

POLYNOMIAL INTERPOLATIONS ON TIME SCALES



Svetlin G. Georgiev
Sorbonne University

Polynomial Interpolations on Time Scales

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (<https://LibreTexts.org>) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s)

before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by [NICE](#) [CXone](#)

[Expert](#) and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptations contact info@LibreTexts.org or visit our main website at <https://LibreTexts.org>.

This text was compiled on
03/29/2026

TABLE OF CONTENTS

[Licensing](#)

[1: Chapters](#)

- [1.0: Lagrange Interpolation](#)
- [1.1: Sigma-Lagrange Interpolation](#)
- [1.2: Hermite Interpolation](#)
- [1.3: Sigma-Hermite Interpolation](#)
- [1.4: Delta Differentiation](#)

[Index](#)

[Glossary](#)

[Detailed Licensing](#)

[Detailed Licensing](#)

LICENSING

A detailed breakdown of this resource's [Matter/Detailed Licensing](#).
licensing can be found in [Back](#)

CHAPTER OVERVIEW

1: Chapters

[1.0: Lagrange Interpolation](#)

[1.1: Sigma-Lagrange Interpolation](#)

[1.2: Hermite Interpolation](#)

[1.3: Sigma-Hermite Interpolation](#)

[1.4: Delta Differentiation](#)

Thumbnail: This image shows, for four points $((-9, 5), (-4, 2), (-1, -2), (7, 9))$, the (cubic) interpolation polynomial $L(x)$ (dashed, black), which is the sum of the scaled basis polynomials $y_1 \ell_1(x)$, $y_2 \ell_2(x)$, $y_3 \ell_3(x)$ and $y_4 \ell_4(x)$. (CC BY-SA 4.0; Glosser.ca via [Wikipedia](#))

1: Chapters is shared under a [not declared](#) license and was authored, remixed, and/or curated by LibreTexts.

Let $\mathbf{T}$ be a time scale with forward jump operator and delta differentiation operator σ and Δ , respectively.

In this section we construct the Lagrange interpolation polynomial on an arbitrary time scale. We present the theoretical background of this construction and solve numerical examples. We assume that $\mathbf{T}$ is a given time scale with forward jump operator σ delta derivative operator Δ and graininess function μ .

Let $\mathcal{P}_n$, $n \in \mathbb{N}_0$, denote the set of all polynomials of degree $\leq n$ defined over the set $\mathbb{R}$ of real numbers. Let $n \in \mathbb{N}$ and $x_i \in \mathbf{T}$, $i \in \{0, 1, \dots, n\}$, be distinct and y_i , $i \in \{0, 1, \dots, n\}$, be given real numbers. We will find $p_n \in \mathcal{P}_n$ such that $p_n(x_i) = y_i$, $i \in \{0, 1, \dots, n\}$. Below we introduce the form of a polynomial taking the given values y_i at the points x_i for $i \in \{0, 1, \dots, n\}$.

THEOREM

Suppose that $n \in \mathbb{N}$. Then there exist polynomials $L_k \in \mathcal{P}_n$, $k \in \{0, 1, \dots, n\}$, such that

$$L_k(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases}$$

$i, k \in \{0, 1, \dots, n\}$. Moreover,

$$p_n(x) = \sum_{k=0}^n L_k(x) y_k, \quad x \in \mathbf{T},$$

satisfies the condition $p_n(x_i) = y_i$, $i \in \{0, 1, \dots, n\}$, $p_n \in \mathcal{P}_n$.

Proof

Define

$$L_k(x) = C_k \sum_{i=0, i \neq k}^n (x - x_i), \quad x \in \mathbf{T},$$

where $C_k \in \mathbb{R}$, $k \in \{0, 1, \dots, n\}$, will be determined below. We have $L_k(x_i) = 0$, $i \in \{0, 1, \dots, n\}$, $i \neq k$, and

$$\begin{aligned} L_k(x_k) &= C_k \prod_{i=0, i \neq k}^n (x_k - x_i) \\ &= 1, \quad k \in \{0, 1, \dots, n\}. \end{aligned}$$

Thus,

$$C_k = \frac{1}{\prod_{i=0, i \neq k}^n (x_k - x_i)}, \quad k \in \{0, 1, \dots, n\},$$

and

$$L_k(x) = \prod_{i=0, i \neq k}^n \frac{x - x_i}{x_k - x_i}, \quad x \in \mathbf{T}, \quad k \in \{0, 1, \dots, n\}.$$

We have that $L_k \in \mathcal{P}_n$, $k \in \{0, 1, \dots, n\}$, and $p_n \in \mathcal{P}_n$. This completes the proof.

The uniqueness of the polynomial given in the previous theorem is proved next.

THEOREM

Assume that $n \in \mathbb{N}_0$. Let $x_i \in \mathbf{T}$, $i \in \{0, 1, \dots, n\}$, be distinct and $y_i \in \mathbb{R}$, $i \in \{0, 1, \dots, n\}$. Then there exists a unique polynomial $p_n \in \mathcal{P}_n$ such that

$$p_n(x_i) = y_i, \quad i \in \{0, 1, \dots, n\}.$$

Proof

The existence of the polynomial p_n follows by the previous theorem. Suppose that there exist two polynomials $p_n, q_n \in \mathcal{P}_n$ such that

$$p_n(x_i) = q_n(x_i) = y_i, \quad i \in \{0, 1, \dots, n\}.$$

Then the polynomial $h_n = p_n - q_n$ has $n + 1$ distinct roots. Therefore $h_n \equiv 0$ or $p_n \equiv q_n$. This completes the proof.

Now, we define formally the polynomial in the above theorems.

DEFINITION:

Assume that $n \in \mathbb{N}_0$. Let $x_i \in \mathbf{T}$, $i \in \{0, 1, \dots, n\}$, be distinct and $y_i \in \mathbb{R}$, $i \in \{0, 1, \dots, n\}$. The polynomial

$$p_n(x) = \sum_{k=0}^n L_k(x) y_k, \quad x \in \mathbf{T},$$

where L_k , $k \in \{0, 1, \dots, n\}$, are defined with (1.0.1), will be called the Lagrange interpolation polynomial of degree n with interpolation points x_i , $i \in \{0, 1, \dots, n\}$.

DEFINITION:

Assume that $n \in \mathbb{N}_0$. Let $x_i \in [a, b] \subset \mathbf{T}$, $i \in \{0, 1, \dots, n\}$, be distinct and $f: [a, b] \rightarrow \mathbb{R}$ be a given function. The polynomial

$$p_n(x) = \sum_{k=0}^n L_k(x) f(x_k), \quad x \in \mathbf{T},$$

where L_k , $k \in \{0, 1, \dots, n\}$, are defined with (1.0.1), will be called the Lagrange interpolation polynomial of degree n with interpolation points x_i , $i \in \{0, 1, \dots, n\}$, for the function f .

EXAMPLE

Let $\mathbf{T} = \mathbb{Z}$. We will construct the Lagrange interpolation polynomial for the set

$$\{(-3, 0), (-2, 2), (0, 1), (1, 0)\}.$$

Here

$$\begin{aligned} x_0 &= -3, & x_1 &= -2, & x_2 &= 0, & x_3 &= 1, \\ y_0 &= 0, & y_1 &= 2, & y_2 &= 1, & y_3 &= 1, & n &= 3. \end{aligned}$$

Then

$$\begin{aligned} L_0(x) &= \prod_{i=1}^3 \frac{x - x_i}{x_0 - x_i} \\ &= \frac{(x + 2)x(x - 1)}{-1 \cdot (-3) \cdot (-4)} \\ &= -\frac{(x - 1)x(x + 2)}{12}, \end{aligned}$$

$$\begin{aligned} L_1(x) &= \prod_{i=0, i \neq 1}^3 \frac{x - x_i}{x_1 - x_i} \\ &= \frac{(x + 3)x(x - 1)}{1 \cdot (-2) \cdot (-3)} \\ &= \frac{(x - 1)x(x + 3)}{6}, \end{aligned}$$

$$\begin{aligned} L_2(x) &= \prod_{i=0, i \neq 2}^3 \frac{x - x_i}{x_2 - x_i} \\ &= \frac{(x + 3)(x + 2)(x - 1)}{3 \cdot 2 \cdot (-1)} \\ &= -\frac{(x - 1)(x + 2)(x + 3)}{6}, \end{aligned}$$

$$\begin{aligned} L_3(x) &= \prod_{i=0}^3 \frac{x - x_i}{x_3 - x_i} \\ &= \frac{(x + 3)(x + 2)x}{4 \cdot 3 \cdot 1} \\ &= \frac{x(x + 2)(x + 3)}{12}, \quad x \in \mathbf{T}. \end{aligned}$$

Hence,

$$\begin{aligned} p_3(x) &= y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) \\ &= 2 \left(\frac{x(x - 1)(x + 3)}{6} \right) - \frac{(x - 1)(x + 2)(x + 3)}{6} + \frac{x(x + 2)(x + 3)}{12} \\ &= \frac{(x - 2)(x - 1)(x + 3)}{6} + \frac{x(x + 2)(x + 3)}{12} \\ &= \frac{(x + 3)(2(x - 1)(x - 2) + x(x + 2))}{12} \\ &= \frac{(x + 3)(3x^2 - 4x + 4)}{12}, \end{aligned}$$

EXERCISE

Let $\mathbf{T} = 2^{\mathbb{N}_0}$. Construct the Lagrange interpolation polynomial for the set

$$\{(1, -1), (4, 0), (8, 1), (16, 2)\}.$$

Suppose that $n \in \mathbb{N}_0$ and $x_j \in \mathbf{T}$, $j \in \{0, 1, \dots, n\}$, are distinct points. For $x \in \mathbf{T}$ we define the polynomials

$\pi_{n+1}(x) = \prod_{j=0}^n (x - x_j)$, $\Pi_{n+1}^k(x) = \pi_{n+1}^{\Delta^k}(x)$, $k \in \mathbb{N}_0$. The following theorem gives the error in approximating a function f by a Lagrange polynomial.

EXAMPLE

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $x_0 = 1$, $x_1 = 2$. Here

$$n = 1, \quad \sigma(x) = 2x, \quad x \in \mathbb{T}.$$

Then

$$\begin{aligned} \pi_2(x) &= (x-1)(x-2) \\ &= x^2 - 3x + 2, \quad x \in \mathbb{T}, \end{aligned}$$

and

$$\begin{aligned} \Pi_2^1(x) &= \pi_2^{\Delta}(x) \\ &= \sigma(x) + x - 3 \\ &= 3x - 3, \\ \Pi_2^2(x) &= \pi_2^{\Delta^2}(x) \\ &= 3, \quad x \in \mathbb{T}. \end{aligned}$$

EXAMPLE

Let $\mathbb{T} = \mathbb{Z}$ and

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3.$$

Then

$$\begin{aligned} \pi_3(x) &= (x-1)(x-2)(x-3) \\ &= (x^2 - 3x + 2)(x-3) \\ &= x^3 - 6x^2 + 11x - 6, \quad x \in \mathbb{T}. \end{aligned}$$

We have

$$\sigma(x) = x + 1, \quad x \in \mathbb{T},$$

and

$$\begin{aligned} \Pi_3^1(x) &= (\sigma(x))^2 + x\sigma(x) + x^2 - 6(\sigma(x) + x) + 11 \\ &= (x+1)^2 + x(x+1) + x^2 - 6(x+1) + 11 \\ &= x^2 + 3x + 1, \\ \Pi_3^2(x) &= 3x^2 - 9x + 6, \\ \Pi_3^3(x) &= 3(\sigma(x) + x) - 9 \\ &= 6x - 6, \\ \Pi_3^4(x) &= 6, \end{aligned}$$

EXERCISE

Let $\mathbb{T} = (\frac{1}{4})^{\mathbb{N}_0}$,

$$x_0 = \frac{1}{64}, \quad x_1 = \frac{1}{16}, \quad x_2 = \frac{1}{4}, \quad x_3 = 1.$$

Find $\pi_4(x)$, $\Pi_4^1(x)$, $\Pi_4^2(x)$, $\Pi_4^3(x)$, $\Pi_4^4(x)$, $x \in \mathbb{T}$.

THEOREM

Suppose that $n \in \mathbb{N}_0$, $a, b \in \mathbb{T}$, $a < b$, $x_j \in [a, b]$, $j \in \{0, 1, \dots, n\}$, are distinct and $f : [a, b] \rightarrow \mathbb{R}$, $f^{\Delta^k}(x)$ exist for any $x \in [a, b]$ and for any $k \in \{1, \dots, n+1\}$. Then for any $x \in [a, b]$ there exists $\xi = \xi(x) \in (a, b)$ such that

$$f(x) - p_n(x) = \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{\Delta^{n+1}}(\xi)} \pi_{n+1}(x), \quad x \in [a, b],$$

or

$$F_{\min, n+1}(\xi) \leq \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \leq F_{\max, n+1}(\xi), \quad x \in [a, b],$$

where

$$F_{\max, n+1}(\xi) = \max \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{\Delta^{n+1}}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{n+1}^{\Delta^{n+1}}(\rho(\xi))} \right\},$$

$$F_{\min, n+1}(\xi) = \min \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{\Delta^{n+1}}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{n+1}^{\Delta^{n+1}}(\rho(\xi))} \right\}.$$

Proof

Let p_n be the Lagrange interpolation polynomial for the function f with interpolation points x_j , $j \in \{0, 1, \dots, n\}$. Define the function

$$\phi(t) = f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t), \quad t \in [a, b].$$

Then

$$\phi(x_j) = f(x_j) - p_n(x_j) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(x_j) = f(x_j) - f(x_j) = 0, \quad j \in \{0, 1, \dots, n\}.$$

Thus, $\phi : [a, b] \rightarrow \mathbb{R}$ has at least $n+1$ generalized zeros (GZs). Hence and the Rolle theorem, it follows that $\phi^{\Delta^{n+1}}$ has at least one GZ on (a, b) . Therefore there exists $\xi = \xi(x) \in (a, b)$ such that

$$\phi^{\Delta^{n+1}}(\xi) = 0 \quad \text{or} \quad \phi^{\Delta^{n+1}}(\rho(\xi)) \phi^{\Delta^{n+1}}(\xi) < 0.$$

Note that

$$\phi^{\Delta^{n+1}}(t) = f^{\Delta^{n+1}}(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}^{\Delta^{n+1}}(t), \quad t \in [a, b].$$

1. Let $\phi^{\Delta^{n+1}}(\xi) = 0$. Then

$$f^{\Delta^{n+1}}(\xi) = \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}^{\Delta^{n+1}}(\xi)$$

or

$$\begin{aligned} f(x) - p_n(x) &= \frac{f^{\Delta^{n+1}}(\xi)}{\pi_{n+1}^{\Delta^{n+1}}(\xi)} \pi_{n+1}(x) \\ &= \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{\Delta^{n+1}}(\xi)} \pi_{n+1}(x). \end{aligned}$$

2. Let

$$\phi^{\Delta^{n+1}}(\rho(\xi)) \phi^{\Delta^{n+1}}(\xi) < 0.$$

Then

$$\begin{aligned} \phi^{\Delta^{n+1}}(\rho(\xi)) &= f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}^{\Delta^{n+1}}(\rho(\xi)) \\ &= f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \Pi_{n+1}^{\Delta^{n+1}}(\rho(\xi)). \end{aligned}$$

and

$$\phi^{\Delta^{n+1}}(\xi) = f^{\Delta^{n+1}}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \Pi_{n+1}^{\Delta^{n+1}}(\xi).$$

Hence,

$$\begin{aligned} 0 &> \phi^{\Delta^{n+1}}(\rho(\xi)) \phi^{\Delta^{n+1}}(\xi) \\ &= \left(f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \Pi_{n+1}^{\Delta^{n+1}}(\rho(\xi)) \right) \left(f^{\Delta^{n+1}}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \Pi_{n+1}^{\Delta^{n+1}}(\xi) \right) \\ &= \left(\frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \right)^2 \Pi_{n+1}^{\Delta^{n+1}}(\rho(\xi)) \Pi_{n+1}^{\Delta^{n+1}}(\xi) \\ &\quad - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \left(\Pi_{n+1}^{\Delta^{n+1}}(\rho(\xi)) f^{\Delta^{n+1}}(\xi) + \Pi_{n+1}^{\Delta^{n+1}}(\xi) f^{\Delta^{n+1}}(\rho(\xi)) \right). \end{aligned}$$

Hence,

$$F_{\min, n+1}(\xi) \leq \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \leq F_{\max, n+1}(\xi).$$

This completes the proof.

Note that as stated in the next remark, the error vanishes if the number of data points increases to infinity.

NOTE

Suppose that all conditions of the previous theorem hold. If

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{\Delta^{n+1}}(\xi)} \pi_{n+1}(x) \right) = 0$$

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{n+1}^{\Delta^{n+1}}(\rho(\xi))} \pi_{n+1}(x) \right) = 0,$$

then

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} |f(x) - p_n(x)| = 0.$$

EXAMPLE

Let $\mathbb{T} = \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4\right\}$. Let

$$a = x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = 1, \quad x_3 = 3, \quad b = 4,$$

and

$$f(x) = \frac{x+1}{x^2+x+6}, \quad x \in \mathbb{T}.$$

We will construct the Lagrange interpolation polynomial p_3 of f for the points x_0, x_1, x_2, x_3 and compare the graphs of

p_3 and f .

First, note that

$$\begin{aligned} f(x_0) &= f(0) = \frac{1}{6}, \\ f(x_1) &= f\left(\frac{1}{4}\right) = \frac{20}{101}, \\ f(x_2) &= f(1) = \frac{1}{4}, \\ f(x_3) &= f(3) = \frac{2}{9}. \end{aligned}$$

Also, we compute

$$\begin{aligned} L_0(x) &= \frac{(x - \frac{1}{4})(x - 1)(x - 3)}{(0 - \frac{1}{4})(0 - 1)(0 - 3)} \\ &= -\frac{4}{3}(x - \frac{1}{4})(x - 1)(x - 3), \\ L_1(x) &= \frac{(x - 0)(x - 1)(x - 3)}{(\frac{1}{4} - 0)(\frac{1}{4} - 1)(\frac{1}{4} - 3)} \\ &= \frac{64}{33}x(x - 1)(x - 3), \\ L_2(x) &= \frac{(x - 0)(x - \frac{1}{4})(x - 3)}{(1 - 0)(1 - \frac{1}{4})(1 - 3)} \\ &= -\frac{2}{3}x(x - \frac{1}{4})(x - 3), \\ L_3(x) &= \frac{(x - 0)(x - \frac{1}{4})(x - 1)}{(3 - 0)(3 - \frac{1}{4})(3 - 1)} \\ &= \frac{2}{33}x(x - \frac{1}{4})(x - 1), \quad x \in \mathbb{T}. \end{aligned}$$

Then the Lagrange interpolation polynomial p_3 of f for the points $0, \frac{1}{4}, 1, 3$ is

$$\begin{aligned} p_3(x) &= f(0)L_0(x) + f\left(\frac{1}{4}\right)L_1(x) + f(1)L_2(x) + f(3)L_3(x) \\ &= \frac{1}{6} \left(-\frac{4}{3}(x - \frac{1}{4})(x - 1)(x - 3) \right) \\ &+ \frac{20}{101} \left(\frac{64}{33}x(x - 1)(x - 3) \right) \\ &+ \frac{1}{4} \left(-\frac{2}{3}x(x - \frac{1}{4})(x - 3) \right) \\ &+ \frac{2}{9} \left(\frac{2}{33}x(x - \frac{1}{4})(x - 1) \right) \\ &= -\frac{2}{9}(x - \frac{1}{4})(x - 1)(x - 3) \\ &+ \frac{1280}{3333}x(x - 1)(x - 3) \\ &- \frac{1}{6}x(x - \frac{1}{4})(x - 3) \\ &+ \frac{4}{297}x(x - \frac{1}{4})(x - 1), \quad x \in \mathbb{T}. \end{aligned}$$

1.1: Sigma-Lagrange Interpolation

In this section we will show that a given function can be approximated with so called σ -Lagrange polynomials. We will show that there are classes of time scales for which the Lagrange interpolation polynomials and the σ -Lagrange interpolation polynomials are different and there are classes of time scales for which the Lagrange interpolation polynomials and the σ -Lagrange interpolation polynomials coincide.

In the following, we define the σ -polynomials. With $\mathcal{P}_n^\sigma$, $n \in \mathbb{N}_0$, we will denote the set of all functions in the form

$$a_n(\sigma(x))^n + a_{n-1}(\sigma(x))^{n-1} + \dots + a_1(x)\sigma(x) + a_0, \quad x \in \mathbb{T},$$

where $a_j \in \mathbb{R}$, $j \in \{0, 1, \dots, n\}$. Let $a, b \in \mathbb{T}$, $a < b$. A function $g \in \mathcal{P}_n^\sigma$ will be called a σ -polynomial.

To define a σ -Lagrange polynomial we have a requirement that the points in the data set should be σ -distinct.

Definition:

Let $n \in \mathbb{N}_0$. The points $x_j \in [a, b]$, $j \in \{0, 1, \dots, n\}$, will be called σ -distinct if $\sigma(x_n) \leq b$ and

$$\sigma(x_0) < \sigma(x_1) < \dots < \sigma(x_n).$$

Example

Let $\mathbb{T} = \{-1, 1\} \cup \{1 + (\frac{1}{2})^n : n \in \mathbb{N}_0\} \cup \{3, 4, 5\}$ and $a = -1$, $b = 5$. Take the points

$$x_0 = -1, \quad x_1 = 1, \quad x_2 = 3.$$

Then

$$\sigma(x_0) = 1, \quad \sigma(x_1) = 1, \quad \sigma(x_2) = 4.$$

Thus, the points $\{x_0, x_1, x_2\}$ are not σ -distinct.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $a = 1$, $b = 16$. Take the points

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 4.$$

Then

$$\sigma(x_0) = 2, \quad \sigma(x_1) = 4, \quad \sigma(x_2) = 8.$$

Therefore $\{x_0, x_1, x_2\}$ are σ -distinct points.

As in the case of Lagrange interpolation, one can prove the following result.

Theorem

Suppose that $n \in \mathbb{N}$ and $x_j \in \mathbb{T}$, $j \in \{0, 1, \dots, n\}$, are σ -distinct. Then there exist unique σ -polynomials $L_{\sigma k} \in \mathcal{P}_n^\sigma$, $k \in \{0, 1, \dots, n\}$, such that

$$L_{\sigma k}(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases}$$

$i, k \in \{0, 1, \dots, n\}$. Moreover,

$$p_{\sigma n}(x) = \sum_{k=0}^n L_{\sigma k}(x) y_k$$

$$= \sum_{k=0}^n \left(\prod_{j=0, j \neq k}^n \frac{\sigma(x) - \sigma(x_j)}{\sigma(x_k) - \sigma(x_j)} \right) y_k, \quad x \in \mathbb{T}$$

satisfies the condition $p_{\sigma n}(x_i) = y_i$, $i \in \{0, 1, \dots, n\}$, $p_{\sigma n} \in \mathcal{P}_n^\sigma$.

Based on the statement of the above theorem we define the σ -Lagrange polynomials as follows.

Definition:

Assume that $n \in \mathbb{N}_0$. Let $x_i \in \mathbb{T}$, $i \in \{0, 1, \dots, n\}$, be σ -distinct and $y_i \in \mathbb{R}$, $i \in \{0, 1, \dots, n\}$. The σ -polynomial

$$p_{\sigma n}(x) = \sum_{k=0}^n L_{\sigma k}(x) y_k, \quad x \in \mathbb{T}$$

will be called the σ -Lagrange interpolation polynomial of degree n with σ -interpolation points x_i , $i \in \{0, 1, \dots, n\}$.

Definition:

Assume that $n \in \mathbb{N}_0$. Let $x_i \in [a, b] \subset \mathbb{T}$, $i \in \{0, 1, \dots, n\}$, be σ -distinct and $f : [a, b] \rightarrow \mathbb{R}$ be a given function. The σ -polynomial

$$p_{\sigma n}(x) = \sum_{k=0}^n L_{\sigma k}(x) f(x_k), \quad x \in \mathbb{T},$$

will be called the σ -Lagrange interpolation polynomial of degree n with σ -interpolation points x_i , $i \in \{0, 1, \dots, n\}$, for the function f .

In the following, we will compute the Lagrange and the σ -Lagrange polynomials for a given set of data on a time scale and compare them.

Example

Let $\mathbb{T} = \{-2, -1, 0, 3, 7\}$,

$$a = -2, \quad b = 7, \quad x_0 = -2, \quad x_1 = 0,$$

$f : \mathbb{T} \rightarrow \mathbb{R}$ is defined by

$$f(t) = t + 3, \quad t \in \mathbb{T}.$$

We will find the σ -Lagrange interpolation polynomial for the function f with σ -interpolation points $\{x_0, x_1\}$. We have

$$\begin{aligned} \sigma(x_0) &= \sigma(-2) = -1, \\ \sigma(x_1) &= \sigma(0) = 3, \\ L_{\sigma 0}(x) &= \frac{\sigma(x) - \sigma(x_1)}{\sigma(x_0) - \sigma(x_1)} \\ &= -\frac{1}{4}(\sigma(x) - 3), \\ L_{\sigma 1}(x) &= \frac{\sigma(x) - \sigma(x_0)}{\sigma(x_1) - \sigma(x_0)} \\ &= \frac{1}{4}(\sigma(x) + 1), \quad x \in \mathbb{T}, \\ f(x_0) &= f(-2) \\ &= 1, \\ &= 3. \end{aligned}$$

Thus,

$$\begin{aligned} p_{\sigma 1}(x) &= f(x_0)L_{\sigma 0}(x) + f(x_1)L_{\sigma 1}(x) \\ &= -\frac{1}{4}(\sigma(x) - 3) + \frac{3}{4}(\sigma(x) + 1) \\ &= \frac{1}{2}(\sigma(x) + 3), \quad x \in [-2, 7]. \end{aligned}$$

Note that,

$$\begin{aligned} p_{\sigma 1}(0) &= \frac{1}{2}(\sigma(0) + 3) \\ &= 3 = f(0), \\ p_{\sigma 1}(-2) &= \frac{1}{2}(\sigma(-2) + 3) \\ &= 1 = f(-2). \end{aligned}$$

Also, we have

$$\begin{aligned} p_{\sigma 1}(-1) &= \frac{1}{2}(\sigma(-1) + 3) \\ &= \frac{3}{2}. \end{aligned}$$

Now, we will find the Lagrange interpolation polynomial for the function f with interpolation points $\{x_0, x_1\}$. We have

$$\begin{aligned} L_0(x) &= \frac{x - x_1}{x_0 - x_1} \\ &= -\frac{1}{2}x, \\ L_1(x) &= \frac{x - x_0}{x_1 - x_0} \\ &= \frac{x + 2}{2}, \quad x \in [-2, 7]. \end{aligned}$$

Therefore

$$\begin{aligned} p_1(x) &= f(x_0)L_0(x) + f(x_1)L_1(x) \\ &= -\frac{1}{2}x + 3\frac{x + 2}{2} \\ &= x + 3, \quad x \in [-2, 7]. \end{aligned}$$

Then

$$p_1(0) = 3 = f(0), \quad p_1(-2) = 1 = f(-2).$$

We also have

$$p_1(-1) = -1 + 3 = 2 \neq p_{\sigma 1}(-1) = \frac{3}{2}.$$

Note

In the above example we see that in general, the Lagrange interpolation polynomial and the σ -Lagrange interpolation polynomial for a function f are different.

Exercise

Let $\mathbf{T} = \{-1, -\frac{1}{4}, -\frac{1}{8}, 0, 2, 3, 7\}$.

$$a = x_0 = -1, \quad b = 7, \quad x_1 = -\frac{1}{8}, \quad x_2 = 3.$$

Find the σ -Lagrange and Lagrange interpolation polynomials for the function $f: \mathbf{T} \rightarrow \mathbb{R}$ defined by

$$f(t) = \frac{t+1}{t^2-t+1} + 3t, \quad t \in [-1, 7],$$

with σ -interpolation and interpolation points x_0, x_1, x_2 .

In the following example we see that on some time scales, the Lagrange and σ -Lagrange polynomials are the same.

Example

Let $\mathbf{T} = \mathbb{Z}$,

$$a = x_0 = -1, \quad x_1 = 1, \quad x_2 = 3,$$

$f: \mathbf{T} \rightarrow \mathbb{R}$ is defined by

$$f(t) = t^2 + t + 1, \quad t \in \mathbf{T}.$$

We will find the σ -Lagrange interpolation polynomial with σ -interpolation points x_0, x_1, x_2 . We have

$$\sigma(x_0) = \sigma(-1) = 0,$$

$$\sigma(x_1) = \sigma(1) = 2,$$

$$\sigma(x_2) = \sigma(3) = 4.$$

Then

$$\sigma(x_0) < \sigma(x_1) < \sigma(x_2),$$

i.e., x_0, x_1, x_2 are σ -distinct points. We have

$$L_{\sigma 0}(x) = \frac{(\sigma(x) - \sigma(x_1))(\sigma(x) - \sigma(x_2))}{(\sigma(x_0) - \sigma(x_1))(\sigma(x_0) - \sigma(x_2))}$$

$$= \frac{(x+1-2)(x+1-4)}{(0-2)(0-4)}$$

$$= \frac{(x-1)(x-3)}{8},$$

$$L_{\sigma 1}(x) = \frac{(\sigma(x) - \sigma(x_0))(\sigma(x) - \sigma(x_2))}{(\sigma(x_1) - \sigma(x_0))(\sigma(x_1) - \sigma(x_2))}$$

$$= \frac{(x+1-0)(x+1-4)}{(2-0)(2-4)}$$

$$= \frac{(x-3)(x+1)}{4},$$

$$L_{\sigma 2}(x) = \frac{(\sigma(x) - \sigma(x_0))(\sigma(x) - \sigma(x_1))}{(\sigma(x_2) - \sigma(x_0))(\sigma(x_2) - \sigma(x_1))}$$

$$= \frac{(x+1-0)(x+1-2)}{(4-0)(4-2)}$$

$$= \frac{(x-1)(x+1)}{8}, \quad x \in [-1, 4],$$

$$f(x_0) = f(-1)$$

$$= 1,$$

$$f(x_1) = f(1)$$

$$= 3,$$

$$f(x_2) = f(3)$$

$$= 13.$$

Hence,

$$p_{\sigma 2}(x) = f(x_0)L_{\sigma 0}(x) + f(x_1)L_{\sigma 1}(x) + f(x_2)L_{\sigma 2}(x)$$

$$= \frac{(x-1)(x-3)}{8} - 3\frac{(x-3)(x+1)}{4} + 13\frac{(x-1)(x+1)}{8}$$

$$= \frac{x^2 - 4x + 3 - 6x^2 + 12x + 13x^2 - 13x + 13}{8}$$

$$= x^2 + x + 1.$$

is the σ -Lagrange interpolation polynomial for the function f .

Now, we will find the Lagrange interpolation polynomial for the function f with interpolation points x_0, x_1, x_2 . We have

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{(x-1)(x-3)}{8},$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$= \frac{(x+1)(x-3)}{4},$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{(x+1)(x-1)}{8}, \quad x \in [-1, 4].$$

Hence,

$$p_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$= \frac{(x-1)(x-3)}{8} - 3\frac{(x+1)(x-3)}{4} + 13\frac{(x-1)(x+1)}{8}$$

$$= \frac{x^2 - 4x + 3 - 6x^2 + 12x + 13x^2 - 13x + 13}{8}$$

$$= x^2 + x + 1.$$

Moreover,

$$p_2(x) = p_{\sigma 2}(x) = f(x), \quad x \in [-1, 4].$$

Note

In the above example we see that there are cases for which the Lagrange interpolation polynomial and the σ -Lagrange interpolation polynomial for a function f coincide.

Exercise

Let $\mathbf{T} = 2^{\mathbb{N}_0}$,

$$a = 1 = x_0, \quad x_1 = 2, \quad x_2 = 8, \quad x_3 = 32 = b,$$

$f: \mathbf{T} \rightarrow \mathbb{R}$ is defined by

$$f(t) = t^3 + t^2 - t + 4, \quad t \in \mathbf{T}.$$

Find the σ -Lagrange interpolation polynomial for the function f with σ -interpolation points x_0, x_1, x_2 .

Now, we will describe the classes of time scales for which the σ -Lagrange and Lagrange interpolation polynomials coincide.

Theorem

Let $\mathbf{T}$ be a time scale such that $\sigma(t) = ct + d$ for any $t \in \mathbf{T}$ and some constants c, d . Let also, $n \in \mathbb{N}$ and

$$a = x_0 < x_1 < \dots < x_n = b, \quad x_j \in \mathbf{T}, \quad j \in \{0, 1, \dots, n\},$$

are σ -interpolation and interpolation points. Then

$$L_k(x) = L_{\sigma k}(x), \quad x \in [a, b], \quad k \in \{0, 1, \dots, n\}.$$

Proof

Since $\sigma(t) = ct + d$ for any $t \in \mathbf{T}$, we get

$$L_{\sigma k}(x) = \prod_{j=0, j \neq k}^n \frac{\sigma(x) - \sigma(x_j)}{\sigma(x_k) - \sigma(x_j)}$$

$$= \prod_{j=0, j \neq k}^n \frac{(cx + d) - (cx_j + d)}{(cx_k + d) - (cx_j + d)}$$

$$= \prod_{j=0, j \neq k}^n \frac{x - x_j}{x_k - x_j}$$

$$= L_k(x), \quad k \in \{0, 1, \dots, n\}, \quad x \in [a, b].$$

This completes the proof.

Note

It is clear that the uniqueness of interpolating polynomial of degree n is not violated. Indeed, if σ is not a linear function, the σ -Lagrange interpolation polynomial of a function f is not a polynomial of degree n or not a polynomial at all. For instance, if $\mathbf{T} = \mathbf{N}_0^2$, then $\sigma(t)(\sqrt{t}+1)^2$, $t \in \mathbf{T}$, and the σ -Lagrange interpolation polynomial is not a polynomial. Therefore, for any function f , defined on a time scale $\mathbf{T}$, there is a unique interpolation polynomial of degree n .

Suppose that $n \in \mathbf{N}_0$ and $x_j \in \mathbf{T}$, $j \in \{0, 1, \dots, n\}$, are σ -distinct points. For $x \in \mathbf{T}$, define the σ -polynomials

$$\pi_{\sigma n+1}(x) = \prod_{j=0}^n (\sigma(x) - \sigma(x_j)), \quad \Pi_{\sigma n+1}^k(x) = \pi_{\sigma n+1}^k(x), \quad x \in \mathbf{T}, \quad k \in \underline{\mathbf{N}}_0.$$

These functions are employed in the error estimate for the σ -Lagrange interpolation.

Example

Let $\mathbf{T} = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1 + \frac{1}{2^n} : n \in \mathbf{N}_0\}$,

$$a = 0, \quad b = 1, \quad x_0 = 0, \quad x_1 = \frac{1}{3}.$$

We will compute $\pi_{\sigma 2}(x)$, $\pi_{\sigma 2}(\frac{1}{2})$ and $\Pi_{\sigma 2}^1(\frac{1}{2})$.

We have

$$\sigma(x_0) = \frac{1}{6},$$

$$\sigma(x_1) = \frac{1}{2},$$

$$\sigma\left(\frac{1}{2}\right) = 1,$$

$$\sigma(1) = 1.$$

Hence,

$$\pi_{\sigma 2}(x) =$$

$$=$$

$$\pi_{\sigma 2}\left(\frac{1}{2}\right) =$$

$$=$$

$$=$$

$$\Pi_{\sigma 2}^1\left(\frac{1}{2}\right) =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$(\sigma(x) - \sigma(x_0))(\sigma(x) - \sigma(x_1))$ proof repeats the corresponding proof for the Lagrange Polynomials. This completes the proof.

$$\left(\sigma\left(\frac{1}{2}\right) - \frac{1}{6}\right)\left(\sigma\left(\frac{1}{2}\right) - \frac{1}{2}\right)$$

$$\left(\sigma\left(\frac{1}{2}\right) - \frac{1}{6}\right)\left(\sigma\left(\frac{1}{2}\right) - \frac{1}{2}\right)$$

$$\left(1 - \frac{1}{6}\right)\left(1 - \frac{1}{2}\right)$$

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{\sigma n+1}^1(\xi)} \pi_{\sigma n+1}(x) \right) = 0$$

and

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{\sigma n+1}^1(\rho(\xi))} \pi_{\sigma n+1}(x) \right) = 0,$$

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} |f(x) - p_{\sigma n}(x)| = 0.$$

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} |f(x) - p_{\sigma n}(x)| = 0.$$

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} |f(x) - p_{\sigma n}(x)| = 0.$$

The last example shows clearly the difference of Lagrange and σ -Lagrange polynomials on a time scale whose forward jump operator is not a linear function.

0.

Exercise

Let $\mathbf{T} = \{0, 2, 3, 5, 9, 16, 18\}$,

$$a = 0, \quad b = 18, \quad x_0 = 0, \quad x_1 = 3, \quad x_2 = 9.$$

Find $\Pi_{\sigma 2}^2(3)$.

The error in the σ -Lagrange interpolation is given in the next theorem.

Theorem

Suppose that $n \in \mathbf{N}_0$, $a, b \in \mathbf{T}$, $a < b$, $x_j \in [a, b]$, $j \in \{0, 1, \dots, n\}$, are σ -distinct and $f : [a, b] \rightarrow \mathbb{R}$, $f^{\Delta^k}(x)$ exist for any $x \in [a, b]$ and for any $k \in \{1, \dots, n+1\}$. Then for any $x \in [a, b]$ there exists $\xi = \xi(x) \in (a, b)$ such that

$$f(x) - p_{\sigma n}(x) = \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{\sigma n+1}^1(\xi)} \pi_{\sigma n+1}(x), \quad x \in [a, b],$$

or

$$F_{\sigma \min, n+1}(\xi) \leq \frac{f(x) - p_{\sigma n}(x)}{\pi_{\sigma n+1}(x)} \leq F_{\sigma \max, n+1}(\xi), \quad x \in [a, b],$$

where

$$F_{\sigma \max, n+1}(\xi) = \max \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{\sigma n+1}^1(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{\sigma n+1}^1(\rho(\xi))} \right\},$$

$$F_{\sigma \min, n+1}(\xi) = \min \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{\sigma n+1}^1(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{\sigma n+1}^1(\rho(\xi))} \right\}.$$

Proof

Let $p_{\sigma n}$ be the σ -Lagrange interpolation polynomial for the function f with σ -interpolation points x_j , $j \in \{0, 1, \dots, n\}$. Define the function

$$\phi(t) = f(t) - p_{\sigma n}(t) - \frac{f(x) - p_{\sigma n}(x)}{\pi_{\sigma n+1}(x)} \pi_{\sigma n+1}(t), \quad t \in [a, b].$$

Example

\begin{example}

Let $\mathbf{T} = \{\sqrt{2n+1}, n \in \mathbf{N}_0\} = \{1, \sqrt{3}, \sqrt{5}, \dots\}$.

Take

$$a = x_0 = \sqrt{3}, \quad x_1 = \sqrt{7}, \quad x_2 = \sqrt{11}, \quad x_3 = \sqrt{17},$$

and

$$f(x) = \frac{x^2 + 1}{2x^2 + 5}, \quad x \in \mathbf{T}.$$

We will write the Lagrange interpolation polynomial p_3 and the σ -Lagrange interpolation polynomial $p_{\sigma 3}$ interpolating the function f at the points x_0, x_1, x_2, x_3 . We will also compare the graphs of the two types interpolating polynomials with the function f . Notice that on the given time scale we have

$$\sigma(x) = \sqrt{x^2 + 2}, \quad x \in \mathbf{T}.$$

Also

$$f(x_0) = f(\sqrt{3}) = \frac{3+1}{6+5} = \frac{4}{11},$$

$$f(x_1) = f(\sqrt{7}) = \frac{7+1}{14+5} = \frac{8}{19},$$

$$f(x_2) = f(\sqrt{11}) = \frac{11+1}{22+5} = \frac{4}{9},$$

$$f(x_3) = f(\sqrt{17}) = \frac{17+1}{34+5} = \frac{6}{13}.$$

First, we compute $L_0(x)$, $L_1(x)$, $L_2(x)$ and $L_3(x)$ as follows,

$$\begin{aligned}
 L_0(x) &= \frac{(x-\sqrt{7})(x-\sqrt{11})(x-\sqrt{17})}{(\sqrt{3}-\sqrt{7})(\sqrt{3}-\sqrt{11})(\sqrt{3}-\sqrt{17})}, & p_{\sigma 3}(x) &= f(\sqrt{3})L_{\sigma 0}(x) + f(\sqrt{7})L_{\sigma 1}(x) + f(\sqrt{11})L_{\sigma 2}(x) + f(\sqrt{17})L_{\sigma 3}(x) \\
 L_1(x) &= \frac{(x-\sqrt{3})(x-\sqrt{11})(x-\sqrt{17})}{(\sqrt{7}-\sqrt{3})(\sqrt{7}-\sqrt{11})(\sqrt{7}-\sqrt{17})}, & &= \frac{4}{11} \frac{(\sqrt{x^2+2}-3)(\sqrt{x^2+2}-\sqrt{13})(\sqrt{x^2+2}-\sqrt{19})}{(\sqrt{5}-3)(\sqrt{5}-\sqrt{13})(\sqrt{5}-\sqrt{19})} \\
 L_2(x) &= \frac{(x-\sqrt{3})(x-\sqrt{7})(x-\sqrt{17})}{(\sqrt{11}-\sqrt{3})(\sqrt{11}-\sqrt{7})(\sqrt{11}-\sqrt{17})}, & &+ \frac{8}{19} \frac{(\sqrt{x^2+2}-\sqrt{5})(\sqrt{x^2+2}-\sqrt{13})(\sqrt{x^2+2}-\sqrt{19})}{(3-\sqrt{5})(3-\sqrt{13})(3-\sqrt{19})} \\
 L_3(x) &= \frac{(x-\sqrt{3})(x-\sqrt{7})(x-\sqrt{11})}{(\sqrt{17}-\sqrt{3})(\sqrt{17}-\sqrt{7})(\sqrt{17}-\sqrt{11})}, & x \in \mathbb{T}. &+ \frac{4}{9} \frac{(\sqrt{x^2+2}-\sqrt{5})(\sqrt{x^2+2}-3)(\sqrt{x^2+2}-\sqrt{19})}{(\sqrt{13}-\sqrt{5})(\sqrt{13}-3)(\sqrt{13}-\sqrt{19})} \\
 & & &+ \frac{6}{13} \frac{(\sqrt{x^2+2}-\sqrt{5})(\sqrt{x^2+2}-3)(\sqrt{x^2+2}-\sqrt{13})}{(\sqrt{19}-\sqrt{5})(\sqrt{19}-3)(\sqrt{19}-\sqrt{13})}, \quad x \in \mathbb{T}.
 \end{aligned}$$

Then the Lagrange interpolation polynomial p_3 interpolating the function f at the points $\sqrt{3}, \sqrt{7}, \sqrt{11}, \sqrt{17}$ is obtained as

It is clear that $p_3(x) \neq p_{\sigma 3}(x)$, $x \in \mathbb{T}$. Moreover, $p_{\sigma 3}$ is not a polynomial.

$$\begin{aligned}
 p_3(x) &= f(\sqrt{3})L_0(x) + f(\sqrt{7})L_1(x) + f(\sqrt{11})L_2(x) + f(\sqrt{17})L_3(x) \\
 &= \frac{4}{11} \frac{(x-\sqrt{7})(x-\sqrt{11})(x-\sqrt{17})}{(\sqrt{3}-\sqrt{7})(\sqrt{3}-\sqrt{11})(\sqrt{3}-\sqrt{17})} \\
 &+ \frac{8}{19} \frac{(x-\sqrt{3})(x-\sqrt{11})(x-\sqrt{17})}{(\sqrt{7}-\sqrt{3})(\sqrt{7}-\sqrt{11})(\sqrt{7}-\sqrt{17})} \\
 &+ \frac{4}{9} \frac{(x-\sqrt{3})(x-\sqrt{7})(x-\sqrt{17})}{(\sqrt{11}-\sqrt{3})(\sqrt{11}-\sqrt{7})(\sqrt{11}-\sqrt{17})} \\
 &+ \frac{6}{13} \frac{(x-\sqrt{3})(x-\sqrt{7})(x-\sqrt{11})}{(\sqrt{17}-\sqrt{3})(\sqrt{17}-\sqrt{7})(\sqrt{17}-\sqrt{11})}, \quad x \in \mathbb{T}.
 \end{aligned}$$

Next, we compute $L_{\sigma 0}(x)$, $L_{\sigma 1}(x)$, $L_{\sigma 2}(x)$ and $L_{\sigma 3}(x)$ as follows,

$$\begin{aligned}
 L_{\sigma 0}(x) &= \frac{(\sigma(x)-\sigma(\sqrt{7}))(\sigma(x)-\sigma(\sqrt{11}))(\sigma(x)-\sigma(\sqrt{17}))}{(\sigma(\sqrt{3})-\sigma(\sqrt{7}))(\sigma(\sqrt{3})-\sigma(\sqrt{11}))(\sigma(\sqrt{3})-\sigma(\sqrt{17}))} \\
 &= \frac{(\sqrt{x^2+2}-3)(\sqrt{x^2+2}-\sqrt{13})(\sqrt{x^2+2}-\sqrt{19})}{(\sqrt{5}-3)(\sqrt{5}-\sqrt{13})(\sqrt{5}-\sqrt{19})}, \\
 L_{\sigma 1}(x) &= \frac{(\sigma(x)-\sigma(\sqrt{3}))(\sigma(x)-\sigma(\sqrt{11}))(\sigma(x)-\sigma(\sqrt{17}))}{(\sigma(\sqrt{7})-\sigma(\sqrt{3}))(\sigma(\sqrt{7})-\sigma(\sqrt{11}))(\sigma(\sqrt{7})-\sigma(\sqrt{17}))} \\
 &= \frac{(\sqrt{x^2+2}-\sqrt{5})(\sqrt{x^2+2}-\sqrt{13})(\sqrt{x^2+2}-\sqrt{19})}{(3-\sqrt{5})(3-\sqrt{13})(3-\sqrt{19})}, \\
 L_{\sigma 2}(x) &= \frac{(\sigma(x)-\sigma(\sqrt{3}))(\sigma(x)-\sigma(\sqrt{7}))(\sigma(x)-\sigma(\sqrt{17}))}{(\sigma(\sqrt{11})-\sigma(\sqrt{3}))(\sigma(\sqrt{11})-\sigma(\sqrt{7}))(\sigma(\sqrt{11})-\sigma(\sqrt{17}))} \\
 &= \frac{(\sqrt{x^2+2}-\sqrt{5})(\sqrt{x^2+2}-3)(\sqrt{x^2+2}-\sqrt{19})}{(\sqrt{13}-\sqrt{5})(\sqrt{13}-3)(\sqrt{13}-\sqrt{19})}, \\
 L_{\sigma 3}(x) &= \frac{(\sigma(x)-\sigma(\sqrt{3}))(\sigma(x)-\sigma(\sqrt{7}))(\sigma(x)-\sigma(\sqrt{11}))}{(\sigma(\sqrt{17})-\sigma(\sqrt{3}))(\sigma(\sqrt{17})-\sigma(\sqrt{7}))(\sigma(\sqrt{17})-\sigma(\sqrt{11}))} \\
 &= \frac{(\sqrt{x^2+2}-\sqrt{5})(\sqrt{x^2+2}-3)(\sqrt{x^2+2}-\sqrt{13})}{(\sqrt{19}-\sqrt{5})(\sqrt{19}-3)(\sqrt{19}-\sqrt{13})}, \quad x \in \mathbb{T}.
 \end{aligned}$$

Hence, the σ -Lagrange polynomial $p_{\sigma 3}$ interpolating the function f at the points $\sqrt{3}, \sqrt{7}, \sqrt{11}, \sqrt{17}$ is obtained as

1.1: Sigma-Lagrange Interpolation is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Svetlin G. Georgiev.

1.2: HERMITE INTERPOLATION

The idea of Lagrange interpolation and σ -Lagrange interpolation can be generalized in various ways. Here, in this section we consider one simple extension where a polynomial p is required to take given values and derivative values at the interpolation points. For given σ -distinct points $x_j, j \in \{0, 1, \dots, n\}$, and two sets of real numbers $y_j, z_j, j \in \{0, 1, \dots, n\}$, $n \in \mathbb{N}_0$, we need to find a polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$ satisfying the conditions

$$p_{2n+1}(x_j) = y_j, \quad p_{2n+1}'(x_j) = z_j, \quad j \in \{0, 1, \dots, n\}.$$

The construction of such a polynomial is similar to that of the Lagrange interpolation polynomial and it is given in the following theorem.

THEOREM (HERMITE INTERPOLATION THEOREM)

Let $n \in \mathbb{N}_0$ and let $a, b \in \mathbb{T}$, $a < b$, and $x_j \in \mathbb{T}$, $j \in \{0, 1, \dots, n\}$, be σ -distinct and $x_j \neq \sigma(x_k)$ for all $j, k \in \{0, 1, \dots, n\}$.

Let also, $y_j, z_j \in \mathbb{R}$, $j \in \{0, 1, \dots, n\}$. Then there exists a unique polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$p_{2n+1}(x_j) = y_j, \quad p_{2n+1}'(x_j) = z_j, \quad j \in \{0, 1, \dots, n\}. \quad (1.2.1)$$

Proof

For $x \in [a, b]$, define the polynomial

$$M_k(x) = \prod_{j=0, j \neq k}^n \frac{x - \sigma(x_j)}{x_k - \sigma(x_j)}, \quad k \in \{0, 1, \dots, n\}.$$

We have

$$M_k(\sigma(x_j)) = 0, \quad M_k(x_k) = 1, \quad j, k \in \{0, 1, \dots, n\}, \quad j \neq k.$$

We will search a polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$ in the following form

$$p_{2n+1}(x) = \sum_{j=0}^n (y_j + (x - x_j)(\alpha_j y_j + \beta_j z_j)) M_j(x) L_j(x),$$

where $\alpha_j, \beta_j \in \mathbb{R}$, $j \in \{0, 1, \dots, n\}$, will be determined by the conditions (1.2.1). We have

$$p_{2n+1}(x_k) = \sum_{j=0}^n (y_j + (x_k - x_j)(\alpha_j y_j + \beta_j z_j)) M_j(x_k) L_j(x_k)$$

=

$$p_{2n+1}^\Delta(x) =$$

$$+ \sum_{j=0}^n (y_j + (x - x_j)(\alpha_j y_j + \beta_j z_j)) (M_j^\Delta(x) L_j(x) + M_j(\sigma(x)) L_j^\Delta(x)),$$

$$p_{2n+1}^\Delta(x_k) =$$

$$+ \sum_{j=0}^n (y_j + (x_k - x_j)(\alpha_j y_j + \beta_j z_j)) (M_j^\Delta(x_k) L_j(x_k) + M_j(\sigma(x_k)) L_j^\Delta(x_k))$$

=

=

or

$$(\alpha_k y_k + \beta_k z_k) M_k(\sigma(x_k)) L_k(\sigma(x_k)) = z_k - y_k (M_k^\Delta(x_k) + M_k(\sigma(x_k)) L_k^\Delta(x_k)) = L_k'(x_k),$$

or

$$\alpha_k y_k + \beta_k z_k = \frac{z_k - y_k (M_k^\Delta(x_k) + M_k(\sigma(x_k)) L_k^\Delta(x_k))}{M_k(\sigma(x_k)) L_k(\sigma(x_k))},$$

and

$$\begin{aligned} p_{2n+1}(x) &= \sum_{j=0}^n \left(y_j + \frac{z_j - y_j (M_j^\Delta(x_j) + M_j(\sigma(x_j)) L_j^\Delta(x_j))}{M_j(\sigma(x_j)) L_j(\sigma(x_j))} (x - x_j) \right) M_j(x) L_j(x) \\ &= \sum_{j=0}^n \left(\left(1 - \frac{M_j^\Delta(x_j) + M_j(\sigma(x_j)) L_j^\Delta(x_j)}{M_j(\sigma(x_j)) L_j(\sigma(x_j))} (x - x_j) \right) y_j \right. \\ &\quad \left. + \frac{z_j}{M_j(\sigma(x_j)) L_j(\sigma(x_j))} (x - x_j) \right) M_j(x) L_j(x), \end{aligned}$$

$x \in [a, b]$. Now, suppose that there are two polynomials such that $p_{2n+1}, q_{2n+1} \in \mathcal{P}_{2n+1}$ and

$$p_{2n+1}(x_k) = q_{2n+1}(x_k) = y_k, \quad p_{2n+1}'(x_k) = q_{2n+1}'(x_k) = z_k$$

Let $h_{2n+1} = p_{2n+1} - q_{2n+1}$. Then $h_{2n+1} \in \mathcal{P}_{2n+1}$ and it has at least $2n + 2$ Zs. Thus,

$$h_{2n+1} \equiv 0 \quad \text{or} \quad p_{2n+1} \equiv q_{2n+1} \quad \text{on} \quad [a, b].$$

This completes the proof.

Next, we give the definition and the general structure of a Hermite interpolation polynomial.

DEFINITION:

Let $n \in \mathbb{N}_0$ and let $a, b \in \mathbb{T}$, $a < b$, and $x_j \in \mathbb{T}$, $j \in \{0, 1, \dots, n\}$, be σ -distinct.

Let also, $y_j, z_j \in \mathbb{R}$, $j \in \{0, 1, \dots, n\}$. Then the polynomial

$$\begin{aligned} x \in [a, b] \Rightarrow p_{2n+1}(x) &= \sum_{j=0}^n \left(\left(1 - \frac{M_j^\Delta(x_j) + M_j(\sigma(x_j)) L_j^\Delta(x_j)}{M_j(\sigma(x_j)) L_j(\sigma(x_j))} (x - x_j) \right) y_j \right. \\ &\quad \left. + \frac{z_j}{M_j(\sigma(x_j)) L_j(\sigma(x_j))} (x - x_j) \right) M_j(x) L_j(x), \end{aligned}$$

is the Hermite interpolation polynomial for the set of values given in

$$\{(x_j, y_j, z_j) : j \in \{0, 1, \dots, n\}\}.$$

We note that the Hermite interpolation polynomial reduces to the classical Hermite interpolation polynomial whenever the time

NOTE

$$\sum_{j=0}^n (\alpha_j y_j + \beta_j z_j) M_j(\sigma(x_k)) L_j(\sigma(x_k))$$

If $\mathbb{T} = \mathbb{R}$, then

$$(\alpha_k y_k + \beta_k z_k) M_k(\sigma(x_k)) L_k(\sigma(x_k))$$

$$+ y_k (M_k^\Delta(x_k) + M_k(\sigma(x_k)) L_k^\Delta(x_k))$$

$$z_k$$

$$M_j^\Delta(x_j) = M_j'(x_j)$$

$$L_j'(x_j),$$

$$L_j^\Delta(x_j) = L_j'(x_j),$$

$$M_j(\sigma(x_j)) = L_j(\sigma(x_j))$$

$$= M_j(x_j)$$

$$L_j(x_j)$$

$$= 1, \quad j \in \{0, 1, \dots, n\}.$$

$x \in [a, b]$. Thus, we get the classical Hermite interpolation polynomial.

EXAMPLE

Let $\mathbb{T} = 2^{\mathbb{N}_0}$,

$$x_0 = 1, \quad x_1 = 4,$$

$$y_0 = 1, \quad y_1 = -1,$$

$$z_0 = 1, \quad z_1 = 2.$$

Here $n = 1$, $\sigma(x) = 2x$, $x \in \mathbb{T}$. Then

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = -\frac{1}{3}(x - 4),$$

$$L_0^\Delta(x) = -\frac{1}{3},$$

$$L_0(\sigma(x_0)) = L_0(2) = \frac{2}{3},$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{1}{3}(x - 1),$$

$$L_1^\Delta(x) = \frac{1}{3}, \quad x \in \mathbb{T},$$

$$L_1(\sigma(x_1)) = L_1(8) = \frac{7}{3},$$

$$M_0(x) = \frac{x - \sigma(x_1)}{x_0 - \sigma(x_1)} = -\frac{1}{7}(x - 8),$$

$$M_0^\Delta(x) = -\frac{1}{7},$$

$$M_0(\sigma(x_0)) = M_0(2) = \frac{6}{7},$$

$$M_1(x) = \frac{x - \sigma(x_0)}{x_1 - \sigma(x_0)} = \frac{1}{2}(x - 2),$$

$$M_1^\Delta(x) = \frac{1}{2},$$

$$M_1(\sigma(x_1)) = M_1(8) = 3, \quad x \in \mathbb{T}.$$

Hence,

$$\begin{aligned}
 p_3(x) &= \left(y_0 + \frac{z_0 - y_0 (M_0^\Delta(x_0) + M_0(\sigma(x_0))L_0^\Delta(x_0))}{M_0(\sigma(x_0))L_0(\sigma(x_0))} (x - \zeta_0) \right) - \sum_{j=1}^n \left(\frac{f(x_j) - p_{2n+1}(x_j)}{M_j(\sigma(x_j))L_j(\sigma(x_j))} (x - \zeta_j) \right) \\
 &+ \left(y_1 + \frac{z_1 - y_1 (M_1^\Delta(x_1) + M_1(\sigma(x_1))L_1^\Delta(x_1))}{M_1(\sigma(x_1))L_1(\sigma(x_1))} (x - \zeta_1) \right) \\
 &= \left(1 + \frac{1 - (-\frac{1}{2} + \frac{6}{7}(-\frac{1}{3}))}{\frac{6}{7} \cdot \frac{2}{3}} (x-1) \right) + \left(-1 + \frac{2 + (\frac{1}{2} + 3 \cdot \frac{1}{3})}{3 \cdot \frac{7}{3}} (x-4) \right) \\
 &= \frac{1}{21} \left(1 + \frac{5}{2}(x-1) \right) (x-4)(x-8) + \frac{1}{6} \left(-1 + \frac{1}{2}(x-4) \right) (x-1)(x-2) \\
 &= \frac{1}{42} (5x-3)(x-4)(x-8) + \frac{1}{12} (x-6)(x-1)(x-2), \quad x \in [1, 8].
 \end{aligned}$$

EXERCISE

Let $\mathbf{T} = 2\mathbb{Z}$. Find the Hermite interpolation polynomial for the set

$$\begin{aligned}
 x_0 &= -4, & x_1 &= 0, & x_2 &= 4, \\
 y_0 &= -1, & y_1 &= 1, & y_2 &= -1, \\
 z_0 &= 1, & z_1 &= -1, & z_2 &= 1.
 \end{aligned}$$

In some applications, instead of a data set of points, function values and delta derivative values at these points a function itself may be given.

The Hermite polynomial for this function, that is, the polynomial which takes the same values as the function and whose delta derivative takes the values of the delta derivative of the function at given points is defined below.

DEFINITION:

Let $n \in \mathbb{N}_0$ and let $a, b \in \mathbf{T}$, $a < b$, and $x_j \in \mathbf{T}$, $j \in \{0, 1, \dots, n\}$, be σ -distinct.

Let also, $f : [a, b] \rightarrow \mathbb{R}$ be delta differentiable on $[a, b]$. Then the polynomial

$$\begin{aligned}
 p_{2n+1}(x) &= \sum_{j=0}^n \left(\left(1 - \frac{M_j^\Delta(x_j) + M_j(\sigma(x_j))L_j^\Delta(x_j)}{M_j(\sigma(x_j))L_j(\sigma(x_j))} (x - \zeta_j) \right) \right. \\
 &\quad \left. + \frac{f^\Delta(x_j)}{M_j(\sigma(x_j))L_j(\sigma(x_j))} (x - x_j) \right) M_j(\sigma(x_j))L_j(\sigma(x_j))
 \end{aligned}$$

$x \in [a, b]$, is called Hermite interpolation polynomial for the function f .

EXERCISE

Let $\mathbf{T} = 3^{\mathbb{N}_0}$, $f : \mathbf{T} \rightarrow \mathbb{R}$ is given by

$$f(x) = x^3 + 3x^2 + e_2(x, 1), \quad x \in \mathbf{T}.$$

Let also,

$$x_0 = 1, \quad x_1 = 9, \quad x_2 = 81.$$

Find the Hermite interpolation polynomial for the function f .

Let $n \in \mathbb{N}_0$ and $a, b \in \mathbf{T}$, $a < b$, and $x_j \in \mathbf{T}$, $j \in \{0, 1, \dots, n\}$, be σ -distinct.

Define the polynomials

THEOREM

Suppose that $n \in \mathbb{N}_0$, $a, b \in \mathbf{T}$, $a < b$, $x_j \in [a, b]$, $j \in \{0, 1, \dots, n\}$, are σ -distinct and $f : [a, b] \rightarrow \mathbb{R}$, $f \in \mathcal{H}_{2n+1}^\Delta([a, b])$ and for any $k \in \{1, \dots, 2n+2\}$. Then for any $x \in [a, b]$ there exists $\xi = \xi(x) \in (a, b)$ such that

$$f(x) - p_{2n+1}(x) = \frac{f^{\Delta^{2n+2}}(\xi)}{\pi_{n+1}(x)\zeta_{n+1}(x)} (\pi_{n+1}(x)\zeta_{n+1}(x))^{\Delta^{2n+2}}(\xi),$$

or

$$G_{\min, 2n+2}(\xi) \leq \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \leq G_{\max, 2n+2}(\xi), \quad x \in [a, b]$$

where

$$\begin{aligned}
 G_{\max, 2n+2}(\xi) &= \max \left\{ \frac{f^{\Delta^{2n+2}}(\xi)}{(\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi)}, \frac{f^{\Delta^{2n+2}}(\rho(\xi))}{(\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\rho(\xi))} \right\}, \\
 G_{\min, 2n+2}(\xi) &= \min \left\{ \frac{f^{\Delta^{2n+2}}(\xi)}{(\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi)}, \frac{f^{\Delta^{2n+2}}(\rho(\xi))}{(\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\rho(\xi))} \right\}.
 \end{aligned}$$

Proof

Let p_{2n+1} be the Hermite interpolation polynomial for the function f with interpolation points x_j , $j \in \{0, 1, \dots, n\}$. Define the function

$$\psi(t) = f(t) - p_{2n+1}(t) - \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \pi_{n+1}(t)\zeta_{n+1}(t), \quad t \in [a, b].$$

Then

$$\begin{aligned}
 \psi(x_j) &= f(x_j) - p_{2n+1}(x_j) - \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \pi_{n+1}(x_j)\zeta_{n+1}(x_j) \\
 &= f(x_j) - p_{2n+1}(x_j) - \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \left((\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\rho(\xi)) f^{\Delta^{2n+2}}(\xi) \right) \\
 &= 0, \quad x \in [a, b], \quad j \in \{0, 1, \dots, n\},
 \end{aligned}$$

and $\psi(x) = 0$, $x \in [a, b]$. Thus, $\psi : [a, b] \rightarrow \mathbb{R}$ has at least $n+1$ Zs. Hence and the Rolle theorem (see the Appendix of this book), it follows that ψ^Δ has at least $n+1$ GZs on (a, b) that do not coincide with x_j , $j \in \{0, 1, \dots, n\}$. Next,

$$\begin{aligned}
 \psi^\Delta(t) &= f^\Delta(t) - p_{2n+1}^\Delta(t) - \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \left(\pi_{n+1}^\Delta(t)\zeta_{n+1}^\Delta(t) + \pi_{n+1}(t)\zeta_{n+1}^\Delta(t) \right) \\
 &= -\frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \left(\pi_{n+1}^\Delta(t)\zeta_{n+1}^\Delta(t) + \pi_{n+1}(t)\zeta_{n+1}^\Delta(t) \right) \\
 &= -\frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \left(\pi_{n+1}^\Delta(t)\zeta_{n+1}^\Delta(t) + \pi_{n+1}(t)\zeta_{n+1}^\Delta(t) \right) \\
 &= -\frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \left(\pi_{n+1}^\Delta(t)\zeta_{n+1}^\Delta(t) + \pi_{n+1}(t)\zeta_{n+1}^\Delta(t) \right)
 \end{aligned}$$

i.e., ψ^Δ has at least $n+1$ GZs at x_j , $j \in \{0, 1, \dots, n\}$. Therefore ψ^Δ has at least $2n+2$ GZs in $[a, b]$.

Then $\psi^{\Delta^{2n+2}}$ has at least one GZ in (a, b) and there exists a $\xi = \xi(x) \in (a, b)$ such that

$$\psi^{\Delta^{2n+2}}(\xi) = 0 \quad \text{or} \quad \psi^{\Delta^{2n+2}}(\rho(\xi))\psi^{\Delta^{2n+2}}(\xi) < 0.$$

Observe that

$$\psi^{\Delta^{2n+2}}(t) = f^{\Delta^{2n+2}}(t) - \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} (\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(t)$$

1. Let $\psi^{\Delta^{2n+2}}(\xi) = 0$. Then

$$f^{\Delta^{2n+2}}(\xi) = \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} (\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi), \quad x \in [a, b]$$

or

$$x \in [a, b] \quad p_{2n+1}(x) = \frac{f^{\Delta^{2n+2}}(\xi)}{(\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi)} \pi_{n+1}(x)\zeta_{n+1}(x)$$

2. Let

$$\psi^{\Delta^{2n+2}}(\rho(\xi))\psi^{\Delta^{2n+2}}(\xi) < 0.$$

Then

$$\begin{aligned}
 0 &> \psi^{\Delta^{2n+2}}(\rho(\xi))\psi^{\Delta^{2n+2}}(\xi) = \left(f^{\Delta^{2n+2}}(\rho(\xi)) - \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} (\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\rho(\xi)) \right) \\
 &\quad \times \left(f^{\Delta^{2n+2}}(\xi) - \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} (\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi) \right) \\
 &= \left(\frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \right)^2 (\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi)
 \end{aligned}$$

Then we have,

$$0 > \left(\frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \right)^2 (\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi)$$

Hence,

$$\begin{aligned}
 G_{\min, 2n+2}(\xi) &\leq \frac{f(x) - p_{2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \leq G_{\max, 2n+2}(\xi), \quad x \in [a, b] \\
 \text{This completes the proof.}
 \end{aligned}$$

NOTE

Suppose that all conditions of Theorem ??? hold. If

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{2n+2}}(\xi)}{(\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi)} \pi_{n+1}(x)\zeta_{n+1}(x) \right) = 0$$

and

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{2n+2}}(\rho(\xi))}{(\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\rho(\xi))} \pi_{n+1}(x)\zeta_{n+1}(x) \right) = 0$$

then

$$0, \quad j \in \{0, 1, \dots, n\}, \quad \lim_{n \rightarrow \infty} \max_{x \in [a, b]} |f(x) - p_{2n+1}(x)| = 0.$$

In the following example we compute the Hermite polynomial for a given function and compare it

graphically with the function.

✓ EXAMPLE

Consider the time scale $\mathbb{T} = 2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$.

Let $a = x_0 = -4, x_1 = 0, x_2 = 4, b = 6$ and $f(x) = \frac{1}{x^2 + 1}, x \in \mathbb{T}$.

We will compute the Hermite polynomial $p_5(x)$ and compare the graphs of p_5 and f as well as the graphs of p_5^Δ and f^Δ .

For this time scale we have $\sigma(x) = x + 2, x \in \mathbb{T}$, so that $\sigma(x_0) = -2, \sigma(x_1) = 2$ and $\sigma(x_2) = 6$.

We have

$$f^\Delta(x) = \frac{\frac{1}{(x+2)^2+1} - \frac{1}{x^2+1}}{x+2-x} = -\frac{2x+2}{(x^2+1)(x^2+4x+5)}, \quad x \in \mathbb{T}.$$

Hence,

$$\begin{aligned} y_0 &= f(x_0) = f(-4) = \frac{1}{17}, \\ y_1 &= f(x_1) = f(0) = 1, \\ y_2 &= f(x_2) = f(4) = \frac{1}{17}, \\ z_0 &= f^\Delta(x_0) = f^\Delta(-4) = \frac{6}{85}, \\ z_1 &= f^\Delta(x_1) = f^\Delta(0) = -\frac{2}{5}, \\ z_2 &= f^\Delta(x_2) = f^\Delta(4) = -\frac{10}{629}. \end{aligned}$$

First we compute L_i and M_i for $i = 0, 1, 2$.

$$\begin{aligned} L_0(x) &= \frac{(x-0)(x-4)}{(-4-0)(-4-4)} = \frac{1}{32}(x^2-4x), \\ L_1(x) &= \frac{(x+4)(x-4)}{(0+4)(0-4)} = -\frac{1}{16}(x^2-16), \\ L_2(x) &= \frac{(x+4)(x-0)}{(4+4)(4-0)} = \frac{1}{32}(x^2+4x), \\ M_0(x) &= \frac{(x-\sigma(0))(x-\sigma(4))}{(-4-\sigma(0))(-4-\sigma(4))} = \frac{(x-2)(x-6)}{(-4-2)(-4-6)} = \frac{1}{60}(x^2-8x+12), \\ M_1(x) &= \frac{(x-\sigma(-4))(x-\sigma(4))}{(0-\sigma(-4))(0-\sigma(4))} = \frac{(x+2)(x-6)}{(0+2)(0-6)} = -\frac{1}{12}(x^2-4x-12), \\ M_2(x) &= \frac{(x-\sigma(-4))(x-\sigma(0))}{(4-\sigma(-4))(4-\sigma(0))} = \frac{(x+2)(x-2)}{(4+2)(4-2)} = \frac{1}{12}(x^2-4), \quad x \in \mathbb{T}. \end{aligned}$$

Let

$$h(x) = x^2, \quad x \in \mathbb{T}.$$

Then,

$$(h(x))^\Delta = x + \sigma(x) = x + x + 2 = 2x + 2, \quad x \in \mathbb{T}.$$

We compute L_i^Δ and M_i^Δ for $i = 0, 1, 2$.

$$\begin{aligned} L_0^\Delta(x) &= \frac{1}{32}(2x+2-4) = \frac{1}{16}(x-1), \\ L_1^\Delta(x) &= -\frac{1}{16}(2x+2) = -\frac{1}{8}(x+1), \\ L_2^\Delta(x) &= \frac{1}{32}(2x+2+4) = \frac{1}{16}(x+3), \\ M_0^\Delta(x) &= \frac{1}{60}(2x+2-8) = \frac{1}{30}(x-3), \\ M_1^\Delta(x) &= -\frac{1}{12}(2x+2-4) = -\frac{1}{6}(x-1), \\ M_2^\Delta(x) &= \frac{1}{12}(2x+2) = \frac{1}{6}(x+1), \quad x \in \mathbb{T}. \end{aligned}$$

The values involved in the Hermite polynomial are computed as

$$\begin{aligned} L_0(\sigma(x_0)) &= L_0(-2) = \frac{1}{32}(4+8) = \frac{3}{8}, \\ L_0^\Delta(x_0) &= L_0^\Delta(-4) = \frac{1}{16}(-4-1) = -\frac{5}{16}, \\ L_1(\sigma(x_1)) &= L_1(2) = -\frac{1}{16}(4-16) = \frac{3}{4}, \\ L_1^\Delta(x_1) &= L_1^\Delta(0) = -\frac{1}{8}(0+1) = -\frac{1}{8}, \\ L_2(\sigma(x_2)) &= L_2(6) = \frac{1}{32}(36+24) = \frac{15}{8}, \\ L_2^\Delta(x_2) &= L_2^\Delta(4) = \frac{1}{16}(4+3) = \frac{7}{16}, \quad x \in \mathbb{T}. \end{aligned}$$

and

$$\begin{aligned} M_0(\sigma(x_0)) &= M_0(-2) = \frac{1}{60}(4+16+12) = \frac{8}{15}, \\ M_0^\Delta(x_0) &= M_0^\Delta(-4) = \frac{1}{30}(-4-3) = -\frac{7}{30}, \\ M_1(\sigma(x_1)) &= M_1(2) = -\frac{1}{12}(4-8-12) = \frac{4}{3}, \\ M_1^\Delta(x_1) &= M_1^\Delta(0) = -\frac{1}{6}(0-1) = \frac{1}{6}, \\ M_2(\sigma(x_2)) &= M_2(6) = \frac{1}{12}(36-4) = \frac{8}{3}, \\ M_2^\Delta(x_2) &= M_2^\Delta(4) = \frac{1}{6}(4+1) = \frac{5}{6}, \quad x \in \mathbb{T}. \end{aligned}$$

The the Hermite polynomial $p_5(x)$ is computed as

$$\begin{aligned} p_5(x) &= \left[\left(1 - \frac{7}{30} - \frac{8}{15} \cdot \frac{5}{16} \right) \frac{1}{17} + \frac{8}{15} \cdot \frac{3}{8} \right] L_0(x) M_0(x) \\ &+ \left[\left(1 - \frac{1}{6} - \frac{4}{3} \cdot \frac{1}{8} \right) \frac{1}{17} + \frac{4}{3} \cdot \frac{3}{4} \right] L_1(x) M_1(x) \\ &+ \left[\left(1 - \frac{5}{6} + \frac{8}{3} \cdot \frac{7}{16} \right) \frac{1}{17} + \frac{10}{629} \cdot \frac{15}{8} \right] L_2(x) M_2(x), \quad x \in \mathbb{T}, \end{aligned}$$

which after simplifying becomes

$$\begin{aligned} p_5(x) &= \left[\frac{1}{17} + \frac{8}{17}(x+4) \right] \frac{1}{32}(x^2-4x) \frac{1}{60}(x^2-8x+12) \\ &+ \left[1 - \frac{2}{5}x \right] \frac{1}{16}(x^2-16) \frac{1}{12}(x^2-4x-12) \\ &+ \left[\frac{1}{17} - \frac{84}{3145}(x-4) \right] \frac{1}{32}(x^2+4x) \frac{1}{12}(x^2-4), \quad x \in \mathbb{T}. \end{aligned}$$

This page titled [1.2: Hermite Interpolation](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

and

$$\begin{aligned}
 p_3(x) &= \frac{z_0}{M_0(\sigma(x_0))L_0(\sigma(x_0))}(x-x_0)M_0(x)L_0(x) \\
 &\quad + \frac{z_1}{M_1(\sigma(x_1))L_1(\sigma(x_1))}(x-x_1)M_1(x)L_1(x) \\
 &= \frac{1}{\frac{2}{3} \cdot \frac{4}{5}}(x+1)\left(-\frac{1}{10}(x-9)\right)\left(-\frac{1}{6}(x-5)\right) \\
 &\quad + \frac{1}{2 \cdot \frac{5}{3}}(x-5)\left(\frac{1}{6}(x+1)\right)\left(\frac{1}{4}(x-1)\right) \\
 &= \frac{1}{32}(x+1)(x-9)(x-5) + \frac{1}{80}(x-5)(x+1)(x-1), \quad x \in \mathbb{T} \\
 p_3(2) &= \frac{1}{32}(2+1)(2-9)(2-5) + \frac{1}{80}(2-5)(2+1)(2-1) \\
 &= \frac{63}{320} \\
 &= \frac{9}{40} \\
 &= \frac{9}{40}
 \end{aligned}$$

Consequently we have

$$p_3(2) \neq p_{\sigma 3}(2).$$

In the above example we see that the Hermite and σ -Hermite interpolation polynomials may be different.

EXERCISE

Let $\mathbb{T} = \{-4, -1, 0, \frac{1}{2}, \frac{5}{6}, 1, \frac{4}{3}, 2, 7\}$,

$$a = -4, \quad b = 7, \quad x_0 = -4, \quad x_1 = 0, \quad x_2 = \frac{5}{6},$$

the function $f: \mathbb{T} \rightarrow \mathbb{R}$ is defined by

$$f(t) = \frac{t+1}{t^2+4t+7} + 1, \quad t \in \mathbb{T}.$$

Find the σ -Hermite interpolation polynomial for the function f .

In the next theorem we will give some criteria for coincidence of the Hermite and σ -Hermite interpolation polynomials.

THEOREM

Assume that $\mathbb{T}$ is a time scale such that $\sigma(t) = ct + d$ for any $t \in \mathbb{T}$ and for some real constants c, d .

Let $n \in \mathbb{N}_0$ and let $a, b \in \mathbb{T}$, $a < b$, and $x_j \in \mathbb{T}$, $j \in \{0, 1, \dots, n\}$, be σ -distinct.

Let also, $y_j, z_j \in \mathbb{R}$, $j \in \{0, 1, \dots, n\}$.

Then

$$p_{\sigma 2n+1} \equiv p_{2n+1}.$$

Proof

Let $\sigma(t) = ct + d$ for any $t \in \mathbb{T}$ and for some real constants c and d . Then $\sigma^A(t) = c$, $t \in \mathbb{T}$ and

$$\begin{aligned}
 p_{\sigma 2n+1}(x) &= \sum_{j=0}^n \left(\left(1 - \frac{M_j^A(x_j) + M_j(\sigma(x_j))L_j^A(x_j)}{\sigma^A(x_j)M_j(\sigma(x_j))L_j(\sigma(x_j))} (\sigma(x) - \sigma(x_j)) \right) \frac{y_j}{g_1(x)} \right. \\
 &\quad \left. + \frac{z_j}{\sigma^A(x_j)M_j(\sigma(x_j))L_j(\sigma(x_j))} (\sigma(x) - \sigma(x_j)) \right) M_j(x)L_j(x) \\
 &= \sum_{j=0}^n \left(\left(1 - \frac{M_j^A(x_j) + M_j(\sigma(x_j))L_j^A(x_j)}{cM_j(\sigma(x_j))L_j(\sigma(x_j))} (cx + d - (cx_j + d)) \right) \frac{y_j}{g_2(x)} \right. \\
 &\quad \left. + \frac{z_j}{cM_j(\sigma(x_j))L_j(\sigma(x_j))} (cx + d - (cx_j + d)) \right) M_j(x)L_j(x) \\
 &= \sum_{j=0}^n \left(\left(1 - \frac{M_j^A(x_j) + M_j(\sigma(x_j))L_j^A(x_j)}{M_j(\sigma(x_j))L_j(\sigma(x_j))} \frac{\sigma^A(x)}{\sigma^A(x_j)} \right) \frac{y_j}{g_2(x)} \right. \\
 &\quad \left. + \frac{z_j}{M_j(\sigma(x_j))L_j(\sigma(x_j))} (\sigma(x) - \sigma(x_j)) \right) M_j(x)L_j(x)
 \end{aligned}$$

This completes the proof.

As we have proved the above theorem, one can deduct the following result regarding the error in σ -Hermite interpolation

THEOREM

Suppose that $n \in \mathbb{N}_0$, $a, b \in \mathbb{T}$, $a < b$, $x_j \in [a, b]$, $j \in \{0, 1, \dots, 2n+1\}$, are σ -distinct and $f: [a, b] \rightarrow \mathbb{R}$, $f^{\Delta^k}(x)$ exist for any $x \in [a, b]$ and for any $k \in \{1, \dots, 2n+2\}$. Then for any $x \in [a, b]$ there exists $\xi = \xi(x) \in (a, b)$ such that

$$f(x) - p_{\sigma 2n+1}(x) = \frac{f^{\Delta^{2n+2}}(\xi)}{\pi_{n+1}(x)\zeta_{n+1}(x)} (\pi_{n+1}\zeta_{n+1})^{\Delta^{2n+2}}(\xi), \quad x \in [a, b],$$

or

$$G_{\min, 2n+2}(\xi) \leq \frac{f(x) - p_{\sigma 2n+1}(x)}{\pi_{n+1}(x)\zeta_{n+1}(x)} \leq G_{\max, 2n+2}(\xi), \quad x \in [a, b].$$

In the following example we consider a time scale with a nonlinear forward jump operator. We show the difference between Hermite and σ -Hermite polynomials graphically.

EXAMPLE

Let $\mathbb{T} = \mathbb{N}_0^2 = \{0, 1, 4, 9, 16, \dots\}$ and $[a, b] = [1, 49]$.

Let

$$x_0 = 1, x_1 = 9, x_2 = 25.$$

We will find the Hermite interpolation polynomial $p_6(x)$ and the σ -Hermite interpolation polynomial $p_{\sigma 6}(x)$ for a function f satisfying

$$f(1) = 2, \quad f(9) = 4, \quad f(25) = 10,$$

and

$$f^A(1) = 3, \quad f^A(9) = 6, \quad f^A(25) = 7.$$

On this time scale we have

$$\begin{aligned}
 \sigma(x) &= (\sqrt{x} + 1)^2, \\
 \sigma^A(x) &= \frac{\sigma(\sigma(x)) - \sigma(x)}{\sigma(x) - x} = \frac{2\sqrt{x} + 3}{2\sqrt{x} + 1}, \quad x \in \mathbb{T}.
 \end{aligned}$$

We first compute the polynomials L_k and M_k for $k = 0, 1, 2$.

$$\begin{aligned}
 L_0(x) &= \frac{(x-9)(x-25)}{(-8)(-24)} = \frac{x^2 - 34x + 225}{192}, \\
 L_1(x) &= \frac{(x-1)(x-25)}{(8)(-16)} = -\frac{x^2 - 26x + 25}{128}, \\
 L_2(x) &= \frac{(x-1)(x-9)}{(24)(16)} = \frac{x^2 - 10x + 9}{384}, \\
 M_0(x) &= \frac{(x-16)(x-36)}{(-15)(-35)} = \frac{x^2 - 52x + 576}{525}, \\
 M_1(x) &= \frac{(x-4)(x-36)}{(5)(-27)} = -\frac{x^2 - 40x + 144}{135}, \\
 M_2(x) &= \frac{(x-4)(x-16)}{(21)(9)} = \frac{x^2 - 20x + 64}{189}, \quad x \in [1, 49].
 \end{aligned}$$

$$\begin{aligned}
 g_1(x) &= x^2, \\
 g_2(x) &= x, \quad x \in \mathbb{T}. \\
 p_{2n+1}(x) &= 2x + 2\sqrt{x} + 1, \\
 p_{\sigma 2n+1}(x) &= 2x + 2\sqrt{x} - 33, \\
 L_1^A(x) &= -\frac{2x + 2\sqrt{x} - 25}{128}, \\
 L_2^A(x) &= \frac{2x + 2\sqrt{x} - 9}{384}, \\
 M_0^A(x) &= \frac{2x + 2\sqrt{x} - 51}{525}, \\
 M_1^A(x) &= -\frac{2x + 2\sqrt{x} - 39}{135}, \\
 M_2^A(x) &= \frac{2x + 2\sqrt{x} - 19}{189}, \quad x \in [1, 49].
 \end{aligned}$$

Then we have

$$\begin{aligned}
 L_0(\sigma(x_0)) &= L_0(4) = \frac{35}{64}, \quad L_0^A(x_0) = L_0^A(1) = -\frac{21}{19}, \\
 L_1(\sigma(x_1)) &= L_1(16) = \frac{135}{128}, \quad L_1^A(x_1) = L_1^A(9) = \frac{1}{12}, \\
 L_2(\sigma(x_2)) &= L_2(36) = \frac{315}{128}, \quad L_2^A(x_2) = L_2^A(25) = \frac{1}{12}, \\
 M_0(\sigma(x_0)) &= M_0(4) = \frac{128}{175}, \quad M_0^A(x_0) = M_0^A(1) = -\frac{1}{175}, \\
 M_1(\sigma(x_1)) &= M_1(16) = \frac{16}{6}, \quad M_1^A(x_1) = M_1^A(9) = \frac{1}{9}, \\
 M_2(\sigma(x_2)) &= M_2(36) = \frac{640}{189}, \quad M_2^A(x_2) = M_2^A(25) = \frac{1}{189}.
 \end{aligned}$$

Using these values we compute

$$\begin{aligned}
 p_6(x) &= \left(2 + \frac{17}{2}(x-1) \right) \left(\frac{x^2 - 34x + 225}{192} \right) \\
 &\quad + \left(4 + \frac{44}{15}(x-9) \right) \left(-\frac{x^2 - 26x + 25}{128} \right) \\
 &\quad + \left(10 + \frac{1}{25}(x-25) \right) \left(\frac{x^2 - 10x + 9}{384} \right) \left(\frac{x^2 - 10x + 9}{384} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 p_{\sigma 6}(x) &= \left(2 + \frac{51}{10}((\sqrt{x} + 1)^2 - 4) \right) \left(\frac{x}{192} \right) \\
 &\quad + \left(4 + \frac{308}{135}((\sqrt{x} + 1)^2 - 16) \right) \left(-\frac{x}{384} \right) \\
 &\quad + \left(10 + \frac{11}{325}((\sqrt{x} + 1)^2 - 36) \right) \left(\frac{x^2 - 10x + 9}{384} \right)
 \end{aligned}$$

The example above shows that the Hermite interpolation polynomial and σ -Hermite interpolation polynomial can be different and moreover, the σ -Hermite interpolation polynomial may not be a polynomial in the classical sense.

This page titled 1.3: Sigma-Hermite Interpolation is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Svetlin G. Georgiev.

1.4: Delta Differentiation

We conclude this chapter with a theoretical discussion of the error in the delta derivative of a function when it is approximated by the delta derivative of the related interpolation polynomial.

Theorem

Suppose that $n \geq 0$, $a, b \in \mathbb{T}$, $a < b$, $x_j \in [a, b]$, $j \in \{0, 1, \dots, n\}$, are distinct and $f : [a, b] \rightarrow \mathbb{R}$, $p_n : [a, b] \rightarrow \mathbb{R}$ is a polynomial of degree n interpolating f at the points $x_j \in [a, b]$, $j \in \{0, 1, \dots, n\}$ and $f^{\Delta^k}(x)$ exist for any $x \in [a, b]$ and for any $k \in \{1, \dots, n+1\}$. Then for any $x \in [a, b]$ there exists $\xi = \xi(x) \in (a, b)$ and distinct points η_j , $j \in \{1, \dots, n\}$, in (a, b) , such that

$$f^{\Delta}(x) - p_n^{\Delta}(x) = \frac{f^{\Delta^{n+1}}(\xi)}{\pi_n^{*\Delta^{n+1}}(\xi)} \pi_n^*(x), \quad x \in [a, b],$$

or

$$H_{\min, n+1}(\xi) \leq \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \leq H_{\max, n+1}(\xi), \quad x \in [a, b],$$

where

$$H_{\max, n+1}(\xi) = \max \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\pi_n^{*\Delta^{n+1}}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\pi_n^{*\Delta^{n+1}}(\rho(\xi))} \right\},$$

$$H_{\min, n+1}(\xi) = \min \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\pi_n^{*\Delta^{n+1}}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\pi_n^{*\Delta^{n+1}}(\rho(\xi))} \right\},$$

$$\pi_n^*(x) = (x - \eta_1) \dots (x - \eta_n), \quad x \in [a, b],$$

Proof

Let p_n be the Lagrange interpolation polynomial for the function f with interpolation points x_j , $j \in \{0, 1, \dots, n\}$. Then the function $f - p_n$ has at least $n + 1$ GZs in $[a, b]$. Hence and the Rolle theorem, it follows that there exist η_j , $j \in \{1, \dots, n\}$, in (a, b) which are GZs of the function $f^{\Delta} - p_n^{\Delta}$. Define the function

$$\chi(t) = f^{\Delta}(t) - p_n^{\Delta}(t) - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \pi_n^*(t), \quad t \in [a, b].$$

Then

$$\begin{aligned} \chi(\eta_j) &= f^{\Delta}(\eta_j) - p_n^{\Delta}(\eta_j) - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \pi_n^*(\eta_j) \\ &= f^{\Delta}(\eta_j) - p_n^{\Delta}(\eta_j) \\ &= 0, \quad x \in [a, b], \quad j \in \{0, 1, \dots, n\}, \end{aligned}$$

and $\chi(x) = 0$, $x \in [a, b]$.

Thus, $\chi : [a, b] \rightarrow \mathbb{R}$ has at least $n + 1$ GZs. Hence and the Rolle theorem, it follows that χ^{Δ^n} has at least one GZ on (a, b) . Therefore there exists an $\xi = \xi(x) \in (a, b)$ such that

$$\chi^{\Delta^n}(\xi) = 0 \quad \text{or} \quad \chi^{\Delta^n}(\rho(\xi)) \chi^{\Delta^n}(\xi) < 0.$$

Note that

$$\chi^{\Delta^n}(t) = f^{\Delta^{n+1}}(t) - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \pi_n^{*\Delta^n}(t), \quad t \in [a, b].$$

Let $\chi^{\Delta^n}(\xi) = 0$. Then

$$f^{\Delta^{n+1}}(\xi) = \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \pi_n^{*\Delta^n}(\xi), \quad x \in [a, b],$$

or

$$\begin{aligned} f^{\Delta}(x) - p_n^{\Delta}(x) &= \frac{f^{\Delta^{n+1}}(\xi)}{\pi_n^{*\Delta^n}(\xi)} \pi_n^*(x) \\ &= \frac{f^{\Delta^{n+1}}(\xi)}{\pi_n^{*\Delta^n}(\xi)} \pi_n^*(x), \quad x \in [a, b]. \end{aligned}$$

2. Let

$$\chi^{\Delta^n}(\rho(\xi)) \chi^{\Delta^n}(\xi) < 0.$$

Then

$$\chi^{\Delta^n}(\rho(\xi)) = f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \pi_n^{*\Delta^n}(\rho(\xi)), \quad x \in [a, b],$$

and

$$\chi^{\Delta^n}(\xi) = f^{\Delta^{n+1}}(\xi) - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \pi_n^{*\Delta^n}(\xi), \quad x \in [a, b].$$

Hence,

$$0 >$$

$$\begin{aligned} &= \left(f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \right) \\ &\quad \times \left(f^{\Delta^{n+1}}(\xi) - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \right) \\ &= \left(\frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \right)^2 \\ &\quad - \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \left(\pi_n^{*\Delta^n}(\rho(\xi)) f^{\Delta^{n+1}}(\xi) + \pi_n^{*\Delta^n}(\xi) f^{\Delta^{n+1}}(\rho(\xi)) \right) \end{aligned}$$

Hence,

$$H_{\min, n+1}(\xi) \leq \frac{f^{\Delta}(x) - p_n^{\Delta}(x)}{\pi_n^*(x)} \leq H_{\max, n+1}(\xi), \quad x \in [a, b].$$

This completes the proof.

Note

Suppose that all conditions of above theorem hold. If

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{n+1}}(\xi)}{\pi_n^{*\Delta^n}(\xi)} \pi_n^*(x) \right) = 0$$

and

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} \left(\frac{f^{\Delta^{n+1}}(\rho(\xi))}{\pi_n^{*\Delta^n}(\rho(\xi))} \pi_n^*(x) \right) = 0,$$

then

$$\lim_{n \rightarrow \infty} \max_{x \in [a, b]} |f^{\Delta}(x) - p_n^{\Delta}(x)| = 0.$$

This page titled [1.4: Delta Differentiation](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

INDEX

1 A

accessible education

[InfoPage](#)

adaptive learning

[InfoPage](#)

2 C

calculus

[1.1: Sigma-Lagrange Interpolation](#)

collaborative platform

[InfoPage](#)

community support

[InfoPage](#)

3 D

delta derivative

[1.4: Delta Differentiation](#)

Delta Differentiation

[1: Chapters](#)

differentiability

[1.4: Delta Differentiation](#)

4 E

error

[1.4: Delta Differentiation](#)

error estimation

[1.4: Delta Differentiation](#)

examples

[1.1: Sigma-Lagrange Interpolation](#)

exercises

[1.1: Sigma-Lagrange Interpolation](#)

5 F

function

[1.1: Sigma-Lagrange Interpolation](#)

function approximation

[1.4: Delta Differentiation](#)

6 H

Hermite Interpolation

[1: Chapters](#)

7 I

interactive textbooks

[InfoPage](#)

interpolation

[1.1: Sigma-Lagrange Interpolation](#)

interpolation points

[1.1: Sigma-Lagrange Interpolation](#)

interpolation polynomial

[1.4: Delta Differentiation](#)

8 L

Lagrange

[1.1: Sigma-Lagrange Interpolation](#)

Lagrange interpolation

[1: Chapters](#)

[1.4: Delta Differentiation](#)

LibreTexts

[InfoPage](#)

9 M

mathematical analysis

[1.1: Sigma-Lagrange Interpolation](#)

10 N

numerical methods

[1.1: Sigma-Lagrange Interpolation](#)

11 O

OER

[InfoPage](#)

12 P

polynomial

[1.1: Sigma-Lagrange Interpolation](#)

polynomial interpolation

[1.4: Delta Differentiation](#)

13 R

real analysis

[1.4: Delta Differentiation](#)

14 T

textbooks

[InfoPage](#)

theorem

[1.4: Delta Differentiation](#)

GLOSSARY

Sample Word 1 | Sample Definition 1

DETAILED LICENSING

1 OVERVIEW

Title: Polynomial Interpolations on Time Scales (Georgiev)

Webpages: 17

All licenses found:

- [CC BY 4.0](#): 82.4% (14 pages)
- [Undeclared](#): 17.6% (3 pages)

2 BY PAGE

- Polynomial Interpolations on Time Scales (Georgiev) - [CC BY 4.0](#)
 - Front Matter - [CC BY 4.0](#)

- TitlePage - [CC BY 4.0](#)
- InfoPage - [CC BY 4.0](#)
- Table of Contents - [Undeclared](#)
- Licensing - [CC BY 4.0](#)
- 1: Chapters - [Undeclared](#)
- 1.0: Lagrange Interpolation - [CC BY 4.0](#)
- 1.1: Sigma-Lagrange Interpolation - [CC BY 4.0](#)
- 1.2: Hermite Interpolation - [CC BY 4.0](#)
- 1.3: Sigma-Hermite Interpolation - [CC BY 4.0](#)
- 1.4: Delta Differentiation - [CC BY 4.0](#)
- Back Matter - [CC BY 4.0](#)
- Index - [CC BY 4.0](#)
- Glossary - [CC BY 4.0](#)
- Detailed Licensing - [CC BY 4.0](#)
- Detailed Licensing - [Undeclared](#)

Detailed Licensing

Overview

Title: Polynomial Interpolations on Time Scales (Georgiev)

Webpages: 17

All licenses found:

- [CC BY 4.0](#): 82.4% (14 pages)
- [Undeclared](#): 17.6% (3 pages)

By Page

- Polynomial Interpolations on Time Scales (Georgiev) - *CC BY 4.0*
 - Front Matter - *CC BY 4.0*
 - TitlePage - *CC BY 4.0*
 - InfoPage - *CC BY 4.0*
 - Table of Contents - *Undeclared*
 - Licensing - *CC BY 4.0*
 - 1: Chapters - *Undeclared*
 - 1.0: Lagrange Interpolation - *CC BY 4.0*
 - 1.1: Sigma-Lagrange Interpolation - *CC BY 4.0*
 - 1.2: Hermite Interpolation - *CC BY 4.0*
 - 1.3: Sigma-Hermite Interpolation - *CC BY 4.0*
 - 1.4: Delta Differentiation - *CC BY 4.0*
 - Back Matter - *CC BY 4.0*
 - Index - *CC BY 4.0*
 - Glossary - *CC BY 4.0*
 - Detailed Licensing - *CC BY 4.0*
 - Detailed Licensing - *Undeclared*