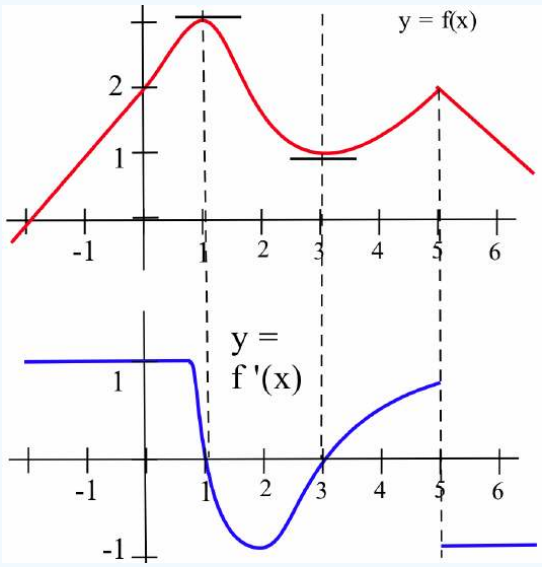
A graph of f appears below:



Sketch a graph of f'. (The graph of f has a "corner" at $x = 5$.)

Answer

A graph of f' appears below:



Notice how the graph of f' is 0 where f has a maximum or minimum.

The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective. It says that if we know something about the values of f', then we can draw some conclusions about the shape of the graph of f.

Second Shape Theorem

For a function f that is differentiable on an interval I:

- if $f'(x) > 0$ for all x in the interval I, then f is increasing on I
- if $f'(x) < 0$ for all x in the interval I, then f is decreasing on I
- if $f'(x) = 0$ for all x in the interval I, then f is constant on I

Proof

This theorem follows directly from the Mean Value Theorem, and the last part is just a restatement of the First Corollary of the Mean Value Theorem.

Case I: Assume that $f'(x) > 0$ for all x in I and pick any points a and b in I with $a < b$. Then, by the Mean Value Theorem, there is a point c between a and b so that $\frac{f(b) - f(a)}{b - a} = f'(c) > 0$ and we can conclude that $f(b) - f(a) > 0$, which means that $f(b) > f(a)$. Because $a < b \Rightarrow f(a) < f(b)$, we know that f is increasing on I.

Case II: Assume that $f'(x) < 0$ for all x in I and pick any points a and b in I with $a < b$. Then there is a point c between a and b so that $\frac{f(b) - f(a)}{b - a} = f'(c) < 0$, and we can conclude that $f(b) - f(a) = (b - a)f'(c) < 0$ so $f(b) < f(a)$. Because $a < b \Rightarrow f(a) > f(b)$, we know f is decreasing on I.

Practice 3.3.4

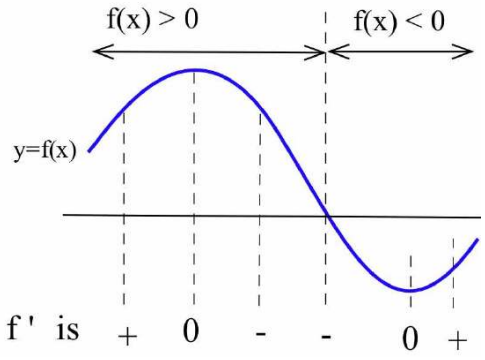
Rewrite the Second Shape Theorem as a statement about the height $h(t)$ and upward velocity $h'(t)$ of a helicopter at time t seconds.

Answer

The Second Shape Theorem for helicopters:

- If the upward velocity h' is positive during time interval I then the height h is increasing during time interval I.
- If the upward velocity h' is negative during time interval I then the height h is decreasing during time interval I.
- If the upward velocity h' is zero during time interval I then the height h is constant during time interval I.

The value of a function f at a number x tells us the height of the graph of f above or below the point $(x, 0)$ on the x-axis. The value of f' at a number x tells us whether the graph of f is increasing or decreasing (or neither) as the graph passes through the point $(x, f(x))$ on the graph of f. If $f(x)$ is positive, it is possible for $f'(x)$ to be positive, negative, zero or undefined: the value of $f(x)$ has absolutely nothing to do with the value of f'. The figure below illustrates some of the possible combinations of values for f and f':



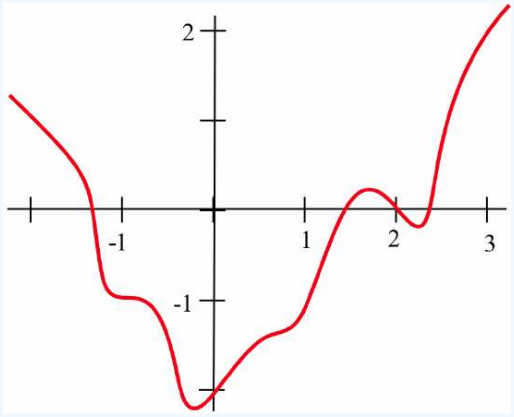
Practice 3.3.5

Graph a continuous function that satisfies the conditions on f and f' given below:

x	-2	-1	0	1	2	3
$f(x)$	1	-1	-2	-1	0	2
$f'(x)$	-1	0	1	2	-1	1

Answer

A graph satisfying the conditions in the table appears below:



The Second Shape Theorem can be particularly useful if we need to graph a function f defined by a formula. Between any two consecutive critical numbers of f, the graph of f is monotonic (why?). If we can find all of the critical numbers of f, then the domain of f will be broken naturally into a number of pieces on which f will be monotonic.

Example 3.3.4

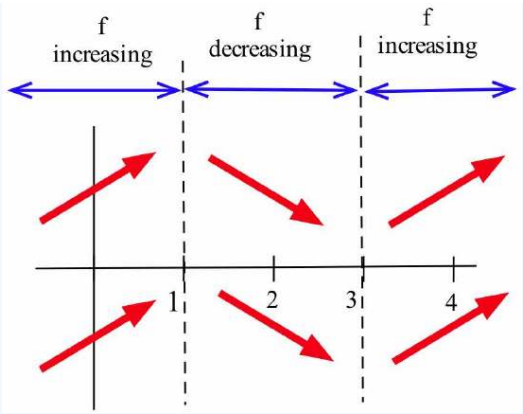
Use information about the values of f' to help graph $f(x) = x^3 - 6x^2 + 9x + 1$.

Solution

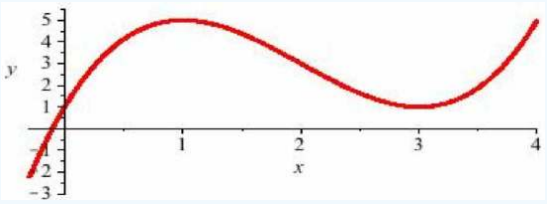
$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ so $f'(x) = 0$ only when $x = 1$ or $x = 3$; f' is a polynomial, so it is always defined. The only critical numbers, $x = 1$ and $x = 3$, break the real number line into three pieces on which f is monotonic: $(-\infty, 1)$, $(1, 3)$ and $(3, \infty)$.

- $x < 1 \Rightarrow f'(x) = 3(\text{negative})(\text{negative}) > 0 \Rightarrow f$ increasing
- $1 < x < 3 \Rightarrow f'(x) = 3(\text{positive})(\text{negative}) < 0 \Rightarrow f$ is decreasing
- $3 < x \Rightarrow f'(x) = 3(\text{positive})(\text{positive}) > 0 \Rightarrow f$ is increasing

Although we don't yet know the value of f anywhere, we do know a lot about the shape of the graph of f: as we move from left to right along the x-axis, the graph of f increases until $x = 1$, then decreases until $x = 3$, after which the graph increases again:



The graph of f "turns" when $x = 1$ and $x = 3$. To plot the graph of f, we still need to evaluate f at a few values of x, but only at a **very** few values: $f(1) = 5$, and $(1, 5)$ is a local maximum of f; $f(3) = 1$, and $(3, 1)$ is a local minimum of f. A graph of f appears below:



Practice 3.3.6

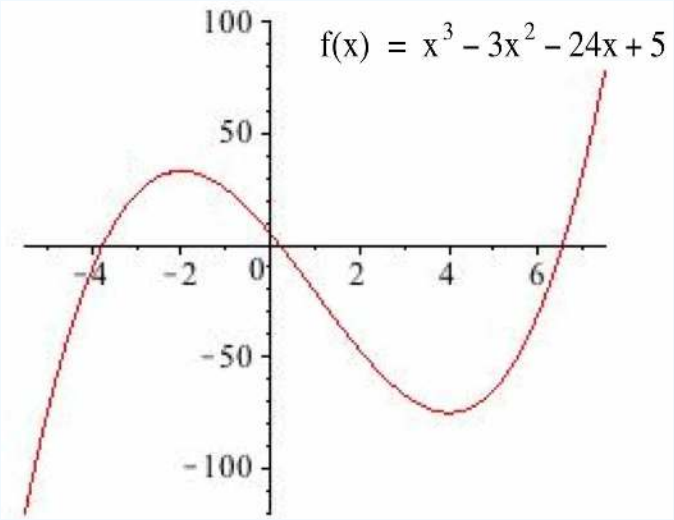
Use information about the values of f' to help graph the function $f(x) = x^3 - 3x^2 - 24x + 5$.

Answer

$f'(x) = 3x^2 - 6x - 24 = 3(x - 4)(x + 2)$ so $f'(x) = 0$ if $x = -2$ or $x = 4$.

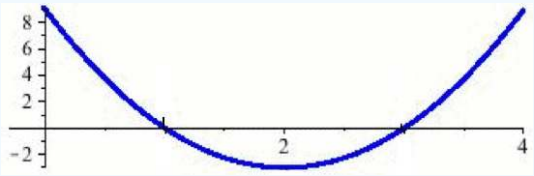
- $x < -2 \Rightarrow f'(x) = 3(\text{negative})(\text{negative}) > 0 \Rightarrow f$ increasing
- \mbox{$-2 < x < 4 \Rightarrow f'(x) = 3(\text{negative})(\text{positive}) < 0 \Rightarrow f$ decreasing}
- $x > 4 \Rightarrow f'(x) = 3(\text{positive})(\text{positive}) > 0 \Rightarrow f$ increasing

Thus f has a relative maximum at $x = -2$ and a relative minimum at $x = 4$. A graph of f below:



Example 3.3.5

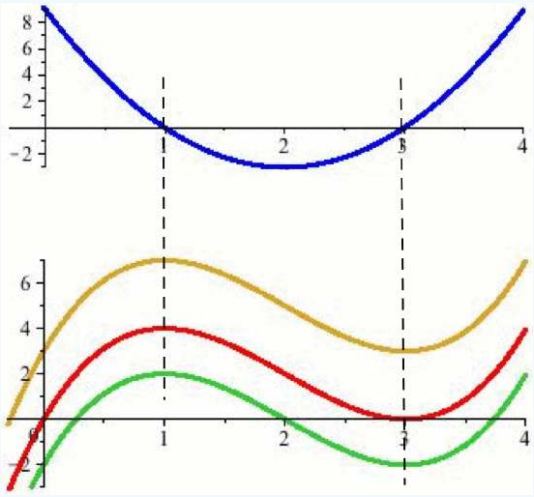
Use the graph of f' below:



to sketch the shape of the graph of f. Why isn't the graph of f' enough to completely determine the graph of f?

Solution

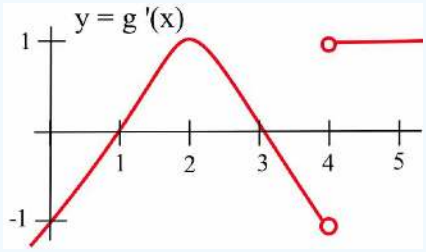
Several functions that have the derivative we want appear below:



and each provides a correct answer. By the Second Corollary to the Mean Value Theorem, we know there is a whole family of "parallel" functions that share the derivative we want, and each of these functions provides a correct answer. If we had additional information about the function—such as a point it passes through—then only one member of the family would satisfy the extra condition and there would be only one correct answer.

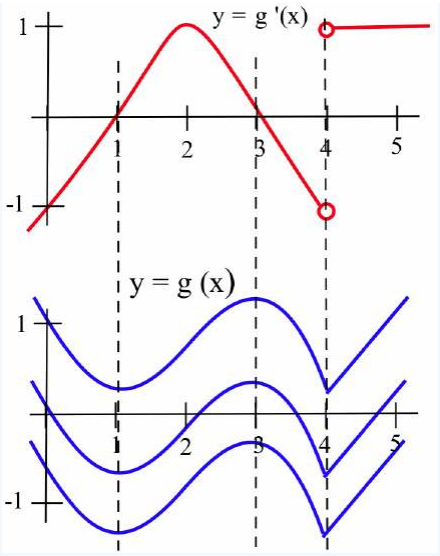
Practice 3.3.7

Use the graph of g' provided below to sketch the shape of a graph of g:



Answer

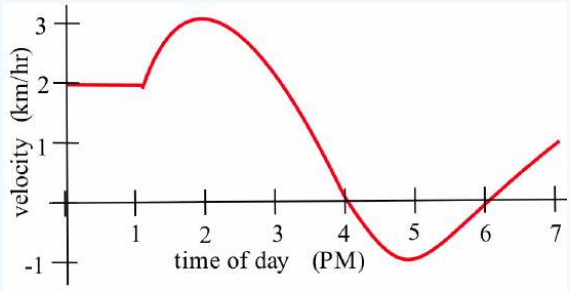
The figure below left shows several possible graphs for g:



Each has the correct shape to give the graph of g'. Notice that the graphs of g are "parallel" (differ by a constant).

Practice 3.3.8

A weather balloon is released from the ground and sends back its upward velocity measurements:

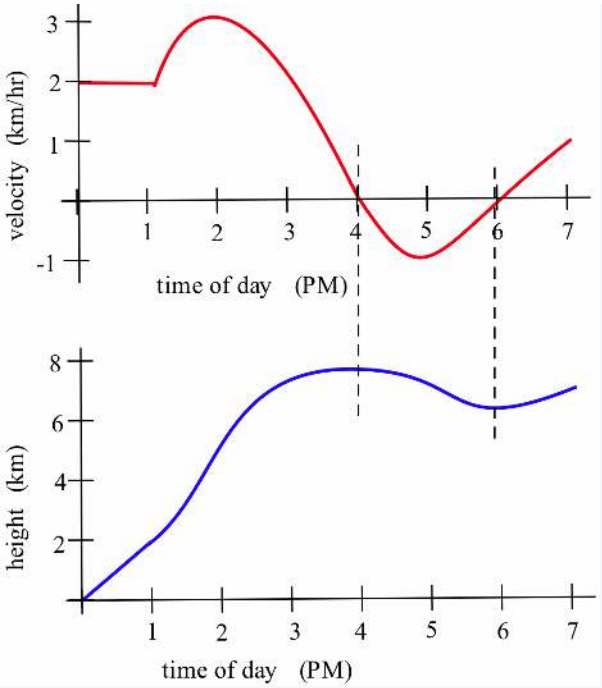


Sketch a graph of the height of the balloon. When was the balloon highest?

Answer

The figure below shows the height graph for the balloon:

https://math.libretexts.org/@go/page/212024



The balloon was highest at 4 p.m. and had a local minimum at 6 p.m.

Using the Derivative to Test for Extremes

The first derivative of a function tells about the general shape of the function, and we can use that shape information to determine whether an extreme point is a (local) maximum or minimum or neither.

First Derivative Test for Local Extremes

Let f be a continuous function with $f'(c) = 0$ or $f'(c)$ undefined.

- If $f'(\text{left of } c) > 0$ and $f'(\text{right of } c) < 0$ then $(c, f(c))$ is a local maximum.
- If $f'(\text{left of } c) < 0$ and $f'(\text{right of } c) > 0$ then $(c, f(c))$ is a local minimum.
- If $f'(\text{left of } c) > 0$ and $f'(\text{right of } c) > 0$ then $(c, f(c))$ is **not** a local extreme.
- If $f'(\text{left of } c) < 0$ and $f'(\text{right of } c) < 0$ then $(c, f(c))$ is **not** a local extreme.

Four graphs: a red inverted-U shape with a black dot at top labeled 'max,' f'(x) 0 to the left and f'(x) < 0 to the right; a red U-shaped graph with black dot at bottom labeled 'min,' f'(x) < 0 to the left and f'(x) > 0 to the right; an increasing blue graph with a black dot at a central point where the graph flattens labeled 'not max, not min,' f'(x) > 0 to the right and left; a decreasing blue graph with a black dot at a central point where the graph flattens labeled 'not max, not min,' f'(x) < 0 to the right and left." class="internal default" style="width: 1021px; height: 200px;" width="1021px" height="200px" src="/@api/deki/files/140441/fig303_19a.png">

Practice 3.3.9

Find all extremes of $f(x) = 3x^2 - 12x + 7$ and use the First Derivative Test to classify them as maximums, minimums or neither.

Answer

$f'(x) = 6x - 12$ so $f'(x) = 0$ only if $x = 2$.

- $x < 2 \Rightarrow f'(x) < 0 \Rightarrow f$ decreasing
- $x > 2 \Rightarrow f'(x) > 0 \Rightarrow f$ increasing

From this we can conclude that f has a minimum when $x = 2$ and has a shape similar to graph below:

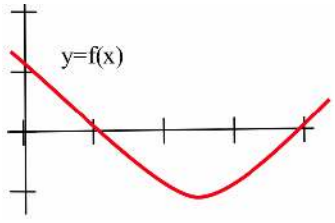
A blue graph of a f(x) = 3x^2- 12x + 7 ona [-1.5]X[-5,25] grid with a dashed-black vertical line at x = 2 denoting the vertex of the parabola. A label to the left of the dashed line reads f'(x)<0 and a label to the right reads f'(x)

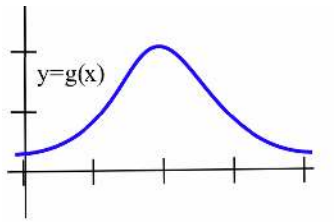
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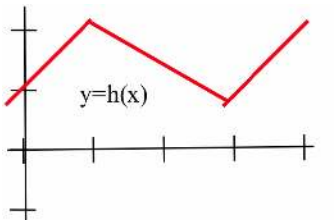
We could also have noticed that the graph of the quadratic function $f(x) = 3x^2 - 12x + 7$ must be an upward-opening parabola.

Problems

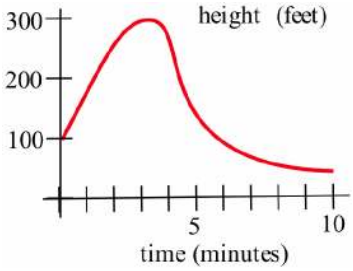
In Problems 1–3, sketch the graph of the **derivative** of each function.

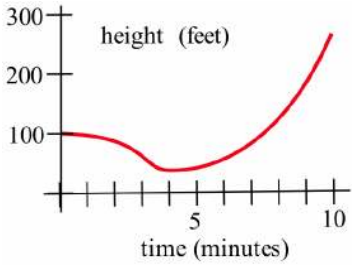
1. 

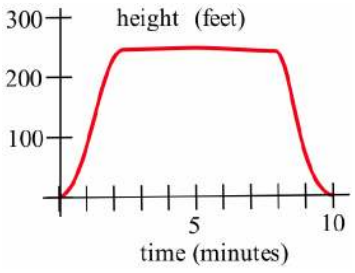
2. 

3. 

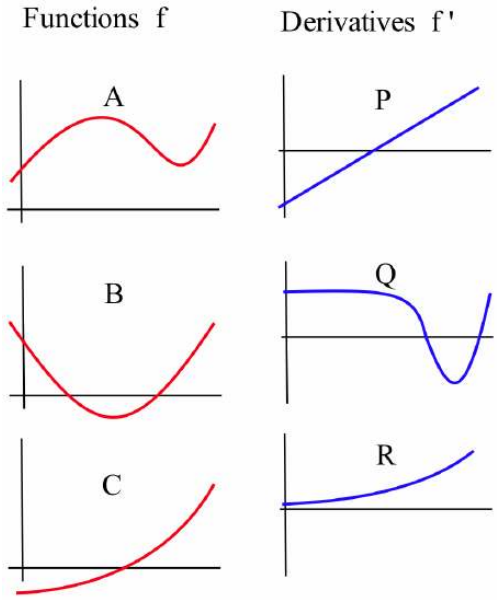
Problems 4–6 show the graph of the height of a helicopter; sketch a graph of its upward velocity.

4. 

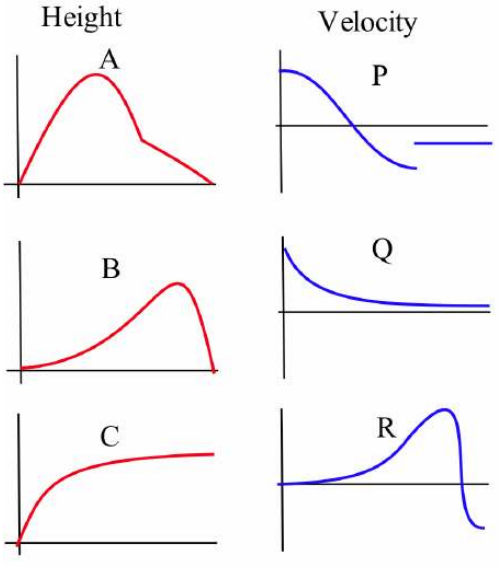
5. 

6. 

7. In the figure below, match the graphs of the functions with those of their derivatives.

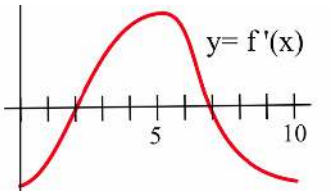


8. Match the graphs showing the heights of rockets with those showing their velocities.



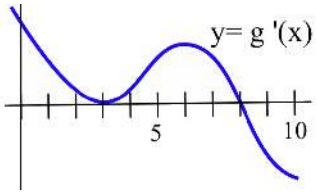
9. Use the Second Shape Theorem to show that $f(x) = \ln(x)$ is monotonic increasing on the interval $(0, \infty)$.
10. Use the Second Shape Theorem to show that $g(x) = e^x$ is monotonic increasing on the entire real number line.
11. A student is working with a complicated function f and has shown that the derivative of f is always positive. A minute later the student also claims that $f(x) = 2$ when $x = 1$ and when $x = \pi$. Without checking the student's work, how can you be certain that it contains an error?

12. The figure below shows the graph of the **derivative** of a continuous function f.



a. List the critical numbers of f.
b. What values of x result in a local maximum?
c. What values of x result in a local minimum?

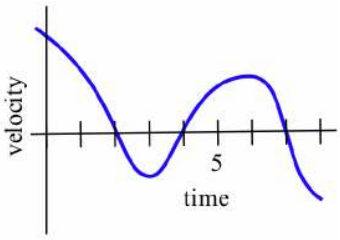
13. The figure below shows the graph of the **derivative** of a continuous function g.



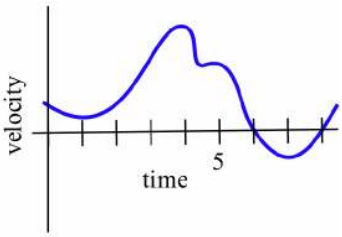
a. List the critical numbers of g.
b. What values of x result in a local maximum?
c. What values of x result in a local minimum?

Problems 14–16 show the graphs of the upward velocities of three helicopters. Use the graphs to determine when each helicopter was at a (relative) maximum or minimum height.

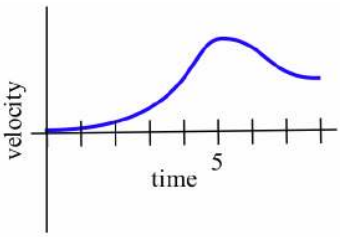
14.



15.



16.



In 17–22, use information from the derivative of each function to help you graph the function. Find all local maximums and minimums of each function.

17. $f(x) = x^3 - 3x^2 - 9x - 5$
18. $g(x) = 2x^3 - 15x^2 + 6$
19. $h(x) = x^4 - 8x^2 + 3$
20. $s(t) = t + \sin(t)$

21. $r(t) = \dfrac{2}{t^2+1}$
22. $f(x) = \dfrac{x^2+3}{x}$
23. $f(x) = 2x + \cos(x)$ so $f(0) = 1$. Without graphing the function, you can be certain that f has how many **positive** roots?
24. $g(x) = 2x - \cos(x)$ so $g(0) = -1$. Without graphing the function, you can be certain that g has how many **positive** roots?
25. $h(x) = x^3 + 9x - 10$ has a root at $x = 1$. Without graphing h, show that h has no other roots.
26. Sketch the graphs of monotonic decreasing functions that have exactly
    a. no roots
    b. one root and
    c. two roots.
27. Each of the following statements is false. Give (or sketch) a counterexample for each statement.
    a. If f is increasing on an interval I, then $f'(x) > 0$ for all x in I.
    b. If f is increasing and differentiable on I, then $f'(x) > 0$ for all x in I.
    c. If cars A and B always have the same speed, then they will always be the same distance apart.
28. a. Find several different functions f that all have the same derivative $f'(x) = 2$.
    b. Determine a function f with derivative $f'(x) = 2$ that also satisfies $f(1) = 5$.
    c. Determine a function g with $g'(x) = 2$ for which the graph of g goes through $(2, 1)$.
29. a. Find several different functions h that all have the same derivative $h'(x) = 2x$.
    b. Determine a function f with derivative $f'(x) = 2x$ that also satisfies $f(3) = 20$.
    c. Determine a function g with $g'(x) = 2x$ for which the graph of g goes through $(2, 7)$.
30. Sketch functions with the given properties to help determine whether each statement is true or false.
    a. If $f'(7) > 0$ and $f'(x) > 0$ for all x near 7, then $f(7)$ is a local maximum of f on $[1, 7]$.
    b. If $g'(7) < 0$ and $g'(x) < 0$ for all x near 7, then $g(7)$ is a local minimum of g on $[1, 7]$.
    c. If $h'(1) > 0$ and $h'(x) > 0$ for all x near 1, then $h(1)$ is a local minimum of h on $[1, 7]$.
    d. If $r'(1) < 0$ and $r'(x) < 0$ for all x near 1, then $r(1)$ is a local maximum of r on $[1, 7]$.
    e. If $s'(7) = 0$, then $s(7)$ is a local maximum of s on $[1, 7]$.

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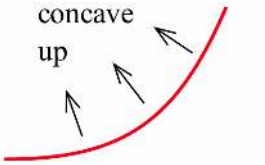
3.4: The Second Derivative and the Shape of f

The first derivative of a function provides information about the shape of the function, so the second derivative of a function provides information about the shape of the first derivative, which in turn will provide additional information about the shape of the original function f.

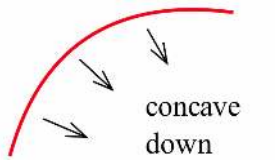
In this section we investigate how to use the second derivative (and the shape of the first derivative) to reach conclusions about the shape of the original function. The first derivative tells us whether the graph of f is increasing or decreasing. The second derivative will tell us about the "concavity" of f: whether f is curving upward or downward.

Concavity

Graphically, a function is **concave up** if its graph is curved with the opening upward:

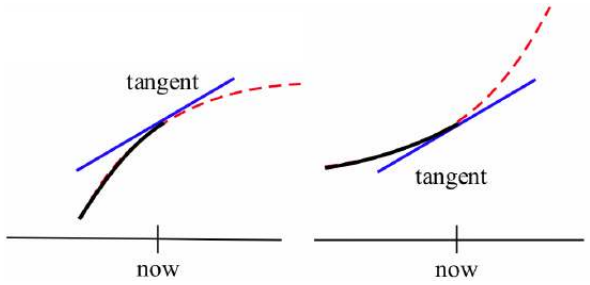


Similarly, a function is **concave down** if its graph opens downward:



The concavity of a function can be important in applied problems and can even affect billion-dollar decisions.

An Epidemic: Suppose you, as an official at the CDC, must decide whether current methods are effectively fighting the spread of a disease—or whether more drastic measures are required. In the figure below:



$f(x)$ represents the number of people infected with the disease at time x in two different situations. In both cases the number of people with the disease, $f(\text{now})$, and the rate at which new people are getting sick, $f'(\text{now})$, are the same. The difference is the concavity of f, and that difference might have a big effect on your decision. In (a), f is concave down at "now," and it appears that the current methods are starting to bring the epidemic under control; in (b), f is concave up, and it appears that the epidemic is growing out of control.

Usually it is easy to determine the concavity of a function by examining its graph, but we also need a definition that does not require a graph of the function, a definition we can apply to a function described by a formula alone.

Definition

Let f be a differentiable function.

- f is **concave up** at a if the graph of f is above the tangent line L to f for all x close to (but not equal to) a:

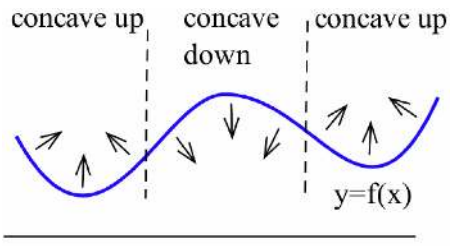
$$f(x) > L(x) = f(a) + f'(a)(x - a)$$

- f is **concave down** at a if the graph of f is below the tangent line L to f for all x close to (but not equal to) a:

- f is **concave down** at a if the graph of f is below the tangent line L to f for all x close to (but not equal to) a.

$$f(x) < L(x) = f(a) + f'(a)(x-a)$$

The figure below shows the concavity of a function at several points:



The next theorem provides an easily applied test for the concavity of a function given by a formula.

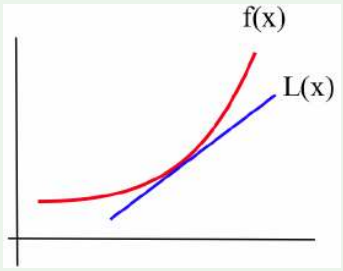
The Second Derivative Condition for Concavity

Let f be a twice differentiable function on an interval I.

a. $f''(x) > 0$ on $I \Rightarrow f'(x)$ increasing on $I \Rightarrow f$ concave up on I
b. $f''(x) < 0$ on $I \Rightarrow f'(x)$ decreasing on $I \Rightarrow f$ concave down on I
c. $f''(a) = 0 \Rightarrow$ no information ($f(x)$ may be concave up or concave down or neither at a)

Proof

a. Assume that $f''(x) > 0$ for all x in I, and let a be any point in I. We want to show that f is concave up at a, so we need to prove that the graph of f:



is above the tangent line to f at a: $f(x) > L(x) = f(a) + f'(a)(x-a)$ for x close to a. Assume that x is in I and apply the Mean Value Theorem to f on the interval with endpoints a and x: there is a number c between a and x so that

$$f'(c) = \frac{f(x) - f(a)}{x - a} \Rightarrow f(x) = f(a) + f'(c)(x-a)$$

Because $f'' > 0$ on I, we know that $f'' > 0$ between a and x, so the Second Shape Theorem tells us that f' is increasing between a and x. We will consider two cases: $x > a$ and $x < a$.

- If $x > a$ then $x - a > 0$ and c is in the interval $[a, x]$ so $a < c$. Because f' is increasing, $a < c \Rightarrow f'(a) < f'(c)$. Multiplying each side of this last inequality by the positive quantity $x - a$ yields $f'(a)(x-a) < f'(c)(x-a)$. Adding $f(a)$ to each side of this last inequality, we have:

$$L(x) = f(a) + f'(a)(x-a) < f(a) + f'(c)(x-a) = f(x)$$

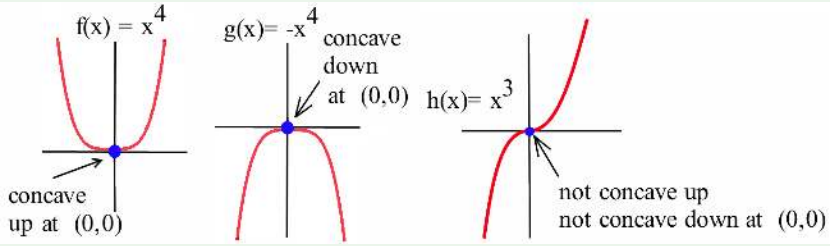
- If $x < a$ then $x - a < 0$ and c is in the interval $[x, a]$ so $c < a$. Because f' is increasing, $c < a \Rightarrow f'(c) < f'(a)$. Multiplying each side of this last inequality by the negative quantity $x - a$ yields $f'(c)(x-a) > f'(a)(x-a)$ so:

$$f(x) = f(a) + f'(c)(x-a) > f(a) + f'(a)(x-a) = L(x)$$

In each case we see that $f(x)$ is above the tangent line $L(x)$.

b. The proof of this part is similar.

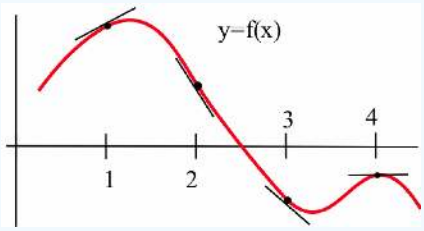
c. Let $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$:



The second derivative of each of these functions is zero at $a = 0$, and at $(0, 0)$ they all have the same tangent line: $L(x) = 0$ (the x-axis). However, at $(0, 0)$ they all have different concavity: f is concave up, while g is concave down and h is neither concave up nor concave down.

Practice 3.4.1

Use the graph of f below:



to finish filling in the table with "+" for positive, "−" for negative or "0."

x	$f(x)$	$f'(x)$	$f''(x)$	concavity
1	+	+	−	down
2	+			
3	−			
4				

Answer

copy of previous table with blanks filled in

x	$f(x)$	$f'(x)$	$f''(x)$	concavity
1	+	+	−	down
2	+	−	−	down
3	−	−	+	up
4	−	0	−	down

"Feeling" the Second Derivative

Earlier we saw that if a function $f(t)$ represents the position of a car at time t, then $f'(t)$ gives the velocity and $f''(t)$ the acceleration of the car at the instant t.

If we are driving along a straight, smooth road, then what we *feel* is the acceleration of the car:

- a large positive acceleration feels like a "push" toward the back of the car

- a large negative acceleration (a deceleration) feels like a “push” toward the front of the car
- an acceleration of 0 for a period of time means the velocity is constant and we do not feel pushed in either direction

In a moving vehicle it is possible to measure these “pushes,” the acceleration, and from that information to determine the velocity of the vehicle, and from the velocity information to determine the position. Inertial guidance systems in airplanes use this tactic: they measure front–back, left–right and up–down acceleration several times a second and then calculate the position of the plane. They also use computers to keep track of time and the rotation of the earth under the plane. After all, in six hours the Earth has made a quarter of a revolution, and Dallas has rotated more than 5,000 miles!

Example 3.4.1

The upward acceleration of a rocket was $a(t) = 30$ m/sec^2 during the first six seconds of flight, $0 \le t \le 6$. The velocity of the rocket at $t = 0$ was 0 m/sec and the height of the rocket above the ground at $t = 0$ was 25 m. Find a formula for the height of the rocket at time t and determine the height at $t = 6$ seconds.

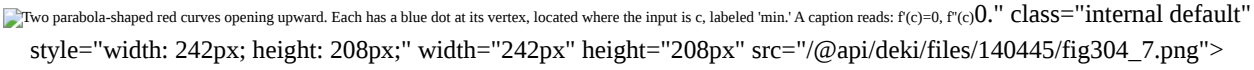
Solution

$v'(t) = a(t) = 30 \Rightarrow v(t) = 30t + K$ for some constant K. We also know $v(0) = 0$ so $30(0) + K = 0 \Rightarrow K = 0$ and this $v(t) = 30t$.

Similarly, $h'(t) = v(t) = 30t \Rightarrow h(t) = 15t^2 + C$ for some constant C. We know that $h(0) = 25$ so $15(0)^2 + C = 25 \Rightarrow C = 25$. Thus $h(t) = 15t^2 + 25$ so $h(6) = 15(6)^2 + 25 = 565$ m.

f'' and Extreme Values of f

The concavity of a function can also help us determine whether a critical point is a maximum or minimum or neither. For example, if a point is at the bottom of a concave-up function then that point is a minimum.


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The Second Derivative Test for Extremes

Let f be a twice differentiable function.

a. If $f'(c) = 0$ and $f''(c) < 0$ then f is concave down and has a local maximum at $x = c$.
b. If $f'(c) = 0$ and $f''(c) > 0$ then f is concave up and has a local minimum at $x = c$.
c. If $f'(c) = 0$ and $f''(c) = 0$ then f may have a local maximum, a local minimum or neither at $x = c$.

Proof

a. Assume that $f'(c) = 0$. If $f''(c) < 0$ then f is concave down at $x = c$ so the graph of f will be below the tangent line $L(x)$ for values of x near c. The tangent line, however, is given by $L(x) = f(c) + f'(c)(x - c) = f(c) + 0(x - c) = f(c)$, so if x is close to c then $f(x) < L(x) = f(c)$ and f has a local maximum at $x = c$.
b. The proof for a local minimum of f is similar.
c. If $f'(c) = 0$ and $f''(c) = 0$, then we cannot immediately conclude anything about local maximums or minimums of f at $x = c$. The functions $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$ all have their first and second derivatives equal to zero at $x = 0$, but f has a local minimum at 0, g has a local maximum at 0, and h has neither a local maximum nor a local minimum at $x = 0$.

The Second Derivative Test for Extremes is very useful when f'' is easy to calculate and evaluate. Sometimes, however, the First Derivative Test—or simply a graph of the function—provides an easier way to determine if the function has a local maximum or a local minimum: it depends on the function and on which tools you have available.

Practice 3.4.2

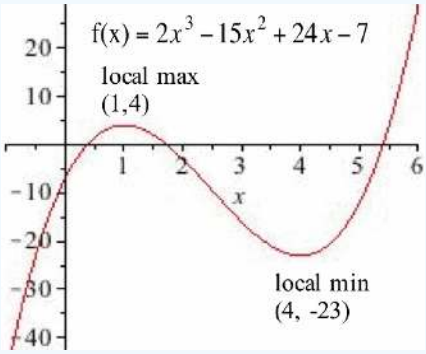
$f(x) = 2x^3 - 15x^2 + 24x - 7$ has critical numbers $x = 1$ and $x = 4$. Use the Second Derivative Test for Extremes to determine whether $f(1)$ and $f(4)$ are maximums or minimums or neither.

Answer

$f'(x) = 6x^2 - 30x + 24$, which is defined for all x.

$f'(x) = 0$ if $x = 1$ or $x = 4$ (critical values).

$f''(x) = 12x - 30$ so $f''(1) = -18 < 0$ tells us that f is concave down at the critical value $x = 1$, so $(1, f(1)) = (1, 4)$ is a relative maximum; and $f''(4) = 18 > 0$ tells us that f is concave up at the critical value $x = 4$, so $(4, f(4)) = (4, -23)$ is a relative minimum. A graph of f appears below:



Inflection Points

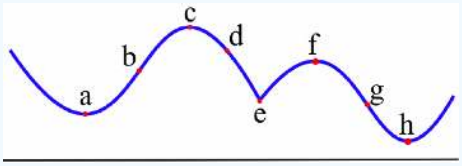
Maximums and minimums typically occur at places where the second derivative of a function is positive or negative, but the places where the second derivative is 0 are also of interest.

Definition

An **inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to concave down or from concave down to concave up.

Practice 3.4.3

Which of the labeled points in the figure below are inflection points?



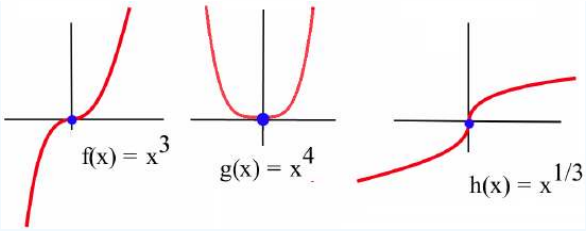
Answer

The points labeled b and g are inflection points.

To find the inflection points of a function we can use the second derivative of the function. If $f''(x) > 0$, then the graph of f is concave up at the point $(x, f(x))$ so $(x, f(x))$ is not an inflection point. Similarly, if $f''(x) < 0$ then the graph of f is concave down at the point $(x, f(x))$ and the point is not an inflection point. The only points left that can possibly be inflection points are the places where $f''(x) = 0$ or where $f''(x)$ does not exist (in other words, where f' is not differentiable). To find the inflection points of a function we need only check the points where $f''(x)$ is 0 or undefined. If $f''(c) = 0$ or is undefined, then the point $(c, f(c))$ **may or may not** be an inflection point—we need to check the concavity of f on each side of $x = c$. The functions in the next example illustrate what can happen at such a point.

Example 3.4.2

Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = \sqrt[3]{x}$.



For which of these functions is the point $(0,0)$ an inflection point?

Solution

Graphically, it is clear that the concavity of $f(x)=x^3$ and $h(x)=\sqrt[3]{x}$ changes at $(0,0)$, so $(0,0)$ is an inflection point for f and h. The function $g(x)=x^4$ is concave up everywhere, so $(0,0)$ is not an inflection point of g.

$f(x)=x^3 \Rightarrow f'(x)=3x^2 \Rightarrow f''(x)=6x$ so the only point at which $f''(x)=0$ or is undefined (f' is not differentiable) is at $x=0$. If $x<0$ then $f''(x)<0$ so f is concave down; if $x>0$ then $f''(x)>0$ so f is concave up. Thus at $x=0$ the concavity of f changes so the point $(0,f(0))=(0,0)$ is an inflection point of $f(x)=x^3$.

$g(x)=x^4 \Rightarrow g'(x)=4x^3 \Rightarrow g''(x)=12x^2$ so the only point at which $g''(x)=0$ or is undefined is at $x=0$. But $g''(x)>0$ (so g is concave up) for any $x\neq 0$. Thus the concavity of g never changes, so the point $(0,g(0))=(0,0)$ is not an inflection point of $g(x)=x^4$.

$h(x)=\sqrt[3]{x}=x^{\frac{1}{3}} \Rightarrow h'(x)=\frac{1}{3}x^{-\frac{2}{3}} \Rightarrow h''(x)=-\frac{2}{9}x^{-\frac{5}{3}}$ so h'' is not defined if $x=0$ (and $h''(x)\neq 0$ elsewhere); $h''(\text{negative number})>0$ and $h''(\text{positive number})<0$, so h changes concavity at $(0,0)$ and $(0,0)$ is an inflection point of $h(x)=\sqrt[3]{x}$.

Practice 3.4.4

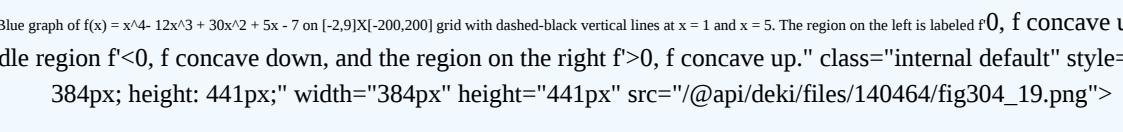
Find all inflection points of $f(x)=x^4-12x^3+30x^2+5x-7$.

Answer

$f'(x)=4x^3-36x^2+60x+5 \Rightarrow f''(x)=12x^2-72x+60=12(x^2-6x+5)=12(x-1)(x-5)$ so the only candidates to be inflection points are $x=1$ and $x=5$.

- If $x<1$ then $f''(x)=12(\text{neg})(\text{neg})>0$
- If $1<x<5$ then $f''(x)=12(\text{pos})(\text{neg})<0$
- If $5<x$ then $f''(x)=12(\text{pos})(\text{pos})>0$

f changes concavity at $x=1$ and $x=5$, so $x=1$ and $x=5$ are both inflection points. A graph of f appears below:

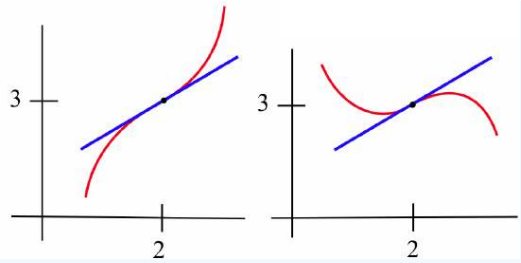


Example 3.4.3

Sketch a graph of a function with $f(2)=3$, $f'(2)=1$ and an inflection point at $(2,3)$.

Solution

Two solutions appear below:



Using f' and f'' to Graph f

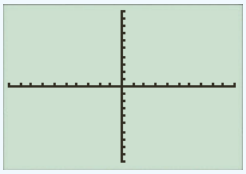
Today you can easily graph most functions of interest using a graphing calculator—and create even nicer graphs using an app on your phone or a Web-based graphing utility. Earlier generations of calculus students did not have these tools, so they relied on calculus to help them draw graphs of unfamiliar functions by hand. While you can create a graph in seconds that your predecessors may have labored over for half an hour or longer, you can still use calculus to help you select an appropriate graphing "window," and to be confident that your window has not missed any points of interest on the graph of a function.

Example 3.4.4

Create a graph of $f(x) = xe^{-9x^2}$ that shows all local and global extrema and all inflection points.

Solution

If you graph $f(x)$ on a calculator using the standard window ($-10 \le x \le 10$ and $-10 \le y \le 10$) you will likely see nothing other than the coordinate axes:



You might consult a table of values for the function to help adjust the window, but this trial-and-error technique will still not guarantee that you have displayed all points of interest. Computing the first derivative of f, we get:

$$f'(x) = x\left[-18xe^{-9x^2}\right] + e^{-9x^2} \cdot 1 = \left[1 - 18x^2\right] e^{-9x^2}$$

which is defined for all values of x; $f'(x) = 0 \Rightarrow 1 - 18x^2 = 0 \Rightarrow x^2 = \frac{1}{18} \Rightarrow x = \pm\frac{1}{3\sqrt{2}}$, so the only critical numbers are $x = -\frac{1}{3\sqrt{2}}$ and $x = \frac{1}{3\sqrt{2}}$. Computing the second derivative of f, we get:

$$\begin{aligned} f''(x) &= \left[1 - 18x^2\right] \cdot \left[-18xe^{-9x^2}\right] + e^{-9x^2} \cdot [-36x] \\ &= \left[324x^3 - 54x\right] e^{-9x^2} = 54x\left[6x^2 - 1\right] e^{-9x^2} \end{aligned}$$

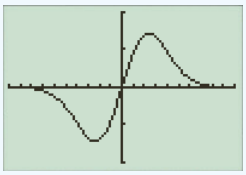
We can check that $f''\left(-\frac{1}{3\sqrt{2}}\right) = 12\sqrt{\frac{2}{e}} > 0$, so f must have a local minimum at $x = -\frac{1}{3\sqrt{2}}$; similarly, $f''\left(\frac{1}{3\sqrt{2}}\right) = -12\sqrt{\frac{2}{e}} < 0$, so f must have a local maximum at $x = \frac{1}{3\sqrt{2}}$.

Furthermore, $f''(x) = 0$ only when $x = 0$ or when $6x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{6}}$, so these three values are candidates for locations of inflection points of f. Noting that:

$$-1 < -\frac{1}{\sqrt{6}} < -\frac{1}{3\sqrt{2}} < 0 < \frac{1}{3\sqrt{2}} < \frac{1}{\sqrt{6}} < 1$$

and that $f''(-1) = -270e^{-9} < 0$ and $f''\left(-\frac{1}{3\sqrt{2}}\right) = 12\sqrt{\frac{2}{e}} > 0$, we observe that f is concave down to the left of $x = -\frac{1}{\sqrt{6}}$ and concave up to the right of $x = -\frac{1}{\sqrt{6}}$, so f does in fact have an inflection point at $x = -\frac{1}{\sqrt{6}}$. Likewise, $f''\left(\frac{1}{3\sqrt{2}}\right) = -12\sqrt{\frac{2}{e}} < 0$ and $f''(1) = 270e^{-9} > 0$, so $f''(x)$ switches sign at $x = 0$ and at $x = \frac{1}{\sqrt{6}}$, and therefore $f(x)$ changes concavity at those points as well.

We have now identified two local extrema of f and three inflection points of f. Equally important, we have used calculus to show that these five points of interest are the *only* places where extrema or inflection points can occur. If we create a graph of f that includes these five points, our graph is guaranteed to include all “interesting” features of the graph of f. A window with $-1 \le x \le 1$ and $-0.2 < y < 0.2$ (because the local extreme values are $f\left(\pm\frac{1}{2\sqrt{3}}\right) \approx \pm 0.14$) should provide a graph:



that includes all five points of interest.

? Practice 3.4.5

Compute the first and second derivatives of the function $g(x) = x^4 + 4x^3 - 90x^2 + 13$, locate all extrema and inflection points of $g(x)$, and create a graph of $g(x)$ that shows these points of interest.

Answer

$g(x) = x^4 + 4x^3 - 90x^2 + 13 \Rightarrow g'(x) = 4x^3 + 12x^2 - 180x \Rightarrow g''(x) = 12x^2 + 24x - 180$; because $g'(x)$ and $g''(x)$ are polynomials, they exist everywhere. The critical numbers for $g(x)$ occur where $g'(x) = 0 \Rightarrow 4x^3 + 12x^2 - 180x = 4x(x^2 + 3x - 45) = 4x(x + 9)(x - 5) = 0 \Rightarrow x = -9$, $x = 0$ or $x = 5$. Using the Second Derivative Test: $g''(-9) = 576 > 0$, so $g(x)$ has a local minimum at $x = -9$; $g''(0) = -180 < 0$, so $g(x)$ has a local maximum at $x = 0$; and $g''(5) = 240 > 0$, so $g(x)$ has a local minimum at $x = 5$.

Candidates for inflection points occur where $g''(x) = 0$:

$$12x^2 + 24x - 180 = 12(x^2 + 2x - 15) = 12(x - 3)(x + 5) = 0$$
$$\Rightarrow x = -5 \text{ or } x = 3$$

Observing that $g''(x) > 0$ for $x < -5$, $g''(x) < 0$ for $-5 < x < 3$ and $g''(x) > 0$ for $x > 3$ confirms that both candidates are in fact inflection points. A graphing window with $-12 \le x \le 8$ (this is only one reasonable possibility) should include all points of interest. Checking that $g(-9) = -3632$, $g(0) = 13$ and $g(5) = -1112$ suggests that a graphing window with $-4000 \le y \le 1000$ should work:

FIGURE

Even with calculus, we will typically need calculators or computers to help solve the equations $f'(x) = 0$ and $f''(x) = 0$ that we use to find critical numbers and candidates for inflection points.

Note

Most problems in calculus textbooks are set up to make solving these equations relatively straightforward, but in general this will not be the case.

Problems

In Problems 1–2, each statement describes a quantity $f(t)$ changing over time. For each statement, tell what f represents and whether the first and second derivatives of f are positive or negative.

1. a. “Unemployment rose again, but the rate of increase is smaller than last month.”
   b. “Our profits declined again, but at a slower rate than last month.”
   c. “The population is still rising and at a faster rate than last year.”
2. a. “The child's temperature is still rising, but more slowly than it was a few hours ago.”

b. “The number of whales is decreasing, but at a slower rate than last year.”

c. “The number of people with the flu is rising and at a faster rate than last month.”

3. Sketch the graphs of functions that are defined and concave up everywhere and have exactly:

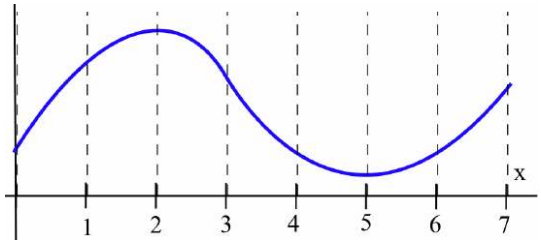
   a. no roots.

   b. 1 root.

   c. 2 roots.

   d. 3 roots.

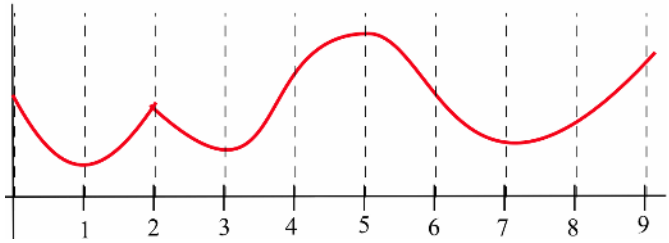
4. On which intervals is the function graphed below:



   a. concave up?

   b. concave down?

5. On which intervals is the function graphed below:

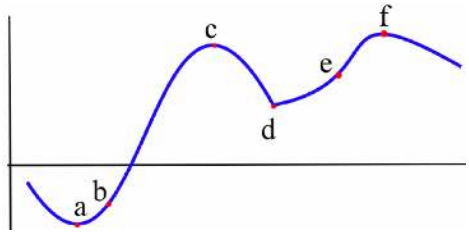


   a. concave up?

   b. concave down?

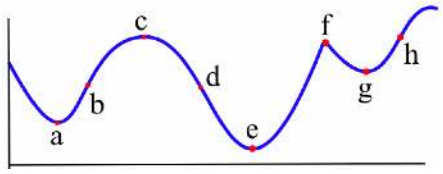
Problems 6–10 give a function and values of x so that $f'(x) = 0$. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither.

6. $f(x) = 2x^3 - 15x^2 + 6$; $x = 0, 5$
7. $g(x) = x^3 - 3x^2 - 9x + 7$; $x = -1, 3$
8. $h(x) = x^4 - 8x^2 - 2$; $x = -2, 0, 2$
9. $f(x) = \sin^5(x)$; $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
10. $f(x) = x \cdot \ln(x)$; $x = \frac{1}{e}$
11. At which values of x labeled in the figure below:



is the point $(x, f(x))$ an inflection point?

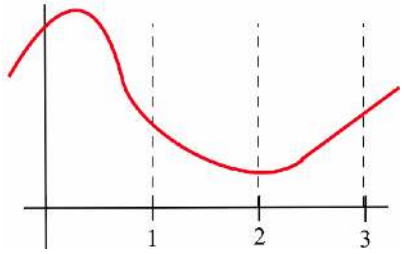
12. At which values of x labeled in the figure below:



is the point $(x, g(x))$ an inflection point?

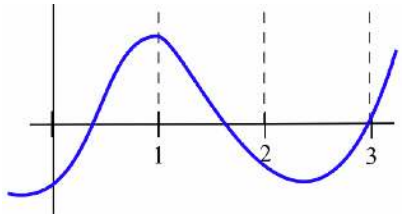
13. How many inflection points can a:
    a. quadratic polynomial have?
    b. cubic polynomial have?
    c. polynomial of degree n have?

14. Fill in the table with "+," "−,"or "0" for the function graphed below:



x	$f(x)$	$f'(x)$	$f''(x)$
0			
1			
2			
3			

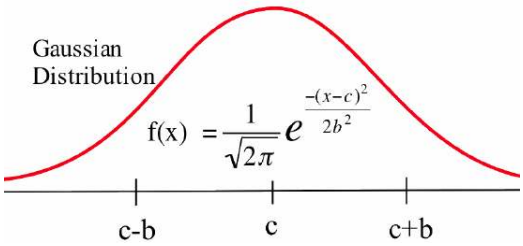
15. Fill in the table with "+," "−,"or "0" for the function graphed below:



x	$g(x)$	$g'(x)$	$g''(x)$
0			
1			
2			
3			

16. Sketch functions f for x-values near 1 so that $f(1) = 2$ and:
    a. $f'(1) > 0$, $f''(1) > 0$
    b. $f'(1) > 0$, $f''(1) < 0$
    c. $f'(1) < 0$, $f''(1) > 0$
    d. $f'(1) > 0$, $f''(1) = 0$, $f''(1^-) < 0$, $f''(1^+) > 0$
    e. $f'(1) > 0$, $f''(1) = 0$, $f''(1^-) > 0$, $f''(1^+) < 0$

17. Some people like to think of a concave-up graph as one that will "hold water" and of a concave-down graph as one which will "spill water." That description is accurate for a concave-down graph, but it can fail for a concave-up graph. Sketch the graph of a function that is concave up on an interval but will not "hold water."
18. The function $f(x) = \frac{1}{2\pi} e^{-\frac{(x-c)^2}{2b^2}}$ defines the **Gaussian distribution** used extensively in statistics and probability; its graph:



is a "bell-shaped" curve.

   a. Show that f has a maximum at $x = c$. (The value c is called the **mean** of this distribution.)
   b. Show that f has inflection points where $x = c + b$ and $x = c - b$. (The value b is called the **standard deviation** of this distribution.)

In Problems 19–36, locate all critical numbers, local extrema and inflection points of the given function, and use these results to sketch a graph of the function showing all points of interest.

19. $f(x) = x^3 - 21x^2 + 144x - 350$
20. $g(x) = \frac{1}{6}x^3 + x^2 - \frac{45}{2}x + 100$
21. $f(x) = e^{7x} - 5x$
22. $g(x) = e^{7x} - 5x$
23. $f(x) = e^{-3x} + x$
24. $g(x) = e^{-3x} - x$
25. $f(x) = xe^{-3x}$
26. $g(x) = xe^{5x}$
27. $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$
28. $g(x) = 6x^{\frac{4}{3}} + 3x^{\frac{1}{3}}$
29. $f(x) = \ln\left(1 + x^2\right)$
30. $g(x) = \ln\left(x^2 - 6x + 10\right)$
31. $f(x) = \sqrt[3]{x^2 + 2x + 2}$
32. $g(x) = \sqrt{x^2 + 2x + 2}$
33. $f(x) = x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}$
34. $g(x) = x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}$
35. $f(\theta) = \sin(\theta) + \sin^2(\theta)$
36. $g(\theta) = \cos(\theta) - \sin^2(\theta)$

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3.5: Applied Maximum and Minimum Problems

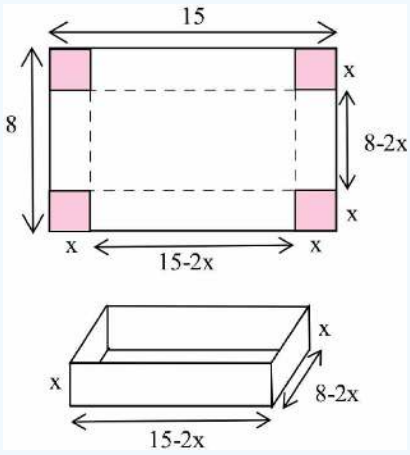
We have used derivatives to find maximums and minimums of functions given by formulas, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme value(s). Typically, someone will describe a problem and ask your help to maximize or minimize a quantity: “What is the largest volume of a package that the post office will accept?”; “What is the quickest way to get from here to there?”; or “What is the least expensive way to accomplish some task?” These problems often involve restrictions—or **constraints**—and sometimes neither the problem nor the constraints are clearly stated.

Before we can use calculus or other mathematical techniques to solve these “**max/min**” problems, we need to understand the situation at hand and translate the problem into mathematical form. After solving the problem using calculus (or other mathematical techniques) we need to check that our mathematical solution really solves the original problem. Often, the most challenging part of this procedure is understanding the problem and translating it into mathematical form.

In this section we examine some problems that require understanding, translation, solution and checking. Most will not be as complicated as those a working scientist, engineer or economist needs to solve, but they represent a step toward developing the required skills.

Example 3.5.1

The company you own has a large supply of 8-inch by 15-inch rectangular pieces of tin, and you decide to use them to make boxes by cutting a square from each corner and folding up the sides:



For example, if you cut a 1-inch square from each corner, the resulting 6-inch by 13-inch by 1-inch box has a volume of 78 cubic inches. The amount of money you can charge for a box depends on how much the box holds, so you want to make boxes with the largest possible volume. What size square should you cut from each corner?

Solution

To help understand the problem, first drawing a diagram can be very helpful. Then we need to translate it into a mathematical problem:

- identify the variables
- label the variable and constant parts of the diagram
- write the quantity to be maximized as a function of the variables

If we label the side of the square to be removed as x inches, then the box is x inches high, $8-2x$ inches wide and $15-2x$ inches long, so the volume is:

$$\begin{aligned}(\text{length})(\text{width})(\text{height}) &= (15-2x)(8-2x)\cdot x \\ &= 4x^3-46x^2+120x \text{ cubic inches}\end{aligned}$$

Now we have a mathematical problem, to maximize the function $V(x)=4x^3-46x^2+120x$, so we use existing calculus techniques, computing $V'(x)=12x^2-92x+120$ to find the critical points.

The maximum volume must occur at the critical point $x = \frac{5}{3}$ or at one of the endpoints ($x = 0$ and $x = 4$): $V(0) = 0$, $V(\frac{5}{3}) = \frac{2450}{27} \approx 90.74$ cubic inches, and $V(4) = 0$, so the maximum volume of the box occurs when we remove a $\frac{5}{3}$-inch by $\frac{5}{3}$-inch square from each corner, resulting in a box $\frac{5}{3}$ inches high, $8 - 2(\frac{5}{3}) = \frac{14}{3}$ inches wide and $15 - 2(\frac{5}{3}) = \frac{35}{3}$ inches long.

- Set $V'(x) = 0$ and solve by factoring or using the quadratic formula:

$$V'(x) = 12x^2 - 92x + 120 = 4(3x - 5)(x - 6) = 0 \Rightarrow x = \frac{5}{3} \text{ or } x = 6$$

  so $x = \frac{5}{3}$ and $x = 6$ are critical points of V.
- $V'(x)$ is a polynomial so it is defined everywhere and there are no critical points resulting from an undefined derivative.
- What are the endpoints for x in this problem? A square cannot have a negative length, so $x \geq 0$. We cannot remove more than half of the width, so $8 - 2x \geq 0 \Rightarrow x \leq 4$. Together, these two inequalities say that $0 \leq x \leq 4$, so the endpoints are $x = 0$ and $x = 4$. (Note that the value $x = 6$ is not in this interval, so $x = 6$ cannot maximize the volume and we do not consider it further.)

Practice 3.5.1

If you start with 7-inch by 15-inch pieces of tin, what size square should you remove from each corner so the box will have as large a volume as possible? [Hint: $12x^2 - 88x + 105 = (2x - 3)(6x - 35)$]

Answer

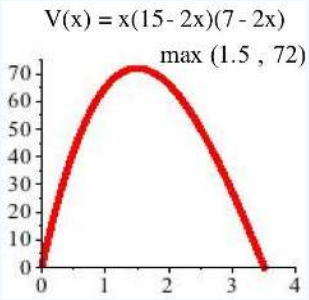
$V(x) = x(15 - 2x)(7 - 2x) = 4x^3 - 44x^2 + 105x$ so:

$$V'(x) = 12x^2 - 88x + 105 = (2x - 3)(6x - 35)$$

which is defined for all x: the only critical numbers are the endpoints $x = 0$ and $x = \frac{7}{2}$ and the places where $V'(x) = 0$: $x = \frac{3}{2}$ and $x = \frac{35}{6}$ (but $\frac{35}{6}$ is not in the interval $[0, \frac{7}{2}]$ so it is not practical). The maximum volume must occur when $x = 0$, $x = \frac{3}{2}$ or $x = \frac{7}{2}$:

$$V(0) = 0 \cdot (15 - 2 \cdot 0) \cdot (7 - 2 \cdot 0) = 0$$
$$V\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \left(15 - 2 \cdot \frac{3}{2}\right) \cdot \left(7 - 2 \cdot \frac{3}{2}\right) = \frac{3}{2}(12)(4) = 72$$
$$V\left(\frac{7}{2}\right) = \frac{7}{2}\left(15 - 2 \cdot \frac{7}{2}\right) \cdot \left(7 - 2 \cdot \frac{7}{2}\right) = \frac{7}{2}(8)(0) = 0$$

The maximum-volume box will result from cutting a 1.5-by-1.5 inch square from each corner. A graph of $V(x)$ appears below:



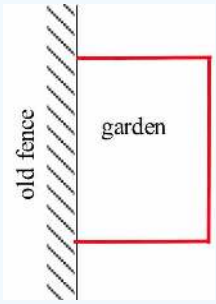
We were fortunate in the previous Example and Practice problem because the functions we created to describe the volume were functions of only one variable. In other situations, the function we get will have more than one variable, and we will need to use additional information to rewrite our function as a function of a single variable. Typically, the constraints will contain the additional information we need.

Example 3.5.2

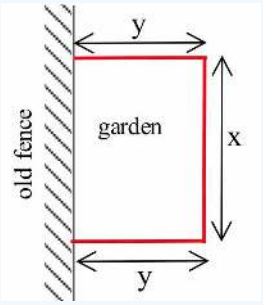
We want to fence a rectangular area in our backyard for a garden. One side of the garden is along the edge of the yard, which is already fenced, so we only need to build a new fence along the other three sides of the rectangle. If a neighbor gives us 80 feet of fencing left over from a home-improvement project, what dimensions should the garden have in order to enclose the largest possible area using all of the available material?

Solution

As a first step toward understanding the problem, we draw a diagram or picture of the situation:



Next, we identify the variables: in this case, the length (call it x) and width (call it y) of the garden. The figure below shows a labeled diagram:



which we can use to write a formula for the function that we want to maximize:

$$A = \text{area of the rectangle} = (\text{length})(\text{width}) = x \cdot y$$

Unfortunately, our function A involves two variables, x and y, so we need to find a relationship between them (an equation containing both x and y) that we can solve for either x or y. The constraint says that we have 80 feet of fencing available, so $x + 2y = 80 \Rightarrow y = 40 - \frac{x}{2}$. Then:

$$A = x \cdot y = x\left(40 - \frac{x}{2}\right) = 40x - \frac{x^2}{2}$$

which is a function of a single variable (x). We want to maximize A.

$A'(x) = 40 - x$ so the only way $A'(x) = 0$ is to have $x = 40$, and $A(x)$ is differentiable for all x so the only critical number (other than the endpoints) is $x = 40$. Finally, $0 \le x \le 80$ (why?) so we also need to check $x = 0$ and $x = 80$: the maximum area must occur at $x = 0$, $x = 40$ or $x = 80$.

$$A(0) = 40(0) - \frac{0^2}{2} = 0 \text{ square feet}$$
$$A(40) = 40(40) - \frac{40^2}{2} = 800 \text{ square feet}$$
$$A(80) = 40(80) - \frac{80^2}{2} = 0 \text{ square feet}$$

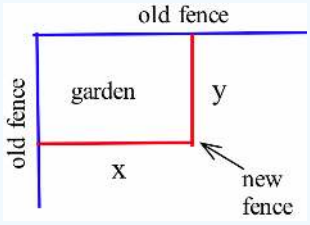
so the largest rectangular garden has an area of 800 square feet, with dimensions $x = 40$ feet by $y = 40 - \frac{40}{2} = 20$ feet

Practice 3.5.2

Suppose you decide to create the rectangular garden in a **corner** of your yard. Then two sides of the garden are bounded by the existing fence, so you only need to use the available 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions if you have F feet of new fencing available?

Answer

a. We have 80 feet of fencing:



Our assignment is to maximize the area of the garden: $A = x \cdot y$ (two variables). Fortunately, we have the constraint that $x + y = 80$, so $y = 80 - x$ and our assignment reduces to maximizing a function of one variable:

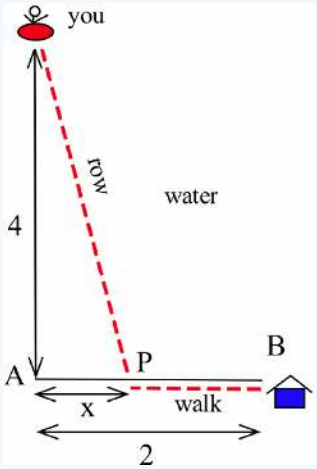
$$A = x \cdot y = x \cdot (80 - x) = 80x - x^2 \Rightarrow A'(x) = 80 - 2x$$

so $A'(x) = 0 \Rightarrow x = 40$. Because $A''(x) = -2 < 0$, the graph of A is concave down, hence A has a maximum at $x = 40$. The maximum area is $A(40) = 40 \cdot 40 = 1600$ ft^2 when $x = 40$ feet and $y = 40$ feet. The maximum-area garden is a square.

b. This is similar to part (a) except we have F feet of fencing instead of 80 feet: $x + y = F \Rightarrow y = F - x$ and we want to maximize $A = xy = x(F - x) = Fx - x^2$. Differentiating, $A'(x) = F - 2x$ so $A'(x) = 0 \Rightarrow x = \frac{F}{2} \Rightarrow y = \frac{F}{2}$. The maximum area is $A\left(\frac{F}{2}\right) = \frac{F^2}{4}$ square feet when the garden is a square with half of the new fence used on each of the two new sides.

Example 3.5.3

You need to reach home as quickly as possible, but you are in a rowboat on a lake 4 miles from shore and your home is 2 miles up the shore:



If you can row at 3 miles per hour and walk at 5 miles per hour, toward which point on the shore should you row? What if your home is 7 miles up the coast?

Solution

The figure above shows a labeled diagram with the variable x representing the distance along the shore from point A, the nearest point on the shore to your boat, to point P, the point you row toward. The total time—rowing and walking—is:

$$
\begin{aligned}
T &= \text{total time} \\
&= (\text{rowing time from boat to } P) + (\text{walking time from } P \text{ to } B) \\
&= \frac{\text{distance from boat to } P}{\text{rate rowing boat}} + \frac{\text{distance from } P \text{ to } B}{\text{rate walking along shore}} \\
&= \frac{\sqrt{x^2+4^2}}{3} + \frac{2-x}{5} = \frac{\sqrt{x^2+16}}{3} + \frac{2-x}{5}
\end{aligned}
$$

It is not reasonable to row to a point below A and then walk home, so $x \geq 0$. Similarly, we can conclude that $x \leq 2$, so our interval is $0 \leq x \leq 2$ and the endpoints are $x = 0$ and $x = 2$.

To find the other critical numbers of T between $x = 0$ and $x = 2$, we need the derivative of T:

$$
T'(x) = \frac{1}{3} \cdot \frac{1}{2}\left(x^2+16\right)^{-\frac{1}{2}}(2x) - \frac{1}{5} = \frac{x}{3\sqrt{x^2+16}} - \frac{1}{5}
$$

This derivative is defined for all values of x (and in particular for all values in the interval $0 \leq x \leq 2$). To find where $T'(x) = 0$ we solve:

$$
\begin{aligned}
\frac{x}{3\sqrt{x^2+16}} - \frac{1}{5} = 0 \;&\Rightarrow\; 5x = 3\sqrt{x^2+16} \\
&\Rightarrow\; 25x^2 = 9x^2 + 144 \\
&\Rightarrow 16x^2 = 144 \;\Rightarrow x^2 = 9 \;\Rightarrow x = \pm 3
\end{aligned}
$$

Neither of these numbers, however, is in our interval $0 \leq x \leq 2$, so neither of them gives a minimum time. The only critical numbers for T on this interval are the endpoints, $x = 0$ and $x = 2$:

$$
\begin{aligned}
T(0) &= \frac{\sqrt{0+16}}{3} + \frac{2-0}{5} = \frac{4}{3} + \frac{2}{5} \approx 1.73 \text{ hours} \\
T(2) &= \frac{\sqrt{2^2+16}}{3} + \frac{2-2}{5} = \frac{\sqrt{20}}{3} \approx 1.49 \text{ hours}
\end{aligned}
$$

The quickest route has P 2 miles down the coast: you should row directly toward home.

If your home is 7 miles down the coast, then the interval for x is $0 \leq x \leq 7$, which has endpoints $x = 0$ and $x = 7$. Our function for the travel time is now:

$$
T(x) = \frac{\sqrt{x^2+16}}{3} + \frac{7-x}{5} \;\Rightarrow\; T'(x) = \frac{x}{3\sqrt{x^2+16}} - \frac{1}{5}
$$

so the only point in our interval where $T'(x) = 0$ is at $x = 3$ and the derivative is defined for all values in this interval. So the only critical numbers for T are $x = 0$, $x = 3$ and $x = 7$:

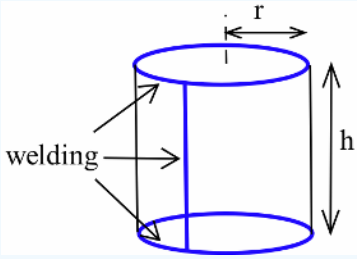
$$
\begin{aligned}
T(0) &= \frac{\sqrt{0+16}}{3} + \frac{7-0}{5} = \frac{4}{3} + \frac{7}{5} && \approx 2.73 \text{ hours} \\
T(3) &= \frac{\sqrt{3^2+16}}{3} + \frac{7-3}{5} = \frac{\sqrt{65}}{3} + \frac{4}{5} && \approx 2.47 \text{ hours} \\
T(7) &= \frac{\sqrt{7^2+16}}{3} + \frac{7-7}{5} = \frac{5}{3} && \approx 2.68 \text{ hours}
\end{aligned}
$$

The quickest way home is to aim for a point P that is 3 miles down the shore, row directly to P, and then walk along the shore to home.

One challenge of max/min problems is that they may require geometry, trigonometry or other mathematical facts and relationships.

Example 3.5.4

Find the height and radius of the least expensive closed cylinder that has a volume of 1,000 cubic inches. Assume that the materials needed to construct the cylinder are free, but that it costs 80¢ per inch to weld the top and bottom onto the cylinder and to weld the seam up the side of the cylinder:



Solution

If we let r be the radius of the cylinder and h be its height, then the volume is $V = \pi r^2 h = 1000$. The quantity we want to minimize is cost, and

$$\begin{aligned} C &= (\text{top seam cost}) + (\text{bottom seam cost}) + (\text{side seam cost}) \\ &= (\text{total seam length}) \left(80 \frac{\text{¢}}{\text{inch}}\right) \\ &= (2\pi r + 2\pi r + h)\,(80) = 320\pi r + 80h \end{aligned}$$

Unfortunately, C is a function of two variables, r and h, but we can use the information in the constraint ($V = \pi r^2 h = 1000$) to solve for h and then substitute this expression for h into the formula for C:

$$1000 = \pi r^2 h \;\Rightarrow\; h = \frac{1000}{\pi r^2} \;\Rightarrow C = 320\pi r + 80h = 320\pi r + 80\left(\frac{1000}{\pi r^2}\right)$$

which is a function of a single variable. Differentiating:

$$C'(r) = 320\pi - \frac{160000}{\pi r^3}$$

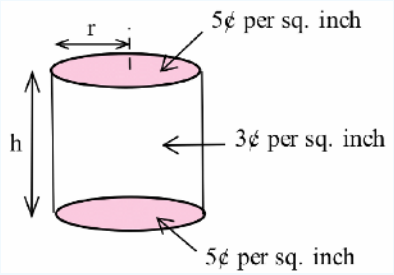
which is defined except when $r = 0$ (a value that does not make sense in the original problem) and there are no restrictions on r (other than $r > 0$) so there are no endpoints to check. Thus C will be at a minimum when $C'(r) = 0$:

$$320\pi - \frac{160000}{\pi r^3} = 0 \;\Rightarrow\; r^3 = \frac{500}{\pi^2} \;\Rightarrow\; r = \sqrt[3]{\frac{500}{\pi^2}}$$

so $r \approx 3.7$ inches and $h = \dfrac{1000}{\pi r^2} = \dfrac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi^2}}\right)^2} \approx 23.3$ inches.

Practice 3.5.3

Find the height and radius of the least expensive closed cylinder that has a volume of 1,000 cubic inches, assuming that the only cost for this cylinder is the price of the materials: the material for the top and bottom costs 5¢~per square inch, while the material for the sides costs 3¢~per square inch:



Answer

The cost C is given by:

$$\begin{aligned} C &= 5(\text{area of top}) + 3(\text{area of sides}) + 5(\text{area of bottom}) \\ &= 5(\pi r^2) + 3(2\pi rh) + 5(\pi r^2) \end{aligned}$$

so our assignment is to minimize $C = 10\pi r^2 + 6\pi rh$, a function of two variables (r and h). Fortunately, we also have the constraint that volume $= 1000$ in$^3 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$. So:

$$C = 10\pi r^2 + 6\pi r\left(\frac{1000}{\pi r^2}\right) = 10\pi r^2 + \frac{6000}{r} \Rightarrow C'(r) = 20\pi r - \frac{6000}{r^2}$$

which exists for $r \neq 0$ ($r = 0$ is not in the domain of $C(r)$).

$$C'(r) = 0 \Rightarrow 20\pi r - \frac{6000}{r^2} = 0 \Rightarrow 20\pi r^3 = 6000 \Rightarrow r = \sqrt[3]{\frac{6000}{20\pi}} \approx 4.57 \text{ inches}$$

When $r = 4.57$, $h = \frac{1000}{\pi(4.57)^2} \approx 15.24$ inches. Examining the second derivative, $C''(r) = 20\pi + \frac{12000}{r^3} > 0$ for all $r > 0$ so C is concave up and we have found the minimum cost.

Example 3.5.5

Find the dimensions of the least expensive rectangular box that is three times as long as it is wide and which holds 100 cubic centimeters of water. The material for the bottom costs 7¢~per cm^2, the sides cost 5¢~per cm^2 and the top costs 2¢~per cm^2.

Solution

Label the box so that $w =$ width, $l =$ length and $h =$ height. Then our cost function C is:

$$\begin{aligned} C &= (\text{bottom cost}) + (\text{cost of front and back}) + (\text{cost of ends}) + (\text{top cost}) \\ &= (\text{bottom area})(7¢) + (\text{front and back area})(5¢) + (\text{ends area})(5¢) + (\text{top area})(2¢) \\ &= (wl)(7) + (2lh)(5) + (2wh)(5) + (wl)(2) \\ &= 7wl + 10lh + 10wh + 2wl \\ &= 9wl + 10lh + 10wh \end{aligned}$$

Unfortunately, C is a function of three variables (w, l and h) but we can use the information from the constraints to eliminate some of the variables: the box is "three times as long as it is wide" so $l = 3w$ and

$$C = 9wl + 10lh + 10wh = 9w(3w) + 10(3w)h + 10wh = 27w^2 + 40wh$$

We also know the volume V is 100 in^3 and $V = lwh = 3w^2 h$ (because $l = 3w$), so $h = \dfrac{100}{3w^2}$. Then:

$$C = 27w^2 + 40wh = 27w^2 + 40w\left(\frac{100}{3w^2}\right) = 27w^2 + \frac{4000}{3w}$$

which is a function of a single variable. Differentiating:

$$C'(w) = 54w - \frac{4000}{3w^2}$$

which is defined everywhere except $w = 0$ (yielding a box of volume 0) and there is no constraint interval, so C is minimized when $C'(w) = 0 \Rightarrow w = \sqrt[3]{\frac{4000}{162}} \approx 2.91$ inches $\Rightarrow l = 3w \approx 8.73$ inches $\Rightarrow h = \frac{100}{3w^2} \approx 3.94$ inches. The minimum cost is approximately \$6.87.

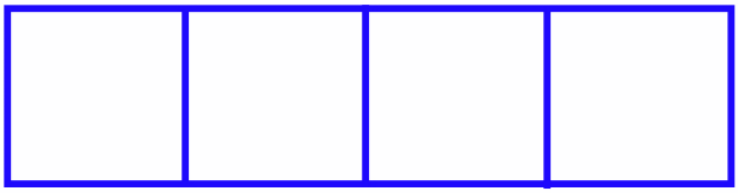
Problems described in words are usually more difficult to solve because we first need to understand and "translate" a real-life problem into a mathematical problem. Unfortunately, those skills only seem to come with practice. With practice, however, you will start to recognize patterns for understanding, translating and solving these problems, and you will develop the skills you need. So read carefully, draw pictures, think hard—and do the best you can.

Problems

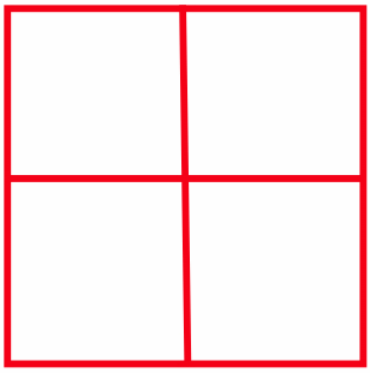
1. a. You have 200 feet of fencing to enclose a rectangular vegetable garden. What should the dimensions of your garden be in order to enclose the largest area?
   b. Show that if you have P feet of fencing available, the garden of greatest area is a square.
   c. What are the dimensions of the largest rectangular garden you can enclose with P feet of fencing if one edge of the garden borders a straight river and does not need to be fenced?
   d. Just thinking—calculus will not help: What do you think is the shape of the largest garden that can be enclosed with P feet of fencing if we do not require the garden to be rectangular? What if one edge of the garden borders a (straight) river?
2. a. You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle:

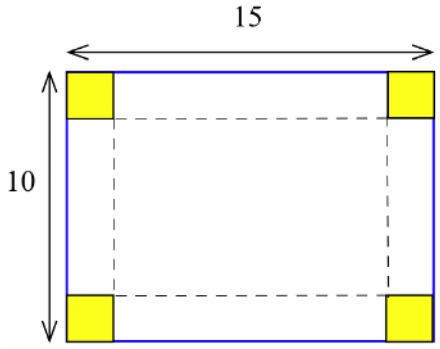
   What dimensions of the pen enclose the largest total area?
   b. If you need two dividers, what dimensions of the pen enclose the largest area?
   c. What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?
3. You have 120 feet of fencing to construct a pen with four equal-sized stalls.
   a. If the pen is rectangular and shaped like the one shown below, what are the dimensions of the pen of largest area and what is that area?

   b. The square pen below uses 120 feet of fencing but encloses a larger area (400 ft^2) than the best design in part (a):

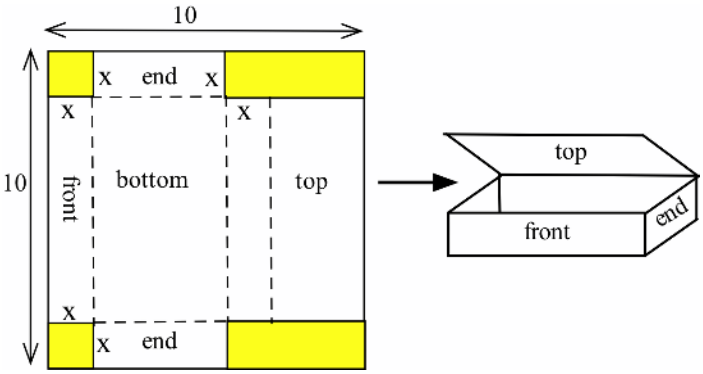
   

   Design a pen that uses only 120 feet of fencing and has four equal-sized stalls but encloses more than 400 ft^2. (Hint: Don't use rectangles and squares.)
4. a. You need to form a 10-inch by 15-inch piece of tin into a box (with no top) by cutting a square from each corner and folding up the sides. How much should you cut so the resulting box has the greatest volume?



b. If the piece of tin is A inches by B inches, how much should you cut from each corner so the resulting box has the greatest volume?

5. Find the dimensions of a box with largest volume formed from a 10-inch by 10-inch piece of cardboard cut and folded as shown below:



6. a. You must construct a square-bottomed box with no top that will hold 100 cubic inches of water. If the bottom and sides are made from the same material, what are the dimensions of the box that uses the least material? (Assume that no material is wasted.)

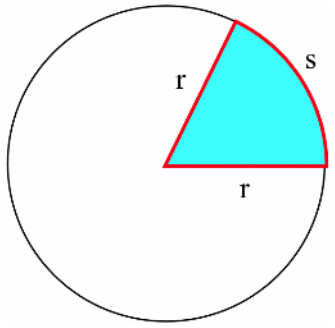
   b. Suppose the box in part (a) uses different materials for the bottom and the sides. If the bottom material costs 5¢~per square inch and the side material costs 3¢~per square inch, what are the dimensions of the least expensive box that will hold 100 cubic inches of water?

   (This is a "classic" problem with many variations. We could require that the box be twice as long as it is wide, or that the box have a top, or that the ends cost a different amount than the front and back, or even that it costs a certain amount to weld each edge. You should be able to set up the cost equations for these variations.)

7. a. Determine the dimensions of the least expensive cylindrical can that will hold 100 cubic inches if the materials cost 2¢, 5¢ and 3¢ per square inch, respectively, for the top, bottom and sides.

   b. How do the dimensions of the least expensive can change if the bottom material costs more than 5¢~per square inch?

8. You have 100 feet of fencing to build a pen in the shape of a circular sector, the "pie slice" shown below:

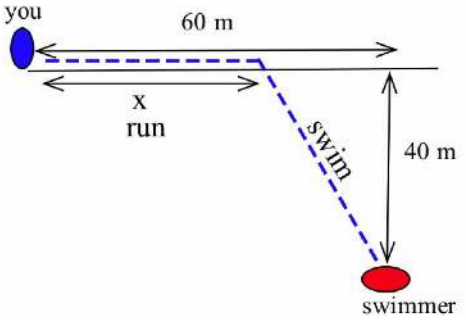


The area of such a sector is $\frac{rs}{2}$.

   a. What value of r maximizes the enclosed area?

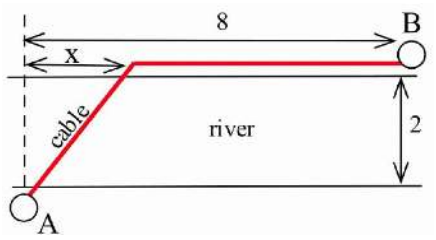
b. What central angle maximizes the area?

9. You are a lifeguard standing at the edge of the water when you notice a swimmer in trouble:



40 m out in the water from a point 60 m down the beach. Assuming you can run at a speed of 8 meters per second and swim at a rate of 2 meters per second, how far along the shore should you run before diving into the water in order to reach the swimmer as quickly as possible?

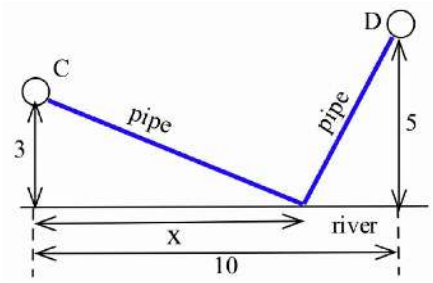
10. You have been asked to determine the least expensive route for a telephone cable that connects Andersonville with Beantown, as shown in the figure below:



a. If it costs $5000 per mile to lay the cable on land and $8000 per mile to lay the cable across the river (with the cost of the cable included), find the least expensive route.

b. What is the least expensive route if the cable costs $7000 per mile in addition to the cost to lay it?

11. You have been asked to determine where a water works should be built along a river between Chesterville and Denton, as shown below:

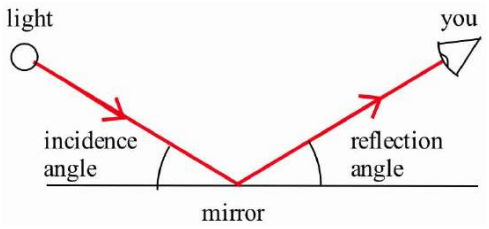


to minimize the total cost of the pipe to the towns.

a. Assume that the same size (and cost) pipe is used to each town. (This part can be done quickly without using calculus.)

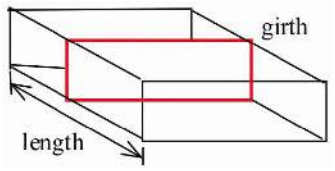
b. Assume instead that the pipe to Chesterville costs $3000 per mile and to Denton it costs $7000 per mile.

12. Light from a bulb at A is reflected off a flat mirror to your eye at point B, as shown below:



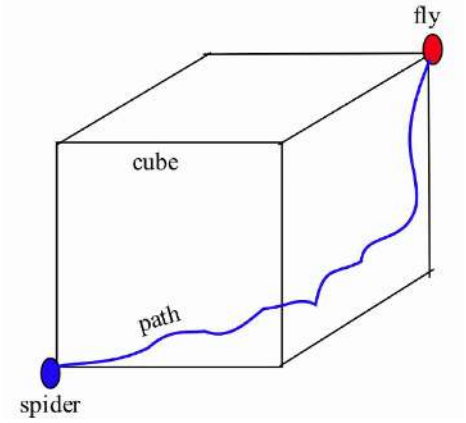
If the time (and length of the path) from A to the mirror and then to your eye is a minimum, show that the angle of incidence equals the angle of reflection. (Hint: This is similar to the previous problem.)

13. U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches, as shown below:



    a. Find the dimensions of the acceptable box with a square end that has the largest volume.
    b. Find the dimensions of the acceptable box that has the largest volume if its end is a rectangle twice as long as it is wide.
    c. Find the dimensions of the acceptable box with a circular end that has the largest volume.

14. Just thinking—you don't need calculus for this problem: A spider and a fly are located on opposite corners of a cube:



    What is the shortest path along the surface of the cube from the spider to the fly?

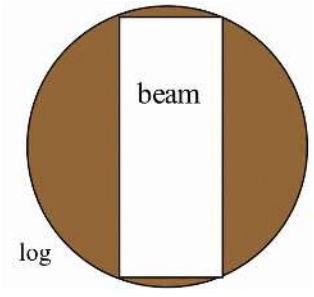
15. Two sides of a triangle are 7 and 10 inches long. What is the length of the third side so the area of the triangle will be greatest? (This problem can be done without using calculus. How? If you do use calculus, consider the angle θ between the two sides.)
16. Find the shortest distance from the point $(2, 0)$ to the curve:

    a. $y = 3x - 1$
    b. $y = x^2$
    c. $x^2 + y^2 = 1$
    d. $y = \sin(x)$

17. Find the dimensions of the rectangle with the largest area if the base must be on the x-axis and its other two corners are on the graph of:

    a. $y = 16 - x^2$, $-4 \le x \le 4$
    b. $x^2 + y^2 = 1$
    c. $|x| + |y| = 1$
    d. $y = \cos(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

18. The strength of a wooden beam is proportional to the product of its width and the square of its height, as shown below:

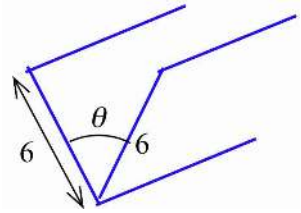


    What are the dimensions of the strongest beam that can be cut from a log with diameter:

    a. 12 inches?

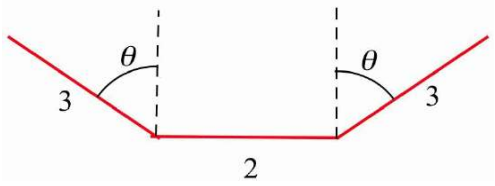
b. d inches?

19. You have a long piece of 12-inch-wide metal that you plan to fold along the center line to form a V-shaped gutter, as shown below:



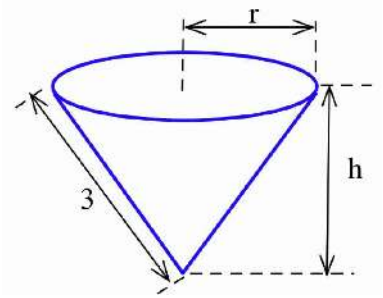
What angle θ will yield a gutter that holds the most water (that is, has the largest cross-sectional area)?

20. You have a long piece of 8-inch-wide metal that you plan to make into a gutter by bending up 3 inches on each side:



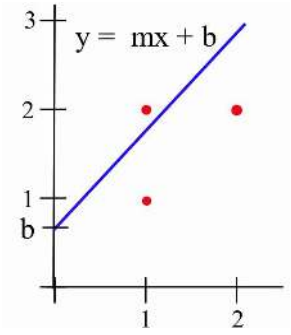
What angle θ will yield a gutter that holds the most water?

21. You have a 6-inch-diameter paper disk that you want to form into a drinking cup by removing a pie-shaped wedge (sector) and then forming the remaining paper into a cone:



Find the height and top radius of the cone so that the volume of the cup is as large as possible.

22. a. What value of b minimizes the sum of the squares of the vertical distances from $y = 2x + b$ to the points $(1, 1)$, $(1, 2)$ and $(2, 2)$?

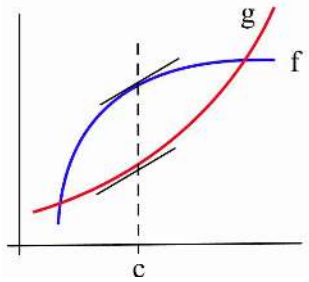


b. What slope m minimizes the sum of the squares of the vertical distances from the line $y = mx$ to the points $(1, 1)$, $(1, 2)$ and $(2, 2)$?

c. What slope m minimizes the sum of the squares of the vertical distances from the line $y = mx$ to the points $(2, 1)$, $(4, 3)$, $(-2, -2)$ and $(-4, -2)$?

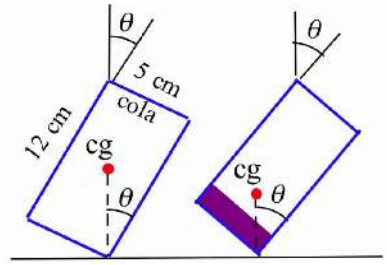
23. You own a small airplane that holds a maximum of 20 passengers. It costs you $100 per flight from St. Thomas to St. Croix for gas and wages plus an additional $6 per passenger for the extra gas required by the extra weight. The charge per passenger is $30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by $1 per passenger for each passenger over 10 who travels (that is, if 11 fly they each pay $29, if 12 fly they each pay $28, etc.). What number of passengers on a flight will maximize your profit?

24. Prove: If f and g are differentiable functions and if the vertical distance between f and g is greatest at $x = c$, then $f'(c) = g'(c)$ and the tangent lines to f and g are parallel when $x = c$.

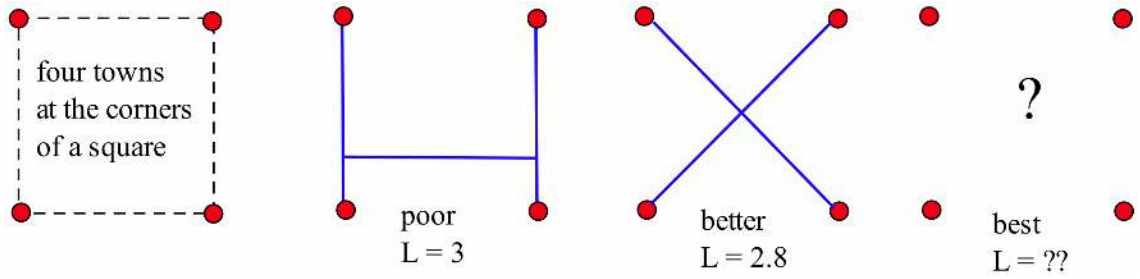


25. Profit = revenue − expenses. Assuming revenue and expenses are differentiable functions, show that when profit is maximized, then marginal revenue $\left(\frac{dR}{dx}\right)$ equals marginal expense $\left(\frac{dE}{dx}\right)$.
26. Dean Simonton claims the "productivity levels" of people in various fields can be described as a function of their "career age" t by $p(t) = e^{-at} - e^{-bt}$ where a and b are constants depending on the field, and career age is approximately 20 less than the actual age of the individual.
    a. Based on this model, at what ages do mathematicians ($a = 0.03$, $b = 0.05$), geologists ($a = 0.02$, $b = 0.04$) and historians ($a = 0.02$, $b = 0.03$) reach their maximum productivity?
    b. Simonton says, "With a little calculus we can show that the curve ($p(t)$) maximizes at $t = \frac{1}{b-a}\ln\left(\frac{b}{a}\right)$." Use calculus to show that Simonton is correct.

    Note: Models of this type have uses for describing the behavior of groups, but it is dangerous—and usually invalid—to apply group descriptions or comparisons to individuals in a group. (*Scientific Genius* by Dean Simonton, Cambridge University Press, 1988, pp. 69–73)
27. After the table was wiped and the potato chips dried off, the question remained: "Just how far could a can of cola be tipped before it fell over?"
    a. For a full can or an empty can the answer was easy: the center of gravity (CG) of the can is at the middle of the can, half as high as the height of the can, and we can tilt the can until the CG is directly above the bottom rim (see figure below left). Find θ if the height of the can is 12 cm and the diameter is 5 cm.



    b. For a partly filled can, more thinking was needed. Some ideas you will see in Chapter 5 tell us that the CG of a can holding x cm of cola is $C(x) = \frac{360 + 9.6x^2}{60 + 19.2x}$ cm above the bottom of the can. Find the height x of cola that will make the CG as low as possible.
    c. Assuming that the cola is frozen solid (so the top of the cola stays parallel to the bottom of the can), how far can we tilt a can containing x cm of cola? (See figure above right.)
    d. If the can contained x cm of liquid cola, could we tilt it farther or less far than the frozen cola before it would fall over?
28. Just thinking—calculus will not help with this one.
    a. Four towns are located at the corners of a square. What is the shortest length of road we can construct so that it is possible to travel along the road from any town to any other town?



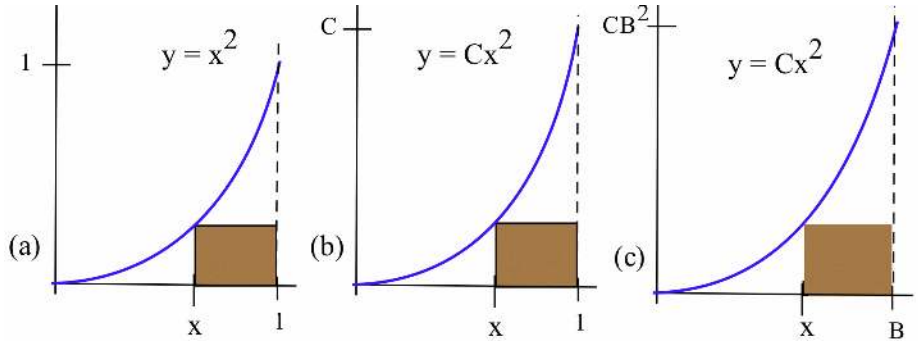
b. What is the shortest connecting path for five towns located on the corners of a pentagon?

c. (The problem of finding the shortest path connecting several points in the plane is called the "Steiner problem." It is important for designing computer chips and telephone networks to be as efficient as possible.)

Generalized Max/Min Problems

The previous max/min problems were mostly numerical problems: the amount of fencing in Problem 2 was 200 feet, the lengths of the piece of tin in Problem 4 were 10 and 15, and the parabola in Problem 17(a) was $y = 16 - x^2$. In working those problems, you might have noticed some patterns among the numbers in the problem and the numbers in your answers, and you might have wondered if the pattern was a coincidence or if there really was a general pattern at work. Rather than trying several numerical examples to see if the "pattern" holds, mathematicians, engineers, scientists and others sometimes resort to generalizing the problem. We free the problem from the particular numbers by replacing the numbers with letters, and then we solve the generalized problem. In this way, relationships between the values in the problem and those in the solution can become more obvious. Solutions to these generalized problems are also useful if you want to program a computer to quickly provide numerical answers.

29. a. Find the dimensions of the rectangle with the greatest area that can be built so the base of the rectangle is on the x-axis between 0 and 1 ($0 \le x \le 1$) and one corner of the rectangle is on the curve $y = x^2$:



What is the area of this rectangle?

b. Generalize the problem in part (a) for the parabola $y = Cx^2$ with $C > 0$ and $0 \le x \le 1$.

c. Generalize for the parabola $y = Cx^2$ with $C > 0$ and $0 \le x \le B$.

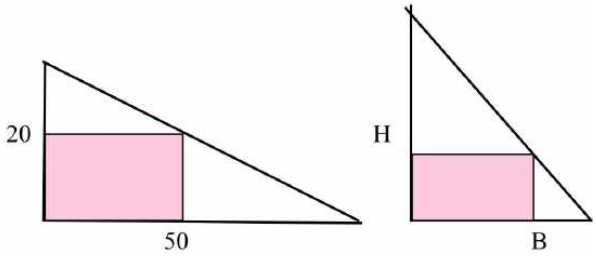
30. a. Find the dimensions of the rectangle with the greatest area that can be built so the base of the rectangle is on the x-axis between 0 and 1 and one corner of the rectangle is on the curve $y = x^3$. What is the area of this rectangle?

b. Generalize the problem in part (a) for the curve $y = Cx^3$ with $C > 0$ and $0 \le x \le 1$.

c. Generalize for the curve $y = Cx^3$ with $C > 0$ and $0 \le x \le B$.

d. Generalize for the curve $y = Cx^n$ with $C > 0$, n a positive integer, and $0 \le x \le B$.

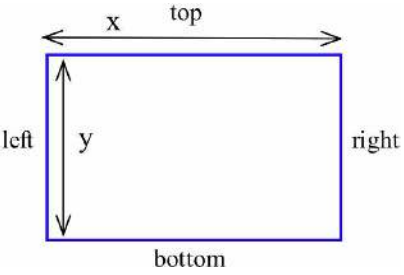
31. a. The base of a right triangle is 50 and the height is 20. Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.



b. The base of a right triangle is B and the height is H. Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.

c. State your general conclusion from part (b) in words.

32. a. You have T dollars to buy fencing material to enclose a rectangular plot of land. The fence for the top and bottom costs \$5 per foot and for the sides it costs \$3 per foot. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom, and for the sides?

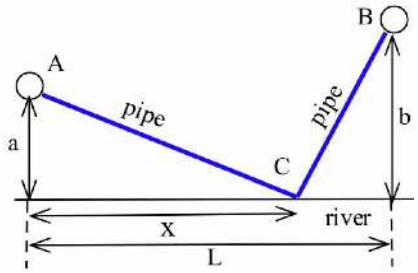


b. You have T dollars to buy fencing material to enclose a rectangular plot of land. The fence for the top and bottom costs \$A per foot and for the sides it costs \$B per foot. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom (together), and for the sides (together)?

c. You have T dollars to buy fencing material to enclose a rectangular plot of land. The fence costs \$A per foot for the top, \$B/foot for the bottom, \$C/ft for the left side and \$D/ft for the right side. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom (together), and for the sides (together)?

33. Determine the dimensions of the least expensive cylindrical can that will hold V cubic inches if the top material costs \$A per square inch, the bottom material costs \$B per square inch, and the side material costs \$C per square inch.

34. Find the location of C in the figure below so that the sum of the distances from A to C and from C to B is a minimum.



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3.6: Asymptotic Behavior of Functions

When you turn on an automobile or a light bulb or a computer, many things happen. Some of them are uniquely part of the start-up process of the system. These “transient” things occur only during start up, and then the system settles down to its steady-state operation. This start-up behavior can be very important, but sometimes we want to investigate the steady-state—or long-term—behavior: how does the system behave “after a long time?” In this section we investigate and describe the long-term behavior of functions and the systems they model: how does a function behave “when x (or $-x$) is arbitrarily large?”

Limits as x Becomes Arbitrarily Large (“Approaches Infinity”)

The same type of questions we considered about a function f as x approached a finite number can also be asked about f as x “becomes arbitrarily large” (or “increases without bound”)—that is, eventually becomes larger than any fixed number.

Example 3.6.1

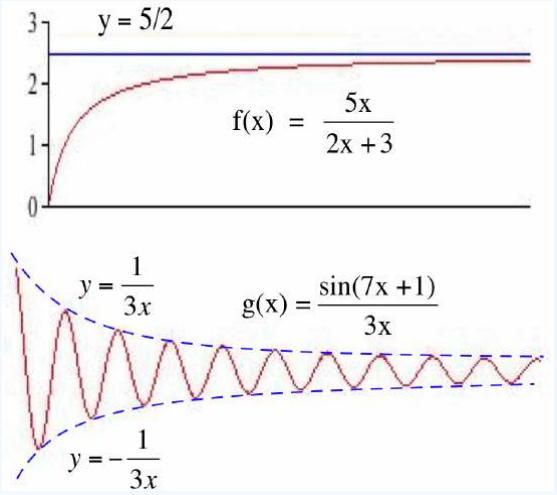
What happens to the values of $f(x) = \dfrac{5x}{2x+3}$ and $g(x) = \dfrac{\sin(7x+1)}{3x}$ as x becomes arbitrarily large (increases without bound)?

Solution

One approach is numerical: evaluate $f(x)$ and $g(x)$ for some “large” values of x and see if there is a pattern to the values of $f(x)$ and $g(x)$. The table below:

x	$\frac{5x}{2x+3}$	$\frac{\sin(7x+1)}{3x}$
10	2.17	0.031702
100	2.463	−0.001374
1000	2.4962	0.000333
10,000	2.4996	0.000001

values of $g(x)$ are close to 0. In fact, we can guarantee that the values of $f(x)$ are as close to $\frac{5}{2}$ as someone wants by taking x to be “big enough.” The values of $f(x) = \dfrac{5x}{2x+3}$ may or may not ever equal $\frac{5}{2}$ (they never do), but if x is “large,” then $f(x)$ is “very close to” $\frac{5}{2}$. Similarly, we can guarantee that the values of $g(x)$ are as close to 0 as someone wants by taking x to be “big enough.” The graphs of f and g for “large” values of x appear below:



Practice 3.6.1

What happens to the values of $f(x) = \dfrac{3x+4}{x-2}$ and $g(x) = \dfrac{\cos(5x)}{2x+7}$ as x becomes arbitrarily large?

Answer

As x becomes arbitrarily large, the values of $f(x)$ approach 3 and the values of $g(x)$ approach 0.

We can express the answers to Example 1 using limits. "As x becomes arbitrarily large, the values of $\dfrac{5x}{2x+3}$ approach $\frac{5}{2}$" can be written:

$$\lim_{x\to\infty} \frac{5x}{2x+3} = \frac{5}{2}$$

and "the values of $\dfrac{\sin(7x+1)}{3x}$ approach 0" can be written:

$$\lim_{x\to\infty} \frac{\sin(7x+1)}{3x} = 0$$

We read $\lim\limits_{x\to\infty}$ as "the limit as x approaches infinity," meaning "the limit as x becomes arbitrarily large" or "as x increases without bound."

Note

During this discussion—and throughout this book—we do not treat "infinity" or "∞" as a number, but only as a useful notation. "Infinity" is not part of the real number system, and we use the common notation "$x \to \infty$" and the phrase "x approaches infinity" only to mean that "x becomes arbitrarily large."

The notation "$x \to -\infty$," read as "x approaches negative infinity," means that the values of $-x$ become arbitrarily large.

Practice 3.6.2

Rewrite your answers to Practice 3.6.1 using limit notation.

Answer

$\lim\limits_{x\to\infty} \dfrac{3x+4}{x-2} = 3$ and $\lim\limits_{x\to\infty} \dfrac{\cos(5x)}{2x+7} = 0$

The expression $\lim\limits_{x\to\infty} f(x)$ asks about the behavior of $f(x)$ as the values of x get larger and larger without any bound. One way to determine this behavior is to look at the values of $f(x)$ for some values of x that are very "large." If the values of the function get arbitrarily close to a single number as x gets larger and larger, then we will say that number is the limit of the function as x approaches infinity. (A more formal definition of the limit as "$x \to \infty$" appears at the end of this section.)

Practice 3.6.3

Fill in the table for $f(x) = \dfrac{6x+7}{3-2x}$ and $g(x) = \dfrac{\sin(3x)}{x}$ and use those values to estimate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$.

x	$\frac{6x+7}{3-2x}$	$\frac{\sin(3x)}{x}$
10		
200		
500		
20,000		

Answer

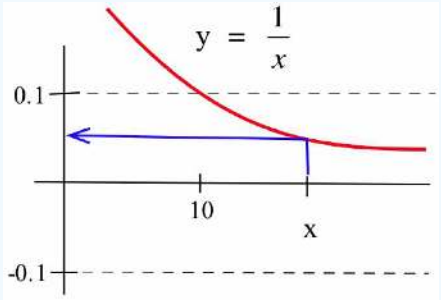
x	$\frac{6x+7}{3-2x}$	$\frac{\sin(3x)}{x}$
10	-3.94117647	-0.09880311
200	-3.04030227	0.00220912
500	-3.00160048	0.00017869
$20,000$	-3.00040003	0.00004787
	$\downarrow$	$\downarrow$
	-3	0

Example 3.6.2

How large must x be to guarantee that $f(x) = \frac{1}{x} < 0.1$? That $f(x) < 0.001$? That $f(x) < E$ (with $E > 0$)?

Solution

If $x > 10$, then $\frac{1}{x} < \frac{1}{10} = 0.1$. If $x > 1000$, then $\frac{1}{x} < \frac{1}{1000} = 0.001$. In general, if E is any positive number, then we can guarantee that $|f(x)| < E$ by picking only values of $x > \frac{1}{E} > 0$: if $x > \frac{1}{E}$, then $\frac{1}{x} < E$. From this we can conclude that $\lim_{x\to\infty} \frac{1}{x} = 0$.



Practice 3.6.4

How large must x be to guarantee that $f(x) = \frac{1}{x^2} < 0.1$? That $f(x) < 0.001$? That $f(x) < E$ (with $E > 0$)? Evaluate $\lim_{x\to\infty} \frac{1}{x^2}$.

Answer

If $x > \sqrt{10} \approx 3.162$ then $f(x) = \frac{1}{x^2} < 0.1$.

If $x > \sqrt{1000} \approx 31.62$ then $f(x) = \frac{1}{x^2} < 0.001$.

If $x > \sqrt{\frac{1}{E}} = \frac{1}{\sqrt{E}}$ then $f(x) = \frac{1}{x^2} < E$.

The Main Limit Theorem (Section 1.2) about limits of combinations of functions still holds true if the limits as "$x \to a$" are replaced with limits as "$x \to \infty$" but we will not prove those results.

Polynomials arise regularly in applications, and we often need the limit, as "$x \to \infty$," of ratios of polynomials or functions containing powers of x. In these situations the following technique is often helpful:

- factor the highest power of x in the denominator from both the numerator and the denominator
- cancel the common factor from the numerator and denominator

The limit of the new denominator is a constant, so the limit of the resulting ratio is easier to determine.

Example 3.6.3

Determine $\lim\limits_{x\to\infty} \dfrac{7x^2+3x-4}{3x^2-5}$ and $\lim\limits_{x\to\infty} \dfrac{9x+2}{3x^2-5x+1}$.

Solution

Factoring x^2 out of the numerator and the denominator of the first rational function results in:

$$\lim_{x\to\infty} \frac{7x^2+3x-4}{3x^2-5} = \lim_{x\to\infty} \frac{x^2(7+\frac{3}{x}-\frac{4}{x^2})}{x^2(3-\frac{5}{x^2})} = \lim_{x\to\infty} \frac{7+\frac{3}{x}-\frac{4}{x^2}}{3-\frac{5}{x^2}} = \frac{7}{3}$$

where we used the facts that $\frac{3}{x}\to 0$, $\frac{4}{x^2}\to 0$ and $\frac{5}{x^2}\to 0$ as $x\to\infty$. Similarly:

$$\lim_{x\to\infty} \frac{9x+2}{3x^2-5x+1} = \lim_{x\to\infty} \frac{x^2(\frac{9}{x}+\frac{2}{x^2})}{x^2(3-\frac{5}{x}+\frac{1}{x^2})} = \lim_{x\to\infty} \frac{\frac{9}{x}+\frac{2}{x^2}}{3-\frac{5}{x}+\frac{1}{x^2}} = \frac{0}{3} = 0$$

because $\frac{k}{x}\to 0$ and $\frac{c}{x^2}\to 0$ as $x\to\infty$ for any constants k and c.

If we need to evaluate a more difficult limit as $x\to\infty$, it is often useful to algebraically manipulate the function into the form of a ratio and then use the previous technique.

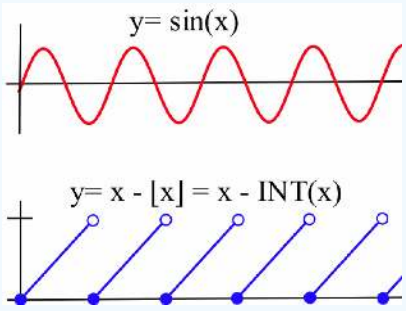
If the values of the function oscillate and do not approach a single number as x becomes arbitrarily large, then the function does not have a limit as x approaches ∞: the limit **does not exist**.

Example 3.6.4

Evaluate $\lim\limits_{x\to\infty} \sin(x)$ and $\lim\limits_{x\to\infty} x-\lfloor x\rfloor$.

Solution

As $x\to\infty$, $f(x)=\sin(x)$ and $g(x)=x-\lfloor x\rfloor$ do not have limits. As x grows without bound, the values of $f(x)=\sin(x)$ oscillate between -1 and $+1$:



and these values do not approach a single number. Similarly, $g(x)=x-\lfloor x\rfloor$ continues to take on all values between 0 and 1, and these values never approach a single number.

Using Calculators to Help Find Limits as "$x\to\infty$" or "$x\to-\infty$"

Calculators only store a limited number of digits for each quantity. This becomes a severe limitation when we deal with extremely large quantities.

Example 3.6.5

The value of $f(x)=(x+1)-x$ is clearly equal to 1 for all values of x, and your calculator will give the right answer if you use it to evaluate $f(4)$ or $f(5)$. Now use it to evaluate f for a big value of x, say $x=10^{40}$: $f(10^{40})=(10^{40}+1)-10^{40}=1$, but most calculators do not store 40 digits of a number, and they will respond that $f(10^{40})=0$, which is **wrong**. In this example the calculator's error is obvious, but similar errors can occur in less obvious ways when using calculators for computations involving very large numbers.

Caution

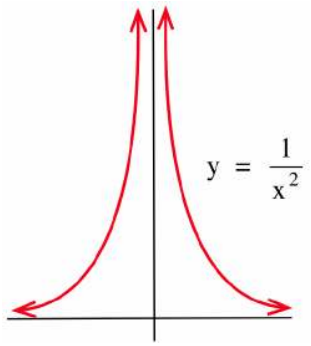
You should be careful with—and somewhat suspicious of—the answers your calculator gives you.

Calculators can still be helpful for examining limits as $x \to \infty$ and $x \to -\infty$ as long as we don't place too much faith in their responses.

Even if you have forgotten some of the properties of the natural logarithm function $\ln(x)$ and the cube root function $\sqrt[3]{x}$, a little experimentation on your calculator can help convince you that $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{\sqrt[3]{x}} = 0$.

Limit Is Infinite

The function $f(x) = \dfrac{1}{x^2}$ is undefined at $x = 0$, but we can still ask about the behavior of $f(x)$ for values of x "close to" 0. The figure below:



indicates that if x is very small (close to 0) then $f(x)$ is very large. As the values of x get closer to 0, the values of $f(x)$ grow larger and can be made as large as we want by picking x to be close enough to 0. Even though the values of f are not approaching any one number, we use the "infinity" notation to indicate that the values of f are growing without bound, and write: $\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$.

The values of $\dfrac{1}{x^2}$ do not *equal* "infinity": the notation $\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$ means that the values of $\dfrac{1}{x^2}$ can be made arbitrarily large by picking values of x very close to 0.

The limit, as $x \to 0$, of $\dfrac{1}{x}$ is slightly more complicated. If x is close to 0, then the value of $f(x) = \dfrac{1}{x}$ can be a large positive number or a large negative number, depending on the sign of x. The function $f(x) = \dfrac{1}{x}$ does not have a (two-sided) limit as x approaches 0, but we can still investigate one-sided limits:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Example 3.6.6

Determine $\lim\limits_{x\to 3^+} \dfrac{x-5}{x-3}$ and $\lim\limits_{x\to 3^-} \dfrac{x-5}{x-3}$.

Solution

As $x \to 3^+$, $x - 5 \to -2$ and $x - 3 \to 0$. Because the denominator is approaching 0, we cannot use the Main Limit Theorem, and we need to examine the function more carefully. When $x \to 3^+$, we know that $x > 3$ so $x - 3 > 0$. So if x is close to 3 and slightly larger than 3, then the ratio of $x - 5$ to $x - 3$ is:

$$\frac{\text{a number close to } -2}{\text{small positive number}} = \text{large negative number}$$

As $x > 3$ gets closer to 3:

$$\frac{x-5}{x-3} = \frac{\text{a number closer to } -2}{\text{positive and closer to } 0} = \text{larger negative number}$$

By taking $x > 3$ even closer to 3, the denominator gets closer to 0 but remains positive, so the ratio gets arbitrarily large and negative: $\lim\limits_{x\to 3^+} \dfrac{x-5}{x-3} = -\infty$.

As $x \to 3^-$, $x-5 \to -2$ and $x-3 \to 0$ as before, but now we know that $x < 3$ so $x-3 < 0$. So if x is close to 3 and slightly smaller than 3, then the ratio of $x-5$ to $x-3$ is:

$$\frac{\text{a number close to } -2}{\text{small negative number}} = \text{large positive number}$$

so $\lim\limits_{x\to 3^-} \dfrac{x-5}{x-3} = \infty$.

Practice 3.6.5

Find:

a. $\lim\limits_{x\to 2^+} \dfrac{7}{2-x}$
b. $\lim\limits_{x\to 2^+} \dfrac{3x}{2x-4}$
c. $\lim\limits_{x\to 2^+} \dfrac{3x^2-6x}{x-2}$.

Answer

a. As $x \to 2^+$, $2-x \to 0$, and $x > 2$ so $2-x < 0$: $2-x$ takes on small negative values.

$$\frac{7}{2-x} = \frac{7}{\text{small negative number}} = \text{large negative number}$$

so we represent the limit as: $\lim\limits_{x\to 2^+} \dfrac{7}{2-x} = -\infty$.

b. As $x \to 2^+$, $2x-4 \to 0$, and $x > 2$ so $2x-4 > 0$: $2x-4$ takes on small positive values. And as $x \to 2^+$, $3x \to 6$ so:

$$\frac{3x}{2x-4} = \frac{\text{number near } 6}{\text{small positive number}} = \text{large positive number}$$

so we represent the limit as: $\lim\limits_{x\to 2^+} \dfrac{3x}{2x-4} = +\infty$.

c. As $x \to 2^+$, $3x^2-6x \to 0$ and $x-2 \to 0$ so we need to do more work. Factoring the numerator as $3x^2-6x = 3x(x-2)$:

$$\lim_{x\to 2^+} \frac{3x^2-6x}{x-2} = \lim_{x\to 2^+} \frac{3x(x-2)}{x-2} = \lim_{x\to 2^+} 3x = 6$$

where we were able to cancel the $x-2$ factor because the limit involves values of x close to (but not equal to) 2.

Horizontal Asymptotes

The limits of f, as "$x \to \infty$" and "$x \to -\infty$," provide information about horizontal asymptotes of f.

Definition

The line $y = K$ is a **horizontal asymptote** of f if:

$$\lim_{x\to\infty} f(x) = K \quad \text{or} \quad \lim_{x\to-\infty} f(x) = K$$

Example 3.6.7

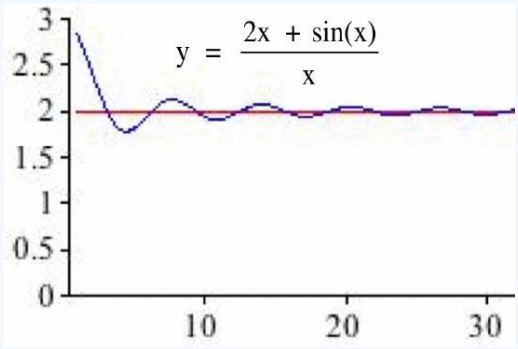
Find any horizontal asymptotes of $f(x) = \dfrac{2x+\sin(x)}{x}$.

Solution

Computing the limit as $x \to \infty$:

$$\lim_{x\to\infty}\frac{2x+\sin(x)}{x}=\lim_{x\to\infty}\left[\frac{2x}{x}+\frac{\sin(x)}{x}\right]=\lim_{x\to\infty}\left[2+\frac{\sin(x)}{x}\right]$$
$$=2+\lim_{x\to\infty}\frac{\sin(x)}{x}=2+0=2$$

so the line $y=2$ is a horizontal asymptote of f. The limit, as "$x\to-\infty$," is also 2 so $y=2$ is the *only* horizontal asymptote of f. The graphs of f and $y=2$ appear below:



A function may or may not cross its asymptote.

You likely explored horizontal asymptotes in a previous course using terms like "end behavior" and investigating only rational functions. The tools of calculus allow us to make the the notion of "end behavior" more precise and investigate a wider variety of functions.

Vertical Asymptotes

As with horizontal asymptotes, you have likely studied vertical asymptotes before (at least for rational functions). We can now define vertical asymptotes using infinite limits.

Definition

The vertical line $x=a$ is a **vertical asymptote** of the graph of f if either or both of the one-sided limits of f, as $x\to a^-$ or $x\to a^+$, is infinite.

If our function f is the ratio of a polynomial $P(x)$ and a polynomial $Q(x)$, $f(x)=\frac{P(x)}{Q(x)}$, then the only **candidates** for vertical asymptotes are the values of x where $Q(x)=0$. However, the fact that $Q(a)=0$ is **not** enough to guarantee that the line $x=a$ is a vertical asymptote of f; we also need to evaluate $P(a)$.

If $Q(a)=0$ and $P(a)\neq 0$, then the line $x=a$ must be a vertical asymptote of f. If $Q(a)=0$ and $P(a)=0$, then the line $x=a$ may or may not be a vertical asymptote.

Example 3.6.8

Find the vertical asymptotes of $f(x)=\frac{x^2-x-6}{x^2-x}$ and $g(x)=\frac{x^2-3x}{x^2-x}$.

Solution

Factoring the numerator and denominator of $f(x)$ yields $f(x)=\frac{(x-3)(x+2)}{x(x-1)}$ so the only values of x that make the denominator 0 are $x=0$ and $x=1$, and these are the only candidates to be vertical asymptotes. Because $\lim_{x\to0^+}f(x)=+\infty$ and $\lim_{x\to1^+}f(x)=-\infty$, both $x=0$ and $x=1$ are vertical asymptotes of f.

Factoring the numerator and denominator of $g(x)$ yields $\frac{x(x-3)}{x(x-1)}$ so the only candidate to be vertical asymptotes are $x=0$ and $x=1$. Because $\lim_{x\to1^+}g(x)=\lim_{x\to1^+}\frac{x(x-3)}{x(x-1)}=\lim_{x\to1^+}\frac{x-3}{x-1}=-\infty$ the line $x=1$ must be a vertical asymptote of g. But $\lim_{x\to0}g(x)$

$= \lim_{x \to 0} \dfrac{x-3}{x-1} = 3 \neq \pm\infty$ so $x = 0$ is **not** a vertical asymptote of g.

? Practice 3.6.6

Find the vertical asymptotes of $f(x) = \dfrac{x^2+x}{x^2+x-2}$ and $g(x) = \dfrac{x^2-1}{x-1}$.

Answer

a. $f(x) = \dfrac{x^2+x}{x^2+x-2} = \dfrac{x(x+1)}{(x-1)(x+2)}$ so f has vertical asymptotes at $x = 1$ and $x = -2$.

b. $g(x) = \dfrac{x^2-1}{x-1} = \dfrac{(x+1)(x-1)}{x-1}$ so the value $x = 1$ is not in the domain of g. If $x \neq 1$, then $g(x) = x + 1$: g has a "hole" when $x = 1$ and no vertical asymptotes.

Other Asymptotes as "$x \to \infty$" and "$x \to -\infty$"

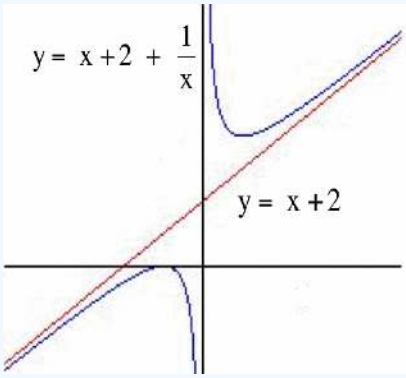
If the limit of $f(x)$, as $x \to \infty$ or $x \to -\infty$, is a constant K, then the graph of f gets arbitrarily close to the horizontal line $y = K$, in which case we call $y = K$ a horizontal asymptote of f. Some functions, however, approach lines that are not horizontal.

✓ Example 3.6.9

Find all asymptotes of $f(x) = \dfrac{x^2+2x+1}{x} = x + 2 + \dfrac{1}{x}$.

Solution

If x is a large positive (or negative) number, then $\dfrac{1}{x}$ is very close to 0, and the graph of $f(x)$ is very close to the line $y = x + 2$:



The line $y = x + 2$ is an asymptote of the graph of f.

If x is a large positive number, then $\dfrac{1}{x}$ is positive, and the graph of f is slightly above the graph of $y = x + 2$. If x is a large negative number, then $\dfrac{1}{x}$ is negative, and the graph of f will be slightly below the graph of $y = x + 2$. The $\dfrac{1}{x}$ piece of f never equals 0, so the graph of f never crosses or touches the graph of the asymptote $y = x + 2$.

The graph of f also has a vertical asymptote at $x = 0$ because $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$.

? Practice 3.6.7

Find all asymptotes of $g(x) = \dfrac{2x^2-x-1}{x+1} = 2x - 3 + \dfrac{2}{x+1}$.

Answer

$g(x) = 2x - 3 + \dfrac{2}{x+1}$ has a vertical asymptote at $x = -1$ and no horizontal asymptotes, but $\lim_{x \to \infty} \dfrac{2}{x+1} = 0$ so g has the linear asymptote $y = 2x - 3$.

Some functions even have **nonlinear asymptotes**: asymptotes that are not straight lines. The graphs of these functions approach some nonlinear function when the values of x become arbitrarily large.

Example 3.6.10

Find all asymptotes of $f(x) = \dfrac{x^4+3x^3+x^2+4x+5}{x^2+1} = x^2+3x+\dfrac{x+5}{x^2+1}$.

Solution

When x is very large, positive or negative, then $\dfrac{x+5}{x^2+1}$ is very close to 0 and the graph of f is very close to the graph of $g(x) = x^2+3x$. The function $g(x) = x^2+3x$ is a nonlinear asymptote of f. The denominator of f is never 0 and f has no vertical asymptotes.

Practice 3.6.8

Find all asymptotes of $f(x) = \dfrac{x^3+2\sin(x)}{x} = x^2+\dfrac{2\sin(x)}{x}$.

Answer

$f(x) = x^2+\dfrac{2\sin(x)}{x}$ is not defined at $x=0$, so f has a vertical asymptote or a "hole" there; $\lim\limits_{x\to 0} x^2+\dfrac{2\sin(x)}{x} = 0+2 = 2$ so f has a "hole" when $x=0$. Because $\lim\limits_{x\to\infty} \dfrac{2\sin(x)}{x} = 0$, f has the nonlinear asymptote $y = x^2$ (but no horizontal asymptotes).

If we can write $f(x)$ as a sum of two functions, $f(x) = g(x)+r(x)$, with $\lim\limits_{x\to\pm\infty} r(x) = 0$, then the graph of f is asymptotic to the graph of g, and g is an asymptote of f. In this situation:

- if $g(x) = K$, then f has a horizontal asymptote $y = K$
- if $g(x) = ax+b$, then f has a linear asymptote $y = ax+b$
- otherwise f has a nonlinear asymptote $y = g(x)$

Formal Definition of $\lim\limits_{x\to\infty} f(x) = K$

The following definition states precisely what we mean by the phrase "we can guarantee that the values of $f(x)$ are arbitrarily close to K by restricting the values of x to be sufficiently large."

Definition

$\lim\limits_{x\to\infty} f(x) = K$ means that, for every given $\epsilon > 0$, there is a number N so that:

if: x is larger than N

then: $f(x)$ is within ϵ units of K.

Equivalently: $|f(x)-K| < \epsilon$ whenever $x > N$.

Example 3.6.11

Show that $\lim\limits_{x\to\infty} \dfrac{x}{2x+1} = \dfrac{1}{2}$.

Solution

Typically, we need to do two things. First we need to find a value of N, often depending on ϵ. Then we need to show that the value of N we found satisfies the conditions of the definition.

Assume that $|f(x)-K|$ is less than ϵ and solve for $x > 0$:

$$\epsilon > \left|\frac{x}{2x+1}\right| = \left|\frac{2x-(2x+1)}{2(2x+1)}\right| = \left|\frac{-1}{4x+2}\right| = \frac{1}{4x+2}$$
$$\Rightarrow 4x+2 > \frac{1}{\epsilon} \Rightarrow x > \frac{1}{4}\left(\frac{1}{\epsilon}-2\right)$$

So, given any $\epsilon > 0$, take $N = \frac{1}{4}\left(\frac{1}{\epsilon}-2\right)$.

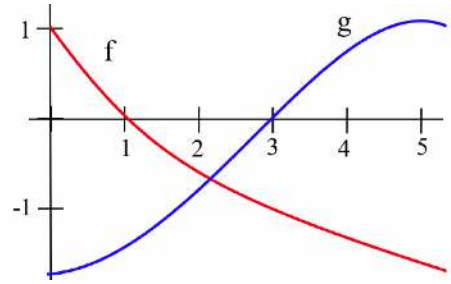
Now we can just reverse the order of the steps above to show that this N satisfies the limit definition. If $x > 0$ and $x > \frac{1}{4}\left(\frac{1}{\epsilon}-2\right)$ then:

$$4x+2 > \frac{1}{\epsilon} \Rightarrow \epsilon > \frac{1}{4x+2} = \left|\frac{x}{2x+1} - \frac{1}{2}\right| = |f(x)-K|$$

We have shown that "for every given ϵ, there is an N" that satisfies the definition.

Problems

1. The figure below shows $f(x)$ and $g(x)$ for $0 \le x \le 5$.



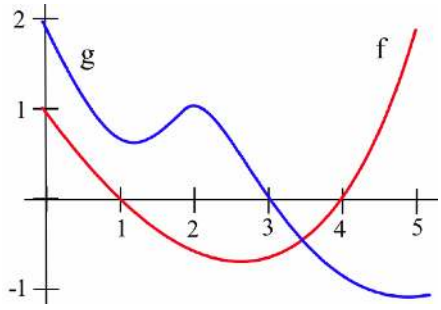
Define a new function $h(x) = \frac{f(x)}{g(x)}$.

a. At what value of x does $h(x)$ have a root?

b. Determine the limits of $h(x)$ as $x \to 1^+$, $x \to 1^-$, $x \to 3^+$ and $x \to 3^-$.

c. Where does $h(x)$ have a vertical asymptote?

2. The figure below shows $f(x)$ and $g(x)$ on the interval $0 \le x \le 5$.

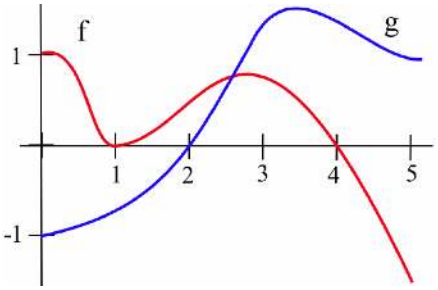


Let $h(x) = \frac{f(x)}{g(x)}$.

a. At what value(s) of x does $h(x)$ have a root?

b. Where does $h(x)$ have vertical asymptotes?

3. The figure below shows $f(x)$ and $g(x)$ for $0 \le x \le 5$.



Let $h(x) = \dfrac{f(x)}{g(x)}$. Determine the limits of $h(x)$ as $x \to 2^+$, $x \to 2^-$, $x \to 4^+$ and $x \to 4^-$.

For Problems 4–24, calculate the limit of each expression as "$x \to \infty$."

4. $\dfrac{6}{x+2}$
5. $\dfrac{28}{3x-5}$
6. $\dfrac{7x+12}{3x-2}$
7. $\dfrac{4-3x}{x+8}$
8. $\dfrac{5\sin(2x)}{2x}$
9. $\dfrac{\cos(3x)}{5x-1}$
10. $\dfrac{2x-3\sin(x)}{5x-1}$
11. $\dfrac{4+x\cdot\sin(x)}{2x-4}$
12. $\dfrac{x^2-5x+2}{x^2+8x-4}$
13. $\dfrac{2x^2-9}{3x^2+10x}$
14. $\dfrac{\sqrt{x+5}}{\sqrt{4x-2}}$
15. $\dfrac{5x^2-7x+2}{2x^3+4x}$
16. $\dfrac{x+\sin(x)}{x-\sin(x)}$
17. $\dfrac{7x^2+x\cdot\sin(x)}{3-x^2+\sin(7x^2)}$
18. $\dfrac{7x^{143}+734x-2}{x^{150}-99x^{83}+25}$
19. $\dfrac{\sqrt{9x^2+16}}{2+\sqrt{x^2+1}}$
20. $\sin\left(\dfrac{3x+5}{2x-1}\right)$
21. $\cos\left(\dfrac{7x+4}{x^2+x+1}\right)$
22. $\ln\left(\dfrac{3x^2+5x}{x^2-4}\right)$
23. $\ln(x+8)-\ln(x-5)$
24. $\ln(3x+8)-\ln(2x+5)$
25. Salt water with a concentration of 0.2 pounds of salt per gallon flows into a large tank that initially contains 50 gallons of pure water.
    a. If the flow rate of salt water into the tank is 4 gallons per minute, what is the volume $V(t)$ of water and the amount $A(t)$ of salt in the tank t minutes after the flow begins?
    b. Show that the salt concentration $C(t)$ at time t is $C(t) = \dfrac{0.8t}{4t+50}$.
    c. What happens to the concentration $C(t)$ after a "long" time?
    d. Redo parts (a)–(c) for a large tank that initially contains 200 gallons of pure water.

26. Under certain laboratory conditions, an agar plate contains $B(t) = 100\,(2 - e^{-t})$ bacteria t hours after the start of the experiment.
    a. How many bacteria are on the plate at the start of the experiment ($t = 0$)?
    b. Show that the population is always increasing. (Show $B'(t) > 0$ for all $t > 0$.)
    c. What happens to the population $B(t)$ after a "long" time?
    d. Redo parts (a)–(c) for $B(t) = A(2 - e^{-t})$.

For Problems 27–41 , calculate the limits.

27. $\lim\limits_{x \to 0} \dfrac{x+5}{x^2}$
28. $\lim\limits_{x \to 3} \dfrac{x-1}{(x-3)^2}$
29. $\lim\limits_{x \to 5} \dfrac{x-7}{(x-5)^2}$
30. $\lim\limits_{x \to 2^+} \dfrac{x-1}{x-2}$
31. $\lim\limits_{x \to 2^-} \dfrac{x-1}{x-2}$
32. $\lim\limits_{x \to 3^+} \dfrac{x-1}{x-2}$
33. $\lim\limits_{x \to 4^+} \dfrac{x+3}{4-x}$
34. $\lim\limits_{x \to 1^-} \dfrac{x^2+5}{1-x}$
35. $\lim\limits_{x \to 3^+} \dfrac{x^2-4}{x^2-2x-3}$
36. $\lim\limits_{x \to 2} \dfrac{x^2-x-2}{x^2-4}$
37. $\lim\limits_{x \to 0} \dfrac{x-2}{1-\cos(x)}$
38. $\lim\limits_{x \to \infty} \dfrac{x^3+7x-4}{x^2+11x}$
39. $\lim\limits_{x \to 5} \dfrac{\sin(x-5)}{(x-5)}$
40. $\lim\limits_{x \to 0} \dfrac{x+1}{\sin^2(x)}$
41. $\lim\limits_{x \to 0^+} \dfrac{1+\cos(x)}{1-e^x}$

In Problems 42–59, write an **equation** of each asymptote for each function and state whether it is a vertical, horizontal or slant asymptote.

42. $f(x) = \dfrac{x+2}{x-1}$
43. $f(x) = \dfrac{x-3}{x^2}$
44. $f(x) = \dfrac{x-1}{x^2-x}$
45. $f(x) = \dfrac{x+5}{x^2-4x+3}$
46. $f(x) = \dfrac{x+\sin(x)}{3x-3}$
47. $f(x) = \dfrac{x^2-4}{x+1}$
48. $f(x) = \dfrac{\cos(x)}{x^2}$
49. $f(x) = 2 + \dfrac{3-x}{x-1}$
50. $f(x) = \dfrac{x \cdot \sin(x)}{x^2-x}$
51. $f(x) = \dfrac{2x^2+x+5}{x}$
52. $f(x) = \dfrac{x^2+x}{x+1}$
53. $f(x) = \dfrac{1}{x-2} + \sin(x)$
54. $f(x) = x + \dfrac{x}{x^2+1}$
55. $f(x) = x^2 + \dfrac{x}{x^2+1}$

56. $f(x) = x^2 + \dfrac{x}{x+1}$
57. $f(x) = \dfrac{x \cdot \cos(x)}{x-3}$
58. $f(x) = \dfrac{x^3 - x^2 + 2x - 1}{x-1}$
59. $f(x) = \sqrt{\dfrac{x^2+3x+2}{x+3}}$

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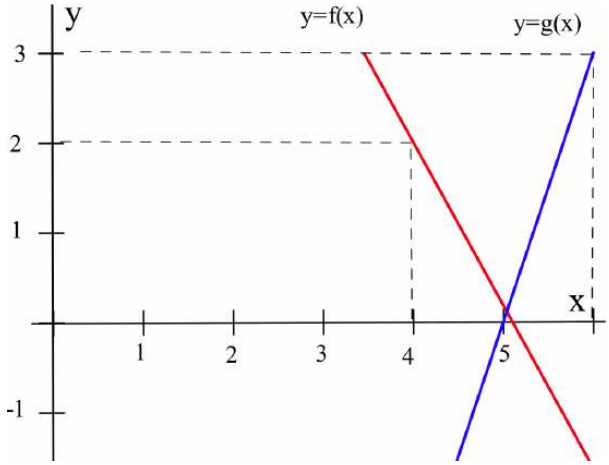
3.7: L'Hôpital's Rule

When taking limits of slopes of secant lines, $m_{\text{sec}} = \dfrac{f(x+h)-f(x)}{h}$ as $h \to 0$, we frequently encountered one difficulty: both the numerator and the denominator approached 0. And because the denominator approached 0, we could not apply the Main Limit Theorem. In many situations, however, we managed to get past this "$\frac{0}{0}$" difficulty by using algebra or geometry or trigonometry to rewrite the expression and then take the limit. But there was no common approach or pattern. The algebraic steps we used to evaluate $\lim\limits_{h\to 0} \dfrac{(2+h)^2-4}{h}$ seem quite different from the trigonometric steps needed for $\lim\limits_{h\to 0} \dfrac{\sin(2+h)-\sin(2)}{h}$.

In this section we consider a single technique, called l'Hôpital's Rule, that enables us to quickly and easily evaluate many limits of the form "$\frac{0}{0}$" as well as several other challenging indeterminate forms. (Although discovered by Johann Bernoulli, this rule was named for the Marquis de l'Hôpital (pronounced low-pee-TALL), who published it in his 1696 calculus textbook, *Analysis of the Infinitely Small for the Understanding of Curved Lines*.)

A Linear Example

The graphs of two linear functions appear below:



and we want to find $\lim\limits_{x\to 5} \dfrac{f(x)}{g(x)}$. Unfortunately, $\lim\limits_{x\to 5} f(x) = 0$ and $\lim\limits_{x\to 5} g(x) = 0$ so we cannot apply the Main Limit Theorem. We do know, however, that f and g are linear, so we can calculate their slopes, and we know that they both lines go through the point $(5,0)$ so we can find their equations: $f(x) = -2(x-5)$ and $g(x) = 3(x-5)$.

Now the limit is easier to compute:

$$\lim_{x\to 5} \frac{f(x)}{g(x)} = \lim_{x\to 5} \frac{-2(x-5)}{3(x-5)} = \lim_{x\to 5} \frac{-2}{3} = -\frac{2}{3} = \frac{\text{slope of } f}{\text{slope of } g}$$

In fact, this pattern works for any two linear functions: If f and g are linear functions with slopes $m \neq 0$ and $n \neq 0$ and a common root at $x = a$, then $f(x) - f(a) = m(x-a)$ and $g(x) - g(a) = n(x-a)$ so $f(x) = m(x-a)$ and $g(x) = n(x-a)$. Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{m(x-a)}{n(x-a)} = \lim_{x\to a} \frac{m}{n} = \frac{m}{n} = \frac{\text{slope of } f}{\text{slope of } g}$$

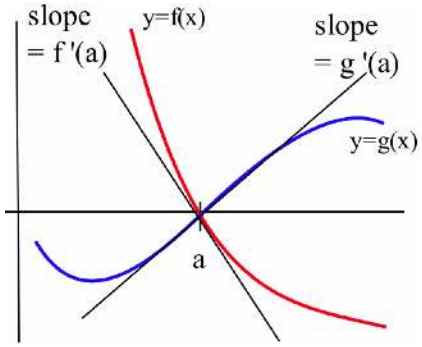
A more powerful result—that the same pattern holds true for differentiable functions even if they are not linear—is called l'Hôpital's Rule.

L'Hôpital's Rule ("$\frac{0}{0}$" Form)

If: f and g are differentiable at $x = a$, $f(a) = 0$, $g(a) = 0$ and $g'(a) \neq 0$

then: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)} = \dfrac{\text{slope of } f \text{ at } a}{\text{slope of } g \text{ at } a}$

Idea for a proof: Even though f and g may not be linear functions, they *are* differentiable. So at the point $x = a$ they are "almost linear" in the sense that we can approximate them quite well using their tangent lines at that point:



Because $f(a) = g(a) = 0$, $f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a)$ and $g(x) \approx g(a) + g'(a)(x-a) = g'(a)(x-a)$. So:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

(Unfortunately, we have ignored some subtle difficulties, such as $g(x)$ or $g'(x)$ possibly being 0 when x is close to, but not equal to, a. Because of these issues, a full-fledged proof of l'Hôpital's Rule is omitted.)

Example 3.7.1

Determine $\lim\limits_{x\to 0} \dfrac{x^2 + \sin(5x)}{3x}$ and $\lim\limits_{x\to 1} \dfrac{\ln(x)}{e^x - e}$.

Solution

We could evaluate the first limit without l'Hôpital's Rule, but let's use it anyway. We can match the pattern of l'Hôpital's Rule by letting $a = 0$, $f(x) = x^2 + \sin(5x)$ and $g(x) = 3x$. Then $f(0) = 0$, $g(0) = 0$, and f and g are differentiable with $f'(x) = 2x + 5\cos(5x)$ and $g'(x) = 3$, so:

$$\lim_{x\to 0} \frac{x^2 + \sin(5x)}{3x} = \frac{f'(0)}{g'(0)} = \frac{2\cdot 0 + 5\cos(5\cdot 0)}{3} = \frac{5}{3}$$

For the second limit, let $a = 1$, $f(x) = \ln(x)$ and $g(x) = e^x - e$. Then $f(1) = 0$, $g(1) = 0$, f and g are differentiable for x near 1 (when $x > 0$), and $f'(x) = \dfrac{1}{x}$ and $g'(x) = e^x$. Then:

$$\lim_{x\to 1} \frac{\ln(x)}{e^x - e} = \frac{f'(1)}{g'(1)} = \frac{\frac{1}{1}}{e^1} = \frac{1}{e}$$

Here no simplification was possible, so we needed l'Hôpital's Rule.

Practice 3.7.1

Evaluate $\lim\limits_{x\to 0} \dfrac{1 - \cos(5x)}{3x}$ and $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x^2 + 2x - 8}$.

Answer

Both numerator and denominator in the first limit are differentiable and both equal 0 when $x = 0$, so we apply l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1 - \cos(5x)}{3x} = \lim_{x\to 0} \frac{5\sin(5x)}{3} = \frac{0}{3} = 0$$

Both numerator and denominator in the second limit are differentiable and both equal 0 when $x = 0$, so we apply l'Hôpital's Rule:

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2+2x-8}=\lim_{x\to 2}\frac{2x+1}{2x+2}=\frac{5}{6}$$

Strong Version of l'Hôpital's Rule

We can strengthen l'Hôpital's Rule to include cases when $g'(a)=0$, and the indeterminate form "$\frac{\infty}{\infty}$" when f and g increase without bound.

L'Hôpital's Rule (Strong "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" Forms)

If: f and g are differentiable on an open interval I containing a, $g'(x)\neq 0$ on I except possibly at a, and $\lim_{x\to a}\frac{f(x)}{g(x)}=$ "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$",

then: $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ if the limit on the right exists.

(Here "a" can represent a finite number or "∞.")

Example 3.7.2

Evaluate $\lim_{x\to\infty}\frac{e^{7x}}{5x}$.

Solution

As "$x\to\infty$," both e^{7x} and $5x$ increase without bound, so we have an "$\frac{\infty}{\infty}$" indeterminate form and can use the Strong Version of l'Hôpital's Rule: $\lim_{x\to\infty}\frac{e^{7x}}{5x}=\lim_{x\to\infty}\frac{7e^{7x}}{5}=\infty$.

The limit of $\frac{f'}{g'}$ may also be an indeterminate form, in which case we can apply l'Hôpital's Rule again to the ratio $\frac{f'}{g'}$. We can continue using l'Hôpital's Rule at each stage as long as we have an indeterminate quotient.

Example 3.7.3

Compute $\lim_{x\to 0}\frac{x^3}{x-\sin(x)}$.

Solution

As $x\to 0$, $f(x)=x^3\to 0$ and $g(x)=x-\sin(x)\to 0$ so:

$$\lim_{x\to 0}\frac{x^3}{x-\sin(x)}=\lim_{x\to 0}\frac{3x^2}{1-\cos(x)}=\lim_{x\to 0}\frac{6x}{\sin(x)}=\lim_{x\to 0}\frac{6}{\cos(x)}=6$$

where we have used l'Hôpital's Rule three times in succession. (At each stage, you should verify the conditions for l'Hôpital's Rule hold.)

Practice 3.7.2

Use l'Hôpital's Rule to find $\lim_{x\to\infty}\frac{x^2+e^x}{x^3+8x}$.

Answer

Both numerator and denominator are differentiable and both become arbitrarily large as $x\to\infty$, so we apply l'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{x^2+e^x}{x^3+8x}=\lim_{x\to\infty}\frac{2x+e^x}{3x^2+8}=\lim_{x\to\infty}\frac{2+e^x}{6x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty$$

(Note that we needed to apply l'Hôpital's Rule three times and that each stage involved an "$\frac{\infty}{\infty}$" indeterminate form.)

Which Function Grows Faster?

Sometimes we want to compare the asymptotic behavior of two systems or functions for large values of x. L'Hôpital's Rule can be useful in such situations. For example, if we have two algorithms for sorting names, and each algorithm takes longer and longer to sort larger collections of names, we may want to know which algorithm will accomplish the task more efficiently for really large collections of names.

Example 3.7.4

Algorithm A requires $n \cdot \ln(n)$ steps to sort n names and algorithm B requires $n^{1.5}$ steps. Which algorithm will be better for sorting very large collections of names?

Solution

We can compare the ratio of the number of steps each algorithm requires, $\frac{n \cdot \ln(n)}{n^{1.5}}$, and then take the limit of this ratio as n grows arbitrarily large: $\lim\limits_{n\to\infty} \frac{n \cdot \ln(n)}{n^{1.5}}$.

If this limit is infinite, we say that $n \cdot \ln(n)$ "grows faster" than $n^{1.5}$. If the limit is 0, we say that $n^{1.5}$ grows faster than $n \cdot \ln(n)$.

Because $n \cdot \ln(n)$ and $n^{1.5}$ both grow arbitrarily large when n becomes large, we can simplify the ratio to $\frac{\ln(n)}{n^{0.5}}$ and then use l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln(n)}{n^{0.5}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{0.5n^{-0.5}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

We conclude that $n^{1.5}$ grows faster than $n \cdot \ln(n)$ so algorithm A requires fewer steps for really large sorts.

Practice 3.7.3

Algorithm A requires e^n operations to find the shortest path connecting n towns, while algorithm B requires $100 \cdot \ln(n)$ operations for the same task and algorithm C requires n^5 operations. Which algorithm is best for finding the shortest path connecting a very large number of towns? The worst?

Answer

Comparing A with e^n operations to B with $100 \cdot \ln(n)$ operations we can apply l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{e^n}{100\ln(n)} = \lim_{n\to\infty} \frac{e^n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n \cdot e^n}{100} = \infty$$

to show that B requires fewer operations than A.

Comparing B with $100\ln(n)$ operations to C with n^5 operations, we again apply l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{100\ln(n)}{n^5} = \lim_{n\to\infty} \frac{\frac{100}{n}}{5n^4} = \lim_{n\to\infty} \frac{20}{n^5} = 0$$

to show that B requires fewer operations than C. So B requires the fewest operations of the three algorithms.

Comparing A to C we must apply l'Hôpital's Rule repeatedly:

$$\lim_{n\to\infty} \frac{e^n}{n^5} = \lim_{n\to\infty} \frac{e^n}{5n^4} = \lim_{n\to\infty} \frac{e^n}{20n^3} = \lim_{n\to\infty} \frac{e^n}{60n^2}$$
$$= \lim_{n\to\infty} \frac{e^n}{120n} = \lim_{n\to\infty} \frac{e^n}{120} = \infty$$

So A requires more operations than C and thus A requires the most operations of the three algorithms.

Other Indeterminate Forms

We call "$\frac{0}{0}$" an indeterminate form because knowing that f approaches 0 and g approaches 0 is not enough to determine the limit of $\frac{f}{g}$, even if that limit exists. The ratio of a "small" number divided by a "small" number can be almost anything as three simple "$\frac{0}{0}$" examples show:

$$\lim_{x\to 0}\frac{3x}{x}=3 \quad \text{while} \quad \lim_{x\to 0}\frac{x^2}{x}=0 \quad \text{and} \quad \lim_{x\to 0}\frac{5x}{x^3}=\infty$$

Similarly, "$\frac{\infty}{\infty}$" is an indeterminate form because knowing that f and g both grow arbitrarily large is not enough to determine the value of the limit of $\frac{f}{g}$ or even if the limit exists:

$$\lim_{x\to \infty}\frac{3x}{x}=3 \quad \text{while} \quad \lim_{x\to \infty}\frac{x^2}{x}=\infty \quad \text{and} \quad \lim_{x\to \infty}\frac{5x}{x^3}=0$$

In addition to the indeterminate quotient forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" there are several other "indeterminate forms." In each case, the resulting limit depends not only on each function's limit but also on how quickly each function approaches its limit.

- **Product**: If f approaches 0 and g grows arbitrarily large, the product $f\cdot g$ has the indeterminate form "$0\cdot\infty$."
- **Exponent**: If f and g both approach 0, the function f^g has the indeterminate form "0^0."
- **Exponent**: If f approaches 1 and g grows arbitrarily large, the function f^g has the indeterminate form "1^∞."
- **Exponent**: If f grows arbitrarily large and g approaches 0, the function f^g has the indeterminate form "∞^0."
- **Difference**: If f and g both grow arbitrarily large, the function $f-g$ has the indeterminate form "$\infty-\infty$."

Unfortunately, l'Hôpital's Rule can only be used directly with an indeterminate quotient ($\frac{0}{0}$ or "$\frac{\infty}{\infty}$"), but we can algebraically manipulate these other forms into quotients and *then* apply l'Hôpital's Rule.

Example 3.7.5

Evaluate $\lim_{x\to 0^+} x\cdot\ln(x)$.

Solution

This limit involves an indeterminate product (of the form "$0\cdot-\infty$") but we need a quotient in order to apply l'Hôpital's Rule. If we rewrite the product $x\cdot\ln(x)$ as a quotient:

$$\lim_{x\to 0^+} x\cdot\ln(x)=\lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}}=\lim_{x\to 0^+}-x=0$$

results from applying the "$\frac{\infty}{\infty}$" version of l'Hôpital's Rule.

To use l'Hôpital's Rule on a product $f\cdot g$ with indeterminate form "$0\cdot\infty$," first rewrite $f\cdot g$ as a quotient: $\frac{f}{\frac{1}{g}}$ or $\frac{g}{\frac{1}{f}}$. Then apply l'Hôpital's Rule.

Example 3.7.6

Evaluate $\lim_{x\to 0^+} x^x$.

Solution

This limit involves the indeterminate form 0^0. We can convert it to a product by recalling a property of exponential and logarithmic functions: for any positive number a, $a=e^{\ln(a)}$ so:

$$f^g=e^{\ln(f^g)}=e^{g\cdot\ln(f)}$$

Applying this to x^x:

$$\lim_{x\to 0^+} x^x=\lim_{x\to 0^+} e^{\ln(x^x)}=\lim_{x\to 0^+} e^{x\cdot\ln(x)}$$

This last limit involves the indeterminate product $x \cdot \ln(x)$. From the previous example we know that $\lim\limits_{x \to 0^+} x \cdot \ln(x) = 0$ so we can conclude that:

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \cdot \ln(x)} = e^{\lim_{x \to 0^+} x \cdot \ln(x)} = e^0 = 1$$

because the function $f(u) = e^u$ is continuous everywhere.

To use l'Hôpital's Rule on an expression involving exponents, f^g with the indeterminate form "0^0," "1^∞" or "∞^0," first convert it to an expression involving an indeterminate product by recognizing that $f^g = e^{g \cdot \ln(f)}$ and then determining the limit of $g \cdot \ln(f)$. The final result is $e^{\text{limit of } g \cdot \ln(f)}$.

Example 3.7.7

Evaluate $\lim\limits_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$.

Solution

This expression has the form 1^∞ so we first use logarithms to convert the problem into a limit involving a product:

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x \to \infty} e^{x \cdot \ln\left(1 + \frac{a}{x}\right)}$$

so now we need to compute $\lim\limits_{x \to \infty} x \cdot \ln\left(1 + \frac{a}{x}\right)$. This limit has the form "$\infty \cdot 0$" so we now convert the product to a quotient:

$$\lim_{x \to \infty} x \cdot \ln\left(1 + \frac{a}{x}\right) = \lim_{x \to \infty} \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}}$$

This last limit has the form "$\frac{0}{0}$" so we can finally apply l'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{\frac{-a}{x^2}}{1 + \frac{a}{x}}}{\frac{-1}{x^2}} = \lim_{x \to \infty} \frac{a}{1 + \frac{a}{x}} = \frac{a}{1} = a$$

and conclude that:

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x \to \infty} e^{x \cdot \ln\left(1 + \frac{a}{x}\right)} = e^{\lim_{x \to \infty} x \cdot \ln\left(1 + \frac{a}{x}\right)} = e^a$$

where we have again used the continuity of the function $f(u) = e^u$.

Problems

In Problems 1–15, evaluate each limit. Be sure to justify any use of l'Hôpital's Rule.

1. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$
2. $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x^5 - 32}$
3. $\lim\limits_{x \to 0} \dfrac{\ln(1 + 3x)}{5x}$
4. $\lim\limits_{x \to \infty} \dfrac{e^x}{x^3}$
5. $\lim\limits_{x \to 0} \dfrac{x \cdot e^x}{1 - e^x}$
6. $\lim\limits_{x \to 0} \dfrac{2^x - 1}{x}$
7. $\lim\limits_{x \to \infty} \dfrac{\ln(x)}{x}$
8. $\lim\limits_{x \to \infty} \dfrac{\ln(x)}{\sqrt{x}}$
9. $\lim\limits_{x \to \infty} \dfrac{\ln(x)}{x^p}$ $(p > 0)$
10. $\lim\limits_{x \to 0} \dfrac{e^{3x} - e^{2x}}{4x}$

11. $\lim\limits_{x\to 0}\dfrac{1-\cos(3x)}{x^2}$
12. $\lim\limits_{x\to 0}\dfrac{1-\cos(2x)}{x}$
13. $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}$
14. $\lim\limits_{x\to 0}\dfrac{\cos(a+x)-\cos(a)}{x}$
15. $\lim\limits_{x\to 0}\dfrac{1-\cos(x)}{x\cdot\cos(x)}$
16. Find a value for p so that $\lim\limits_{x\to\infty}\dfrac{3x}{px+7}=2$.
17. Find a value for p so that $\lim\limits_{x\to 0}\dfrac{e^{px}-1}{3x}=5$.
18. The limit $\lim\limits_{x\to\infty}\dfrac{\sqrt{3x+5}}{\sqrt{2x-1}}$ has the indeterminate form "$\frac{\infty}{\infty}$." Why doesn't l'Hôpital's Rule work with this limit? (Hint: Apply l'Hôpital's Rule twice and see what happens.) Evaluate the limit without using l'Hôpital's Rule.
19. a. Evaluate $\lim\limits_{x\to\infty}\dfrac{e^x}{x}$, $\lim\limits_{x\to\infty}\dfrac{e^x}{x^2}$ and $\lim\limits_{x\to\infty}\dfrac{e^x}{x^5}$.
    b. An algorithm is "exponential" if it requires $a\cdot e^{bn}$ steps (a, and b are positive constants). An algorithm is "polynomial" if it requires $c\cdot n^d$ steps. (c and d are positive constants). Show that polynomial algorithms require fewer steps than exponential ones for large values of n.
20. The problem $\lim\limits_{x\to 0}\dfrac{x^2}{3x^2+x}$ appeared on a test. One student determined the limit was an indeterminate "$\frac{0}{0}$" form and applied l'Hôpital's Rule to get:

$$\lim_{x\to 0}\frac{x^2}{3x^2+x}=\lim_{x\to 0}\frac{2x}{6x+1}=\lim_{x\to 0}\frac{2}{6}=\frac{1}{3}$$

    Another student also determined the limit was an indeterminate "$\frac{0}{0}$" form and wrote:

$$\lim_{x\to 0}\frac{x^2}{3x^2+x}=\lim_{x\to 0}\frac{2x}{6x+1}=\frac{0}{0+1}=0$$

    Which student is correct? Why?

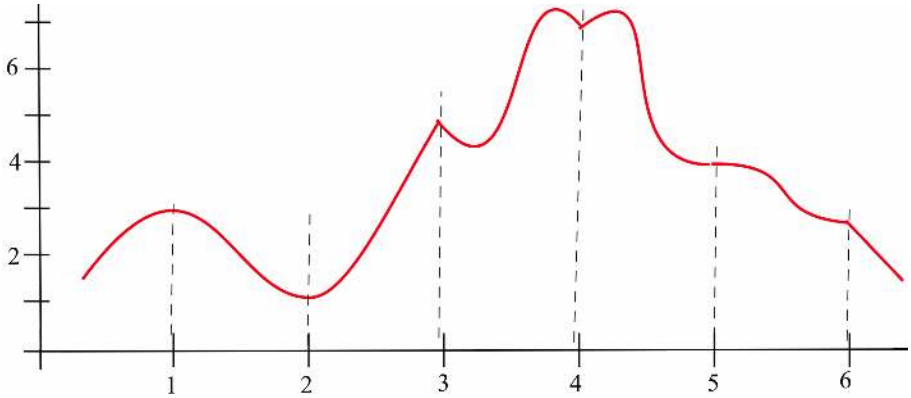
In Problems 21–30, evaluate each limit. Be sure to justify any use of l'Hôpital's Rule.

21. $\lim\limits_{x\to 0^+}\sin(x)\cdot\ln(x)$
22. $\lim\limits_{x\to\infty}x^3e^{-x}$
23. $\lim\limits_{x\to 0^+}\sqrt{x}\cdot\ln(x)$
24. $\lim\limits_{x\to 0^+}x^{\sin(x)}$
25. $\lim\limits_{x\to\infty}\left(1-\dfrac{3}{x^2}\right)^x$
26. $\lim\limits_{x\to 0}(1-\cos(3x))^x$
27. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin(x)}\right)$
28. $\lim\limits_{x\to\infty}[x-\ln(x)]$
29. $\lim\limits_{x\to\infty}\left(\dfrac{x+5}{x}\right)^{\frac{1}{x}}$
30. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{\frac{2}{x}}$

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3.A: Answers

Section 3.1

1. Local maximums at $x = 3$, $x = 5$, $x = 9$ and $x = 13$; global maximum at $x = 13$. Local minimums at $x = 1$, $x = 4.5$, $x = 7$ and $x = 10.5$; global minimum at $x = 7$.
3. $f(x) = x^2 + 8x + 7 \Rightarrow f'(x) = 2x + 8$, which is defined for all values of x; $f'(x) = 0$ when $x = -4$, so $x = -4$ is a critical number. There are no endpoints. The only critical number is $x = -4$, so the only critical point is $(-4, f(-4)) = (-4, -9)$, which is the global (and local) minimum.
5. $f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$, which is defined for all values of x; $f'(x) = 0$ when $x = \frac{\pi}{2} + k\pi$ so those values are critical numbers. There are no endpoints. $f(x) = \sin(x)$ has local and global maximums at $x = \frac{\pi}{2} + 2k\pi$ and global and local minimums at $x = \frac{3\pi}{2} + 2k\pi$.
7. $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$, which is defined for all values of x except $x = 0$, and $f'(x) \neq 0$ for all x, so $x = 0$ is the only critical number. There are no endpoints. the function $f(x) = \sqrt[3]{x}$ has no global or local extrema.
9. $f(x) = xe^{5x} \Rightarrow f'(x) = e^{5x}[5x + 1]$, which is defined for all values of x; $f'(x) = 0$ when $5x + 1 = 0 \Rightarrow x = -\frac{1}{5}$ so this is the only critical number. There are no endpoints. The function $f(x) = xe^{5x}$ has a local and global minimum at $x = -\frac{1}{5}$ and no other extrema.
11. $f(x) = (x-1)^2(x-3) \Rightarrow f'(x) = (x-1)^2 + 2(x-1)(x-3) = (x-1)(3x-7)$, which is defined for all values of x; $f'(x) = 0$ when $x = 1$ and $x = \frac{7}{3}$ so those values are critical numbers. There are no endpoints. The only critical points are $(1, 0)$, which is a local maximum, and $\left(\frac{7}{3}, -\frac{32}{27}\right)$, which is a local minimum. When the interval is the entire real number line, this function does not have any global extrema.
13. $f(x) = 2x^3 - 96x + 42 \Rightarrow f'(x) = 6x^2 - 96$, which is defined for all values of x; $f'(x) = 6(x+4)(x-4) = 0$ when $x = -4$ and $x = 4$ so those values are critical numbers. There are no endpoints. The only critical points are $(-4, 298)$, which is a local maximum, and $(4, -214)$, which is a local minimum. When the interval is the entire real number line, this function does not have a global maximum or global minimum.
15. $f(x) = e^{-(x-2)^2} \Rightarrow f'(x) = -2(x-2)e^{-(x-2)^2}$, which is defined for all x; $f'(x) = 0$ only when $x = 2$, so that is a critical number. There are no endpoints. The only critical point is $(2, 1)$, which is a local and global maximum. When the interval is the entire real number line, this function does not have a local or global minimum.
17. $f(x) = \dfrac{x}{1+x^2} \Rightarrow f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$, which is defined for all x; $f'(x) = 0$ only when $x = \pm 1$, so these are the only critical numbers. There are no endpoints. The critical point $\left(-1, -\frac{1}{2}\right)$ is a local and global minimum; the critical point $\left(1, \frac{1}{2}\right)$ is a local and global maximum.
19. $f(x) = (x-2)^{\frac{2}{3}} \Rightarrow f'(x) = \frac{2}{3}(x-2)^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x-2}}$, which is defined for all values of x except $x = 2$, and $f'(x) \neq 0$ for all x so $x = 2$ is the only critical number. There are no endpoints. The function $f(x) = (x-2)^{\frac{2}{3}}$ has a global and local minimum at $x = 2$ and no other extrema.
21. $f(x) = (x^2-4)^{\frac{1}{3}} \Rightarrow f'(x) = \frac{2}{3}(x^2-4)^{-\frac{2}{3}} \cdot 2x = \frac{2x}{3\sqrt[3]{(x^2-4)^2}}$, which is defined for all values of x except $x = \pm 2$, and $f'(x) = 0$ only when $x = 0$, so $x = -2$, $x = 0$ and $x = 2$ are the only critical numbers. There are no endpoints. The function $f(x) = \sqrt[3]{x^2-4}$ has a global and local minimum at $x = 0$ and no other global or local extrema.
23. 

25. $f(x) = x^2 - 6x + 5 \Rightarrow f'(x) = 2x - 6$, which is defined for all values of x; $f'(x) = 0$ only when $x = 3$, so that is a critical number. The endpoints are $x = -2$ and $x = 5$. The critical points are $(3, -4)$, which is the local and global minimum, $(-2, 21)$, which is a local and global maximum, and $(5, 0)$, which is a local maximum.
27. $f(x) = 2 - x^3 \Rightarrow f'(x) = -3x^2$, which is defined for all values of x; $f'(x) = 0$ only when $x = 0$, so that is a critical number. The endpoints are $x = -2$ and $x = 1$, which are also critical numbers. The critical points are $(-2, 10)$, which is a local and global maximum, $(0, 2)$, which is not a local or global maximum or minimum, and $(1, 1)$, which is a local and global minimum.
29. $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$, which is defined for all x; $f'(x) = 0$ only when $x = -1$ and $x = 1$, so these are critical numbers. The endpoints $x = -2$ and $x = 1$ are also critical numbers. The critical points are $(-2, 3)$, which is a local and global minimum on $[-2, 1]$, the point $(-1, 7)$, which is a local and global maximum on $[-2, 1]$, and the point $(1, 3)$, which is a local and global minimum on $[-2, 1]$.
31. $f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x-3)(x-1)$, which is defined for all x; $f'(x) = 0$ only when $x = 0$ or $x = 1$ in the interval $[0, 2]$, so each of these values is a critical number. The endpoints $x = 0$ and $x = 2$ are also critical numbers. The critical points are $(0, 7)$, which is a local minimum, $(1, 8)$ which is a local and global maximum, and $(2, -1)$, which is a local and global minimum. (It is also true that $f'(3) = 0$, but $x = 3$ is not in the interval $[0, 2]$.)
33. $f(x) = \frac{1}{x^2+1} \Rightarrow f'(x) = \frac{-2x}{(x^2+1)^2}$, which is defined for all values of x; $f'(x) = 0$ only when $x = 0$, which is not in the interval $[1, 3]$ so $x = 0$ is a not a critical number. The endpoints $x = 1$ and $x = 3$ are critical numbers. The critical points are $\left(1, \frac{1}{2}\right)$, which is a local and global maximum, and $\left(3, \frac{1}{10}\right)$, which is a local and global minimum.
35. $f'(x) = \frac{3}{2}(x^2+4)^{-\frac{1}{2}} \cdot 2x - 1 = \frac{3x}{\sqrt{x^2+4}} - 1$, which exists for all x; $f'(x) = 0$ only when $3x = \sqrt{x^2+4} \Rightarrow x = \frac{1}{\sqrt{2}}$. The endpoints are $x = 0$ and $x = 2$, so the critical numbers are $x = 0$, $x = \frac{1}{\sqrt{2}}$ and $x = 2$: $f(0) = 6$, $f\left(\frac{1}{\sqrt{2}}\right) \approx 5.66$ and $f(2) \approx 6.49$, so $f(x) = 3\sqrt{x^2+4} - x$ has a local max at $x = 0$, a global and local min at $x = \frac{1}{\sqrt{2}}$, and a global and local max at $x = 2$.
37. $f(x) = x^3 - \ln(x) \Rightarrow f'(x) = 3x^2 - \frac{1}{x}$, which exists for all x in the interval $\left[\frac{1}{2}, 2\right]$, and $f'(x) = 0 \Rightarrow x = \frac{1}{\sqrt[3]{3}}$. The endpoints are $x = \frac{1}{2}$ and $x = 2$, so the three critical numbers are $x = \frac{1}{2}$, $x = \frac{1}{\sqrt[3]{3}}$ and $x = 2$: $f\left(\frac{1}{2}\right) \approx 0.82$, $f\left(\frac{1}{\sqrt[3]{3}}\right) \approx 0.70$ and $f(2) \approx 7.3$ so $f(x) = x^3 - \ln(x)$ has a local max at $x = \frac{1}{2}$, a global and local min at $x = \frac{1}{\sqrt[3]{3}}$, and a global and local max at $x = 2$.
39. $A(x) = 4x\sqrt{1-x^2}$ for $0 < x < 1$, so:

$$\begin{aligned} A'(x) &= 4\left[\frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right] \\ &= 4\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right] \end{aligned}$$

hence $A'(x) > 0$ if $0 < x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ if $\frac{1}{\sqrt{2}} < x < 1$. Max when $x = \frac{1}{\sqrt{2}}$: $A\left(\frac{1}{\sqrt{2}}\right) = 2$

41. $V = x(8-2x)^2$ for $0 \le x \le 4$, so:

$$\begin{aligned} V' &= x(2)(8-2x)(-2) + (8-2x)^2 \\ &= (8-2x)(-4x + 8 - 2x) \\ &= (8-2x)(8-6x) = 4(4-x)(4-3x) \end{aligned}$$

hence $V' < 0$ if $x > \frac{4}{3}$ and $V' > 0$ if $0 < x < \frac{4}{3}$. $V\left(\frac{4}{3}\right) = \frac{4}{3}\left(8 - \frac{8}{3}\right)^2 = \frac{1024}{27} \approx 37.926$ cubic units is the largest volume. Smallest volume is 0, which occurs when $x = 0$ and $x = 4$.

43. a. 4: the endpoints and two values of x for which $f'(x) = 0$
    b. 2: the endpoints
    c. At most $n + 1$: the two endpoints and the $n - 1$ points x for which $f'(x) = 0$; at least 2: the endpoints.
45. a. local minimum at $(1, 5)$
    b. no extreme at $(1, 5)$
    c. local maximum at $(1, 5)$
    d. no extreme at $(1, 5)$
47. a. 0, 2, 6, 8, 11, 12
    b. 0, 6, 11
    c. 2, 8, 12
49. If f does not attain a maximum on $[a, b]$ or f does not attain a minimum on $[a, b]$, then f must have a discontinuity on $[a, b]$.

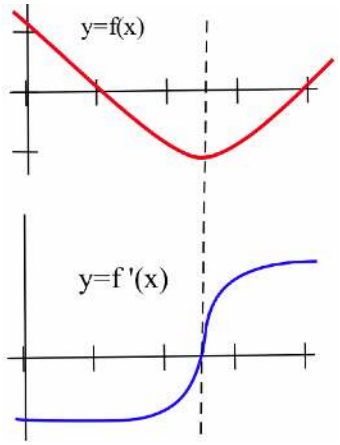
51. a. yes, -1
    b. no
    c. yes, -1
    d. no
    e. yes, $1-\pi$

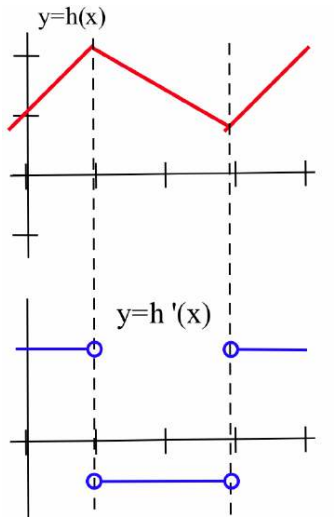
53. a. yes, 0
    b. yes, 0
    c. yes, 0
    d. yes, 0
    e. yes, 0

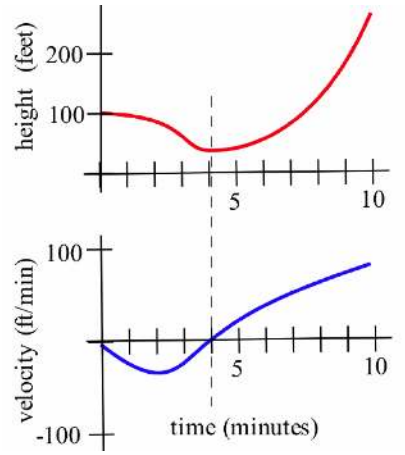
55. a. $S(x)$ is minimum when $x \approx 8$
    b. $S(x)$ is maximum when $x = 2$.

Section 3.2

1. $c \approx 3$, 10 and 13
3. a. $c = \dfrac{\pi}{2}$
   b. $c = \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}$
5. Rolle's Theorem asserts that the velocity $h'(t)$ will equal 0 at some point between the time the ball is tossed and the time you catch it. The ball is not moving as fast when it reaches the balcony after being thrown from below.
7. The function does not violate Rolle&trsquo;s Theorem because the function does not satisfy the hypotheses of the theorem: f is not differentiable at 0, a point in the interval $-1 < x < 1$.
9. No. Velocity is not the same as the rate of change of altitude, which is only one of the components of position. Rolle's Theorem only says there was a time when altitude was not changing.
11. $f'(x) = 3x^2 + 5$, so $f'(x) = 0$ has no real roots. If $f(x) = 0$ for a value of x other than 2, then by the corollary from Problem 10, we would have an immediate contradiction.
13. a. $f(0) = 0$, $f(2) = 4$, $f'(c) = 2c$, so $\frac{4-0}{2-0} = 2c \Rightarrow c = 1$
    b. $f(1) = 4$, $f(5) = 8$, $f'(c) = 2c - 5$, so $\frac{8-4}{5-1} = 2c - 5 \Rightarrow c = 3$
15. a. $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 1$, $f'(c) = \cos(c)$, so $\frac{1-0}{\frac{\pi}{2}-0} = \cos(c) \Rightarrow c = \arccos\left(\frac{2}{\pi}\right) \approx 0.88$
    b. $f(-1) = -1$, $f(3) = 27$, $f'(c) = 3c^2$, so $\frac{27+1}{3+1} = 3c^2 \Rightarrow c^2 = \frac{7}{3}$, hence $c = \sqrt{\frac{7}{3}}$ (we know $c > -1$, which eliminates $c = -\sqrt{\frac{7}{3}}$)
17. a. $f(1) = 4$, $f(9) = 2$, $f'(c) = \frac{-1}{2\sqrt{c}}$, so $\frac{2-4}{1-9} = \frac{-1}{2\sqrt{c}} \Rightarrow -\frac{1}{4} = -\frac{1}{2\sqrt{c}} \Rightarrow c = 4$
    b. $f(1) = 3$, $f(7) = 15$, $f'(c) = 2$, so $\frac{15-3}{7-1} = 2$ and any c between 1 and 7 will do.
19. The hypotheses are not all satisfied: $f'(x)$ does not exist at $x = 0$, which is between -1 and 3.
21. $f'(c) = 17$ at some time c does not prove the motorist "could not have been speeding."
23. $f(x) = x^3 + x^2 + 5x + c$, so $f(1) = 7 + c = 10$ when $c = 3$. Therefore $f(x) = x^3 + x^2 + 5x + 3$.
25. a. $f'(x) = 2Ax$. We need $A \cdot 1^2 + B = 9$ and $2A \cdot 1 = 4$ so $A = 2$ and $B = 7$, hence $f(x) = 2x^2 + 7$.
    b. $A \cdot 2^2 + B = 3$ and $2A \cdot 2 = -2$ so $A = -\frac{1}{2}$ and $B = 5$, hence $f(x) = -\frac{1}{2}x^2 + 5$.
    c. $A \cdot 0^2 + B = 2$ and $2A \cdot 0 = 3$, so there is no such A. The point $(0, 2)$ is not on the parabola $y = x^2 + 3x - 2$.
27. $f(x) = x^3 + C$, a family of "parallel" curves
29. $v(t) = 300$: Assuming the rocket left the ground at $t = 0$, we have $y(1) = 300$ ft, $y(2) = 600$ ft, $y(5) = 1500$ ft.
31. $f''(x) = 6$, $f'(0) = 4$, $f(0) = -5 \Rightarrow f(x) = 3x^2 + 4x - 5$
33. a. $A(x) = 3x$
    b. $A'(x) = 3$
35. a. $A(x) = x^2 + x$
    b. $A'(x) = 2x + 1$
37. $a_1 = 5$, $a_2 = a_1 + 3 = 5 + 3 = 8$, $a_3 = a_2 + 3 = (5 + 3) + 3 = 11$, $a_4 = a_3 + 3 = (5 + 3 + 3) + 3 = 14$: in general, $a_n = 5 + 3(n - 1) = 2 + 3n$

1. 

3. 

5. 

7. A: Q, B: P, C: R

9. $f'(x) = \frac{1}{x} > 0$ for $x > 0$ so $f(x) = \ln(x)$ is increasing on $(0, \infty)$.

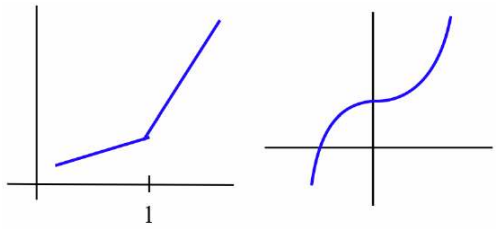
11. If f is increasing then $f(1) < f(\pi)$ so $f(1)$ and $f(\pi)$ cannot both equal 2.

13. a. $x = 3$, $x = 8$
    b. maximum at $x = 8$
    c. none (or only at right endpoint)

14. Relative maximum height at $x = 2$ and $x = 7$; relative minimum height at $x = 4$.

15. Relative maximum height at $x = 6$; relative minimum height at $x = 8$.

17. $f(x) = x^3 - 3x^2 - 9x - 5$ has a relative minimum at $(3, -32)$ and a relative maximum at $(-1, 0)$.
19. $h(x) = x^4 - 8x^2 + 3$ has a relative max at $(0, 3)$ and relative minimums at $(2, -13)$ and $(-2, -13)$.
21. $r(t) = 2(t^2 + 1)^{-1}$ has a relative maximum at $(0, 2)$ and no relative minimums.
23. No positive roots. $f(x) = 2x + \cos(x)$ is continuous and $f(0) = 1 > 0$. Because $f'(x) = 2 - \sin(x) \geq 1 > 0$ for all x, f is increasing and never decreases back to the x-axis (a root).
24. One positive root. $g(x) = 2x - \cos(x)$ is continuous, $g(0) = -1 < 0$ and $g(1) = 2 - \cos(1) \geq 1 > 0$ so by the Intermediate Value Theorem, g has a root between 0 and 1. Because $g'(x) = 2 + \sin(x) \geq 1 > 0$ for all x, g is always increasing and can have only that one root.
25. $h(x) = x^3 + 9x - 10 \Rightarrow h'(x) = 3x^2 + 9 = 3(x^2 + 3) > 0$ for all x so h is always increasing and can cross the x-axis at most at one place. Because the $h(1) = 0$, the graph of h crosses the x-axis at $x = 1$, which is the only root of h.
27. a. See below left: f is always increasing but $f'(1)$ is undefined.



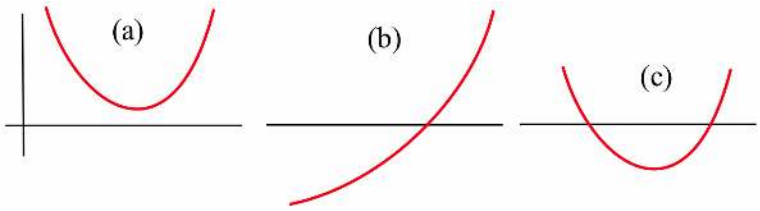
b. See above right: $f(x) = x^3 + 1$ is always increasing but $f'(0) = 0$.
c. If they travel at the same positive speed in different directions, then the distance between them will not remain constant.

29. a. $h(x) = x^2$ or $x^2 + 1$ or $x^2 - 7$ or, in general, $x^2 + C$ for any constant C.
b. $f(x) = x^2 + C$ and $20 = f(3) = 3^2 + C \Rightarrow C = 20 - 9 = 11$ so $f(x) = x^2 + 11$.
c. $g(x) = x^2 + C$ and $7 = g(2) = 2^2 + C \Rightarrow C = 7 - 4 = 3$ so $g(x) = x^2 + 3$.

Section 3.4

1. a. $f(t)$ = number of workers unemployed at time t; $f'(t) > 0$ and $f''(t) < 0$
b. $f(t)$ = profit at time t; $f'(t) < 0$ and $f''(t) > 0$
c. $f(t)$ = population at time t; $f'(t) > 0$ and $f''(t) > 0$

3.



d. Not possible.

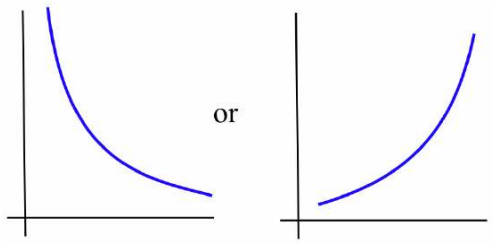
5. a. Concave up on $(0, 2)$, $(2, 3+)$, $(6, 9)$.
b. Concave down on $(3+, 6)$. (A small technical note: we have defined concavity only at points where the function is differentiable, so we exclude the endpoints and points where the function is not differentiable from the intervals of concave up and concave down.)

7. $g''(x) = 6x - 6 \Rightarrow g''(-1) < 0$ so $(-1, 12)$ is a local max; $g''(3) > 0$ so $(3, -20)$ is a local min.
9. $f''(x) = 5\left[-\sin^5(x) + 4\sin^3(x) \cdot \cos^2(x)\right] \Rightarrow f''\left(\frac{\pi}{2}\right) < 0$ so $\left(\frac{\pi}{2}, 1\right)$ is a local maximum; $f''\left(\frac{3\pi}{2}\right) > 0$ so $\left(\frac{3\pi}{2}, -1\right)$ is a local minimum; $f''(\pi) = 0$ and f changes concavity at $x = \pi$ so $(\pi, 0)$ is an inflection point.
11. d and e
13. a. 0
b. at most 1
c. at most $n - 2$

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15.

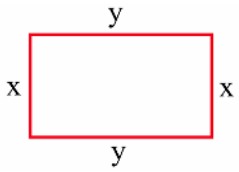
x	$f(x)$	$f'(x)$	$f''(x)$
0	−	+	+
1	+	0	−
2	−	−	+
3	0	+	+

17. 

19. $f'(x) = 3x^2 - 42x + 144 = 3(x^2 - 14x + 48) = 3(x-6)(x-8)$, which exists everywhere and $f'(x) = 0 \Rightarrow x = 6$ or $x = 8$; meanwhile, $f''(x) = 6x - 42 = 6(x-7)$, so $f''(6) = -6 < 0$ means that f has a local max at $x = 6$ and $f''(8) = 6 > 0$ means that f has a local min at $x = 8$. The only candidate for an inflection point is where $f''(x) = 0 \Rightarrow x = 7$; $f''(x) < 0$ for $x < 7$ and $f''(x) > 0$ for $x > 7$, so f does have an inflection point at $x = 7$. A window with $2 \le x \le 12$ and $-50 \le y \le 10$ should work.
21. $f'(x) = 7e^{7x} - 5 \Rightarrow f''(x) = 49e^{7x}$ so $f''(x) > 0$ for all x, hence f is always concave up and has no inflection points. Its only critical number is where $7e^{7x} - 5 = 0 \Rightarrow e^{7x} = \frac{5}{7} \Rightarrow x = \frac{1}{7} \cdot \ln\left(\frac{5}{7}\right) \approx -0.05$, which must be a local min. A window with $-1 \le x \le \frac{1}{2}$ and $0 \le y \le 20$ should work.
23. $f'(x) = -3e^{-3x} + 1 \Rightarrow f''(x) = 9e^{-3x} > 0$ for all x, hence f is always concave up and has no inflection points. Its only critical number: $-3e^{-3x} + 1 = 0 \Rightarrow e^{-3x} = \frac{1}{3} \Rightarrow x = \frac{1}{3} \cdot \ln(3) \approx 0.37$, which must be a local min. Try a window with $-1 \le x \le 2$ and $0 \le y \le 3$.
25. $f'(x) = (1-3x)e^{-3x} \Rightarrow f''(x) = (9x-6)e^{-3x}$ so the only critical number is $x = \frac{1}{3}$ and $f''\left(\frac{1}{3}\right) - 3e^{-1} < 0$, so it must be a local max. The only inflection point is where $x = \frac{2}{3}$. Try a window with $-1 \le x \le 2$ and $-2 \le y \le 1$.
27. $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}(4x-1)$, so the critical numbers are $x = 0$ and $x = \frac{1}{4}$; $f''(x) = \frac{2}{9}x^{-\frac{5}{3}}(2x+1)$ so the only candidates for an inflection point are $x = 0$ and $x = -\frac{1}{2}$. Because $f''(x) < 0$ when $x < -\frac{1}{2}$ and $x > 0$, the graph of f is concave up on those intervals and concave down for $-\frac{1}{2} < x < 0$; because $f\left(\frac{1}{4}\right) > 0$, f has a local min at $x = \frac{1}{4}$. Try $-1.5 \le x \le 2.5$ and $-1 \le y \le 3$.
29. Critical number at $x = 0$, which is a local min; inflection points at $x = \pm 1$; concave up on $(-1, 1)$, concave down elsewhere; try a window with $-3 \le x \le 3$ and $-1 \le y \le 3$.
31. Critical number at $x = -1$, which is a local min; inflection points at $x = -1 \pm \sqrt{3}$; concave up between those values, concave down elsewhere; try a window with $-6 \le x \le 6$ and $-1 \le y \le 4$.

Section 3.5

1. a. Using a rectangle with sides of length x and y as in the figure below:

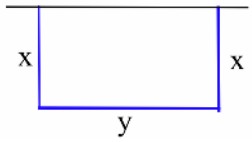


$2x + 2y = 200 \Rightarrow y = 100 - x$. We want to maximize $A = x \cdot y = x(100 - x) = 100x - x^2$ and $A'(x) = 100 - 2x$ so $A'(x) = 0 \Rightarrow x = 50 \Rightarrow y = 100 - x = 50$; $A''(x) = -2 < 0$ for all x, so $x = 50$ yields the maximum enclosed area. When $x = 50$, $A(50) = 50(100 - 50) = 2500$ square feet.

b. Using a rectangle with sides of length x and y (see figure above) $2x + 2y = P \Rightarrow y = \frac{P}{2} - x$. We want to maximize $A = x \cdot y = x\left(\frac{P}{2} - x\right) = \left(\frac{P}{2}\right)x - x^2$ and $A'(x) = \frac{P}{2} - 2x$ so $A'(x) = 0$ when $x = \frac{P}{4} \Rightarrow y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$; $A''(x) = -2 < 0$ for all x, so $x = \frac{P}{4}$ yields the maximum enclosed area. This garden is a $\frac{P}{4}$ by $\frac{P}{4}$ square.

https://math.libretexts.org/@go/page/211923

c. Using a rectangle with sides of length x and y, as in the figure below:

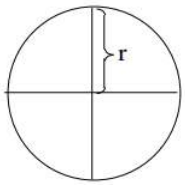


$2x + y = P \Rightarrow y = P - 2x$. We want to maximize $A = xy = x(P - 2x) = Px - 2x^2$ and $A'(x) = P - 4x$ so $A'(x) = 0$ when $x = \frac{P}{4} \Rightarrow y = P - 2\left(\frac{P}{4}\right) = \frac{P}{2}$.

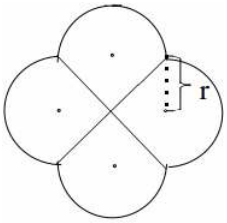
d. A circle. A semicircle.

3. a. If x is the horizontal length and y is the vertical length, then $120 = 2x + 5y \Rightarrow y = 24 - \frac{2}{5}x$. We want to maximize $A = xy = x\left(24 - \frac{2}{5}x\right) = 24x - \frac{2}{5}x^2$ and $A'(x) = 24 - \frac{4}{5}x$ so $A'(x) = 0$ when $x = 30$ (and $y = 12$); $A''(x) = -\frac{4}{5} < 0$ for all x so $x = 30$ yields the maximum enclosed area. Area is $(30\,\text{ft})(12\,\text{ft}) = 360$ square feet.

b. A circular pen divided into four equal stalls by two diameters:



does a better job than a square with area 400 square feet. If the radius is r, then $4r + 2\pi r = 120 \Rightarrow r = \frac{120}{4+2\pi} \approx 11.67$; the resulting enclosed area is $A = \pi r^2 \approx \pi(11.67)^2 \approx 427.8$ square feet. The pen shown here:



does even better. If each semicircle has radius r, then the figure uses $4\sqrt{2}r + 4\pi r = 120$ feet of fencing so $r = \frac{120}{4\sqrt{2}+4\pi} \approx 6.585$ feet; the resulting enclosed area is

$$\begin{aligned} A &= (\text{square}) + (\text{four semicircles}) \\ &= (2r)^2 + 4\left(\frac{1}{2}\pi r^2\right) \approx 445.90\,\text{ft}^2 \end{aligned}$$

5. The dimensions of the bottom of the box are $10 - 2x$ and $5 - x$ so we want to maximize $V(x) = x(10 - 2x)(5 - x) = 50x - 20x^2 + 2x^3$; $V'(x) = 50 - 40x + 6x^2 = 2(3x - 5)(x - 5)$ so $V'(x) = 0$ when $x = 5$ or $x = \frac{5}{3}$. When $x = 5$, then $V = 0$ (clearly not a maximum!) so $x = \frac{5}{3}$. The dimensions of the box with the largest volume are $\frac{5}{3}$ inches by $\frac{10}{3}$ inches by $\frac{20}{3}$ inches.

7. a. Let r and h be the radius and height of the cylindrical can, so the volume is $V = \pi r^2 h = 100 \Rightarrow h = \frac{100}{\pi r^2}$. We want to minimize the cost:

$$\begin{aligned} C &= 2(\text{top area}) + 5(\text{bottom area}) + 3(\text{side area}) \\ &= 2\left(\pi r^2\right) + 5\left(\pi r^2\right) + 3\left(2\pi r h\right) \\ &= 7\pi r^2 + 6\pi r\left(\frac{100}{\pi r^2}\right) = 7\pi r^2 + \frac{600}{r} \end{aligned}$$

So $C'(r) = 14\pi r - \frac{600}{r^2} \Rightarrow C'(r) = 0$ when $r = \sqrt[3]{\frac{600}{14\pi}} \approx 2.39 \Rightarrow h \approx \frac{100}{\pi(2.39)^2} \approx 5.57$.

b. Let k = top material cost + bottom material cost = 2¢ + bottom cost > 2¢ + 5¢ = 7¢. We want to minimize the cost $C = k\pi r^2 + \frac{600}{r}$; $C'(r) = 2k\pi r - \frac{600}{r^2}$ so $C'(r) = 0$ when $r = \sqrt[3]{\frac{600}{2k\pi}}$. If $k = 8$, then $r \approx 2.29$; if $k = 9$, then $r \approx 2.20$; if $k = 10$, then $r \approx 2.12$. As the cost of the bottom material increases, the radius of the least expensive cylindrical can decreases: the least expensive can becomes narrower and taller.

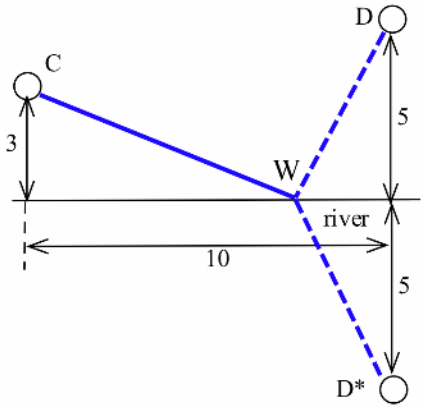
9. Recall that time $= \frac{\text{distance}}{\text{rate}}$. Let the run distance $= x$ (so $0 \le x \le 60$; why?), hence the run time $= \frac{x}{8}$ and the swim distance $= \sqrt{40^2 + (60 - x)^2}$ so the swim time $= \frac{1}{2}\sqrt{40^2 + (60 - x)^2}$ and the total time is $T(x) = \frac{x}{8} + \frac{1}{2}\sqrt{40^2 + (60 - x)^2}$. Differentiating

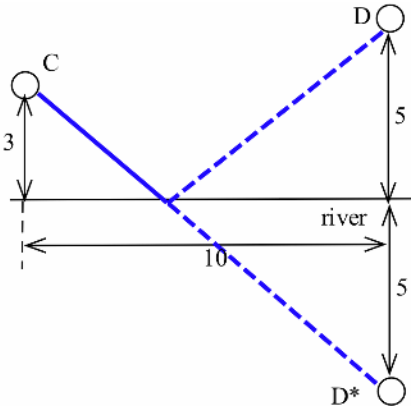
to compute $T'(x)$ yields:

$$\frac{1}{8}+\frac{1}{2}\cdot\frac{1}{2}\left(40^2+(60-x)^2\right)^{-\frac{1}{2}}\cdot 2(60-x)(-1)$$
$$=\frac{1}{8}-\frac{60-x}{2\sqrt{40^2+(60-x)^2}}$$

so $T'(x)=0$ when $x=60\pm\frac{40}{\sqrt{15}}$. The value $x=60+\frac{40}{\sqrt{15}}>60$ so the least total time occurs when $x=60-\frac{40}{\sqrt{15}}\approx 49.7$ meters. In this situation, the lifeguard should run about $\frac{5}{6}$ of the way along the beach before going into the water.

11. a. Consider a similar problem with a new town D^* located at the "mirror image" of D across the river (below left):





If the water works is built at any location W along the river, then the distances are the same from W to D and from W to D^*: $\text{dist}(W,D)=\text{dist}(W,D^*)$. Then $\text{dist}(C,W)+\text{dist}(W,D)=\text{dist}(C,W)+\text{dist}(W,D^*)$ so the shortest distance from C to D^* is a straight line (see figure above right), and this straight line yields similar triangles with equal side ratios: $\frac{x}{3}=\frac{10-x}{5}\Rightarrow x=\frac{15}{4}=3.75$ miles. A consequence of this "mirror image" view of the problem is that "at the best location W the angle of incidence α equals the angle of reflection β."

b. We want to minimize the cost:

$$C(x)=3000\text{dist}(C,W)+7000\text{dist}(W,D)$$
$$=3000\sqrt{x^2+9}+7000\sqrt{(10-x)^2+25}$$

so, differentiating gives:

$$C'(x)=\frac{3000x}{\sqrt{x^2+9}}+\frac{-7000(10-x)}{\sqrt{(10-x)^2+25}}$$

and (after much algebra) it turns out that $C'(x)=0$ when $x\approx 7.82$ miles. As it becomes more expensive to build the pipe from W to D, the cheapest route tends to shorten the distance from W to D.

13. a. Let x be the length of one edge of the square end. The volume is $V(x)=x^2(108-4x)=108x^2-4x^3\Rightarrow V'(x)=216x-12x^2=6x(18-x)$ so $V'(x)=0$ when $x=0$ or $x=18$. Clearly $x=0$ results in a box of no volume, so the dimensions of the acceptable box with a square end and greatest volume are 18 in by 18 in by 36 in; the volume is 11,664 in^3.

b. Let x be the length of the shorter edge of the end. The volume $V(x)=2x^2(108-6x)=216x^2-12x^3\Rightarrow V'(x)=432x-36x^2=36x(12-x)$ so $V'(x)=0$ if $x=0$ or $x=12$. Clearly $x=0$ results in a box of no volume, so the dimensions of the box with an acceptable shape and largest volume are 12 in by 24 in by 36 in; the volume is 10,368 in^3.

c. Let x be the radius of the circular end. The volume is $V(x)=\pi x^2(108-2\pi x)=108\pi x^2-2\pi^2x^3\Rightarrow V'(x)=216\pi x-6\pi^2x^2=6\pi x(36-\pi x)$ so $V'(x)=0$ when $x=0$ or $x=\frac{36}{\pi}\approx 11.46$ in. The dimensions of the acceptable box with circular end and largest volume are a radius of 11.46 in and a length of 36 in; the volume is 14,851 in^3.

15. **Without calculus**: The area of the triangle is $\frac{1}{2}(\text{base})(\text{height})=\frac{1}{2}(7)(\text{height})$ and the height is maximum when the angle between the sides is a right angle.
Using calculus: Let θ be the angle between the sides. Then the area of the triangle is $A=\frac{7}{2}(10\sin(\theta))=35\sin(\theta)\Rightarrow A'(\theta)=35\cos(\theta)$ so $A'(\theta)=0$ when $\theta=\frac{\pi}{2}$, hence the triangle is a right triangle with sides 7 and 10.

Using either approach, the maximum area of the triangle is $\frac{1}{2}(7)(10) = 35$ square inches, and the third side is the hypotenuse with length $\sqrt{7^2 + 10^2} = \sqrt{149} \approx 12.2$ inches.

17. a. $A(x) = 2x(16 - x^2) = 32x - 2x^3 \Rightarrow A'(x) = 32 - 6x^2$ so $A'(c) = 0$ when $x = \frac{32}{6} = \frac{16}{3} \approx 2.31$. The dimensions of the rectangle are $2\left(\frac{16}{3}\right) \approx 4.62$ and $16 - \left(\frac{16}{3}\right)^2 = \frac{32}{3} \approx 10.67$.

b. $A(x) = 2x\sqrt{1 - x^2}$ so, using the Product Rule:

$$A'(x) = 2\left(\sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}}\right)$$

and $A'(x) = 0$ when $x = \frac{1}{\sqrt{2}} \approx 0.707$. The dimensions of the rectangle are $2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \approx 1.414$ and $\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}} \approx 0.707$.

c. The graph of $|x| + |y| = 1$ is a "diamond" (a square) with corners at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. For $0 \le x \le 1$:

$$A(x) = 2x \cdot 2(1 - x) = 4x - 4x^2$$

so $A'(x) = 4 - 8x$ and $A'(x) = 0$ when $x = \frac{1}{2}$; $A''(x) = -8 < 0$ for all x, so $x = \frac{1}{2}$ must be a local max. The dimensions of the rectangle are $2\left(\frac{1}{2}\right) = 1$ and $2\left(1 - \frac{1}{2}\right) = 1$.

d. $A(x) = 2x\cos(x)$, where $0 \le x \le \frac{\pi}{2}$, so $A'(x) = 2\cos(x) - 2x \cdot sin(x)$ and $A'(x) = 0$ when $x \approx 0.86$. The dimensions of the rectangle are $2(0.86) = 1.72$ and $\cos(0.86) \approx 0.65$.

19. The cross-sectional area is:

$$\begin{aligned} A &= 6\sin\left(\frac{\theta}{2}\right) \cdot 6\cos\left(\frac{\theta}{2}\right) \\ &= 36 \cdot \frac{1}{2}\sin(\theta) = 18\sin(\theta) \end{aligned}$$

which is a maximum when $\theta = \frac{\pi}{2}$. Then the maximum area is $A = 18\sin\left(\frac{\pi}{2}\right) = 18$ in^2. (This problem is similar to Problem 15.)

21. $V = \frac{1}{3}\pi r^2 h$ and $h = \sqrt{9 - r^2}$ so:

$$V(r) = \frac{1}{3}\pi r^2\sqrt{9 - r^2} = \frac{\pi}{3}\sqrt{9r^4 - r^6}$$

Differentiating and simplifying:

$$V'(r) = \frac{\pi}{6} \cdot \frac{36r^3 - 6r^5}{\sqrt{9r^4 - r^6}}$$

so $V'(r) = 0$ when $36r^3 - 6r^5 = 6r^3(6 - r^2) = 0$; $r = 0$ results in zero volume, so $r = \sqrt{6} \approx 2.45$ in $\Rightarrow h = \sqrt{9 - (\sqrt{6})^2} = \sqrt{3} \approx 1.73$ in.

23. Let $n \ge 10$ be the number of passengers. The income is $I(n) = n(30 - (n - 10)) = 40n - n^2$ and the cost is $C(n) = 100 + 6n$, so the profit is $P(n) = I(n) - C(n) = (40n - n^2) - (100 + 6n) = 34n - n^2 - 100 \Rightarrow P'(n) = 34 - 2n$ so $P'(n) = 0$ when $n = 17$: 17 passengers on the flight maximize your profit. (This is an example of treating a naturally discrete variable, the number of passengers, as a continuous variable.)

25. Apply Problem 24 with $R = f$ and $E = g$.

27. a. Let D be diameter of the can's base and H = the can's height. Then:

$$\begin{aligned} \theta &= \arctan\left(\frac{\text{radius of can}}{\text{height of CG}}\right) \\ &= \arctan\left(\frac{D/2}{H/2}\right) = \arctan\left(\frac{D}{H}\right) \end{aligned}$$

For this can, $D = 5$ cm and $H = 12$ cm so $\theta = \arctan\left(\frac{2.5}{6}\right) \approx 0.395$ radians. The can can be tilted about $22.6°$ before it falls over.

b. Differentiating $C(x)$ yields:

$$\frac{(60 + 19.2x)(19.2x) - (360 + 9.6x^2)(19.2)}{(60 + 19.2x)^2}$$

so $C'(x) = 0 \Rightarrow (19.2)(9.6x^2 + 60x - 360) = 0 \Rightarrow x = 3.75$ or $x = -10$: the height of the cola is 3.75 cm.

c. $C(3.75) = 3.75$ (The center of gravity is exactly at the top edge of the cola. It turns out that when the CG of a can-and-liquid system is as low as possible then the CG is at the top edge of the liquid.) Then $\theta = \arctan\left(\frac{2.5}{3.75}\right) \approx 0.588$ radians: in this situation, the can can be tilted about $33.7°$ before it falls over.

d. Less far.

29. $A = (\text{base})(\text{height}) = (1-x)(x^2) = x^2 - x^3$ for $0 \le x \le 1$, so $A'(x) = 2x - 3x^2$ and $A'(x) = 0 \Rightarrow x = \frac{2}{3}$ (clearly the endpoints $x = 0$ and $x = 1$ will not yield the largest area). So $A\left(\frac{2}{3}\right) = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$.

30. a. $A(x) = (1-x)(Cx^2) = Cx^2 - Cx^3$ where $0 \le x \le 1$, so $A'(x) = 2Cx - 3Cx^2$ and $A'(x) = 0$ if $x = \frac{2}{3}$ (as before, neither endpoint works). So $A\left(\frac{2}{3}\right) = \frac{1}{3} \cdot C\left(\frac{2}{3}\right)^2 = \frac{4C}{27}$.

b. $A(x) = (B-x)(Cx^2) = BCx^2 - Cx^3$, where $0 \le x \le B$, so $A'(x) = 2BCx - 3Cx^2 = Cx(2B - 3x)$ and $A'(x) = 0$ if $x = \frac{2}{3}B$. So $A\left(\frac{2B}{C}\right) = \left(\frac{B}{3}\right)(C)\left(\frac{2B}{3}\right)^2 = \frac{4}{27}B^3C$.

31. a. Position the lower-left vertex of the triangle at $(0,0)$ so the hypotenuse sits on $y = 20 - \frac{20}{50}x$ and the area of the rectangle is $A = xy = x\left(20 - \frac{2}{5}x\right) = 20x - \frac{2}{5}x^2 \Rightarrow A'(x) = 20 - \frac{4}{5}x$ and $A'(x) = 0$ when $x = 25$. Then $y = 10$ and the area of the rectangle is $25 \cdot 10 = 250$.

b. Proceeding as in part (a), $y = H - \frac{H}{B}x$ and $A(x) = x\left(H - \frac{H}{B}x\right) = Hx - \frac{H}{B}x^2 \Rightarrow A'(x) = H - \frac{2H}{B}x$ and $A'(x) = 0$ when $x = \frac{B}{2}$. Then $y = \frac{H}{2}$ and area of rectangle is $\frac{B}{2} \cdot \frac{H}{2} = \frac{BH}{4}$.

33. Let r and h be radius and height of can, and $F = (\text{top cost}) + (\text{bottom cost}) + (\text{cost of sides}) = A(\pi r)^2 + B(\pi r)^2 + C(2\pi rh)$. But volume $V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$ so $F(r) = (A+B)\pi r^2 + \frac{2CV}{r} \Rightarrow F'(r)2(A+B)\pi r - \frac{2CV}{r^2}$, so $F'(r) = 0$ when $r = \sqrt[3]{\frac{CV}{\pi(A+B)}}$. Now you can find h and F.

Section 3.6

1. a. $h(x)$ has a root at $x = 1$.

b. $\lim_{x\to 1^+} h(x) = 0$, $\lim_{x\to 1^-} h(x) = 0$
$\lim_{x\to 3^+} h(x) = -\infty$, $\lim_{x\to 3^-} h(x) = +\infty$

c. $y = h(x)$ has a vertical asymptote at $x = 3$

3. $\lim_{x\to 2^+} h(x) = +\infty$, $\lim_{x\to 2^-} h(x) = -\infty$
$\lim_{x\to 4^+} h(x) = 0$, $\lim_{x\to 4^-} h(x) = 0$
5. 0
7. -3
9. 0
11. DNE
13. $\frac{2}{3}$
15. 0
17. -7
19. 3
21. $\cos(0) = 1$
23. $\ln(1) = 0$
25. a. $V(t) = 50 + 4t$ gals, $A(t) = 0.8t$ lbs of salt

b. $C(t) =$

c. as $t \to \infty$, $C(t) \to \frac{0.8}{4} = 0.2$ lbs per gal

d. $V(t) = 200 + 4t$ and $A(t) = 0.8t$, so $C(t) = \frac{0.8t}{200+4t} \to \frac{0.8}{4} = 0.2$ lbs per gal

27. $+\infty$
29. $-\infty$
31. $-\infty$
33. $-\infty$
35. $+\infty$
37. $-\infty$
39. 1

41. $-\infty$
43. horizontal: $y=0$; vertical: $x=0$
44. horizontal: $y=0$; vertical: $x=0$; "hole": $(1,1)$
45. horizontal: $y=0$; vertical: $x=3$ and $x=1$
47. vertical: $x=-1$
49. horizontal: $y=1$; vertical: $x=1$
51. slant: $y=2x+1$; vertical: $x=0$
53. other: $y=\sin(x)$; vertical: $x=2$
55. other: $y=x^2$
57. other: $y=\cos(x)$; vertical: $x=3$
59. $y=\sqrt{x}$; $x=-3$

Section 3.7

1. $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\lim\limits_{x\to 1}\dfrac{3x^2}{2x}=\lim\limits_{x\to 1}\dfrac{3}{2}x=\dfrac{3}{2}$
3. $\lim\limits_{x\to 0}\dfrac{\ln(1+3x)}{5x}=\lim\limits_{x\to 0}\dfrac{\frac{3}{1+3x}}{5}=\dfrac{3}{5}$
5. $\lim\limits_{x\to 0}\dfrac{x\cdot e^x}{1-e^x}=\lim\limits_{x\to 0}\dfrac{x\cdot e^x+e^x\cdot 1}{-e^x}=\lim\limits_{x\to 0}\dfrac{x+1}{-1}=-1$
7. $\lim\limits_{x\to\infty}\dfrac{\ln(x)}{x}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{1}=0$
9. $\lim\limits_{x\to\infty}\dfrac{\ln(x)}{x^p}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{px^{p-1}}=\lim\limits_{x\to\infty}\dfrac{1}{px^p}=0$
11. $\lim\limits_{x\to 0}\dfrac{1-\cos(3x)}{x^2}=\lim\limits_{x\to 0}\dfrac{3\sin(3x)}{2x}=\lim\limits_{x\to 0}\dfrac{9\cos(3x)}{2}=\dfrac{9}{2}$
13. For $a\neq 0$: $\dfrac{f'}{g'}=\dfrac{m}{n}x^{m-n}\to\dfrac{m}{n}a^{m-n}$. For $a=0$:

$$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\begin{cases}0 & \text{if } m>n\\ 1 & \text{if } m=n\\ +\infty & \text{if } m<n \text{ and } (m-n) \text{ is even}\\ \text{DNE} & \text{if } m<n \text{ and } (m-n) \text{ is odd}\end{cases}$$

15. $\lim\limits_{x\to 0}\dfrac{1-\cos(x)}{x\cdot\cos(x)}=\lim\limits_{x\to 0}\dfrac{\sin(x)}{-x\cdot\sin(x)+\cos(x)}=0$
17. $\dfrac{f'}{g'}=\dfrac{pe^{px}}{3}\to\dfrac{p}{3}$ so $p=3(5)=15$.
19. a. All three limits are $+\infty$.
    b. After applying L'Hôpital's Rule m times:

$$\frac{f^{(m)}}{g^{(m)}}=\frac{a\cdot b^m\cdot e^{bn}}{c(m)(m-1)(m-2)\cdots(2)(1)}=\frac{\text{constant}\cdot e^{bn}}{\text{another constant}}\to+\infty$$

21. $\lim\limits_{x\to 0^+}\dfrac{\ln(x)}{\csc(x)}=\lim\limits_{x\to 0^+}\dfrac{-\sin(x)}{x}\cdot\tan(x)=-1\cdot 0=0$
23. $\lim\limits_{x\to 0^+}\dfrac{\ln(x)}{x^{-\frac{1}{2}}}=\lim\limits_{x\to 0^+}\dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}=\lim\limits_{x\to 0^+}-2\sqrt{x}=0$
25. Write $\left(1-\dfrac{3}{x^2}\right)^x=e^{x\cdot\ln\left(1-\frac{3}{x^2}\right)}$ and compute:

$$\lim_{x\to\infty}\frac{\ln\left(1-\frac{3}{x^2}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{-6}{x-\frac{3}{x}}=0$$

so that the original limit equals $e^0=1$.
27. $\lim\limits_{x\to 0}\dfrac{\sin(x)-x}{x\cdot\sin(x)}=\lim\limits_{x\to 0}\dfrac{\cos(x)-1}{x\cdot\cos(x)+\sin(x)}$

$$=\lim_{x\to 0}\frac{-\sin(x)}{-x\cdot\sin(x)+2\cos(x)}=0$$

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29. Write $\left(\frac{x+5}{x}\right)^{\frac{1}{x}} = e^{\frac{\ln\left(\frac{x+5}{x}\right)}{x}}$ and compute:

$$\lim_{x\to\infty} \frac{\ln\left(\frac{x+5}{x}\right)}{x} = \lim_{x\to\infty} \frac{-5}{x(x+5)} = 0$$

so that the original limit equals $e^0 = 1$.

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CHAPTER OVERVIEW

4: The Integral

Previous chapters dealt with **differential calculus**. They started with the “simple” geometrical idea of the slope of a tangent line to a curve, developed it into a combination of theory about derivatives and their properties, examined techniques for calculating derivatives, and applied these concepts and techniques to real-life situations. This chapter begins the development of integral calculus and starts with the “simple” geometric idea of area— an idea that will spawn its own combination of theory, techniques and applications.

One of the most important results in mathematics, the Fundamental Theorem of Calculus, appears in this chapter. It unifies differential and integral calculus into a single grand structure. Historically, this unification marked the beginning of modern mathematics, and it provided important tools for the growth and development of the sciences.

The chapter begins with a look at area, some geometric properties of areas, and some applications. First we will examine ways of approximating the areas of regions such as tree leaves bounded by curved edges and the areas of regions bounded by graphs of functions. Then we will develop ways to calculate the areas of some of these regions exactly. Finally, we will explore the rich variety of uses of “areas.”

4.0: Area
4.1: Sigma Notation and Riemann Sums
4.2: The Definite Integral
4.3: Properties of the Definite Integral
4.4: Areas, Integrals and Antiderivatives
4.5: The Fundamental Theorem of Calculus
4.6: Finding Antiderivatives
4.7: First Applications of Definite Integrals
4.8: Using Tables [and Technology] to Find Antiderivatives
4.9: Approximating Definite Integrals
4.A: Answers

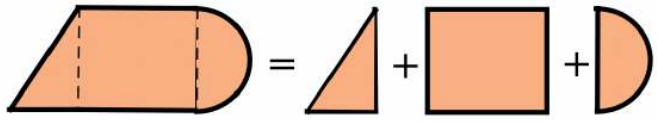
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4.0: Area

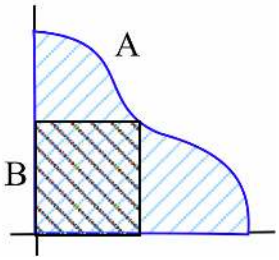
The primary purpose of this introductory section is to help develop your intuition about areas and your ability to reason using geometric arguments about area. This type of reasoning will appear often in the rest of this book and is very helpful for applying the ideas of calculus.

The basic shape we will use is the rectangle: the area of a rectangle is $(\text{base}) \cdot (\text{height})$. If the units for each side of the rectangle are "meters," then the area will have units $(\text{meters}) \cdot (\text{meters}) =$ "square meters" $= \text{m}^2$. The only other area formulas needed for this section are for triangles (area $= \frac{1}{2} b \cdot h$) and for circles (area $= \pi \cdot r^2$). In addition, we will use (and assume to be true) three other familiar properties of area:

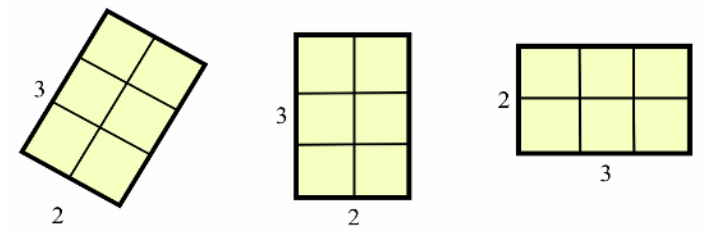
- **Addition Property**: The total area of a region is the sum of the areas of the non-overlapping pieces that comprise the region:



- **Inclusion Property**: If region B is inside region A (see margin), then the area of region B is less than or equal to the area of region A:

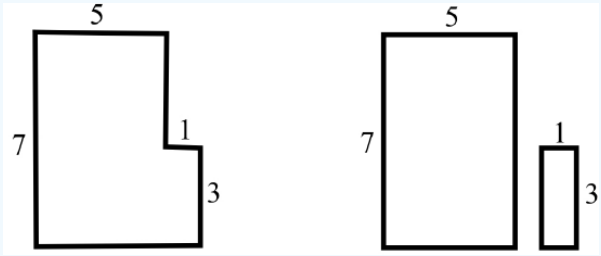


- **Location-Independence Property**: The area of a region does not depend on its location:



Example 4.0.1

Determine the area of the region shown below left.

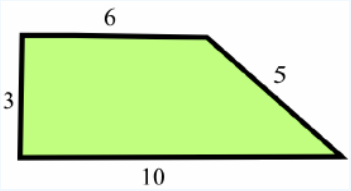


Solution

We can easily break the region into two rectangles (shown above right), with areas of 35 square inches and 3 square inches respectively, so the area of the original region is 38 square inches.

Practice 4.0.1

Determine the area of the trapezoidal region shown below:



by cutting it in two ways:

a. into a rectangle and triangle and
b. into two triangles.

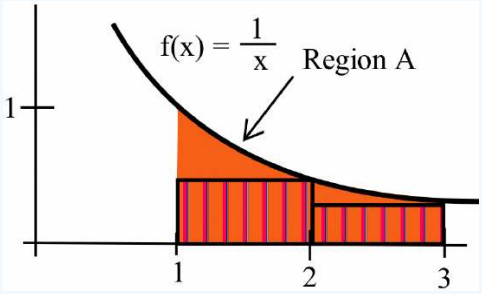
Answer

a. $3(6)+\frac{1}{2}(4)(3)=24$
b. $\frac{1}{2}(3)(10)+\frac{1}{2}(6)(3)=24$

We can use our three area properties to deduce information about areas that are difficult to calculate exactly. Let A be the region bounded by the graph of $f(x)=\frac{1}{x}$, the x-axis, and the vertical lines $x=1$ and $x=3$. Because the two rectangles in the margin figure sit inside region A and do not overlap each other, the area of the rectangles, $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, is less than the area of region A.

Practice 4.0.2

Build two rectangles, each with base 1 unit, with boundaries that extend outside the shaded region in the figure below:



and use their areas to make another valid statement about the area of region A.

Answer

outside rectangular area $=(1)(1)+(1)\left(\frac{1}{2}\right)=1.5$

Practice 4.0.3

What can you say about the area of region A from Practice 4.0.2 if we use "inside" and "outside" rectangles each with base $\frac{1}{2}$ unit?

Answer

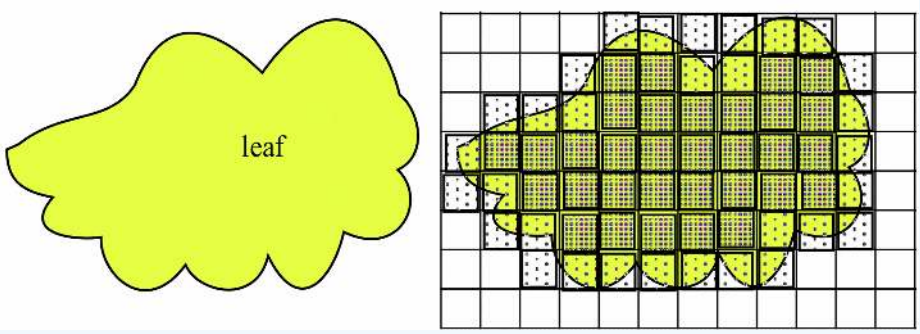
Using rectangles with base $=\frac{1}{2}$:

$$\text{inside area}=\frac{1}{2}\left(\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}\right)=\frac{57}{60}\approx 0.95$$
$$\text{outside area}=\frac{1}{2}\left(1+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}\right)=\frac{72}{60}=1.2$$

so the area of the region is between 0.95 and 1.2.

Example 4.0.2

The figure below right includes 32 dark squares, each 1 centimeter on a side, and 31 lighter squares of the same size:



We can be sure that the area of the leaf below left is smaller than what number?

Solution

The area of the leaf is smaller than $32 + 31 = 63\,\text{cm}^2$.

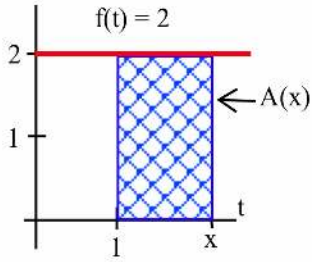
Practice 4.0.4

We can be sure that the area of the leaf from Example $4.0.2$ is at least how large?

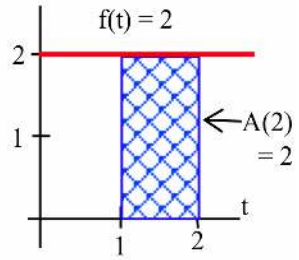
Answer

The leaf's area is larger than the area of the dark rectangles, $32\,\text{cm}^2$.

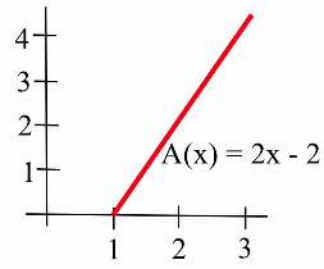
Functions can be defined in terms of areas. For the constant function $f(t) = 2$, define $A(x)$ to be the area of the rectangular region:



bounded by the graph of f, the t-axis, and the vertical lines at $t = 1$ and $t = x$; we can easily see that $A(2) = 2$, as indicated by the shaded region:



Similarly, $A(3) = 4$ and $A(4) = 6$. In general, $A(x) = (\text{base})(\text{height}) = (x - 1)(2) = 2x - 2$ for any $x \geq 1$. From the graph of $y = A(x)$:

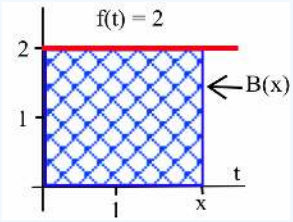


we can see that $A'(x) = 2$ for every value of $x > 1$.

(The fact that $A'(x) = f(x)$ in the preceding discussion is not a coincidence, as we shall soon learn.)

? Practice 4.0.5

For $f(t) = 2$, define $B(x)$ to be the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = 0$ and $t = x$:



Fill in the table below with the requested values of B. How are the graphs of $y = A(x)$ and $y = B(x)$ related?

x	$B(x)$
0	
0.5	
1	
2	

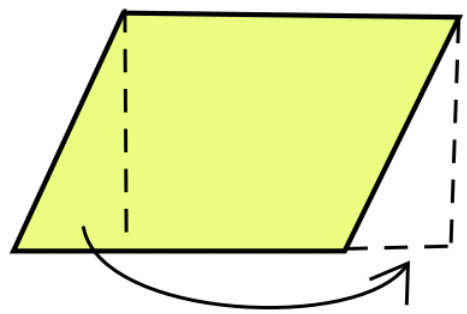
Answer

$y = B(x) = 2x$ is a line with slope 2, so it is parallel to the line $y = A(x) = 2x - 2$.

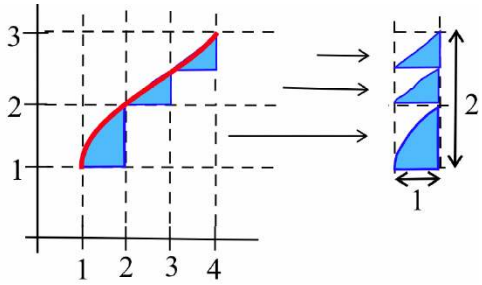
version of previous table with values of B filled in

x	$B(x)$
0	0
0.5	1
1	2
2	4

Sometimes it is useful to move regions around. The area of a parallelogram is obvious if we move the triangular region from one side of the parallelogram to fill the region on the other side, resulting in with a rectangle:



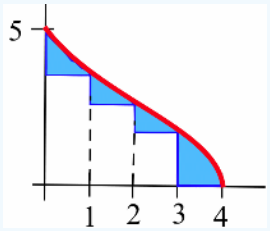
At first glance, it is difficult to estimate the total area of the shaded regions shown below left:



but if we slide all of them into a single column (above right), then becomes easy to determine that the shaded area is less than the area of the enclosing rectangle $= (\text{base})(\text{height}) = (1)(2) = 2$.

? Practice 4.0.6

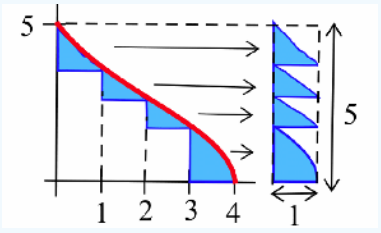
The total area of the shaded regions in the figure below:



is less than what number?

Answer

Area $<$ area of the rectangle enclosing the shifted regions $= 5$:

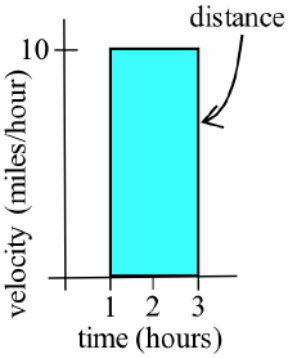


Some Applications of "Area"

One reason "areas" are so useful is that they can represent quantities other than sizes of simple geometric shapes. For example, if the units of the base of a rectangle are "hours" and the units of the height are "$\frac{\text{miles}}{\text{hour}}$," then the units of the "area" of the rectangle are:

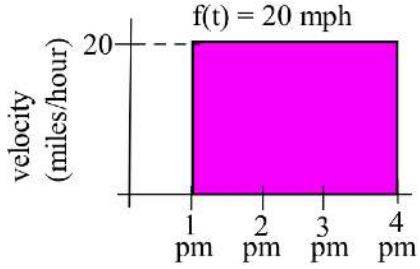
$$(\text{hours}) \cdot \left(\frac{\text{miles}}{\text{hour}}\right) = \text{miles}$$

a measure of distance:



Similarly, if the base units are "pounds" and the height units are "feet," then the "area" units are "foot-pounds," a measure of work.

In the figure below, $f(t)$ is the velocity of a car in "miles per hour," and t is the time in "hours":



So the shaded "area" will be $(\text{base}) \cdot (\text{height}) = (3 \text{ hours}) \cdot \left(20 \frac{\text{miles}}{\text{hour}}\right) = 60$ miles, the distance traveled by the car in the 3 hours from 1:00 p.m. until 4:00 p.m.

Distance as an "Area"

If $f(t)$ is the (positive) forward velocity of an object at time t, then the "area" between the graph of f and the t-axis and the vertical lines at times $t = a$ and $t = b$ will equal the distance that the object has moved forward between times a and b.

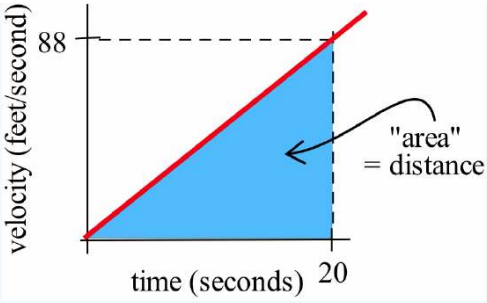
This "area as distance" concept can make some difficult distance problems much easier.

Example 4.0.3

A car starts from rest (velocity $= 0$) and steadily speeds up so that 20 seconds later its speed is 88 feet per second (60 miles per hour). How far did the car travel during those 20 seconds?

Solution

We could answer the question using the techniques of Chapter 3 (try this). But if "steadily" means that the velocity increases linearly, then it is easier to use the figure below:



and the concept of “area as distance.”

The “area” of the triangular region represents the distance traveled:

$$\text{distance} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(20\,\text{sec})\left(88\,\frac{\text{ft}}{\text{sec}}\right) = 880\,\text{ft}$$

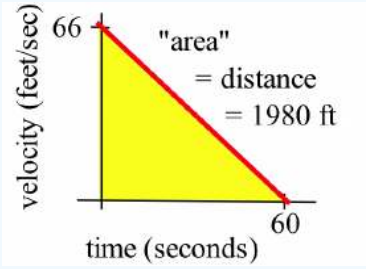
The car travels a total of 880 feet during those 20 seconds.

Practice 4.0.7

A train initially traveling at 45 miles per hour (66 feet per second) takes 60 seconds to decelerate to a complete stop. If the train slowed down at a steady rate (the velocity decreased linearly), how many feet did the train travel before coming to a stop?

Answer

Draw a graph of the velocity function:

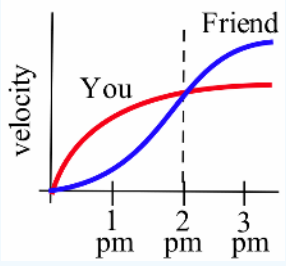


and then use the concept of “area as distance”:

$$\begin{aligned}\text{distance} &= \text{area of shaded region}\\ &= \frac{1}{2}(\text{base})(\text{height})\\ &= \frac{1}{2}(60\,\text{sec})\left(66\,\frac{\text{ft}}{\text{sec}}\right) = 1980\,\text{feet}\end{aligned}$$

Practice 4.0.8

You and a friend start off at noon and walk in the same direction along the same path at the rates shown below:



a. Who is walking faster at 2:00 p.m.? Who is ahead at 2:00 p.m.?

b. Who is walking faster at 3:00 p.m.? Who is ahead at 3:00 p.m.?
c. When will you and your friend be together? (Answer in words.)

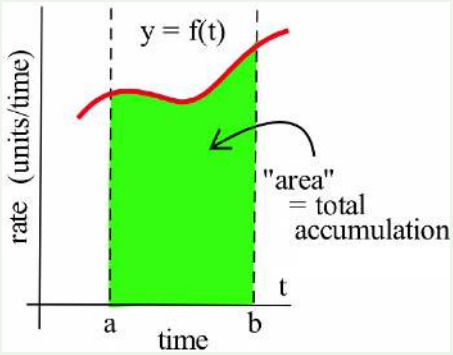
Answer

a. At 2:00 p.m. both are walking at the same velocity. You are ahead.
b. At 3:00 p.m. your friend is walking faster than you, but you are still ahead. (The "area" under your velocity curve is larger than the "area" under your friend's.)
c. You and your friend will be together on the trail when the "areas" (distances) under the two velocity graphs are equal.

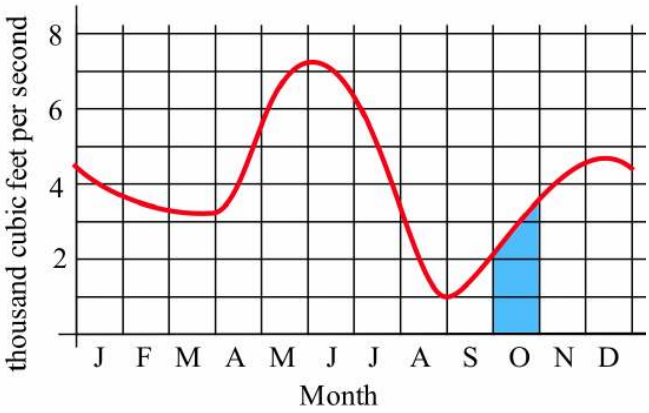
In the preceding Example and Practice problems, a function represented a rate of travel (in miles per hour, for instance) and the area represented the total distance traveled. For functions representing other rates, such as the production of a factory (bicycles per day) or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute) or the spread of a disease (newly sick people per week), the area will still represent the total amount of something.

"Area" as a Total Accumulation

If $f(t)$ represents a positive rate (in units per time interval) at time t, then the "area" between the graph of f and the t-axis and the vertical lines at times $t = a$ and $t = b$ will be the total amount of {something} that accumulates between times a and b:



For example, the figure below shows the flow rate (in cubic feet per second) of water in the Skykomish River near the town of Gold Bar, Washington:



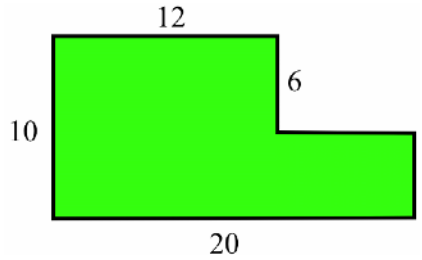
The area of the shaded region represents the total volume (cubic feet) of water flowing past the town during the month of October:

$$\begin{aligned}\text{total water} = \text{“area”} &= \text{area of rectangle} + \text{area of triangle} \\ &\approx \left(2000\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) + \frac{1}{2}\left(1500\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) \\ &= \left(2750\,\frac{\text{ft}^3}{\text{sec}}\right)(30\,\text{days}) = \left(2750\,\frac{\text{ft}^3}{\text{sec}}\right)(2592000\,\text{sec}) \\ &\approx 7.128 \times 10^9\,\text{ft}^3\end{aligned}$$

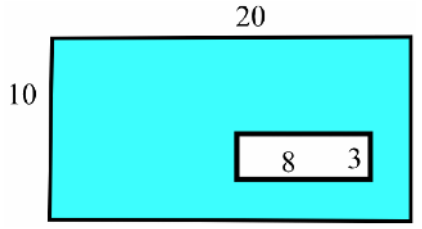
For comparison, the flow over Niagara Falls is about $2.12 \times 10^5\,\frac{\text{ft}^3}{\text{sec}}$.

Problems

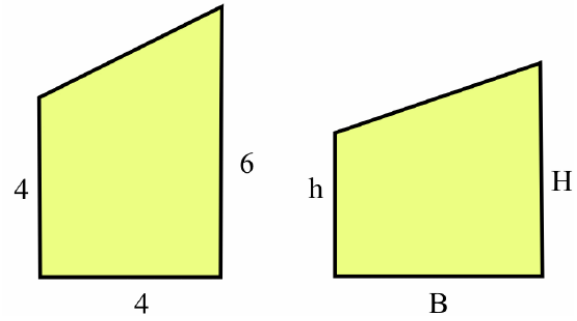
1. a. Calculate the area of the shaded region:



b. Calculate the area of the shaded region:

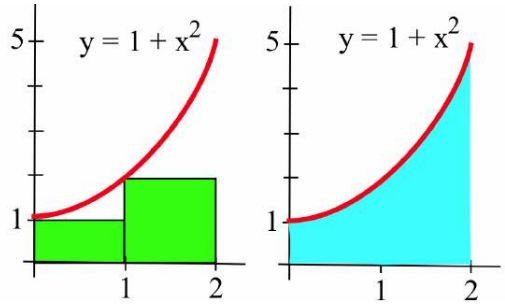


2. Calculate the area of the trapezoidal region in the figure below left by breaking it into a triangle and a rectangle.



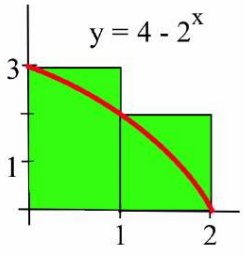
3. Break the region shown above right into a triangle and rectangle and verify that the total area of the trapezoid is $b \cdot \left(\frac{h+H}{2}\right)$.

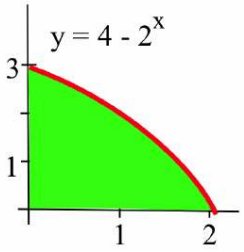
4. a. Calculate the sum of the rectangular areas in the region shown below left.



b. What can you say about the area of the shaded region shown above right?

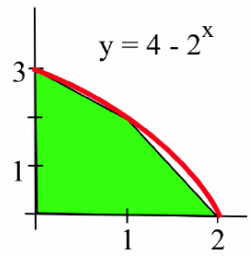
5. a. Calculate the sum of the areas of the rectangles shown below left.

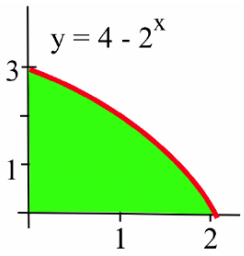




   b. What can you say about the area of the shaded region shown above right?

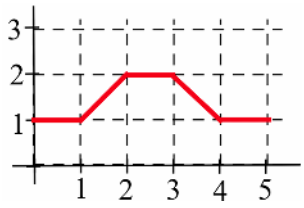
6. a. Calculate the sum of the areas of the trapezoids shown below left.





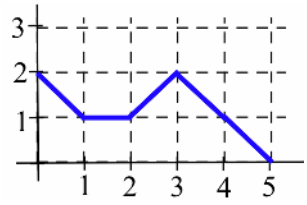
   b. What can you say about the area of the shaded region shown above right?

7. Consider the region bounded by the graph of $y = 2 + x^3$, the positive x-axis, the positive y-axis and the line $x = 2$. Use two well-placed rectangles to estimate the area of this region.
8. Consider the region bounded by the graph of $y = 9 - 3^x$, the positive x-axis and the positive y-axis. Use two well-placed trapezoids to estimate the area of this region.
9. Let $A(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$:



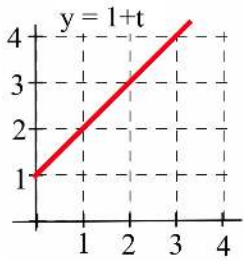
   Evaluate $A(x)$ for $x = 1, 2, 3, 4$ and 5.

10. Let $B(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$:



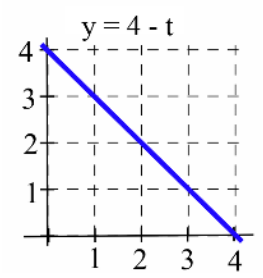
    Evaluate $B(x)$ for $x = 1, 2, 3, 4$ and 5.

11. Let $C(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$:



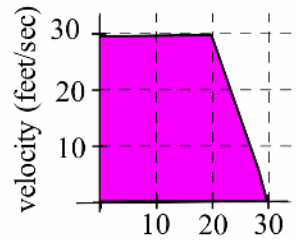
Evaluate $C(x)$ for $x = 1$, 2 and 3, and use that information to deduce a formula for $C(x)$.

12. Let $A(x)$ represent the area bounded by the graph of the function shown below, the horizontal axis, and vertical lines at $t = 0$ and $t = x$:



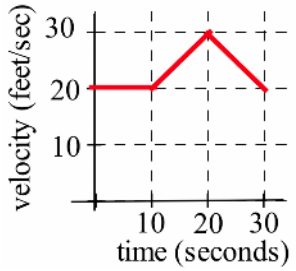
Evaluate $A(x)$ for $x = 1$, 2 and 3, and find a formula for $A(x)$.

13. The figure below shows the velocity of a car during a 30-second time frame:



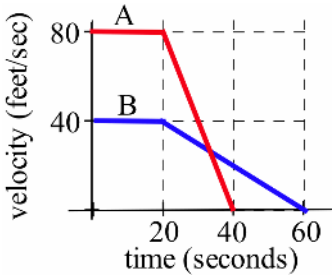
How far did the car travel between $t = 0$ to $t = 30$ seconds?

14. The figure below shows the velocity of a car during a 30-second time frame:



How far did the car travel between $t = 0$ to $t = 30$ seconds?

15. The figure below shows the velocities of two cars:

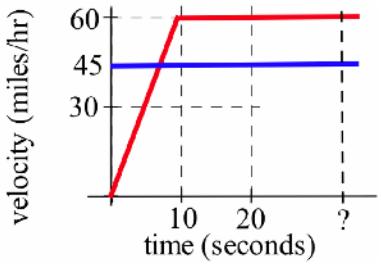


From the time the brakes were applied:

a. how long did it take each car to stop?

b. which car traveled farther before stopping?

16. A speeder traveling 45 miles per hour (in a 25-mph zone) passes a stopped police car, which immediately takes off after the speeder. If the police car speeds up steadily to 60 mph over a 10-second interval and then travels at a constant 60 mph, how long—and how far—will it be before the police car catches the speeder, who continued traveling at 45 mph?



17. Fill in the table with the units for “area” of a rectangle with the given base and height units.

base	**height**	**“area”**
miles per second	seconds	
hours	dollars per hour	
square feet	feet	
kilowatts	hours	
houses	people per house	
meals	meals	

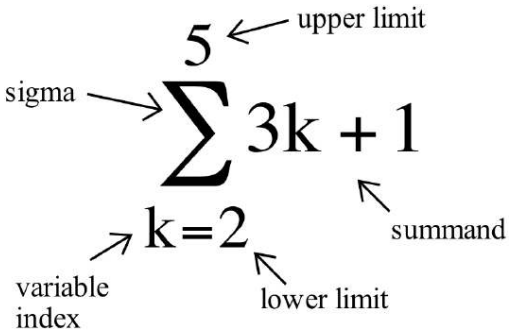
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4.1: Sigma Notation and Riemann Sums

One strategy for calculating the area of a region is to cut the region into simple shapes, calculate the area of each simple shape, and then add these smaller areas together to get the area of the whole region. When you use this approach with many sub-regions, it will be useful to have a notation for adding many values together: the sigma (Σ) notation.

summation	sigma notation	how to read it
$1^2+2^2+3^2+4^2+5^2$	$\sum_{k=1}^{5} k^2$	the sum of k squared, from k equals 1 to k equals 5
$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$	$\sum_{k=3}^{7} \frac{1}{k}$	the sum of 1 divided by k, from k equals 3 to k equals 7
$2^0+2^1+2^2+2^3+2^4+2^5$	$\sum_{j=0}^{5} 2^j$	the sum of 2 to the j-th power, from j equals 0 to j equals 5
$a_2+a_3+a_4+a_5+a_6+a_7$	$\sum_{n=2}^{7} a_n$	the sum of a sub n, from n equals 2 to n equals 7

The variable (typically i, j, k, m or n) used in the summation is called the **counter** or **index** variable. The function to the right of the sigma is called the **summand**, while the numbers below and above the sigma are called the **lower** and **upper limits** of the summation.



? Practice 4.1.1

Write the summation denoted by each of the following:

a. $\sum_{k=1}^{5} k^3$

b. $\sum_{j=2}^{7} (-1)^j \frac{1}{j}$

c. $\sum_{m=0}^{4} (2m+1)$

Answer

a. $\sum_{k=1}^{5} k^3 = 1+8+27+64+125$

b. $\sum_{j=2}^{7} (-1)^j \cdot \frac{1}{j} = \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}$

c. $\sum_{m=0}^{4} (2m+1) = 1+3+5+7+9$

In practice, we frequently use sigma notation together with the standard function notation:

$$\sum_{k=1}^{3} f(k+2) = f(1+2) + f(2+2) + f(3+2)$$
$$= f(3) + f(4) + f(5)$$
$$\sum_{j=1}^{4} f(x_j) = f(x_1) + f(x_2) + f(x_3) + f(x_4)$$

Example 4.1.1

Use the table:

x	$f(x)$	$g(x)$	$h(x)$
1	2	4	3
2	3	1	3
3	1	−2	3
4	0	3	3
5	3	5	3

to evaluate $\sum_{k=2}^{5} 2 \cdot f(k)$ and $\sum_{j=3}^{5} [5 + f(j-2)]$.

Solution

Writing out the sum and using the table values:

$$\sum_{k=2}^{5} 2 \cdot f(k) = 2 \cdot f(2) + 2 \cdot f(3) + 2 \cdot f(4) + 2 \cdot f(5)$$
$$= 2 \cdot 3 + 2 \cdot 1 + 2 \cdot 0 + 2 \cdot 3 = 14$$

while:

$$\sum_{j=3}^{5} [5 + f(j-2)] = [5 + f(3-2)] + [5 + f(4-2)] + [5 + f(5-2)]$$
$$= [5 + f(1)] + [5 + f(2)] + [5 + f(3)]$$
$$= [5 + 2] + [5 + 3] + [5 + 1]$$

which adds up to **21**.

Practice 4.1.2

Use the values in the preceding the table from Example **4.1.1** to evaluate:

a. $\sum_{k=2}^{5} g(k)$

b. $\sum_{j=1}^{4} h(j)$

c. $\sum_{k=3}^{5} [g(k) + f(k-1)]$

Answer

a. $\sum_{k=2}^{5} g(k) = g(2) + g(3) + g(4) + g(5) = 1 + (-2) + 3 + 5 = 7$

b. $\sum_{j=1}^{4} h(j) = h(1) + h(2) + h(3) + h(4) = 3 + 3 + 3 + 3 = 12$

c. $\sum_{k=3}^{5} (g(k) + f(k-1)) = (g(3) + f(2)) + (g(4) + f(3)) + (g(5) + f(4)) = (-2 + 3) + (3 + 1) + (5 + 0) = 10$

Example 4.1.2

For $f(x) = x^2 + 1$, evaluate $\sum_{k=0}^{3} f(k)$.

Solution

Writing out the sum and using the function values:

$$\begin{aligned}\sum_{k=0}^{3} f(k) &= f(0) + f(1) + f(2) + f(3) \\ &= (0^2 + 1) + (1^2 + 1) + (2^2 + 1) + (3^2 + 1) \\ &= 1 + 2 + 5 + 10\end{aligned}$$

which adds up to 18.

Practice 4.1.3

For $g(x) = \frac{1}{x}$, evaluate $\sum_{k=2}^{4} g(k)$ and $\sum_{k=1}^{3} g(k+1)$.

Answer

$\sum_{k=2}^{4} g(k) = g(2) + g(3) + g(4) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$

$\sum_{k=1}^{3} g(k+1) = g(2) + g(3) + g(4) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$

The summand need not contain the index variable explicitly: you can write a sum from $k = 2$ to $k = 4$ of the constant function $f(k) = 5$ as $\sum_{k=2}^{4} f(k)$ or $\sum_{k=2}^{4} 5 = 5 + 5 + 5 = 3 \cdot 5 = 15$. Similarly:

$$\sum_{k=3}^{7} 2 = 2 + 2 + 2 + 2 + 2 = 5 \cdot 2 = 10$$

Because the sigma notation is simply a notation for addition, it possesses all of the familiar properties of addition.

Summation Properties

Sum of Constants: $\sum_{k=1}^{n} C = C + C + C + \cdots + C = n \cdot C$

Addition: $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Subtraction: $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Constant Multiple: $\sum_{k=1}^{n} C \cdot a_k = C \cdot \sum_{k=1}^{n} a_k$

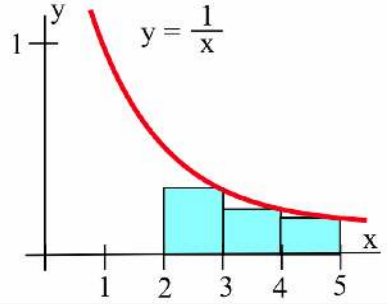
(Problems 16 and 17 illustrate that similar patterns for sums of products and quotients are *not* valid.)

Sums of Areas of Rectangles

In Section 4.2, we will approximate areas under curves by building rectangles as high as the curve, calculating the area of each rectangle, and then adding the rectangular areas together.

Example 4.1.3

Evaluate the sum of the rectangular areas in the figure below:



then write the sum using sigma notation.

Solution

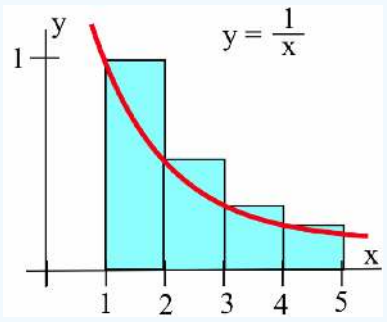
The sum of the rectangular areas is equal to the sum of $(\text{base}) \cdot (\text{height})$ for each rectangle:

$$(1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{5}\right) = \frac{47}{60}$$

which we can rewrite as $\sum_{k=3}^{5} \frac{1}{k}$ using sigma notation.

Practice 4.1.4

Evaluate the sum of the rectangular areas in the figure below:

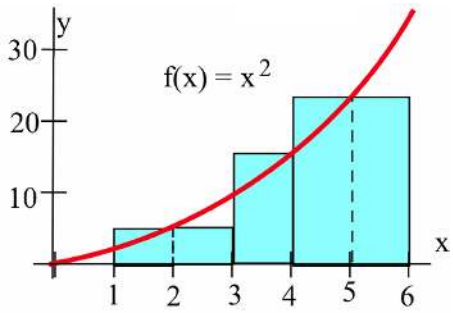


then write the sum using sigma notation.

Answer

$$\text{Rectangular areas} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = \sum_{j=1}^{4} \frac{1}{j}$$

The bases of these rectangles need not be equal. For the rectangular areas associated with $f(x) = x^2$ in the figure below:



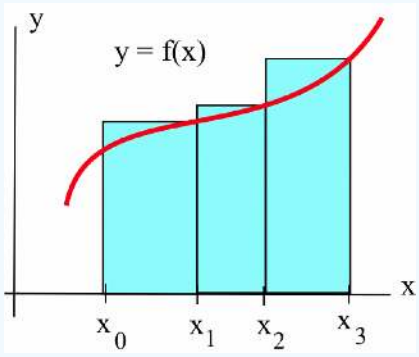
we have:

rectangle	base	height	area
1	$3-1=2$	$f(2)=4$	$2\cdot 4=8$
2	$4-3=1$	$f(4)=16$	$1\cdot 16=16$
3	$6-4=2$	$f(5)=25$	$2\cdot 25=50$

so the sum of the rectangular areas is $8+16+50=74$.

Example 4.1.4

Write the sum of the areas of the rectangles in the figure below:



using sigma notation.

Solution

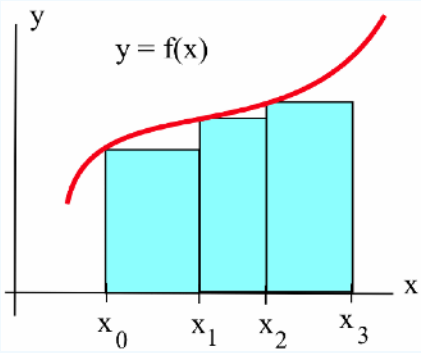
The area of each rectangle is (base) · (height):

rectangle	base	height	area
1	x_1-x_0	$f(x_1)$	$(x_1-x_0)\cdot f(x_1)$
2	x_2-x_1	$f(x_2)$	$(x_2-x_1)\cdot f(x_2)$
3	x_3-x_2	$f(x_3)$	$(x_3-x_2)\cdot f(x_3)$

The area of the k-th rectangle is $(x_k-x_{k-1})\cdot f(x_k)$, so we can express the total area of the three rectangles as $\sum_{k=1}^{3}(x_k-x_{k-1})\cdot f(x_k)$.

Practice 4.1.5

Write the sum of the areas of the shaded rectangles in the figure below:



using sigma notation.

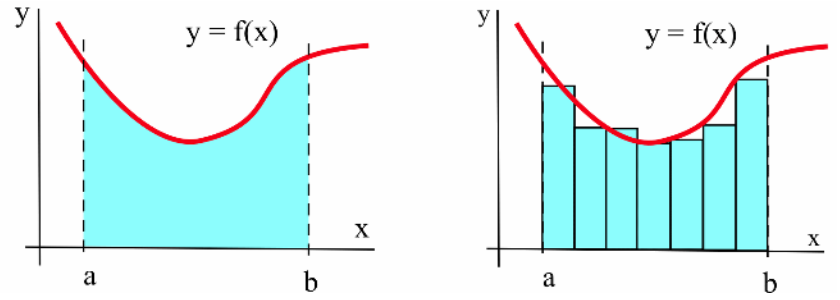
Answer

$$f(x_0)\cdot(x_1-x_0)+f(x_1)\cdot(x_2-x_1)+f(x_2)\cdot(x_3-x_2)=\sum_{j=1}^{3} f(x_{j-1})\cdot(x_j-x_{j-1})$$

$$\sum_{k=0}^{2} f(x_k)\cdot(x_{k+1}-x_k)$$

Area Under a Curve: Riemann Sums

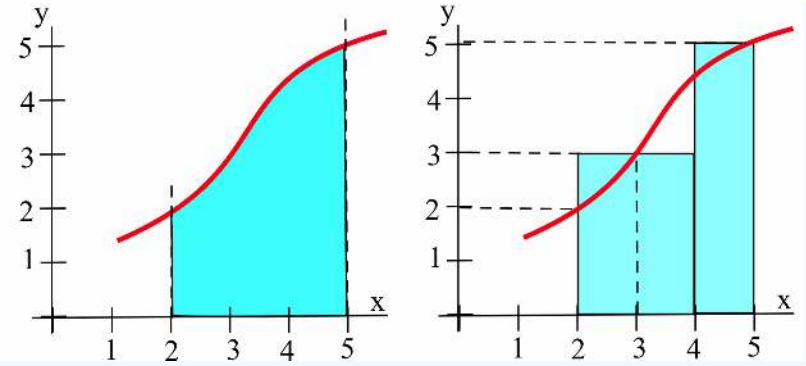
Suppose we want to calculate the area between the graph of a positive function f and the x-axis on the interval $[a,b]$, as shown below left:



One method to approximate the area involves building several rectangles with bases on the x-axis spanning the interval $[a,b]$ and with sides that reach up to the graph of f (as shown above right). We then compute the areas of the rectangles and add them up to get a number called a **Riemann sum** of f on $[a,b]$. The area of the region formed by the rectangles provides an approximation of the area we want to compute.

Example 4.1.5

Approximate the area shown below left between the graph of f and the x-axis spanning the interval $[2,5]$:



by summing the areas of the rectangles shown in the figure above right.

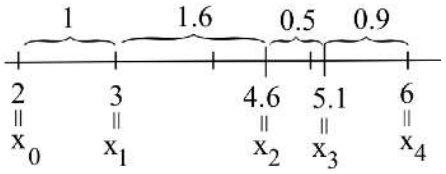
Solution

The total area is $(2)(3)+(1)(5)=11$ square units.

In order to effectively describe this process, some new vocabulary is helpful: a **partition** of an interval and the **mesh** of a partition.

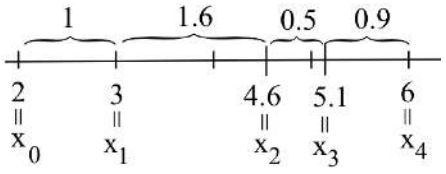
A **partition** $\mathcal{P}$ of a closed interval $[a,b]$ into n subintervals consists of a set of $n+1$ points $\{x_0=a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n=b\}$ listed in increasing order, so that $a=x_0<x_1<x_2<x_3<\ldots<x_{n-1}<x_n=b$. (A partition is merely a collection of points on the horizontal axis, unrelated to the function f in any way.)

The points of the partition $\mathcal{P}$ divide $[a,b]$ into n subintervals:



These intervals are $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$,…, $[x_{n-1},x_n]$ with lengths $\Delta x_1=x_1-x_0$, $\Delta x_2=x_2-x_1$, $\Delta x_3=x_3-x_2$, …, $\Delta x_n=x_n-x_{n-1}$. The points x_k of the partition $\mathcal{P}$ mark the locations of the vertical lines for the sides of the rectangles, and the bases of the rectangles have lengths Δx_k for $k=1, 2, 3,\ldots, n$. The **mesh** or **norm** of a partition $\mathcal{P}$ is the length of the longest of the subintervals $[x_{k-1},x_k]$ or, equivalently, the maximum value of Δx_k for $k=1, 2, 3,\ldots, n$.

For example, the set $\mathcal{P}=\{2,3,4.6,5.1,6\}$ is a partition of the interval $[2,6]$:



that divides the interval $[2,6]$ into four subintervals with lengths $\Delta x_1=1$, $\Delta x_2=1.6$, $\Delta x_3=0.5$ and $\Delta x_4=0.9$, so the mesh of this partition is 1.6, the maximum of the lengths of the subintervals. (If the mesh of a partition is "small," then the length of each one of the subintervals is the same or smaller.)

? Practice 4.1.6

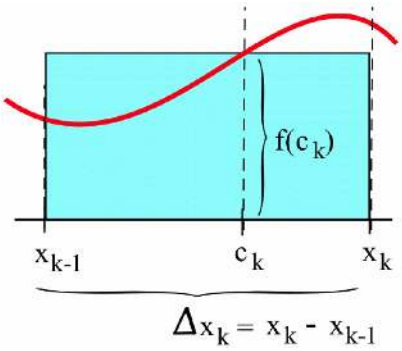
$\mathcal{P}=\{3, 3.8, 4.8, 5.3, 6.5, 7, 8\}$ is a partition of what interval? How many subintervals does it create? What is the mesh of the partition? What are the values of x_2 and Δx_2?

Answer

Interval is $[3,8]$; six subintervals; mesh $=1.2$; $x_2=4.8$; $\Delta x_2=x_2-x_1=4.8-3.8=1$.

A function, a partition and a point chosen from each subinterval determine a **Riemann sum**. Suppose f is a positive function on the interval $[a,b]$ (so that $f(x) > 0$ when $a \le x \le b$), $\mathcal{P} = \{x_0 = a, x_1, x_2, x_3, \dots, x_{n-1}, x_n = b\}$ is a partition of $[a,b]$, and c_k is an x-value chosen from the k-th subinterval $[x_{k-1}, x_k]$ (so $x_{k-1} \le c_k \le x_k$). Then the area of the k-th rectangle is:

$$f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k$$



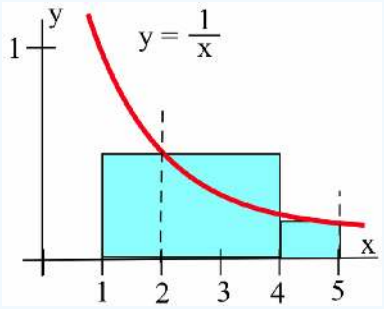
Definition

A summation of the form $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called a **Riemann sum** of f for the partition $\mathcal{P}$ and the chosen points $\{c_1, c_2, \dots, c_n\}$.

This Riemann sum is the total of the areas of the rectangular regions and provides an approximation of the area between the graph of f and the x-axis on the interval $[a,b]$.

Example 4.1.6

Find the Riemann sum for $f(x) = \frac{1}{x}$ using the partition $\{1,4,5\}$ and the values $c_1 = 2$ and $c_2 = 5$:



Solution

The two subintervals are $[1,4]$ and $[4,5]$, hence $\Delta x_1 = 3$ and $\Delta x_2 = 1$. So the Riemann sum for this partition is:

$$\sum_{k=1}^{2} f(c_k) \cdot \Delta x_k = f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2$$
$$= f(2) \cdot 3 + f(5) \cdot 1 = \frac{1}{2} \cdot 3 + \frac{1}{5} \cdot 1 = \frac{17}{10}$$

The value of the Riemann sum is 1.7.

Practice 4.1.7

Calculate the Riemann sum for $f(x) = \frac{1}{x}$ on the partition $\{1,4,5\}$ using the chosen values $c_1 = 3$ and $c_2 = 4$.

Answer

RS $= (3)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{4}\right) = 1.25$

? Practice 4.1.8

What is the smallest value a Riemann sum for $f(x) = \frac{1}{x}$ can have using the partition $\{1, 4, 5\}$? (You will need to choose values for c_1 and c_2.) What is the largest value a Riemann sum can have for this function and partition?

Answer

smallest RS $= (3)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{5}\right) = 0.95$ BREAK largest RS $= (3)(1) + (1)\left(\frac{1}{4}\right) = 3.25$

The table below shows the output of a computer program that calculated Riemann sums for the function $f(x) = \frac{1}{x}$ with various numbers of subintervals (denoted n) and different ways of choosing the points c_k in each subinterval.

n	mesh	c_k = left edge = x_{k-1}	c_k = "random" point	c_k = right edge = x_k
4	1.0	2.083333	1.473523	1.283333
8	0.5	1.828968	1.633204	1.428968
16	0.25	1.714406	1.577806	1.514406
40	0.10	1.650237	1.606364	1.570237
400	0.01	1.613446	1.609221	1.605446
4000	0.001	1.609838	1.609436	1.609038

(As the mesh gets smaller, all of the Riemann Sums seem to be approaching the same value, approximately 1.609. As we shall soon see, these values are all approaching $\ln(5) \approx 1.609437912$.)

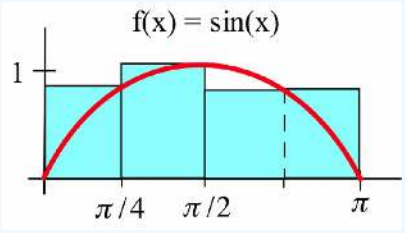
When the mesh of the partition is small (and the number of subintervals, n, is large), it appears that all of the ways of choosing the c_k locations result in approximately the same value for the Riemann sum. For this decreasing function, using the left endpoint of the subinterval always resulted in a sum that was larger than the area approximated by the sum. Choosing the right endpoint resulted in a value smaller than that area. Why?

✓ Example 4.1.7

Find the Riemann sum for the function $f(x) = \sin(x)$ on the interval $[0, \pi]$ using the partition $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\right\}$ and the chosen points $c_1 = \frac{\pi}{4}$, $c_2 = \frac{\pi}{2}$ and $c_3 = \frac{3\pi}{4}$.

Solution

The three subintervals are $\left[0, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$:



so $\Delta x_1 = \frac{\pi}{4}$, $\Delta x_2 = \frac{\pi}{4}$ and $\Delta x_3 = \frac{\pi}{2}$. The Riemann sum for this partition is:

$$\sum_{k=1}^{3} f(c_k)\cdot\Delta x_k = \sin\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{4} + \sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{4} + \sin\left(\frac{3\pi}{4}\right)\cdot\frac{\pi}{2}$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} + 1\cdot\frac{\pi}{4} + \frac{\sqrt{2}}{2}\cdot\frac{\pi}{2} = \frac{(2+3\sqrt{2})\pi}{8}$$

or approximately 2.45148.

Practice 4.1.9

Find the Riemann sum for the function and partition in the previous Example, but this time choose $c_1 = 0$, $c_2 = \frac{\pi}{2}$ and $c_3 = \frac{\pi}{2}$.

Answer

$\text{RS} = (0)\left(\frac{\pi}{4}\right) + (1)\left(\frac{\pi}{4}\right) + (1)\left(\frac{\pi}{2}\right) \approx 2.356$

Two Special Riemann Sums: Lower and Upper Sums

Two particular Riemann sums are of special interest because they represent the extreme possibilities for a given partition.

Note

We need f to be continuous in order to assure that it attains its minimum and maximum values on any closed subinterval of the partition. If f is bounded—but not necessarily continuous—we can generalize this definition by replacing $f(m_k)$ with the **greatest lower bound** of all $f(x)$ on the interval and $f(M_k)$ with the **least upper bound** of all $f(x)$ on the interval.

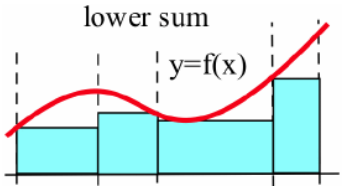
Definition

If f is a positive, continuous function on $[a,b]$ and $\mathcal{P}$ is a partition of $[a,b]$, let m_k be the x-value in the k-th subinterval so that $f(m_k)$ is the minimum value of f on that interval, and let M_k be the x-value in the k-th subinterval so that $f(M_k)$ is the maximum value of f on that subinterval. Then:

$\text{LS}_{\mathcal{P}} = \sum_{k=1}^{n} f(m_k)\cdot\Delta x_k$ is the **lower sum** of f for $\mathcal{P}$

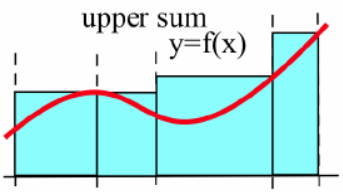
$\text{US}_{\mathcal{P}} = \sum_{k=1}^{n} f(M_k)\cdot\Delta x_k$ is the **upper sum** of f for $\mathcal{P}$

Geometrically, a lower sum arises from building rectangles under the graph of f:



and *every* lower sum is less than or equal to the exact area A of the region bounded by the graph of f and the x-axis on the interval $[a,b]$: $\text{LS}_{\mathcal{P}} \le A$ for every partition $\mathcal{P}$.

Likewise, an upper sum arises from building rectangles over the graph of f:



and *every* upper sum is greater than or equal to the exact area A of the region bounded by the graph of f and the x-axis on the interval $[a, b]$: $\text{US}_{\mathcal{P}} \geq A$ for every partition $\mathcal{P}$.

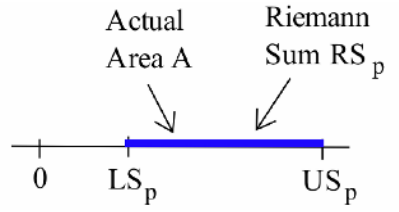
Together, the lower and upper sums provide bounds on the size of the exact area: $\text{LS}_{\mathcal{P}} \leq A \leq \text{US}_{\mathcal{P}}$.

For any c_k value in the k-th subinterval, $f(m_k) \leq f(c_k) \leq f(M_k)$, so, for *any* choice of the c_k values, the Riemann sum $\text{RS}_{\mathcal{P}} = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ satisfies the inequality:

$$\sum_{k=1}^{n} f(m_k) \cdot \Delta x_k \quad \leq \quad \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \quad \leq \quad \sum_{k=1}^{n} f(M_k) \cdot \Delta x_k$$

or, equivalently, $\text{LS}_{\mathcal{P}} \leq \text{RS}_{\mathcal{P}} \leq \text{US}_{\mathcal{P}}$. The lower and upper sums provide bounds on the size of *all* Riemann sums for a given partition.

The exact area A and every Riemann sum $\text{RS}_{\mathcal{P}}$ for partition $\mathcal{P}$ and any choice of points $\{c_k\}$ both lie between the lower sum and the upper sum for $\mathcal{P}$:



Therefore, if the lower and upper sums are close together, then the area and *any* Riemann sum for $\mathcal{P}$ (regardless of how you choose the points c_k) must also be close together. If we know that the upper and lower sums for a partition $\mathcal{P}$ are within 0.001 units of each other, then we can be sure that every Riemann sum for partition $\mathcal{P}$ is within 0.001 units of the exact area A.

Unfortunately, finding minimums and maximums for each subinterval of a partition can be a time-consuming (and tedious) task, so it is usually not practical to determine lower and upper sums for "wiggly" functions. If f is monotonic, however, then it is *easy* to find the values for m_k and M_k, and sometimes we can even explicitly calculate the limits of the lower and upper sums.

For a **monotonic**, bounded function we can guarantee that a Riemann sum is within a certain distance of the exact value of the area it is approximating.

(Recall from Section 3.3 that "monotonic" means "always increasing or always decreasing" on the interval in question.)

Theorem

If: f is a positive, monotonic, bounded function on $[a, b]$

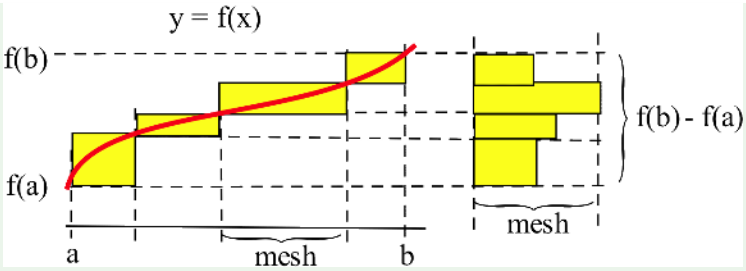
then: for any partition $\mathcal{P}$ and any Riemann sum for f using $\mathcal{P}$,

$$|\text{RS}_{\mathcal{P}} - A| \leq \text{US}_{\mathcal{P}} - \text{LS}_{\mathcal{P}} \leq |f(b) - f(a)| \cdot (\text{mesh of } \mathcal{P})$$

(In words, this string of inequalities says that the distance between any Riemann sum and the area being approximated is no bigger than the difference between the upper and lower Riemann sums for the same partition, which in turn is no bigger than the distance between the values of the function at the endpoints of the interval times the mesh of the partition.)

Proof

The Riemann sum and the exact area are both between the upper and lower sums, so the distance between the Riemann sum and the exact area is no bigger than the distance between the upper and lower sums. If f is monotonically increasing, we can slide the areas representing the difference of the upper and lower sums into a rectangle:



whose height equals $f(b) - f(a)$ and whose base equals the mesh of $\mathcal{P}$. So the total difference of the upper and lower sums is smaller than the area of that rectangle, $[f(b) - f(a)] \cdot (\text{mesh of } \mathcal{P})$.

(See Problem 56 for the monotonically decreasing case.)

Problems

In Problems 1–6 , rewrite the sigma notation as a summation and perform the indicated addition.

1. $\sum_{k=2}^{4} k^2$
2. $\sum_{j=1}^{5} (1+j)$
3. $\sum_{n=1}^{3} (1+n)^2$
4. $\sum_{k=0}^{5} \sin(\pi k)$
5. $\sum_{j=0}^{5} \cos(\pi j)$
6. $\sum_{k=1}^{3} \frac{1}{k}$

In Problems 7–12, rewrite each summation using the sigma notation. Do not evaluate the sums.

7. $3+4+5+\cdots+93+94$
8. $4+6+8+\cdots+24$
9. $9+16+25+36+\cdots+144$
10. $\frac{3}{4}+\frac{3}{9}+\frac{3}{16}+\cdots+\frac{3}{100}$
11. $1 \cdot 2^1+2 \cdot 2^2+3 \cdot 2^3+\cdots+7 \cdot 2^7$
12. $3+6+9+\cdots+30$

In Problems 13--15, use this table:

k	a_k	b_k
1	1	2
2	2	2
3	3	2

to verify the equality for these values of a_k and b_k.

13. $\sum_{k=1}^{3} (a_k+b_k) = \sum_{k=1}^{3} a_k + \sum_{k=1}^{3} b_k$
14. $\sum_{k=1}^{3} (a_k-b_k) = \sum_{k=1}^{3} a_k - \sum_{k=1}^{3} b_k$
15. $\sum_{k=1}^{3} 5a_k = 5 \cdot \sum_{k=1}^{3} a_k$

For Problems 16–18, use the values of a_k and b_k in the table above to verify the inequality.

16. $\sum_{k=1}^{3} a_k \cdot b_k \neq \left(\sum_{k=1}^{3} a_k\right)\left(\sum_{k=1}^{3} b_k\right)$
17. $\sum_{k=1}^{3} a_k^2 \neq \left(\sum_{k=1}^{3} a_k\right)^2$
18. $\sum_{k=1}^{3} \frac{a_k}{b_k} \neq \frac{\sum_{k=1}^{3} a_k}{\sum_{k=1}^{3} b_k}$

For Problems 19–30, $f(x) = x^2$, $g(x) = 3x$ and $h(x) = \frac{2}{x}$. Evaluate each sum.

19. $\sum_{k=0}^{3} f(k)$
20. $\sum_{k=0}^{3} f(2k)$
21. $\sum_{j=0}^{3} 2 \cdot f(j)$
22. $\sum_{i=0}^{3} f(1+i)$
23. $\sum_{m=1}^{3} g(m)$
24. $\sum_{k=1}^{3} g\left(f(k)\right)$
25. $\sum_{j=1}^{3} g^2(j)$
26. $\sum_{k=1}^{3} k \cdot g(k)$
27. $\sum_{k=2}^{4} h(k)$
28. $\sum_{i=1}^{4} h(3i)$
29. $\sum_{n=1}^{3} f(n) \cdot h(n)$
30. $\sum_{k=1}^{7} g(k) \cdot h(k)$

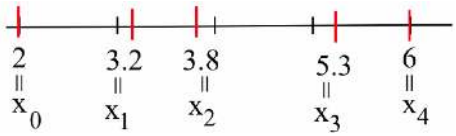
In Problems 31–36, write out each summation and simplify the result. These are examples of “telescoping sums.”

31. $\sum_{k=1}^{7} \left[k^2 - (k-1)^2\right]$
32. $\sum_{k=1}^{6} \left[k^3 - (k-1)^3\right]$
33. $\sum_{k=1}^{5} \left[\frac{1}{k} - \frac{1}{k+1}\right]$
34. $\sum_{k=0}^{4} \left[(k+1)^3 - k^3\right]$
35. $\sum_{k=0}^{8} \left[\sqrt{k+1} - \sqrt{k}\right]$
36. $\sum_{k=1}^{5} \left[x_k - x_{k-1}\right]$

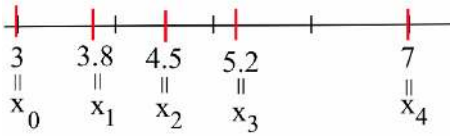
In Problems 37–43:

a. list the subintervals determined by the partition $\mathcal{P}$
b. find the values of Δx_k,
c. find the mesh of $\mathcal{P}$ and (d) calculate $\sum_{k=1}^{n} \Delta x_k$.

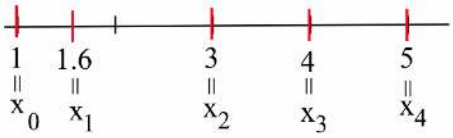
37. $\mathcal{P} = \{2, 3, 4.5, 6, 7\}$
38. $\mathcal{P} = \{3, 3.6, 4, 4.2, 5, 5.5, 6\}$
39. $\mathcal{P} = \{-3, -1, 0, 1.5, 2\}$
40. $\mathcal{P}$ as shown below:



41. $\mathcal{P}$ as shown below:



42. $\mathcal{P}$ as shown below:



43. For $\Delta x_k = x_k - x_{k-1}$, verify that:

$$\sum_{k=1}^{n} \Delta x_k = \text{length of the interval } [a, b]$$

For Problems 44–48, sketch a graph of f, draw vertical lines at each point of the partition, evaluate each $f(c_k)$ and sketch the corresponding rectangle, and calculate and add up the areas of those rectangles.

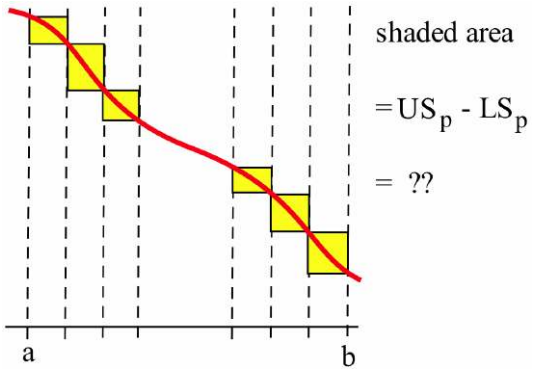
44. $f(x) = x + 1$, $\mathcal{P} = \{1, 2, 3, 4\}$
    a. $c_1 = 1$, $c_2 = 3$, $c_3 = 3$
    b. $c_1 = 2$, $c_2 = 2$, $c_3 = 4$
45. $f(x) = 4 - x^2$, $\mathcal{P} = \{0, 1, 1.5, 2\}$
    a. $c_1 = 0$, $c_2 = 1$, $c_3 = 2$
    b. $c_1 = 1$, $c_2 = 1.5$, $c_3 = 1.5$
46. $f(x) = \sqrt{x}$, $\mathcal{P} = \{0, 2, 5, 10\}$
    a. $c_1 = 1$, $c_2 = 4$, $c_3 = 9$
    b. $c_1 = 0$, $c_2 = 3$, $c_3 = 7$
47. $f(x) = \sin(x)$, $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$
    a. $c_1 = 0$, $c_2 = \dfrac{\pi}{4}$, $c_3 = \dfrac{\pi}{2}$
    b. $c_1 = \dfrac{\pi}{4}$, $c_2 = \dfrac{\pi}{2}$, $c_3 = \pi$
48. $f(x) = 2^x$, $\mathcal{P} = \{0, 1, 3\}$
    a. $c_1 = 0$, $c_2 = 2$
    b. $c_1 = 1$, $c_2 = 3$

For Problems 49–52, sketch the function and find the smallest possible value and the largest possible value for a Riemann sum for the given function and partition.

49. $f(x) = 1 + x^2$
    a. $\mathcal{P} = \{1, 2, 4, 5\}$
    b. $\mathcal{P} = \{1, 2, 3, 4, 5\}$
    c. $\mathcal{P} = \{1, 1.5, 2, 3, 4, 5\}$
50. $f(x) = 7 - 2x$

a. $\mathcal{P} = \{0, 2, 3\}$
b. $\mathcal{P} = \{0, 1, 2, 3\}$
c. $\mathcal{P} = \{0, .5, 1, 1.5, 2, 3\}$

51. $f(x) = \sin(x)$
    a. $\mathcal{P} = \{0, \frac{\pi}{2}, \pi\}$
    b. $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$
    c. $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi\}$
52. $f(x) = x^2 - 2x + 3$
    a. $\mathcal{P} = \{0, 2, 3\}$
    b. $\mathcal{P} = \{0, 1, 2, 3\}$
    c. $\mathcal{P} = \{0, 0.5, 1, 2, 2.5, 3\}$
53. Suppose $\text{LS}_{\mathcal{P}} = 7.362$ and $\text{US}_{\mathcal{P}} = 7.402$ for a positive function f and a partition $\mathcal{P}$ of $[1, 5]$.
    a. You can be certain that every Riemann sum for the partition $\mathcal{P}$ is within what distance of the exact value of the area between the graph of f and the x-axis on the interval $[1, 5]$?
    b. What if $\text{LS}_{\mathcal{P}} = 7.372$ and $\text{US}_{\mathcal{P}} = 7.390$?
54. Suppose you divide the interval $[1, 4]$ into 100 equally wide subintervals and calculate a Riemann sum for $f(x) = 1 + x^2$ by randomly selecting a point c_k in each subinterval.
    a. You can be certain that the value of the Riemann sum is within what distance of the exact value of the area between the graph of f and the x-axis on interval $[1, 4]$?
    b. What if you use 200 equally wide subintervals?
55. If you divide $[2, 4]$ into 50 equally wide subintervals and calculate a Riemann sum for $f(x) = 1 + x^3$ by randomly selecting a point c_k in each subinterval, then you can be certain that the Riemann sum is within what distance of the exact value of the area between f and the x-axis on the interval $[2, 4]$?
56. If f is monotonic decreasing on $[a, b]$ and you divide $[a, b]$ into n equally wide subintervals:



then you can be certain that the Riemann sum is within what distance of the exact value of the area between f and the x-axis on the interval $[a, b]$?

Summing Powers of Consecutive Integers

Formulas for some commonly encountered summations can be useful for explicitly evaluating certain special Riemann sums. (The formulas below are included here for your reference. They will not be used in the following sections, except for a handful of exercises in Section 4.2.)

The summation formula for the first n positive integers is relatively well known, has several easy but clever proofs, and even has an interesting story behind it.

$$1 + 2 + 3 + \cdots + (n-1) + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proof

Let $S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$, which we can also write as $S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$. Adding these two representations of S together:

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ +\ S &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ 2S &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \end{aligned}$$

So $2S = n \cdot (n+1) \Rightarrow S = \dfrac{n(n+1)}{2}$, the desired formula.

Note

Karl Friedrich Gauss (1777–1855), a German mathematician sometimes called the "prince of mathematics," supposedly discovered this formula for himself at the age of five when his teacher, planning to keep the class busy for a while, asked the students to add up the integers from 1 to 100. Gauss thought a few minutes, wrote his answer on his slate, and turned it in, then sat smugly while his classmates struggled with the problem.

57. Find the sum of the first 100 positive integers in two ways:
    a. using Gauss' formula, and
    b. using Gauss' method (from the proof).
58. Find the sum of the first 10 odd integers. (Each odd integer can be written in the form $2k-1$ for $k = 1, 2, 3, \ldots$.)
59. Find the sum of the integers from 10 to 20.

Formulas for other integer powers of the first n integers are also known:

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \left(\frac{n(n+1)}{2}\right)^2 \\ \sum_{k=1}^{n} k^4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \end{aligned}$$

In Problems 60–62, use the properties of summation and the formulas for powers given above to evaluate each sum.

60. $\displaystyle\sum_{k=1}^{10} (3 + 2k + k^2)$
61. $\displaystyle\sum_{k=1}^{10} k \cdot (k^2 + 1)$
62. $\displaystyle\sum_{k=1}^{10} k^2 \cdot (k - 3)$

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4.2: The Definite Integral

Each particular Riemann sum depends on several things: the function f, the interval $[a, b]$, the partition $\mathcal{P}$ of that interval, and the chosen values c_k from each subinterval of that partition. Fortunately—for most of the functions needed for applications—as the approximating rectangles get thinner (and as the meshes of the partitions $\mathcal{P}$ approach 0 and the number of subintervals n in those partitions approaches ∞) the values of the Riemann sums approach the same number, independent of the particular partitions P and the chosen points c_k in the subintervals of those partitions.

This limit of the Riemann sums will become the next big topic in calculus: the **definite integral**. Integrals arise throughout the rest of this book and in applications in almost every field that uses mathematics.

Definition of the Definite Integral

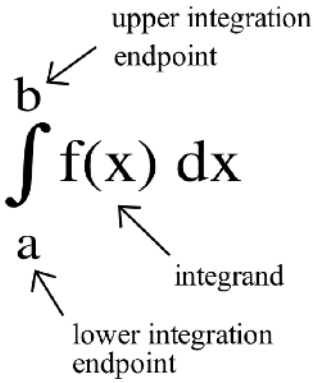
If $\lim\limits_{\|\mathcal{P}\|\to 0}\left(\sum\limits_{k=1}^{n} f(c_k)\cdot\Delta x_k\right)$ equals a finite number I, where each $\mathcal{P}$ is a partition of the interval $[a, b]$, then we say a bounded function f is **integrable** on the interval $[a, b]$ and call the number I the **definite integral** of f on $[a, b]$ and write it as $\int_a^b f(x)\,dx$.

Note

Here we use the notation $\|\mathcal{P}\|$ to mean "the mesh of $\mathcal{P}$" and we assume $b > a$ so that $[a, b]$ is not a single point.

The dx is a differential (see Section 3.6), the limit of the discrete quantity Δx in the Riemann sum.

We read the symbol $\int_a^b f(x)\,dx$ as "the integral from a to b of 'eff' of x 'dee' x" or "the integral from a to b of $f(x)$ with respect to x." Furthermore, we call $f(x)$ the **integrand**, a the **lower endpoint of integration** and b the **upper endpoint** of integration. (We will sometimes also call a and b the **upper and lower limits** of integration.)



Example 4.2.1

Describe the area between the graph of $f(x) = \frac{1}{x}$, the x-axis, and the vertical lines at $x = 1$ and $x = 5$ as a limit of Riemann sums and as a definite integral.

Solution

Here $f(x) = \frac{1}{x}$, $a = 1$ and $b = 5$, so:

$$\text{area} = \lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}\frac{1}{c_k}\cdot\Delta x_k\right) = \int_1^5 \frac{1}{x}\,dx$$

which, according to estimations made in Section 4.1, is approximately equal to 1.609.

Note

You may have noticed that we did not precisely define what $\lim_{\|\mathcal{P}\|\to 0}$ means or how to compute this limit. Providing a definition turns out to be more complicated than the limits we have encountered so far, and in practice we will rarely need to compute such a limit, so a formal definition is left to more advanced textbooks.

? Practice 4.2.1

Describe the area between the graph of $f(x) = \sin(x)$, the x-axis, and the vertical lines at $x = 0$ and $x = \pi$ as a limit of Riemann sums and as a definite integral.

Answer

$$\text{area} = \lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} \sin(c_k) \cdot \Delta x_k \right) = \int_0^{\pi} \sin(x)\, dx$$

✓ Example 4.2.2

Using the concept of area, determine the values of:

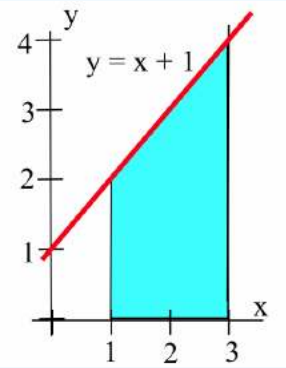
a. $\lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} (1 + c_k) \cdot \Delta x_k \right)$ on the interval $[1, 3]$

b. $\int_0^4 (5 - x)\, dx$

c. $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$

Solution

a. The limit of the Riemann sums represents the area between the graph of $f(x) = 1 + x$, the x-axis, and the vertical lines at $x = 1$ and $x = 3$:



This area equals 6 square units.

b. The definite integral represents the area between $f(x) = 5 - x$, the x-axis and the vertical lines at $x = 0$ and $x = 4$, which is a trapezoid with area 12 square units.

c. The definite integral represents the area of the upper half of the circle $x^2 + y^2 = 1$, which has radius 1 and center at $(0, 0)$. The area of this semicircle is $\frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi \cdot 1^2 = \frac{\pi}{2}$.

? Practice 4.2.2

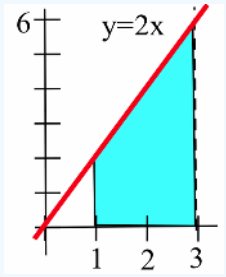
Using the concept of area, determine the values of:

a. $\lim_{\|\mathcal{P}\|\to 0} \left(\sum_{k=1}^{n} (2c_k) \cdot \Delta x_k \right)$ on the interval $[1, 3]$

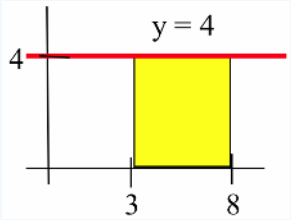
b. $\int_3^8 4\, dx$

Answer

$\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}(2c_k)\cdot\Delta x_k\right)$ = area of trapezoid in the figure below $= 8$



$\int_3^8 4\,dx$ = area of rectangle in figure below $= 20$



Example 4.2.3

Represent each limit of Riemann sums as a definite integral.

a. $\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}(3+c_k)\,\Delta x_k\right)$ on $[1,4]$

b. $\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n}\sqrt{c_k}\,\Delta x_k\right)$ on $[0,9]$

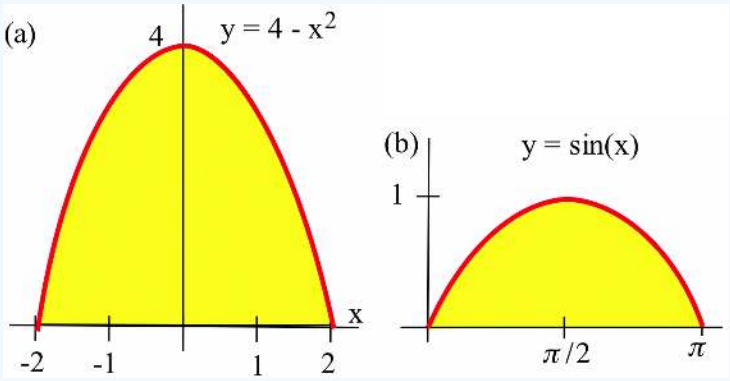
Solution

a. $\int_1^4 (3+x)\,dx$

b. $\int_0^9 \sqrt{x}\,dx$

Example 4.2.4

Represent each shaded area in the figure below as a definite integral:



(Do not attempt to evaluate the definite integral, just translate the picture into symbols.)

Solution

a. $\int_{-2}^{2} (4 - x^2)\, dx$

b. $\int_{0}^{\pi} \sin(x)\, dx$

The value of a definite integral $\int_a^b f(x)\, dx$ depends only on the function f being integrated and on the interval $[a, b]$. Replacing the variable x that appears in $\int_a^b f(x)\ dx$, sometimes called a "dummy variable," does not change the value of the integral. The following definite integrals each represent the integral of the function f on the interval $[a, b]$, and they are all equal:

$$\int_a^b f(x)\, dx \quad = \quad \int_a^b f(t)\, dt \quad = \quad \int_a^b f(u)\, du \quad = \quad \int_a^b f(w)\, dw$$

Definite Integrals of Negative Functions

A definite integral is a limit of Riemann sums, and you can form Riemann sums using any integrand function f, positive or negative (or both), continuous or discontinuous. The definite integral of an integrable function will still have a geometric meaning even if the function is sometimes (or always) negative, and definite integrals of negative functions also have meaningful interpretations in applications.

Example 4.2.5

Find the definite integral of $f(x) = -2$ on $[1, 4]$.

Solution

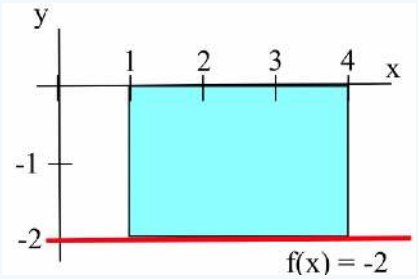
Writing a Riemann sum for $f(x) = -2$ on the interval $[1, 4]$:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} (-2) \cdot \Delta x_k = -2 \cdot \sum_{k=1}^{n} \Delta x_k = -2\,(3) = -6$$

for every partition $\mathcal{P}$ and every choice of values for c_k so:

$$\int_1^4 -2\, dx = \lim_{\|\mathcal{P}\| \to 0} \left(\sum_{k=1}^{n} (-2) \cdot \Delta x_k \right) = \lim_{\|\mathcal{P}\| \to 0} -6 = -6$$

The **area** of the region in the figure below is **6** units:



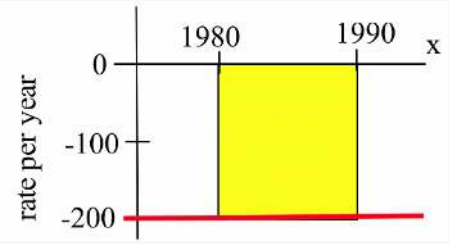
but because the region is below the x-axis, the value of the **integral** is -6.

If the graph of $f(x)$ is below the x-axis for $a \le x \le b$ (f is negative) then $\int_a^b f(x)\, dx$ is -1 times the area of the region below the x-axis and above the graph of $f(x)$ between $x = a$ and $x = b$.

If $f(t)$ represents the rate of population change (people per year) for a town, then negative values of f for a given time interval would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would represent the change in the population—a decrease—during that time interval.

Example 4.2.6

In 1980 there were 12,000 ducks nesting around a lake. The **rate** of population change is shown below:



Write a definite integral to represent the **total change** in the duck population between 1980 and 1990, then estimate the population in 1990.

Solution

The total change in population is given by $\int_{1980}^{1990} f(t)\,dt$ and this definite integral is equal to -1 times the area of the rectangle in the figure below: FIGURE

$$-200\,\frac{\text{ducks}}{\text{year}}\cdot 10\,\text{years} = -2000\,\text{ducks}$$

so:

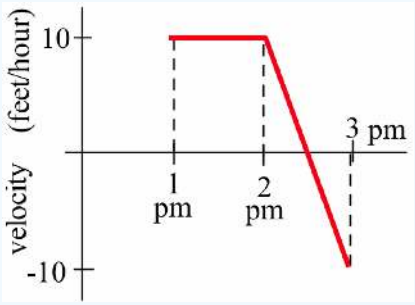
$$\begin{aligned}[1990\text{ population}] &= [1980\text{ population}] + [\text{change from 1980 to 1990}]\\ &= 12000 + (-2000) = 10000\end{aligned}$$

Approximately 10,000 ducks were nesting around the lake in 1990.

If $f(t)$ represents the velocity of a car (in miles per hour) moving in the positive direction along a straight line at time t, then negative values of f indicate that the car was travelling in the negative direction (that is, backwards). The definite integral of f (the integral will be a negative number) represents the change in position of the car during the time interval: how far the car travelled in the negative direction.

Practice 4.2.3

A bug starts at the location $x = 12$ on the x-axis at 1:00 p.m. and walks along the axis with the velocity shown below:



How far does the bug travel between 1:00 p.m. and 3:00 p.m.? Where is the bug at 3:00 p.m.?

Answer

a. 12.5 feet forward and 2.5 feet backward $= 15$ feet total

b. The bug ends up 10 feet forward of its starting position at $x = 12$, so the bug's final location is at $x = 22$.

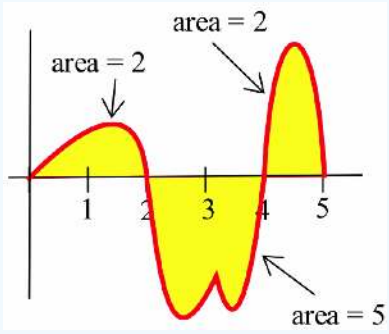
Frequently an integrand function will be positive some of the time and negative some of the time. If f represents a rate of population increase, then the integral of the positive parts of f will give the increase in population and the integral of the negative parts of f will give the decrease in population. Altogether, the integral of f over the entire time interval will give the **total (net) change** in the population.

Geometrically, we can now interpret a definite integral as a difference of areas of the region(s) between the graph of f and the horizontal axis:

$$\int_a^b f(x)\,dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$

Example 4.2.7

Use the figure below:



to calculate $\int_0^2 f(x)\,dx$, $\int_2^4 f(x)\,dx$, $\int_4^5 f(x)\,dx$ and $\int_0^5 f(x)\,dx$.

Solution

Using the areas indicated in the figure, $\int_0^2 f(x)\,dx = 2$, $\int_2^4 f(x)\,dx = -5$ and $\int_4^5 f(x)\,dx = 2$, while

$$\begin{aligned}\int_0^5 f(x)\,dx &= [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}] \\ &= [2+2] - [5] = -1\end{aligned}$$

where we added the areas of the regions above the x-axis and subtracted the area of the region below the x-axis.

Practice 4.2.4

Use geometric reasoning to evaluate $\int_0^{2\pi} \sin(x)\,dx$.

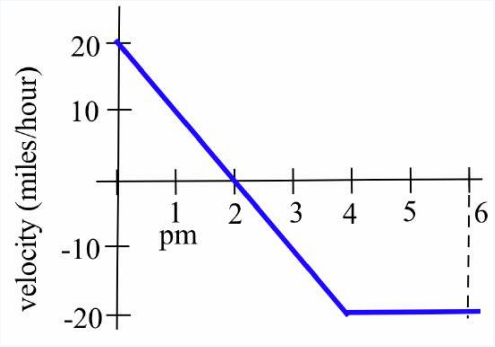
Answer

Between $x = 0$ and $x = 2\pi$, the graph of $y = \sin(x)$ has the same area above the x-axis as below the x-axis so the two areas cancel and the definite integral is 0: $\int_0^{2\pi} \sin(x)\,dx = 0$.

If f represents a velocity, then integrals on the intervals where f is positive measure distances moved in the forward direction and integrals on the intervals where f is negative measure distances moved in the backward direction. The integral over the whole time interval gives the **total (net) change** in position: the distance moved forward minus the distance moved backward.

? Practice 4.2.5

A car travels west with the velocity shown below:



a. How far does the car travel between noon and 6:00 p.m.?
b. At 6:00 p.m., where is the car relative to its position at noon?

Answer

a. 20 miles west (from noon to 2:00 p.m.) plus 60 miles east (from 2:00 p.m. to 6:00 p.m.) yields a total travel distance of 80 miles. (At 4:00 p.m. the driver is back at the starting position after driving 40 miles: 20 miles west and then 20 miles east.)
b. The car is 40 miles east of the starting location. (East is the "negative" of west.)

Units for the Definite Integral

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if x measures time in "seconds" and $f(x)$ gives a velocity with units "feet per second," then Δx has the units "seconds" and $f(x) \cdot \Delta x$ has units:

$$\left(\frac{\text{feet}}{\text{second}}\right)(\text{seconds}) = \text{feet}$$

which is a measure of distance. Because each Riemann sum $\sum f(x) \cdot \Delta x$ is a sum of "feet" and the definite integral is a limit of these Riemann sums, the definite integral has the same units, "feet."

If the units of $f(x)$ are "square feet" and the units of x are "feet," then $\int_a^b f(x)\,dx$ is a number with the units $(\text{feet}^2) \cdot (\text{feet}) = \text{feet}^3$, or cubic feet, a measure of volume. If $f(x)$ represents a force in pounds and x is a distance in feet, then $\int_a^b f(x)\,dx$ is a number with the units foot-pounds, a measure of work.

In general, the units for $\int_a^b f(x)\,dx$ are $(\text{units for } f(x)) \cdot (\text{units for } x)$. A quick check of the units when computing a definite integral can help avoid errors when solving an applied problem.

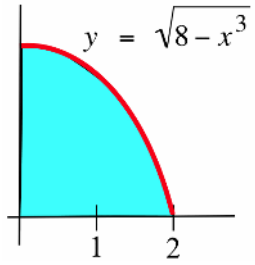
Problems

In Problems 1–4 , rewrite each limit of Riemann sums as a definite integral.

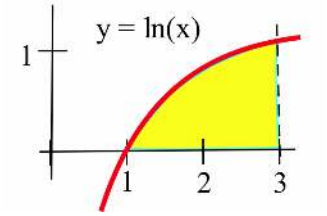
1. $\lim\limits_{\|\mathcal{P}\|\to 0}\left(\sum\limits_{k=1}^{n}(2+3c_k)\cdot\Delta x_k\right)$ on $[0,4]$
2. $\lim\limits_{\|\mathcal{P}\|\to 0}\left(\sum\limits_{k=1}^{n}(c_k)^3\cdot\Delta x_k\right)$ on $[0,11]$
3. $\lim\limits_{\|\mathcal{P}\|\to 0}\left(\sum\limits_{k=1}^{n}\cos(5c_k)\cdot\Delta x_k\right)$ on $[2,5]$
4. $\lim\limits_{\|\mathcal{P}\|\to 0}\left(\sum\limits_{k=1}^{n}\sqrt{c_k}\cdot\Delta x_k\right)$ on $[1,4]$

In Problems 5–10, represent the area of each bounded region as a definite integral. (Do not attempt to evaluate the integral, just translate the area into an integral.)

5. The region bounded by $y = x^3$, the x-axis, and the lines $x = 1$ and $x = 5$.
6. The region bounded by $y = \sqrt{x}$, the x-axis and the line $x = 9$.
7. The region bounded by $y = x \cdot \sin(x)$, the x-axis, and the lines $x = \frac{1}{2}$ and $x = 2$.
8. The shaded region shown below:



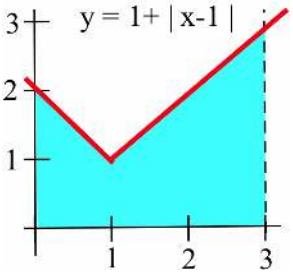
9. The shaded region shown below:



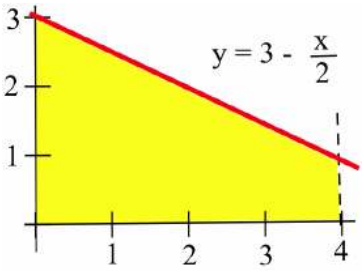
10. The shaded region shown above for $2 \leq x \leq 3$.

In Problems 11–15, represent the area of each bounded region as a definite integral, then use geometry to determine the value of that definite integral.

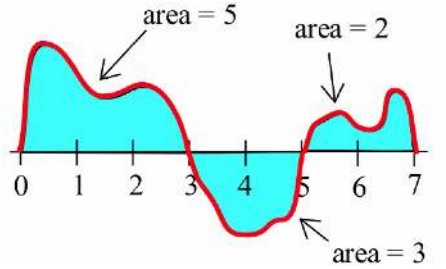
11. The region bounded by $y = 2x$, the x-axis, and the lines $x = 1$ and $x = 3$.
12. The region bounded by $y = 4 - 2x$, the x-axis and the y-axis.
13. The region bounded by $y = |x|$, the x-axis and the line $x = -1$.
14. The shaded region shown below:



15. The shaded region shown below:

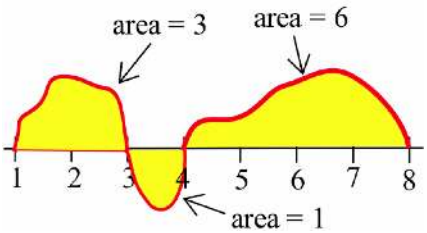


16. Evaluate each integral using the figure below showing the graph of f and various areas:



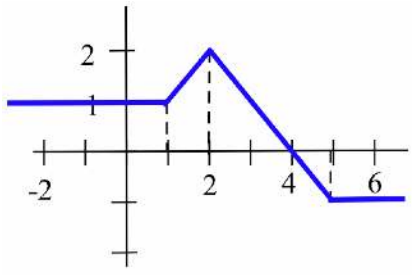
a. $\int_0^5 f(x)\,dx$
b. $\int_3^5 f(x)\,dx$
c. $\int_5^7 f(x)\,dx$
d. $\int_0^5 |f(x)|\,dx$
e. $\int_3^7 f(x)\,dx$

17. Evaluate each integral using the figure below showing the graph of g and various areas:



a. $\int_1^3 g(x)\,dx$
b. $\int_3^4 g(x)\,dx$
c. $\int_4^8 g(x)\,dx$
d. $\int_1^8 g(x)\,dx$
e. $\int_3^8 |g(x)|\,dx$

18. Evaluate each integral using the figure below showing the graph of h:



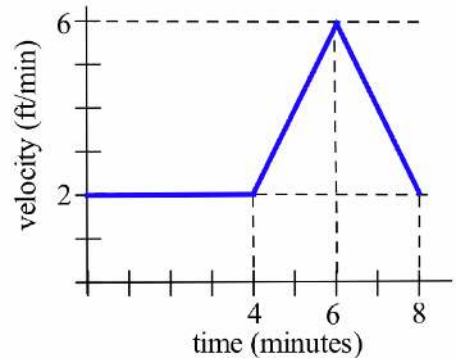
a. $\int_{-1}^1 h(x)\,dx$
b. $\int_4^6 h(x)\,dx$
c. $\int_{-2}^6 h(x)\,dx$
d. $\int_{-2}^4 h(x)\,dx$
e. $\int_{-2}^4 |h(x)|\,dx$

f. $\int_{-2}^{6} |h(x)|\, dx$

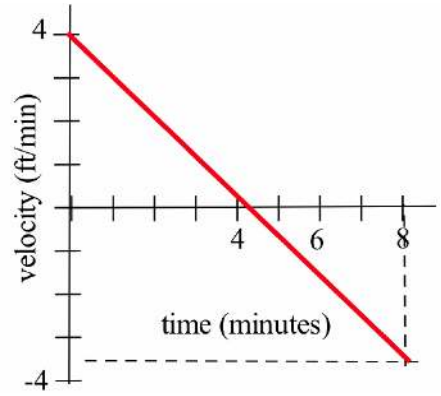
For Problems 19–20, the figure shows your velocity (in feet per minute) along a straight path.

a. Sketch a graph of your location.
b. How many feet did you walk in 8 minutes?
c. Where, relative to your starting location, are you after 8 minutes?

19. See figure below:



20. See figure below:



Problems 21–27 give the units for x and $f(x)$. Specify the units of the definite integral $\int_a^b f(x)\, dx$.

21. x is time in "seconds"; $f(x)$ is velocity in "meters per second"
22. x is time in "hours"; $f(x)$ is a flow rate in "gallons per hour"
23. x a position in "feet"; $f(x)$ area in "square feet"
24. x is a time in "days"; $f(x)$ is a temperature in "degrees Celsius"
25. x a height in "meters"; $f(x)$ force in "grams"
26. x is a position in "inches"; $f(x)$ is a density in "pounds per inch"
27. x is a time in "seconds"; $f(x)$ is an acceleration in "feet per second per second" $\left(\frac{\text{ft}}{\text{sec}^2}\right)$

The remaining problems use the summation formulas given at the end of Section 4.1, as demonstrated in the following Example.

Example 4.2.8

For $f(x) = x^2$, divide the interval $[0, 2]$ into n equally wide subintervals, evaluate the lower sum, and compute the limit of that lower sum as $n \to \infty$.

Solution

The width of the interval is $b - a = 2 - 0 = 2$ so each of the n subintervals should have width $\Delta x = \frac{b-a}{n} = \frac{2}{n}$. The endpoints of the k-th interval in the partition are therefore $(k-1) \cdot \frac{2}{n}$ and $k \cdot \frac{2}{n}$ for $k = 1, 2, \ldots, n$.

Because $f(x) = x^2$ is increasing on $[0,2]$ the minimum value of the function on each subinterval occurs at the left endpoint of the subinterval, hence we need to choose $c_k = (k-1)\cdot\frac{2}{n}$. So:

$$\begin{aligned}\text{LS} &= \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \sum_{k=1}^{n}\left((k-1)\cdot\frac{2}{n}\right)^2\cdot\frac{2}{n} = \frac{8}{n^3}\cdot\sum_{k=1}^{n}(k-1)^2 \\ &= \frac{8}{n^3}\cdot\sum_{k=1}^{n}\left(k^2-2k+1\right) = \frac{8}{n^3}\left[\sum_{k=1}^{n}k^2 - 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1\right] \\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\cdot\frac{n(n+1)}{2} + n\right] = \frac{8}{n^3}\left[\frac{2n^3-3n^2+n}{6}\right] \\ &= \frac{8}{6}\left[2-\frac{3}{n}+\frac{1}{n^2}\right]\end{aligned}$$

As $n\to\infty$, LS $\to \frac{8}{6}(2) = \frac{8}{3}$ so we can be certain that $\int_0^2 x^2\,dx \geq \frac{8}{3}$.

? Practice 4.2.6

Redo Example 8 but find the upper Riemann sum for n equally wide partition intervals and show that the limit of these upper sums, as $n\to\infty$, is $\frac{8}{3}$.

Answer

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $M_k = \frac{2}{n}\cdot k$ so $f(M_k) = \left(\frac{2}{n}\cdot k\right)^2 = \frac{4}{n^2}\cdot k^2$. Then:

$$\begin{aligned}\text{US} &= \sum_{k=1}^{n} f(M_k)\cdot\Delta x = \sum_{k=1}^{n}\frac{4}{n^2}\cdot k^2\cdot\frac{2}{n} \\ &= \frac{8}{n^3}\sum_{k=1}^{n}k^2 = \frac{8}{n^3}\left[\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right] = \frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\end{aligned}$$

so the limit of these upper sums as $n\to\infty$ is $\frac{8}{3}$.

From the previous Example and Practice problem, we know that

$$\frac{8}{3} \leq \int_0^2 x^2\,dx \leq \frac{8}{3}$$

so we can conclude that $\int_0^2 x^2 = \frac{8}{3}$. We will discover a much easier method for evaluating this integral in Section 4.4.

28. For $f(x) = 3+x$, partition the interval $[0,2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.
    a. Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n\to\infty$.
    b. Compute the upper sum for this function and partition and find the limit of the upper sum as $n\to\infty$.
29. For $f(x) = x^3$, partition the interval $[0,2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.
    a. Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n\to\infty$.
    b. Compute the upper sum for this function and partition and find the limit of the upper sum as $n\to\infty$.
30. For $f(x) = \sqrt{x}$, partition the interval $[0,9]$ into n subintervals by taking $x_k = \frac{9}{n^2}\cdot k^2$ for $k = 1,2,\ldots,n$.
    a. Choose $c_k = x_k$ for each subinterval and compute the upper sum for this function and partition, then calculate the limit of that upper sum as $n\to\infty$.
    b. Compute the lower sum for this function and partition and find the limit of the lower sum as $n\to\infty$.

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4.3: Properties of the Definite Integral

We have defined definite integrals as limits of Riemann sums, which can often be interpreted as "areas" of geometric regions. These two powerful concepts of the definite integral can help us understand integrals and use them in a variety of applications.

This section continues to emphasize this dual view of definite integrals and presents several properties of definite integrals. We will justify these properties using the properties of summations and the definition of a definite integral as a Riemann sum, but they also have natural interpretations as properties of areas of regions.

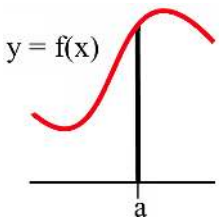
We will then use these properties to help understand functions that are defined by integrals, and later to help calculate the values of definite integrals.

Properties of the Definite Integral

As you read each statement about definite integrals, draw a sketch or examine the accompanying figure to interpret the property as a statement about areas.

$$\int_a^a f(x)\,dx = 0$$

This property says that the definite integral of a function over an interval consisting of a single point must be 0. Geometrically, we can see that the area under the graph of a a function above a single point should be 0 because the "width" of a point is 0:



In terms of Riemann sums, we can't partition a single point, so instead we must *define* the value of any definite integral over a non-existent "interval" to be 0.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

In words, this property says that if we reverse the limits of integration, we must multiply the value of the definite integral by -1.

Geometrically, if $a < b$, then the x-values in the first integral are moving "backwards" from $x = b$ to $x = a$, so it might seem reasonable that we should get a negative answer.

In terms of Riemann sums, if we move from right to left, each Δx_k in any partition $\mathcal{P}$ will be negative:

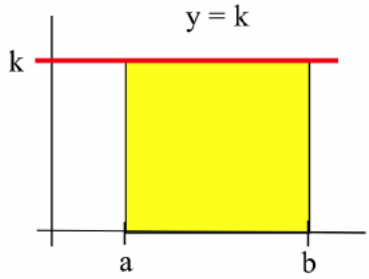
$$\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \sum_{k=1}^{n} f(c_k)\cdot(-|\Delta x_k|) = -1\cdot\sum_{k=1}^{n} f(c_k)\cdot|\Delta x_k|$$

resulting in -1 times the Riemann sum we would use for $\int_a^b f(x)\,dx$.

(Our definition of a Riemann sum only allows each Δx_k to be positive, however, so we can simply treat this integral property as another definition.)

$$\int_a^b k\,dx = k(b-a)\ (k \text{ is any constant})$$

Thinking geometrically, if $k > 0$, then $\int_a^b k\,dx$ represents the area of a rectangle with base $b-a$ and height k:



so:

$$\int_a^b k\,dx = (\text{height}) \cdot (\text{base}) = k \cdot (b-a)$$

Alternatively, for any $\mathcal{P} = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ that partitions the interval $[a, b]$, and every choice of points c_j from the subintervals of that partition, the Riemann sum is:

$$\sum_{j=1}^{n} f(c_j) \cdot \Delta x_j = \sum_{j=1}^{n} k \cdot \Delta x_j = k \sum_{j=1}^{n} \Delta x_j = k \cdot (b-a)$$

Because every Riemann sum equals $k \cdot (b-a)$, the limit of those sums, as $\|\mathcal{P}\| \to 0$, must also be $k \cdot (b-a)$.

(Here we use the fact that the sum of the lengths of the subinterval of any partition of the interval $[a, b]$ is equal to the width of $[a, b]$, which is $b-a$.)

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx \text{ (k is any constant)}$$

In words, this property says that multiplying an integrand by a constant k has the same result as multiplying the value of the definite integral by that constant.

Geometrically, multiplying a function by a positive constant k stretches the graph of $y = f(x)$ by a factor of k in the vertical direction, which should multiply the area of the region between that graph and the x-axis by the same factor.

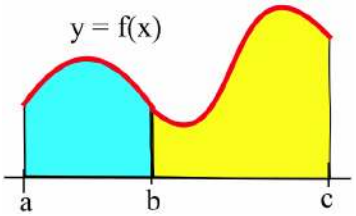
Thinking in terms of Riemann sums:

$$\sum_{j=1}^{n} k \cdot f(c_j) \cdot \Delta x_j = k \cdot \sum_{j=1}^{n} f(c_j) \cdot \Delta x_j$$

so the limit of the sum on the left over all possible partitions $\mathcal{P}$, as $\|\mathcal{P}\| \to 0$, is $\int_a^b k \cdot f(x)\,dx$, while the corresponding limit of the sums on the right yields $k \cdot \int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \;=\; \int_a^c f(x)\,dx$$

This property is most easily understood (and believed) in terms of a picture:



We can also justify this property using Riemann sums by restricting our partitions to include the point $x = b$ between $x = a$ and $x = c$ and then splitting that partition into two sub-partitions that partition $[a, b]$ and $[b, c]$, respectively.

This property remains true, however, even when $b \geq c$ or $b \leq a$.

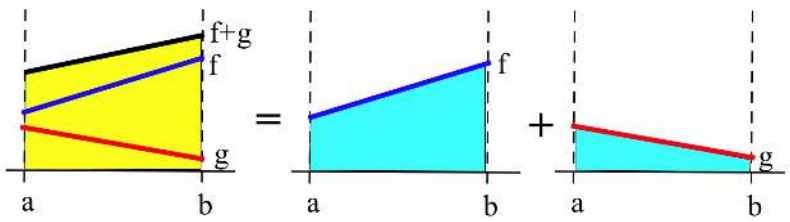
Properties of Definite Integrals of Combinations of Functions

The next two properties relate the values of integrals of sums and differences of functions to the sums and differences of integrals of the individual functions. You will find these properties very useful when computing integrals of functions that involve the sum or difference of several terms (such as a polynomial): you can integrate each term and then add or subtract the individual results to get the answer. Both properties have natural interpretations as statements about areas.

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx \; + \; \int_a^b g(x)\, dx$$

(In words, this says "the integral of a sum is the sum of the integrals.")

The following graph supplies a geometrical justification:



Using Riemann sums, we can write:

$$\sum_{j=1}^{n} [f(c_j) + g(c_j)] \cdot \Delta x_j = \sum_{j=1}^{n} f(c_j) \cdot \Delta x_j \; + \; \sum_{j=1}^{n} g(c_j) \cdot \Delta x_j$$

and then take the limit on each side as $\|\mathcal{P}\| \to 0$.

$$\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx \; - \; \int_a^b g(x)\, dx$$

(In words, this says "the integral of a difference is the difference of the integrals.")

The justification for this difference property is quite similar to the justification of the sum property. (Or we can combine the sum property with the constant-multiple property, setting $k = -1$.)

? Practice 4.3.1

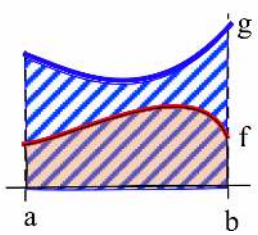
Given that $\int_1^4 f(x)\, dx = 7$ and that $\int_1^4 g(x)\, dx = 3$, evaluate the definite integral $\int_1^4 [f(x) - g(x)]\, dx$.

Answer

$$\int_1^4 [f(x) - g(x)]\, dx = 7 - 3 = 4$$

If $f(x) \leq g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$

Geometrically, the figure below illustrates that if f and g are both positive and that $f(x) \leq g(x)$ on the interval $[a, b]$:



then the area of region between the graph of f and the x-axis is smaller than the area of region between the graph of g and the x-axis.

Similar sketches for the situations where f or g are sometimes or always negative illustrate that the property holds in other situations as well, but we can avoid all of those different cases using Riemann sums.

If we use the same partition $\mathcal{P}$ and chosen points c_j for Riemann sums for f and g, then $f(c_j) \le g(c_j)$ for each j, so:

$$\sum_{j=1}^{n} f(c_j) \cdot \Delta x_j \le \sum_{j=1}^{n} g(c_j) \cdot \Delta x_j$$

Taking the limit over all such partitions as the mesh of those partitions approaches 0, we get $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Note

You may have noticed that we haven't called the justifications of these properties "proofs," in part because we haven't precisely defined what $\lim_{\|\mathcal{P}\| \to 0}$ means, but also because of some other technical details left to more advanced textbooks.

If: $m \le f(x) \le M$ for all x in $[a, b]$

then: $m \cdot (b - a) \le \int_a^b f(x)\,dx \le M \cdot (b - a)$

This property follows easily from the previous one. First let $g(x) = M$ so that $f(x) \le M = g(x)$ for all x in $[a, b]$, hence

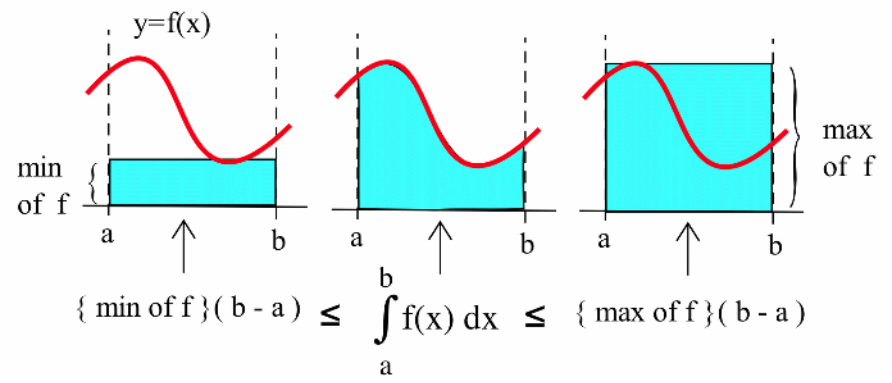
$$\int_a^b f(x)\,dx \le \int_a^b M\,dx = M \cdot (b - a)$$

(using one of our previous properties). Likewise, taking $g(x) = m$ so that $f(x) \ge m = g(x)$ for all x in $[a, b]$:

$$\int_a^b f(x)\,dx \ge \int_a^b m\,dx = m \cdot (b - a)$$

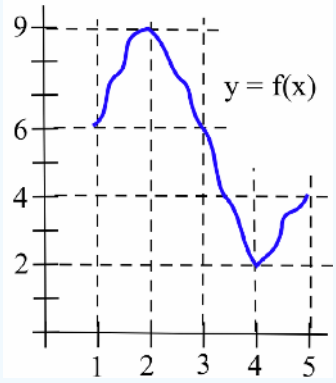
Geometrically, this says that if we can "trap" the output values of a function on the interval $[a, b]$ between two upper and lower bounds, m and M, then the value of the definite integral must lie between the areas of the rectangles with heights m and M.

If f is continuous on the closed interval $[a, b]$, then we know that f takes on a minimum value on that interval (call it m) and a maximum value (call it M), in which case this property just uses the lower and upper Riemann sums for the simplest possible partition of $[a, b]$:



Example 4.3.1

Determine lower and upper bounds for the value of $\int_1^5 f(x)\,dx$ with $f(x)$ given graphically below:



Solution

If $1 \le x \le 5$, then we can estimate (from the graph) that $2 \le f(x) \le 9$ so a lower bound for $\int_1^5 f(x)\,dx$ is

$$(b-a)\cdot(\text{minimum of } f \text{ on } [a,b]) = (4)(2) = 8$$

and an upper bound is:

$$(b-a)\cdot(\text{maximum of } f \text{ on } [a,b]) = (4)(9) = 36$$

We can conclude that $8 \le \int_1^5 f(x)\,dx \le 36$.

Knowing that the value of a definite integral is somewhere between 8 and 36 is not useful for finding its exact value, but the preceding estimation property is very easy to use and provides a "ballpark estimate" that will help you avoid reporting an unreasonable value.

Practice 4.3.2

Determine a lower bound and an upper bound for the value of $\int_3^5 f(x)\,dx$ with f as in the previous Example.

Answer

$m = 2$ and $M = 6$ so $(2)(5-3) = 4 \le \int_3^5 f(x)\,dx \le 12 = (6)(5-3)$

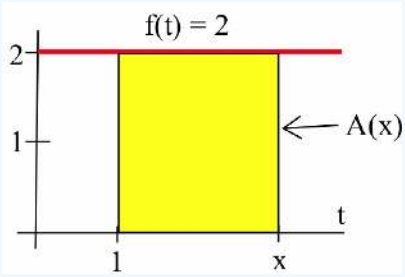
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Functions Defined by Integrals

If one of the endpoints a or b of the interval $[a, b]$ changes, then the value of the integral $\int_a^b f(t)\,dt$ typically changes. A definite integral of the form $\int_a^x f(t)\,dt$ defines a function of x that possesses interesting and useful properties. The next examples illustrate one such property: the derivative of a function defined by an integral is closely related to the integrand, the function "inside" the integral.

Example 4.3.2

For the function $f(t) = 2$, define $A(x)$ to be the area of the region bounded by f, the t-axis, and vertical lines at $t = 1$ and $t = x$:



a. Evaluate $A(1)$, $A(2)$, $A(3)$ and $A(4)$.
b. Find an algebraic formula for $A(x)$ valid for $x \geq 1$.
c. Calculate $A'(x)$.
d. Express $A(x)$ as a definite integral.

Solution

a. Referring to the graph above, we can see that $A(1) = 0$, $A(2) = 2$, $A(3) = 4$ and $A(4) = 6$.
b. Using the same area idea to compute a more general area:

$$A(x) = \text{area of a rectangle} = (\text{base})\,(\text{height}) = (x-1)(2) = 2x - 2$$

c. $A'(x) = \dfrac{d}{dx}(2x - 2) = 2$
d. $A(x) = \int_1^x 2\,dt$

Practice 4.3.3

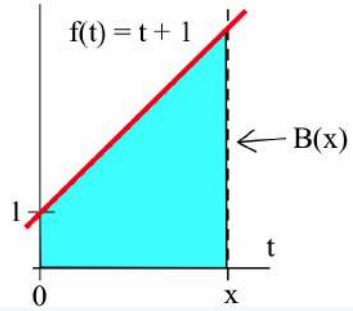
Answer the questions in the previous Example for $f(x) = 3$.

Answer

a. $A(1) = 0$, $A(2) = 3$, $A(3) = 6$, $A(4) = 9$
b. $A(x) = (x-1)(3) = 3x - 3$
c. $A'(x) = 3$
d. $A(x) = \int_1^x 3\,dt$

Example 4.3.3

For the function $f(t) = 1 + t$, define $B(x)$ to be the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = 0$ and $t = x$, as shown in the figure below:



a. Evaluate $B(0)$, $B(1)$, $B(2)$ and $B(3)$.
b. Find an algebraic formula for $B(x)$ valid for $x \geq 0$.
c. Calculate $B'(x)$.
d. Express $B(x)$ as a definite integral.

Solution

a. From the graph, $B(0) = 0$, $B(1) = 1.5$, $B(2) = 4$ and $B(3) = 7.5$.
b. Using the same area concept:

$$B(x) = \text{area of trapezoid} = (\text{base}) \cdot (\text{average height})$$
$$= (x) \cdot \left(\frac{1 + (1 + x)}{2} \right) = x + \frac{1}{2}x^2$$

c. $B'(x) = \dfrac{d}{dx}\left(x + \dfrac{1}{2}x^2\right) = 1 + x$
d. $B(x) = \displaystyle\int_0^x [1 + t]\ dt$

Practice 4.3.4

Answer the questions in the previous Example for $f(t) = 2t$.

Answer

a. $B(0) = 0$, $B(1) = 1$, $B(2) = 4$, $B(3) = 9$
b. $B(x) = \frac{1}{2}(x)(2x) = x^2$
c. $B'(x) = 2x$
d. $B(x) = \displaystyle\int_0^x 2t\, dt$

A curious "coincidence" appeared in each of these Examples and Practice problems: the derivative of the function defined by the integral was the same as the integrand, the function "inside" the integral. Stated another way, the function defined by the integral was an "antiderivative" of the function "inside" the integral. In Section 4.4 we will see that this "coincidence" is actually a property shared by all functions defined by an integral in this way. And it is such an important property that it is part of a result called the Fundamental Theorem of Calculus. Before we study the Fundamental Theorem of Calculus, however, we need to consider an "existence" question: Which functions can be integrated?

Which Functions Are Integrable?

This important question was finally answered in the 1850s by Bernhard Riemann, a name that should be familiar to you by now. Riemann proved that a function must be *badly* discontinuous in order to not be integrable.

Theorem

Every continuous function is integrable.

Note

Due to our inexact definition of the limit involved in the definition of the definite integral, we defer a proof of this theorem to more advanced textbooks.

This result says that if f is continuous on the interval $[a, b]$, then $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ approaches the same finite number, $\int_a^b f(x)\,dx$, as $\|\mathcal{P}\| \to 0$, no matter how we choose the partitions $\mathcal{P}$.

In fact, we can generalize this result to functions that have a finite number of breaks or jumps, as long as the function is bounded:

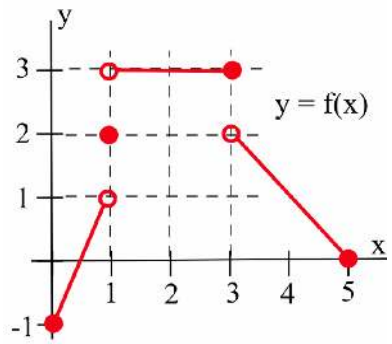
Theorem

If: f is defined on an interval $[a, b]$ and bounded ($|f(x)| \leq M$ for some number M for all x in $[a, b]$)

and: continuous except at a finite number of points in $[a, b]$

then & f is integrable on $[a, b]$.

The function f graphed below is always between -3 and 3 (in fact, always between -1 and 3), so it is bounded, and it is continuous except at $x = 1$ and $x = 3$:

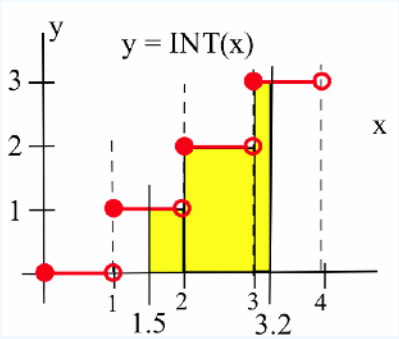


As long as the values of $f(1)$ and $f(3)$ are finite numbers, their actual values will not affect the value of the definite integral, and we can compute the value of the integral by computing the areas of the (triangular and rectangular) regions between the graph of f and the x-axis:

$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = 0 + 6 + 2 = 8$$

Practice 4.3.5

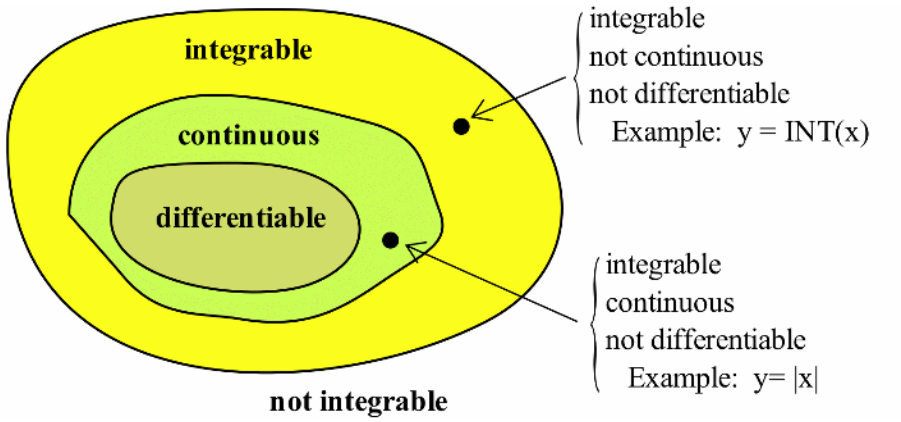
Evaluate $\int_{1.5}^{3.2} \lfloor x \rfloor\,dx$:



Answer

$(0.5)(1) + (1)(2) + (0.2)(3) = 3.1$

The figure below depicts graphically the relationships between differentiable, continuous and integrable functions:



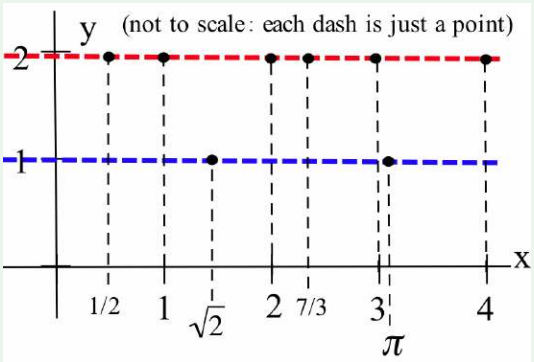
- Every differentiable function is continuous, but there are continuous functions that are not differentiable: a simple example of the latter is $f(x) = |x|$, which is continuous but not differentiable at $x = 0$.
- Every continuous function is integrable, but there are integrable functions that are not continuous: a simple example of the latter situation is the function $f(x)$ graphed in the margin, which is integrable on $[0, 5]$ but discontinuous at $x = 2$ and $x = 3$.
- Finally, as demonstrated by the next example, there are functions that are not integrable.

A Non-integrable Function

If f is continuous or piecewise continuous on $[a, b]$, then f is integrable on $[a, b]$. Fortunately, nearly all of the functions we will use throughout the rest of this book are integrable, as are the functions you are likely to need for common applications.

There are functions, however, for which the limit of the Riemann sums does not exist and hence, by definition, are not integrable. Recall the "holey" function from Section 0.4:

The function $h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$ is **not** integrable on $[0, 3]$.



Proof

For any partition $\mathcal{P}$ of $[0, 3]$, suppose that you, a very rational person, always choose values of c_k that are rational numbers. (Any open interval on the real-number line contains rational numbers and irrational numbers, so for each subinterval of the partition $\mathcal{P}$ you can always choose c_k to be a rational number.)

Then $h(c_k) = 2$, so for your Riemann sum:

$$\text{YS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} 2 \cdot \Delta x_k = 2 \cdot \sum_{k=1}^{n} \Delta x_k = 2 \cdot (3 - 0) = 6$$

Suppose your friend, however, always selects values of c_k that are irrational numbers. Then $h(c_k) = 1$ for each c_k, so for your friend's Riemann sum:

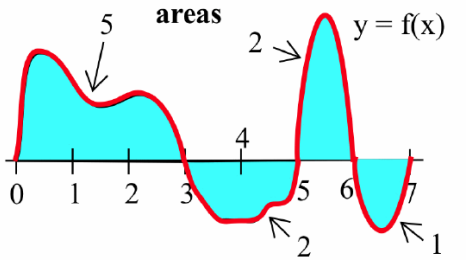
$$\text{FS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = 1 \cdot \sum_{k=1}^{n} \Delta x_k = 1 \cdot (3 - 0) = 3$$

So the limit of your Riemann sums, as the mesh of $\mathcal{P}$ approaches 0, will be 6, while the limit of your friend's sums will be 3. This means that $\lim\limits_{\|\mathcal{P}\| \to 0} \left(\sum\limits_{k=1}^{n} h(c_k) \cdot \Delta x_k \right)$ does not exist (because there is no single limiting value of the Riemann sums as $\|\mathcal{P}\| \to 0$) so $h(x)$ is not integrable on $[0, 3]$.

A similar argument shows that $h(x)$ is not integrable on *any* interval of the form $[a, b]$ (where $a < b$).

Problems

In Problems 1–20, refer to the graph of f given below:

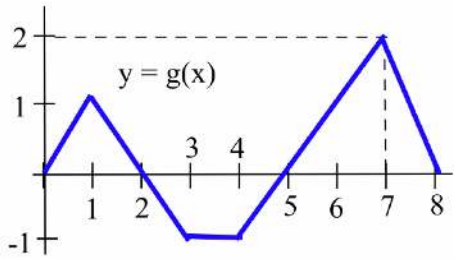


to determine the value of each definite integral.

1. $\int_0^3 f(x)\,dx$
2. $\int_3^5 f(x)\,dx$
3. $\int_2^2 f(x)\,dx$
4. $\int_6^7 f(w)\,dw$
5. $\int_0^5 f(x)\,dx$
6. $\int_0^7 f(x)\,dx$
7. $\int_3^6 f(t)\,dt$
8. $\int_5^7 f(x)\,dx$
9. $\int_3^0 f(x)\,dx$
10. $\int_5^3 f(x)\,dx$
11. $\int_6^0 f(x)\,dx$
12. $\int_0^3 2 \cdot f(x)\,dx$
13. $\int_4^4 f^2(s)\,ds$
14. $\int_0^3 [1 + f(x)]\,dx$
15. $\int_0^3 [x + f(x)]\,dx$

16. $\int_3^5 [3 + f(x)]\ dx$
17. $\int_0^5 [2 + f(x)]\ dx$
18. $\int_3^5 |f(x)|\ dx$
19. $\int_0^5 |f(x)|\ dx$
20. $\int_7^3 [1 + |f(x)|]\ dx$

Problems 21–30 refer to the graph of g given below:



Use the graph to evaluate each integral.

21. $\int_0^2 g(x)\,dx$
22. $\int_1^3 g(t)\,dt$
23. $\int_0^5 g(x)\,dx$
24. $\int_4^2 g(x)\,dx$
25. $\int_0^8 g(s)\,ds$
26. $\int_1^4 |g(x)|\,dx$
27. $\int_0^3 2 \cdot g(t)\,dt$
28. $\int_5^8 [1 + g(x)]\,dx$
29. $\int_6^3 g(u)\,du$
30. $\int_0^8 [t + g(t)]\,dt$

For Problems 31–34 , use the constant functions $f(x) = 4$ and $g(x) = 3$ on the interval $[0, 2]$. Calculate the value of each integral and verify that the value obtained in part (a) is **not** equal to the value in part (b).

31. a. $\int_0^2 f(x)\,dx \cdot \int_0^2 g(x)\,dx$
    b. $\int_0^2 f(x) \cdot g(x)\,dx$
32. a. $\dfrac{\int_0^2 f(x)\,dx}{\int_0^2 g(x)\,dx}$
    b. $\int_0^2 \dfrac{f(x)}{g(x)}\,dx$
33. a. $\int_0^2 [f(x)]^2\,dx$
    b. $\left(\int_0^2 f(x)\,dx\right)^2$
34. a. $\int_0^2 \sqrt{f(x)}\,dx$

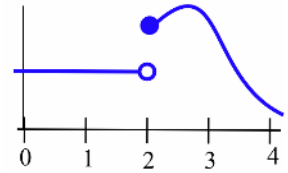
b. $\sqrt{\int_0^2 f(x)\,dx}$

For Problems 35–42, sketch a graph of the integrand function and use it to help evaluate the integral.

35. $\int_0^4 |x|\,dx$
36. $\int_0^4 [1+|t|]\,dt$
37. $\int_{-1}^2 |x|\,dx$
38. $\int_0^2 [|x|-1]\,dx$
39. $\int_1^3 \lfloor u \rfloor\,du$
40. $\int_1^{3.5} \lfloor x \rfloor\,dx$
41. $\int_1^3 [2+\lfloor t \rfloor]\,dt$
42. $\int_3^1 \lfloor x \rfloor\,dx$

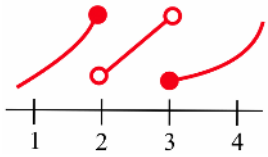
For Problems 43–46, sketch (a) a graph of $y = A(x) = \int_0^x f(t)\,dt$ and (b) a graph of $y = A'(x)$.

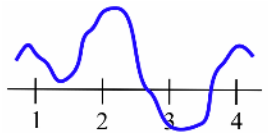
43. $f(x) = x$
44. $f(x) = x - 2$
45. 

46. 

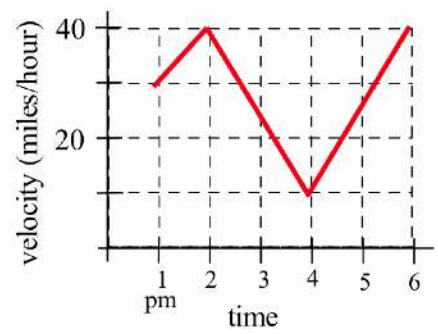
For Problems 47–50, state whether or not each function is:

a. continuous on $[1, 4]$
b. differentiable on $[1, 4]$
c. integrable on $[1, 4]$

47. $f(x)$ from Problem 45.
48. $f(x)$ from Problem 46.
49. 

50. 

51. The figure below shows the velocity of a car:



Write the total distance traveled by the car between 1:00 p.m. and 4:00 p.m. as a definite integral and estimate the value of that integral.

52. Write the total distance traveled by the car in the previous problem between 3:00 p.m. and 6:00 p.m. as a definite integral and estimate the value of that integral.

53. Define $g(x) = 7$ for $x \neq 2$ and $g(2) = 5$.

    a. Show that the Riemann sum for $g(x)$ for any partition $\mathcal{P}$ of the interval $[1, 4]$ is equal to $5w + 7(3 - w)$, where w is the width of the subinterval that includes $x = 2$.

    b. Compute the limit of these sums, as $\|\mathcal{P}\| \to 0$

    c. Compare the values of $\int_1^4 g(x)\,dx$ and $\int_1^4 7\,dx$.

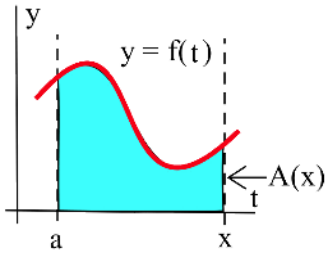
    d. What can you conclude about how changing the value of an integrable function at a single point affects the value of its definite integral?

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4.4: Areas, Integrals and Antiderivatives

This section explores properties of functions defined as areas and examines some connections among areas, integrals and antiderivatives. In order to focus on these connections and their geometric meaning, all of the functions in this section are nonnegative, but in the next section we will generalize (and prove) the results for all continuous functions. This section also introduces examples showing how you can use the relationships between areas, integrals and antiderivatives in various applications.

When f is a continuous, nonnegative function, the "area function" $A(x) = \int_a^x f(t)\,dt$ represents the area of the region bounded by the graph of f, the t-axis, and vertical lines at $t = a$ and $t = x$:



and the derivative of $A(x)$ represents the rate of change (growth) of $A(x)$ as the vertical line $t = x$ moves rightward. Examples 2 and 3 of Section 4.3 showed that for certain functions f, $A'(x) = f(x)$ so that $A(x)$ was an antiderivative of $f(x)$. The next theorem says the result is true for every continuous, nonnegative function f.

Theorem: The Area Function Is an Antiderivative

If: f is a continuous, nonnegative function and $A(x) = \int_a^x f(t)\,dt$ for $x \geq a$

then: $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = A'(x) = f(x)$

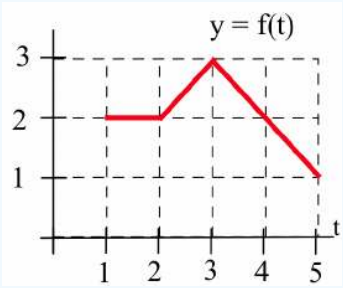
so : $A(x)$ is an antiderivative of $f(x)$.

This result relating integrals and antiderivatives is a special case (for nonnegative functions f) of the first part of the Fundamental Theorem of Calculus (FTC[1]), which we will prove in Section 4.5. This result is important for two reasons:

- It says that a large collection of functions have antiderivatives.
- It leads to an **easy** way to **exactly** evaluate definite integrals.

Example 4.4.1

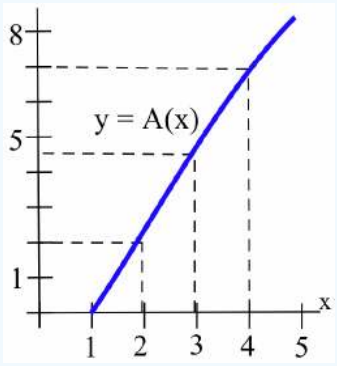
Define $A(x) = \int_1^x f(t)\,dt$ for the function $f(t)$ shown below:



Estimate the values of $A(x)$ and $A'(x)$ for $x = 2, 3, 4$ and 5 and use these values to sketch a graph of $y = A(x)$.

Solution

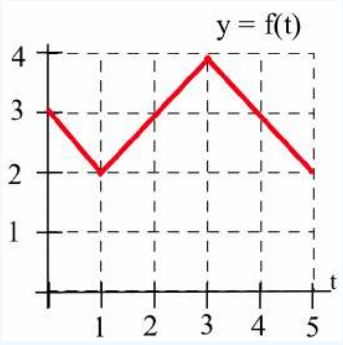
Dividing the region into squares and triangles, it is easy to see that $A(2) = 2$, $A(3) = 4.5$, $A(4) = 7$ and $A(5) = 8.5$. Because $A'(x) = f(x)$, we know that $A'(2) = f(2) = 2$, $A'(3) = f(3) = 3$, $A'(4) = f(4) = 2$ and $A'(5) = f(5) = 1$. A graph of $y = A(x)$ appears below:



It is important to recognize that f is not differentiable at $x = 2$ or $x = 3$ but that the values of A change smoothly near $x = 2$ and $x = 3$, and the function A *is* differentiable at those points and at every other point between $x = 1$ and $x = 5$. Also note that $f'(4) = -1$ (f is clearly decreasing near $x = 4$) but that $A'(4) = f(4) = 2$ is positive (the area A is growing even though f is getting smaller).

Practice 4.4.1

Let $B(x)$ be the area bounded by the horizontal axis, vertical lines at $t = 0$ and $t = x$, and the graph of $f(t)$, shown below:



Estimate the values of $B(x)$ and $B'(x)$ for $x = 1, 2, 3, 4$ and 5.

Answer

$B(1) = 2.5$, $B(2) = 5$, $B(3) = 8.5$, $B(4) = 12$, $B(5) = 14.5$

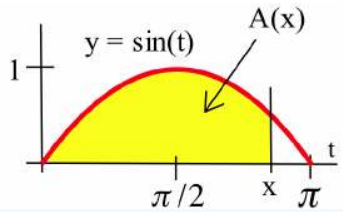
$B(x) = \int_0^x f(t)\,dt \;\Rightarrow\; B'(x) = \dfrac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$ (by the **Area Function Is an Antiderivative** Theorem), hence: $B'(1) = f(1) = 2$, $B'(2) = f(2) = 3$, $B'(3) = 4$, $B'(4) = 3$ and $B'(5) = 2$.

Example 4.4.2

Let $G(x) = \dfrac{d}{dx}\left(\int_0^x \sin(t)\,dt\right)$. Evaluate $G(x)$ for $x = \dfrac{\pi}{4}$, $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{4}$.

Solution

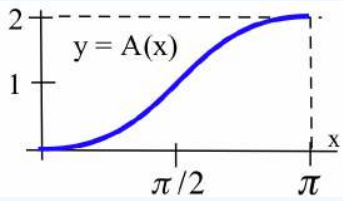
The figure below shows $A(x) = \int_0^x \sin(t)\,dt$ graphically:



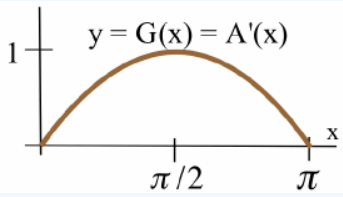
By the theorem, $A'(x) = \sin(x)$, so:

$$G\left(\frac{\pi}{4}\right) = A'\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707$$
$$G\left(\frac{\pi}{2}\right) = A'\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$
$$G\left(\frac{3\pi}{4}\right) = A'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707$$

The next figure shows a graph of $y = A(x)$:



and this figure shows the graph of $y = A'(x) = G(x)$:



Using Antiderivatives to Evaluate $\int_a^b f(x)\,dx$

Now we combine the ideas of areas and antiderivatives to devise a technique for evaluating definite integrals that is exact—and often easy.

If $A(x) = \int_a^x f(t)\,dt$, then we know that $A(a) = \int_a^a f(t)\,dt = 0$, $A(b) = \int_a^b f(t)\,dt$ and that $A(x)$ is an antiderivative of f, so $A'(x) = f(x)$. We also know that if $F(x)$ is **any** antiderivative of f, then $F(x)$ and $A(x)$ have the same derivative so $F(x)$ and $A(x)$ are “parallel” functions and differ by a constant: $F(x) = A(x) + C$ for all x and some constant C. As a consequence:

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a)$$
$$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt$$

This result says that, to evaluate a definite integral $A(b) = \int_a^b f(t)\,dt$, we can find **any** antiderivative F of f and simply evaluate $F(b) - F(a)$.

This result is a special case of the second part of the Fundamental Theorem of Calculus (FTC[2]), stated and proved in Section 4.5), which you will use hundreds of times over the next several chapters.

Theorem: Antiderivatives and Definite Integrals

If: f is a continuous, nonnegative function and F is any antiderivative of f (so that $F'(x) = f(x)$) on $[a, b]$

then: $\int_a^b f(t)\,dt = F(b) - F(a)$

The problem of finding the exact value of a definite integral has been reduced to finding some (any) antiderivative F of the integrand and then evaluating $F(b) - F(a)$. Even finding one antiderivative can be difficult, so for now we will restrict our attention to functions that have "easy" antiderivatives. Later we will explore some methods for finding antiderivatives of more "difficult" functions.

Because an evaluation of the form $F(b) - F(a)$ will occur quite often, we represent it symbolically as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

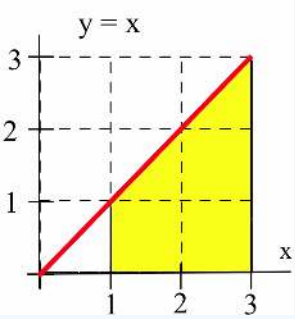
Example 4.4.3

Evaluate $\int_1^3 x\,dx$ in two ways:

a. by sketching a graph of $y = x$ and finding the area represented by the definite integral.
b. by finding an antiderivative $F(x)$ of $f(x) = x$ and evaluating $F(3) - F(1)$.

Solution

a. A graph of $y = x$ appears below:



the area of the trapezoidal region in question has area 4.

b. One antiderivative of x is $F(x) = \frac{1}{2}x^2$ (you should check for yourself that $\mathbf{D}\left(\frac{x^2}{2}\right) = x$), so:

$$F(x)\Big|_1^3 = F(3) - F(1) = \frac{1}{2}(3)^2 - \frac{1}{2}(1)^2 = \frac{9}{2} - \frac{1}{2} = 4$$

which agrees with the area from part (a).

If someone chose another antiderivative of x, say $F(x) = \frac{1}{2}x^2 + 7$ (you should check for yourself that $\mathbf{D}\left(\frac{x^2}{2} + 7\right) = x$), then:

$$F(x)\Big|_1^3 = F(3) - F(1) = \left[\frac{1}{2}(3)^2 + 7\right] - \left[\frac{1}{2}(1)^2 + 7\right] = \frac{23}{2} - \frac{15}{2} = 4$$

No matter which antiderivative F we choose, $F(3) - F(1) = 4$.

Practice 4.4.2

Evaluate $\int_1^3 (x - 1)\,dx$ in the two ways specified in the previous Example.

Answer

a. $\int_1^3 (x - 1)\,dx$ gives the area of the triangular region between the graph of $y = x - 1$ and the x-axis for $1 \le x \le 3$:

$$\text{area} = \frac{1}{2}\,(\text{base})\,(\text{height}) = \frac{1}{2}(2)(2) = 2$$

b. $F(x) = \frac{1}{2}x^2 - x$ is an antiderivative of $f(x) = x - 1$ so:

$$\int_1^3 (x-1)\,dx = F(3) - F(1) = \left[\frac{1}{2}\cdot 3^3 - 3\right] - \left[\frac{1}{2}\cdot 1^3 - 1\right] = 2$$

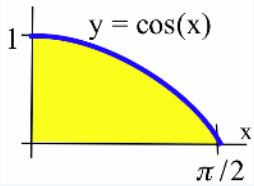
This antiderivative method provides an extremely powerful way to evaluate some definite integrals, and we will use it often.

Example 4.4.4

Find the area of the region in the first quadrant bounded by the graph of $y = \cos(x)$, the horizontal axis, and the vertical line $x = 0$.

Solution

The area we want:



is $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$ so we need an antiderivative of $f(x) = \cos(x)$. $F(x) = \sin(x)$ is one such antiderivative (you should check that $\mathrm{D}\,(\sin(x)) = \cos(x)$), so

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \sin(x)\Big|_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$$

is the area of the region in question.

Practice 4.4.3

Find the area of the region bounded by the graph of $y = 3x^2$, the horizontal axis and the vertical lines $x = 1$ and $x = 2$.

Answer

$$\text{Area} = \int_1^2 3x^2\,dx = x^3\Big|_1^2 = 2^3 - 1^3 = 8 - 1 = 7$$

Integrals, Antiderivatives and Applications

The antiderivative method for evaluating definite integrals can also be used when we need to find a more general "area," so it is often useful for solving applied problems.

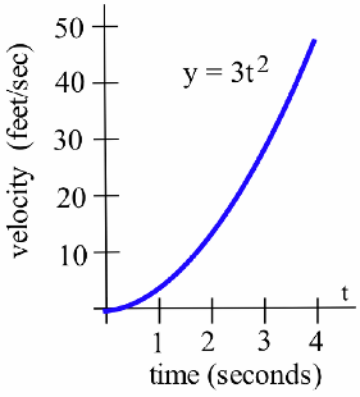
Example 4.4.5

A robot has been programmed so that when it starts to move, its velocity after t seconds will be $3t^2$ feet per second.

a. How far will the robot travel during its first four seconds of movement?
b. How far will the robot travel during its next four seconds of movement?
c. How long will it take for the robot to move 729 feet from its starting place?

Solution

a. The distance during the first four seconds will be the area under the graph of the velocity function:



from $t = 0$ to $t = 4$, an area we can compute with the definite integral $\int_0^4 3t^2\,dt$. One antiderivative of $3t^2$ is t^3 so:

$$\int_0^4 3t^2\,dt = \left[t^3\right]_0^4 = 4^3 - 0^3 = 64$$

and we can conclude that the robot will be 64 feet away from its starting position after four seconds.

b. Proceeding similarly:

$$\int_4^8 3t^2\,dt = \left[t^3\right]_4^8 = 8^3 - 4^3 = 512 - 64 = 448 \text{ feet}$$

c. This question is different from the first two. Here we know the lower integration endpoint, $t = 0$, and the total distance, 729 feet, and need to find the upper integration endpoint (the time when the robot is 729 feet away from its starting position). Calling this upper endpoint T, we know that:

$$729 = \int_0^T 3t^2\,dt = \left[t^3\right]_0^T = T^3 - 0^3 = T^3$$

so $T = \sqrt[3]{729} = 9$. The robot is 729 feet away after 9 seconds.

? Practice 4.4.4

Refer to the robot from Example **4.4.5**.

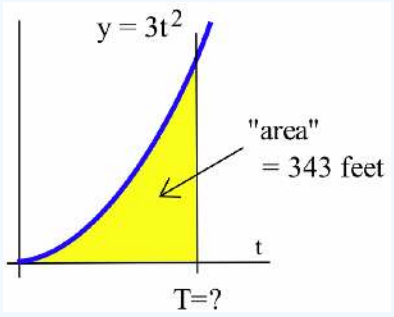
a. How far will the robot travel between $t = 1$ and $t = 5$ seconds?

b. How long will it take for the robot to move 343 feet from its starting place?

Answer

a. distance $= \int_1^5 3t^2\,dt = t^3\Big|_1^5 = 125 - 1 = 124$ feet.

b. We know the starting point is $x = 0$ and the total distance ("area" under the velocity curve) is 343 feet. We need to find the time T:



so that 343 feet $= \int_0^T 3t^2\,dt$:

$$343 = \int_0^T 3t^2\,dt = t^3\Big|_0^T = T^3 - 0 = T^3$$

hence $T = \sqrt[3]{343} = 7$ seconds.

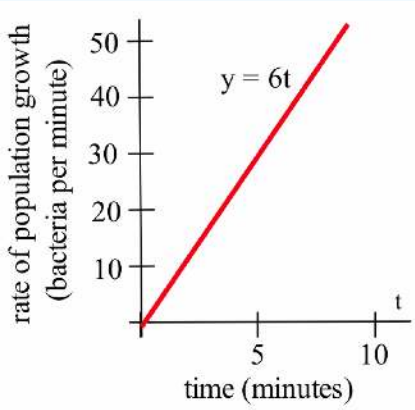
Example 4.4.6

Suppose that t minutes after placing 1,000 bacteria on a Petri plate the rate of growth of the bacteria population is $6t$ bacteria per minute.

a. How many new bacteria are added to the population during the first seven minutes?
b. What is the total population after seven minutes?
c. When will the total population reach 2,200 bacteria?

Solution

a. The number of new bacteria is represented by the area under the rate-of-growth graph:



and one antiderivative of $6t$ is $3t^2$ (check that $\mathrm{D}\left(3t^2\right) = 6t$) so:

$$\text{new bacteria} = \int_0^7 6t\,dt = \left[3t^2\right]_0^7 = 3(7)^2 - 3(0)^2 = 147$$

b. $[\text{old population}] + [\text{new bacteria}] = 1000 + 147 = 1147$ bacteria.
c. When the total population reaches 2,200 bacteria, then there are $2200 - 1000 = 1200$ new bacteria, hence we need to find the time T required for that many new bacteria to grow:

$$1200 = \int_0^T 6t\,dt = \left[3t^2\right]_0^T = 3(T)^2 - 3(0)^2 = 3T^2$$

so $T^2 = 400 \Rightarrow T = 20$. After 20 minutes, the total bacteria population will be $1000 + 1200 = 2200$.

Practice 4.4.5

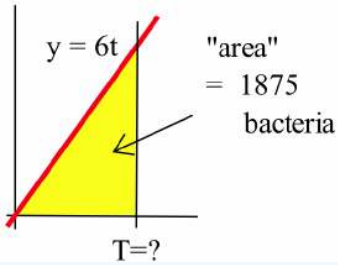
Refer to the bacteria population from the previous Example.

a. How many new bacteria will be added to the population between $t = 4$ and $t = 8$ minutes?
b. When will the total population reach 2,875 bacteria?

Answer

a. new bacteria $= \int_4^8 6t\,dt = 3t^2\Big|_4^8 = 3 \cdot 64 - 3 \cdot 16 = 144$ bacteria.

b. We know the total new population ("area" under the rate-of-change graph):



is $2875 - 1000 = 1875$ so:

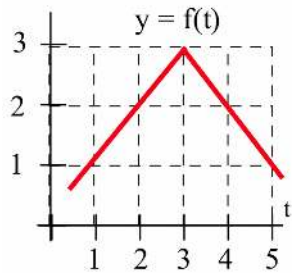
$$1875 = \int_0^T 6t\,dt = 3t^2\Big|_0^T = 3T^2 - 0 = 3T^2 \;\Rightarrow\; T^2 = 625$$

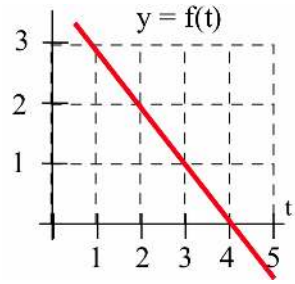
hence $T = \sqrt{625} = 25$ minutes.

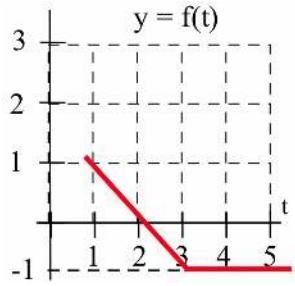
Problems

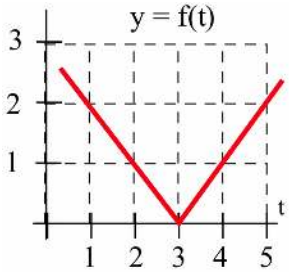
In Problems 1–8, $A(x) = \int_1^x f(t)\,dt$ with $f(t)$ given.

a. Graph $y = A(x)$ for $1 \le x \le 5$.
b. Estimate the values of $A(1)$, $A(2)$, $A(3)$ and $A(4)$.
c. Estimate $A'(1)$, $A'(2)$, $A'(3)$ and $A'(4)$.

1. 

2. 

3. 

4. 

5. $f(t) = 2$
6. $f(t) = 1 + t$
7. $f(t) = 6 - t$
8. $f(t) = 1 + 2t$

In Problems 9–18, use the **Antiderivatives and Definite Integrals** Theorem to evaluate each integral.

9. a. $\int_0^3 2x\,dx$
   b. $\int_1^3 2x\,dx$
   c. $\int_0^1 2x\,dx$
10. a. $\int_0^2 4x^3\,dx$
    b. $\int_0^1 4x^3\,dx$
    c. $\int_1^2 4x^3\,dx$
11. a. $\int_1^3 6x^2\,dx$
    b. $\int_1^2 6x^2\,dx$
    c. $\int_0^3 6x^2\,dx$
12. a. $\int_{-2}^2 2x\,dx$
    b. $\int_{-2}^{-1} 2x\,dx$
    c. $\int_{-2}^0 2x\,dx$
13. a. $\int_0^3 4x^3\,dx$
    b. $\int_1^3 4x^3\,dx$
    c. $\int_0^1 4x^3\,dx$
14. a. $\int_0^5 4x^3\,dx$
    b. $\int_0^2 4x^3\,dx$
    c. $\int_2^5 4x^3\,dx$
15. a. $\int_{-3}^3 3x^2\,dx$
    b. $\int_{-3}^0 3x^2\,dx$
    c. $\int_0^3 3x^2\,dx$
16. a. $\int_0^3 5\,dx$

b. $\int_0^2 5\,dx$

c. $\int_2^3 5\,dx$

17. a. $\int_0^2 3x^2\,dx$

b. $\int_1^3 3x^2\,dx$

c. $\int_3^1 3x^2\,dx$

18. a. $\int_{-2}^2 \left[12 - 3x^2\right]\,dx$

b. $\int_0^2 \left[12 - 3x^2\right]\,dx$

c. $\int_1^2 \left[12 - 3x^2\right]\,dx$

In 19–21, use the given velocity of a car (in feet per second) after t seconds to find:

a. how far the car travels during the first 10 seconds.

b. how many seconds it takes the car to travel half the distance in part (a).

19. $v(t) = 2t$

20. $v(t) = 3t^2$

21. $v(t) = 4t^3$

Problems 22–23 give the velocity of a car (in feet per second) after t seconds.

a. How many seconds does it take for the car to come to a stop (velocity $= 0$)?

b. How far does the car travel before coming to a stop?

c. How many seconds does it take the car to travel half the distance in part (b)?

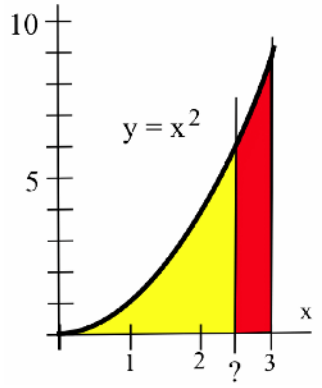
22. $v(t) = 20 - 2t$

23. $v(t) = 75 - 3t^2$

24. Find the exact area under half of one arch of the sine curve: $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$.

25. An artist you know wants to make a figure consisting of the region between the curve $y = x^2$ and the x-axis for $0 \le x \le 3$.

a. Where should the artist divide the region with a vertical line:



so that each piece has the same area?

b. Where should she divide the region with vertical lines to get three pieces with equal areas?

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4.5: The Fundamental Theorem of Calculus

This section contains the most important and most frequently used theorem of calculus, **THE** Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz during the late 1600s, it establishes a connection between derivatives and integrals, provides a way to easily calculate many definite integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is the most important brick in that beautiful structure.

Prior sections have emphasized the meaning of the definite integral, defined it, and began to explore some of its applications and properties. In this section, the emphasis shifts to the Fundamental Theorem of Calculus. You will use this theorem often in later sections.

The Fundamental Theorem has two parts. They resemble results in the previous section but apply to more general situations. The first part (FTC^1) says that every continuous function has an antiderivative and shows how to differentiate a function defined as an integral. The second part (FTC^2) shows how to evaluate the definite integral of any function if we know a formula for an antiderivative of that function.

Part 1: Antiderivatives

Every continuous function has an antiderivative, even functions with "corners," such as the absolute value function $f(x) = |x|$, that fail to be differentiable at one or more points.

The Fundamental Theorem of Calculus Part 1 (FTC^1)

If: f is continuous and $A(x) = \int_a^x f(t)\,dt$

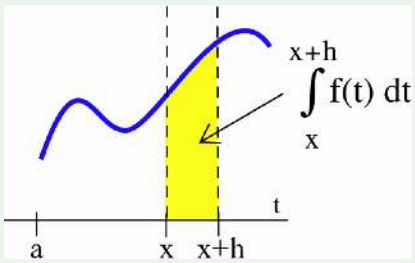
then: $A'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

so: $A(x)$ is an antiderivative of $f(x)$.

Proof

For a continuous function f, let $A(x) = \int_a^x f(t)\,dt$. By the definition of derivative,

$$A'(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$



Using one of the integral properties from Section 4.3, we know that:

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$$
$$\Rightarrow \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

Assume for the moment that $h > 0$. Because f is continuous on $[x, x+h]$ we know that f attains a maximum and minimum on that interval, so there are values m_h and M_h with $x < m_h < x+h$ and $x < M_h < x+h$ so that $f(m_h) \le f(t) \le f(M_h)$ when $x \le t \le x+h$. Hence:

$$\int_x^{x+h} f(m_h)\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} f(M_h)\,dt$$
$$\Rightarrow f(m_h)\cdot h \le \int_x^{x+h} f(t)\,dt \le f(M_h)\cdot h$$
$$\Rightarrow f(m_h) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le f(M_h)$$

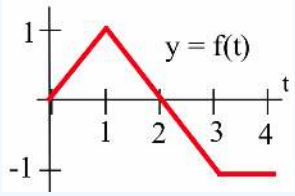
Because $x < m_h < x+h$, we know $\lim_{h\to 0^+} m_h = x$; consequently—because $f(t)$ is continuous—we also know that $\lim_{h\to 0^+} f(m_h) = f(x)$. Likewise, $\lim_{h\to 0^+} f(M_h) = f(x)$, so the Squeezing Theorem tells us that:

$$\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$$

Repeating this argument for $h < 0$ is relatively straightforward.

Example 4.5.1

Define $A(x) = \int_0^x f(t)\,dt$ for f in the figure below:



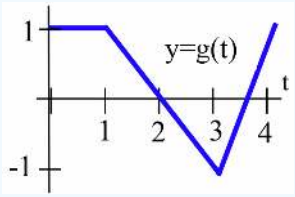
Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.

Solution

$A(1) = \int_0^1 f(t)\,dt = \frac{1}{2}$, $A(2) = \int_0^2 f(t)\,dt = 1$, $A(3) = \int_0^3 f(t)\,dt = \frac{1}{2}$ and $A(4) = \int_0^4 f(t)\,dt = -\frac{1}{2}$. Because f is continuous, FTC[1] tells us that $A'(x) = f(x)$, so $A'(1) = f(1) = 1$, $A'(2) = f(2) = 0$, $A'(3) = f(3) = -1$ and $A'(4) = f(4) = -1$.

Practice 4.5.1

Define $A(x) = \int_0^x g(t)\,dt$ for g in the figure below:



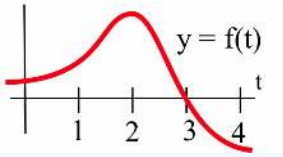
Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.

Answer

$A(1) = 1$, $A(2) = 1.5$, $A(3) = 1$, $A(4) = 1$; $A'(x) = g(x)$ so $A'(1) = g(1) = 1$, $A'(2) = g(2) = 0$, $A'(3) = -1$, $A'(4) = 1$.

Example 4.5.2

Define $A(x) = \int_0^x f(t)\,dt$ for f in the figure below:



For which value of x is $A(x)$ maximum? For which x is the rate of change of A maximum?

Solution

Because A is differentiable, the only critical points are where $A'(x) = 0$ or at endpoints. $A'(x) = f(x) = 0$ at $x = 3$, and A has a maximum at $x = 3$. Notice that the values of $A(x)$ increase as x goes from 0 to 3 and then the values of A decrease. The rate of change of $A(x)$ is $A'(x) = f(x)$, and $f(x)$ appears to have a maximum at $x = 2$, so the rate of change of $A(x)$ is maximum when $x = 2$. Near $x = 2$, a slight increase in the value of x yields the maximum increase in the value of $A(x)$.

Part 2: Evaluating Definite Integrals

If we know a formula for an antiderivative of a function, then we can compute any definite integral of that function.

The Fundamental Theorem of Calculus Part 2 (FTC^2)

If: $f(x)$ is continuous and $F(x)$ is any antiderivative of f (so that $F'(x) = f(x)$)

then: $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

Proof

Define $A(x) = \int_a^x f(t)\,dt$. If F is an antiderivative of f, then $F'(x) = f(x)$ and by FTC^1 we know that $A'(x) = f(x)$ so $F'(x) = A'(x)$, hence $F(x)$ and $A(x)$ differ by a constant: $A(x) - F(x) = C$ for all x and some constant C. At $x = a$, we have $C = A(a) - F(a) = 0 - F(a) = -F(a)$ so $C = -F(a)$ and the equation $A(x) - F(x) = C$ becomes $A(x) - F(x) = -F(a)$. Then $A(x) = F(x) - F(a)$ for all x, so setting $x = b$ yields $A(b) = F(b) - F(a)$, hence $\int_a^b f(x)\,dx = F(b) - F(a)$, the formula we wanted.

We can evaluate the definite integral of a continuous function f by finding an antiderivative of f (*any* antiderivative of f will work) and then doing some arithmetic with this antiderivative. FTC^2 does not tell us *how* to find an antiderivative of f, and it does not tell us how to find the definite integral of a discontinuous function. It is possible to evaluate definite integrals of some discontinuous functions (as we saw in Section 4.3) but not by using FTC^2 directly.

Example 4.5.3

Evaluate $\int_0^2 (x^2 - 1)\,dx$.

Solution

$F(x) = \frac{1}{3}x^3 - x$ is an antiderivative of $f(x) = x^2 - 1$ (you should check that $\mathbf{D}\left(\frac{1}{3}x^3 - x\right) = x^2 - 1$), so:

$$\int_0^2 (x^2 - 1)\,dx = \left[\frac{1}{3}x^3 - x\right]_0^2 = \left[\frac{1}{3}\cdot 2^3 - 2\right] - \left[\frac{1}{3}\cdot 0^3 - 0\right] = \frac{2}{3}$$

If your friend had picked a different antiderivative of $x^2 - 1$, say $G(x) = \frac{1}{3}x^3 - x + 4$,
then her calculations would be slightly different:

$$\begin{aligned}\int_0^2 (x^2 - 1)\,dx &= \left[\frac{1}{3}x^3 - x + 4\right]_0^2 \\ &= \left[\frac{1}{3}\cdot 2^3 - 2 + 4\right] - \left[\frac{1}{3}\cdot 0^3 - 0 + 4\right] = \frac{2}{3} + 4 - 4 = \frac{2}{3}\end{aligned}$$

but the result would be the same.

Practice 4.5.2

Evaluate $\int_1^3 (3x^2 - 1)\, dx$.

Answer

$F(x) = x^3 - x$ is an antiderivative of $f(x) = 3x^2 - 1$ so:

$$\int_1^3 [3x^2 - 1]\, dx = [x^3 - x]_1^3 = [3^3 - 3] - [1^3 - 1] = 24$$

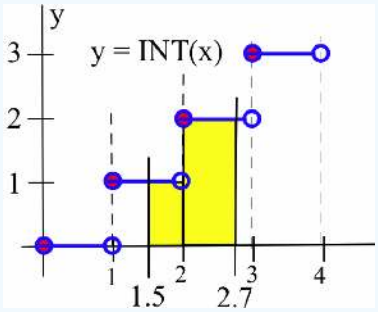
\)F(x) = x^3 - x + 7\) is another antiderivative of $f(x) = 3x^2 - 1$ so:

$$\int_1^3 [3x^2 - 1]\, dx = [x^3 - x + 7]_1^3 = [3^3 - 3 + 7] - [1^3 - 1 + 7] = 24$$

No matter which antiderivative of $f(x) = 3x^2 - 1$ you use, the value of the definite integral $\int_1^3 [3x^2 - 1]\, dx$ is 24.

Example 4.5.4

Evaluate $\int_{1.5}^{2.7} \lfloor x \rfloor\, dx$ where $\lfloor x \rfloor = \text{INT}(x)$ is the largest integer less than or equal to x, as in the figure below:



Solution

$f(x) = \lfloor x \rfloor$ is not continuous at $x = 2$ in the interval $[1.5, 2.7]$, so we cannot employ the Fundamental Theorem of Calculus directly. We can, however, use our understanding of the geometric meaning of a definite integral to compute:

$$\begin{aligned}\int_{1.5}^{2.7} \lfloor x \rfloor\, dx &= (\text{area below } y = \lfloor x \rfloor \text{ for } 1.5 \le x \le 2) + (\text{area below } y = \lfloor x \rfloor \text{ for } 2 \le x \le 2.7) \\ &= (\text{first base})\,(\text{first height}) + (\text{second base})\,(\text{second height}) \\ &= (0.5)(1) + (0.7)(2) = 1.9\end{aligned}$$

We could also split the integral into two pieces:

$$\begin{aligned}\int_{1.5}^{2.7} \lfloor x \rfloor\, dx &= \int_{1.5}^{2.0} \lfloor x \rfloor\, dx + \int_{2.0}^{2.7} \lfloor x \rfloor\, dx \\ &= \int_{1.5}^{2.0} 1\, dx + \int_{2.0}^{2.7} 2\, dx = \Big[x\Big]_{1.5}^{2.0} + \Big[x\Big]_{2.0}^{2.7} \\ &= [2.0 - 1.5] + [2(2.7) - 2(2.0)] = 0.5 + 1.4 = 1.9\end{aligned}$$

using the fact that $\lfloor x \rfloor = 1$ for $1.5 \le x < 2.0$ and the fact that $\lfloor x \rfloor = 2$ for $2.0 \le x \le 2.7$. (We also need to redefine the first integrand to equal 1 at its right endpoint and the second integrand to equal 2 at its right endpoint so that each integrand is continuous on a closed interval).

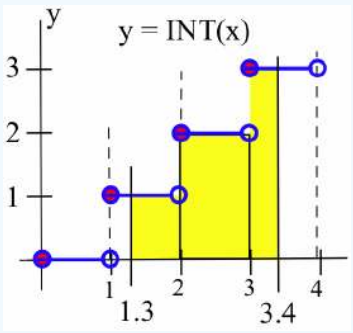
(Problem 53 in Section 4.3 indicates that this redefinition is perfectly legal.)

? Practice 4.5.3

Evaluate $\int_{1.3}^{3.4} \lfloor x \rfloor \, dx$.

Answer

Because $f(x) = \lfloor x \rfloor$ is not continuous on $[1.3, 3.4]$ we cannot use the Fundamental Theorem of Calculus. Instead, we can think of the definite integral as an area:



and compute:

$$\int_{1.3}^{3.4} \lfloor x \rfloor \, dx = 3.9$$

Calculus is the study of derivatives and integrals, their meanings and their applications. The Fundamental Theorem of Calculus demonstrates how differentiation and integration are closely related processes: integration is really anti-differentiation, the inverse of differentiation.

Applications: The Future

Calculus is important for many reasons, but students are usually required to study calculus because they will need to *apply* calculus concepts in a variety of fields. Most applied problems in integral calculus require the following steps to get from a real-life problem to a numerical answer:

$$\boxed{\text{applied problem}} \xrightarrow{1} \boxed{\text{Riemann sum}} \xrightarrow{2} \boxed{\text{definite integral}} \xrightarrow{3} \boxed{\text{number}}$$

Step 1 is absolutely vital. If we can not translate the ideas of an applied problem into an area or a Riemann sum or a definite integral, then we can not use integral calculus to solve the problem. For a few special types of applied problems, we will be able to move directly from the problem to an integral, but usually it will be easier to first break the problem into smaller pieces and to build a Riemann sum. Section 4.7 and all of Chapter 5 focus on translating different types of applied problems into Riemann sums and definite integrals. **Computers and calculators are seldom of any help with Step 1.**

Step 2 is usually easy. If we have a Riemann sum $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ on an interval $[a, b]$, then the limit of the sum (as $n \to \infty$) is simply the definite integral $\int_a^b f(x)\,dx$.

Step 3 can be handled in several ways.

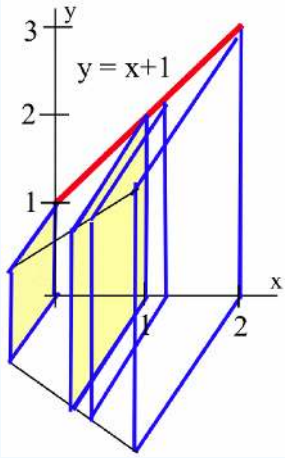
- If the function f is relatively simple, we may be able to find an antiderivative for f (using techniques from Section 4.6 and Chapter 8) and then apply FTC^2 to get a numerical answer.
- If the function f is more complicated, then integral tables or computers (Section 4.8) may help us find an antiderivative for f, in which case we can apply FTC^2 to get a numerical answer.
- If we cannot find an antiderivative for f, we can compute approximate numerical answers for the definite integral using various approximation methods (Sections 4.9 and 8.7); we typically employ computers to carry out the heavy-duty arithmetic.

Usually any difficulties in solving an applied problem arise in the first and third steps. There are techniques and details to master and understand, but it is also important to keep in mind where these techniques and details fit into the bigger picture.

The next Example illustrates these steps for the problem of finding a volume of a solid. We will explore techniques for finding volumes of solids in greater detail in Chapter 5.

Example 4.5.5

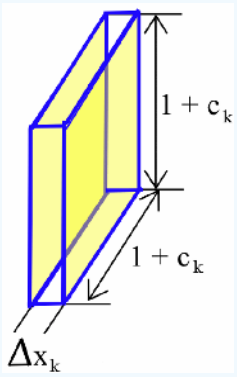
Find the volume of the solid shown in the figure below:



for $0 \le x \le 2$. (Each "slice" perpendicular to the xy-plane is a square.)

Solution

Step 1: Going from the figure to a Riemann sum. If we break the solid into n "slices" with cuts perpendicular to the x-axis (and the xy-plane) using a partition $\mathcal{P}$ with cuts at x_1, x_2, x_3,…, x_{n-1} (like slicing a block of cheese or a loaf of bread), then the volume of the original solid is equal to the sum of the volumes of the "slices."



The volume of the k-th slice is *approximately* equal to the volume of a thin, rectangular box:

$$(\text{height}) \cdot (\text{base}) \cdot (\text{thickness}) \approx (c_k + 1)\,(c_k + 1) \cdot \Delta x_k$$

where c_k is any chosen value between x_{k-1} and x_k. Therefore:

$$\text{total volume} = \sum_{k=1}^{n} (\text{volume of the } k\text{-th slice}) = \sum_{k=1}^{n} (c_k+1)^2\,\Delta x_k$$

which is a Riemann sum.

Step 2: Going from the Riemann sum to a definite integral. We can improve the Riemann sum approximation of the total volume from Step 1 by taking thinner slices (making all of the Δx_k smaller and smaller) so that the mesh of the partition $\mathcal{P}$ approaches 0:

$$\lim_{\|\mathcal{P}\|\to 0}\sum_{k=1}^{n}(c_k+1)^2\,\Delta x_k=\int_0^2(x+1)^2\,dx=\int_0^2\left[x^2+2x+1\right]\,dx$$

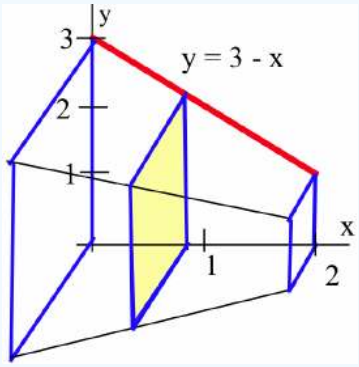
Step 3: Going from the definite integral to a numerical answer. We can now use FTC[2] to evaluate the integral: $F(x)=\frac{1}{3}x^3+x^2+x$ is an antiderivative of x^2+2x+1 (check this by differentiating $F(x)$), so:

$$\begin{aligned}\int_0^2\left[x^2+2x+1\right]\,dx&=\left[\frac{1}{3}x^3+x^2+x\right]_0^2\\&=\left[\frac{1}{3}\cdot 2^3+2^2+2\right]-\left[\frac{1}{3}\cdot 0^3+0^2+0\right]=\frac{26}{3}\end{aligned}$$

The volume of the solid shape is exactly $\frac{26}{3}$ cubic inches.

Practice 4.5.4

Find the volume of the solid shape in the figure below:



for $0\le x\le 2$. (Each “slice” perpendicular to the xy-plane is a square.)

Answer

First break the solid into “slices” and approximate the volume of the k-th slice by $(3-c_k)^2\cdot\Delta x_k$ where c_k is any point in the k-th subinterval. Next add up these approximate volumes to get a Riemann Sum:

$$\sum_{k=1}^{n}(3-c_k)^2\cdot\Delta x_k$$

and then take the limit of these Riemann sums as the mesh of the partitions approaches 0 (and $n\to\infty$, where n is the number of subintervals in the partition):

$$\begin{aligned}\lim_{\|\mathcal{P}\|\to 0}\left[\sum_{k=1}^{n}(3-c_k)^2\cdot\Delta x_k\right]&=\int_0^2(3-x)^2\,dx\\&=\int_0^2\left(9-6x+x^2\right)\,dx\\&=\left[9x-3x^2+\frac{1}{3}x^3\right]_0^2\\&=\left[18-12+\frac{8}{3}\right]-[0-0+0]=\frac{26}{3}\end{aligned}$$

Leibniz’s Rule For Differentiating Integrals

If the endpoint of an integral is a function of x rather than simply x, then we need to use the Chain Rule together with FTC[1] to calculate the derivative of the integral. For example:

$$A'(x) = f(x) \quad \Rightarrow \quad \frac{d}{dx}\left[A\left(x^2\right)\right] = A'(x)\cdot 2x = f\left(x^2\right)\cdot 2x$$

We can generalize this result by applying the Chain Rule to the derivative of the integral:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = \frac{d}{dx}\left[A\left(g(x)\right)\right] = f\left(g(x)\right)\cdot g'(x)$$

and combine this with some integral properties to further extend FTC[1].

Leibniz's Rule

If: f is a continuous function, $A(x) = \int_a^x f(t)\,dt$ and $g_1(x)$ and $g_2(x)$ are both differentiable functions

then: $\frac{d}{dx}\left[\int_{g_1(x)}^{g_2(x)} f(t)\,dt\right] = f\left(g_2(x)\right)\cdot g_2'(x) - f\left(g_1(x)\right)\cdot g_1'(x)$

Proof

Assume for simplicity that f, g_1 and g_2 are continuous on $(-\infty,\infty)$ and let c be any number. Then:

$$\begin{aligned}\int_{g_1(x)}^{g_2(x)} f(t)\,dt &= \int_c^{g_2(x)} f(t)\,dt + \int_{g_1(x)}^{c} f(t)\,dt \\ &= \int_c^{g_2(x)} f(t)\,dt - \int_c^{g_1(x)} f(t)\,dt\end{aligned}$$

Now apply the preceding result.

Example 4.5.6

If a is any constant, compute the derivatives $\frac{d}{dx}\left[\int_a^{5x} t^2\,dt\right]$, $\frac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right]$ and $\frac{d}{dw}\left[\int_{\pi w}^{\sin w} z^3\,dz\right]$.

Solution

Applying Leibniz's Rule:

$$\begin{aligned}\frac{d}{dx}\left[\int_a^{5x} t^2\,dt\right] &= (5x)^2\cdot 5 = 125x^2 \\ \frac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right] &= \cos(x^2)\cdot 2x = 2x\cos(x^2) \\ \frac{d}{dw}\left[\int_{\pi w}^{\sin(w)} z^3\,dz\right] &= (\sin(w))^3\cdot\cos(w) - (\pi w)^3\cdot\pi\end{aligned}$$

The last quantity simplifies to $\sin^3(w)\cos(w) - \pi^4 w^3$.

Practice 4.5.5

Compute $\frac{d}{dx}\left[\int_0^{x^3} \sin(t)dt\right]$.

Answer

$$\frac{d}{dx}\left[\int_0^{x^3} \sin(t)\,dt\right] = \sin(x^3)\cdot\frac{d}{dx}\left[x^3\right] = 3x^2\sin(x^3)$$

Problems

In Problems 1–2:

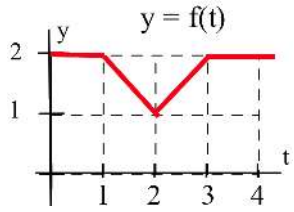
a. Use FTC[2] to find a formula for $A(x)$, differentiate $A(x)$ to obtain a formula for $A'(x)$, and evaluate $A'(x)$ at $x = 1$, 2 and 3.
b. Use FTC[1] to evaluate $A'(x)$ at $x = 1$, 2 and 3.

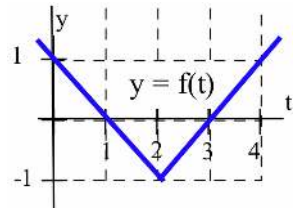
1. $A(x) = \int_0^x 3t^2\,dt$
2. $A(x) = \int_1^x (1 + 2t)\,dt$

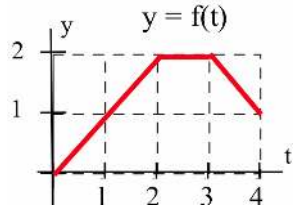
In Problems 3–8, compute $A'(1)$, $A'(2)$ and $A'(3)$.

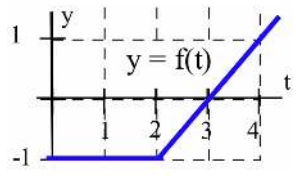
3. $A(x) = \int_0^x 2t\,dt$
4. $A(x) = \int_1^x 2t\,dt$
5. $A(x) = \int_{-3}^x 2t\,dt$
6. $A(x) = \int_0^x (3 - t^2)\,dt$
7. $A(x) = \int_0^x \sin(t)\,dt$
8. $A(x) = \int_1^x |t - 2|\,dt$

In Problems 9–12, $A(x) = \int_0^x f(t)\,dt$, with $f(t)$ given graphically. Evaluate $A'(1)$, $A'(2)$ and $A'(3)$.

9. 

10. 

11. 

12. 

In Problems 13–33, verify that $F(x)$ is an antiderivative of the integrand and use FTC[2] to evaluate the integral.

13. $\int_0^1 2x\,dx,\ F(x) = x^2 + 5$
14. $\int_1^4 3x^2\,dx,\ F(x) = x^3 + 2$

15. $\int_1^3 x^2\,dx,\ F(x)=\frac{1}{3}x^3$
16. $\int_0^3 \left[x^2+4x-3\right]\,dx,\ F(x)=\frac{1}{3}x^3+2x^2-3x$
17. $\int_1^5 \frac{1}{x}\,dx,\ F(x)=\ln(x)$
18. $\int_2^5 \frac{1}{x}\,dx,\ F(x)=\ln(x)+4$
19. $\int_{\frac{1}{2}}^3 \frac{1}{x}\,dx,\ F(x)=\ln(x)$
20. $\int_1^3 \frac{1}{x}\,dx,\ F(x)=\ln(x)+2$
21. $\int_0^{\frac{\pi}{2}} \cos(x)\,dx,\ F(x)=\sin(x)$
22. $\int_0^{\pi} \sin(x)\,dx,\ F(x)=-\cos(x)$
23. $\int_0^1 \sqrt{x}\,dx,\ F(x)=\frac{2}{3}x^{\frac{3}{2}}$
24. $\int_1^4 \sqrt{x}\,dx,\ F(x)=\frac{2}{3}x^{\frac{3}{2}}$
25. $\int_1^7 \sqrt{x}\,dx,\ F(x)=\frac{2}{3}x^{\frac{3}{2}}$
26. $\int_1^4 \frac{1}{2\sqrt{x}}\,dx,\ F(x)=\sqrt{x}$
27. $\int_1^9 \frac{1}{2\sqrt{x}}\,dx,\ F(x)=\sqrt{x}$
28. $\int_2^5 \frac{1}{x^2}\,dx,\ F(x)=-\frac{1}{x}$
29. $\int_{-2}^3 e^x\,dx,\ F(x)=e^x$
30. $\int_0^3 \frac{2x}{1+x^2}\,dx,\ F(x)=\ln(1+x^2)$
31. $\int_0^{\frac{\pi}{4}} \sec^2(x)\,dx,\ F(x)=\tan(x)$
32. $\int_1^e \ln(x)\,dx,\ F(x)=x\cdot\ln(x)-x$
33. $\int_0^3 2x\sqrt{1+x^2}\,dx,\ F(x)=\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}$

For Problems 34–48, find an antiderivative of the integrand and use FTC^2 to evaluate the definite integral.

34. $\int_2^5 3x^2\,dx$
35. $\int_{-1}^2 x^2\,dx$
36. $\int_3^1 \left[x^2+4x-3\right]\,dx$
37. $\int_1^e \frac{1}{x}\,dx$
38. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)\,dx$
39. $\int_{25}^{100} \sqrt{x}\,dx$
40. $\int_3^5 \sqrt{x}\,dx$
41. $\int_1^{10} \frac{1}{x^2}\,dx$
42. $\int_{1000}^1 \frac{1}{x^2}\,dx$
43. $\int_0^1 e^x\,dx$
44. $\int_{-2}^2 \frac{2x}{1+x^2}\,dx$

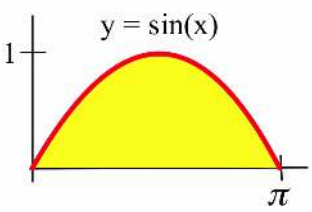
45. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2(x)\,dx$

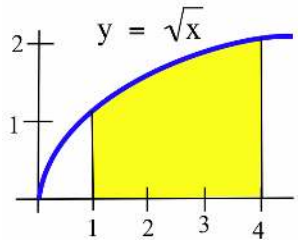
46. $\int_{0}^{1} e^{2x}\,dx$

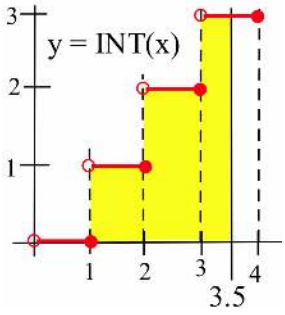
47. $\int_{3}^{3} \sin(x)\cdot\ln(x)\,dx$

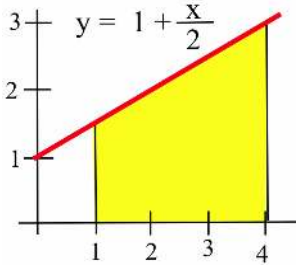
48. $\int_{2}^{4} (x-2)^3\,dx$

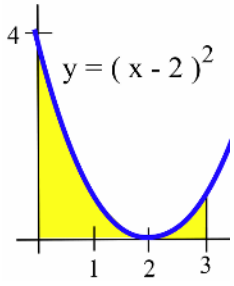
In Problems 49–54, find the area of the shaded region.

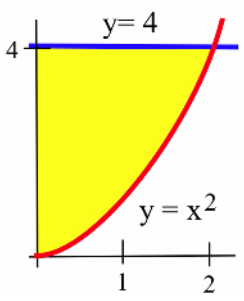
49.


50.


51.


52.


53.


54. 

55. Given that $A'(x) = \tan(x)$, find $\mathbf{D}\left(A(3x)\right)$, $\mathbf{D}\left(A(x^2)\right)$ and $\mathbf{D}\left(A(\sin(x))\right)$.

56. Given that $B'(x) = \sec(x)$, find $\mathbf{D}\left(B(3x)\right)$, $\mathbf{D}\left(B(x^2)\right)$ and $\mathbf{D}\left(B(\sin(x))\right)$.

In Problems 57–68, apply Leibniz's Rule.

57. $\dfrac{d}{dx}\left[\displaystyle\int_1^{5x} \sqrt{1+t}\,dt\right]$

58. $\dfrac{d}{dx}\left[\displaystyle\int_2^{x^2} \sqrt{1+t}\,dt\right]$

59. $\dfrac{d}{dx}\left[\displaystyle\int_0^{\sin(x)} \sqrt{1+t}\,dt\right]$

60. $\dfrac{d}{dx}\left[\displaystyle\int_1^{2+3x} \left(t^2+5\right)\,dt\right]$

61. $\dfrac{d}{dx}\left[\displaystyle\int_0^{1-2x} \left(3t^2+2\right)\,dt\right]$

62. $\dfrac{d}{dx}\left[\displaystyle\int_x^{9} \left(3t^2+2\right)\,dt\right]$

63. $\dfrac{d}{dx}\left[\displaystyle\int_x^{\pi} \cos(3t)\,dt\right]$

64. $\dfrac{d}{dx}\left[\displaystyle\int_{7x}^{\pi} \cos(2t)\,dt\right]$

65. $\dfrac{d}{dx}\left[\displaystyle\int_x^{x^2} \tan(t)\,dt\right]$

66. $\dfrac{d}{dx}\left[\displaystyle\int_0^{\pi} \cos(3t)\,dt\right]$

67. $\dfrac{d}{dx}\left[\displaystyle\int_2^{\ln(x)} 5t\cdot\cos(3t)\,dt\right]$

68. $\dfrac{d}{dx}\left[\displaystyle\int_0^{\pi} \tan(7t)\,dt\right]$

69. $\dfrac{d}{dy}\left[\displaystyle\int_0^{y^2} \tan(\theta)\,d\theta\right]$

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4.6: Finding Antiderivatives

In order to use the second part of the Fundamental Theorem of Calculus, we need an antiderivative of the integrand, but sometimes it is not easy to find one. This section collects some of the information we already know about general properties of antiderivatives and about antiderivatives of particular functions. It shows how to use this information to find antiderivatives of more complicated functions and introduces a "change of variable" technique to make that job easier.

Indefinite Integrals and Antiderivatives

Antiderivatives arise so often that there is a special notation to indicate the antiderivative of a function:

Definition

$\int f(x)\,dx$, read as "the **indefinite integral** of f" or as "the antiderivatives of f," represents the collection (or family) of all functions whose derivatives are f.

If you've been wondering why we called $\int_a^b f(t)\,dt$ a **definite** integral, now you know. A definite integral has specific upper and lower limits, while an indefinite integral does not.

If F is an antiderivative of f, then any member of the family $\int f(x)\,dx$ has the form $F(x)+C$ for some constant C. We write $\int f(x)\,dx = F(x)+C$, where C represents an arbitrary constant. There are no small families in the world of antiderivatives: if f has one antiderivative F, then f has an *infinite* number of antiderivatives and each has the form $F(x)+C$, which means there are many ways to write a particular indefinite integral and some of them may look very different. You can check that $F(x)=\sin^2(x)$, $G(x)=-\cos^2(x)$ and $H(x)=2\sin^2(x)+\cos^2(x)$ all have the same derivative, $f(x)=2\sin(x)\cos(x)$, so the indefinite integral of $2\sin(x)\cos(x)$, $\int 2\sin(x)\cos(x)\,dx$, can be written in several ways: $\sin^2(x)+C$ or $-\cos^2(x)+K$ or $2\sin^2(x)+\cos^2(x)+C$.

Practice 4.6.1

Verify that $\int 2\tan(x)\cdot\sec^2(x)\,dx=\tan^2(x)+C$ and that $\int 2\tan(x)\cdot\sec^2(x)\,dx=\sec^2(x)+K$.

Answer

$\mathrm{D}\left(\tan^2(x)+C\right) = 2\tan^1(x)\cdot\mathrm{D}\left(\tan(x)\right) = 2\tan(x)\sec^2(x)$ BREAK $\mathrm{D}\left(\sec^2(x)+C\right) = 2\sec^1(x)\cdot\mathrm{D}\left(\sec(x)\right) = 2\sec(x)\cdot\sec(x)\tan(x)$

Properties of Antiderivatives (Indefinite Integrals)

These sum, difference and constant-multiple properties follow directly from corresponding properties for derivatives.

Theorem

If f and g are integrable functions, then:

- $\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\int k\cdot f(x)\,dx = k\cdot\int f(x)\,dx$

Although we know general rules for *derivatives* of products and quotients, unfortunately there are no easy general patterns for *antiderivatives* of products and quotients—we will only be able to add one more general property to this list (in Section 8.2).

We already know antiderivatives for several important functions.

Theorem

Constant Functions: $\int k\,dx = kx + C$

Powers of x: $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C$ if $p \neq -1$, $\int \dfrac{1}{x}\,dx = \ln|x| + C$

Exponential Functions: $\int e^x\,dx = e^x + C$

Trig Functions: $\int \cos(x)\,dx = \sin(x) + C$, $\int \sin(x)\,dx = -\cos(x) + C$

$\int \sec^2(x)\,dx = \tan(x) + C$, $\int \csc^2(x)\,dx = -\cot(x) + C$

$\int \sec(x)\cdot\tan(x)\,dx = \sec(x) + C$, $\int \csc(x)\cdot\cot(x)\,dx = -\csc(x) + C$

All of these antiderivatives can be verified by differentiating. For $\int \frac{1}{x}\,dx$ you may be wondering about the presence of the absolute value signs in the antiderivative. If $x > 0$, you can check that:

$$\mathrm{D}\left(\ln(|x|)\right) = \mathrm{D}\left(\ln(x)\right) = \frac{1}{x}$$

If $x < 0$, then you can check that:

$$\mathrm{D}\left(\ln(|x|)\right) = \mathrm{D}\left(\ln(-x)\right) = \frac{-1}{-x} = \frac{1}{x}$$

When computing a definite integral of the form $\int_a^b \frac{1}{x}\,dx$, either a and b will both be positive or both be negative, because the integrand is not defined at $x = 0$, so $x = 0$ cannot be included in the interval of integration.

Our list of antiderivatives of particular functions will grow in coming chapters and will eventually include antiderivatives of additional trigonometric functions, the inverse trigonometric functions, logarithms, rational functions and more. (See Appendix I.)

Antiderivatives of More Complicated Functions

Antiderivatives are very sensitive to small changes in the integrand, so we should be very careful. (Fortunately, an antiderivative can always be checked by differentiating, so even though we may not find the correct antiderivative, we should be able to determine whether or not an antiderivative candidate is actually an antiderivative.)

Example 4.6.1

We know $\mathrm{D}\left(\sin(x)\right) = \cos(x)$, so $\int \cos(x)\,dx = \sin(x) + C$. Find:

a. $\int \cos(2x+3)\,dx$

b. $\int \cos(5x-7)\,dx$

c. $\int \cos(x^2)\,dx$

Solution

a. Because $\sin(x)$ is an antiderivative of $\cos(x)$, it is reasonable to hope that $\sin(2x+3)$ will be an antiderivative of $\cos(2x+3)$. Unfortunately, we see that $\mathrm{D}\left(\sin(2x+3)\right) = \cos(2x+3)\cdot 2$, exactly twice the result we want. Let's try again by modifying our "guess" to be half the original guess:

$$\mathrm{D}\left(\frac{1}{2}\sin(2x+3)\right) = \frac{1}{2}\cos(2x+3)\cdot 2 = \cos(2x+3)$$

which is what we want, so $\int \cos(2x+3)\,dx = \frac{1}{2}\sin(2x+3) + C$.

b. $\mathrm{D}\left(\sin(5x-7)\right) = \cos(5x-7)\cdot 5$, so dividing the original guess by 5 we get $\mathrm{D}\left(\frac{1}{5}\sin(5x-7)\right) = \frac{1}{5}\cos(5x-7)\cdot 5 = \cos(5x-7)$ and conclude that $\int \cos(5x-7)\,dx = \frac{1}{5}\sin(5x-7) + C$.

c. $\mathrm{D}\left(\sin(x^2)\right) = \cos(x^2)\cdot 2x$. It was easy enough in parts (a) and (b) to modify our "guesses" to eliminate the constants 2 and 5, but here the x is much harder to eliminate:

$$\begin{aligned}\mathrm{D}\left(\frac{1}{2x}\sin\left(x^2\right)\right) &= \mathrm{D}\left(\frac{\sin(x^2)}{2x}\right)\\ &= \frac{2x\cdot \mathrm{D}\left(\sin(x^2)\right) - \sin(x^2)\cdot \mathrm{D}(2x)}{(2x)^2}\\ &= \frac{(2x)^2\cos(x^2) - 2\sin(x^2)}{(2x)^2}\\ &= \cos(x^2) - \frac{\sin(x^2)}{2x^2} \neq \cos(x^2)\end{aligned}$$

Our guess did not check out—we're stuck.

The value of a definite integral of $\cos(x^2)$ could still be approximated as accurately as needed by using Riemann sums or one of the numerical techniques in Sections 4.9 and 8.7, but no matter how hard we try, we cannot find a concise formula for an antiderivative of $\cos(x^2)$ in order to use the Fundamental Theorem of Calculus. Even a simple-looking integrand can be very difficult. At this point, there is no quick way to tell the difference between an "easy" indefinite integral and a "difficult" or "impossible" one.

Note

Advanced mathematical techniques beyond the scope of this text can show that $\cos(x^2)$ does not have an "elementary" antiderivative composed of polynomials, roots, trigonometric functions, exponential functions or their inverses.

Getting the Constants Right

The previous example illustrated one technique for finding antiderivatives: "guess" the form of the answer, differentiate your "guess" and then modify your original "guess" so its derivative is exactly what you want it to be.

Example 4.6.2

Knowing that $\int \sec^2(x)\,dx = \tan(x) + C$ and $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$, find:

a. $\int \sec^2(3x+7)\,dx$

b. $\int \frac{1}{\sqrt{5x+3}}\,dx$

Solution

a. If we "guess" an answer of $\tan(3x+7)$ and then differentiate it, we get $\mathrm{D}\left(\tan(3x+7)\right) = \sec^2(3x+7)\cdot \mathrm{D}(3x+7) = 3\sec^2(3x+7)$, which is three times what we want. If we divide our original guess by 3 and try again, we have:

$$\begin{aligned}\mathrm{D}\left(\frac{1}{3}\tan(3x+7)\right) &= \frac{1}{3}\mathrm{D}\left(\tan(3x+7)\right) = \frac{1}{3}\sec^2(3x+7)\cdot 3\\ &= \sec^2(3x+7)\end{aligned}$$

so $\int \sec^2(3x+7)\,dx = \frac{1}{3}\tan(3x+7) + C$.

b. If we "guess" $2\sqrt{5x+3}$ and then differentiate it, we get:

$$\mathrm{D}\left(2\,(5x+3)^{\frac{1}{2}}\right) = 2\cdot\frac{1}{2}(5x+3)^{-\frac{1}{2}}\,\mathrm{D}(5x+3) = 5\cdot(5x+3)^{-\frac{1}{2}}$$

which is five times what we want. Dividing our guess by 5 and differentiating, we have:

$$\mathrm{D}\left(\frac{2}{5}(5x+3)^{\frac{1}{2}}\right) = \frac{2}{5}\cdot\frac{1}{2}(5x+3)^{-\frac{1}{2}}\cdot 5 = \frac{1}{\sqrt{5x+3}}$$

so $\int \frac{1}{\sqrt{5x+3}}\,dx = \frac{2}{5}\sqrt{5x+3} + C$.

? Practice 4.6.2

Find $\int \sec^2(7x)\,dx$ and $\int \frac{1}{\sqrt{3x+8}}\,dx$.

Answer

We know $\mathbf{D}(\tan(x)) = \sec^2(x)$, so it is reasonable to try $\tan(7x)$: \)\mbox{D}\left(\tan(7x)\right) = \sec^2(7x)\cdot\mbox{D}(7x) = 7\sec^2(7x)\), a result seven times the result we want, so divide the original "guess" by 7 and try again:

$$\mathbf{D}\left(\frac{1}{7}\tan(7x)\right) = \frac{1}{7}\sec^2(7x)\cdot 7 = \sec^2(7x)$$

so $\int \sec^2(7x)\,dx = \frac{1}{7}\tan(7x) + C$.

$\mathbf{D}\left((3x+8)^{\frac{1}{2}}\right) = \frac{1}{2}(3x+8)^{-\frac{1}{2}}\mathbf{D}(3x+8) = \frac{3}{2}(3x+8)^{-\frac{1}{2}}$ so multiply our original "guess" by $\frac{2}{3}$:

$$\mathbf{D}\left(\frac{2}{3}(3x+8)^{\frac{1}{2}}\right) = \frac{2}{3}\cdot\frac{1}{2}\cdot(3x+8)^{-\frac{1}{2}}\cdot\mathbf{D}(3x+8) = \frac{2}{3}\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{3x+8}}$$

hence $\int \frac{1}{\sqrt{3x+8}}\,dx = \frac{2}{3}\sqrt{3x+8} + C$.

The "guess and check" method is a very effective technique if you can make a good first guess, one that misses the desired result only by a constant multiple. In that situation, just divide the first guess by the unwanted constant multiple. If the derivative of your guess misses by something other than a constant multiple, then more drastic modifications are needed. Sometimes the next technique can help.

Making Patterns More Obvious: Changing the Variable

Successful integration is mostly a matter of recognizing patterns. The "change of variable" technique can make some underlying patterns of an integral easier to recognize. Essentially, the technique involves rewriting an integral that is originally in terms of one variable, say x, in terms of another variable, say u, with the hope that it will be easier to find an antiderivative of the new integrand.

For example, we can rewrite $\int \cos(5x+1)\,dx$ by setting $u = 5x+1$. Then $\cos(5x+1)$ becomes $\cos(u)$ but we must also convert the dx in the original integral. We know that $\frac{du}{dx} = 5$, so rewriting this last expression in differential notation, we get $du = 5\,dx$; isolating dx yields $dx = \frac{1}{5}\,du$ so:

$$\int \cos(5x+1)\,dx = \int \cos(u)\cdot\frac{1}{5}\,du = \frac{1}{5}\int \cos(u)\,du$$

This new integral is easier:

$$\frac{1}{5}\int \cos(u)\,du = \frac{1}{5}\sin(u) + C$$

but our original problem was in terms of x and our answer is in terms of u, so we must "resubstitute" using the relationship $u = 5x + 1$:

$$\frac{1}{5}\sin(u) + C = \frac{1}{5}\sin(5x+1) + C$$

We can now conclude that:

$$\int \cos(5x+1)\,dx = \frac{1}{5}\sin(5x+1) + C$$

(As always, you can check this result by differentiating.)

Note

We first discussed differential notation in Section 2.8; although you may not have used them much in differential calculus, you will now use them extensively.

We can summarize the steps of this "change of variable" (or "u-substitution") method as:

- set a new variable, say u, equal to some function of the original variable x
- calculate the differential du in terms of x and dx
- rewrite the original integral in terms of u and du
- integrate the new integral to get an answer in terms of u
- resubstitute for u to get a result in terms of the original variable x

Often u is set equal to some "interior" part of the original integrand function.

Example 4.6.3

Make the suggested change of variable, rewrite each integral in terms of u and du, and evaluate the integral.

a. $\int \cos(x) \cdot e^{\sin(x)}\, dx$ with $u = \sin(x)$

b. $\int \dfrac{2x}{5+x^2}\, dx$ with $u = 5 + x^2$

Solution

a. $u = \sin(x) \Rightarrow du = \cos(x)\, dx$ and $e^{\sin(x)} = e^u$:

$$\int \cos(x) e^{\sin(x)}\, dx = \int e^u\, du = e^u + C = e^{\sin(x)} + C$$

b. $u = 5 + x^2 \Rightarrow du = 2x\, dx$, so:

$$\int \frac{2x}{5+x^2}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln\left|5 + x^2\right| + C$$

Because $5 + x^2 > 0$, we can also write the answer as $\ln(5 + x^2)$.

In each example, the change of variable did not find the antiderivative, but it did make the pattern of the integrand more obvious, which in turn made it easier to determine an antiderivative.

Practice 4.6.3

Make the suggested change of variable, rewrite each integral in terms of u and du and evaluate the integral.

a. $\int (7x+5)^3\, dx$ with $u = 7x + 5$

b. $\int 3x^2 \cdot \sin\left(x^3 - 1\right)\, dx$ with $u = x^3 - 1$

Answer

a. $u = 7x + 5 \Rightarrow du = 7\, dx \Rightarrow dx = \frac{1}{7}\, du$ so:

$$\int (7x+5)^3\, dx = \int u^3 \cdot \frac{1}{7}\, du = \frac{1}{7} \cdot \frac{1}{4} u^4 + C = \frac{1}{28}(7x+5)^4 + C$$

b. $u = x^3 - 1 \Rightarrow du = 3x^2\, dx$ so $\int \sin(x^3 - 1) \cdot 3x^2\, dx$ becomes:

$$\int \sin(u)\, du = -\cos(u) + C = -\cos\left(x^3 - 1\right) + C$$

The previous examples have supplied a suggested substitution, but in the future *you* will need to decide what u should equal. Unfortunately there are no rules that guarantee your choice will lead to an easier integral—sometimes you will need to resort to

trial and error until you find a particular u-substitution that works for your integrand. There is, however, a "rule of thumb" that frequently results in easier integrals. Even though the following suggestion comes with no guarantees, it is often worth trying.

A "Rule of Thumb" for Changing the Variable

If part of the integrand consists of a composition of functions, $f(g(x))$, try setting $u = g(x)$, the "inner" function.

If part of the integrand is being raised to a power, try setting u equal to the part being raised to the power. For example, if the integrand includes $(3+\sin(x))^5$, try $u = 3 + \sin(x)$. If part of the integrand involves a trigonometric (or exponential or logarithmic) function of another function, try setting u equal to the "inside" function: if the integrand includes the function $\sin(3 + x^2)$, try $u = 3 + x^2$.

The key to becoming skilled at selecting a good u and correctly making the substitution is **practice**.

Example 4.6.4

Select a u for each integrand and rewrite the associated integral in terms of u and du.

a. $\int \cos(3x)\sqrt{2+\sin(3x)}\,dx$

b. $\int \frac{5e^x}{2+e^x}\,dx$

c. $\int e^x \cdot \sin(e^x)\,dx$

Solution

a. If $u = 2 + \sin(3x)$, $du = 3\cos(3x)\,dx \Rightarrow \frac{1}{3}\,du = \cos(3x)\,dx$ so the integral becomes $\int \frac{1}{3}\sqrt{u}\,du$.

b. With $u = 2 + e^x \Rightarrow du = e^x\,dx$, the integral becomes $\int \frac{5}{u}\,du$.

c. With $u = e^x \Rightarrow du = e^x\,dx$, the integral becomes $\int \sin(u)\,du$.

Changing Variables with Definite Integrals

If we need to change variables in a *definite* integral, we have two choices:

- First work out the corresponding *indefinite* integral and then use that antiderivative and FTC[2] to evaluate the definite integral.
- Change variables in the definite integral, which requires changing the limits of integration from x limits to u limits.

For the second option, if the original integral had endpoints $x = a$ and $x = b$, and we make the substitution $u = g(x) \Rightarrow du = g'(x)\,dx$, then the new integral will have endpoints $u = g(a)$ and $u = g(b)$:

$$\int_{x=a}^{x=b} f(g(x)) \cdot g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Example 4.6.5

Evaluate $\int_0^1 (3x-1)^4\,dx$.

Solution

Using the first option with $u = 3x - 1 \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$, the corresponding indefinite integral becomes:

$$\int (3x-1)^4\,dx = \int u^4 \cdot \frac{1}{3}\,du = \frac{1}{3}\cdot\frac{1}{5}u^5 + C = \frac{1}{15}(3x-1)^5 + C$$

We now use this result to evaluate the original definite integral:

$$\int_0^1 (3x-1)^4\,dx = \left[\frac{1}{15}(3x-1)^5\right]_0^1 = \left[\frac{1}{15}\cdot 2^5\right] - \left[\frac{1}{15}\cdot(-1)^5\right]$$
$$= \frac{32}{15} - \frac{-1}{15} = \frac{33}{15} = \frac{11}{5}$$

For the second option, we make the same substitution $u = 3x-1 \Rightarrow \frac{1}{3}\,du = dx$ while also computing $x = 0 \Rightarrow u = 3\cdot 0 - 1 = -1$ and $x = 1 \Rightarrow u = 3\cdot 1 - 1 = 2$:

$$\int_{x=0}^{x=1} (3x-1)^4\,dx = \int_{u=-1}^{u=2} \frac{1}{3}u^4\,du = \frac{1}{3}\cdot\frac{1}{5}u^5\Big|_{-1}^{2} \qquad = \frac{2^5}{15} - \frac{(-1)^5}{15} = \frac{33}{15}$$

We arrive at the same answer either way.

(Both options require roughly the same amount of work and computation. In practice you should choose the option that seems easiest for you and poses the least risk of error.)

? Practice 4.6.4

If the original integrals in Example 4 had endpoints:

a. $x = 0$ to $x = \pi$
b. $x = 0$ to $x = 2$ or
c. $x = 0$ to $x = \ln(3)$

then the new integrals should have what endpoints?

Answer

a. $u = 2 + \sin(3x)$ so $x = 0 \Rightarrow u = 2 + \sin(3\cdot 0) = 2$ and $x = \pi \Rightarrow u = 2 + \sin(3\pi) = 2$. (This integral is now easy; why?)
b. $u = 2 + e^x$ so $x = 0 \Rightarrow u = 2 + e^0 = 3$ and $x = 2 \Rightarrow u = 2 + e^2$
c. $u = e^x$ so $x = 0 \Rightarrow u = e^0 = 1$ and $x = \ln(3) \Rightarrow u = e^{\ln(3)} = 3$

Special Transformations for $\int \sin^2(x)\,dx$ and $\int \cos^2(x)\,dx$

The integrals of $\sin^2(x)$ and $\cos^2(x)$ arise often, and we can find their antiderivatives with the help of some trigonometric identities:

$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\sin(2x) = 2\sin(x)\cos(x)$$

Solving the first identity in the margin for $\sin^2(x)$, we get:

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and solving the second identity for $\cos^2(x)$, we get:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Integrating the first of these new identities yields:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right] dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

Using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we can also write:

$$\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C$$

Similarly, using $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ yields:

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C = \frac{1}{2}x + \frac{1}{2}\sin(x)\cos(x) + C$$

(In practice, it's easier to remember the new trig identities and use them to work out these antiderivatives, rather than memorizing the antiderivatives directly.)

Problems

For Problems 1–4:

a. put $f(x) = x^2$ and
b. $g(x) = x$ to verify the inequality.

1. $\int_1^2 f(x)\cdot g(x)\,dx \neq \left(\int_1^2 f(x)\,dx\right)\left(\int_1^2 g(x)\,dx\right)$
2. $\int_1^2 \frac{f(x)}{g(x)}\,dx \neq \frac{\int_1^2 f(x)\,dx}{\int_1^2 g(x)\,dx}$
3. $\int_0^1 f(x)\cdot g(x)\,dx \neq \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right)$
4. $\int_1^4 \frac{f(x)}{g(x)}\,dx \neq \frac{\int_1^4 f(x)\,dx}{\int_1^4 g(x)\,dx}$

For Problems 5–14, use the suggested u to find du and rewrite the integral in terms of u and du. Then find an antiderivative in terms of u and, finally, rewrite your answer in terms of x.

5. $\int \cos(3x)\,dx,\ u = 3x$
6. $\int \sin(7x)\,dx,\ u = 7x$
7. $\int e^x \sin(2 + e^x)\,dx,\ u = 2 + e^x$
8. $\int e^{5x}\,dx,\ u = 5x$
9. $\int \cos(x)\sec^2(\sin(x))\,dx,\ u = \sin(x)$
10. $\int \frac{\cos(x)}{\sin(x)}\,dx,\ u = \sin(x)$
11. $\int \frac{5}{3 + 2x}\,dx,\ u = 3 + 2x$
12. $\int x^2\left(5+x^3\right)^7\,dx,\ u = 5 + x^3$
13. $\int x^2 \sin\left(1 + x^3\right)\,dx,\ u = 1 + x^3$
14. $\int \frac{e^x}{1 + e^x}\,dx,\ u = 1 + e^x$

For Problems 15–26, use the change-of-variable technique to evaluate the indefinite integral.

15. $\int \cos(4x)\,dx$
16. $\int e^{3x}\,dx$
17. $\int x^3\left(5+x^4\right)^{11}\,dx$
18. $\int x\cdot\sin\left(x^2\right)\,dx$
19. $\int \frac{3x^2}{2 + x^3}\,dx$
20. $\int \frac{\sin(x)}{\cos(x)}\,dx$
21. $\int \frac{\ln(x)}{x}\,dx$
22. $\int x\sqrt{1 + x^2}\,dx$

23. $\int (1+3x)^7\,dx$
24. $\int \frac{1}{x}\cdot \sin(\ln(x))\,dx$
25. $\int e^x\cdot \sec\left(e^x\right)\cdot \tan\left(e^x\right)\,dx$
26. $\int \frac{1}{\sqrt{x}}\cos\left(\sqrt{x}\right)\,dx$

In Problems 27–42, evaluate the integral.

27. $\int_0^{\frac{\pi}{2}} \cos(3x)\,dx$
28. $\int_0^{\pi} \cos(4x)\,dx$
29. $\int_0^1 e^x\cdot \sin\left(2+e^x\right)\,dx$
30. $\int_0^1 e^{5x}\,dx$
31. $\int_{-1}^1 x^2\left(1+x^3\right)^5\,dx$
32. $\int_0^1 x^4\left(x^5-1\right)^{10}\,dx$
33. $\int_0^2 \frac{5}{3+2x}\,dx$
34. $\int_0^{\ln(3)} \frac{e^x}{1+e^x}\,dx$
35. $\int_0^1 x\sqrt{1-x^2}\,dx$
36. $\int_2^5 \frac{2}{1+x}\,dx$
37. $\int_0^1 \sqrt{1+3x}\,dx$
38. $\int_0^1 \frac{1}{\sqrt{1+3x}}\,dx$
39. $\int \sin^2(5x)\,dx$
40. $\int \cos^2(3x)\,dx$
41. $\int \left[\frac{1}{2}-\sin^2(x)\right]\,dx$
42. $\int \left[e^x+\sin^2(x)\right]\,dx$
43. Find the area under one arch of $y=\sin^2(x)$.
44. Evaluate $\int_0^{2\pi} \sin^2(x)\,dx$.

In Problems 45–53, expand the integrand first.

45. $\int \left(x^2+1\right)^3\,dx$
46. $\int \left(x^3+5\right)^2\,dx$
47. $\int \left(e^x+1\right)^2\,dx$
48. $\int \left(x^2+3x-2\right)^2\,dx$
49. $\int (x^2+1)(x^3+5)\,dx$
50. $\int (7+\sin(x))^2\,dx$
51. $\int e^x\left(e^x+e^{3x}\right)\,dx$
52. $\int (2+\sin(x))\sin(x)\,dx$
53. $\int \sqrt{x}\left(x^2+3x-2\right)\,dx$

In Problems 54–64, divide, then find an antiderivative.

54. $\int \frac{x+1}{x}\,dx$
55. $\int \frac{3x}{x+1}\,dx$
56. $\int \frac{x-1}{x+2}\,dx$
57. $\int \frac{x^2-1}{x+1}\,dx$
58. $\int \frac{2x^2-13x+15}{x-1}\,dx$
59. $\int \frac{2x^2-13x+18}{x-1}\,dx$
60. $\int \frac{2x^2-13x+11}{x-1}\,dx$
61. $\int \frac{x+2}{x-1}\,dx$
62. $\int \frac{e^x+e^{3x}}{e^x}\,dx$
63. $\int \frac{x+4}{\sqrt{x}}\,dx$
64. $\int \frac{\sqrt{x}+3}{x}\,dx$

The definite integrals in Problems 65–70 involve areas associated with parts of circles; use your knowledge of circles and their areas to evaluate them. (Suggestion: Sketch a graph of the integrand function.)

65. $\int_{-1}^{1} \sqrt{1-x^2}\,dx$
66. $\int_{0}^{1} \sqrt{1-x^2}\,dx$
67. $\int_{-3}^{3} \sqrt{9-x^2}\,dx$
68. $\int_{-4}^{0} \sqrt{16-x^2}\,dx$
69. $\int_{-1}^{1} \left[2+\sqrt{1-x^2}\right]\,dx$
70. $\int_{0}^{2} \left[3-\sqrt{1-x^2}\right]\,dx$

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4.7: First Applications of Definite Integrals

The development of calculus by Newton and Leibniz was a vital step in the advancement of pure mathematics, but Newton also advanced the sciences and applied mathematics. Not only did he discover theoretical results, he immediately used those results to answer important questions about gravity and motion. The success of these applications of mathematics to the physical sciences helped establish what we now take for granted: mathematics can and should be used to answer questions about the world.

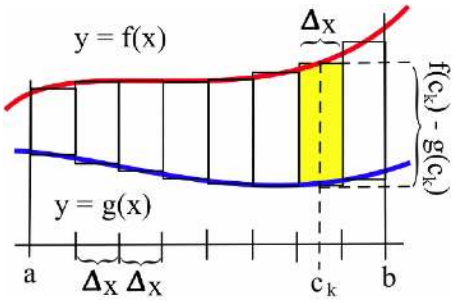
Newton applied mathematics to the outstanding problems of his day, problems primarily in the field of physics. During the intervening 300-plus years, thousands upon thousands of people have continued these theoretical and applied traditions, using mathematics to help develop our understanding of the physical and biological sciences, as well as the behavioral sciences and economics. Mathematics is still used to answer new questions in physics and engineering, but it is also important for modeling ecological processes, for understanding the behavior of DNA, for determining how the brain works, and even for devising financial strategies. The mathematics you are learning now can help you become part of this tradition, and you might even use it to add to our understanding of the world.

It is important to understand the special applications of integration we will study in case you need to use those particular applications. But it is also important that you understand the *process* of building models with integrals so you can apply that process to other situations in a variety of fields of study. Conceptually, converting an applied problem to a Riemann sum is the most valuable step. (Typically, it is also the most challenging.)

Area between Two Curves

We have already used integrals to find the area between the graph of a function and the horizontal axis. We can also use integrals to find the area between the graphs of two functions.

If $f(x) \geq g(x)$ for all x in $[a, b]$, then we can approximate the area between the graphs of f and g by partitioning the interval $[a, b]$ and forming a Riemann sum:



The height of each rectangle is $f(c_k) - g(c_k)$ so the area of the k-th rectangle is:

$$(\text{height}) \cdot (\text{base}) = [f(c_k) - g(c_k)] \cdot \Delta x_k$$

and an approximation of the total area is given by

$$\sum_{k=1}^{n} [f(c_k) - g(c_k)] \cdot \Delta x_k$$

which is a Riemann sum.

The limit of this Riemann sum, as the mesh of the partitions approaches 0, is a definite integral:

$$\int_a^b [f(x) - g(x)]\, dx$$

We will sometimes use an arrow to indicate "the limit of the Riemann sum as the mesh of the partitions approaches 0," writing:

$$\sum_{k=1}^{n} [f(c_k) - g(c_k)] \cdot \Delta x_k \longrightarrow \int_a^b [f(x) - g(x)]\, dx$$

If: $T(x) \geq B(x)$ for $a \leq x \leq b$

then: the area of the region bounded by the graphs of the “top” function $T(x)$, the “bottom” function $B(x)$, and the lines $x = a$ and $x = b$ is given by:

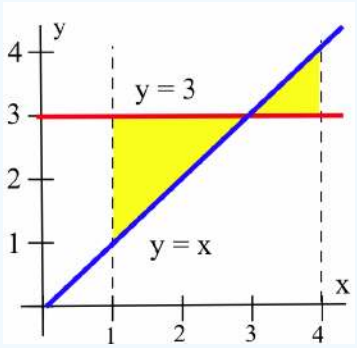
$$\int_a^b [T(x) - B(x)]\, dx$$

Example 4.7.1

Find the area bounded between the graphs of $f(x) = x$ and $g(x) = 3$ for $1 \leq x \leq 4$.

Solution

It is clear from the figure below:



that the area between f and g is 2.5 square units. Using the integration procedure above, we need to identify a "top" function and a "bottom" function. For $1 \leq x \leq 3$, $g(x) = 3 \geq x = f(x)$ so the area of the left-hand triangle is given by the integral:

$$\int_1^3 [3 - x]\, dx = \left[3x - \frac{1}{2}x^2\right]_1^3 = \left[9 - \frac{9}{2}\right] - \left[3 - \frac{1}{2}\right] = 2$$

For the interval $3 \leq x \leq 4$, $g(x) = 3 \leq x = f(x)$ so the area of the right-hand triangle is given by the integral:

$$\int_3^4 [x - 3]\, dx = \left[\frac{1}{2}x^2 - 3x\right]_3^4 = [8 - 12] - \left[\frac{9}{2} - 9\right] = \frac{1}{2}$$

Adding these two areas, we get $2 + 0.5 = 2.5$.

If we had mindlessly integrated in the previous Example without consulting a graph:

$$\int_1^4 [3 - x]\, dx = \left[3x - \frac{1}{2}x^2\right]_1^4 = [12 - 8] - \left[3 - \frac{1}{2}\right] = \frac{3}{2}$$

we would have arrived at an incorrect answer.

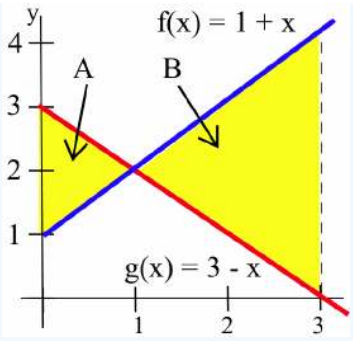
Graphing the region in question to determine which function is on "top" and which is on "bottom" is often crucial to getting the right answer to a problem involving the area between two curves.

Practice 4.7.1

Use integrals and the graphs of $f(x) = 1 + x$ and $g(x) = 3 - x$ to determine the area between the graphs of f and g for $0 \leq x \leq 3$.

Answer

Referring to a graph:



and using geometry: \)\displaystyle A = \frac12(2)(1) = 1\) and $B = \frac{1}{2}(4)(2) = 4$ so the total area is $1 + 4 = 5$.

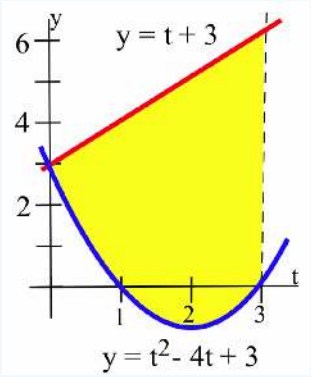
Referring to a graph of the functions and using integrals:

$$A = \int_0^1 [(3-x) - (1+x)]\, dx = \int_0^1 [2-2x]\, dx$$
$$= [2x - x^2]_0^1 = [2-1] - [0-0] = 1$$
$$B = \int_1^3 [(1+x) - (3-x)]\, dx = \int_1^3 [2x-2]\, dx$$
$$= [x^2 - 2x]_1^3 = [9-6] - [1-2)] = 4$$

which also results in a total area of $1 + 4 = 5$.

Example 4.7.2

Objects A and B start from the same location at the same time and travel along the same path with respective velocities $v_A(t) = t + 3$ and $v_B(t) = t^2 - 4t + 3$ meters per second:



How far ahead is A after 3 seconds? After 5 seconds?

Solution

From the graph, it appears that $v_A(t) \geq v_B(t)$, at least for $0 \leq t \leq 3$, but for the second question we need to know whether this holds for $3 \leq t \leq 5$ as well. Setting $v_A(t) = v_B(t)$ to see where the graphs intersect:

$$t + 3 = t^2 - 4t + 3 \quad \Rightarrow \quad t^2 - 5t = 0 \quad \Rightarrow \quad t = 0 \text{ or } t = 5$$

Checking that $v_A(1) = 4 > 0 = v_B(1)$ (or referring to the graph), we can conclude that $v_A(t) \geq v_B(t)$ on the interval $[0, 5]$.

Because $v_A(t) \geq v_B(t)$, the “area” between the graphs of v_A and v_B over an interval $[0, x]$ represents the distance between the objects after x seconds. After three seconds, the distance apart is:

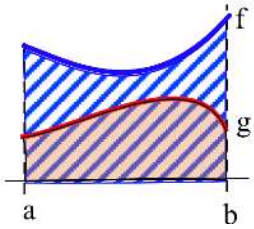
$$\int_0^3 [v_A(t) - v_B(t)]\, dt = \int_0^3 \left[(t+3) - (t^2 - 4t + 3)\right]\, dt = \int_0^3 \left[5t - t^2\right]\, dt$$
$$= \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^3 = \left[\frac{45}{2} - 9\right] - [0 - 0] = \frac{27}{2}$$

or 13.5 meters. After five seconds, the distance apart is

$$\int_0^5 [v_A(t) - v_B(t)]\, dt = \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^5 = \frac{125}{6}$$

or approximately 20.83 meters.

If $f(x) \geq g(x) \geq 0$ on an interval $[a, b]$, as illustrated in the figure below:



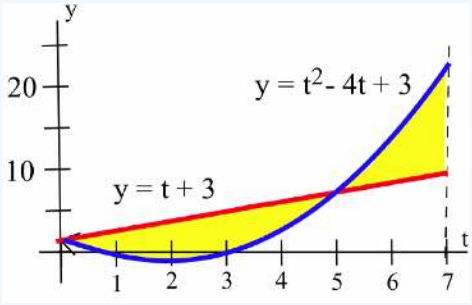
we could have used a simpler geometric argument that the area between the graphs of f and g is just the area below the graph of f **minus** the area below the graph of g:

$$\int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx$$

which agrees with our previous result. We took a different approach at the beginning of this section, however, because it provides a nice (yet simple) example of translating a geometric or physical problem into a Riemann sum and then into a definite integral.

Example 4.7.3

Find the area of the shaded region in the figure below:



Solution

These are the same two functions from our previous Example; in our previous solution we observed that $t + 3 \geq t^2 - 4t + 3$ for $0 \leq t \leq 5$, and it is straightforward to check that $t + 3 \leq t^2 - 4t + 3$ for $t \geq 5$ (and, in particular, for $5 \leq t \leq 7$).

We therefore need to split our problem into two pieces and subtract the "bottom" function from the "top" function on each interval. The area of the left region is:

$$\int_0^5 \left[(t+3) - (t^2 - 4t + 3)\right]\, dt = \left[\frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^5 = \frac{125}{6}$$

(as worked out in the previous example), while the area of the region on the right is:

$$\int_5^7 \left[(t^2-4t+3)-(t+3)\right]\,dt = \left[\frac{1}{3}t^3-\frac{5}{2}t^2\right]_5^7 = \frac{38}{3}$$

so the total area is $\frac{125}{6}+\frac{38}{3}=\frac{67}{2}=33.5$.

Average Value of a Function

We compute the **average** (or **mean value**) of n numbers, $a_1, a_2, \ldots, a_n$ by adding them up and dividing by n:

$$\text{average} = \overline{a} = \frac{1}{n}\sum_{k=1}^{n} a_k$$

but computing the average value of a *function* requires an integral. (A "bar" above a quantity typically indicates the **mean** of that quantity.)

To estimate the average value of f on the interval $[a,b]$, we can partition $[a,b]$ into n equally long subintervals of length $\Delta x = \frac{b-a}{n}$, then choose a value c_k in each subinterval, and find the average of the function values $f(c_k)$ at those n points:

$$\overline{f} = \text{average of } f \;\approx\; \frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} \;=\; \sum_{k=1}^{n} f(c_k)\cdot\frac{1}{n}$$

While this last term resembles a Riemann sum, it does not have the form $\sum f(c_k)\cdot\Delta x_k$, because $\frac{1}{n} \neq \Delta x = \frac{b-a}{n}$. But multiplying and dividing by $b-a$ yields:

$$\sum_{k=1}^{n} f(c_k)\cdot\frac{1}{n} = \sum_{k=1}^{n} f(c_k)\cdot\frac{b-a}{n}\cdot\frac{1}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\cdot\frac{b-a}{n}$$

This last (Riemann) sum converges to a definite integral:

$$\frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\cdot\frac{b-a}{n} = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\cdot\Delta x \longrightarrow \frac{1}{b-a}\int_a^b f(x)\,dx$$

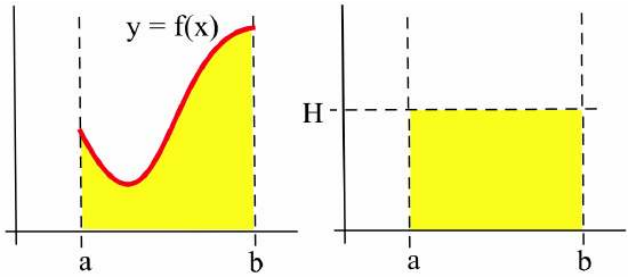
as the number of subintervals n gets larger and the mesh, $\Delta x = \frac{b-a}{n}$, approaches 0.

Definition: Average (Mean) Value of a Function

The **average value** of an integrable function f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx \tag{4.7.1}$$

The average value of a positive function has a nice geometric interpretation. Imagine that the area under f represents a liquid trapped above by the graph of f and on the other sides by the x-axis and the lines $x=a$ and $x=b$:



If we remove the "lid" (the graph of f), the liquid would settle into the shape of a rectangle with the same area as the region under the graph of f. If the height of this rectangle is H, then the area of the rectangle is $H\cdot(b-a)$, so:

$$H\cdot(b-a) = \int_a^b f(x)\,dx \quad\Rightarrow\quad H = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The average value of a positive function f is the height H of the rectangle whose area is the same as the area under f.

Example 4.7.4

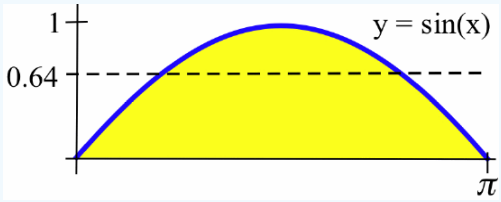
Find the average value of $\sin(x)$ on the interval $[0, \pi]$.

Solution

Using our definition, the average value is:

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{1}{\pi}[(1) - (-1)] = \frac{2}{\pi} \approx 0.6366$$

A rectangle with height $\frac{2}{\pi} \approx 0.64$ on the interval $[0, \pi]$ encloses the same area as one arch of the sine curve.



If the interval in the previous Example had been $[0, 2\pi]$, the average value would be 0. (Why?)

Practice 4.7.2

During a nine-hour work day, the production rate at time t hours was $r(t) = 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate.

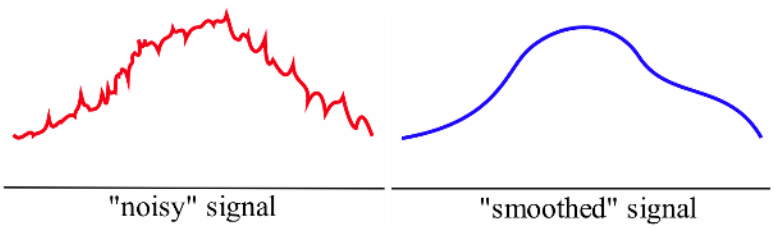
Answer

Using the average value formula:

$$\frac{1}{9-0}\int_0^9 \left[5 + \sqrt{t}\right]\,dt = \frac{1}{9}\int_0^9 \left[5 + t^{\frac{1}{2}}\right]\,dt = \frac{1}{9}\left[5t + \frac{2}{3}t^{\frac{3}{2}}\right]_0^9$$
$$= \frac{1}{9}\left[\left(45 + \frac{2}{3}\cdot 27\right) - (0+0)\right] = \frac{45+18}{9} = 7$$

so the average hourly production rate is 7 cars per hour.

Function averages, involving means as well as more complicated techniques, are used to "smooth" data so that underlying patterns become more obvious and to remove high frequency "noise" from signals. In these situations, the value of the original function f at a point is replaced by some "average of f" over an interval including that point. If f is the graph of rather jagged data:

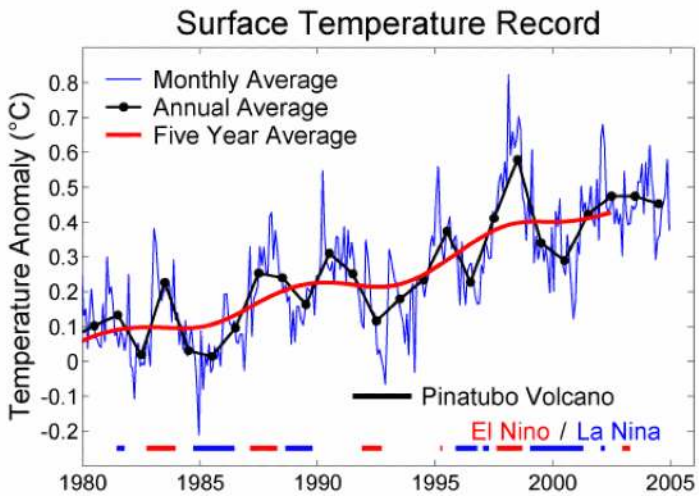


then the 10-year average of f is the integral:

$$g(x) = \frac{1}{10}\int_{x-5}^{x+5} f(t)\,dt$$

an average of f over a timespan of five years on either side of x.

The figure below shows the graphs of a monthly average (rather "noisy" data) of surface-temperature data, an annual average (still rather "jagged") and a five-year average (a much smoother function):



Typically this "moving average" function reveals a pattern much more clearly than the original data. (This "moving average" of "noisy" data is frequently used with data such as weather information and stock prices.)

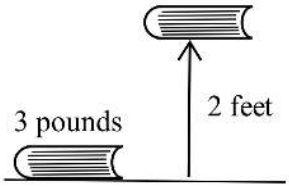
Work

In physics, the amount of work done on an object is defined as the force applied to the object times the displacement of the object (the distance the object is moved while the force is applied). Or, more succinctly:

Definition

$\textbf{work} = (\textbf{force}) \cdot (\textbf{displacement})$

If you lift a three-pound book two feet, then the force is 3 pounds (the weight of the book), and the displacement is 2 feet, so you have done $(3 \text{ pounds}) \cdot (2 \text{ feet}) = 6$ foot-pounds of work.



When the applied force and the displacement are both constants, calculating work is simply a matter of multiplication.

Practice 4.7.3

How much work is done lifting a 10-pound object from the ground to the top of a 30-foot building?

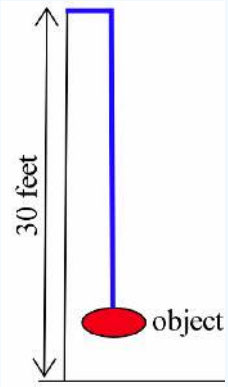
Answer

$(\text{force}) \cdot (\text{displacement}) = (10 \text{ pounds}) \cdot (30 \text{ feet}) = 300 \text{ foot-pounds}$

If either the force or the displacement varies, however, we need to use integration.

Example 4.7.5

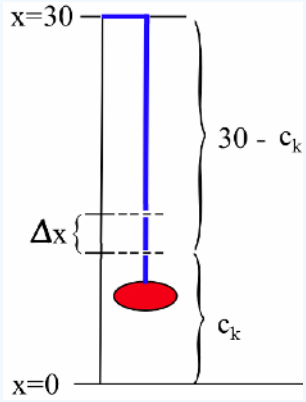
How much work is done lifting a 10-pound object from the ground to the top of a 30-foot building using a cable that weighs 2 pounds per foot?



Solution

This is more challenging situation. We know the work needed to move the object is $(10)(30) = 300$ foot-pounds, but once we start pulling the cable onto the roof, we need to do less and less work to pull the remaining part of the cable.

Let's partition the cable into small increments so the displacement of each small piece of the cable is roughly constant.



If we break the cable into n small pieces, each piece has length $\Delta x = \frac{30}{n}$, so its weight (the force required to move it) is:

$$(\Delta x \text{ ft}) \cdot \left(2 \frac{\text{lbs}}{\text{ft}}\right) = 2\Delta x \text{ lbs}$$

If this small piece of cable is initially c_k feet above the ground, then its displacement is $30 - c_k$ feet, so the work done on this small piece is $2(30 - c_k)\Delta x$ ft-lbs and the total work done on the entire cable is (approximately):

$$\sum_{k=1}^{n} 2(30 - c_k)\Delta x \quad \longrightarrow \quad \int_0^{30} 2(30 - x)\, dx$$

Once again we have formed a Riemann sum, which converges to a definite integral as we chop the cable into smaller and smaller pieces. This integral represents the work needed to lift the cable to the roof:

$$\begin{aligned} \int_0^{30} 2(30 - x)\, dx = \int_0^{30} (60 - 2x)\, dx &= 60x - x^2 \Big|_0^{30} \\ &= [1800 - 900] - [0 - 0] = 900 \text{ ft-lbs} \end{aligned}$$

so the total work required to lift the object and the cable to the roof is $300 + 900 = 1200$ ft-lbs.

? Practice 4.7.4

Suppose the building in Example 5 is 50 feet tall and the cable weighs 3 pounds per foot.

a. Compute the work done raising the object from the ground to a height of 10 feet.
b. From a height of 10 feet to a height of 20 feet.

Answer

a. The work required to move the object a distance of 10 feet is $(10 \text{ pounds}) \cdot (10 \text{ feet}) = 100 \text{ foot-pounds}$. The work required to move the top 10 feet of the cable onto the roof is:

$$\int_0^{10} (10 - x) \cdot 3 \, dx = \left[30x - \frac{3}{2}x^2\right]_0^{10} = [300 - 150] - [0] = 150 \text{ ft-lbs}$$

and the force required to move the remaining 40 feet of cable is:

$$(40 \text{ ft}) \cdot \left(3 \frac{\text{lbs}}{\text{ft}}\right) (10 \text{ ft}) = 1200 \text{ ft-lbs}$$

so the total work required is $100 + 150 + 1200 = 1450$ foot-pounds.

b.

c. The work required to move the object a distance of 10 feet is again $(10 \text{ pounds}) \cdot (10 \text{ feet}) = 100 \text{ foot-pounds}$. The work required to move the top 10 feet of the cable onto the roof is again 150 foot-pounds, and the force required to move the remaining 30 feet of cable is:

$$(30 \text{ ft}) \cdot \left(3 \frac{\text{lbs}}{\text{ft}}\right) (10 \text{ ft}) = 900 \text{ ft-lbs}$$

so the total work required is $100 + 150 + 900 = 1150$ foot-pounds.

The situation in the previous Example and Practice problems is but one of many that arise when computing work. We will examine others in Section 5.4.

Summary

The area, average and work applications in this section merely introduce a few of the many applications of definite integrals. They illustrate the pattern of moving from an applied problem to a Riemann sum, to a definite integral and, finally, to a numerical answer. We will explore many more applications in Chapter 5.

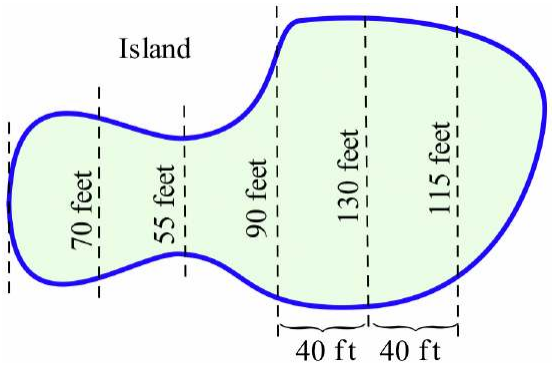
Problems

In Problems 1–4, use the values in the table below to estimate the indicated areas.

x	$f(x)$	$g(x)$	$h(x)$
0	5	2	5
1	6	1	6
2	6	2	8
3	4	2	6
4	3	3	5
5	2	4	4
6	2	5	2

1. Estimate the area between f and g for $1 \le x \le 4$.
2. Estimate the area between f and g for $1 \le x \le 6$.

3. Estimate the area between f and h for $0 \le x \le 4$.
4. Estimate the area between g and h for $0 \le x \le 6$.
5. Estimate the area of the island in the figure below:



6. Estimate the area of the island in figure above if the distances between the lines is 50 feet instead of 40 feet.

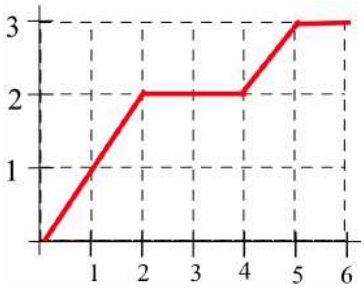
In Problems 7–18, sketch a graph of each function and find the area between the graphs of f and g for x in the given interval.

7. $f(x) = x^2 + 3,\ g(x) = 1,\ -1 \le x \le 2$
8. $f(x) = x^2 + 3,\ g(x) = 1 + x,\ 0 \le x \le 3$
9. $f(x) = x^2,\ g(x) = x,\ 0 \le x \le 2$
10. $f(x) = 4 - x^2,\ g(x) = x + 2,\ 0 \le x \le 2$
11. $f(x) = \dfrac{1}{x},\ g(x) = x,\ 1 \le x \le e$
12. $f(x) = \sqrt{x},\ g(x) = x,\ 0 \le x \le 4$
13. $f(x) = x + 1,\ g(x) = \cos(x),\ 0 \le x \le \dfrac{\pi}{4}$
14. $f(x) = (x-1)^2,\ g(x) = x + 1,\ 0 \le x \le 3$
15. $f(x) = e^x,\ g(x) = x,\ 0 \le x \le 2$
16. $f(x) = \cos(x),\ g(x) = \sin(x),\ 0 \le x \le \dfrac{\pi}{4}$
17. $f(x) = 3,\ g(x) = \sqrt{1 - x^2},\ 0 \le x \le 1$
18. $f(x) = 2,\ g(x) = \sqrt{4 - x^2},\ -2 \le x \le 2$

In Problems 19–22, use the values of f in the table at the beginning of the page to estimate the average value of f on the indicated interval.

19. $[0.5, 4.5]$
20. $[0.5, 6.5]$
21. $[1.5, 3.5]$
22. $[3.5, 6.5]$

In Problems 23–26, find the average value of the function whose graph appears below:



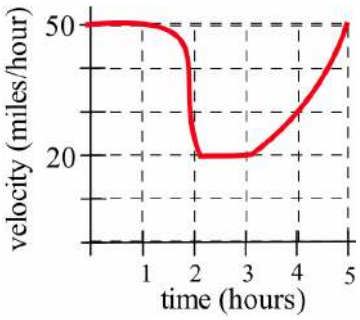
on the given interval.

23. $[0, 2]$
24. $[0, 4]$
25. $[1, 6]$

26. $[4, 6]$

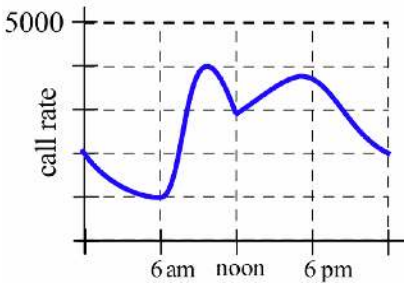
In Problems 27–32, find the average value of the given function on the indicated interval.

27. $f(x) = 2x + 1,\ 0 \le x \le 4$
28. $f(x) = x^2,\ 0 \le x \le 2$
29. $f(x) = x^2,\ 1 \le x \le 3$
30. $f(x) = \sqrt{x},\ 0 \le x \le 4$
31. $f(x) = \sin(x),\ 0 \le x \le \pi$
32. $f(x) = \cos(x),\ 0 \le x \le \pi$
33. Calculate the average value of $f(x) = \sqrt{x}$ on the interval $[0, C]$ for $C = 1, 9, 81, 100$. What is the pattern?
34. Calculate the average value of $f(x) = x$ on the interval $[0, C]$ for $C = 1, 10, 80, 100$. What is the pattern?
35. The figure below shows the velocity of a car during a five-hour trip:



    a. Estimate how far the car traveled.
    b. At what constant velocity should you drive in order to travel the same distance in five hours?

36. The figure below shows the number of telephone calls per minute at a large company:



    Estimate the average number of calls per minute:

    a. from 8:00 a.m. to 5:00 p.m.
    b. from 9:00 a.m. to 1:00 p.m.

37. a. How much work is done lifting a 20-pound bucket from the ground to the top of a 30-foot building with a cable that weighs three pounds per foot?
    b. How much work is done lifting the same bucket from the ground to a height of 15 feet with the same cable?
38. a. How much work is done lifting a 60-pound chair from the ground to the top of a 20-foot building with a cable that weighs 1 pound per foot?
    b. How much work is done lifting the same chair from the ground to a height of 5 feet with the same cable?
39. a. How much work is done lifting a 10-pound calculus book from the ground to the top of a 30-foot building with a cable that weighs 2 pounds per foot?
    b. From the ground to a height of 10 feet?
    c. From a height of 10 feet to a height of 20 feet?
40. How much work is done lifting an 80-pound injured child to the top of a 20-foot hole using a stretcher weighing 14 pounds and a cable that weighs 1 pound per foot?

41. How much work is done lifting a 60-pound injured child to the top of a 15-foot hole using a stretcher weighing 10 pounds and a cable that weighs 2 pound per foot?
42. How much work is done lifting a 120-pound injured adult to the top of a 30-foot hole using a stretcher weighing 10 pounds and a cable that weighs 2 pound per foot?

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4.8: Using Tables [and Technology] to Find Antiderivatives

Appendix I shows patterns for many antiderivatives—some of which you should already know based on your work in this chapter. Many reference books and web sites contain far more than the ones listed in the appendix. A table of integrals helps you while you are learning calculus and serves as a reference later when you are using calculus.

Think of an integral table as a dictionary: something to use when you need to spell a challenging word or need the meaning of a new word. It would be difficult to write a report if you had to look up the spelling of *every* word, and it will be difficult to learn and use calculus if you have to look up every antiderivative. Tables of antiderivatives are limited by necessity and often take longer to use than finding an antiderivative from scratch, but they can also be very valuable and useful.

This section shows how to transform some integrals into forms found in Appendix I and how to use "recursion" formulas found in integral tables. The first Examples and Practice problems illustrate some of the techniques used to change an integral into a standard form.

These techniques are useful whether that standard form resides in a table or in your head.

Note

Appendix I (like some other integral tables) omits the "$+C$" arbitrary constant for conciseness, but you need to remember to include it when using the results of the table to find an indefinite integral.

Example 4.8.1

Use Appendix I to find $\int \frac{1}{9+x^2}\,dx$.

Solution

The integrand is a rational function, and the first entry you see listed in the "Rational Functions" section of Appendix I should be:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

This resembles the pattern we need, so replacing the a with 3 we have:

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{3^2+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C$$

You can (and should) check this answer by differentiating.

Practice 4.8.1

Use Appendix I to find $\int \frac{1}{25+x^2}\,dx$ and $\int \frac{1}{25-x^2}\,dx$.

(Notice that a small change in the form of the integrand (from $+$ to $-$ here) can lead to a very different result.)

Answer

The integral $\int \frac{1}{25+x^2}\,dx$ resembles the pattern from Example 1:

$$\int \frac{1}{25+x^2}\,dx = \int \frac{1}{5^2+x^2}\,dx = \frac{1}{5}\arctan\left(\frac{x}{5}\right)+C$$

The integrand in $\int \frac{1}{25-x^2}\,dx$ is also a rational function, but we need a different pattern from Appendix I:

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| \tag{4.8.1}$$

with $a = 5$:

$$\int \frac{1}{25 - x^2}\,dx = \int \frac{1}{5^2 - x^2}\,dx = \frac{1}{10}\ln\left|\frac{x+5}{x-5}\right| + C$$

Example 4.8.2

Use Appendix I to find $\int \frac{1}{5 + x^2}\,dx$.

Solution

The integrand is again a rational function, and the general form is the same as in the previous Example:

$$\int \frac{1}{5 + x^2}\,dx = \int \frac{1}{(\sqrt{5})^2 + x^2}\,dx = \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

but here we needed to put $a = \sqrt{5}$.

(The constant in the denominator of this integrand was not a perfect square, but the process is exactly the same—even if the result looks a bit "messier" due to the presence of the radical.)

Practice 4.8.2

Use Appendix I to find $\int \frac{1}{7 + x^2}\,dx$ and $\int \frac{1}{7 - x^2}\,dx$.

Answer

The integral $\int \frac{1}{7 + x^2}\,dx$ matches the pattern in Example 2:

$$\int \frac{1}{7 + x^2}\,dx = \int \frac{1}{(\sqrt{7})^2 + x^2}\,dx = \frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C$$

For $\int \frac{1}{7 - x^2}\,dx$ we need the pattern in the margin with $a = \sqrt{7}$:

$$\int \frac{1}{7 - x^2}\,dx = \int \frac{1}{(\sqrt{7})^2 - x^2}\,dx = \frac{1}{2\sqrt{7}}\ln\left|\frac{x+\sqrt{7}}{x-\sqrt{7}}\right| + C$$

We often need to perform some algebraic manipulations to change an integrand into one that exactly matches a pattern in the table.

Example 4.8.3

Use Appendix I to find $\int \frac{1}{9 + 4x^2}\,dx$.

Solution

The integrand is again a rational function, and the general form resembles the one used in the previous Examples, but here we have a $4x^2$ where we only see x^2 in the table pattern. To get the integrand in the form we want, we can factor a 4 out of the denominator:

$$\begin{aligned}\int \frac{1}{9 + 4x^2}\,dx &= \int \frac{1}{4\left(\frac{9}{4} + x^2\right)}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{3}{2}\right)^2 + x^2}\,dx \\ &= \frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\cdot\arctan\left(\frac{x}{\frac{3}{2}}\right) + C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + C\end{aligned}$$

Another approach involves a change of variable. First write:

$$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{3^2+(2x)^2}\,dx$$

We have $2x$ where we would like to see a standalone variable. To get that pattern, put $u = 2x \Rightarrow du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$:

$$\begin{aligned}\int \frac{1}{3^2+(2x)^2}\,dx &= \int \frac{1}{3^2+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)+C \\ &= \frac{1}{6}\arctan\left(\frac{2x}{3}\right)+C\end{aligned}$$

which yields the same result as our previous method.

? Practice 4.8.3

Use Appendix I to find $\int \frac{1}{25+9x^2}\,dx$ and $\int \frac{1}{25-9x^2}\,dx$.

Answer

For the integral $\int \frac{1}{25+9x^2}\,dx$ we can factor 9 from the denominator:

$$\begin{aligned}\int \frac{1}{25+9x^2}\,dx &= \int \frac{1}{9\left(\frac{25}{9}+x^2\right)}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{5}{3}\right)^2+x^2}\,dx \\ &= \frac{1}{9}\cdot\frac{1}{\frac{5}{3}}\arctan\left(\frac{x}{\frac{5}{3}}\right)+C = \frac{1}{15}\arctan\left(\frac{3x}{5}\right)+C\end{aligned}$$

and proceed as before. We could proceed similarly for $\int \frac{1}{25-9x^2}\,dx$ or we could substitute $u = 3x$:

$$u = 3x \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}\,du$$

to get:

$$\begin{aligned}\int \frac{1}{25-9x^2}\,dx &= \int \frac{1}{5^2-(3x)^2}\,dx = \int \frac{1}{5^2-u^2}\cdot\frac{1}{3}\,du \\ &= \frac{1}{3}\cdot\frac{1}{2\cdot 5}\ln\left|\frac{u+5}{u-5}\right|+C = \frac{1}{30}\cdot\ln\left|\frac{3x+5}{3x-5}\right|+C\end{aligned}$$

Sometimes a change of variable is absolutely necessary.

✓ Example 4.8.4

Use Appendix I to find $\int \frac{e^x}{9+e^{2x}}\,dx$.

Solution

Here the integrand is *not* a rational function, but we can transform it into one by using the substitution $u = e^x \Rightarrow du = e^x\,dx$ so that $u^2 = (e^x)^2 = e^{2x}$:

$$\begin{aligned}\int \frac{e^x}{9+e^{2x}}\,dx &= \int \frac{1}{3^2+(e^x)^2}\cdot e^x\,dx = \int \frac{1}{3^2+u^2}\,du \\ &= \frac{1}{3}\arctan\left(\frac{u}{3}\right)+C = \frac{1}{3}\arctan\left(\frac{e^x}{3}\right)+C\end{aligned}$$

If you don't see the exact pattern you need in an integral table, try a substitution first.

? Practice 4.8.4

Evaluate $\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$ and $\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$.

Answer

For $\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$, first use the substitution:

$$u=\sin(x) \Rightarrow du=\cos(x)\,dx$$

toget:

$$\int \frac{\cos(x)}{25+\sin^2(x)}\,dx=\int \frac{1}{25+u^2}\,du$$

followed by the result of the first part of Practice 1:

$$\int \frac{1}{25+u^2}\,du=\frac{1}{5}\arctan\left(\frac{u}{5}\right)+C=\frac{1}{5}\arctan\left(\frac{\sin(x)}{5}\right)+C$$

For $\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$ use the same substitution, followed by the result from the second part of Practice 1:

$$\int \frac{1}{25-u^2}\,du=\frac{1}{10}\ln\left|\frac{u+5}{u-5}\right|+C=\frac{1}{10}\ln\left|\frac{\sin(x)+5}{\sin(x)-5}\right|+C$$

$$u=7x \Rightarrow du=7\,dx \Rightarrow dx=\frac{1}{7}\,du$$

$$\int \cos^4(7x)\,dx=\frac{1}{7}\int \cos^4(u)\,du$$

How should you recognize whether algebra or a change of variable is needed? Experience and practice, practice, practice.

Using "Recursion" Formulas

A **recursion formula** gives one antiderivative in terms of another antiderivative. Usually the new antiderivative is somehow simpler than the original one. For example, the first recursion formula for a trigonometric function listed in Appendix I states:

$$\int \sin^n(x)\,dx=-\frac{1}{n}\sin^{n-1}(x)\cos(x)+\frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

This formula would allow us to write $\int \sin^8(x)\,dx$, for instance, in terms of $\int \sin^6(x)\,dx$, which should (theoretically, at least) be easier to compute than the original integral.

(We will develop this formula from scratch in Problem 25 of Section 8.2. For now, you can check that it works by comparing the derivative of your answer to the original integrand for an integration problem that uses this—or any other—recursion formula.)

✔ Example 4.8.5

Use a recursion formula to evaluate $\int \sin^4(x)\,dx$.

Solution

Applying the formula given in the discussion above:

$$\int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx$$

This new integral is one we already know how to evaluate:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + K$$

Putting this together with the result of the recursion formula, we get:

$$\begin{aligned}\int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right] + C \\ &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C\end{aligned}$$

We could have used Appendix I to find $\int \sin^2(x)\,dx$ instead — or even applied the recursion formula a second time to rewrite $\int \sin^2(x)\,dx$ in terms of $\int \sin^0(x)\,dx = \int 1\,dx$.

On the line

$$\int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right] + C$$

above, we could have included the "$+K$" but then the result at the next stage would have included the constant terms

$$\cdots + \frac{3}{4}K + C$$

which is also an arbitrary constant.

? Practice 4.8.5

Use Appendix I to evaluate $\int \cos^4(x)\,dx$ and $\int \cos^4(7x)\,dx$.

Answer

For $\int \cos^4(x)\,dx$ we need the recursion formula:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

with $n = 4$:

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx \\ &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \left[\frac{1}{2} + \frac{1}{2}\cos(2x)\right]\,dx \\ &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right] + C\end{aligned}$$

For $\int \cos^4(7x)\,dx$, first use a substitution:

$$u = 7x \Rightarrow du = 7\,dx \Rightarrow dx = \frac{1}{7}\,du$$

$$\int \cos^4(7x)\,dx = \frac{1}{7}\int \cos^4(u)\,du$$

and then the result of the previous integration:

$$\int \cos^4(7x)\,dx = \frac{1}{28}\cos^3(7x)\sin(7x) + \frac{3}{28}\left[\frac{1}{2}(7x) + \frac{1}{4}\sin(14x)\right] + C$$

Using Technology

Many web sites (such as Wolfram|Alpha, www.wolframalpha.com), computer programs (wxMaxima is a good free one) and calculators (such as the TI-89 or TI-Nspire CAS) feature computer algebra systems that can find antiderivatives of a wide variety of functions. For example, typing integral sin^4(x) into Wolfram|Alpha yields:

$$\int \sin^4(x)\,dx = \frac{1}{32}(12x - 8\sin(2x) + \sin(4x)) + \text{constant}$$

which (applying some trig identities) agrees with our result above.

> Although technology can help us find an antiderivative and evaluate a definite integral, in an application problem **you** still need to set up the Riemann sum that leads to the definite integral.

Problems

In Problems 1–48, use patterns and recursion formulas from the integral table in Appendix I as necessary (along with any other antiderivatives and integration techniques you already know) to evaluate each integral.

1. $\int \frac{1}{4+x^2}\,dx$
2. $\int \frac{5}{4+x^2}\,dx$
3. $\int \left[2x + \frac{2}{25+x^2}\right]\,dx$
4. $\int \frac{1}{4-x^2}\,dx$
5. $\int \frac{2}{9-x^2}\,dx$
6. $\int \left[\cos(x) + \frac{3}{25-x^2}\right]\,dx$
7. $\int \frac{1}{3+x^2}\,dx$
8. $\int \frac{5}{7+x^2}\,dx$
9. $\int \left[e^x + \frac{7}{2+x^2}\right]\,dx$
10. $\int \frac{1}{\sqrt{4-x^2}}\,dx$
11. $\int \frac{3}{\sqrt{5-x^2}}\,dx$
12. $\int \frac{3}{\sqrt{4-x^2}}\,dx$
13. $\int \frac{1}{4+25x^2}\,dx$
14. $\int \frac{2}{\sqrt{9-16x^2}}\,dx$
15. $\int \frac{5}{\sqrt{1-4x^2}}\,dx$
16. $\int \sec(x+5)\,dx$
17. $\int \frac{2}{\sqrt{1+9x^2}}\,dx$
18. $\int x\cdot\sec(2x^2+7)\,dx$
19. $\int \ln(x+1)\,dx$
20. $\int \ln(3x-1)\,dx$
21. $\int 3x\cdot\ln(5x^2+7)\,dx$
22. $\int e^x \ln(e^x-3)\,dx$

23. $\int \cos(x) \cdot \ln(\sin(x))\, dx$
24. $\int \frac{2}{\sqrt{x^2-9}}\, dx$
25. $\int \sqrt{4+x^2}\, dx$
26. $\int \sqrt{9+x^2}\, dx$
27. $\int \sqrt{16+x^2}\, dx$
28. $\int_0^1 \frac{1}{4+x^2}\, dx$
29. $\int_1^3 \left[2x + \frac{2}{25+x^2}\right] dx$
30. $\int_0^2 \frac{2}{9-x^2}\, dx$
31. $\int_{-1}^1 \frac{1}{3+x^2}\, dx$
32. $\int_0^1 \left[e^x + \frac{7}{2+x^2}\right] dx$
33. $\int_1^2 \frac{3}{\sqrt{5-x^2}}\, dx$
34. $\int_0^1 \frac{1}{4+25x^2}\, dx$
35. $\int_0^{0.1} \frac{5}{\sqrt{1-4x^2}}\, dx$
36. $\int_0^1 \frac{1}{\sqrt{9-4x^2}}\, dx$
37. $\int_0^6 \ln(x+1)\, dx$
38. $\int_0^3 3x \cdot \ln(5x^2+7)\, dx$
39. $\int_0^{\frac{\pi}{2}} \cos(x) \cdot \ln(2+\sin(x))\, dx$
40. $\int_0^2 \sqrt{4+x^2}\, dx$
41. $\int_{-3}^3 \sqrt{9+x^2}\, dx$
42. $\int_0^1 \sqrt{16+x^2}\, dx$
43. $\int \sin^3(x)\, dx$
44. $\int \cos^3(x)\, dx$
45. $\int \cos^5(x)\, dx$
46. $\int \sec^5(x)\, dx$
47. $\int x^2 \cos(x)\, dx$
48. $\int x^2 \sin^5(x)\, dx$
49. Before doing any calculations, predict which you expect to be larger:
    a. the average value of $\sin(x)$ on $[0, \pi]$
    b. the average value of $\sin^2(x)$ on $[0, \pi]$

    Then calculate each average to see if your prediction was correct.
50. Find the area of the region bounded by the graph of $f(x) = \ln(x)$, the x-axis and the lines $x = 1$ and $x = C$ when $C = e$, 10, 100 and 200.
51. Find the average value of $f(x) = \ln(x)$ on the interval $1 \le x \le C$ when $C = e$, 10, 100, 200.
52. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.
    a. $\int_0^1 e^x\, dx$

b. $\int_0^1 xe^x\,dx$

c. $\int_0^1 x^2e^x\,dx$

53. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

a. $\int_1^2 e^x\,dx$

b. $\int_1^2 xe^x\,dx$

c. $\int_1^2 x^2e^x\,dx$

54. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

a. $\int_0^\pi \sin(x)\,dx$

b. $\int_0^\pi x\sin(x)\,dx$

c. $\int_0^\pi x^2\sin(x)\,dx$

55. Evaluate $\int_0^C \frac{2}{1+x^2}\,dx$ for $C = 1$, 10, 20 and 30. Before doing the calculation, estimate the value of the integral when $C = 40$.

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4.9: Approximating Definite Integrals

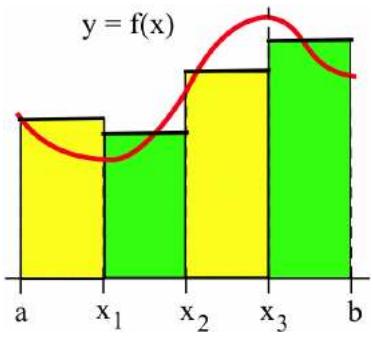
The Fundamental Theorem of Calculus tells how to calculate the exact value of a definite integral *if* the integrand is continuous and *if* we can find a formula for an antiderivative of the integrand. In practice, however, we may need to compute the definite integral of a function for which we only have table values or a graph—or of a function that does not have an elementary antiderivative. This section presents several techniques for getting approximate numerical values for definite integrals without using antiderivatives. Mathematically, exact answers are preferable and satisfying, but for most applications a numerical answer accurate to several digits is just as useful.

> **Note**
>
> The ideas behind these methods are geometric and rather simple, but using the methods to get good approximations typically requires lots of arithmetic, something calculators and computers are very good (and quick) at doing.

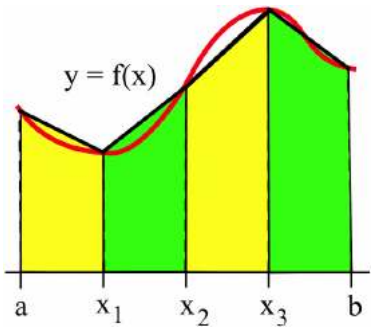
The General Approach

The methods in this section approximate the definite integral of a function f by partitioning the interval of integration and building an "easy" function with values close to those of f on each interval, then evaluating the definite integrals of the "easy" functions exactly. If the "easy" functions are close to f, then the sum of the definite integrals of the "easy" functions should be close to the definite integral of f.

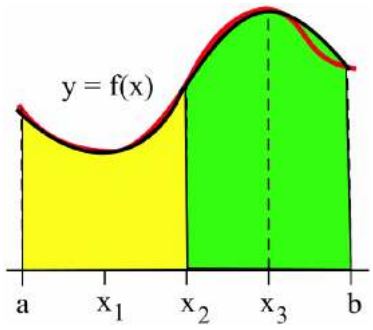
The **Left**, **Right** and **Midpoint Rules** approximate f with horizontal lines on each partition interval so the "easy" functions are constant functions, and the approximating regions are rectangles:



The **Trapezoidal Rule** approximates f with slanted lines, so the "easy" functions are linear and the approximating regions are trapezoids:



Simpson's Rule approximates f with parabolas, so the "easy" functions are quadratic polynomials:



The Left and Right approximation rules are simply Riemann sums with the point c_k in the k-th subinterval chosen to be the left or right endpoint of that subinterval. They typically require a large number of computations to get even mediocre approximations to the definite integral of f and are seldom used in practice. Along with the Midpoint Rule (which chooses each c_k to be the midpoint of the k-th subinterval), they are discussed near the end of the Problems for this section.

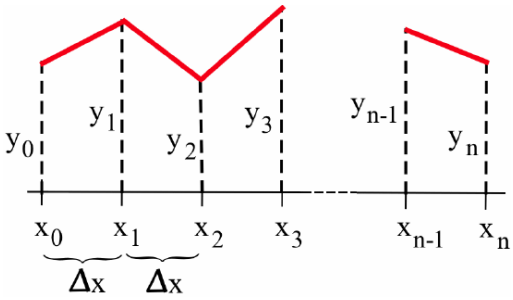
All of these methods partition the interval $[a, b]$ into n subintervals of equal width, so each subinterval has width $h = \Delta x_k = \dfrac{b-a}{n}$. The points of the partition are $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2 \cdot h$, $x_3 = a + 3 \cdot h$, and so on. The k-th point in the partition is given by the formula $x_k = a + k \cdot h$ and the last (n-th) point is thus:

$$x_n = a + n \cdot h = a + n\left(\frac{b-a}{n}\right) = a + b - a = b$$

Approximating a Definite Integral Using Trapezoids

If the graph of f is curved, then slanted lines typically come closer to the graph of f than horizontal ones do. These slanted lines lead to trapezoidal approximating regions.

The area of a trapezoid is $(\text{base}) \cdot (\text{average height})$ so the area of the first trapezoid in the figure below:



is:

$$(\Delta x) \cdot \left(\frac{y_0 + y_1}{2}\right)$$

Similarly, the areas of the next few trapezoids are:

$$(\Delta x) \cdot \left(\frac{y_1 + y_2}{2}\right), \quad (\Delta x) \cdot \left(\frac{y_2 + y_3}{2}\right), \quad (\Delta x) \cdot \left(\frac{y_3 + y_4}{2}\right)$$

and so on, with the area of the last region being

$$(\Delta x) \cdot \left(\frac{y_{n-1} + y_n}{2}\right)$$

The sum of these n trapezoidal areas is:

$$T_n = (\Delta x)\left(\frac{y_0+y_1}{2}\right) + (\Delta x)\left(\frac{y_1+y_2}{2}\right) + (\Delta x)\left(\frac{y_2+y_3}{2}\right) + \cdots + (\Delta x)\left(\frac{y_{n-1}+y_n}{2}\right)$$
$$= \left(\frac{\Delta x}{2}\right)[(y_0+y_1)+(y_1+y_2)+(y_2+y_3)+\cdots+(y_{n-1}+y_n)]$$
$$= \left(\frac{h}{2}\right)[y_0+2y_1+2y_2+2y_3+\cdots+2y_{n-1}+y_n]$$
$$= \left(\frac{h}{2}\right)[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{n-1})+f(x_n)]$$

Each $f(x_k)$ value, except the first ($k=0$) and the last ($k=n$), is the right-endpoint height of one trapezoid and the left-endpoint height of the next, so it shows up in the calculation for two trapezoids and is multiplied by 2 in the formula for the trapezoidal approximation.

Trapezoidal Approximation Rule

If: f is integrable on $[a,b]$ and $[a,b]$ is partitioned into n subintervals of width $h = \frac{b-a}{n}$

then: the Trapezoidal approximation of $\int_a^b f(x)\,dx$ is:

$$T_n = \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{n-1})+f(x_n)]$$

Example 4.9.1

Compute T_4, the Trapezoidal approximation of $\int_1^3 f(x)\,dx$ for $n=4$, with the values of f in the table below:

x	$f(x)$
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Solution

The step size is $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$ so:

$$T_4 = \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)]$$
$$= \frac{0.5}{2}[4.2+2(3.4)+2(2.8)+2(3.6)+(3.2)] = (0.25)(27) = 6.75$$

so we can say that $\int_1^3 f(x)\,dx \approx 6.75$.

Let's see how well the Trapezoidal Rule approximates an integral whose value we can compute exactly:

$$\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}[27-1] = \frac{26}{3} \approx 8.6666667$$

Example 4.9.2

Calculate T_4 for $\int_1^3 x^2\,dx$.

Solution

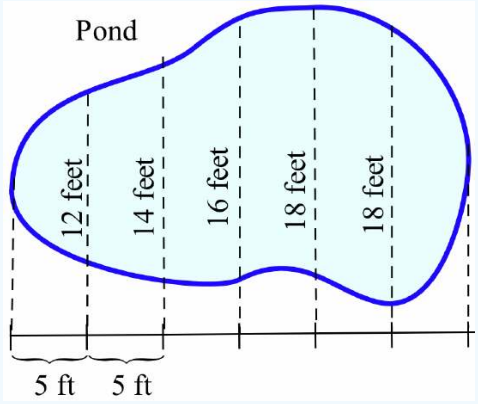
The step size is $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$ so:

$$\begin{aligned} T_4 &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \\ &= \frac{0.5}{2}\left[(1.0)^2 + 2(1.5)^2 + 2(2.0)^2 + 2(2.5)^2 + (3.0)^2\right] \\ &= (0.25)\left[1 + 2(2.25) + 2(4) + 2(6.25) + 9\right] = 8.75 \end{aligned}$$

which is within 0.1 of the exact answer. Larger values for n give better approximations: $T_{20} = 8.67$ and $T_{100} = 8.6668$.

Practice 4.9.1

On a summer day, the level of the pond shown below:



fell 0.1 feet because of evaporation. Use the Trapezoidal Rule to approximate the surface area of the pond and then estimate how much water evaporated.

Answer

Using the Trapezoidal Rule to approximate the pond’s surface area:

$$T \approx \frac{5\text{ ft}}{2} \cdot [(0 + 2\cdot 12 + 2\cdot 14 + 2\cdot 16 + 2\cdot 18 + 2\cdot 18 + 0)\ \text{ft}] = 390\ \text{ft}^2 \tag{4.9.1}$$

so the volume is $(\text{surface area})(\text{depth}) \approx (390\ \text{ft}^2)\,(0.1\ \text{ft}) = 39\ \text{ft}^3$. FIGURE SOMEWHERE

Approximating a Definite Integral Using Parabolas

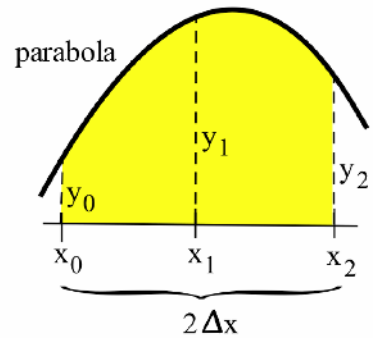
If the graph of f is curved, the slanted lines from the Trapezoidal Rule may not fit the graph of f as closely as we would like, requiring a large number of subintervals to achieve a good approximation of the definite integral. Curves typically fit the graph of f better than straight lines in such situations, and the easiest nonlinear curves we know are parabolas.

Note

This parabolic method is known as **Simpson's Rule**, named after British mathematician and inventor Thomas Simpson (1710–1761); Germans call it *Kepler'sche Fassregel*, after Johannes Kepler, who developed it 100 years before Simpson.

Just as we need two points to determine an equation of a line, we will need three points to determine an equation of a parabola.

Calling these points (x_0, y_0), (x_1, y_1) and (x_2, y_2), the area under a parabolic region with evenly spaced x_k values:



is:

$$(2\Delta x) \cdot \left[\frac{y_0 + 4y_1 + y_2}{6}\right] = \frac{\Delta x}{3}\left[y_0 + 4y_1 + y_2\right]$$

(This result is not obvious; see Problem 32 for the necessary algebra.)

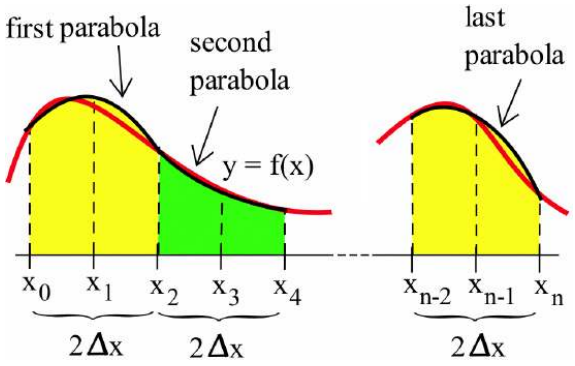
Taking the subintervals in pairs, the areas of the next few parabolic regions are:

$$\frac{\Delta x}{3}\left[y_2 + 4y_3 + y_4\right], \quad \frac{\Delta x}{3}\left[y_4 + 4y_5 + y_6\right], \quad \frac{\Delta x}{3}\left[y_6 + 4y_7 + y_8\right]$$

and so on, with the area of the last pair of regions being:

$$\frac{\Delta x}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

so the sum of all n parabolic areas:



is:

$$\begin{aligned} S_n &= \frac{\Delta x}{3}\left[y_0 + 4y_1 + y_2\right] + \frac{\Delta x}{3}\left[y_2 + 4y_3 + y_4\right] + \cdots \\ &\quad + \frac{\Delta x}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right] \\ &= \left(\frac{h}{3}\right)\left[(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4\right. \\ &\quad \left. \cdots + y_{n-2} + 4y_{n-1} + y_n\right] \\ &= \left(\frac{h}{3}\right)\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots\right. \\ &\quad \left. + 2y_{n-1} + 4y_{n-1} + y_n)\right] \\ &= \left(\frac{h}{3}\right)\left[f(x_0) + 4f(x_1) + 2f(x_2)\right. \\ &\quad + 4f(x_3) + 2f(x_4) + \cdots \\ &\quad \left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right] \end{aligned}$$

$$+ \frac{\Delta x}{3}\left[y_2 + 4y_3 + y_4\right] + \cdots$$
$$+ \frac{\Delta x}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

In order to use **pairs** of subintervals, the number n of subintervals must be **even**. Notice that the coefficient pattern for the area under a single parabolic region is 1–4–1, but when we put several parabolas next to each other, they share some edges and the pattern becomes 1–4–2–4–2–⋯–2–4–1 with the shared edges getting counted twice.

Parabolic Approximation Rule (Simpson's Rule)

If: f is integrable on $[a,b]$ and $[a,b]$ is partitioned into n subintervals of width $h = \dfrac{b-a}{n}$

then: the Parabolic approximation of $\int_a^b f(x)\,dx$ is:

$$S_n = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)$$

Example 4.9.3

Calculate S_4, the Simpson's Rule approximation of $\int_1^3 f(x)\,dx$ for the function f with values in this table:

x	$f(x)$
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Solution

The step size is $h = \dfrac{b-a}{n} = \dfrac{3-1}{4} = \dfrac{1}{2}$, so:

$$\begin{aligned} S_4 &= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &= \frac{\frac{1}{2}}{3}\left[4.2 + 4(3.4) + 2(2.8) + 4(3.6) + (3.2)\right] = \frac{1}{6}(41) = \frac{41}{6} \end{aligned}$$

or approximately **6.833**.

Example 4.9.4

Calculate S_4 for $\int_1^3 2^x\,dx$.

Solution

As in the previous Examples, $h = \frac{b-a}{n} = 0.5$ and $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$ and $x_4 = 3$.

$$\begin{aligned} S_4 &= \frac{h}{3} \cdot \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &= \frac{\frac{1}{2}}{3} \cdot \left[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right] \\ &= \left(\frac{1}{6}\right)\left[2^1 + 4\left(2^{1.5}\right) + 2\left(2^2\right) + 4\left(2^{2.5}\right) + \left(2^3\right)\right] \\ &= \left(\frac{1}{6}\right)\left[2 + 4(2\sqrt{2}) + 2(4) + 4(4\sqrt{2}) + 8\right] = \left(\frac{1}{6}\right)\left[18 + 20\sqrt{2}\right] \end{aligned}$$

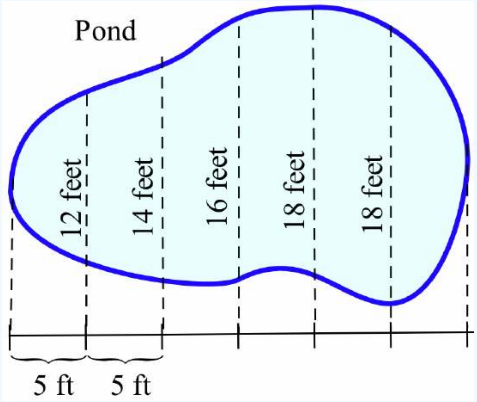
or approximately **8.656854**. The exact value of the integral is:

$$\int_1^3 2^x\,dx = \left[\frac{2^x}{\ln(2)}\right]_1^3 = \frac{8}{\ln(2)} - \frac{2}{\ln(2)} = \frac{6}{\ln(2)} \approx 8.65617024533$$

Larger values of n give better approximations: $S_{20} = 8.656171$ and $S_{100} = 8.656170$.

? Practice 4.9.2

Use Simpson's Rule to estimate the surface area of the pond in the figure below:



Answer

Using Simpson's Rule to approximate the pond's surface area:

$$S \approx \frac{5 \text{ ft}}{3} \cdot [(0 + 4 \cdot 12 + 2 \cdot 14 + 4 \cdot 16 + 2 \cdot 18 + 4 \cdot 18 + 0) \text{ ft}] \approx 413 \text{ ft}^2 \tag{4.9.2}$$

Which Method Is Best?

The most difficult and time-consuming part of these approximations, whether done by hand or by computer, is the evaluation of the function at the x_k values. For n subintervals, all of the methods require about the same number of function evaluations. The table on the next page illustrates how closely each method approximates $\int_1^5 \frac{1}{x}\,dx = \ln(5) \approx 1.609437912$ using several values of n. The results in the table also show how quickly the actual error shrinks as the value of n increases: just doubling n from 4 to 8 cuts the actual error of the Simpson's Rule approximation of this definite integral by a factor of 9—a good reward for our extra work.

method	approximation	error bound	actual error
T_4	1.683333333	0.6666666	0.07389542
S_4	1.622222222	0.5333333	0.01278431
L_4	2.083333333	2.0000000	0.47389542
R_4	1.283333333	2.0000000	0.32610458
M_4	1.574603175	0.3333333	0.03483474
T_8	1.628968254	0.1666666	0.01953034
S_8	1.610846561	0.0333333	0.00140865
L_8	1.828968254	1.0000000	0.21953034
R_8	1.428968254	1.0000000	0.18046966
M_8	1.599844394	0.0833333	0.00959352
T_{20}	1.612624844	0.0266667	0.00318693
S_{20}	1.609486789	0.0008533	0.00004888
L_{20}	1.692624844	0.4000000	0.08318693
R_{20}	1.532624844	0.4000000	0.07681307
M_{20}	1.607849324	0.0133333	0.00158859

The "error bounds" in the third column are discussed below. Notice that for each value of n, the Simpson's Rule approximation S_n has the smallest error, and that the error for the Midpoint Rule approximation M_n (discussed in the Problems) is roughly half the error for the Trapezoidal Rule T_n. L_n and R_n denote the Left and Right approximations, respectively.

How Good Are the Approximations?

The approximation rules are valuable by themselves, but they are particularly useful because we can find "error bound" formulas that guarantee how close these approximations come to the exact values of the definite integral. It is useful to know that the value of an integral is "about 3.7," but we can have more confidence in our approximation if we know that value is "within 0.0001 of 3.7." Then we can decide if our approximation is good enough for the job at hand or if we need to improve it.

We can also solve the formulas for the error bounds provided below to determine how many subintervals we need to guarantee that our approximation is within some specified distance of the exact answer. There is no reason to use 1000 subintervals if 18 will give the needed accuracy. Unfortunately, the formulas for the error bounds require information about the derivatives of the integrands, so we cannot use these error bound formulas for the approximations of integrals of functions defined only by tables or graphs—or of continuous (hence integrable) functions that fail to have continuous derivatives.

The "error bound" formula for the Trapezoidal Rule approximation given at the top of the next page is just a "guarantee": the actual error is guaranteed to be no larger than the error bound. In fact, the actual error is usually much smaller than the error bound (compare the error bounds with the actual error for T_4, T_8 and T_{20} in the table above to see this principle in action). The word "error" does not indicate a mistake, it simply means the deviation or distance of the approximate answers from the exact answer.

Error Bound for Trapezoidal Approximation

If: f'' is continuous on $[a,b]$ and $|f''(x)| \le B_2$

then: the "error" of the T_n approximation of $\int_a^b f(x)\,dx$ satisfies:

$$|\text{"error"}| = \left|\int_a^b f(x)\,dx - T_n\right| \le \frac{(b-a)^3 B_2}{12n^2}$$

While it's possible to prove this error bound formula using mathematics you've already learned, the proof is highly technical and sheds little or no insight into the workings of the Trapezoidal Rule, so we (like the authors of most calculus books) have omitted it.

Example 4.9.5

You can be certain that the T_{10} approximation of $\int_0^2 \sin(x^2)\,dx$ is within what distance of the exact value of the integral?

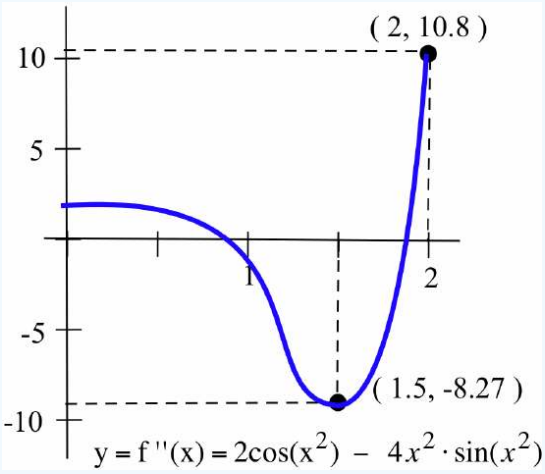
Solution

We know that $b-a=2$, $n=10$ and $f(x)=\sin(x^2)$, so $f''(x)=-4x^2\cdot\sin(x^2)+2\cdot\cos(x^2)$ is continuous on $[0,2]$. (Practice your differentiation skills by verifying this.)

We now need an "upper bound" for $|f''(x)|$. If $f''(x)$ is differentiable (it is here) then we could use the techniques of Chapter 3 to find its maximum value on $[0,2]$ but that would require finding a *third* derivative of f, as well as some challenging algebra. Using the triangle inequality and the facts that $-1\le\sin(\theta)\le 1$ and $-1\le\cos(\theta)\le 1$, we can conclude:

$$|f''(x)|=\left|-4x^2\cdot\sin(x^2)+2\cdot\cos(x^2)\right|\le 4x^2\left|sin(x^2)\right|+2\left|cos(x^2)\right|\le 4\cdot 2^2\cdot 1+2\cdot 1=18$$

so we could take $B_2=18$. We can do a bit better, however, by consulting a graph of $f''(x)$ on $[0,2]$:



it appears clear from the graph that $|f''(x)|\le 11$, so we take $B_2=11$ instead.

Using these values for a, b, n and B_2 in the "error bound" formula:

$$|\text{"error"}|=\left|\int_0^2 \sin(x^2)\,dx-T_{10}\right|\le\frac{2^3\cdot 11}{12\cdot 10^2}=\frac{88}{1200}=\frac{11}{150}<0.074$$

so we can be certain that our T_{10} approximation of the definite integral is within 0.074 of the exact value:

$$T_{10}-0.074\le\int_0^2 \sin(x^2)\,dx\le T_{10}+0.074$$

Computing $T_{10}=0.7959247$, we can be certain that the value of the integral $\int_0^2 \sin(x^2)\,dx$ is somewhere between 0.722 and 0.870.

Note

Notice that (a bound for) the "error" depends on three things: the size of the interval of integration (the bigger the interval, the bigger the potential error); the number of subintervals in the partition (the more subintervals, the smaller the potential error); and the size of the second derivative of the integrand. We've already seen that the second derivative of a function is related to the concavity of its graph—later on we will learn that the second derivative helps measure the "curvature" of the graph of f; it should make sense that the more "curvy" a function is, the less effective a linear approximation technique would be.

Practice 4.9.3

Find an error bound for the T_{12} approximation of $\int_2^5 \frac{1}{x}\,dx$.

Answer

$b-a=3$, $n=12$ and $f(x)=\frac{1}{x} \Rightarrow f'(x)=-\frac{1}{x^2} \Rightarrow f''(x)=\frac{2}{x^3}$, so on the interval $[2,5]$:

$$|f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{2^3} = \frac{1}{4} \tag{4.9.3}$$

We can therefore take $B_2=\frac{1}{4}$, so:

$$|\text{error}| \le \frac{(b-a)^3 \cdot B_2}{12n^2} \le \frac{3^3 \cdot \frac{1}{4}}{12(12)^2} = \frac{27}{6912} \approx 0.004 \tag{4.9.4}$$

Example 4.9.6

How large must n be to be certain that T_n is within 0.001 of $\int_0^2 \sin(x^2)\,dx$?

Solution

Here we know the "allowable error" of 0.001 and we must find n. From Example 5 we know that $b-a=2$ and $B_2=11$, so we want the error bound to be less than the allowable error of 0.001:

$$\frac{2^3 \cdot 11}{12 \cdot n^2} < 0.001 \quad \Rightarrow \quad \frac{12 \cdot n^2}{88} > 1000 \quad \Rightarrow \quad n^2 > \frac{88000}{12}$$
$$\Rightarrow \quad n > \sqrt{\frac{22000}{3}} \approx 85.6$$

Because n must be an integer, we can take $n=86$. Computing $T_{86} \approx 0.80465$, we can be certain that the exact value of the integral is between 0.80365 and 0.80565.

As often happens, T_{86} is even closer than 0.001 to the exact value of the integral:

$$\left|T_{86} - \int_0^2 \sin\left(x^2\right)\,dx\right| \approx 0.00012$$

Practice 4.9.4

Determine how large n must be in order to ensure that T_n is within 0.001 of $\int_2^5 \frac{1}{x}\,dx$.

Answer

We want:

$$|\text{error}| \le \frac{(b-a)^3 \cdot B_2}{12n^2} \le \frac{3^3 \cdot \frac{1}{4}}{12 \cdot n^2} = \frac{27}{48n^2} < 0.001 \tag{4.9.5}$$

so solving for n:

$$\frac{48n^2}{27} > 1000 \Rightarrow n^2 > \frac{27000}{48} = \frac{1125}{2} \Rightarrow n > \sqrt{562.5} \approx 23.7 \tag{4.9.6}$$

Using $n=24$ will work. We can be certain that T_{24} is within 0.001 of the exact value of the integral. (We cannot guarantee that T_{23} is within 0.001 of the exact value of the integral, but it probably is.)

Error Bound for Simpson's Parabolic Approximation

If: $f^{(4)}$ is continuous on $[a,b]$ and $|f^{(4)}(x)| \le B_4$

then: the "error" of the S_n approximation of $\int_a^b f(x)\,dx$ satisfies:

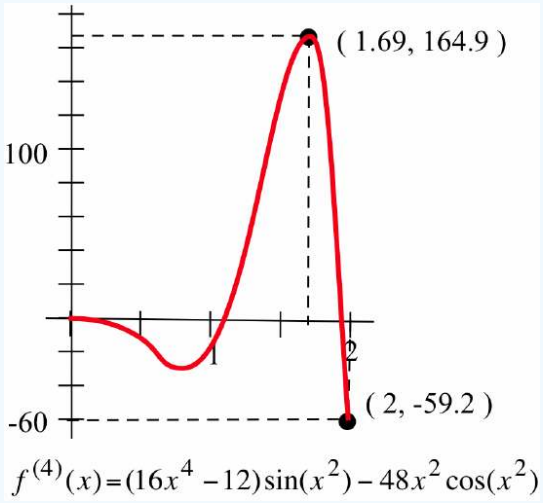
$$|\text{"error"}| = \left|\int_a^b f(x)\,dx - S_n\right| \le \frac{(b-a)^5 B_4}{180n^4}$$

Example 4.9.7

Find an error bound for the S_{10} approximation of $\int_0^2 \sin(x^2)\,dx$.

Solution

We have $b-a=2$, $n=10$ and $f(x)=\sin(x^2)$, so $f^{(4)}(x) = (16x^4-12)\sin(x^2) - 48x^2\cos(x^2)$ is continuous on $[0,2]$. From a graph of $f^{(4)}(x)$ on $[0,2]$:



we can estimate that $B_4 = 165$, so

$$|\text{"error"}| = \left|\int_0^2 \sin(x^2)\,dx - S_{10}\right| \le \frac{2^5 \cdot 165}{180 \cdot 10^4} = \frac{5280}{1800000} < 0.003$$

and we can be certain that our S_{10} approximation of $\int_0^2 \sin(x^2)\,dx$ is within 0.003 of the exact value:

$$S_{10} - 0.003 \le \int_0^2 \sin(x^2)\,dx \le S_{10} + 0.003$$

Computing $S_{10} = 0.80537615$, we are certain that the exact value of $\int_0^2 \sin(x^2)\,dx$ is between 0.80237615 and 0.80837615. Notice that we achieved a much narrower guarantee using S_{10} compared to using T_{10} to approximate the same integral.

Example 4.9.8

Determine how large n must be to ensure that S_n is within 0.001 of the exact value of $\int_0^2 \sin(x^2)\,dx$.

Solution

We want the "error bound" to be less than 0.001 and need to find n. We know that $b-a=2$ and $B_4 = 165$

$$\frac{2^5 \cdot 165}{180 \cdot n^4} < 0.001 \quad \Rightarrow \quad \frac{180 \cdot n^2}{5280} > 1000 \quad \Rightarrow \quad n^2 > \frac{5280000}{180} = \frac{88000}{3}$$

$$\Rightarrow \quad n > \sqrt[4]{\frac{88000}{3}} \approx 13.09$$

Because n must be an even integer, we can take $n = 14$ and be certain that S_{14} is within 0.001 of $\int_0^2 \sin(x^2)\,dx$.

As we have come to expect, S_{14} is even closer than 0.001 to the exact value of the integral; using advanced methods, we can show that:

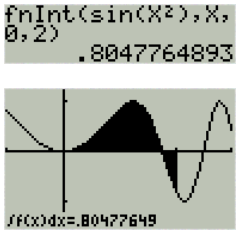
$$\left|\int_0^2 \sin(x^2)\,dx - S_{14}\right| \approx 0.00015$$

Alternative Methods

In Section 8.7 and in Chapter 10, you will learn how to approximate a function f over an entire interval $[a, b]$ using a single polynomial $p(x)$ of degree n; you can then approximate $\int_a^b f(x)\,dx$ with $\int_a^b p(x)\,dx$, which is relatively easy to compute. One advantage of this method is that (once we have found $p(x)$), we only need to evaluate another polynomial ($P(x)$ where $P'(x) = p(x)$) at two values ($P(a)$ and $P(b)$) to compute $\int_a^b p(x)\,dx \approx \int_a^b f(x)\,dx$ and we can get better approximations by increasing n and using polynomials of higher and higher degree; using the Trapezoidal Rule or Simpson's Rule requires us to evaluate $f(x)$ at $n + 1$ points. A disadvantage of this approach is that our original $f(x)$ must have n continuous derivatives, which is not always the case, and we need to be able to compute those n derivatives at a single point. Most textbooks on Numerical Analysis offer more sophisticated techniques for approximating definite integrals.

Using Technology

If you have written even the most basic computer code, you should be able to write a program to compute any Trapezoidal Rule or Simpson's Rule approximation you want (accurate up to the floating-point limitations of the machine running your code). If you have a graphing calculator, it likely has one or more numerical integration utilities:



The Web site Wolfram|Alpha (`www.wolframalpha.com`) can approximate definite integrals to any desired accuracy; typing `integral sin(x^2) from x=0 to x=2` yields:

$$\int_0^2 \sin(x^2)\,dx = \sqrt{\frac{\pi}{2}}\,S\left(2\sqrt{\frac{2}{\pi}}\right) \approx 0.804776$$

Wolfram|Alpha can also be used to quickly apply Simpson's Rule: `use Simpson's rule sin(x^2) from 0 to 2 with 10 intervals` yields an approximation of 0.804811 for $\int_0^2 \sin(x^2)\,dx$.

Problems

1. Use the values in the table below to approximate $\int_2^6 f(x)\,dx$ by calculating T_4 and S_4.

x	$f(x)$
2.0	2.1
2.5	2.7
3.0	3.8
3.5	2.3
4.0	0.3
4.5	-1.8
5.0	-0.9
5.5	0.5
6.0	2.2

2. Use the values in the table above to approximate $\int_2^6 f(x)\,dx$ by calculating T_8 and S_8.
3. Use the values in the table below to approximate $\int_{-3}^1 g(x)\,dx$ by calculating T_8 and S_8.

x	$g(x)$
-3.0	4.2
-2.5	1.8
-2.0	0.7
-1.5	1.5
-1.0	3.4
-0.5	4.3
0.0	3.5
0.5	-0.3
1.0	-4.6

4. Use the values in the table above to approximate $\int_{-3}^1 g(x)\,dx$ by calculating T_4 and S_4.

For Problems 5–10, calculate:

a. T_4
b. S_4
c. the exact value of the integral.

5. $\int_1^3 x\,dx$
6. $\int_0^2 [1-x]\,dx$
7. $\int_{-1}^1 x^2\,dx$
8. $\int_{2\pi}^6 \frac{1}{x}\,dx$
9. $\int_0^1 \sin(x)\,dx$
10. $\int_0^1 \sqrt{x}\,dx$

For Problems 11–16, calculate:

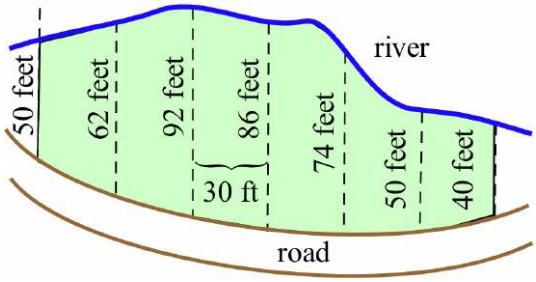
a. T_6
b. S_6

11. $\int_0^2 \frac{1}{1+x^3}\,dx$
12. $\int_1^2 2^x\,dx$
13. $\int_{-1}^1 \sqrt{4-x^2}\,dx$
14. $\int_0^1 e^{-x^2}\,dx$
15. $\int_1^4 \frac{\sin(x)}{x}\,dx$
16. $\int_0^1 \sqrt{1+\sin(x)}\,dx$

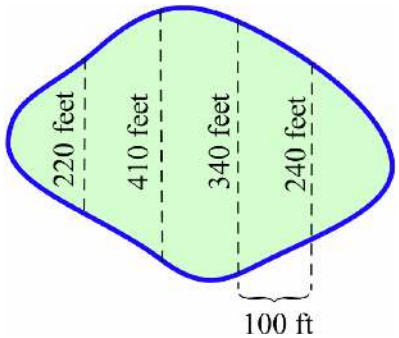
For Problems 17–22, calculate:

a. the error bound for T_4
b. the error bound for S_4
c. the value of n so that the error bound for T_n is less than 0.001
d. the value of n so that the error bound for S_n is less than 0.001.

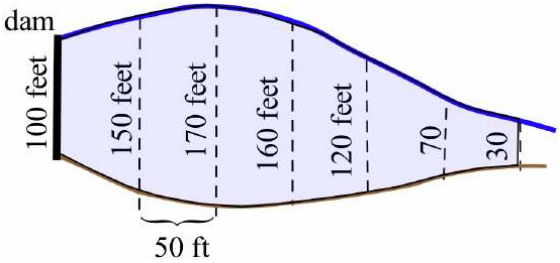
17. $\int_1^3 x\,dx$
18. $\int_0^2 [1-x]\,dx$
19. $\int_{-1}^1 x^3\,dx$
20. $\int_{2\pi}^6 \frac{1}{x}\,dx$
21. $\int_0^{2\pi} \sin(x)\,dx$
22. $\int_0^1 \sqrt{x}\,dx$
23. Estimate the area of the piece of land located between the river and the road in the figure below:



24. Estimate the area of the island in the figure below:



25. Estimate the volume of water in the reservoir shown below if the average depth is 22 feet:



26. The table below shows the speedometer readings (in feet per minute) for a car at one-minute intervals:

t	$v(t)$	t	$v(t)$
0	0	6	5200
1	2000	7	4400
2	3000	8	3000
3	5000	9	2000
4	5000	10	1200
5	6000		

Estimate how far the car traveled:

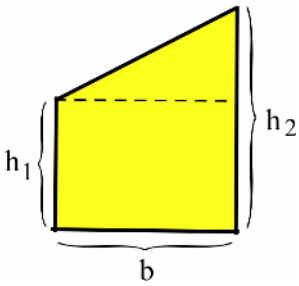
a. during the first 5 minutes of the trip and
b. during the first 10 minutes of the trip.

27. The table below shows the speed (in feet per minute) of a jogger at one-minute intervals:

t	$v(t)$	t	$v(t)$
0	0	6	520
1	420	7	440
2	540	8	360
3	300	9	260
4	500	10	180
5	580		

Estimate how far the jogger ran during her workout.

28. Use the error-bound formula for Simpson's Rule to show that the parabolic approximation gives the exact value of $\int_a^b f(x)\,dx$ if $f(x) = Ax^3 + Bx^2 + Cx + D$ is a polynomial of degree 3 or less.
29. A trapezoidal region with base b and heights h_1 and h_2 (assume $h_1 \neq h_2$) can be cut into a rectangle with base b and height h_1 and a triangle with base b and height $h_1 - h_2$:



Show that the sum of the area of the rectangle and the area of the triangle is $b \cdot \left[\frac{h_1 + h_2}{2}\right]$.

30. Let $f(m)$ be the minimum value of f on the interval $[x_0, x_1]$, $f(M)$ be the maximum value of f on $[x_0, x_1]$, and $h = x_1 - x_0$. Show that:

$$h \cdot f(m) \leq b \cdot \left[\frac{f(x_0) + f(x_1}{2}\right] \leq h \cdot f(M)$$

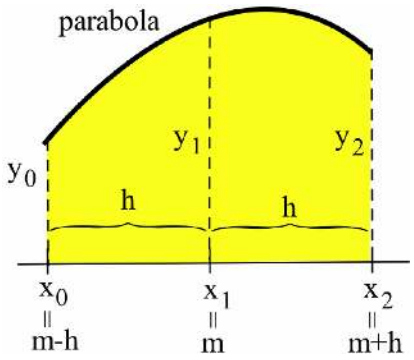
and use this result to show that the trapezoidal approximation is between the lower and upper Riemann sums for f. Because the limit (as $h \to 0$) of these Riemann sums is $\int_a^b f(x)\,dx$, conclude that the limit of the trapezoidal sums must equal $\int_a^b f(x)\,dx$.

31. Let $f(m)$ be the minimum value of f on the interval $[x_0, x_2]$, $f(M)$ the maximum of f on $[x_0, x_2]$ and $h = x_1 - x_0 = x_2 - x_1$. Show that the value

$$2h \cdot \left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right]$$

is between $2h \cdot f(m)$ and $2h \cdot f(M)$ and use this result to show that the parabolic approximation of $\int_a^b f(x)\,dx$ is between the lower and upper Riemann sums for f. Conclude that the limit of the parabolic sums must equal $\int_a^b f(x)\,dx$.

32. This problem guides you through the steps to show that the area under a parabolic region with evenly spaced x_k values:



(which, for the purposes of this problem we will call $x_0 = m - h$, $x_1 = m$ and $x_2 = m + h$) is:

$$\frac{h}{3} \cdot [f(x_0) + 4f(x_1) + f(x_2)] = \frac{h}{3} \cdot [y_0 + 4y_1 + y_2]$$

a. For $f(x) = Ax^2 + Bx + C$, verify that:

$$\int_{m-h}^{m+h} f(x)\,dx = \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{m-h}^{m+h} = 2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch$$

b. Expand each of the polynomials:

$$\begin{aligned} y_0 = f(m-h) &= A(m-h)^2 + B(m-h) + C \\ y_1 = f(m) &= Am^2 + Bm + C \\ y_2 = f(m+h) &= A(m+h)^2 + B(m+h) + C \end{aligned}$$

and use the results to verify that:

$$\frac{h}{3}[y_0+4y_1+y_2] = 2h\left[\frac{f(m-h)+4f(m)+f(m+h)}{6}\right]$$
$$= 2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch$$

c. Compare the results of parts (a) and (b) to conclude that for any quadratic function $f(x) = Ax^2 + Bx + C$:

$$\int_{m-h}^{m+h} f(x)\,dx = \frac{h}{3}[y_0+4y_1+y_2]$$

Left-Endpoint, Right-Endpoint and Midpoint Rules

The rectangular approximation methods approximate an integrand with horizontal lines, so that the approximating regions are rectangles and the sum of the areas of these rectangular regions is a Riemann sum. The Left- and Right-Endpoint Rules are easy to understand and use, but they typically require a very large number of subintervals to ensure good approximations of a definite integral. The Midpoint Rule uses the value of the integrand at the midpoint of each subinterval: if these midpoint values of f are available (for example, when f is given by a formula) then the Midpoint Rule is often more efficient than the Trapezoidal rule. The rectangular approximation rules are:

$$L_n = h\cdot[f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})]$$
$$R_n = h\cdot[f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)]$$
$$M_n = h\cdot[f(c)+f(c+h)+f(c+2h)+\cdots+f(c+(n-1)h)]$$

where $c = x_0 + \frac{h}{2}$ so that the points c, $c+h$, $c+2h$, etc. are the midpoints of the subintervals. The "error bounds" for these methods are:

$$|\text{“error” for } L_n \text{ or } R_n| \le \frac{(b-a)^2B_1}{2n}$$
$$|\text{“error” for } M_n| \le \frac{(b-a)^3B_2}{24n^2}$$

where $B_1 \ge |f'(x)|$ on $[a,b]$ and $B_2 \ge |f''(x)|$ on $[a,b]$. Notice that the error bound for M_n is half the error bound of T_n, the trapezoidal approximation.

For Problems 33–38, calculate:

a. L_4
b. R_4
c. M_4
d. the exact value of the integral.

33. $\int_1^3 x\,dx$
34. $\int_0^2 [1-x]\,dx$
35. $\int_{-1}^1 x^2\,dx$
36. $\int_2^6 \frac{1}{x}\,dx$
37. $\int_0^\pi \sin(x)\,dx$
38. $\int_0^1 \sqrt{x}\,dx$
39. Show that the Trapezoidal approximation is the average of the Left- and Right-Endpoint approximations: $T_n = \frac{1}{2}(L_n + R_n)$.

The integrals in Problems 40–43 will arise in applications from Chapter 5. Use technology to approximate each integral by applying Simpson's Rule with $n = 10$ and $n = 40$ to approximate their values. (Is S_{40} very different from S_{10}?)

40. $\frac{1}{\sqrt{2\pi}}\int_{-2}^2 e^{-\frac{1}{2}x^2}\,dx$
41. $\int_{-1}^2 \sqrt{1+4x^2}\,dx$

42. $\int_0^{\pi} \sqrt{1+\cos^2(x)}\,dx$
43. $\int_0^{2\pi} \sqrt{16\sin^2(t)+9\cos^2(t)}\,dt$

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4.A: Answers

Section 4.0

1. a. $(10)(12)+(8)(4)=152$
   b. $(10)(20)-(3)(8)=176$
3. $bh+\frac{1}{2}b(H-h)=bh+\frac{1}{2}bH-\frac{1}{2}bh=b\left(\frac{h+H}{2}\right)$
5. a. $1\cdot 3+1\cdot 2=5$
   b. area <5
7. Answers will vary from 5 to 13.
9. $A(1)=1$, $A(2)=2.5$, $A(3)=4.5$, $A(4)=6$, $A(5)=7$
11. $C(1)=1.5$, $C(2)=4$, $C(3)=7.5$, $C(x)$ is sum of rectangular and triangular areas:

$$C(x)=x+\frac{1}{2}x\cdot x=x+\frac{1}{2}x^2$$

13. $(20)(30)+\frac{1}{2}(10)(30)=600+150=750$ feet
15. a. A: 20 seconds to stop; B: 40 seconds to stop
    b. A: $\frac{1}{2}(20)(80)=800$ ft; B: $\frac{1}{2}(40)(40)=800$ ft
17. miles, \$, ft^3, kilowatt-hours, people, square meals

Section 4.1

1. $2^2+3^2+4^2=29$
3. $(1+1)^2+(1+2)^2+(1+3)^2=29$
5. $\cos(0)+\cos(\pi)+\cos(2\pi)+\cos(3\pi)+\cos(4\pi)+\cos(5\pi)=1+(-1)+1+(-1)+1+(-1)=0$
7. $\sum_{k=3}^{94}k$
9. $\sum_{k=3}^{12}k^2$
11. $\sum_{k=1}^{7}k\cdot 2^k$
13. a. $(1+2)+(2+2)+(3+2)=3+4+5=12$
    b. $(1+2+3)+(2+2+2)=12$
15. a. $5\cdot 1+5\cdot 2+5\cdot 3=5+10+15=30$
    b. $5\cdot(1+2+3)=5\cdot 6=30$
17. a. $1^2+2^2+3^2=1+4+9=14$
    b. $(1+2+3)^2=6^2=36$
19. $f(0)+f(1)+f(2)+f(3)=0^2+1^2+2^2+3^2=14$
21. $2\cdot f(0)+2\cdot f(1)+2\cdot f(2)+2\cdot f(3)=2\cdot 0+2\cdot 1+2\cdot 4+2\cdot 9=28$
23. $g(1)+g(2)+g(3)=3+6+9=18$
25. $g^2(1)+g^2(2)+g^2(3)=3^2+6^2+9^2=126$
27. $h(2)+h(3)+h(4)=\frac{2}{2}+\frac{2}{3}+\frac{2}{4}=\frac{13}{6}$
29. $f(1)h(1)+f(2)h(2)+f(3)h(3)=(1)(2)+(4)(1)+(9)\left(\frac{2}{3}\right)=12$
31. $(1^2-0^2)+(2^2-1^2)+(3^2-2^2)+(4^2-3^2)+\cdots+(7^2-6^2)=7^2-0^2=49$
33. $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)=1-\frac{1}{6}=\frac{5}{6}$
35. $\left(\sqrt{1}-\sqrt{0}\right)+\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{4}-\sqrt{3}\right)+\cdots+\left(\sqrt{9}-\sqrt{8}\right)=3-0=3$
37. a. $[2,3]$, $[3,4.5]$, $[4.5,6]$, $[6,7]$
    b. 1, 1.5, 1.5, 1
    c. mesh $=1.5$
    d. $1+1.5+1.5+1=5$
39. a. $[-3,-1]$, $[-1,0]$, $[0,1.5]$, $[1.5,2]$

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b. $2, 1, 1.5, 0.5$

c. mesh $= 2$

d. $2 + 1 + 1.5 + 0.5 = 5$

41. a. $[3, 3.8]$, $[3.8, 4.5]$, $[4.5, 5.2]$, $[5.2, 7]$

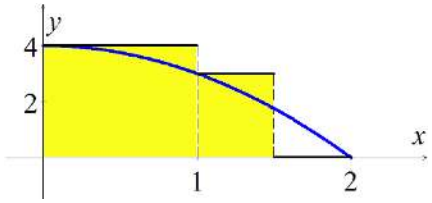
b. $0.8, 0.7, 0.7, 1.8$

c. mesh $= 1.8$

d. $0.8 + 0.7 + 0.7 + 1.8 = 4$

43. $\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) = x_n - x_0$

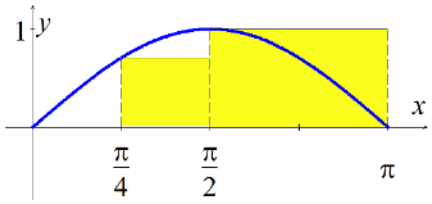
45. a.



$f(0)(1) + f(1)(0.5) + f(2)(0.5) = (4)(1) + (3)(0.5) + (0)(0.5) = 5.5$

b. $f(1)(1) + f(1.5)(0.5) + f(1.5)(0.5) = (3)(1) + (1.75)(0.5) + (1.75)(0.5) = 4.75$

47. a.



$\left(\frac{\pi}{4}\right)(0) + \left(\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\pi}{2}\right)(1) \approx 2.13$

b. $\left(\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\pi}{4}\right)(1) + \left(\frac{\pi}{2}\right)(0) \approx 1.34$

49. a. $(2)(1) + (5)(2) + (17)(1) \le \text{RS} \le (5)(1) + (17)(2) + (26)(1) \Rightarrow 29 \le \text{RS} \le 65$

b. $(2)(1) + (5)(1) + (10)(1) + (17)(1) \le \text{RS} \le (5)(1) + (10)(1) + (17)(1) + (26)(1) \Rightarrow 34 \le \text{RS} \le 58$

c. $2(0.5) + 3.25(0.5) + 5(1) + 10(1) + 17(1) \le \text{RS} \le 3.25(0.5) + 5(0.5) + 10(1) + 17(1) + 26(1) \Rightarrow 34.625 \le \text{RS} \le 57.125$

51. a. $(0)\left(\frac{\pi}{2}\right) + (0)\left(\frac{\pi}{2}\right) \le \text{RS} \le (1)\left(\frac{\pi}{2}\right) + (1)\left(\frac{\pi}{2}\right) \Rightarrow 0 \le \text{RS} \le \pi$

b. $0\left(\frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) + 0\left(\frac{\pi}{2}\right) \le \text{RS} \le \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) + 1\left(\frac{\pi}{4}\right) + 1\left(\frac{\pi}{2}\right) \Rightarrow 0.56 \le \text{RS} \le 2.91$

c. $0\left(\frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) + 0\left(\frac{\pi}{4}\right) \le \text{RS} \le \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}\right) + 1\left(\frac{\pi}{4}\right) + 1\left(\frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) \Rightarrow 1.11 \le \text{RS} \le 2.68$

53. a. $|7.402 - 7.362| = 0.04$

b. $|7.390 - 7.372| = 0.018$

55. $|\text{error}| = (\text{base})(\text{height}) = \frac{4-2}{50}(65 - 9) = \frac{56}{25} = 2.24$

57. a. $\frac{100(101)}{2} = 5050$

b.

$S =$	1	+	2	+	3	+	⋯	+	100
$+ S =$	100	+	99	+	98	+	⋯	+	1
$2S =$	101	+	101	+	101	+	⋯	+	101

so $2S = 100(101) = 10100 \Rightarrow S = 5050$

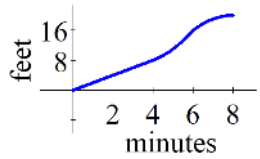
59. $10 + 11 + 12 + \cdots + 20 = (1 + 2 + 3 + \cdots + 20) - (1 + 2 + 3 + \cdots + 9) = \frac{20(21)}{2} - \frac{9(10)}{2} = 210 - 45 = 165$

61. $\sum_{k=1} 10\,(k^3 + k) = \sum_{k=1} 10\,k^3 + \sum_{k=1} 10\,k = \left[\frac{10(11)}{2}\right]^2 + \left[\frac{10(11)}{2}\right] = (55)^2 + 55 = 3080$

Section 4.2

1. $\int_0^4 [2 + 3x]\,dx$

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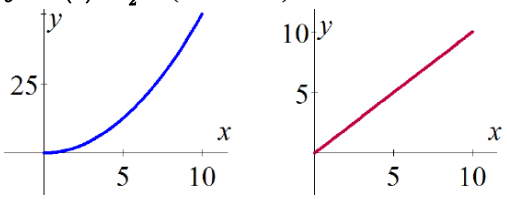
3. $\int_2^5 \cos(5x)\,dx$
5. $\int_1^5 x^3\,dx$
7. $\int_{0.5}^2 x \cdot \sin(x)\,dx$
9. $\int_1^3 \ln(x)\,dx$
11. $\int_1^3 2x\,dx = 8$
13. $\int_{-1}^0 |x|\,dx = \frac{1}{2}$
15. $\int_0^4 \left[3 - \frac{x}{2}\right]\,dx = 8$
17. a. 3
    b. -1
    c. 6
    d. 8
    e. 7
19. a. 

    b. 24 ft
    c. 24 feet from starting point
21. meters
23. ft^3
25. gram-meters
27. ft/sec
29. $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $m_k = \frac{2}{n}(k-1)$, $M_k = \frac{2}{n}k$, so $f(m_k) = \left[\frac{2}{n}(k-1)\right]^3$ and $f(M_k) = \left[\frac{2}{n}k\right]^3$
    a. $\text{LS} = \sum_{k=1}^n f(m_k)\,\Delta x = \sum_{k=1}^n \left[\frac{2}{n}(k-1)\right]^3 \Delta x = \frac{2}{n} \cdot \frac{8}{n^3}\left[\sum_{k=1}^n k^3 - 3\sum_{k=1}^n k^2 + 3\sum_{k=1}^n k - \sum_{k=1}^n 1\right]$
    $= \frac{16}{n^4}\left[\left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{3}{12}n^2\right) - 3\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{2}{12}n\right) + 3\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) - n\right]$
    $= \frac{16}{n^4}\left[\frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2\right] = 4 - \frac{8}{n} + \frac{4}{n^2} \longrightarrow 4$
    b. $\text{US} = \sum_{k=1}^n f(M_k)\,\Delta x = \sum_{k=1}^n \left[\frac{2}{n}(k)\right]^3 \frac{2}{n} = \frac{16}{n^4}\left[\sum_{k=1}^n k^3\right] = \frac{16}{n^4}\left[\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{3}{12}n^2\right] = 4 + \frac{8}{n} + \frac{4}{n^2} \longrightarrow 4$

Section 4.3

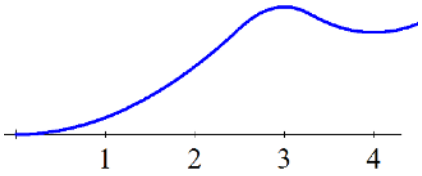
1. 5
3. 0
5. 3
7. 0
9. -5
11. -5
13. 0
15. $4.5 + 5 = 9.5$
17. $10 + 3 = 13$
19. $5 + 2 = 7$
21. 1
23. -1
25. 2

27. $\mathbf{1}$
29. $\mathbf{1}$
31. a. $\mathbf{8 \cdot 6 = 48}$
    b. $\mathbf{24}$
33. a. $\mathbf{32}$
    b. $\mathbf{8^2 = 64}$
35. $\mathbf{8}$
37. $\mathbf{2.5}$
39. $\mathbf{3}$
41. $\mathbf{7}$
43. a. $y = A(x) = \frac{1}{2}x^2$ (below left)

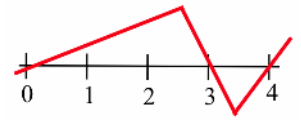


    b. $y = A'(x) = x$ (above right)

45. a.



    b.

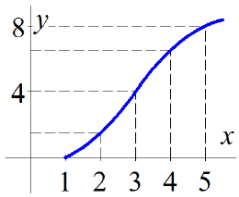


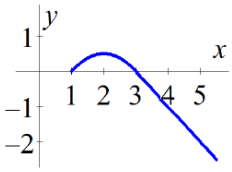
47. a. f is continuous on $[1, 4]$
    b. f is not differentiable on $[1, 4]$ (not differentiable at $x \approx 2.5$ and $x \approx 3.3$)
    c. f is integrable on $[1, 4]$
49. a. f is not continuous on $[1, 4]$ (not continuous at $x = 2$)
    b. f is not differentiable on $[1, 4]$ (not differentiable at $x = 2$)
    c. f is integrable on $[1, 4]$
51. $\int_1^4 v(t)\,dt = \int_1^2 v(t)\,dt + \int_2^4 v(t)\,dt = 35 + 50 = 85$ miles
53. a. The rectangle associated with the interval containing $x = 2$ has width w and height 5, so its area is $5w$; all of the other rectangles have the same height (7) and the sum of their widths is $(4 - 1) - w = 3 - w$, so their total area is $7(3 - w)$: $\mathrm{RS} = 5w + 7(3 - w)$
    b. Because $0 < w \leq \|\mathcal{P}\|$, as $\|\mathcal{P}\| \to 0 \Rightarrow w \to 0$ so $\lim\limits_{\|\mathcal{P}\| \to 0} \mathrm{RS} = \lim\limits_{w \to 0} [5w + 7(3 - w)] = 21$
    c. $\int_1^4 g(x)\,dx = \lim\limits_{\|\mathcal{P}\| \to 0} \mathrm{RS} = 21$
    $\int_1^4 7\,dx = 7(4 - 1) = 7(3) = 21$
    d. A (very) similar argument shows that redefining any constant function f at a single point does not alter the value of $\int_a^b f(x)\,dx$. A (somewhat) similar argument can extend this result to all integrable functions.

Section 4.4

1. a. See figure below left.
   b. $A(1) = 0$, $A(2) = 1.5$, $A(3) = 4$, $A(4) = 6.5$
   c. $A'(1) = 1$, $A'(2) = 2$, $A'(3) = 3$, $A'(4) = 2$

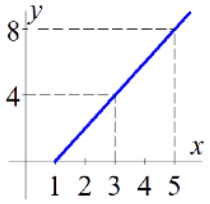
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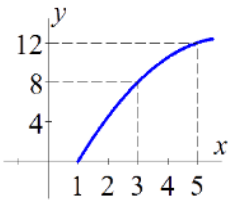




3. a. See figure above right.
   b. $A(1) = 0,\ A(2) = 0.5,\ A(3) = 0,\ A(4) = -1$
   c. $A'(1) = 1,\ A'(2) = 0,\ A'(3) = -1,\ A'(4) = -1$

5. a. See figure below left.
   b. $A(1) = 0,\ A(2) = 2,\ A(3) = 4,\ A(4) = 6$
   c. $A'(1) = 2,\ A'(2) = 2,\ A'(3) = 2,\ A'(4) = 2$

6. 



7. a. See figure above right.
   b. $A(1) = 0,\ A(2) = 4.5,\ A(3) = 8,\ A(4) = 10.5$
   c. $A'(1) = 5,\ A'(2) = 4,\ A'(3) = 3,\ A'(4) = 2$

9. a. $x^2\Big|_0^3 = 9$
   b. $x^2\Big|_1^3 = 8$
   c. $x^2\Big|_0^1 = 1$

11. a. $2x^3\Big|_1^3 = 52$
    b. $2x^3\Big|_1^2 = 14$
    c. $2x^3\Big|_0^3 = 54$

13. a. $x^4\Big|_0^3 = 81$
    b. $x^4\Big|_1^3 = 80$
    c. $x^4\Big|_0^1 = 1$

15. a. $x^3\Big|_{-3}^3 = 54$
    b. $x^3\Big|_{-3}^0 = 27$
    c. $x^3\Big|_0^3 = 27$

17. a. $x^3\Big|_0^2 = 8$
    b. $x^3\Big|_1^3 = 26$
    c. $x^3\Big|_3^1 = -26$

19. a. $\displaystyle\int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100$ ft
    b. $\displaystyle 50 = \int_0^T 2t\,dt = T^2 \Rightarrow T = \sqrt{50} \approx 7.07$ sec

21. a. $\displaystyle\int_0^{10} 4t^3\,dt = t^4\Big|_0^{10} = 10000$ ft
    b. $\displaystyle 5000 = \int_0^T 4t^3\,dt = T^4 \Rightarrow T =$ sec

23. a. $75 - 3t^2 = 0 \Rightarrow t = 5$ sec

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b. $\int_0^5 \left[75-3t^2\right]\,dt = 75t - t^3\Big|_0^5 = 250$ ft

c. $125 = \int_0^T \left[75-3t^2\right]\,dt = 75T - T^3$, so use a graph of $y = x^3 - 75x + 125$ (or Newton's method) to solve for $T \approx 1.74$ sec

25. The total area is $\int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = \frac{1}{3}\cdot 27 = 9$.

a. $\frac{1}{2}\cdot 9 = \frac{1}{3}x^3\Big|_0^T = \frac{1}{3}T^3 \Rightarrow T = \sqrt[3]{\frac{27}{2}} \approx 2.38$

b. $\frac{1}{3}\cdot 9 = \int_0^T x^2\,dx = \frac{1}{3}T^3 \Rightarrow T = \sqrt[3]{9} \approx 2.08$

$\frac{2}{3}\cdot 9 = \int_0^T x^2\,dx = \frac{1}{3}T^3 \Rightarrow T = \sqrt[3]{18} \approx 2.62$

Section 4.5

1. a. $A(x) = x^3 \Rightarrow A'(x) = 3x^2$ so $A'(1) = 3$, $A'(2) = 12$ and $A'(3) = 27$

b. $A'(x) = \mathrm{D}\left[\int_0^x 3t^2\,dt\right] = 3x^2$ so $A'(1) = 3$, $A'(2) = 12$ and $A'(3) = 27$

3. $A'(x) = 2x$ so $A'(1) = 2$, $A'(2) = 4$, $A'(3) = 6$
5. $A'(x) = 2x$ so $A'(1) = 2$, $A'(2) = 4$, $A'(3) = 6$
7. $A'(1) \approx 0.84$, $A'(2) \approx 0.91$, $A'(3) \approx 0.14$
9. $A'(x) = f(x)$, $A'(1) = 2$, $A'(2) = 1$, $A'(3) = 2$
11. $A'(x) = f(x)$, $A'(1) = 1$, $A'(2) = 2$, $A'(3) = 2$
13. $F(1) - F(0) = 6 - 5 = 1$
15. $F(3) - F(1) = 9 - \frac{1}{3} = \frac{26}{3}$
17. $F(5) - F(1) \approx 1.61 - 0 = 1.61$
19. $F(3) - F\left(\frac{1}{2}\right) \approx 1.10 - (-0.69) = 1.79$
21. $F\left(\frac{\pi}{2}\right) - F(0) = 1 - 0 = 1$
23. $F(1) - F(0) \approx 0.67 - 0 = 0.67$
25. $F(7) - F(1) = \frac{2}{3}(7)^{\frac{3}{2}} - \frac{2}{3} \approx 11.68$
27. $F(9) - F(1) = 3 - 1 = 2$
29. $F(3) - F(-2) \approx 20.09 - 0.14 = 19.95$
31. $F\left(\frac{\pi}{4}\right) - F(0) = 1 - 0 = 1$
33. $F(3) - F(0) = \frac{2}{3}(10)^{\frac{3}{2}} - \frac{2}{3} \approx 20.42$
35. $F(x) = \frac{1}{3}x^3 \Rightarrow F(2) - F(-1) = \frac{8}{3} - \left(-\frac{1}{3}\right) = 3$
37. $F(x) = \ln(x) \Rightarrow F(e) - F(1) = 1 - 0 = 1$
39. $F(x) = \frac{2}{3}x^{\frac{3}{2}} \Rightarrow F(100) - F(25) = \frac{2000}{3} - \frac{250}{3} = \frac{1750}{3}$
41. $F(x) = -\frac{1}{x} \Rightarrow F(10) - F(1) = -0.1 - (-1) = 0.9$
43. $F(x) = e^x \Rightarrow F(1) - F(0) = e - 1 \approx 1.718$
45. $F(x) = \tan(x) \Rightarrow F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right) = 1 - \frac{1}{\sqrt{3}} \approx 0.423$
47. $\int_3^3 f(x)\,dx = 0$ for any integrable f
49. $\int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = -(-1) - (-1) = 2$
51. $\int_0^{3.5} \lfloor x \rfloor\,dx = 0 + 1 + 2 + \frac{1}{2}(3) = 4.5$
53. $\int_0^3 (x-2)^2\,dx = \int_0^3 (x^2 - 4x + 4)\,dx = \frac{1}{3}x^3 - 2x^2 + 4x\Big|_0^3 = 3$
55. $\mathrm{D}\left(A(3x)\right) = 3\tan(3x)$, $\mathrm{D}\left(A(x^2)\right) = 2x\tan(x^2)$, $\mathrm{D}\left(A(\sin(x))\right) = \cos(x)\tan(\sin(x))$
57. $\sqrt{1+5x}(5)$
59. $\sqrt{1+\sin(x)}\cdot\cos(x)$
61. $\left[3(1-2x)^2 + 2\right](-2)$
63. $-\cos(3x)$
65. $\tan(x^2)\cdot 2x - \tan(x)$
67. $5\ln(x)\cos(3\ln(x))\cdot\frac{1}{x}$

Section 4.6

1. $\frac{1}{4}x^4\Big|_1^2 = \frac{15}{4} \neq \frac{7}{2} = \left[\frac{1}{3}x^3\Big|_1^2\right]\left[\frac{1}{2}x^2\Big|_1^2\right]$
3. $\frac{1}{4} \neq \frac{1}{3}\cdot\frac{1}{2}$
5. $\frac{1}{3}\sin(3x)+C$
7. $-\cos(2+e^x)+C$
9. $\tan(\sin(x))+C$
11. $\frac{5}{2}\ln|3+2x|+C$
13. $-\frac{1}{3}\cos(1+x^3)+C$
15. $\frac{1}{4}\sin(4x)+C$
17. $\frac{1}{48}(5+x^4)^{12}+C$
19. $\ln|2+x^3|+C$
21. $\frac{1}{2}(\ln(x))^2+C$
23. $\frac{1}{24}(1+3x)^8+C$
25. $\sec(e^x)+C$
27. $\frac{1}{3}\sin(3x)\Big|_0^{\frac{\pi}{2}} = -\frac{1}{3}$
29. $-\cos(2+e^x)\Big|_0^1 \approx -0.996$
31. $\frac{1}{18}(1+x^3)^6\Big|_{-1}^1 = \frac{32}{9}$
33. $\frac{5}{2}\ln|3+2x|\Big|_0^2 = \frac{5}{2}\ln\left(\frac{7}{3}\right)$
35. $-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{1}{3}$
37. $\frac{2}{9}(1+3x)^{\frac{3}{2}}\Big|_0^1 = \frac{14}{9}$
39. $\frac{1}{2}x-\frac{1}{20}\sin(10x)+C$
41. $\frac{1}{4}\sin(2x)+C$
43. $\frac{1}{2}x-\frac{1}{4}\sin(2x)\Big|_0^{\pi} = \frac{\pi}{2}$
45. $\frac{1}{7}x^7+\frac{3}{5}x^5+x^3+x+C\}$
47. $\frac{1}{2}e^{2x}+2e^x+x+C$
49. $\frac{1}{6}x^6+\frac{1}{4}x^4+\frac{5}{3}x^3+5x+C$
51. $\frac{1}{2}e^{2x}+\frac{1}{4}e^{4x}+C$
53. $\frac{2}{7}x^{\frac{7}{2}}+\frac{6}{5}x^{\frac{5}{2}}-\frac{4}{3}x^{\frac{3}{2}}+C$
55. $3x-3\ln|x+1|+C$
57. $\frac{1}{2}x^2-x+C$
59. $x^2-11x+7\ln|x-1|+C$
61. $x+3\ln|x-1|+C$
63. $\frac{2}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$
65. area of semicircle with radius $1 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$
67. area of semicircle with radius $3 = \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$
69. $(2)(2)+\frac{1}{2}\pi(1)^2 = 4+\frac{\pi}{2}$

Section 4.7

1. Answers will vary between **11** (using left endpoints) and **6** (using right endpoints).
3. Between **4** (left endpoints) and **6** (right).
5. Using left-hand widths:

$$(40)\,[0+70+55+90+130+115] = 18400\ \text{ft}^2$$

   Right-hand widths (**70**, **55**,…) and average widths ($\frac{70}{2}$, $\frac{125}{2}$,…) yield the same result.
7. $\int_{-1}^{2}\left[(x^2+3)-1\right]\,dx = 9$

9. $\int_0^1 \left[x - x^2\right]\,dx + \int_1^2 \left[x^2 - x\right]\,dx = 1$
11. $\int_1^e \left[x - \frac{1}{x}\right]\,dx = \frac{1}{2}e^2 - \frac{3}{2}$
13. $\int_0^{\frac{\pi}{4}} \left[(x+1) - \cos(x)\right]\,dx = \frac{1}{32}\pi^2 + \frac{1}{4}\pi - \frac{\sqrt{2}}{2}$
15. $\int_0^2 \left[e^x - x\right]\,dx = e^2 - 3$
17. $\int_0^1 \left[3 - \sqrt{1-x^2}\right]\,dx = 3 - \frac{\pi}{4}$
19. Using $\mathcal{P} = \{0.5, 1.5, 2.5, 3.5, 4.5\}$, so that $\Delta x = 1$, and $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_4 = 4$:

$$\frac{1}{4.5 - 0.5}\int_{0.5}^{4.5} f(x)\,dx \approx \frac{1}{4}\left[6 + 6 + 4 + 3\right](1) = \frac{19}{4}$$

21. With $\mathcal{P} = \{1.5, 2.5, 3.5\}$, $\Delta x = 1$, $c_1 = 2$, $c_2 = 3$:

$$\frac{1}{3.5 - 1.5}\int_{0.5}^{3.5} f(x)\,dx \approx \frac{1}{2}\left[6 + 4\right](1) = \frac{10}{2} = 5$$

23. $\frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{2}{2} = 1$
25. $\frac{1}{6-1}\int_1^6 f(x)\,dx = \frac{11}{5}$
27. $\frac{1}{4-0}\int_0^4 \left[2x + 1\right]\,dx = 5$
29. $\frac{1}{3-1}\int_1^3 x^2\,dx = \frac{13}{3}$
31. $\frac{1}{\pi - 0}\int_0^{\pi} \sin(x)\,dx = \frac{2}{\pi}$
33. $C = 1$: $\overline{f} = \frac{2}{3}$; $C = 9$: $\overline{f} = 2$; $C = 81$: $\overline{f} = 6$; $C = 100$: $\overline{f} = \frac{20}{3}$. In general, $\overline{f} = \frac{2}{3}\sqrt{C}$.
35. a. About 180 miles
    b. About 36 mph
37. a. 1,950 foot-pounds
    b. 1,312.5 foot-pounds
39. a. 1,200 ft-lbs
    b. 600 ft-lbs
    c. 400 foot-lbs
41. 1,275 foot-pounds

Section 4.8

1. $\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$
3. $x^2 + \frac{2}{5}\arctan\left(\frac{x}{5}\right) + C$
5. $\frac{1}{3}\ln\left|\frac{x+3}{x-3}\right| + C$
7. $\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$
9. $e^x + \frac{7}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$
11. $3\arcsin\left(\frac{x}{\sqrt{5}}\right) + C$
13. $\frac{1}{10}\arctan\left(\frac{5x}{2}\right) + C$
15. $\frac{5}{2}\arcsin\left(2x\right) + C$
17. $\frac{2}{3}\ln\left|3x + \sqrt{1 + 9x^2}\right| + C$
19. $(x+1)\ln(x+1) - x + K$
21. $\frac{3}{10}\left(5x^2 + 7\right)\left[\ln\left(5x^2 + 7\right) - 1\right] + C$

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23. $\sin(x)\,[\ln|\sin(x)|-1]+C$
25. $\frac{x}{2}\sqrt{x^2+4}+2\ln\left|x+\sqrt{x^2+4}\right|+C$
27. $\frac{x}{2}\sqrt{x^2+16}+8\ln\left|x+\sqrt{x^2+16}\right|+C$
29. $8+\frac{2}{5}\left[\arctan\left(\frac{3}{5}\right)-\arctan\left(\frac{1}{5}\right)\right]$
31. $\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\Big|_{-1}^{1}=\frac{\pi}{3\sqrt{3}}$
33. $3\left[\arcsin\left(\frac{2}{\sqrt{5}}\right)-\arcsin\left(\frac{1}{\sqrt{5}}\right)\right]$
35. $\frac{5}{2}\arcsin(2x)\Big|_{0}^{0.1}=\frac{5}{2}\arcsin(0.2)$
37. $7\ln(7)-6$
39. $3\ln(3)-2\ln(2)-1$
41. $3\sqrt{18}+\frac{9}{2}\ln\left(\frac{3+\sqrt{18}}{-3+\sqrt{18}}\right)$
43. $-\frac{1}{3}\sin^2(x)\cos(x)-\frac{2}{3}\cos(x)+C$
45. $\frac{1}{5}\cos^4(x)\sin(x)+\frac{4}{5}\int\cos^3(x)\,dx$
47. $x^2\sin(x)+2x\cos(x)-2\sin(x)+C$
49. average of $\sin(x)=\frac{2}{\pi}>\frac{1}{2}=$ average of $\sin^2(x)$
51. $C=e$: $\frac{1}{e-1}$; $C=10$: $\frac{1}{9}\,[10\ln(10)-9]$; $C=100$: $\frac{1}{99}\,[100\ln(100)-99]$; $C=1000$: $\frac{1}{999}\,[1000\ln(1000)-999]$
53. a. $e^2-e\approx 4.67$
    b. $e^2\approx 7.39$
    c. $2e^2-e\approx 12.06$
55. $2\arctan(C)\approx 1.57,\ 2.94,\ 3.04,\ 3.07,\ 3.09$

Section 4.9

1. $T_4=\frac{1}{2}\,[2.1+2(3.8)+2(0.3)+2(-0.9)+2.2]=5.35$
   $S_4=\frac{1}{3}\,[2.1+4(3.8)+2(0.3)+4(-0.9)+2.2]=5.5\}$
3. $T_8=7.35$, $S_8=\frac{22}{3}\approx 7.3333$
5. a. $T_4=4$
   b. $S_4=4$
   c. 4
7. a. $T_4=0.75$
   b. $S_4=\frac{2}{3}\approx 0.67$
   c. $\frac{2}{3}$
9. a. $T_4=1.896118898$
   b. $S_4=2.004559755$
   c. 2
11. a. $T_6=1.088534906$
    b. $S_6=1.090560447$
13. a. $T_6=3.815780054$
    b. $S_6=3.826350295$
15. a. $T_6=0.8159928163$
    b. $S_6=0.8120491229$
17. a. $f(x)=x\Rightarrow f''(x)=0\Rightarrow B_2=0$, so the error bound is 0 (the Trapezoidal approximation is exact)
    b. $f^{(4)}(x)=0\Rightarrow B_4=0$, so the error bound is 0 (the Simpson's Rule approximation is exact)
    c. $n=1$
    d. $n=2$ (must be an even integer)
19. a. $f(x)=x^3\Rightarrow f''(x)=6x$ so when $|x|\le 1$, $|f''(x)|\le 6$; taking $B_2=6$, $|\text{error}|\le\frac{2^3\cdot 6}{12\cdot 4^2}=0.25$

b. $f^{(4)}(x) = 0 \Rightarrow B_4 = 0$, so error bound is 0

c. $\frac{2^3 \cdot 6}{12 \cdot n^2} \leq 0.001 \Rightarrow n^2 \geq 4000 \Rightarrow n \geq 63.25$, so take $n = 64$

d. $n = 2$

21. a. $f''(x) = -\sin(x) \Rightarrow |f''(x)| \leq 1 \Rightarrow B_2 = 1$, so $|\text{error}| \leq \frac{\pi^3 \cdot 1}{12 \cdot 4^2} \approx 0.1612$

b. $f^{(4)}(x) = \sin(x) \Rightarrow \left|f^{(4)}(x)\right| \leq 1 = B_4$, so $|\text{error}| \leq \frac{\pi^5 \cdot 1}{180 \cdot 4^4} \approx 0.0066$

c. $\frac{\pi^3 \cdot 1}{12 \cdot n^2} \leq 0.001 \Rightarrow n^2 \geq \frac{1000\pi^3}{12} \Rightarrow n \geq 50.83$, so take $n = 51$

d. $\frac{\pi^5 \cdot 1}{180 \cdot n^4} \leq 0.001 \Rightarrow n^4 \geq \frac{1000\pi^3}{180} \Rightarrow n \geq 6.42$, so take $n = 8$

23. $S_6 = \frac{30}{3}[50 + 4(62) + 2(92) + 4(86) + 2(74); +4(50) + 40] = 12140$ ft\)^2\)

25. area: $S_6 \approx 37166.7$ ft^2; volume: $(37166.7)(22) =$ 817,667 ft^3

27. distance $\approx T_{10} =$ 4,010 ft

29. On your own.

31. On your own.

33. a. $L_4 = 3.5$

b. $R_4 = 4.5$

c. $M_4 = 4$

d. 4

35. a. $L_4 = 0.75$

b. $R_4 = 0.75$

c. $M_4 = 0.625$

d. $\frac{2}{3}$\

37. a. 1.8961

b. 1.8961

c. 2.0523

d. 2

39. On your own.

41. $S_{10} = 6.12572$; $S_{40} = 6.12573$

43. $S_{10} = 22.1035$; $S_{40} = 22.1035$

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CHAPTER OVERVIEW

5: Applications of Definite Integrals

The previous chapter introduced the concepts of a definite integral as an “area” and as a limit of Riemann sums, demonstrated some of the properties of integrals, introduced some methods to compute values of definite integrals, and began to examine a few of their uses. This chapter focuses on several common applications of definite integrals.

An obvious goal of the chapter is to enable you to use integration when you encounter these particular applications later in mathematics or in other fields. A deeper goal is to illustrate the process of going from a problem to an integral, a process much broader than these particular applications. If you understand the process, then you can understand the use of integrals in many other fields and can even develop the integrals needed to solve problems in new areas. Another goal is to give you additional practice evaluating definite integrals.

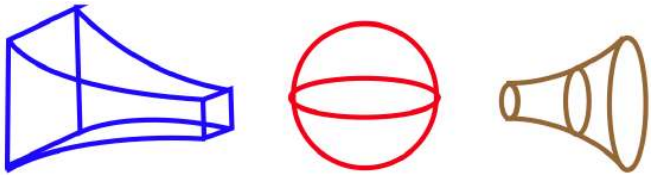
Each section in this chapter follows the same basic format. First we describe a problem and present some background information. Then we approximate the solution to the basic problem using a Riemann sum. An exact answer comes from taking a limit of the Riemann sum, and we get a definite integral. After looking at several examples of the same basic application, we will examine some variations.

5.1: Volumes by Slicing
5.2: Disks and Washers
5.3: Arclength and Surface Area
5.4: More Work
5.5: Volumes: Tubes
5.6: Moments and Centers of Mass
5.7: Improper Integrals
5.8: Additional Applications
5.A: Answers

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5.1: Volumes by Slicing

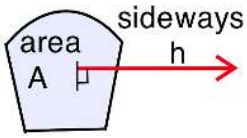
The previous chapter emphasized a geometric interpretation of definite integrals as "areas" in two dimensions. This section emphasizes another geometrical use of integration, computing volumes of solid three-dimensional objects such as those shown below:



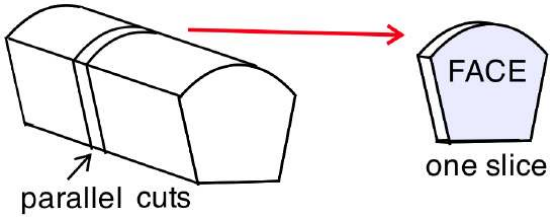
Our basic approach will involve cutting the whole solid into thin "slices" whose volumes we can approximate, adding the volumes of these "slices" together (to get a Riemann sum), and finally obtaining an exact answer by taking a limit of those sums to get a definite integral.

The Building Blocks: Right Solids

A **right solid** is a three-dimensional shape swept out by moving a planar region A some distance h along a line perpendicular to the plane of A:



We call the region A a **face** of the solid and use the word "right" to indicate that the movement occurs along a line perpendicular—at a right angle—to the plane of A. Two parallel cuts produce one slice with two faces:



Note

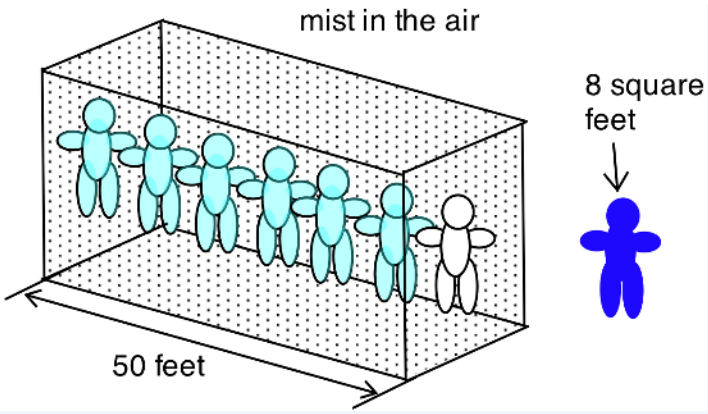
A slice has volume, and a face has area.

Example 5.1.1

Suppose a fine, uniform mist is suspended in the air and that every cubic foot of mist contains 0.02 ounces of water droplets. If you run 50 feet in a straight line through this mist, how wet do you get? Assume that the front (or a cross section) of your body has an area of 8 square feet.

Solution

As you run, the front of your body sweeps out a "tunnel" through the mist:

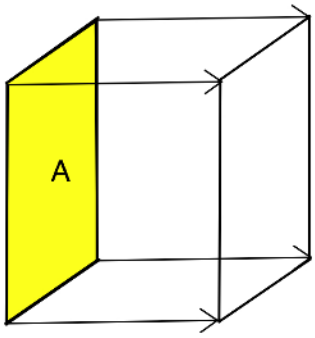


The volume of the "tunnel" is the area of the front of your body multiplied by the length of the tunnel:

$$\text{volume} = (8\ \text{ft}^2)\,(50\ \text{ft}) = 400\ \text{ft}^3$$

Because each cubic foot of mist held 0.02 ounces of water (which is now on you), you swept out a total of $(400\ \text{ft}^3)\left(0.02\ \frac{\text{oz}}{\text{ft}^3}\right) = 8$ ounces of water. If the water were truly suspended and not falling, would it matter how fast you ran?

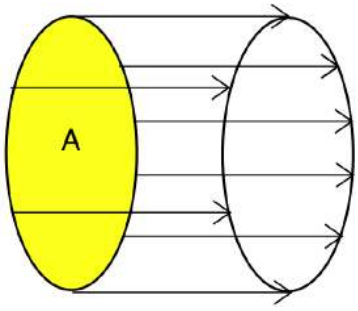
If A is a rectangle, then the "right solid" formed by moving A along a line:



is a 3-dimensional solid box B. The volume of B is:

$$(\text{area of } A)\,(\text{distance along the line}) = (\text{base})\,(\text{height})\,(\text{width})$$

If A is a circle with radius r meters:



then the "right solid" formed by moving A along a line a distance of h meters is a right circular cylinder with volume equal to:

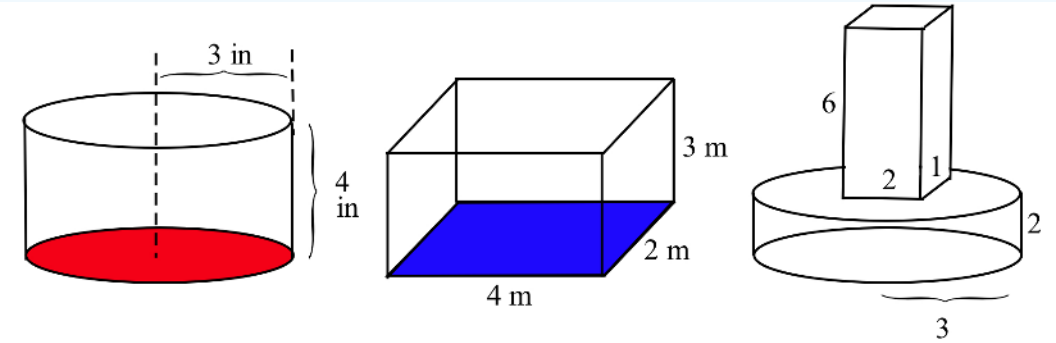
$$(\text{area of } A)\,(\text{distance along the line}) = \left[\pi\,(r\ \text{ft})^2\right] \cdot [h\ \text{ft}] = \pi r^2 h\ \text{ft}^3$$

If we cut a right solid perpendicular to its axis (like slicing a block of cheese), then each face (cross-section) has the same two-dimensional shape and area. In general, if a 3-dimensional right solid B is formed by moving a 2-dimensional shape A along a line perpendicular to A, then the **volume** of B is defined to be:

$$(\text{area of } A) \cdot (\text{distance moved along the line perpendicular to } A)$$

Example 5.1.2

Calculate the volumes of the right solids shown below:



Solution

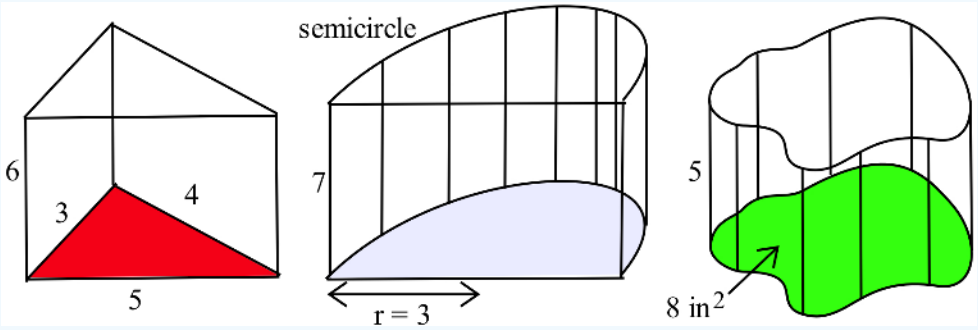
The cylinder is formed by moving the circular base with cross-sectional area $\pi r^2 = 9\pi \text{ in}^2$ a distance of 4 inches along a line perpendicular to the base, so the volume is $(9\pi \text{ in}^2) \cdot (4 \text{ in}) = 36\pi \text{ in}^3$.

The volume of the box is $(\text{base area}) \cdot (\text{distance base is moved}) = (8 \text{ m}^2) \cdot (3 \text{ m}) = 24 \text{ m}^3$. We can also simply multiply "length times width times height" to get the same answer.

The last shape consists of two "easy" right solids with volumes $V_1 = (\pi \cdot 3^2) \cdot (2) = 18\pi \text{ cm}^3$ and $V_2 = (6)(1)(2) = 12 \text{ cm}^3$, so the total volume is $(18\pi + 12) \text{ cm}^3 \approx 68.5 \text{ cm}^3$.

Practice 5.1.1

Calculate the volumes of the right solids shown below:



Answer

triangular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2} \cdot 3 \cdot 4\right)(6) = 36$

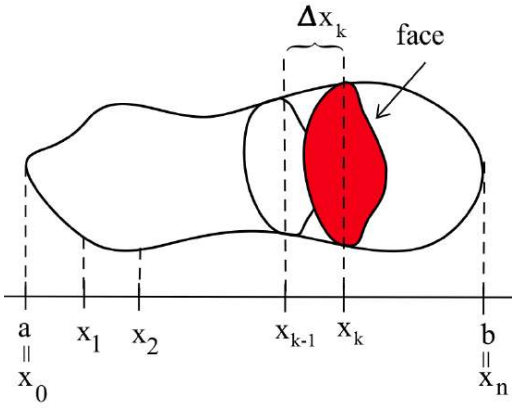
\mbox{semicircular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2}\pi \cdot 3^2\right)(7) \approx 98.96$}

"blob"-shaped base: $V = (\text{base area}) \cdot (\text{height}) = (8)(5) = 40 \text{ in}^3$

Volumes of General Solids

We can cut a general solid into "slices," each of which is "almost" a right solid if the cuts are close together. The volume of each slice will then be approximately equal to the volume of a right solid, so we can approximate the total volume of the entire solid by adding up the approximate volumes of the right-solid "slices."

First we position an x-axis below the solid shape:



and let $A(t)$ be the area of the face formed when we cut the solid perpendicular to the x-axis where $x = t$. If $\mathcal{P} = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ is a partition of $[a, b]$ and we cut the solid at each x_k, then each slice of the solid is "almost" a right solid and the volume of each slice is approximately

$$(\text{area of a face of the slice})\,(\text{thickness of the slice}) \approx A(x_k) \cdot \Delta x_k$$

The total volume V of the solid is approximately equal to the sum of the volumes of the slices:

$$V = \sum (\text{volume of each slice}) \approx \sum A(x_k) \cdot \Delta x_k$$

which is a Riemann sum.

The limit, as the mesh of the partitions approaches 0 (taking thinner and thinner slices), of the Riemann sum is the definite integral of $A(x)$:

$$V \approx \sum A(x_k) \cdot \Delta x_k \longrightarrow \int_a^b A(x)\,dx \tag{5.1.1}$$

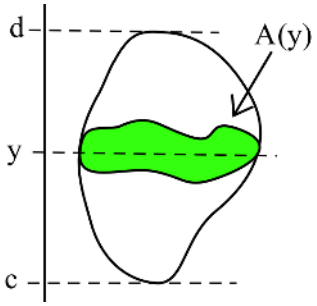
Volume by Slices Formula

If: S is a solid and $A(x)$ is the area of the face formed by a cut at x made perpendicular to the x-axis

then: the volume V of the part of S sitting above $[a, b]$ is:

$$V = \int_a^b A(x)\,dx$$

If S is a solid:



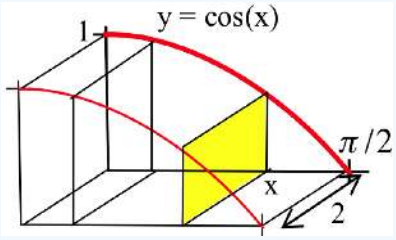
and $A(y)$ is the area of a face formed by a cut at y perpendicular to the y-axis, then the volume of a slice with thickness Δy_k is approximately $A(y_k) \cdot \Delta y_k$. The volume of the part of S between cuts at $y = c$ and $y = d$ on the y-axis is therefore:

$$V = \int_c^d A(y)\,dy$$

Whether you slice a region with cuts perpendicular to the x-axis or cuts perpendicular to the y-axis depends on which slicing method results in slices with cross-sectional areas that are easiest to compute. Furthermore, slicing one way may result in a definite integral that is difficult to compute, while slicing the other way may result in a much easier definite integral (although you often can't tell which method will result in an easier integration process until you actually set up the integrals).

Example 5.1.3

For the solid shown below, the cross-section formed by a cut at x is a rectangle with a base of 2 inches:



a. Find a formula for the approximate volume of the slice between x_{k-1} and x_k.
b. Compute the volume of the solid for x between 0 and $\frac{\pi}{2}$.

Solution

a. The volume of a "slice" is approximately:

$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= (2 \text{ in}) \left(\cos(x_k) \text{ in}\right) \cdot (\Delta x_k \text{ in}) \\ &= 2\cos(x_k)\Delta x_k \text{ in}^3\end{aligned}$$

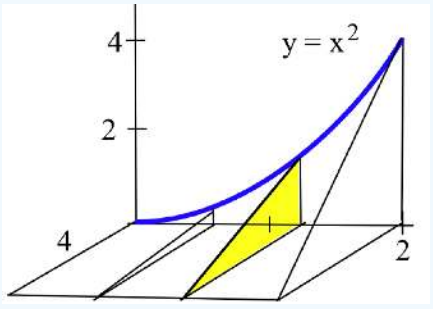
b. If we cut the solid into n slices of equal thickness Δx and add up the approximate volumes of the slices, we get a Riemann sum

$$\sum_{k=1}^{n} 2\cos(x_k)\Delta x \longrightarrow \int_0^{\frac{\pi}{2}} 2\cos(x)\,dx = 2\sin(x)\Big|_0^{\frac{\pi}{2}} = 2$$

so the volume of the solid is 2 in^3.

Practice 5.1.2

For the solid shown below, the face formed by a cut at x is a triangle with a base of 4 inches:



a. Find a formula for the approximate volume of the slice between x_{k-1} and x_k.
b. Use a definite integral to compute the volume of the solid for x between 1 and 2.

Answer

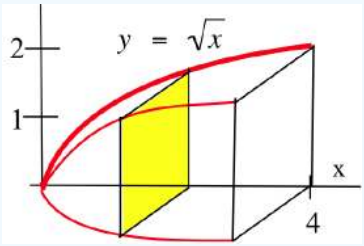
a. The base of each triangular slice is 4 and the height is approximately $x_k{}^2$ so $A(x_k) \approx \frac{1}{2}(4)\left(x_k{}^2\right) = 2x_k{}^2$ and the volume of the k-th slice is this approximately $2x_k{}^2 \cdot \Delta x_k$.

b. Adding up the approximate volumes of all n slices yields $\sum_{n=1}^{\infty} 2x_k{}^2 \cdot \Delta x_k$, which is a Riemann sum with limit:

$$\int_0^2 2x^2\,dx = \frac{2}{3}x^3\Big|_1^2 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

Example 5.1.4

For the solid shown below, each face formed by a cut at x is a square:



Compute the volume of the solid.

Solution

The volume of a "slice" is approximately:

$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base})^2 \cdot (\text{thickness}) \\ &= (\sqrt{x_k})^2 \cdot \Delta x_k = x_k \cdot \Delta x_k\end{aligned}$$

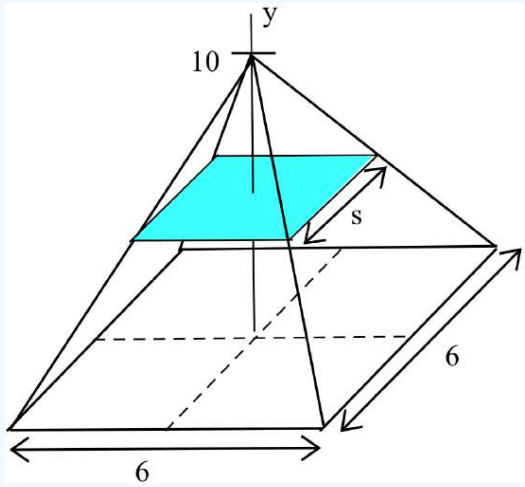
Adding up the approximate volumes of n slices, we get a Riemann sum that approximates the volume of the entire solid:

$$\sum_{k=1}^{n} x_k \cdot \Delta x_k \longrightarrow \int_0^4 x\,dx = \frac{1}{2}x^2\Big|_0^4 = 8$$

so the volume of the solid is 8. You can check that this answer is reasonable by noticing that the solid is contained in a rectangular box with dimensions 2 by 2 by 4, which has a volume of $(2)(2)(4) = 16$.

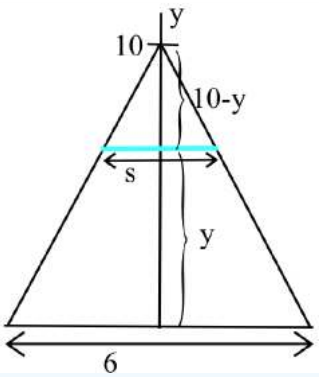
Example 5.1.5

Find the volume of the square-based pyramid shown below:



Solution

Each cut perpendicular to the y-axis yields a square face, but in order to find the area of each square we need a formula for the length of one side s of the square as a function of y, the location of the cut. Using similar triangles:



we know that:

$$\frac{s}{10-y}=\frac{6}{10} \quad \Rightarrow \quad s=\frac{6}{10}(10-y)=6-\frac{3}{5}y$$

The rest of the solution is straightforward:

$$A(y)=(\text{side})^2=\left[\frac{3}{5}(10-y)\right]^2=\frac{9}{25}\left(100-20y+y^2\right)$$

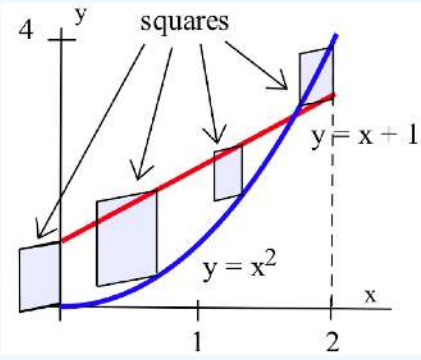
so the volume of the solid is:

$$\begin{aligned} V &= \int_0^{10} A(y)\,dy=\int_0^{10}\frac{9}{25}\left(100-20y+y^2\right)\,dy \\ &= \frac{9}{25}\left[100y-10y^2+\frac{1}{3}y^3\right]_0^{10} \\ &= \frac{9}{25}\left[\left(1000-1000+\frac{1000}{3}\right)-(0-0+0)\right]=120 \end{aligned}$$

You may recall from geometry that the formula for the volume of a pyramid is $\frac{1}{3}Bh$ where B is the area of the base, which yields the same result as the definite integral: $\frac{1}{3}\left(6^2\right)(10)=120$.

Example 5.1.6

Form a solid with a base that is the region between the graphs of $f(x)=x+1$ and $g(x)=x^2$ for $0\le x\le 2$ by building squares with heights (sides) equal to the vertical distance between the graphs of f and g:



Find the volume of this solid.

Solution

The area of a square face is $A(x)=(\text{side})^2$ and the length of a side is either $f(x)-g(x)$ or $g(x)-f(x)$, depending on whether $f(x)\ge g(x)$ or $g(x)\ge f(x)$. We can express this side length as $|f(x)-g(x)|$ but the side length is squared in the area formula, so $A(x)=|f(x)-g(x)|^2=(f(x)-g(x))^2$. Then:

$$V = \int_a^b A(x)\,dx = \int_0^2 (f(x)-g(x))^2\,dx = \int_0^2 \left[(x+1)-x^2\right]^2\,dx$$
$$= \int_0^2 \left[1 + 2x - x^2 - 2x^3 + x^4\right]\,dx$$
$$= \left[x+x^2-\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_0^2$$

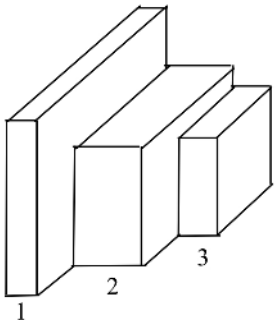
which results in a volume of $\frac{26}{15}$.

Wrap-Up

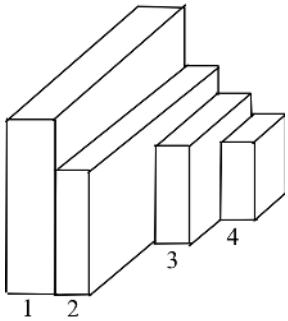
At first, all of these volumes may seem overwhelming—there are so many possible solids and formulas and different cases. If you concentrate on the differences, things can indeed seem very complicated. Instead, focus on the pattern of cutting, finding areas of faces, volumes of slices, and adding those volumes. Then reason your way to a definite integral. Try to make cuts so the resulting faces have regular shapes (rectangles, triangles, circles) whose areas you can calculate easily. Try not to let the complexity of the whole solid confuse you. Sketch the shape of one face and label its dimensions. If you can find the area of one face in the middle of the solid, you can usually find the pattern for all of the faces—and then you can easily set up the integral.

Problems

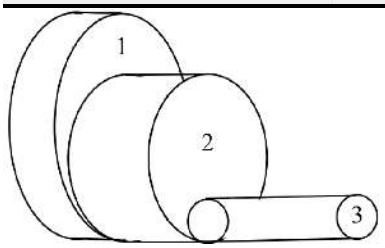
In Problems 1–5, compute the volume of the solid using the values provided in the table.

1. 

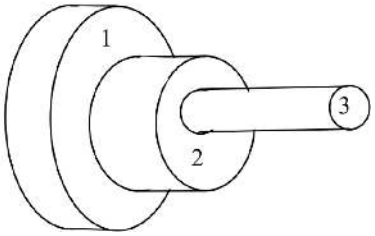
box	base	height	width
1	8	6	1
2	6	4	2
3	3	3	1

2. 

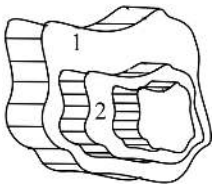
box	base	height	width
1	8	6	1
2	8	4	2
3	4	3	2
4	2	2	1

3. 

disk	radius	width
1	4	0.5
2	3	1.0
3	1	2.0

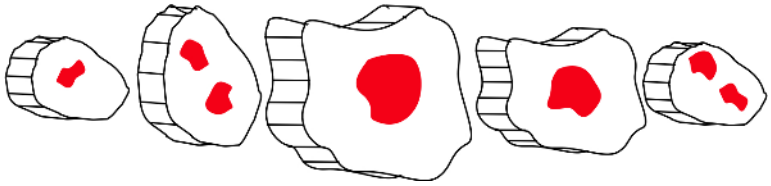
4. 

disk	diameter	width
1	8	0.5
2	6	1.0
3	2	2.0

5. 

slice	face area	width
1	9	0.2
2	6	0.2
3	2	0.2

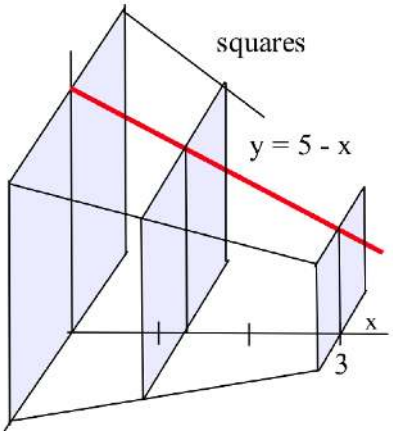
6. Five rock slices are embedded with mineral deposits. Use the information in the table to estimate the total rock volume.



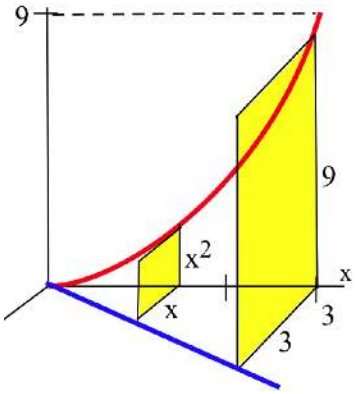
slice	face area	min. area	width
1	4	1	0.6
2	12	2	0.6
3	20	4	0.6
4	10	3	0.6
5	8	2	0.6

In Problems 7–12, represent the volume of each solid as a definite integral, then evaluate the integral.

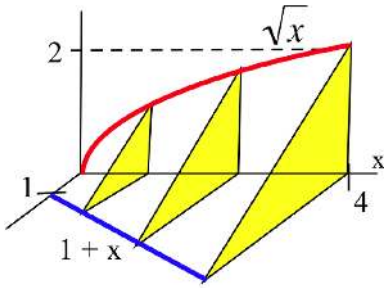
7. For $0 \le x \le 3$, each face is a square with height $5 - x$ inches.



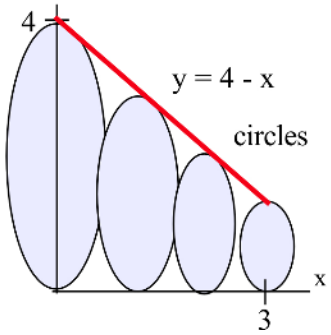
8. For $0 \le x \le 3$, each face is a rectangle with base x inches and height x^2 inches.



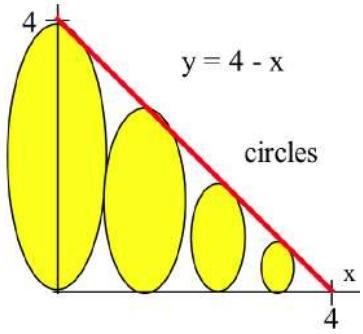
9. For $0 \le x \le 4$, each face is a triangle with base $x+1$ m and height $\sqrt{x}$ m.



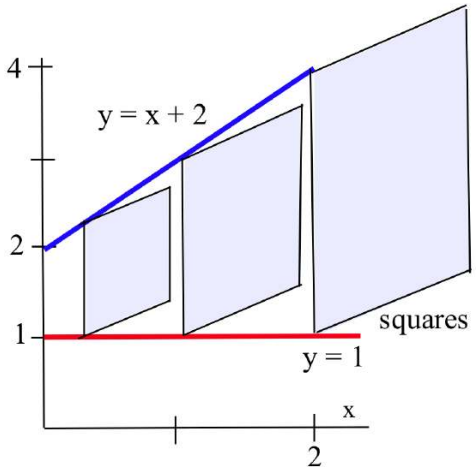
10. For $0 \le x \le 3$, each face is a circle with height (diameter) $4-x$ m.



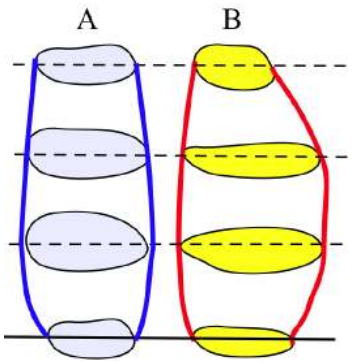
11. For $0 \le x \le 4$, each face is a circle with height (diameter) $4-x$ m.



12. For $0 \le x \le 2$, each face is a square with a side extending from $y=1$ to $y=x+2$.

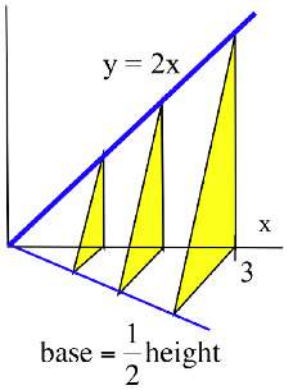


13. Suppose A and B are solids (see below) so that every horizontal cut produces faces of A and B that have equal areas. What can we conclude about the volumes of A and B? Justify your answer.

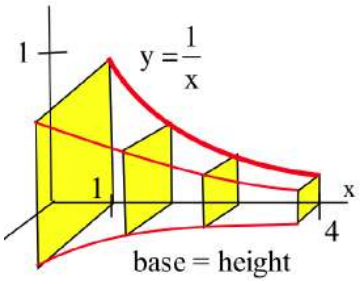


In Problems 14–18, represent the volume of each solid as a definite integral, then evaluate the integral.

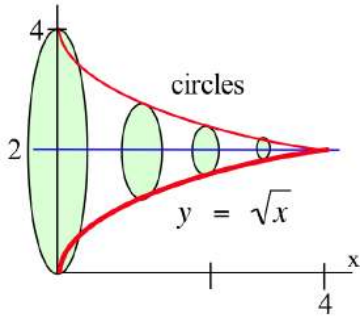
14.



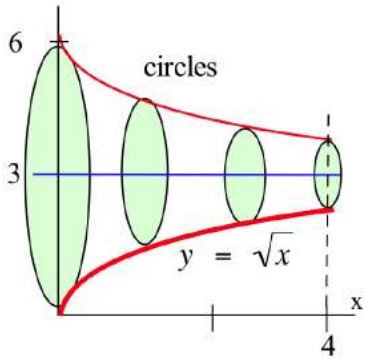
15.



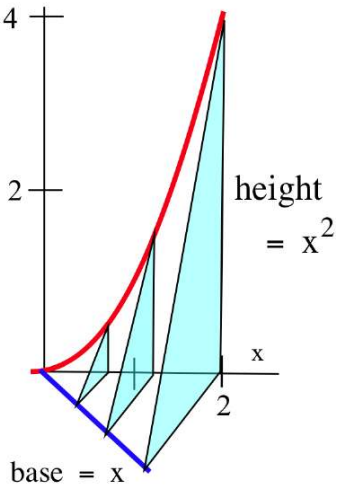
16.



17.



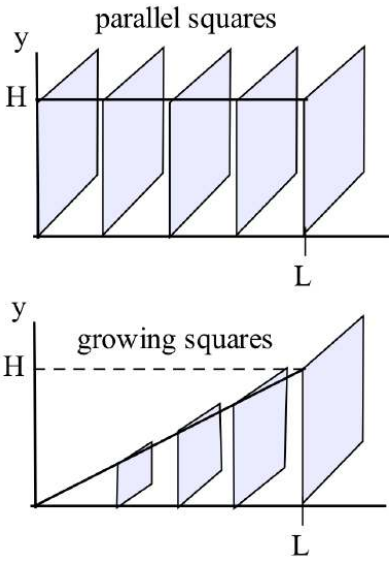
18.



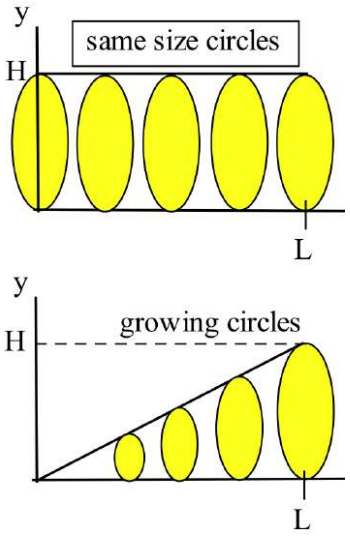
In Problems 19–28, represent the volume of each solid as a definite integral, then evaluate the integral.

19. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.
20. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.
21. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are semicircles.
22. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and slices perpendicular to the base (and to the x-axis) are equilateral triangles.
23. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 3 + x - x^2$, and slices perpendicular to the base (and to the x-axis) are:
    a. squares.
    b. semicircles.
    c. rectangles twice as tall as they are wide.
    d. isosceles right triangles with a hypotenuse in the base of the solid.
24. The base of a solid is the first-quadrant region bounded by the y-axis, the curve $y = \sin(x)$ and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are:
    a. squares.
    b. semicircles.
    c. rectangles twice as tall as they are wide.
    d. isosceles right triangles with a hypotenuse in the base of the solid.
25. The base of a solid is the region bounded by the x-axis, the y-axis and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.

26. The base of a solid is the region bounded by the x-axis, the line $y = 3$ and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.
27. The base of a solid is the region bounded below by the line $y = 1$, on the left by the line $x = 2$ and above by the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are semicircles.
28. The base of a solid is the region bounded below by the line $y = 1$, on the left by the line $x = 2$ and above by the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the x-axis) are semicircles.
29. Calculate:
    a. the volume of the right solid in the figure:

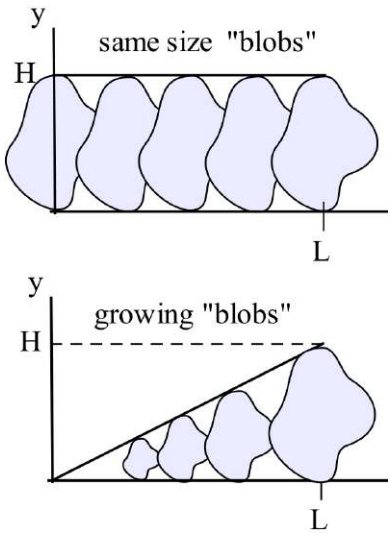


    b. the volume of the "right cone" in the bottom figure and
    c. the ratio of the "right cone" volume to the right solid volume.
30. Calculate:
    a. the volume of the right solid in the top figure below:



    b. the volume of the "right cone" in the bottom figure above.
    c. the ratio of the "right cone" volume to the right solid volume.
31. Calculate

a. the volume of the right solid in the top figure below:



if each “blob” has area B

b. the volume of the “right cone” in the bottom figure above, using “similar blobs” to conclude that the cross-section x units from the y-axis has area $A(x) = \frac{B}{L^2}x^2$

c. the ratio of the “right cone” volume to the right solid volume.

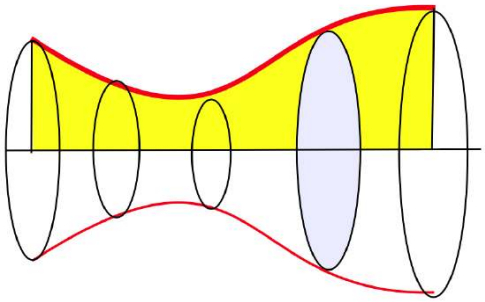
32. “Personal calculus”: Describe a practical way to determine the volume of your hand and arm up to the elbow.

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5.2: Disks and Washers

In the previous section, we computed volumes of solids for which we could determine the area of a cross-section or "slice." In this section, we restrict our attention to a special case in which the solid is generated by *rotating* a region in the xy-plane about a horizontal or vertical line. We call a solid formed in this way a **solid of revolution** and we call the line an **axis of rotation**.

If the axis of rotation coincides with a boundary of the region:



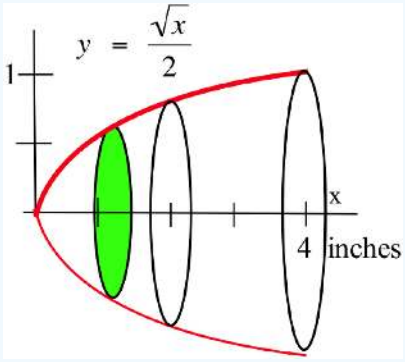
then the cross-sections of the region perpendicular to the axis of rotation will be disks, making it relatively easy to find a formula for the area of a cross-section:

$$A(x) = \text{area of a disk} = \pi(\text{radius})^2$$

The radius is often a function of x, the location of the cross-section.

Example 5.2.1

Find the volume of the solid formed by rotating the region in the first quadrant bounded by the curve $y = \frac{\sqrt{x}}{2}$ and the line $x = 4$ about the x-axis:



Solution

Any slice perpendicular to the x-axis (and to the xy-plane) will yield a circular cross-section with radius equal to the distance between the curve $y = \frac{\sqrt{x}}{2}$ and the x-axis, so the volume of the region is given by:

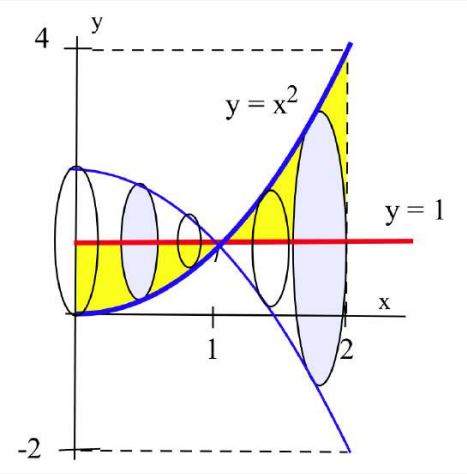
$$V = \int_0^4 \pi \left[\frac{\sqrt{x}}{2}\right]^2 dx = \int_0^4 \pi \cdot \frac{x}{4}\, dx = \frac{\pi}{8}x^2\Big|_0^4 = 2\pi$$

or about 6.28 cubic inches.

Sometimes the boundary curve intersects the axis of rotation.

Example 5.2.2

The region between the graph of $f(x) = x^2$ and the horizontal line $y = 1$ for $0 \le x \le 2$ is revolved about the horizontal line $y = 1$ to form a solid:



Compute the volume of the solid.

Solution

The margin figure shows cross-sections for several values of x, all of them disks. If $0 \le x \le 1$, the radius of the disk is $r(x) = 1 - x^2$; if $1 \le x \le 2$, then $r(x) = x^2 - 1$. We could split up the volume computation into two separate integrals, using $A(x) = \pi [r(x)]^2 = \pi [1-x^2]^2$ for $0 \le x \le 1$ and $A(x) = \pi [r(x)]^2 = \pi [x^2-1]^2$ for $1 \le x \le 2$, but:

$$\pi [x^2-1]^2 = \pi [-(1-x^2)]^2 = \pi [1-x^2]^2$$

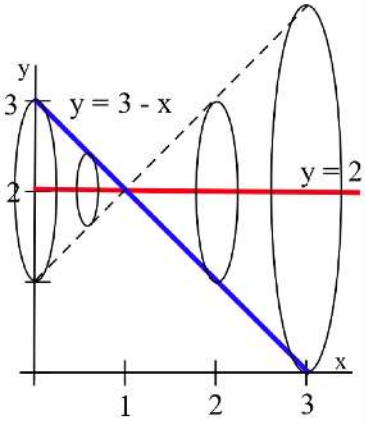
for all x so we can instead compute the volume with a single integral:

$$V = \int_0^2 \pi [x^2-1]^2 \, dx = \pi \int_0^2 [x^4 - 2x^2 + 1] \, dx$$
$$= \pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_0^2 = \pi \left[\frac{32}{5} - \frac{16}{3} + 2\right] = \frac{46\pi}{15}$$

or about 9.63.

Practice 5.2.1

Find the volume of the solid formed by revolving the region between $f(x) = 3 - x$ and the horizontal line $y = 2$ about the line $y = 2$ for $0 \le x \le 3$:



Answer

$$\int_0^3 \pi\Big[\,|(3-x)-2)|\,\Big]^2\,dx = \pi\int_0^3 (1-x)^2\,dx = \pi\int_0^3 \left[1-2x+x^2\right]\,dx = \pi\left[x-x^2+\frac{1}{3}x^3\right]_0^3 = 3\pi$$

Volumes of Revolved Regions ("Disk Method")

If: the region constrained by the graph of $y = f(x)$, the horizontal line $y = L$ and the interval $[a, b]$ is revolved about the horizontal line $y = L$

then: then the volume of the resulting solid is:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\,(\text{radius})^2\,dx = \int_a^b \pi\,[f(x)-L]^2\,dx$$

We often refer to this technique as the "disk" method because revolving a thin rectangular slice of the region (that we might use in a Riemann sum to approximate the area of the region) results in a disk. If the region between the graph of f and the x-axis ($L = 0$) is revolved about the x-axis, then the previous formula reduces to:

$$V = \int_a^b \pi\,[f(x)]^2\,dx$$

Example 5.2.3

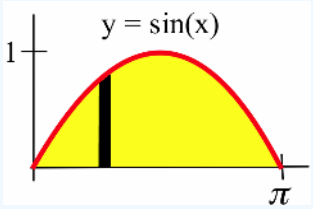
Find the volume generated when the region between one arch of the sine curve (for $0 \le x \le \pi$) and

a. the x-axis is revolved about the x-axis and

b. the line $y = \frac{1}{2}$ is revolved about the line $y = \frac{1}{2}$.

Solution

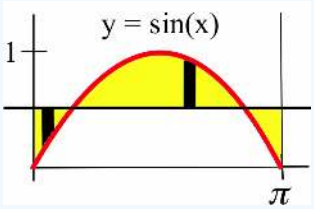
a. The radius of each circular slice:



is just the height of the function $y = \sin(x)$:

$$V = \int_0^\pi \pi\,[\sin(x)]^2\,dx = \pi\int_0^\pi \sin^2(x)\,dx = \pi\int_0^\pi \left[\frac{1}{2}-\frac{1}{2}\cos(2x)\right]\,dx$$
$$= \pi\left[\frac{1}{2}x-\frac{1}{4}\sin(2x)\right]_0^\pi = \pi\left[\frac{\pi}{2}-0\right]-\pi\,[0-0] = \frac{\pi^2}{2} \approx 4.93$$

b. Here we use the more general disk formula:



with $L = \frac{1}{2}$:

$$V = \int_0^{\pi} \pi \left[\sin(x) - \frac{1}{2}\right]^2 dx = \pi \int_0^{\pi} \left[\sin^2(x) - \sin(x) + \frac{1}{4}\right] dx$$
$$= \pi \int_0^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos(2x) - \sin(x) + \frac{1}{4}\right] dx$$
$$= \pi \left[\frac{3}{4}x - \frac{1}{4}\sin(2x) + \cos(x)\right]_0^{\pi}$$
$$= \pi \left[\frac{3\pi}{4} - 0 - 1\right] - \pi[0 - 0 + 1] = \frac{3\pi^2}{4} - 2\pi$$

or approximately 1.12.

Practice 5.2.2

Find the volume generated when:

a. the region between the parabola $y = x^2$ (for $0 \le x \le 2$) and the x-axis is revolved about the x-axis and
b. the region between the parabola $y = x^2$ (for $0 \le x \le 2$) and the line $y = 2$ is revolved about the line $y = 2$.

Answer

a. Slicing the region vertically and rotating the slice about the x-axis results in disks, so the volume of the solid is:

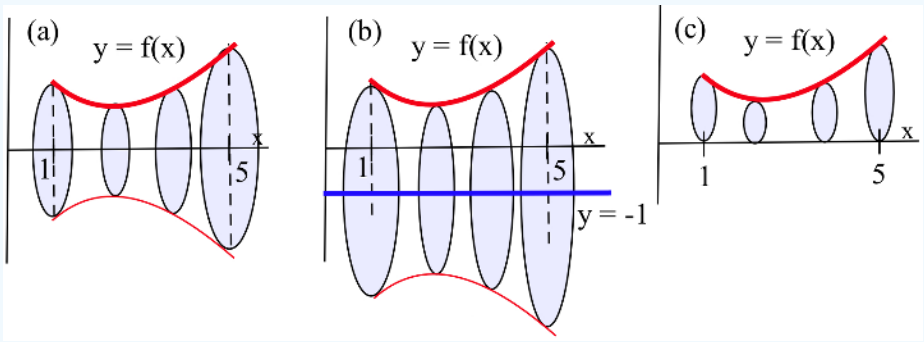
$$\int_0^2 \pi \left[x^2\right]^2 dx = \pi \int_0^2 x^4 \, dx = \pi \left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}$$

b. Here the slices extend from $y = x^2$ to $y = 2$ so the radius of each disk is $2 - x^2$ and the volume is:

$$\int_0^2 \pi \left[2 - x^2\right]^2 dx = \pi \int_0^2 \left[4 - 4x^2 + x^4\right] dx = \pi \left[4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{56\pi}{15}$$

Example 5.2.4

Given that $\int_1^5 f(x)\,dx = 4$ and $\int_1^5 [f(x)]^2\,dx = 7$, represent the volume of each solid shown below:



as a definite integral, and evaluate those integrals.

Solution

a. Here the axis of rotation is $y = 0$ so:

$$V = \int_1^5 \pi\,(\text{radius})^2\,dx = \int_1^5 \pi\,[f(x)]^2\,dx = \pi\int_1^5 [f(x)]^2\,dx = 7\pi$$

b. Here the axis of rotation is $y = -1$ so:

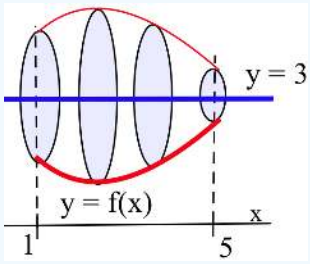
$$\begin{aligned} V &= \int_1^5 \pi\,(\text{radius})^2\,dx = \int_1^5 \pi\,[f(x)-(-1)]^2\,dx \\ &= \pi\int_1^5 [f(x)+1]^2\,dx = \pi\int_1^5 \left[(f(x))^2 + 2f(x) + 1\right]\,dx \\ &= \pi\left[\int_1^5 (f(x))^2\,dx + 2\int_1^5 f(x)\,dx + \int_1^5 1\,dx\right] \\ &= \pi\,[7 + 2\cdot 4 + (5-1)] = 19\pi \end{aligned}$$

c. This is not a solid of revolution, even though the cross-sections are disks. Each disk has diameter equal to the function height, so the radius of each disk is half that height, and the volume is:

$$V = \int_1^5 \pi\left[\frac{f(x)}{2}\right]^2\,dx = \frac{\pi}{4}\int_1^5 [f(x)]^2\,dx = \frac{\pi}{4}\cdot 7 = \frac{7\pi}{4}$$

? Practice 5.2.3

Set up and evaluate an integral to compute the volume of the solid shown below:



Answer

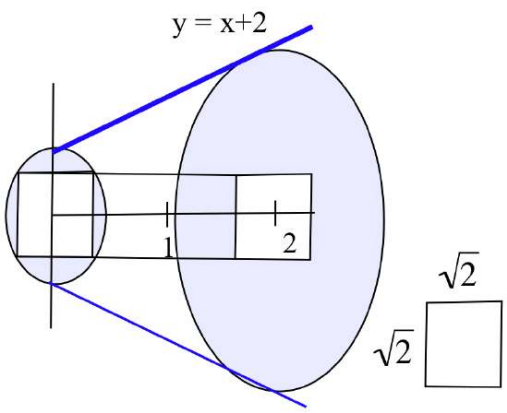
$$\int_1^5 \pi\,[3-f(x)]^2\,dx = \int_1^5 \pi\left[9 - 6f(x) + (f(x))^2\right] = \pi\left[\int_1^5 9\,dx - 6\int_1^5 f(x)\,dx + \int_1^5 [f(x)]^2\,dx\right] = \pi\,[36 - 6\cdot 4 + 7] = 19\pi$$

Solids with Holes

Some solids have “holes”: for example, we might drill a cylindrical hole through a spherical solid (such as a ball bearing) to create a part for an engine. One approach involves using an integral (or using geometry) to compute the volume of the “outer” solid, then use another integral (or geometry) to compute the volume of the “hole” cut out of the original solid, and finally subtracting the second result from the first. You should be able to use this approach in the next problem.

? Practice 5.2.4

Compute the volume of the solid shown below:



Answer

The volume we want can be obtained by subtracting the volume of the "box" from the volume of the truncated cone generated by the rotated line segment. The volume of the truncated cone is:

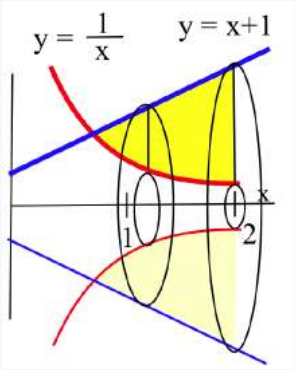
$$\int_0^2 \pi [x+2]^2 \, dx = \pi \int_0^2 \left[x^2+4x+4\right] \, dx = \pi \left[\frac{1}{3}x^3+2x^2+4x\right]_0^2 = \frac{56\pi}{3}$$

while the volume of the box is $\left[\sqrt{2}\right]^2 (2) = 4$ so the volume of the solid shown in the graph is $\frac{56\pi}{3} - 4 \approx 54.64$.

A special case of a solid with a hole results from rotating a region bounded by two curves around an axis that does not intersect the region.

Example 5.2.5

Compute the volume of the solid shown below:



Solution

The face for a slice made at x has area:

$$\begin{aligned} A(x) &= [\text{area of BIG circle}] - [\text{area of small circle}] \\ &= \pi [\text{BIG radius}]^2 - \pi [\text{small radius}]^2 \end{aligned}$$

Here the BIG radius is the distance from the line $y = x + 1$ to the x-axis, or $R(x) = (x+1) - 0 = x + 1$; similarly, the small radius is the distance from the curve $y = \frac{1}{x}$ to the x-axis, or $r(x) = \frac{1}{x} - 0 = \frac{1}{x}$, hence the cross-sectional area is:

$$A(x) = \pi [x+1]^2 - \pi \left[\frac{1}{x}\right]^2 = \pi \left[x^2+2x+1-\frac{1}{x^2}\right]$$

The curves intersect where:

$$x+1=\frac{1}{x} \Rightarrow x^2+x=1 \Rightarrow x^2+x-1=0$$
$$\Rightarrow x=\frac{-1\pm\sqrt{1-4(-1)}}{2}=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}$$

Clearly we need $x>0$ for this region, so the left endpoint of integration must be $x=\frac{-1+\sqrt{5}}{2}$ while the right endpoint is $x=2$, so the volume of the solid is:

$$V=\int_{\frac{-1+\sqrt{5}}{2}}^{2}\pi\left[x^2+2x+1-\frac{1}{x^2}\right]dx=\pi\left[\frac{1}{3}x^3+x^2+x+\frac{1}{x}\right]_{\frac{-1+\sqrt{5}}{2}}^{2}$$
$$=\pi\left[\frac{2^3}{3}+2^2+2+\frac{1}{2}\right]-\pi\left[\frac{1}{3}\left(\frac{-1+\sqrt{5}}{2}\right)^3+\left(\frac{-1+\sqrt{5}}{2}\right)^2+\frac{-1+\sqrt{5}}{2}+\frac{2}{-1+\sqrt{5}}\right]$$

which simplifies to $\frac{\pi}{6}\left[50-5\sqrt{5}\right]\approx 20.33$.

The previous Example extends the "disk" method to a more general technique often called the "washer" method because a big disk with a smaller disk cut out of the middle resembles a washer (a small flat ring used with nuts and bolts).

Volumes of Revolved Regions ("Washer Method")

If: the region constrained by the graphs of $y=f(x)$ and $y=g(x)$ and the interval $[a,b]$ is revolved about a horizontal line

then: the volume of the resulting solid is:

$$V=\int_a^b\left[\pi\left(R(x)\right)^2-\pi\left(r(x)\right)^2\right]dx$$

where $R(x)$ represents the distance from the axis of rotation to the farthest curve from that axis, and $r(x)$ represents the distance from the axis to the closest curve.

If $r(x)=0$, the "washer" method becomes the "disk" method.

When applying the washer method, you should:

- graph the region
- draw a representative rectangular "slice" of that region
- check that revolving the slice about the axis of rotation results in a "washer"
- locate the limits of integration
- set up an integral
- evaluate the integral

If you are unable to find an antiderivative for the integrand of your integral, you can consult an integral table or use numerical methods to approximate the volume of the solid. You might also need to use numerical methods to locate where the boundary curves of the region intersect.

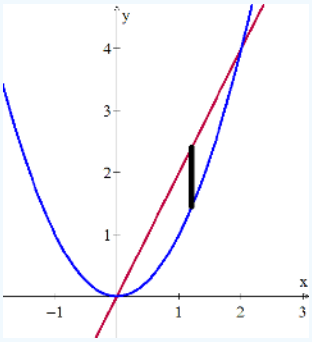
Example 5.2.6

Find the volume of the solid generated by rotating the region between the curves $y=2x$ and $y=x^2$ about the

a. x-axis
b. y-axis
c. the line $x=-1$
d. the line $y=5$.

Solution

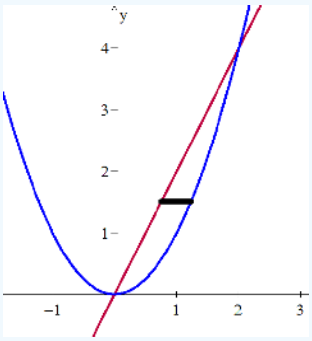
a. The curves intersect where $x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$, so the limits of integration should involve $x = 0$ and $x = 2$. Revolving a vertical slice of the region with width Δx about the x-axis:



yields a "washer" with big radius $R(x) = 2x - 0 = 2x$ (the line $y = 2x$ is farthest from the x-axis) and small radius $r(x) = x^2 - 0 = x^2$ (the parabola is closest to the x-axis when $0 \le x \le 2$). So the volume of the solid is:

$$V = \int_0^2 \left[\pi(2x)^2 - \pi(x^2)^2\right]\,dx = \pi\int_0^2 \left[4x^2 - x^4\right]\,dx$$
$$= \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \pi\left[\frac{32}{3} - \frac{32}{5}\right] - \pi[0-0] = \frac{64\pi}{15}$$

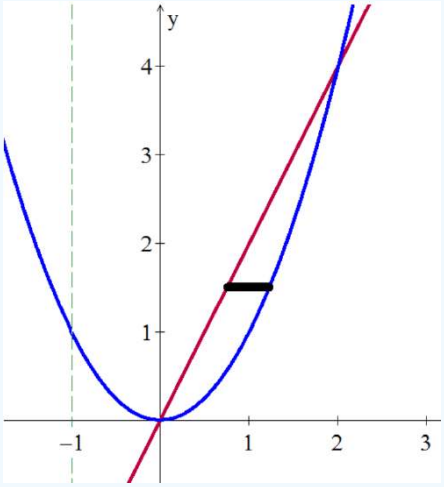
b. A vertical slice revolved around the y-axis does **not** result in a "washer" so instead we try slicing horizontally. A horizontal slice of thickness Δy revolved around the y-axis does result in a washer:



The big radius is the x-distance from the parabola (where $x = \sqrt{y}$) to the y-axis (where $x = 0$) so $R(y) = \sqrt{y}$. Similarly, the small radius is the distance from the line (where $x = \frac{y}{2}$) to the y-axis (where $x = 0$), so $r(y) = \frac{y}{2}$. Because the variable of integration is now y, we need y-values for the limits of integration. At the lower intersection point of the two curves, $x = 0 \Rightarrow y = 0$; at the upper intersection point, $x = 2 \Rightarrow y = x^2 = 2^2 = 4$. So the volume of the solid is:

$$V = \int_{y=0}^{y=4} \left[\pi\left(\sqrt{y}\right)^2 - \pi\left(\frac{y}{2}\right)^2\right]\,dy = \pi\int_0^4 \left[y - \frac{1}{4}y^2\right]\,dy$$
$$= \pi\left[\frac{1}{2}y^2 - \frac{1}{12}y^3\right]_0^4 = \pi\left[8 - \frac{16}{3}\right] = \frac{8\pi}{3}$$

c. This solid resembles the one from part (b) except now the radii are both bigger:



because the region (and the curves that form the boundary of the region) are farther away from the axis of rotation: $R(x) = \sqrt{y} - (-1) = \sqrt{y} + 1$ and $r(x) = \frac{y}{2} - (-1) = \frac{y}{2} + 1$:

$$\begin{aligned} V &= \int_{y=0}^{y=4} \left[\pi\left(\sqrt{y}+1\right)^2 - \pi\left(\frac{y}{2}+1\right)^2\right] dy \\ &= \pi \int_0^4 \left[\left(y + 2\sqrt{y} + 1\right) - \left(\frac{1}{4}y^2 + y + 1\right)\right] dy \\ &= \pi \int_0^4 \left[2y^{\frac{1}{2}} - \frac{1}{4}y^2\right] dy = \pi\left[\frac{4}{3}y^{\frac{3}{2}} - \frac{1}{12}y^3\right]_0^4 = \pi\left[\frac{32}{3} - \frac{16}{3}\right] = \frac{16\pi}{3} \end{aligned}$$

d. For this solid, slicing the region vertically as in part (a) results in washers, but here the "near" and "far" roles of the curves are reversed: the parabola is farthest away from $y = 5$ while the line is closest. The radii are $R(x) = 5 - x^2$ and $r(x) = 5 - 2x$:

$$V = \int_{x=0}^{x=2} \pi\left[(5 - x^2)^2 - (5 - 2x)^2\right] dx = \frac{136\pi}{15}$$

The details of evaluating this definite integral are left to you.

Practice 5.2.5

Find the volume of the solid generated by rotating the region between the curves $y = 2x$ and $y = x^2$ about the:

a. the line $x = 5$
b. the line $y = -5$.

Answer

a. Slicing the region vertically and rotating the slice about the line $x = 5$ results in something other than a washer, so we instead slice the region horizontally. The slice extends from $x = \frac{y}{2}$ (farthest from the axis of rotation) to $x = \sqrt{y}$ (closest), so the volume of the solid is:

$$\int_0^4 \left[\pi\left(5 - \frac{y}{2}\right)^2 - \pi\left(5 - \sqrt{y}\right)^2\right] dy = \frac{32\pi}{3}$$

b. Slicing the region vertically and rotating the slice about the line $y = -5$ results in washers, so the volume is:

$$\int_0^2 \left[\pi\left(2x+5\right)^2 - \pi\left(x^2+5\right)^2\right] dx = \frac{88\pi}{5}$$

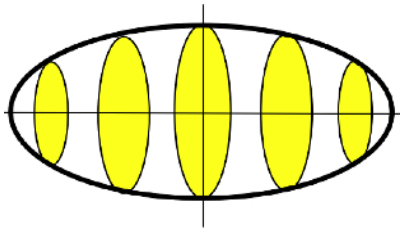
Problems

In Problems 1–12, find the volume of the solid generated when the region in the first quadrant bounded by the given curves is rotated about the x-axis.

1. $y = x$, $x = 5$
2. $y = \sin(x)$, $x = \pi$
3. $y = \cos(x)$, $x = \frac{\pi}{3}$
4. $y = 3 - x$
5. $y = \sqrt{7 - x}$
6. $y = \sqrt[4]{9 - x}$
7. $y = 5 - x^2$
8. $x = 9 - y^2$
9. $x = 121 - y^2$
10. $x^2 + y^2 = 4$
11. $9x^2 + 25y^2 = 225$
12. $3x^2 + 5y^2 = 15$

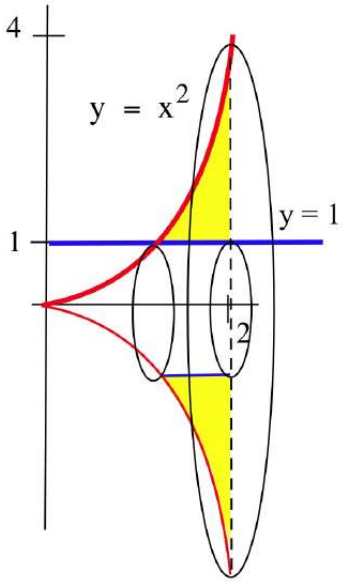
In Problems 13–30, compute the volume of the solid formed when the region between the given curves is rotated about the specified axis.

13. $y = x$, $y = x^4$ about the x-axis
14. $y = x$, $y = x^4$ about the y-axis
15. $y = x^2$, $y = x^4$ about the y-axis
16. $y = x^2$, $y = x^4$ about the x-axis
17. $y = x^2$, $y = x^3$ about the x-axis
18. $y = \sec(x)$, $y = 2\cos(x)$, $x = \frac{\pi}{3}$ about the x-axis
19. $y = \sec(x)$, $y = \cos(x)$, $x = \frac{\pi}{3}$ about the x-axis
20. $y = x$, $y = x^4$ about $y = 3$
21. $y = x$, $y = x^4$ about $y = -4$
22. $y = x$, $y = x^4$ about $x = -4$
23. $y = x$, $y = x^4$ about $x = 3$
24. $y = x$, $y = x^4$ about $x = 1$
25. $y = \sin(x)$, $y = x$, $x = 1$ about $y = 3$
26. $y = \sin(x)$, $y = x$, $x = \frac{\pi}{2}$ about $y = -2$
27. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $x = -2$
28. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $x = 4$
29. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $y = 2$
30. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $y = -\sqrt{3}$
31. Use calculus to compute the volume of a sphere of radius 2. (A sphere is formed when the region bounded by the x-axis and the top half of the circle $x^2 + y^2 = 2^2$ is revolved about the x-axis.)
32. Use calculus to determine the volume of a sphere of radius r. (Revolve the region bounded by the x-axis and the top half of the circle $x^2 + y^2 = r^2$ about the x-axis.)
33. Compute the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ is revolved around the x-axis:

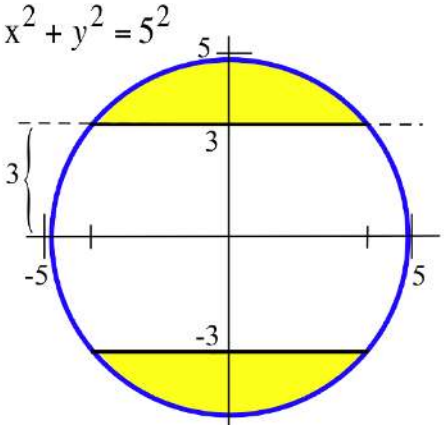


34. Compute the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved around the x-axis.

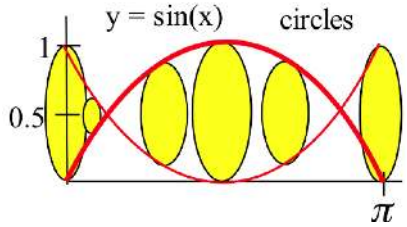
35. Compute the volume of the region shown below:



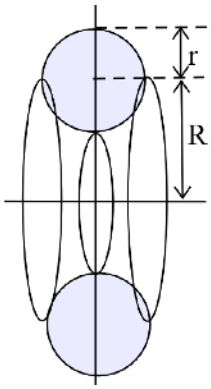
36. Compute the volume of a sphere of radius 5 with a hole of radius 3 drilled through its center:



37. Compute the volume of the region shown below:



38. Determine the volume of the “doughnut” (called a “torus”):



generated by rotating a disk of radius r with center R units away from the x-axis about the x-axis.

39. a. Find the **area** between $f(x) = \frac{1}{x}$ and the x-axis for $1 \leq x \leq 10$, $1 \leq x \leq 100$ and $1 \leq x \leq M$. What is the limit of the area for $1 \leq x \leq M$ when $M \to \infty$?

    b. Find the **volume** swept out when the region in part (a) is revolved about the x-axis for $1 \leq x \leq 10$, $1 \leq x \leq 100$ and $1 \leq x \leq M$. What is the limit of the volume for $1 \leq x \leq M$ when $M \to \infty$?

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5.3: Arclength and Surface Area

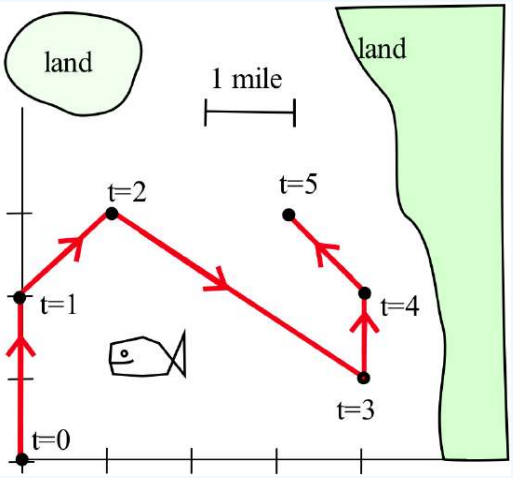
This section introduces two additional geometric applications of integration: finding the length of a curve and finding the area of a surface generated when you revolve a curve about a line. The general strategy remains the same: partition the problem into small pieces, approximate the solution on each small piece, add the small solutions together to form a Riemann sum and, finally, take the limit of the Riemann sum to get a definite integral.

Arclength: How Long Is a Curve?

In order to better understand an animal, biologists need to know how it moves through its environment and how far it travels. We need to know the length of the path it moves along. If we know the object's location a successive times, then we can easily calculate the distances between those locations and add them together to get a total (approximate) distance.

Example 5.3.1

In order to study the movement of whales, marine biologists implant a small transmitter on selected whales and track the location of a whale via satellite. Position data at one-hour time intervals over a five-hour period appears in the figure below:



How far did the whale swim during the first three hours?

Solution

In moving from the point $(0,0)$ to the point $(0,2)$, the whale traveled *at least* 2 miles. Similarly, the whale traveled at least $\sqrt{(1-0)^2+(3-2)^2}=\sqrt{2}\approx 1.4$ miles during the second hour and at least $\sqrt{(4-1)^2+(1-3)^2}=\sqrt{13}\approx 3.6$ miles during the third hour. The scientist concluded that the whale swam at least $2+1.4+3.6=7$ miles during the three-hour period.

Practice 5.3.1

How far did the whale swim during the entire five-hour time period?

Answer

At least $2+\sqrt{2}+\sqrt{13}+1+\sqrt{2}\approx 9.43$ miles.

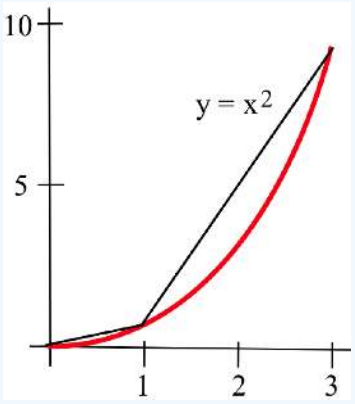
Our strategy for finding the length of a curve will resemble the one the scientist used, and if the locations are given by a formula, then we can calculate the successive locations over very short intervals and get very good approximations of the total path length.

Example 5.3.2

Use the points $(0,0)$, $(1,1)$ and $(3,9)$ to approximate the length of $y = x^2$ for $0 \le x \le 3$.

Solution

The lengths of the two line segments:



are:

$$\sqrt{(1-0)^2+(1-0)^2} = \sqrt{1+1} = \sqrt{2} \approx 1.41$$

and:

$$\sqrt{(3-1)^2+(9-1)^2} = \sqrt{4+64} = \sqrt{68} \approx 8.25$$

so the length of the curve is approximately $1.41 + 8.25 = 9.66$.

Practice 5.3.2

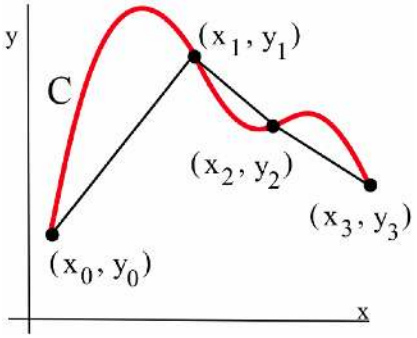
Get a better approximation of the length of $y = x^2$ for $0 \le x \le 3$ by using the points $(0,0)$, $(1,1)$, $(2,4)$ and $(3,9)$. Is your approximation longer or shorter than the actual length?

Answer

$L \approx \sqrt{2} + \sqrt{10} + \sqrt{26} <$ actual length

By using more and more points, we can get better and better approximations of the length of a curve. We can get the *exact* length of the path by evaluating a definite integral.

For a curve $\mathcal{C}$, pick some points (x_k, y_k) along $\mathcal{C}$ and connect those points with line segments:



Then the sum of the lengths of the line segments will approximate the length of $\mathcal{C}$. We can think of this as pinning a string to the curve at the selected points, and then measuring the length of the string as an approximation of the length of the curve. Of course, if

we only pick a few points (as in the image above), then the total length approximation will probably be rather poor, so eventually we want lots of points (x_k, y_k) close together all along $\mathcal{C}$.

Label these points so that (x_0, y_0) is one endpoint of $\mathcal{C}$ and (x_n, y_n) is the other endpoint, and so that the subscripts increase as we move along $\mathcal{C}$. Then the distance between the successive points (x_{k-1}, y_{k-1}) and (x_k, y_k) is:

$$\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

and the total length of these line segments is simply the sum of the successive lengths:

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

This summation does not have the form $\sum g(c_k) \cdot \Delta x_k$ so it is not a Riemann sum. It is, however, algebraically equivalent to an expression very much like a Riemann sum that will lead us to a definite integral representation for the length of $\mathcal{C}$.

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, so that y is a function of x, we can factor $(\Delta x_k)^2$ from inside the radical and simplify:

$$\begin{aligned} \text{length of } C &\approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 \left[1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}\right]} \\ &= \sum_{k=1}^{n} (\Delta x_k) \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k \end{aligned}$$

to get an expression that looks more like a Riemann sum. The $\frac{\Delta y_k}{\Delta x_k}$ inside the radical should remind you of two things: the slope of a line segment (it is, in fact, the slope of the k-th line segment in our approximation of the curve $\mathcal{C}$) and a derivative, $\frac{dy}{dx}$. If $f(x)$ is both continuous and differentiable, then the Mean Value Theorem (review Section 3.2 if you need to refresh your memory about the hypotheses and conclusions of the Mean Value Theorem) guarantees that there is some number c_k between x_{k-1} and x_k so that:

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y_k}{\Delta x_k}$$

in which case we can write:

$$\sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k$$

This last expression *is* a Riemann sum, so it converges to a definite integral:

$$\sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k \longrightarrow \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

This definite integral provides us with a formula for the length of a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$.

(In order to be sure that the sum converges to the integral, we need the resulting integrand to be an integrable function. If we require $f'(x)$ to be continuous on $[a, b]$ then the integrand will be a composition of continuous functions, hence continuous, and we know that a function that is continuous on a closed interval is integrable.)

Arclength Formula: $y = f(x)$ version

If: $\mathcal{C}$ is a curve given by $y = f(x)$ for $a \le x \le b$

and: $f'(x)$ exists and is continuous on $[a, b]$

then: the length L of $\mathcal{C}$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example 5.3.3

Compute the length of $y = x^2$ for $0 \le x \le 3$.

Solution

Here $f(x) = x^2 \Rightarrow f'(x) = 2x$ so the length of this curve is:

$$\int_0^3 \sqrt{1 + [2x]^2}\, dx = \int_0^3 \sqrt{1 + 4x^2}\, dx$$

Unfortunately we do not (yet) have a technique to find an antiderivative of this integrand, but we can use numerical methods (such as Simpson's Rule, or a calculator or computer) to determine that the value of the integral is approximately 9.7471 (compare this with the answers from Example 2 and Practice 2).

(You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.)

Practice 5.3.3

Compute the length of $y = x^2$ between $(1, 1)$ and $(4, 16)$.

Answer

$$\int_1^4 \sqrt{1 + [2x]^2}\, dx = \int_1^4 \sqrt{1 + 4x^2}\, dx \approx 15.34$$

Practice 5.3.4

Represent the length of one period of $y = \sin(x)$ as a definite integral, then find the length of this curve (using technology to approximate the value of the definite integral, if necessary).

Answer

$$\int_0^{2\pi} \sqrt{1 + [\cos(x)]^2}\, dx \approx 7.64$$

More Arclength Formulas

Not all interesting curves are graphs of functions of the form $y = f(x)$. For a curve given by $x = g(y)$ we can mimic the previous argument (or simply swap x and y) to arrive at another arclength formula:

Arclength Formula: $x = g(y)$ version

If: $\mathcal{C}$ is a curve given by $x = g(y)$ for $c \le y \le d$

and: $g'(y)$ exists and is continuous on $[c, d]$

then: the length L of $\mathcal{C}$ is given by:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Practice 5.3.5

Compute the length of $x = \sqrt{y}$ between $(1, 1)$ and $(4, 16)$.

Answer

Here $g(y) = \sqrt{y} = y^{\frac{1}{2}} \Rightarrow g'(y) = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$ so the arclength is:

$$\int_1^{16} \sqrt{1+\left[\frac{1}{2\sqrt{y}}\right]^2}\,dy = \int_1^{16} \sqrt{1+\frac{1}{4y}}\,dy \approx 15.34$$

(This answer is the same as the answer to Practice 3. Should that surprise you?)

A curve $\mathcal{C}$ can also be described using parametric equations, where functions $x(t)$ and $y(t)$ give the coordinates of a point on the curve specified by a parameter t. (Review Section 2.5 to refresh your memory about parametric equations.) We often think of t as "time," so that $(x(t), y(t))$ represents the position of a particle in the xy-plane t seconds (or minutes or hours) after time $t = 0$. In Section 2.5, we discovered that the speed of such a particle at time t is given by:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

To find the distance the particle travels between times $t = \alpha$ and $t = \beta$, we could then integrate this speed function, which would also tell us the length of the curve. (The distance traveled by the particle and the length of the curve will be equal as long as the particle does not traverse any part of the curve more than once on the interval $\alpha \le t < \beta$.)

Arclength Formula: parametric version

If: $\mathcal{C}$ is a curve given by $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$

and: $x'(t)$ and $y'(t)$ exist and are continuous on $[\alpha, \beta]$

then: the length L of $\mathcal{C}$ is given by:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Practice 5.3.6

Compute the length of the parametric curve given by the functions $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$.

Answer

Here $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$ so the arclength is:\[\int_0^{2\pi} \sqrt{\left[-\sin(t)\right]^2+\left[\cos(t)\right]^2} \, dt = \int_0^{2\pi} \sqrt{1} \, dt = 2\pi]

(The curve in question is a circle of radius 1. Does the answer from the integral formula agree with the answer you can obtain using simple geometry?)

Practice 5.3.7

Compute the length of the parametric path given by the functions $x(t) = 1 + 3t$ and $y(t) = 4t$ for $1 \le t \le 3$.

Answer

Here $x'(t) = 3$ and $y'(t) = 4$ so the arclength is:

$$\int_1^3 \sqrt{[3]^2 + [4]^2}\,dt = \int_1^3 5\,dt = 10$$

(The "curve" is a line segment from $(4, 4)$ to $(10, 12)$.

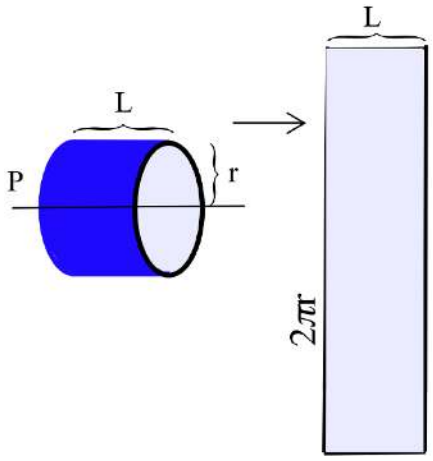
(Does the answer from the integral formula agree with the answer you can obtain using simple geometry?)

Areas of Surfaces of Revolution

In the previous section, we revolved a *region* in the xy-plane about a horizontal or vertical axis to create a *solid*, then used an integral to compute the volume of that solid. If we instead rotate a *curve* about an axis, we get a *surface* whose *surface area* we can

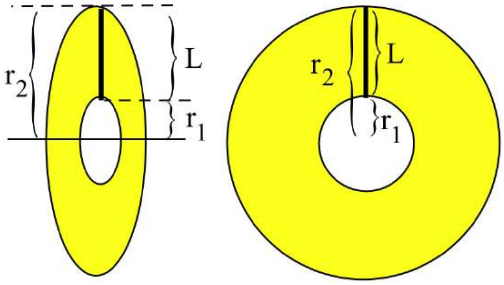
also compute using an integral. Just as the integral formulas for arclength came from the simple distance formula, the integral formulas for the area of a surface of revolution come from the formula for revolving a single line segment.

If we rotate a line segment of length L parallel to a line P about the line P:



then the resulting surface (a cylinder) can be "unrolled" and laid flat. This flattened surface is a rectangle with area $A = 2\pi \cdot r \cdot L$.

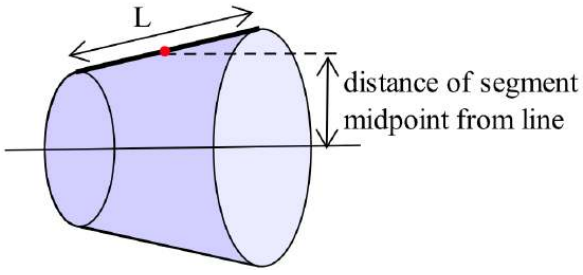
If we rotate a line segment of length L perpendicular to a line P and not intersecting P about the line P:



then the resulting surface is the region between two concentric circles (an "annulus") and its area is:

$$\begin{aligned} A &= (\text{area of large circle}) - (\text{area of small circle}) \\ &= \pi (r_2)^2 - \pi (r_1)^2 = \pi \left[(r_2)^2 - (r_1)^2 \right] = \pi (r_2 + r_1)(r_2 - r_1) \\ &= 2\pi \left(\frac{r_2 + r_1}{2} \right) L \end{aligned}$$

The expression $\frac{r_2+r_1}{2}$ represents the distance of the *midpoint* of the line segment L from the axis of rotation P and $2\pi\left(\frac{r_2+r_1}{2}\right)$ is the *distance* this midpoint travels when we revolve the line segment about the axis. It turns out that this pattern holds when we revolve *any* line segment of length L that does not intersect a line P about the line P:

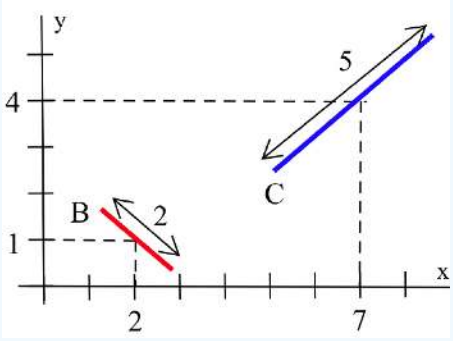


$$\begin{aligned} A &= (\text{distance traveled by segment midpoint}) \cdot (\text{length of line segment}) \\ &= 2\pi\,(\text{distance of segment midpoint from line } P) \cdot L \end{aligned}$$

(See Problem 52 for a proof.)

Example 5.3.4

Compute the area of the surface generated when each line segment in the figure below:



is rotated about the x-axis and the y-axis.

Solution

Line segment B has length $L = 2$ and its midpoint is at $(2, 1)$, which is 1 unit from the x-axis and 2 units from the y-axis. When B is rotated about the x-axis, the surface area is therefore:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 2 = 2\pi(1)2 = 4\pi$$

and when B is rotated about the y-axis, the surface area is:

$$2\pi \cdot (\text{distance of midpoint from } y\text{-axis}) \cdot 2 = 2\pi(2)2 = 8\pi$$

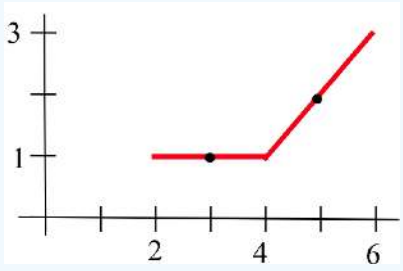
Line segment C has length 5 and its midpoint is at $(7, 4)$. When C is rotated about the x-axis, the resulting surface area is:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 5 = 2\pi(4)5 = 40\pi$$

When C is rotated about the y-axis, the distance of the midpoint from the axis is 7, so the surface area is $2\pi(7)5 = 70\pi$.

Practice 5.3.8

Find the area of the surface generated when the graph below:

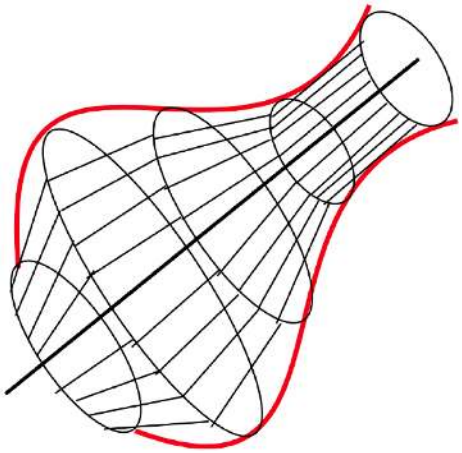


is rotated about each coordinate axis.

Answer

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When we rotate a *curve* C (that does not intersect a line P, as shown in the figure below) about the line P, we also get a surface:



To approximate the area of that surface, we can use the same strategy we used to approximate the length of a curve: select some points (x_k, y_k) along the curve, connect the points with line segments, calculate the surface area of each rotated line segment, and add together the surface areas of the rotated line segments.

The rotated line segment with endpoints (x_{k-1}, y_{k-1}) and (x_k, y_k) has midpoint:

$$(\overline{x}_k, \overline{y}_k) = \left(\frac{x_{k-1}+x_k}{2}, \frac{y_{k-1}+y_k}{2}\right)$$

and length:

$$L = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

If we rotate $\mathcal{C}$ about the x-axis, the distance from the midpoint of the k-th line segment to the x-axis is $\overline{y}_k$ so the surface area of the k -th rotated line segment will be:

$$\begin{aligned} 2\pi\left(\overline{y}_k\right) L &= 2\pi\left(\frac{y_{k-1}+y_k}{2}\right)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= 2\pi\left(\frac{y_{k-1}+y_k}{2}\right)\sqrt{1+\left[\frac{\Delta y_k}{\Delta x_k}\right]^2}\,\Delta x_k \end{aligned}$$

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, and $f'(x)$ is continuous on $[a, b]$, we can appeal to the Mean Value Theorem to find a c_k with $x_{k-1} < c_k < x_k$ and $f'(c_k) = \frac{\Delta y_k}{\Delta x_k}$ so that our last expression becomes:

$$2\pi\left(\frac{f(x_{k-1})+f(x_k)}{2}\right)\sqrt{1+\left[f'(c_k)\right]^2}\,\Delta x_k$$

Adding up these approximations, we get:

$$\sum_{k=1}^{n} 2\pi\left(\frac{f(x_{k-1})+f(x_k)}{2}\right)\sqrt{1+\left[f'(c_k)\right]^2}\,\Delta x_k$$

which converges to a definite integral:

$$\int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

that gives us a formula for the surface area of the revolved curve.

(This is not actually a Riemann sum, because the values x_{k-1}, x_k and c_k are not all the same. Proving that this sum converges to a definite integral requires some more advanced techniques.)

Example 5.3.5

Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the x-axis.

Solution

Here $f(x) = 2 + x^2 \;\Rightarrow\; f'(x) = 2x$ so, using the integral formula we just obtained:

$$\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx = \int_0^3 2\pi\left(2 + x^2\right)\sqrt{1 + 4x^2}\,dx$$

We do not (yet) know how to find an antiderivative for this integrand, but numerical approximation yields a result of 383.8.

(You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.)

More Surface Area Formulas

If a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$ is instead rotated about the y-axis, then the distance from the midpoint of the k-th line segment to the axis of rotation is $\overline{x}_k$. Replacing $\overline{y}_k$ with $\overline{x}_k$ in our work on the previous page yields the formula:

$$\int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\,dx$$

for the area of the surface generated by revolving $\mathcal{C}$ about the y-axis (assuming, as before, that $f'(x)$ is continuous for $a \le x \le b$).

Example 5.3.6

Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the y-axis.

Solution

Here again $f(x) = 2 + x^2 \;\Rightarrow\; f'(x) = 2x$ so, using our newest integral formula:

$$\int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\,dx = \int_0^3 2\pi x\sqrt{1 + 4x^2}\,dx$$

We can find an antiderivative of this integrand using substitution:

$$u = 1 + 4x^2 \;\Rightarrow\; du = 8x\,dx \;\Rightarrow\; \frac{1}{8}\,du = x\,dx$$

The integral limits become $u = 1 + 4(0)^2 = 1$ and $u = 1 + 4(3)^2 = 37$, so the surface area is:

$$\int_{u=1}^{u=37} 2\pi \cdot \frac{1}{8}\sqrt{u}\,du = \frac{\pi}{4}\int_1^{37} u^{\frac{1}{2}}\,du = \frac{\pi}{4}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{37} = \frac{\pi}{6}\left[37\sqrt{37} - 1\right]$$

or approximately 117.3.

Wrap-Up

Developing formulas for the area of a surface generated by rotating a curve $x = g(y)$ for $c \le y \le d$ (or by parametric equations) present little additional difficulty. (See Problems 48–51.) In future chapters, however, we will develop much more general—yet simpler—formulas for arclength and surface area. While the integral formulas developed in this section can be useful, more importantly their development served to illustrate yet again how relatively simple approximation formulas can lead us—via Riemann sums—to integral formulas. We will see this process again and again.

Problems

The graphs of the functions in Problems 5–8 are line segments. Calculate each length:

1. The locations (in feet, relative to an oak tree) at various times (in minutes) for a squirrel spotted in a back yard appear in the table below:

time	north	east
0	10	7
5	25	27
10	1	45
15	13	33
20	24	40
25	10	23
30	0	14

At least how far did the squirrel travel during the first 15 minutes?

2. The squirrel in the previous Problem traveled at least how far during the first 30 minutes?
3. Use the partition $\{0,1,2\}$ to estimate the length of $y=2^x$ between the points $(0,1)$ and $(2,4)$.
4. Use the partition $\{1,2,3,4\}$ to estimate the length of $y=\frac{1}{x}$ between the points $(1,1)$ and $\left(4,\frac{1}{4}\right)$.
   a. using the distance formula between two points and
   b. by setting up and evaluating an appropriate arclength integral.
5. $y=1+2x$ for $0\le x\le 2$
6. $y=5-x$ for $1\le x\le 4$
7. $x=2+t$, $y=1-2t$ for $0\le t\le 3$
8. $x=-1-4t$, $y=2+t$ for $1\le t\le 4$
9. Calculate the length of $y=\frac{2}{3}x^{\frac{3}{2}}$ for $0\le x\le 4$
10. Calculate the length of $y=4x^{\frac{3}{2}}$ for $1\le x\le 9$

Very few functions of the form $y=f(x)$ lead to integrands of the form $\sqrt{1+[f'(x)]^2}$ that have elementary antiderivatives. In Problems 11–14, $1+[f'(x)]^2$ ends up being a perfect square, so you can evaluate the resulting arclength integral using antiderivatives.

11. $y=\frac{x^3}{3}+\frac{1}{4x}$ for $1\le x\le 5$.
12. $y=\frac{x^4}{4}+\frac{1}{8x^2}$ for $1\le x\le 9$.
13. $y=\frac{x^5}{5}+\frac{1}{12x^3}$ for $1\le x\le 5$.
14. $y=\frac{x^6}{6}+\frac{1}{16x^4}$ for $4\le x\le 25$.

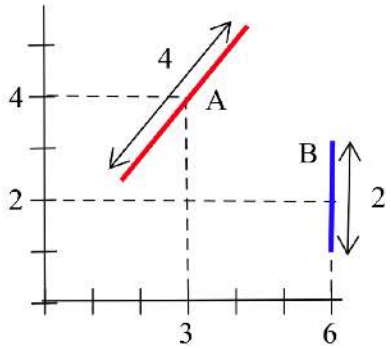
In Problems 15–23, represent each length as a definite integral, then evaluate the integral (using technology, if necessary).

15. The length of $y=x^2$ from $(0,0)$ to $(1,1)$.
16. The length of $y=x^3$ from $(0,0)$ to $(1,1)$.
17. The length of $y=\sqrt{x}$ from $(1,1)$ to $(9,3)$.
18. The length of $y=\ln(x)$ from $(1,0)$ to $(e,1)$.
19. The length of $y=\sin(x)$ from $(0,0)$ to $\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$ and from $\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$ to $\left(\frac{\pi}{2},1\right)$.
20. The length of the ellipse $x(t)=3\cos(t)$, $y(t)=4\sin(t)$ for $0\le t\le 2\pi$.
21. The length of the ellipse $x(t)=5\cos(t)$, $y(t)=2\sin(t)$ for $0\le t\le 2\pi$.
22. A robot programmed to be at location $x(t)=t\cos(t)$, $y(t)=t\sin(t)$ at time t will travel how far between $t=0$ and $t=2\pi$?
23. How far will the robot in the previous problem travel between $t=10$ and $t=20$?
24. As a tire of radius R rolls, a pebble stuck in the tread will travel a "cycloid" path, given by $x(t)=R\cdot(t-\sin(t))$, $y(t)=R\cdot(1-\cos(t))$. As t increases from 0 to 2π, the tire makes one complete revolution and travels forward $2\pi R$ units. How far does the pebble travel?

25. Referring to the previous problem, as a tire with a 1-foot radius rolls forward 1 mile, how far does a pebble stuck in the tire tread travel?
26. Graph $y = x^n$ for $n = 1$, 3, 10 and 20. As the value of n becomes large, what happens to the graph of $y = x^n$? Estimate the value of:

$$\lim_{n\to\infty} \int_{x=0}^{x=1} \sqrt{1 + [n \cdot x^{n-1}]^2}\, dx$$

27. Find the point on the curve $f(x) = x^2$ for $0 \le x \le 4$ that will divide the curve into two equally long pieces. Find the points that will divide the segment into three equally long pieces.
28. Find the pattern for the functions in Problems 11–14. If $y = \frac{x^n}{n} + \frac{1}{Ax^p}$, how must A and p be related to n?
29. Use the formulas for A and p from the previous problem with $n = \frac{3}{2}$ and find a new function $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{Ax^p}$ so that $1 + \left[\frac{dy}{dx}\right]^2$ is a perfect square.
30. Find the surface area when each line segment in the figure below:



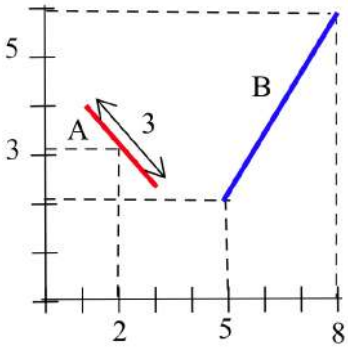
is rotated about the

a. x-axis and
b. y-axis.

31. Find the surface area when each line segment in the figure from the previous Problem is rotated about the line

a. $y = 1$ and
b. $x = -2$.

32. Find the surface area when each line segment in the figure below:



is rotated about the line

a. $y = 1$ and
b. $x = -2$.

33. Find the surface area when each line segment in the figure from the previous Problem is rotated about the:

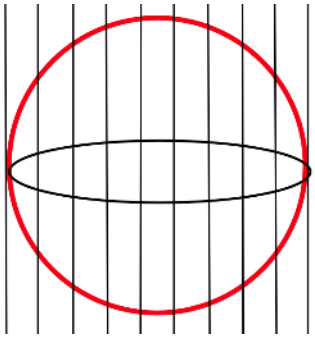
a. x-axis and
b. y-axis.

34. A line segment of length 2 with midpoint $(2, 5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the y-axis? Explain your reasoning.

35. A line segment of length 2 with one end at $(2,5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the x-axis? Explain your reasoning.

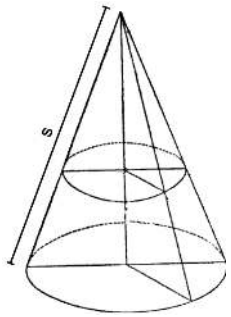
In Problems 36–43, when the given curve is rotated about the given axis, represent the area of the resulting surface as a definite integral, then evaluate that integral using technology.

36. $y = x^3$ for $0 \le x \le 2$ about the y-axis
37. $y = 2x^3$ for $0 \le x \le 1$ about the y-axis
38. $y = x^2$ for $0 \le x \le 2$ about the x-axis
39. $y = 2x^2$ for $0 \le x \le 1$ about the x-axis
40. $y = \sin(x)$ for $0 \le x \le \pi$ about the x-axis
41. $y = x^3$ for $0 \le x \le 2$ about the x-axis
42. $y = \sin(x)$ for $0 \le x \le \frac{\pi}{2}$ about the y-axis
43. $y = x^2$ for $0 \le x \le 2$ about the y-axis
44. Find the area of the surface formed when the graph of $y = \sqrt{4 - x^2}$ is rotated about the x-axis:
    a. for $0 \le x \le 1$.
    b. for $1 \le x \le 2$.
    c. for $2 \le x \le 3$.
45. Show that if a thin hollow sphere is sliced into pieces by equally spaced parallel cuts:



    then each piece has the same weight. (Hint: Does each piece have the same surface area?)
46. Interpret the result of the previous Problem for an orange sliced by equally spaced parallel cuts.
47. A hemispherical cake with a uniformly thick layer of frosting is sliced with equally spaced parallel cuts. Does everyone get the same amount of cake? The same amount of frosting?
48. Devise a formula for the area of the surface generated by revolving the curve $x = g(y)$ for $c \le y \le d$ about the:
    a. x-axis and
    b. y-axis.
49. Use the answer to the previous problem to find the area of the surface generated by revolving $x = e^y$ for $0 \le y \le 1$ about:
    a. the x-axis and
    b. the y-axis.
50. Devise a formula for the area of the surface generated by revolving the curve given by parametric equations $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$ about the:
    a. x-axis and
    b. y-axis.
51. Use the answer to the previous Problem to find the area of the surface generated by revolving the curve given by $x = \cos(t)$ and $y = \sin(t)$ for $0 \le t \le \frac{\pi}{2}$ about:
    a. the x-axis and
    b. the y-axis.
52. The surface generated by revolving a line segment of length L about a line P (that does not intersect the line segment) is the **frustrum** of a cone: the surface that results from taking a larger cone of radius r_2 and removing a smaller cone of radius r_1

("chopping off the top"). We know from geometry that the surface area of a cone is πrs where r is the radius of the cone and s is the **slant height**:



a. If s_1 is the slant height of the smaller cone that is removed from the bigger cone, show that:

$$s_1 + L = \frac{r_2 L}{r_2 - r_1}$$

(Hint: Use similar triangles.)

b. Show that the surface area of the frustrum is:

$$\pi r_2 (s_1 + L) - \pi r_1 s_1$$

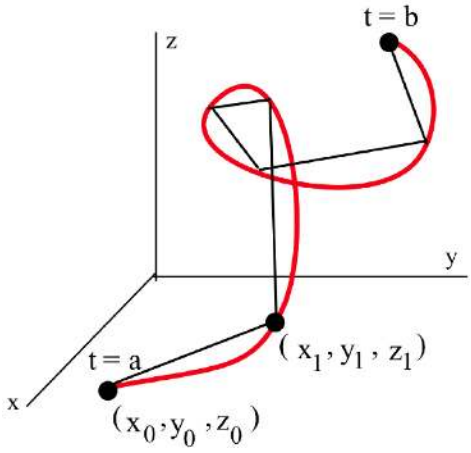
c. Show that this quantity equals:

$$\pi (r_1 + r_2) L$$

d. Show that this last quantity is the product of the distance traveled by the midpoint and the length of the line segment.

3-D Arclength

If a 3-dimensional curve $\mathcal{C}$ is given parametrically by $x = x(t)$, $y = y(t)$ and $z = z(t)$ for $\alpha \le t \le \beta$:

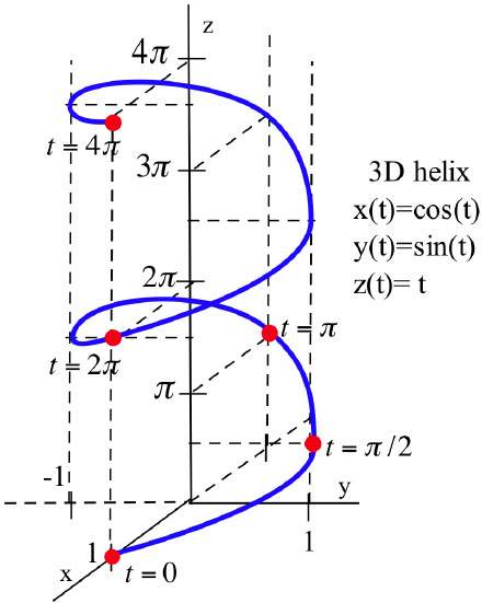


then we can easily extend the arclength formula to three dimensions:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The remaining problems in this section use this formula to provide you with a preview of calculus in higher dimensions.

53. Find the length of the helix given by $x = \cos(t)$, $y = \sin(t)$, $z = t$ for $0 \le t \le 4\pi$:



54. Find the length of the line segment given by $x = t$, $y = t$, $z = t$ for $0 \le t \le 1$.
55. Find the length of the curve given by $x = t$, $y = t^2$, $z = t^3$ for $0 \le t \le 1$.
56. Find the length of the "stretched helix" given by $x = \cos(t)$, $y = \sin(t)$, $z = t^2$ for $0 \le t \le 2\pi$.
57. Find the length of the curve given by $x = 3\cos(t)$, $y = 2\sin(t)$, $z = \sin(7t)$ for $0 \le t \le 2\pi$.

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5.4: More Work

In Section 4.7 we investigated the problem of calculating the work done in lifting an object using a cable. This section continues that investigation and extends the process to handle situations in which the applied force or the distance—or both—may vary. The method we used before turns up again here. The first step is to divide the problem into small "slices" so that the force and distance vary only slightly on each slice. Then we calculate the work done for each slice, approximate the total work by adding together the work for each slice (to get a Riemann sum) and, finally, take a limit of that Riemann sum to get a definite integral representing the total work.

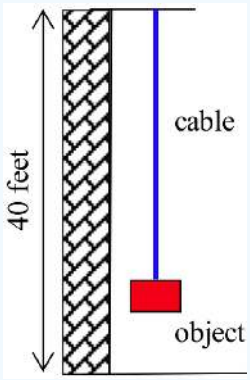
> There are so many possible variations in work problems that it is vital you understand the process.

Recall that the work done on an object by a constant force is defined to be the magnitude of the force applied to the object multiplied by the distance over which the force is applied:

$$\text{work} = (\text{force}) \cdot (\text{distance})$$

Example 5.4.1

A 10-pound object is lifted 40 feet from the ground to the top of a building using a cable that weighs $\frac{1}{2}$ pound per foot:



How much work is done?

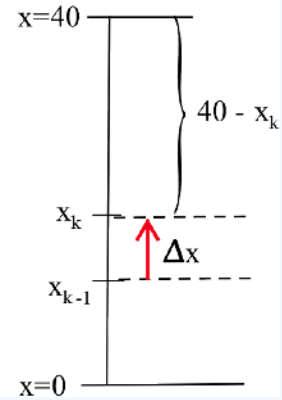
(A similar example appeared in Section 4.7, but it provides a good illustration of the process of dividing a problem into pieces and analyzing each piece.)

Solution

The work done on the object is simply:

$$W = F \cdot d = (10 \text{ lbs}) \cdot (40 \text{ ft}) = 400 \text{ ft-lbs}$$

For the rope, we can partition it into n small pieces, each with length Δx:



Each small piece of rope weighs:

$$\left(\frac{1}{2}\,\frac{\text{lb}}{\text{ft}}\right)(\Delta x\ \text{ft}) = \frac{1}{2}\Delta x\ \text{lb}$$

and the k-th slice of rope is lifted a distance of (approximately) $40 - x_k$ feet, so the work done on the k-th slice of rope is (approximately):

$$W_k = F_k \cdot d_k = \left(\frac{1}{2}\Delta x\ \text{lb}\right) \cdot \left((40 - x_k)\ \text{ft}\right) = \frac{1}{2}\,(40 - x_k)\,\Delta x\ \text{ft-lbs}$$

and the total work done to lift the rope is therefore:

$$\sum_{k=1}^{n} \frac{1}{2}\,(40 - x_k)\,\Delta x \ \longrightarrow\ \int_0^{40} \frac{1}{2}\,(40 - x)\,dx$$

Evaluating this integral yields:

$$\frac{1}{2}\left[40x - \frac{1}{2}x^2\right]_0^{40} = \frac{1}{2}\,[1600 - 800] = 400\ \text{ft-lbs}$$

so the total work done to lift the object is $400 + 400 = 800$ ft-lbs.

Practice 5.4.1

How much work is done lifting a 130-pound injured person to the top of a 30-foot cliff using a stretcher that weighs 10 pounds and a cable weighing 2 pounds per foot?

Answer

The work done lifting the person and the stretcher is:

$$(130\ \text{lb} + 10\ \text{lb}) \cdot (30\ \text{ft}) = 4200\ \text{ft-lbs}$$

The work done lifting a small piece of cable with length Δx ft at an initial height of x feet above the ground is:

$$\left(2\,\frac{\text{lb}}{\text{ft}}\right)(\Delta x\ \text{ft})\left((30 - x)\ \text{ft}\right) = (60 - 2x)\Delta x\ \text{ft-lbs}$$

so the work done lifting the cable is:

$$\sum_{k=1}^{n} (60 - 2x)\Delta x \ \longrightarrow\ \int_0^{30} (60 - 2x)\,dx = 900\ \text{ft-lbs}$$

and the total work is $4200 + 900 = 5100$ ft-lbs.

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Work in the Metric System

All of the work problems we have considered so far measured force in pounds and distance in feet, so that work was measured in "foot-pounds." Virtually all countries other than the United States—along with U.S. scientists and engineers—use the metric system, so you need to know how to solve work problems using metric units.

In the metric system, we often measure distance in meters (m) and force in **newton**} (N). According to Newton's second law of motion:

$$\text{force} = (\text{mass}) \cdot (\text{acceleration})$$

or, more succinctly, $F = ma$. The force in many work problems is the weight of an object, so the acceleration in question is the acceleration due to gravity, denoted by g. Near sea level on Earth, $g \approx 9.80665 \ \frac{\text{m}}{\text{sec}^2}$, although the value 9.81 is commonly used in computations. An object with a mass of 10 kg would thus have a weight of:

$$mg = (10 \text{ kg}) \cdot \left(9.81 \frac{\text{m}}{\text{sec}^2}\right) = 98.1 \frac{\text{kg} \cdot \text{m}}{\text{sec}^2} = 98.1 \text{ N}$$

Note

Sir Isaac Newton (1643–1727) not only invented calculus, he formulated the laws of motion and universal gravitation in physics (among many other accomplishments).

Example 5.4.2

An object with a mass of 10 kg is lifted 40 m from the ground to the top of a building using a 40-meter cable with a mass of 20 kg. How much work is done?

Solution

The work done on the object is:

$$W = F \cdot d = mg \cdot d = (10 \text{ kg}) \left(9.81 \frac{\text{m}}{\text{sec}^2}\right) \cdot (40 \text{ m}) = 3924 \text{ N-m}$$

or 3,924 joules (a **joule**, abbreviated "J," is 1 N-m). The cable has total mass 20 kg and is 40 m long, so it has a linear density of:

$$\frac{20 \text{ kg}}{40 \text{ m}} = \frac{1}{2} \frac{\text{kg}}{\text{m}}$$

We can partition the cable into n small pieces, each with length Δx, so each small piece of cable has a mass of:

$$\left(\frac{1}{2} \frac{\text{kg}}{\text{m}}\right) (\Delta x \text{ m}) = \frac{1}{2} \Delta x \text{ kg}$$

and thus has a weight of:

$$F = mg = \left(\frac{1}{2} \Delta x \text{ kg}\right) \left(9.81 \frac{\text{m}}{\text{sec}^2}\right) = 4.905 \Delta x \frac{\text{kg} \cdot \text{m}}{\text{sec}^2} = 4.905 \Delta x \text{ N}$$

The k-th slice of cable is lifted a distance of approximately $40 - x_k$ m, so the work done on the k-th slice of cable is:

$$W_k = F_k \cdot d_k = (4.905 \Delta x \text{ N}) \cdot \left((40 - x_k) \text{ m}\right) = 4.905 \left(40 - x_k\right) \Delta x \text{ N-m}$$

and the total work done lifting the cable is therefore:

$$\sum_{k=1}^{n} 4.905 \left(40 - x_k\right) \Delta x \longrightarrow \int_0^{40} 4.905 \left(40 - x\right) dx = 3924 \text{ J}$$

so the total work done to lift the object is $3924 + 3924 = 7848$ J.

(Much of this process should look familiar. Compare the solution of Example 2 to that of Example 1 above.)

Note

This metric unit for work, the joule, is named after another English physicist, James Prescott Joule (1818–1889).

Practice 5.4.2

How much work is done lifting an injured person of mass 50 kg to the top of a 30-meter cliff using a stretcher of mass 5 kg and a 30-meter cable of mass 10 kg?

Answer

The work done lifting the person and the stretcher is:

$$(50\text{ kg} + 5\text{ kg}) \cdot \left(9.81\ \frac{\text{m}}{\text{sec}^2}\right) \cdot (30\text{ m}) = (55\text{ N})\,(30\text{ m}) = 16186.5\text{ J}$$

The work done lifting a small piece of cable with length Δx m at an initial height of x m above the ground is:

$$\left(\frac{1}{3}\ \frac{\text{kg}}{\text{m}}\right)\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)(\Delta x\text{ m})\left((30-x)\text{ m}\right) = 3.27(30-x)\Delta x\text{ J}$$

so the work done lifting the cable is:

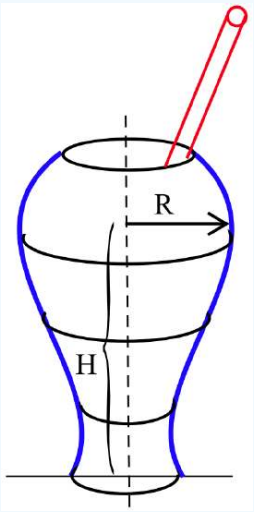
$$\sum_{k=1}^{n} 3.27(30-x)\Delta x \longrightarrow 3.27\int_0^{30}(30-x)\,dx = 1471.5\text{ J}$$

and the total work is $16186.5 + 1471.5 = 17658$ J.

Lifting Liquids

Example 5.4.3

A cola glass:



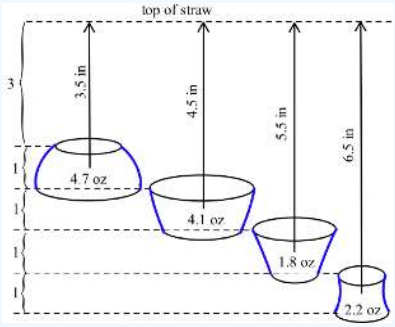
has dimensions given in the table below:

height	radius
4	1.4
3	1.6
2	1.5
1	1.0
0	1.1

Approximately how much work do you do when you drink a cola glass full of water by sucking it through a straw to a point 3 inches above the top edge of the glass?

Solution

The table partitions the water into 1-inch “slices”:



The work needed to move each slice is approximately the weight of the slice times the distance the slice is moved. We can use the radius at the bottom of each slice to approximate the volume—and then the weight—of the slice, and a point halfway up each slice to calculate the distance the slice is moved. For the top slice:

$$\text{weight} = (\text{volume})\,(\text{density}) \approx \pi\,(1.6\,\text{in})^2\,(1\,\text{in})\left(0.5787\,\frac{\text{oz}}{\text{in}^3}\right) \approx 4.7\,\text{oz}$$

(water’s density is $62.5\frac{\text{lb}}{\text{ft}^3} = 0.5787\frac{\text{oz}}{\text{in}^3}$ and the distance this slice travels is roughly 3.5 inches, so:

$$W = F \cdot d \approx (4.7\ \text{oz})\,(3.5\ \text{in}) \approx 16.4\ \text{oz-in}$$

For the next slice:

$$\text{weight} = (\text{volume})\,(\text{density}) \approx \pi\,(1.5\,\text{in})^2\,(1\,\text{in})\left(0.5787\,\frac{\text{oz}}{\text{in}^3}\right) \approx 4.1\,\text{oz}$$

and the distance this slice travels is roughly 4.5 inches, so:

$$W = F \cdot d \approx (4.1\ \text{oz})\,(4.5\ \text{in}) \approx 18.4\ \text{oz-in}$$

The work for the last two slices is $(1.8\,\text{oz})\,(5.5\,\text{in}) = 9.9$ oz-in and $(2.2\,\text{oz})\,(6.5\,\text{in}) = 14.3$ oz-in. The total work is then sum of the work needed to raise each slice of water:

$$(16.4\,\text{oz-in}) + (18.4\,\text{oz-in}) + (9.9\,\text{oz-in}) + (14.3\,\text{oz-in}) = 59.0\,\text{oz-in}$$

or about 0.31 ft-lbs.

You might wonder why the displacement is not computed by taking the distance from the bottom of the straw up to the top of the straw, but when computing work we need to use the *net* displacement.

Practice 5.4.3

Approximate the total work needed to raise the water in Example 3 by using the top radius of each slice to approximate its weight and the midpoint of each slice to approximate the distance the slice is raised.

Answer

The total work done is approximately

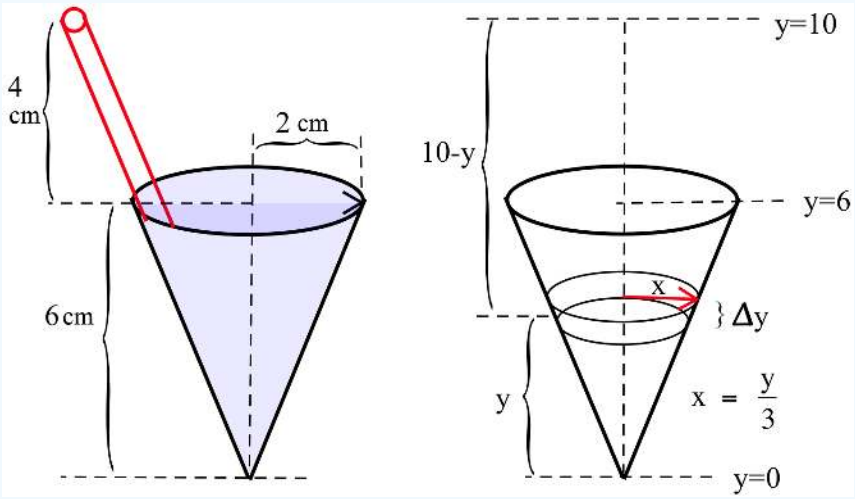
$$\left[\pi(1.4)^2(3.5)+\pi(1.6)^2(4.5)+\pi(1.5)^2(5.5)+\pi(1.0)^2(6.5)\right](0.5787)$$

or 67.73 oz-in ≈ 0.35 ft-lbs.

If we knew the radius of the cola glass at *every* height, then we could improve our approximation by taking thinner and thinner slices. In fact, we could have formed a Riemann sum, taken the limit of the Riemann sum as the thickness of the slices approached 0, and obtained a definite integral. In the next Example we *do* know the radius of the container at every height.

Example 5.4.4

Find the work needed to raise the water in the cone shown below:



to the top of the straw.

(In this example, both the force and the distance vary, and each depends on the height of the "slice" above the bottom of the cone.)

Solution

We can partition the cone to get n "slices" of water. The work done raising the k-th slice is the product of the distance the slice is raised and the force needed to move the slice (the weight of the slice). For any c_k in the subinterval $[y_{k-1}, y_k]$, the slice is raised a distance of approximately $(10-c_k)$ cm. (If you want, you can choose $c_k = y_k$ like you did in Practice 3.)

Each slice is approximately a right circular cylinder, so its volume is:

$$\pi\,(\text{radius})^2\,\Delta y$$

At a height y above the bottom of the cone, the radius of the cylinder is $x = \frac{y}{3}$ so at a height c_k the radius is $\frac{1}{3}c_k$. To see this, use similar triangles in the right-hand figure above:

$$\frac{x}{y} = \frac{2}{6} \Rightarrow x = \frac{y}{3}$$

The mass of each slice is therefore:

$$\begin{aligned}(\text{volume})\,(\text{density}) &\approx \pi\,(\text{radius})^2\,(\Delta y)\left(1\,\frac{\text{g}}{\text{cm}^3}\right)\\ &= \pi\left(\frac{1}{3}c_k\ \text{cm}\right)^2(\Delta y\ \text{cm})\left(1\,\frac{\text{g}}{\text{cm}^3}\right)\\ &= \frac{\pi}{9}(c_k)^2\,\Delta y\ \text{g}\end{aligned}$$

Note

In the metric system, a **gram** (abbreviated "g") is defined as the mass of one cubic centimeter of water, so the density of water is:

$$1\,\frac{\text{g}}{\text{cm}^3} = 1{,}000\,\frac{\text{kg}}{\text{m}^3}$$

So the force required to raise the k-th slice is:

$$F_k = m_k \cdot g \approx \left[\frac{\pi}{9}(c_k)^2\,\Delta y\ \text{g}\right]\cdot\left[981\,\frac{\text{cm}}{\text{sec}^2}\right] = 109\pi\,(c_k)^2\,\Delta y\,\frac{\text{g-cm}}{\text{sec}^2}$$

and the work required to lift the k-th slice is:

$$\begin{aligned}W_k = F_k \cdot d_k &\approx \left[109\pi\,(c_k)^2\,\Delta y\ \text{dyn}\right]\cdot[(10-c_k)\ \text{cm}]\\ &= 109\pi\,(c_k)^2\,(10-c_k)\,\Delta y\ \text{dyn-cm}\end{aligned}$$

Note

In the g–cm–sec version of the metric system, the standard unit of force is a **dyne** (abbreviated "dyn"), which is $1\,\frac{\text{g-cm}}{\text{sec}^2}$:

$$1\ \text{N} = 100{,}000\ \text{dyn}$$

We can then add the work done on all n slices to get a Riemann sum:

$$W \approx \sum_{k=1}^{n} 109\pi\,(c_k)^2\,(10-c_k)\,\Delta y \ \longrightarrow\ \int_{y=0}^{y=6} 109\pi y^2(10-y)\,dy$$

Evaluating this integral is relatively straightforward:

$$\begin{aligned}W &= 109\pi\int_0^6 (10y^2-y^3)\,dy = 109\pi\left[\frac{10}{3}y^3-\frac{1}{4}y^4\right]_0^6\\ &= 109\pi\,[720-324] = 43164\pi\ \text{erg}\end{aligned}$$

or about $135{,}604\ \text{erg} = 0.0135604\ \text{J}$.

Note

We integrate from $y=0$ to $y=6$ because the bottom slice of water is at a height of 0 cm and the top slice of water is at a height of 6 cm. In the g–cm–sec version of the metric system, the standard unit of work is called an **erg**, which is: 1 dyn-cm:

$$1\ \text{J} = 10{,}000{,}000\ \text{erg}$$

Practice 5.4.4

How much work is done drinking just the top 3 cm of the water in Example 4?

Answer

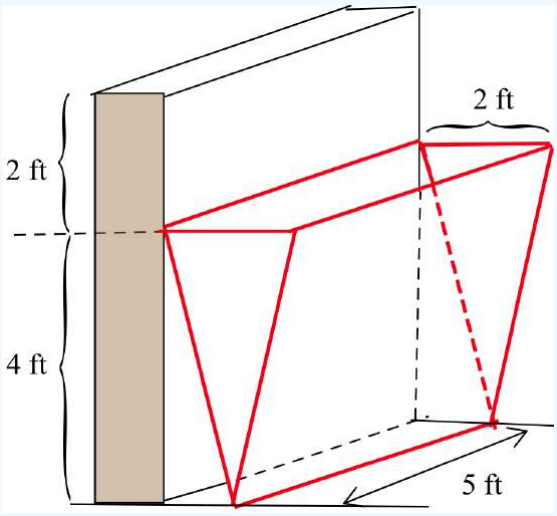
We can use the same integral as in the solution to Example 4, but instead integrate from $y=3$ to $y=6$:

$$W = 109\pi \int_3^6 (10y^2 - y^3)\, dy = 109\pi \left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_3^6$$
$$= 109\pi \left[(720 - 324) - \left(90 - \frac{81}{4}\right)\right] = 35561.25\pi \text{ erg}$$

or about $111,719 \text{ erg} = 0.0111719 \text{ J}$.

Example 5.4.5

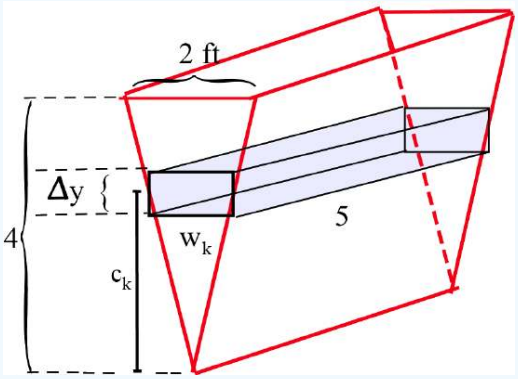
The trough shown below:



is filled with a liquid weighing 70 pounds per cubic foot. How much work is done pumping the liquid over the wall next to the trough?

Solution

As before, we can partition the height of the trough to get n "slices" of liquid:



To form a Riemann sum for the total work, we need the weight of a typical slice and the distance that slice is raised. The weight of the k-th slice is:

$$(\text{volume}) \cdot (\text{density}) \approx (\text{length})\,(\text{width})\,(\text{height}) \cdot \left(70 \frac{\text{lb}}{\text{ft}^3}\right)$$

The length of each slice is 5 feet, and the height of each slice is Δy feet, but the width of each slice (w_k) varies and depends on how far the slice is above the bottom of the trough (c_k). Using similar triangles on the edge of the trough, we can observe that:

$$\frac{w_k}{c_k} = \frac{2}{4} \Rightarrow w_k = \frac{c_k}{2}$$

so the weight of the k-th slice is therefore:

$$(5 \text{ ft})\left(\frac{c_k}{2} \text{ ft}\right)(\Delta y \text{ ft}) \cdot \left(70 \frac{\text{lb}}{\text{ft}^3}\right) = 175 c_k \Delta y \text{ lb}$$

The k-th slice is raised from a height of c_k feet to a height of 6 feet, through a distance of $6 - c_k$ feet, so the work done on the k-th slice is:

$$W_k = F_k \cdot d_k \approx \left[175 c_k \Delta y \text{ lb}\right] \cdot \left[(6 - c_k) \text{ ft}\right] = 175 c_k (6 - c_k) \Delta y \text{ lb-ft}$$

Adding up the work done on all n slices yields a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} 175 c_k (6 - c_k) \Delta y \longrightarrow \int_0^4 175 y (6 - y)\, dy$$

Evaluating the integral is straightforward:

$$175 \int_0^4 \left(6y - y^2\right)\, dy = 175 \left[3y^2 - \frac{1}{3} y^3\right]_0^4 = \frac{14000}{3}$$

or about $4,667$ ft-lbs.

(We integrate from $y = 0$ to $y = 4$ because the bottom slice of liquid is at a height of 0 feet and the top slice of liquid is at a height of 4 feet.)

You can generally handle "raise the liquid" problems by partitioning the height of the container and then focusing on a typical slice. If you can calculate the weight of a typical slice and the distance it is raised, the rest of the steps are straightforward: form a Riemann sum, let it converge to a definite integral, and evaluate the integral to get the total work.

? Practice 5.4.5

How much work is done pumping half of the liquid over the wall in Example 5?

Answer

The total amount of liquid in the trough is $\frac{1}{2} \cdot 4 \cdot 2 \cdot 5 = 20 \text{ ft}^3$, so we need to lift the top 10 ft^3 of liquid out of the trough. To find the height separating the bottom 10 ft^3 of liquid from the rest, we can recall that (from our similar-triangles computation), the width at height h is $w = \frac{h}{2}$, so the volume of liquid between height $y = 0$ and height $y = h$ is:

$$10 = \frac{1}{2} \cdot h \cdot \frac{h}{2} \cdot 5 \;\Rightarrow\; h^2 = 8 \;\Rightarrow\; h = 2\sqrt{2}$$

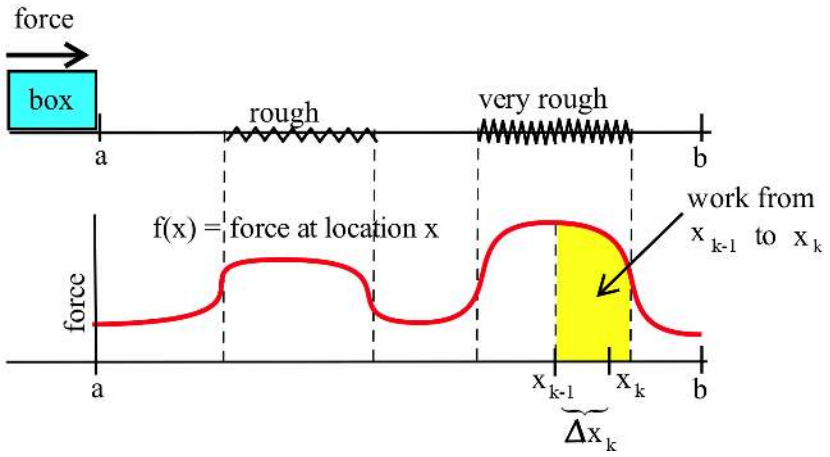
The work to lift the top 10 ft^3 of liquid is thus:

$$\begin{aligned} 175 \int_{2\sqrt{2}}^4 \left(6y - y^2\right)\, dy &= 175 \left[3y^2 - \frac{1}{3} y^3\right]_{2\sqrt{2}}^4 \\ &= 175 \left[\left(48 - \frac{64}{3}\right) - \left(24 - \frac{16\sqrt{2}}{3}\right)\right] \end{aligned}$$

or about $1,786.6$ ft-lbs.

Work Moving an Object Along a Straight Path

If you push a box along a flat surface, as in the figure below:



that is smooth in some places and rough in others, at some places you only need to push the box lightly and in other places you have to push hard. If $f(x)$ is the amount of force needed at location x, and you want to push the box along a straight path from $x = a$ to $x = b$, then we can partition the interval $[a, b]$ into n pieces, $[a, x_1]$, $[x_1, x_2], \ldots, [x_{n-1}, b]$.

(The force $f(x)$ discussed here is the minimum force required to counteract the **kinetic friction** between the box and the surface at any point. You will learn more about friction in physics and engineering classes.)

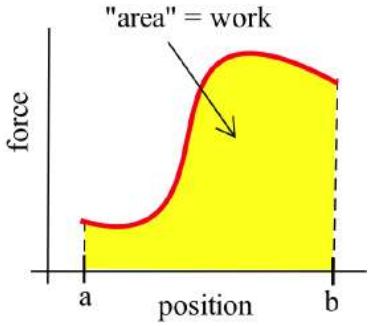
The work required to move the box through the k-th subinterval, from x_{k-1} to x_k, is approximately:

$$(\text{force}) \cdot (\text{distance}) \approx f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k$$

for any c_k in the subinterval $[x_{k-1}, x_k]$. The total work is the sum of the work along these n pieces, which is a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \longrightarrow \int_a^b f(x)\, dx$$

This has a simple geometric interpretation. If $f(x)$ is the force applied at position x, then the work done to move the object from position $x = a$ to position $x = b$ is the area under the graph of f between $x = a$ and $x = b$:



This formula applies in more general situations, as demonstrated in the next Example.

Example 5.4.6

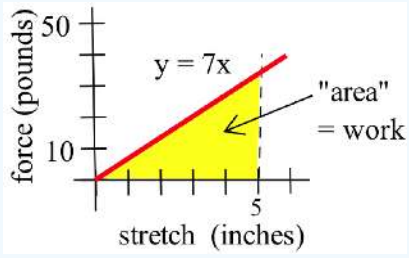
If a force of $7x$ pounds is required to keep a spring stretched x inches past its natural length, how much work will be done stretching the spring from its natural length ($x = 0$) to five inches beyond its natural length ($x = 5$)?

Solution

According to the formula we just developed:

$$\text{work} = \int_a^b f(x)\, dx = \int_0^5 7x\, dx = \left[\frac{7}{2}x^2\right]_0^5 = \frac{175}{2} = 87.5 \text{ in-lbs}$$

or about 7.29 ft-lbs. See the figure below for a graphical interpretation:



? Practice 5.4.6

How much work is done to stretch the spring in Example 6 from 5 inches past its natural length to 10 inches past its natural length?

Answer

We can use the same integral as in the solution to Example 6, but instead integrate from $x = 5$ to $x = 10$:

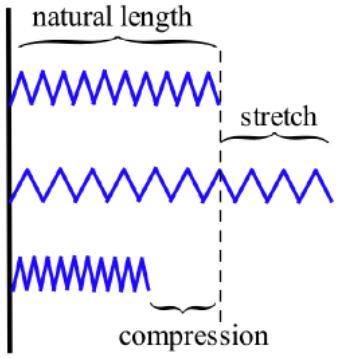
$$\int_5^{10} 7x\,dx = \left[\frac{7}{2}x^2\right]_5^{10} = 350 - \frac{175}{2} = 262.5 \text{ in-lbs} = 21.875 \text{ ft-lbs}$$

The preceding spring example is an application of a physical principle discovered by English physicist Robert Hooke (1635–1703), a contemporary of Newton.

Hooke's Law

The force $f(x)$ needed to keep a spring stretched (or compressed) x units beyond its natural length is proportional to the distance x: $f(x) = kx$ for some constant k.

We call the "k" in Hooke's Law a "spring constant." It varies from spring to spring (depending on the materials and dimensions of the spring—and even on the temperature of the spring), but remains constant for each spring as long as the spring is not overextended or overcompressed.



(In fact, Hooke's Law holds for most solid objects, at least for limited forces: "Nor is it observable in these bodies only, but in all other springy bodies whatsoever, whether metal, wood, stones, baked earth, hair, horns, silk, bones, sinews, glass and the like."—Robert Hooke, *De Potentia Restitutiva, or Of Spring*)

Most bathroom scales use compressed springs—and Hooke's Law—to measure a person's weight.

Example 5.4.7

A spring has a natural length of 43 cm when hung from a ceiling. A mass of 40 grams stretches it to a length of 75 cm. How much work is done stretching the spring from a length of 63 cm to a length of 93 cm?

Solution

First we need to use the given information to find the value of k, the spring constant. A mass of 40 g produces a stretch of $75 - 43 = 32$ cm. Substituting $x = 32$ cm and $f(x) = 40\,\text{g} \cdot 981 \frac{\text{cm}}{\text{sec}^2}$ into Hooke's Law $f(x) = kx$, we have:

$$40(981) = k(32) \;\Rightarrow\; k = \frac{4905}{4}$$

The length of 63 cm represents a stretch of 20 cm beyond the spring's natural length, while the length of 93 cm represents a 50-cm stretch. The work done is therefore:

$$\int_{20}^{50} \frac{4905}{4} x\,dx = \left[\frac{4905}{8}x^2\right]_{20}^{50} = 613.125\left[50^2 - 20^2\right] = 1287562.5 \text{ ergs}$$

or about 0.129 joules.

Practice 5.4.7

A spring has a natural length of 3 inches when hung from a ceiling, and a force of 2 pounds stretches it to a length of 8 inches. How much work is done stretching the spring from a length of 5 inches to a length of 10 inches?

Answer

According to Hooke's Law, $2\text{ lb} = k \cdot (8\text{ in} - 3\text{ in}) \;\Rightarrow\; k = \frac{2}{5}$, so stretching the spring from $5 - 3 = 2$ in to $10 - 3 = 7$ in beyond its natural length requires:

$$\int_2^7 \frac{2}{5}x\,dx = \left[\frac{1}{5}x^2\right]_2^7 = 9 \text{ in-lb} = \frac{3}{4} \text{ ft-lb}$$

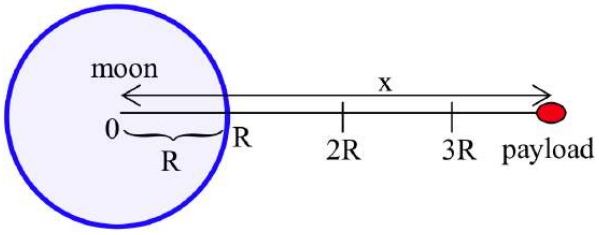
Lifting a Payload

Calculating the work required to lift a payload from the surface of a moon (or any body with no atmosphere) can be accomplished using a similar computation. Newton's Law of Universal Gravitation says that the gravitational force between two bodies of mass M and m is:

$$F = \frac{GMm}{x^2}$$

where $G \approx 6.673 \times 10^{-11}\ \text{N}\left(\frac{\text{m}}{\text{kg}}\right)^2$ is the **gravitational constant** and x is the distance between (the centers of) the two bodies.

If the moon has a radius of R m and mass M:



the payload has mass m and x measures the distance (in meters) from payload to the center of the moon (so $x \geq R$), then the total amount of work done lifting the payload from the surface of the moon (an altitude of 0, where $x = R$) to an altitude of R (where $x = R + R = 2R$) is:

$$\int_R^{2R} \frac{GMm}{x^2}\,dx = GMm\left[\frac{-1}{x}\right]_R^{2R} = GMm\left[\frac{-1}{2R} + \frac{1}{R}\right] = \frac{GMm}{2R}$$

? Practice 5.4.8

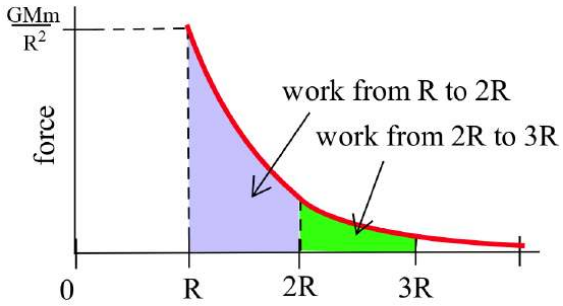
How much work is required to lift the payload from an altitude of R m above the surface ($x = 2R$) to an altitude of $2R$ m?

Answer

The work required to lift the payload from $x = 2R$ to $x = 3R$ is:

$$\int_{2R}^{3R} \frac{GMm}{x^2}\,dx = GMm\left[\frac{-1}{x}\right]_{2R}^{3R} = GMm\left[\frac{-1}{3R} + \frac{1}{2R}\right] = \frac{GMm}{6R}$$

The appropriate areas under the force graph:



illustrate why the work to lift the payload from $x = R$ to $x = 2R$ is much larger than the work to lift it from $x = 2R$ to $x = 3R$. In fact, the work to lift the payload from $x = 2R$ to x = $100R$ is $0.49GMmR^{-1}$, which is less than the $0.5GMmR^{-1}$ needed to lift it from $x = R$ to $x = 2R$.

The real-world problem of lifting a payload turns out to be much more challenging, because the rocket doing the lifting must also lift itself (more work) and the mass of the rocket keeps changing as it burns up fuel. Lifting a payload from a moon (or planet) with an atmosphere is even more difficult: the atmosphere produces friction, and the frictional force depends on the density of the atmosphere (which varies with height), the speed of the rocket and the shape of the rocket. Life can get complicated.

Power

In physics, **power** is defined as the rate of work done per unit of time. One traditional measurement of power, **horsepower** (abbreviated “hp”), originated with James Watt’s determination in 1782 that a horse could turn a mill wheel of radius 12 feet 144 times in an hour while exerting a force of 180 pounds. Such a horse would travel:

$$144\ \frac{\text{rev}}{\text{hr}} \cdot 2\pi(12)\ \frac{\text{ft}}{\text{rev}} \cdot \frac{1}{60}\ \frac{\text{hr}}{\text{min}} = \frac{288\pi}{5}\ \frac{\text{ft}}{\text{min}}$$

and so it would produce work at a rate of:

$$(180\text{ lb})\left(\frac{288\pi}{5}\ \frac{\text{ft}}{\text{min}}\right) = 10368\pi\ \frac{\text{ft-lb}}{\text{min}} \approx 32572\ \frac{\text{ft-lb}}{\text{min}}$$

which Watt subsequently rounded to:

$$33000\ \frac{\text{ft-lb}}{\text{min}} = 550\ \frac{\text{ft-lb}}{\text{sec}} = 1\text{ horsepower}$$

(Scottish engineer James Watt (1736–1819) devised horsepower to compare the output of steam engines with the power of draft horses.)

The metric unit of power, called a **watt** (abbreviated “W”) in Watt''s honor, is equivalent to 1 joule per second: $1\text{ hp} \approx 746\text{ W}$.

Example 5.4.8

How long will it take for a 1-horsepower electric pump to pump all of the liquid in the trough from Example 5 over the wall?

Solution

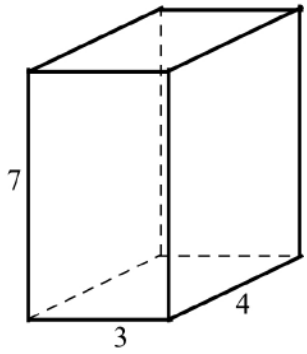
Power (P) is the rate at which work (W) is done, so:

$$P = \frac{W}{t} \Rightarrow t = \frac{W}{P} = \frac{\frac{14000}{3}\text{ ft-lbs}}{1\text{ hp}} = \frac{\frac{14000}{3}\text{ ft-lbs}}{550\ \frac{\text{ft-lbs}}{\text{sec}}} = \frac{280}{33}\text{ sec}$$

or about 8.5 seconds.

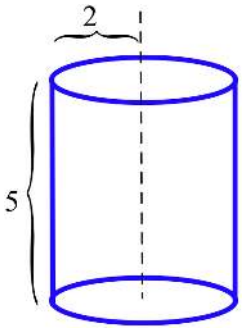
Problems

1. A tank 4 feet long, 3 feet wide and 7 feet tall is filled with water:



How much work is required to pump the water out over the top edge of the tank?

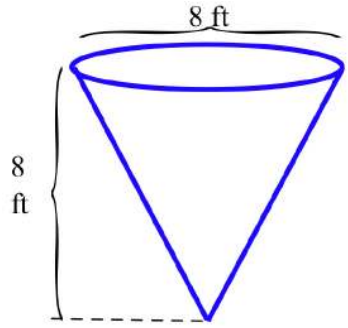
2. A tank 4 feet long, 3 feet wide and 6 feet tall is filled with a oil with a density of 60 pounds per cubic foot.
   a. How much work is needed to pump all of the oil over the top edge of the tank?
   b. How much work is needed to pump the top 3 feet of oil over the top edge of the tank?
3. A tank 5 m long, 2 m wide and 4 m tall is filled with an oil of density 900 kg/m^3.
   a. How much work is needed to pump all of the oil over the top edge of the tank?
   b. How much work is needed to pump the top 10 m^3 of oil over the top edge of the tank?
   c. How long does it take for a 200-watt pump to empty the tank?
4. A cylindrical aquarium with radius 2 feet and height 5 feet:



is filled with salt water (which has a density of 64 pounds per cubic foot).
   a. How much work is done pumping all of the water over the top edge of the aquarium?
   b. How long does it take for a $\frac{1}{2}$-horsepower pump to empty the tank? A $\frac{1}{4}$-horsepower pump? Which pump does more work?
   c. If the aquarium is only filled to a height of 4 feet with sea water, how much work is required to empty it?
5. A cylindrical barrel with a radius of 1 m and a height of 6 m is filled with water.
   a. How much work is done pumping all of the water over the top edge of the barrel?
   b. How much work is done pumping the top 1 m of water to a point 2 m above the top edge of the barrel?

c. How long will it take a $\frac{1}{2}$-horsepower pump to remove half of the water from the barrel?

6. The conical container shown below:



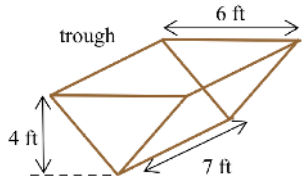
is filled with oats that weigh 25 pounds per ft^3.

a. How much work is done lifting all of the grain over the top edge of the cone?

b. How much work is required to lift the top 2 feet of grain over the top edge of the cone?

7. If you and a friend share the work equally in emptying the conical container in the previous Problem, what depth of grain should the first person leave for the second person to empty?

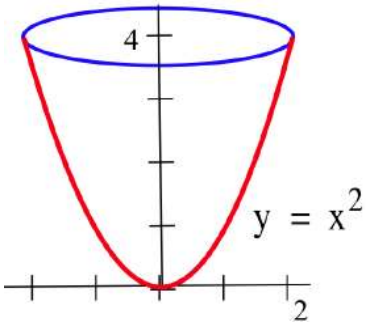
8. A trough is filled with pig slop weighing 80 pounds per ft^3:



How much work is done lifting all the slop over the top of the trough?

9. In the preceding problem, how much work is done lifting the top 14 ft^3 of slop over the top edge of the trough?

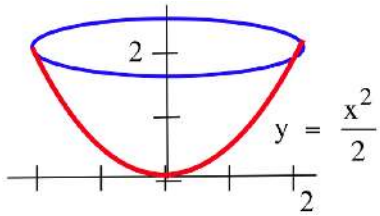
10. The parabolic container shown below (with a height of 4 m) is filled with water:



a. How much work is done pumping all of the water over the top edge of the tank?

b. How much work is done pumping all of the water to a point 3 m above the top of the tank?

11. The parabolic container shown below (with a height of 2 m) is filled with water:

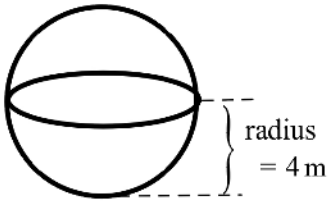


a. How much work is done pumping all of the water over the top edge of the tank?

b. How much work is done pumping all of the water to a point 3 m above the top of the tank?

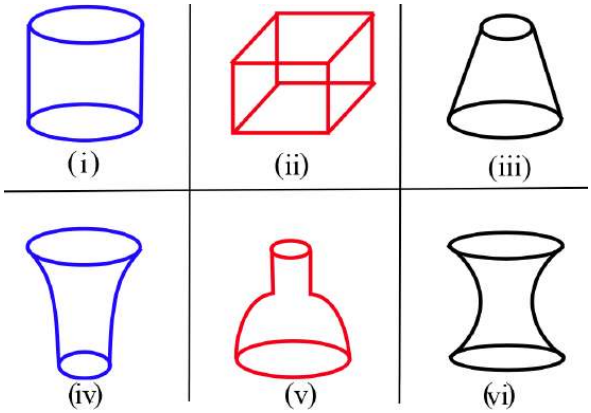
12. A spherical tank with radius 4 m is full of water. How much work is done lifting all of the water to the top of the tank?

13. The spherical tank shown below is filled with water to a depth of 2 m:



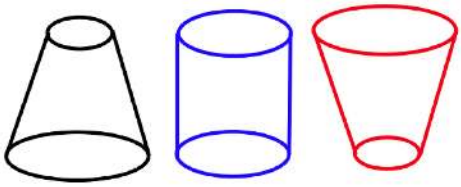
How much work is done lifting all of that water to the top of the tank?

14. A student said, “I’ve got a shortcut for these tank problems, but it doesn’t always work. I figure the weight of the liquid and multiply that by the distance I have to move the ‘middle point’ in the water. It worked for the first five problems and then it didn't.”



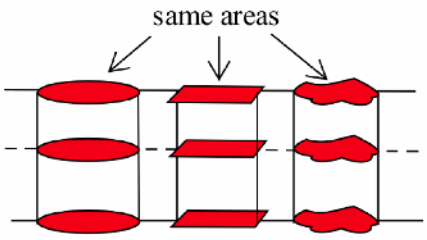
a. Does this “shortcut” really give the right answer for the first five problems?
b. How do the containers in the first five problems differ from the others?
c. For which of the containers shown below will the “shortcut” work?

15. All of the containers shown below have the same height and hold the same volume of water:



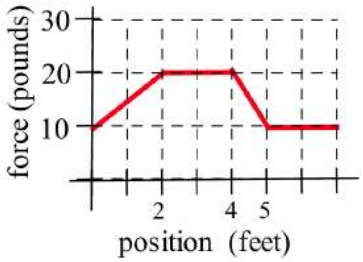
a. Which requires the most work to empty? Justify your response with a detailed explanation.
b. Which requires the least work to empty?

16. All of the containers shown below have the same total height and at each height x above the ground they all have the same cross-sectional area:



a. Which requires the most work to empty? Justify your response with a detailed explanation.
b. Which requires the least work to empty?

17. The figure below shows the force required to move a box along a rough surface:



How much work is done pushing the box:

a. from $x = 0$ to $x = 5$ feet?
b. from $x = 3$ to $x = 5$ feet?

18. How much work is done pushing the box in the figure from the previous Problem:

a. from $x = 3$ to $x = 7$ feet?
b. from $x = 0$ to $x = 7$ feet?

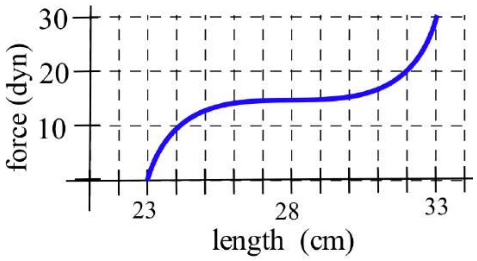
19. A spring requires a force of $6x$ ounces to keep it stretched x inches past its natural length. How much work is done stretching the spring:

a. from its natural length ($x = 0$) to 3 inches beyond its natural length?
b. from its natural length to 6 inches beyond its natural length?

20. A spring requires a force of $5x$ dyn to keep it compressed x cm from its natural length. How much work is done compressing the spring:

a. 7 cm from its natural length?
b. 10 cm from its natural length?

21. The figure below shows the force required to keep a spring that does not obey Hooke's Law stretched beyond its natural length of 23 cm:



About how much work is done stretching it:

a. from a length of 23 cm to a length 33 cm?
b. from a length of 28 cm to a length 33 cm?

22. Approximately how much work is done stretching the defective spring in the previous Problem:

a. from a length of 23 cm to a length 26 cm?
b. from a length of 30 cm to a length 35 cm?

23. A 3-kg object attached to a spring hung from the ceiling stretches the spring 15 cm. How much work is done stretching the spring 4 more cm?
24. A 2-lb fish stretches a spring 3 in. How much work is done stretching the spring 3 more inches?
25. A payload of mass 100 kg sits on the surface of the asteroid Ceres, a dwarf planet that is the largest object in the asteroid belt between Mars and Jupiter. Ceres has diameter 950 km and mass 896×10^{18} kg. How much work is required to lift the payload from the asteroid's surface to an altitude of:

a. 10 km?
b. 100 km?
c. 500 km?

26. Calculate the amount of work required to lift *you* from the surface of the Earth's moon to an altitude of 100 km above the moon's surface. (The moon's radius is approximately 1,737.5 km and its mass is about 7.35×10^{22} kg.)
27. Calculate the amount of work required to lift *you* from the surface of the Earth's moon (see previous Problem) to an altitude of:
    a. 200 km.
    b. 400 km.
    c. 10,000 km.
28. An object located at the origin repels you with a force inversely proportional to your distance from the object (so that $f(x) = \frac{k}{x}$ where x is your distance from the object, measured in feet). When you are 10 feet away from the origin, the repelling force is 0.1 pound. How much work must you do to move:
    a. from $x = 20$ to $x = 10$?
    b. from $x = 10$ to $x = 1$?
    c. from $x = 1$ to $x = 0.1$?
29. An object located at the origin repels you with a force inversely proportional to the square of your distance from the object (so that $f(x) = \frac{k}{x^2}$ where x is your distance from the object, measured in meters). When you are 10 m away from the origin, the repelling force is 0.1 N. How much work must you do to move:
    a. from $x = 20$ to $x = 10$?
    b. from $x = 10$ to $x = 1$?
    c. from $x = 1$ to $x = 0.1$?
30. A student said "I've got a 'work along a line' shortcut that always seems to work. I figure the average force and then multiply by the total distance. Will it always work?"
    a. Will it? Justify your answer. (Hint: What is the formula for "average force"?)
    b. Is this a shortcut?

Work Along a Curved Path

If the location of a moving object is defined parametrically as $x = x(t)$ and $y = y(t)$ for $a \le t \le b$ (where t often represents time), and the force required to overcome friction at time t is given as $f(t)$, we can represent the work done moving along the (possibly curved) path as a definite integral. Partitioning $[a, b]$ into n subintervals of the form $[t_{k-1}, t_k]$, we can choose any c_k in $[t_{k-1}, t_k]$ and approximate the force required on $[t_{k-1}, t_k]$ by $f(c_k)$ so that the work done between $t = t_{k-1}$ and $t = t_k$ is approximately:

$$f(c_k) \cdot \sqrt{[\Delta x_k]^2 + [\Delta y_k]^2} = f(c_k) \cdot \sqrt{\left[\frac{\Delta x_k}{\Delta t_k}\right]^2 + \left[\frac{\Delta y_k}{\Delta t_k}\right]^2} \cdot \Delta t_k$$

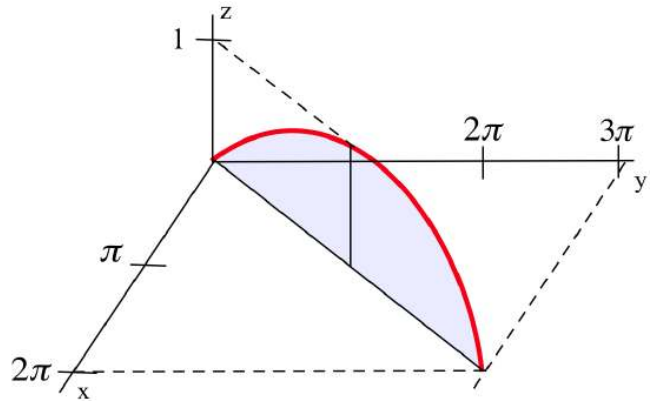
The total work done between times $t = a$ and $t = b$ is then:

$$\sum_{k=1}^{n} f(c_k) \cdot \sqrt{\left[\frac{\Delta x_k}{\Delta t_k}\right]^2 + \left[\frac{\Delta y_k}{\Delta t_k}\right]^2} \cdot \Delta t_k \longrightarrow \int_a^b f(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

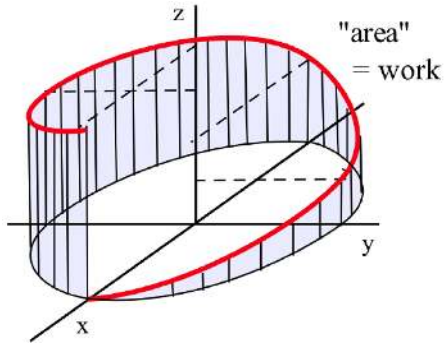
In Problems 31–35, find the work done as an object is moved along the given parametric path (with distance measured in meters), where $f(t)$ (in newtons) is the force required at time t (in seconds). If necessary, approximate the value of the integral using technology.

31. $f(t) = t$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$
32. $f(t) = t$, $x(t) = t$, $y(t) = t^2$, $0 \le t \le 1$
33. $f(t) = t$, $x(t) = t^2$, $y(t) = t$, $0 \le t \le 1$

34. $f(t) = \sin(t)$, $x(t) = 2t$, $y(t) = 3t$, $0 \le t \le \pi$:



35. $f(t) = t$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$:

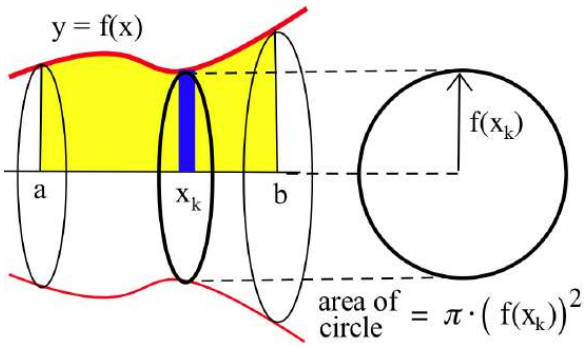


(Can you find a geometric way to calculate the shaded area?)

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5.5: Volumes: Tubes

In Section 5.2, we devised the "disk" method to find the volume swept out when a region is revolved about a line. To find the volume swept out when revolving a region about the x-axis:



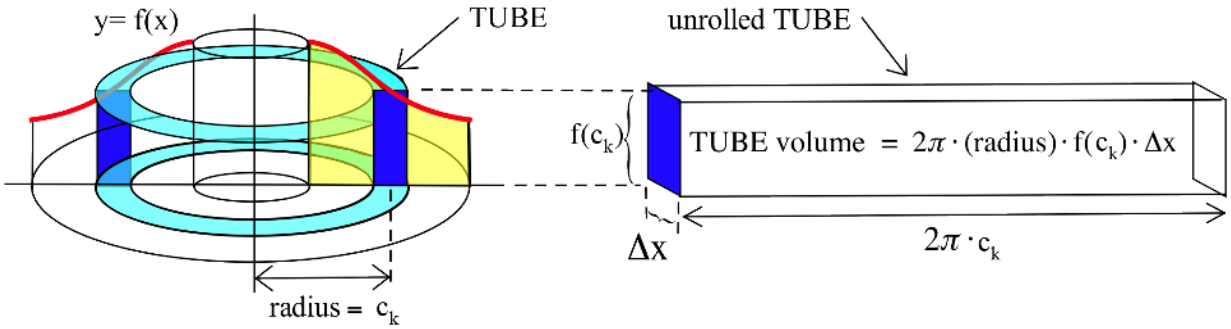
we made cuts perpendicular to the x-axis so that each slice was (approximately) a "disk" with volume $\pi\,(\text{radius})^2 \cdot (\text{thickness})$. Adding the volumes of these slices together yielded a Riemann sum. Taking a limit as the thicknesses of the slices approached 0, we obtained a definite integral representation for the exact volume that had the form:

$$\int_a^b \pi\,[f(x)]^2\,dx$$

The disk method, while useful in many circumstances, can be cumbersome if we want to find the volume when a region defined by a curve of the form $y = f(x)$ is revolved about the y-axis or some other vertical line. To revolve the region about the y-axis, the disk method requires that we rewrite the original equation $y = f(x)$ as $x = g(y)$. Sometimes this is easy: if $y = 3x$ then $x = \frac{y}{3}$. But sometimes it is not easy at all: if $y = x + e^x$, then we cannot solve for x as an elementary function of y. (Refer to Examples 6(b) and 6(c) from Section 5.2 to refresh your memory.)

The "Tube" Method

Partition the x-axis (as we did in the "disk" method) to cut the region into thin, almost-rectangular vertical "slices." When we revolve one of these slices about the y-axis (see below), we can approximate the volume of the resulting "tube" by cutting the "wall" of the tube and rolling it out flat:



to get a thin, solid rectangular box. The volume of the tube is approximately the same as the volume of the solid box:

$$\begin{aligned} V_{\text{tube}} \approx V_{\text{box}} &= (\text{length}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= (2\pi \cdot [\text{radius}]) \cdot (\text{height}) \cdot (\Delta x_k) \\ &= (2\pi c_k)\left(f(c_k)\right) \cdot \Delta x_k \end{aligned}$$

where c_k is (as usual) any point chosen from the interval $[x_{k-1}, x_k]$.

The volume swept out when we revolve the whole region about the y-axis is (approximately) the sum of the volumes of these "tubes," which is a Riemann sum that converges to a definite integral:

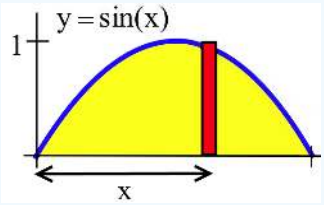
$$\sum_{k=1}^{n} (2\pi c_k)\left(f(c_k)\right)\cdot \Delta x_k \longrightarrow \int_a^b 2\pi x \cdot f(x)\, dx$$

Example 5.5.1

Use a definite integral to represent the volume of the solid generated by rotating the region between the graph of $y = \sin(x)$ (for $0 \le x \le \pi$) and the x-axis around the y-axis.

Solution

Slicing this region vertically yields slices with width Δx and height $\sin(x)$:



Rotating a slice located x units away from the y-axis results in a "tube" with volume:

$$2\pi\,(\text{radius})\,(\text{height})\,(\text{thickness}) = 2\pi\,(x)\left(\sin(x)\right)\Delta x$$

where the radius of the tube (x) is the distance from the slice to the y-axis and the height of the tube is the height of the slice ($\sin(x)$). Adding the volumes of all such tubes yields a Riemann sum that converges to a definite integral:

$$\int_0^{\pi} 2\pi\,(\text{radius})\,(\text{height})\,dx = \int_0^{\pi} 2\pi x \sin(x)\, dx$$

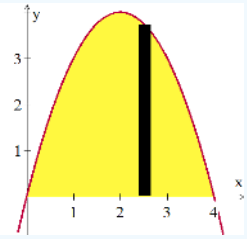
We don't (yet) know how to find an antiderivative for $x\sin(x)$ but we can use technology (or a numerical method from Section 4.9) to compute the value of the integral, which turns out to be $2\pi^2 \approx 19.74$. (You'll learn how to find an antiderivative for $x\sin(x)$ in Section 8.2 or, for now, you can look for this pattern in Appendix I.)

Practice 5.5.1

Use a definite integral to compute the volume of the solid generated by rotating the region in the first quadrant bounded by $y = 4x - x^2$ about the y-axis.

Answer

Graph the region:



and note that the curve $y = 4x - x^2$ intersects the x-axis where $4x - x^2 = 0 \Rightarrow x(4 - x) = 0 \Rightarrow x = 0$ or $x = 4$. Rotating a vertical slice around the y-axis results in a tube with radius x (the distance between the slice and the y-axis) and height $4x - x^2$ so the volume of the solid is:

$$\begin{aligned}\int_0^4 2\pi x\left(4x - x^2\right)\, dx &= 2\pi \int_0^4 \left[4x^2 - x^3\right]\, dx \\ &= 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = \frac{128\pi}{3} \approx 134\end{aligned}$$

If we had sliced the region in Example 1 horizontally instead of vertically, the rotated slices would have resulted in "washers"; applying the "washer" method from Section 5.2 yields the integral:

$$\int_0^1 \pi \left[(\pi - \arcsin(y))^2 - (\arcsin(y))^2\right]\, dy$$

The value of this integral is also $2\pi^2$, but finding an antiderivative for this integrand will be much more challenging than finding an antiderivative for $x\sin(x)$. (Furthermore, the washer-method integral in this situation is more challenging to set up than the integral using the tube method, so the tube method is the most efficient choice on all counts.)

Rotating About Other Axes

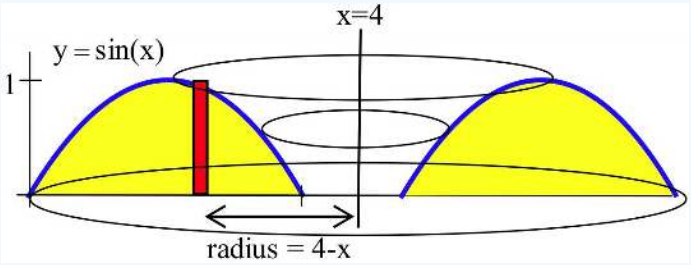
The "tube" method extends easily to solids generated by rotating a region about any vertical line (not just the y-axis).

Example 5.5.2

Use a definite integral to represent the volume of the solid generated by rotating the region between the graph of $y = \sin(x)$ (for $0 \le x \le \pi$) and the x-axis around the line $x = 4$.

Solution

The region is the same as the one in Example 1, but here we're rotating that region about a different vertical line:



Vertical slices again generate tubes when rotated about $x = 4$; the only difference here is that the radius for a slice located x units away from y-axis is now $4 - x$ (the distance from the axis of rotation to the slice). The volume integral becomes:

$$\int_0^{\pi} 2\pi\,(\text{radius})\,(\text{height})\, dx = \int_0^{\pi} 2\pi(4 - x)\cdot\sin(x)\, dx$$

which turns out to be $2\pi(8 - \pi) \approx 4.8584$. (Use technology—or a table of integrals—to verify this numerical result.)

Practice 5.5.2

Use a definite integral to compute the volume of the solid generated by rotating the region in the first quadrant bounded by $y = 4x - x^2$ about the line $x = -7$.

Answer

The region here is identical to the region in Practice 1, but we are now rotating a slice around the axis $x = -7$, so the radius of the resulting tube is $x - (-7) = x + 7$ (the distance from the slice at location x to the axis of rotation). The volume of the solid is therefore:

$$\int_0^4 2\pi(x + 7)\left(4x - x^2\right)\, dx = 2\pi \int_0^4 \left[28x - 3x^2 - x^3\right]\, dx$$
$$= 2\pi\left[14x^2 - x^3 - \frac{1}{4}x^4\right]_0^4 = 192\pi \approx 603$$

More General Regions

The “tube” method also extends easily to more general regions. (Many textbooks refer to this method as the “method of cylindrical shells” or the “shell method,” but “cylindrical shells” is a mouthful compared with “tube,” and “shell method” is not precise, as shells are not necessarily cylindrical.)

Volumes of Revolved Regions (“Tube Method”)

If: the region constrained by the graphs of $y = f(x)$ and $y = g(x)$ and the interval $[a, b]$ is revolved about a vertical line $x = c$ that does not intersect the region

then: the volume of the resulting solid is:

$$V = \int_a^b 2\pi \cdot |x - c| \cdot |f(x) - g(x)| \, dx$$

Note

The absolute values appear in the general formula because the radius and the height are both distances, hence both must be positive. You can ensure that these ingredients in your tube-method integral will be positive by always subtracting smaller values from larger values: think “$\textbf{right} - \textbf{left}$” for x-values and “$\textbf{top} - \textbf{bottom}$” for y-values.

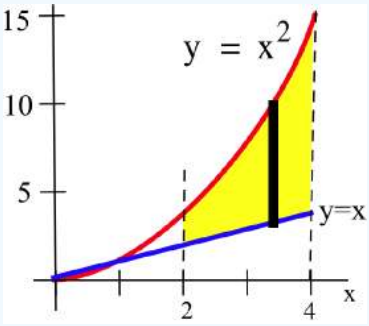
Example 5.5.3

Compute the volume of the solid generated by rotating the region between the graphs of $y = x$ and $y = x^2$ for $2 \le x \le 4$ around the y-axis using:

a. vertical slices and
b. horizontal slices.

Solution

a. Vertical slices result in tubes when rotated about the y-axis:

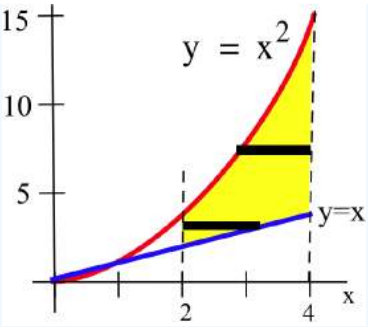


and a slice x units away from the y-axis results in a tube of radius x and height $x^2 - x$, so the volume of the solid is:

$$\int_2^4 2\pi x \left[x^2 - x\right] \, dx = 2\pi \int_2^4 \left[x^3 - x^2\right] \, dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_2^4$$
$$= 2\pi \left[\left(64 - \frac{64}{3}\right) - \left(4 - \frac{8}{3}\right)\right] = \frac{248\pi}{3}$$

or about 259.7.

b. Horizontal slices result in washers when rotated about the y-axis, but we have a new problem: the lower slices (where $2 \le y \le 4$) extend from the line $x = 2$ on the left to the line $y = x$ on the right, while the upper slices (where $4 \le y \le 16$) extend from the parabola $y = x^2$ on the left to the line $x = 4$ on the right.



This requires us to use *two* integrals to compute the volume:

$$\int_{y=2}^{y=4} \pi \left[y^2 - 2^2\right]\, dy + \int_{y=4}^{y=16} \pi \left[4^2 - (\sqrt{y})^2\right]\, dy$$

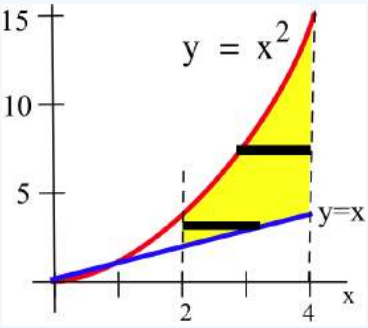
Evaluating these integrals also results in a volume of $\frac{248\pi}{3} \approx 259.7$. (Evaluating these integrals is straightforward, but setting them up was more time-consuming than using the tube method.)

Practice 5.5.3

Find the volume of the solid formed by rotating the region between the graphs of $y = x$ and $y = x^2$ for $2 \le x \le 4$ around $x = 13$.

Answer

Rotating a vertical slice about the line $x = 13$ results in a tube with radius $13 - x$ and height $x^2 - x$:



so the volume of the solid is:

$$\int_2^4 2\pi(13 - x)(x^2 - x)\, dx = 2\pi \int_2^4 \left[-13x + 14x^2 - x^3\right]\, dx$$
$$= 2\pi \left[-\frac{13}{2}x^2 + \frac{14}{3}x^3 - \frac{1}{4}x^4\right]_2^4 = 2\pi \left[\frac{392}{3} - \frac{10}{3}\right] = \frac{764\pi}{3} \approx 800$$

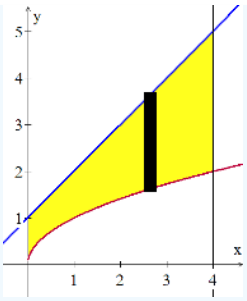
Practice 5.5.4

Compute the volume of the solid generated by rotating the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$, $y = x + 1$ and $x = 4$ around:

a. the y-axis
b. the x-axis.

Answer

Graph the region and draw a representative vertical slice:



(Horizontal slices would require splitting the region into two pieces—why?)

a. Rotating the vertical slice about the y-axis results in a tube of radius x (the distance from the slice to the y-axis) and height $(x+1)-\sqrt{x}$, and the region sits between $x=0$ and $x=4$ so the volume of the solid is:

$$\int_0^4 2\pi x\left[x+1-x^{\frac{1}{2}}\right]\,dx = 2\pi\int_0^4\left[x^2+x-x^{\frac{3}{2}}\right]\,dx$$
$$= 2\pi\left[\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \frac{496\pi}{15}\approx 104$$

b. Rotating the vertical slice around the x-axis results in a washer with big radius $x+1$ (the distance from the x-axis to the curve farthest from the x-axis) and small radius $\sqrt{x}$ (the distance from the x-axis to the closer curve) so the volume of the solid is:

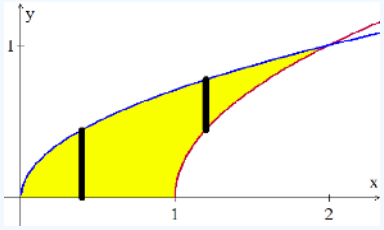
$$\int_0^4 \pi\left[(x+1)^2-\left(\sqrt{x}\right)^2\right]\,dx = \pi\int_0^4\left[x^2+x+1\right]\,dx = \tag{5.5.1}$$

Example 5.5.4

Compute the volume of the solid swept out by rotating the region in the first quadrant between the graphs of $y=\sqrt{\frac{x}{2}}$ and $y=\sqrt{x-1}$ about the x-axis.

Solution

Graphing the region:



it is apparent that the curves intersect where:

$$\sqrt{\frac{x}{2}}=\sqrt{x-1}\;\Rightarrow\;\frac{x}{2}=x-1\;\Rightarrow\;x=2$$

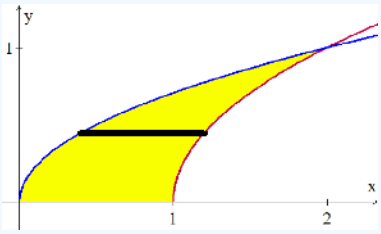
Slicing the region vertically results in two cases: when $0\le x\le 1$, the slice extends from the x-axis to the curve $y=\sqrt{\frac{x}{2}}$; when $1\le x\le 2$, the slice extends from $y=\sqrt{x-1}$ to $y=\sqrt{\frac{x}{2}}$. Rotating the first type of slice about the x-axis results in a disk; rotating the second type of slice about the x-axis results in a washer. Using the disk method for the first interval and the washer method for the second interval, the volume of the solid is:

$$\int_0^1 \pi\left[\sqrt{\frac{x}{2}}\right]^2\,dx+\int_1^2\pi\left[\left(\sqrt{\frac{x}{2}}\right)^2-\left(\sqrt{x-1}\right)^2\right]\,dx$$

(Both types of slices are perpendicular to the x-axis, so the width of each slice is of the form Δx and our integrals should involve dx.) Evaluating these integrals is straightforward:

$$\pi\int_0^1 \frac{x}{2}\,dx + \pi\int_1^2 \left[\frac{x}{2} - (x-1)\right]\,dx = \pi\left[\frac{x^2}{4}\right]_0^1 + \pi\left[x - \frac{x^2}{4}\right]_1^2 = \frac{\pi}{2}$$

If you had instead sliced the region horizontally, you would only need one type of slice:



Rotating a horizontal slice around the x-axis results in a tube. Because this slice is perpendicular to the y-axis, the thickness of the slice is of the form Δy, so the tube-method integral will include a dy and we will need to formulate the radius and "height" of the tube in terms of y. The radius of the slice is merely y, the distance between the slice and the x-axis. The "height" of the slice is its length, which is the distance between the two curves. The left-hand curve is:

$$y = \sqrt{\frac{x}{2}} \;\Rightarrow\; y^2 = \frac{x}{2} \;\Rightarrow x = 2y^2$$

and the right-hand curve is:

$$y = \sqrt{x-1} \;\Rightarrow\; y^2 = x - 1 \;\Rightarrow\; x = y^2 + 1$$

so the distance between the two curves is:

$$(y^2+1) - (2y^2) = 1 - y^2$$

The curves intersect where: $y^2 + 1 = 2y^2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$; from the graph we can see that the bottom of the region corresponds to $y = 0$ and the top of the region is at $y = 1$. Applying the tube method, the volume of the solid is:

$$\int_{y=0}^{y=1} 2\pi y \cdot \left[1 - y^2\right]\,dy = 2\pi\int_0^1 \left[y - y^3\right]\,dy = 2\pi\left[\frac{y}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

which agrees with the result above from the disk+washer method.

(This application of the tube method rotates a horizontal slice around a horizontal axis; in previous tube-method applications we have only rotated a vertical slice about a vertical axis. Either option results in a tube, and the general formula on page 433 can be further extended to this new situation—as we have done here—by swapping the roles of x and y.)

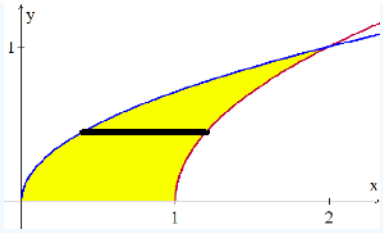
? Practice 5.5.5

Compute the volume of the solid swept out by rotating the region in the first quadrant between the graphs of $y = \sqrt{\frac{x}{2}}$ and $y = \sqrt{x-1}$ about:

a. the line $x = 5$
b. the line $y = 5$.

Answer

This region is the same as the one in Example 4, where it was apparent that slicing horizontally resulted in a single type of slice (compared with vertical slices, which required us to split the region into two pieces):



a. Rotating a horizontal slice around the vertical line $x = 5$ results in washers with thickness Δy (so our integral will involve dy), big radius $5 - 2y^2$ (the distance between the axis of rotation and the farthest curve) and small radius $5 - (y^2 + 1) = 4 - y^2$ (the distance between the axis of rotation and closest curve). Applying the washer method, the volume of the solid is:

$$\int_0^1 \pi \left[(5-2y^2)^2 - (4-y^2)^2 \right] dy = \frac{14\pi}{5} \approx 8.8$$

b. Rotating a horizontal slice around the horizontal line $y = 5$ results in a tube of radius $5 - y$ (the distance between the slice and the axis of rotation) and "height" $1 - y^2$ (the length of the slice). Applying the tube method, the volume of the solid is:

$$\int_0^1 2\pi(5-y)\left(1-y^2\right)\, dy = \frac{37\pi}{6} \approx 19.4$$

Which Method Is Best?

In theory, both the washer method and the tube method will work for any volume-of-revolution problem involving a horizontal or vertical axis. (We will investigate a method for computing volumes of solids formed by rotating a region around "tilted" axes in Section 5.6.)

In practice, however, one of these methods is usually easier to use than the other—but which one is easier depends on the particular region and type of axis. As we have seen, challenges may include:

- The necessity to split the region into two (or more) pieces, resulting in two (or more) integrals.
- The difficulty (or impossibility) of solving an equation of the form $y = f(x)$ for x or an equation of the form $x = g(y)$ for y.
- The difficulty (or impossibility) of finding an antiderivative for the resulting integrand.

With experience (and lots of practice) you will begin to develop an intuition for which method might be the best choice for a particular situation. Sketching the region along with representative horizontal and vertical slices is a vital first step.

The method that avoids the need to split the region up into more than one piece is often—but not always—the superior choice. Avoiding the need to find an inverse function for a boundary curve should also be a priority. Finally, if you need an exact value and one method results in a challenging antiderivative search, start over and try the other method.

Problems

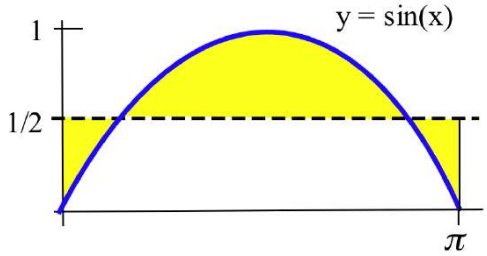
In Problems 1–6, sketch the region and calculate the volume swept out when the region is revolved about the specified vertical line.

1. The region in the first quadrant between the curve $y = \sqrt{1 - x^2}$ and the x-axis is rotated about the y-axis.
2. The region in the first quadrant between the curve $y = 2x - x^2$ and the x-axis is rotated about the y-axis.
3. The region in the first quadrant between between $y = 2x$ and $y = x^2$ for $0 \le x \le 3$ is rotated about the line $x = 4$.
4. The region in the first quadrant between the curve $y = \dfrac{1}{1 + x^2}$, the x-axis and the line $x = 3$ is rotated about the y-axis.
5. The region between $y = \dfrac{1}{x}$, $y = \dfrac{1}{3}$ and $x = 1$ is rotated about the line $x = 5$.
6. The region between $y = x$, $y = 2x$, $x = 1$ and $x = 3$ is rotated about the line $x = 1$.

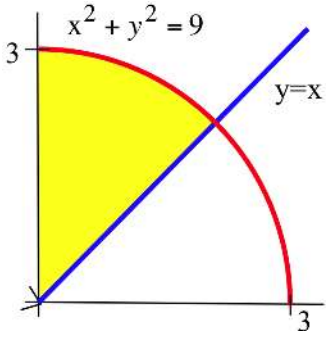
In Problems 7–11, use a definite integral to represent the volume swept out when the given region is revolved about the y-axis, then use technology to evaluate the integral.

7. The region in the first quadrant between the graphs of $y = \ln(x)$, $y = x$ and $x = 4$.

8. The region in the first quadrant between the graphs of $y = e^x$, $y = x$ and $x = 2$.
9. The region between $y = x^2$ and $y = 6 - x$ for $1 \le x \le 4$.
10. The shaded region in the figure below:



11. The shaded region in the figure below:



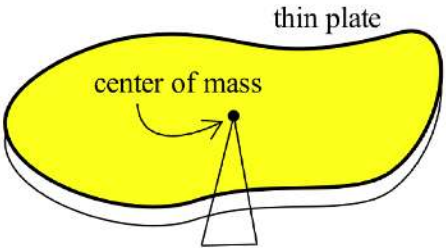
In Problems 12–30, set up an integral to calculate the volume swept out when the region between the given curves is rotated about the specified axis, using any appropriate method (disks, washers, tubes). If possible, work out an exact value of the integral; otherwise, use technology to find an approximate numerical value.

12. $y = x$, $y = x^4$, about the y-axis
13. $y = x^2$, $y = x^4$, about the y-axis
14. $y = x^2$, $y = x^4$, about the x-axis
15. $y = \sin(x^2)$, $y = 0$, $x = 0$, $x = \sqrt{\pi}$, about $x = 0$
16. $y = \cos(x^2)$, $y = 0$, $x = 0$, $x = \frac{\sqrt{\pi}}{2}$, about $x = 0$
17. $y = \dfrac{1}{\sqrt{1 - x^2}}$, $y = 0$, $x = 0$, $x = \frac{1}{2}$, about $x = 0$
18. $y = \dfrac{1}{\sqrt{1 - x^2}}$, $y = 0$, $x = 0$, $x = \frac{1}{2}$, about $y = 0$
19. $y = x$, $y = x^4$, about $x = 3$
20. $y = x$, $y = x^4$, about $y = 3$
21. $y = x$, $y = x^4$, about $y = -3$
22. $y = x$, $y = x^4$, about $x = -3$
23. $y = \dfrac{1}{1 + x^2}$, $y = 0$, $x = 0$, $x = 1$ about $x = 2$
24. $y = \dfrac{1}{1 + x^2}$, $y = 0$, $x = 1$, $x = \sqrt{3}$, about $x = 2$
25. $y = \dfrac{1}{1 + x^2}$, $y = 1$, $x = 1$, about $x = -2$
26. $y = \dfrac{1}{1 + x^2}$, $y = \frac{1}{2}$, about $x = 1$
27. $y = \sqrt{x - 2}$, $y = \sqrt{x - 1}$, $y = 0$, $x = 3$, about $x = 4$
28. $y = \sqrt{x - 2}$, $y = \sqrt{x - 1}$, $y = 0$, $x = 3$, about $x = -4$
29. $y = \sqrt{x - 2}$, $y = \sqrt{x - 1}$, $y = 0$, $x = 3$, about $y = 4$
30. $y = \sqrt{x - 2}$, $y = \sqrt{x - 1}$, $y = 0$, $x = 3$, about $y = -4$

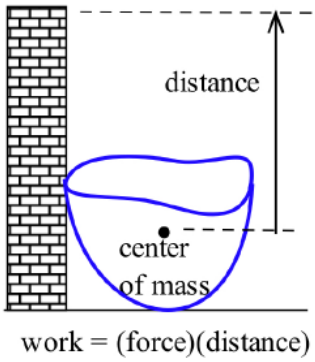
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5.6: Moments and Centers of Mass

This section develops a method for finding the center of mass of a thin, flat shape—the point at which the shape will balance without tilting:



Centers of mass are important because in many applied situations an object behaves as though its entire mass is located at its center of mass. For example, the work required to pump the water in a tank to a higher point is the same as the work required to move a small object with the same mass located at the tank's center of mass to the higher point, a much easier problem (if we know the mass and the center of mass of the water):

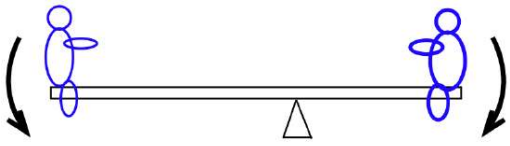


Volumes and surface areas of solids of revolution can also become easy to calculate if we know the center of mass of the region being revolved.

Point-Masses in One Dimension

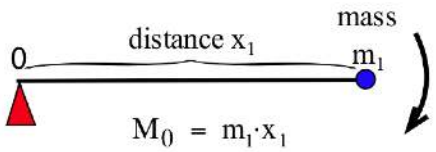
Before investigating the centers of mass of complicated regions, we consider **point-masses** (and systems of point-masses), first in one dimension and then in two dimensions.

Two people with different masses can position themselves on a seesaw so that the seesaw balances:



The person on the right causes the seesaw to "want to turn" clockwise about the fulcrum, and the person on the left causes it to "want to turn" counterclockwise. If these two "tendencies" are equal, the seesaw will balance on the fulcrum. A measure of this tendency to turn about the fulcrum is called the **moment** about the fulcrum of the system, and its magnitude is the product of the mass and the distance from the mass to the fulcrum. (In this seesaw example, we need to imagine that the seesaw is constructed using a very lightweight—yet sturdy—substance, so that its mass is negligible compared with the masses of the two people.)

In general, the **moment about the origin**, M_0, produced by a mass m_1 at a location x_1 is $m_1 \cdot x_1$, the product of the mass and the "signed distance" of the point-mass from the origin:



For a **system** of n masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively, the total mass of the system is:

$$m = m_1 + m_2 + \cdots + m_n = \sum_{k=1}^{n} m_k$$

and the moment about the origin of the system is:

$$M_0 = m_1 \cdot x_1 + m_2 \cdot x_2 + \cdots + m_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot x_k$$

If the moment about the origin is positive, then the system "tends to rotate" clockwise about the origin. If the moment about the origin is negative, then the system "tends to rotate" counterclockwise about the origin. If the moment about the origin is zero, then the system does not tend to rotate in either direction about the origin: it balances on a fulcrum located at the origin.

The **moment about a point $x = p$**, M_p, produced by a mass m_1 at location $x = x_1$ is the product of the mass and the signed distance of x_1 from the point p: $m_1 \cdot (x_1 - p)$. The moment about a point $x = p$ produced by masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively, is:

$$M_p = m_1 (x_1 - p) + m_2 (x_2 - p) + \cdots + m_n (x_n - p) = \sum_{k=1}^{n} m_k (x_k - p)$$

The point at which a system of point-masses balances is called the **center of mass** of the system, written $\overline{x}$ (pronounced "x-bar"). Because the system balances at $x = \overline{x}$, the moment about $\overline{x}$, $M_{\overline{x}}$, must be 0. (You have seen this "bar" notation before, in conjunction with the average value of a function. Here we can think of $\overline{x}$ as a "weighted average.")

Using this fact (and summation properties), we obtain a formula for $\overline{x}$:

$$\begin{aligned} 0 = M_{\overline{x}} &= \sum_{k=1}^{n} m_k \cdot (x_k - \overline{x}) = \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \left[\sum_{k=1}^{n} m_k \cdot \overline{x}\right] \\ &= \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \overline{x} \cdot \left[\sum_{k=1}^{n} m_k\right] = M_0 - \overline{x} \cdot m \end{aligned}$$

(we can factor $\overline{x}$ out of the second sum because it is constant) so $\overline{x} \cdot m = M_0$ and solving for $\overline{x}$ yields the following formula.

The **center of mass of a system of point-masses** $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$ is:

$$\overline{x} = \frac{M_0}{m} = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k}$$

A single point-mass with mass m (the total mass of the system) located at $\overline{x}$ (the center of mass of the system) produces the same moment about any point on the line as the whole system:

$$\begin{aligned} M_p &= \sum_{k=1}^{n} m_k (x_k - p) = \left[\sum_{k=1}^{n} m_k x_k\right] - p\left[\sum_{k=1}^{n} m_k\right] = M_0 - pm \\ &= m\left(\frac{M_0}{m} - p\right) = m(\overline{x} - p) \end{aligned}$$

For many purposes, we can think of the mass of the entire system as being "concentrated at $\overline{x}$."

Example 5.6.1

Find the center of mass of the system consisting of the first three point-masses listed in the table below:

k	m_k	x_k
1	2	-3
2	3	4
3	1	6
4	5	-2
5	3	4

Solution

$m = 2 + 3 + 1 = 6$ and $M_0 = (2)(-3) + (3)(4) + (1)(6) = 12$ so:

$$\overline{x} = \frac{M_0}{m} = \frac{12}{6} = 2$$

The system of three point-masses will balance on a fulcrum at $x = 2$.

Practice 5.6.1

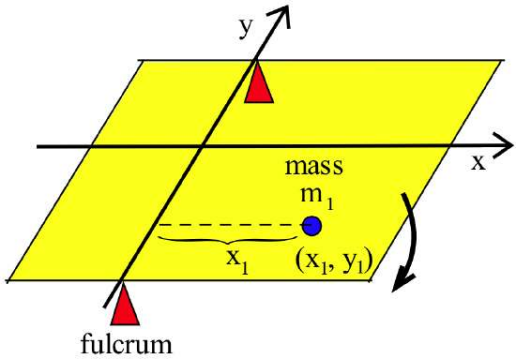
Find the center of mass of the system consisting of the last three point-masses listed in the table from Example 5.6.1.

Answer

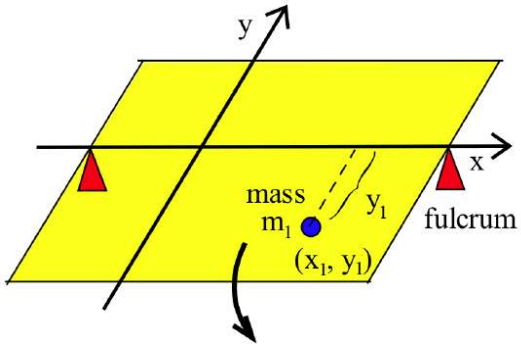
$m = 1 + 5 + 3 = 9$; $M_0 = (1)(6) + (5)(-2) + (3)(4) = 8$; \mbox{\)\displaystyle \overline{x} = \frac{M_0}{m} = \frac89\);} the three point-masses will balance on a fulcrum located at $\overline{x} = \frac{8}{9}$.

Point-Masses in Two Dimensions

The concepts of moments and centers of mass extend nicely from one dimension to a system of masses located at points in a plane. For a "knife edge" fulcrum located along the y-axis:



the moment of a point-mass with mass m_1 located at the point (x_1, y_1) is the product of the mass and the signed distance of the point-mass from the y-axis: $m_1 \cdot x_1$. This "tendency to rotate about the y-axis" is called the **moment about the y-axis**, written M_y. Here, $M_y = m_1 \cdot x_1$. Similarly, a point-mass with mass m_1 located at the point (x_1, y_1) has a **moment about the x-axis**: $M_x = m_1 \cdot y_1$



Note

As in the seesaw example, we need to imagine that the point-masses are sitting on a thin—yet strong—plate of negligible mass compared with the point-masses.

For a **system of masses** m_k located at the points (x_k, y_k), the total mass of the system is (as before):

$$m = m_1 + m_2 + \cdots + m_n = \sum_{k=1}^{n} m_k$$

while the moment about the y-axis is:

$$M_y = m_1 \cdot x_1 + m_2 \cdot x_2 + \cdots + m_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot x_k$$

and the moment about the x-axis is:

$$M_x = m_1 \cdot y_1 + m_2 \cdot y_2 + \cdots + y_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot y_k$$

At first, it may seem confusing that the formula for M_y would involve x and the formula for M_x would involve y, but keep in mind that an equation for the y-axis is $x = 0$, so we could write the moment about the y-axis as $M_{x=0}$ and the moment about the x-axis as $M_{y=0}$.

The **center of mass** of this two-dimensional system is a point $(\overline{x}, \overline{y})$ such that any line that passes through this point is a "balancing fulcrum" for the system. So we need the moment about any such line—including $x = \overline{x}$ and $y = \overline{y}$—to be zero:

$$0 = M_{x=\overline{x}} = \sum_{k=1}^{n} m_k \left(x_k - \overline{x}\right) = \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \overline{x}\left[\sum_{k=1}^{n} m_k\right] = M_y - \overline{x}m$$

so $\overline{x} = \dfrac{M_y}{m}$, and similar arithmetic shows that $\overline{y} = \dfrac{M_x}{m}$.

If we can find such a point, then the system will balance on a single "point-fulcrum" located at the center of mass.

The **center of mass of a system of point-masses** m_1, m_2,…, m_n at locations (x_1, y_1), (x_2, y_2),…, (x_n, y_n) is the point $(\overline{x}, \overline{y})$ where:

$$\overline{x} = \frac{M_y}{m} = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{\sum_{k=1}^{n} m_k \cdot y_k}{\sum_{k=1}^{n} m_k}$$

A single point-mass with mass m (the total mass of the system) located at $(\overline{x}, \overline{y})$ (the center of mass of the system) produces the same moment about any line as the whole system does about that line. (The arithmetic needed to prove this statement is similar to

arithmetic we did to prove the corresponding assertion for a one-dimensional system.) For many purposes, we can think of the mass of the entire system being "concentrated at $(\overline{x}, \overline{y})$."

Example 5.6.2

Find the center of mass of the system consisting of the first three point-masses listed in the table below:

k	m_k	x_k	y_k
1	2	-3	4
2	3	4	-7
3	1	6	-2
4	5	-2	1
5	3	4	-6

Solution

$m = 2+3+1 = 6$ and $M_y = (2)(-3)+(3)(4)+(1)(6) = 12$ while $M_x = (2)(4)+(3)(-7)+(1)(-2) = -15$ so:

$$\overline{x} = \frac{M_y}{m} = \frac{12}{6} = 2 \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{-15}{6} = -2.5$$

The system of three point-masses will balance on any fulcrum passing through the point $(2, -2.5)$.

Practice 5.6.2

Find the center of mass of the system consisting of all five point-masses listed in the table from Example 5.6.2.

Answer

$m = 2+3+1+5+3 = 14$

$M_y = (2)(-3)+(3)(4)+(1)(6)+(5)(-2)+(3)(4) = 14$

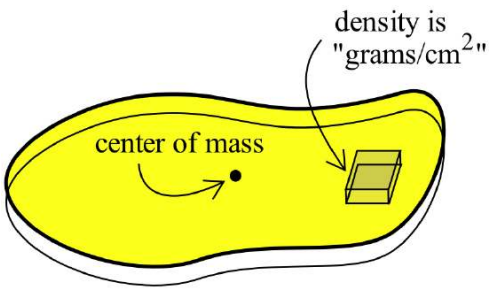
$M_x = (2)(4)+(3)(-7)+(1)(-2)+(5)(1)+(3)(-6) = -28$

$\overline{x} = \frac{M_y}{m} = \frac{14}{14} = 1$ and $\overline{y} = \frac{M_x}{m} = \frac{-28}{14} = -2$

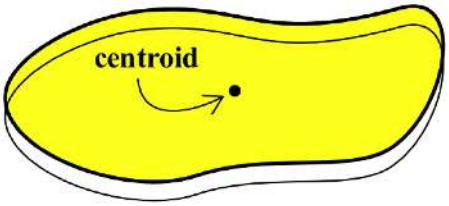
The five point-masses balance at the point $(1, -2)$.

Centroid of a Region

When we move from discrete point-masses to continuous regions in a plane, we move from finite sums and arithmetic to limits of Riemann sums, definite integrals and calculus. The following discussion extends ideas and calculations from point-masses to uniformly thin, flat plates (called **lamina**) that have a uniform density throughout (given as mass per area, such as "grams per cm ^{2}"). The center of mass of one of these plates is the point $(\overline{x}, \overline{y})$ at which the plate balances without tilting.

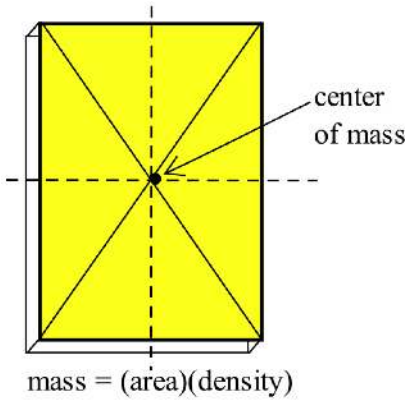


It turns out that for plates with uniform density, the center of mass $(\overline{x}, \overline{y})$ depends only on the shape (and location) of the region of the plane covered by the plate and not on the (constant) density. In these uniform-density situations, we call the center of mass the **centroid** of the region. Throughout the following discussion, you should notice that each finite sum that appeared in the discussion of point-masses has an integral counterpart for these thin plates.



Rectangles

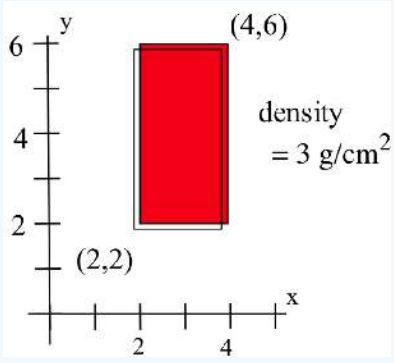
The components of a Riemann sum typically involve areas of rectangles, so it should come as no surprise that the basic shape used to extend point-mass concepts to regions is the rectangle. The total mass of a rectangular plate is the product of the area of the plate and its (constant) density: $m = \text{mass} = (\text{area}) \cdot (\text{density})$. We will assume that the center of mass of a thin, rectangular plate is located halfway up and halfway across the rectangle, at the point where the diagonals of the rectangle cross:



The moments of the rectangle about an axis can be found by treating the rectangle as a single point-mass with mass m located at the center of mass of the rectangle.

Example 5.6.3

Find the moments about the x-axis, y-axis and the line $x = 5$ of the thin, rectangular plate shown below:



Solution

The density of the plate is 3 g/cm² and the area of the plate is $(2\,\text{cm})\,(4\,\text{cm}) = 8\,\text{cm}^2$ so the total mass is:

$$m = \left(8\ \text{cm}^2\right)\left(3\ \frac{\text{g}}{\text{cm}^2}\right) = 24\ \text{g}$$

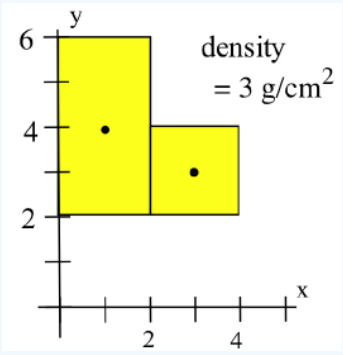
https://math.libretexts.org/@go/page/212049

The center of mass of the rectangular plate is $(\overline{x}, \overline{y}) = (3, 4)$. The moment about the x-axis is the product of the mass and the signed distance of the mass from the x-axis: $M_x = (24\,\text{g})\,(4\,\text{cm}) = 96\,\text{g-cm}$. Similarly, $M_y = (24\,\text{g})\,(3\,\text{cm}) = 72\,\text{g-cm}$. The moment about the line $x = 5$ is $M_{x=5} = (24\,\text{g})\,([5-3]\,\text{cm}) = 48\,\text{g-cm}$.

To find the moments and center of mass of a plate made up of several rectangular regions, we can simply treat each of the rectangular pieces as a point-mass concentrated at its center of mass, then treat the plate as a system of discrete point-masses.

Example 5.6.4

Find the centroid of the region in the figure below:



Solution

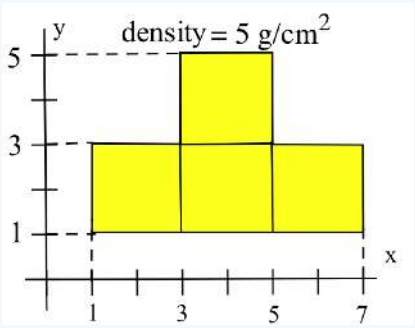
We can divide the plate into two rectangular plates, one with mass $24\ \text{g}$ and center of mass $(1, 4)$, and the other with mass $12\ \text{g}$ and center of mass $(3, 3)$. The total mass of the pair of point-masses is $m = 24 + 12 = 36\ \text{g}$, and the moments about the axes are $M_x = (24\ \text{g})\,(4\ \text{cm}) + (12\ \text{g})\,(3\ \text{cm}) = 132\ \text{g-cm}$ and $M_y = (24\ \text{g})\,(1\ \text{cm}) + (12\ \text{g})\,(3\ \text{cm}) = 60\ \text{g-cm}$. So:

$$\overline{x} = \frac{M_y}{m} = \frac{60\ \text{g-cm}}{36\ \text{g}} = \frac{5}{3}\ \text{cm} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{132\ \text{g-cm}}{36\ \text{g}} = \frac{11}{3}\ \text{cm}$$

The centroid of the plate is located at $\left(\frac{5}{3}, \frac{11}{3}\right)$.

Practice 5.6.3

Find the centroid of the region in the figure below:



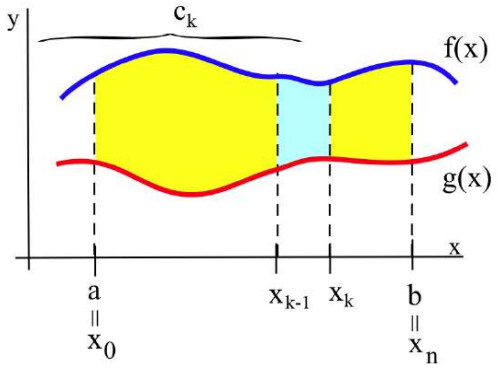
Answer

There are several ways to break the region into "easy" pieces—one way is to consider the four 2 cm-by-2 cm squares. The center of mass of each square is located at the center of the square (at $(2, 2)$, $(4, 2)$, $(6, 2)$ and $(4, 4)$), and each square has mass $(4\ \text{cm}^2)\left(5\ \frac{\text{g}}{\text{cm}^2}\right) = 20\ \text{g}$ so: $m = 4\,(20\,\text{g}) = 80\,\text{g}$, \)M_y = 2(20) + 4(20) + 6(20) + 4(20) = 320\, \mbox{g-cm}\) and $M_x = 2(20) + 2(20) + 2(20) + 4(20) = 200\ \text{g-cm}$. Therefore $\overline{x} = \frac{M_y}{m} = \frac{320\ \text{g-cm}}{80\ \text{g}} = 4\ \text{cm}$ and $\overline{y} = \frac{M_x}{m} = \frac{200\ \text{g-cm}}{80\ \text{g}} = 2.5\ \text{cm}$ so the center of mass is located at $(4, 2.5)$.

To find the center of mass of a thin, non-rectangular plate, we will "slice" the plate into narrow, almost-rectangular plates and treat the collection of almost-rectangular plates as a system of point-masses located at the centers of mass of the almost-rectangles. The total mass and moments about the axes for the system of point-masses will be Riemann sums. By taking limits as the widths of the almost-rectangles approach 0, we will obtain exact values for the mass and moments as definite integrals.

$\overline{x}$ for a Region

Suppose $f(x) \geq g(x)$ on the interval $[a, b]$ and $\mathcal{R}$ is a plate of uniform density $(= \rho)$ sitting on the region between the graphs of $f(x)$ and $g(x)$ and the lines $x = a$ and $x = b$:



If we partition the interval $[a, b]$ into n subintervals of the form $[x_{k-1}, x_k]$ and choose the points c_k to be the midpoints of these subintervals, then the slice between vertical cuts at $x = x_{k-1}$ and $x = x_k$ is approximately rectangular and has mass approximately equal to:

$$\begin{aligned} \text{(area) (density)} &= \text{(height) (width) (density)} \\ &\approx [f(c_k) - g(c_k)] \cdot (x_{k-1} - x_k) \cdot \rho \\ &= \rho [f(c_k) - g(c_k)] \, \Delta x_k \end{aligned}$$

So the mass of the whole plate is approximately

$$m = \sum_{k=1}^{n} \rho [f(c_k) - g(c_k)] \, \Delta x_k \longrightarrow \int_a^b \rho [f(x) - g(x)] \, dx = \rho \cdot A$$

where A is the area of the region $\mathcal{R}$.

(The Greek letter ρ (pronounced "row," as in "row your boat") is often used to represent the density of a region.)

The moment about the y-axis of each almost-rectangular slice is the product of the mass of the slice (m) and the distance from the centroid of the almost-rectangle to the y-axis. The x-coordinate of that centroid is located at $x = c_k$, so the distance from the centroid to the y-axis is $c_k - 0 = c_k$. The moment of the almost-rectangle about the y-axis is therefore:

$$m_k \cdot c_k = (\rho [f(c_k) - g(c_k)] \, \Delta x_k) \cdot c_k$$

so the moment of the entire plate about the y-axis is (approximately):

$$M_y = \sum_{k=1}^{n} \rho c_k \cdot [f(c_k) - g(c_k)] \, \Delta x_k \longrightarrow \int_a^b \rho x \cdot [f(x) - g(x)] \, dx$$

The x-coordinate of the centroid of the plate is therefore:

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x \cdot [f(x) - g(x)] \, dx}{\rho \int_a^b [f(x) - g(x)] \, dx} = \frac{\int_a^b x \cdot [f(x) - g(x)] \, dx}{\int_a^b [f(x) - g(x)] \, dx}$$

The density constant ρ is a factor of both M_y and m, so it cancels and has no effect on the value of $\overline{x}$. The value of $\overline{x}$ depends only on the shape and location of the region $\mathcal{R}$.

If the bottom boundary of $\mathcal{R}$ is the x-axis, then $g(x) = 0$ and the previous formulas simplify to:

$$m = \rho \int_a^b f(x)\,dx, \quad M_y = \rho \int_a^b x f(x)\,dx \quad \text{and} \quad \overline{x} = \frac{M_y}{m} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} \tag{5.6.1}$$

? Practice 5.6.4

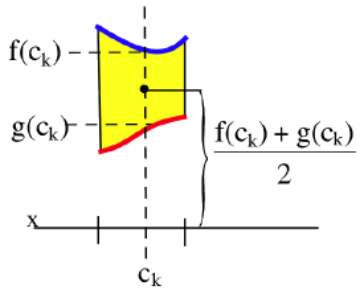
Find the x-coordinate of the centroid of the region between $f(x) = x^2$, the x-axis and $x = 2$.

Answer

For simplicity, let $\rho = 1$. Then the mass is $m = \int_0^2 x^2\,dx = \frac{8}{3}$ while \)\displaystyle M_y = \int_0^2 x\cdot x^2 \, dx = \int_0^2 x^3 \, dx = 4\) so \)\displaystyle \overline{x} = \frac{4}{\frac83} = \frac32 = 1.5\).

$\overline{y}$ for a Region

To find $\overline{y}$, the y-coordinate of the centroid of $\mathcal{R}$, we need to find M_x, the moment of $\mathcal{R}$ about the x-axis. For vertical partitions of $\mathcal{R}$:



the moment of each narrow strip about the x-axis, M_x, is the product of the strip's mass and the signed distance between the centroid of the strip and the x-axis. We've already computed the mass:

$$m_k = \rho\,[f(c_k) - g(c_k)]\,\Delta x_k$$

Because each strip is nearly rectangular, the centroid of the k-th strip is roughly halfway up the strip, at a point midway between $f(c_k)$ and $g(c_k)$, so we can average those function values to compute:

$$\overline{y}_k \approx \frac{f(c_k) + g(c_k)}{2}$$

The moment about the x-axis for this strip is thus:

$$\rho\,[f(c_k) - g(c_k)]\,\Delta x_k \cdot \left[\frac{f(c_k) + g(c_k)}{2}\right] = \frac{\rho}{2}\left[(f(c_k))^2 - (g(c_k))^2\right]\Delta x_k$$

Adding up the moments of all n strips yields:

$$M_x = \sum_{k=1}^{n} \frac{\rho}{2}\left[(f(c_k))^2 - (g(c_k))^2\right]\Delta x_k \longrightarrow \int_a^b \frac{\rho}{2}\left[(f(x))^2 - (g(x))^2\right]dx$$

The y-coordinate of the centroid of the plate is therefore:

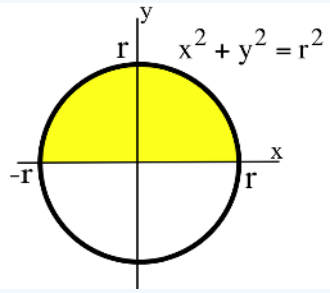
$$\overline{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]dx}{\rho \int_a^b [f(x) - g(x)]\,dx} = \frac{\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]dx}{\int_a^b [f(x) - g(x)]\,dx}$$

If the bottom boundary of $\mathcal{R}$ is the x-axis, then $g(x) = 0$ and the previous formulas simplify to:

$$M_x = \frac{\rho}{2}\int_a^b x\,[f(x)]^2\,dx \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}$$

Example 5.6.5

Find the y-coordinate of the centroid of the region $\mathcal{R}$ bounded below by the x-axis and above by the top half of a circle of radius r centered at the origin:



Solution

An equation for the circle is $x^2+y^2=r^2$ so the top half is given by $f(x)=y=\sqrt{r^2-x^2}$, and $g(x)=0$. The mass of the region is:

$$m=\int_{-r}^{r}\rho\sqrt{r^2-x^2}\,dx=\rho\int_{-r}^{r}\sqrt{r^2-x^2}\,dx=\rho\cdot[\text{area of }\mathcal{R}]=\rho\cdot\frac{\pi r^2}{2}$$

The moment of $\mathcal{R}$ about the y-axis is:

$$M_y=\int_{-r}^{r}\rho x\cdot\sqrt{r^2-x^2}\,dx=\left[-\frac{\rho}{3}\left(r^2-x^2\right)^{\frac{3}{2}}\right]_{x=-r}^{x=r}=0$$

so $\bar{x}=0$. (Could you have guessed this result merely by looking at the region?)

The moment of $\mathcal{R}$ about the x-axis is:

$$\begin{aligned}M_x&=\int_{-r}^{r}\frac{\rho}{2}\cdot\left[\sqrt{r^2-x^2}\right]^2dx=\frac{\rho}{2}\int_{-r}^{r}\left[r^2-x^2\right]dx\\&=\frac{\rho}{2}\left[r^2x-\frac{1}{3}x^3\right]_{-r}^{r}=\frac{\rho}{2}\cdot\frac{4}{3}r^3=\frac{2\rho}{3}r^3\end{aligned}$$

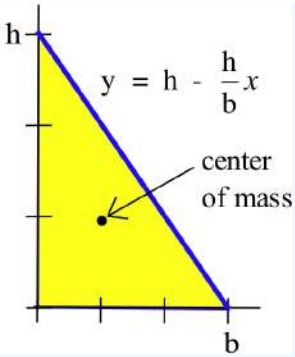
so $\bar{y}=\dfrac{\frac{2\rho}{3}r^3}{\frac{\rho\pi}{2}r^2}=\dfrac{4}{3\pi}r\approx 0.4244r$. (Could you have guessed that centroid would be located a bit less than halfway above the bottom edge of the semicircle, merely by looking at the region?)

Practice 5.6.5

Show that the centroid of a triangular region with vertices $(0,0)$, $(0,h)$ and $(b,0)$ is located at $(\bar{x},\bar{y})=\left(\frac{b}{3},\frac{h}{3}\right)$.

Answer

The triangular region appears below:



Here $f(x) = h - \frac{h}{b}x$ for $0 \le x \le b$ and $g(x) = 0$. The "mass" is just the area of the triangle, so $m = \frac{1}{2} \cdot b \cdot h$ while:

$$M_y = \int_0^b x\left[h - \frac{h}{b}x\right]\,dx = \int_0^b \left[hx - \frac{h}{b}x^2\right]\,dx = \left[\frac{h}{2}x^2 - \frac{h}{3b}x^3\right]_0^b = \frac{b^2h}{6}$$

and:

$$M_x = \int_0^b \frac{1}{2}\left[h - \frac{h}{b}x\right]^2\,dx = \left[\frac{1}{6}\left(-\frac{b}{h}\right)\left(h - \frac{h}{b}x\right)^3\right]_0^b = 0 + \frac{b}{6h} \cdot h^3 = \frac{bh^2}{6}$$

So $(\overline{x}, \overline{y}) = \left(\frac{\frac{b^2h}{6}}{\frac{bh}{2}}, \frac{\frac{bh^2}{6}}{\frac{bh}{2}}\right) = \left(\frac{b}{3}, \frac{h}{3}\right)$.

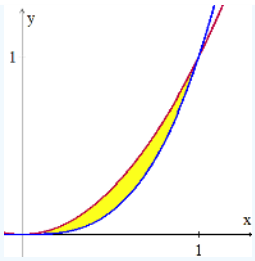
The following table summarizes and compares formulas for computing moments and centers of mass for a system of point-masses in a plane (using sums) and for a region in a plane (using integrals). The integral formulas appear in a form for calculating moments of a region $\mathcal{R}$ bounded by the graphs of two functions, $f(x)$ and $g(x)$, and two vertical lines, $x = a$ and $x = b$, where $f(x) \ge g(x)$ for $a \le x \le b$.

	point-masses in plane	**region $\mathcal{R}$ between f and g**
total mass:	$m = \sum_{k=1}^{n} m_k$	$m = \int_a^b \rho\left[f(x) - g(x)\right]\,dx = \rho \cdot \text{Area}\,(\mathcal{R})$
moment about y-axis ($x = 0$):	$M_y = \sum_{k=1}^{n} m_k \cdot x_k$	$M_y = \int_a^b \rho x \cdot \left[f(x) - g(x)\right]\,dx$
moment about x-axis ($y = 0$):	$M_x = \sum_{k=1}^{n} m_k \cdot y_k$	$M_x = \int_a^b \frac{\rho}{2}\left[(f(x))^2 - (g(x))^2\right]\,dx$
center of mass (ρ constant):	$\overline{x} = \frac{M_y}{m}$, $\overline{y} = \frac{M_x}{m}$	$\overline{x} = \frac{M_y}{m}$, $\overline{y} = \frac{M_x}{m}$

While the integral formulas above are often useful, it is important that you understand the process used to obtain these formulas in order to compute moments and centroids of more general regions. (With the knowledge of Riemann sums you have developed, you should be able to set up integrals to compute masses and moments for regions bounded by curves of the form $x = g(y)$, and deal with situations where the density of a thin plate is a function of x or y.)

Example 5.6.6

Find the centroid of the region $\mathcal{R}$ bounded by the graphs of $y = x^2$ and $y = x^3$:



Solution

The curves intersect where $x^2 = x^3 \Rightarrow x^2 - x^3 = 0 \Rightarrow x^2(1-x) = 0 \Rightarrow x = 0$ or $x = 1$. A graph (see margin) helps confirm that $x^2 \geq x^3$ on $[0, 1]$. If the density of $\mathcal{R}$ is ρ then the mass of $\mathcal{R}$ is:

$$m = \int_0^1 \rho \left[x^2 - x^3\right]\, dx = \rho \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\rho}{12}$$

The moment of $\mathcal{R}$ about the y-axis is:

$$M_y = \rho \int_0^1 x\left[x^2 - x^3\right]\, dx = \rho \int_0^1 \left[x^3 - x^4\right]\, dx = \rho \left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{\rho}{20}$$

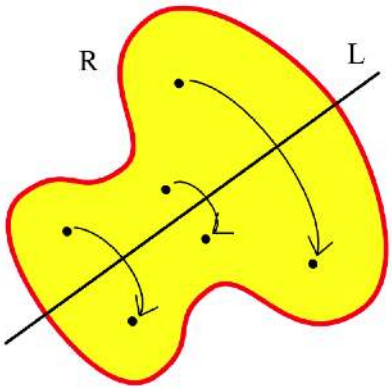
And the moment of $\mathcal{R}$ about the x-axis is:

$$M_x = \frac{\rho}{2} \int_0^1 \left[\left(x^2\right)^2 - \left(x^3\right)^2\right]\, dx = \frac{\rho}{2}\left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1 = \frac{\rho}{35}$$

so $\overline{x} = \dfrac{M_y}{m} = \dfrac{\frac{\rho}{20}}{\frac{\rho}{12}} = \dfrac{3}{5}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{\frac{\rho}{35}}{\frac{\rho}{12}} = \dfrac{12}{35}$. Plotting the point $\left(\frac{3}{5}, \frac{12}{35}\right) \approx (0.60, 0.34)$ along with $\mathcal{R}$ confirms that it sits inside $\mathcal{R}$ (just barely) and appears to be a reasonable candidate for the centroid.

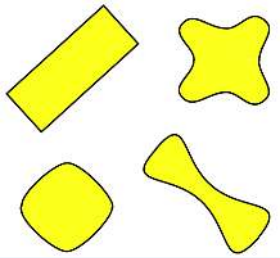
Symmetry

Symmetry is a very powerful geometric concept that can simplify many mathematical and physical problems, including the task of finding centroids of regions. For some regions, we can use symmetry alone to determine the centroid. Geometrically, a region $\mathcal{R}$ is **symmetric about a line L** if, when $\mathcal{R}$ is folded along L, each point of $\mathcal{R}$ on one side of the fold matches up with exactly one point of $\mathcal{R}$ on the other side of the fold:



Example 5.6.7

Sketch two lines of symmetry for each region shown in the figure below:



Solution

See solution to Practice 5.6.6.

A very useful fact about symmetric regions is that the centroid $(\overline{x},\overline{y})$ of a symmetric region must lie on every line of symmetry of the region.

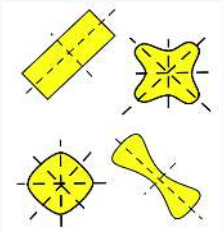
(In Example 5.6.5, the half-disk was symmetric with respect to the y-axis, so we could have avoided setting up and evaluating the M_y integral by noticing that $(\overline{x},\overline{y})$ must be located on the y-axis (the line $x=0$) and concluding that $\overline{x}=0$.} If a region has two different lines of symmetry, then the centroid must lie on each of them, so the centroid must be located at the point where the lines of symmetry intersect.)

? Practice 5.6.6

Locate the centroid of each region in Example 5.6.7.

Answer

The centroid of each region is located at the point where the lines of symmetry intersect:

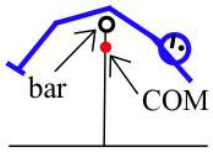


Work

In a uniform gravitational field, the **center of gravity** of an object is located at the same point as its center of mass, and the work done to lift an object is the product of the object's weight and the distance that the center of gravity of the object is raised:

$$\text{work} = (\text{object's weight})\,(\text{distance object's center of gravity is raised})$$

In the high jump, this explains the effectiveness of the "Fosbury Flop," a technique where the jumper assumes an inverted **U** position while going over the bar:

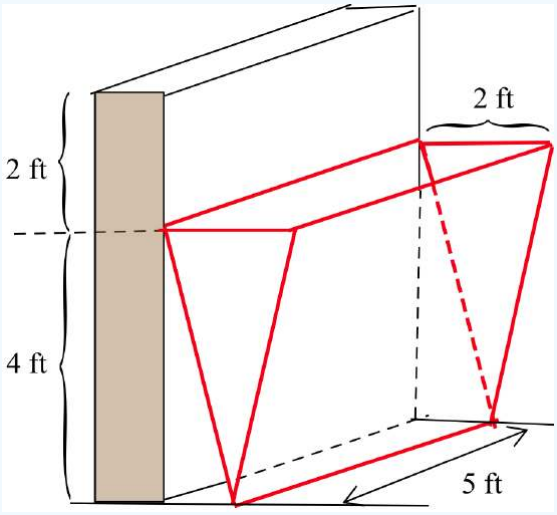


The jumper's body goes over the bar while the jumper's center of gravity goes under it, allowing the jumper to clear a higher bar with no additional upward thrust.

If you know the center of gravity of an object being lifted, some work problems become much easier.

Example 5.6.8

The trough shown below is filled with a liquid weighing 70 pounds per cubic foot:



How much work is done pumping the liquid over the wall next to the trough?

Solution

(We've already solved this problem, as Example 5 in Section 5.4, but here we try a new approach using centroids.)

This is a 3-D problem, but symmetry tells us the centroid of the liquid must be at a point 2.5 feet from either end of the trough, and 1 foot away from the wall. The vertical coordinate of the centroid will be the same as the centroid of the trough's triangular end region.

Using the result of Practice **5.6.5**, we can conclude that the centroid of the triangle is at a height of $\frac{2}{3} \cdot 4 = \frac{8}{3}$. The weight of the liquid is:

$$(\text{density}) \cdot (\text{volume}) = \left(70 \ \frac{\text{lb}}{\text{ft}^3}\right) \cdot \frac{1}{2} \ (5 \text{ ft}) \cdot (2 \text{ ft}) \cdot (4 \text{ ft}) = 1400 \text{ lbs}$$

and the distance the center of gravity must be moved is $6 - \frac{8}{3} = \frac{10}{3}$ ft so the total work required is:

$$(1400 \text{ lbs}) \cdot \left(\frac{10}{3} \text{ ft}\right) = \frac{14000}{3} \text{ ft-lbs} \approx 4666.7 \text{ ft-lbs}$$

which agrees with the answer obtained in Section 5.4.

Theorems of Pappus

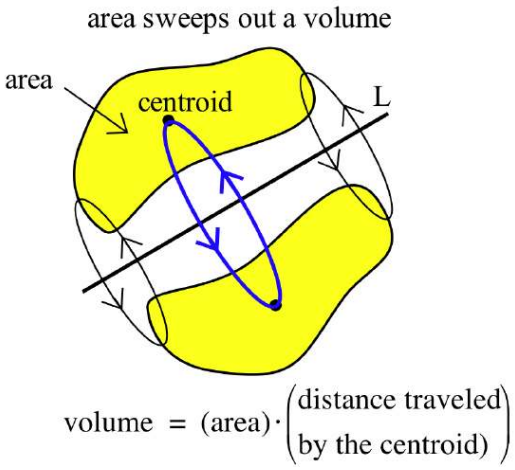
Two theorems due to Pappus of Alexandria can make some volume and surface area calculations relatively easy. (Pappus, the last of the great Greek geometers, flourished during the first half of the fourth century.)

Theorem of Pappus: Volume of Revolution

If: a plane region $\mathcal{R}$ with area A and centroid $(\overline{x}, \overline{y})$ is revolved around a line L in the plane that does not pass through $\mathcal{R}$ (touching the boundary is OK)

then: the volume swept out by one revolution of $\mathcal{R}$ is the product of A and the distance traveled by the centroid.

The distance from the centroid to the line will be the radius of the circle swept out by the centroid, so the distance traveled by the centroid is 2π times this radius. When L is the x-axis, the volume of the solid is $A \cdot 2\pi\overline{y}$; when L is the y-axis, the volume of the solid is $A \cdot 2\pi\overline{x}$.



Example 5.6.9

Find the volume swept out when the region $\mathcal{R}$ bounded by the graphs of $y = x^2$ and $y = x^3$ is revolved around the line $x = 2$.

Solution

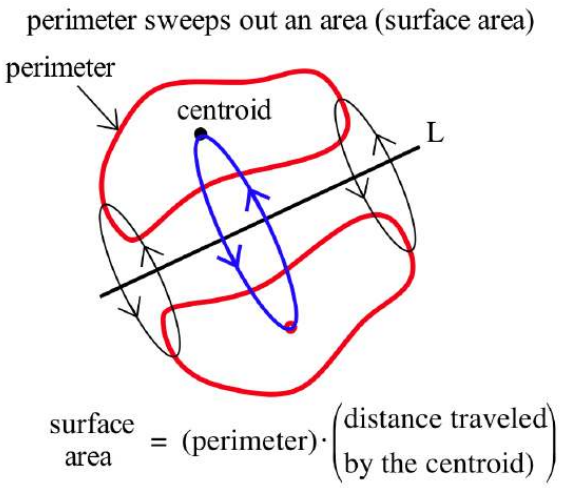
From Example 6, we know the area of $\mathcal{R}$ is $\frac{1}{12}$ and its centroid is $\left(\frac{3}{5}, \frac{12}{35}\right)$. The distance from this point to the line $x = 2$ is $2 - \frac{3}{5} = \frac{7}{5}$, so the distance traveled by the centroid is $2\pi \cdot \frac{7}{5} = \frac{14\pi}{5}$. The volume of the solid of revolution is therefore $\frac{1}{12} \cdot \frac{14\pi}{5} = \frac{7\pi}{30}$.

Theorem of Pappus: Surface Area of Revolution

If: a plane region $\mathcal{R}$ with perimeter P and centroid $(\overline{x}, \overline{y})$ is revolved around a line L in the plane that does not pass through $\mathcal{R}$ (touching the boundary is OK)

then: the surface area swept out by one revolution of $\mathcal{R}$ is the product of P and the distance traveled by the centroid.

When L is the x-axis, the surface area of the solid is $P \cdot 2\pi\overline{y}$; when L is the y-axis, the surface area is $P \cdot 2\pi\overline{x}$.



Example 5.6.10

Find the surface area of the solid swept out when the square region $\mathcal{R}$ with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ is revolved around the line $y = 3$.

Solution

By symmetry, the centroid of the square is $(0,0)$ and its distance from $y=3$ is 3. The perimeter of the square is $4\sqrt{2}$, so the surface area of the solid of revolution is $4\sqrt{2}\cdot 2\pi\cdot 3 = 24\pi\sqrt{3}$.

Problems

1. a. Find the total mass and the center of mass for a system consisting of the three point-masses in the table below:

m_k	2	5	5
x_k	4	2	6

   b. Where should you locate a new object with mass 8 so the new system has its center of mass at $x=5$?
   c. What mass should you put at $x=10$ so the original system plus the new mass has its center of mass at $x=6$?

2. a. Find the total mass and the center of mass for a system consisting of the four point-masses in the table below:

m_k	5	3	2	6
x_k	1	7	5	5

   b. Where should you locate a new object with mass 10 so the new system has its center of mass at $x=6$?
   c. What mass should you put at $x=14$ so the original system plus the new mass has its center of mass at $x=6$?

3. a. Find the total mass and the center of mass for a system consisting of the three point-masses in the table below.

m_k	2	5	5
x_k	4	2	6
y_k	3	4	2

   b. Where should you locate a new object with mass 10 so the new system has its center of mass at $(5,2)$?

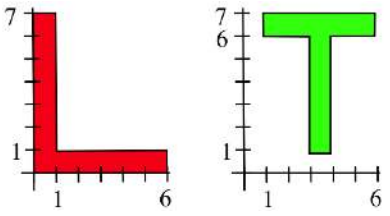
4. a. Find the total mass and the center of mass for a system consisting of the four point-masses in the table below.

m_k	5	3	2	6
x_k	1	7	5	5
y_k	4	7	0	8

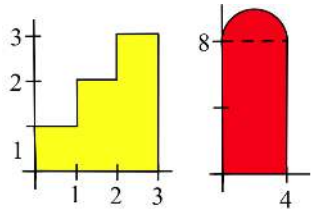
   b. Where should you locate a new object with mass 12 so the new system has its center of mass at $(3,5)$?

In Problems 5–10, divide the plate shown into rectangles and semicircles, calculate the mass, moments and centers of mass of each piece, then find the center of mass of the plate. Assume the density of the plate is $\rho=1$. Plot the location of the center of mass for each shape. (Refer to Example $5.6.5$ for centroids of semicircular regions.)

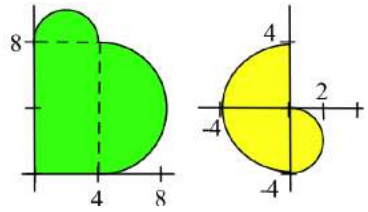
5. Use the figure below left.



6. Use the figure above right.
7. Use the figure below left.



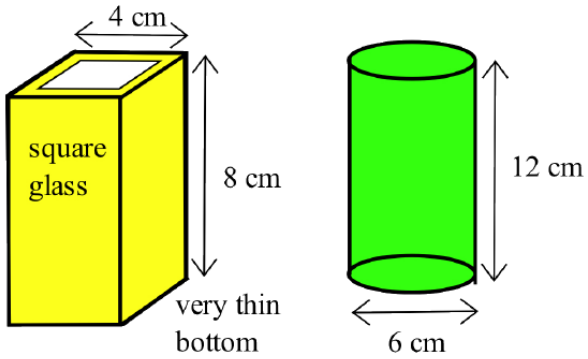
8. Use the figure above right.
9. Use the figure below left.



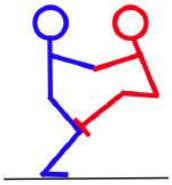
10. Use the figure above right.

In Problems 11–26, sketch the region bounded by the the given curves and find the centroid of each region (use technology to evaluate integrals, if necessary). Plot the location of the centroid on your sketch of the region.

11. $y = x$, the x-axis, $x = 3$
12. $y = x^2$, the x-axis, $x = -2$, $x = 2$
13. $y = x^2$, $y = 4$
14. $y = \sin(x)$, the x-axis, the y-axis, $x = \pi$
15. $y = 4 - x^2$ and the x-axis for $-2 \le x \le 2$
16. $y = x^2$, $y = x$
17. $y = 9 - x$, $y = 3$, $x = 0$, $x = 3$
18. $y = \sqrt{1 - x^2}$, the x-axis, $x = 0$, $x = 1$
19. $y = \sqrt{x}$, the x-axis, $x = 9$
20. $y = \ln(x)$, the x-axis, $x = e$
21. $y = e^x$, $y = e$ and the y-axis
22. $y = x^2$ and $y = 2x$
23. An empty box in the shape of a cube measuring 1 foot on each side weighs 10 pounds. By symmetry, we know its center of mass is 6 inches above its bottom. When the box is full of a liquid with density 60 lb/ft^3, the center of mass of the box-liquid system is again (due to symmetry) 6 inches above the bottom of the box.
    a. Find the height of the center of mass of the box-liquid system as a function of h, the height of water in the box.
    b. To what height should you fill the box so that the box-liquid system has the lowest center of gravity (and the greatest stability)?
24. The empty glass shown below left has a mass of 100 g when empty. Find the height of the center of mass of the glass-water system as a function of the height of water in the glass.



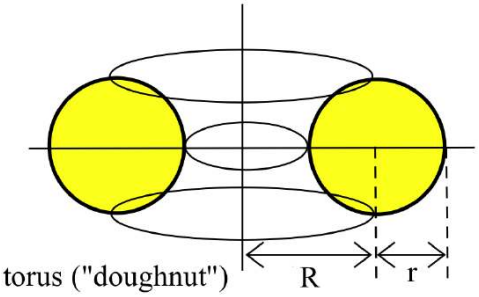
25. The empty soda can shown above right has a mass of 15 g when empty and 400 g when full of soda. Find the height of the center of mass of the can-soda system as a function of the height of the soda in the can.
26. Give a practical set of directions someone could actually use to find the height of the center of gravity of their body with their arms at their sides. How will the height of the center of gravity change if they lift their arms?
27. Try the following experiment. Stand straight with your back and heels against a wall. Slowly raise one leg, keeping it straight, in front of you. What happened? Why?
28. Why can't two dancers stand in the position shown below?



29. If a shape has exactly two lines of symmetry, the lines can meet at right angles. Must they meet at right angles?
30. Sketch regions with exactly two lines of symmetry, exactly three lines of symmetry, and exactly four lines of symmetry.
31. A rectangular box is filled to a depth of 4 feet with 300 pounds of water. How much work is done pumping the water to a point 10 feet above the bottom of the box?
32. A cylinder is filled to a depth of 2 feet with 40 pounds of water. How much work is done pumping the water to a point 7 feet above the bottom of the cylinder?
33. A sphere of radius 2 m is filled with water. How much work is done pumping the water to a point 3 m above the top of the sphere?
34. A sphere of radius 2 feet is filled with water. How much work is done pumping the water to a point 5 feet above the top of the sphere?
35. The center of a square region with sides of length 2 cm is located at the point $(3,4)$. Find the volume swept out when the square region is rotated:
    a. about the x-axis.
    b. about the y-axis.
    c. about the line $y=6$
    d. about the line $x=6$
    e. about the line $2x+3y=6$
36. The lower-left corner of a rectangular region with an 8-inch base and a 4-inch height is located at the point $(3,5)$. Find the volume swept out when the rectangular region is rotated:
    a. about the x-axis.
    b. about the y-axis.
    c. about the line $y=x+5$
37. The center of a square region with sides of length 2 cm is located at the point $(3,4)$. Find the surface area swept out when the square region is rotated:
    a. about the x-axis.
    b. about the y-axis.

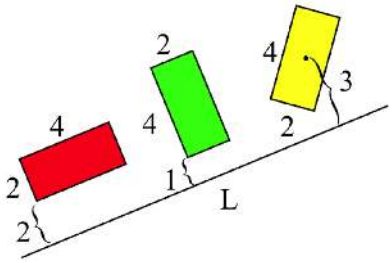
c. about the line $y = 6$
d. about the line $x = 6$
e. about the line $2x + 3y = 6$

38. The lower-left corner of a rectangular region with an 8-inch base and a 4-inch height is located at the point $(3, 5)$. Find the surface area swept out when the rectangular region is rotated:
    a. about the x-axis.
    b. about the y-axis.
    c. about the line $y = x + 5$
39. Find the volume and surface area swept out when the region inside the circle $(x - 3)^2 + (y - 5)^2 = 4$ is rotated:
    a. about the x-axis.
    b. about the y-axis.
    c. about the line $y = 9$
    d. about the line $x = 6$
    e. about the line $2x + 3y = 6$
40. Find the volume and surface area swept out when the center of a circle with radius r and center $(R, 0)$ is rotated about the y-axis:



torus ("doughnut")

41. Find the volumes and surface areas swept out when the rectangles shown below are rotated about the line L:

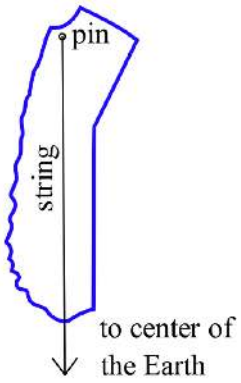


(Measurements are in feet.)

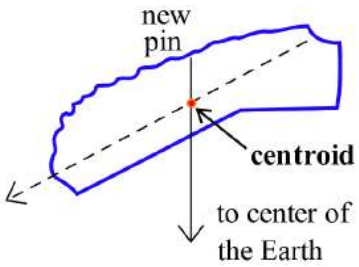
Physically Approximating Centroids of Regions

You can approximate the location of a centroid of a region experimentally, even if the region—such as a state or country—is not described by a formula.

Cut the shape out of a piece of some uniformly thick material, such as paper or cardboard, and pin an edge to a wall. The shape will pivot about the pin until its center of mass is directly below the pin so the center of mass of the shape must lie directly below the pin, on the line connecting the pin with the center of mass of Earth.



Repeat the process using a different point near the edge of the shape to find a different line. The center of mass also lies on the new line, so you can conclude that the centroid of the shape is located where the two lines intersect:



It is a good idea to pick a third point near the edge and plot a third line to check that this third line also passes through the point of intersection of the first two lines.

You can experimentally approximate the “population center” of a region by attaching masses proportional to the populations of the cities and then repeating the “pin” process with this weighted model. The point on the new model where the lines intersect is the approximate “population center” of the region.

42. Determine the centroid of your state.
43. Which state would result in the easiest centroid problem? The most difficult centroid problem?

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5.7: Improper Integrals

In Section 5.4, we computed the work required to lift a payload of mass m from the surface of a moon of mass M and radius R to a height H above the surface of the moon:

$$\int_R^{R+H} \frac{GMm}{x^2}\,dx = \left[-\frac{GMm}{x}\right]_R^{R+H} = \frac{GMm}{R} - \frac{GMm}{R+H}$$

Notice that as the height H grows very large, the second term in this answer becomes very small and the total work approaches $\frac{GMm}{R}$. We can write:

$$\lim_{H\to\infty}\left[\frac{GMm}{R} - \frac{GMm}{R+H}\right] = \frac{GMm}{R}$$

Here we're taking a limit of an expression that arose as the value of a definite integral, so we can also write:

$$\frac{GMm}{R} = \lim_{H\to\infty}\left[\frac{GMm}{R} - \frac{GMm}{R+H}\right] = \lim_{H\to\infty}\int_R^{R+H}\frac{GMm}{x^2}\,dx$$

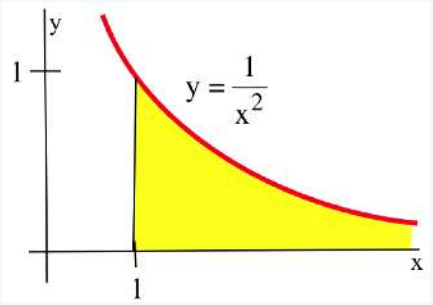
We could write this last integral, at least informally, as:

$$\int_R^{\infty}\frac{GMm}{x^2}\,dx$$

We call this new type of integral an **improper integral** because the interval of integration is infinite, violating an assumption we made when originally developing the definite integral $\int_a^b f(x)\,dx$ using Riemann sums that the length of the interval of integration, $[a, b]$, was finite.

Example 5.7.1

Represent the area of the infinite region between $f(x) = \frac{1}{x^2}$ and the x-axis for $x \geq 1$:



as an improper integral.

Solution

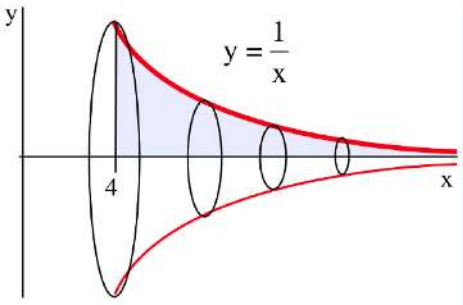
We can represent the area of region (which has infinite length) as:

$$\int_1^{\infty}\frac{1}{x^2}\,dx$$

We don't yet know whether this area is finite or infinite.

Practice 5.7.1

Represent the volume swept out when the infinite region between $f(x) = \frac{1}{x}$ and the x-axis for $x \geq 4$ is revolved about the x-axis:



using an improper integral.

Answer

$$\int_4^\infty \pi \cdot \left(\frac{1}{x}\right)^2 dx = \pi \int_4^\infty \frac{1}{x^2}\, dx$$

General Strategy for Improper Integrals

In the lifting-a-payload example above, we defined our first improper integral as the limit of a "proper" integral over a finite interval as the length of the interval became larger and larger.

Our general approach to evaluate improper integrals over infinitely long intervals—as well as another type of improper integral introduced later in this section—will mimic this strategy: Shrink the interval of integration so you have a (proper) definite integral you can evaluate, then let the interval grow to approach the desired interval of integration. The value of the improper integral will be the limiting value of the (proper) definite integrals as the intervals grow to the interval you want, provided that this limit exists.

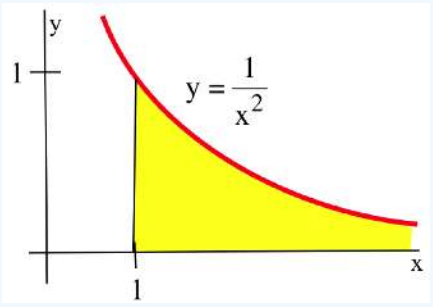
Infinitely Long Intervals of Integration

To evaluate an improper integral on an infinitely long interval:

- replace the infinitely long interval with a finite interval
- evaluate the integral on the finite interval
- let the finite interval grow longer and longer, approaching the original infinitely long interval

Example 5.7.2

Evaluate $\int_1^\infty \frac{1}{x^2}\, dx$:



Solution

The interval $[1, \infty)$ is infinitely long, but we can evaluate the integral on finite intervals such as $[1, 2]$, $[1, 10]$, $[1, 1000]$ and, more generally, $[1, M]$ where M is some massive positive number:

$$\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = \left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{2} = \frac{1}{2}$$
$$\int_1^{10} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{10} = \left[-\frac{1}{10}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{10} = \frac{9}{10}$$
$$\int_1^{1000} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{1000} = \left[-\frac{1}{1000}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{1000} = \frac{999}{1000}$$

and, more generally, $\int_1^M \frac{1}{x^2}\,dx = 1 - \frac{1}{M}$ so:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{x^2}\,dx = \lim_{M\to\infty} \left[1 - \frac{1}{M}\right] = 1$$

The value of the improper integral is $\mathbf{1}$.

We say that the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ in the Example 2 "is **convergent**" and that it "**converges to 1**."

Furthermore, from Example 1, we know that this improper integral represents the area of an infinitely long region. We now have an example—which you may find highly counterintuitive—of a region with infinite length but finite area.

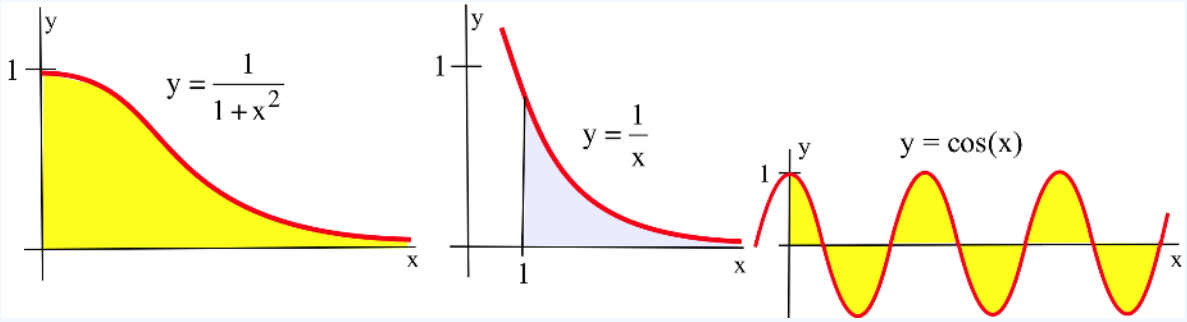
Not all improper integrals converge, however.

Example 5.7.3

Evaluate each improper integral.

a. $\int_0^\infty \frac{1}{1+x^2}\,dx$
b. $\int_1^\infty \frac{1}{x}\,dx$
c. $\int_0^\infty \cos(x)\,dx$

See below for graphical interpretations of these integrals as areas of unbounded regions:



Solution

a. Replacing the upper limit of the improper integral with a massive positive number M:

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \int_0^M \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \Big[\arctan(x)\Big]_0^M$$
$$= \lim_{M\to\infty} \Big[\arctan(M) - 0\Big] = \frac{\pi}{2}$$

so the improper integral is convergent and converges to $\frac{\pi}{2}$.

b. Replacing the upper limit of the improper integral with a massive positive number M:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{x}\,dx = \lim_{M\to\infty} \Big[\ln(x)\Big]_1^M = \lim_{M\to\infty} \ln(M) = \infty$$

Because this limit diverges, we say the improper integral **is divergent** or that it **diverges**.

c. Once again replacing ∞ with M in the upper limit of the integral:

$$\lim_{M\to\infty}\int_0^M \cos(x)\,dx = \lim_{M\to\infty}\Big[\sin(x)\Big]_0^M = \lim_{M\to\infty}\sin(M)$$

As M grows without bound, the values of $\sin(M)$ oscillate between -1 and 1, never approaching a single value, so the limit does not exist; we say that this improper integral diverges.

? Practice 5.7.2

Evaluate:

a. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

b. $\displaystyle\int_0^\infty \sin(x)\,dx$

Answer

a. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx = \lim_{M\to\infty}\int_1^M x^{-3}\,dx = \lim_{M\to\infty}\left[-\frac{1}{2}x^{-2}\right]_1^M = \lim_{M\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{M^2}+\frac{1}{2}\right] = \frac{1}{2}$

b. Replacing ∞ with M in the upper limit of the integral:

$$\begin{aligned}\int_0^\infty \sin(x)\,dx &= \lim_{M\to\infty}\int_0^M \sin(x)\,dx = \lim_{M\to\infty}\Big[-\cos(x)\Big]_0^M \\ &= \lim_{M\to\infty}\Big[-\cos(M)+1\Big] = \lim_{M\to\infty}\big[1-\cos(M)\big]\end{aligned}$$

This limit does not exist (the values of $1-\cos(M)$ oscillate between 0 and 2 and never approach any fixed number) so the improper integral diverges.

Definition

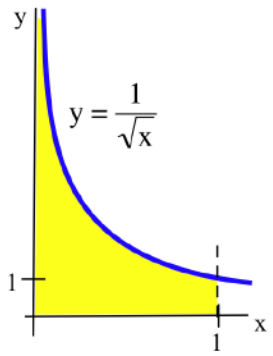
For any integrable function $f(x)$ defined for all $x \ge a$ and any integrable function $g(x)$ defined for all $x \le b$:

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty}\int_a^M f(x)\,dx$$
$$\int_{-\infty}^b g(x)\,dx = \lim_{N\to-\infty}\int_N^b g(x)\,dx$$

If the limit in question exists and is finite, we say that the corresponding improper integral **converges** or **is convergent** and define the value of the improper integral to be the value of the limit. If the limit in question does not exist, we say that the corresponding improper integral **diverges** or **is divergent**.

Functions Undefined at an Endpoint of the Interval of Integration

Consider the graph of $\frac{1}{\sqrt{x}}$ on the interval $(0,1]$:



and compare this region to the graph from Example 2. It appears we can generate the new region by reflecting the old region across $y = x$ and adding a rectangle (of area 1) at the bottom, so we might reasonably assume that the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is a finite number. This integral is over a finite interval, $[0, 1]$, but we have a new problem: the integrand is undefined at $x = 0$, one of the endpoints of the interval of integration. This violates another assumption we made when developing the definition of a definite integral as a limit of Riemann sums.

If the function you want to integrate is unbounded at one of the endpoints of an interval of finite length, as in this situation, you can shrink the interval of integration so that the function is bounded at both endpoints of the new, smaller interval, then evaluate the integral over the smaller interval, and finally let the smaller interval grow to approach the original interval.

Example 5.7.4

Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution

The function $\frac{1}{\sqrt{x}}$ is not defined at $x = 0$, the lower endpoint of integration, but the function is bounded on intervals such as $[0.36, 1]$, $[0.09, 1]$ and, more generally, on the interval $[c, 1]$ for any $c > 0$:

$$\int_{0.36}^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{0.36}^1 = 2\sqrt{1} - 2\sqrt{0.36} = 2 - 1.2 = 0.8$$
$$\int_{0.09}^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{0.09}^1 = 2\sqrt{1} - 2\sqrt{0.09} = 2 - 0.6 = 1.4$$

and, in general:

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_c^1 = 2\sqrt{1} - 2\sqrt{c} = 2 - 2\sqrt{c}$$

so, taking the limit as c decreases toward 0:

$$\lim_{c\to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \left[2 - 2\sqrt{c}\right] = 2$$

which is what you should have expected based on the graph.

NOTE SOMEWHERE A region of area 1 plus a rectangle of area 1 should have an area of $1 + 1 = 2$.

Definition

For any function $f(x)$ defined and continuous on $(a, b]$ and any function $g(x)$ defined and continuous on $[a, b)$:

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$
$$\int_a^b g(x)\,dx = \lim_{c\to b^-} \int_a^c g(x)\,dx$$

If the limit exists, we say the integral **converges** and define the value of the integral to be the value of the limit. If the limit does not exist, we say that the integral **diverges**.

Practice 5.7.3

Show that:

a. $\int_1^{10} \frac{1}{\sqrt{10 - x}}\,dx = 6$ and

b. $\int_0^1 \frac{1}{x}\,dx$ diverges.

Answer

a. The integral is improper at its upper limit, where $x = 10$, so:

$$\int_1^{10} \frac{1}{\sqrt{10-x}}\,dx = \lim_{c\to 10^-} \int_1^c (10-x)^{-\frac{1}{2}}\,dx = \lim_{c\to 10^-} \left[-2\sqrt{10-x}\right]_1^c$$
$$= \lim_{c\to 10^-} \left[-2\sqrt{10-c} + 2\sqrt{9}\right] = 0 + 2\cdot 3 = 6$$

b. The integral is improper at its lower limit, where $x = 0$, so:

$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \left[\ln(|x|)\right]_c^1 = \lim_{c\to 0^+} [\ln(1) - \ln(c)] = \infty \tag{5.7.1}$$

so the integral diverges.

If an integrand is unbounded at one or more points *inside* the interval of integration, you can split the original improper integral into two or more improper integrals over subintervals where the integrand is unbounded at only one endpoint of each subinterval. (See Problems 22–26 for practice with integrals of this type.)

Testing for Convergence: The P-Test and the Comparison Test

Sometimes we care only whether or not an improper integral converges. We now consider two methods for testing the convergence of an improper integral. Neither method gives you the actual value of the integral, but each enables you to determine whether or not certain improper integrals converge. The **Comparison Test for Integrals** enables you to determine the convergence (or divergence) of certain integrals by comparing them with other (easier) integrals. The **P-Test** involves special cases often used with the Comparison Test for Integrals.

P-Test for integrals

For any $a > 0$, the improper integral $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \le 1$.

Proof

It is easiest to consider three cases rather than two: $p = 1$, $p > 1$ and $p < 1$. If $p = 1$ then:

$$\int_a^\infty \frac{1}{x^p}\,dx = \int_a^\infty \frac{1}{x}\,dx = \lim_{M\to\infty} \int_a^M \frac{1}{x}\,dx = \lim_{M\to\infty} \left[\ln(|x|)\right]_a^M$$
$$= \lim_{M\to\infty} \left[\ln(M) - \ln(a)\right] = \infty$$

so the improper integral diverges. For the other two cases, $p \neq 1$, so:

$$\lim_{M\to\infty} \int_a^M x^{-p}\,dx = \lim_{M\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^M = \lim_{M\to\infty} \left[\frac{M^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right]$$

If $p > 1$, then $1 - p < 0$ so $\lim_{M\to\infty} \frac{M^{1-p}}{1-p} = 0$ and:

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{M\to\infty} \left[\frac{M^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right] = \frac{a^{1-p}}{p-1}$$

which is a finite number, so the improper integral converges. If $p < 1$, then $1 - p > 0$ so $\lim_{M\to\infty} \frac{M^{1-p}}{1-p} = \infty$ and:

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{M\to\infty} \left[\frac{M^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right] = \infty$$

so the improper integral diverges.

Example 5.7.5

Determine the convergence or divergence of each integral:

a. $\int_5^\infty \frac{1}{x^2}\,dx$
b. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$
c. $\int_1^8 \frac{1}{\sqrt[3]{x}}\,dx$

Solution

a. The integral matches the form required by the P-Test with $p = 2 > 1$, so the improper integral converges. The P-Test does *not* tell us the value of the integral.
b. The integral matches the form required by the P-Test with $p = \frac{1}{2} < 1$, so the improper integral diverges.
c. This is not an improper integral, so the P-Test does not apply, but:

$$\int_1^8 \frac{1}{\sqrt[3]{x}}\,dx = \int_1^8 x^{-\frac{1}{3}}\,dx = \left[\frac{3}{2}x^{\frac{2}{3}}\right]_1^8 = \frac{3}{2}\left[8^{\frac{2}{3}} - 1^{\frac{2}{3}}\right] = \frac{3}{2}[4-1] = \frac{9}{2}$$

so the value of the integral is **4.5**.

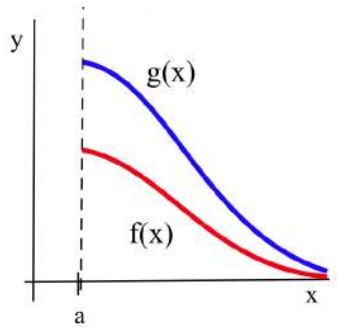
The following **Comparison Test** enables us to determine the convergence or divergence of an improper integral of a positive function by comparing this function with functions whose improper integrals we already know converge or diverge.

Comparison Test for Integrals of Positive Functions

Suppose $f(x)$ and $g(x)$ are defined and integrable for all $x \geq a$ with $0 \leq f(x) \leq g(x)$. Then:

- $\int_a^\infty g(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ converges.
- $\int_a^\infty f(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ diverges.

The proof involves a straightforward application of the definition of an improper integral and various facts about limits, but the graph below provides a geometrically intuitive way of understanding why these results must hold:



If $\int_a^\infty g(x)\,dx$ converges, then the area under the graph of $g(x)$ is finite, so the (smaller) area under the graph of $f(x)$ must also be finite, and $\int_a^\infty f(x)\,dx$ must converge as well. If $\int_a^\infty f(x)\,dx$ diverges, then the area under the graph of $f(x)$ is infinite, so the (bigger) area under the graph of $g(x)$ must also be infinite, and $\int_a^\infty g(x)\,dx$ must also diverge.

Just as important as understanding what this Comparison Test *does* tell us is realizing what the Comparison Test does *not* tell us. If $\int_a^\infty g(x)\,dx$ diverges, or if $\int_a^\infty f(x)\,dx$ converges, the Comparison Test *tells us absolutely nothing* about the convergence or divergence of the other integral. Geometrically, if $\int_a^\infty g(x)\,dx$ diverges, then the area under the graph of $g(x)$ is infinite, but the (smaller) area under the graph of $f(x)$ could be either finite or infinite, so we can't conclude anything about the convergence or divergence of $\int_a^\infty f(x)\,dx$. Likewise, if $\int_a^\infty f(x)\,dx$ converges, then the area under the graph of $f(x)$ is finite, but the (bigger) area under the graph of $g(x)$ could be either finite or infinite, so we can't conclude anything about the convergence or divergence of $\int_a^\infty g(x)\,dx$.

Example 5.7.6

Determine whether each of these integrals is convergent or divergent by comparing it with an appropriate integral that you already know converges or diverges:

a. $\int_1^\infty \frac{7}{x^3+5}\,dx$
b. $\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$
c. $\int_6^\infty \frac{9}{\sqrt{x-5}}\,dx$

Solution

a. We know that $5 > 0$ and $x \geq 1$ so:

$$x^3+5 > x^3 \;\Rightarrow\; 0 < \frac{1}{x^3+5} < \frac{1}{x^3} \;\Rightarrow\; 0 < \frac{7}{x^3+5} < \frac{7}{x^3}$$

We also know, by the P-Test with $p = 3 > 1$, that $\int_1^\infty \frac{1}{x^3}\,dx$ converges, so $\int_1^\infty \frac{7}{x^3}\,dx = 7 \cdot \int_1^\infty \frac{1}{x^3}\,dx$ also converges. By the Comparison Test, the smaller integral $\int_1^\infty \frac{7}{x^3+5}\,dx$ must converge as well. (The numerator of the original integrand is constant and the dominant term in the denominator of that integrand is x^3, so it should make sense to compare the original integrand with $\frac{1}{x^3}$.)

b. We know that $-1 \leq \sin(x) \leq 1$, so:

$$2 \leq 3+\sin(x) \leq 4 \;\Rightarrow\; 0 < \frac{3+\sin(x)}{x^2} \leq \frac{4}{x^2} = 4 \cdot \frac{1}{x^2}$$

By the P-Test with $p = 2 > 1$, $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so $\int_1^\infty \frac{4}{x^2}\,dx = 4 \cdot \int_1^\infty \frac{1}{x^2}\,dx$ also converges. By the Comparison Test, the smaller integral $\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$ must converge as well. (The numerator of the original integrand fluctuates between 2 and 4 while the dominant (and only) term in its denominator is x^2, so it should make sense to compare the original integrand with $\frac{1}{x^2}$.)

c. We know that $\sqrt{u}$ is an increasing function, so:

$$x-5 < x \;\Rightarrow\; \sqrt{x-5} < \sqrt{x} \;\Rightarrow\; \frac{1}{\sqrt{x-5}} > \frac{1}{\sqrt{x}}$$

By the P-Test with $p = \frac{1}{2} < 1$, $\int_6^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, so the bigger integral $\int_6^\infty \frac{1}{\sqrt{x-5}}\,dx$ must also diverge.

Problems

In Problems 1–26, evaluate each improper integral, or show why it diverges.

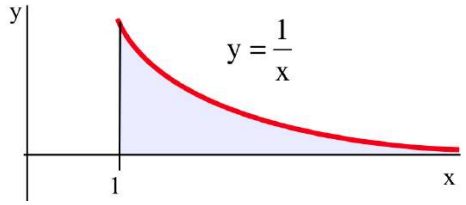
1. $\int_{10}^\infty \frac{1}{x^3}\,dx$
2. $\int_e^\infty \frac{5}{x \cdot [\ln(x)]^2}\,dx$
3. $\int_{\sqrt{3}}^\infty \frac{1}{1+x^2}\,dx$
4. $\int_1^\infty \frac{2}{e^x}\,dx$
5. $\int_e^\infty \frac{5}{x \cdot \ln(x)}\,dx$
6. $\int_0^\infty \frac{x}{1+x^2}\,dx$
7. $\int_3^\infty \frac{1}{x-2}\,dx$
8. $\int_3^\infty \frac{1}{(x-2)^2}\,dx$
9. $\int_3^\infty \frac{1}{(x-2)^3}\,dx$

10. $\int_3^\infty \frac{1}{x+2}\,dx$
11. $\int_3^\infty \frac{1}{(x+2)^2}\,dx$
12. $\int_3^\infty \frac{1}{(x+2)^3}\,dx$
13. $\int_0^4 \frac{1}{\sqrt{x}}\,dx$
14. $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$
15. $\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx$
16. $\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$
17. $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$
18. $\int_0^2 \frac{3x^2}{\sqrt{8-x^3}}\,dx$
19. $\int_{-2}^\infty \sin(x)\,dx$
20. $\int_\pi^\infty \sin(x)\,dx$
21. $\int_0^{\frac{\pi}{2}} \tan(x)\,dx$
22. $\int_0^3 \frac{1}{x-2}\,dx$
23. $\int_0^\pi \tan(x)\,dx$
24. $\int_3^\infty \frac{1}{x\sqrt{x}}\,dx$
25. $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$
26. $\int_{-\infty}^0 \frac{1}{x^2}\,dx$

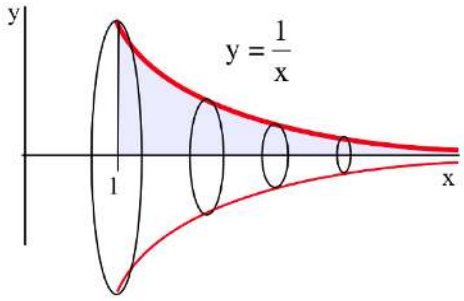
In Problems 27–44, determine whether each improper integral converges or diverges, but do not evaluate the integral.

27. $\int_1^\infty \frac{1}{x^5}\,dx$
28. $\int_2^\infty \frac{1}{\sqrt[5]{x}}\,dx$
29. $\int_3^\infty \frac{1}{\sqrt[5]{x^6}}\,dx$
30. $\int_4^\infty \frac{1}{\sqrt[5]{x^4}}\,dx$
31. $\int_5^\infty \frac{1}{\sqrt{x\sqrt[3]{x}}}\,dx$
32. $\int_6^\infty x^{-\frac{4}{7}}\,dx$
33. $\int_7^\infty x^{-\frac{7}{4}}\,dx$
34. $\int_8^\infty \frac{1}{1+x^2}\,dx$
35. $\int_3^\infty \frac{1}{x^2+5}\,dx$
36. $\int_4^\infty \frac{7}{x^2+5}\,dx$
37. $\int_5^\infty \frac{1}{x^3+x}\,dx$
38. $\int_6^\infty \frac{1}{x-2}\,dx$
39. $\int_e^\infty \frac{7}{x+\ln(x)}\,dx$
40. $\int_2^\infty \frac{1}{x^2-1}\,dx$

41. $\int_{\pi}^{\infty} \frac{1+\cos(x)}{x^2}\,dx$
42. $\int_{0}^{\infty} \frac{x^4}{x^6+1}\,dx$
43. $\int_{0}^{\infty} \frac{x^4}{x^5+1}\,dx$
44. $\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+1}\,dx$
45. Example 3(b) showed that $\int_{1}^{M} \frac{1}{x}\,dx$ grew arbitrarily large as M grew arbitrarily large, so no finite amount of paint would cover the region bounded by the x-axis and the graph of $f(x) = \frac{1}{x}$ for $x > 1$:

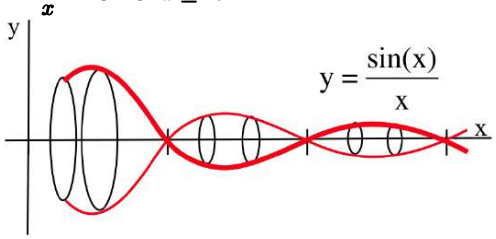


Show that the volume of the solid obtained when the region graphed above is revolved about the x-axis:



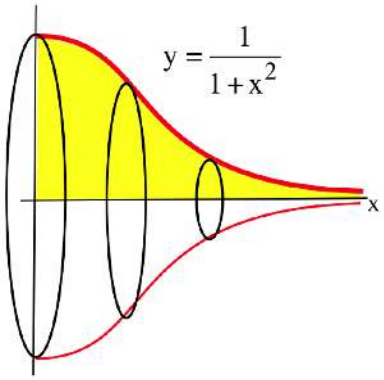
is finite, so the 3-dimensional trumpet-shaped region can be filled with a finite amount of paint. Does this present a contradiction?

46. Determine whether or not the volume of the solid obtained by revolving the region between the x-axis and the graph of $f(x) = \frac{\sin(x)}{x}$ for for $x \geq 1$:



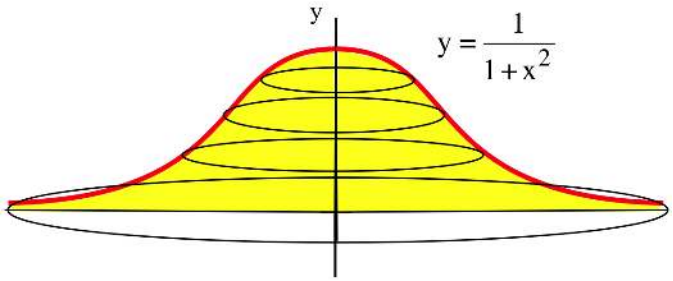
about the x-axis is finite.

47. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = \frac{1}{x^2+1}$:



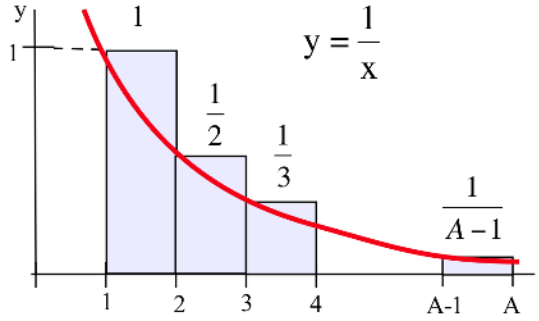
is revolved about the x-axis.

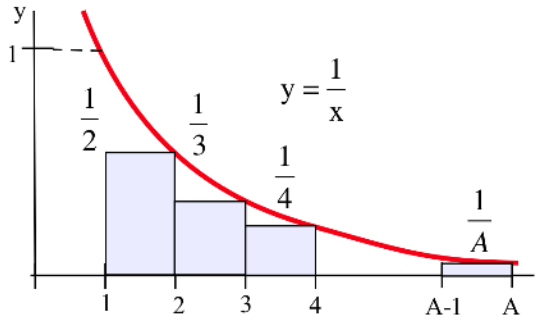
48. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = e^{-x}$ is revolved about the x-axis.
49. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = \dfrac{1}{x^2+1}$:



is revolved about the y-axis.

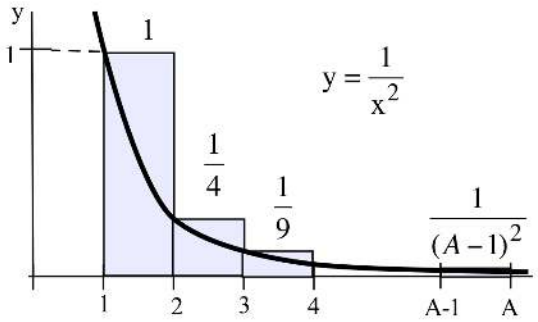
50. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = e^{-x}$ is revolved about the y-axis.
51. Use the figure below left to help determine which is larger: $\int_1^A \frac{1}{x}\,dx$ or $\sum_{k=1}^{A-1} \frac{1}{k}$.

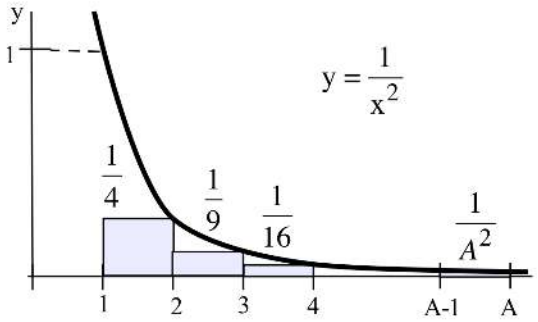




52. Use the figure above right to help determine which is larger: $\int_1^A \frac{1}{x}\,dx$ or $\sum_{k=2}^{A} \frac{1}{k}$.

53. Use the figure below left to help determine which is larger: $\int_1^A \frac{1}{x^2}\,dx$ or $\sum_{k=1}^{A-1} \frac{1}{k^2}$.





54. Use the figure above right to help determine which is larger: $\int_1^A \frac{1}{x^2}\,dx$ or $\sum_{k=2}^{A} \frac{1}{k^2}$.

The **Laplace transform** of a function $f(t)$ is defined using an improper integral involving a parameter s:

$$F(s) = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

Laplace transforms are often used to solve differential equations.

55. Compute the Laplace transform of the constant function $f(t) = 1$.
56. Compute the Laplace transform of $f(t) = e^{4t}$.
57. Define a function $g(t)$ by:

$$g(t) = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

Compute the Laplace transform of $g(t)$.

58. Define a function $h(t)$ by:

$$h(t) = \begin{cases} 1 & \text{if } t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

Compute the Laplace transform of $h(t)$.

59. Devise a "Q-Test" to determine whether $\int_0^b \frac{1}{x^q}\,dx$ converges or diverges for any number $b > 0$.
60. Use the result of the previous Problem to test the convergence of $\int_0^e \frac{1}{\sqrt[3]{x}}\,dx$ and $\int_0^\pi \frac{1}{x \cdot \sqrt[3]{x}}\,dx$.

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5.8: Additional Applications

This section introduces two additional applications of integrals that once again illustrate the process of going from an applied problem to a Riemann sum and on to a definite integral. A third application does not follow this process: it uses the idea of "area" to model an election and to qualitatively understand why certain election outcomes occur.

The main point of this section is to demonstrate the power of definite integrals to solve a wide variety of applied problems. Each of these new applications is treated more briefly than those in the previous sections. These are far from the only applications that could be included here. By now, however, you should have developed enough of an understanding of the Riemann-sum process so that when you encounter other applications (in physics, engineering, biology, statistics, probability, economics, computer graphics…) you will be able to use that process to set up an integral to compute or approximate a desired quantity.

Fluid Pressures and Forces

In physics, **pressure** is defined as force per unit of area. The **hydrostatic pressure** on an object immersed in a fluid (such as water) is the product of the density of that fluid and the depth of the object:

$$\text{pressure} = (\text{density})(\text{depth})$$

The total **hydrostatic force** applied against an immersed object is the sum of the hydrostatic forces against each part of the object. (Fluids exert pressure in all possible directions, and the forces due to this pressure act on solid objects in a direction perpendicular to the object.)

Note

In the metric system, the standard unit of force is a **pascal** (abbreviated "Pa"):

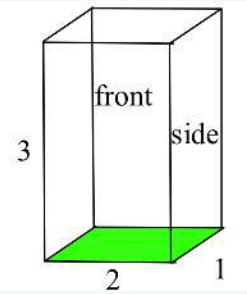
$$1\,\text{Pa} = 1\,\frac{\text{N}}{\text{m}}$$

and named after Blaise Pascal (1623–1662), a French mathematician, physicist, inventor, writer and philosopher.

If an entire object is at the same depth, we can determine the total hydrostatic force against that (necessarily flat) object simply by multiplying the density of the fluid times the depth of the object times the object's area. If the unit of density is "pounds per cubic foot" and depth is measured in "feet," then the unit of pressure is "pounds per square foot," a measure of force per unit of area. If pressure, with the units "pounds per square foot," is multiplied by an area with units "square feet," the result is a force, measured in "pounds."

Example 5.8.1

Find the total hydrostatic force against the bottom of the freshwater aquarium shown below:



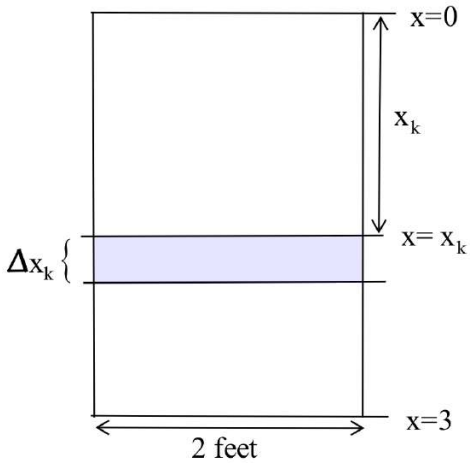
Solution

Water's desity is $62.5\,\frac{\text{lb}}{\text{ft}^3}$, so the total hydrostatic force is:

$$(\text{density}) \cdot (\text{depth}) \cdot (\text{area}) = \left(62.5\,\frac{\text{lb}}{\text{ft}^3}\right) \cdot (3\,\text{ft}) \cdot (2\,\text{ft}^2)$$

or 375 lbs. Finding the total hydrostatic force against the *front* of the aquarium is a very different problem, because different parts of that front face are located at different depths and subject to different pressures.

To compute the force against the *front* of the aquarium, we can partition it into n thin horizontal slices:



and focus on one of them. Because the slice is very thin, every part of the k-th slice is at (almost) the same depth, so every part of that slice is subject to (almost) the same pressure. We can approximate the total hydrostatic force against the slice at the depth x_k as:

$$(\text{density}) \cdot (\text{depth}) \cdot (\text{area}) = \left(62.5 \frac{\text{lb}}{\text{ft}^3}\right) \cdot (x_k \,\text{ft}) \cdot (2\,\text{ft})\,(\Delta x_k \,\text{ft})$$

or $125 x_k \cdot \Delta x_k$ lbs. The total hydrostatic force against the front is the sum of the forces against each slice:

$$\text{total hydrostatic force} \approx \sum_{k=0}^{n} 125 x_k \cdot \Delta x_k$$

which is a Riemann sum. The limit of this Riemann sum as the slices get thinner ($\Delta x_k \to 0$) is a definite integral:

$$\sum_{k=0}^{n} 125 x_k \cdot \Delta x_k \;\longrightarrow\; \int_{x=0}^{x=3} 125x\,dx = \Big[62.5x^2\Big]_{x=0}^{x=3} = 562.5\,\text{lbs}$$

? Practice 5.8.1

Find the total hydrostatic force against one side of the aquarium and the total force against the entire aquarium.

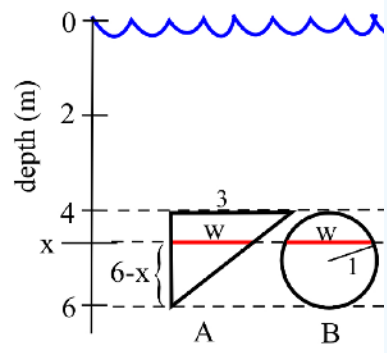
Answer

The reasoning for a side of the aquarium is exactly the same as for the front, except a side is 1 foot long instead of 2, so the force is half of that against the front: 281.25 lbs. The total force against all sides (and the bottom) is:

$$2(281.25) + 2(562.5) + 375 = 2062.5 \text{ lbs}$$

✓ Example 5.8.2

Find the total hydrostatic force against viewing windows A and B in the freshwater aquarium shown below:



Solution

For window A, using similar triangles, the width w of a slice at depth x m satisfies:

$$\frac{w}{6-x} = \frac{3}{2} \;\Rightarrow\; w = \frac{3}{2}(6-x) = 9 - \frac{3}{2}x$$

so the area of a slice of height Δx_k m at depth x_k m is $\left(9 - \frac{3}{2}x_k\right)\Delta x_k\ \mathrm{m}^2$. The density of water is $1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$. Multiplying this density by area (with units m^2) would give kg per m, but pressure is measured in N per m, so we need to multiply by the acceleration due to gravity, $g \approx 9.81\ \frac{\mathrm{m}}{\mathrm{sec}^2}$. The hydrostatic force applied to the k-th slice is thus:

$$1000(9.81)x_k\left(9 - \frac{3}{2}x_k\right)\Delta x_k$$

and the total hydrostatic force applied to the window is therefore:

$$\begin{aligned}\int_{x=4}^{x=6} 9810\left[9x - \frac{3}{2}x^2\right]\,dx &= 9810\left[\frac{9}{2}x^2 - \frac{1}{2}x^3\right]_4^6 \\ &= 9810\left[(162-108) - (72-32)\right] = 137340\,\mathrm{N}\end{aligned}$$

For window B, applying the Pythagorean Theorem yields:

$$(5-x)^2 + \left(\frac{w}{2}\right)^2 = 1 \;\Rightarrow\; w = 2\sqrt{1-(5-x)^2}$$

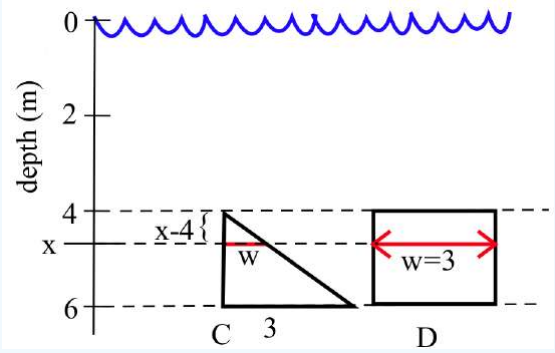
The total hydrostatic force is thus:

$$\int_{x=4}^{x=6} 1000(9.81)x \cdot 2\sqrt{1-(5-x)^2}\,dx$$

which (using technology) is approximately 154,095 N.

? Practice 5.8.2

Find the total hydrostatic force against viewing windows C and D of the freshwater aquarium shown below:



Answer

For window C, using similar triangles the width w of a slice at depth x m satisfies:

$$\frac{w}{x-4} = \frac{3}{2} \;\Rightarrow\; w = \frac{3}{2}(x-4) = \frac{3}{2}x - 6$$

so the area of a slice of height Δx_k m at depth x_k m is $\left(\frac{3}{2}x_k - 6\right)\Delta x_k\,\text{m}^2$. The density of water is $1000\,\frac{\text{kg}}{\text{m}^3}$ and $g \approx 9.81\,\frac{\text{m}}{\text{sec}^2}$, so the hydrostatic force applied to the k-th slice is:

$$1000(9.81)x_k\left(\frac{3}{2}x_k - 6\right)\Delta x_k$$

and the total hydrostatic force applied to the window is therefore:

$$\begin{aligned}\int_{x=4}^{x=6} 9810\left[\frac{3}{2}x^2 - 6x\right]\,dx &= 9810\left[\frac{1}{2}x^3 - 3x^2\right]_4^6 \\ &= 9810\Big[(108-108)-(32-48)\Big] = 156960\,\text{N}\end{aligned}$$

For window D, the width is 3 at all depths, so the total hydrostatic force against the window is:

$$\int_{x=4}^{x=6} 1000(9.81)x\cdot 3\,dx = 14715x^2\Big|_4^6 = 294300\text{ N}$$

(Windows A (from Example 2) and C (with a horizontal flip) fit together to form window D, so it is encouraging that the sum of the total hydrostatic forces against A and C is $137340 + 156960 = 294300$ N, the total hydrostatic force against window D.)

Because the total force at even moderate depths is so large, underwater windows are made of thick glass or plastic and strongly secured to their frames. Similarly, the bottom of a dam is much thicker than the top in order to withstand the greater force against the bottom.

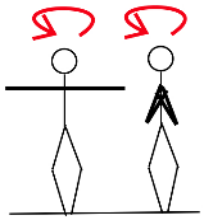
Kinetic Energy

Physicists define the **kinetic energy** (energy of motion) of an object with mass m and velocity v to be:

$$\text{KE} = \frac{1}{2}m\cdot v^2$$

The greater the mass of an object or the faster it is moves, the greater its kinetic energy. If every part of the object has the same velocity, computing its kinetic energy becomes relatively easy.

Sometimes, however, different parts of an object move with different velocities. For example, if an ice skater is spinning with an angular velocity of 2 revolutions per second, her arms travel further in one second (have a greater *linear* velocity) when they are extended than when drawn in close to her body:



So the ice skater, spinning at 2 revolutions per second, has greater kinetic energy when her arms are extended. Similarly, the tip of a rotating propeller (or the barrel of a swinging baseball bat) has a greater linear velocity than other parts of the propeller (or the bat's handle).

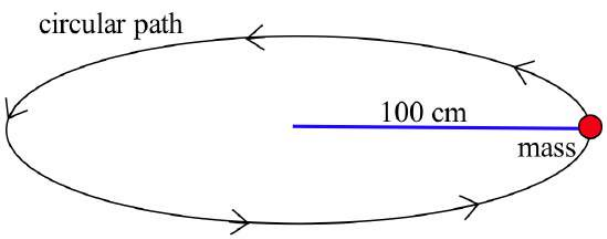
If the units of mass are kg and the units of velocity are m/sec^2, then:

$$\text{KE} = \frac{1}{2}(m\,\text{kg}) \cdot \left(v\,\frac{\text{m}}{\text{sec}}\right)^2 = \frac{1}{2}mv^2\ \text{kg} \cdot \text{m} \cdot \frac{\text{m}}{\text{sec}^2}$$

so the units of kinetic energy are N-m, or Joules, the same as work. Similarly, if the units of mass are g and the units of velocity are cm/sec², then the units of kinetic energy are dyn-cm, or ergs.

Example 5.8.3

A point-mass of 1 gram at the end of a (massless) 100-cm string rotates at a rate of 2 revolutions per second:



a. Find the kinetic energy of the point-mass.
b. Find its kinetic energy if the string is 200 cm long.

Solution

a. In one second, the mass travels twice around a circle with radius 100 cm so it travels $2 \cdot (2\pi \cdot 100) = 400\pi$ cm. Its velocity is thus $v = 400\pi$ cm/sec, and:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(1\ \text{g}) \cdot \left(400\pi\,\frac{\text{cm}}{\text{sec}}\right)^2 = 80000\pi^2\ \text{ergs}$$

or about 0.079 J.

b. If the string is 200 cm long, then the velocity is $2 \cdot (2\pi \cdot 400) = 800\pi$ cm/sec and:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(1\ \text{g}) \cdot \left(800\pi\,\frac{\text{cm}}{\text{sec}}\right)^2 = 320000\pi^2\ \text{ergs}$$

or about 0.316 J.

(When the length of the string doubles, the velocity doubles and the kinetic energy quadruples.)

Practice 5.8.3

A 1-gram point-mass at the end of a 2-meter (massless) string rotates at a rate of 4 revolutions per second. Find the kinetic energy of the point mass.

Answer

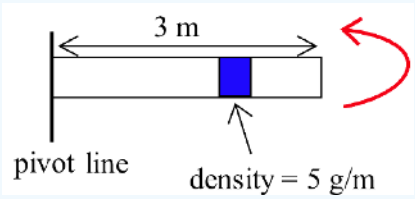
The object travels $2\pi\,(2\ \text{m}) = 4\pi$ m during one revolution, so during the 1 second it takes to make 4 revolutions, the object travels 16π m; its velocity is thus $v = 1600\pi\ \frac{\text{cm}}{\text{sec}}$ and its kinetic energy is:

$$\frac{1}{2}m \cdot v^2 = \frac{1}{2}(1\ \text{g})\left(1600\pi\,\frac{\text{cm}}{\text{sec}}\right)^2 = 1280000\pi^2\ \text{ergs} \approx 12633094\ \text{ergs}$$

If different parts of a rotating object are different distances from the axis of rotation, then those parts have different linear velocities, and it becomes more difficult to calculate the total kinetic energy of the object. By now the method should seem very familiar: partition the object into small pieces, approximate the kinetic energy of each piece, and add the kinetic energies of the small pieces (a Riemann sum) to approximate the total kinetic energy of the object. The limit of the Riemann sum as the pieces get smaller is a definite integral.

Example 5.8.4

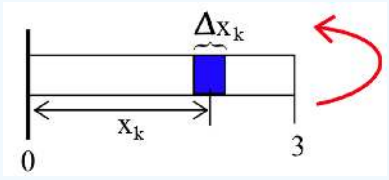
The density of a narrow bar:



is 5 grams per meter of length. Find the kinetic energy of the 3-meter-long bar when it rotates at a rate of 2 revolutions per second.

Solution

Partition the bar into n pieces:



so that the mass of the k-th piece is:

$$m_k \approx (\text{length}) \cdot (\text{density}) = (\Delta x_k \text{ m}) \left(5 \frac{\text{g}}{\text{m}}\right) = 5 \cdot \Delta x_k \text{ g}$$

During one second, the k-th piece, located at a distance of x_k m from the pivot line, will make two revolutions, traveling approximately:

$$2\,(2\pi\,[\text{radius}]) = 4\pi\left[100 x_k \text{ cm}\right] = 400\pi x_k \text{ cm}$$

so $v_k \approx 400\pi x_k$ cm/sec. The kinetic energy of the k-th piece is:

$$\frac{1}{2} m_k \cdot v_k^2 \approx \frac{1}{2}\,(5\Delta x_k \text{ g}) \left(400\pi x_k \frac{\text{cm}}{\text{sec}}\right)^2 = 400000\pi^2 x_k^2 \text{ ergs}$$

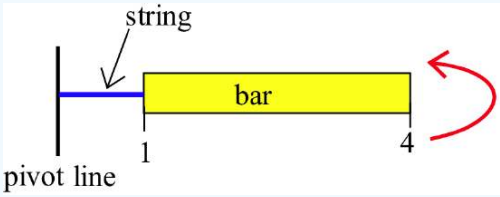
and the total kinetic energy of the rotating bar is therefore:

$$\sum_{k=1}^{n} 400000\pi^2 x_k^2 \cdot \Delta x_k \longrightarrow \int_{x=0}^{x=3} 400000\pi^2 x^2\, dx = 400000\pi^2 \left[\frac{1}{3} x^3\right]_0^3$$

which equals $3600000\pi^2$ ergs, or about 3.55 J.

Practice 5.8.4

Find the kinetic energy of the bar in the previous Example if it rotates at 2 revolutions per second at the end of a 100-centimeter (massless) string:



Answer

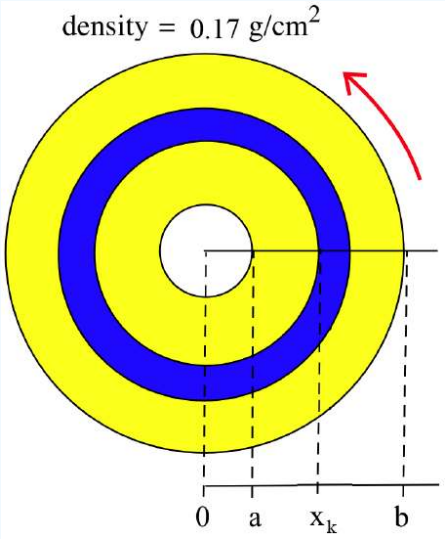
Everything remains the same as in Example 4, except for the endpoints of integration:

$$\int_{x=1}^{x=4} 400000\pi^2 x^2\,dx = 400000\pi^2\left[\frac{1}{3}x^3\right]_1^4 = 8400000\pi^2 \text{ ergs}$$

which is approximately $82,904,677$ ergs, or 8.29 J.

Example 5.8.5

Find the kinetic energy of the thin, flat object with density 0.17 g/cm^2 shown below:



when it rotates at 45 revolutions per minute.

Solution

We can partition the object along one radial line and form n annular "slices" each Δx cm wide. Then the "slice" between x_k and $x_k + \Delta x$ is a thin annulus (a disk with a smaller disk removed from its center) with area:

$$\begin{aligned}\pi\,(x_k+\Delta x)^2 - \pi\,(x_k)^2 &= \pi\left[x_k^2 + 2x_k\Delta x + (\Delta x)^2 - x_k^2\right] \\ &= 2\pi x_k\Delta x + \pi\,(\Delta x)^2 \approx 2\pi x_k\Delta x\end{aligned}$$

and mass $(0.17)2\pi x_k\Delta x$. During one revolution, a point on this slice travels approximately $2\pi x_k$ cm and 45 rev/min is equivalent to $\frac{3}{4}$ rev/sec, so the linear velocity of the point is $2\pi x_k \cdot \frac{3}{4} = \frac{3}{2}\pi x_k$ cm/sec. The kinetic energy of this slice is therefore:

$$\left((0.17)2\pi x_k\Delta x\right)\left(\frac{3}{2}\pi x_k\right)^2 = \frac{9}{2}(0.17)\pi^2 x_k^3\Delta x$$

so the total kinetic energy of the object is:

$$\sum_{k=1}^{n} \frac{9}{2}(0.17)\pi^2 x_k^3 \cdot \Delta x \;\longrightarrow\; \int_a^b \frac{9}{2}(0.17)\pi^2 x^3\,dx$$

Evaluating this integral yields:

$$\frac{9}{2}(0.17)\pi^2\left[\frac{1}{4}x^4\right]_a^b = \frac{9}{8}(0.17)\pi^2\left[b^4 - a^4\right]$$

Because b is raised to the fourth power, a small increase in the value of b (if $b > 1$) leads to a large increase in the object's kinetic energy.

Note

(The "slices" that give rise to the Riemann sum in this problem are—unlike most examples we have seen previously—*not* rectangles. We also use here the notion that if Δx is small, then $(\Delta x)^2$ is *very* small, so we can essentially ignore it in our

approximation of area.)

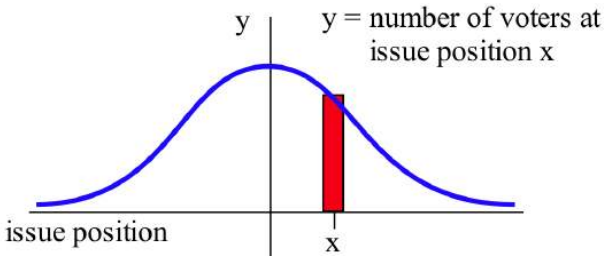
If $a = 0.75$ in ≈ 1.905 cm and $b = 3.75$ in ≈ 9.525 cm, the total mass of the object is 42 g and its total kinetic energy is about 15,512 ergs.

(In the not-so-distant past your grandparents (and perhaps even your parents) used such objects to listen to music—and each one only held two songs!)

Areas and Elections

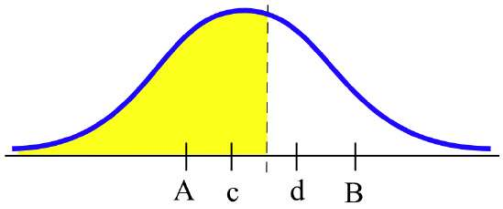
The previous applications in this chapter have used definite integrals to determine areas, volumes, pressures and energies precisely. But exactness and numerical precision are not the same as "understanding," and sometimes we can gain insight and understanding simply by determining which of two areas or integrals is larger. One situation of this type involves models of elections.

Suppose the voters of a state have been surveyed about their positions on a single issue, with their responses recorded on a quantitative scale. The distribution of voters who place themselves at each position on this issue appears below:



Suppose also that each voter casts his or her vote for the candidate whose position on this issue is closest to his or her position.

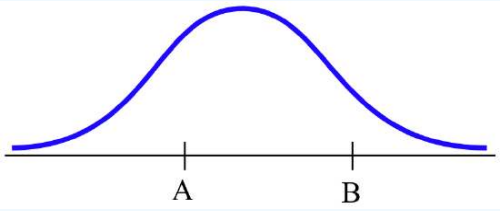
If two candidates have taken the positions labeled A and B, then a voter at position c votes for the candidate at A because A is closer to c than B is to c. Similarly, a voter at position d votes for the candidate at B. The total votes for the candidate at A in this election is represented by the shaded area under the curve:

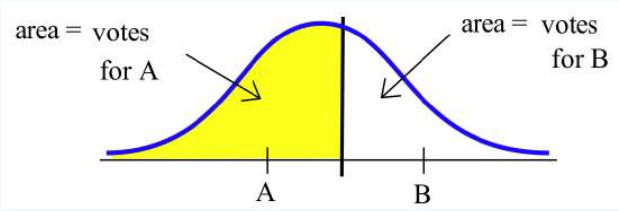


and the candidate with the larger number of votes—the larger area—wins the election. In this illustration, the candidate at A wins.

Example 5.8.6

The distribution of voters on an issue appears below left. If these voters decide between candidates on the basis of that single issue, which candidate will win the election?



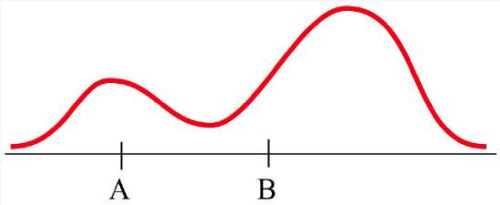


Solution

The figure above right illustrates that A corresponds to a larger area (more votes) than B: A will win.

Practice 5.8.5

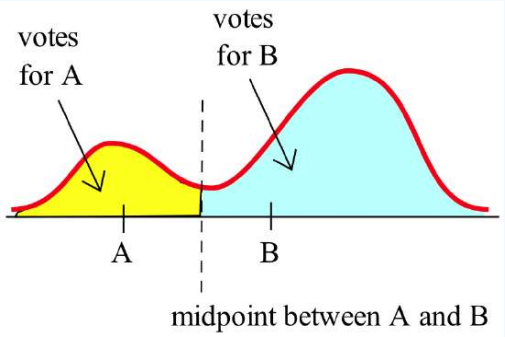
In an election between candidates with positions A and B in the figure below:



who will win?

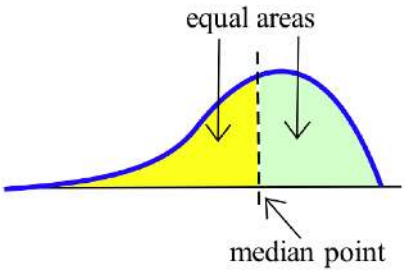
Answer

The shaded regions in the figure below:



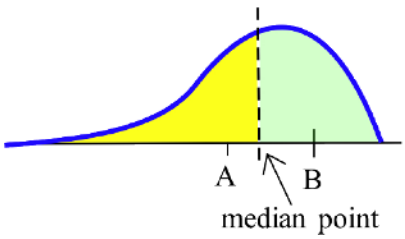
show the total votes for each candidate: B wins.

If voters behave as described and if the election is between two candidates, then we can give the candidates some advice. The best position for a candidate is at the "median point," the location that divides the voters into two equal-sized (equal-area) groups so that half of the voters are on one side of the median point and half are on the other side:



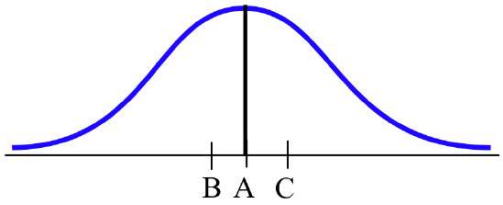
A candidate at the median point gets more votes than a candidate at any other point. (Why?)

If two candidates have positions on opposite sides of the median point:

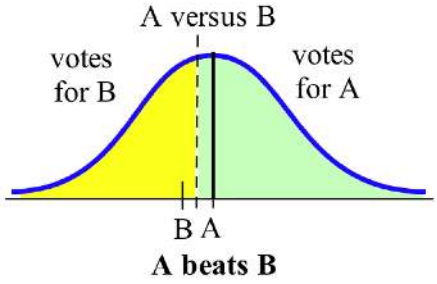


then a candidate can get more votes by moving a bit toward the median point. This "move toward the middle ground" commonly occurs in elections as candidates attempt to sell themselves as "moderates" and their opponents as "extremists."

If more than two candidates are running in an election, the situation changes dramatically. A candidate at the median position, the unbeatable place in a two-candidate election can even get the fewest votes. If the figure below:

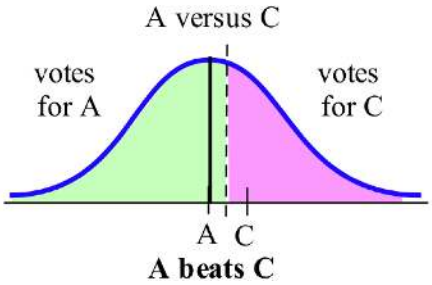


represents the distribution of voters on the single issue in the election, then candidate A would beat B in an election just between A and B:



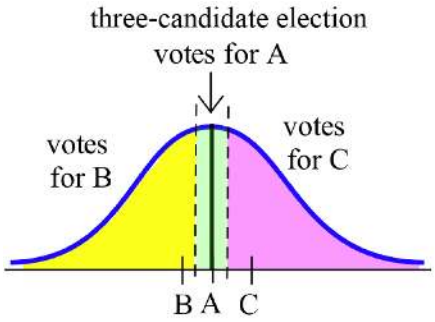
A beats B

and A would beat C in an election just between A and C:



A beats C

But in an election among all three candidates, A would get the fewest votes:



B wins, A is last

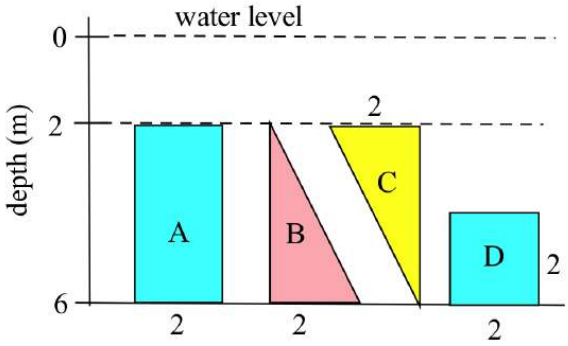
This type of situation really does occur. It leads to the political saying about a primary election with many candidates and a general election between the final nominees from two parties: “extremists can win primaries, but moderates are elected to office.”

The previous discussion of elections and areas is greatly oversimplified. Most elections involve several issues of different importance to different voters, and the views of the voters are seldom completely known before the election. Many candidates take “fuzzy” positions on issues. And it is not even certain that real voters vote for the candidate with the “closest” position: perhaps they don't vote at all unless some candidate is “close enough” to their position. But this very simple model of elections can still help us understand how and why some things happen in elections. It is also a starting place for building more sophisticated models to help understand more complicated election situations and to test assumptions about how voters really do make voting decisions.

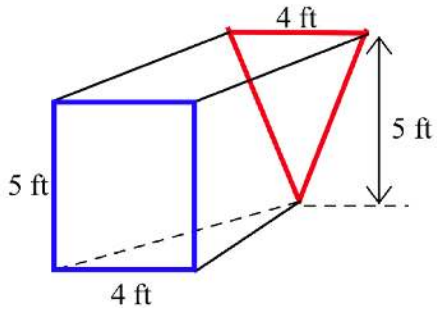
Problems

In Problems 1–5, use ρ for the density of the fluid in the given container.

1. Calculate the force against windows A and B in the figure below:

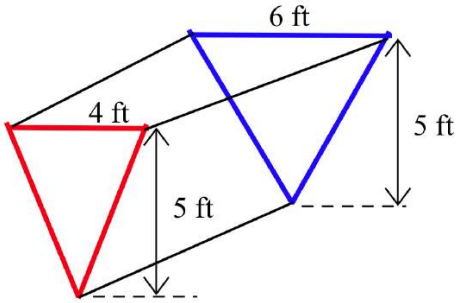


2. Calculate the force against windows C and D in the figure above.
3. Calculate the total force against each end of the tank shown below:

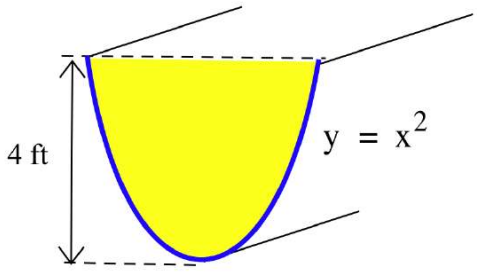


How does the total force against the ends of the tank change if the length of the tank is doubled?

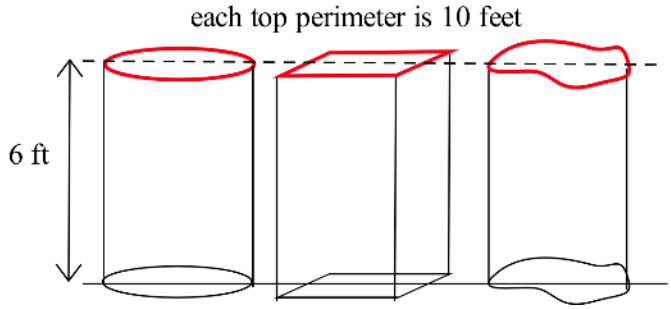
4. Calculate the total force against each end of the tank shown below:



5. Calculate the total force against the end of the tank shown below:

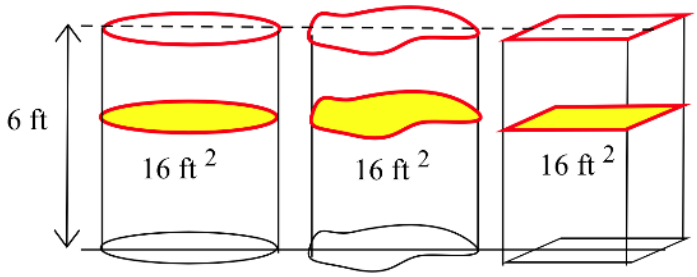


6. The three tanks shown below are all 6 feet tall and the top perimeter of each tank is 10 feet:



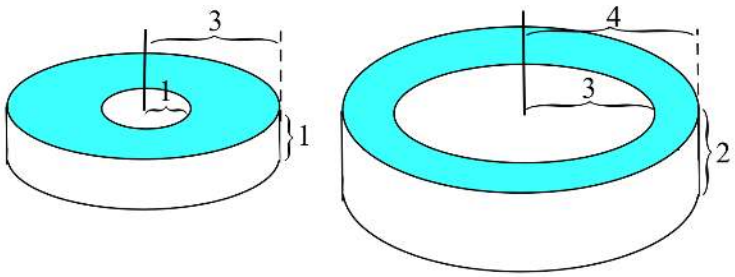
Which tank has the greatest total force against its sides?

7. The three tanks shown below are all 6 feet tall and the cross-sectional area of each tank is 16 ft^2:



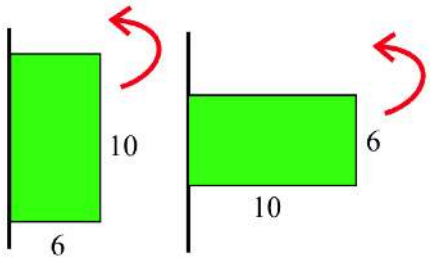
Which tank has the greatest total force against its sides?

8. Calculate the total force against the bottom 2 feet of the sides of a tank with a square 40-foot by 40-foot base that is filled with water:
   a. to a depth of 30 feet.
   b. to a depth of 35 feet.
9. Calculate the total force against the bottom 2 feet of the side of a cylindrical tank with a radius of 20 feet that is filled with water:
   a. to a depth of 30 feet.
   b. to a depth of 35 feet.
10. Calculate the total force against the side and bottom of a cylindrical aluminum soda can with diameter 6 cm and height 12 cm if it is filled with 385 g of soda. (Assume the can has been opened so carbonization is not a factor.)
11. Find the kinetic energy of a 20-gram object rotating at 3 revolutions per second at the end of:
    a. a 15-cm (massless) string and
    b. a 20-cm string.
12. Each centimeter of a metal bar has a mass of 3 grams. Calculate the kinetic energy of the 50-centimeter bar if it is rotating at a rate of 2 revolutions per second about one of its ends.
13. Each centimeter of a metal bar has a mass of 3 grams. Calculate the kinetic energy of the 50-centimeter bar if it is rotating at a rate of 2 revolutions per second at the end of a 10-cm cable.
14. Calculate the kinetic energy of a 20-gram meter stick if it is rotating at a rate of 1 revolution per second about one of its ends.
15. Calculate the kinetic energy of a 20-gram meter stick if it is rotating at a rate of 1 revolution per second about its center point.
16. A flat, circular plate is made from material that has a density of 2 grams per cubic centimeter. The plate is 5 centimeters thick, has a radius of 30 centimeters and is rotating about its center at a rate of 2 revolutions per second.
    a. Calculate its kinetic energy.
    b. Find the radius of a plate that would have twice the kinetic energy of the first plate, assuming the density, thickness and rotation rate are the same.
17. Each “washer” in the figure below is made from material with density of 1 gram per cm^3, and each is rotating about its center at a rate of 3 revolutions per second:



Calculate the kinetic energy of each washer (dimensions are in cm).

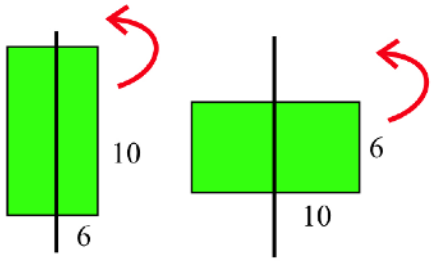
18. The rectangular plate shown below is 1 cm thick, 10 cm long and 6 cm wide and is made of a material with a density of 3 grams per cm^3:



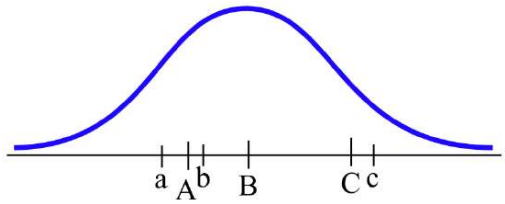
Calculate the kinetic energy of of the plate if it is rotating at a rate of 2 revolutions per second:

a. about its 10-cm side and
b. about its 6-cm side.

19. Calculate the kinetic energy of the plate in Problem 18 if it is rotating at a rate of 2 revolutions per second about a vertical line through the center of the plate, as shown below left:

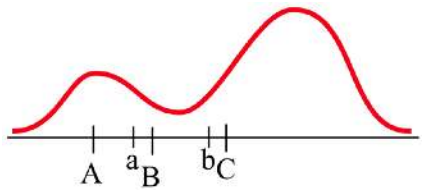


20. Calculate the kinetic energy of the plate in Problem 18 if it is rotating at a rate of 2 revolutions per second about a vertical line through the center of the plate, as shown above right.
21. For the voter distribution shown below:



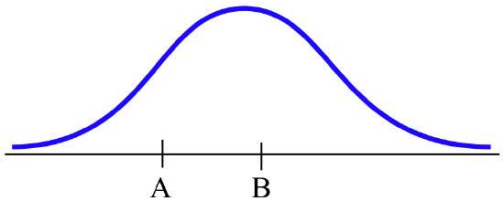
which candidates would the voters at positions a, b and c vote for?

22. For the voter distribution shown below:



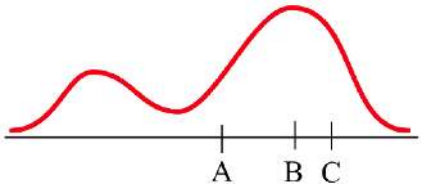
which candidates would the voters at positions a, b and c vote for?

23. Shade the region representing votes for candidate A in the distribution shown below:



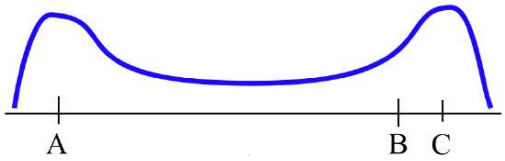
Which candidate wins?

24. Shade the region representing votes for candidate A in the distribution shown below:



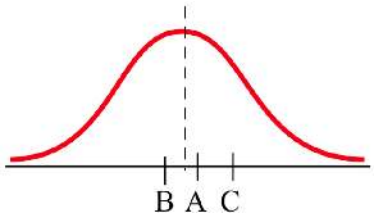
Which candidate wins?

25. Refer to the voter distribution shown below:



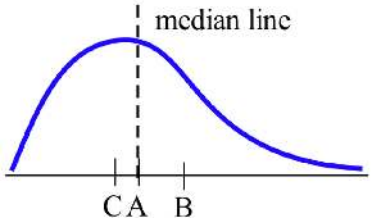
a. Which candidate wins?
b. If candidate B withdraws before the election, which candidate will win?
c. If candidate B stays in the election but C withdraws, then who wins?

26. Refer to the voter distribution shown below:



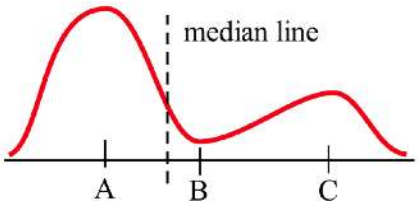
a. Which candidate wins?
b. If candidate B withdraws before the election, which candidate will win?
c. If candidate B stays in the election but C withdraws, then who wins?

27. Refer to the voter distribution shown below:



a. If the election is between A and B, who wins?
b. If the election is between A and C, who wins?
c. If the election is among A, B and C, who wins?

28. Refer to the voter distribution shown below:



a. If the election is between A and B, who wins?
b. If the election is between A and C, who wins?
c. If the election is among A, B and C, who wins?

29. Sketch a distribution for a two-issue election.

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5.A: Answers

Section 5.1

1. $(8)(6)(1)+(6)(4)(2)+(3)(3)(1)=105$
3. $\pi(4)^2(0.5)+\pi(3)^2(1.0)+\pi(1)^2(2.0)=19\pi$
5. $(9)(0.2)+(6)(0.2)+(2)(0.2)=3.4$
7. $\int_0^3 (5-x)^2\,dx = 39$
9. $\int_0^4 \frac{1}{2}(x+1)\sqrt{x}\,dx = \frac{136}{15}$
11. $\int_0^4 \pi\left[\frac{4-x}{2}\right]^2 dx = \frac{16\pi}{3}$
13. If $A(y)$ is the cross-sectional area of object A at height y above the base and $B(y)$ is the cross-sectional area of object B at height y above its base, then we know that $A(y)=B(y)$ for all values of y, so Vol$(A) = \int_a^b A(y)\,dy = \int_a^b B(y)\,dy =$ Vol (B): the volumes are equal.
15. $\int_1^4 \left[\frac{1}{x}\right]^2 dx = \frac{3}{4}$
17. $\int_0^4 \pi\left[3-\sqrt{x}\right]^2 dx = 12\pi$
19. $\int_0^\pi [\sin(x)]^2\,dx = \frac{\pi}{2}$
21. $\int_0^{\frac{\pi}{2}} \frac{1}{2}\cdot\pi\left[\frac{1}{2}\cos(x)\right]^2 dx = \frac{\pi^2}{32}$
23. a. $\int_{-1}^{\frac{3}{2}} \left[(3+x-x^2)-x^2\right]^2 dx = \frac{625}{48} \approx 13.021$
    b. $\int_{-1}^{\frac{3}{2}} \frac{1}{2}\cdot\pi\left[\frac{1}{2}(3+x-2x^2)\right]^2 dx = \frac{625\pi}{384} \approx 5.1$
    c. $\int_{-1}^{\frac{3}{2}} 2\left[3+x-2x^2\right]^2 dx = \frac{625}{24} \approx 26.0$
    d. $\int_{-1}^{\frac{3}{2}} \frac{1}{2}\left[\frac{3+x-2x^2}{\sqrt{2}}\right]^2 dx = \frac{625}{192} \approx 3.3$
25. $\int_0^8 \left[\sqrt{8-y}\right]^2 dy = 32$
27. $\int_1^4 \frac{\pi}{2}\left[\frac{\sqrt{8-y}-2}{2}\right]^2 dy = \frac{\pi}{48}\left[299-112\sqrt{7}\right]$
29. a. H^2L
    b. $\int_0^L \left(\frac{H}{L}x\right)^2 dx = \frac{1}{3}H^2L$
    c. $\frac{1}{3}$
31. a. BL
    b. $\int_0^L \frac{B}{L^2}x^2\,dx = \frac{1}{3}BL$
    c. $\frac{1}{3}$

Section 5.2

1. $\int_0^5 \pi x^2\,dx = \frac{125\pi}{3}$
3. $\int_0^{\frac{\pi}{3}} \pi\cos^2(x)\,dx = \frac{\pi}{2}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]$ (There are actually *two* such regions.)
5. $\int_0^7 \pi\left[\sqrt{7-x}\right]^2 dx = \frac{49\pi}{2}$
7. $\int_0^{\sqrt{5}} \pi\left[5-x^2\right]^2 dx = \frac{40\pi\sqrt{5}}{3}$
9. $\int_0^{121} \pi\left[\sqrt{121-x}\right]^2 dx = \frac{14641\pi}{2}$

11. $\int_0^5 \pi \left[\sqrt{\frac{225-9x^2}{25}}\right]^2 dx = 30\pi$

13. $\int_0^1 \pi \left[x^2 - \left(x^4\right)^2\right] dx = \frac{2\pi}{9}$

15. $\int_0^1 \pi \left[\left(\sqrt[4]{y}\right)^2 - \left(\sqrt{y}\right)^2\right] dy = \frac{\pi}{6}$

17. $\int_0^1 \pi \left[\left(x^2\right)^2 - \left(x^3\right)^2\right] dx = \frac{2\pi}{35}$

19. $\int_0^{\frac{\pi}{3}} \pi \left[\sec^2(x) - \cos^2(x)\right] dx = \pi \left[\sqrt{3} - \frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$

21. $\int_0^1 \pi \left[(x-(-4))^2 - \left(x^4-(-4)\right)^2\right] dx = \frac{118\pi}{45}$

23. $\int_0^1 \pi \left[(3-y)^2 - \left(3-\sqrt[4]{y}\right)^2\right] dy = \frac{22\pi}{15}$

25. $\int_0^1 \pi \left[(3-\sin(x))^2 - (3-x)^2\right] dx = \pi \left[6\cos(1) - \frac{1}{4}\sin(2) - \frac{17}{6}\right]$

27. $\int_0^1 \pi \left[\left(y^2-(-2)\right)^2 - \left(y^3-(-2)\right)^2\right] dy = \frac{41\pi}{105}$

29. $\int_0^1 \pi \left[\left(2-\sqrt{x}\right)^2 - \left(2-\sqrt[3]{x}\right)^2\right] dx = \frac{7\pi}{30}$

31. $\int_{-2}^2 \pi \left[\sqrt{4-x^2}\right]^2 dx = \frac{32\pi}{3} = \frac{4}{3}\pi(2)^3$

33. $\int_{-5}^5 \pi \left[3\sqrt{1-\frac{x^2}{25}}\right]^2 dx = 60\pi$

35. $\int_1^2 \pi \left[\left(x^2\right)\right)^2 - (1)^2\right] dx = \frac{26\pi}{5}$

37. $\int_0^{\pi} \pi \left|\frac{1}{2}-\sin(x)\right|^2 dx = \pi \left[\frac{3\pi}{4} - 2\right]$

39. a. $\int_1^{10} \frac{1}{x}\,dx = \ln(10)$; $\int_1^{100} \frac{1}{x}\,dx = \ln(100)$; $\int_1^M \frac{1}{x}\,dx = \ln(M)$; $\lim_{M\to\infty} \ln(M) = \infty$

    b. $\int_1^{10} \frac{\pi}{x^2}\,dx = \frac{9}{10}\pi$; $\int_1^{100} \frac{\pi}{x^2}\,dx = \frac{99}{100}\pi$; $\int_1^M \frac{\pi}{x^2}\,dx = \left[1 - \frac{1}{M}\right]\pi$; $\lim_{M\to\infty} \left[1 - \frac{1}{M}\right]\pi = \pi$

Section 5.3

1. $\sqrt{15^2+20^2} + \sqrt{(-24)^2+18^2} + \sqrt{12^2+(-12)^2} \approx 71.97$ ft

3. $\sqrt{(1-0)^2+(2-1)^2} + \sqrt{(2-1)^2+(4-2)^2} \approx 3.65$

5. a. $\sqrt{(2-0)^2+(5-1)^2} = 2\sqrt{5}$

    b. $\int_0^2 \sqrt{1+[2]^2}\,dx = 2\sqrt{5}$

7. a. $\sqrt{(5-2)^2+(-5-1)^2} = 3\sqrt{5}$

    b. $\int_0^3 \sqrt{[1]^2+[-2]^2}\,dx = 3\sqrt{5}$

9. $y' = x^{\frac{1}{2}} = \sqrt{x}$, so $L = \int_0^4 \sqrt{1+\left[\sqrt{x}\right]^2}\,dx = \int_0^4 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^4 = \frac{2}{3}\left[5\sqrt{5}-1\right]$

11. $y' = x^2 - \frac{1}{4}x^{-2} \Rightarrow 1+[y']^2 = 1+x^4-\frac{1}{2}+\frac{1}{16}x^{-4} = x^4+\frac{1}{2}+\frac{1}{16}x^{-4} = \left(x^2+\frac{1}{4}x^{-2}\right)^2$, so $L = \int_1^5 \sqrt{1+[y']^2}\,dx$
    $= \int_1^5 \left[x^2+\frac{1}{4}x^{-2}\right] dx = \left[\frac{1}{3}x^3-\frac{1}{4}x^{-1}\right]_1^5 = \left[\frac{125}{3}-\frac{1}{20}\right] - \left[\frac{1}{3}-\frac{1}{4}\right] = \frac{623}{15} \approx 41.53$

13. $y' = x^4 - \frac{1}{4}x^{-4} \Rightarrow 1+[y']^2 = 1+x^8-\frac{1}{2}+\frac{1}{16}x^{-8} = x^8+\frac{1}{2}+\frac{1}{16}x^{-8} = \left(x^4+\frac{1}{4}x^{-4}\right)^2$, so $L = \int_1^5 \sqrt{1+[y']^2}\,dx$
    $= \int_1^5 \left[x^4+\frac{1}{4}x^{-4}\right] dx = \left[\frac{1}{5}x^5-\frac{1}{12}x^{-3}\right]_1^5 = \left[625-\frac{1}{1500}\right] - \left[\frac{1}{5}-\frac{1}{12}\right] \approx 624.88$

15. $L = \int_0^1 \sqrt{1+[2x]^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx \approx 1.4789$ (using technology)

17. $L = \int_1^9 \sqrt{1+\left[\frac{1}{2\sqrt{x}}\right]^2}\,dx = \int_1^9 \sqrt{1+\frac{1}{4x}}\,dx \approx 8.2681$

19. $L = \int_0^{\frac{\pi}{4}} \sqrt{1+[\cos(x)]^2}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{1+\cos^2(x)}\,dx \approx 1.058$; $L = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1+\cos^2(x)}\,dx \approx 0.852$

21. $\int_0^{2\pi} \sqrt{[-5\sin(t)]^2 + [2\cos(t)]^2}\, dt = \int_0^{2\pi} \sqrt{25\sin^2(t) + 4\cos^2(t)}\, dt \approx 23.018$
23. $\int_{10}^{20} \sqrt{[-t\cdot\sin(t)+\cos(t)]^2 + [t\cdot\cos(t)+\sin(t)]^2}\, dt = \int_{10}^{20} \sqrt{1+t^2}\, dt \approx 150.346$
24. $\int_0^{2\pi} \sqrt{[R(1-\cos(t))]^2 + [R\sin(t)]^2}\, dt = R\int_0^{2\pi} \sqrt{2\,(1-\cos(t))}\, dt \approx 8R$
25. $\frac{5280\text{ ft}}{2\pi\ \frac{\text{ft}}{\text{rev}}} \cdot 8\ \frac{\text{ft}}{\text{rev}} \approx 6723 \text{ feet} \approx 1.27 \text{ miles}$
27. $L = \int_0^4 \sqrt{1+[2x]^2}\, dx = \int_0^4 \sqrt{1+4x^2}\, dx \approx 16.8186$ so we need a such that $\int_0^a \sqrt{1+4x^2}\, dx \approx \frac{1}{2}(16.8186) = 8.4093$ and b and c such that $\int_0^b \sqrt{1+4x^2}\, dx \approx \frac{1}{3}(16.8186) = 5.6062$ and $\int_0^c \sqrt{1+4x^2}\, dx \approx \frac{2}{3}(16.8186) = 11.2124$. Guessing and checking using technology yields $a \approx 2.77$, $b \approx 2.22$ and $c \approx 3.23$.
29. $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2x^{\frac{1}{2}}}$
31. a. A: $2\pi(3)(4) = 24\pi$; B: $2\pi(1)(2) = 4\pi$
    b. A: $2\pi(5)(4) = 40\pi$; B: $2\pi(8)(2) = 32\pi$
33. a. A: $2\pi(3)(3) = 18\pi$; B: $2\pi(4)(5) = 40\pi$
    b. A: $2\pi(2)(3) = 12\pi$; B: $2\pi(6.5)(5) = 65\pi$
35. $\theta = \frac{\pi}{2} = 90°$ results in the midpoint of the line segment being furthest from the x-axis.
37. $\int_0^1 2\pi x\sqrt{1+[6x^2]^2}\, dx \approx 10.207$
39. $\int_0^1 2\pi\,(2x^2)\sqrt{1+[4x]^2}\, dx \approx 13.306$
41. $\int_0^2 2\pi x^3\sqrt{1+[3x^2]^2}\, dx = \frac{\pi}{27}\left[145\sqrt{145}-1\right]$
43. $\int_0^2 2\pi x\sqrt{1+[2x]^2}\, dx = \frac{\pi}{4}\left[17\sqrt{17}-1\right]$
45. Rotate $y = \sqrt{R^2-x^2}$ about the x-axis to get a sphere of radius R. Then:

$$y = \left[R^2-x^2\right]^{\frac{1}{2}} \Rightarrow y' = -x\left[R^2-x^2\right]^{-\frac{1}{2}} \Rightarrow 1+[y']^2 = 1+\frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

so the slice from $x=a$ to $x=b$ has surface area:

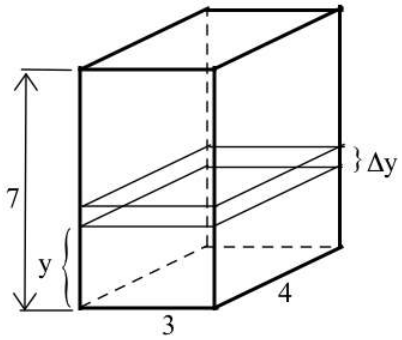
$$\int_a^b 2\pi\sqrt{R^2-x^2}\cdot\sqrt{\frac{1}{R^2-x^2}}\, dx = 2\pi(b-a)$$

which depends only on the width of the slice.
47. Cake: no; frosting: yes.
49. a. $\int_c^d 2\pi\cdot y\sqrt{1+[g'(y)]^2}\, dy$
    b. $\int_c^d 2\pi\cdot g(y)\sqrt{1+[g'(y)]^2}\, dy$
51. a. $\int_0^1 2\pi\cdot y\sqrt{1+[e^y]^2}\, dy \approx 7.055$
    b. $\int_0^1 2\pi\cdot e^y\sqrt{1+[e^y]^2}\, dy \approx 22.943$
53. a. $\int_\alpha^\beta 2\pi\cdot y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\, dt$
    b. $\int_\alpha^\beta 2\pi\cdot x(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\, dt$
55. a. $\int_0^{\frac{\pi}{2}} 2\pi\sin(t)\sqrt{[-\sin(t)]^2+[\cos(t)]^2}\, dt = 2\pi$
    b. $\int_0^{\frac{\pi}{2}} 2\pi\cos(t)\sqrt{[-\sin(t)]^2+[\cos(t)]^2}\, dt = 2\pi$
57. $\int_0^{4\pi} \sqrt{[-\sin(t)]^2+[\cos(t)]^2+[1]^2}\, dt = 4\pi\sqrt{2}$
59. $\int_0^1 \sqrt{[1]^2+[2t]^2+[3t^2]^2}\, dt \approx 1.863$

61. $\int_0^{2\pi} \sqrt{[-3\sin(t)]^2 + [2\cos(t)]^2 + [7\cos(7t)]^2}\, dt \approx 33.293$

Section 5.4

1. A "slice" of water y ft high with thickness Δy ft:



has volume $3 \cdot 4 \cdot \Delta y = 12\Delta y$ ft^3, so its weight is $62.5(12)\Delta y$ lb; the work required to lift this slice $(7-y)$ feet is $750(7-y)\Delta y$ ft-lbs. The total work required to pump out all of the water is:

$$\int_0^7 750(7-y)\, dy = 18375 \text{ ft-lbs}$$

3. a. A slice of oil at initial height y m with thickness Δy m has volume $5 \cdot 2 \cdot \Delta y$ m^3, so its mass is $900\,(10\Delta y) = 9000\Delta y$ kg and the force required to lift it is $(9000\Delta y)(9.81) = 88290\Delta y$ N. Lifting this slice $(4-y)$ m requires $88290(4-y)\Delta y$ N-m of work, so the total work needed to empty the tank is:

$$\int_0^4 88290(4-y)\, dy = 706320 \text{ J}$$

b. The top $10\,\text{m}^3$ corresponds to the top 1 m of oil, so the work required is:

$$\int_3^4 88290(4-y)\, dy = 44145 \text{ J}$$

c. $\dfrac{706320 \text{ J}}{200 \frac{\text{J}}{\text{sec}}} = 3531.15 \text{ sec} \approx 59 \text{ min}$

5. a. A slice of water at initial height y m with thickness Δy m has volume $\pi(1)^2\,\Delta y$ m^3, so its mass is $1000\,(\pi\Delta y) = 1000\pi\Delta y$ kg and the force required to lift it is $(1000\pi\Delta y)(9.81) = 9810\pi\Delta y$ N. Lifting this slice $(6-y)$ m requires $9810\pi(6-y)\Delta y$ N-m of work, so the total work needed to empty the tank is:

$$\int_0^6 9810\pi(6-y)\, dy = 176580\pi \approx 554742 \text{ J}$$

b. The distance each slice must be lifted is now $(6-y)+2 = 8-y$ so the total work done is:

$$\int_5^6 9810\pi(8-y)\, dy = 24525\pi \approx 77048 \text{ J}$$

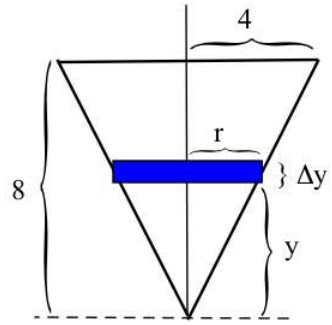
c. The work needed to remove half the water is:

$$\int_3^6 9810\pi(6-y)\, dy = 44145\pi \approx 138686 \text{ J}$$

and $\frac{1}{2}$ hp $= \frac{1}{2}\,(746 \text{ watts}) = 373$ watts so:

$$\frac{138686 \text{ J}}{373 \frac{\text{J}}{\text{sec}}} = 372 \text{ sec} \approx 6.2 \text{ min}$$

6. a. Using similar triangles on a cross-section of the cone and a slice of radius r at height y:



reveals that $\frac{r}{y} = \frac{4}{8} \Rightarrow r = \frac{1}{2}y$ so the volume of the slice is $\pi\left[\frac{y}{2}\right]^2 \Delta y$ ft^3 and its weight is $25 \cdot \pi\frac{y^2}{4}\Delta y$ lb. The work required to lift this slice $8-y$ ft to the top edge of the cone is thus $\frac{25\pi}{4}y^2(8-y)\,\Delta y$ ft-lb, so the total work needed to empty the container is:

$$\int_0^8 \frac{25\pi}{4}y^2(8-y)\,dy = \frac{6400\pi}{3} \approx 6702 \text{ ft-lbs}$$

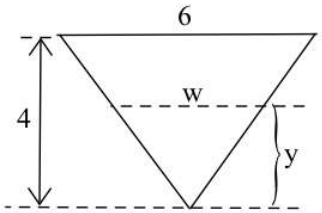
b. $\int_6^8 \frac{25\pi}{4}y^2(8-y)\,dy = \frac{1675\pi}{3} \approx 1754$ ft-lbs

7. Half the work is 3351 ft-lbs, so find h so that:

$$3351 = \int_0^h \frac{25\pi}{4}y^2(8-y)\,dy = \frac{25\pi}{48}\left[32h^3 - 3h^4\right]$$

Solving this equation using technology yields $h \approx 4.9$ so you should dig from the top down to a depth of 3.1 ft and leave the rest for your friend.

8. Using similar triangles on a cross-section of the trough and a slice of width w at height y:



with thickness Δy reveals that $\frac{w}{y} = \frac{6}{4} \Rightarrow w = \frac{3}{2}y$ so the weight of this slice, which must be lifted $(4-y)$ ft, is $80\left(\frac{3}{2}y\right)(7)\Delta y$ lb and the total work needed is:

$$\int_0^4 840y(4-y)\,dy = 8960 \text{ ft-lbs}$$

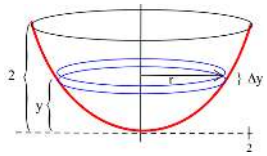
9. If the bottom 70 ft^3 of slop has depth h ft:

$$\frac{1}{2}h\left(\frac{3}{2}h\right)(7) = 70 \Rightarrow h = \sqrt{\frac{40}{3}} \approx 3.65 \text{ ft}$$

so the work needed to remove the top 14 ft^3 is:

$$\int_{3.65}^4 840y(4-y)\,dy \approx 192.21 \text{ ft-lbs}$$

11. a. A horizontal slice at height $y = \frac{1}{2}x^2$ is (approximately) a disk with radius $x = \sqrt{2y}$ and thickness Δy:



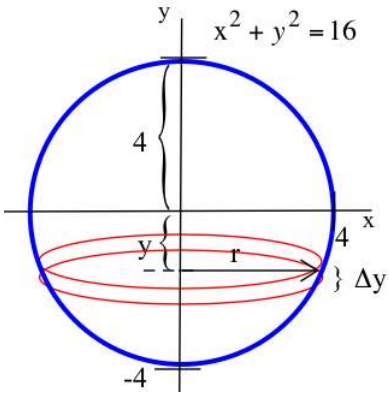
has volume $\pi\left[\sqrt{2y}\right]^2 \Delta y = 2\pi y \Delta y \text{ m}^3$ and mass $2000\pi y \Delta y \text{ m}^3$; that slice must be lifted $(2-y)$ m so the total work required is:

$$\int_0^2 2000\pi y(9.81)(2-y)\,dy \approx 82184 \text{ J}$$

b. Each slice is lifted $(2-y)+3=(5-y)$ m, so the work needed is now:

$$\int_0^2 2000\pi y(9.81)(5-y)\,dy \approx 452012 \text{ J}$$

13. Rotate the circle $x^2+y^2=16$ about the x-axis to generate the sphere. A slice at height y (for $-4 \le y \le 4$) and thickness Δy is (approximately) a disk with radius $x=\sqrt{16-y^2}$:



so the slice has volume $\pi\left[\sqrt{16-y^2}\right]^2 \Delta y$ and mass $1000\pi\left[16-y^2\right]\Delta y$; the slice must be lifted $(4-y)$ m, so the total work is:

$$\int_{-4}^{-2} (9.81)1000\pi\left(16-y^2\right)(4-y)\,dy = 876360\pi \approx 2753166 \text{ J}$$

15. a. The leftmost container: more water is near the bottom, so more of the water must be lifted a greater distance.
b. The rightmost container.

17. a. 85 ft-lbs
b. 35 ft-lbs

19. a. $\int_0^3 6x\,dx = 27$ in-oz ≈ 0.14 ft-lbs
b. $\int_0^6 6x\,dx = 108$ in-oz ≈ 0.56 ft-lbs

21. a. The average force between $x=23$ and $x=33$ is ≈ 15 dyn so the work done is about:

$$15(33-23) = 150 \text{ dyn-cm} = 0.000015 \text{ J}$$

b. The area under the graph between $x=28$ and $x=33$ is about 90 dyn-cm $= 0.000009$ J.

23. Converting cm to m and applying Hooke's Law yields $3(9.8) = k(0.15) \Rightarrow k = 196$ so the total work is:

$$\int_{0.15}^{0.19} 196x\,dx = \left[98x^2\right]_{0.15}^{0.19} = 1.3328 \text{ J}$$

25. a. Ceres' radius is $475 \text{ km} = 475000$ m and:

$$GMm = \left(6.673 \times 10^{-11}\right)\left(896 \times 10^{18}\right)(100)$$

or 5.979×10^{12}; the work to lift the payload from $x=475000$ m to $x=485000$ m is:

$$\int_{475000}^{485000} \frac{5.979 \times 10^{12}}{x^2}\,dx \approx 259530 \text{ J}$$

b. $\int_{475000}^{575000} \frac{5.979 \times 10^{12}}{x^2}\,dx \approx 2189110 \text{ J}$

c. $\int_{475000}^{975000} \frac{5.979 \times 10^{12}}{x^2}\,dx \approx 6455070$ J

27. a. Answers will depend on your mass. If your mass (including a space suit to ensure your survivial) is 100 kg then:

$$GMm = \left(6.673 \times 10^{-11}\right)\left(7.35 \times 10^{22}\right)(100)$$

or 4.905×10^{14}; the work done lifting you from $x = 1737500$ m to $x = 1937500$ m is:

$$\int_{1737500}^{1937500} \frac{4.905 \times 10^{14}}{x^2}\,dx \approx 29138818 \text{ J}$$

b. $\int_{1737500}^{2137500} \frac{4.905 \times 10^{14}}{x^2}\,dx \approx 52824758$ J

c. $\int_{1737500}^{11737500} \frac{4.905 \times 10^{14}}{x^2}\,dx \approx 240496104$ J

29. We know that $-0.1 = \frac{k}{10^2} \Rightarrow k = -10$.

a. $\int_{20}^{10} \frac{-10}{x^2}\,dx = 0.5$ J

b. $\int_{10}^{1} \frac{-10}{x^2}\,dx = 9$ J

c. $\int_{1}^{0.1} \frac{-10}{x^2}\,dx = 90$ J

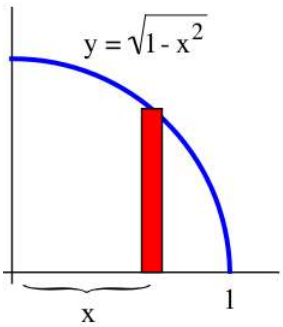
31. $\int_0^{2\pi} t\sqrt{[-\sin(t)]^2 + [\cos(t)]^2}\,dt = \int_0^{2\pi} t\,dt = 2\pi^2$

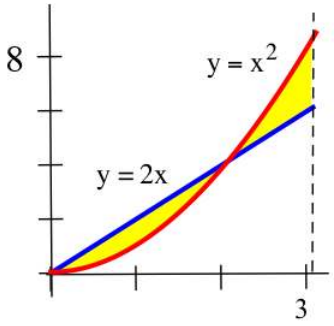
33. $\int_0^1 t\sqrt{[2t]^2 + [1]^2}\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt = \frac{5\sqrt{5}-1}{12}$ }

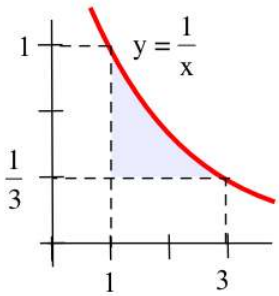
35. This is the same as Problem 31. To solve without calculus, "unroll" the region to get a triangle with base 2π (the circumference of the circle upon which the region was originally sitting) and height 2π. The area of the triangle is then $\frac{1}{2}(2\pi)(2\pi) = 2\pi^2$.

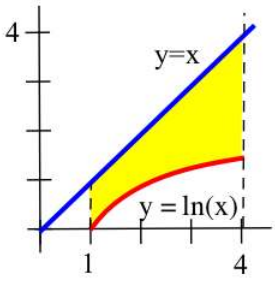
Section 5.5

Graphs of the regions from Problems 1, 3, 5 and 7 appear below:









1. A vertical slice rotated around the y axis results in a tube: $\int_0^1 2\pi x\sqrt{1-x^2}\,dx = \frac{2\pi}{3} \approx 2.09$

3. We need to split this region into two pieces. The curves intersect where $x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. For $0 \le x \le 2$, $y = 2x$ is the top curve and $y = x^2$ is the bottom curve, so the height of a vertical slice is $2x - x^2$; for $2 \le x \le 3$, the roles are reversed, so the height is $x^2 - 2x$:

$$\int_0^2 2\pi(4-x)\left[2x-x^2\right]\,dx + \int_2^3 2\pi(4-x)\left[x^2-2x\right]\,dx = 8\pi + \frac{7\pi}{2} = \frac{23\pi}{2} \approx 36.13$$

5. Vertical slices yield tubes: $\int_1^3 2\pi(5-x)\left[\frac{1}{x} - \frac{1}{3}\right]\,dx = [10\ln(3) - 8]\,\pi \approx 9.38$

7. Vertical slices yield tubes, but the resulting integral requires numerical integration or tables (for now):

$$\int_1^4 2\pi x\left[x - \ln(x)\right]\,dx \approx 85.83$$

Horizontal slices result in washers and require us to split the region into three pieces, but finding antiderivatives is easier:

$$\int_0^1 \pi\left[(e^y)^2-1^2\right]\,dy+\int_1^{\ln(4)} \pi\left[(e^y)^2-y^2\right]\,dy+\int_{\ln(4)}^4 \pi\left[4^2-y^2\right]\,dy$$
$$=\frac{\left(e^2-3\right)\pi}{2}+\frac{\left(50-3e^2-2\left(\ln(4)\right)^3\right)\pi}{6}$$
$$+\frac{(4-\ln(4))^2\left(8+\ln(4)\right)\pi}{3}\approx 85.83$$

9. Washers: $\int_1^2 2\pi x\left[(6-x)-x^2\right]\,dx+\int_2^4 2\pi x\left[x^2-(6-x)\right]\,dx=\frac{35\pi}{6}+\frac{256\pi}{3}=\frac{541\pi}{6}\approx 283.27$
11. Tubes: $\int_0^{\frac{3}{\sqrt{2}}} 2\pi x\left[\sqrt{9-x^2}-x\right]\,dx=9\pi\left(2-\sqrt{2}\right)\approx 16.56$
13. $\int_0^1 2\pi x\left[x^2-x^4\right]\,dx=\frac{\pi}{6}\approx 0.52$
15. $\int_0^{\sqrt{\pi}} 2\pi x\cdot\sin\left(x^2\right)\,dx=2\pi\approx 6.28$
17. $\int_0^{\frac{1}{2}} 2\pi x\cdot\frac{1}{\sqrt{1-x^2}}\,dx=\pi\left(2-\sqrt{3}\right)\approx 0.84$
19. $\int_0^1 2\pi\left(3-x\right)\left[x-x^4\right]\,dx=\frac{22\pi}{15}\approx 4.61$
21. $\int_0^1 \pi\left[(x-(-3))^2-\left(x^4-(-3)\right)^2\right]\,dx=\frac{91\pi}{45}$
23. $\int_0^1 2\pi\left(2-x\right)\cdot\frac{1}{1+x^2}\,dx=\pi\left[\pi-\ln(2)\right]\approx 7.69$
25. $\int_0^1 2\pi\left(x+2\right)\left[1-\frac{1}{1+x^2}\right]\,dx=\pi\left[5-\pi-\ln(2)\right]\approx 3.66$
27. Slicing horizontally results in washers and requires us to split the region into two pieces:

$$\int_{y=0}^{y=1} \pi\left[\left(4-\left(1+y^2\right)\right)^2-\left(4-\left(2+y^2\right)\right)^2\right]\,dy+\int_{y=1}^{y=\sqrt{2}} \pi\left[\left(4-\left(1+y^2\right)\right)^2-(4-3)^2\right]\,dy$$

which evaluates to $=\frac{13\pi}{3}+\frac{\left(24\sqrt{2}-31\right)\pi}{5}\approx 15.46$.

29. Slicing horizontally results in tubes and requires us to split the region into two pieces:

$$\int_{y=0}^{y=1} 2\pi\left(4-y\right)\left[\left(2+y^2\right)-\left(1+y^2\right)\right]\,dy+\int_{y=1}^{y=\sqrt{2}} 2\pi\left(4-y\right)\left[3-\left(1+y^2\right)\right]\,dy$$

which evaluates to $=7\pi+\frac{\left(64\sqrt{2}-83\right)\pi}{6}\approx 25.92$.

Section 5.6

1. a. The total mass is $m=2+5+5=12$ and $M_0=2\cdot 4+5\cdot 2+5\cdot 6=48$ so $\overline{x}=\frac{48}{12}=4$.
   b. The total mass is now $m=2+5+5+8=20$ and if the new object is located at $x=b$ then $M_0=2\cdot 4+5\cdot 2+5\cdot 6+8\cdot b=48+8b$ so:

$$5=\overline{x}=\frac{48+8b}{20}=2.4+0.4b\ \Rightarrow\ b=6.5$$

   c. If the new mass is μ, the total mass becomes $m=12+\mu$ and $M_0=48+\mu\cdot 10$ so:

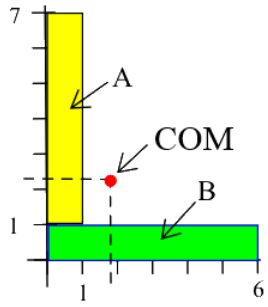
$$6=\overline{x}=\frac{48+10\mu}{12+\mu}\ \Rightarrow \mu=6$$

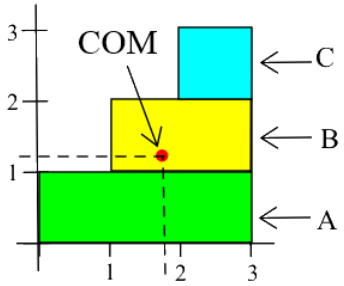
3. a. $m=2+5+5=12$, while $M_y=2\cdot 4+5\cdot 2+5\cdot 6=48$ and $M_x=2\cdot 3+5\cdot 4+5\cdot 2=36$, so $\overline{x}=\frac{48}{12}=4$ and $\overline{x}=\frac{36}{12}=3$
   b. The total mass is now $m=2+5+5+10=22$ and if the new object is located at (b,c) then $M_y=48+10b$ and $M_x=36+10c$ so:

$$5=\overline{x}=\frac{48+10b}{22}=\frac{24}{11}+\frac{5}{11}b\ \Rightarrow\ b=6.2$$
$$2=\overline{y}=\frac{36+10c}{22}=\frac{18}{11}+\frac{5}{11}c\ \Rightarrow\ c=0.8$$

You should locate the new object at $(6.2, 0.8)$.

5. Split the region into two rectangles, A and B (see figure below left). Then A has mass 6 and center of mass $(0.5, 4)$, while B has mass 6 and center of mass $(3, 0.5)$. The total mass is $m = 12$, $M_y = (6)(0.5) + (6)(3) = 21$ and $M_x = (6)(4) + (6)(0.5) = 27$, so $\bar{x} = \frac{21}{12} = 1.75$ and $\bar{y} = \frac{27}{12} = 2.25$. Note the center of mass $(1.75, 2.25)$ is not in the region!



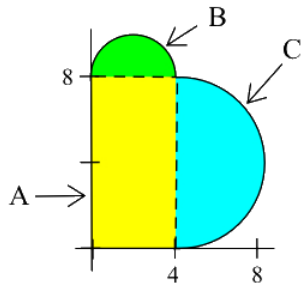


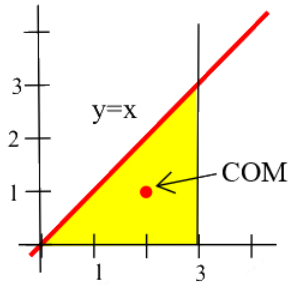
7. Split the region into three rectangles, A, B and C (see figure above right). Then A has mass 3 and center of mass $(1.5, 0.5)$, while B has mass 2 and center of mass $(2, 1.5)$, and C has mass 1 and center of mass $(2.5, 2.5)$. The total mass is $m = 6$, $M_y = (3)(1.5) + (2)(2) + (1)(2.5) = 11$ and $M_x = (3)(0.5) + (2)(1.5) + (1)(2.5) = 7$, so $\bar{x} = \frac{11}{6}$ and $\bar{y} = \frac{7}{6}$.

9. Split the region into three pieces, A, B and C (see below left); assume $\rho = 1$. A has mass $(8)(4) = 32$ and center of mass $(2, 4)$, while B has mass $\frac{1}{2}\pi(2^2) = 2\pi$ and center of mass $\left(2, 8 + \frac{4}{3\pi}(2)\right)$, and C has mass $\frac{1}{2}\pi(4^2) = 8\pi$ and center of mass $\left(4 + \frac{4}{3\pi}(4), 4\right)$. The total mass is $32 + 10\pi$, while:

$$M_y = (32)(2) + (2\pi)(2) + (8\pi)\left(4 + \frac{16}{3\pi}\right) \approx 219.76$$
$$M_x = (32)(4) + (2\pi)\left(8 + \frac{8}{3\pi}\right) + (8\pi)(4) = 284.13$$

so $\bar{x} \approx \frac{219.76}{63.42} \approx 3.47$ and $\bar{y} \approx \frac{284.13}{63.42} \approx 4.48$



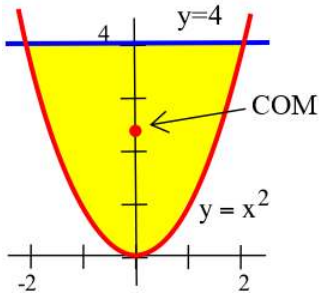


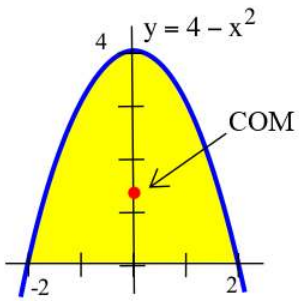
11. See above right. With $\rho = 1$, $m = \int_0^3 x\,dx = \frac{9}{2}$, $M_y = \int_0^3 x \cdot x\,dx = 9$ and $M_x = \int_0^3 \frac{1}{2} \cdot [x]^2\,dx = \frac{9}{2}$ so $\bar{x} = \frac{9}{4.5} = 2$ and $\bar{y} = \frac{4.5}{4.5} = 1$.

13. See below left. The region is symmetric about the y-axis, so $\bar{x} = 0$. With $\rho = 1$, $m = \int_{-2}^{2} [4 - x^2]\,dx = \frac{32}{3}$ and:

$$M_x = \int_{-2}^{2} \frac{1}{2}\left[(4)^2 - (x^2)^2\right]\,dx = \frac{128}{5}$$

so $\bar{y} = \frac{\frac{128}{5}}{\frac{32}{3}} = \frac{12}{5} = 2.4$.



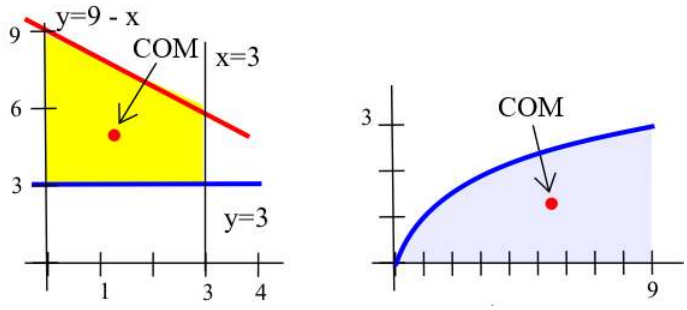


15. See above right. By symmetry, $\bar{x} = 0$. With $\rho = 1$, $m = \int_{-2}^{2} (4 - x^2)\, dx = \frac{32}{3}$ and:

$$M_x = \int_{-2}^{2} \frac{1}{2}\left[4 - x^2\right]^2 dx = \frac{256}{15}$$

so $\bar{y} = \frac{\frac{256}{15}}{\frac{32}{3}} = \frac{8}{5} = 1.6$.

17. If $\rho = 1$, $m = \int_0^3 [(9 - x) - 3]\, dx = \frac{27}{2}$, while $M_y = \int_0^3 x \cdot [(9 - x) - 3]\, dx = 18$ and $M_x = \int_0^3 \frac{1}{2}\left[(9 - x)^2 - 3^2\right] dx = 72$ so $\bar{x} = \frac{18}{\frac{27}{2}} = \frac{4}{3}$ and $\bar{y} = \frac{72}{\frac{27}{2}} = \frac{16}{3}$. (See below left.)

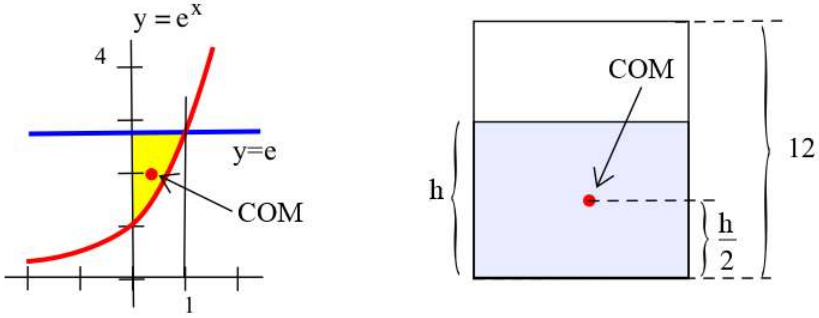


19. See above right. If $\rho = 1$, $m = \int_0^9 \sqrt{x}\, dx = 18$, $M_y = \int_0^9 x \cdot \sqrt{x}\, dx = 97.2$ and $M_x = \int_0^9 \frac{1}{2}\left[\sqrt{x}\right]^2 dx = 20.25$ so $\bar{x} = \frac{97.2}{18} = 5.4$ and $\bar{y} = \frac{20.25}{18} = 1.125$.

21. If $\rho = 1$, $m = \int_0^1 [e - e^x]\, dx = 1$, while:

$$M_y = \int_0^1 x \cdot [e - e^x]\, dx \approx 0.359$$
$$M_x = \int_0^1 \frac{1}{2}\left[e^2 - (e^x)^2\right] dx = \frac{1}{4}\left(e^2 + 1\right) \approx 2.097$$

so $\bar{x} \approx 0.359$ and $\bar{y} \approx 2.097$. (See below left.)



23. a. The box has weight 10 and center of mass located 6 inches above the center of its base. If the box is filled with liquid to a height h inches, then the weight of the liquid is $\left(\frac{h}{12}\right)(60) = 5h$ and the liquid's center of mass is located $\frac{h}{2}$ inches above the center of the box's base (see figure above right). The total weight is thus $10 + 5h$ and the moment of the system about the base of the box is $(10)(6) + (5h)\left(\frac{h}{2}\right) = 60 + \frac{5}{2}h^2$, making the height of the system's center of gravity:

$$H = \frac{60 + \frac{5}{2}h^2}{10 + 5h}$$

b. To minimize H, compute:

$$\frac{dH}{dh} = \frac{(10 + 5h) \cdot 5h - \left(60 + \frac{5}{2}h^2\right) \cdot 5}{(10+5h)^2}$$

Set this equal to 0 to get $h = -2 \pm 2\sqrt{7}$. The $-$ option is unrealistic, so the only critical point occurs where $h = -2 + 2\sqrt{7} \approx 3.3$ inches. Both "endpoints" (empty and full) have center of mass at 6 inches, so $h \approx 3.3$ must yield a minimum.

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25. The can has mass 15 g and center of mass located at height 6 cm. If the soda in the can has height h, the mass of the soda is $(400\text{ g})\left(\frac{h}{12}\right) = \frac{100h}{3}$ g and its center of mass is at height $\frac{h}{2}$. The mass of the can-soda system is $15 + \frac{100h}{3}$ and the moment of the system about the bottom of the can is $(15)(6) + \left(\frac{100h}{3}\right)\left(\frac{h}{2}\right)$ so the height of the system's center of mass is:

$$H = \frac{90 + \frac{50h^2}{3}}{15 + \frac{100h}{3}} = \frac{54 + 10h^2}{9 + 20h}$$

27. On your own.
29. Yes. (What are the only shapes that have exactly two lines of symmetry?)
31. The center of gravity of the water is 2 feet above the bottom of the box, so the total work is $(300\text{ lb})(8\text{ ft}) = 2400\text{ ft-lbs}$.
33. The center of gravity of the water is at the center of the sphere. The total volume of water is $\frac{4}{3}\pi(2)^3 = \frac{32\pi}{3}\text{ m}^3$, so the total mass of water is $\frac{32000\pi}{3}\text{ kg}$ and the total work is $\left(\frac{32000\pi}{3}\text{ kg}\right)\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)(5\text{ m}) \approx 1643681\text{ J}$.
35. The area of the square is 4 cm^2. Using Pappus' Theorem about solids of revolution:
    a. $(4\text{ cm}^2)(2\pi \cdot 4\text{ cm}) = 32\pi\text{ cm}^3$
    b. $(4\text{ cm}^2)(2\pi \cdot 3\text{ cm}) = 24\pi\text{ cm}^3$
    c. $(4\text{ cm}^2)(2\pi \cdot 2\text{ cm}) = 16\pi\text{ cm}^3$
    d. $(4\text{ cm}^2)(2\pi \cdot 3\text{ cm}) = 24\pi\text{ cm}^3$
    e. $(4\text{ cm}^2)\left(2\pi\sqrt{\left(\frac{18}{11}\right)^2 + \left(\frac{32}{11}\right)^2}\text{ cm}\right) \approx 20.97\text{ cm}^3$
37. The perimeter of the square is 4 cm. Using Pappus' Theorem about surface areas of revolution:
    a. $(4\text{ cm})(2\pi \cdot 4\text{ cm}) = 32\pi\text{ cm}^2$
    b. $(4\text{ cm})(2\pi \cdot 3\text{ cm}) = 24\pi\text{ cm}^2$
    c. $(4\text{ cm})(2\pi \cdot 2\text{ cm}) = 16\pi\text{ cm}^2$
    d. $(4\text{ cm})(2\pi \cdot 3\text{ cm}) = 24\pi\text{ cm}^2$
    e. $(4\text{ cm})\left(2\pi\sqrt{\left(\frac{18}{11}\right)^2 + \left(\frac{32}{11}\right)^2}\text{ cm}\right) \approx 20.97\text{ cm}^2$
39. The area of the circle is $4\pi\text{ cm}^2$ and the perimeter is 4π cm. Using Pappus' Theorems:
    a. volume $= (4\pi\text{ cm}^2)(2\pi \cdot 5\text{ cm}) = 40\pi^2\text{ cm}^3$; surface area $= (4\pi\text{ cm})(2\pi \cdot 5\text{ cm}) = 40\pi^2\text{ cm}^2$
    b. V $= (4\pi\text{ cm}^2)(2\pi \cdot 3\text{ cm}) = 24\pi^2\text{ cm}^3$; SA $= (4\pi\text{ cm})(2\pi \cdot 3\text{ cm}) = 24\pi^2\text{ cm}^2$
    c. V $= (4\pi\text{ cm}^2)(2\pi \cdot 4\text{ cm}) = 32\pi^2\text{ cm}^3$; SA $= (4\pi\text{ cm})(2\pi \cdot 4\text{ cm}) = 32\pi^2\text{ cm}^2$
    d. V $= (4\pi\text{ cm}^2)(2\pi \cdot 3\text{ cm}) = 24\pi^2\text{ cm}^3$; SA $= (4\pi\text{ cm})(2\pi \cdot 3\text{ cm}) = 24\pi^2\text{ cm}^2$
    e. V $= (4\text{ cm}^2)\left(2\pi \cdot \sqrt{\left(\frac{24}{11}\right)^2 + \left(\frac{36}{11}\right)^2}\text{ cm}\right) \approx 98.86\text{ cm}^3\}$
    f. SA $= (4\text{ cm})\left(2\pi \cdot \sqrt{\left(\frac{24}{11}\right)^2 + \left(\frac{36}{11}\right)^2}\text{ cm}\right) \approx 98.86\text{ cm}^2\}$
41. Each rectangle has area 8, perimeter 12 and a centroid 3 ft away from the axis, so each rectangle has volume $8 \cdot 2\pi \cdot 3 = 48\pi\text{ ft}^3$ and surface area $12 \cdot 2\pi \cdot 3 = 72\pi\text{ ft}^2$.
43. Colorado? Hawaii?

Section 5.7

1. Replace the upper (infinite) limit of the integral with a massive number M:

$$\int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{M\to\infty}\int_{10}^{M} x^{-3}\,dx = \lim_{M\to\infty}\left[-\frac{1}{2}x^{-2}\right]_{10}^{M}$$
$$= \lim_{M\to\infty}\left[-\frac{1}{2M^2} + \frac{1}{200}\right] = \frac{1}{200}$$

   so the improper integral converges to 0.05.
3. Replacing the upper limit of the integral with M:

$$\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{M\to\infty}\Big[\arctan(x)\Big]_{\sqrt{3}}^{M}$$
$$= \lim_{M\to\infty}\left[\arctan(M) - \frac{\pi}{3}\right] = \frac{\pi}{6}$$

   so the improper integral converges to $\frac{\pi}{6}$.

5. Replacing the upper limit of the integral with M:

$$\int_e^{\infty} \frac{5}{x \cdot \ln(x)}\, dx = \lim_{M\to\infty} \int_e^{M} \frac{5}{x \cdot \ln(x)}\, dx$$

Substitute $u = \ln(x) \Rightarrow du = \frac{1}{x}\, dx$, so $x = e \Rightarrow u = \ln(e) = 1$ and $x = M \Rightarrow u = \ln(M)$:

$$\begin{aligned}\int_e^{M} \frac{5}{x \cdot \ln(x)}\, dx = \int_1^{\ln(M)} \frac{5}{u}\, du &= \Big[5\ln(u)\Big]_1^{\ln(M)} \\ &= 5\ln\left(\ln(M)\right) - 0\end{aligned}$$

Returning to the improper integral:

$$\lim_{M\to\infty} \int_e^{M} \frac{5}{x \cdot \ln(x)}\, dx = \lim_{M\to\infty} \left[5\ln\left(\ln(M)\right)\right] = \infty$$

so the improper integral diverges.

7. Replacing the upper limit of the integral with M:

$$\begin{aligned}\lim_{M\to\infty} \int_3^{M} \frac{1}{x-2}\, dx &= \lim_{M\to\infty} \Big[\ln\left(|x-2|\right)\Big]_3^{M} \\ &= \lim_{M\to\infty} \left[\ln(M-2) - 0\right] = \infty\end{aligned}$$

so the improper integral diverges.

9. Replacing the upper limit of the integral with M:

$$\begin{aligned}\lim_{M\to\infty} \int_3^{M} (x-2)^{-3}\, dx &= \lim_{M\to\infty} \left[-\frac{1}{2}(x-2)^{-2}\right]_3^{M} \\ &= \lim_{M\to\infty} \left[-\frac{1}{2(M-2)^2} + \frac{1}{2}\right]\end{aligned}$$

so the improper integral converges to 0.5.

11. Replacing the upper limit of the integral with M:

$$\begin{aligned}\lim_{M\to\infty} \int_3^{M} (x+2)^{-2}\, dx &= \lim_{M\to\infty} \left[-(x+2)^{-1}\right]_3^{M} \\ &= \lim_{M\to\infty} \left[-\frac{1}{M+2} + \frac{1}{5}\right]\end{aligned}$$

so the improper integral converges to 0.2.

13. Replacing the lower limit of the integral with a:

$$\begin{aligned}\lim_{a\to 0^+} \int_a^{4} x^{-\frac{1}{2}}\, dx &= \lim_{a\to 0^+} \left[2\sqrt{x}\right]_a^{4} \\ &= \lim_{a\to 0^+} \left[4 - 2\sqrt{a}\right] = 4\end{aligned}$$

so the improper integral converges to 4.

15. Replacing the lower limit of the integral with a:

$$\begin{aligned}\lim_{a\to 0^+} \int_a^{16} x^{-\frac{1}{4}}\, dx &= \lim_{a\to 0^+} \left[\frac{4}{3}x^{\frac{3}{4}}\right]_a^{16} \\ &= \lim_{a\to 0^+} \left[\frac{4}{3}\cdot 8 - \frac{4}{3}a^{\frac{3}{4}}\right] = \frac{32}{3}\end{aligned}$$

so the improper integral converges to $\frac{32}{3}$.

17. Replacing the upper limit of the integral with b and using Appendix I:

$$\begin{aligned}\lim_{b\to 2^-} \int_0^{b} \frac{1}{\sqrt{4-x^2}}\, dx &= \lim_{b\to 2^-} \left[\arcsin\left(\frac{x}{2}\right)\right]_0^{b} \\ &= \lim_{b\to 2^-} \arcsin\left(\frac{b}{2}\right) = \frac{\pi}{2}\end{aligned}$$

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so the improper integral converges to $\frac{\pi}{2}$.

19. Replacing the upper limit of the integral with M:

$$\begin{aligned}\lim_{M\to\infty}\int_{-2}^{M}\sin(x)\,dx &= \lim_{M\to\infty}\Big[-\cos(x)\Big]_{-2}^{M}\\ &= \lim_{M\to\infty}\left[-\cos(M)+\cos(-2)\right]\end{aligned}$$

This limit does not exist, so the integral diverges.

21. Replacing the upper limit of the integral with b:

$$\begin{aligned}\lim_{b\to\frac{\pi}{2}^-}\int_{0}^{b}\tan(x)\,dx &= \lim_{b\to\frac{\pi}{2}^-}\Big[-\ln\left(|\cos(x)|\right)\Big]_{0}^{b}\\ &= \lim_{b\to\frac{\pi}{2}^-}\left[-\ln\left(\cos(b)\right)\right]=\infty\end{aligned}$$

so the improper integral diverges.

23. The issue for this integrand occurs not an endpoint but at the point $x=\frac{\pi}{2}$, so we need to split the original integral into two pieces, each of which has only one improper endpoint:

$$\int_{0}^{\pi}\tan(x)\,dx=\int_{0}^{\frac{\pi}{2}}\tan(x)\,dx+\int_{\frac{\pi}{2}}^{\pi}\tan(x)\,dx$$

From Problem 21, we know the first of these two new integrals diverges, so the original integral must diverge as well.

25. This integral is improper at both endpoints, so we need to split it into two pieces. We can choose any value we like at which to split the interval of integration, but $x=0$ works nicely:

$$\int_{\infty}^{\infty}\frac{1}{1+x^2}\,dx=\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx+\int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

The first of these new integrals converges (to $\frac{\pi}{2}$):

$$\lim_{N\to-\infty}\int_{N}^{0}\frac{1}{1+x^2}\,dx=\lim_{N\to-\infty}\left[-\arctan\left(N\right)\right]=\frac{\pi}{2}$$

Similarly, the second integral converges (to $\frac{\pi}{2}$):

$$\lim_{M\to\infty}\int_{0}^{M}\frac{1}{1+x^2}\,dx=\lim_{M\to\infty}\left[\arctan\left(M\right)\right]=\frac{\pi}{2}$$

so the original integral converges to $\frac{\pi}{2}+\frac{\pi}{2}=\pi$.

27. Converges (by the P-Test, with $p=5>1$).

29. Converges (by the P-Test, with $p=\frac{6}{5}>1$).

31. Diverges (by the P-Test, with $p=\frac{2}{3}<1$).

33. Converges (by the P-Test, with $p=\frac{7}{4}>1$).

35. Using the comparison $0<\frac{1}{x^2+5}<\frac{1}{x^2}$ and the fact that $\int_3^\infty\frac{1}{x^2}\,dx$ converges (P-Test, with $p=2>1$), the smaller integral $\int_3^\infty\frac{1}{x^2+5}\,dx$ converges.

37. When $x>5$, $x^3+x>x^3$ so $0<\frac{1}{x^3+x}<\frac{1}{x^3}$. By the P-Test with $p=3>1$, we know that $\int_3^\infty\frac{1}{x^3}\,dx$ converges, so the smaller integral $\int_3^\infty\frac{1}{x^3+x}\,dx$ also converges.

39. For $x\geq e$, $\ln(x)<x$, so $x+\ln(x)<2x$ and:

$$\frac{1}{x+\ln(x)}>\frac{1}{2x}\;\Rightarrow\;\frac{7}{x+\ln(x)}>\frac{7}{2}\cdot\frac{1}{x}$$

By the P-Test with $p=1$, we know that $\int_e^\infty\frac{1}{x}\,dx$ diverges, so $\frac{7}{2}\int_e^\infty\frac{1}{x}\,dx$ diverges and the bigger integral $\int_e^\infty\frac{7}{x+\ln(x)}\,dx$ must diverge as well.

41. For $x\geq\pi$, $-1\leq\cos(x)\leq1$ so:

$$0\leq1+\cos(x)\leq2\;\Rightarrow\;0\leq\frac{1+\cos(x)}{x^2}\leq\frac{2}{x^2}$$

We also know that $\int_\pi^\infty \frac{1}{x^2}\,dx$ converges (by the P-Test with $p=2>1$), hence $2\cdot\int_\pi^\infty \frac{1}{x^2}\,dx$ converges, so the smaller integral $\int_\pi^\infty \frac{1+\cos(x)}{x^2}\,dx$ converges as well.

43. For $x\geq 1$, $x^5+1\leq x^5+x^5=2x^5$ so:

$$\frac{1}{x^5+1}\geq\frac{1}{2x^5}\;\Rightarrow\;\frac{x^4}{x^5+1}\geq\frac{x^4}{2x^5}=\frac{1}{2}\cdot\frac{1}{x}$$

Because $\int_1^\infty \frac{1}{x}\,dx$ diverges (by the P-Test with $p=1$), we know that $\int_1^\infty \frac{1}{2}\cdot\frac{1}{x}\,dx$ diverges, so the bigger integral $\int_1^\infty \frac{x^4}{x^5+1}\,dx$ diverges as well. We can write the original integral as:

$$\int_0^1 \frac{x^4}{x^5+1}\,dx+\int_1^\infty \frac{x^4}{x^5+1}\,dx$$

Because the second of these integrals diverges, the original integral diverges.

45. Using the disk method, the volume is:

$$\begin{aligned}\int_1^\infty \pi\cdot\left[\frac{1}{x}\right]^2 dx&=\lim_{M\to\infty}\pi\int_1^M x^{-2}\,dx\\&=\lim_{M\to\infty}\pi\left[-\frac{1}{x}\right]_1^M=\lim_{M\to\infty}\pi\left[-\frac{1}{M}+1\right]\end{aligned}$$

so the volume is finite and is equal to π.

47. Using the disk method, the volume is:

$$\int_0^\infty \pi\cdot\left[\frac{1}{1+x^2}\right]^2 dx=\lim_{M\to\infty}\pi\int_0^M\frac{1}{(x^2+1)^2}\,dx$$

Using Appendix I to find an antiderivative:

$$\begin{aligned}&\lim_{M\to\infty}\frac{\pi}{2}\left[\frac{x}{x^2+1}+\arctan(x)\right]_0^M\\&=\lim_{M\to\infty}\frac{\pi}{2}\left[\frac{M}{M^2+1}+\arctan(M)\right]=\frac{\pi}{2}\cdot\frac{\pi}{2}\end{aligned}$$

so the volume is finite and is equal to $\frac{\pi^2}{4}$.

49. Using the tube method, the volume is:

$$\int_0^\infty 2\pi x\left[\frac{1}{1+x^2}\right]dx=\lim_{M\to\infty}\pi\int_0^M\frac{2x}{(x^2+1)}\,dx$$

Using the substitution $u=x^2+1$, this becomes:

$$\lim_{M\to\infty}\pi\left[\ln(x^2+1)\right]_0^M=\lim_{M\to\infty}\pi\left[\ln(M^2+1)\right]=\infty$$

so the volume is infinite.

51. The sum (which is a left-endpoint Riemann sum for the integral) is larger.

53. The sum is larger.

55. As long as $s>0$:

$$\begin{aligned}F(s)&=\int_0^\infty e^{-st}\cdot 1\,dt=\lim_{M\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^M\\&=\lim_{M\to\infty}-\frac{1}{s}\left[e^{-sM}-1\right]=\frac{1}{s}\end{aligned}$$

57. As long as $s>0$:

$$\begin{aligned}G(s)&=\int_2^\infty e^{-st}\cdot 1\,dt=\lim_{M\to\infty}\left[-\frac{1}{s}e^{-st}\right]_2^M\\&=\lim_{M\to\infty}-\frac{1}{s}\left[e^{-sM}-e^{-2s}\right]=\frac{e^{-2s}}{s}\end{aligned}$$

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59. If $b > 0$, the integral $\int_0^b \frac{1}{x^q}\,dx$ converges if $q < 1$ and diverges if $q \geq 1$. The proof mimics the proof of the P-Test. Rewrite the integral as:

$$\lim_{a \to 0^+} \int_a^b \frac{1}{x^q}\,dx$$

then consider the cases $q = 1$, $q < 1$ and $q > 1$.

Section 5.8

1. For window A, the total hydrostatic force is:

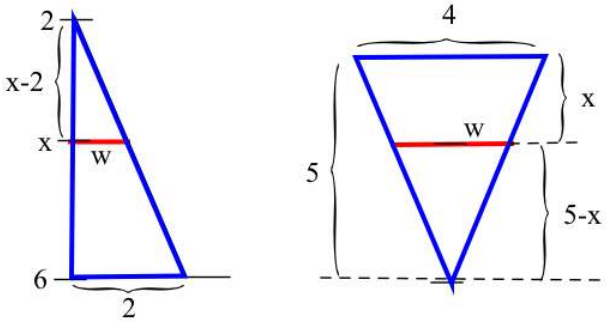
$$\int_2^6 2\rho x\,dx = \rho\left[x^2\right]_2^6 = 32\rho$$

For window B, the width w of a horizontal slice at depth x (see below left) satisfies:

$$\frac{w}{x-2} = \frac{2}{4} \;\Rightarrow\; w = \frac{1}{2}(x-2)$$

so the total hydrostatic force is:

$$\int_2^6 2\rho x \cdot \frac{1}{2}(x-2)\,dx = \frac{\rho}{2}\left[\frac{1}{3}x^3 - x^2\right]_2^6 = \frac{56\rho}{3}$$



3. For the triangular end, the width w of a horizontal slice at depth x (see above right) satisfies:

$$\frac{w}{5-x} = \frac{4}{5} \;\Rightarrow\; w = \frac{4}{5}(5-x)$$

so the total hydrostatic force is:

$$\int_0^5 \rho x \cdot \frac{4}{5}(5-x)\,dx = \frac{4\rho}{5}\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^5 = \frac{50\rho}{3}$$

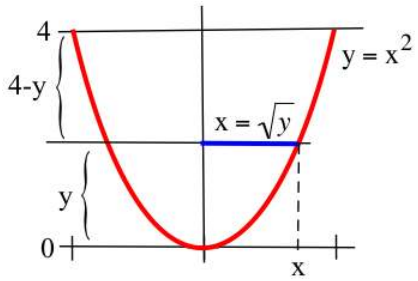
For the rectangular end, the total force is:

$$\int_0^5 \rho x \cdot 4\,dx = 2\rho\left[x^2\right]_0^5 = 50\rho$$

The length plays no role in the above computations, so doubling the length does not change the value of either hydrostatic force.

5. A slice at height y above the bottom of the tank has width $2\sqrt{y}$ (see below) so the total force is:

$$\begin{aligned}\int_0^4 \rho(4-y)\cdot 2\sqrt{y}\,dy &= 2\rho\int_0^4 \left[4y^{\frac{1}{2}} - y^{\frac{3}{2}}\right]\,dy \\ &= 2\rho\left[\frac{8}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right]_0^4 = \frac{256\rho}{15}\end{aligned}$$



7. The one with the largest perimeter (not the left one, probably the middle one).

9. a. A "slice" of the cylinder at depth x of thickness Δx has area $2\pi \cdot 20 \cdot \Delta x$ so the total force against the bottom 2 feet of the cylinder is:

$$\int_{28}^{30} \rho x \cdot 40\pi \, dx = 20\pi\rho \left[x^2\right]_{28}^{30} = 2320\pi\rho$$

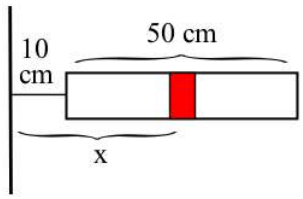
b. Only the limits of integration change:

$$\int_{33}^{35} \rho x \cdot 40\pi \, dx = 20\pi\rho \left[x^2\right]_{33}^{35} = 2720\pi\rho$$

11. a. $m = 20$ g and $v = 3 \cdot 2\pi \cdot 15 = 90\pi$ cm/sec, so KE $= \frac{1}{2} \cdot 20\,(90\pi)^2 = 81000\pi^2 \approx 799438$ ergs ≈ 0.08 J.
b. $m = 20$ g, $v = 3 \cdot 2\pi \cdot 20 = 120\pi$ cm/sec, so KE $= \frac{1}{2} \cdot 20\,(120\pi)^2 = 144000\pi^2 \approx 1421223$ ergs

13. The kinetic energy of a vertical slice of the bar (see below) of width Δx cm located x cm from the axis has kinetic energy $\frac{1}{2}\,(3\,\Delta x)\,(2 \cdot 2\pi x)^2 = 24\pi^2 x^2\,\Delta x$ so the object's total kinetic energy is:

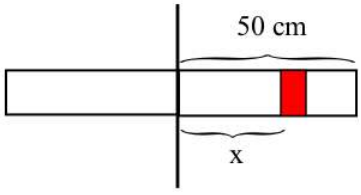
$$\int_{10}^{60} 24\pi^2 x^2 \, dx = 8\pi^2 x^3 \Big|_{10}^{60} \approx 1.7 \text{ J}$$



15. The density of the bar is 0.2 g/cm so the kinetic energy of a vertical slice of the bar (see below) of width Δx cm located x cm from the axis has kinetic energy $\frac{1}{2}\,(0.2\,\Delta x)\,(2\pi x)^2 = 0.4\pi^2 x^2\,\Delta x$ so the kinetic energy of the right half of the object is:

$$\int_0^{50} 0.4\pi^2 x^2 \, dx = \frac{0.4}{3}\pi^2 x^3 \Big|_0^{50} \approx 164493.4 \text{ ergs}$$

By symmetry, the total kinetic energy of the object is $2 \cdot 164493.4 = 328,986.8$ ergs.



17. a. Proceeding as in Example 5, an annular "slice" of the first washer of thickness Δx and radius x has (approximate) mass $1 \cdot 2\pi x \Delta x \cdot 1$ g and velocity $6\pi x$ cm/sec so the kinetic energy of the slice is (approximately) $\frac{1}{2}\,(2\pi x \delta x)\,(6\pi x)^2 = 36\pi^3 x^3 \Delta x$ and the total kinetic energy of the washer is thus:

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$$\int_1^3 36\pi^3 x^3\,dx = 9\pi^3 x^4\Big|_1^3 = 720\pi^3 \text{ ergs}$$

b. Only the height (2 cm instead of 1 cm) and the radii of the second washer differ from the first washer, so the total kinetic energy of the second washwer is:

$$\int_3^4 72\pi^3 x^3\,dx = 18\pi^3 x^4\Big|_3^4 = 3150\pi^3 \text{ ergs}$$

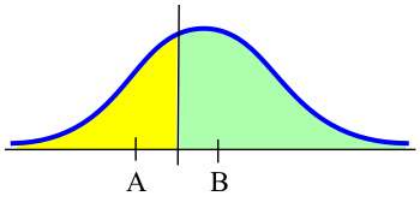
19. The mass of a vertical slice of the place of thickness Δx cm located x cm right of the axis is $3 \cdot 10 \cdot \Delta x$ g and its velocity is $2 \cdot 2\pi x = 4\pi x$ cm/sec so its kinetic energy is $\frac{1}{2} \cdot 30\Delta x\,(4\pi x)^2 = 240\pi^2 x^2 \delta x$ and the kinetic energy of the right half of the plate is:

$$\int_0^3 240\pi^2 x^2\,dx = 80\pi^2 x^3\Big|_0^3 = 2160\pi^2 \text{ ergs}$$

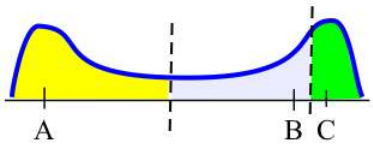
and the kinetic energy of the entire plate is $2 \cdot 2160\pi^2 = 4320\pi^2$ ergs.

21. a votes for candidate A, b for A and c for C.

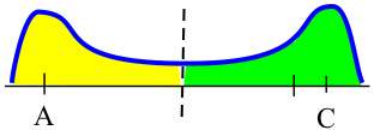
23. B wins:



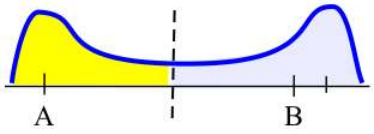
25. a. A wins: FIGURE



b. C wins:



c. B wins:



27. a. A

b. A

c. looks like C

29. On your own.

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CHAPTER OVERVIEW

6: Differential Equations

This chapter introduces you to differential equations, a major field in applied and theoretical mathematics that provides useful tools for engineers, scientists and others studying changing phenomena.

Physical laws of motion, heat and electricity can be expressed using differential equations. The growth of a population, the changing gene frequencies in that population, and the spread of a disease can be described by differential equations. Economic and social models use differential equations, and the earliest examples of “chaos” came from differential equations used for modeling atmospheric behavior. Some scientists even assert that the main purpose of a calculus course should be to teach people to understand and solve differential equations. (This chapter merely provides an introduction. In the near future you may very well take one or more classes solely devoted to solving differential equations.)

The purpose of this chapter is to introduce some basic ideas, vocabulary and techniques for differential equations and to explore additional applied problems that can be solved using calculus. Applications in this chapter include exponential population growth, calculating how long a population takes to double in size, radioactive decay and its use for dating ancient objects and detecting fraud, describing the motion of an object, and chemical mixtures and rates of reaction.

6.1: Introduction to Differential Equations

6.2: The Differential Equation y = f'(x)

6.3: Separable Equations

6.4: Exponential Growth and Decay

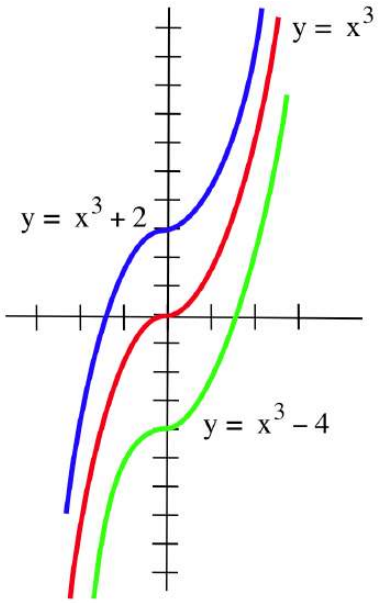
6.5: Heating, Cooling and Mixing

6.A: Answers

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6.1: Introduction to Differential Equations

Algebraic equations involve constants and variables, and solutions of algebraic equations typically involve numbers. For example, $x = 3$ and $x = -2$ are solutions of the algebraic equation $x^2 = x + 6$. **Differential equations** contain derivatives (or differentials) of functions and solutions of differential equations are functions. The differential equation $y' = 3x^2$ has infinitely many solutions, and two of those solutions are the functions $y = x^3 + 2$ and $y = x^3 - 4$:



You have already solved lots of differential equations: every time you found an antiderivative of a function $f(x)$, you solved the differential equation $y' = f(x)$ to get a solution y. You have also used differential equations in applications. Areas, volumes, work and motion problems all involved integration and finding antiderivatives, so they all involved solving a differential equation. The differential equation $y' = f(x)$, however, is just the beginning. Other applications generate differential equations that may involve higher-order derivatives of y (such as y'') and functions of y as well as x.

Checking Solutions of Differential Equations

Whether a differential equation is easy or difficult to solve, it is important to be able to check that a possible solution actually satisfies the differential equation. A possible solution of an algebraic equation can be checked by putting the solution into the equation to see if it results in true statement: $x = 3$ is a solution of $5x + 1 = 16$ because $5(3) + 1 = 16$ is true; $x = 4$ is not a solution, because $5(4) + 1 \neq 16$.

Similarly, a solution of a differential equation can be checked by substituting the function (and its appropriate derivatives) into the original equation to see if the result is true: $y = x^2$ is a solution of $xy' = 2y$ because $y = x^2 \Rightarrow y' = 2x$ and $x \cdot 2x = 2 \cdot x^2$ is a true statement for all values of x.

Example 6.1.1

Check that:

a. $y = x^2 + 5$ is a solution of $y'' + y = x^2 + 7$ and

b. $y = x + \dfrac{5}{x}$ is a solution of $y' + \dfrac{y}{x} = 2$.

Solution

a. $y = x^2 + 5 \Rightarrow y' = 2x \Rightarrow y'' = 2$. Substituting these functions for y and y'' into the left side of the differential equation $y'' + y = x^2 + 7$ yields:

$$y'' + y = (2) + (x^2 + 5) = x^2 + 7 \tag{6.1.1}$$

so $y = x^2 + 5$ is a solution of the differential equation.

b. $y = x + \frac{5}{x} \Rightarrow y' = 1 - \frac{5}{x^2}$. Substituting these functions for y and y' into the left side of the differential equation $y' + \frac{y}{x} = 2$, we have:

$$y' + \frac{y}{x} = \left[1 - \frac{5}{x^2}\right] + \frac{1}{x}\left[x + \frac{5}{x}\right] = 1 - \frac{5}{x^2} + 1 + \frac{5}{x^2} = 2 \tag{6.1.2}$$

which matches the right side of the original differential equation.

? Practice 6.1.1

Check that:

a. $y = 2x + 6$ is a solution of $y - 3y' = 2x$ and

b. $y = e^{3x}$, $y = 5e^{3x}$ and $y = Ae^{3x}$ (where A is any constant) are all solutions of $y'' - 2y' - 3y = 0$.

Answer

a. $y = 2x + 6 \Rightarrow y' = 2$; $y - 3y' = (2x + 6) - 3(2) = 2x$ (OK)

b. $y = e^{3x} \Rightarrow y' = 3e^{3x} \Rightarrow y'' = 9e^{3x}$ so:

$$y'' - 2y' - 3y = 9e^{3x} - 2(3e^{3x}) - 3(e^{3x}) = 0 \quad \text{(OK)} \tag{6.1.3}$$

\)y = 5e^{3x} \Rightarrow y' = 15e^{3x} \Rightarrow y'' = 45e^{3x}\) so:

$$y'' - 2y' - 3y = 45e^{3x} - 2(15e^{3x}) - 3(5e^{3x}) = 0 \quad \text{(OK)} \tag{6.1.4}$$

\)y = Ae^{3x} \Rightarrow y' = 3Ae^{3x} \Rightarrow y'' = 9Ae^{3x}\) so:

$$y'' - 2y' - 3y = 9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 0 \quad \text{(OK)} \tag{6.1.5}$$

A solution of a differential equation with the **initial condition** $y(x_0) = y_0$ is a function that satisfies the differential equation as well as the initial condition. To check the solution of an **initial value problem** (or **IVP**), we must check that a solution function satisfies both the equation and the initial condition.

✓ Example 6.1.2

Which of the given functions is a solution of the initial value problem $y' = 3y$, $y(0) = 5$?

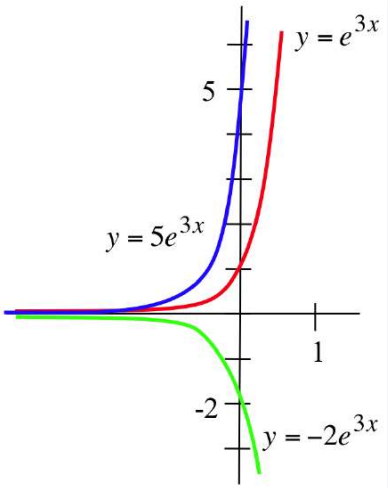
a. $y = e^{3x}$

b. $y = 5e^{3x}$

c. $y = -2e^{3x}$

Solution

All three functions satisfy the differential equation, but only one of them satisfies the initial condition that $y(0) = 5$. If $y = e^{3x}$, then $y(0) = e^{3(0)} = 1 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition:



If $y = 5e^{3x}$, then $y(0) = 5e^{3(0)} = 5$ so $y = 5e^{3x}$ does satisfy the initial condition. If $y = -2e^{3x}$, then $y(0) = -2e^{3(0)} = -2 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition.

? Practice 6.1.2

Which function is a solution of the initial value problem $y'' + 9y = 0$, $y(0) = 2$?

a. $y = \sin(3x)$
b. $y = 2\sin(3x)$
c. $y = 2\cos(3x)$

Answer

We want $y'' + 9y = 0$ and \)y(0) = 2\).

a. $y = \sin(3x) \Rightarrow y' = 3\cos(3x) \Rightarrow y'' = -9\sin(3x)$ so we have $y'' + 9y = -9\sin(3x) + 9 \cdot \sin(3x) = 0$ (OK) but checking the initial condition: $y(0) = \sin(0) = 0 \neq 2$.
b. $y = 2\sin(3x) \Rightarrow y' = 6\cos(3x) \Rightarrow y'' = -18\sin(3x)$ so we have $y'' + 9y = -18\sin(3x) + 9 \cdot 2\sin(3x) = 0$ (OK) but checking the initial condition: $y(0) = 2\sin(0) = 0 \neq 2$
c. $y = 2\cos(3x) \Rightarrow y' = -6\sin(3x) \Rightarrow y'' = -18\cos(3x)$ so $y'' + 9y = -18\cos(3x) + 9 \cdot 2\cos(3x) = 0$ (OK) and checking the initial condition: $y(0) = 2\cos(0) = 2$ (OK).

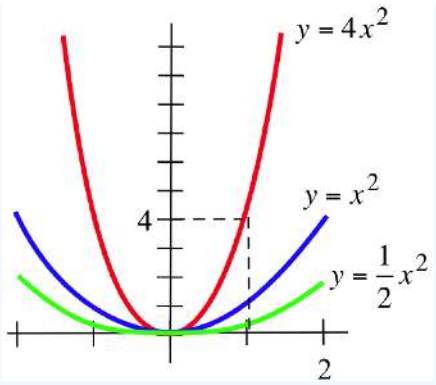
Only $y = 2\cos(3x)$ satisfies both the ODE and the initial condition.

Finding the Value of the Constant

Differential equations usually have many solutions, typically a whole "family" of them, with each solution in the family satisfying a different initial condition. To find which solution of a differential equation also satisfies a given initial condition of the form $y(x_0) = y_0$, we replace x and y in an equation describing the solution family with the values x_0 and y_0, then algebraically solve for the value of an unknown constant.

✓ Example 6.1.3

For every value of C, the function $y = Cx^2$ is a solution of $xy' = 2y$:



Find the value of C so that $y(5) = 50$.

Solution

Substituting the initial condition $x = 5$ and $y = 50$ into the solution $y = Cx^2$:

$$50 = C(5^2) \Rightarrow C = \frac{50}{25} = 2 \tag{6.1.6}$$

so the function $y = 2x^2$ satisfies both the differential equation and the initial condition.

? Practice 6.1.3

For every value of C, the function $y = e^{2x} + C$ is a solution of $y' = 2e^{2x}$. Find the value of C so that $y(0) = 7$.

Answer

$y = e^{2x} + C \Rightarrow y' = 2e^{2x}$ (OK) so, plugging in the initial values:

$$7 = y(0) = e^{2\cdot 0} + C \Rightarrow 7 = 1 + C \Rightarrow C = 6\backslash) \tag{6.1.7}$$

so $y = e^{2x} + 6$.

Types of Differential Equations

In Chapter 14, we will begin studying **partial derivatives** of functions of more than one variable, which can appear in differential equations called **partial differential equations** (or **PDE**s). Because of this, we will call differential equations involving ordinary derivatives, such as $\frac{dy}{dx}$, $\frac{dy}{dt}$ or $\frac{d^2y}{dt^2}$, **ordinary differential equations** (or **ODE**s).

Problems

In Problems 1–10, check that the function y is a solution of the given differential equation.

1. $y' + 3y = 6$; $y = e^{-3x} + 2$
2. $y' - 2y = 8$; $y = e^{2x} - 4$
3. $y'' - y' + y = x^2$; $y = x^2 + 2x$
4. $3y'' + y' + y = x^2 - 4x$; $y = x^2 - 6x$
5. $xy' - 3y = x^2$; $y = 7x^3 - x^2$
6. $xy'' - y' = 3$; $y = x^2 - 3x + 5$
7. $y' + y = e^x$; $y = \frac{1}{2}e^x + 2e^{-x}$
8. $y'' + 25y = 0$; $y = \sin(5x) + 2\cos(5x)$
9. $y' = -\frac{x}{y}$; $y = \sqrt{7 - x^2}$
10. $y' = x - y$; $y = x - 1 + 2e^{-x}$

In Problems 11–20, check that the function y is a solution of the given initial value problem.

11. $y' = 6x^2 - 3$, $y(1) = 2$; $y = 2x^3 - 3x + 3$
12. $y' = 6x + 4$, $y(2) = 3$; $y = 3x^2 + 4x - 17$
13. $y' = 2\cos(2x)$, $y(0) = 1$; $y = \sin(2x) + 1$
14. $y' = 1 + 6\sin(2x)$, $y(0) = 2$; $y = x - 3\cos(2x) + 5$
15. $y' = 5y$, $y(0) = 7$; $y = 7e^{5x}$
16. $y' = -2y$, $y(0) = 3$; $y = 3e^{-2x}$
17. $xy' = -y$, $y(1) = -4$; $y = -\dfrac{4}{x}$
18. $y \cdot y' = -x$, $y(0) = 3$; $y = \sqrt{9 - x^2}$
19. $y' = \dfrac{5}{x}$, $y(e) = 3$; $y = 5\ln(x) - 2$
20. $y' + y = e^x$, $y(0) = 5$; $y = \frac{1}{2}e^x + \frac{9}{2}e^{-x}$

In Problems 21–30, find the value of the constant C so a function from the given family of solutions satisfies the given initial value problem.

21. $y' = 2x$, $y(3) = 7$; $y = x^2 + C$
22. $y' = 3x^2 - 5$, $y(1) = 2$; $y = x^3 - 5x + C$
23. $y' = 3y$, $y(0) = 5$; $y = Ce^{3x}$
24. $y' = -2y$, $y(0) = 3$; $y = Ce^{-2x}$
25. $y' = 6\cos(3x)$, $y(0) = 4$; $y = 2\sin(3x) + C$
26. $y' = 3 - 2\sin(2x)$, $y(0) = 1$; $y = 3x + \cos(2x) + C$
27. $y' = \dfrac{1}{x}$, $y(e) = 2$; $y = \ln(x) + C$
28. $y' = \dfrac{1}{x^2}$, $y(1) = 3$; $y = -\frac{1}{x} + C$
29. $y' = -\dfrac{y}{x}$, $y(2) = 10$; $y = -\frac{C}{x}$
30. $y' = -\frac{x}{y}$, $y(3) = 4$; $y = \sqrt{C - x^2}$

In Problems 31–40, find the find the function y that satisfies the given initial value problem.

31. $y' = 4x^2 - x$, $y(1) = 7$
32. $y' = x - e^x$, $y(0) = 3$
33. $y' = \dfrac{3}{x}$, $y(1) = 2$
34. $xy' = 1$, $y(e) = 7$
35. $y' = 6e^{2x}$, $y(0) = 1$
36. $y' = 36(3x - 2)^2$, $y(1) = 8$
37. $y' = x \cdot \sin(x^2)$, $y(0) = 3$
38. $y' = \dfrac{6}{x^2}$, $y(1) = 2$
39. $xy' = 6x^3 - 10x^2$, $y(2) = 5$
40. $x^2y' = 6x^3 - 1$, $y(1) = 10$
41. Show that if $y = f(x)$ and $y = g(x)$ are both solutions to $y' + 5y = 0$, then $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$ and $y = A \cdot f(x) + B \cdot g(x)$ are solutions for any constants A and B.
42. Show that if $f(x)$ and $g(x)$ are both solutions to $y'' + 2y' - 3y = 0$, then so are $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$ and $y = A \cdot f(x) + B \cdot g(x)$ for any constants A and B.
43. Show that $y = \sin(x) + x$ and $y = \cos(x) + x$ are both solutions of $y'' + y = x$. Are $y = 3[\sin(x) + x]$ and $y = [\sin(x) + x] + [\cos(x) + x]$ solutions of $y'' + y = x$?
44. Show that $y = e^{3x} - 2$ and $y = 5e^{3x} - 2$ are both solutions of $y' - 3y = 6$. Are $y = 7[e^{3x} - 2]$ and $y = [e^{3x} - 2] + [5e^{3x} - 2]$ also solutions?
45. The ODE $\dfrac{dy}{dt} = A - By$ (where A and B are positive constants) describes the concentration y of glucose in a person's blood at time t. Check that $y = \dfrac{A}{B} - C \cdot e^{-Bt}$ is a solution of the ODE for any value of the constant C.
46. The ODE $\dfrac{dy}{dt} = Ay$ (where A is a positive constant) is used to model “exponential" growth and decay. Check that $y = C \cdot e^{At}$ is a solution of the differential equation for any value of the constant C.

47. The ODE $L \cdot \frac{dI}{dt} + RI = E$ (where L, R and E are positive constants) describes the current $I(t)$ in an electrical circuit. Show that $I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ is a solution of the ODE.
48. The ODE $m \cdot y'' + C \cdot y = 0$ (where C is a positive constant) describes the position y of an object hung from a spring as it moves up and down. Show that $y = A \cdot \sin(\omega t) + B \cdot \cos(\omega t)$ with $\omega = \sqrt{\frac{C}{m}}$ is a solution of the ODE for all values of the constants A and B.

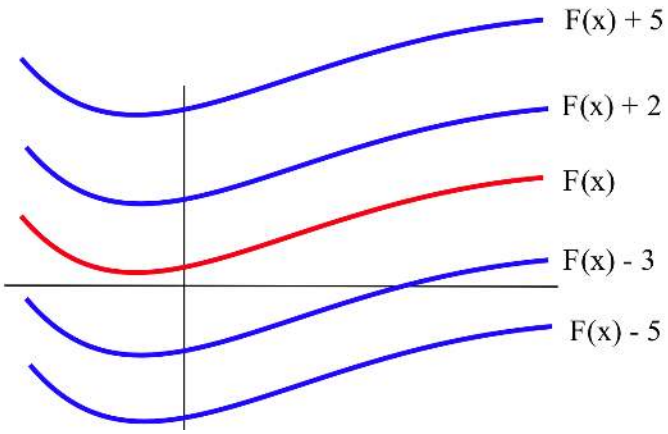
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6.2: The Differential Equation y = f'(x)

This section introduces some basic concepts and vocabulary of the study of ODEs as they apply to the familiar problem, $y' = f(x)$: the notion of a general solution of an ODE, the (possibly unique) solution to an IVP and the direction field of an ODE. We'll use these concepts in later sections as we examine more complicated differential equations and their applications.

Solving $y' = f(x)$

The solution of the ODE $y' = f(x)$ is the collection of all antiderivatives of f: $y = \int f(x)\,dx$. (This is due to Corollary 2 to the Mean Value Theorem in Section 3.2.) If $y = F(x)$ is one antiderivative of f, then we have essentially found *all* antiderivatives of f because any antiderivative of f has the form $F(x) + C$, for some value of the constant C. If F is one particular antiderivative of f, the collection of functions $F(x) + C$ is called the **general solution** of $y' = f(x)$. The general solution consists of a **one-parameter family** of functions (the parameter here is C).



Example 6.2.1

Find the general solution of the ODE $y' = 2x + e^{3x}$.

Solution

$$y = \int \left[2x + e^{3x}\right]\,dx = x^2 + \frac{1}{3}e^{3x} + C.$$

Practice 6.2.1

Find general solutions for $y' = x + \dfrac{3}{x+2}$ and $y' = \dfrac{6}{x^2+1}$.

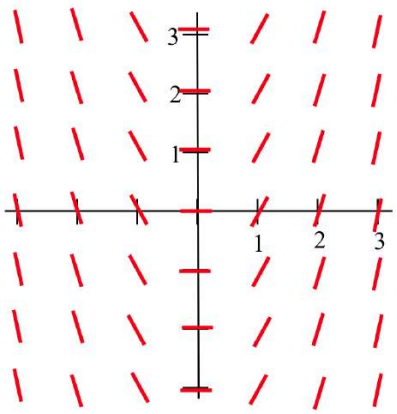
Answer

$$y' = x + \frac{3}{x+2} \;\Rightarrow\; y = \int \left[x + \frac{3}{x+2}\right]\,dx = \frac{1}{2}x^2 + 3\ln\left(|x+2|\right) + C$$

$$y' = \frac{6}{x^2+1} \;\Rightarrow\; y = \int \frac{6}{x^2+1}\,dx = 6\arctan(x) + C$$

Direction Fields

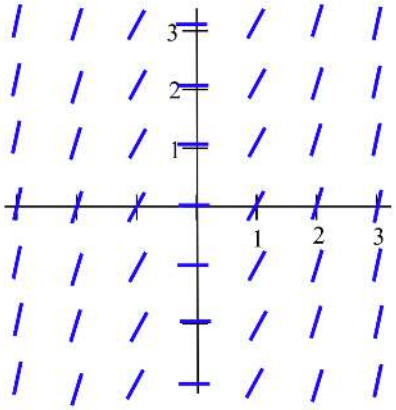
Geometrically, a derivative tells us the slope of the tangent line to a curve, so we can interpret the ODE $y' = f(x)$ as a geometric condition: at each point (a, b) on the graph of the solution function y, the slope of the tangent line is $f(a)$. The ODE $y' = 2x$ says that at each point (a, b) on the graph of y, the slope of the line tangent to the graph is $2a$: if the point $(5, 3)$ is on the graph of y, then the slope of the tangent line there is $2 \cdot 5 = 10$. We can present this information graphically as a **direction field**: a collection of short line segments through some sample points in the plane so that the slope of the segment through (a, b) is $f(a)$. A direction field for $y' = 2x$ appears below:



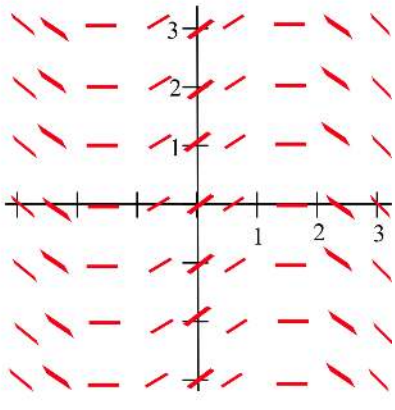
y' = 2x

At a point (a, b), the slope is $2a$. What are the solutions to $y' = 3x^2$? To $y' = \cos(x)$? Can you "see" those solution curves in the direction fields?

Direction fields for $y' = 3x^2$ and $y' = \cos(x)$ appear below:

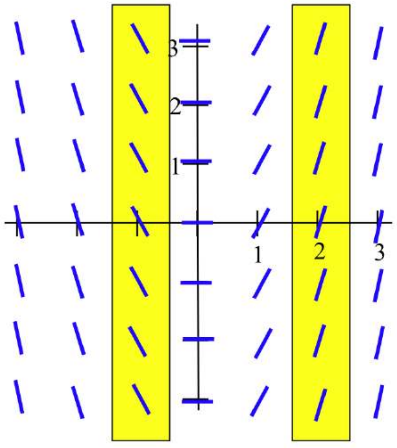


y' = 3x^2



y' = cos(x)

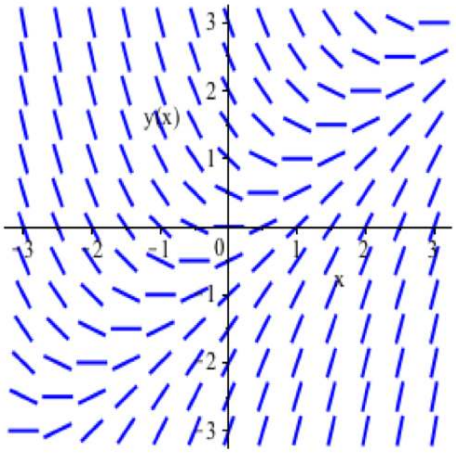
For any ODE of the form $y' = f(x)$, the values of y' depend only on x, so along any vertical line (where x is fixed) all the small line segments have the same y', hence the same slope, and they are parallel:



Direction field for y' = f(x)

If y' depends on both x and y, then the slopes of the line segments will depend on both x and y, and the slopes of the small line segments along a vertical line are not all the same. The figure below shows a direction field for $y' = x - y$, where y' is a function of

both x and y:



Direction field for y' = x - y

A direction field of an ODE $y' = g(x, y)$ is a collection of short line segments with slope $g(a, b)$ at the point (a, b).

Practice 6.2.2

Construct direction fields for:

a. $y' = x + 1$ and
b. $y' = x + y$

by sketching a short line segment with slope y' at each point (a, b) with integer coordinates from -3 to 3.

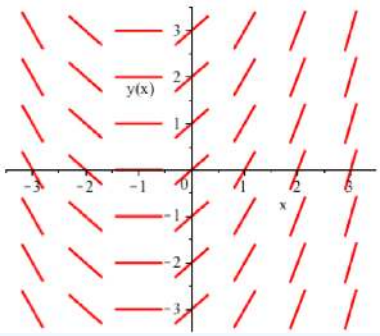
Answer

a. If $y' = x + 1$ then the table below lists values of y' for integer values of x and y from -3 to 3 (notice that y' does not depend on the value of y):

described above

	$x = -3$	-2	-1	0	1	2	3
$y = 3$	-2	-1	0	1	2	3	4
2	-2	-1	0	1	2	3	4
1	-2	-1	0	1	2	3	4
0	-2	-1	0	1	2	3	4
-1	-2	-1	0	1	2	3	4
-2	-2	-1	0	1	2	3	4
-3	-2	-1	0	1	2	3	4

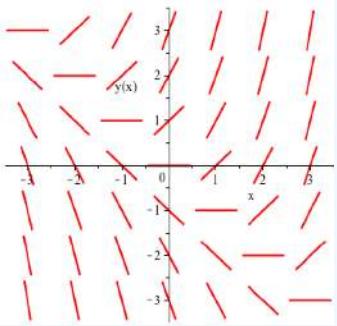
The figure below shows the direction field:



b. If $y' = x + y$ then the table below lists values of y' for integer values of x and y from -3 to 3 (notice that y' here {\em does} depend on both x and y):

	$x=-3$	-2	-1	0	1	2	3
$y=3$	0	1	2	3	4	5	6
2	-1	0	1	2	3	4	5
1	-2	-1	0	1	2	3	4
0	-3	-2	-1	0	1	2	3
-1	-4	-3	-2	-1	0	1	2
-2	-5	-4	-3	-2	-1	0	1
-3	-6	-5	-4	-3	-2	-1	0

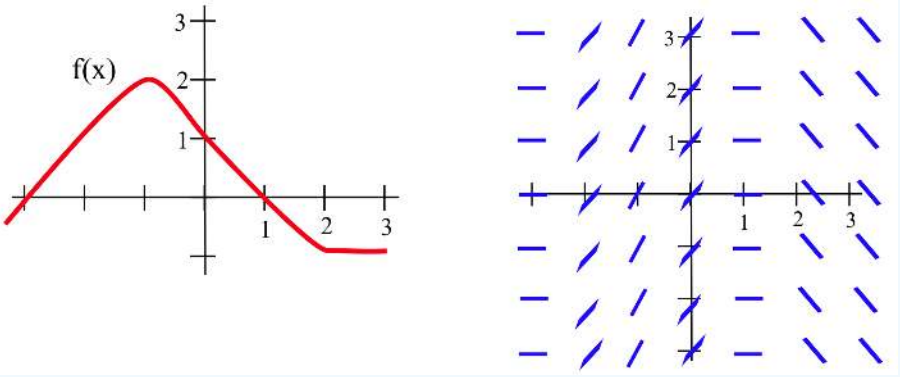
The figure below shows the direction field:



As you discovered in the previous Practice problem, direction fields are usually tedious to plot by hand, but computers (and some calculators) can plot them quickly. If you only have a graph of the function f in the differential equation $y' = f(x)$, you can construct an approximate direction field using the information you have about f from its graph.

Example 6.2.2

Construct a direction field for the differential equation $y' = f(x)$ for the f given graphically below left:

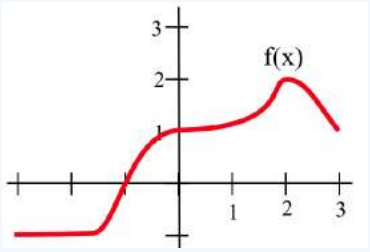


Solution

If $x = 0$ then $y' = f(0) = 1$, so at every point on the vertical line where $x = 0$ (the y-axis) the line segments of the direction field have slope $y' = 1$ (above right). Similarly, if $x = 1$ then $y' = f(1) = 0$, so the line segments of the direction field have slope $y' = 0$ at every point on the vertical line where $x = 1$. The small line segments along any vertical line are parallel.

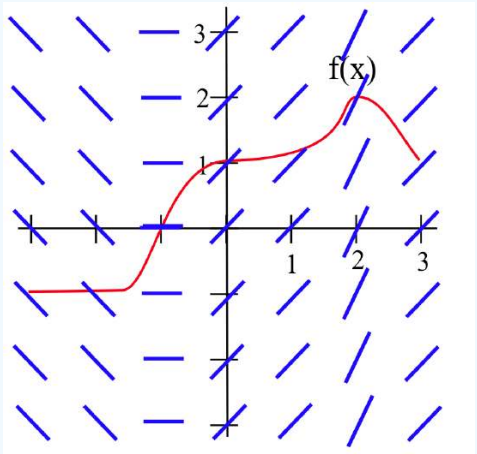
? Practice 6.2.3

Construct a direction field for the differential equation $y' = f(x)$ for the f given graphically below:



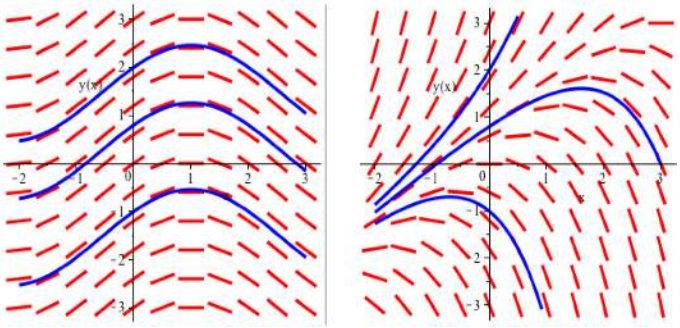
Answer

An approximate direction field for the ODE $y' = f(x)$ appears below:



(The function shown is $f(x)$, not a solution to the ODE.)

Once you have a direction field for an ODE, you can sketch curves that have the appropriate tangent-line slopes so you can "see" the shapes of the solution curves even if you do not have formulas for them:



These shapes can be useful for estimating which initial conditions lead to straight-line solutions or periodic solutions or solutions with other properties, and they can help us understand the behavior of machines and organisms in applied problems.

Initial Value Problems

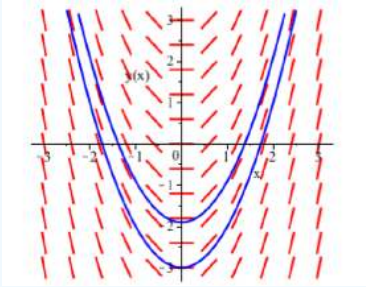
An initial condition $y(x_0) = y_0$ specifies that the solution y of the differential equation should go through the point (x_0, y_0) in the plane. To solve a differential equation with an initial condition, you typically use integration to find the general solution (a family of solutions containing an arbitrary constant) and then you use algebra to find the one value for the constant so the solution satisfies the initial condition.

Example 6.2.3

Solve the differential equation $y' = 2x$ with the initial condition $y(2) = 1$.

Solution

The general solution is $y = \int 2x\,dx = x^2 + C$. Substituting $x_0 = 2$ and $y_0 = 1$ into the general solution: $1 = (2)^2 + C \Rightarrow C = -3$. So the solution we want is $y = x^2 - 3$. (A quick check verifies that $[x^2 - 3]' = 2x$ and $2^2 - 3 = 1$.) The figure below shows a direction field for $y' = 2x$ and the solution curve that goes through $(2, 1)$, $y = x^2 - 3$, along with the solution of the ODE that satisfies $y(1) = -1$:



Example 6.2.4

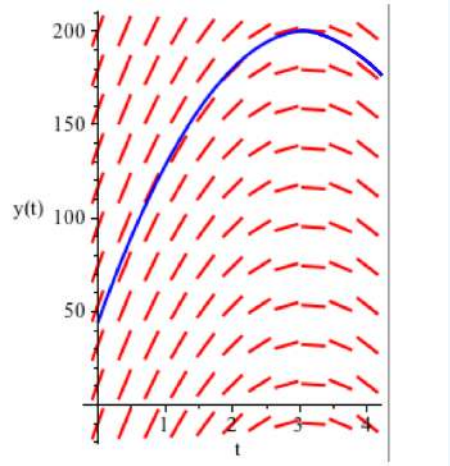
If you toss a ball upward with an initial velocity of 100 feet per second, its height y (in feet) at time t (in seconds) satisfies the differential equation $y' = 100 - 32t$. Sketch the direction field for y (for $0 \le t \le 4$) and then sketch the solution that satisfies the condition that the ball is 200 feet high after 3 seconds.

Solution

$y = \int [100 - 32t]\,dt = 100t - 16t^2 + C$; if $y(3) = 200$:

$$200 = 100(3) - 16(3)^2 + C \;\Rightarrow\; 200 = 156 + C \;\Rightarrow C = 44$$

The function we want is $y = 100t - 16t^2 + 44$. The direction field and the solution satisfying $y(3) = 200$ appear below:



Practice 6.2.4

Find the solution of $y' = 9x^2 - 6\sin(2x) + e^x$ that goes through the point $(0, 6)$.

Answer

$y = \int \left[9x^2 - 6\sin(2x) + e^x\right]\, dx = 3x^3 + 3\cos(2x) + e^x + C$ so:

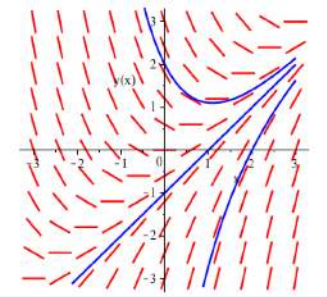
$$6 = y(0) = 3 \cdot 0^3 + 3\cos(2 \cdot 0) + e^0 + C = 0 + 3 + 1 + C \;\Rightarrow\; C = 2$$

and therefore $y = 3x^3 + 3\cos(2x) + e^x + 2$.

Example 6.2.5

Sketch the three solutions of the ODE $y' = x - y$ that satisfy the initial conditions $y(0) = 2$, $y(0) = -1$ and $y(1) = -2$.

Solution



Existence and Uniqueness

When solving IVPs, three questions often present themselves:

- Does a solution to the IVP exist?
- Is that solution unique?
- On what interval(s) is that solution valid?

We typically hope the answer the first two questions is "yes," but (as we will see in the next section) that will not always be true. (Surprisingly, for reasonably "nice" IVPs, we will eventually be able to answer these questions without actually finding the solution.)

For IVPs of the form $y' = f(x)$, $y(x_0) = y_0$, where $f(x)$ is continuous on some interval (a, b) with $a < x_0 < b$, the answer to the first two questions is "yes" and the answer to third question is "on the interval (a, b), and possibly on some larger interval." In this

situation, define:

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$

The Fundamental Theorem of Calculus tells us that $y'(x) = f(x)$, so this $y(x)$ solves the ODE, and:

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t)\,dt = y_0 + 0 = y_0$$

so it also satisfies the initial value condition. If $\tilde{y}(x)$ is any other solution to the IVP, then $\tilde{y}'(x) = f(x) = y'(x)$ so $\tilde{y}(x) = y(x) + C$, but we also know that $\tilde{y}(x_0) = y_0 = y(x_0)$:

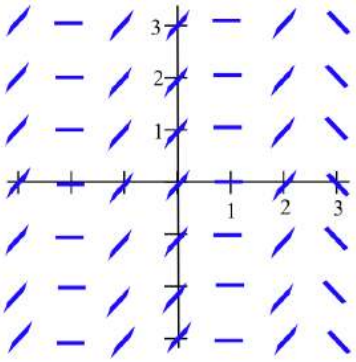
$$\tilde{y}(x_0) = y(x_0) + C \;\Rightarrow\; y_0 = y_0 + C \;\Rightarrow\; C = 0 \;\Rightarrow \tilde{y}(x) = y(x)$$

which tells us that $y(x)$ is the *only* solution. Finally, $f(x)$ is continuous on (a, b), hence integrable on (a, b), so the integral definition of $y(x)$ is defined (and solves the IVP) on (a, b).

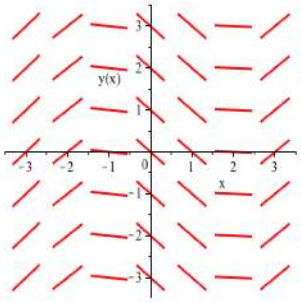
Problems

In Problems 1–6, use the given direction field to sketch the solutions of the underlying ODE that satisfy the given initial conditions.

1. See figure below; $y(0) = 1$, $y(1) = -2$ and $y(1) = 3$.

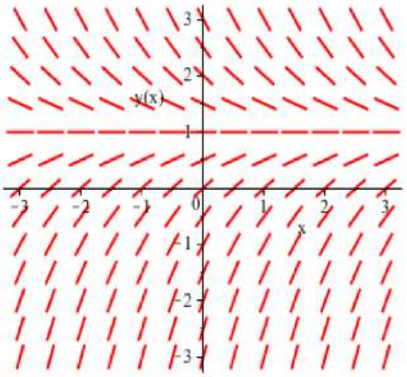


2. See figure above; $y(0) = 2$, $y(1) = -1$ and $y(1) = -2$.
3. See figure below; $y(-2) = 1$, $y(0) = 1$ and $y(2) = 1$



4. See figure above; $y(-2) = -1$, $y(0) = -1$, $y(2) = -1$.

5. See figure below; $y(0) = -2$, $y(0) = 0$ and $y(0) = 2$.



6. See figure above; $y(2) = -2$, $y(2) = 0$ and $y(2) = 2$.
7. How do the three solutions in Problem 5 behave for large values of x?
8. How do the three solutions in Problem 6 behave for large values of x?

In Problems 9–14:

a. sketch a direction field for the given ODE and
b. without solving the ODE, sketch solutions that go through the points $(0, 1)$ and $(2, 0)$.

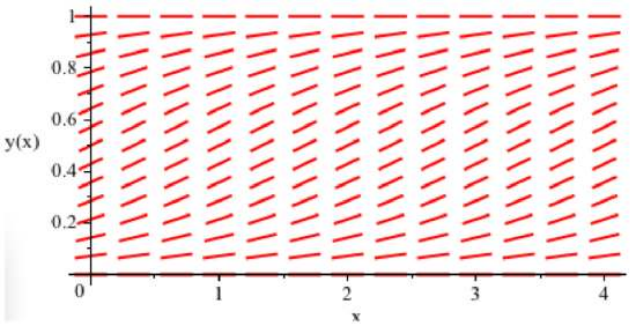
9. $y' = 2x$
10. $y' = 2 - x$
11. $y' = 2 + \sin(x)$
12. $y' = e^x$
13. $y' = 2x + y$
14. $y' = 2x - y$

In Problems 15–20:

a. find the family of functions that solve the given ODE,
b. find the member of the family that satisfies the given IVP and
c. report the interval on which the solution to the IVP is valid.

15. $y' = 2x - 3$, $y(1) = 4$
16. $y' = 1 - 2x$, $y(2) = -3$
17. $y' = e^x + \cos(x)$, $y(0) = 7$
18. $y' = \sin(2x) - \cos(x)$, $y(0) = -5$
19. $y' = \dfrac{6}{2x + 1} + \sqrt{x}$, $y(1) = 4$
20. $y' = \dfrac{e^x}{1 + e^x}$, $y(0) = 0$

Problems 21–22 concern a direction field (shown below) that comes from an ODE called the **logistic equation**, $y' = y(1 - y)$, used to model the growth of a population in an environment with renewable but limited resources.

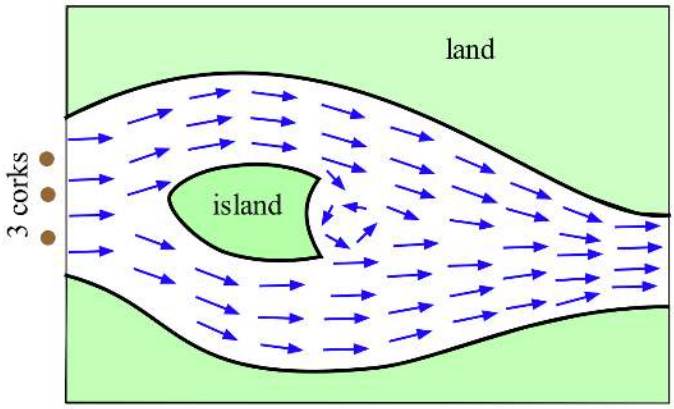


(It is also used to describe the spread of a rumor or disease through a population.)

21. Sketch the solution that satisfies the initial condition $y(0) = 0.1$. What letter of the alphabet does this solution resemble?
22. Sketch several solutions that have different initial values for $y(0)$. What appears to happen to all of these solutions after a "long time" (for large values of x)?

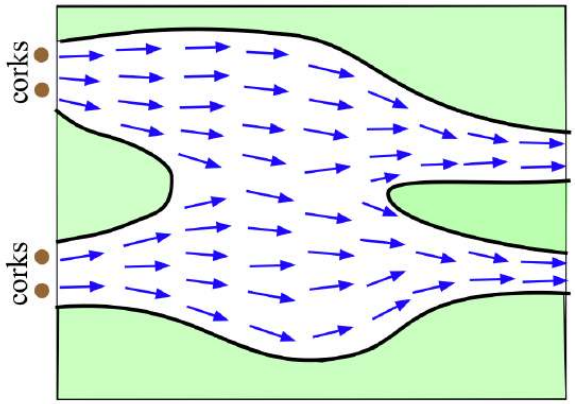
In Problems 23–24, the given figures show the direction of surface flow at different locations along a river. Sketch the paths small corks will follow if they are put into the river at the dots in each figure. (Because they indicate both the magnitude and the direction of flow, each diagram is called a **vector field**.) Notice that corks that start close to each other can drift far apart, and corks that start far apart can drift close together.

23. See figure below.:



Surface flow along a river

24. See figure below.

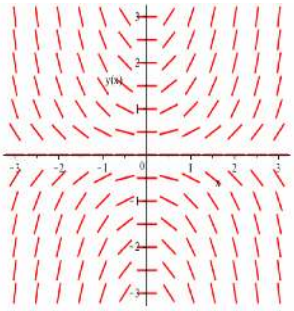


Surface flow along a river

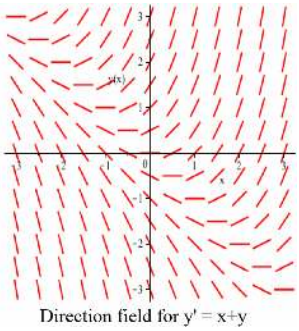
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6.3: Separable Equations

So far, we have only learned how to solve ODEs of the form $y' = f(x)$ (which you already knew how to solve), where y' depended only on x and the slopes of the line segments of a corresponding direction field did not depend on the y-coordinate of the location of the line segment. In many situations, however, y' depends on both x and y, for example, $y' = xy$:



or $y' = x + y$:



In Example 2 of Section 2.9, we considered the problem of finding a tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. You can solve this problem geometrically. Or you can solve the equation for y to get $y = \sqrt{25 - x^2}$ and then compute $\frac{dy}{dx}$. Or you can implicitly differentiate the equation $x^2 + y^2 = 25$, keeping in mind that y is a function of x:

$$x^2 + y^2 = 25 \;\Rightarrow\; 2x + 2y \cdot \frac{dy}{dx} = 0 \;\Rightarrow\; 2y \cdot \frac{dy}{dx} = -2x \;\Rightarrow\; \frac{dy}{dx} = -\frac{x}{y}$$

The result is a differential equation (although we didn't call it that in Section 2.9) and putting $x = 3$ and $y = 4$ into the right side of that ODE yields $\frac{dy}{dx} = -\frac{3}{4}$, which gives us the slope of the line tangent to the circle at the point $(3, 4)$.

What if we start with $\frac{dy}{dx} = -\frac{x}{y}$ and try to solve this ODE? It is not of the form $y' = f(x)$, but we can try to solve it by beginning to reverse the steps in the implicit differentiation process above:

$$\frac{dy}{dx} = -\frac{x}{y} \quad\Rightarrow\quad y \cdot \frac{dy}{dx} = -x$$

Each side of this new ODE is a function of x because y and $\frac{dy}{dx}$ are functions of x, so we can integrate both sides of this equation to get:

$$\int y \cdot \frac{dy}{dx}\, dx = \int -x\, dx \;\Rightarrow\; \frac{1}{2}y^2 + C_1 = -\frac{1}{2}x^2 + C_2$$

(Each indefinite integral yields a constant, but these are not necessarily equal.)

The integration on the left-hand side of this equation uses the fact that:

$$\frac{d}{dx}\left[\frac{1}{2}y^2\right] = \frac{1}{2} \cdot 2y \cdot \frac{dy}{dx} = y \cdot \frac{dy}{dx}$$

but this is the same result as if we had "cancelled" the dx in the original left-hand integral:

$$\int y \cdot \frac{dy}{dx}\,dx = \int y\,dy = \frac{1}{2}y^2 + C_1$$

so in our first steps above we could have instead rewritten the ODE using differentials:

$$\frac{dy}{dx} = -\frac{x}{y} \;\Rightarrow\; y \cdot \frac{dy}{dx} = -x \;\Rightarrow y\,dy = -x\,dx \;\Rightarrow \int y\,dy = \int -x\,dx$$

and then integrated both sides. (We will follow this procedure as we solve similar ODEs from now on.) We can now move both constants of integration to the same side of the equation:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + [C_2 - C_1] \;\Rightarrow\; \frac{1}{2}y^2 = -\frac{1}{2}x^2 + K$$

where K is another arbitrary constant. Because we can *always* move the left-hand constant to the right side of the equation and combine it with the right-hand constant, in the future we will use a single constant on the right side when integrating both sides of the differential equation.

At this stage we can write:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + K \;\Rightarrow\; y^2 = -x^2 + 2K \;\Rightarrow x^2 + y^2 = C$$

where C is yet another arbitrary constant. We recognize this as an equation for a circle centered at the origin and we call a solution of this type an **implicit solution** to the ODE because we have not explicitly solved for y. If we know our solution curve passes through $(3, 4)$, we can put this information into our solution equation to get:

$$3^2 + 4^2 = C \;\Rightarrow\; C = 25 \;\Rightarrow\; x^2 + y^2 = 25$$

Solving for y, we get $y = \sqrt{25 - x^2}$, an **explicit solution** to the ODE. (We first get $y = \pm\sqrt{25 - x^2}$ but knowing that $y(3) = 4$ tells us to use the $+$ symbol rather than the $-$ symbol.)

Separable Equations

In the ODE we solved above, we were able to "separate" the variables so that only y appeared on one side and only x appeared on the other, allowing us to integrate each side separately to solve the ODE.

Definition

A differential equation is called *separable* if we can separate the variables algebraically so that the equation has the form:

$$g(y) \cdot y' = f(x)$$

Example 6.3.1

If possible, separate the variables in each ODE by writing the given differential equation in the form $g(y) \cdot y' = f(x)$:

a. $y' = xy$
b. $xy' = \dfrac{y+1}{x}$
c. $y' = \dfrac{1+\sin(x)}{y^2+y}$
d. $y' = y$
e. $y' = x + y$

Solution

a. Divide each side of $y' = xy$ by y (to do so we must assume $y \neq 0$) to get $\frac{1}{y} \cdot y' = x$ so $g(y) = \frac{1}{y}$ and $f(x) = x$.

b. Divide each side by $x(y+1)$ (to do so we must assume $x \neq 0$ and $y \neq -1$) to get $\frac{1}{y+1} \cdot y' = \frac{1}{x^2}$ so $g(y) = \frac{1}{y+1}$ and $f(x) = \frac{1}{x^2}$.

c. Multiply each side by y^2+y to get $(y^2+y)\cdot y' = 1+\sin(x)$ so $g(y) = y^2+y$ and $f(x) = 1+\sin(x)$.

d. Divide each side by y (to do so we must assume $y \neq 0$) to get $\frac{1}{y}\cdot y' = 1$ so $g(y) = \frac{1}{y}$ and $f(x) = 1$.

e. We cannot write this ODE in the form $g(y)\cdot y' = f(x)$.

The first four ODEs are separable, but the last one is not.

? Practice 6.3.1

If possible, separate the variables in each ODE by writing the given differential equation in the form $g(y)\cdot y' = f(x)$:

a. $y' = x^3(y-5)$

b. $y' = \dfrac{3}{2x + x\cdot\sin(y+2)}$

Answer

a. If $y \neq 5$, divide the ODE $y' = x^3(y-5)$ by $y-5$ to get $\frac{1}{y-5}\cdot\frac{dy}{dx} = x^3$ so $g(y) = \frac{1}{y-5}$ and $f(x) = x^3$.

b. Factor x out of the denominator on the right-hand side and multiply both sides of the ODE by $2+\sin(y+2)$ to get:

$$\frac{dy}{dx} = \frac{3}{x\,[2+\sin(y+2)]} \quad\Rightarrow\quad [2+\sin(y+2)]\cdot\frac{dy}{dx} = \frac{3}{x}$$

so $g(y) = 2+\sin(y+2)$ and $f(x) = \frac{3}{x}$.

Solving Separable ODEs

To solve a separable differential equation, we will follow the steps outlined above when solving $\frac{dy}{dx} = -\frac{x}{y}$:

- Use algebra to separate the variables in the ODE: $g(y)\cdot y' = f(x)$
- Put into an equivalent form using differentials: $g(y)\,dy = f(x)\,dx$
- Integrate each side of the equation: $\int g(y)\,dy = \int f(x)\,dx$
- Find antiderivatives, if possible: $G(y) = F(x) + C$
- If given an initial condition (x_0, y_0), find C: $C = G(y_0) - F(x_0)$
- If possible, solve explicitly for y.

✓ Example 6.3.2

Find the general solution of $\frac{1}{x}\cdot y' = \frac{x}{2y}$.

Solution

Multiply each side by $2xy$ to get $2y\cdot y' = x^2$, so the ODE is separable and we can translate it into differential form:

$$2y\cdot\frac{dy}{dx} = x^2 \quad\Rightarrow\quad 2y\,dy = x^2\,dx$$

Integrating each side, we get:

$$\int 2y\,dy = \int x^2\,dx \quad\Rightarrow\quad y^2 = \frac{1}{3}x^3 + C$$

which is an implicit form of the general solution. Solving for y, we get:

$$y = \pm\sqrt{\frac{1}{3}x^3 + C}$$

which is an explicit form of the general solution.

Example 6.3.3

Find the solution of $y' = \dfrac{6x+1}{2y}$ that satisfies $y(2) = 3$.

Solution

We can rewrite this ODE as $2y \cdot y' = 6x + 1$, so it is separable. Next rewrite using differentials:

$$2y \cdot \frac{dy}{dx} = 6x + 1 \quad \Rightarrow \quad 2y\,dy = (6x+1)\,dx$$

and then integrate each side:

$$\int 2y\,dy = \int (6x+1)\,dx \quad \Rightarrow \quad y^2 = 3x^2 + x + C$$

Putting $x = 2$ and $y = 3$ into the general solution $y^2 = 3x^2 + x + C$:

$$3^2 = 3 \cdot 2^2 + 2 + C \quad \Rightarrow \quad 9 = 12 + 2 + C \quad \Rightarrow \quad C = -5$$

(In an initial value problem, it is often easier to solve for C immediately after finding the antiderivatives.) So $y^2 = 3x^2 + x - 5$ $\Rightarrow$ $y = \pm\sqrt{3x^2 + x - 5}$. Because $y(2) = 3$, we need the + value of the square root: $y = \sqrt{3x^2 + x - 5}$.

Practice 6.3.2

Find the general solution of $y' = \dfrac{1 - \sin(x)}{3y^2}$ and then find the solution that passes through $(0, 2)$.

Answer

Multiply both sides of the ODE by $3y^2$ to get:

$$3y^2 \cdot \frac{dy}{dx} = 1 - \sin(x) \quad \Rightarrow \quad 3y^2\,dy = [1 - \sin(x)]\,dx$$

and integrate:

$$\int 3y^2\,dy = \int [1 - \sin(x)]\,dx \quad \Rightarrow \quad y^3 = x + \cos(x) + C$$

This is an implicit form of the general solution; an explicit general solution is $y = \sqrt[3]{x + \cos(x) + C}$.

Using the initial condition $y(0) = 2$ with the implicit form of the general solution:

$$2^3 = 0 + \cos(0) + C \Rightarrow 8 = 1 + C \quad \Rightarrow \quad C = 7$$

so the solution to the IVP is $y = \sqrt[3]{x + \cos(x) + 7}$.

Sometimes algebra is the hardest part of solving an ODE, and often logarithms crop up when solving separable equations.

Example 6.3.4

Solve $x \cdot y' = y + 3$.

Solution

To write the ODE in the form $g(y) \cdot y' = f(x)$ we can divide by x and by $y + 3$ (so we must assume that $x \neq 0$ and $y \neq -3$):

$$x \cdot \frac{dy}{dx} = y + 3 \Rightarrow \frac{1}{y+3}\frac{dy}{dx} = \frac{1}{x} \Rightarrow \frac{1}{y+3}\,dy = \frac{1}{x}\,dx$$

Now integrate both sides:

$$\int \frac{1}{y+3}\,dy = \int \frac{1}{x}\,dx \quad \Rightarrow \quad \ln(|y+3|) = \ln(|x|) + C$$

which is an implicit form of the general solution. To explicitly solve for y, recall that $e^{\ln(a)} = a$ so:

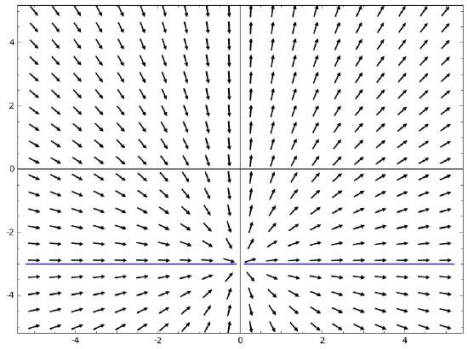
$$e^{\ln(|y+3|)} = e^{\ln(|x|)+C} = e^{\ln(|x|)} \cdot e^{C} \quad \Rightarrow \quad |y+3| = e^{C} \cdot |x|$$

Removing the absolute value signs we get:

$$y+3 = \pm e^{C} \cdot x \quad \Rightarrow \quad y = \pm e^{C} x - 3 \quad \Rightarrow \quad y = Ax - 3$$

where A is any nonzero constant. In the first step above, we had to assume that $y \neq -3$. You can check that the constant function $y = -3$ is also a solution to the ODE: $x \cdot [-3]' = -3 + 3$. So $y = Ax - 3$ is a solution of the ODE even when $A = 0$, and the general solution of the ODE is $y = Ax - 3$ for any value of A.

Study the direction field for $x \cdot y' = y + 3$ shown below:



What do you notice about its behavior near the lines $x = 0$ and $y = -3$?

Note

The domain of any particular solution of the form $y = Ax - 3$ is either $(-\infty, 0)$ or $(0, \infty)$, which is related to the assumption that $x \neq 0$ in our solution. Putting $x = 0$ into the original ODE yields $0 = y + 3 \Rightarrow y - 3$, so the only solution for which $x = 0$ is the single point $(0, -3)$ rather than a function defined on an open interval.

Existence and Uniqueness

In Section 6.2, we saw that any IVP of the form $y' = f(x)$, $y(x_0) = y_0$ had a unique solution: that is, at least one solution to the IVP exists, and this solution is the only possible solution. Unfortunately, for other types of IVPs things can be more complicated.

The ODE $y^2 + (y')^2 = -1$ has *no* solution, no matter what initial value condition we might specify. (Can you see why?) The ODE $y^2 + (y')^2 = 0$ has only one solution (the constant solution $y = 0$) so the IVP $y^2 + (y')^2 = 0$, $y(0) = 0$ has one unique solution, but the IVP $y^2 + (y')^2 = 0$, $y(0) = 5$ has none.

Even separable equations can behave in unexpected ways. Consider the IVP $y' = 3y^{\frac{2}{3}}$, $y(0) = 0$. The ODE is separable (if $y \neq 0$):

$$\frac{dy}{dx} = 3y^{\frac{2}{3}} \quad \Rightarrow \quad \frac{1}{3} y^{-\frac{2}{3}} \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{1}{3} y^{-\frac{2}{3}}\, dy = dx$$

Integrating both sides:

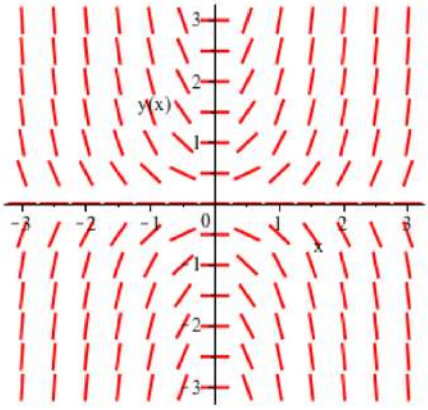
$$\int \frac{1}{3} y^{-\frac{2}{3}}\, dy = \int 1\, dx \quad \Rightarrow \quad y^{\frac{1}{3}} = x + C \quad \Rightarrow \quad y = (x + C)^3$$

Using the initial condition, $0 = y(0) = (0 + C)^3 = C^3 \Rightarrow C = 0$ so $y = x^3$ solves the IVP. But we had to assume that $y \neq 0$; checking the constant function $y = 0$, $[0]' = 3 \cdot 0^{\frac{2}{3}}$ and $y(0) = 0$, so $y(x) = 0$ also satisfies the IVP, giving the IVP *two* solutions: $y(x) = 0$ and $y(x) = x^3$.

(In a course on differential equations, you will learn about a theorem that specifies restrictions on a first-order IVP that will guarantee a unique solution.)

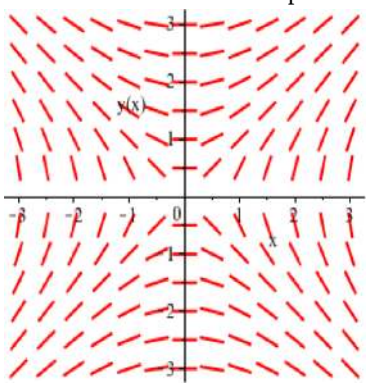
Problems

1. The direction field of the separable ODE $y' = 2xy$ appears below:



   Sketch solutions of the ODE that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$ and $y(1) = 2$.

2. The direction field of the separable ODE $y' = x/y$ appears below:



   Sketch solutions of the ODE that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$ and $y(1) = 2$.

In Problems 3–10, solve the separable ODE.

3. $y' = 2xy$
4. $y' = \dfrac{x}{y}$
5. $(1 + x^2) \cdot y' = 3$
6. $xy' = y + 3$
7. $y' \cdot \cos(x) = e^y$
8. $y' = x^2 y + 3y$
9. $y' = 4y$
10. $y' = 5(2 - y)$

In Problems 11–18, solve the separable ODEs subject to the given initial conditions.

11. $y' = 2xy$ for $y(0) = 3$, $y(0) = 5$ and $y(1) = 2$.
12. $y' = \dfrac{x}{y}$ for $y(0) = 3$, $y(0) = 5$ and $y(1) = 2$.
13. $y' = 3y$ for $y(0) = 4$, $y(0) = 7$ and $y(1) = 3$.
14. $y' = -2y$ for $y(0) = 4$, $y(0) = 7$ and $y(1) = 3$.
15. $y' = 5(2 - y)$ for $y(0) = 5$ and $y(0) = -3$.
16. $y' = 7(1 - y)$ for $y(0) = 4$ and $y(0) = -2$.
17. $(1 + x^2) \cdot y' = 3$ for $y(1) = 4$ and $y(0) = 2$.
18. $xy' = y + 3$ for $y(1) = 20$.
19. For $xy' = y + 3$, can $y(0) = 2$?
20. Find all solutions to the IVP $y' = \sqrt[3]{y}$, $y(0) = 0$.

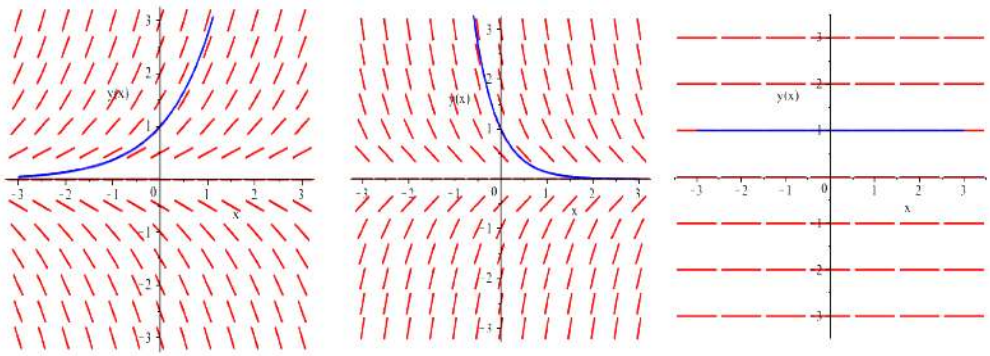
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6.4: Exponential Growth and Decay

The separable differential equation $y' = ky$ is relatively simple to solve, but it can model a wealth of important situations, including population growth, radioactive decay and drug absorption in the bloodstream. In this section we will solve this ODE and explore some related applications.

The Differential Equation $y' = ky$

The differential equation $y' = ky$ says that the rate of change of a quantity y is proportional to the value of y. The figures below show direction fields for $y' = 1y$ (growth) and $y' = -2y$ (decay):



Direction fields for y ' = ky

The ODE $y' = ky$ can model the behavior of populations (the rate at which babies are born is proportional to the number of people currently in the population), radioactive decay (the rate at which atoms decay is proportional to the number of atoms present), the absorption of some medicines by our bodies, and many other situations. The solutions of $y' = ky$ will help us determine how long it takes a population to double in size, the age of some prehistoric artifacts, and even how often some medicines should be taken in order to maintain a safe and effective concentration of that medicine in a patient's body.

Theorem

If $y' = ky$ (for $y > 0$), then $y(t) = y(0) \cdot e^{kt}$.

Proof

The ODE $y' = ky$ is separable, so we can employ the method of Section 6.3 to solve it:

$$\frac{dy}{dt} = k \cdot y \;\Rightarrow\; \frac{1}{y} \cdot \frac{dy}{dt} = k \;\Rightarrow\; \frac{1}{y}\,dy = k\,dt \;\Rightarrow\; \int \frac{1}{y}\,dy = \int k\,dt$$
$$\Rightarrow\; \ln(|y|) = kt + C \;\Rightarrow e^{\ln(|y|)} = e^{kt+C}$$
$$\Rightarrow\; |y| = e^{kt} \cdot e^{C} \;\Rightarrow y = \pm e^{C} e^{kt}$$

Because we assumed that $y > 0$, we didn't need to worry about dividing by y, and we didn't really need the absolute values (or the $\pm$) in the solution above. But $y = 0$ is also a solution to $y' = ky$, so $y = Ae^{kt}$ solves $y' = ky$ for any value of A.

We've found an infinite family of solutions for $y' = ky$, but how do we know that we've found *all* solutions to that ODE? Let $f(t)$ be any solution to $y' = ky$ so that $f'(t) = k \cdot f(t)$. Then define another function $g(t) = f(t) \cdot e^{-kt}$ so that:

$$g(t) = \frac{f(t)}{e^{kt}} \;\Rightarrow\; g'(t) = \frac{e^{kt} \cdot f'(t) - f(t) \cdot e^{kt}}{[e^{kt}]^2} = \frac{e^{kt}[f'(t) - k \cdot f(t)]}{e^{2kt}}$$

The last expression in brackets is 0 (because $f'(t) = k \cdot f(t)$), so:

$$g'(t) = 0 \;\Rightarrow\; g(t) = C \;\Rightarrow\; \frac{f(t)}{e^{kt}} = C \;\Rightarrow\; f(t) = Ce^{kt}$$

We now know that any function of the form $y = Ae^{kt}$ solves $y' = ky$ *and* that any solution of $y' = ky$ must have the form $y = Ae^{kt}$.

Finally, putting $t = 0$ into the general solution:

$$y(0) = Ae^{k\cdot 0} = A \;\Rightarrow\; A = y(0) \;\Rightarrow y(t) = y(0)\cdot e^{kt}$$

which holds for any value of $y(0)$, but in particular for any $y(0) > 0$.

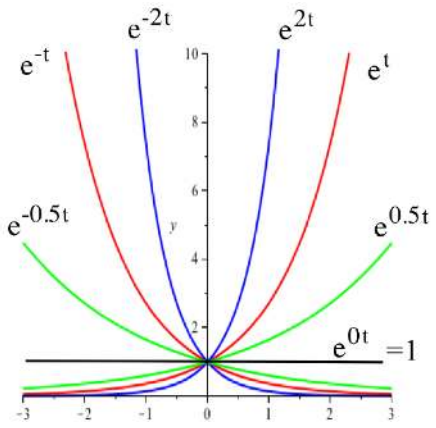
Exponential Growth

A population of people, a chunk of radioactive material and the amount of money in a bank account can all share a common trait. In each situation, the rate at which an amount changes at a particular time is often proportional to the value of that amount at that time. For example:

- the number of births per year is proportional to the number of people in the population
- the number of atoms per hour that release a particle is proportional to the number of atoms present
- the number of dollars of interest per year added to a bank account is proportional to the amount of money in that bank account

These situations can all be modeled with the separable ODE solved above. Our focus in this section will be on using those equations and their solutions to answer questions about applied problems. The applications here all involve the rate of change of some quantity with respect to time, so the input variable will generally be time t (instead of x). We might also write the output quantity as $f(t)$ (instead of y). The ODE $y' = ky$ then becomes $f'(t) = k\cdot f(t)$ and the solution $y = y_0\cdot e^{kx}$ becomes $f(t) = f(0)\cdot e^{kt}$.

When $k > 0$, $f(t) = f(0)\cdot e^{kt}$ represents **exponential growth** and we call k the **growth constant**. When $k < 0$, $f(t) = f(0)\cdot e^{kt}$ represents **exponential decay** and we call k the **decay constant**. The figure below shows the graphs of $f(t) = e^{kt}$ for several values of k:



When you know the initial population $f(0)$ and the growth constant k, you can write an equation for $f(t)$, the population at any time t, and use it to answer questions about the population.

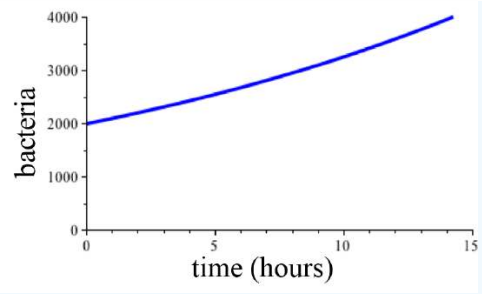
Example 6.4.1

The number of bacteria on a Petri plate t hours after an experiment starts is $2000\cdot e^{0.0488t}$.

a. How many bacteria are on the plate after one hour? Two hours?
b. What is the percentage growth of the population from $t = 0$ to $t = 1$? From $t = 1$ to $t = 2$?
c. How long does it take for the population to reach 3000? To double?

Solution

See figure below for a graph of $f(t) = 2000\cdot e^{0.0488t}$:



a. $f(1) = 2000 \cdot e^{0.0488} \approx 2100$; $f(2) = 2000 \cdot e^{0.0976} \approx 2205$

b. The percentage growth from $t = 0$ to $t = 1$ is:

$$\frac{f(1) - f(0)}{f(0)} = \frac{2100 - 2000}{2000} = \frac{100}{2000} = 0.05 = 5\%$$

The percentage growth from $t = 1$ to $t = 2$ is:

$$\frac{f(2) - f(1)}{f(1)} = \frac{2205 - 2100}{2100} = \frac{105}{2100} = 0.05 = 5\%$$

During the first hour, the population grows by 100 and during the second hour it grows by 105, but the percentage growth during each hour remains constant at 5%.

c. We need the value of T so that $3000 = f(T) = 2000 \cdot e^{0.0488T}$:

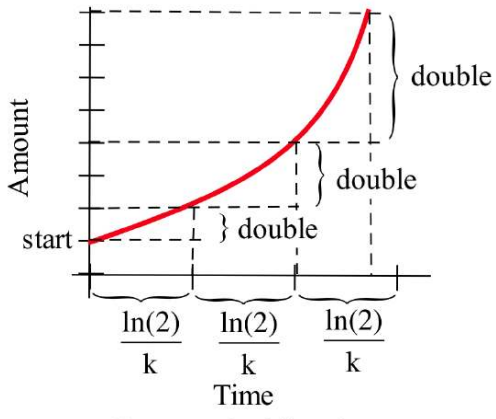
$$1.5 = e^{0.0488T} \Rightarrow \ln(1.5) = \ln\left(e^{0.0488T}\right) = 0.0488T$$
$$\Rightarrow T = \frac{\ln(1.5)}{0.0488} \approx 8.31 \text{ hours}$$

The original population is 2000, so the doubled population is 4000 and the doubling time is $\frac{\ln(2)}{0.0488} \approx 14.2$ hours.

When we know the growth constant k, the doubling time is simple to find (as in the preceding Example). If $f(t) = f(0) \cdot e^{kt}$ then the doubling time is the time t_d so that:

$$2f(0) = f(0) \cdot e^{kt_d} \Rightarrow 2 = e^{kt_d} \Rightarrow \ln(2) = kt_d \Rightarrow t_d = \frac{\ln(2)}{k}$$

An important aspect of exponential growth is that the doubling time depends only on the growth constant k.



Constant doubling times

? Practice 6.4.1

Use the information from the previous Example to:

a. determine the population at $t = 5$.

b. find how long it takes for the population to reach 5,000. To triple.

If you do not know the value of the growth constant k, your first step will typically be to use other information to find it.

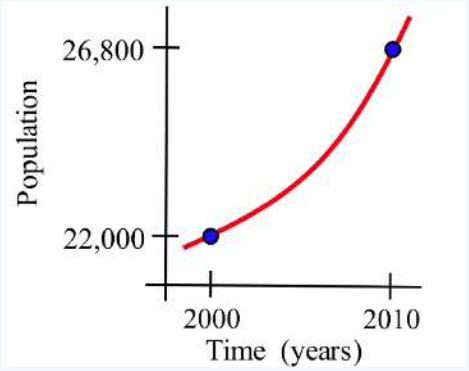
Answer

a. $f(5) = 2000e^{0.0488(5)} \approx 2552$

b. $f(T) = 5000 \Rightarrow 5000 = 2000e^{0.0488T} \Rightarrow 2.5 = e^{0.0488T} \Rightarrow \ln(2.5) = 0.0488T \Rightarrow T = \dfrac{\ln(2.5)}{0.0488} \approx 18.78$ hours; tripling time is $\dfrac{\ln(3)}{0.0488} \approx 22.51$ hours.

Example 6.4.2

The population of a community was 22,000 in 2000 and 26,800 in 2010:



Assuming that the community maintains the same rate of exponential growth:

a. Find a formula for the population t years after 2000.
b. Find the annual percentage rate of growth of the community.

Solution

Let t represent the number of years since 2000, so the year 2000 corresponds to $t = 0$ and the year 2010 corresponds to $t = 10$. Then $f(0) = 22000$, $f(10) = 26800$ and $f(t) = 22000 \cdot e^{kt}$.

a. To find the value for k:

$$26800 = f(10) = 22000 \cdot e^{k(10)} \Rightarrow 1.218 = e^{10k} \Rightarrow \ln(1.218) = 10k$$
$$\Rightarrow k = 0.1\ln(1.218) \approx 0.0197$$

so $f(t) \approx 22000 \cdot e^{0.0197t}$.

b. $f(0) = 22000$ and $f(1) \approx 22000 \cdot e^{(0.0197)1} \approx 22438$ so the annual percentage increase was

$$\frac{f(1) - f(0)}{f(0)} = \frac{438}{22000} \approx 0.0199 = 1.99\%$$

during the first year.

Practice 6.4.2

A scientist released 12,000 free neutrons into a material. Two seconds later, the material contained 18,000 free neutrons. If the number of free neutrons grows exponentially:

Answer

a. Find a formula for the number of neutrons present t seconds after the beginning of the experiment.
b. Find the doubling time for the number of free neutrons.

a. $f(0) = 12000$ so $f(t) = 12000e^{kt}$ and $f(2) = 18000$ so:

$$18000 = 12000e^{2k} \Rightarrow 1.5 = e^{2k} \Rightarrow \ln(1.5) = 2k \Rightarrow k = \frac{1}{2}\ln(1.5) \approx 0.2027$$

and thus $f(t) \approx 12000e^{0.2027t}$.

b. Doubling time is $t_d = \dfrac{\ln(2)}{k} \approx \dfrac{\ln(2)}{0.2027} \approx 3.42$ seconds.

Compound interest provides another example of exponential growth.

Example 6.4.3

How long does it take $1,000 to double when invested in a savings account with interest compounded continuously at an annual rate of 5%? At an effective annual rate of return of 5% (compounded continuously)?

Solution

If the interest is compounded continuously at an annual rate of 5% and $A(t)$ is the amount of money in the account t years after the initial deposit of $1,000, then:

$$A'(t) = 0.05A(t),\ A(0) = 1000 \quad \Rightarrow \quad A(t) = 1000e^{0.05t}$$

and the doubling time is $t_d = \frac{\ln(2)}{0.05} \approx 13.86$ years.

If the effective annual rate of return is 5%, then $A(1) = 1000 + 0.05(1000) = 1.05(1000) = 1050$ so:

$$1050 = 1000e^{k\cdot 1} \quad \Rightarrow \quad 1.05 = e^{k} \quad \Rightarrow \quad k = \ln(1.05) \approx 0.0488$$

hence the doubling time is $t_d = \frac{\ln(2)}{\ln(1.05)} \approx 14.21$ years. (An effective annual return of 5% corresponds to a continuously compounded annual interest rate of 4.88%.)

Practice 6.4.3

How long does it take an investment to double if the effective annual rate of return is 2%?

Answer

After 1 year, each \)1 invested will become

)1 + (0.02)(

)1) =

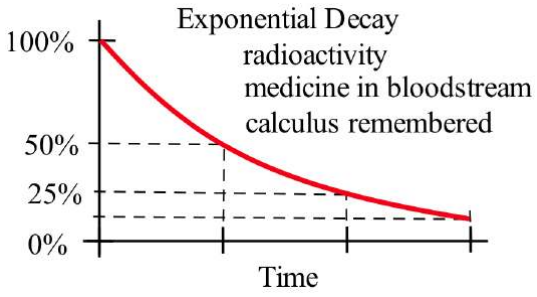
)1.02 so:

$$f(1) = 1.02 = 1 \cdot e^{k\cdot 1} \Rightarrow \ln(1.02) = k \Rightarrow k \approx 0.0198$$

The doubling time is therefore: $t_d = \dfrac{\ln(2)}{0.0198} \approx 35$ years.

Exponential Decay

Exponential decay occurs when the rate of loss of something is proportional to the amount present. One example of exponential decay is radioactive decay: the number of atoms of a radioactive substance that “decay” (split into nonradioactive atoms and release particles) during a short time interval is proportional to the number of radioactive atoms present at that time. Exponential decay:



also models how quickly some medicines are absorbed from the bloodstream—and even how quickly you forget calculus concepts.

Exponential decay calculations are similar to those for growth, but the value of k is negative and we talk about "**half-life**," the time for half of the material to decay or be absorbed, instead of the doubling time. The table below shows the half-lives of some isotopes.

element	half-life
iodine-131	8.07 days
strontium-90	29 years
argon-39	265 years
carbon-14	5700 years
plutonium-239	24400 years
uranium-238	4.51×10^5 years
uranium-234	2.47×10^9 years

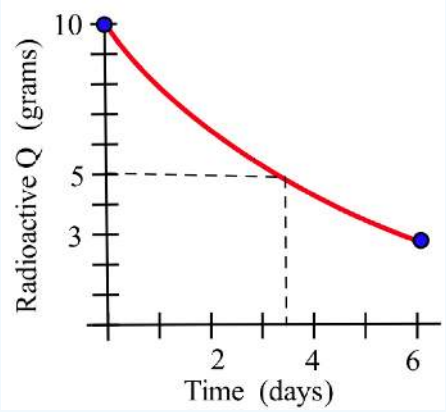
Example 6.4.4

You started an experiment with 10 g of a radioactive substance, but after 6 days of decay only 3 g remained.

a. Find a formula for the amount of radioactive material present t days after beginning the experiment.
b. Find the half-life for the radioactive substance.

Solution

Let $f(t)$ represent the amount of the radioactive substance present after t days. Then $f(t) = 10e^{kt}$:



a. $3 = f(6) = 10e^{6k} \Rightarrow 0.3 = e^{6k} \Rightarrow \ln(0.3) = 6k$ so $k = \frac{1}{6}\ln(0.3) \approx -0.2007$ and $f(t) = 10e^{-0.2007t}$.
b. The half-life t_h is the time required for half of the material to decay, so we need to solve $5 = 10e^{-0.2007t_h}$ for t_h:

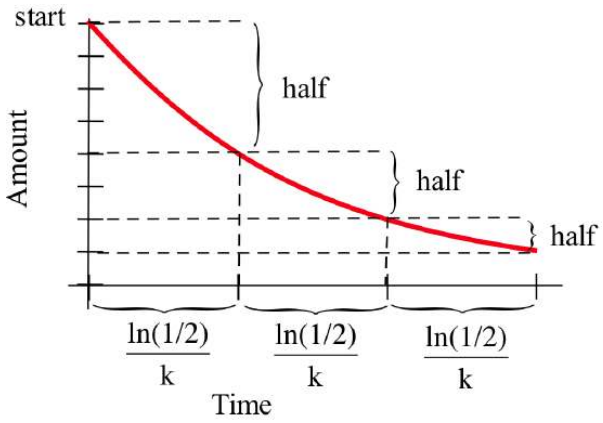
$$0.5 = e^{-0.2007t_h} \Rightarrow \ln(0.5) = -0.2007t_h \Rightarrow t_h = \frac{\ln(0.5)}{-0.2007} \approx 3.5 \text{ days}$$

Note that $t_h = \frac{\ln(0.5)}{k}$, which will hold true generally.

When you know the decay constant k, the half-life t_h is simple to find, as in the preceding Example:

$$t_h = \frac{\ln(0.5)}{k} = \frac{\ln\left(\frac{1}{2}\right)}{k} = \frac{-\ln(2)}{k}$$

The half-life depends only on the decay constant k:

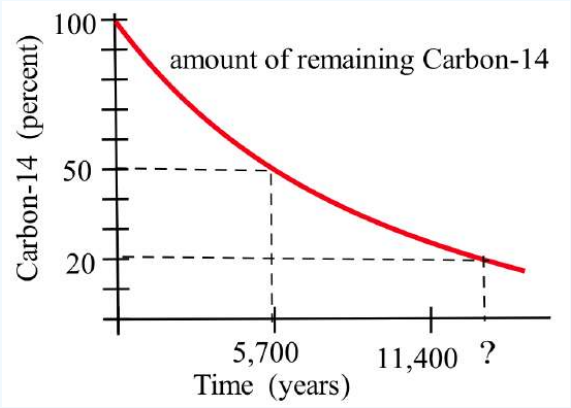


If you know the half-life of a substance and you know how much of the substance is present in a sample now, you can determine how much was present at some past time or determine how long ago the sample contained a particular amount of the substance.

Scientists use radioactive **carbon-14**, with a half-life of about 5,700 years, in this way to estimate how long ago plants and animals lived. A living plant continually exchanges carbon-14 and ordinary carbon with the atmosphere so that the ratio of carbon-14 to non-radioactive carbon remains relatively constant. But once the plant dies, this exchange stops. The ordinary carbon remains in the material, but the carbon-14 decays, so the ratio of carbon-14 to ordinary carbon decreases at a known rate. By measuring the ratio of carbon-14 to ordinary carbon in a sample of plant tissue, scientists can determine how long ago the plant died and obtain an estimate for the age of the sample.

Example 6.4.5

The amount of carbon-14 in plant fiber of a woven basket is 20% of the amount present in a living plant:



Estimate the age of the basket.

Solution

Let $f(t)$ represent the relative amount of carbon-14 in a sample with age t years. Because the half-life of carbon-14 is 5,700 years:

$$t_h = \frac{-\ln(2)}{k} \Rightarrow 5700 = \frac{-\ln(2)}{k} \Rightarrow k = \frac{-\ln(2)}{5700} \approx -0.0001216$$

so that $f(t) = f(0) \cdot e^{-0.0001216t}$. Because 20% of the carbon-14 remains in our sample, we want the value of T so that:

$$\begin{aligned} 0.20f(0) = f(T) &\Rightarrow 0.20f(0) = f(0) \cdot e^{-0.0001216T} \\ &\Rightarrow 0.20 = e^{-0.0001216T} \Rightarrow \ln(0.20) = -0.0001216T \\ &\Rightarrow T = \frac{\ln(0.2)}{-0.0001216} \approx 13235 \end{aligned}$$

We can conclude that the basket was made from a plant that died about 13,200 years ago. (Does that mean the basket was made then?)

> This dating method is very sensitive to small changes in the measured amount of carbon-14.

? Practice 6.4.4

The half-life of an isotope is 8 days. Write a formula for the amount of the isotope present t days after you begin an experiment with 10 mg of the isotope.

Answer

$8 = t_h = -\frac{\ln(2)}{k} \Rightarrow k = -\frac{\ln(2)}{8} \approx -0.0866$ so $f(t) = 10e^{-0.0866t}$.

The rate at which many medicines are absorbed from the blood is proportional to the concentration of the medicine in the blood: the higher the concentration in the blood, the faster it is absorbed.

✓ Example 6.4.6

Suppose a certain medicine has an absorption (decay) constant of -0.17 (determined experimentally) and that the lowest "effective"concentration of the medicine is 0.3 mg/l (milligrams of medicine per liter of blood). If a patient who has 5 liters of blood is injected with 20 mg of the medicine, how long will the medicine be effective?

Solution

Because the patient is starting with 20 mg of the medicine in 5 liters of blood, the initial concentration is $\frac{20 \text{ mg}}{5 \text{ L}} = 4$ mg/L. The amount of medicine in the blood t hours later is thus $f(t) = 4e^{-0.17t}$ and we want to find T so that $f(T) = 0.3$ mg/L

$$0.3 = 4e^{-0.17T} \Rightarrow \frac{0.3}{4} = e^{-0.17T} \Rightarrow \ln\left(\frac{0.3}{4}\right) = -0.17T \Rightarrow T \approx 15.2$$

so a patient should receive a new does of the medicine about 15 hours after the first dose.

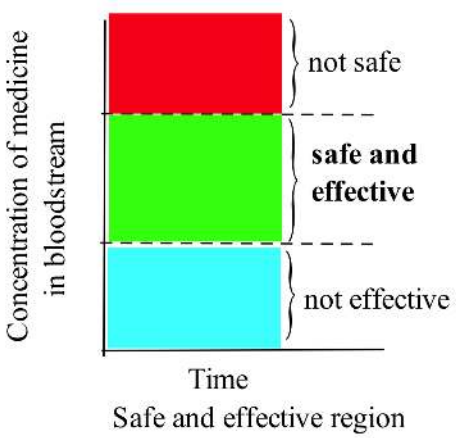
? Practice 6.4.5

Should the amount of the second dose in the preceding Example be the same as the initial dose?

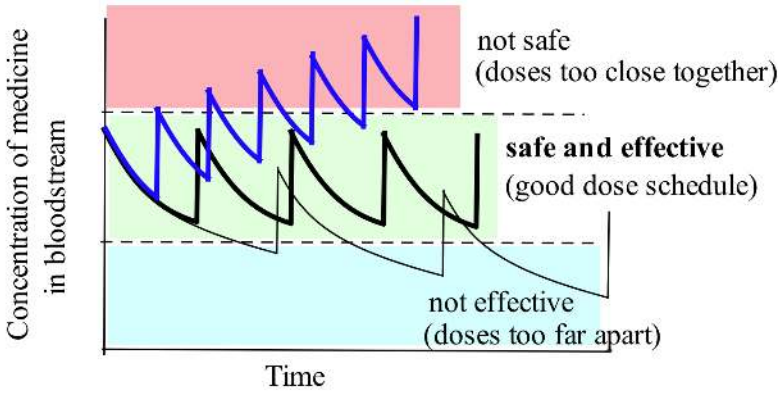
Answer

No. After 15.2 hours, the patient still has 0.3 mg/l of medicine in his blood, or $5(0.3) = 1.5$ mg. A dose of $20 - 1.5 = 18.5$ mg would return the medicine in his blood to the original level of 20 mg.

Many medicines have a “safe and effective” interval of concentrations:



Safe and effective region

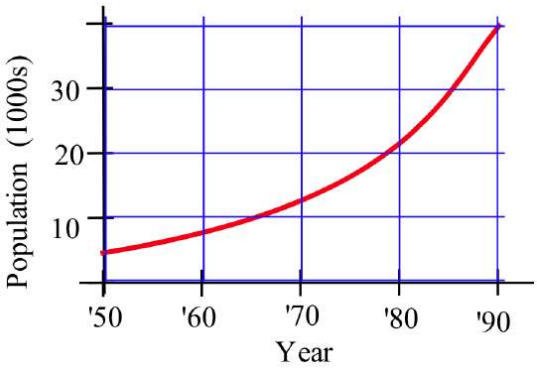


Safe and effective dosing schedule

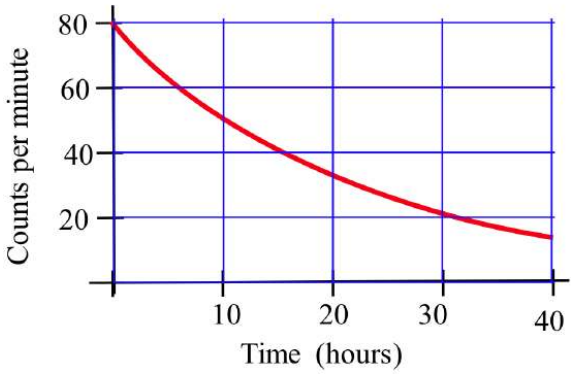
so the goal of a schedule for taking the medicine is to keep the concentration near the middle of that range. Taking doses too close together in time can result in an overdose, while taking them too far apart is eventually ineffective.

Problems

1. How long did it take the population of a city (see figure below) to double from 10,000 to 20,000? From 15,000 to 30,000? Approximate the doubling time.



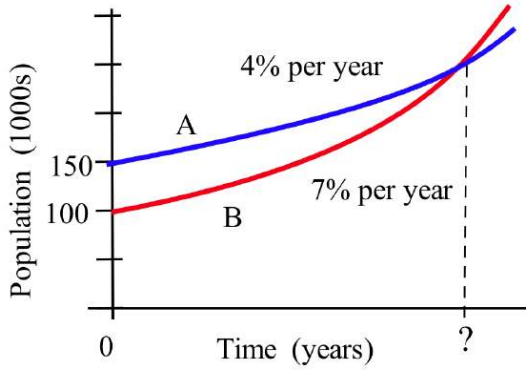
2. How long did it take the counts for a radioactive material (see figure below) to decay from 80 per minute to 40? From 60 to 30?



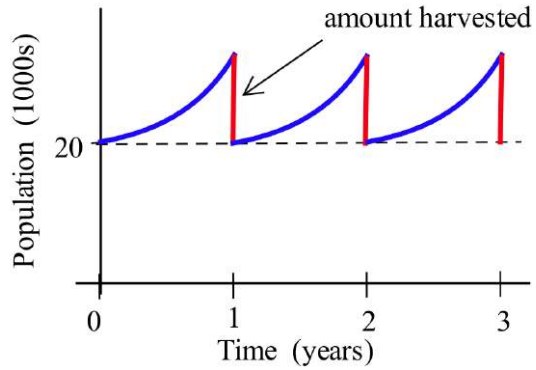
From 40 to 20? What is the half-life?

3. The population of a community in 1990 was 48,000 people and in 2010 it was 64,000 people.
   a. Write a formula for the population of the community t years after 1990.
   b. Estimate the population in the year 2020.
   c. When will the population be 100,000?
   d. What is the doubling time of the population?
4. Repeat Problem 3 if the population of another community was was 40,000 people in 1990 and 60,000 people in 2010.

5. A terrific investment pays interest at an effective annual rate of 15%. How long will it take for a $5,000 investment to double? To triple?
6. You have invested $3,000 for 10 years at an effective annual rate of 7.5% and a friend has invested the same amount invested at an effective annual interest rate of 7.75%. Your friend will get back how much more money than you at the end of 10 years? At the end of 20 years?
7. Find a formula for the population in Problem 1.
8. Each bacterium of a certain species splits into two bacteria at the end of each minute. If we start with a few bacteria in a bowl at 3:00 p.m. and the bowl is full of bacteria at 4:30 p.m., when was the bowl half full? (Calculus is not required.)
9. A newscaster reports that the world's population is now doubling every 50 years. What annual rate of growth results in a 50-year doubling time?
10. Group A has a population of 150,000 and an annual growth rate of 4%; group B has a population of 100,000 and an annual growth rate of 7% (see figure below). After how many years will the two groups be the same size?



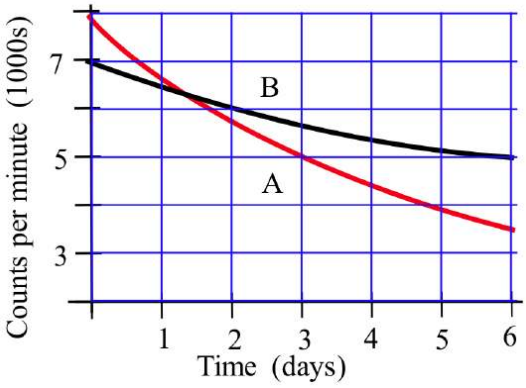
11. Group A has a population of 600,000 and an annual growth rate of 3%; group B has a population of 400,000 and an annual growth rate of 6%. After how many years will the two groups be the same size?
12. The unregulated population of fish in a certain lake grows by 30% per year under optimum conditions. A fish census reveals there are approximately 20,000 fish in the lake. How many fish can be harvested (see figure below) at the end of each year in order to maintain a stable population? (This is an example of calculating the yield for a "renewable resource." In practice, more sophisticated calculations also take into account the distribution of species, ages and genders.)



13. The annual exponential growth constant for a population of snails is $k = 0.14$. Currently you have 8,000 snails.
    a. Determine the size of the population over the next 20 months if you harvest 2,000 snails every 2 months.
    b. What happens if you harvest 3,000 snails every 2 months?
    c. How many can we harvest every 2 months in order to maintain a stable population?
14. An exponential function $f(t) = Ae^{kt}$ has constant doubling time, but some non-exponential functions also have constant doubling times.
    a. Show that the exponential function $f(t) = 2^t = e^{\ln(2)t}$ has a constant doubling time of 1. (Show that $f(t+1) = 2f(t)$.)
    b. Graph $g(t) = 2^t\,[1 + A\sin(2\pi t)]$ for $A = 0.5$ and $A = 1.5$. Show that g has a constant doubling time 1 for any choice of A.
15. An experiment starts with 10 grams of a radioactive material and 14 days later 2 grams remain.

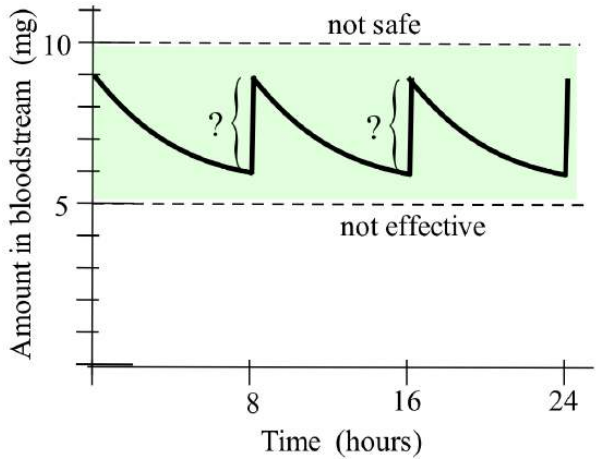
a. Find a formula for the amount of material remaining t days after the experiment starts.
b. Find the half-life of the material.
c. When will 0.7 grams of the material remain?

16. You start with 8 mg of a radioactive substance and 10 days later determine that 6.3 mg remains.
    a. Find a formula for the amount left t days later.
    b. Find the half-life of the substance.
    c. When will 1 mg of the substance remain?
17. A Geiger counter initially recorded 187 counts per minute from a radioactive material, but 2 days later the count was down to 143 counts per minute. (The count per minute is proportional to the amount of radioactive material present.)
    a. What is the half-life of the material?
    b. When will the count be down to 20 counts per minute?
18. The initial Geiger counter measurement from a radioactive substance was 540 counts per minute, and a week later it was 500 counts per minute.
    a. What is the half-life of the substance?
    b. When will the count be down to 100 counts per minute?
19. Find a formula for the counts per minute for the radioactive material A in the figure below.



20. Find a formula for the counts per minute for the radioactive material B in the figure above.
21. Your friend plans to purchase a letter reputedly written by Isaac Newton (1642–1727), but an analysis of the paper shows that it contains 97.5% of the proportion of carbon-14 present in new paper of the same type. Can you be certain the letter is a forgery? If the age of the paper is consistent with time frame during which Newton lived, can you be certain the letter is genuine?
22. For several centuries, the Shroud of Turin was widely believed to be the shroud of Jesus. Three independent laboratories in England, Switzerland and the United States used carbon-14 dating on a few square centimeters of the cloth, and in 1988 they reported (*Science*, October 21, 1988) that the Shroud of Turin was probably made during the early 1300s and certainly after 1200 A.D.
    a. If the Shroud was made in 1300 A.D., what percentage of the original carbon-14 was still present in 1988?
    b. If the Shroud was made in 30 A.D., what percentage of the original carbon-14 was still present in 1988?
23. Half of a particular medicine is used up by the body every 6 hours, and the medicine is not effective if the concentration in the blood is less than 10 mg/l. If an ill person is given an initial dose of medicine to raise the concentration to 30 mg/l, for how long will the medicine be effective?
24. A particular controlled substance has a half-life of 12 hours, and it can be detected in concentrations as low as 0.002 mg/l in the blood.
    a. If a person has an initial concentration of the substance of 15 mg/l in the blood, for how long can it be detected?
    b. If the detection test is improved by a factor of 100, so it can detect a concentration of 0.00002 mg/l, for how long can an initial concentration of 15 mg/l be detected?
25. A doctor gave a patient 9 mg of a medicine that has half-life of 15 hours in the body. How much of the medicine does the patient need to take every 8 hours in order to maintain between 6 and 9 mg of the medicine in the body all of the time? (See

figure below.)



26. Each layer of a dark film transmits 40% of the light that strikes it.
    a. How many layers are needed for an eye shield to transmit only 10% of the light?
    b. How many layers are needed to transmit only 2% of the light?
27. A region has been contaminated with radioactive iodine-131 to a level five times the safe level. How long will it be until the area is safe?
28. A region has been contaminated with radioactive strontium-90 to a level five times the safe level. How long will it take until the area is safe?
29. The population of a country is 4 million and is growing at 5% per year. Currently the country has 10 million acres of forests that are being cut down (and not replanted) at a rate of 300,000 acres per year.
    a. Find a formula for the number of acres of forest per person.
    b. At what rate is the number of acres of forest per person changing?
    c. If the population and harvest rates remain constant, in approximately how many years will there be one acre of forest per person?

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6.5: Heating, Cooling and Mixing

The previous section examined applications of exponential growth and decay. This section examines other applications involving separable IVPs that can be solved using the methods of Section 6.3.

Newton's Law of Cooling

Some rates of change depend on how far the value of a variable quantity is from a fixed value. The rate at which a hot cup of soup cools (or a cool cup of water warms up) is proportional to the difference in temperature between the soup (or water) and the surrounding air. This principle, called Newton's Law of Cooling (or Warming), provides an approximate model for temperature change when the object in question is subject to forced convection (a steady breeze, for example). It also applies to natural convection (no breeze or draft) if the temperature difference between the object and the surrounding air is not too great.

Newton's Law of Cooling/Warming

If: $f(t)$ is the temperature at time t of an object in an atmosphere with constant temperature a

then: the rate of change of f is proportional to the difference between f and a: $f'(t) = k[f(t) - a]$ and the solution to this ODE is:

$$f(t) = a + [f(0) - a] \cdot e^{kt}$$

Note

The statement that the rate of change is proportional to the difference:

$$f'(t) = k[f(t) - a]$$

can be observed by experimentation.

Proof

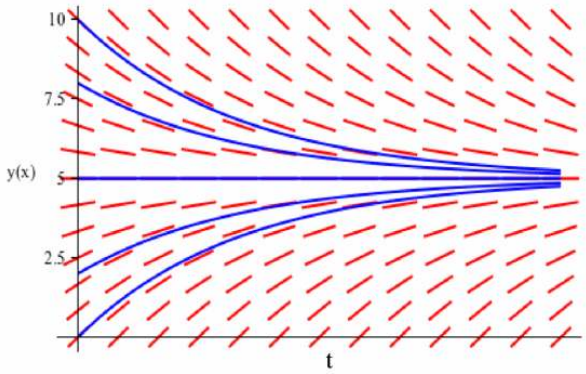
This ODE is separable, so we can solve it using the methods of Section 6.3. Or we can let $g(t) = f(t) - a$ so that $g'(t) = f'(t)$ and $g(0) = f(0) - a$, transforming the ODE for $f(t)$ into the IVP:

$$g'(t) = kg(t), \qquad g(0) = f(0) - a$$

From Section 6.4 we know that $g(t) = g(0) \cdot e^{kt}$ so that:

$$f(t) - a = [f(0) - a]\, e^{kt} \quad \Rightarrow \quad f(t) = a + [f(0) - a]\, e^{kt}$$

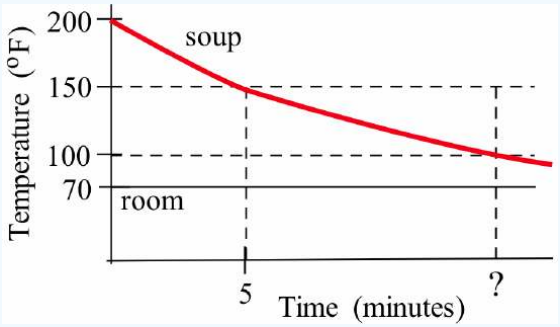
The figure below shows some functions with different initial values that all satisfy the differential equation $f'(t) = f(t) - 5$:



Solutions of f '(t)= (-1)*(f(t) - 5)

Example 6.5.1

A cup of soup sits on a counter in a room with temperature 70° F. When first poured into the cup, the soup had a temperature of 200° F, and five minutes later its temperature was 150° F:



a. Find a formula for the soup's temperature at any time $t > 0$.
b. How long does it take for the soup to cool to 100° F?
c. What will the temperature of the soup be after a "long" time?

Solution

a. If $f(t)$ is the soup's temperature t minutes after being poured and the room temperature is $a = 70°$ F, then we know that $f(0) = 200°$ F so $f'(t) = k[f(t) - 70]$ and hence:

$$f(t) = 70 + [200 - 70] \cdot e^{kt} = 70 + 130 \cdot e^{kt}$$

We can now use the information that $f(5) = 150°$ F to find the value of k and an equation for $f(t)$:

$$150 = f(5) = 70 + 130.e^{k \cdot 5} \;\Rightarrow\; k = \frac{1}{5} \cdot \ln\left(\frac{80}{130}\right) \approx -0.0971$$

so $f(t) \approx 70 + 130 \cdot e^{-0.0971t}$.

b. We now want to find the time t so that $f(t) = 100$:

$$100 = f(t) \approx 70 + 130 \cdot e^{-0.0971t} \;\Rightarrow\; 30 = 130 \cdot e^{-0.0971t} \;\Rightarrow\; t = \frac{\ln\left(\frac{30}{130}\right)}{-0.0971} \approx 15.1 \text{ minutes}$$

c. After a "long time" means for very large values of t:

$$\lim_{t \to \infty} \left[70 + 130 \cdot e^{-0.0971t}\right] = 70$$

so (eventually) the temperature of the soup approaches 70° F, the temperature of the room.

Practice 6.5.1

After eating some of the soup in the preceding Example, you refrigerate the leftover soup, cooling it to a temperature of 37° F. At noon the next day, you move the container of soup from the refrigerator back to the 70° F room. One hour later, the temperature of the soup is 48° F. At what time will the temperature of the soup be 60° F?

Answer

The temperature of the soup t minutes after being removed from the refrigerator is:

$$f(t) = 70 + [37 - 70] \cdot e^{kt} = 70 - 33e^{kt}$$

Using the fact that $f(60) = 48$:

$$48 = 70 - 33e^{kt} \;\Rightarrow\; \frac{22}{33} = e^{60k} \;\Rightarrow\; k = \frac{1}{60} \cdot \ln\left(\frac{22}{33}\right) \approx -0.006758$$

so that $f(t) \approx 70 - 33e^{-0.006758t}$. If $60 = 70 - 33e^{-0.006758T}$ then:

$$\frac{10}{33} = e^{-0.006758T} \Rightarrow T = -\frac{1}{0.006758} \cdot \ln\left(\frac{10}{33}\right) \approx 176.7$$

so the soup will warm up to 60\)^{\circ}\) F after roughly three hours.

Mixing

Many important applications involve one substance being mixed into another: chlorine being slowly introduced into a swimming pool, a toxic gas being mixed into the air in a closed room, or (in a classic example) salt (NaCl) being mixed into water (H_2O).

Example 6.5.2

A tank contains 350 grams of salt dissolved in 10 liters of water. Pure water begins flowing into the tank at 2 liters per hour, and the well-mixed solution flows out of the tank at the same rate.

a. Find a formula for the amount of salt in the tank t hours after this process begins.
b. Find a formula for the concentration salt in the tank.
c. What will the concentration be after a "long" time?

Solution

a. Let $A(t)$ be the amount of salt (in grams) t hours after this process begins. We know that $A(0) = 350$. Now consider $A'(t)$, the rate at which amount of salt in the tank is changing with respect to time. No salt is flowing in, but the rate at which salt is leaving the tank is:

$$\frac{A(t)\ \text{g}}{10\ \text{L}} \cdot 2\ \frac{\text{L}}{\text{hr}} = 0.2A(t)\ \frac{\text{g}}{\text{hr}}$$

which gives us the IVP:

$$\frac{dA}{dt} = -0.2A, \quad A(0) = 350$$

This is a separable IVP with solution $A(t) = 350e^{-0.2t}$.

b. The concentration is:

$$C(t) = \frac{A(t)\ \text{g}}{10\ \text{L}} = \frac{350e^{-0.2t}\ \text{g}}{10\ \text{L}} = 35e^{-0.2t}\ \frac{\text{g}}{\text{L}}$$

c. After a "long" time:

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} 35e^{-0.2t} = 0$$

This seems reasonable: after a very long time, most of the original salt in the tank will be flushed out, replaced by pure water.

Practice 6.5.2

A tank contains 300 grams of salt dissolved in 10 liters of water. Water containing 15 grams of salt per liter begins flowing into the tank at a rate of 4 liters per hour, and the well-mixed solution flows out of the tank at the same rate.

a. Find a formula for the amount of salt in the tank t hours after this process begins.
b. Find a formula for the concentration salt in the tank.
c. What will the concentration be after a "long" time?

Answer

If $A(t)$ is the amount of salt (in grams) t hours after this process begins, then:

$$\frac{dA}{dt} = 15\ \frac{\text{g}}{\text{L}} \cdot 4\ \frac{\text{L}}{\text{hr}} - \frac{A\ \text{g}}{10\ \text{L}} \cdot 4\ \frac{\text{L}}{\text{hr}}$$

so $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = 60 - \frac{2}{5}A, \quad A(0) = 300$$

a. This IVP is separable, so we can solve it:

$$\int \frac{1}{60 - \frac{2}{5}A}\, dA = \int 1\, dt \;\Rightarrow\; -\frac{5}{2}\ln\left(\left|60 - \frac{2}{5}A\right|\right) = t + C$$
$$\Rightarrow\; \ln\left(\left|60 - \frac{2}{5}A\right|\right) = -\frac{2}{5}t + K$$

Using the initial condition:

$$\ln\left(\left|60 - \frac{2}{5}\cdot 300\right|\right) = -\frac{2}{5}\cdot 0 + K \;\Rightarrow\; K = \ln(60)$$

so that:

$$\ln\left(\left|60 - \frac{2}{5}A\right|\right) = -\frac{2}{5}t + \ln(60) \;\Rightarrow\; 60 - \frac{2}{5}A = -60e^{-\frac{2}{5}t}$$

where we need to use a $-$ sign when "undoing" the absolute value to ensure that the initial condition $A(0) = 300$ will still hold. Solving for A yields $A(t) = 150\left[1 + e^{-\frac{2}{5}t}\right]$.

b. Divide by the volume to get $C(t) = 15\left[1 + e^{-\frac{2}{5}t}\right]$.

c. $\lim_{t\to\infty} 15\left[1 + e^{-\frac{2}{5}t}\right] = 15$, so the limiting concentration is 15 g/L (the concentration of solution flowing into the tank, as expected).

If the volume of liquid in the tank does not remain constant, the differential equation can become a bit more complicated.

Example 6.5.3

A 20-liter tank contains 250 grams of salt dissolved in 10 liters of water. Pure water begins flowing into the tank at a rate of 5 liters per hour, and the well-mixed solution flows out of the tank at a rate of 3 liters per hour.

a. Find a formula for the amount of salt in the tank t hours after this process begins.
b. Find a formula for the concentration salt in the tank.
c. What will the concentration be at the moment the tank overflows?

Solution

a. Let $A(t)$ be the amount of salt (in grams) t hours after this process begins. We know that $A(0) = 250$ and that the volume of water in the tank is $V(t) = 10 + 2t$ (because 5 liters of water flow in each hour, but only 3 liters flow out). No salt is flowing into the tank, but the rate at which salt is leaving the tank is:

$$\frac{A(t)\ \text{g}}{10 + 2t\ \text{L}} \cdot 3\ \frac{\text{L}}{\text{hr}} = \frac{3A(t)}{10 + 2t}\ \frac{\text{g}}{\text{hr}}$$

which gives us the IVP:

$$\frac{dA}{dt} = -\frac{3A}{10 + 2t}, \quad A(0) = 250$$

This is a separable IVP:

$$\frac{1}{A}\cdot\frac{dA}{dt} = -\frac{3}{10 + 2t} \;\Rightarrow\; \ln(A) = -\frac{3}{2}\ln(10 + 2t) + C$$

Using the initial condition:

$$\ln(250) = -\frac{3}{2}\ln(10) + C \;\Rightarrow\; C = \ln(250) + \frac{3}{2}\ln(10)$$

Solving for A:

$$\ln(A) = \ln(10+2t)^{-\frac{3}{2}} + \ln(250) + \ln(10^{\frac{3}{2}})$$
$$\Rightarrow \quad A(t) = (10+2t)^{-\frac{3}{2}} \cdot 250 \cdot 10\sqrt{10} = \frac{2500\sqrt{10}}{(10+2t)^{\frac{3}{2}}}$$

b. The concentration is:

$$C(t) = \frac{A(t)\ \text{g}}{10+2t\ \text{L}} = \frac{2500\sqrt{10}}{(10+2t)^{\frac{5}{2}}}\ \frac{\text{g}}{\text{L}}$$

c. The tank overflows when $20 = 10 + 2t \Rightarrow t = 5$, at which time the concentration is:

$$C(5) = \frac{2500\sqrt{10}}{(10+2\cdot 5)^{\frac{5}{2}}} = \frac{25}{4\sqrt{2}}\ \frac{\text{g}}{\text{L}}$$

or about 4.42 grams per liter.

? Practice 6.5.3

A 20-liter tank contains 250 grams of salt dissolved in 10 liters of water. Water containing 35 grams of salt per liter begins flowing into the tank at a rate of 5 liters per hour, and the well-mixed solution flows out of the tank at a rate of 3 liters per hour. Set up an initial value problem to model the amount of salt in the tank t hours after this process begins. Is the ODE separable?

Answer

If $A(t)$ is the amount of salt (in grams) t hours after this process begins, then:

$$\frac{dA}{dt} = 35\ \frac{\text{g}}{\text{L}} \cdot 5\ \frac{\text{L}}{\text{hr}} - \frac{A\ \text{g}}{(10+2t)\ \text{L}} \cdot 3\ \frac{\text{L}}{\text{hr}}$$

so $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = 175 - \frac{3A}{10+2t}, \quad A(0) = 250$$

This is not a separable equation. (You will learn how to solve equations of this type in a course about differential equations. For now, you could construct a direction field to analyze the behavior of solutions to the ODE.)

Problems

1. You remove a cheesecake from the oven with an ideal internal temperature of 165° F and place it into a 35° F refrigerator. After 10 minutes, the cheesecake has cooled to 150° F. If you must wait until the cheesecake has cooled to 70° F before you eat it, how long will you have to wait?
2. When a pot of hot (200° F) water is removed from the stove in a 70° F kitchen, it takes 15 minutes to cool to a temperature of 150° F.
   a. Find a formula for the temperature of the water t minutes after it is removed from the stove.
   b. When will the water be 100° F?
   c. When will the water be 80° F?
   d. When will the water be 60° F?
3. When you remove the pot of 200° F water from the previous problem from the stove and take it outside on a 40° F day, it only takes 4 minutes to cool to a temperature of 150° F.
   a. Find a formula for the temperature of the water t minutes after it is removed from the stove.
   b. When will the water be 100° F?
   c. When will the water be 80° F?
   d. When will the water be 60° F?
4. When you remove a pitcher of orange juice from a 40° F refrigerator into a 70° F kitchen, it takes 25 minutes to warm up to a temperature of 60° F.

a. Estimate the temperature of the juice t minutes after you removed it from the refrigerator.
b. When was the juice 50° F?
c. When will the juice be 65° F?

5. A tank initially holds 100 gallons of brine (water mixed with salt) containing 50 pounds of dissolved salt. Pure water flows into the tank at a rate of 3 gallons per minute and flows out at the same rate, with the brine concentration kept roughly uniform by constant stirring. How much salt remains in the tank after one hour?
6. A tank initially holds 1000 liters of brine containing 8 kg of dissolved salt. Pure water enters the tank at a rate of 10 liters per minute and brine (with concentration kept roughly uniform by constant stirring) leaves the tank at the same rate.
   a. Find the concentration of salt in the tank after half an hour.
   b. Determine the limiting concentration of salt in the tank after "a very long time."
7. A large tank initially contains 100 gallons of brine (water mixed with salt) containing 50 pounds of dissolved salt. Pure water flows into the tank at a rate of 3 gallons per minute and flows out at 2 gallons per minute, with the brine concentration kept roughly uniform by constant stirring. How much salt remains in the tank after one hour?
8. A tank initially holds 1000 liters of pure water. Brine containing 8 g of dissolved salt per liter enters the tank at a rate of 10 liters per minute and the well-mixed solution leaves the tank at the same rate.
   a. Find the concentration of salt in the tank after half an hour.
   b. Determine the limiting concentration of salt in the tank after "a very long time."

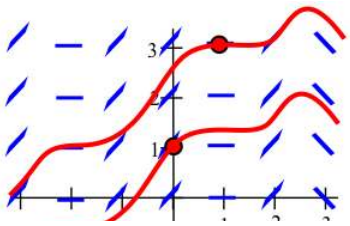
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6.A: Answers

Section 6.1

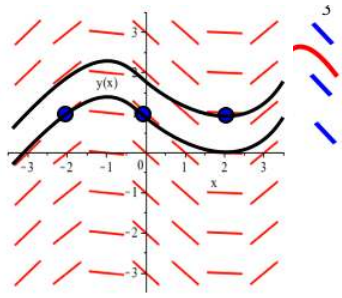
1. $y = e^{-3x} + 2 \Rightarrow y' = -3e^{-3x}$ so $y' + 3y = \left(-3e^{-3x}\right) + 3\left(e^{-3x} + 2\right) = -3e^{-3x} + 3e^{-3x} + 6 = 6$
3. $y = x^2 + 2x \Rightarrow y' = 2x + 2 \Rightarrow y' = 2$ so $y'' - y' + y = 2 - (2x + 2) + \left(x^2 + 2x\right) = 2 - 2x - 2 + x^2 + 2x = x^2$
5. $y = 7x^3 - x^2 \Rightarrow y' = 21x^2 - 2x$ so $xy' - 3y = x\left(21x^2 - 2x\right) - 3\left(7x^3 - x^2\right) = 21x^3 - 2x^2 - 21x^3 + 3x^2 = x^2$
7. $y = \frac{1}{2}e^x + 2e^{-x} \Rightarrow y' = \frac{1}{2}e^x - 2e^{-x}$ so $y' + y = \left(\frac{1}{2}e^x - 2e^{-x}\right) + \left(\frac{1}{2}e^x + 2e^{-x}\right) = e^x$
9. $y = \left(7 - x^2\right)^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2}\left(7 - x^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{7 - x^2}} = -\frac{x}{y}$
11. $y = 2x^3 - 3x + 3 \Rightarrow y(1) = 2 \cdot 1^3 - 3 \cdot 1 + 3 = 2$ (OK) and $y' = 6x^2 - 3$ (OK)
13. $y = \sin(2x) + 1 \Rightarrow y(0) = \sin(0) + 1 = 1$ (OK) and $y' = 2\cos(2x)$ (OK)
15. $y = 7e^{5x} \Rightarrow y(0) = 7e^{5 \cdot 0} = 7$ (OK) and $y' = 7e^{5x} \cdot 5 - 5y$ (OK)
17. $y = -\frac{4}{x} \Rightarrow y(1) = -\frac{4}{1} = -4$ (OK) and $y' = \frac{4}{x^2}$ so $x \cdot y' = x\left(\frac{4}{x^2}\right) = \frac{4}{x} = -y$
19. $y = 5\ln(x) - 2 \Rightarrow y(e) = 5\ln(e) - 2 = 5 - 2 = 3$ (OK) and $y' = 5 \cdot \frac{1}{x}$ (OK)
21. $7 = y(3) = 3^2 + C = 9 + C \Rightarrow C = -2$
23. $5 = y(0) = Ce^{3 \cdot 0} = C \cdot 1 \Rightarrow C = 5$
25. $4 = y(0) = 2\sin(3 \cdot 0) + C = 0 + C \Rightarrow C = 4$
27. $2 = y(e) = \ln(e) + C = 1 + C \Rightarrow C = 1$
29. $10 = y(2) = -\frac{C}{2} \Rightarrow C = -20$
31. $y = \int \left[4x^2 - x\right]\, dx = \frac{4}{3}x^3 - \frac{1}{2}x^2 + C$ so $7 = y(1) = \frac{4}{3} - \frac{1}{2} + C \Rightarrow C = \frac{37}{6}$ and $y = \frac{4}{3}x^3 - \frac{1}{2}x^2 + \frac{37}{6}$
33. $y = \int \frac{3}{x}\, dx = 3\ln\left(|x|\right) + C$ so $2 = y(1) = 3\ln(1) + C = 0 + C$ so $C + 2$ and $y = 3\ln\left(|x|\right) + 2$
35. $y = \int 6e^{2x}\, dx = 3e^{2x} + C$ so $1 = y(0) = 3e^{2 \cdot 0} + C = 3 + C \Rightarrow C = -2$ and $y = 3e^{2x} - 2$
37. $y = \int x \cdot \sin\left(x^2\right)\, dx = -\frac{1}{2}\cos\left(x^2\right) + C$ so $3 = y(0) = -\frac{1}{2}\cos(0) + C = -\frac{1}{2} + C \Rightarrow C = \frac{7}{2}$ and $y = -\frac{1}{2}\cos\left(x^2\right) + \frac{7}{2}$
39. $y' = \frac{1}{x}\left(6x^3 - 10x^2\right) = 6x^2 - 10x \Rightarrow y = \int \left[6x^2 - 10x\right]\, dx = 2x^3 - 5x^2 + C$ so $5 = y(2) = 2 \cdot 2^3 - 5 \cdot 2^2 + C = 16 - 20 + C = -4 + C \Rightarrow C = 9$ and $y = 2x^3 - 5x^2 + 9$
41. We know that $f'(x) + 5 \cdot f(x) = 0$ and $g'(x) + 5 \cdot g(x) = 0$, so: $y = 3f(x) \Rightarrow y' = 3f'(x) \Rightarrow y' + 5y = 3f'(x) + 5 \cdot 3f(x) = 3\left[f'(x) + 5 \cdot f(x)\right] = 3 \cdot 0 = 0$
    $y = 7g(x) \Rightarrow y' = 7g'(x) \Rightarrow y' + 5y = 7g'(x) + 5 \cdot 7g(x) = 7\left[g'(x) + 5 \cdot g(x)\right] = 3 \cdot 0$
    $y = f(x) + g(x) \Rightarrow y' = f'(x) + g'(x) \Rightarrow y' + 5 \cdot y = \left[f'(x) + g'(x)\right] + 5 \cdot \left[f(x) + g(x)\right] \Rightarrow y = f'(x) + g'(x) + 5f(x) + 5g(x) = \left[f'(x) + 5f(x)\right] + \left[g'(x) + 5g(x)\right] = 0 + 0 = 0$
    $y = A \cdot f(x) + B \cdot g(x) \Rightarrow y' = A \cdot f'(x) + B \cdot g'(x) \Rightarrow y' + 5 \cdot y = \left[A \cdot f'(x) + B \cdot g'(x)\right] + 5 \cdot \left[A \cdot f(x) + B \cdot g(x)\right] \Rightarrow y = Af'(x) + Bg'(x) + 5Af(x) + 5Bg(x) = A\left[f'(x) + 5f(x)\right] + B\left[g'(x) + 5g(x)\right] = A \cdot 0 + B \cdot 0 = 0$
43. $y = \sin(x) + x \Rightarrow y' = \cos(x) + 1 \Rightarrow y'' = -\sin(x) \Rightarrow y'' + y = \left[-\sin(x)\right] + \left[\sin(x) + x\right] = x$
    $y = \cos(x) + x \Rightarrow y' = -\sin(x) + 1 \Rightarrow y'' = -\cos(x) \Rightarrow y'' + y = \left[-\cos(x)\right] + \left[\cos(x) + x\right] = x$ (OK)
    $y = 3\left[\sin(x) + x\right] \Rightarrow y' = 3\cos(x) + 3 \Rightarrow y'' = -3\sin(x) \Rightarrow y'' + y = -3\sin(x) + 3\left[\sin(x) + x\right] = 3x$ (NO)
    Similarly, $y = \left[\sin(x) + x\right] + \left[\cos(x) + x\right] \Rightarrow y'' + y = 2x$ (NO)
45. $y = \frac{A}{B} - C \cdot e^{-Bt} \Rightarrow \frac{dy}{dt} = 0 - C\left[-Be^{-Bt}\right] = BCe^{-Bt} \Rightarrow A - By = A - B\left[\frac{A}{B} - C \cdot e^{-Bt}\right] = A - A + BCe^{-Bt} = BCe^{-Bt} = \frac{dy}{dt}$ (OK)
47. $I = \frac{E}{R}\left[1 - \cdot e^{-\frac{Rt}{L}}\right] \Rightarrow \frac{dI}{dt} = \frac{E}{R}\left[0 - \left(-\frac{R}{L}\right)e^{-\frac{Rt}{L}}\right] = \frac{E}{R}e^{-\frac{Rt}{L}} \Rightarrow L \cdot \frac{dI}{dt} + R \cdot I = l\left[\frac{E}{R}e^{-\frac{Rt}{L}}\right] + r \cdot \frac{E}{R}\left[1 - \cdot e^{-\frac{Rt}{L}}\right] = E \cdot e^{-\frac{Rt}{L}} + E\left[1 - e^{-\frac{Rt}{L}}\right] = E$ (OK)

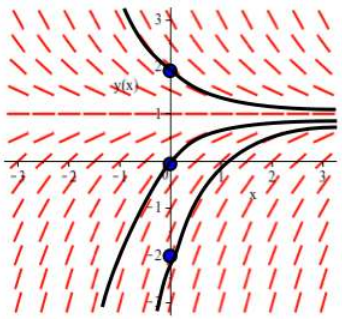
Section 6.2



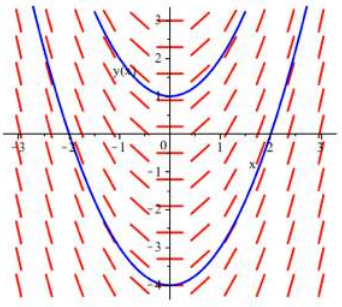
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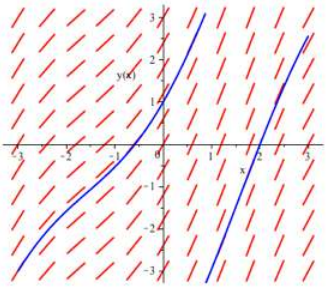
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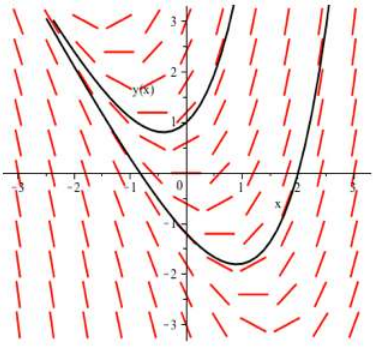
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5. 

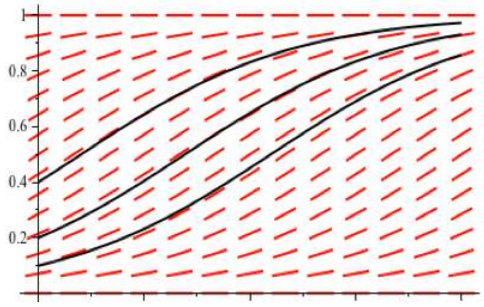
6. All solutions appear to approach the horizontal line $y = 1$: for any solution $y(x)$, $\lim_{x \to \infty} y(x) = 1$.

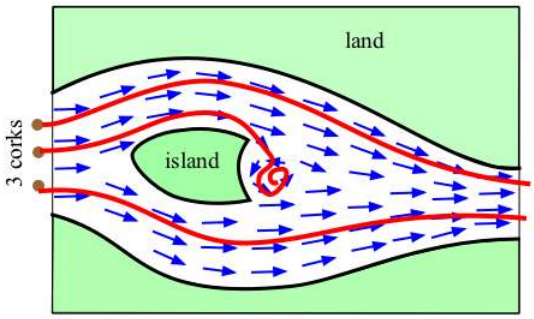
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11. 

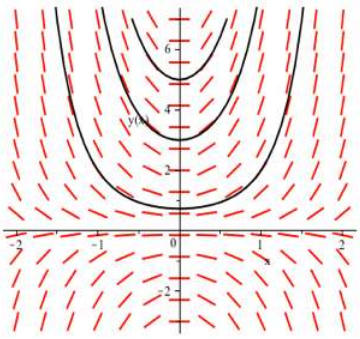
13. 

15. a. $y = x^2 - 3x + C$
    b. $4 = y(1) = 1^2 - 3 \cdot 1 + C \Rightarrow 4 = C - 2 \Rightarrow C = 6$ so $y = x^2 - 3x + 6$
    c. $(-\infty, \infty)$

17. a. $\Rightarrow y = e^x + \sin(x) + C$
    b. $7 = y(0) = 1 + 0 + C \Rightarrow C = 6$ so $y = e^x + \sin(x) + 6$
    c. $(-\infty, \infty)$
19. a. $y' = \dfrac{6}{2x+1} + \sqrt{x} \Rightarrow y = 3\ln(|2x+1|) + \dfrac{2}{3}x^{\frac{3}{2}} + C$
    b. $4 = y(1) = 3\ln(3) + \frac{2}{3} + C \Rightarrow C = \frac{10}{3} - 3\ln(3)$ so $3\ln(|2x+1|) + \frac{2}{3}x^{\frac{3}{2}} + \frac{10}{3} - 3\ln(3)$
    c. $(0, \infty)$
21. 

23. 

Section 6.3

1. 

3. If $y \neq 0$, divide both sides by y to separate:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x \Rightarrow \frac{1}{y}\,dy = 2x\,dx$$

and then integrate both sides:

$$\int \frac{1}{y}\,dy = \int 2x\,dx \Rightarrow \ln(|y|) = x^2 + C$$

then exponentiate both sides to solve for y:

$$e^{\ln(|y|)} = e^{x^2+C} \Rightarrow |y| = e^C \cdot e^{x^2} \Rightarrow y = \pm e^C \cdot e^{x^2}$$

so $y = Ae^{x^2}$ is a solution for any $A \neq 0$. But the constant function $y(x) = 0$ is also a solution to $y' = 2xy$ so the general solution is $y = Ae^{x^2}$ for any constant A.

5. Divide both sides of the ODE by $1 + x^2$ to separate, then integrate:

$$\frac{dy}{dx} = \frac{3}{1+x^2} \Rightarrow y = 3\arctan(x) + C$$

7. Divide both sides by $e^y \cdot \cos(x)$ to separate:

$$\frac{1}{e^y} \cdot \frac{dy}{dx} = \frac{1}{\cos(x)} \Rightarrow e^{-y}\,dy = \sec(x)\,dx$$

and use Appendix I to assist with the integration:

$$\int e^{-y}\,dy = \int \sec(x)\,dx$$
$$\Rightarrow -e^{-y} = \ln\left(|\sec(x) + \tan(x)|\right) + C$$

then solve for y:

$$e^{-y} = K - \ln\left(|\sec(x) + \tan(x)|\right) \Rightarrow$$
$$-y = \ln\left(K - \ln\left(|\sec(x) + \tan(x)|\right)\right)$$

so the general solution is:

$$y = -\ln\left(K - \ln\left(|\sec(x) + \tan(x)|\right)\right)$$

9. Note that $y = 0$ is a solution to $y' = 4y$. If $y \neq 0$, divide both sides of the ODE by y to separate:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 4 \Rightarrow \frac{1}{y}\,dy = 4\,dx$$

then integrate both sides:

$$\int \frac{1}{y}\,dy = \int 4\,dx \Rightarrow \ln(|y|) = 4x + C$$

and solve for y:

$$e^{\ln(|y|)} = e^{4x+C} \Rightarrow |y| = e^C \cdot e^{4x} \Rightarrow y = \pm e^C \cdot e^{4x}$$

so $y = Ae^{4x}$ is a solution for any constant $A \neq 0$ but $y(x) = 0$ is also a solution, so the general solution is $y(x) = Ae^{4x}$ for any constant A.

11. From Problem 3, we know that $y = Ae^{x^2}$, so using each initial condition:

$$3 = y(0) = A \cdot e^{0^2} = A \Rightarrow A = 3 \Rightarrow y = 3e^{x^2}$$
$$5 = y(0) = A \cdot e^{0^2} = A \Rightarrow A = 5 \Rightarrow y = 5e^{x^2}$$
$$2 = y(1) = A \cdot e \Rightarrow A = 2e^{-1} \Rightarrow y = 2e^{x^2-1}$$

13. Mimic the solution to Problem 9 to arrive at the general solution: $y = Ae^{3x}$. Then use each initial condition:

$$4 = y(0) = A \cdot e^{3\cdot 0} = A \Rightarrow A = 4 \Rightarrow y = 4e^{3x}$$
$$7 = y(0) = A \cdot e^{3\cdot 0} = A \Rightarrow A = 7 \Rightarrow y = 7e^{3x}$$
$$3 = y(1) = A \cdot e^{3\cdot 1} = Ae^3 \Rightarrow A = 3e^{-3} \Rightarrow y = 3e^{3x-3}$$

15. From Problem 10, we know $y = 2 + Ae^{-5x}$, so using each initial condition:

$$5 = y(0) = 2 + A \cdot 1 \Rightarrow A = 3 \Rightarrow y = 2 + 3e^{-5x}$$
$$-3 = y(0) = 2 + A \cdot 1 \Rightarrow A = -5 \Rightarrow y = 2 - 5e^{-5x}$$

17. From Problem 5, we know $y = 3\arctan(x) + C$, so using each initial condition:

$$4 = y(1) = 3 \cdot \frac{\pi}{4} + C \Rightarrow y = 3\arctan(x) + 4 - \frac{3\pi}{4}$$
$$2 = y(0) = 3 \cdot 0 + C \Rightarrow y = 3\arctan(x) + 2$$

19. No. Putting $x = 0$ and $y = 2$ into the ODE:

$$0 \cdot y'(0) = 2 + 3 \Rightarrow 0 = 5$$

which is a contradiction.

Section 6.4

1. The population is 10,000 around 1966 and is 20,000 around 1978, so $1978-1966=12$ years. The population grows from 15,000 around 1974 to 30,000 around 1985, so $1985-1974=11$ years. The doubling time is approximately 12 years.

3. a. If $P(t)$ represents the population (in thousands) t years after 1990, then $P(t)=48e^{kt}$ for some constant k. We also know that:

$$64=48e^{20k} \Rightarrow \frac{4}{3}=e^{20k} \Rightarrow k=\frac{1}{20}\ln\left(\frac{4}{3}\right)$$

so that $P(t)=48\left(\frac{4}{3}\right)^{\frac{t}{20}}\approx 48e^{0.01438t}$.

b. $P(30)=48\left(\frac{4}{3}\right)^{\frac{30}{20}}\approx 73.901$, so if the model holds, in 2020 the community's population will be approximately 74,000.

c. Solving $P(t)=100$ for t:

$$100=48\left(\frac{4}{3}\right)^{\frac{t}{20}} \Rightarrow \frac{100}{48}=\left(\frac{4}{3}\right)^{\frac{t}{20}}$$
$$\Rightarrow \ln\left(\frac{25}{12}\right)=\frac{t}{20}\ln\left(\frac{4}{3}\right) \Rightarrow t=\frac{20\ln\left(\frac{25}{12}\right)}{\ln\left(\frac{4}{3}\right)}$$

so about 51 years later (in 2041).

d. $\frac{\ln(2)}{k}=\frac{20\ln(2)}{\ln\left(\frac{4}{3}\right)}\approx 48.19$ years

5. If $A(t)$ is the value of the investment t years later:

$$A(t)=5000(1.15)^t=5000e^{t\cdot\ln(1.15)}$$

so the doubling time is $\frac{\ln(2)}{\ln(1.15)}\approx 4.96$ years and the tripling time is $\frac{\ln(3)}{\ln(1.15)}\approx 7.86$ years.

7. In 1950 the population is approximately 5,000, so if $P(t)$ represents the size of the population t years after 1950, then $P(t)=5000e^{kt}$ for some constant k. If the doubling time is 12 years:

$$\frac{\ln(2)}{k}=12\Rightarrow k=\frac{\ln(2)}{12}\Rightarrow P(t)=5000e^{\frac{t}{12}\ln(2)}$$

so $P(t)=5000\,(2)^{\frac{t}{12}}$.

9. $k=\frac{\ln(2)}{50}\Rightarrow P(t)=P(0)e^{\frac{t}{50}\cdot\ln(2)}=P(0)\cdot 2^{\frac{t}{50}}$ so $\frac{P(1)-P(0)}{P(0)}=2^{\frac{1}{50}-1}\approx 0.014=1.4\%$

11. $6(1.03)^t=4(1.06)^t\Rightarrow t=\frac{\ln\left(\frac{3}{2}\right)}{\ln\left(\frac{1.06}{1.03}\right)}\approx 14.12$ years

13. After t months, you have $S(t)=8000e^{0.14\left(\frac{t}{12}\right)}$ snails before harvest.

a. $S(2)=8000e^{0.14\left(\frac{2}{12}\right)}\approx 8,188$, so after harvest $S=6188$; $S(4)=6188e^{0.14\left(\frac{2}{12}\right)}\approx 6334$, so after harvest $S=4,334$; after third harvest, 2436 remain; after fourth harvest, 494 remain; after fifth harvest, no snails remain.

b. No snails remain after the third harvest.

c. The population growth is 188 snails after two months, so you can harvest 188 snails every two months and maintain a stable population (between 8,000 and 8,188).

15. $A(t)=A(0)e^{kt}=10e^{kt}$ and $A(14)=2$

a. $2=10e^{14k}\Rightarrow k\approx -0.115\Rightarrow A(t)=10e^{-0.115t}$

b. $\frac{-\ln(2)}{-0.115}\approx 6$ days

c. $0.7=10e^{-0.115t}\Rightarrow t=\frac{\ln(0.7)}{-0.115}\approx 23$ days

17. a. $143=187e^{2k}\Rightarrow k=\frac{\ln\left(\frac{143}{187}\right)}{2}\approx -0.134$, hence $\frac{-\ln(2)}{k}\approx 5.17$ days

b. $20=187e^{-0.134t}\Rightarrow t=\frac{\ln\left(\frac{20}{187}\right)}{-0.134}\approx 16.7$ days

19. $3.5=8e^{6k}\Rightarrow k=\frac{\ln\left(\frac{3.5}{8}\right)}{6}\approx -0.138$, so $A(t)=8000e^{-0.138t}$ counts per minute after t days.

21. Carbon-14 has a half-life of 5,700 years, so $k = \dfrac{-\ln(2)}{5700} \approx -0.00012$, hence $0.975 = e^{-0.00012t} \Rightarrow t = \dfrac{\ln(0.975)}{-0.00012} \approx 211$ years, but Newton died in 1727, over 288 years ago, so the letter is a fake.

23. $k = \dfrac{-\ln(2)}{6} \approx -0.116$, so $A(t) = 30e^{-0.116t}$ and $A(t) \geq 10 \Rightarrow -0.116t \geq \ln\left(\dfrac{10}{30}\right) \Rightarrow t \leq \dfrac{-1.009}{-0.116} \approx 9.47$ hours. After about 9.5 hours, the concentration of medicine is no longer effective.

25. $k = \dfrac{-\ln(2)}{15} \approx -0.046$, so $A(t) = 9e^{-0.046t}$, hence $A(8) = 9e^{-0.046(8)} \approx 6.23$ mg, resulting in a "decay" of $9 - 6.23 = 2.77$ mg during these 8 hours. Taking a 2.77 mg dose every 8 hours keeps level of the medicine in the safe and effective range over a long period of time.

27. The half-life of iodine-131 is 8.07 days, hence $k = \dfrac{-\ln(2)}{8.07} \approx -0.086$, thus $A(t) = 5Se^{-0.086t}$. If S is highest safe level, $S = 5Se^{-0.086t} \Rightarrow 0.2 = e^{-0.086t}$ so that $t = \dfrac{\ln(0.2)}{-0.086} \approx 18.7$ days.

29. For the population $P(t)$, $P(0) = 4$ and $P(1) = (1.05)(4) = 4e^{k(1)} \Rightarrow k = \ln(1.05) \approx 0.049$, so $P(t) = 4e^{0.049t}$ (in millions). The size of the forest is $F(t) = 10000000 - 300000t$ acres after t years (the entire forest will be gone in 33.3 years).
    a. $\dfrac{100 - 3t}{40e^{0.049t}}$ acres per person
    b. $\mathbf{D}\left(\dfrac{100 - 3t}{40e^{0.049t}}\right) = \dfrac{-7.9 + 0.147t}{40e^{0.049t}}$
    c. Solve $\dfrac{100 - 3t}{40e^{0.049t}} = 1$ using technology to determine that $t \approx 10.75$ years.

Section 6.5

1. The temperature of the cheesecake t minutes later is given by:

$$f(t) = 35 + [165 - 35]\, e^{kt} = 35 + 130e^{kt}$$

Solving $150 = 35 + 130e^{10k}$ for k yields:

$$k = \frac{1}{10}\ln\left(\frac{115}{130}\right) \approx -0.01226$$

so that $f(t) = 35 + 130e^{-0.01226t}$. Solving $f(T) = 70$ for T then yields:

$$70 = 35 + 130e^{-0.01226t} \Rightarrow T = -\frac{1}{0.01226}\ln\left(\frac{35}{130}\right)$$

so you will need to wait about 107 minutes.

3. a. The temperature of the water t minutes later is given by:

$$f(t) = 40 + [200 - 40]\, e^{kt} = 40 + 160e^{kt}$$

Solving $150 = 40 + 160e^{4k}$ for k yields:

$$k = \frac{1}{4}\ln\left(\frac{110}{160}\right) \approx -0.09367$$

so that $f(t) = 40 + 160e^{-0.09367t}$.

b. Solving $100 = 40 + 160e^{-0.09367T}$ for T yields:

$$T = -\frac{1}{0.09367}\ln\left(\frac{60}{160}\right) \approx 10.5 \text{ minutes}$$

c. $-\dfrac{1}{0.09367}\ln\left(\dfrac{40}{160}\right) \approx 14.8$ minutes
d. ≈ 22.2 minutes

5. If $A(t)$ represents the amount of salt in the tank t minutes later, then $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = -\frac{A}{100} \cdot 3, \quad A(0) = 50$$

Solving this separable ODE yields:

$$\int \frac{1}{A}\,dA = \int -0.03\,dt \Rightarrow \ln(A) = -0.03t + C$$

The initial condition $A(0) = 50$ tells us that $\ln(50) = C$, so:

$$\ln(A) = -0.03t + \ln(50) \Rightarrow A(t) = 50e^{-0.03t}$$

Hence $A(60) = 50e^{-0.03(60)} \approx 8.26$ pounds.

7. If $A(t)$ represents the amount of salt in the tank t minutes later, then $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = -\frac{A}{100+t} \cdot 2, \quad A(0) = 50$$

Solving this separable ODE yields:

$$\int \frac{1}{A}\,dA = \int -\frac{2}{100+t}\,dt$$

so that $\ln(A) = \ln(100+t)^{-2} + C$. The initial condition $A(0) = 50$ then tells us that:

$$\ln(50) = -2\ln(100) + C \Rightarrow C = \ln(500000)$$

hence $A(t) = \frac{500000}{(100+t)^2}$, so $A(60) = \frac{500000}{160^2} \approx 19.53$ pounds.

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CHAPTER OVERVIEW

7: Transcendental Functions

An **algebraic number** is any number that can be a solution to a polynomial equation with integer coefficients. Any rational number is algebraic: for example, $\frac{5}{7}$ is a solution of $7x - 5 = 0$. Any square root or cube root is an algebraic number: $\sqrt{3}$ is a solution of $x^2 - 3 = 0$ and $-\sqrt[3]{5}$ is a solution of $x^3 + 5 = 0$. In fact, any number that can be expressed as the sum, difference, product, quotient or rational power of integers is an algebraic number, including something as horrible-looking as:

$$\sqrt[5]{\frac{\sqrt[4]{93} - \sqrt[3]{436 + \sqrt{2}}}{17 + \sqrt{37}}} \tag{7.1}$$

Any real number that is *not* an algebraic number is called a **transcendental number**. We have already met (and used extensively) two very important transcendental numbers: π and e.

> It turns out that proving that these numbers are transcendental is rather difficult, although by the end of Chapter 8 we will have most of the tools necessary to prove that e is transcendental.

Functions defined using sums, differences, products, quotients or rational powers of rational coefficients and a real-valued variable x are called **algebraic functions**. Any non-algebraic functions of a real variable are called **transcendental functions**. Examples of transcendental functions with which you should be very familiar are $\sin(x)$, $\cos(x)$, $\tan(x)$ and the other trigonometric functions, as well as e^x. (The geometric definitions of the trigonometric functions involve π, while the exponential function obviously involves e.)

The inverse functions of these transcendental functions are also transcendental functions: $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$ and the other inverse trigonometric functions, as well as $\ln(x)$.

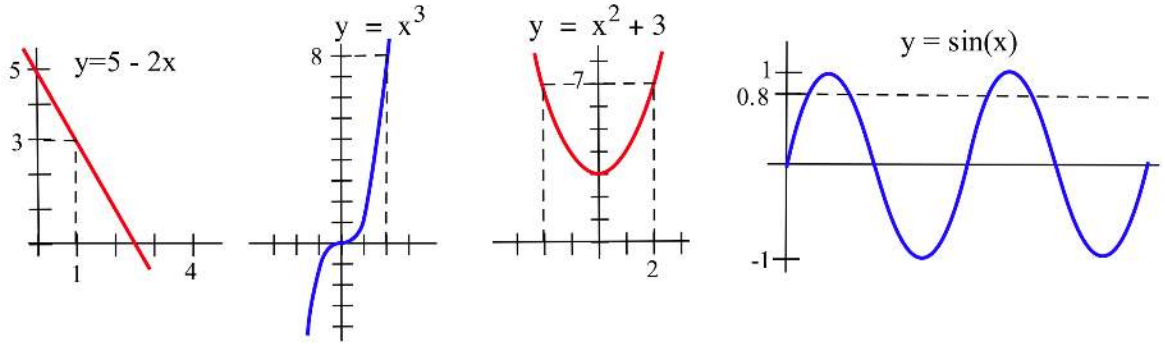
Because many important transcendental functions are defined as inverse functions, this chapter begins with a review of inverse functions and a discussion of finding derivatives of inverse functions. It continues with a review of inverse trigonometric functions and a discussion of their derivatives (and the usefulness of these derivative patterns in finding certain antiderivatives), then ties up some (very important) loose ends related to the definitions and properties of e^x and $\ln(x)$. The chapter concludes by introducing some new transcendental functions: the hyperbolic functions and their inverses, which play important roles in calculus and in applications.

7.1: One-to-One Functions
7.2: Inverse Functions
7.3: Inverse Trigonometric Functions
7.4: Derivatives of Inverse Trigonometric Functions
7.5: Integrals Involving Inverse Trigonometric Functions
7.A: Answers

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7.1: One-to-One Functions

You've seen that some equations have only one solution (for example, $5-2x=3$ and $x^3=8$), while some have two solutions ($x^2+3=7$) and some even have an infinite number of solutions ($\sin(x)=0.8$). The graphs of $y=5-2x$, $y=x^3$, $y=x^2+3$ and $y=\sin(x)$ and the solutions of the equations mentioned above appear below:



Functions f for which equations of the form $f(x)=k$ have at most one solution for each value of k (that is, each outcome k comes from only one input x) arise often in applications and possess a number of useful mathematical properties. This brief section focuses on those functions and examines some of their properties.

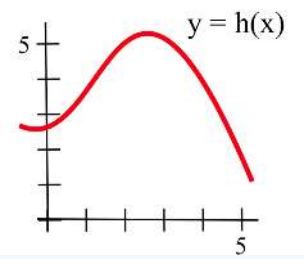
Example 7.1.1

a. $f(x)=0$ for $f(x)=x(x-4)$

b. $g(x)=3$ for g given in the table below:

x	0	1	2	3	4	5
$g(x)$	5	7	3	5	0	7

c. $h(x)=4$ for h given by the graph below:



d. $f(x)=k$ for $f(x)=e^x$.

Solution

a. Two: $x(4-x)=0 \Rightarrow x=0$ or $x=4$.

b. One: $g(x)=3$ only if $x=2$.

c. Two: $h(x)=4$ if $x\approx 1.2$ or if $x\approx 4$.

d. If $k>0$, it has one solution: $x=\ln(k)$. If $k\leq 0$, it has no solutions.

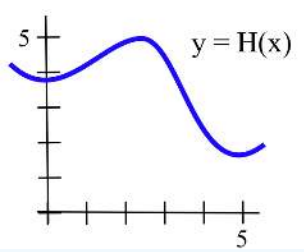
Practice 7.1.1

How many solutions does each equation have?

a. $f(x)=4$ for $f(x)=x(4-x)$

b. $g(x)=7$ for g given by the table from Example 7.1.1.

c. $H(x) = 3$ for H given by the graph below:



d. $f(x) = 5$ for $f(x) = \ln(x)$

Answer

a. One: solve $x(4 - x) = 4$ to get $x = 2$.
b. Two: $x = 1$ and $x = 5$.
c. One: $x \approx 3.5$.
d. One: solve $5 = \ln(x)$ to get $x = e^5 \approx 148.4$.

Horizontal Line Test

You should be familiar with the Vertical Line Test, a graphical tool you can use to help determine whether or not a curve in the xy-plane is the graph of a function. (If not, review Section 0.3.) A similar geometrical test leads to the definition of a **one-to-one function** and provides a tool for helping to determine when a function is one-to-one.

Horizontal Line Test (Definition of One-to-One)

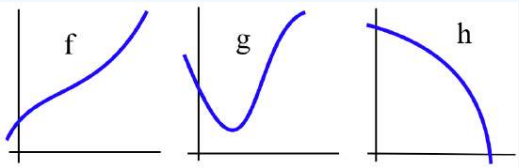
A function is **one-to-one** if each horizontal line intersects the graph of the function at most once.

Equivalently, a function $y = f(x)$ is one-to-one if two *distinct* x-values always produce two *distinct* y-values: that is, $a \neq b \Rightarrow f(a) \neq f(b)$. This immediately tells us that every strictly increasing function is one-to-one, and that every strictly decreasing function is one-to-one. (Why?)

For any function, if we know an input value we can calculate the output, but an output may arise from any of several different inputs. With a one-to-one function, each output comes from only one input.

Example 7.1.2

a. Which functions in the figure below are one-to-one?



b. Which functions in the table below are one-to-one?

x	$f(x)$	$g(x)$	$h(x)$
0	5	7	2
1	2	3	-1
2	3	0	5
3	5	1	4
4	0	6	3
5	1	3	0

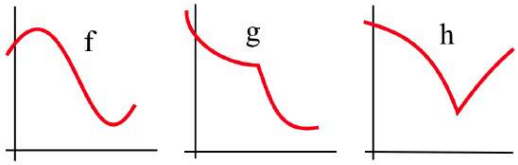
Solution

a. In the figure, f and h are one-to-one; g fails the Horizontal Line Test, so g is not one-to-one.

b. In the table, h is one-to-one, while f and g are not one-to-one because $f(0) = f(3)$ and $g(1) = g(5)$.

Practice 7.1.2

a. Which functions graphed below are one-to-one?



b. Which functions in the table below are one-to-one?

x	$f(x)$	$g(x)$	$h(x)$
0	4	2	-2
1	2	3	5
2	-2	0	1
3	5	4	14
4	3	6	3
5	1	7	1

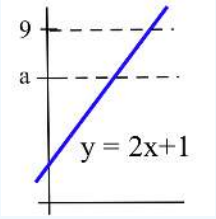
Answer

a. Only g is one-to-one; f and h fail the Horizontal Line Test.

b. Both f and g are one-to-one; h is not, because $h(2) = h(5)$.

Example 7.1.3

Let $f(x) = 2x + 1$:



Find the values of x so that:

a. $f(x) = 9$ and
b. $f(x) = a$ and then
c. solve $f(y) = x$ for y.

Solution

a. $9 = f(x) = 2x + 1 \Rightarrow 8 = 2x \Rightarrow x = \frac{8}{2} = 4$
b. $a = 2x + 1 \Rightarrow 2x = a - 1 \Rightarrow x = \frac{a-1}{2}$
c. $x = f(y) = 2y + 1 \Rightarrow 2y = x - 1$ so $y = \dfrac{x-1}{2}$. Notice that this new function reverses the operations of $f(x)$, applied in reverse order: $f(x)$ multiplies x by 2, then adds 1; the new function subtracts 1, then divides by 2.

? Practice 7.1.3

Let $g(x) = 3x - 5$. Find the values of x so that:

a. $g(x) = 7$ and
b. $g(x) = b$ and then
c. solve $g(y) = x$ for y.

Answer

a. $3x - 5 = 7 \Rightarrow 3x = 12 \Rightarrow x = 4$
b. $3x - 5 = a \Rightarrow 3x = a + 5 \Rightarrow x = \dfrac{a+5}{3}$
c. $f(x) = 3x - 5 \Rightarrow f(y) = 3y - 5$ so $f(y) = x \Rightarrow 3y - 5 = x \Rightarrow 3y = x + 5 \Rightarrow y = \dfrac{x+5}{3}$

? Practice 7.1.4

Show that exponential growth, for example $f(x) = e^{3x}$, and exponential decay, for example $g(x) = e^{-2x}$, are both one-to-one.

Answer

If $f(x) = e^{kx}$ where $k > 0$ then $f'(x) = k \cdot e^{kx} > 0$ so $f(x)$ is strictly increasing, hence one-to-one. If $g(x) = e^{rx}$ where $r < 0$ then $g'(x) = r \cdot e^{rx} < 0$ so $g(x)$ is strictly decreasing, hence one-to-one.

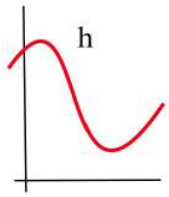
Problems

In Problems 1–4, explain why each given function is (or is not) one-to-one.

1. $f(x) = 3x - 5$, $y = 3 - x$, $g(x)$ given by the table below:

x	$g(x)$
0	3
1	4
2	5
3	2
4	4

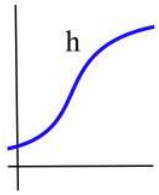
and $h(x)$ given by the graph below:



2. $f(x) = \frac{x}{4}$, $y = x^2 + 3$, $g(x)$ given by the table below:

x	$g(x)$
0	3
1	2
2	0
3	-2
4	1

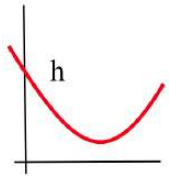
and $h(x)$ given by the graph below:



3. $f(x) = \sin(x)$, $y = e^x - 2$, $g(x)$ given by the table below:

x	$g(x)$
0	-1
1	5
2	3
3	1
4	0

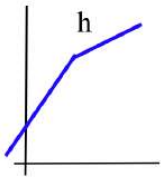
and $h(x)$ given by the graph below:



4. $f(x) = 17$, $y = x^3 - 1$, $g(x)$ given by the table below:

x	$g(x)$
0	2
1	5
2	4
3	1
4	2

and $h(x)$ given by the graph below:



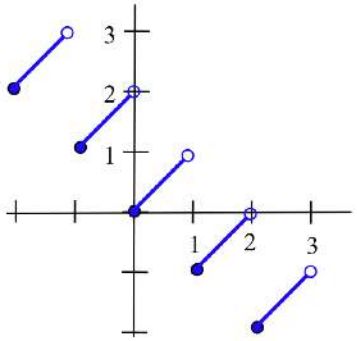
5. Is the relation between people and Social Security numbers a function? A one-to-one function?
6. Is the relation between people and phone numbers a function? If so, is it one-to-one?
7. What would it mean if the scores on a calculus test were one-to-one?
8. The relation given below represents "y is married to x":

x	A	B	C	D
y	P	Q	P	R

   a. Is this relation a function?
   b. Is it one-to-one?
   c. Is P breaking the law?
   d. Is A breaking the law?
9. In how many places can a one-to-one function touch the x-axis?
10. Can a continuous one-to-one function have the values given below? Explain.

x	1	3	5
$f(x)$	2	7	3

11. The graph of $f(x) = x - 2 \cdot \lfloor x \rfloor$ for $-2 \leq x \leq 3$ appears below:

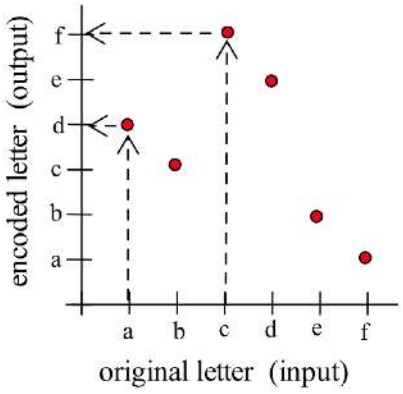


   a. Is f a one-to-one function?
   b. Is f an increasing function?
   c. Is f a decreasing function?

12. Is every linear function $L(x) = ax + b$ one-to-one? If not, which linear functions *are* one-to-one?
13. Show that $f(x) = \ln(x)$ is one-to-one for $x > 0$.
14. Show that $g(x) = e^x$ is one-to-one.
15. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	c	f	e	b	a

a. Is the encoding rule a function?
b. Is the encoding rule one-to-one?
c. Encode the word “bad.”
d. Create a table for decoding the encoded letters and use it to decode your answer to part (c).
e. A graph of the encoding rule appears below:



Create a graph of the decoding rule.
f. Compare the encoding and decoding graphs.

16. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
b	d	b	b	a	c

a. Is the encoding rule a function?
b. Is the encoding rule one-to-one?
c. Encode the word “bad.”
d. Create a table for decoding the encoded letters and use it to decode your answer to part (c).
e. Create a graph of the encoding rule.
f. Create a graph of the decoding rule.
g. Compare the encoding and decoding graphs.

17. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	f	e	a	c	b

a. Is the encoding rule a function?
b. Is the encoding rule one-to-one?
c. Encode the word “bad.”

d. Create a table for decoding the encoded letters and use it to decode your answer to part (c).
e. Create a graph of the encoding rule.
f. Create a graph of the decoding rule.
g. Compare the encoding and decoding graphs.
h. What happens if you encode a word, then encode the encoded word? For example, $\text{encode}(\text{encode}(\text{“bad”})) = ?$

18. The table below gives an encoding rule for a six-letter alphabet:

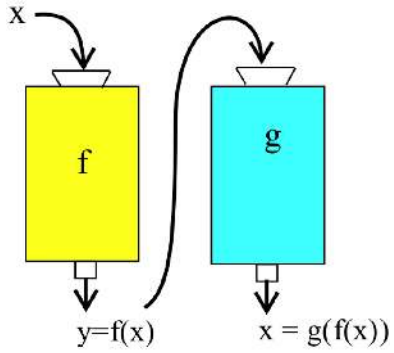
a	b	c	d	e	f
e	a	f	c	b	d

a. Is the encoding rule a function?
b. Is the encoding rule one-to-one?
c. Encode the word “bad.”
d. Create a table for decoding the encoded letters and use it to decode your answer to part (c).
e. Create a graph of the encoding rule.
f. Create a graph of the decoding rule.
g. Compare the encoding and decoding graphs.
h. What happens if you apply this encoding rule three times in succession? For example: $\text{encode}(\text{encode}(\text{encode}(\text{“bad”}))) = ?$

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7.2: Inverse Functions

If f is any one-to-one function then equations of the form $f(x) = k$ have (at most) a single solution. Such functions can be uniquely "undone": if f is a one-to-one function, then there is another function g that "undoes" the effect of f, so that $g(f(x)) = x$. When g and f are composed in this manner, g retrieves the original input of f:



We call this function g that "undoes" the effect of f the **inverse function** of f or simply **the inverse** of f.

If f is a function that encodes a message, then the inverse of f is the function that decodes an encoded message to retrieve the original message. The functions e^x and $\ln(x)$ "undo" the effects of each other:

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x \text{ (for } x > 0)$$

so the functions e^x and $\ln(x)$ are inverses of each other.

This section will examine some of the properties of inverse functions and explain how to find the inverse of a function given by a table of data, a graph or a formula.

Definition

If f and g are functions that satisfy both $g(f(x)) = x$ and $f(g(y)) = y$ for all x in the domain of f and all y in the domain of g, then g is the **inverse** of f, f is the inverse of g, and f and g are a **pair of inverse functions**.

We often write the inverse function of f as f^{-1} (pronounced "eff inverse") but *you must be very careful*: f^{-1} does **not** mean $\frac{1}{f}$.

Example 7.2.1

The values of a function f appear in the table below:

x	$f(x)$
0	3
1	2
2	0
3	1

Evaluate $f^{-1}(0)$ and $f^{-1}(1)$.

Solution

For every x, $f^{-1}(f(x)) = x$ so the value of $f^{-1}(0)$ is the solution x of the equation $f(x) = 0$. The value we want is $x = 2$, and we can check that $f^{-1}(0) = f^{-1}(f(2)) = 2$.

The value of $f^{-1}(1)$ is the solution x of the equation $f(x) = 1$, which is $x = 3$, and we can check that $f^{-1}(1) = f^{-1}(f(3)) = 3$. Similarly, $f^{-1}(2) = 1$ and $f^{-1}(3) = 0$. These results appear in the table below:

x	$f^{-1}(x)$
0	2
1	3
2	1
3	0

You should notice that if the ordered pair (a, b) is in the table for f, then the reversed pair (b, a) is in the table for f^{-1}.

Practice 7.2.1

The values of the function g appear in the table below:

x	$g(x)$
0	2
1	1
2	3
3	4
4	0

Create a table of values for g^{-1}.

Answer

x	$g^{-1}(x)$
0	4
1	1
2	0
3	2
4	3

The method of interchanging the coordinates of a point on the graph (or in a table of values) of f to get a point on the graph (or in a table of values) of f^{-1} provides an efficient way to graph f^{-1}.

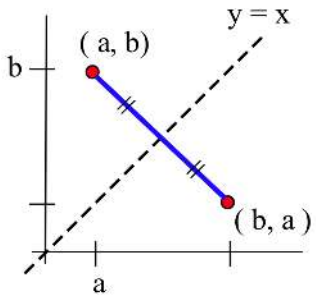
Theorem

If the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.

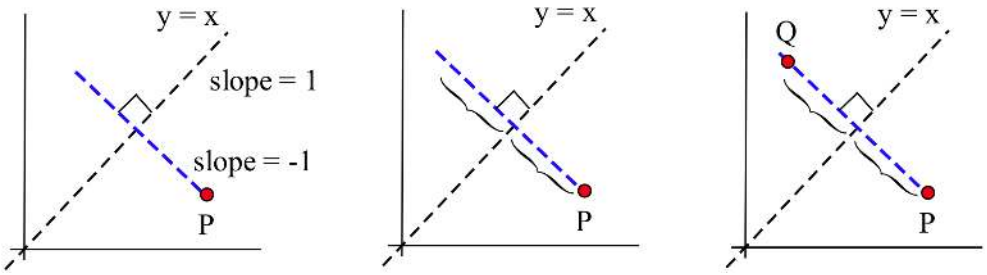
Proof

If (a, b) is on the graph of f, then $b = f(a)$, so $f^{-1}(b) = f^{-1}(f(a))$. By definition, $f^{-1}(f(a)) = a$, so $f^{-1}(b) = a$, which tells us that (b, a) is on the graph of f^{-1}.

Graphically, when you interchange the coordinates of a point (a, b) to get a new point (b, a), the old point and the new point are symmetric about the line $y = x$. If you put a spot of wet ink at the point (a, b) on a piece of graph paper:



and fold the paper along the line $y = x$, a new spot of ink will appear at the point (b, a). The figures below demonstrate another graphical method for finding the location of the point (b, a):



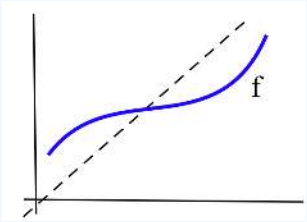
Draw the line $y = x$ along with a line of slope -1 that passes through the given point P (above left), then measure the distance from P to the line $y = x$ and move that same distance on the other side of $y = x$ (above center), and plot the new point Q at that location.

Corollary

The graphs of f and f^{-1} are symmetric about the line $y = x$.

Example 7.2.2

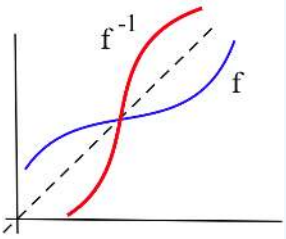
A graph of f appears below:



Sketch a graph of f^{-1}.

Solution

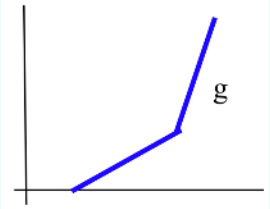
Imagine the graph of f is drawn with wet ink and fold the xy-plane along the line $y = x$. When you unfold the plane, the new graph is f^{-1}:



You could also proceed point by point: Pick several points (a, b) on the graph of f and plot the symmetric points (b, a), then use the new (b, a) points as a guide for sketching the graph of f^{-1}.

? Practice 7.2.2

A graph of g appears below:



Sketch a graph of g^{-1}. What happens to points on the graph of g that lie on the line $y = x$?

Answer

FIGURE

If a point lies on the line $y = x$, it must have the form (a, a) so that $g(a) = a \Rightarrow a = g^{-1}(a)$: the point (a, a) is also on the graph of g^{-1}.

Finding a Formula for $f^{-1}(x)$

When you have a table of values for f, you can create a table of values for f^{-1} by interchanging the values of x and y in the table for f (as in Example 7.2.1). When you have a graph of f, you can sketch a graph of f^{-1} by reflecting the graph of f about the line $y = x$ (as in Example 7.2.2). When you have a formula for f, you can try to find a formula for f^{-1}.

✓ Example 7.2.3

The steps for wrapping a gift are:

1. put the gift in a box,
2. cover the box with paper and
3. attach a ribbon.

What are the steps for opening the gift—the inverse of the wrapping operation?

Solution

1. Remove the ribbon (undo step 3),
2. remove the paper (undo step 2) and
3. remove the gift from the box (undo step 1

(Then show happiness and gratitude.)

The reason for the preceding trivial example is to point out that the first unwrapping step undoes the last wrapping step, the second unwrapping step undoes the second-to-last wrapping step…and the last unwrapping step undoes the first wrapping step. This pattern holds for functions and their inverses too.

Example 7.2.4

The steps to evaluate $f(x) = 9x + 6$ are:

1. multiply the input by **9** and
2. add **6** to the result.

Write the steps, in words, for the inverse of this function, and then translate the verbal steps for the inverse into a formula for the inverse function.

Solution

1. Subtract **6**, undoing (2), and
2. divide by **9**, undoing (1):

$$x \longrightarrow x-6 \longrightarrow \frac{x-6}{9}$$

so $f^{-1}(x) = \frac{x-6}{9}$ provides a formula for the inverse function.

If (x, y) is a point on the graph of f, we know that (y, x) is on the graph of f^{-1}, so interchanging the roles of x and y in the formula for f should lead us to a formula for f^{-1}. Applying this idea to the function $f(x) = 9x + 6$ from the previous Example, we swap x and y in the formula $y = 9x + 6$ to get $x = 9y + 6 \Rightarrow x - 6 = 9y \Rightarrow y = \frac{x-6}{9}$, yielding the formula $f^{-1}(x) = \frac{x-6}{9}$ as in Example 4.

The following algorithm provides a general "recipe" for finding a formula for an inverse function:

- Start with a formula for f: $y = f(x)$.
- Interchange the roles of x and y: $x = f(y)$.
- Solve $x = f(y)$ for y.
- The resulting formula for y is the inverse of f: $y = f^{-1}(x)$.

The "interchange" and "solve" steps in the algorithm effectively undo the original operations in reverse order.

Example 7.2.5

Find formulas for the inverses $f^{-1}(x)$ and $g^{-1}(x)$ of $f(x) = \frac{7x-5}{4}$ and $g(x) = 2e^{5x}$.

Solution

Starting with $y = f(x) = \frac{7x-5}{4}$ and interchanging the roles of x and y yields:

$$x = \frac{7y-5}{4} \Rightarrow 4x = 7y - 5 \Rightarrow 4x + 5 = 7y \Rightarrow y = \frac{4x+5}{7}$$

so $f^{-1}(x) = \frac{4x+5}{7}$. Starting with $y = g(x) = 2e^{5x}$ and interchanging the roles of x and y yields:

$$x = 2e^{5y} \Rightarrow \frac{x}{2} = e^{5y} \Rightarrow \ln\left(\frac{x}{2}\right) = 5y \Rightarrow y = \frac{1}{5}\ln\left(\frac{x}{2}\right)$$

so $g^{-1}(x) = \frac{1}{5}\ln\left(\frac{x}{2}\right)$.

Practice 7.2.3

Find formulas for the inverses $f^{-1}(x)$, $g^{-1}(x)$ and $h^{-1}(x)$ of $f(x) = 2x - 5$, $g(x) = \frac{2x-1}{x+7}$ and $h(x) = 2 + \ln(3x)$.

Answer

a. Swapping x and y in $y = 2x - 5$ yields $x = 2y - 5 \Rightarrow x + 5 = 2y \Rightarrow y = \frac{x+5}{2}$ so $f^{-1}(x) = \frac{x+5}{2}$.

b. Swapping x and y in $y = \frac{2x-1}{x+7}$: $x = \frac{2y-1}{y+7} \Rightarrow xy + 7x = 2y - 1 \Rightarrow 7x + 1 = 2y - xy \Rightarrow y = \frac{7x+1}{2-x}$ so $g^{-1}(x) = \frac{7x+1}{2-x}$.

c. Interchanging x and y in $y = 2 + \ln(3x)$ yields $x = 2 + \ln(3y) \Rightarrow x - 2 = \ln(3y) \Rightarrow e^{x-2} = 3y \Rightarrow y = \frac{1}{3}e^{x-2}$ so $h^{-1}(x) = \frac{1}{3}e^{x-2}$.

Sometimes it is easy to "solve $x = f(y)$ for y," but often it is not. When we try to find a formula for the inverse of $y = f(x) = x + e^x$, the first step is easy: interchanging the roles of x and y yields $x = y + e^y$. At this point, unfortunately, there is no way to algebraically solve the equation $x = y + e^y$ explicitly for y. The function $y = x + e^x$ has an inverse function, but we cannot find an explicit formula for that inverse.

Which Functions Have Inverse Functions?

We have seen how to find the inverse function for some functions given by tables of values, by graphs and by formulas, but there are functions that do not have inverse functions. The only way a graph and its reflection about the line $y = x$ can *both* be function graphs (so that f and f^{-1} are both functions) is if the graph of f passes both the Vertical Line Test (so that f is a function) and the Horizontal Line Test (so that the graph of f^{-1} passes the Vertical Line Test, verifying that f^{-1} is a function). This idea leads to the next theorem (stated without proof).

Theorem

The function f has an inverse function if and only if f is one-to-one.

Two useful corollaries follow from this theorem.

Corollary 7.2.1

If f is strictly increasing or strictly decreasing, then f has an inverse function.

Corollary 7.2.2

If $f'(x) > 0$ for all x or $f'(x) < 0$ for all x, then f has an inverse function.

Applying this last Corollary to $f(x) = x + e^x$, we know that $f'(x) = 1 + e^x > 0$ for all x, so f must have an inverse (even though we can't find an explicit formula for f^{-1}).

Slopes of Inverse Functions

When a function f has an inverse, the symmetry of the graphs of f and f^{-1} also provides us with information about slopes and derivatives.

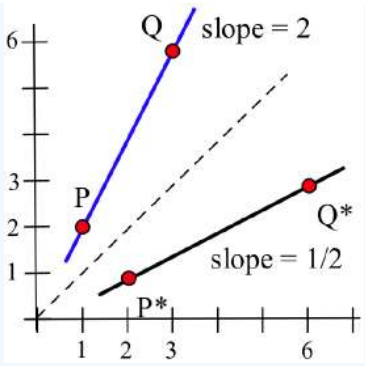
Example 7.2.6

Suppose the points $P = (1, 2)$ and $Q = (3, 6)$ are on the graph of a function f.

a. Sketch the line passing through P and Q.
b. Compute the slope of that line.
c. Graph the reflected points P^* and Q^* on the graph of f^{-1}.
d. Sketch the line passing through P^* and Q^*. Find the slope of the line through P^* and Q^*.

Solution

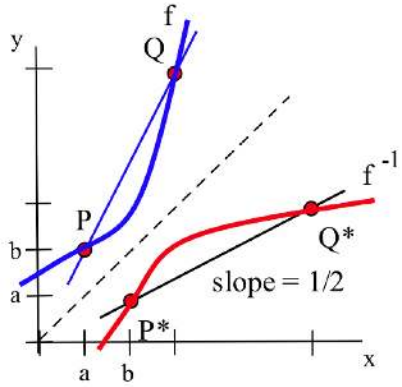
a. See the figure below.
b. The slope through P and Q is $m = \frac{6-2}{3-1} = \frac{4}{2} = 2$.
c. The reflected points, obtained by interchanging the first and second coordinates of each point on the graph of f, are $P^* = (2, 1)$ and $Q^* = (6, 3)$.
d. See the figure below.
e. The slope of the line though P^* and Q^* is $\frac{3-1}{6-2} = \frac{1}{2}$



In the preceding Example, you may have noticed that:

$$\text{slope of line through } P^* \text{ and } Q^* = \frac{1}{\text{slope of line through } P \text{ and } Q}$$

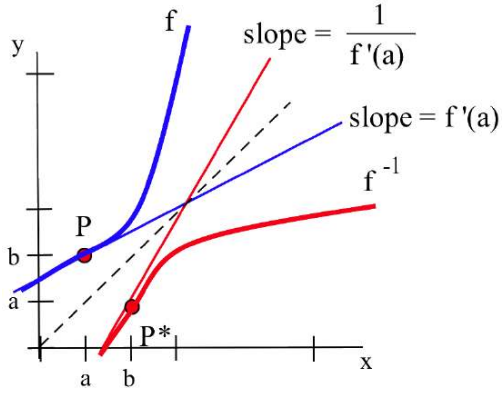
This was not a coincidence. In general, if $P = (a, b)$ and $Q = (x, y)$ are points on the graph of f:



then the reflected points $P^* = (b, a)$ and $Q^* = (y, x)$ are on the graph of f^{-1} and:

$$\text{slope of segment } P^*Q^* = \frac{1}{\text{slope of segment } PQ}$$

Because the slope of a tangent line is the limit of slopes of secant lines, a similar relationship holds between the slope of the tangent line to f at the point (a, b) and slope of the tangent line to f^{-1} at the point (b, a). If we let the point Q^* approach the point P^* along the graph of f^{-1}:



then:

$$\left(f^{-1}\right)'(b) = \lim_{Q^* \to P^*} [\text{slope of segment } P^*Q^*]$$
$$= \lim_{Q \to P} \frac{1}{\text{slope of segment } PQ} = \frac{1}{f'(a)}$$

This geometric idea leads to the following result:

Theorem: Derivative of an Inverse Function

If: $b = f(a)$, f is differentiable at the point (a, b) and $f'(a) \neq 0$

then: $a = f^{-1}(b)$, f^{-1} is differentiable at the point (b, a) and:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

Example 7.2.7

The point $(e^2, 2)$ is on the graph of $f(x) = \ln(x)$ and $f'(x) = \frac{1}{x} \Rightarrow f'\left(e^2\right) = \frac{1}{e^2}$. Let g be the inverse function of f. Give one point on the graph of g and evaluate g' at that point.

Solution

The point $(2, e^2)$ is on the graph of g and:

$$g'(2) = \frac{1}{f'(e^2)} = \frac{1}{\frac{1}{e^2}} = e^2$$

In fact, the inverse of $f(x) = \ln(x)$ is the exponential function $g(x) = e^x$ and we can check that $g'(x) = e^x \Rightarrow g'(2) = e^2$.

Example 7.2.8

In the table below, fill in the values of $f^{-1}(x)$ and $\left(f^{-1}\right)'(x)$ for $x = 0$ and $x = 1$:

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$\left(f^{-1}\right)'(x)$
0	2	3		
1	3	−2		
2	1	−1		
3	0	2		

Solution

$f(3) = 0$, so $f^{-1}(0) = 3$, while $\left(f^{-1}\right)'(0) = \frac{1}{f'(3)} = \frac{1}{2}$; $f(2) = 1$, so $f^{-1}(1) = 2$ while $\left(f^{-1}\right)'(1) = \frac{1}{f'(2)} = \frac{1}{-1} = -1$.

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$\left(f^{-1}\right)'(x)$
0	2	3	3	0.5
1	3	−2	2	−1
1	3	−2	2	−1

Practice 7.2.4

Fill in the missing values of $f^{-1}(x)$ and $(f^{-1})'(x)$ in the table from Example 7.2.8 for $x = 2$ and $x = 3$.

Answer

$f^{-1}(2) = 0$ and $f^{-1}(3) = 1$ while $(f^{-1})'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{3}$ and $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{-2}$.

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$(f^{-1})'(x)$
2	1	−1	0	0.333
3	0	2	1	−0.5

Problems

1. Given the values of f and f' in the table below, compute the specified values of f^{-1} and $(f^{-1})'$.

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$(f^{-1})'(x)$
1	3	−3		
2	1	2		
3	2	3		

2. Given the values of g and g' in the table below, compute the specified values of g^{-1} and $(g^{-1})'$.

x	$g(x)$	$g'(x)$	$g^{-1}(x)$	$(g^{-1})'(x)$
1	2	−2		
2	1	4		
3	3	2		

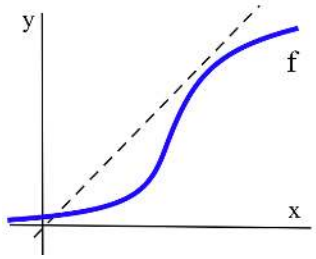
3. Given the values of h and h' in the table below, compute the specified values of h^{-1} and $(h^{-1})'$.

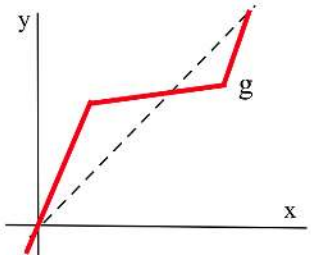
x	$h(x)$	$h'(x)$	$h^{-1}(x)$	$(h^{-1})'(x)$
1	2	2		
2	3	−2		
3	1	0		

4. Given the values of w and w' in the table below, compute the specified values of w^{-1} and $(w^{-1})'$.

x	$w(x)$	$w'(x)$	$w^{-1}(x)$	$(w^{-1})'(x)$
1	1	2		
2	3	0		
3	2	5		

5. The figure below left shows a graph of f. Sketch a graph of f^{-1}.





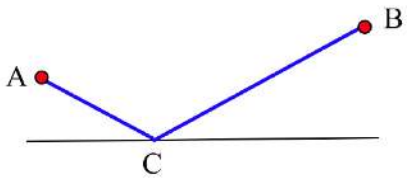
6. The figure above right shows a graph of g. Sketch a graph of g^{-1}.
7. If the graphs of f and f^{-1} intersect at the point (a, b), how are a and b related?
8. If the graph of f intersects the line $y = x$ at $x = a$, does the graph of f^{-1} intersect the line $y = x$? If so, where?
9. The steps to evaluate the function $f(x) = \dfrac{7x - 5}{4}$ are
   a. multiply by 7
   b. subtract 5 and
   c. divide by 4.

   Write the steps, in words, for the inverse of this function, and then translate these verbal steps into a formula for the inverse function.
10. Find a formula for the inverse function of $f(x) = 3x - 2$. Verify that $f^{-1}(f(5)) = 5$ and $f(f^{-1}(2)) = 2$.
11. Find a formula for the inverse function of $g(x) = 2x + 1$. Verify that $g^{-1}(g(1)) = 1$ and $g(g^{-1}(7)) = 7$.
12. Find a formula for the inverse of $h(x) = 2e^{3x}$. Verify that $h^{-1}(h(0)) = 0$.
13. Find a formula for the inverse of $w(x) = 5 + \ln(x)$. Verify that $w^{-1}(w(1)) = 1$.
14. If the graph of f goes through the point $(2, 5)$ and $f'(2) = 3$, then the graph of f^{-1} goes through the point (____, ____) and $(f^{-1})'(5) =$ ____.
15. If the graph of f goes through the point $(1, 3)$ and $f'(1) > 0$, then the graph of f^{-1} goes through the point (____, ____). What can you say about the value of $(f^{-1})'(3)$?
16. If $f(6) = 2$ and $f'(6) < 0$, then the graph of f^{-1} goes through the point (____, ____). What can you say about $(f^{-1})'(2)$?
17. If $f'(x) > 0$ for all values of x, what can you say about $(f^{-1})'(x)$? What does this tell you about the graphs of f and f^{-1}?
18. If $f'(x) < 0$ for all values of x, what can you say about $(f^{-1})'(x)$? What does this tell you about the graphs of f and f^{-1}?
19. Find a linear function $f(x) = ax + b$ so the graphs of f and f^{-1} are parallel and do not intersect.
20. Does $f(x) = 3 + \sin(x)$ have an inverse function? Justify your answer.
21. Does $f(x) = 3x + \sin(x)$ have an inverse function? Justify your answer.
22. For which positive integers n is $f(x) = x^n$ one-to-one? Justify your answer.
23. Some functions are their own inverses. For which four of these functions does $f^{-1}(x) = f(x)$?
    a. $f(x) = \dfrac{1}{x}$
    b. $f(x) = \dfrac{x + 1}{x - 1}$
    c. $f(x) = \dfrac{3x - 5}{7x - 3}$
    d. $f(x) = \dfrac{ax + b}{cx - a}$
    e. $f(x) = x + a \ (a \neq 0)$

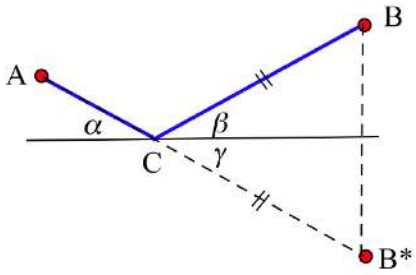
Reflections on Folding

The symmetry of the graphs of a function and its inverse about the line $y = x$ make sketching a graph of the inverse function relatively easy if you have a graph of f: fold your graph paper along the line $y = x$ so the graph of f^{-1} is the "folded" image of f. This simple idea of obtaining a new image of something by folding along a line can enable us to quickly "see" solutions to some otherwise difficult problems.

The minimization problem of finding the shortest path from town A to town B with an intermediate stop at a (straight) river, as depicted in below:



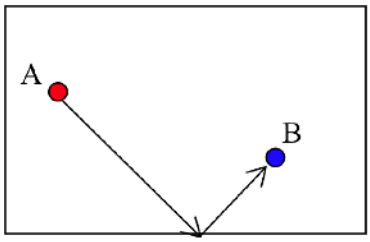
is moderately difficult to solve using derivatives (see Problem 11 in Section 3.5). Geometry allows us to solve the problem much more easily:



- obtain the point B^* by folding the image of B across the river line
- connect A and B^* with a line segment (the shortest path connecting A and B^*)
- fold the C-to-B^* segment back across the river to obtain the A-to-C-to-B solution.

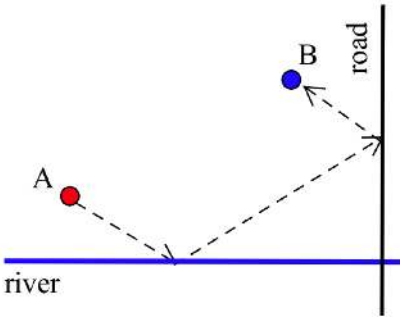
As an almost-free bonus, we see that—for the minimum path—the angle of incidence at the river equals the angle of reflection.

24. Devise an algorithm using "folding" to locate the point at the bottom edge of the billiards table you should aim ball A in the figure below:



so that ball A will hit ball B after bouncing off the bottom edge of the table. (Assume that the angle of incidence equals the angle of reflection.)

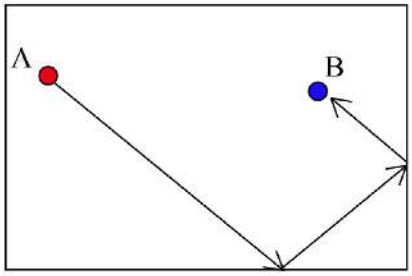
25. Devise an algorithm using "folding" to sketch the shortest path in the figure below:



from town A to town B that includes a stop at the river *and* at the road. (One fold may not be enough.)

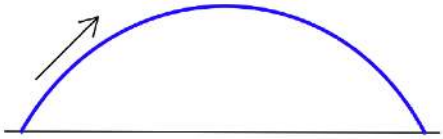
26. Devise an algorithm using "folding" to find where you should aim ball A at the bottom edge of the billiards table in the figure below so that ball A will hit ball B after bouncing off the bottom edge *and* the right edge of the table. (Assume that the angle of incidence equals the angle of reflection. Unfortunately, in a real-life game of billiards, the ball picks up a spin, called "English,"

when it bounces off the first bank, so at the second bounce the angle of incidence does not equal the angle of reflection.)

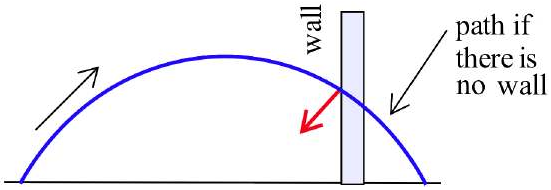


The “folding” idea can even be useful if the path is not a straight line.

26. The figure below shows the parabolic path of a thrown ball:



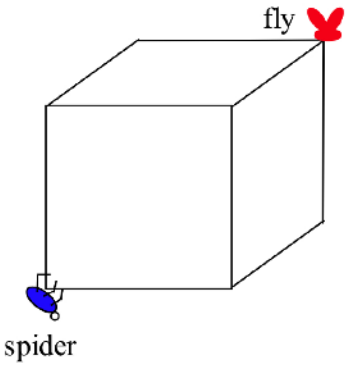
If the ball bounces off the tall vertical wall in this figure:



where will it hit the ground? (Assume that the angle of incidence equals the angle of reflection and that the ball does not lose energy during the bounce.)

Sometimes “unfolding” a problem is useful too.

27. A spider and a fly are located at opposite corners of the cube as shown below:



Sketch the shortest path the spider can travel along the surface of the cube to reach the fly.

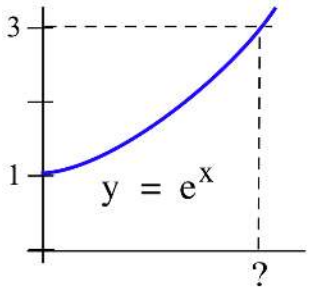
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7.3: Inverse Trigonometric Functions

We now turn our attention to the inverse trigonometric functions, their properties and their graphs, focusing on properties and techniques needed to investigate derivatives and integrals of these functions. We will concentrate on the inverse sine and inverse tangent functions, the two inverse trigonometric functions that arise most often in calculus.

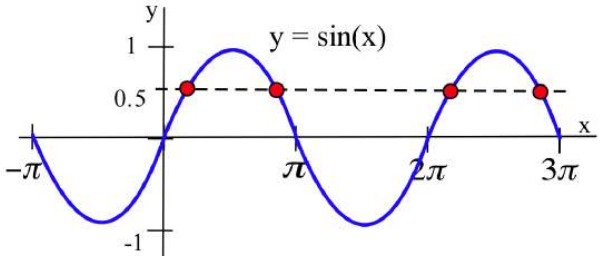
Inverse Sine: Solving $k = \sin(x)$ for x

It is straightforward to solve the equation $3 = e^x$:

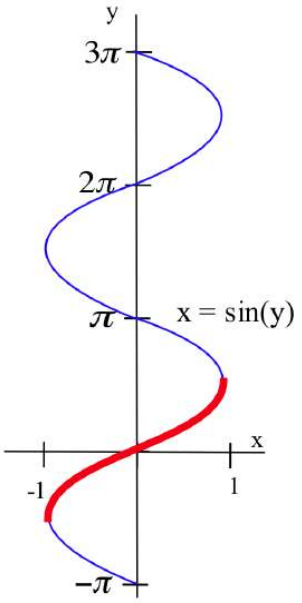


Simply apply the natural logarithm function, the inverse of the exponential function e^x, to each side of the equation to get $\ln(3) = \ln(e^x) = x$. Because the function $f(x) = e^x$ is one-to-one, the equation $3 = e^x$ has only the one solution $x = \ln(3) \approx 1.1$.

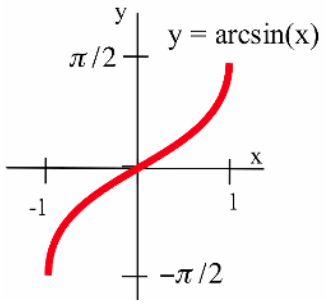
The solution of the equation $0.5 = \sin(x)$ presents more difficulties. As the figure below illustrates:



the function $f(x) = \sin(x)$ is *not* one-to-one: its graph reflected about the line $y = x$:



is *not* the graph of a function. Sometimes, however, it is necessary to "undo" the sine function, and we can do so by restricting its domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, the function $f(x) = \sin(x)$ is one-to-one and has an inverse function—and the graph of the inverse function



is the reflection about the line $y = x$ of the (restricted) graph of $y = \sin(x)$.

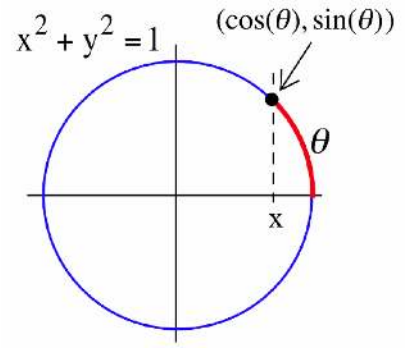
Caution

Many textbooks—and most calculators—use the notation $\sin^{-1}(x)$ for $\arcsin(x)$. You must be very careful to **never** interpret $\sin^{-1}(x)$ to mean:

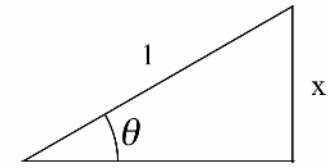
$$(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x) \tag{7.3.1}$$

We avoid the $\sin^{-1}(x)$ notation for this reason and suggest that you do as well.

We call this inverse of the (restricted) sine function the **arcsine** and denote it $\arcsin(x)$. The name "arcsine" comes from the unit-circle definition of the sine function. On the unit circle:



if θ is the length of the arc whose sine is x, then $\sin(\theta) = x$ and $\theta = \arcsin(x)$. Using the right-triangle definition of sine:



$\sin(\theta) = x, \quad \theta = \arcsin(x)$

θ represents an angle whose sine is x.

Definition: Inverse Sine

For $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$:

$$y = \arcsin(x) \iff x = \sin(y) \tag{7.3.2}$$

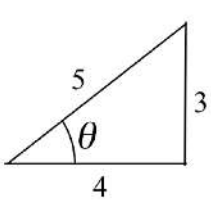
The domain of $\arcsin(x)$ is $[-1, 1]$ and its range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

The (restricted) sine function and the arcsine are inverses of each other:

$$-1 \le x \le 1 \;\Rightarrow\; \sin(\arcsin(x)) = x$$
$$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \;\Rightarrow\; \arcsin(\sin(y)) = y$$

Right Triangles and Arcsine

For the right triangle shown below:



$$\sin(\theta) = \frac{3}{5}$$
$$\theta = \arcsin\left(\frac{3}{5}\right)$$
$$\theta \approx 0.6435$$

$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$ so $\theta = \arcsin\left(\frac{3}{5}\right)$. It is possible to evaluate other trigonometric functions (such as cosine and tangent) of an angle expressed as an arcsine without explicitly solving for the value of the angle. For example:

$$\cos\left(\arcsin\left(\frac{3}{5}\right)\right) = \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\tan\left(\arcsin\left(\frac{3}{5}\right)\right) = \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

Once you know the sides of the right triangle, you can compute the values of the other trigonometric functions using their standard right-triangle definitions:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adjacent}}{\text{opposite}}$$

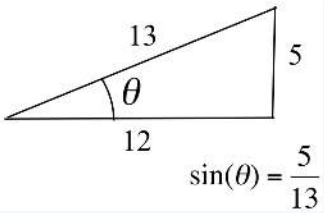
If you are given an angle θ as the arcsine of a number, but not given the sides of a right triangle, you can construct your own triangle with the given angle: select values for the opposite side and hypotenuse so the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is the value whose arcsine we want: $\arcsin\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$. You can calculate the length of the third ("adjacent") side using the Pythagorean Theorem.

Example 7.3.1

Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin\left(\frac{5}{13}\right)$. Use the triangle to determine the values of $\tan\left(\arcsin\left(\frac{5}{13}\right)\right)$ and $\csc\left(\arcsin\left(\frac{5}{13}\right)\right)$.

Solution

We want the sine of θ, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, to be $\frac{5}{13}$ so we can choose the opposite side to be 5 and the hypotenuse to be 13:



Then $\sin(\theta) = \frac{5}{13}$, as desired. Using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{13^2 - 5^2} = 12$. So:

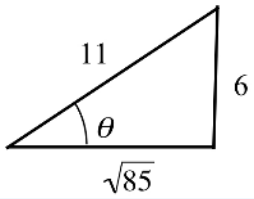
$$\tan(\theta) = \tan\left(\arcsin\left(\frac{5}{13}\right)\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$$
$$\csc(\theta) = \csc\left(\arcsin\left(\frac{5}{13}\right)\right) = \frac{1}{\sin\left(\arcsin\left(\frac{5}{13}\right)\right)} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$$

Any choice of values for the opposite side and the hypotenuse will work (for example opposite $= 500$ and hypotenuse $= 1300$), as long as the ratio of the opposite side to the hypotenuse is $\frac{5}{13}$.

Practice 7.3.1

Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin\left(\frac{6}{11}\right)$. Use the triangle to determine the values of $\tan\left(\arcsin\left(\frac{6}{11}\right)\right)$, $\csc\left(\arcsin\left(\frac{6}{11}\right)\right)$ and $\cos\left(\arcsin\left(\frac{6}{11}\right)\right)$.

Answer



With hypotenuse 11 and opposite side 6, the adjacent side must have length $\sqrt{11^2 - 6^2} = \sqrt{85}$, so:

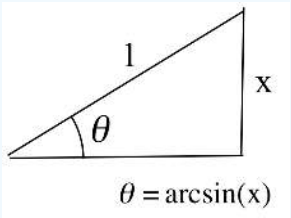
$$\tan\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{6}{\sqrt{85}}$$
$$\csc\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{11}{6}$$
$$\cos\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{\sqrt{85}}{11}$$

Example 7.3.2

Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin(x)$. Use the triangle to determine the values of $\tan(\arcsin(x))$ and $\cos(\arcsin(x))$.

Solution

We want the sine of θ, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, to be x so we can choose the opposite side to be x and the hypotenuse to be 1:



Then $\sin(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{1 - x^2}$ so that:

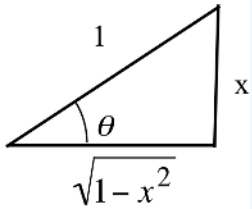
$$\tan(\arcsin(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}$$
$$\cos(\arcsin(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Other choices for the lengths of the opposite side and hypotenuse, such as $3x$ and 3, will work, but x and 1 are the simplest choices.

? Practice 7.3.2

Evaluate $\sec(\arcsin(x))$ and $\csc(\arcsin(x))$.

Answer

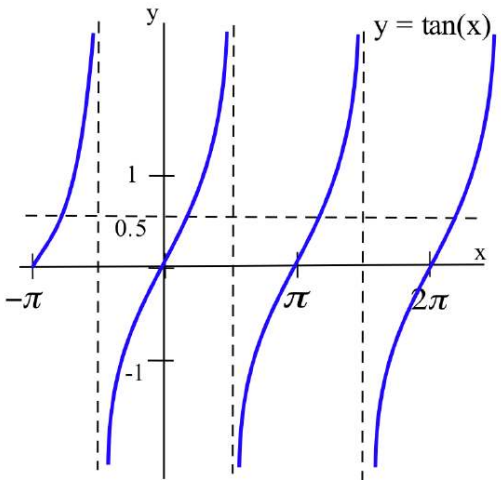


With hypotenuse 1 and opposite side x, the adjacent side must have length $\sqrt{1^2-x^2} = \sqrt{1-x^2}$, so:

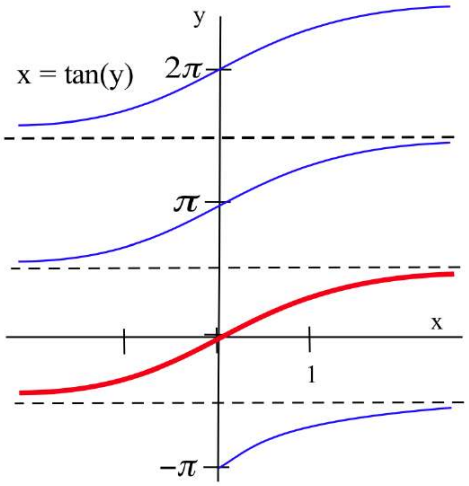
$$\sec(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$
$$\csc(\arcsin(x)) = \frac{1}{x}$$

Inverse Tangent: Solving $k = \tan(x)$ for x

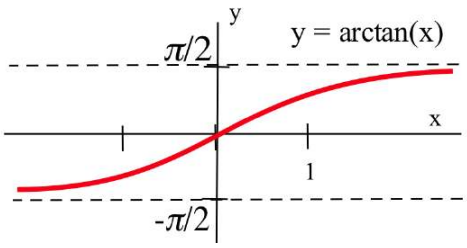
The equation $0.5 = \tan(x)$:



has many solutions: the function $f(x) = \tan(x)$ is not one-to-one, and its graph reflected across the line $y = x$:



is not the graph of a function. If, however, we restrict the domain of the tangent function to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then the restricted $f(x) = \tan(x)$ *is* one-to-one and has an inverse function. The graph of this inverse tangent function:



is the reflection about the line $y = x$ of the (restricted) graph of $y = \tan(x)$. We call this inverse of the (restricted) tangent function the **arctangent** and denote it $\arctan(x)$. For $x > 0$, the number $\arctan(x)$ is the length of the arc on the unit circle whose tangent is x, and $\arctan(x)$ is the angle whose tangent is x: $\tan(\arctan(x)) = x$.

Caution

Many textbooks—and most calculators—use the notation $\tan^{-1}(x)$ for $\arctan(x)$. You must be very careful to **never** interpret $\tan^{-1}(x)$ to mean:

$$(\tan(x))^{-1} = \frac{1}{\tan(x)} = \cot(x) \tag{7.3.3}$$

We avoid the $\tan^{-1}(x)$ notation for this reason and suggest that you do as well.

Definition: Inverse Tangent

For all x and $-\frac{\pi}{2} < y < \frac{\pi}{2}$:

$$y = \arctan(x) \iff x = \tan(y) \tag{7.3.4}$$

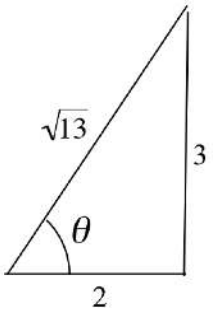
The domain of $\arctan(x)$ is $(-\infty, \infty)$ and its range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

The (restricted) tangent function and the arctangent are inverses:

$$\begin{aligned} -\infty < x < \infty &\Rightarrow \tan(\arctan(x)) = x \\ -\frac{\pi}{2} < y < \frac{\pi}{2} &\Rightarrow \arctan(\tan(y)) = y \end{aligned}$$

Right Triangles and Arctangent

For the right triangle in the figure below:



$$\tan(\theta) = \frac{3}{2}$$
$$\theta = \arctan\left(\frac{3}{2}\right)$$
$$\theta \approx 0.9828$$

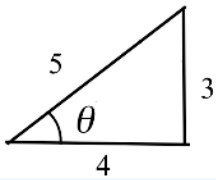
$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{2}$ so that $\theta = \arctan\left(\frac{3}{2}\right)$, hence:

$$\sin\left(\arctan\left(\frac{3}{2}\right)\right) = \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{\sqrt{13}} \approx 0.832$$
$$\cot\left(\arctan\left(\frac{3}{2}\right)\right) = \frac{1}{\tan\left(\arctan\left(\frac{3}{2}\right)\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3} \approx 0.667$$

Practice 7.3.3

Determine the lengths of the sides of a right triangle so that one angle is $\theta = \arctan\left(\frac{3}{4}\right)$, then use the triangle to determine the values of $\sin\left(\arctan\left(\frac{3}{4}\right)\right)$, $\cot\left(\arctan\left(\frac{3}{4}\right)\right)$ and $\cos\left(\arctan\left(\frac{3}{4}\right)\right)$.

Answer

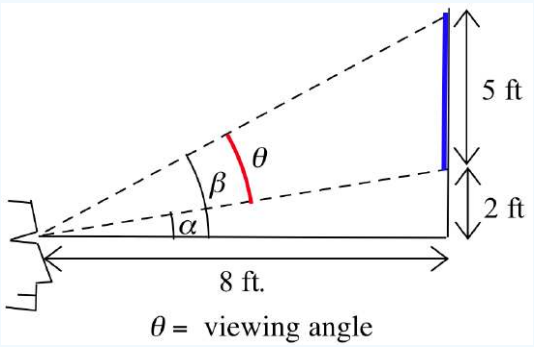


With opposite side 3 and adjacent side 4, the hypotenuse must have length $\sqrt{3^2+4^2} = \sqrt{25} = 5$, so:

$$\sin\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{3}{5}$$
$$\cot\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{4}{3}$$
$$\cos\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{4}{5}$$

Example 7.3.3

On a wall 8 feet in front of you, the lower edge of a 5-foot-tall painting rests 2 feet above your eye level:



https://math.libretexts.org/@go/page/212065

Represent your viewing angle θ using arctangents.

Solution

The viewing angle α to the bottom of the painting satisfies:

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{8} \Rightarrow \alpha = \arctan\left(\frac{1}{4}\right) \tag{7.3.5}$$

Similarly, the angle β to the top of the painting satisfies:

$$\tan(\beta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{7}{8} \Rightarrow \alpha = \arctan\left(\frac{7}{8}\right) \tag{7.3.6}$$

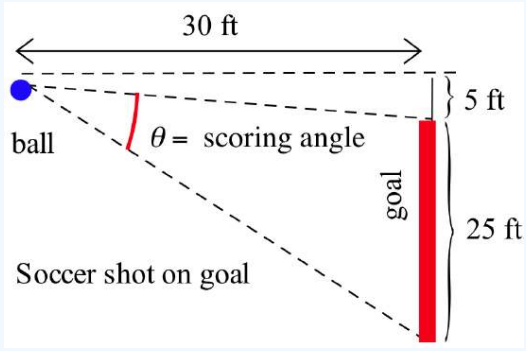
The viewing angle θ for the painting is therefore:

$$\theta = \beta - \alpha = \arctan\left(\frac{7}{8}\right) - \arctan\left(\frac{1}{4}\right) \approx 0.719 - 0.245 = 0.474 \tag{7.3.7}$$

or about $27°$.

Practice 7.3.4

Determine the scoring angle for the soccer player in the figure below:



Answer

$\tan(\alpha) = \frac{5}{30} = \frac{1}{6}$ so $\alpha = \arctan\left(\frac{1}{6}\right) \approx 0.165$ (or about $9.46°$). Likewise, $\tan(\alpha + \theta) = \frac{30}{30} = 1$ so $\alpha + \theta = \arctan(1) = \frac{\pi}{4} \approx 0.785$ (or $45°$). Finally:

$$\theta = (\alpha + \theta) - \alpha \approx 0.785 - 0.165 = 0.62 \tag{7.3.8}$$

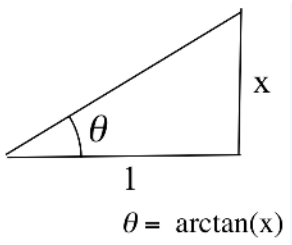
or about $35.54°$.

Example 7.3.4

Determine the lengths of the sides of a right triangle so that one angle is $\theta = \arctan(x)$, then use the triangle to determine the values of $\sin(\arctan(x))$ and $\cos(\arctan(x))$.

Solution

We want the tangent of θ (the ratio of opposite to adjacent) to be x, so we can choose the opposite side to be x and the adjacent side to be 1:



Then $\tan(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1+x^2}$ so that:

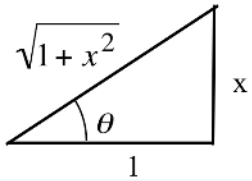
$$\sin(\arctan(x)) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}}$$
$$\cos(\arctan(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}$$

We could have chosen other values for the opposite and adjacent sides (such as x^2 and x), but x and 1 provide the simplest option.

? Practice 7.3.5

Evaluate $\sec(\arctan(x))$ and $\cot(\arctan(x))$.

Answer

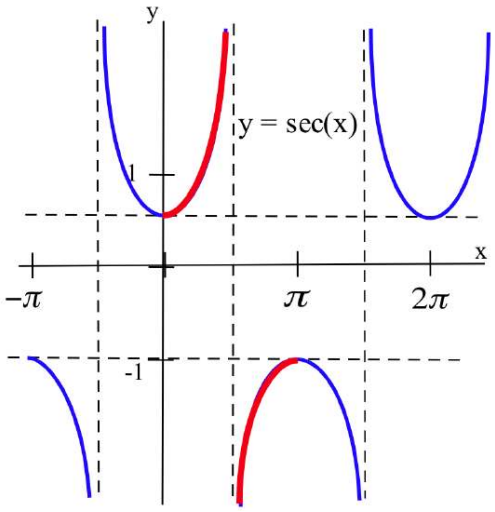


With opposite side x and adjacent side 1, the hypotenuse must have length $\sqrt{1^2+x^2} = \sqrt{1+x^2}$, so:

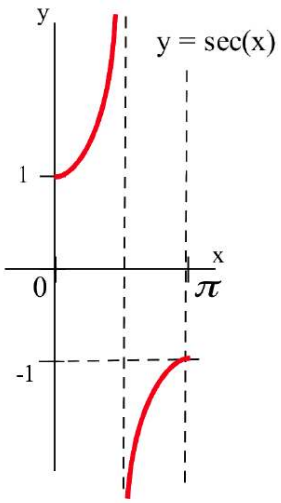
$$\sec(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$
$$\cot(\arctan(x)) = \frac{1}{x}$$

Inverse Secant: Solving $k = \sec(x)$ for x

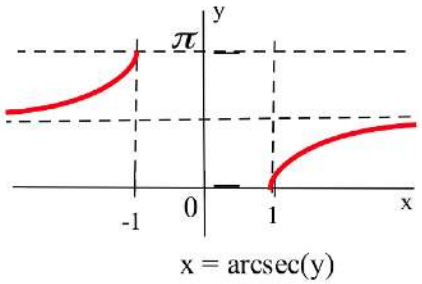
The equation $2 = \sec(x)$:



has many solutions, but we can create an inverse function for secant—much the same way we did for sine and tangent—by suitably restricting the domain of the secant function so that it becomes a one-to-one function:



The figure above shows the restriction $0 \leq x \leq \pi$ $(x \neq \frac{\pi}{2})$, which results in a one-to-one function that has an inverse. The graph of the inverse function:



is the reflection about the line $y = x$ of the (restricted) graph of $y = \sec(x)$. We call this inverse of the (restricted) secant function the **arcsecant** and denote it $\operatorname{arcsec}(x)$.

Definition: Inverse Secant

If $|x| \geq 1$ and $0 \leq y \leq \pi$ with $y \neq \frac{\pi}{2}$:

$$y = \operatorname{arcsec}(x) \iff x = \sec(y) \tag{7.3.9}$$

The domain of $\operatorname{arcsec}(x)$ is $(-\infty, -1] \cup [1, \infty)$ and the range of $\operatorname{arcsec}(x)$ is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

Note

There are alternate ways to restrict the secant function to get a one-to-one function, and they lead to slightly different definitions of the inverse secant. We chose to use this restriction because it seems more "natural" than the alternatives, it is easier to evaluate on a calculator, and it is the most commonly used.

Caution

Many textbooks use the notation $\sec^{-1}(x)$ for $\operatorname{arcsec}(x)$. You must be very careful to **never** interpret $\sec^{-1}(x)$ to mean:

$$(\sec(x))^{-1} = \frac{1}{\sec(x)} = \cos(x) \tag{7.3.10}$$

We avoid the $\sec^{-1}(x)$ notation for this reason and suggest that you do as well.

The (restricted) secant function and the arcsecant are inverses:

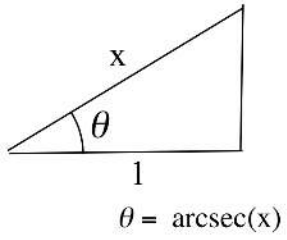
$$\begin{aligned} |x| \geq 1 &\Rightarrow \sec(\operatorname{arcsec}(x)) = x \\ 0 \leq y \leq \pi \left(y \neq \frac{\pi}{2}\right) &\Rightarrow \operatorname{arcsec}(\sec(y)) = y \end{aligned}$$

Example 7.3.5

Evaluate $\tan(\operatorname{arcsec}(x))$.

Solution

We want the secant of θ (the ratio of hypotenuse to adjacent) to be x, so we can choose the hypotenuse to be x and the adjacent side to be 1:



Then $\sec(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the opposite side is $\sqrt{x^2 - 1}$, so:

$$\tan(\operatorname{arcsec}(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2 - 1} \tag{7.3.11}$$

As usual, x and 1 are the simplest—but not the only—choices.

Practice 7.3.6

Evaluate $\sin(\operatorname{arcsec}(x))$ and $\cot(\operatorname{arcsec}(x))$.

Answer

FIGURE With hypotenuse x and adjacent side 1, the opposite side must have length $\sqrt{x^2 - 1^2} = \sqrt{x^2 - 1}$, so:

$$\sin\left(\arcsec\left(x\right)\right) = \frac{\sqrt{x^2-1}}{x}$$
$$\cot\left(\arcsec\left(x\right)\right) = \frac{1}{\sqrt{x^2-1}}$$

The Other Inverse Trigonometric Functions

The inverse tangent and inverse sine functions are by far the most commonly used of the six inverse trigonometric functions in calculus. The inverse secant function turns up less often. The other three inverse trigonometric functions ($\arccos(x)$, $\operatorname{arccot}(x)$ and $\operatorname{arccsc}(x)$) can be defined as the inverses of restricted versions of $\cos(x)$, $\cot(x)$ and $\csc(x)$, respectively, but these functions are almost dispensable in calculus. (The reasons for this will become apparent in the next section.)

Calculators and Inverse Trigonometric Functions

Most calculators only have keys for $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$, but the following identities allow you to compute values of the other inverse trigonometric functions.

Theorem

If: $x \neq 0$ and x is in the appropriate domain

then: $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$, $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

Proof

If $x \neq 0$, then:

$$\tan\left(\operatorname{arccot}(x)\right) = \frac{1}{\cot\left(\operatorname{arccot}(x)\right)} = \frac{1}{x} \tag{7.3.12}$$

Applying the arctangent function to each side of this equation:

$$\arctan\left(\tan\left(\operatorname{arccot}(x)\right)\right) = \arctan\left(\frac{1}{x}\right) \;\Rightarrow\; \operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right) \tag{7.3.13}$$

Proofs of the other two identities are left to you.

Theorem

If: $x \neq 0$ and x is in the appropriate domain

then: $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$, $\arctan(x) + \operatorname{arccot}(x) = \frac{\pi}{2}$ and $\operatorname{arcsec}(x) + \operatorname{arccsc}(x) = \frac{\pi}{2}$.

Proof

If α and β are complementary angles in a right triangle, so that $\alpha + \beta = \frac{\pi}{2}$, then $\sin(\alpha) = \cos(\beta)$. Let $x = \sin(\alpha) = \cos(\beta)$ so that $\alpha = \arcsin(x)$ and $\beta = \arccos(x)$, hence:

$$\alpha + \beta = \arcsin(x) + \arccos(x) = \frac{\pi}{2} \tag{7.3.14}$$

This proves the first result for $0 < x < 1$. You can easily check that the result also holds for $x = 0$, $x = 1$ and $x = -1$. To check that it holds for $-1 < x < 0$, we need the the next set of identities listed below. Proofs of the other two identities above are left to you.

Theorem

If: x is in the appropriate domain

then: $\arcsin(-x) = -\arcsin(x)$, $\arccos(-x) = \pi - \arccos(x)$, $\arctan(x) = -\arctan(x)$, $\operatorname{arcsec}(x) = \pi - \operatorname{arcsec}(x)$, $\operatorname{arccsc}(-x) = -\operatorname{arccsc}(x)$ and $\operatorname{arccot}(-x) = -\operatorname{arccot}(x)$.

Proof

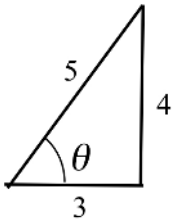
If $-1 \le x \le 1$, let $\theta = \arcsin(-x)$ so that $\sin(\theta) = -x \Rightarrow x = -\sin(\theta) = \sin(-\theta) \Rightarrow \arcsin(x) = -\theta \Rightarrow -\arcsin(x) = \theta = \arcsin(-x)$. This proves the first identity; the others are left to you.

Some programming languages only include a *single* inverse trigonometric function, $\arctan(x)$, but it suffices to enable you to evaluate the other five inverse trigonometric functions:

- $\arcsin(x) = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$
- $\arccos(x) = \dfrac{\pi}{2} - \arcsin(x) = \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$
- $\operatorname{arccot}(x) = \arctan\left(\dfrac{1}{x}\right)$
- $\operatorname{arcsec}(x) = \arctan\left(\sqrt{x^2-1}\right)$
- $\operatorname{arccsc}(x) = \dfrac{\pi}{2} - \operatorname{arcsec}(x) = \dfrac{\pi}{2} - \arctan\left(\sqrt{x^2-1}\right)$

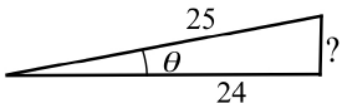
Problems

1. a. List the three smallest positive angles θ that are solutions of the equation $\sin(\theta) = 1$.
   b. Evaluate $\arcsin(1)$ and $\operatorname{arccsc}(1)$.
2. a. List the three smallest positive angles θ that are solutions of the equation $\tan(\theta) = 1$.
   b. Evaluate $\arctan(1)$ and $\operatorname{arccot}(1)$.
3. Find all x between 1 and 7 so that:
   a. $\sin(x) = 0.3$
   b. $\sin(x) = -0.4$
   c. $\sin(x) = 0.5$
4. Find all values of x between 1 and 7 so that:
   a. $\sin(x) = 0.3$
   b. $\sin(x) = -0.4$
5. Find all values of x between 2 and 7 so that:
   a. $\tan(x) = 3.2$
   b. $\tan(x) = -0.2$
6. Find all values of x between 1 and 5 so that:
   a. $\tan(x) = 8$
   b. $\tan(x) = -3$
7. In the figure below:



   angle θ is
   a. the arcsine of what number?
   b. the arctangent of what number?
   c. the arcsecant of what number?
   d. the arccosine of what number?

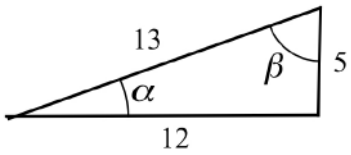
8. In the figure below:



angle θ is

   a. the arcsine of what number?
   b. the arctangent of what number?
   c. the arcsecant of what number?
   d. the arccosine of what number?

9. For the angle α in the triangle below:



evaluate:

   a. $\sin(\alpha)$
   b. $\tan(\alpha)$
   c. $\sec(\alpha)$
   d. $\cos(\alpha)$

10. For the angle β in the triangle Problem 9, evaluate:

   a. $\sin(\beta)$
   b. $\tan(\beta)$
   c. $\sec(\beta)$
   d. $\cos(\beta)$

11. For $\theta = \arcsin\left(\frac{2}{7}\right)$, find the exact values of:

   a. $\tan(\theta)$
   b. $\cos(\theta)$
   c. $\csc(\theta)$
   d. $\cot(\theta)$

12. For $\theta = \arctan\left(\frac{9}{2}\right)$, find the exact values of:

   a. $\sin(\theta)$
   b. $\cos(\theta)$
   c. $\csc(\theta)$
   d. $\cot(\theta)$

13. For $\theta = \arccos\left(\frac{1}{5}\right)$, find the exact values of:

   a. $\tan(\theta)$
   b. $\sin(\theta)$
   c. $\csc(\theta)$
   d. $\cot(\theta)$

14. For $\theta = \arcsin\left(\frac{a}{b}\right)$ with $0 < a < b$, find the exact values of:

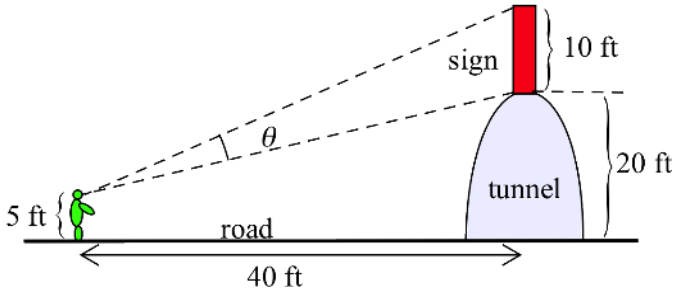
   a. $\tan(\theta)$
   b. $\cos(\theta)$
   c. $\csc(\theta)$
   d. $\cot(\theta)$

15. For $\theta = \arctan\left(\frac{a}{b}\right)$ with $0 < a < b$, find the exact values of:

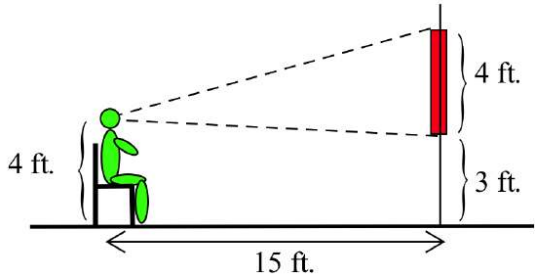
   a. $\tan(\theta)$

b. $\sin(\theta)$
c. $\cos(\theta)$
d. $\cot(\theta)$

16. For $\theta = \arctan(x)$, find the exact values of:
    a. $\sin(\theta)$
    b. $\cos(\theta)$
    c. $\sec(\theta)$
    d. $\cot(\theta)$
17. Find the exact values of
    a. $\sin(\arccos(x))$
    b. $\cos(\arcsin(x))$
    c. $\sec(\arccos(x))$
18. Find the exact values of
    a. $\tan(\arccos(x))$
    b. $\cos(\arctan(x))$
    c. $\sec(\arcsin(x))$
19. a. Does $\arcsin(1) + \arcsin(1) = \arcsin(2)$?
    b. Does $\arccos(1) + \arccos(1) = \arccos(2)$?
20. a. What is the viewing angle for the tunnel sign in the figure below?



    b. Use arctangents to describe the viewing angle when the observer is x feet from the entrance of the tunnel.
21. a. What is the viewing angle for the whiteboard in the figure below?



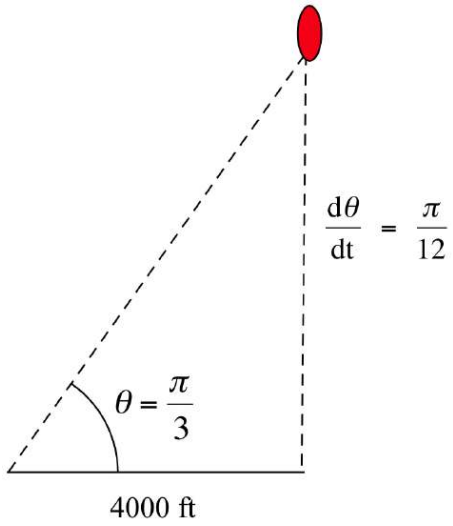
    b. Use arctangents to describe the viewing angle when the student is x feet from the wall.
22. Graph $y = \arcsin(2x)$ and $y = \arctan(2x)$.
23. Graph $y = \arcsin\left(\frac{x}{2}\right)$ and $y = \arctan\left(\frac{x}{2}\right)$.
24. Which curve is longer, $y = \sin(x)$ from $x = 0$ to $x = \pi$, or $y = \arcsin(x)$ from $x = -1$ to $x = 1$?

For Problems 25–28, $\left.\frac{d\theta}{dt}\right|_{\theta=1.3} = 12$, and θ and h are related by the given formula. Find $\left.\frac{dh}{dt}\right|_{\theta=1.3}$.

25. $\sin(\theta) = \frac{h}{20}$
26. $\tan(\theta) = \frac{h}{50}$
27. $\cos(\theta) = 3h + 20$
28. $3 + \tan(\theta) = 7h$

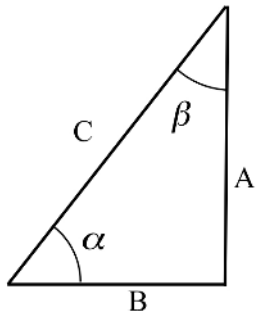
For Problems 29–32, $\left.\frac{dh}{dt}\right|_{\theta=1.3} = 4$, and θ and h are related by the given formula. Find $\left.\frac{d\theta}{dt}\right|_{\theta=1.3}$.

29. $\sin(\theta) = \frac{h}{38}$
30. $\tan(\theta) = \frac{h}{40}$
31. $\cos(\theta) = 7h - 23$
32. $\tan(\theta) = h^2$
33. You are observing a rocket launch from a position located 4000 feet from the launch pad:



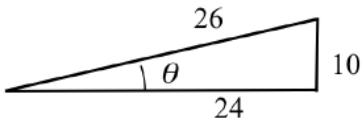
When your observation angle of the rocket is $\frac{\pi}{3}$, the angle is increasing at $\frac{\pi}{12}$ feet per second. How fast is the rocket traveling?

34. You are observing a rocket launch from a position 3000 feet from the launch pad. NASA's Twitter feed reports that when the rocket is 5000 feet high, its velocity is 100 feet per second.
    a. What is the angle of elevation of the rocket when it is 5000 feet above the launch pad?
    b. How fast is the angle of elevation increasing when the rocket is 5000 feet high?
35. Refer to the right triangle shown below:



    a. Angle α is arcsine of what number?
    b. Angle β is arccosine of what number?
    c. For positive numbers A and C, evaluate $\arcsin\left(\frac{A}{C}\right) + \arccos\left(\frac{A}{C}\right)$.
36. Refer to the right triangle in Problem 35.
    a. Angle α is arctangent of what number?
    b. Angle β is arccotangent of what number?
    c. For positive numbers A and B, evaluate $\arctan\left(\frac{A}{B}\right) + \operatorname{arccot}\left(\frac{A}{B}\right)$.
37. Refer to the triangle from Problems 35–36.
    a. Angle α is arcsecant of what number?
    b. Angle β is arccosecant of what number?
    c. For positive numbers B and C, evaluate $\operatorname{arcsec}\left(\frac{C}{B}\right) + \operatorname{arccsc}\left(\frac{C}{B}\right)$.

38. Describe the pattern apparent in your results from the previous three problems.
39. Refer to the right triangle shown below:



a. Angle θ is arctangent of what number?
b. Angle θ is arccotangent of what number?

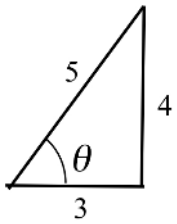
40. Refer to the right triangle from the previous Problem.
a. Angle θ is arcsine of what number?
b. Angle θ is arccosecant of what number?

41. Refer to the triangle from Problems 39–40.
a. Angle θ is arccosine of what number?
b. Angle θ is arcsecant of what number?

42. Describe the pattern apparent in your results from the previous three problems.

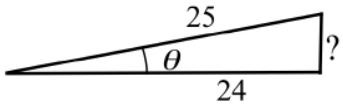
In Problems 43–51, use a calculator (as necessary) and appropriate identities to compute the given values.

43. $\operatorname{arcsec}(3)$
44. $\operatorname{arcsec}(-2)$
45. $\operatorname{arcsec}(-1)$
46. $\operatorname{arccos}(0.5)$
47. $\operatorname{arccos}(-0.5)$
48. $\operatorname{arccos}(1)$
49. $\operatorname{arccot}(1)$
50. $\operatorname{arccot}(0.5)$
51. $\operatorname{arccot}(-3)$
52. For the triangle shown below:



a. $\theta = \operatorname{arctan}(\_\_\_\_)$
b. $\theta = \operatorname{arccot}(\_\_\_\_)$
c. $\operatorname{arccot}(\_\_\_\_) = \operatorname{arctan}(\_\_\_\_)$

53. For the triangle shown below:



a. $\theta = \operatorname{arcsin}(\_\_\_\_)$
b. $\theta = \operatorname{arccos}(\_\_\_\_)$
c. $\operatorname{arccos}(\_\_\_\_) = \operatorname{arcsin}(\_\_\_\_)$

54. Prove that $\operatorname{arcsec}(x) = \operatorname{arccos}\left(\frac{1}{x}\right)$.
55. Prove that $\operatorname{arccsc}(x) = \operatorname{arcsin}\left(\frac{1}{x}\right)$.

Using a right triangle you can show that:

$$\tan\left(\arcsin(x)\right) = \frac{x}{\sqrt{1-x^2}}$$
$$\Rightarrow \arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Imitate this reasoning in Problems 56–57.

56. Evaluate $\tan\left(\operatorname{arccot}(x)\right)$ and use the result to find a formula for $\operatorname{arccot}(x)$ in terms of arctangent.
57. Evaluate $\tan\left(\operatorname{arcsec}(x)\right)$ and use the result to find a formula for $\operatorname{arcsec}(x)$ in terms of arctangent.
58. Let $a = \arctan(x)$ and $b = \arctan(y)$. Use the identity:

$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\cdot\tan(b)} \tag{7.3.15}$$

to show that:

$$\arctan(x)+\arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right) \tag{7.3.16}$$

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7.4: Derivatives of Inverse Trigonometric Functions

The three previous sections introduced the ideas of one-to-one functions and inverse functions, then used those concepts to define arcsine, arctangent and the other inverse trigonometric functions. In this section, we obtain derivative formulas for the inverse trigonometric functions and consider an important application and some of its variations.

Derivative Formulas for Inverse Trigonometric Functions

- $\mathbf{D}\left(\arcsin(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ (for $|x| < 1$)
- $\mathbf{D}\left(\arccos(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$ (for $|x| < 1$)
- $\mathbf{D}\left(\arctan(x)\right) = \dfrac{1}{1+x^2}$ (for all x)
- $\mathbf{D}\left(\operatorname{arccot}(x)\right) = \dfrac{-1}{1+x^2}$ (for all x)
- $\mathbf{D}\left(\operatorname{arcsec}(x)\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$ (for $|x| > 1$)
- $\mathbf{D}\left(\operatorname{arccsc}(x)\right) = \dfrac{-1}{|x|\sqrt{x^2-1}}$ (for $|x| > 1$)

Proof

If $y = \arctan(x)$ then $\tan(y) = x$, so differentiating with respect to x and applying the Chain Rule yields:

$$\sec^2(y) \cdot \frac{dy}{dx} = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

Recalling the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, we can write:

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

proving the second formula listed above. To prove the first formula, put $y = \arcsin(x)$ so that $\sin(y) = x$. Implicitly differentiating with respect to x yields:

$$\cos(y) \cdot \frac{dy}{dx} = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Solving the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for $\cos(\theta)$:

$$\cos^2(\theta) = 1 - \sin^2(\theta) \;\Rightarrow\; \cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}$$

The range of $y = \arcsin(x)$ is $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$, so $\cos(y) \ge 0$ and:

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

For the third formula, if $y = \operatorname{arcsec}(x)$ then $\sec(y) = x$, so differentiation yields:

$$\sec(y)\cdot\tan(y) \cdot \frac{dy}{dx} = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\sec(y)\cdot\tan(y)}$$

Solving the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ for $\tan(\theta)$:

$$\tan^2(\theta) = \sec^2(\theta) - 1 \;\Rightarrow\; \tan(\theta) = \pm\sqrt{\sec^2(\theta)-1}$$

If $x \ge 1$ then $0 \le y < \dfrac{\pi}{2}$ and on this interval $\tan(y) \ge 0$, so:

$$\frac{dy}{dx} = \frac{1}{\sec(y)\cdot\tan(y)} = \frac{1}{\sec(y)\cdot\sqrt{\sec^2(y)-1}} = \frac{1}{x\cdot\sqrt{x^2-1}}$$

If $x \le -1$ then $\dfrac{\pi}{2} < y \le \pi$ and on this interval $\tan(y) \le 0$ so:

$$\frac{dy}{dx} = \frac{1}{\sec(y)\cdot\tan(y)} = \frac{1}{\sec(y)\cdot\left[-\sqrt{\sec^2(y)-1}\right]} = \frac{1}{-x\cdot\sqrt{x^2-1}}$$

Recognizing that $x = |x|$ when $x \geq 1$ and that $-x = |x|$ when $x \leq -1$, we can combine these two cases into a single compact form:

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$$

Proofs of the the remaining three formulas are nearly identical to those given above and are left for you as Problems 24–26.

Note

Notice that:

- the derivative of $\arctan(x)$ is defined for all x, which corresponds to the domain of $\arctan(x)$
- the derivative of $\arcsin(x)$ is defined only when $1-x^2 > 0$ or, equivalently, if $|x| < 1$, corresponding to the domain of $\arcsin(x)$ (omitting endpoints)
- the derivative of $\operatorname{arcsec}(x)$ is defined only when $x^2-1 > 0$ or, equivalently, if $|x| > 1$, corresponding to the domain of $\operatorname{arcsec}(x)$ (omitting endpoints)

Example 7.4.1

Compute $\mathrm{D}\left(\arcsin\left(e^x\right)\right)$, $\mathrm{D}\left(\arctan\left(x-3\right)\right)$, $\mathrm{D}\left(\arctan^3\left(5x\right)\right)$ and $\mathrm{D}\left(\ln\left(\arcsin(x)\right)\right)$.

Solution

Each of these functions involves a composition of an inverse trigonometric function with other functions, so we need to use the Chain Rule:

$$\begin{aligned}
\mathrm{D}\left(\arcsin\left(e^x\right)\right) &= \frac{1}{\sqrt{1-\left(e^x\right)^2}}\cdot\mathrm{D}\left(e^x\right) = \frac{e^x}{\sqrt{1-e^{2x}}}\\
\mathrm{D}\left(\arctan\left(x-3\right)\right) &= \frac{1}{1+(x-3)^2}\cdot\mathrm{D}\left(x-3\right)\\
&= \frac{1}{1+(x-3)^2} = \frac{1}{x^2-6x+10}\\
\mathrm{D}\left(\arctan^3\left(5x\right)\right) &= 3\arctan^2\left(5x\right)\cdot\mathrm{D}\left(\arctan(5x)\right)\\
&= 3\arctan^2\left(5x\right)\cdot\frac{1}{1+(5x)^2}\cdot 5 = \frac{15\arctan^2\left(5x\right)}{1+25x^2}\\
\mathrm{D}\left(\ln\left(\arcsin(x)\right)\right) &= \frac{1}{\arcsin(x)}\cdot\mathrm{D}\left(\arcsin(x)\right) = \frac{1}{\arcsin(x)}\cdot\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

You have already practiced some of these differentiation patterns in Chapter 2 (see Problems 72–83 in Section 2.4).

Practice 7.4.1

Compute $\mathrm{D}\left(\arcsin\left(5x\right)\right)$, $\mathrm{D}\left(\arctan\left(x+2\right)\right)$, $\mathrm{D}\left(\operatorname{arcsec}\left(7x\right)\right)$ and $\mathrm{D}\left(e^{\arctan(7x)}\right)$.

Answer

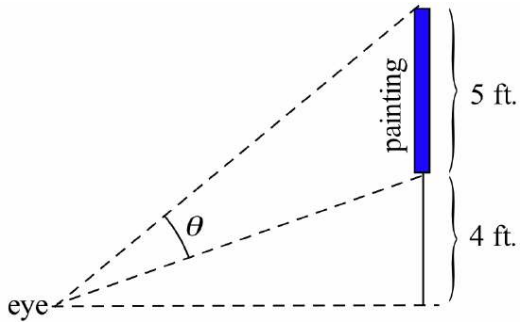
Applying the Chain Rule:

$$\mathrm{D}\left(\arcsin(5x)\right) = \frac{1}{\sqrt{1-(5x)^2}} \cdot 5 = \frac{5}{\sqrt{1-25x^2}}$$
$$\mathrm{D}\left(\arctan(x+2)\right) = \frac{1}{1+(x+2)^2} = \frac{1}{x^2+4x+5}$$
$$\mathrm{D}\left(\operatorname{arcsec}(7x)\right) = \frac{1}{|7x| \cdot \sqrt{(7x)^2-1}} \cdot 7 = \frac{1}{|x| \cdot \sqrt{49x^2-1}}$$
$$\mathrm{D}\left(e^{\arctan(7x)}\right) = e^{\arctan(7x)} \cdot \mathrm{D}\left(\arctan(7x)\right)$$
$$= e^{\arctan(7x)} \cdot \frac{1}{1+(7x)^2} \cdot 7 = \frac{7e^{\arctan(7x)}}{1+49x^2}$$

A Classic Application

Mathematics is the study of patterns, and one of the pleasures of mathematics is that patterns can crop up in some unexpected places. The mathematician Johannes Müller first posed the version of the classical Museum Problem below in 1471; it is one of the oldest known maximization problems.

Museum Problem: The lower edge of a 5-foot-tall painting is 4 feet above your eye level:

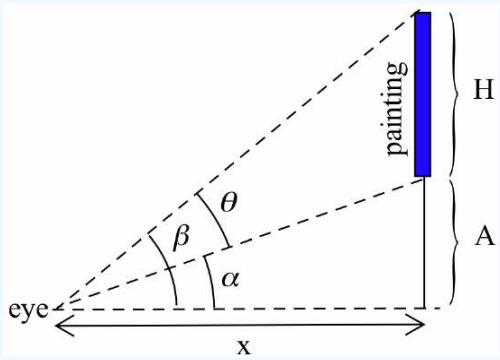


At what distance should you stand from the wall so that your viewing angle of the painting is maximum?

On a typical sunny weekend, many people might rather be watching or playing a football or soccer game than visiting a museum or solving a calculus problem about a painting. But the pattern of the Museum Problem even appears in football and soccer. Because we also want to examine the Museum Problem in these other contexts, let's solve the general version.

Example 7.4.2

The lower edge of an H-foot-tall painting is A feet above your eye level:



At what distance x should you stand from the painting so that your viewing angle is maximum?

Solution

Let $B = A + H$. Then $\tan(\alpha) = \frac{A}{x}$ and $\tan(\beta) = \frac{B}{x}$ so $\alpha = \arctan\left(\frac{A}{x}\right)$ and $\beta = \arctan\left(\frac{B}{x}\right)$. The viewing angle is thus:

$$\theta = \beta - \alpha = \arctan\left(\frac{B}{x}\right) - \arctan\left(\frac{A}{x}\right)$$

We can maximize θ by calculating the derivative $\frac{d\theta}{dx}$ and finding where that derivative is 0. Differentiation yields:

$$\begin{aligned}\frac{d\theta}{dx} &= \mathrm{D}\left[\arctan\left(\frac{B}{x}\right)\right] - \mathrm{D}\left[\arctan\left(\frac{A}{x}\right)\right] \\ &= \frac{1}{1+\left(\frac{B}{x}\right)^2}\cdot\left(\frac{-B}{x^2}\right) - \frac{1}{1+\left(\frac{A}{x}\right)^2}\cdot\left(\frac{-A}{x^2}\right) \\ &= \frac{-B}{x^2+B^2} + \frac{A}{x^2+A^2}\end{aligned}$$

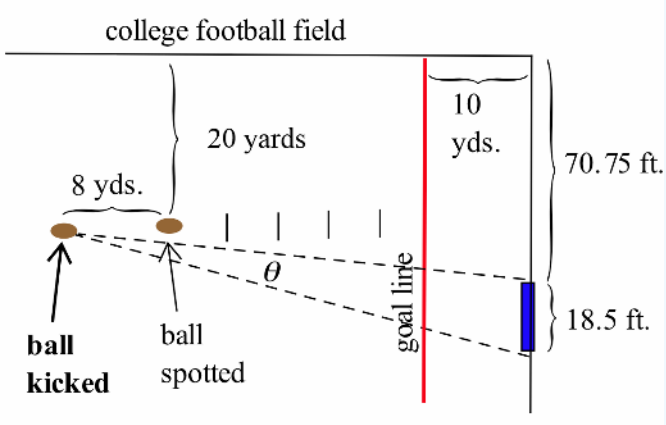
Setting $\frac{d\theta}{dx} = 0$ and solving for x yields $x = \sqrt{AB} = \sqrt{A(A+H)}$.

(We can disregard the endpoints, as you clearly do not have a maximum viewing angle with your nose pressed against the wall, or from an infinite distance far away from the wall.)

Now the original Museum Problem above becomes straightforward (as do the football and soccer variations below). With $A = 4$ and $H = 5$, the maximum viewing angle occurs when $x = \sqrt{4(4+5)} = 6$ feet. The maximum angle is therefore $\theta = \arctan\left(\frac{9}{6}\right) - \arctan\left(\frac{4}{6}\right) \approx 0.983 - 0.588 \approx 0.395$, or about $22.6°$.

? Practice 7.4.2

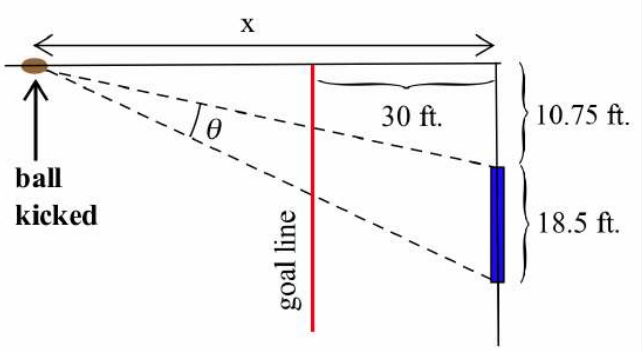
In a football game, a kicker is attempting a field goal by kicking the football between the goal posts:



At what distance from the goal line should the ball be "spotted" so the kicker has the largest angle for making the field goal? The goal posts are 18.5 feet wide and located 10 yards beyond the goal line. (Assume that the ball is placed on a "hash mark" that is 20 yards from the edge of the field and is actually kicked from a point about 8 yards farther from the goal line than where the ball is spotted.)

Answer

$A = 10.75$ ft.\ and $H = 18.5$ ft.\ (see below) so $x = \sqrt{A(A+H)} = \sqrt{314.4375} \approx 17.73$ feet from the back edge of the endzone. Unfortunately, that is still more than 12 feet into the endzone. Our mathematical analysis shows that to maximize the angle for kicking a field goal, the ball should be kicked from a position 12 feet into the endzone, a touchdown!



If the ball is placed on a hash mark just outside the endzone, then the kicking distance is 54 feet (10 yards for the depth of the endzone plus 8 yards that the ball is hiked) and the scoring angle is:

$$\theta = \arctan\left(\frac{29.25}{54}\right) - \arctan\left(\frac{10.75}{54}\right) \approx 17.2^\circ$$

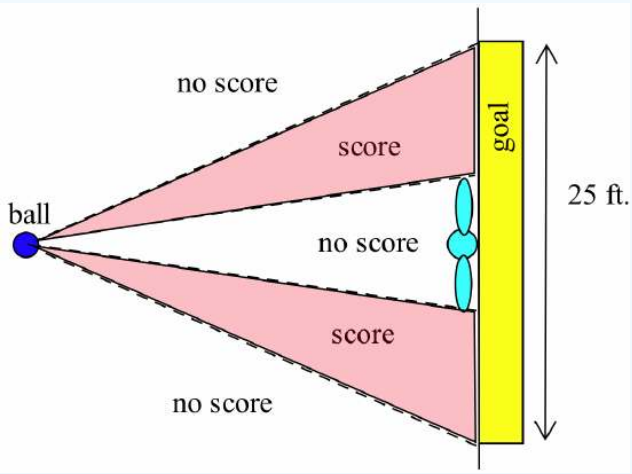
It is somewhat interesting to see investigate how the scoring angle:

$$\theta = \arctan\left(\frac{29.25}{x}\right) - \arctan\left(\frac{10.75}{x}\right)$$

changes with the distance x (in feet), and also to compare the scoring angle for balls placed on the hash mark with those placed in the center of the field.

Practice 7.4.3

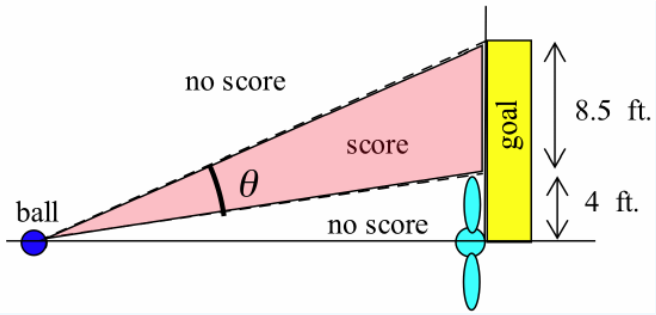
Kelcey is bringing a ball down the middle of a soccer field toward the 25-foot wide goal, which is defended by a goalie:



positioned in the center of the goal. The goalie can stop a shot that is within four feet of the center of the goal. At what distance from the goal should Kelcey shoot so the scoring angle is maximum?

Answer

$A = 4$ ft.\ and $H = 8.5$ ft.\ so $x = \sqrt{A(A+H)} = \sqrt{50} \approx 7.1$ ft.:



From 7.1 feet, the scoring angle on one side of the goalie is:

$$\theta = \arctan\left(\frac{12.5}{7.1}\right) - \arctan\left(\frac{4}{7.1}\right) \approx 31^\circ$$

For comparison, the table below gives values of the general scoring angle $\theta = \arctan\left(\frac{12.5}{x}\right) - \arctan\left(\frac{4}{x}\right)$ for some other distances x.

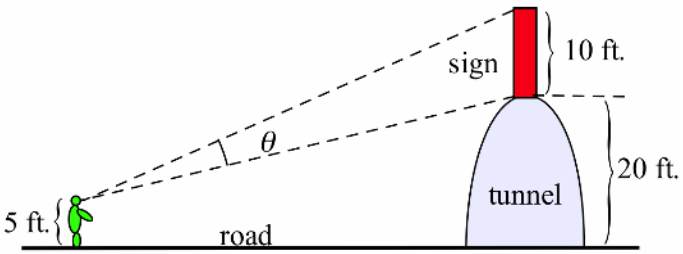
x	θ
5	29.5°
7	31.0°
10	29.5°
15	24.9°
20	20.7°
30	15.0°
40	11.6°

Problems

In Problems 1–21, compute the derivative.

1. $\mathrm{D}\left(\arcsin(3x)\right)$
2. $\mathrm{D}\left(\arctan(7x)\right)$
3. $\mathrm{D}\left(\arctan(x+5)\right)$
4. $\mathrm{D}\left(\arcsin\left(\frac{x}{2}\right)\right)$
5. $\mathrm{D}\left(\arctan\left(\sqrt{x}\right)\right)$
6. $\mathrm{D}\left(\operatorname{arcsec}\left(x^2\right)\right)$
7. $\mathrm{D}\left(\ln\left(\arctan(x)\right)\right)$
8. $\mathrm{D}\left(x \cdot \arctan\left(x\right)\right)$
9. $\mathrm{D}\left(\left(\operatorname{arcsec}(x)\right)^3\right)$
10. $\mathrm{D}\left(\arctan\left(\frac{5}{x}\right)\right)$
11. $\mathrm{D}\left(\arctan\left(\ln\left(x\right)\right)\right)$
12. $\mathrm{D}\left(\arcsin\left(x+2\right)\right)$
13. $\mathrm{D}\left(e^x \cdot \arctan\left(2x\right)\right)$
14. $\mathrm{D}\left(\tan(x) \cdot \arctan\left(x\right)\right)$
15. $\mathrm{D}\left(\arcsin(x) + \arccos(x)\right)$
16. $\mathrm{D}\left(\frac{1}{\arcsin(x)}\right)$
17. $\mathrm{D}\left(\sqrt{\arcsin(x)}\right)$

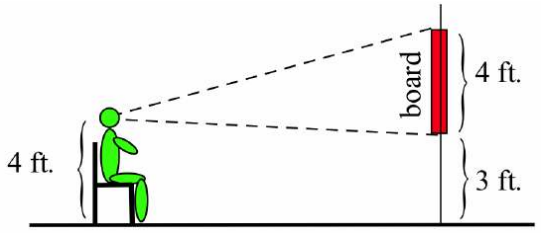
18. $\mathbf{D}\left((1+\operatorname{arcsec}(x))^3\right)$
19. $\mathbf{D}\left(\sin\left(3+\arctan(x)\right)\right)$
20. $\mathbf{D}\left(\dfrac{\arcsin(x)}{\arccos(x)}\right)$
21. $\mathbf{D}\left(x\cdot\arctan\left(\dfrac{1}{x}\right)\right)$
22. a. Use arctangents to describe the viewing angle for the sign in the figure below:



when the observer is x feet from the tunnel entrance.

b. At what distance x is the angle maximized?

23. a. Use arctangents to describe the viewing angle for the whiteboard in figure below:



when the student is x feet from the front wall.

b. At what distance x is the angle maximized?

24. Mimic the first part of the proof at the beginning of this section to show that (for all x):

$$\mathbf{D}\left(\operatorname{arccot}(x)\right)=\frac{-1}{1+x^2}$$

25. Mimic the second part of the proof at the beginning of this section to show that (for $|x|<1$):

$$\mathbf{D}\left(\arccos(x)\right)=\frac{-1}{\sqrt{1-x^2}}$$

26. Mimic the third part of the proof at the beginning of this section to show that (for $|x|>1$):

$$\mathbf{D}\left(\operatorname{arccsc}(x)\right)=\frac{-1}{|x|\cdot\sqrt{x^2-1}}$$

27. Revisit Problem 15 and draw a triangle to arrive at your answer without using any of the derivative patterns for inverse trigonometric functions developed in this section.

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7.5: Integrals Involving Inverse Trigonometric Functions

Aside from the Museum Problem and its sporting variations introduced in the previous section, the primary use of the inverse trigonometric functions in calculus involves their role as antiderivatives of rational and algebraic functions. Each of the six differentiation patterns from the previous section provides us with an integral formula, but they give rise to only three essentially different patterns:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \qquad \text{(valid for: } -1 < x < 1\text{)}$$
$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \qquad \text{(valid for: } -\infty < x < \infty\text{)}$$
$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C \qquad \text{(valid for: } |x| > 1\text{)}$$

Most of the related antiderivative patterns you will need in practice arise from variations of these basic ones. Typically, you need to transform an integrand so that it exactly matches one of the basic patterns. (Why are the derivative patterns for **arccos**, **arccot** and **arccsc** of little use to us when finding antiderivatives of algebraic functions?)

Example 7.5.1

Evaluate $\int \frac{1}{16+x^2}\,dx$.

Solution

We can transform this integrand into the arctangent pattern by factoring **16** from the denominator

$$\int \frac{1}{16+x^2}\,dx = \int \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,dx = \frac{1}{16}\int \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

and then using the substitution $u = \frac{x}{4} \;\Rightarrow du = \frac{1}{4}\,dx \;\Rightarrow\; 4\,du = dx$:

$$\frac{1}{16}\int \frac{1}{1+u^2}\cdot 4\,du = \frac{4}{16}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\arctan(u) + C$$

Replacing u with $\frac{x}{4}$ we get $\frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$ as our final answer.

Practice 7.5.1

Evaluate $\int \frac{1}{1+9x^2}\,dx$ and $\int \frac{1}{\sqrt{25-x^2}}\,dx$.

Answer

For the first integral, write:

$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$$

and substitute $u = 3x \;\Rightarrow du = 3\,dx \;\Rightarrow\; \frac{1}{3}\,du = dx$ to get:

$$\int \frac{1}{1+u^2}\cdot\frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan(3x) + C$$

For the second integral, factor out **25** to get:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{5\sqrt{1-\frac{x^2}{25}}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\,dx$$

and then substitute $u = \frac{x}{5} \;\Rightarrow\; du = \frac{1}{5}\,dx \;\Rightarrow\; 5\,du = dx$:

$$\frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\cdot 5\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u)+K$$

Replacing u with $\frac{x}{5}$ yields:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right)+K$$

The integrands that arise most often contain patterns with the forms a^2-x^2, a^2+x^2 and x^2-a^2, where a is some positive constant, so it is worthwhile to develop general integral patterns for these forms, list them in Appendix I, and refer to them when necessary:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C \qquad (\text{valid for: } -a<x<a)$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C \qquad (\text{valid for: } -\infty<x<\infty)$$

$$\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right)+C \qquad (\text{valid for: } |x|>a)$$

You can arrive at each of these general formulas by factoring the a^2 out of the denominator and making a suitable change of variable (as in Example 1, with a in place of 4). You can then check the result by differentiating. The **arctan** pattern is, by far, the most common. The **arcsin** pattern appears occasionally, and the **arcsec** pattern only rarely.

Example 7.5.2

Develop the general formula for $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ from the known formula for $\int \frac{1}{\sqrt{1-x^2}}\,dx$. (Assume that $a>0$.)

Solution

Using Example 1 as a guide, factor a^2 out of the denominator:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}\,dx = \frac{1}{a}\int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx$$

Now substitute $u=\frac{x}{a} \;\Rightarrow du=\frac{1}{a}\,dx \;\Rightarrow\; a\,du=dx$ to get:

$$\frac{1}{a}\int \frac{1}{\sqrt{1-u^2}}\cdot a\,du = \frac{a}{a}\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u)+C$$

and replace u with $\frac{x}{a}$ to get $\arcsin\left(\frac{x}{a}\right)+C$, the desired result.

Practice 7.5.2

Verify that the derivative of $\frac{1}{a}\cdot\arctan\left(\frac{x}{a}\right)$ is $\frac{1}{a^2+x^2}$.

Answer

Using the Chain Rule:

$$\mathrm{D}\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right] = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1}{a^2\left[1+\left(\frac{x}{a}\right)^2\right]} = \frac{1}{a^2+x^2}$$

Example 7.5.3

Evaluate $\int \frac{1}{\sqrt{5-x^2}}\,dx$ and $\int_1^3 \frac{1}{5+x^2}\,dx$.

Solution

The constant a needn't be an integer, so take $a^2 = 5 \Rightarrow a = \sqrt{5}$:

$$\int \frac{1}{\sqrt{5-x^2}}\,dx = \arcsin\left(\frac{x}{\sqrt{5}}\right) + C$$

using the pattern from Example 2, while the general **arctan** pattern yields:

$$\begin{aligned}\int_1^3 \frac{1}{5+x^2}\,dx &= \left[\frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)\right]_1^3 \\ &= \frac{1}{\sqrt{5}}\left[\arctan\left(\frac{3}{\sqrt{5}}\right) - \arctan\left(\frac{1}{\sqrt{5}}\right)\right]\end{aligned}$$

or about **0.228**.

The easiest way to integrate certain rational functions is to split the original integrand into two pieces.

Example 7.5.4

Evaluate $\int \frac{6x+7}{25+x^2}\,dx$.

Solution

The integrand splits nicely into the sum of two other functions that you can integrate more easily:

$$\int \frac{6x+7}{25+x^2}\,dx = \int \frac{6x}{25+x^2}\,dx + \int \frac{7}{25+x^2}\,dx$$

In the first integral, use the subsitution $u = 25 + x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$:

$$\int \frac{6x}{25+x^2}\,dx = \frac{6}{2}\int \frac{1}{u}\,du = 3\ln(|u|) + C_1 = 3\ln\left(25+x^2\right) + C_1$$

Meanwhile, the second integral matches the general arctangent pattern with $a = 5$:

$$\int \frac{7}{25+x^2}\,dx = 7\int \frac{1}{5^2+x^2}\,dx = 7\cdot\frac{1}{5}\arctan\left(\frac{x}{5}\right) + C_2$$

Combining these two results yields:

$$\int \frac{6x+7}{25+x^2}\,dx = \ln\left(25+x^2\right)^3 + \frac{7}{5}\arctan\left(\frac{x}{5}\right) + C$$

as our final answer. (The two constants C_1 and C_2 add up to another arbitrary constant, which we can simply call C.)

Why can we remove the absolute value signs in the last step?

The antiderivative of a linear function divided by an irreducible quadratic polynomial will typically result in the sum of a logarithm and an arctangent.

Practice 7.5.3

Evaluate $\int \frac{4x+3}{x^2+7}\,dx$.

Answer

Split the integrand into two pieces:

$$\int \frac{4x+3}{x^2+7}\,dx = \int \frac{4x}{x^2+7}\,dx + \int \frac{3}{x^2+7}\,dx$$

For the first integral, let $u = x^2+7 \;\Rightarrow\; du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$:

$$\int \frac{4x}{x^2+7}\,dx = \int \frac{2}{u}\,du = 2\ln(|u|) + C_1 = 2\ln\left(x^2+7\right) + C_1$$

The second integral matches the arctangent pattern with $a^2 = 7 \Rightarrow a = \sqrt{7}$:

$$\int \frac{3}{x^2+7}\,dx = 3\int \frac{1}{\left(\sqrt{7}\right)^2 + x^2}\,dx = 3\cdot\frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C_2$$

Combining these results yields:

$$\int \frac{4x+3}{x^2+7}\,dx = \ln\left(\left[x^2+7\right]^2\right) + \frac{3}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C \tag{7.5.1}$$

Problems

In Problems 1–24, evaluate the integral.

1. $\int \frac{7}{\sqrt{9-x^2}}\,dx$
2. $\int \frac{9}{\sqrt{7-y^2}}\,dy$
3. $\int_0^1 \frac{3}{x^2+25}\,dx$
4. $\int_5^7 \frac{5}{x\sqrt{x^2-16}}\,dx$
5. $\int \frac{9}{\sqrt{49-x^2}}\,dx$
6. $\int_1^4 \frac{2}{7+x^2}\,dx$
7. $\int_6^{10} \frac{3}{x\sqrt{x^2-25}}\,dx$
8. $\int \frac{7}{(x-5)^2+9}\,dx$
9. $\int \frac{1}{(x-1)^2+1}\,dx$
10. $\int \frac{1}{x^2-2x+2}\,dx$
11. $\int_{-1}^1 \frac{e^x}{1+e^{2x}}\,dx$
12. $\int_1^e \frac{1}{x}\cdot\frac{3}{1+[\ln(x)]^2}\,dx$
13. $\int \frac{\cos(\theta)}{\sqrt{9-\sin^2(\theta)}}\,d\theta$
14. $\int \frac{8x}{16+x^2}\,dx$
15. $\int \frac{3x}{\sqrt{9+x^2}}\,dx$
16. $\int \frac{3x}{\sqrt{9-x^2}}\,dx$
17. $\int \frac{6x}{9+x^4}\,dx$
18. $\int \frac{6x}{\sqrt{9-x^4}}\,dx$
19. $\int \frac{1}{1+4x^2}\,dx$
20. $\int \frac{1}{\sqrt{1-9x^2}}\,dx$

21. $\int_0^{\infty} \frac{1}{3+x^2}\,dx$
22. $\int_0^{2} \frac{1}{\sqrt{4-x^2}}\,dx$
23. $\int_0^{\sqrt{7}} \frac{1}{\sqrt{7-x^2}}\,dx$
24. $\int_0^{\infty} \frac{x}{1+x^4}\,dx$

In Problems 25–28, solve the initial value problem.

25. $\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}},\ y(0) = e$
26. $\frac{dy}{dx} = \frac{1}{y(1+x^2)},\ y(0) = 4$
27. $\frac{dy}{dx} = \frac{y^2}{9+x^2},\ y(1) = 2$
28. $\frac{dy}{dx} \cdot \sqrt{16-x^2} = y,\ y(4) = 1$

In Problems 29–32, evaluate the integral by splitting the integrand into two simpler functions.

29. $\int \frac{8x+5}{x^2+9}\,dx$
30. $\int \frac{1-4x}{x^2+1}\,dx$
31. $\int \frac{7x+3}{x^2+10}\,dx$
32. $\int \frac{x+5}{x^2+16}\,dx$

Problems 33–40 illustrate how we can sometimes decompose a difficult integral into simpler ones. (Hints: For 33, complete the square in the denominator; for 34, let $u =$ denominator; for 35, write $4x+20 = (4x+12)+8$.)

33. $\int \frac{8}{x^2+6x+10}\,dx$
34. $\int \frac{4x+12}{x^2+6x+10}\,dx$
35. $\int \frac{4x+20}{x^2+6x+10}\,dx$
36. $\int \frac{7}{x^2+4x+5}\,dx$
37. $\int \frac{12x+24}{x^2+4x+5}\,dx$
38. $\int \frac{12x+31}{x^2+4x+5}\,dx$
39. $\int \frac{6x+15}{x^2+4x+20}\,dx$
40. $\int \frac{2x+5}{x^2-4x+13}\,dx$

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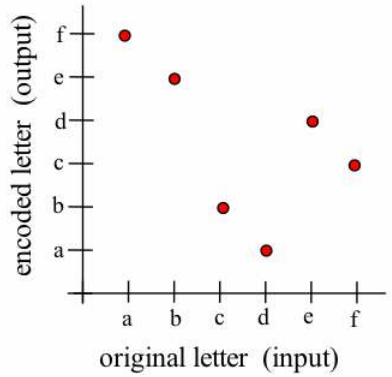
7.A: Answers

Section 7.1

1. $f(x)$ and $y = 3 - x$ are one-to-one because their graphs pass the horizontal line test (HLT); $g(x)$ is not one-to-one because $g(1) = g(4)$; h is not because its graph fails the HLT.
3. The graph of $f(x)$ fails the HLT; $y = e^x - 2$ is one-to-one because $y' = e^x > 0$ (see Problem 13); $g(x)$ is one-to-one because the output values are all distinct; the graph of h passes the HLT.
5. The relation is a function if the domain is all people with a Social Security number; ideally, it is a one-to-one function (but due to identity theft and other issues may not be so in practice).
7. No two students received the same score.
9. At most one.
11. a. Yes (graph passes HLT).
    b. No, $f(0) > f(1)$.
    c. No, $f(0) < f(0.5)$.
13. $f'(x) = \frac{1}{x} > 0$ for all $x > 0$. If $0 < a < b$ with $f(a) = f(b)$, then Rolle's Theorem guarantees a c with $a < c < b$ and $f'(c) = 0$, which is impossible. Hence $f(a) \neq f(b)$for any $a \neq b$ on $(0, \infty)$, which proves that $f(x)$ is one-to-one.
15. a. Yes (each input has exactly one output).
    b. Yes (each input appears only once).
    c. cde
    d. Interchanging inputs and outputs:

a	b	c	d	e	f
f	e	b	a	d	c

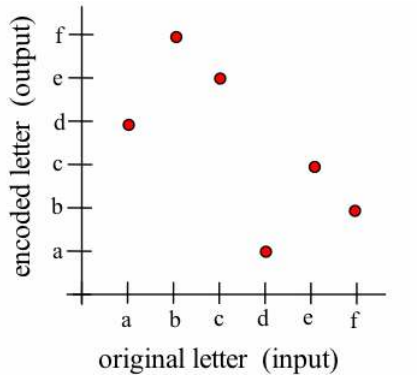
    e. Interchanging inputs and outputs:



    f. They are reflections of each other.
17. a. Yes (each output appears only once).
    b. Yes (each input appears only once).
    c. fda
    d. Interchanging inputs and outputs:

a	b	c	d	e	f
d	c	f	e	b	a

e. Interchanging inputs and outputs:



f. It's the same! (Function is its own inverse.)

Section 7.2

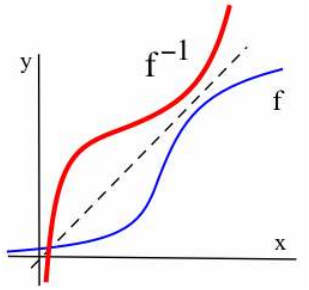
1.

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$(f^{-1})'(x)$
1	3	-3	2	$\frac{1}{2}$
2	1	2	3	$\frac{1}{3}$
3	2	3	1	$-\frac{1}{3}$

3.

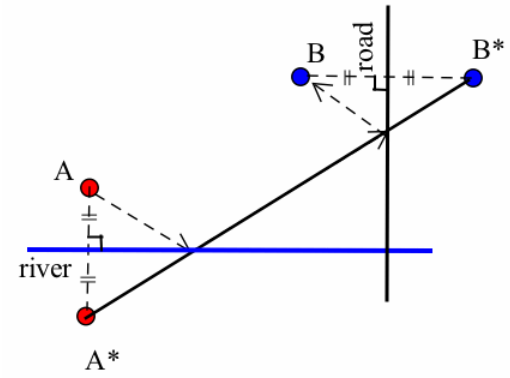
x	$h(x)$	$h'(x)$	$h^{-1}(x)$	$(h^{-1})'(x)$
1	2	2	3	undefined
2	3	-2	1	$\frac{1}{2}$
3	1	0	2	$-\frac{1}{2}$

5. The graphs of f and f^{-1} appear below:

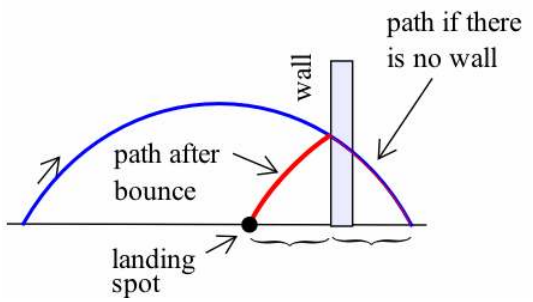


7. $a = b$
9. (1) Multiply by 4, (2) add 5 and (3) divide by 7: $f^{-1}(x) = \dfrac{4x+5}{7}$
11. $g^{-1}(x) = \frac{x-1}{2}$, hence $g^{-1}(g(1)) = \frac{3-1}{2} = 1$ and $g^{-1}(g(7)) = \frac{15-1}{2} = 7$.
13. $w^{-1}(x) = e^{x-5}$, so $w^{-1}(w(1)) = e^{5-5} = 1$.
15. The graph of f^{-1} goes through $(3,1)$ and $(f^{-1})'(3) = \dfrac{1}{f'(1)} > 0$.
17. $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} > 0$, so both f and f^{-1} are increasing functions.
19. $f(x) = x + 2 \Rightarrow f^{-1}(x) = x - 2$ both have slope 1 (hence are parallel) but different y-intercepts.
21. $f'(x) = 3 + \cos(x) \geq 2 > 0$, so $f(x)$ is increasing, hence one-to-one.

23. $x = \dfrac{ay+b}{cy-a} \Rightarrow cxy - ax = ay + b \Rightarrow (cx-a)y = ax + b \Rightarrow y = \dfrac{ax+b}{cx-a}$ so the function in (d) is its own inverse, and the functions in (a), (b) and (c) are all special cases of (d).
25. Fold across the river to generate point A^* (see figure below) then unfold; fold across the road to generate the point B^*, then unfold; connect A^* and B^* with a line; fold the line across the river and the road to get the shortest path:



27. Fold across the wall:

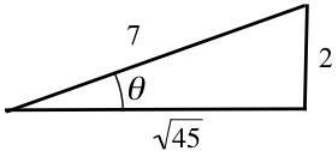


The distance from the wall to the landing spot is the same whether the ball bounces off the wall or passes through it.

Section 7.3

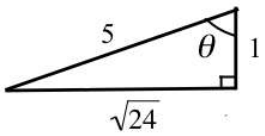
1. $\dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$
3. a. $\arcsin(0.3) < 1$ but $\pi - \arcsin(0.3) \approx 2.837$ and $\arcsin(0.3) + 2\pi \approx 6.588$
   b. $\arcsin(-0.4) < 1$ but $\pi - \arcsin(-0.4) \approx 3.553$ and $\arcsin(-0.4) + 2\pi \approx 5.872$
   c. $\dfrac{5\pi}{6}, \dfrac{13\pi}{6}$
5. a. $\arctan(3.2) \approx 1.268, \pi + 1.268 \approx 4.410$
   b. $\arctan(-0.2) + \pi \approx 2.944, \pi + 2.944 \approx 6.086$
7. a. $\dfrac{4}{5}$
   b. $\dfrac{4}{3}$
   c. $\dfrac{5}{3}$
   d. $\dfrac{3}{5}$
9. a. $\dfrac{5}{13}$
   b. $\dfrac{5}{12}$
   c. $\dfrac{13}{12}$
   d. $\dfrac{12}{13}$

11. The triangle below should help:



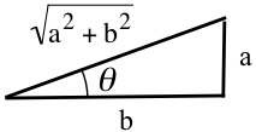
a. $\frac{2}{\sqrt{45}}$
b. $\frac{\sqrt{45}}{7}$
c. $\frac{7}{2}$
d. $\frac{\sqrt{45}}{2}$

13. The triangle below should help:



a. $\sqrt{24}$
b. $\frac{\sqrt{24}}{5}$
c. $\frac{5}{\sqrt{24}}$
d. $\frac{1}{\sqrt{24}}$

15. The triangle below should help:



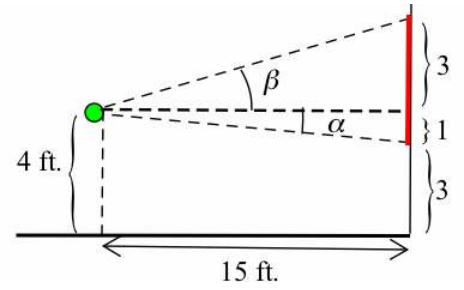
a. $\frac{a}{b}$
b. $\frac{a}{\sqrt{a^2+b^2}}$
c. $\frac{b}{\sqrt{a^2+b^2}}$
d. $\frac{b}{a}$

17. For $0 < \theta < \frac{\pi}{2}$, $\cos^2(\theta) + \sin^2(\theta) = 1 \Rightarrow \sin(\theta) = \sqrt{1-\cos^2(\theta)}$ and $\cos(\theta) = \sqrt{1-\sin^2(\theta)}$ so:

a. $\sin(\arccos(x)) = \sqrt{1-x^2}$
b. $\cos(\arcsin(x)) = \sqrt{1-x^2}$
c. $\sec(\arccos(x)) = \frac{1}{x}$

19. a. No: $2\arcsin(1) = 2 \cdot \frac{\pi}{2} = \pi$, but $\arcsin(2)$ is not defined.
b. No: $2\arccos(1) = 2 \cdot 0 = 0$, but $\arccos(2)$ is not defined.

21. Let α represent the angle of declination from the viewer to the bottom of the whiteboard, and β the angle of elevation to the top:

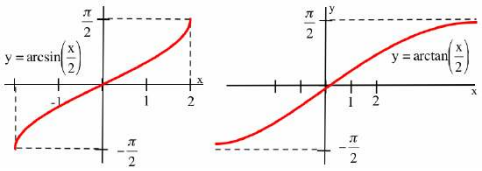


a. $\alpha+\beta=\arctan\left(\frac{1}{15}\right)+\arctan\left(\frac{3}{15}\right)\approx 0.264$, or about $15.12°$.

b. Replacing 15 with x:

$$\alpha+\beta=\arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{3}{x}\right)$$

23. The graphs appear below:



25. Differentiating implicitly, $\frac{dh}{dt}=20\cos(\theta)\cdot\frac{d\theta}{dt}$, so $\left.\frac{dh}{dt}\right|_{\theta=1.3}=20\cos(1.3)\cdot 12\approx 64.20$.

27. Differentiating implicitly, $-\sin(\theta)\cdot\frac{d\theta}{dt}=3\cdot\frac{dh}{dt}$, so $\left.\frac{dh}{dt}\right|_{\theta=1.3}=-\frac{1}{3}\sin(1.3)\cdot 12\approx -3.85$.

29. Differentiating implicitly, $\cos(\theta)\cdot\frac{d\theta}{dt}=\frac{1}{38}\cdot\frac{dh}{dt}$, so $\left.\frac{d\theta}{dt}\right|_{\theta=1.3}=\frac{4}{38}\sec(1.3)\approx 0.394$.

31. Differentiating implicitly, $-\sin(\theta)\cdot\frac{d\theta}{dt}=7\cdot\frac{dh}{dt}$, so $\left.\frac{d\theta}{dt}\right|_{\theta=1.3}=-28\csc(1.3)\cdot 12\approx -29.059$.

33. If h is the height of the rocket, then $\tan(\theta)=\frac{h}{4000}\Rightarrow h=4000\tan(\theta)\Rightarrow\frac{dh}{dt}=4000\sec^2(\theta)\cdot\frac{d\theta}{dt}$, so $\left.\frac{dh}{dt}\right|_{\theta=\frac{\pi}{3}}=4000\cdot 4\cdot\frac{\pi}{12}\approx 4189$ ft/sec.

35. a. $\alpha=\arcsin\left(\frac{A}{C}\right)$

b. $\beta=\arccos\left(\frac{A}{C}\right)$

c. $\arcsin\left(\frac{A}{C}\right)+\arccos\left(\frac{A}{C}\right)=\alpha+\beta=\frac{\pi}{2}$

37. a. $\alpha=\operatorname{arcsec}\left(\frac{C}{B}\right)$

b. $\beta=\operatorname{arccsc}\left(\frac{C}{B}\right)$

c. $\operatorname{arcsec}\left(\frac{C}{B}\right)+\operatorname{arccsc}\left(\frac{C}{B}\right)=\alpha+\beta=\frac{\pi}{2}$

39. a. $\frac{5}{12}$

b. $\frac{12}{5}$

41. a. $\frac{12}{13}$

b. $\frac{13}{12}$

43. 1.231

45. π

47. $\frac{2\pi}{3}$

49. $\frac{\pi}{4}$

51. 0.322

53. a. $\frac{7}{25}$

b. $\frac{24}{25}$

c. $\arcsin\left(\frac{7}{25}\right)=\arccos\left(\frac{24}{25}\right)$

55. If $x\neq 0$ then:

$$\sin\left(\operatorname{arccsc}(x)\right)=\frac{1}{\csc\left(\operatorname{arccsc}(x)\right)}=\frac{1}{x}$$

Applying $\arcsin$ to each side yields the result.

57. We know that $\tan\left(\operatorname{arcsec}(x)\right)=\sqrt{x^2-1}$ from Example 5, so $\operatorname{arcsec}(x)=\arctan\left(\sqrt{x^2-1}\right)$.

Section 7.4

1. $\dfrac{1}{\sqrt{1-(3x)^2}}\cdot 3 = \dfrac{3}{\sqrt{1-9x^2}}$
3. $\dfrac{1}{1+(x+5)^2} = \dfrac{1}{x^2+10x+26}$
5. $\dfrac{1}{1+\left(\sqrt{x}\right)^2}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}\,(1+x)}$
7. $\dfrac{1}{\arctan(x)}\cdot\dfrac{1}{1+x^2}$
9. $3\left(\operatorname{arcsec}(x)\right)^2\cdot\dfrac{1}{|x|\sqrt{x^2-1}}$
11. $\dfrac{1}{1+(\ln(x))^2}\cdot\dfrac{1}{x}$
13. $e^x\cdot\dfrac{2}{1+4x^2}+\arctan(2x)\cdot e^x$
15. $\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-x^2}} = 0$
17. $\dfrac{1}{2\sqrt{\arcsin(x)}}\cdot\dfrac{1}{\sqrt{1-x^2}}$
19. $\cos\left(3+\arctan(x)\right)\cdot\dfrac{1}{1+x^2}$
21. $\arctan\left(\dfrac{1}{x}\right)-\dfrac{x}{x^2+1}$
23. a. Use the result of Section 7.3 Problem 21:

$$\theta = \arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{3}{x}\right)$$

b. Differentiating yields:

$$\frac{d\theta}{dx} = \frac{-1}{x^2+1}-\frac{3}{x^2+9} = \frac{-4\left(x^2+3\right)}{(x^2+1)(x^2+9)} < 0$$

so there are no critical points. As $x\to\infty$, $\theta\to 0$ (a minimum), and as $x\to 0$, $\theta\to\pi$: the viewer's nose touches the whiteboard!

25. With $y=\arccos(x)$, $\cos(y)=x$ so that:

$$-\sin(y)\cdot\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin(y)}$$

The range of $\arccos(y)$ is $0\le y\le\pi$, and on this interval $\sin(y)\ge 0$, so:

$$\cos^2(y)+\sin^2(y)=1 \Rightarrow \sin(y)=\sqrt{1-\cos^2(y)} \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}}$$

27. For $0<x<\dfrac{\pi}{2}$: $\arccos(x)+\arcsin(x) = \dfrac{\pi}{2}$

Section 7.5

1. $7\cdot\displaystyle\int\frac{1}{\sqrt{3^2-x^2}}\,dx = 7\arcsin\left(\frac{x}{3}\right)+C$
3. $3\cdot\displaystyle\int_0^1\frac{1}{5^2+x^2}\,dx = \left[\frac{3}{5}\arctan\left(\frac{x}{5}\right)\right]_0^1 = \frac{3}{5}\arctan\left(\frac{1}{5}\right)\approx 0.1184$
5. $9\cdot\displaystyle\int\frac{1}{\sqrt{7^2-x^2}}\,dx = 9\arcsin\left(\frac{x}{7}\right)+C$
7. $3\cdot\displaystyle\int_6^{10}\frac{1}{x\sqrt{x^2-5^2}}\,dx = \left[\frac{3}{5}\operatorname{arcsec}\left(\frac{x}{5}\right)\right]_6^{10} = \frac{3}{5}\left[\operatorname{arcsec}(2)-\operatorname{arcsec}\left(\frac{6}{5}\right)\right]\approx 0.2769$
9. With $u=x-1\Rightarrow du=dx$ the integral becomes:

$$\int\frac{1}{1+u^2}\,du = \arctan(u)+C = \arctan(x-1)+C$$

11. With $u=e^x\Rightarrow du=e^x\,dx$ the integral becomes:

$$\int_{e^{-1}}^{e} \frac{1}{1+u^2}\,du = \arctan(e) - \arctan\left(e^{-1}\right) \approx 0.8658$$

13. With $u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$ this becomes:

$$\begin{aligned}\int \frac{1}{\sqrt{3^2-u^2}}\,du &= \arcsin\left(\frac{u}{3}\right) + C \\ &= \arcsin\left(\frac{\sin(\theta)}{3}\right) + C\end{aligned}$$

15. With $u = 9 + x^2 \Rightarrow du = 2x\,dx$ this becomes:

$$\frac{3}{2}\int u^{-\frac{1}{2}}\,du = 3\sqrt{u} + C = 3\sqrt{9+x^2} + C$$

17. With $u = x^2 \Rightarrow du = 2x\,dx$ the integral becomes:

$$3\int \frac{1}{3^2+u^2}\,du = \arctan\left(\frac{u}{3}\right) + C = \arctan\left(\frac{x^2}{3}\right) + C$$

19. With $u = 2x \Rightarrow du = 2\,dx$ the integral becomes:

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(2x) + C$$

21. This is an improper integral:

$$\lim_{M\to\infty}\int_0^M \frac{1}{\left(\sqrt{3}\right)^2+x^2}\,dx = \lim_{M\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^M = \lim_{M\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{M}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}$$

23. This is an improper integral:

$$\lim_{b\to\sqrt{7}^-}\int_0^b \frac{1}{\sqrt{\left(\sqrt{7}\right)^2-x^2}}\,dx = \lim_{b\to\sqrt{7}^-}\left[\arcsin\left(\frac{x}{\sqrt{7}}\right)\right]_0^b = \lim_{b\to\sqrt{7}^-}\arcsin\left(\frac{b}{\sqrt{7}}\right) = \arcsin(1) = \frac{\pi}{2}$$

25. Separating variables:

$$\int \frac{1}{y}\,dy = \int \frac{1}{\sqrt{1-x^2}}\,dx \Rightarrow \ln\left(|y|\right) = \arcsin(x) + C$$

Using $y(0) = e$:

$$\ln(e) = \arcsin(0) + C \Rightarrow 1 = 0 + C \Rightarrow C = 1$$

Solving for y:

$$\ln\left(|y|\right) = \arcsin(x) + 1 \Rightarrow |y| = e^{\arcsin(x)+1}$$

so $y = e^{\arcsin(x)+1}$ (because $y(0) = e > 0$).

27. Separating variables:

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{3^2+x^2}\,dx \Rightarrow -\frac{1}{y} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

Using $y(1) = 2$:

$$-\frac{1}{2} = \frac{1}{3}\arctan\left(\frac{1}{3}\right) + C \Rightarrow C = -\frac{1}{2} - \frac{1}{3}\arctan\left(\frac{1}{3}\right)$$

Solving for y yields:

$$y = \frac{-1}{\frac{1}{3}\arctan\left(\frac{x}{3}\right) - \frac{1}{2} - \frac{1}{3}\arctan\left(\frac{1}{3}\right)}$$

29. Rewrite the integral as:

$$\int \frac{4 \cdot 2x}{3^2 + x^2}\, dx + \int \frac{5}{3^2 + x^2}\, dx$$

and integrate each term separately to get:

$$4 \ln\left(x^2 + 9\right) + \frac{5}{3} \arctan\left(\frac{x}{3}\right) + C$$

31. Rewrite the integral as:

$$\frac{7}{2} \int \frac{2x}{\left(\sqrt{10}\right)^2 + x^2}\, dx + 3 \int \frac{1}{\left(\sqrt{10}\right)^2 + x^2}\, dx$$

and integrate each term separately to get:

$$\frac{7}{2} \ln\left(x^2 + 10\right) + \frac{3}{\sqrt{10}} \arctan\left(\frac{x}{\sqrt{10}}\right) + C$$

33. Completing the square in the denominator, the integral becomes:

$$8 \int \frac{1}{1 + (x+3)^2}\, dx = 8 \arctan(x+3) + C$$

35. Rewrite the integral as:

$$\int \frac{2(2x+6)}{x^2 + 6x + 10}\, dx + \int \frac{8}{1 + (x+3)^2}\, dx$$

and integrate each term separately to get:

$$2 \ln(x^2 + 6x + 10) + 8 \arctan(x+3) + C$$

37. Use $u = x^2 + 4x + 5 \Rightarrow du = (2x+4)\, dx$:

$$\int \frac{6(2x+4)}{x^2 + 4x + 5}\, dx = 6 \int \frac{1}{u}\, du = 6 \ln\left(|u|\right) + C$$

and resubstitute to get $6 \ln(x^2 + 4x + 5) + C$

39. Rewrite the integral as:

$$\int \frac{3(2x+4)}{x^2 + 4x + 20}\, dx + \int \frac{3}{4^2 + (x+2)^2}\, dx$$

and integrate each term separately to get:

$$3 \ln(x^2 + 4x + 20) + \frac{3}{4} \arctan\left(\frac{x+2}{4}\right) + C$$

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Index

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  - 1: Limits and Continuity - *CC BY-NC-SA 4.0*
    - 1.0: Tangent Lines, Velocities and Graphs - *CC BY-NC-SA 4.0*
    - 1.1: The Limit of a Function - *CC BY-NC-SA 4.0*
    - 1.2: Properties of Limits - *CC BY-NC-SA 4.0*
    - 1.3: Continuous Functions - *CC BY-NC-SA 4.0*
    - 1.4: Definition of Limit - *CC BY-NC-SA 4.0*
    - 1.A: Answers - *CC BY-NC-SA 4.0*
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    - 2.0: Introduction to Derivatives - *CC BY-NC-SA 4.0*
    - 2.1: The Definition of Derivative - *CC BY-NC-SA 4.0*
    - 2.2: Derivatives: Properties and Formulas - *CC BY-NC-SA 4.0*
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    - 2.4: The Chain Rule - *CC BY-NC-SA 4.0*
    - 2.5: Applications of the Chain Rule - *CC BY-NC-SA 4.0*
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    - 2.7: Newton's Method - *CC BY-NC-SA 4.0*
    - 2.8: Linear Approximation and Differentials - *CC BY-NC-SA 4.0*
    - 2.9: Implicit and Logarithmic Differentiation - *CC BY-NC-SA 4.0*
    - 2.A: Answers - *CC BY-NC-SA 4.0*
  - 3: Derivatives and Graphs - *CC BY-NC-SA 4.0*
    - 3.1: Finding Maximums and Minimums - *CC BY-NC-SA 4.0*
    - 3.2: Mean Value Theorem - *CC BY-NC-SA 4.0*
    - 3.3: The First Derivative and the Shape of f - *CC BY-NC-SA 4.0*
    - 3.4: The Second Derivative and the Shape of f - *CC BY-NC-SA 4.0*
    - 3.5: Applied Maximum and Minimum Problems - *CC BY-NC-SA 4.0*
    - 3.6: Asymptotic Behavior of Functions - *CC BY-NC-SA 4.0*
    - 3.7: L'Hôpital's Rule - *CC BY-NC-SA 4.0*
    - 3.A: Answers - *CC BY-NC-SA 4.0*
  - 4: The Integral - *CC BY-NC-SA 4.0*
    - 4.0: Area - *CC BY-NC-SA 4.0*
    - 4.1: Sigma Notation and Riemann Sums - *CC BY-NC-SA 4.0*
    - 4.2: The Definite Integral - *CC BY-NC-SA 4.0*
    - 4.3: Properties of the Definite Integral - *CC BY-NC-SA 4.0*
    - 4.4: Areas, Integrals and Antiderivatives - *CC BY-NC-SA 4.0*
    - 4.5: The Fundamental Theorem of Calculus - *CC BY-NC-SA 4.0*
    - 4.6: Finding Antiderivatives - *CC BY-NC-SA 4.0*
    - 4.7: First Applications of Definite Integrals - *CC BY-NC-SA 4.0*
    - 4.8: Using Tables [and Technology] to Find Antiderivatives - *CC BY-NC-SA 4.0*
    - 4.9: Approximating Definite Integrals - *CC BY-NC-SA 4.0*

- 4.A: Answers - *CC BY-NC-SA 4.0*
- 5: Applications of Definite Integrals - *CC BY-NC-SA 4.0*
  - 5.1: Volumes by Slicing - *CC BY-NC-SA 4.0*
  - 5.2: Disks and Washers - *CC BY-NC-SA 4.0*
  - 5.3: Arclength and Surface Area - *CC BY-NC-SA 4.0*
  - 5.4: More Work - *CC BY-NC-SA 4.0*
  - 5.5: Volumes: Tubes - *CC BY-NC-SA 4.0*
  - 5.6: Moments and Centers of Mass - *CC BY-NC-SA 4.0*
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  - 5.8: Additional Applications - *CC BY-NC-SA 4.0*
  - 5.A: Answers - *CC BY-NC-SA 4.0*
- 6: Differential Equations - *CC BY-NC-SA 4.0*
  - 6.1: Introduction to Differential Equations - *CC BY-NC-SA 4.0*
  - 6.2: The Differential Equation y = f'(x) - *CC BY-NC-SA 4.0*
  - 6.3: Separable Equations - *CC BY-NC-SA 4.0*
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  - 6.5: Heating, Cooling and Mixing - *CC BY-NC-SA 4.0*
  - 6.A: Answers - *CC BY-NC-SA 4.0*
- 7: Transcendental Functions - *CC BY-NC-SA 4.0*
  - 7.1: One-to-One Functions - *CC BY-NC-SA 4.0*
  - 7.2: Inverse Functions - *CC BY-NC-SA 4.0*
  - 7.3: Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
  - 7.4: Derivatives of Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
  - 7.5: Integrals Involving Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
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  - 8.5: Integrals of Trigonometric Functions - *CC BY-NC-SA 4.0*
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  - 9.1: Sequences - *Undeclared*
  - 9.2: Limits of Sequences - *Undeclared*
  - 9.3: Infinite Series - *Undeclared*
  - 9.4: Geometric and Harmonic Series - *Undeclared*
  - 9.5: An Interlude - *Undeclared*
  - 9.6: Integral Test and P-Test - *Undeclared*
  - 9.7: Comparison Tests - *Undeclared*
  - 9.8: Alternating Series - *Undeclared*
  - 9.9: New Page - *Undeclared*
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16: New Page - *Undeclared*

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- 17.8: New Page - *Undeclared*
- 17.9: New Page - *Undeclared*

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  - Front Matter - *CC BY-NC-SA 4.0*
    - TitlePage - *CC BY-NC-SA 4.0*
    - InfoPage - *CC BY-NC-SA 4.0*
    - Table of Contents - *CC BY-NC-SA 4.0*
    - Licensing - *CC BY-NC-SA 4.0*
  - 1: Limits and Continuity - *CC BY-NC-SA 4.0*
    - 1.0: Tangent Lines, Velocities and Graphs - *CC BY-NC-SA 4.0*
    - 1.1: The Limit of a Function - *CC BY-NC-SA 4.0*
    - 1.2: Properties of Limits - *CC BY-NC-SA 4.0*
    - 1.3: Continuous Functions - *CC BY-NC-SA 4.0*
    - 1.4: Definition of Limit - *CC BY-NC-SA 4.0*
    - 1.A: Answers - *CC BY-NC-SA 4.0*
  - 2: The Derivative - *CC BY-NC-SA 4.0*
    - 2.0: Introduction to Derivatives - *CC BY-NC-SA 4.0*
    - 2.1: The Definition of Derivative - *CC BY-NC-SA 4.0*
    - 2.2: Derivatives: Properties and Formulas - *CC BY-NC-SA 4.0*
    - 2.3: More Differentiation Patterns - *CC BY-NC-SA 4.0*
    - 2.4: The Chain Rule - *CC BY-NC-SA 4.0*
    - 2.5: Applications of the Chain Rule - *CC BY-NC-SA 4.0*
    - 2.6: Related Rates - *CC BY-NC-SA 4.0*
    - 2.7: Newton's Method - *CC BY-NC-SA 4.0*
    - 2.8: Linear Approximation and Differentials - *CC BY-NC-SA 4.0*
    - 2.9: Implicit and Logarithmic Differentiation - *CC BY-NC-SA 4.0*
    - 2.A: Answers - *CC BY-NC-SA 4.0*
  - 3: Derivatives and Graphs - *CC BY-NC-SA 4.0*
    - 3.1: Finding Maximums and Minimums - *CC BY-NC-SA 4.0*
    - 3.2: Mean Value Theorem - *CC BY-NC-SA 4.0*
    - 3.3: The First Derivative and the Shape of f - *CC BY-NC-SA 4.0*
    - 3.4: The Second Derivative and the Shape of f - *CC BY-NC-SA 4.0*
    - 3.5: Applied Maximum and Minimum Problems - *CC BY-NC-SA 4.0*
    - 3.6: Asymptotic Behavior of Functions - *CC BY-NC-SA 4.0*
    - 3.7: L'Hôpital's Rule - *CC BY-NC-SA 4.0*
    - 3.A: Answers - *CC BY-NC-SA 4.0*
  - 4: The Integral - *CC BY-NC-SA 4.0*
    - 4.0: Area - *CC BY-NC-SA 4.0*
    - 4.1: Sigma Notation and Riemann Sums - *CC BY-NC-SA 4.0*
    - 4.2: The Definite Integral - *CC BY-NC-SA 4.0*
    - 4.3: Properties of the Definite Integral - *CC BY-NC-SA 4.0*
    - 4.4: Areas, Integrals and Antiderivatives - *CC BY-NC-SA 4.0*
    - 4.5: The Fundamental Theorem of Calculus - *CC BY-NC-SA 4.0*
    - 4.6: Finding Antiderivatives - *CC BY-NC-SA 4.0*
    - 4.7: First Applications of Definite Integrals - *CC BY-NC-SA 4.0*
    - 4.8: Using Tables [and Technology] to Find Antiderivatives - *CC BY-NC-SA 4.0*
    - 4.9: Approximating Definite Integrals - *CC BY-NC-SA 4.0*

- 4.A: Answers - *CC BY-NC-SA 4.0*
- 5: Applications of Definite Integrals - *CC BY-NC-SA 4.0*
  - 5.1: Volumes by Slicing - *CC BY-NC-SA 4.0*
  - 5.2: Disks and Washers - *CC BY-NC-SA 4.0*
  - 5.3: Arclength and Surface Area - *CC BY-NC-SA 4.0*
  - 5.4: More Work - *CC BY-NC-SA 4.0*
  - 5.5: Volumes: Tubes - *CC BY-NC-SA 4.0*
  - 5.6: Moments and Centers of Mass - *CC BY-NC-SA 4.0*
  - 5.7: Improper Integrals - *CC BY-NC-SA 4.0*
  - 5.8: Additional Applications - *CC BY-NC-SA 4.0*
  - 5.A: Answers - *CC BY-NC-SA 4.0*
- 6: Differential Equations - *CC BY-NC-SA 4.0*
  - 6.1: Introduction to Differential Equations - *CC BY-NC-SA 4.0*
  - 6.2: The Differential Equation y = f'(x) - *CC BY-NC-SA 4.0*
  - 6.3: Separable Equations - *CC BY-NC-SA 4.0*
  - 6.4: Exponential Growth and Decay - *CC BY-NC-SA 4.0*
  - 6.5: Heating, Cooling and Mixing - *CC BY-NC-SA 4.0*
  - 6.A: Answers - *CC BY-NC-SA 4.0*
- 7: Transcendental Functions - *CC BY-NC-SA 4.0*
  - 7.1: One-to-One Functions - *CC BY-NC-SA 4.0*
  - 7.2: Inverse Functions - *CC BY-NC-SA 4.0*
  - 7.3: Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
  - 7.4: Derivatives of Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
  - 7.5: Integrals Involving Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
  - 7.A: Answers - *CC BY-NC-SA 4.0*
- 8: Integration Techniques - *CC BY-NC-SA 4.0*
  - 8.1: Finding Antiderivatives: A Review - *CC BY-NC-SA 4.0*
  - 8.2: Integration by Parts - *CC BY-NC-SA 4.0*
  - 8.3: Partial Fraction Decomposition - *CC BY-NC-SA 4.0*
  - 8.4: Trigonometric Substitution - *CC BY-NC-SA 4.0*
  - 8.5: Integrals of Trigonometric Functions - *CC BY-NC-SA 4.0*
  - 8.6: Integration Tactics - *CC BY-NC-SA 4.0*
  - 8.7: Approximating Antiderivatives - *CC BY-NC-SA 4.0*
  - 8.A: Answers - *CC BY-NC-SA 4.0*
- 9: Sequences and Series - *Undeclared*
  - 9.1: Sequences - *Undeclared*
  - 9.2: Limits of Sequences - *Undeclared*
  - 9.3: Infinite Series - *Undeclared*
  - 9.4: Geometric and Harmonic Series - *Undeclared*
  - 9.5: An Interlude - *Undeclared*
  - 9.6: Integral Test and P-Test - *Undeclared*
  - 9.7: Comparison Tests - *Undeclared*
  - 9.8: Alternating Series - *Undeclared*
  - 9.9: New Page - *Undeclared*
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- 17.9: New Page - *Undeclared*

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