

THE LAPLACE AND FOURIER TRANSFORMS ON TIME SCALES



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The Laplace and Fourier Transforms on Time Scales

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Chapter 1: Definition of the Laplace Transform. Properties

Let $\mathbb{T}$ be a time scale with forward jump operator, backward jump operator, delta differentiation operator and nabla differentiation operator σ , ρ , Δ and ∇ , respectively. Let also, $\sup \mathbb{T} = \infty$. Take $s \in \mathbb{T}$. If $z \in \mathbb{C}$ and $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}^\kappa$, then

$$\Theta z = -\frac{z}{1 + \mu(t)z}$$

and

$$\begin{aligned} 1 + \mu(t)(\Theta z) &= 1 - \frac{\mu(t)z}{1 + \mu(t)z} \\ &= \frac{1}{1 + \mu(t)z} \end{aligned}$$

for any $t \in \mathbb{T}^\kappa$, and hence, $e_{\Theta z}(\cdot, 0)$ is well defined on $\mathbb{T}^\kappa$.

Definition

\label{definition5.1} Suppose that $s \in \mathbb{T}$, $f : \mathbb{T} \rightarrow \mathbb{R}$ is regulated. Then the Laplace transform of f is defined by

$$\mathcal{L}(f)(z, s) = \int_s^\infty e_{\Theta z}^\sigma(t, s) f(t) \Delta t \quad (\text{Chapter 1.1})$$

for $z \in \mathbb{C}$ for which $1 + \mu(t)z \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral [\(Chapter 1.1\)](#) exists. When $s = 0$, we will write $\mathcal{L}(f)(z)$. \vbox{\index{Laplace Transform}}

Note

\label{remark5.3} Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be regulated. With $\mathcal{D}\{f\}$ we will denote the set of all complex numbers z for which for which $1 + \mu(t)z \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral [\(Chapter 1.1\)](#) exists.

Theorem (Linearity)

Assume that $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are regulated. Then for any constants α and β , we have

$$\mathcal{L}(\alpha f + \beta g)(z, s) = \alpha \mathcal{L}(f)(z, s) + \beta \mathcal{L}(g)(z, s)$$

for $z \in \mathcal{D}\{f\} \cap \mathcal{D}\{g\}$, $t \in \mathbb{T}$.

Proof

Let $z \in \mathcal{D}\{f\} \cap \mathcal{D}\{g\}$, $t \in \mathbb{T}$.

Then, we have

$$\mathcal{L}(\alpha f + \beta g)(z, s) = \int_s^\infty e_{\Theta z}^\sigma(t, s) (\alpha f + \beta g)(t) \Delta t$$

$$\begin{aligned}
 &= \int_s^\infty e_{\Theta z}^\sigma(t, s) (\alpha f(t) + \beta g(t)) \Delta t \\
 &= \int_s^\infty e_{\Theta z}^\sigma(t, s) \alpha f(t) \Delta t + \int_s^\infty e_{\Theta z}^\sigma(t, s) \beta g(t) \Delta t \\
 &= \alpha \int_s^\infty e_{\Theta z}^\sigma(t, s) f(t) \Delta t + \beta \int_s^\infty e_{\Theta z}^\sigma(t, s) g(t) \Delta t \\
 &= \alpha \mathcal{L}(f)(z, s) + \beta \mathcal{L}(g)(z, s).
 \end{aligned}$$

This completes the proof.

Lemma

Let $z \in \mathbb{C}$. Then

$$\begin{aligned}
 e_{\Theta z}^\sigma(t, s) &= \frac{e_{\Theta z}(t, s)}{1 + \mu(t)z} \\
 &= -\frac{(\Theta z)(t)}{z} e_{\Theta z}(t, s), \quad t \in \mathbb{T}.
 \end{aligned}$$

Proof

For $z \in \mathbb{C}$ and $t \in \mathbb{T}$, we have

$$\begin{aligned}
 e_{\Theta z}^\sigma(t, s) &= e_{\Theta z}(t, s) + \mu(t) e_{\Theta z}^\Delta(t, s) \\
 &= e_{\Theta z}(t, s) + (\Theta z)(t) \mu(t) e_{\Theta z}(t, s) \\
 &= (1 + (\Theta z)(t) \mu(t)) e_{\Theta z}(t, s) \\
 &= \left(1 - \frac{\mu(t)z}{1 + \mu(t)z}\right) e_{\Theta z}(t, s) \\
 &= \frac{1}{1 + \mu(t)z} e_{\Theta z}(t, s) \\
 &= -\frac{1}{z} \left(-\frac{z}{1 + \mu(t)z}\right) e_{\Theta z}(t, s) \\
 &= -\frac{(\Theta z)(t)}{z} e_{\Theta z}(t, s).
 \end{aligned}$$

This completes the proof.

Theorem

Assume $f: \mathbb{T} \rightarrow \mathbb{C}$ is such that f^{Δ^l} , $l \in \{0, \dots, k\}$, is regulated. Then

$$\mathcal{L}(f^{\Delta^k})(z, s) = z^k \mathcal{L}(f)(z, s) - \sum_{l=0}^{k-1} z^l f^{\Delta^{k-1-l}}(s) \quad (\text{Chapter 1.2})$$

for those $z \in \mathcal{D}\{f\} \cap \mathcal{D}\{f^\Delta\} \cap \dots \cap \mathcal{D}\{f^{\Delta^k}\}$ satisfying

$$\lim_{t \rightarrow \infty} (f^{\Delta^l}(t) e_{\Theta z}(t, s)) = 0, \quad l = 0, \dots, k-1, \quad (\text{Chapter 1.3})$$

for any $t \in \mathbb{T}$.

Proof

Let $t \in \mathbb{T}$.

We will use induction.

1. Let $k = 1$.

Integrating by parts and using the condition (Chapter 1.3), we get

$$\begin{aligned}
 \mathcal{L}(f^\Delta)(z, s) &= \int_s^\infty f^\Delta(t) e_{\ominus z}^\sigma(t, s) \Delta t \\
 &= f(t) e_{\ominus z}^\sigma(t, s) \Big|_{t=s}^{t \rightarrow \infty} - \int_s^\infty f(t) (\ominus z)(t) e_{\ominus z}(t, s) \Delta t \\
 &= -f(s) + z \int_s^\infty f(t) e_{\ominus z}^\sigma(t, s) \Delta t \\
 &= -f(s) + z \mathcal{L}(f)(z, s).
 \end{aligned}$$

2. Assume that (Chapter 1.2) holds for some $k \in \mathbb{N}$.

3. We will prove

$$\mathcal{L}(f^{\Delta^{k+1}})(z, s) = z^{k+1} \mathcal{L}(f)(z, s) - \sum_{l=0}^k z^l f^{\Delta^{k-l}}(s).$$

Integrating by parts and using (Chapter 1.2), we get

$$\begin{aligned}
 \mathcal{L}(f^{\Delta^{k+1}})(z, s) &= \int_s^\infty f^{\Delta^{k+1}}(t) e_{\ominus z}^\sigma(t, s) \Delta t \\
 &= f^{\Delta^k}(t) e_{\ominus z}(t, s) \Big|_{t=s}^{t \rightarrow \infty} - \int_s^\infty f^{\Delta^k}(t) (\ominus z)(t) e_{\ominus z}(t, s) \Delta t \\
 &= z \int_s^\infty f^{\Delta^k}(t) e_{\ominus z}^\sigma(t, s) \Delta t - f^{\Delta^k}(s) \\
 &= z^{k+1} \mathcal{L}(f)(z, s) - \sum_{l=0}^{k-1} z^{l+1} f^{\Delta^{k-1-l}}(s) - f^{\Delta^k}(s) \\
 &= z^{k+1} \mathcal{L}(f)(z, s) - \sum_{l=0}^k z^l f^{\Delta^{k-l}}(s).
 \end{aligned}$$

This completes the proof.

✓ Example

Let $z \in \mathbb{C}$ be such that

$$1 + \mu(t)z \neq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} e_{\ominus z}(t, s) = 0.$$

Then

$$\begin{aligned}
 \mathcal{L}(1)(z) &= \int_s^\infty e_{\Theta z}^\sigma(t, s) \Delta t \\
 &= \int_s^\infty (1 + \mu(t) (\Theta z)(t)) e_{\Theta z}(t, s) \Delta t \\
 &= \int_s^\infty \left(1 - \frac{z\mu(t)}{1 + z\mu(t)}\right) e_{\Theta z}(t, s) \Delta t \\
 &= -\frac{1}{z} \int_s^\infty \left(-\frac{z}{1 + z\mu(t)}\right) e_{\Theta z}(t, s) \Delta t \\
 &= -\frac{1}{z} \int_s^\infty (\Theta z)(t) e_{\Theta z}(t, s) \Delta t \\
 &= -\frac{1}{z} \int_s^\infty e_{\Theta z}^\Delta(t, s) \Delta t \\
 &= -\frac{1}{z} e_{\Theta z}(t, s) \Big|_{t=0}^{t \rightarrow \infty} \\
 &= \frac{1}{z}.
 \end{aligned}$$

✓ Example

Let $\mathbb{T} = \mathbb{Z}$, $s = 0$ and

$$f(t) = t^2 + 1, \quad t \in \mathbb{T}.$$

We will find $\mathcal{L}(f)(z)$ for $z \in \mathcal{D}\{f\}$.

We have

$$\sigma(t) = t + 1, \quad \mu(t) = 1, \quad t \in \mathbb{T}.$$

We set

$$f_1(t) = t, \quad t \in \mathbb{T}.$$

By (Chapter 1.2), we have

$$\mathcal{L}(f_1^\Delta)(z) = z\mathcal{L}(f_1)(z) - f_1(0),$$

or

$$\mathcal{L}(1)(z) = z\mathcal{L}(f_1)(z),$$

or

$$\frac{1}{z} = z\mathcal{L}(f_1)(z),$$

or

$$\mathcal{L}(f_1)(z) = \frac{1}{z^2}.$$

Note that

$$\begin{aligned}
 f^\Delta(t) &= \sigma(t) + t \\
 &= t + 1 + t \\
 &= 2t + 1, \quad t \in \mathbb{T}.
 \end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}(f^\Delta)(z) &= 2\mathcal{L}(f_1)(z) + \mathcal{L}(1)(z) \\ &= 2\frac{1}{z^2} + \frac{1}{z} \\ &= \frac{z+2}{z^2}.\end{aligned}$$

Now, we use (Chapter 1.2) to we get

$$\mathcal{L}(f^\Delta)(z) = z\mathcal{L}(f)(z) - f(0),$$

or

$$\frac{z+2}{z^2} = z\mathcal{L}(f)(z) - 1,$$

or

$$\frac{z+2}{z^2} + 1 = z\mathcal{L}(f)(z),$$

or

$$\frac{z^2 + z + 2}{z^2} = z\mathcal{L}(f)(z),$$

or

$$\mathcal{L}(f)(z) = \frac{z^2 + z + 2}{z^3}.$$

✓ Example

Let $\mathbb{T} = \mathbb{N}_0^2$, $s = 0$ and

$$f(t) = \frac{2t + 2\sqrt{t} + 3}{(t+1)^2(t+2\sqrt{t}+2)^2}, \quad t \in \mathbb{T}.$$

We will find $\mathcal{L}(f)(z)$ for $z \in \mathcal{D}\{f\}$.

Let

$$h(t) = \frac{1}{(t+1)^2}, \quad t \in \mathbb{T}.$$

We have

$$\sigma(t) = t + 2\sqrt{t} + 1, \quad \mu(t) = 2\sqrt{t} + 1, \quad t \in \mathbb{T}.$$

Note that

$$\begin{aligned}
 e_{\Theta z}^{\sigma}(t, 0) &= e_{-\frac{z}{1+z\mu(t)}}(\sigma(t), 0) \\
 &= e^{-\int_0^{\sigma(t)} \text{Log}(1+z\mu(\tau))\Delta\tau} \\
 &= e^{-\sum_{\tau \in [0,t]} \mu(\tau) \text{Log}(1+z\mu(\tau))} \\
 &= \prod_{\tau \in [0,t]} \frac{1}{(1+z\mu(\tau))^{\mu(\tau)}} \\
 &= \prod_{\tau \in [0,t]} \frac{1}{(1+z(1+2\sqrt{\tau}))^{1+2\sqrt{\tau}}}, \quad t \in \mathbb{T}.
 \end{aligned}$$

Then

$$\begin{aligned}
 \mathcal{L}(h)(z) &= \int_0^{\infty} \frac{1}{(1+t)^2} e_{\Theta z}^{\sigma}(t, 0) \Delta t \\
 &= \sum_{t \in [0, \infty)} \frac{2\sqrt{t}+1}{(t+1)^2} \prod_{\tau \in [0,t]} \frac{1}{(1+z(1+2\sqrt{\tau}))^{1+2\sqrt{\tau}}}, \quad t \in \mathbb{T}.
 \end{aligned}$$

Next,

$$\begin{aligned}
 h^{\Delta}(t) &= -\frac{\sigma(t)+t+2}{(t+1)^2(\sigma(t)+1)^2} \\
 &= -\frac{2t+2\sqrt{t}+3}{(t+1)^2(t+2\sqrt{t}+2)^2} \\
 &= -f(t), \quad t \in \mathbb{T}.
 \end{aligned}$$

Hence and (Chapter 1.2), we obtain

$$\mathcal{L}(h^{\Delta})(z) = z\mathcal{L}(h)(z) - h^{\Delta}(0),$$

or

$$-\mathcal{L}(f)(z) = z\mathcal{L}(h)(z) - \frac{3}{4},$$

or

$$\mathcal{L}(f)(z) = -z \sum_{t \in [0, \infty)} \frac{2\sqrt{t}+1}{(t+1)^2} \prod_{\tau \in [0,t]} \frac{1}{(1+z(1+2\sqrt{\tau}))^{1+2\sqrt{\tau}}} + \frac{3}{4}.$$

✓ Example

Let $\mathbb{T} = 3^{\mathbb{N}_0} \cup \{0\}$, $s = 0$ and

$$f(t) = \begin{cases} \frac{4}{(t+1)(3t+1)(9t+1)}, & t \neq 0, \quad t \in \mathbb{T}, \\ \frac{3}{8}, & t = 0. \end{cases}$$

We will find $\mathcal{L}(f)(z)$ for $z \in \mathcal{D}\{f\}$.

Let

$$h(t) = \frac{1}{1+t}, \quad t \in \mathbb{T}.$$

We have

$$\sigma(t) = 3t, \quad \mu(t) = 2t, \quad \sigma(0) = 1, \quad \mu(0) = 1, \quad t \in 3^{\mathbb{N}_0}.$$

Next,

$$\begin{aligned} e_{\Theta z}^{\sigma}(t, 0) &= e^{\int_0^{\sigma(t)} \text{Log}(1+(\Theta z)(t)\mu(\tau))\Delta\tau} \\ &= e^{\int_0^{\sigma(t)} \text{Log}\left(1-\frac{z\mu(\tau)}{1+z\mu(\tau)}\right)\Delta\tau} \\ &= e^{\int_0^{\sigma(t)} \text{Log}\frac{1}{1+z\mu(\tau)}\Delta\tau} \\ &= e^{\sum_{\tau \in [0,t]} \mu(\tau) \text{Log}\frac{1}{1+z\mu(\tau)}} \\ &= \prod_{\tau \in [0,t]} \frac{1}{(1+z\mu(\tau))^{\mu(\tau)}} \\ &= \begin{cases} \frac{1}{1+z} \prod_{\tau \in [1,t]} \frac{1}{(1+2z\tau)^{2\tau}}, & t \in 3^{\mathbb{N}_0}, \\ \frac{1}{z+1}, & t = 0. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}(h)(z) &= \int_0^{\infty} \frac{1}{t+1} e_{\Theta z}^{\sigma}(t, 0) \Delta t \\ &= \sum_{t \in [0, \infty)} \frac{\mu(t)}{t+1} e_{\Theta z}^{\sigma}(t, 0) \\ &= \frac{1}{1+z} \left(1 + \sum_{t \in [1, \infty)} \frac{2t}{t+1} \prod_{\tau \in [1,t]} \frac{1}{(1+2z\tau)^{2\tau}} \right). \end{aligned}$$

Note that

$$\begin{aligned} h^{\Delta}(t) &= -\frac{1}{(1+t)(1+\sigma(t))} \\ &= \begin{cases} -\frac{1}{2}, & t = 0, \\ -\frac{1}{(1+t)(1+3t)}, & t \neq 0, \end{cases} \end{aligned}$$

for $t = 0$ we get

$$\begin{aligned} h^{\Delta^2}(0) &= \frac{h^{\Delta}(\sigma(0)) - h^{\Delta}(0)}{\sigma(0) - 0} \\ &= h^{\Delta}(1) + \frac{1}{2} \\ &= -\frac{1}{8} + \frac{1}{2} \\ &= \frac{3}{8} \\ &= f(0), \end{aligned}$$

for $t \neq 0$ we obtain

$$\begin{aligned} h^{\Delta^2}(t) &= \frac{h^{\Delta}(\sigma(t)) - h^{\Delta}(t)}{\sigma(t) - t} \\ &= \frac{-\frac{1}{(3t+1)(9t+1)} + \frac{1}{(t+1)(3t+1)}}{\frac{2t}{4}} \\ &= \frac{1}{(t+1)(3t+1)(9t+1)} \\ &= f(t), \quad t \in \mathbb{T}. \end{aligned}$$

Hence and (Chapter 1.2), we get

$$\begin{aligned}\mathcal{L}(h^{\Delta^2})(z) &= \mathcal{L}(f)(z) \\ &= z^2 \mathcal{L}(h)(z) - h^{\Delta}(0) - zh(0) \\ &= \frac{z^2}{1+z} \left(1 + \sum_{t \in [1, \infty)} \frac{2t}{1+t} \prod_{\tau \in [1, t]} \frac{1}{(1+2z\tau)^{2\tau}} \right) \\ &\quad + \frac{1}{2} - z.\end{aligned}$$

? Exercise

Let $\mathbb{T} = 2^{\mathbb{N}_0} \cup \{0\}$, $s = 0$ and

$$f(t) = \frac{1}{2t^3 + 1}, \quad t \in \mathbb{T}.$$

Find $\mathcal{L}(f)(z)$ for $z \in \mathcal{D}\{f\}$.

🔗 Theorem

Assume $f: \mathbb{T} \rightarrow \mathbb{C}$ is regulated. If

$$F(x) = \int_s^x f(y) \Delta y$$

for $x \in \mathbb{T}$, then

$$\mathcal{L}(F)(z, s) = \frac{1}{z} \mathcal{L}(f)(z, s)$$

for those $z \in \mathcal{D}\{f\} \setminus \{s\}$ satisfying

$$\lim_{x \rightarrow \infty} \left(e_{\ominus z}(x, s) \int_s^x f(y) \Delta y \right) = 0.$$

Proof

Using integration by parts, we get

$$\begin{aligned}
 \mathcal{L}(F)(z, s) &= \int_s^\infty F(y) e_{\Theta z}(\sigma(y), s) \Delta y \\
 &= \int_s^\infty F(y) (1 + (\Theta z)(y) \mu(y)) e_{\Theta z}(y, s) \Delta y \\
 &= \int_s^\infty F(y) \frac{1}{1 + z \mu(y)} e_{\Theta z}(y, s) \Delta y \\
 &= -\frac{1}{z} \int_s^\infty F(y) \frac{-z}{1 + z \mu(y)} e_{\Theta z}(y, s) \Delta y \\
 &= -\frac{1}{z} \int_s^\infty F(y) (\Theta z)(y) e_{\Theta z}(y, s) \Delta y \\
 &= -\frac{1}{z} \int_s^\infty F(y) e_{\Theta z}^\Delta(y, s) \Delta y \\
 &= -\frac{1}{z} \left(\lim_{y \rightarrow \infty} F(y) e_{\Theta z}(y, s) - F(s) e_{\Theta z}(s, s) \right) \\
 &\quad + \frac{1}{z} \int_s^\infty F^\Delta(y) e_{\Theta z}(\sigma(y), s) \Delta y \\
 &= \frac{1}{z} \int_s^\infty f(y) e_{\Theta z}(\sigma(y), s) \Delta y \\
 &= \frac{1}{z} \mathcal{L}(f)(z, s),
 \end{aligned}$$

which completes the proof.

Theorem

Let $\alpha \in \mathbb{C}$ and $1 + \alpha \mu(x) \neq 0$ for $x \in \mathbb{T}$. Then

$$\mathcal{L}(e_\alpha(x, s))(z, s) = \frac{1}{z - \alpha}, \quad x \in \mathbb{T},$$

provided

$$\lim_{x \rightarrow \infty} e_{\alpha \Theta z}(x, s) = 0.$$

Proof

We have

$$\begin{aligned}
 \mathcal{L}(e_\alpha(x, s))(z, s) &= \int_s^\infty e_\alpha(y, s) e_{\Theta z}(\sigma(y), s) \Delta y \\
 &= \int_s^\infty e_\alpha(y, s) (1 + (\Theta z)(y) \mu(y)) e_{\Theta z}(y, s) \Delta y \\
 &= \int_s^\infty \frac{1}{1 + z \mu(y)} e_\alpha(y, s) e_{\Theta z}(y, s) \Delta y \\
 &= \int_s^\infty \frac{1}{1 + z \mu(y)} e_{\alpha \Theta z}(y, s) \Delta y \\
 &= \frac{1}{\alpha - z} \int_s^\infty \frac{\alpha - z}{1 + z \mu(y)} e_{\alpha \Theta z}(y, s) \Delta y,
 \end{aligned}$$

i.e.,

$$\mathcal{L}(e_\alpha(x, s))(z, s) = \frac{1}{\alpha - z} \int_s^\infty \frac{\alpha - z}{1 + z \mu(y)} e_{\alpha \Theta z}(y, s) \Delta y. \quad (\text{Chapter 1.4})$$

We note that

$$\begin{aligned}
 \alpha \ominus z &= \alpha \oplus (\ominus z) \\
 &= \alpha + (\ominus z) + \alpha(\ominus z)\mu(y) \\
 &= \alpha - \frac{z}{1 + z\mu(y)} - \frac{\alpha z\mu(y)}{1 + z\mu(y)} \\
 &= \frac{\alpha - z}{1 + z\mu(y)}.
 \end{aligned}$$

Hence and (Chapter 1.4), we obtain

$$\begin{aligned}
 \mathcal{L}(e_\alpha(x, s))(z, s) &= \frac{1}{\alpha - z} \int_s^\infty \frac{\alpha - z}{1 + z\mu(y)} e_{\alpha \ominus z}(y, s) \Delta y \\
 &= \frac{1}{\alpha - z} \int_s^\infty \alpha \ominus z(y) e_{\alpha \ominus z}(y, s) \Delta y \\
 &= \frac{1}{\alpha - z} \int_s^\infty e_{\alpha \ominus z}^\Delta(y, s) \Delta y \\
 &= \frac{1}{\alpha - z} \left(\lim_{y \rightarrow \infty} e_{\alpha \ominus z}(y, s) - e_{\alpha \ominus z}(s, s) \right) \\
 &= \frac{1}{z - \alpha}
 \end{aligned}$$

Corollary

We have

$$\mathcal{L}(\cos_\alpha(x, s))(z, s) = \frac{z}{z^2 + \alpha^2},$$

provided that

$$\lim_{x \rightarrow \infty} e_{i\alpha \ominus z}(x, s) = \lim_{x \rightarrow \infty} e_{-i\alpha \ominus z}(x, s) = 0.$$

Proof

By the definition of $\cos_\alpha(x, s)$ we have

$$\cos_\alpha(x, s) = \frac{e_{i\alpha}(x, s) + e_{-i\alpha}(x, s)}{2}.$$

Hence,

$$\begin{aligned}
 \mathcal{L}(\cos_\alpha(x, s))(z, s) &= \mathcal{L}\left(\frac{e_{i\alpha}(x, s) + e_{-i\alpha}(x, s)}{2}\right)(z, s) \\
 &= \frac{1}{2} \mathcal{L}(e_{i\alpha}(x, s) + e_{-i\alpha}(x, s))(z, s) \\
 &= \frac{1}{2} (\mathcal{L}(e_{i\alpha}(x, s))(z, s) + \mathcal{L}(e_{-i\alpha}(x, s))(z, s)) \\
 &= \frac{1}{2} \left(\frac{1}{z - i\alpha} + \frac{1}{z + i\alpha} \right) \\
 &= \frac{1}{2} \frac{z - i\alpha + z + i\alpha}{(z - i\alpha)(z + i\alpha)} \\
 &= \frac{z}{z^2 + \alpha^2}.
 \end{aligned}$$

The usual convolution of two functions f and g on the real interval $[0, \infty)$ is defined by

$$(f \star g)(x) = \int_0^x f(x-y)g(y)ds \quad \text{for } x \geq 0.$$

However, this definition does not work for general time scales because $x, y \in \mathbb{T}_0$ does not imply that $x - y \in \mathbb{T}_0$.

Definition

Assume that f is one of the functions $e_\alpha(x, 0)$, $\sinh_\alpha(x, 0)$, $\cosh_\alpha(x, 0)$, $\cos_\alpha(x, 0)$, $\sin_\alpha(x, 0)$, or $h_k(x, 0)$, $k \in \mathbb{N}_0$. If g is a regulated function on $\mathbb{T}_0$, then we define the convolution of f with g by

$$(f \star g)(x) = \int_0^x f(x, \sigma(y))g(y)\Delta y \quad \text{for } x \in \mathbb{T}_0.$$

\vbox{\index{Convolution of Functions}}

Theorem (The Convolution Theorem)

Assume that $\alpha \in \mathbb{R}$ and f is one of the functions $e_\alpha(x, 0)$, $\sinh_\alpha(x, 0)$, $\cosh_\alpha(x, 0)$, $\cos_\alpha(x, 0)$, $\sin_\alpha(x, 0)$, or $h_k(x, 0)$, $k \in \mathbb{N}_0$. If g is a regulated function on $\mathbb{T}_0$ such that

$$\lim_{x \rightarrow \infty} e_{\ominus \alpha}(x, 0)(f \star g)(x) = 0,$$

then

$$\mathcal{L}(f \star g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z). \quad (\text{Chapter 1.5})$$

Proof

1. $f(x, 0) = e_\alpha(x, 0)$. Consider the initial value problem for the dynamic equation

$$l^\Delta - \alpha l = g(x), \quad l(0) = 0. \quad (\text{Chapter 1.6})$$

The solution of this problem is given by

$$l(x) = \int_0^x e_\alpha(x, \sigma(y))g(y)\Delta y,$$

which can be rewritten in the form

$$l(x) = (e_\alpha(x, \sigma(y)) \star g)(x).$$

Now we apply the Laplace transform to both sides of the equation ([Chapter 1.6](#)) to get

$$\mathcal{L}(l^\Delta - \alpha l)(z) = \mathcal{L}(g)(z)$$

or

$$\mathcal{L}(l^\Delta)(z) - \alpha \mathcal{L}(l)(z) = \mathcal{L}(g)(z),$$

or

$$z\mathcal{L}(l)(z) - \alpha \mathcal{L}(l)(z) = \mathcal{L}(g)(z),$$

whereupon

$$\mathcal{L}(l)(z) = \frac{1}{z - \alpha} \mathcal{L}(g)(z).$$

Since

$$\mathcal{L}(e_\alpha(x, 0))(z) = \frac{1}{z - \alpha},$$

we conclude that

$$\mathcal{L}(l)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z)$$

or we get (Chapter 1.5).

2. Let $f(x, 0) = \cosh_{\alpha}(x, 0)$. We have

$$\cosh_{\alpha}(x, 0) = \frac{e_{\alpha}(x, 0) + e_{-\alpha}(x, 0)}{2}.$$

Then

$$\begin{aligned} (f \star g)(x) &= \int_0^x \cosh_{\alpha}(x, \sigma(y))g(y)\Delta y \\ &= \frac{1}{2} \int_0^x (e_{\alpha}(x, \sigma(y)) + e_{-\alpha}(x, \sigma(y)))g(y)\Delta y \\ &= \frac{1}{2} \int_0^x e_{\alpha}(x, \sigma(y))g(y)\Delta y + \frac{1}{2} \int_0^x e_{-\alpha}(x, \sigma(y))g(y)\Delta y \\ &= \frac{1}{2}(e_{\alpha}(x, \sigma(y)) \star g)(x) + \frac{1}{2}(e_{-\alpha}(x, \sigma(y)) \star g)(x). \end{aligned}$$

From here,

$$\begin{aligned} \mathcal{L}(f \star g)(z) &= \mathcal{L}\left(\frac{1}{2}(e_{\alpha}(x, \sigma(y)) \star g)(x) + \frac{1}{2}(e_{-\alpha}(x, \sigma(y)) \star g)(x)\right)(z) \\ &= \frac{1}{2}\mathcal{L}(e_{\alpha}(x, \sigma(y)) \star g)(z) + \frac{1}{2}\mathcal{L}(e_{-\alpha}(x, \sigma(y)) \star g)(z) \\ &= \frac{1}{2}\mathcal{L}(e_{\alpha}(x, \sigma(y)))(z)\mathcal{L}(g)(z) + \frac{1}{2}\mathcal{L}(e_{-\alpha}(x, \sigma(y)))(z)\mathcal{L}(g)(z) \\ &= \frac{1}{2}\frac{1}{z - \alpha}\mathcal{L}(g)(z) + \frac{1}{2}\frac{1}{z + \alpha}\mathcal{L}(g)(z) \\ &= \frac{1}{2}\left(\frac{1}{z - \alpha} + \frac{1}{z + \alpha}\right)\mathcal{L}(g)(z) \\ &= \frac{z}{z^2 - \alpha^2}\mathcal{L}(g)(z) \\ &= \mathcal{L}(\cosh_{\alpha}(x, 0))(z)\mathcal{L}(g)(z) \\ &= \mathcal{L}(f)(z)\mathcal{L}(g)(z). \end{aligned}$$

3. Let $f(x, 0) = \sinh_{\alpha}(x, 0)$. Then

$$\sinh_{\alpha}(x, 0) = \frac{e_{\alpha}(x, 0) - e_{-\alpha}(x, 0)}{2},$$

and

$$\begin{aligned} (f \star g)(x) &= \int_0^x \sinh_{\alpha}(x, \sigma(y))g(y)\Delta y \\ &= \frac{1}{2} \int_0^x (e_{\alpha}(x, \sigma(y)) - e_{-\alpha}(x, \sigma(y)))g(y)\Delta y \\ &= \frac{1}{2} \int_0^x e_{\alpha}(x, \sigma(y))g(y)\Delta y - \frac{1}{2} \int_0^x e_{-\alpha}(x, \sigma(y))g(y)\Delta y \\ &= \frac{1}{2}(e_{\alpha}(x, \sigma(y)) \star g)(x) - \frac{1}{2}(e_{-\alpha}(x, \sigma(y)) \star g)(x). \end{aligned}$$

Hence,

$$\begin{aligned}
 \mathcal{L}(f \star g)(z) &= \mathcal{L} \left(\frac{1}{2} (e_{\alpha}(x, \sigma(y)) \star g)(x) - \frac{1}{2} (e_{-\alpha}(x, \sigma(y)) \star g)(x) \right) (z) \\
 &= \frac{1}{2} \mathcal{L}(e_{\alpha}(x, \sigma(y)) \star g)(z) - \frac{1}{2} \mathcal{L}(e_{-\alpha}(x, \sigma(y)) \star g)(z) \\
 &= \frac{1}{2} \mathcal{L}(e_{\alpha}(x, \sigma(y)))(z) \mathcal{L}(g)(z) - \frac{1}{2} \mathcal{L}(e_{-\alpha}(x, \sigma(y)))(z) \mathcal{L}(g)(z) \\
 &= \frac{1}{2} \frac{1}{z - \alpha} \mathcal{L}(g)(z) - \frac{1}{2} \frac{1}{z + \alpha} \mathcal{L}(g)(z) \\
 &= \frac{1}{2} \left(\frac{1}{z - \alpha} - \frac{1}{z + \alpha} \right) \mathcal{L}(g)(z) \\
 &= \frac{\alpha}{z^2 - \alpha^2} \mathcal{L}(g)(z) \\
 &= \mathcal{L}(\sinh_{\alpha}(x, 0))(z) \mathcal{L}(g)(z) \\
 &= \mathcal{L}(f)(z) \mathcal{L}(g)(z).
 \end{aligned}$$

4. Let $f(x, 0) = \cos_{\alpha}(x, 0)$. We have

$$\cos_{\alpha}(x, 0) = \frac{e_{i\alpha}(x, 0) + e_{-i\alpha}(x, 0)}{2}.$$

Then

$$\begin{aligned}
 (f \star g)(x) &= \int_0^x \cos_{\alpha}(x, \sigma(y)) g(y) \Delta y \\
 &= \frac{1}{2} \int_0^x (e_{i\alpha}(x, \sigma(y)) + e_{-i\alpha}(x, \sigma(y))) g(y) \Delta y \\
 &= \frac{1}{2} \int_0^x e_{i\alpha}(x, \sigma(y)) g(y) \Delta y + \frac{1}{2} \int_0^x e_{-i\alpha}(x, \sigma(y)) g(y) \Delta y \\
 &= \frac{1}{2} (e_{i\alpha}(x, \sigma(y)) \star g)(x) + \frac{1}{2} (e_{-i\alpha}(x, \sigma(y)) \star g)(x).
 \end{aligned}$$

From here,

$$\begin{aligned}
 \mathcal{L}(f \star g)(z) &= \mathcal{L} \left(\frac{1}{2} (e_{i\alpha}(x, \sigma(y)) \star g)(x) + \frac{1}{2} (e_{-i\alpha}(x, \sigma(y)) \star g)(x) \right) (z) \\
 &= \frac{1}{2} \mathcal{L}(e_{i\alpha}(x, \sigma(y)) \star g)(z) + \frac{1}{2} \mathcal{L}(e_{-i\alpha}(x, \sigma(y)) \star g)(z) \\
 &= \frac{1}{2} \mathcal{L}(e_{i\alpha}(x, \sigma(y)))(z) \mathcal{L}(g)(z) + \frac{1}{2} \mathcal{L}(e_{-i\alpha}(x, \sigma(y)))(z) \mathcal{L}(g)(z) \\
 &= \frac{1}{2} \frac{1}{z - i\alpha} \mathcal{L}(g)(z) + \frac{1}{2} \frac{1}{z + i\alpha} \mathcal{L}(g)(z) \\
 &= \frac{1}{2} \left(\frac{1}{z - i\alpha} + \frac{1}{z + i\alpha} \right) \mathcal{L}(g)(z) \\
 &= \frac{z}{z^2 + \alpha^2} \mathcal{L}(g)(z) \\
 &= \mathcal{L}(\cos_{\alpha}(x, 0))(z) \mathcal{L}(g)(z) \\
 &= \mathcal{L}(f)(z) \mathcal{L}(g)(z).
 \end{aligned}$$

5. Let $f(x, 0) = \sin_{\alpha}(x, 0)$. Then

$$\sin_{\alpha}(x, 0) = \frac{e_{i\alpha}(x, 0) - e_{-i\alpha}(x, 0)}{2i}, \quad (\text{Chapter 1.7})$$

and

$$\begin{aligned}
 (f \star g)(x) &= \int_0^x \sin_{\alpha}(x, \sigma(y)) g(y) \Delta y \\
 &= \frac{1}{2i} \int_0^x (e_{i\alpha}(x, \sigma(y)) - e_{-i\alpha}(x, \sigma(y))) g(y) \Delta y \\
 &= \frac{1}{2i} \int_0^x e_{i\alpha}(x, \sigma(y)) g(y) \Delta y - \frac{1}{2i} \int_0^x e_{-i\alpha}(x, \sigma(y)) g(y) \Delta y \\
 &= \frac{1}{2i} (e_{i\alpha}(x, \sigma(y)) \star g)(x) - \frac{1}{2i} (e_{-i\alpha}(x, \sigma(y)) \star g)(x).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \mathcal{L}(f \star g)(z) &= \mathcal{L}\left(\frac{1}{2i}(e_{i\alpha}(x, \sigma(y)) \star g)(x) - \frac{1}{2i}(e_{-i\alpha}(x, \sigma(y)) \star g)(x)\right)(z) \\
 &= \frac{1}{2i}\mathcal{L}(e_{i\alpha}(x, \sigma(y)) \star g)(z) - \frac{1}{2i}\mathcal{L}(e_{-i\alpha}(x, \sigma(y)) \star g)(z) \\
 &= \frac{1}{2i}\mathcal{L}(e_{i\alpha}(x, \sigma(y)))(z)\mathcal{L}(g)(z) - \frac{1}{2i}\mathcal{L}(e_{-i\alpha}(x, \sigma(y)))(z)\mathcal{L}(g)(z) \\
 &= \frac{1}{2i}\frac{1}{z - i\alpha}\mathcal{L}(g)(z) - \frac{1}{2i}\frac{1}{z + i\alpha}\mathcal{L}(g)(z) \\
 &= \frac{1}{2i}\left(\frac{1}{z - i\alpha} - \frac{1}{z + i\alpha}\right)\mathcal{L}(g)(z) \\
 &= \frac{\alpha}{z^2 + \alpha^2}\mathcal{L}(g)(z) \\
 &= \mathcal{L}(\sin_\alpha(x, 0))(z)\mathcal{L}(g)(z) \\
 &= \mathcal{L}(f)(z)\mathcal{L}(g)(z).
 \end{aligned}$$

6. $f(x, 0) = h_k(x, 0)$, $k \in \mathbb{N}_0$. Consider the initial value problem for the dynamic equation

$$l^{\Delta^{k+1}}(x) = g(x), \quad l^{\Delta^j}(0) = 0, \quad j = 0, 1, \dots, k. \quad (\text{Chapter 1.8})$$

The solution of the equation ([Chapter 1.8](#)) is given by

$$l(x) = \int_0^x h_k(x, \sigma(y))g(y)\Delta y$$

or

$$l(x) = (h_k(x, \sigma(y)) \star g)(x).$$

Taking the Laplace transform of both sides of ([Chapter 1.8](#)), gives

$$\mathcal{L}(l^{\Delta^{k+1}})(z) = \mathcal{L}(g)(z),$$

whereupon

$$z^{k+1}\mathcal{L}(l)(z) = \mathcal{L}(g)(z)$$

or

$$\begin{aligned}
 \mathcal{L}(l)(z) &= \frac{1}{z^{k+1}}\mathcal{L}(g)(z) \\
 &= \mathcal{L}(h_k(x, 0))(z)\mathcal{L}(g)(z) \\
 &= \mathcal{L}(f)(z)\mathcal{L}(g)(z).
 \end{aligned}$$

Theorem

Assume f and g are each one of the functions $a_\alpha(x, 0)$, $\cosh_\alpha(x, 0)$, $\sinh_\alpha(x, 0)$, $\cos_\alpha(x, 0)$, $\sin_\alpha(x, 0)$, or $h_k(x, 0)$, not both $h_k(x, 0)$. Then

$$f \star g = g \star f.$$

Proof

1. Let $f(x, 0) = e_\alpha(x, 0)$ and $g(x, 0) = e_\beta(x, 0)$. Let also,

$$p(x) = e_\alpha(x, 0) \star e_\beta(x, 0) \quad \text{and} \quad q(x) = e_\beta(x, 0) \star e_\alpha(x, 0).$$

Note that

$$p(0) = q(0) = 0.$$

Also, $p(x)$ and $q(x)$ are solutions to the initial value problems

$$p^\Delta - \alpha p = e_\beta(x, 0), \quad p(0) = 0,$$

and

$$q^\Delta - \beta q = e_\alpha(x, 0), \quad q(0) = 0,$$

respectively.

Then

$$\begin{aligned} p^\Delta(0) &= \alpha p(0) + e_\beta(0, 0) = 1, \\ q^\Delta(0) &= \beta q(0) + e_\alpha(0, 0) = 1. \end{aligned}$$

We claim that $p(x)$ and $q(x)$ are solutions to the initial value problem

$$m^{\Delta^2} - (\alpha + \beta)m^\Delta + \alpha\beta m = 0, \quad m(0) = 0, \quad m^\Delta(0) = 1. \quad (\text{Chapter 1.9})$$

We have

$$p^\Delta = \alpha p + e_\beta(x, 0),$$

from where, after differentiating, we obtain

$$p^{\Delta^2} = \alpha p^\Delta + \beta e_\beta(x, 0).$$

Hence,

$$\begin{aligned} p^{\Delta^2} - (\alpha + \beta)p^\Delta + \alpha\beta p &= \alpha p^\Delta + \beta e_\beta(x, 0) - (\alpha + \beta)p^\Delta + \alpha\beta p \\ &= \beta e_\beta(x, 0) - \beta p^\Delta + \alpha\beta p \\ &= \beta e_\beta(x, 0) - \beta(\alpha p + e_\beta(x, 0)) + \alpha\beta p \\ &= \beta e_\beta(x, 0) - \alpha\beta p - \beta e_\beta(x, 0) + \alpha\beta p \\ &= 0. \end{aligned}$$

Also,

$$q^\Delta = \beta q + e_\alpha(x, 0),$$

whereupon, differentiating, we obtain

$$q^{\Delta^2} = \beta q^\Delta + \alpha e_\alpha(x, 0).$$

Hence,

$$\begin{aligned}
 q^{\Delta^2} - (\alpha + \beta)q^{\Delta} + \alpha\beta q &= \beta q^{\Delta} + \alpha e_{\alpha}(x, 0) - (\alpha + \beta)q^{\Delta} + \alpha\beta q \\
 &= \alpha e_{\alpha}(x, 0) - \alpha q^{\Delta} + \alpha\beta q \\
 &= \alpha e_{\alpha}(x, 0) - \alpha(\beta q + e_{\alpha}(x, 0)) + \alpha\beta q \\
 &= \alpha e_{\alpha}(x, 0) - \alpha\beta q - \alpha e_{\alpha}(x, 0) + \alpha\beta q \\
 &= 0.
 \end{aligned}$$

Since the problem (Chapter 1.9) has unique solution, we conclude that

$$p(x) = q(x).$$

2. Next we consider

$$\begin{aligned}
 e_{\alpha}(x, 0) \star \cosh_{\beta}(x, 0) &= e_{\alpha}(x, 0) \star \frac{e_{\beta}(x, 0) + e_{-\beta}(x, 0)}{2} \\
 &= \frac{1}{2} (e_{\alpha}(x, 0) \star (e_{\beta}(x, 0) + e_{-\beta}(x, 0))) \\
 &= \frac{1}{2} (e_{\alpha}(x, 0) \star e_{\beta}(x, 0) + e_{\alpha}(x, 0) \star e_{-\beta}(x, 0)) \\
 &= \frac{1}{2} (e_{\beta}(x, 0) \star e_{\alpha}(x, 0) + e_{-\beta}(x, 0) \star e_{\alpha}(x, 0)) \\
 &= \frac{1}{2} (e_{\beta}(x, 0) + e_{-\beta}(x, 0)) \star e_{\alpha}(x, 0) \\
 &= \cosh_{\beta}(x, 0) \star e_{\alpha}(x, 0).
 \end{aligned}$$

3. Let

$$z(x) = e_{\alpha}(x, 0) \star h_k(x, 0) \quad \text{and} \quad q(x) = h_k(x, 0) \star e_{\alpha}(x, 0).$$

We have that $z(x)$ is the solution to the initial value problem

$$z^{\Delta}(x) - \alpha z(x) = h_k(x, 0), \quad z(0) = 0.$$

Differentiating this equation i times gives

$$z^{\Delta^{i+1}}(x) - \alpha z^{\Delta^i}(x) = h_{k-i}(x, 0), \quad i = 1, 2, \dots, k.$$

Also,

$$z^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k, \quad z^{\Delta^{k+1}}(0) = 1.$$

Thus we get that $z(x)$ is the solution to the initial value problem

$$\begin{aligned}
 z^{\Delta^{k+2}}(x) - \alpha z^{\Delta^{k+1}}(x) &= 0, \\
 z^{\Delta^i}(0) &= 0, \quad 0 \leq i \leq k, \quad z^{\Delta^{k+1}}(0) = 1.
 \end{aligned}$$

Since $h_k(0, 0) = 0$, $k > 0$, we obtain that

$$q^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k.$$

Also, q is the solution of the initial value problem

$$q^{\Delta^{k+1}}(x) = e_{\alpha}(x, 0), \quad q^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k.$$

Hence, differentiating the last equation, we obtain

$$\begin{aligned} q^{\Delta^{k+2}}(x) &= e_{\alpha}^{\Delta}(x, 0) \\ &= \alpha e_{\alpha}(x, 0) \\ &= \alpha q^{\Delta^{k+1}}(x) \end{aligned}$$

and

$$q^{\Delta^{k+1}}(0) = e_{\alpha}(0, 0) = 1.$$

Consequently $q(x)$ is the solution of the initial value problem

$$q^{\Delta^{k+2}}(x) - \alpha q^{\Delta^{k+1}}(x) = 0, \quad q^{\Delta^i}(0) = 0, \quad q^{\Delta^{k+1}}(0) = 1, \quad 0 \leq i \leq k.$$

Therefore $z(x)$ and $q(x)$ are solutions to the same initial value problem. Hence, they must be equal.

? Exercise

Prove that

1. $e_{\alpha}(x, 0) \star \sinh_{\beta}(x, 0) = \sinh_{\beta}(x, 0) \star e_{\alpha}(x, 0),$
2. $e_{\alpha}(x, 0) \star \cos_{\beta}(x, 0) = \cos_{\beta}(x, 0) \star e_{\alpha}(x, 0),$
3. $e_{\alpha}(x, 0) \star \sin_{\beta}(x, 0) = \sin_{\beta}(x, 0) \star e_{\alpha}(x, 0),$
4. $\cosh_{\alpha}(x, 0) \star \cosh_{\beta}(x, 0) = \cosh_{\beta}(x, 0) \star \cosh_{\alpha}(x, 0),$
5. $\cosh_{\alpha}(x, 0) \star \sinh_{\beta}(x, 0) = \sinh_{\beta}(x, 0) \star \cosh_{\alpha}(x, 0),$
6. $\cosh_{\alpha}(x, 0) \star \cos_{\beta}(x, 0) = \cos_{\beta}(x, 0) \star \cosh_{\alpha}(x, 0),$
7. $\cosh_{\alpha}(x, 0) \star \sin_{\beta}(x, 0) = \sin_{\beta}(x, 0) \star \cosh_{\alpha}(x, 0),$
8. $\cosh_{\alpha}(x, 0) \star h_k(x, 0) = h_k(x, 0) \star \cosh_{\alpha}(x, 0),$
9. $\sinh_{\alpha}(x, 0) \star \sinh_{\beta}(x, 0) = \sinh_{\beta}(x, 0) \star \sinh_{\alpha}(x, 0),$
10. $\sinh_{\alpha}(x, 0) \star \cos_{\beta}(x, 0) = \cos_{\beta}(x, 0) \star \sinh_{\alpha}(x, 0),$
11. $\sinh_{\alpha}(x, 0) \star \sin_{\beta}(x, 0) = \sin_{\beta}(x, 0) \star \sinh_{\alpha}(x, 0),$
12. $\sinh_{\alpha}(x, 0) \star h_k(x, 0) = h_k(x, 0) \star \sinh_{\alpha}(x, 0),$
13. $\cos_{\alpha}(x, 0) \star \cos_{\beta}(x, 0) = \cos_{\beta}(x, 0) \star \cos_{\alpha}(x, 0),$
14. $\cos_{\alpha}(x, 0) \star \sin_{\beta}(x, 0) = \sin_{\beta}(x, 0) \star \cos_{\alpha}(x, 0),$
15. $\cos_{\alpha}(x, 0) \star h_k(x, 0) = h_k(x, 0) \star \cos_{\alpha}(x, 0),$
16. $\sin_{\alpha}(x, 0) \star \sin_{\beta}(x, 0) = \sin_{\beta}(x, 0) \star \sin_{\alpha}(x, 0),$
17. $\sin_{\alpha}(x, 0) \star h_k(x, 0) = h_k(x, 0) \star \sin_{\alpha}(x, 0).$

Theorem

Let $\alpha, \beta \in \mathbb{R}$ and $1 + \alpha\mu(x) \neq 0, 1 + \beta\mu(x) \neq 0$ for all $x \in \mathbb{T}$. Then

$$\mathcal{L}\left(e_{\alpha}(x, 0) \sin_{\frac{\beta}{1+\alpha\mu}}(x, 0)\right)(z) = \frac{\beta}{(z - \alpha)^2 + \beta^2}$$

provided

$$\lim_{x \rightarrow \infty} e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} e_\alpha(x, 0) \left(\sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) \right)^\Delta = 0.$$

Proof

Let

$$p(x) = e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0).$$

Then

$$\begin{aligned} p^\Delta(x) &= e_\alpha^\Delta(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) + e_\alpha(\sigma(x), 0) \sin_{\frac{\beta}{1+\mu\alpha}}^\Delta(x, 0) \\ &= \alpha e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) + (1 + \alpha\mu(x)) e_\alpha(x, 0) \frac{\beta}{1 + \alpha\mu(x)} \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0) \\ &= \alpha e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) + \beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0). \end{aligned}$$

We differentiate the last equation and we get

$$\begin{aligned} z^{\Delta^2}(x) &= \alpha e_\alpha^\Delta(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) + \alpha e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}^\Delta(x, 0) \\ &\quad + \beta e_\alpha^\Delta(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0) + \beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}^\Delta(x, 0) \\ &= \alpha^2 e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) + \alpha(1 + \alpha\mu(x)) e_\alpha(x, 0) \frac{\beta}{1 + \alpha\mu(x)} \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0) \\ &\quad + \alpha\beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0) - \beta(1 + \alpha\mu(x)) e_\alpha(x, 0) \frac{\beta}{1 + \alpha\mu(x)} \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) \\ &= \alpha e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) + \alpha\beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0) \\ &\quad + \alpha\beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0) - \beta^2 e_\alpha(x, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(x, 0) \\ &= (\alpha^2 - \beta^2) p(x) + 2\alpha\beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0). \end{aligned}$$

In this way we obtain the system

$$\begin{cases} p^\Delta(x) = \alpha p(x) + \beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0) \\ p^{\Delta^2}(x) = (\alpha^2 - \beta^2) p(x) + 2\alpha\beta e_\alpha(x, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(x, 0). \end{cases}$$

Hence,

$$p^{\Delta^2}(x) - 2\alpha p^\Delta(x) = -(\alpha^2 + \beta^2) p(x)$$

or

$$p^{\Delta^2}(x) + 2\alpha p^\Delta(x) + (\alpha^2 + \beta^2) p(x) = 0.$$

Also,

$$\begin{aligned} p(0) &= e_\alpha(0, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(0, 0) = 0, \\ p^\Delta(0) &= \alpha e_\alpha(0, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(0, 0) + \beta e_\alpha(0, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(0, 0) = \beta. \end{aligned}$$

Consequently we obtain the following initial value problem

$$\begin{cases} p^{\Delta^2}(x) - 2\alpha p^{\Delta}(x) + (\alpha^2 + \beta^2)p(x) = 0 \\ p(0) = 0, \quad p^{\Delta}(0) = \beta. \end{cases} \quad (\text{Chapter 1.10})$$

Now we apply the Laplace transform to both sides of the dynamic equation (Chapter 1.10) and we obtain

$$\mathcal{L}(p^{\Delta^2}(x) - 2\alpha p^{\Delta}(x) + (\alpha^2 + \beta^2)p(x))(z) = 0$$

or

$$\mathcal{L}(p^{\Delta^2}(x))(z) - 2\alpha \mathcal{L}(p^{\Delta}(x))(z) + (\alpha^2 + \beta^2)\mathcal{L}(p(x))(z) = 0,$$

or

$$z^2 \mathcal{L}(p)(z) - p^{\Delta}(0) - zp(0) - 2\alpha z \mathcal{L}(p)(z) + 2\alpha p(0) + (\alpha^2 + \beta^2)\mathcal{L}(p)(z) = 0,$$

or

$$z^2 \mathcal{L}(p)(z) - \beta - 2\alpha z \mathcal{L}(p)(z) + (\alpha^2 + \beta^2)\mathcal{L}(p)(z) = 0$$

or

$$(z^2 - 2\alpha z + \alpha^2 + \beta^2) \mathcal{L}(p)(z) = \beta,$$

whereupon

$$\mathcal{L}(p)(z) = \frac{\beta}{(z - \alpha)^2 + \beta^2},$$

which completes the proof.

? Exercise

Let $\alpha, \beta \in \mathbb{R}$ and $1 + \alpha\mu(x) \neq 0, 1 + \beta\mu(x) \neq 0$ for all $x \in \mathbb{T}$. Prove that

$$\mathcal{L}\left(e_{\alpha}(x, 0) \cos_{\frac{\beta}{1+\alpha\mu}}(x, 0)\right)(z) = \frac{z - \alpha}{(z - \alpha)^2 + \beta^2},$$

provided that

$$\lim_{x \rightarrow \infty} e_{\alpha}(x, 0) \cos_{\frac{\beta}{1+\alpha\mu}}(x, 0) = \lim_{x \rightarrow \infty} e_{\alpha}(x, 0) \cos_{\frac{\beta}{1+\alpha\mu}}^{\Delta}(x, 0) = 0.$$

? Exercise

Prove the following relations.

1. If $\alpha \neq \beta$, then

$$e_{\alpha}(x, 0) \star e_{\beta}(x, 0) = \frac{1}{\beta - \alpha} (e_{\beta}(x, 0) - e_{\alpha}(x, 0)).$$

2.

$$e_{\alpha}(x, 0) \star e_{\alpha}(x, 0) = e_{\alpha}(x, 0) \int_0^x \frac{1}{1 + \alpha\mu(y)} \Delta y.$$

Item If $\alpha^2 + \beta^2 \neq 0$, then

$$e_{\alpha}(x, 0) \star \sin_{\beta}(x, 0) = \frac{\beta e_{\alpha}(x, 0) - \alpha \sin_{\beta}(x, 0) - \beta \cos_{\beta}(x, 0)}{\alpha^2 + \beta^2}.$$

3. If $\alpha \neq 0, \alpha \neq \beta$, then

$$\cos_{\alpha}(x, 0) \star \cos_{\beta}(x, 0) = \frac{-\beta \sin_{\beta}(x, 0) + \alpha \sin_{\alpha}(x, 0)}{\alpha^2 - \beta^2}.$$

4. If $\alpha \neq 0$, then

$$\cos_{\alpha}(x, 0) \star \cos_{\alpha}(x, 0) = \frac{1}{\alpha} \sin_{\alpha}(x, 0) + \frac{1}{2} x \cos_{\alpha}(x, 0).$$

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Chapter 2: Decay of the Exponential Function

For $h > 0$, set

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\},$$

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h} \right\},$$

and $\mathbb{C}_0 = \mathbb{C}$, $\mathbb{Z}_0 = \mathbb{Z}$. For $h > 0$, the Hilger real part and imaginary part of a complex number z are defined as follows.

$$\text{Re}_h(z) = \frac{1}{h} (|1 + zh| - 1), \quad \text{Im}_h(z) = \text{Arg}(1 + hz),$$

where $\text{Arg}(\cdot)$ denotes the principal argument function, i.e., $\text{Arg} : \mathbb{C} \rightarrow (-\pi, \pi]$, and $\text{Re}_0(z) = \text{Re}(z)$, $\text{Im}_0(z) = \text{Im}(z)$. For any fixed complex number z and $h \in [0, \infty)$, define

$$u(h) = \text{Re}_h(z).$$

Proposition

The function $u : [0, \infty) \rightarrow \mathbb{R}$ is a nondecreasing function.

Proof

We have

$$\begin{aligned} u(h) &= \frac{|1 + hz| - 1}{h} \\ &\geq \frac{\text{Re}(1 + hz) - 1}{h} \\ &= \frac{1 + h\text{Re}(z) - 1}{h} \\ &= \text{Re}(z) \\ &= u(0). \end{aligned}$$

Therefore

$$\text{Re}_h(z) \geq \text{Re}(z).$$

Let $h_2 > h_1 > 0$. Then

$$\begin{aligned} u(h_2) - u(h_1) &= \text{Re}_{h_2}(z) - \text{Re}_{h_1}(z) \\ &= \frac{|1 + h_2 z| - 1}{h_2} - \frac{|1 + h_1 z| - 1}{h_1} \\ &= \frac{|h_1 + h_1 h_2 z| - h_1 - |h_2 + h_1 h_2 z| + h_2}{h_1 h_2} \\ &\geq \frac{-|h_1 - h_2| + h_2 - h_1}{h_1 h_2} \\ &= 0. \end{aligned}$$

This completes the proof.

Lemma

Let $s \in \mathbb{T}$ and $\lambda \in \mathcal{R}_c^+([s, \infty), \mathbb{R})$. Then

1. $m_\lambda^\Delta(t, s) \geq 0$ for all $t \in (s, \infty)$, where the differentiation is with respect to the first argument.
2. $\lim_{t \rightarrow \infty} m_\lambda(t, s) = \infty$.

Proof

1. Since $\lambda \in \mathcal{R}_c^+([s, \infty), \mathbb{R})$, we have

$$m_\lambda^\Delta(t, s) = \frac{1}{1 + \lambda\mu(t)} > 0, \quad t \in [s, \infty).$$

2. We will consider two cases.

- a) Let $\sup \mu(\mathbb{T}) < \infty$. Then

$$1 + \mu(t)\lambda \leq 1 + \mu(t)|\lambda| \leq 1 + |\lambda| \sup \mu(\mathbb{T}), \quad t \in [s, \infty).$$

Hence,

$$\begin{aligned} m_\lambda(t, s) &= \int_s^t \frac{\Delta\tau}{1 + \lambda\mu(\tau)} \\ &\geq \int_s^t \frac{\Delta\tau}{1 + |\lambda| \sup \mu(\mathbb{T})} \\ &= \frac{t - s}{1 + |\lambda| \sup \mu(\mathbb{T})} \\ &\rightarrow \infty, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

- b) Suppose that $\sup \mu(\mathbb{T}) = \infty$. Then, using that $\lambda \in \mathcal{R}_c^+([s, \infty), \mathbb{R})$, we get that $\lambda \geq 0$. If $\lambda = 0$, then

$$m_\lambda(t, s) = t - s \rightarrow \infty, \quad \text{as } t \rightarrow \infty.$$

Assume that $\lambda > 0$. Because $\mu(\mathbb{T}) = \infty$, there exists an increasing sequence $\{\xi_k\}_{k \in \mathbb{N}} \subset [s, \infty)$ such that $\{\mu(\xi_k)\}_{k \in \mathbb{N}}$ is an increasing and divergent sequence. Observe that

$$\lim_{k \rightarrow \infty} \frac{\mu(\xi_k)}{1 + \lambda\mu(\xi_k)} = \frac{1}{\lambda}.$$

Hence,

$$\begin{aligned} m_\lambda(t, s) &= \int_s^t \frac{\Delta\tau}{1 + \lambda\mu(\tau)} \\ &\geq \sum_{\sigma(\xi_k) \leq t, \xi_k \geq s} \int_{\xi_k}^{\sigma(\xi_k)} \frac{\Delta\tau}{1 + \lambda\mu(\tau)} \\ &= \sum_{\sigma(\xi_k) \leq t, \xi_k \geq s} \frac{\mu(\xi_k)}{1 + \lambda\mu(\xi_k)} \\ &\rightarrow \infty, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This completes the proof.

The set of regressive functions and the set of positively regressive functions are denoted by $\mathcal{R}(\mathbb{T}, \mathbb{C})$ and $\mathcal{R}^+(\mathbb{T}, \mathbb{C})$, respectively, and $\mathcal{R}^-(\mathbb{T}, \mathbb{C})$ is defined similarly. For simplicity, we denote by $\mathcal{R}_c(\mathbb{T}, \mathbb{C})$ the set of complex regressive constants, and similarly, we define the sets $\mathcal{R}_c^+(\mathbb{T}, \mathbb{C})$ and $\mathcal{R}_c^-(\mathbb{T}, \mathbb{C})$. Let us set

$$m_\lambda(t, s) = \int_s^t \frac{1}{1 + \mu(\tau)\lambda} \Delta\tau \quad \text{for } s, t \in \mathbb{T} \quad \text{and } \lambda \in \mathcal{R}_c(\mathbb{T}, \mathbb{R}).$$

For $h \geq 0$, define

$$\mathbb{R}_h = \mathbb{C}_h \cap \mathbb{R} = \left\{ \lambda \in \mathbb{R} : \lambda \neq -\frac{1}{h} \right\}$$

and

$$\Psi_h(z, \lambda) = \begin{cases} \frac{1}{h} \log \left| \frac{1+h\lambda}{1+hz} \right| & \text{if } h > 0 \\ \lambda - \operatorname{Re}(z) & \text{if } h = 0. \end{cases}$$

Lemma

Let $h \geq 0$ and $\lambda \in \mathbb{R}_h$.

1. For all $z \in \mathbb{C}_h$, we have

$$\Psi_h(z, \lambda) = \Psi_h(\operatorname{Re}_h(z), \lambda).$$

2. Suppose $1 + h\lambda > 0$ and $x_1, x_2 \in \mathbb{R}$, $1 + hx_1 > 0$, $1 + hx_2 > 0$. If $x_1 < x_2$, then

$$\Psi_h(x_1, \lambda) > \Psi_h(x_2, \lambda).$$

Proof

1. If $h = 0$, then

$$\operatorname{Re}_0(z) = \operatorname{Re}(z)$$

and

$$\begin{aligned} \Psi_0(z, \lambda) &= \lambda - \operatorname{Re}(z) \\ &= \lambda - \operatorname{Re}_0(z) \\ &= \Psi_0(\operatorname{Re}_0(z), \lambda). \end{aligned}$$

Now, suppose that $h > 0$. Then

$$\begin{aligned} \Psi_h(\operatorname{Re}_h(z), \lambda) &= \frac{1}{h} \log \left| \frac{1+h\lambda}{1+h\operatorname{Re}_h(z)} \right| \\ &= \frac{1}{h} \log \left| \frac{1+h\lambda}{1+|1+zh|-1} \right| \\ &= \frac{1}{h} \log \left| \frac{1+h\lambda}{|1+zh|} \right| \\ &= \frac{1}{h} \log \left| \frac{1+h\lambda}{1+zh} \right| \\ &= \Psi_h(z, \lambda). \end{aligned}$$

2. If $h = 0$, then

$$\begin{aligned} \Psi_0(x_1, \lambda) &= \lambda - x_1 \\ &> \lambda - x_2 \\ &= \Psi_0(x_2, \lambda). \end{aligned}$$

Suppose that $h > 0$. Then

$$\begin{aligned}\Psi_h(x_1, \lambda) &= \frac{1}{h} \log \left| \frac{1 + h\lambda}{1 + hx_1} \right| \\ &= \frac{1}{h} \log \left(\frac{1 + h\lambda}{1 + hx_1} \right) \\ &> \frac{1}{h} \log \left(\frac{1 + h\lambda}{1 + hx_2} \right) \\ &= \Psi_h(x_2, \lambda).\end{aligned}$$

This completes the proof.

Define $\mu_* : \mathbb{T} \rightarrow \mathbb{R}_0^+$ by

$$\mu_*(s) = \inf_{\tau \in [s, \infty)} \mu(\tau), \quad s \in \mathbb{T}.$$

For $h > 0$ and $\lambda \in \mathbb{R}$, define

$$\mathbf{C}_h(\lambda) = \{z \in \mathbf{C}_h : \operatorname{Re}_h(z) > \lambda\}.$$

Theorem (Decay of the Exponential Function)

Let $s \in \mathbb{T}$ and $\lambda \in \mathcal{R}_c^+([s, \infty), \mathbb{R})$. Then, for any $z \in \mathbf{C}_{\mu_*(s)}(\lambda)$, we have the following properties.

1.

$$|e_{\lambda \ominus z}(t, s)| \leq e_{\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z)}(t, s), \quad t \in [s, \infty).$$

2.

$$\lim_{t \rightarrow \infty} e_{\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z)}(t, s) = 0.$$

3.

$$\lim_{t \rightarrow \infty} e_{\lambda \ominus z}(t, s) = 0.$$

Proof

1. Set

$$\psi_h(z) = \begin{cases} \frac{1}{h} \log |1 + hz| & \text{if } h > 0 \\ \operatorname{Re}(z) & \text{if } h = 0 \end{cases}$$

for $z \in \mathbf{C}_h$. Then

$$|e_f(t, s)| = \left| e^{\int_s^t \psi_{\mu(\tau)}(f(\tau)) \Delta \tau} \right|, \quad t \in [s, \infty),$$

and

$$\begin{aligned}\psi_{\mu(\tau)}((\lambda \ominus z)(\tau)) &= \begin{cases} \frac{1}{\mu(\tau)} \log |1 + \mu(\tau)((\lambda \ominus z)(\tau))| & \text{if } \mu(\tau) > 0 \\ \operatorname{Re}(\lambda - z) & \text{if } \mu(\tau) = 0 \end{cases} \\ &= \begin{cases} \frac{1}{\mu(\tau)} \log \left| \frac{1 + \lambda\mu(\tau)}{1 + z\mu(\tau)} \right| & \text{if } \mu(\tau) > 0 \\ \lambda - \operatorname{Re}(z) & \text{if } \mu(\tau) = 0. \end{cases}\end{aligned}$$

Therefore

$$\psi_{\mu(\tau)}((\lambda \ominus z)(\tau)) = \Psi_{\mu(\tau)}(z, \lambda).$$

Hence, for $\lambda \in \mathcal{R}_c^+([s, \infty), \mathbb{R})$ and $z \in \mathbb{C}_{\mu_*(s)}(\lambda)$, using the above lemma, and $\operatorname{Re}_{\mu_*(s)}(z) > \lambda$,

$$\begin{aligned} 1 + \mu(\tau) \operatorname{Re}_{\mu(\tau)}(z) &\geq 1 + \mu(\tau) \operatorname{Re}_{\mu_*(s)}(z) \\ &\geq 1 + \lambda \mu(\tau) \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} |e_{\lambda \ominus z}(t, s)| &= e^{\int_s^t \Psi_{\mu(\tau)}(\operatorname{Re}_{\mu(\tau)}(z, \lambda)) \Delta \tau} \\ &\leq e^{\int_s^t \Psi_{\mu(\tau)}(\operatorname{Re}_{\mu_*(s)}(z, \lambda)) \Delta \tau} \\ &= e_{\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z, \lambda)}(t, s) \end{aligned}$$

for any $t \in [s, \infty)$.

2. Observe that

$$\begin{aligned} (\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z))(t) &= \frac{\lambda - \operatorname{Re}_{\mu_*(s)}(z)}{1 + \mu(t) \operatorname{Re}_{\mu_*(s)}(z)} \\ &= \frac{\lambda - \operatorname{Re}_{\mu_*(s)}(z)}{1 + \lambda \mu(t) + \mu(t) (\operatorname{Re}_{\mu_*(s)}(z) - \lambda)} \end{aligned}$$

and

$$\begin{aligned} 1 + \mu(t) (\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z))(t) &= 1 + \mu(t) \frac{\lambda - \operatorname{Re}_{\mu_*(s)}(z)}{1 + \lambda \mu(t) + \mu(t) (\operatorname{Re}_{\mu_*(s)}(z) - \lambda)} \\ &= \frac{1 + \lambda \mu(t)}{1 + \lambda \mu(t) + \mu(t) (\operatorname{Re}_{\mu_*(s)}(z) - \lambda)}. \end{aligned}$$

Therefore

$$\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z) < 0$$

and

$$\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z) \in \mathcal{R}_c^+([s, \infty), \mathbb{R}).$$

Now, applying the above lemma, we have

$$\begin{aligned} e_{\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z)}(t, s) &\leq e^{\int_s^t (\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z))(\tau) \Delta \tau} \\ &= e^{-\frac{(\operatorname{Re}_{\mu_*(s)}(z) - \lambda)}{1 + \mu(\tau) \operatorname{Re}_{\mu_*(s)}(z)} \int_s^t \Delta \tau} \\ &= e^{-(\operatorname{Re}_{\mu_*(s)}(z) - \lambda) m_{\operatorname{Re}_{\mu_*(s)}(z)}(t, s)} \\ &\rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

3. By 1 and 2, we obtain

$$\begin{aligned} |e_{\lambda \ominus z}(t, s)| &\leq e_{\lambda \ominus \operatorname{Re}_{\mu_*(s)}(z)}(t, s) \\ &\rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This completes the proof.

Definition (Exponential Order)

A function $f \in \mathcal{C}_{rd}(\mathbb{T})$ has exponential order α on $[s, \infty)$, if

1. $\alpha \in \mathcal{R}_c^+([s, \infty), \mathbb{R})$,
2. there exists a constant K such that

$$|f(t)| \leq K e_\alpha(t, s), \quad t \in [s, \infty).$$

✓ Example

$e_\alpha(\cdot, s)$, $s \in \mathbb{T}$, $\alpha \in \mathcal{R}_c^+([s, \infty), \mathbb{R})$ is a function of exponential order α .

✓ Example

Let $\epsilon > 0$ and $n \in \mathbb{N}_0$. Then, for all $s, t \in \mathbb{T}$, $t \geq s$, we have

$$\begin{aligned} h_n(t, s) &= \frac{1}{\epsilon^n} \epsilon^n h_n(t, s) \\ &\leq \frac{1}{\epsilon^n} \sum_{k=0}^n \epsilon^k h_k(t, s) \\ &= \frac{1}{\epsilon^n} e_\epsilon(t, s). \end{aligned}$$

Thus, $h_n(\cdot, s)$ is a function of exponential order ϵ on $[s, \infty)$.

Lemma

Let $s \in \mathbb{T}$ and $f \in \mathcal{C}_{rd}([s, \infty))$ be a function of exponential order α . Then

$$\lim_{t \rightarrow \infty} f(t) e_{\Theta z}(t, s) = 0$$

for any $z \in \mathbb{C}_{\mu_\bullet(s)}(\alpha)$.

Proof

Since f is of exponential order α , there exists a constant $K > 0$ such that

$$|f(t)| \leq K e_\alpha(t, s), \quad t \in [s, \infty).$$

Hence and the above theorem, we get

$$\begin{aligned} |f(t) e_{\Theta z}(t, s)| &= |f(t)| |e_{\Theta z}(t, s)| \\ &\leq K e_\alpha(t, s) e_{\Theta \operatorname{Re}_{\mu_\bullet(s)}(z)}(t, s) \\ &= K e_{\alpha \ominus \operatorname{Re}_{\mu_\bullet(s)}(z)}(t, s) \\ &\rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This completes the proof.

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Chapter 3: Convergence of the Laplace Transform

Theorem (Absolute Convergence of the Laplace Transform)

Let $f \in \mathcal{C}_{rd}([s, \infty))$ be of exponential order α . Then the Laplace transform $\mathcal{L}(f)(\cdot, s)$ exists on $\mathbb{C}_{\mu_*(s)}(\alpha)$ and converges absolutely.

Proof

Since $f \in \mathcal{C}_{rd}([s, \infty))$ is of exponential order α on $[s, \infty)$, there exists a constant $K > 0$ such that

$$|f(t)| \leq K e_{\alpha}(t, s), \quad t \in [s, \infty).$$

Note that

$$\operatorname{Re}_{\mu(t)}(z) \geq \operatorname{Re}_{\mu_*(s)}(z), \quad t \in [s, \infty).$$

Thus,

$$\begin{aligned} |1 + \mu(t)z| &= 1 + \mu(t)\operatorname{Re}_{\mu(t)}(z) \\ &\geq 1 + \mu(t)\operatorname{Re}_{\mu_*(s)}(z), \quad t \in [s, \infty). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \left| \int_s^t f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta \tau \right| &\leq \int_s^t |f(\tau) e_{\Theta z}(\sigma(\tau), s)| \Delta \tau \\ &\leq K \int_s^t |e_{\alpha}(\tau, s) e_{\Theta z}(\sigma(\tau), s)| \Delta \tau \\ &= K \int_s^t \frac{1}{|1 + \mu(\tau)z|} |e_{\alpha \Theta z}(\tau, s)| \Delta \tau \\ &\leq K \int_s^t \frac{e_{\alpha \Theta \operatorname{Re}_{\mu_*(s)}(z)}(\tau, s)}{1 + \mu(\tau)\operatorname{Re}_{\mu_*(s)}(z)} \Delta \tau \\ &= \frac{K}{\alpha - \operatorname{Re}_{\mu_*(s)}(z)} \int_s^t \frac{\alpha - \operatorname{Re}_{\mu_*(s)}(z)}{1 + \mu(\tau)\operatorname{Re}_{\mu_*(s)}(z)} e_{\alpha \Theta \operatorname{Re}_{\mu_*(s)}(z)}(\tau, s) \Delta \tau \\ &= \frac{K}{\alpha - \operatorname{Re}_{\mu_*(s)}(z)} \int_s^t (\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z)) e_{\alpha \Theta \operatorname{Re}_{\mu_*(s)}(z)}(\tau, s) \Delta \tau \\ &= \frac{K}{\operatorname{Re}_{\mu_*(s)}(z) - \alpha} (1 - e_{\alpha \Theta \operatorname{Re}_{\mu_*(s)}(z)}(t, s)) \\ &\rightarrow \frac{K}{\operatorname{Re}_{\mu_*(s)}(z) - \alpha}, \quad \text{as } t \rightarrow \infty, \quad z \in \mathbb{C}_{\mu_*(s)}(\alpha). \end{aligned}$$

Thus,

$$\begin{aligned} \left| \int_s^{\infty} f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta \tau \right| &\leq \int_s^{\infty} |f(\tau) e_{\Theta z}(\sigma(\tau), s)| \Delta \tau \\ &\leq \frac{K}{\operatorname{Re}_{\mu_*(s)}(z) - \alpha}, \quad z \in \mathbb{C}_{\mu_*(s)}(\alpha). \end{aligned} \quad (\text{Chapter 3.1})$$

Consequently the Laplace transform of the function f converges absolutely on $\mathbb{C}_{\mu_*(s)}(z)$. This completes the proof.

By the last theorem, we get the following corollary.

Corollary

Let $f \in \mathcal{C}_{rd}([s, \infty))$ be of some exponential order. Then, by (Chapter 3.1), we have that

$$\lim_{\operatorname{Re}(z) \rightarrow \infty} \mathcal{L}(f)(z, s) = 0.$$

Therefore

$$\lim_{|z| \rightarrow \infty} \mathcal{L}(f)(z, s) = 0$$

in the case when $\mu_*(s) > 0$.

Theorem (Uniform Convergence of the Laplace Transform)

Let $f \in \mathcal{C}_{rd}([s, \infty))$ be of exponential order α . Then, the Laplace transform $\mathcal{L}(f)$ converges uniformly in the half plane $\mathbb{C}_{\mu_*(s)}(\beta)$, where $\beta > \alpha$.

Proof

Since f is of exponential order α on $[s, \infty)$, there exists a constant $K > 0$ such that

$$|f(t)| \leq K e_{\alpha}(t, s), \quad t \in [s, \infty).$$

For $t \geq s$, we have

$$\int_t^{\infty} |f(\tau) e_{\ominus z}(\sigma(\tau), s)| \Delta\tau \leq \frac{K}{\operatorname{Re}_{\mu_*(s)}(z) - \alpha} e_{\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z)}(t, s).$$

Thus,

$$\int_t^{\infty} |f(\tau) e_{\ominus z}(\sigma(\tau), s)| \Delta\tau \leq \frac{K}{\beta - \alpha} e_{\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z)}(t, s), \quad t \in [s, \infty). \quad (\text{Chapter 3.2})$$

Because

$$1 + \mu(\tau) \operatorname{Re}_{\mu_*(s)}(z) \geq 1 + \mu(\tau) \alpha, \quad \tau \in [s, t],$$

we have

$$m_{\operatorname{Re}_{\mu_*(s)}(z)}(t, s) \leq m_{\alpha}(t, s), \quad z \in \mathbb{C}_{\mu_*(s)}(\alpha), \quad t \in [s, \infty).$$

Hence, we arrive at

$$\begin{aligned} e_{\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z)}(t, s) &\leq e^{\int_s^t (\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z))(\tau) \Delta\tau} \\ &= e^{-\frac{(\operatorname{Re}_{\mu_*(s)}(z) - \alpha) \int_s^t \frac{\Delta\tau}{1 + \mu(\tau) \operatorname{Re}_{\mu_*(s)}(z)}}} \\ &= e^{-(\operatorname{Re}_{\mu_*(s)}(z) - \alpha) m_{\operatorname{Re}_{\mu_*(s)}(z)}(t, s)} \\ &\rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Therefore, for any $\epsilon > 0$, there exists an $r \in [s, \infty)$ such that

$$\left| \int_t^{\infty} f(\tau) e_{\ominus z}(\sigma(\tau), s) \Delta\tau \right| \leq \epsilon, \quad t \in [r, \infty), \quad z \in \mathbb{C}_{\mu_*(s)}(\beta).$$

This completes the proof.

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Chapter 4: Inversion of the Laplace Transform

Theorem

Let $f \in \mathcal{C}_{rd}([s, \infty))$ be of exponential order α . Consider $\mathcal{L}(f)(\cdot, s)$ on $\mathbb{C}_{\mu(s)}(\beta)$ for some $\beta > \alpha$ and suppose that it has finitely many regressive poles of finite order $\{z_1, z_2, \dots, z_n\}$ and $\tilde{F}_{\mathbf{R}}(z)$ is the Laplace transform of the function $\tilde{f}$ on $\mathbb{R}$ that corresponds to the transform $\mathcal{L}(f)(z, s)$ of f on $\mathbb{T}$. If

$$\int_{c-i\infty}^{c+i\infty} |\tilde{F}_{\mathbf{R}}(z)| |dz| < \infty,$$

then

$$f(t) = \sum_{i=1}^n \text{res}_{z=z_i} e_z(t, s) \mathcal{L}(f)(z, s)$$

has Laplace transform $\mathcal{L}(f)(z, s)$ for all $z \in \mathbb{C}_{\mu(s)}(\beta) \cap \mathbb{C}_0(c)$.

Proof

Without loss of generality, suppose that $s = 0$.

We have that $\mathcal{L}(f)(\cdot)$ converges uniformly on $\mathbb{C}_{\mu(s)}(\beta) \cap \mathbb{C}_0(c)$ and hence, it is analytic in this region. Next, we have that

$$\lim_{|z| \rightarrow \infty} \mathcal{L}(f)(z) = 0.$$

Let $\mathcal{C}$ be the collection of the Laplace transforms over $\mathbb{R}$, $\mathcal{D}$ be the collection of the Laplace transforms over $\mathbb{T}$, i.e., $\mathcal{C} = \{\tilde{F}_{\mathbf{R}}(z)\}$, $\mathcal{D} = \{\mathcal{L}(f)(z)\}$, where

$$\begin{aligned} \tilde{F}_{\mathbf{R}}(z) &= G(z) e^{-z\tau}, \\ \mathcal{L}(f)(z) &= G(z) e_{\Theta z}(\tau, 0) \end{aligned}$$

for G a rational function and for $\tau \in \mathbb{T}$ a constant. Let $C_{p-c0}(\mathbb{R}, \mathbb{R})$ denotes the space of piecewise continuous functions of exponential order, $C_{prd-cz}(\mathbb{T}, \mathbb{R})$ denotes the space of piecewise right-dense continuous functions of double exponential type in which the exponential function coincides with a Hilger exponential function. Each of $\theta, \gamma, \theta^{-1}, \gamma^{-1}$ map functions involving the continuous exponential to the time scale exponential and vice versa. For example, γ maps the function

$$\tilde{F}_{\mathbf{R}}(z) = \frac{e^{-za}}{z}$$

to the function

$$\mathcal{L}(f)(z) = \frac{e_{\Theta z}(a, 0)}{z},$$

while γ^{-1} maps $\mathcal{L}(f)(z)$ back to $\tilde{F}_{\mathbf{R}}(z)$ in the obvious manner. If the representation of $\mathcal{L}(f)(z)$ is independent of the exponential, that is $\tau = 0$, then γ and its inverse γ^{-1} will act as the identity. For example,

$$\gamma\left(\frac{1}{1+z^2}\right) = \gamma^{-1}\left(\frac{1}{1+z^2}\right) = \frac{1}{1+z^2}.$$

The map θ will set the continuous exponential function to the time scale exponential function in the following manner: if we

write $\tilde{f} \in C_{p-\mathcal{A}0}(\mathbb{R}, \mathbb{R})$ as

$$\tilde{f}(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e^{zt} \tilde{F}_{\mathbf{R}}(z),$$

then

$$\theta(\tilde{f}(t)) = \sum_{i=1}^n \text{Res}_{z=z_i} e_z(t, 0) \mathcal{L}(f)(z).$$

To go from $\tilde{F}_{\mathbf{R}}(z)$ to $\mathcal{L}(f)(z)$, we simply switch expressions involving the continuous exponential in $\tilde{F}_{\mathbf{R}}$ with the time scale exponential giving $\mathcal{L}(f)(z)$ as was done for γ and its inverse θ^{-1} will then act on $g \in C_{prd-ez}(\mathbb{T}, \mathbb{R})$,

$$g(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e_z(t, 0) G_{\mathbf{T}}(z)$$

as

$$\theta^{-1}(g(t)) = \sum_{i=1}^n \text{Res}_{z=z_i} e^{zt} \tilde{G}_{\mathbf{R}}(z).$$

For example, for the unit step function $\tilde{u}_a(t)$ on $\mathbb{R}$, we know from the continuous result that we may write the step function as

$$\tilde{u}_a(t) = \text{Res}_{z=0} e^{zt} \frac{e^{-az}}{z},$$

so that if $a \in \mathbb{T}$, then

$$\theta(\tilde{u}_a(t)) = \text{Res}_{z=0} e_z(t, 0) \frac{e_{\ominus z}(a, 0)}{z}.$$

Now, for a given time scale Laplace transform $\mathcal{L}(f)(z)$, we begin by mapping to $\tilde{F}_{\mathbf{R}}(z)$ via γ^{-1} . We have that the inverse of $\tilde{F}_{\mathbf{R}}(z)$ exists for all z with $z \in \mathbb{C}_{\mu_+(a)}(\beta) \cap \mathbb{C}_0(c)$ and it is given by

$$\tilde{f}(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e^{zt} \tilde{F}_{\mathbf{R}}(z).$$

Applying θ to $\tilde{f}(t)$ to retrieve the time scale function, we get

$$f(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e_z(t, 0) \mathcal{L}(f)(z),$$

whereby

$$(\gamma \circ \tilde{F}_{\mathbf{R}} \circ \theta^{-1})(f(t)) = \mathcal{L}(f)(z).$$

This completes the proof.

Theorem

If the functions $f, g : \mathbb{T} \rightarrow \mathbb{R}$ satisfy all conditions of the previous theorem and have the same Laplace transform, then $f = g$ a.e.

Proof

Suppose that

$$\int_s^\infty e_{\ominus z}^\sigma(t, s) f(t) \Delta t = \int_s^\infty e_{\ominus z}^\sigma(t, s) g(t) \Delta t.$$

Hence, $h = f - g$ has Laplace transform zero and $h \in \ker \mathcal{L}$. If we denote by $\mathcal{L}^{-1}$ the inversion of $\mathcal{L}$, then

$$\begin{aligned} \mathcal{L}^{-1} \circ \mathcal{L}(h) &= \mathcal{L}^{-1}(0) \\ &= 0 \\ &= h. \end{aligned}$$

Therefore $f = g$ a.e. This completes the proof.

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Chapter 5: The Laplace transform of Power Series

Theorem

Assume that there exist $M, \alpha > 0$ such that

$$|a_k| \leq M\alpha^k, \quad k \in \mathbb{N}_0.$$

Then, for any $z \in \mathbb{C}_{\mu_\alpha(s)}(\alpha)$, we have

$$\begin{aligned} \mathcal{L} \left(\sum_{k=0}^{\infty} a_k h_k(\cdot, s) \right) (z, s) &= \sum_{k=0}^{\infty} a_k \mathcal{L}(h_k(\cdot, s))(z, s) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}. \end{aligned}$$

Proof

Let

$$f(t) = \sum_{k=0}^{\infty} a_k h_k(t, s), \quad t \in [s, \infty),$$

which is a convergent series. Since $h_k(\cdot, s)$ is a continuous function on $[s, \infty)$, we have that f is a continuous function on $[s, \infty)$. Next,

$$\begin{aligned} |f(t)| &= \left| \sum_{k=0}^{\infty} a_k h_k(t, s) \right| \\ &\leq \sum_{k=0}^{\infty} |a_k| h_k(t, s) \\ &\leq M \sum_{k=0}^{\infty} \alpha^k h_k(t, s) \\ &= M e_\alpha(t, s), \quad t \in [s, \infty). \end{aligned}$$

Thus, f is of exponential order α . Observe that

$$|e_{\Theta z}(t, s)| \leq e_{\Theta \operatorname{Re}_{\mu_\alpha(s)}(z)}(t, s), \quad t \in [s, \infty), \quad z \in \mathbb{C}_{\mu_\alpha(s)}(\alpha),$$

and

$$\begin{aligned} \left| f(t) - \sum_{k=0}^{n-1} a_k h_k(t, s) \right| &= \left| \sum_{k=n}^{\infty} a_k h_k(t, s) \right| \\ &\leq \sum_{k=n}^{\infty} |a_k| h_k(t, s) \\ &\leq M \sum_{k=n}^{\infty} \alpha^k h_k(t, s) \\ &= M \left(\sum_{k=0}^{\infty} \alpha^k h_k(t, s) - \sum_{k=0}^{n-1} \alpha^k h_k(t, s) \right) \\ &= M \left(e_\alpha(t, s) - \sum_{k=0}^{n-1} \alpha^k h_k(t, s) \right), \quad t \in [s, \infty). \end{aligned}$$

Hence,

$$\begin{aligned}
 & \left| \mathcal{L}(f)(z, s) - \sum_{k=0}^{n-1} a_k \mathcal{L}(h_k(\cdot, s))(z, s) \right| \\
 &= \left| \mathcal{L} \left(f - \sum_{k=0}^{n-1} a_k h_k(\cdot, s) \right) (z, s) \right| \\
 &\leq \mathcal{L} \left(\left| f - \sum_{k=0}^{n-1} a_k h_k(\cdot, s) \right| \right) (\operatorname{Re}_{\mu_*(s)}(z), s) \\
 &\leq M \mathcal{L} \left(e_\alpha(t, s) - \sum_{k=0}^{n-1} \alpha^k h_k(t, s) \right) (\operatorname{Re}_{\mu_*(s)}(z), s) \\
 &= M \left(\frac{1}{\operatorname{Re}_{\mu_*(s)}(z) - \alpha} - \frac{1}{\operatorname{Re}_{\mu_*(s)}(z)} \sum_{k=0}^{n-1} \left(\frac{\alpha}{\operatorname{Re}_{\mu_*(s)}(z)} \right)^k \right) \\
 &= M \left(\frac{1}{\operatorname{Re}_{\mu_*(s)}(z) - \alpha} - \frac{1}{\operatorname{Re}_{\mu_*(s)}(z)} \frac{1 - \left(\frac{\alpha}{\operatorname{Re}_{\mu_*(s)}(z)} \right)^n}{1 - \frac{\alpha}{\operatorname{Re}_{\mu_*(s)}(z)}} \right) \\
 &= \frac{M}{\operatorname{Re}_{\mu_*(s)}(z) - \alpha} \left(\frac{\alpha}{\operatorname{Re}_{\mu_*(s)}(z)} \right)^n \\
 &\rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad z \in \mathbb{C}_{\mu_*(s)}(\alpha).
 \end{aligned}$$

This completes the proof.

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Chapter 6: Differentiation of the Laplace Transform

Assume that there exist $a, b \in \mathbb{T}$, $b > a$, and a bounded domain $D \subset \mathbb{C}$ such that $s, t \in [a, b]$ and for $z \in D$ we have $z \in \mathcal{R}_c(\mathbb{T}, \mathbb{C})$. Then, for any $t, s \in [a, b]$, $t \geq s$, and for all $z \in D$, we have

$$\begin{aligned}
 \frac{d}{dz} e_z(t, s) &= \frac{d}{dz} \sum_{k=0}^{\infty} h_k(t, s) z^k \\
 &= \sum_{k=1}^{\infty} k h_k(t, s) z^{k-1} \\
 &= \sum_{k=0}^{\infty} (k+1) h_{k+1}(t, s) z^k \\
 &= \sum_{k=0}^{\infty} \left(\sum_{l=0}^k h_{k+1}(t, s) \right) z^k \\
 &= \sum_{k=0}^{\infty} \left(\sum_{l=0}^k \int_s^t h_{k-l}(t, \sigma(\tau)) h_l(\tau, s) \Delta\tau \right) z^k \\
 &= \int_s^t \sum_{k=0}^{\infty} \left(\sum_{l=0}^k h_{k-l}(t, \sigma(\tau)) h_l(\tau, s) \right) z^k \Delta\tau \\
 &= \int_s^t \left(\sum_{k=0}^{\infty} h_k(t, \sigma(\tau)) z^k \right) \left(\sum_{k=0}^{\infty} h_k(\tau, s) z^k \right) \Delta\tau \\
 &= \int_s^t e_z(t, \sigma(\tau)) e_z(\tau, s) \Delta\tau \\
 &= \int_s^t \frac{e_z(t, \tau) e_z(\tau, s)}{1 + z\mu(\tau)} \Delta\tau \\
 &= m_z(t, s) e_z(t, s), \quad t, s \in [a, b], \quad t \geq s, \quad z \in D.
 \end{aligned}$$

Because $a, b \in \mathbb{T}$ and $D \subset \mathbb{C}$ was arbitrarily chosen, we conclude that

$$\frac{d}{dz} e_z(t, s) = m_z(t, s) e_z(t, s), \quad z \in \mathcal{R}_c(\mathbb{T}, \mathbb{C}), \quad t \in [s, \infty).$$

Now, we differentiate with respect to z the equation

$$e_z^\Delta(t, s) = z e_z(t, s), \quad e_z(s, s) = 1, \quad t \geq s,$$

and we get

$$\left(\frac{d}{dz} e_z(t, s) \right)^\Delta = e_z(t, s) + z \frac{d}{dz} e_z(t, s), \quad \frac{d}{dz} e_z(s, s) = 0.$$

Hence, using the variation of parameters, we find

$$\begin{aligned}
 \frac{d}{dz} e_z(t, s) &= \int_s^t e_z(t, \sigma(\tau)) e_z(\tau, s) \Delta\tau \\
 &= e_z(t, s) m_z(t, s), \quad t \in [s, \infty), \quad z \in \mathcal{R}_c(\mathbb{T}, \mathbb{C}).
 \end{aligned}$$

Also,

$$\begin{aligned}
 \frac{d}{dz} e_{\Theta z}(t, s) &= \frac{d}{dz} \left(\frac{1}{e_z(t, s)} \right) \\
 &= - \frac{m_z(t, s) e_z(t, s)}{(e_z(t, s))^2} \\
 &= - \frac{m_z(t, s)}{e_z(t, s)} \\
 &= - m_z(t, s) e_{\Theta z}(t, s), \quad t, s \in [a, b].
 \end{aligned}$$

Theorem

Let $f \in \mathcal{C}_{rd}(\mathbb{T})$ be of exponential order α with a constant K . Then for any $k \in \mathbb{N}$ and $z \in \mathbb{C}_{\mu(s)}(\alpha)$, $\frac{d^k}{dz^k} \mathcal{L}_{\mathbb{T}}(f)(z; s)$ exists with

$$\frac{d^k}{dz^k} \mathcal{L}^b(f)(z, s) = \mathcal{L}(f u_k(z, \sigma(\cdot), s))(z, s),$$

where

$$u_n(z, t, s) = \begin{cases} 1 & \text{if } n = 0 \\ \frac{d}{dz} u_{n-1}(z, t, s) - u_{n-1}(z, t, s) m_z(t, s) & \text{if } n \in \mathbb{N}. \end{cases}$$

Proof

We will prove that for $k \in \mathbb{N}$, the integrals

$$\int_s^\infty f(t) \frac{d^k}{dz^k} e_{\Theta z}(\sigma(t), s) \Delta t$$

are uniformly convergent with respect to $z \in \mathbb{C}_{\mu(s)}(\alpha)$. It suffices to prove that for $l, r \in \mathbb{N}_0$, the integrals

$$\int_0^\infty f(t) (m_z(\sigma(t), s))^l \frac{d^r}{dz^r} [m_z(\sigma(t), s)] e_{\Theta z}(\sigma(t), s) \Delta t$$

are uniformly convergent with respect to $z \in \mathbb{C}_{\mu(s)}(\alpha)$. For $t, s \in \mathbb{T}$ and $r \in \mathbb{N}$, we compute

$$\frac{d^r}{dz^r} m_z(t, s) = (-1)^r r! \int_s^t \frac{(\mu(\tau))^r}{(1 + z\mu(\tau))^{r+1}} \Delta \tau.$$

For $t \geq s$, set $p_1(x) = \frac{x}{1+\gamma x}$ for $x \in \mathbb{R}$ such that $x, 1 + \gamma x > 0$. Then $p_1'(x) = \frac{1}{(1+\gamma x)^2} > 0$. Since $\lim_{x \rightarrow \infty} p_1(x) = \frac{1}{\gamma}$, there exists a constant $P_1 > 0$ such that $p_1(x) \leq P_1$ for $x > 0$ and $1 + \gamma x > 0$. Hence,

$$\begin{aligned}
 \left| \frac{d^r}{dz^r} m_z(t, s) \right| &\leq r! \int_s^t \frac{(\mu(\tau))^r}{|1 + z\mu(\tau)|^{r+1}} \Delta \tau \\
 &= r! \int_s^t \frac{(\mu(\tau))^r}{(1 + \mu(\tau) \operatorname{Re}_{\mu(\tau)}(z))^{r+1}} \Delta \tau \\
 &\leq r! \int_s^t \frac{(\mu(\tau))^r}{(1 + \mu(\tau) \operatorname{Re}_{\mu(s)}(z))^r (1 + \mu(\tau) \operatorname{Re}_{\mu(\tau)}(z))} \Delta \tau \\
 &\leq r! \int_s^t \frac{(\mu(\tau))^r}{(1 + \mu(\tau) \alpha)^r (1 + \mu(\tau) \operatorname{Re}_{\mu(\tau)}(z))} \Delta \tau \\
 &\leq P_1^r r! \int_s^t \frac{1}{1 + \mu(\tau) \operatorname{Re}_{\mu(\tau)}(z)} \Delta \tau \\
 &= P_1^r r! m_{\operatorname{Re}_{\mu(z)}}(t, s),
 \end{aligned}$$

and

$$\begin{aligned}
 |m_z(t, s)| &= \left| \int_s^t \frac{1}{1 + \mu(\tau)z} \Delta\tau \right| \\
 &\leq \int_s^t \frac{1}{|1 + \mu(\tau)z|} \Delta\tau \\
 &= \int_s^t \frac{1}{1 + \mu(\tau)\operatorname{Re}_{\mu(\tau)}(z)} \Delta\tau \\
 &= m_{\operatorname{Re}_{\mu(z)}}(t, s).
 \end{aligned}$$

Therefore we have to estimate the integral

$$J_1 = \int_s^\infty f(t) (m_z(\sigma(t), s))^l e_{\Theta z}(\sigma(t), s) \Delta t.$$

Using the Potzsche chain rule, we get

$$\begin{aligned}
 \left((m_{\operatorname{Re}_{\mu(z)}}(t, s))^l \right)^\Delta &= l \left(\int_0^1 \left(m_{\operatorname{Re}_{\mu(z)}}(t, s) + h\mu(t) m_{\operatorname{Re}_{\mu(z)}}^\Delta(t, s) \right)^{l-1} dh \right) m_{\operatorname{Re}_{\mu(z)}}^\Delta(t, s) \\
 &= \frac{l}{1 + \mu(t)\operatorname{Re}_{\mu(z)}} \int_0^1 \left(m_{\operatorname{Re}_{\mu(z)}}(t, s) + h\mu(t) m_{\operatorname{Re}_{\mu(z)}}^\Delta(t, s) \right)^{l-1} dh \\
 &\leq \frac{l}{1 + \mu(t)\operatorname{Re}_{\mu(z)}} \left(m_{\operatorname{Re}_{\mu(z)}}(t, s) + \mu(t) m_{\operatorname{Re}_{\mu(z)}}^\Delta(t, s) \right)^{l-1} \\
 &= \frac{l}{1 + \mu(t)\operatorname{Re}_{\mu(z)}} \left(m_{\operatorname{Re}_{\mu(z)}}(\sigma(t), s) \right)^{l-1}.
 \end{aligned}$$

Then, for $s_1 \geq s$, after multiple applications of integration by parts, we find

$$\begin{aligned}
 \left| \int_s^{s_1} f(t) (m_z(\sigma(t), s))^l e_{\Theta z}(\sigma(t), s) \Delta t \right| &\leq \frac{K}{\alpha - \operatorname{Re}_{\mu_*(s)}(z)} e_{\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z)}(s_1, s) \left(m_{\operatorname{Re}_{\mu_*(s)}(z)}(s_1, s) \right)^l \\
 &\quad + \frac{Kl}{(\operatorname{Re}_{\mu_*(s)}(z) - \alpha)(\alpha - \operatorname{Re}_{\mu_*(s)}(z))} \\
 &\quad \left(m_{\operatorname{Re}_{\mu_*(s)}(z)}(s_1, s) \right)^{l-1} e_{\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z)}(s_1, s) \\
 &\quad + \dots - \frac{Kl!}{(\operatorname{Re}_{\mu_*(s)}(z) - \alpha)^{l+1}} \left(e_{\alpha \ominus \operatorname{Re}_{\mu_*(s)}(z)}(s_1, s) - 1 \right).
 \end{aligned}$$

Therefore

$$|J_1| \leq \frac{Kl!}{(\operatorname{Re}_{\mu_*(s)}(z) - \alpha)^{l+1}}.$$

Hence,

$$\begin{aligned}
 \frac{d}{dz} \mathcal{L}(f)(z, s) &= \frac{d}{dz} \int_s^\infty f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta \tau \\
 &= \int_s^\infty f(\tau) \frac{d}{dz} e_{\Theta z}(\sigma(\tau), s) \Delta \tau \\
 &= - \int_s^\infty f(\tau) m_z(\sigma(\tau), s) e_{\Theta z}(\sigma(\tau), s) \Delta \tau \\
 &= - \mathcal{L}(m_z(\sigma(\cdot), s) f)(z, s) \\
 &= \mathcal{L}(u_1(z, \sigma(\cdot), s))(z, s).
 \end{aligned}$$

for all $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$. Assume that

$$\frac{d^k}{dz^k} \mathcal{L}(f)(z, s) = \mathcal{L}(u_{k-1}(z, \sigma(\cdot), s) f)(z, s)$$

for all $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$. We will prove that

$$\frac{d^{k+1}}{dz^{k+1}} \mathcal{L}(f)(z, s) = \mathcal{L}(u_k(z, \sigma(\cdot), s) f)(z, s)$$

for all $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$. Really,

$$\begin{aligned}
 \frac{d^{k+1}}{dz^{k+1}} \mathcal{L}(f)(z, s) &= \frac{d}{dz} \mathcal{L}(u_{k-1}(z, \sigma(\cdot), s) f)(z, s) \\
 &= \frac{d}{dz} \int_s^\infty u_{k-1}(z, \sigma(t), s) f(t) e_{\Theta z}(\sigma(t), s) \Delta t \\
 &= \int_s^\infty \left(\frac{d}{dz} u_{k-1}(z, \sigma(t), s) - m_z(\sigma(t), s) u_{k-1}(z, \sigma(t), s) \right) f(t) e_{\Theta z}(\sigma(t), s) \Delta t \\
 &= \int_s^\infty u_k(z, \sigma(t), s) f(t) e_{\Theta z}(\sigma(t), s) \Delta t \\
 &= \mathcal{L}(u_k(z, \sigma(\cdot), s) f)(z, s)
 \end{aligned}$$

for all $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$. This completes the proof.

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Chapter 7: The Dirac Delta Functional

Definition

The Hilger Dirac delta function is defined as follows.

$$\delta_{\tau}^H(t) = \begin{cases} \frac{\delta[t-\tau]}{\mu(t)} & \text{if } \mu(t) > 0 \\ \delta(t-\tau) & \text{otherwise,} \end{cases}$$

where $\delta[\cdot]$ denotes the Kronecker delta, i.e., $\delta[0] = 1$ and $\delta[t] = 0$ for $t \neq 0$, $\delta(\cdot)$ is the Dirac delta function.

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a given function having unit area. With $\mathcal{C}_c^\infty(\mathbb{T})$ we denote the $\mathcal{C}^\infty(\mathbb{T})$ functions with compact support.

Definition

For $g^\sigma \in \mathcal{C}_c^\infty(\mathbb{T})$ and for all $\epsilon > 0$, define the time scale Dirac delta functional as follows.

$$\langle \delta_a(t), g^\sigma \rangle = \begin{cases} \int_0^\infty \delta_a^H(x) g^\sigma(\sigma(x)) \Delta x & \text{if } a \text{ is right-scattered} \\ \lim_{\epsilon \rightarrow 0} \int_0^\infty \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) g^\sigma(\sigma(x)) \Delta x & \text{if } a \text{ is right-dense.} \end{cases}$$

Below we will list some of the properties of the Dirac delta functional.

1. Suppose that a is right-dense. Consider

$$f(x) = \begin{cases} \frac{1}{\epsilon} & \text{if } a \leq x \leq a + \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \langle \delta_a(x), g^\sigma \rangle &= \lim_{\epsilon \rightarrow 0} \int_0^\infty \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) g^\sigma(\sigma(x)) \Delta x \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_a^{a+\epsilon} g^\sigma(\sigma(x)) \Delta x \\ &= g^\sigma(\sigma(a)) \\ &= g^\sigma(a), \end{aligned}$$

where $g^\sigma \in \mathcal{C}_c^\infty(\mathbb{T})$.

2. Let a is right-scattered. Then

$$\begin{aligned} \langle \delta_a(x), g^\sigma \rangle &= \int_0^\infty \delta_a^H(x) g^\sigma(\sigma(x)) \Delta x \\ &= g^\sigma(\sigma(a)) \\ &= g^\sigma(a), \end{aligned}$$

where $g^\sigma \in \mathcal{C}_c^\infty(\mathbb{T})$.

3. We have

$$\begin{aligned} \mathcal{L}(\delta_0)(z) &= \int_0^\infty e_{\ominus z}^\sigma(t, 0) \delta_0 \Delta t \\ &= 1. \end{aligned}$$

4. We have

$$\begin{aligned}
 \mathcal{L}(\delta_0^\Delta)(z) &= \int_0^\infty \delta_0^\Delta(t) e_{\ominus z}^\sigma(t, 0) \Delta t \\
 &= \delta_0 e_{\ominus z}(t, 0) \Big|_{t=0}^{t=\infty} - \int_0^\infty \ominus z e_{\ominus z}(t, 0) \delta_0 \Delta t \\
 &= z \int_0^\infty e_{\ominus z}^\sigma(t, 0) \delta_0 \Delta t + 1 \\
 &= z \mathcal{L}(\delta_0)(z) + 1 \\
 &= z + 1.
 \end{aligned}$$

5. Let H be the Heaviside function. Then

$$\begin{aligned}
 \langle H^\Delta, g^\sigma \rangle &= \int_0^\infty H^\Delta(t) g^\sigma(t) \Delta t \\
 &= H(t) g(t) \Big|_{t=0}^{t=\infty} - \int_0^\infty H(t) g^\Delta(t) \Delta t \\
 &= - \int_0^\infty g^\Delta(t) \\
 &= g(0) \\
 &= \langle \delta_0, g^\sigma \rangle,
 \end{aligned}$$

where $g^\sigma \in \mathcal{C}_c^\infty(\mathbb{T})$.

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Chapter 8: The Bilateral Laplace transform

Suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ , respectively, and $\inf \mathbb{T} = -\infty$, $\sup \mathbb{T} = \infty$. For $s \in \mathbb{T}$, define

$$\begin{aligned}\mu_*(s) &= \inf_{t \in [s, \infty)} \mu(t), \\ \mu^*(s) &= \sup_{t \in (-\infty, s]} \mu(t), \\ \bar{\mu}(s) &= \inf_{t \in (-\infty, s]} \mu(t).\end{aligned}$$

For $s, t \in \mathbb{T}$ and $\lambda \in \mathcal{R}(\mathbb{T})$, set

$$M_\lambda(t, s) = \int_t^s \frac{1}{1 + \lambda \mu(\tau)} \Delta \tau.$$

Theorem

Let $s \in \mathbb{T}$, $\lambda \in \mathcal{R}^+((-\infty, s])$. Then

1. $M_\lambda^\Delta(t, s) < 0$ for all $t \in (-\infty, s)$, where the differentiation is with respect to t .
2. $\lim_{t \rightarrow -\infty} M_\lambda(t, s) = \infty$.

Proof

1. By the definition of the function M_λ , it follows

$$\begin{aligned}M_\lambda^\Delta(t, s) &= -\frac{1}{1 + \lambda \mu(t)} \\ &< 0, \quad t \in (-\infty, s].\end{aligned}$$

2. We will consider two cases.

- a) Let $\sup_{t \in (-\infty, s]} \mu(t) < \infty$. Then

$$\begin{aligned}1 + \lambda \mu(t) &\leq 1 + |\lambda| \mu(t) \\ &\leq 1 + |\lambda| \sup_{t \in (-\infty, s]} \mu(t), \quad t \in (-\infty, s].\end{aligned}$$

Hence,

$$\begin{aligned}M_\lambda(t, s) &= \int_t^s \frac{\Delta \tau}{1 + \lambda \mu(\tau)} \\ &\geq \int_t^s \frac{\Delta \tau}{1 + |\lambda| \sup_{t \in (-\infty, s]} \mu(t)} \\ &= \frac{s - t}{1 + |\lambda| \sup_{t \in (-\infty, s]} \mu(t)} \\ &\rightarrow \infty, \quad \text{as } t \rightarrow -\infty.\end{aligned}$$

- b) Suppose that $\sup_{t \in (-\infty, s]} \mu(t) = \infty$. Because $\lambda \in \mathcal{R}^+((-\infty, s])$, we have that $\lambda \geq 0$. If $\lambda = 0$, then

$$\begin{aligned}M_\lambda(t, s) &= s - t \\ &\rightarrow \infty, \quad \text{as } t \rightarrow -\infty.\end{aligned}$$

Assume that $\lambda > 0$. Since $\sup_{t \in (-\infty, s]} \mu(t) = \infty$, there exists a decreasing and divergent sequence $\{\xi_k\}_{k \in \mathbb{N}} \subset (-\infty, s]$ of right-scattered points such that

$\{\mu(\xi_k)\}_{k \in \mathbb{N}}$

is an increasing and divergent sequence. Hence, using that

$$\lim_{k \rightarrow \infty} \frac{\mu(\xi_k)}{1 + \lambda\mu(\xi_k)} = \frac{1}{\lambda},$$

we get

$$\begin{aligned} M_\lambda(t, s) &= \int_t^s \frac{\Delta\tau}{1 + \lambda\mu(\tau)} \\ &\geq \sum_{\sigma(\xi_k) \geq t, \xi_k \leq s} \int_{\xi_k}^{\sigma(\xi_k)} \frac{\Delta\tau}{1 + \lambda\mu(\tau)} \\ &= \sum_{\sigma(\xi_k) \geq t, \xi_k \leq s} \frac{\mu(\xi_k)}{1 + \lambda\mu(\xi_k)} \\ &\rightarrow \infty, \quad \text{as } t \rightarrow -\infty. \end{aligned}$$

This completes the proof.

Theorem (Decay of the Exponential Function)

Let $s \in \mathbb{T}$, $\lambda \in \mathcal{R}^+((-\infty, s])$ and

$$\overline{\mathcal{C}}_{\mu^*(s)}(\lambda) = \{z \in \mathbb{C} : \operatorname{Re}_{\mu^*(s)}(z) < \lambda\}.$$

Then, for any $z \in \overline{\mathcal{C}}_{\mu^*(s)}(\lambda)$, we have the following properties.

1.

$$|e_{\lambda \ominus z}(t, s)| \leq e_{\lambda \ominus \operatorname{Re}_{\mu^*(s)}(z)}(t, s), \quad t \in (-\infty, s],$$

\item

$$\lim_{t \rightarrow -\infty} e_{\lambda \ominus \operatorname{Re}_{\mu^*(s)}(z)}(t, s) = 0,$$

2. $\lim_{t \rightarrow -\infty} e_{\lambda \ominus z}(t, s) = 0$.

Proof

Let $\Psi_h(z, \lambda)$ be the function, defined in the previous sections.

1. We have

$$\begin{aligned} |e_{\lambda \ominus z}(t, s)| &= e^{\int_s^t \Psi_{\mu(r)}(\operatorname{Re}_{\mu^*(s)}(z), \lambda) \Delta\tau} \\ &\leq e^{\int_s^t \Psi_{\mu(r)}(\operatorname{Re}_{\mu^*(s)}(z), \lambda) \Delta\tau} \\ &= e_{\lambda \ominus \operatorname{Re}_{\mu^*(s)}(z)}(t, s), \quad t \in (-\infty, s]. \end{aligned}$$

2. For $z \in \overline{\mathcal{C}}_{\mu^*(s)}(\lambda)$, we have

$$\begin{aligned} \lambda \ominus \operatorname{Re}_{\mu^*(s)}(z) &= \frac{\lambda - \operatorname{Re}_{\mu^*(s)}(z)}{1 + \mu(t) \operatorname{Re}_{\mu^*(s)}(z)} \\ &> \frac{\lambda - \operatorname{Re}_{\mu^*(s)}(z)}{1 + \lambda\mu(t)} \\ &> 0, \quad t \in (-\infty, s], \end{aligned}$$

and

$$\begin{aligned}
 1 + \mu(t) (\lambda \ominus \operatorname{Re}_{\mu^*(s)}(z)) &= \frac{1 + \lambda\mu(t)}{1 + \mu(t)\operatorname{Re}_{\mu^*(s)}(z)} \\
 &> \frac{1 + \lambda\mu(t)}{1 + \lambda\mu(t)} \\
 &= 1, \quad t \in (-\infty, s].
 \end{aligned}$$

Thus,

$$\lambda \ominus \operatorname{Re}_{\mu^*(s)}(z) \in \mathcal{R}^+((-\infty, s])$$

and

$$\begin{aligned}
 e_{\lambda \ominus \operatorname{Re}_{\mu^*(s)}(z)}(t, s) &= e^{(\lambda - \operatorname{Re}_{\mu^*(s)}(z)) \int_s^t \frac{1}{1 + \mu(\tau)\operatorname{Re}_{\mu^*(s)}(z)} \Delta\tau} \\
 &= e^{(\operatorname{Re}_{\mu^*(s)}(z) - \lambda) M_\lambda(t, s)} \\
 &\rightarrow 0, \quad \text{as } t \rightarrow -\infty.
 \end{aligned}$$

Item By 1 and 2, we obtain

$$\begin{aligned}
 |e_{\lambda \ominus z}(t, s)| &\leq e_{\lambda \ominus \operatorname{Re}_{\mu^*(s)}(z)}(t, s) \\
 &\rightarrow 0, \quad \text{as } t \rightarrow -\infty.
 \end{aligned}$$

This completes the proof.

Definition

Suppose that $s \in \mathbb{T}$, $f: \mathbb{T} \rightarrow \mathbb{R}$ is regulated. Then the bilateral Laplace transform of f is defined by

$$\mathcal{L}^b(f)(z, s) = \int_{-\infty}^{\infty} f(t) e_{\ominus z}^\sigma(t, s) \Delta t$$

for $z \in \mathbb{C}$ for which $1 + \mu(t)z \neq 0$ for any $t \in \mathbb{T}^*$ and the improper integral exists.

Definition (Double Exponential Order)

Let $s \in \mathbb{T}$. A function $f \in \mathcal{C}_{rd}(\mathbb{T})$ has double exponential order (α, γ) on $\mathbb{T}$ if

1. $\alpha \in \mathcal{R}^+([s, \infty))$, $\gamma \in \mathcal{R}^+((-\infty, s])$,
2. there exist positive constants K_α, K_γ such that

$$\begin{aligned}
 |f(t)| &\leq K_\alpha e_\alpha(t, s), \quad t \in [s, \infty), \\
 |f(t)| &\leq K_\gamma e_\gamma(t, s), \quad t \in (-\infty, s].
 \end{aligned}$$

Example

Let $\beta > 0$. Consider the function

$$f(t) = e_\beta(t, s), \quad t \in \mathbb{T}.$$

If $s \leq t$, then $\alpha = \beta$. If $s > t$, then $0 \leq f(t) \leq 1$ and $e_{\ominus\beta}(t, s) > 1$. Thus,

$$f(t) \leq e_{\ominus\beta}(t, s).$$

Therefore f is of double exponential order $(\beta, \Theta\beta)$.

Lemma

Let $s \in \mathbb{T}$ and $f \in \mathcal{C}_{rd}(\mathbb{T})$ be a function of double exponential order (α, γ) . Then

1. $\lim_{t \rightarrow \infty} f(t)e_{\Theta z}(t, s) = 0$ for any $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$.
2. $\lim_{t \rightarrow -\infty} f(t)e_{\Theta z}(t, s) = 0$ for any $z \in \overline{\mathbb{C}}_{\mu^*(s)}(\gamma)$.

Proof

Since f is of double exponential order (α, γ) , there exist positive constants K_α, K_γ such that

$$\begin{aligned} |f(t)| &\leq K_\alpha e_\alpha(t, s), \quad t \in [s, \infty), \\ |f(t)| &\leq K_\gamma e_\gamma(t, s), \quad t \in (-\infty, s]. \end{aligned}$$

1. The first statement follows by the computations for the Laplace transform.
2. By the above theorem, we have

$$\begin{aligned} |f(t)e_{\Theta z}(t, s)| &\leq K_\gamma |e_{\alpha\Theta z}(t, s)| \\ &\leq K_\gamma e_{\alpha\Theta \operatorname{Re}_{\mu^*(s)}(z)}(t, s) \\ &\rightarrow 0, \quad \text{as } t \rightarrow -\infty. \end{aligned}$$

This completes the proof.

For $z \in \mathbb{C}$, denote

$$\overline{\overline{\mu}}(s) = \mu^*(s) \quad \text{if } \operatorname{Re}_{\overline{\mu}(s)}(z) \leq 0$$

and

$$\overline{\overline{\mu}}(s) = \overline{\mu}(s) \quad \text{if } \operatorname{Re}_{\overline{\mu}(s)}(z) \geq 0.$$

Definition

Let $s \in \mathbb{T}$, $\alpha \in \mathcal{R}^+([s, \infty))$, $\gamma \in \mathcal{R}^+((-\infty, s])$. We say that (s, α, γ) is an admissible triple if

$$\mathbb{C}_{s, \alpha, \gamma} = \left\{ z \in \mathbb{C} : \operatorname{Re}_{\mu^*(s)}(z) < \gamma, \quad \operatorname{Re}_{\mu_*(s)}(z) > \alpha, \right. \\ \left. 1 + \overline{\overline{\mu}}(s) \operatorname{Re}_{\overline{\mu}(s)}(z) \neq 0 \right\}.$$

Definition

Let $s \in \mathbb{T}$, $\alpha \in \mathcal{R}^+([s, \infty))$, $\gamma \in \mathcal{R}^+((-\infty, s])$. We say that (s, α, γ) is an admissible triple if

$$\mathbb{C}_{s, \alpha, \gamma} = \left\{ z \in \mathbb{C} : \operatorname{Re}_{\mu^*(s)}(z) < \gamma, \quad \operatorname{Re}_{\mu_*(s)}(z) > \alpha, \right. \\ \left. 1 + \overline{\overline{\mu}}(s) \operatorname{Re}_{\overline{\mu}(s)}(z) \neq 0 \right\}.$$

Theorem (Absolute Convergence of the Bilateral Laplace Transform)

Let (s, α, γ) is an admissible triple, $f \in \mathcal{C}_{rd}(\mathbb{T})$ is of double exponential order (α, γ) . Then $\mathcal{L}^b(f)(\cdot, s)$ exists on $\mathbb{C}_{s, \alpha, \gamma}$ and converges absolutely.

Proof

Since f is of double exponential order (α, γ) , there exist positive constants K_α, K_γ such that

$$\begin{aligned} |f(t)| &\leq K_\alpha e_\alpha(t, s), \quad t \in [s, \infty), \\ |f(t)| &\leq K_\gamma e_\gamma(t, s), \quad t \in (-\infty, s]. \end{aligned}$$

Next, for $t \leq s$, we have

$$\begin{aligned} |1 + \mu(t)z| &= 1 + \mu(t)\operatorname{Re}_{\mu(t)}(z) \\ &\geq 1 + \bar{\mu}(s)\operatorname{Re}_{\bar{\mu}(s)}(z). \end{aligned}$$

For $z \in \mathbb{C}_{s, \alpha, \gamma}$, we find

$$\begin{aligned} |\mathcal{L}^b(f)(z, s)| &= \left| \int_s^\infty f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta\tau + \int_{-\infty}^s f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta\tau \right| \\ &\leq \int_s^\infty |f(\tau) e_{\Theta z}(\sigma(\tau), s)| \Delta\tau + \int_{-\infty}^s |f(\tau) e_{\Theta z}(\sigma(\tau), s)| \Delta\tau \\ &= J_1 + J_2. \end{aligned}$$

By the computations for the Laplace transform, we have

$$J_1 \leq \frac{K}{\operatorname{Re}_{\mu^*(s)}(z) - \alpha}, \quad z \in \mathbb{C}_{s, \alpha, \gamma}.$$

Consider the integral

$$J = \int_t^s |f(\tau) e_{\Theta z}(\sigma(\tau), s)| \Delta\tau, \quad t \leq s.$$

Using the corresponding computations for the Laplace transform, we obtain

$$\begin{aligned} J &\leq K_\gamma \int_t^s \frac{|e_{\alpha\Theta z}(\tau, s)|}{|1 + \mu(\tau)z|} \Delta\tau \\ &= K_\gamma \int_t^s \frac{|e_{\gamma\Theta z}(\tau, s)|}{1 + \mu(\tau)\operatorname{Re}_{\mu(\tau)}(z)} \Delta\tau \\ &\leq K_\gamma \int_t^s \frac{e_{\gamma\Theta \operatorname{Re}_{\mu(\tau)}(z)}(\tau, s)}{1 + \mu(\tau)\operatorname{Re}_{\mu(\tau)}(z)} \Delta\tau \\ &= K_\gamma \int_t^s \frac{1}{\gamma - \operatorname{Re}_{\mu(\tau)}(z)} e^{-\gamma \ominus \operatorname{Re}_{\mu(\tau)}(z)^\Delta(\tau, s)} \Delta\tau \\ &\leq K_\gamma \gamma - \operatorname{Re}_{\mu(\tau)}(z) (1 - e_{\gamma\Theta \operatorname{Re}_{\mu(\tau)}(z)}(t, s)) \end{aligned}$$

Therefore

$$J_2 \leq \frac{K_\gamma}{\gamma - \operatorname{Re}_{\mu^*(s)}(z)}$$

and

$$|\mathcal{L}^b(f)(z, s)| \leq \frac{K_\alpha}{\operatorname{Re}_{\mu^*(s)}(z) - \alpha} + \frac{K_\gamma}{\gamma - \operatorname{Re}_{\mu^*(s)}(z)}.$$

Consequently the bilateral Laplace transform of the function f converges absolutely on $\mathbb{C}_{s,\alpha,\gamma}$. This completes the proof.

Corollary

Let (s, α, γ) is an admissible triple, $f \in \mathcal{C}_{rd}(\mathbb{T})$ is of double exponential order (α, γ) . Then

$$\lim_{|z| \rightarrow \infty} \mathcal{L}^b(f)(z, s) = 0.$$

Proof

Note that $|z| \rightarrow \infty$ implies $\operatorname{Re}_{\mu^*(s)}(z) \rightarrow \infty$, $\operatorname{Re}_{\mu^*(s)}(z) \rightarrow \infty$ and $\operatorname{Re}_{\bar{\mu}(s)}(z) \rightarrow \infty$. This completes the proof.

Theorem (Uniform Convergence of the Bilateral Laplace Transform)

Let (s, α, γ) is an admissible triple, $f \in \mathcal{C}_{rd}(\mathbb{T})$ is of double exponential order (α, γ) . Then $\mathcal{L}^b(f)(\cdot, s)$ converges uniformly in $\mathbb{C}_{s,\alpha,\gamma}$.

Proof

Let $\epsilon > 0$ be arbitrarily chosen. By the corresponding computations for the Laplace transform, we have that there exists an $r \in [s, \infty)$ such that

$$\left| \int_t^\infty f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta \tau \right| \leq \epsilon, \quad t \in [r, \infty), \quad z \in \mathbb{C}_{s,\alpha,\gamma}.$$

Let $t \in (-\infty, s]$ and $a \leq t$.

By the proof of the previous theorem, we get

$$\begin{aligned} \left| \int_a^t f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta \tau \right| &\leq \frac{K_\gamma}{\gamma - \operatorname{Re}_{\mu^*(s)}(z)} \left(e_{\gamma \ominus \operatorname{Re}_{\mu(r)}(z)}(t, s) - e_{\gamma \ominus \operatorname{Re}_{\mu(r)}(z)}(a, s) \right) \\ &\leq \frac{K_\gamma}{\gamma - \operatorname{Re}_{\mu^*(s)}(z)} e_{\gamma \ominus \operatorname{Re}_{\mu(r)}(z)}(t, s) \\ &\leq \frac{K_\gamma}{\gamma - \operatorname{Re}_{\mu^*(s)}(z)} e_{\gamma \ominus \operatorname{Re}_{\mu^*(s)}(z)}(t, s). \end{aligned}$$

Thus, there exists an $r_1 \in (-\infty, s]$, such that

$$\left| \int_{-\infty}^t f(\tau) e_{\Theta z}(\sigma(\tau), s) \Delta \tau \right| \leq \epsilon, \quad t \in (-\infty, r_1], \quad z \in \mathbb{C}_{s,\alpha,\gamma}.$$

This completes the proof.

Theorem

Let (s, α, γ) be an admissible triple and $f \in \mathcal{C}_{rd}([s, \infty))$ be of double exponential order (α, γ) . Consider $\mathcal{L}^b(f)(\cdot, s)$ on $\mathbb{C}_{s,\alpha,\gamma}$ and suppose that it has finitely many regressive poles of finite order $\{z_1, z_2, \dots, z_n\}$ and $\tilde{F}_{\mathbb{R}}^b(z)$ is the bilateral Laplace transform of the function $\tilde{f}$ on $\mathbb{R}$ that corresponds to the transform $\mathcal{L}^b(f)(z, s)$ of f on $\mathbb{T}$. If

$$\int_{c-i\infty}^{c+i\infty} \left| \tilde{F}_{\mathbb{R}}^b(z) \right| |dz| < \infty,$$

then

$$f(t) = \sum_{i=1}^n \text{res}_{z=z_i} e_z(t, s) \mathcal{L}^b(f)(z, s)$$

has bilateral Laplace transform $\mathcal{L}^b(f)(z, s)$ for all $z \in \mathbb{C}_{s, \alpha, \gamma} \cap \mathbb{C}_0(c)$.

Proof

Without loss of generality, suppose that $s = 0$.

By the previous theorem, we have that $\mathcal{L}(f)(\cdot)$ converges uniformly on $\mathbb{C}_{\mu, (s)}(\beta) \cap \mathbb{C}_0(c)$ and hence, it is analytic in this region.

Next, we have that

$$\lim_{|z| \rightarrow \infty} \mathcal{L}^b(f)(z) = 0.$$

Let $\mathcal{C}$ be the collection of the bilateral Laplace transforms over $\mathbb{R}$, $\mathcal{D}$ be the collection of the bilateral Laplace transforms over $\mathbb{T}$, i.e., $\mathcal{C} = \{\tilde{F}_{\mathbb{R}}^b(z)\}$, $\mathcal{D} = \{\mathcal{L}^b(f)(z)\}$, where

$$\begin{aligned} \tilde{F}_{\mathbb{R}}^b(z) &= G(z) e^{-z\tau}, \\ \mathcal{L}^b(f)(z) &= G(z) e_{\Theta z}(\tau, 0) \end{aligned}$$

for G a rational function and for $\tau \in \mathbb{T}$ a constant. Let $C_{p-\alpha}(\mathbb{R}, \mathbb{R})$ denotes the space of piecewise continuous functions of exponential order, $C_{prd-\alpha}(\mathbb{T}, \mathbb{R})$ denotes the space of piecewise right-dense continuous functions of double exponential type in which the exponential function coincides with a Hilger exponential function. Each of $\theta, \gamma, \theta^{-1}, \gamma^{-1}$ map functions involving the continuous exponential to the time scale exponential and vice versa. For example, γ maps the function

$$\tilde{F}_{\mathbb{R}}^b(z) = \frac{e^{-za}}{z}$$

to the function

$$\mathcal{L}^b(f)(z) = \frac{e_{\Theta z}(a, 0)}{z},$$

while γ^{-1} maps $\mathcal{L}^b(f)(z)$ back to $\tilde{F}_{\mathbb{R}}^b(z)$ in the obvious manner. If the representation of $\mathcal{L}^b(f)(z)$ is independent of the exponential, that is $\tau = 0$, then γ and its inverse γ^{-1} will act as the identity. For example,

$$\gamma\left(\frac{1}{1+z^2}\right) = \gamma^{-1}\left(\frac{1}{1+z^2}\right) = \frac{1}{1+z^2}.$$

The map θ will set the continuous exponential function to the time scale exponential function in the following manner: if we write $\tilde{f} \in C_{p-\alpha}(\mathbb{R}, \mathbb{R})$ as

$$\tilde{f}(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e^{zt} \tilde{F}_{\mathbb{R}}^b(z),$$

then

$$\theta(\tilde{f}(t)) = \sum_{i=1}^n \text{Res}_{z=z_i} e_z(t, 0) \mathcal{L}^b(f)(z).$$

To go from $\tilde{F}_{\mathbb{R}}^b(z)$ to $\mathcal{L}^b(f)(z)$, we simply switch expressions involving the continuous exponential in $\tilde{F}_{\mathbb{R}}^b$ with the time scale

exponential giving $\mathcal{L}^b(f)(z)$ as was done for γ and its inverse θ^{-1} will then act on $g \in C_{prd-ez}(\mathbb{T}, \mathbb{R})$,

$$g(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e_z(t, 0) G_{\mathbb{T}}(z)$$

as

$$\theta^{-1}(g(t)) = \sum_{i=1}^n \text{Res}_{z=z_i} e^{zt} \tilde{G}_{\mathbb{R}}(z).$$

For example, for the unit step function $\tilde{u}_a(t)$ on $\mathbb{R}$, we know from the continuous result that we may write the step function as

$$\tilde{u}_a(t) = \text{Res}_{z=0} e^{zt} \frac{e^{-az}}{z},$$

so that if $a \in \mathbb{T}$, then

$$\theta(\tilde{u}_a(t)) = \text{Res}_{z=0} e_z(t, 0) \frac{e_{\ominus z}(a, 0)}{z}.$$

Now, for a given time scale Laplace transform $\mathcal{L}^b(f)(z)$, we begin by mapping to $\tilde{F}_{\mathbb{R}}^b(z)$ via γ^{-1} . We have that the inverse of $\tilde{F}_{\mathbb{R}}^b(z)$ exists for all z with $z \in \mathbb{C}_{\mu_\alpha(s)}(\beta) \cap \mathbb{C}_0(c)$ and it is given by

$$\tilde{f}(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e^{zt} \tilde{F}_{\mathbb{R}}^b(z).$$

Applying θ to $\tilde{f}(t)$ to retrieve the time scale function, we get

$$f(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e_z(t, 0) \mathcal{L}^b(f)(z),$$

whereby

$$(\gamma \circ \tilde{F}_{\mathbb{R}}^b \circ \theta^{-1})(f(t)) = \mathcal{L}^b(f)(z).$$

This completes the proof.

Theorem

If the functions $f, g : \mathbb{T} \rightarrow \mathbb{R}$ satisfy the previous theorem and have the same bilateral Laplace transform, then $f = g$ a.e.

Proof

Suppose that

$$\int_s^\infty e_{\ominus z}^\sigma(t, s) f(t) \Delta t = \int_s^\infty e_{\ominus z}^\sigma(t, s) g(t) \Delta t.$$

Hence, $h = f - g$ has Laplace transform zero and $h \in \ker \mathcal{L}^b$. If we denote by $\mathcal{L}^{b^{-1}}$ the inversion of $\mathcal{L}^b$, then

$$\begin{aligned} \mathcal{L}^{b^{-1}} \circ \mathcal{L}(h) &= \mathcal{L}^{b^{-1}}(0) \\ &= 0 \\ &= h. \end{aligned}$$

Therefore $f = g$ a.e. This completes the proof.

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Chapter 9: The Fourier Transform

Suppose that $\mathbf{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ , respectively. Let also, $\inf \mathbf{T} = -\infty, \sup \mathbf{T} = \infty$ and $s \in \mathbf{T}$. Denote

$$\begin{aligned}\mu_*(s) &= \inf_{t \in [s, \infty)} \mu(t), \quad \mu^*(s) = \sup_{t \in (-\infty, s]} \mu(t), \quad \bar{\mu}(s) = \inf_{t \in (-\infty, s]} \mu(t), \\ \bar{\bar{\mu}}(s) &= \mu^*(s) \quad \text{if } \operatorname{Re}_{\bar{\mu}(s)}(x) \leq 0, \\ \bar{\bar{\mu}}(s) &= \bar{\mu}(s) \quad \text{if } \operatorname{Re}_{\bar{\mu}(s)}(x) \geq 0,\end{aligned}$$

where $x \in \mathbb{R}$.

Definition

Suppose that $f : \mathbf{T} \rightarrow \mathbb{R}$ is regulated. Then the Fourier transform of the function f is defined by

$$\mathcal{F}(f)(x, s) = \int_{-\infty}^{\infty} f(t) e_{\ominus ix}^{\sigma}(t, s) \Delta t$$

for $x \in \mathbb{R}$ for which $1 + ix\mu(t) \neq 0$ for any $t \in \mathbf{T}^{\kappa}$ and the improper integral exists.

Definition

Let $\alpha \in \mathcal{R}^+([s, \infty)), \gamma \in \mathcal{R}^+((-\infty, s])$. We say that (s, α, γ) is a real admissible triple if

$$\begin{aligned}R_{s, \alpha, \gamma} = \left\{ x \in \mathbb{R} : \operatorname{Re}_{\mu^*(s)}(ix) < \gamma, \quad \operatorname{Re}_{\mu_*(s)}(ix) > \alpha, \right. \\ \left. 1 + \bar{\bar{\mu}}(s) \operatorname{Re}_{\bar{\mu}(s)}(ix) \neq 0 \right\} \neq \emptyset.\end{aligned}$$

If $f \in \mathcal{C}_{rd}(\mathbf{T})$, then the triple (s, α, γ) is a real admissible triple and f is of double exponential order (α, γ) , then by the computations for the bilateral Laplace transform, it follows that $\mathcal{F}(f)(\cdot, s)$ exists on $R_{s, \alpha, \gamma}$ and converges absolutely and uniformly on $R_{s, \alpha, \gamma}$. Below we will list some of the properties of the Fourier transform.

Theorem

Let $f, g : \mathbf{T} \rightarrow \mathbb{R}, \alpha, \beta \in \mathbb{C}$. Then

$$\mathcal{F}(\alpha f + \beta g)(x, s) = \alpha \mathcal{F}(f)(x, s) + \beta \mathcal{F}(g)(x, s)$$

for those $x \in \mathbb{R}$ for which $1 + x\mu(t) \neq 0, t \in \mathbf{T}^{\kappa}$, and the respective integrals exist.

Proof

We have

$$\begin{aligned}\mathcal{F}(\alpha f + \beta g)(x, s) &= \int_{-\infty}^{\infty} (\alpha f + \beta g)(t) e_{\ominus ix}^{\sigma}(t, s) \Delta t \\ &= \alpha \int_{-\infty}^{\infty} f(t) e_{\ominus ix}^{\sigma}(t, s) \Delta t + \beta \int_{-\infty}^{\infty} g(t) e_{\ominus ix}^{\sigma}(t, s) \Delta t \\ &= \alpha \mathcal{F}(f)(x, s) + \beta \mathcal{F}(g)(x, s).\end{aligned}$$

This completes the proof.

Theorem

Let $f: \mathbb{T} \rightarrow \mathbb{R}$. Then

$$\mathcal{F}(e_y^\sigma(\cdot, s)f(\cdot))(x, s) = \mathcal{F}(f)(z, s),$$

where $z = \frac{x+iy}{1+\mu y}$, for those $x, y \in \mathbb{R}$ for which $1 + \mu(t)y \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the respective integrals exist.

Proof

Note that

$$\begin{aligned} iz &= i \frac{x+iy}{1+\mu y} \\ &= -\frac{x+iy}{i(1+\mu y)} \end{aligned}$$

and

$$\begin{aligned} \Theta iz &= \frac{-iz}{1+i\mu z} \\ &= \frac{\frac{x+iy}{i(1+\mu y)}}{1-\mu \frac{x+iy}{i(1+\mu y)}} \\ &= \frac{x+iy}{i+i\mu y-\mu x-i\mu y} \\ &= \frac{x+iy}{i-\mu x} \\ &= \frac{i(x+iy)}{-1-i\mu x} \\ &= \frac{y-ix}{1+i\mu x} \\ &= y \ominus ix. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{F}(e_y^\sigma(\cdot, s)f(\cdot))(x, s) &= \int_{-\infty}^{\infty} e_{\Theta ix}^\sigma(t, s)e_y^\sigma(t, s)f(t)\Delta t \\ &= \int_{-\infty}^{\infty} e_{y \ominus ix}^\sigma(t, s)f(t)\Delta t \\ &= \mathcal{F}(f)(z, s). \end{aligned}$$

This completes the proof.

Theorem

Let $f: \mathbb{T} \rightarrow \mathbb{R}$. For any $k \in \mathbb{N}$, we have

$$\mathcal{F}(f^{\Delta^k})(x, s) = (ix)^k \mathcal{F}(f)(x, s)$$

for those $x \in \mathbb{R}$ for which $1 + x\mu(t) \neq 0, t \in \mathbb{T}^\kappa$, and the respective integrals exist and

$$\lim_{t \rightarrow \pm\infty} f^{\Delta^l}(t)e_{\Theta ix}(t, s) = 0, \quad l \in \{0, \dots, k-1\}.$$

Proof

We will use the principle of mathematical induction.

1. For $k = 1$, we have

$$\begin{aligned}\mathcal{F}(f^\Delta)(x, s) &= \int_{-\infty}^{\infty} f^\Delta(t) e_{\ominus ix}^\sigma(t, s) \Delta t \\ &= \lim_{t \rightarrow \infty} f(t) e_{\ominus ix}(t, s) - \lim_{t \rightarrow -\infty} f(t) e_{\ominus ix}(t, s) \\ &\quad - \int_{-\infty}^{\infty} (\ominus ix) f(t) e_{\ominus ix}(t, s) \Delta t \\ &= ix \int_{-\infty}^{\infty} f(t) e_{\ominus ix}^\sigma(t, s) \Delta t \\ &= ix \mathcal{F}(f)(x, s).\end{aligned}$$

2. Assume that

$$\mathcal{F}(f^{\Delta^k})(x, s) = (ix)^k \mathcal{F}(f)(x, s)$$

for some $k \in \mathbb{N}$.

3.

We will prove that

$$\mathcal{F}(f^{\Delta^{k+1}})(x, s) = (ix)^{k+1} \mathcal{F}(f)(x, s).$$

Really, we have

$$\begin{aligned}\mathcal{F}(f^{\Delta^{k+1}})(x, s) &= ix \mathcal{F}(f^{\Delta^k})(x, s) \\ &= (ix)^{k+1} \mathcal{F}(f)(x, s).\end{aligned}$$

This completes the proof.

Theorem

Let $f: \mathbb{T} \rightarrow \mathbb{R}$. Then

$$\overline{\mathcal{F}(f)(x, s)} = \mathcal{F}(f)(-x, s)$$

for those $x \in \mathbb{R}$ for which $1 \pm x\mu(t) \neq 0, t \in \mathbb{T}^\kappa$, and the respective integrals exist.

Proof

Let $t \in \mathbb{T}^\kappa$ and $x \in \mathbb{R}$ be such that $1 \pm x\mu(t) \neq 0$. Then, we have

$$\begin{aligned}
 (\Theta(ix))(t) &= -\frac{ix}{1+i\mu(t)x} \\
 &= -\frac{ix(1-i\mu(t)x)}{(1+i\mu(t)x)(1-i\mu(t)x)} \\
 &= -\frac{ix+\mu(t)x^2}{1+(\mu(t))^2x^2}, \\
 1+\mu(t)(\Theta(ix))(t) &= 1-\mu(t)\frac{ix+\mu(t)x^2}{1+(\mu(t))^2x^2} \\
 &= \frac{1+(\mu(t))^2x^2-i\mu(t)x-(\mu(t))^2x^2}{1+(\mu(t))^2x^2} \\
 &= \frac{1-i\mu(t)x}{1+(\mu(t))^2x^2} \\
 &= \frac{1}{\sqrt{1+(\mu(t))^2x^2}} \left(\frac{1}{\sqrt{1+(\mu(t))^2x^2}} - i\frac{\mu(t)x}{\sqrt{1+(\mu(t))^2x^2}} \right)
 \end{aligned}$$

and

$$(\Theta(i(-x)))(t) = \frac{1}{\sqrt{1+(\mu(t))^2x^2}} \left(\frac{1}{\sqrt{1+(\mu(t))^2x^2}} + i\frac{\mu(t)x}{\sqrt{1+(\mu(t))^2x^2}} \right).$$

Let

$$r(t) = \frac{1}{\sqrt{1+(\mu(t))^2x^2}}, \quad \cos \theta(t) = \frac{1}{\sqrt{1+(\mu(t))^2x^2}}.$$

Then

$$1+\mu(t)(\Theta(ix))(t) = r(t)e^{-i\theta(t)}$$

and

$$1+\mu(t)(\Theta(i(-x)))(t) = r(t)e^{i\theta(t)}.$$

Also,

$$\begin{aligned}
 \log(1+\mu(t)(\Theta(ix))(t)) &= \ln(r(t)) - i(\theta + 2k\pi), \\
 \log(1+\mu(t)(\Theta(i(-x)))(t)) &= \ln(r(t)) + i(\theta + 2k\pi), \quad k \in \mathbb{Z}.
 \end{aligned}$$

From here and from the definition of the Fourier transform, we have

$$\begin{aligned}
 \mathcal{F}(f)(x, s) &= \int_{-\infty}^{\infty} f(t)e^{\sigma_{\Theta(ix)}(t, s)} \Delta t \\
 &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} \log(1+\mu(\tau)(\Theta(ix))(\tau)) \Delta \tau} f(\tau) \Delta \tau \\
 &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} (\ln r(\tau) - i(\theta(\tau) + 2k\pi)) \Delta \tau} f(\tau) \Delta t
 \end{aligned}$$

and

$$\begin{aligned}
 \overline{\mathcal{F}(f)(x, s)} &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} (\ln r(\tau) - i(\theta(\tau) + 2k\pi)) \Delta \tau} f(t) \Delta t \\
 &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} (\ln r(\tau) - i(\theta(\tau) + 2k\pi)) \Delta \tau} f(t) \Delta t \\
 &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} (\ln r(\tau) - i(\theta(\tau) + 2k\pi)) \Delta \tau} f(t) \Delta t \\
 &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} (\ln r(\tau) - i(\theta(\tau) + 2k\pi)) \Delta \tau} f(t) \Delta t \\
 &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} (\ln r(\tau) - i(\theta(\tau) + 2k\pi)) \Delta \tau} f(t) \Delta t \\
 &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)(\Theta(i(-x))))(\tau) \Delta \tau} f(t) \Delta t \\
 &= \int_{-\infty}^{\infty} f(t) e^{\sigma_{\Theta(i(-x))}}(t, s) \Delta t \\
 &= \mathcal{F}(f)(-x, s).
 \end{aligned}$$

This completes the proof.

Theorem

Let $a \in \mathbb{T}$. Then

$$\mathcal{F}_a(\delta_a^H)(x, s) = e_{\Theta ix}^{\sigma}(a, s)$$

for those $x \in \mathbb{R}$ for which $1 + x\mu(t) \neq 0, t \in \mathbb{T}^{\kappa}$.

Proof

We have

$$\begin{aligned}
 \mathcal{F}(\delta_a^H)(x, s) &= \int_{-\infty}^{\infty} \delta_a(t - a) e_{\Theta ix}^{\sigma}(t, s) \Delta t \\
 &= e_{\Theta ix}^{\sigma}(a, s)
 \end{aligned}$$

for those $x \in \mathbb{R}$ such that $1 + x\mu(t) \neq 0, t \in \mathbb{T}^{\kappa}$.

Note

1. If $\mathbb{T} = \mathbb{R}$, then we take $s = a$ and we get

$$\mathcal{F}_a(\delta_a^H)(x, s) = 1.$$

2. If $a \in \mathbb{T}$ is right-scattered and $s = a$, then

$$\begin{aligned}
 e_{\Theta ix}^{\sigma}(a, a) &= e^{\log \frac{1}{1 + i\mu(a)s}} \\
 &= \frac{1}{1 + i\mu(a)x}
 \end{aligned}$$

for those $x \in \mathbb{R}$ for which $1 + i\mu(a)x \neq 0$.

Theorem

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be regulated and

$$F(t) = \int_a^t f(\tau) \Delta \tau, \quad t \in \mathbb{T},$$

for some fixed $a \in \mathbb{T}$. Then

$$\mathcal{F}(F)(x, s) = -\frac{i}{x} \mathcal{F}(f)(x, s)$$

for those $x \in \mathbb{R}, x \neq 0$, for which

$$\lim_{t \rightarrow \pm\infty} F(t) e_{\ominus ix}(t, s) = 0.$$

Proof

We have

$$\begin{aligned} \mathcal{F}(F)(x, s) &= \int_{-\infty}^{\infty} F(t) e_{\ominus ix}^{\sigma}(t, s) \Delta t \\ &= \int_{-\infty}^{\infty} F(t) (1 + \mu(t)(\ominus ix)(t)) e_{\ominus ix}(t, s) \Delta t \\ &= \int_{-\infty}^{\infty} F(t) \frac{1}{1 + i\mu(t)x} e_{\ominus ix}(t, s) \Delta t \\ &= -\frac{1}{ix} \int_{-\infty}^{\infty} F(t) \frac{-ix}{1 + i\mu(t)x} e_{\ominus ix}(t, s) \Delta t \\ &= \frac{i}{x} \int_{-\infty}^{\infty} F(t) (\ominus ix)(t) e_{\ominus ix}(t, s) \Delta t \\ &= \frac{i}{x} \int_{-\infty}^{\infty} F(t) e_{\ominus ix}^{\Delta}(t, s) \Delta t \\ &= \frac{i}{x} \left(\lim_{t \rightarrow \infty} F(t) e_{\ominus ix}(t, s) - \lim_{t \rightarrow -\infty} F(t) e_{\ominus ix}(t, s) \right) \\ &\quad - \frac{i}{x} \int_{-\infty}^{\infty} f(t) e_{\ominus ix}^{\sigma}(t, s) \Delta t \\ &= -\frac{i}{x} \mathcal{F}(f)(x, s) \end{aligned}$$

for those $x \in \mathbb{R}, x \neq 0$, for which

$$\lim_{t \rightarrow \pm\infty} F(t) e_{\ominus ix}(t, s) = 0.$$

This completes the proof.

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