INTRODUCTION TO DIFFERENTIAL EQUATIONS (IVC MATH 24)



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1: FIRST ORDER DIFFERENTIAL EQUATIONS

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1.1: Integration Review

Learning Objectives

- Review all Integration Techniques

Solving differential equations inherently relies on integration. This rapid refresher covers the core integration methods—substitution, integration by parts, and partial fractions—that will be used extensively throughout the course, ensuring you have the computational skills needed for success.

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1.1: Integration Review

Integration and differentiation are two of the most important skills necessary to solve differential equation problems. Recall that the derivative of a function $f(x)$ at a point x in its domain is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

whenever it exists. While these limits are clearly important as the building blocks of a first calculus course, students are quickly taught a few shortcuts to avoid these cumbersome calculations. Below is a table of the standard derivatives that should already be known and memorized.

Table of Known Derivatives

$f(x)$	$f'(x)$
1	0
x^n for any $n \in \mathbb{R}$	nx^{n-1}
$\ln(x)$	x^{-1}
e^x	e^x
b^x for any $b > 0$	$b^x \ln(b)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\cot^{-1}(x)$	$-\frac{1}{1+x^2}$
$\sec^{-1}(x)$	$\frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1}(x)$	$-\frac{1}{x\sqrt{x^2-1}}$

Taking derivatives is a *linear operation*, meaning that the derivative "distributes" across sums and scalar multiplication:

$$(af(x) + bg(x))' = af'(x) + bg'(x) \text{ for any constants } a, b \in \mathbb{R}$$

This property allows us to take derivatives of all linear combinations of the functions in the table above, for example:

$$h(x) = 2\sin(x) - 3e^x \implies h'(x) = 2(\sin(x))' - 3(e^x)' = 2\cos(x) - 3e^x$$

Other functions can be created by multiplying, dividing, or composing known differentiable functions. Each of these operations yields a differentiation formula.

Known Differentiation Rules

Assuming $f(x)$ and $g(x)$ are both differentiable functions at a given point, we know:

The derivative of their product obeys the *product rule*.

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

The derivative of their quotient obeys the *quotient rule*.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

The derivative of their composition obeys the *chain rule*.

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Derivatives are useful for describing rates of change of a function at a given point, but they gain more utility when we realize that we can go in the opposite direction to find *anti-derivatives* of a function as well. We say an anti-derivative of $f(x)$ is any function $F(x)$ satisfying $F'(x) = f(x)$. The shorthand notation of this function is quickly defined to be written as the *indefinite integral* of $f(x)$, written $F(x) = \int f(x)\,dx$.

It is worth remembering that anti-derivatives are not unique. Since the derivative of any constant function is zero, any two functions that differ by a constant will have the same derivative (thanks to linearity). For this reason, we say the *family of anti-derivatives* is a function written $F(x) + C$:

$$(7x^2)' = 14x \implies \int 14x\,dx = 7x^2 + C$$

This relationship is the bridge between differentiation properties and integration properties. We summarize the famous anti-derivatives that most students have memorized below (it should look very similar to the previous table of derivatives!)

Table of Known Anti-Derivatives

$f(x)$	$\int f(x)\,dx$
1	$x + C$
x^n for any $n \neq -1$	$\frac{1}{n+1}x^{n+1} + C$
x^{-1}	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
b^x for any $b > 0$	$\frac{1}{\ln(b)}b^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\csc^2(x)$	$-\cot(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$

$\csc(x)\cot(x)$	$-\csc(x)+C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)+C$
$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)+C$
$\dfrac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}(x)+C$

Since the derivative is a linear operation, the anti-derivative is also linear (as long as you are willing to use the convention that linear combinations of the $+C$ terms always yield some $+C$ term). This means that for any constants $a, b^{\text{int}}\mathbb{R}$, we have:

$$\int f(x)\,dx = F(x)+C \quad \text{and} \quad \int g(x)\,dx = G(x)+C \implies \int (af(x)+bg(x))\,dx = aF(x)+bG(x)+C$$

This allows us to easily find the anti-derivative of linear combinations of any of the functions in the table above. Unfortunately, there are many more functions that we wish to integrate that are not precisely in this form. For such integrals, we have different integration methods based on the more complicated differentiation rules mentioned above.

When trying to take the anti-derivative of some function $h(x) = f'(g(x)) \cdot g'(x)$, it is apparent that this is the result of a chain rule. "Un-doing" the chain rule can be difficult in your head, so we use the method of **substitution**.

Integration Method: Substitution

When evaluating the integral $\int f'(g(x)) \cdot g'(x)\,dx$, we let $u = g(x)$ be the function "inside" of the composition.

Differentiating this new equation gives $\dfrac{du}{dx} = g'(x)$, which we rewrite as $du = g'(x)dx$.

Replacing *all* of the instances of the variable x inside of the integral with its equivalent in terms of u yields $\int f'(u)\,du$

This (hopefully) easier anti-derivative can be calculated to be $f(u)+C$, after which we can back-substitute our original relationship $u = g(x)$ to get the final answer $f(g(x))+C$.

Substitution temporarily changes the variable in our integral to eliminate some of the clutter that comes from the chain rule when taking derivatives.

Example 1.1.1

Evaluate each integral below by using substitution.

(a) $\int 4x\sec^2(2x^2)\,dx$

(b) $\int 3^{4x}\,dx$

(c) $\int \tan(x)\,dx$

(d) $\int \sec(x)\,dx$

(e) $\int \frac{e^x}{9+e^{2x}}\,dx$

Solution

(a) Let $u = 2x^2$, which yields $du = 4xdx$

$$\int 4x\sec^2(2x^2)\,dx = \int \sec^2(2x^2)\cdot 4xdx = \int \sec^2(u)\,du = \tan(u) + C = \boxed{\tan(2x^2) + C}$$

(b) Let $u = 4x$, which yields $du = 4dx \leftrightarrow \frac{du}{4} = dx$

$$\int 3^{4x}\,dx = \int 3^u\,\frac{du}{4} = \frac{1}{4}\int 3^u\,du = \frac{1}{4}\cdot\frac{1}{\ln(3)}3^u + C = \boxed{\frac{1}{4\ln(3)}3^{4x} + C}$$

(c) It helps to rewrite $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Let $u = \cos(x)$, which yields $du = -\sin(x)dx \leftrightarrow -du = \sin(x)dx$

$$\int \tan(x)\,dx = \int \frac{1}{\cos(x)}\cdot\sin(x)dx = \int \frac{1}{u}\cdot(-du) = -\int u^{-1}\,du = -\ln|u| + C = \boxed{-\ln|\cos(x)| + C}$$

(d) It helps to rewrite $\sec(x) = \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)} = \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}$

Let $u = \sec(x) + \tan(x)$, which yields $du = \left(\sec(x)\tan(x) + \sec^2(x)\right)dx$

$$\int \sec(x)\,dx = \int \frac{1}{\sec(x)+\tan(x)}\cdot\left(\sec(x)\tan(x)+\sec^2(x)\right)dx = \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\sec(x)+\tan(x)| + C}$$

(e) It helps to rewrite $\frac{e^x}{9+e^{2x}} = \frac{1}{9}\cdot\frac{e^x}{1+\left(\frac{e^x}{3}\right)^2}$

Let $u = \frac{e^x}{3}$, which yields $du = \frac{1}{3}e^x dx \leftrightarrow 3du = e^x dx$

$$\int \frac{e^x}{9+e^{2x}}\,dx = \int \frac{1}{9}\cdot\frac{1}{1+\left(\frac{e^x}{3}\right)^2}\cdot e^x dx = \frac{1}{9}\int \frac{1}{1+u^2}\cdot(3du) = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\tan^{-1}(u) + C = \boxed{\frac{1}{3}\tan^{-1}\left(\frac{e^x}{3}\right) + C}$$

Substitution is a powerful tool for turning one integral into another (that is hopefully easier to solve). The anti-derivative of $\tan(x)$ and $\sec(x)$ above are common enough that students should memorize the results for ease of use.

Integration involving trigonometric functions often leads to substitutions, especially in cases of the forms $\int \sin^n(x)\cos^m(x)\;dx$

These are commonly referred to as *trigonometric integrals* and they often make use of the many forms of the Pythagorean Identity to simplify.

Integration Method: Trigonometric Integrals

When evaluating the integral $\int \sin^n(x)\cos^m(x)\;dx$, the following substitutions are used (depending on n and m):

Case 1: $n = 2k+1$ for some integer $k \geq 0$

Let $u = \cos(x)$, yielding $du = -\sin(x)\;dx$. This gives us

$$\int \sin^n(x)\cos^m(x)\,dx = \int \sin^{2k}(x)\cos^m(x)\cdot\sin(x)dx = \int \left(1-\cos^2(x)\right)^k \cos^m(x)\cdot\sin(x)dx = \int (1-u^2)^k u^m(-du)$$

This becomes a polynomial multiplied by a power function, which is easily evaluated using the power rule and linearity.

Case 2: $m = 2k+1$ **for some integer** $k \geq 0$

Let $u = \sin(x)$, yielding $du = \cos(x)\ dx$. This gives us

$$\int \sin^n(x)\cos^m(x)\,dx = \int \sin^n(x)\cos^{2k}(x)\cdot\cos(x)dx = \int \sin^n(x)\left(1-\sin^2(x)\right)^k\cdot\cos(x)dx = \int u^n(1-u^2)^k\,du$$

This becomes a polynomial multiplied by a power function, which is easily evaluated using the power rule and linearity.

Case 3: $n = 2k$ **and** $m = 2j$ **for some integers** $j, k \geq 0$

We use the half-angle identities $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$ and $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$ to rewrite:

$$\int \sin^n(x)\cos^m(x)\,dx = \int \left(\sin^2(x)\right)^k\left(\cos^2(x)\right)^j\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)^k\left(\frac{1+\cos(2x)}{2}\right)^j\,dx$$

This reduces the powers of sine and cosine into smaller integer powers (of cosine exclusively), which can be turned into Case 1 or 2 above (or 3 again).

Case 4: $n+m = -2k-2$ **for some integer** $k \geq 0$ **and** $m \geq 0$

Rewrite $\sin^n(x) = \csc^{-n}(x) = \csc^{2k+m+2}(x)$ and let $u = \cot(x)$, yielding $du = -\csc^2(x)dx$. This gives us

$$\int \sin^n(x)\cos^m(x)\,dx = \int \csc^{2k+m}(x)\cos^m(x)\cdot\csc^2(x)\,dx = \int \csc^{2k}(x)\cot^m(x)\cdot\csc^2(x)\,dx = \int \left(1+\cot^2(x)\right)^k\cot^m(x)\cdot\csc^2(x)\,dx$$

This becomes a polynomial multiplied by a power function, which is easily evaluated using the power rule and linearity.

Case 5: $n+m = -2k-2$ **for some integer** $k \geq 0$ **and** $n \geq 0$

Rewrite $\cos^m(x) = \sec^{-m}(x) = \sec^{2k+n+2}(x)$ and let $u = \tan(x)$, yielding $du = \sec^2(x)dx$. This gives us

$$\int \sin^n(x)\cos^m(x)\,dx = \int \sin^n(x)\sec^{2k+n}(x)\cdot\sec^2(x)\,dx = \int \tan^n(x)\sec^{2k}(x)\cdot\sec^2(x)\,dx = \int \tan^n(x)\left(1+\tan^2(x)\right)^k\cdot\sec^2(x)\,dx =$$

This becomes a polynomial multiplied by a power function, which is easily evaluated using the power rule and linearity.

Using these various forms of the Pythagorean Identity and half angle identity can lead to greatly simplified expressions inside of integrals.

Example 1.1.2

Use substitution to evaluate each integral below.

(a) $\int \sin^3(x)\,dx$

(b) $\int \sin^2(2x)\cos^5(2x)\,dx$

(c) $\int \sin^2(3x)\,dx$

(d) $\int \dfrac{\cos^2(x)}{\sin^4(x)}\,dx$

(e) $\int \sec^4\left(\dfrac{x}{2}\right)\,dx$

Solution

(a) We have an odd number of sines (Case 1), so we will use $u = \cos(x)$ and $du = -\sin(x)dx$:

$$\int \sin^3(x)\,dx = \int (1-\cos^2(x))\cdot\sin(x)dx = \int (1-u^2)\cdot(-du) = \int u^2 - 1\,du = \frac{1}{3}u^3 - u + C = \boxed{\frac{1}{3}\cos^3(x) - \cos(x) + C}$$

(b) We have an odd number of cosines (Case 2), so we will use $u = \sin(2x)$ and $du = 2\cos(2x)dx \leftrightarrow \dfrac{du}{2} = \cos(2x)dx$:

$$\int \sin^2(2x)\cos^5(2x)\,dx = \int \sin^2(2x)\left(1-\cos^2(2x)\right)^2\cdot\cos(2x)dx = \int u^2(1-u^2)^2\cdot\frac{du}{2} = \frac{1}{2}\int u^2 - 2u^4 + u^6\,du = \frac{1}{2}\left[\frac{1}{3}u^3 - \frac{2}{5}u^5 -$$

(c) We have even sines (and cosines, so Case 3) and must start with the half angle identity $\sin^2(3x) = \dfrac{1+\cos(2\cdot 3x)}{2}$:

$$\int \sin^2(3x)\,dx = \int \frac{1-\cos(6x)}{2}\,dx = \frac{1}{2}\int 1-\cos(6x)\,dx = \frac{1}{2}\left[x - \frac{1}{6}\sin(6x)\right] + C = \boxed{\frac{1}{2}x - \frac{1}{12}\sin(6x) + C}$$

Notice that in this problem, we used a simple substitution to evaluate the anti-derivative of $\cos(6x)$. When the chain rule is "easy" to see (like multiplying by a constant), we can typically reverse it (by dividing by that constant) without explicitly writing out the substitution we used.

(d) This integral involves negative powers of sine ($n = -4, m = 2 \implies n+m = -2$ which is Case 4).

Therefore, we benefit from re-writing it in terms of $\csc(x)$ and $\cot(x)$, giving $\dfrac{\cos^2(x)}{\sin^4(x)} = \dfrac{\cos^2(x)}{\sin^2(x)}\cdot\dfrac{1}{\sin^2(x)} = \cot^2(x)\csc^2(x)$

This means we can use the substitution $u = \cot(x)$ and $du = -\csc^2(x)dx$:

$$\int \frac{\cos^2(x)}{\sin^4(x)}\,dx = \int \cot^2(x)\csc^2(x)\,dx = \int u^2\cdot(-du) = -\frac{1}{3}u^3 + C = \boxed{-\frac{1}{3}\cot^3(x) + C}$$

(e) Notice that $\sec^4(\theta) = \dfrac{1}{\cos^4(\theta)}$ so we have negative powers of cosine with $n = -4, m = 0 \implies n+m = -4$ (which is Case 5).

Therefore, we will use the substitution $u = \tan\left(\dfrac{x}{2}\right)$ and $du = \dfrac{1}{2}\sec^2\left(\dfrac{x}{2}\right) \leftrightarrow du = \sec^2\left(\dfrac{x}{2}\right)dx$:

$$\int \sec^4\left(\frac{x}{2}\right)\,dx = \int \left(1+\tan^2\left(\frac{x}{2}\right)\right)\cdot\sec^2\left(\frac{x}{2}\right)dx = \int (1+u^2)\cdot(2du) = 2\int 1+u^2\,du = 2\left[u + \frac{1}{3}u^3\right] + C = \boxed{2\tan\left(\frac{x}{2}\right) + \frac{2}{3}\tan}$$

Substitution does not have to be defined in the form $u = g(x)$ directly, sometimes it is beneficial to use $x = h(u)$. This allows us to ensure that the $dx = h'(u)du$ term is always easy to replace in the original integral (you just have to pick a nice choice of $h(u)$ that simplifies the integrand (function inside of the integral). This is most commonly used when substituting x as a trigonometric function of u, known as *trigonometric substitution*.

Integration Method: Trigonometric Substitution

Whenever the integrand contains a term of the form $\sqrt{1-x^2}$, we can use the substitution $x=\sin(u)$, since $\sqrt{1-x^2}=\sqrt{1-\sin^2(u)}=\sqrt{\cos^2(u)}$

Whenever the integrand contains a term of the form $\sqrt{1+x^2}$, we can use the substitution $x=\tan(u)$, since $\sqrt{1+x^2}=\sqrt{1+\tan^2(u)}=\sqrt{\sec^2(u)}$

Whenever the integrand contains a term of the form $\sqrt{x^2-1}$, we can use the substitution $x=\sec(u)$, since $\sqrt{x^2-1}=\sqrt{\sec^2(u)-1}=\sqrt{\tan^2(u)}$

Each of these substitutions uses the Pythagorean identity to turn the inside of a radical term into a perfect square, which typically results in transforming the integral into a trigonometric integral.

Example 1.1.3

Evaluate each integral below by using a trigonometric substitution.

(a) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$

(c) $\displaystyle\int \frac{1}{(9x^2-1)^{\frac{3}{2}}}\,dx$

Solution

(a) We see $\sqrt{1-x^2}$, so let $x=\sin(u)$ and $dx=\cos(u)du$:

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx=\int \frac{\sqrt{1-(\sin(u))^2}}{(\sin(u))^2}\cdot\cos(u)du=\int \frac{\cos^2(u)}{\sin^2(u)}\,du=\int \cot^2(u)\,du=\int \csc^2(u)-1\,du=-\cot(u)-u+C=-\cot\left(\si\right.$$

(b) We can rewrite $\sqrt{x^2+4}=2\sqrt{\left(\frac{x}{2}\right)^2+1}$ so let $\frac{x}{2}=\tan(u)\implies x=2\tan(u)$ and $dx=2\sec^2(u)du$:

$$\int \frac{1}{\sqrt{x^2+4}}\,dx=\int \frac{1}{2\sqrt{(\tan(u))^2+1}}\cdot 2\sec^2(u)du=\int \frac{\sec^2(u)}{\sec(u)}\,du=\int \sec(u)\,du=\ln|\sec(u)+\tan(u)|+C=\ln\left|\sec\left(\tan^{-1}\left(\frac{x}{2}\right)\right.\right.$$

(c) We can rewrite $(9x^2-1)^{\frac{3}{2}}=\left[\sqrt{(3x)^2-1}\right]^3$, so let $3x=\sec(u)\implies x=\frac{1}{3}\sec(u)$ and $dx=\frac{1}{3}\sec(u)\tan(u)du$:

$$\int \frac{1}{(9x^2-1)^{\frac{3}{2}}}\,dx=\int \frac{1}{\left((\sec(u))^2-1\right)^{\frac{3}{2}}}\cdot\frac{1}{3}\sec(u)\tan(u)du=\frac{1}{3}\int \frac{\sec(u)\tan(u)}{\tan^3(u)}\,du=\frac{1}{3}\int \cot(u)\csc(u)\,du=-\frac{1}{3}\csc(u)+C=-$$

While substitution expands our ability to evaluate anti-derivatives, we can do even more by studying the relationship between the product rule and anti-derivatives.

Integration Method: Integrating By Parts

When evaluating an integral of the form $\int f(x) \cdot g(x)dx$, you can think of it as part of the result of a product rule:

$$(u(x) \cdot v(x))' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

If we integrate both sides, we get the equation

$$\int (u(x) \cdot v(x))' \, dx = \int u'(x) \cdot v(x) \, dx + \int u(x) \cdot v'(x) \, dx$$

And this simplifies to the commonly taught formula:

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx$$

So if we pick the easier to integrate function $f(x) = v'(x)$ and the other function $g(x) = u(x)$, we transform the original integral into a new one (that is hopefully easier to solve.)

A good tool for knowing which function is "easier to integrate" is to use the mnemonic device **dETAIL**. The letters each stand for a family of functions that are increasingly difficult to integrate:

Exponentials < **T**rigonometric < **A**lgebraic < **I**nverse Trigonometric < **L**ogarithmic

This is why we pick the leftmost of these letters as our v' term and the rightmost letter as our u term.

Example 1.1.4

Evaluate each integral below by using integration by parts.

(a) $\int x5^x \, dx$

(b) $\int x^2 \ln(x) \, dx$

(c) $\int \tan^{-1}(x) \, dx$

Solution

(a) The integrand is the product of $f(x) = x$ (which is algebraic) and $g(x) = 5^x$ (which is exponential).

Therefore, we pick $v'(x) = 5^x \implies v(x) = \frac{1}{\ln(5)}5^x$ and $u(x) = x \implies u'(x) = 1$:

$$\int x5^x \, dx = x \cdot \frac{1}{\ln(5)}5^x - \int \frac{1}{\ln(5)}5^x \cdot 1 \, dx = \frac{1}{\ln(5)}x5^x - \frac{1}{\ln(5)} \int 5^x \, dx = \frac{1}{\ln(5)}x5^x - \frac{1}{\ln(5)} \cdot \frac{1}{\ln(5)}5^x + C = \boxed{\frac{1}{\ln(5)}x5^x - \frac{1}{(\ln(5))}}$$

(b) The integrand is the product of $f(x) = x^2$ (which is algebraic) and $g(x) = \ln(x)$ (which is logarithmic).

Therefore, we pick $v'(x) = x^2 \implies v(x) = \frac{1}{3}x^3$ and $u(x) = \ln(x) \implies u'(x) = \frac{1}{x}$:

$$\int x^2 \ln(x) \, dx = \ln(x) \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3} \int x^2 \, dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3} \cdot \frac{1}{3}x^3 + C = \boxed{\frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C}$$

(c) The integrand is the product of $f(x) = \tan^{-1}(x)$ (which is inverse trigonometric) and $g(x) = 1$ (which is algebraic).

Therefore, we pick $v'(x) = 1 \implies v(x) = x$ and $u(x) = \tan^{-1}(x) \implies u'(x) = \dfrac{1}{1+x^2}$:

$$\int \tan^{-1}(x)\,dx = \tan^{-1}(x)\cdot x - \int x\cdot\frac{1}{1+x^2}\,dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx = \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln\left|1+x^2\right| + C}$$

Note that in the final integral of this example, we used a substitution $w = 1 + x^2$ to find the resulting anti-derivative.

Sometimes integration methods can be combined together (as we saw in part (c) of the example above, which combined integration by parts with substitution). Sometimes integration by parts results in a new integral that we can use integration by parts on again! There is no limit to the number of applications of this formula you can chain together.

Example 1.1.5

Evaluate each integral below by using multiple integration by parts.

(a) $\displaystyle\int e^x \sin(2x)\,dx$

(b) $\displaystyle\int x^3\cos(x)\,dx$

Solution

(a) The integrand is the product of $f(x) = e^x$ (which is exponential) and $g(x) = \sin(2x)$ (which is trigonometric).

Therefore, we pick $v'(x) = e^x \implies v(x) = e^x$ and $u(x) = \sin(2x) \implies u'(x) = 2\cos(2x)$:

$$\int e^x\sin(2x)\,dx = \sin(2x)\cdot e^x - \int e^x\cdot 2\cos(2x)\,dx = e^x\sin(2x) - 2\int e^x\cos(2x)\,dx$$

The new integral is of the same type as our original integral, so we can use integration by parts on it again.

We pick $v'(x) = e^x \implies v(x) = e^x$ and $u(x) = \cos(2x) \implies u'(x) = -2\sin(2x)$:

$$\int e^x\cos(2x)\,dx = \cos(2x)\cdot e^x - \int e^x\cdot(-2\sin(2x))\,dx = e^x\cos(2x) + 2\int e^x\sin(2x)\,dx$$

Notice this returns us back to the original integral we wanted to solve!

$$\int e^x\sin(2x)\,dx = e^x\sin(2x) - 2\left[e^x\cos(2x) + 2\int e^x\sin(2x)\,dx\right]$$

$$\int e^x\sin(2x)\,dx = e^x\sin(2x) - 2e^x\cos(2x) - 4\int e^x\sin(2x)\,dx$$

$$5\int e^x\sin(2x)\,dx = e^x\sin(2x) - 2e^x\cos(2x)$$

$$\int e^x\sin(2x)\,dx = \boxed{\frac{1}{5}e^x\sin(2x) - \frac{2}{5}e^x\cos(2x) + C}$$

(b) The integrand is the product of $f(x) = x^3$ (which is algebraic) and $g(x) = \cos(x)$ (which is trigonometric).

Therefore, we pick $v'(x) = \cos(x) \implies v(x) = \sin(x)$ and $u(x) = x^3 \implies u'(x) = 3x^2$:

$$\int x^3\cos(x)\,dx = x^3\cdot\sin(x) - \int \sin(x)\cdot 3x^2\,dx = x^3\sin(x) - 3\int x^2\sin(x)\,dx$$

The new integral needs integration by parts again, we pick $v'(x) = \sin(x) \implies v(x) = -\cos(x)$ and $u(x) = x^2 \implies u'(x) = 2x$:

$$\int x^2 \sin(x)\,dx = x^2 \cdot (-\cos(x)) - \int (-\cos(x)) \cdot 2x\,dx = -x^2\cos(x) + 2\int x\cos(x)\,dx$$

Once again the resulting integral uses integration by parts, with $v'(x) = \cos(x) \implies v(x) = \sin(x)$ and $u(x) = x \implies u'(x) = 1$:

$$\int x\cos(x)\,dx = x \cdot \sin(x) - \int \sin(x) \cdot 1\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C$$

We substitute each of these back into the previous ones to get our final answer:

$$\int x^3\cos(x) = x^3\sin(x) - 3\left[-x^2\cos(x) + 2\left[x\sin(x) + \cos(x) + C\right]\right] = \boxed{x^3\sin(x) + 3x^2\cos(x) - 6x\sin(x) - 6\cos(x) + C}$$

There is a shortcut for integrals that you know will require many iterations of integration by parts, called the *tabular method*. If you ever have a polynomial for $f(x)$ (so that its derivative will ultimately become zero), and an exponential or sine/cosine function for $g(x)$ (so that its anti-derivatives are easy to compute in your head), you can write the table:

$u(x)$	$v'(x)$
$u'(x)$	$v(x)$
$u''(x)$	$\int v(x)\,dx$
$\vdots$	$\vdots$
0	$\int \cdots \int v(x)\,dx$

Then the nth entry in the u column multiplies by the $(n+1)^{st}$ entry in the v' column and the product is multiplied by $(-1)^{n+1}$. Each of these (until you reach 0) is summed up to get your final answer.

Looking at part (b) of the example above, this looks like:

x^3	$\cos(x)$
$3x^2$	$\sin(x)$
$6x$	$-\cos(x)$
6	$-\sin(x)$
0	$\cos(x)$

Summing the alternating products as described gives $(-1)^2 \cdot x^3 \cdot \sin(x) + (-1)^3 \cdot 3x^2 \cdot (-\cos(x)) + (-1)^4 \cdot 6x \cdot (-\sin(x)) + (-1)^5 \cdot 6 \cdot \cos(x) + C$, which is exactly the same answer we found above.

One last tool we will find very useful for integrating (and throughout this course) is the art of decomposing a rational function from one term with a common denominator into separate rational terms.

Integration Method: Partial Fraction Decomposition

When integrating a rational function $\int \frac{p(x)}{q(x)}\,dx$ where the degree of the polynomial $q(x)$ in the denominator is greater than the degree of the polynomial $p(x)$ in the numerator, we factor $q(x)$ and attempt to write an equivalent expression in terms of fractions that would combine to be $\frac{p(x)}{q(x)}$.

When $q(x) = (x-r_1)(x-r_2)\cdots(x-r_m)$ for distinct real numbers $r_1, r_2, \ldots, r_m$ this looks like

$$\frac{p(x)}{(x-r_1)(x-r_2)\cdots(x-r_m)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_m}{x-r_m} \quad \text{for some constants } A_1, A_2, \ldots, A_m$$

When $q(x) = (x-r_1)^{n_1}(x-r_2)^{n_2}\cdots(x-r_m)^{n_m}$ for distinct real numbers $r_1, r_2, \ldots, r_m$ this looks like

$$\frac{p(x)}{(x-r_1)^{n_1}(x-r_2)^{n_2}\cdots(x-r_m)^{n_m}} = \frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \cdots + \frac{A_{n_1}}{(x-r_1)^{n_1}} + \frac{B_1}{x-r_2} + \frac{B_2}{(x-r_2)^2} + \cdots + \frac{B_{n_2}}{(x-r_2)^{n_2}} + \cdots + \frac{M_1}{x-r_m}$$

When $q(x)$ has a factor that is an irreducible quadratic, like $(x^2+r^2)^n$, the decomposition uses terms of the form

$$\frac{p(x)}{(x^2+r^2)^n} = \frac{A_1x+B_1}{x^2+r^2} + \frac{A_2x+B_2}{(x^2+r^2)^2} + \cdots + \frac{A_nx+B_n}{(x^2+r^2)^n} \quad \text{for some constants } A_1, A_2, \ldots, A_n, B_1, B_2, \ldots B_n$$

Once the shape of the decomposition is known, we find the common denominator and solve for the coefficients by setting up systems of equations or plugging in specific values of x. The resulting decomposition should be easier to integrate by using simple substitutions.

The most important part of using partial fraction decomposition is knowing what the final form should look like. Once you have that, it is just calculation intensive (but not difficult) to find the coefficients and integrate.

Example 1.1.6

Evaluate each integral below by using partial fraction decomposition.

(a) $\int \frac{2x+11}{2x^2+7x+3}\,dx$

(b) $\int \frac{x^3-x^2+3x+1}{x^4-2x^2+1}\,dx$

(c) $\int \frac{x^2-4x+12}{x^3+4x}\,dx$

Solution

(a) We factor the denominator $2x^2+7x+3=(2x+1)(x+3)$ into linear terms with no repeated roots, so

$$\frac{2x+11}{(2x+1)(x+3)}=\frac{A}{2x+1}+\frac{B}{x+3}=\frac{A(x+3)+B(2x+1)}{(2x+1)(x+3)}$$

This means the numerators must be equivalent: $2x+11=A(x+3)+B(2x+1)$

Plugging in $x=-3$ this becomes $5=A\cdot 0+B\cdot(-5) \implies -5B=5 \implies B=-1$

Plugging in $x=-\frac{1}{2}$ this becomes $10=A\cdot\frac{5}{2}+B\cdot 0 \implies \frac{5}{2}A=10 \implies A=4$

Therefore, $\frac{2x+11}{(2x+1)(x+3)}=\frac{4}{2x+1}-\frac{1}{x+3}$ and our integral becomes:

$$\int \frac{2x+11}{2x^2+7x+3}\,dx=\int \frac{4}{2x+1}-\frac{1}{x+3}\,dx=\boxed{2\ln|2x+1|-\ln|x+3|+C}$$

(b) We factor the denominator $x^4-2x^2+1=(x-1)^2(x+1)^2$ into linear terms with repeated roots, so

$$\frac{x^3-x^2+3x+1}{(x-1)^2(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}=\frac{A(x-1)(x+1)^2+B(x+1)^2+C(x+1)(x-1)^2+D(x-1)^2}{(x-1)^2(x+1)^2}$$

This means the numerators must be equivalent: $x^3-x^2+3x+1=A(x-1)(x+1)^2+B(x+1)^2+C(x+1)(x-1)^2+D(x-1)^2$

Plugging in $x=1$ this becomes $4=A\cdot 0+B\cdot 4+C\cdot 0+D\cdot 0 \implies 4B=4 \implies B=1$

Plugging in $x=-1$ this becomes $-4=A\cdot 0+B\cdot 0+C\cdot 0+D\cdot 4 \implies 4D=-4 \implies D=-1$

No other values of x make zero coefficients in the numerator equation, so we can pick whatever number seems reasonable.

Plugging in $x=0$ this becomes $1=A\cdot(-1)+B\cdot 1+C\cdot 1+D\cdot 1 \implies -A+B+C+D=1 \implies -A+C=1$

Plugging in $x=2$ this becomes $11=A\cdot 9+B\cdot 9+C\cdot 3+D\cdot 1 \implies 9A+9B+3C+D=11 \implies 9A+3C=3$

Solving $\begin{cases} -A+C=1 \\ 9A+3C=3 \end{cases} \implies A=0, C=1$

Therefore, $\frac{x^3-x^2+3x+1}{(x-1)^2(x+1)^2}=\frac{1}{(x-1)^2}+\frac{1}{x+1}-\frac{1}{(x+1)^2}$

$$\int \frac{x^3-x^2+3x+1}{x^4-2x^2+1}\,dx=\int \frac{1}{(x-1)^2}+\frac{1}{x+1}-\frac{1}{(x+1)^2}\,dx=\boxed{-(x-1)^{-1}+\ln|x+1|+(x+1)^{-1}+C}$$

(c) We factor the denominator $x^3+4x = x(x^2+4)$ into a linear term and irreducible quadratic term, so

$$\frac{x^2-4x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)}$$

This means the numerators must be equivalent: $x^2-4x+12 = A(x^2+4)+(Bx+C)x$

Plugging in $x=0$ this becomes $12 = A\cdot 4 + (B\cdot 0 + C)\cdot 0 \implies 4A = 12 \implies A = 3$

There is no other value of x that creates zero coefficients, so we again pick whatever numbers seem reasonable.

Plugging in $x=1$ this becomes $9 = A\cdot 5 + (B+C)\cdot 1 \implies 5A+B+C = 9 \implies B+C = -6$

Plugging in $x=-1$ this becomes $17 = A\cdot 5 + (-B+C)\cdot(-1) \implies 5A+B-C = 17 \implies B-C = 2$

Solving $\begin{cases} B+C=-6 \\ B-C=2 \end{cases} \implies B=-2, C=-4$

Therefore, $\dfrac{x^2-4x+12}{x(x^2+4)} = \dfrac{3}{x} - \dfrac{2x}{x^2+4} - \dfrac{4}{x^2+4}$

$$\int \frac{x^2-4x+12}{x^3+4x}\,dx = \int \frac{3}{x} - \frac{2x}{x^2+4} - \frac{4}{x^2+4}\,dx = \boxed{3\ln|x| - \ln|x^2+4| - 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Integration is *the* key skill necessary for successfully solving many differential equations, so mastery of these methods will prove invaluable throughout the rest of this text.

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1.1: Practice Set

Exercise 1.1.1

Evaluate $\int x \sec(24x^2)\ dx$

Answer available at the end of this section.

Exercise 1.1.2

Evaluate $\int \cos^6\left(\frac{x}{2}\right)\ dx$

Answer available at the end of this section.

Exercise 1.1.3

Evaluate $\int \frac{1}{x^3\sqrt{x^2-25}}\ dx$

Answer available at the end of this section.

Exercise 1.1.4

Evaluate $\int x^4 \sin(2x)\ dx$

Answer available at the end of this section.

Exercise 1.1.5

Evaluate $\int \sin^5(x)\cos^3(x)\ dx$

Answer available at the end of this section.

Exercise 1.1.6

Evaluate $\int \frac{\sqrt{x^2+16}}{x}\ dx$

Answer available at the end of this section.

Exercise 1.1.7

Evaluate $\int \sin^{-1}(x)\ dx$

Answer available at the end of this section.

Exercise 1.1.8

Evaluate $\int \sin^4(x)\cos^2(x)\ dx$

Answer available at the end of this section.

Exercise 1.1.9

Evaluate $\int \frac{1}{(x^2-4)^{\frac{3}{2}}}\ dx$

Answer available at the end of this section.

Exercise 1.1.10

Evaluate $\int x\tan^{-1}(x)\ dx$

Answer available at the end of this section.

Exercise 1.1.11

Evaluate $\int \sec^3(x)\ dx$

Answer available at the end of this section.

Exercise 1.1.12

Evaluate $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\ dx$

Answer available at the end of this section.

Exercise 1.1.13

Evaluate $\int x^3 e^{-x^2}\ dx$

Answer available at the end of this section.

Exercise 1.1.14

Evaluate $\int \frac{2x^2-2x-42}{2x^3-x^2+8x-4}\ dx$

Answer available at the end of this section.

Exercise 1.1.15

Evaluate $\int e^{4x}\cos(5x)\ dx$

Answer available at the end of this section.

Exercise 1.1.16

Evaluate $\int \frac{x^3}{\sqrt{x^2+1}}\ dx$

Answer available at the end of this section.

Exercise 1.1.17

Evaluate $\int \frac{3x^3-18x^2+29x-4}{x^4-5x^3+6x^2+4x-8}\ dx$

Answer available at the end of this section.

Exercise 1.1.18

Evaluate $\int \sin^3(4x)\ dx$

Answer available at the end of this section.

Exercise 1.1.19

Evaluate $\int \tan^4(x)\ dx$

Answer available at the end of this section.

Exercise 1.1.20

Evaluate $\int \frac{8x^2-45x+4}{x^3-4x^2}\ dx$

Answer available at the end of this section.

Exercise 1.1.21

Evaluate $\int \cos^5(6x)\ dx$

Answer available at the end of this section.

Exercise 1.1.22

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\ dx$

Answer available at the end of this section.

Exercise 1.1.23

Evaluate $\int \sin^4(5x)\ dx$

Answer available at the end of this section.

Exercise 1.1.24

Evaluate $\int \frac{1}{x^2\sqrt{x^2+3}}\ dx$

Answer available at the end of this section.

Answers

Exercise 1.1.1

We use substitution to simplify, let $u = 24x^2$ and $du = 48x\ dx \leftrightarrow \frac{du}{48}$:

$$\int x\sec(24x^2)\,dx = \frac{1}{48}\int \sec(u)\,du = \frac{1}{48}\ln|\sec(u)+\tan(u)| + C = \frac{1}{48}\ln\left|\sec\left(24x^2\right)+\tan\left(24x^2\right)\right| + C$$

Exercise 1.1.2

We use the half angle identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$$\begin{aligned}
\int \cos^6\left(\frac{x}{2}\right)\,dx &= \int \left(\frac{1+\cos(x)}{2}\right)^3\,dx \\
&= \int \frac{1}{8}\left(1+3\cos(x)+3\cos^2(x)+\cos^3(x)\right)\,dx \\
&= \int \frac{1}{8}\left[1+3\cos(x)+\frac{3}{2}(1+\cos(2x))+\cos(x)(1-\sin^2(x))\right]\,dx \\
&= \int \frac{5}{16}+\frac{1}{2}\cos(x)+\frac{3}{16}\cos(2x)-\frac{1}{8}\sin^2(x)\cos(x)\,dx \\
&= \frac{5x}{16}+\frac{1}{2}\sin(x)+\frac{3}{32}\sin(2x)-\frac{1}{24}\sin^3(x)+C
\end{aligned}$$

Exercise 1.1.3

Re-writing $\sqrt{x^2-25} = 5\sqrt{\left(\frac{x}{5}\right)^2-1}$, we use substitution with $x = 5\sec(\theta)$ so $dx = 5\sec(\theta)\tan(\theta)\ d\theta$:

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{x^2-25}} &= \int \frac{1}{(5\sec(\theta))^3\cdot 5\sqrt{\sec^2(\theta)-1}}\cdot 5\sec(\theta)\tan(\theta)d\theta \\
&= \frac{1}{125}\int \frac{\sec(\theta)\tan(\theta)}{\sec^3(\theta)\tan(\theta)}d\theta \\
&= \frac{1}{125}\int \cos^2(\theta)\,d\theta \\
&= \frac{1}{125}\int \frac{1}{2}\left(1+\cos(2\theta)\right)\,d\theta \\
&= \frac{1}{250}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C \\
&= \frac{1}{250}\left(\theta+\sin(\theta)\cos(\theta)\right)+C \\
&= \frac{\sqrt{x^2-25}}{50x^2}+\frac{1}{250}\sec^{-1}\left(\frac{x}{5}\right)+C
\end{aligned}$$

Exercise 1.1.4

We use the tabular method:

x^4	$\sin(2x)$
$4x^3$	$-\frac{1}{2}\cos(2x)$
$12x^2$	$-\frac{1}{4}\sin(2x)$
$24x$	$\frac{1}{8}\cos(2x)$
24	$\frac{1}{16}\sin(2x)$
0	$-\frac{1}{32}\cos(2x)$

Therefore, our integral becomes:

$$\begin{aligned}\int x^4\sin(2x)\,dx &= (-1)^2\cdot x^4\cdot\left(-\frac{1}{2}\cos(2x)\right)+(-1)^3\cdot 4x^3\cdot\left(-\frac{1}{4}\sin(2x)\right)\\ &\quad+(-1)^4\cdot 12x^2\cdot\frac{1}{8}\cos(2x)+(-1)^5\cdot 24x\cdot\frac{1}{16}\sin(2x)+(-1)^6\cdot 24\cdot\left(-\frac{1}{32}\cos(2x)\right)+C\\ &= -\frac{1}{2}x^4\cos(2x)+x^3\sin(2x)+\frac{3}{2}x^2\cos(2x)-\frac{3}{2}x\sin(2x)-\frac{3}{4}\cos(2x)+C\end{aligned}$$

Exercise 1.1.5

$$\int \sin^5(x)\cos^3(x)\,dx = \int \sin^5(x)\cos^2(x)\cos(x)\,dx = \int \sin^5(x)(1-\sin^2(x))\cos(x)\,dx$$

We substitute using $u=\sin(x)$ and $du=\cos(x)\;dx$

$$\int \sin^5(x)\cos^3(x)\,dx = \int u^5(1-u^2)\,du = \int (u^5-u^7)\,du = \frac{1}{6}u^6-\frac{1}{8}u^8+C = \frac{1}{6}\sin^6(x)-\frac{1}{8}\sin^8(x)+C$$

Exercise 1.1.6

We rewrite $\sqrt{x^2+16}=4\sqrt{\left(\frac{x}{4}\right)^2+1}$ and substitute $x=4\tan(\theta)$ so $dx=4\sec^2(\theta)\;d\theta$:

$$\begin{aligned}\int \frac{\sqrt{x^2+16}}{x}\,dx &= \int \frac{4\sqrt{\tan^2(\theta)+1}}{4\tan(\theta)}\cdot 4\sec^2(\theta)\,d\theta\\ &= 4\int \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta\\ &= 4\int \frac{\sec^2(\theta)}{\sin(\theta)}\,d\theta\\ &= 4\int \frac{1+\tan^2(\theta)}{\sin(\theta)}\,d\theta\\ &= 4\int (\csc(\theta)+\sec(\theta)\tan(\theta))\,d\theta\\ &= 4\left(-\ln|\csc(\theta)+\cot(\theta)|+\sec(\theta)\right)+C\\ &= -4\ln\left|\frac{\sqrt{x^2+16}}{x}+\frac{4}{x}\right|+\sqrt{x^2+16}+C\end{aligned}$$

Exercise 1.1.7

We start with integration by parts, letting $u=\sin^{-1}(x) \implies u'=\frac{1}{\sqrt{1-x^2}}$ and $v'=1 \implies v=x$:

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x)-\int \frac{x}{\sqrt{1-x^2}}\,dx$$

For the second integral, we use substitution with $w=1-x^2$ and $dw=-2x\;dx$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\cdot x\,dx = \int \frac{1}{\sqrt{w}}\,\frac{dw}{2} = \frac{1}{2}\int w^{-1/2}\,dw = w^{\frac{1}{2}}+C = \sqrt{1-x^2}+C$$

Therefore, our final answer is

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x)+\sqrt{1-x^2}+C$$

Exercise 1.1.8

We use the half angle identities to rewrite:

$$\begin{aligned}\int \sin^4(x)\cos^2(x)\,dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^2\left(\frac{1+\cos(2x)}{2}\right)\,dx\\ &= \frac{1}{8}\int \left(1-2\cos(2x)+\cos^2(2x)\right)\left(1+\cos(2x)\right)\,dx\\ &= \frac{1}{8}\int \left(1-\cos(2x)-\cos^2(2x)+\cos^3(2x)\right)\,dx\\ &= \frac{1}{8}\int \left(1-\cos(2x)-\frac{1+\cos(4x)}{2}+(1-\sin^2(2x))\cos(2x)\right)\,dx\\ &= \frac{1}{16}\int 1-\cos(4x)-2\sin^2(2x)\cos(2x)\,dx\\ &= \frac{1}{16}x-\frac{1}{64}\sin(4x)-\frac{1}{48}\sin^3(2x)+C\end{aligned}$$

Exercise 1.1.9

We rewrite $(x^2-4)^{\frac{3}{2}} = 8\left(\left(\frac{x}{2}\right)^2-1\right)^{\frac{3}{2}}$ and use substitution $x = 2\sec(\theta)$ and $dx = 2\sec(\theta)\tan(\theta)\ d\theta$:

$$\begin{aligned}\int \frac{1}{(x^2-4)^{\frac{3}{2}}}\,dx &= \int \frac{1}{8(\sec^2(\theta)-1)^{\frac{3}{2}}}\cdot 2\sec(\theta)\tan(\theta)d\theta\\ &= \frac{1}{4}\int \frac{\sec(\theta)\tan(\theta)}{\tan^3(\theta)}\,d\theta\\ &= \frac{1}{4}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta\\ &= \frac{1}{4}\int \cot(\theta)\csc(\theta)\,d\theta\\ &= -\frac{1}{4}\csc(\theta)+C\\ &= -\frac{x}{4\sqrt{x^2-4}}+C\end{aligned}$$

Exercise 1.1.10

We use integration by parts with $u = \tan^{-1}(x) \implies u' = \frac{1}{1+x^2}$ and $v' = x \implies v = \frac{1}{2}x^2$:

$$\begin{aligned}\int x\tan^{-1}(x)\,dx &= \frac{1}{2}x^2\cdot\tan^{-1}(x)-\int \frac{1}{1+x^2}\cdot\frac{1}{2}x^2\,dx\\ &= \frac{1}{2}x^2\tan^{-1}(x)-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)\,dx\\ &= \frac{1}{2}x^2\tan^{-1}(x)-\frac{1}{2}x+\frac{1}{2}\tan^{-1}(x)+C\\ &= \frac{1}{2}(x^2+1)\tan^{-1}(x)-\frac{1}{2}x+C\end{aligned}$$

Exercise 1.1.11

We rewrite $\sec^3(x) = \sec^2(x)\cdot\sec(x)$ and use integration by parts.

Let $u = \sec(x) \implies u' = \sec(x)\tan(x)$ and $v' = \sec^2(x) \implies v = \sec(x)$:

$$\begin{aligned}\int \sec^3(x)\,dx &= \sec(x)\tan(x)-\int \sec(x)\tan^2(x)\,dx\\ &= \sec(x)\tan(x)-\int \sec(x)(\sec^2(x)-1)\,dx\\ &= \sec(x)\tan(x)-\int \sec^3(x)\,dx+\int \sec(x)\,dx\\ &= \sec(x)\tan(x)+\ln|\sec(x)+\tan(x)|-\int \sec^3(x)\,dx\end{aligned}$$

We add $\int \sec^3(x)\ dx$ to both sides and solve for it algebraically:

$$\begin{aligned}2\int \sec^3(x)\,dx &= \sec(x)\tan(x)+\ln|\sec(x)+\tan(x)|\\ \int \sec^3(x)\,dx &= \frac{1}{2}\sec(x)\tan(x)+\frac{1}{2}\ln|\sec(x)+\tan(x)|+C\end{aligned}$$

Exercise 1.1.12

We factor the denominator for partial fraction decomposition: $x^3+2x^2-3x = x(x+3)(x-1)$

$$\begin{aligned}\frac{4x^2+13x-9}{x(x+3)(x-1)} &= \frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}\\ 4x^2+13x-9 &= A(x+3)(x-1)+Bx(x-1)+Cx(x+3)\end{aligned}$$

Plugging in $x = 0$ gives $-9 = A\cdot(-3)+B\cdot 0+C\cdot 0 \implies -3A = -9 \implies A = 3$

Plugging in $x = -3$ gives $-12 = A\cdot 0+B\cdot 12+C\cdot 0 \implies 12B = -12 \implies B = -1$

Plugging in $x = 1$ gives $8 = A\cdot 0+B\cdot 0+C\cdot 4 \implies 4C = 8 \implies C = 2$

Therefore, we have

$$\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx = \int \frac{3}{x}-\frac{1}{x+3}+\frac{2}{x-1}\,dx = 3\ln|x|-\ln|x+3|+2\ln|x-1|+C$$

Exercise 1.1.13

We start with substitution, using $w = -x^2$ and $dw = -2x\ dx$:

$$\int x^3 e^{-x^2}\,dx = \int x^2 e^{-x^2}\cdot x\,dx = \int (-w)e^w \cdot \frac{dw}{-2} = \frac{1}{2}\int we^w\,dw$$

Now we use integration by parts, with $u = w \implies u' = 1$ and $v' = e^w \implies v = e^w$:

$$\int we^w\,dw = w\cdot e^w - \int e^w\cdot 1\,dw = we^w - \int e^w\,dw = we^w - e^w + C$$

Substituting back into terms of x gives our final answer:

$$\int x^3 e^{-x^2}\,dx = \frac{1}{2}\left((-x^2)e^{-x^2} - e^{-x^2}\right) + C = -\frac{1}{2}(x^2+1)e^{-x^2} + C$$

Exercise 1.1.14

We factor the denominator for partial fraction decomposition: $2x^3 - x^2 + 8x - 4 = (2x-1)(x^2+4)$

$$\begin{aligned}\frac{2x^2-2x-42}{(2x-1)(x^2+4)} &= \frac{A}{2x-1} + \frac{Bx+C}{x^2+4}\\ 2x^2-2x-42 &= A(x^2+4) + (Bx+C)(2x-1)\\ &= (A+2B)x^2 + (-B+2C)x + (4A-C)\end{aligned}$$

Setting each coefficient equal in the numerator gives:

$$\begin{cases} A+2B = 2\\ -B+2C = -2\\ 4A-C = -42\end{cases} \implies \begin{cases} A = -10\\ B = 6\\ C = 2\end{cases}$$

Therefore, we have

$$\int \frac{2x^2-2x-42}{2x^3-x^2+8x-4}\,dx = \int \frac{-10}{2x-1} + \frac{6x}{x^2+4} + \frac{2}{x^2+4}\,dx = -5\ln|2x-1| + 3\ln|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right) + C$$

Exercise 1.1.15

We use integration by parts twice, first using $u = \cos(5x) \implies u' = -5\sin(5x)$ and $v' = e^{4x} \implies v = \frac{1}{4}e^{4x}$:

$$\int e^{4x}\cos(5x)\,dx = \cos(5x)\cdot\frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\cdot(-5\sin(5x))\,dx = \frac{1}{4}e^{4x}\cos(5x) + \frac{5}{4}\int e^{4x}\sin(5x)\,dx$$

The next integration by parts uses $u = \sin(5x) \implies u' = 5\cos(5x)$ and $v' = e^{4x} \implies v = \frac{1}{4}e^{4x}$:

$$\int e^{4x}\sin(5x)\,dx = \sin(5x)\cdot\frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\cdot(5\cos(5x))\,dx = \frac{1}{4}e^{4x}\sin(5x) - \frac{5}{4}\int e^{4x}\cos(5x)\,dx$$

Putting these together we get:

$$\begin{aligned}\int e^{4x}\cos(5x)\,dx &= \frac{1}{4}e^{4x}\cos(5x) + \frac{5}{4}\left[\frac{1}{4}e^{4x}\sin(5x) - \frac{5}{4}\int e^{4x}\cos(5x)\,dx\right]\\ &= \frac{1}{4}e^{4x}\cos(5x) + \frac{5}{16}e^{4x}\sin(5x) - \frac{25}{16}\int e^{4x}\cos(5x)\,dx\end{aligned}$$

Adding $\frac{25}{16}\int e^{4x}\cos(5x)\,dx$ to both sides gives:

$$\begin{aligned}\frac{41}{16}\int e^{4x}\cos(5x)\,dx &= \frac{1}{4}e^{4x}\cos(5x) + e^{4x}\sin(5x) + C\\ \int e^{4x}\cos(5x)\,dx &= \left(\frac{4}{41}\cos(5x) + \frac{5}{41}\sin(5x)\right)e^{4x} + C\end{aligned}$$

Exercise 1.1.16

Re-writing $\frac{x^3}{\sqrt{x^2+1}} = \frac{x^2}{\sqrt{x^2+1}}\cdot x$ we see that we can avoid trigonometric substitution and use regular substitution with $u = x^2+1$ and $du = 2x\ dx$:

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{u-1}{\sqrt{u}}\cdot\frac{du}{2}\\ &= \frac{1}{2}\int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du\\ &= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) + C\\ &= \frac{1}{3}(x^2+1)^{\frac{3}{2}} - (x^2+1)^{\frac{1}{2}} + C\end{aligned}$$

Exercise 1.1.17

We factor the denominator for partial fraction decomposition: $x^4 - 5x^3 + 6x^2 + 4x - 8 = (x-2)^3(x+1)$

$$\begin{aligned}\frac{3x^3-18x^2+29x-4}{x^4-5x^3+6x^2+4x-8} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{x+1}\\ 3x^3-18x^2+29x-4 &= A(x-2)^2(x+1) + B(x-2)(x+1) + C(x+1) + D(x-2)^3\end{aligned}$$

Plugging in $x = 2$ gives: $6 = A\cdot 0 + B\cdot 0 + C\cdot 3 + D\cdot 0 \implies 3C = 6 \implies C = 2$

Plugging in $x = -1$ gives: $-54 = A\cdot 0 + B\cdot 0 + C\cdot 0 + D\cdot(-27) \implies -27D = -54 \implies D = 2$

Plugging in $x = 0$ gives: $-4 = A\cdot 4 + B\cdot(-2) + C\cdot 1 + D\cdot(-8) \implies 4A - 2B + C - 8D = -4 \implies 4A - 2B = 10$

Plugging in $x = 1$ gives: $10 = A\cdot 2 + B\cdot(-2) + C\cdot 2 + D\cdot(-1) \implies 2A - 2B + 2C - D = 10 \implies 2A - 2B = 8$

$$\begin{cases} 4A-2B = 10\\ 2A-2B = 8\end{cases} \implies \begin{cases} A = 1\\ B = -3\end{cases}$$

Therefore, we have

$$\begin{aligned}\int \frac{3x^3-18x^2+29x-4}{x^4-5x^3+6x^2+4x-8}\,dx &= \int \frac{1}{x-2}-\frac{3}{(x-2)^2}+\frac{2}{(x-2)^3}+\frac{2}{x+1}\,dx\\ &= \ln|x-2|-3(x-2)^{-1}+(x-2)^{-2}+2\ln|x+1|+C\end{aligned}$$

Exercise 1.1.18

We rewrite $\sin^3(4x)=(1-\cos^2(4x))\sin(4x)$ and use the substitution $u=\cos(4x)$ and $du=-4\sin(4x)\ dx$:

$$\begin{aligned}\int \sin^3(4x)\,dx &= \int (1-u^2)\cdot\left(\frac{du}{-4}\right)\\ &= \frac{1}{4}\int u^2-1\,du\\ &= \frac{1}{4}\left(\frac{1}{3}u^3-u\right)+C\\ &= \frac{1}{12}\cos^3(4x)-\frac{1}{4}\cos(4x)+C\end{aligned}$$

Exercise 1.1.19

We rewrite $\tan^4(x)=(\sec^2(x)-1)\cdot\tan^2(x)=\sec^2(x)\tan^2(x)-\tan^2(x)=\sec^2(x)\tan^2(x)-(\sec^2(x)-1)$:

$$\int \tan^4(x)\,dx=\int \sec^2(x)\tan^2(x)-\sec^2(x)+1\,dx=\int \tan^2(x)\sec^2(x)\,dx-\tan(x)+x+C$$

To evaluate the remaining integral, we substitute $u=\tan(x)$ and $du=\sec^2(x)\ dx$:

$$\int \tan^2(x)\sec^2(x)\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}\tan^3(x)+C$$

Therefore, our final answer is:

$$\int \tan^4(x)\,dx=\frac{1}{3}\tan^3(x)-\tan(x)+x+C$$

Exercise 1.1.20

We factor the denominator for partial fraction decomposition: $x^3-4x^2=x^2(x-4)$

$$\begin{aligned}\frac{8x^2-45x+4}{x^2(x-4)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-4}\\ 8x^2-45x+4 &= Ax(x-4)+B(x-4)+Cx^2\end{aligned}$$

Plugging in $x=0$ gives: $4=A\cdot 0+B\cdot(-4)+C\cdot 0 \implies -4B=4 \implies B=-1$

Plugging in $x=4$ gives: $-48=A\cdot 0+B\cdot 0+C\cdot 16 \implies 16C=-48 \implies C=-3$

Plugging in $x=1$ gives: $-33=A\cdot(-3)+B\cdot(-3)+C\cdot 1 \implies -3A-3B+C=-33 \implies A=11$

Therefore, we have

$$\begin{aligned}\int \frac{8x^2-45x+4}{x^3-4x^2}\,dx &= \int \frac{11}{x}-\frac{1}{x^2}-\frac{3}{x-4}\,dx\\ &= 11\ln|x|+x^{-1}-3\ln|x-4|+C\end{aligned}$$

Exercise 1.1.21

We rewrite $\cos^5(6x)=\left(1-\sin^2(6x)\right)^2\cos(6x)$ to substitute $u=\sin(6x)$ and $du=6\cos(6x)\ dx$:

$$\begin{aligned}\int \cos^5(6x)\,dx &= \int \left(1-u^2\right)^2\cdot\frac{du}{6}\\ &= \frac{1}{6}\int 1-2u^2+u^4\,du\\ &= \frac{1}{6}\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)+C\\ &= \frac{1}{6}\sin(6x)-\frac{1}{9}\sin^3(6x)+\frac{1}{30}\sin^5(6x)+C\end{aligned}$$

Exercise 1.1.22

Re-writing $\sqrt{9-x^2}=3\sqrt{1-\left(\frac{x}{3}\right)^2}$ we substitute $x=3\sin(\theta)$ and $dx=3\cos(\theta)\ d\theta$:

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\sqrt{1-\sin^2(\theta)}}{\left(3\sin(\theta)\right)^2}\cdot 3\cos(\theta)\,d\theta\\ &= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta\\ &= \int \cot^2(\theta)\,d\theta\\ &= \int \left(\csc^2(\theta)-1\right)\,d\theta\\ &= -\cot(\theta)-\theta+C\\ &= -\frac{\sqrt{9-x^2}}{x}-\sin^{-1}\left(\frac{x}{3}\right)+C\end{aligned}$$

Exercise 1.1.23

Using the half-angle identity we see:

$$\begin{aligned}\int \sin^4(5x)\,dx &= \int \left(\frac{1-\cos(10x)}{2}\right)^2 dx \\ &= \frac{1}{4}\int \left(1-2\cos(10x)+\cos^2(10x)\right)\,dx \\ &= \frac{1}{4}\int \left(1-2\cos(10x)+\frac{1+\cos(20x)}{2}\right)\,dx \\ &= \frac{1}{8}\int 3-4\cos(10x)+\cos(20x)\,dx \\ &= \frac{3}{8}x-\frac{1}{20}\sin(10x)+\frac{1}{160}\sin(20x)+C\end{aligned}$$

Exercise 1.1.24

Re-writing $\sqrt{x^2+3}=\sqrt{3}\cdot\sqrt{\left(\frac{x}{\sqrt{3}}\right)^2+1}$ we substitute $x=\sqrt{3}\ \tan(\theta)$ and $dx=\sqrt{3}\ \sec^2(\theta)\ d\theta$:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+3}}\,dx &= \int \frac{1}{\left(\sqrt{3}\tan(\theta)\right)^2\cdot\sqrt{3}\sqrt{\tan^2(\theta)+1}}\cdot\sqrt{3}\,\sec^2(\theta)\,d\theta \\ &= \frac{1}{3}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta \\ &= \frac{1}{3}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta \\ &= \frac{1}{3}\int \csc(\theta)\cot(\theta)\,d\theta \\ &= -\frac{1}{3}\csc(\theta)+C \\ &= -\frac{\sqrt{x^2+3}}{3x}+C\end{aligned}$$

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1.2: Basic Terminology

Learning Objectives

- Solve linear systems of equations
- Calculate the determinant of a 2×2 or 3×3 matrix
- Use determinants to detect linearly independent or dependent sets
- Use Cramer's Rule to solve systems of equations
- Classify types of differential equations

Before diving deeper into solution methods, we build the necessary linear algebra toolkit. This section provides a focused review of solving linear systems, determinants, Cramer's Rule, and the concept of linear independence, all of which form the mathematical backbone for solving systems of differential equations and understanding solution structures. This subchapter also establishes the fundamental vocabulary of the subject, teaching you to classify equations by type, order, and linearity. Understanding terms like *ordinary vs. partial*, *linear vs. nonlinear*, and *homogeneous vs. nonhomogeneous* is the critical first step in choosing the correct solution method for any problem you encounter.

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1.2: Linear Algebra Review

The study of differential equations is deeply connected to linear algebra in many ways. Students with a firm understanding of linear algebra will be able to see many similarities in the structure of solutions to systems of equations and many common differential equations. Beyond that, as we have seen with the method of partial fraction decomposition, solving systems of linear equations is important for solving integrals (and other situations) we may encounter.

A system of linear equations is a collection of one or more equations involving one or more variables.

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

Any combination of numbers $(x_1, x_2, \ldots, x_n)$ that satisfy *all equations simultaneously* is called a **particular solution** to the system of equations. The collection of all such particular solutions together form the **solution set** for the system of equations. For example, in the system

$$\begin{cases} x + y = 2 \\ 3x + 3y = 6 \end{cases}$$

We see that $x = 1, y = 1$ satisfies both equations at once, so it is a particular solution. It is not the only particular solution to this system, we see that $x = 0, y = 2$ and $x = 5, y = -3$ also make both equations true. The solution set for this system of equations includes all possible pairs $x = t, y = 2 - t$ when t is a real number. This contains infinitely many different possible solutions to the system.

Alternatively, consider the system

$$\begin{cases} x + y = 2 \\ 3x - 3y = 6 \end{cases}$$

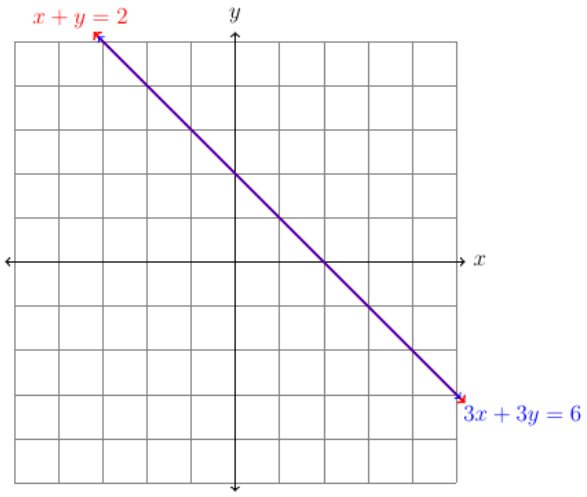
We see that $x = 2, y = 0$ satisfies both equations at once, but cannot find another pair of numbers that does so. In this case, we have found the *only* particular solution to this system of equations, so the solution set contains only that one pair.

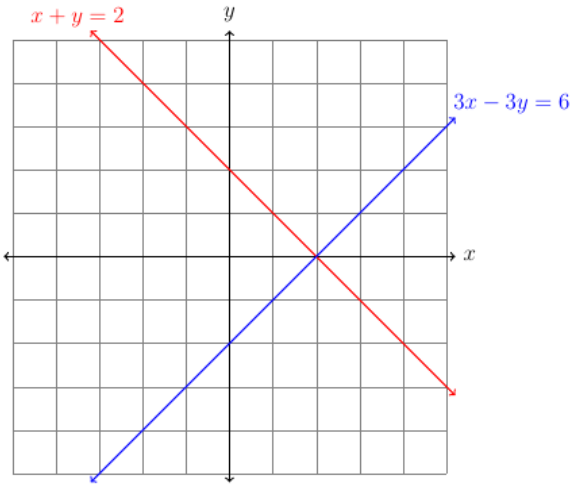
One last time, consider the system

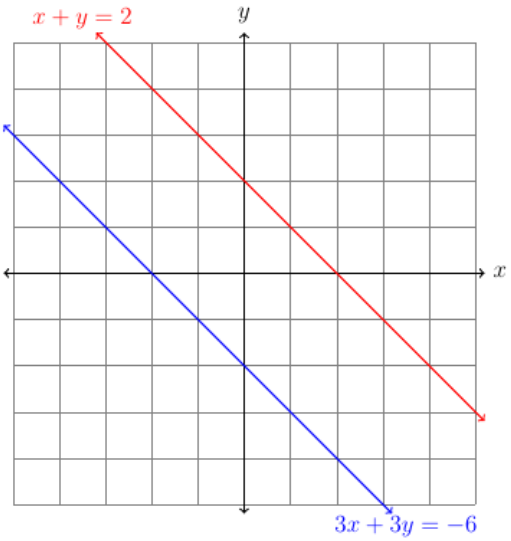
$$\begin{cases} x + y = 2 \\ 3x + 3y = -6 \end{cases}$$

No matter how you try, there is no possible way to make both equations satisfied at once. There are *no particular solutions* and we say that the solution set is empty.

Graphically, we can see why these three situations arise:







The intersection of two lines can yield infinitely many points, exactly one point, or no points at all. To determine which of these cases we are in, we use a process known as **Gaussian elimination.** This is the mathematical art of adding or subtracting multiples of one equation into another. For each of our three examples above, we can subtract the second equation from three times the first equation to eliminate the variable x:

$$3\,(x + y = 2) - (3x + 3y = 6) \implies 0y = 0$$
this is true for infinitely many values of y

$$3\,(x + y = 2) - (3x - 3y = 6) \implies 6y = 0$$
this is true for exactly one value, $y = 0$

$$3\,(x + y = 2) - (3x + 3y = -6) \implies 0y = 12$$
this is never true for any values of y

Illustration 1.2.1

We will use elimination to determine how many solutions exist for each system of equations below:

(a) $\begin{cases} 2x+y=2 \\ 3x-y=13 \end{cases}$

(b) $\begin{cases} 6x-4y=12 \\ -\frac{1}{2}x+\frac{1}{3}y=-1 \end{cases}$

(c) $\begin{cases} 3x-6y=8 \\ 5x-10y=12 \end{cases}$

For (a), we can eliminate y by adding the equations together:

$$(2x+y=2)+(3x-y=13) \implies 5x=15 \implies x=3$$

There is exactly one solution to this system of equations.

In fact, substituting in $x=3$ gives $y=-4$, so we can see this is our only particular solution to this system.

In (b), we can eliminate x by adding the first equation to 12 times the second equation:

$$(6x-4y=12)+12\left(-\frac{1}{2}x+\frac{1}{3}y=-1\right) \implies 0y=0$$

This has infinitely many solutions.

In (c), we can eliminate x by subtracting 5 times the first equation from 3 times the second equation:

$$3(5x-10y=12)-5(3x-6y=8) \implies 0y=-4$$

This has no solutions.

A system of equations is called **homogeneous** when the answers on the right side (the values of $b_1, b_2, \ldots, b_m$) are all zero. In this case, we know that no matter what the coefficients $a_{1,1} \ldots, a_{m,n}$ are in the system, there will always be at least one solution:

$$\begin{cases} a_{1,1}x_1+a_{1,2}x_2+\cdots+a_{1,n}x_n=0 \\ a_{2,1}x_1+a_{2,2}x_2+\cdots+a_{2,n}x_n=0 \\ \vdots \\ a_{m,1}x_1+a_{m,2}x_2+\cdots+a_{m,n}x_n=0 \end{cases} \quad \text{always holds when } x_1=0, x_2=0, \ldots, x_n=0$$

The solution $x_1=0, x_2=0, \ldots, x_n=0$ is known as the **trivial solution**. The word trivial here means "obvious". A homogeneous system of linear equations is always satisfied by the trivial solution, so its solution set is never empty. Remember though, it is entirely possible for there to be more than one solution to a system (in which case there will be infinitely many solutions).

Consider the homogeneous version of our starting examples:

$$\begin{cases} x+y=0 \\ 3x+3y=0 \end{cases} \quad \text{and} \quad \begin{cases} x+y=0 \\ 3x-3y=0 \end{cases}$$

If we use elimination to solve them, we see that:

$$3\,(x+y=0)-(3x+3y=0) \implies 0y=0$$
this is true for infinitely many values of y

$$3\,(x+y=0)-(3x-3y=0) \implies 6y=0$$
this is true for exactly one value, $y=0$

These homogeneous systems yield the same *number* of solutions as the original non-homogeneous systems of equations. (It is important to note the solutions themselves *are not the same points,* just that there are the same number of them!) This is always

the case when solutions exist.

Note

For any non-homogeneous system of linear equations, consider its homogeneous counterpart:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases} \quad \text{and} \quad \begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = 0 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = 0 \end{cases}$$

When there is exactly one solution to the non-homogeneous system, there will be exactly one solution to the homogeneous system.

When there are infinitely many solutions to the non-homogeneous system, there will be infinitely many solutions to the homogeneous system.

Therefore, it seems appropriate to study what causes a homogeneous system to have more than one solution.

For any homogeneous system of equations, since we already know the answer column is full of zeros, we can collect all of the important information in a **matrix**:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = 0 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = 0 \end{cases} \implies \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

The columns inside of this matrix will tell us exactly when our homogeneous system has more than one solution. When one column can be made by adding up scalar multiples of the others, this will lead to infinitely many solutions. For our initial example again, this looks like:

$$\begin{cases} x + y = 0 \\ 3x + 3y = 0 \end{cases} \implies A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{which is why we had infinitely many solutions}$$

Alternatively, our second example gave:

$$\begin{cases} x + y = 0 \\ 3x - 3y = 0 \end{cases} \implies A = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq k \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ for any } k^{\text{int}}\mathbb{R}, \text{ which is why we had a unique solution}$$

Whenever one column of a matrix can be made by scaling and adding the other columns, we say that those columns are **linearly dependent**. (Alternatively, if this is not the case we say the columns are **linearly independent**.)

Illustration 1.2.2

We will determine if the columns of each matrix below are linearly independent or linearly dependent:

(a) $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

(b) $B = \begin{bmatrix} -1 & 4 \\ 2 & -2 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & -2 & 5 \\ -1 & -1 & -2 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

(a) We check if $k \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$:

Setting both entries equal, we solve $\begin{cases} 2k = 3 \\ 4k = 6 \end{cases} \implies \begin{cases} k = \dfrac{3}{2} \\ k = \dfrac{3}{2} \end{cases}$

Since there is some k that satisfies both equations, we know these columns are **linearly dependent**.

(b) We check if $k \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$:

Setting both entries equal, we solve $\begin{cases} -k = 2 \\ 2k = -2 \end{cases} \implies \begin{cases} k = -2 \\ k = -1 \end{cases}$

Since there is no k that satisfies both equations, we know these columns are **linearly independent**.

(c) We check if $k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$:

Setting both entries equal, we solve $\begin{cases} k_1 - 2k_2 = 5 \\ -k_1 - k_2 = -2 \end{cases} \implies \begin{cases} k_1 = 3 \\ k_2 = -1 \end{cases}$

Since there is some k_1, k_2 that satisfies both equations, we can make the third column of C out of the first two, so these columns are **linearly dependent**.

(d) We check if $k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$:

Setting all three entries equal, we solve $\begin{cases} k_1 = 1 \\ k_2 = -1 \\ k_1 + k_2 = 1 \end{cases}$ which is clearly impossible since $1 + (-1) \neq 1$.

This does not mean the columns are necessarily independent! We still have to check if we could make the first column out of the last two, or if we could make the middle column out of the outer two.

We check if $k_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$:

Setting all three entries equal, we solve $\begin{cases} k_1 = 1 \\ -k_1 + k_2 = 0 \\ k_1 + k_2 = 1 \end{cases}$ which is again impossible since $k_2 = 1$ for the middle equation and $k_2 = 0$ for the bottom equation.

Lastly, we check if $k_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$:

Setting all three entries equal, we solve $\begin{cases} k_1 + k_2 = 0 \\ -k_1 = 1 \\ k_1 + k_2 = 1 \end{cases}$ which is clearly impossible since $k_1 + k_2$ cannot equal both 0 and 1 at the same time.

Finally we have shown that you cannot make any of these three columns out of the other two so they are **linearly independent**.

While the concept of linear independence and dependence is not complicated, we can see that checking whether the columns of a matrix are independent takes a lot of work. Thankfully there is a formula that gives you this information much more quickly for square matrices.

Definition: Determinant (2×2)

The ***determinant*** of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The columns of A will be *linearly dependent* when the determinant is equal to zero, and they will be *linearly independent* when the determinant is nonzero.

Notice that this matches with what we have discovered in our previous examples:

$$\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 1 \cdot 3 - 3 \cdot 1 = 0 \quad \text{since the columns of } \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \text{ are linearly dependent.}$$

$$\begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = 1 \cdot (-3) - 3 \cdot 1 = -6 \quad \text{since the columns of } \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} \text{ are linearly independent.}$$

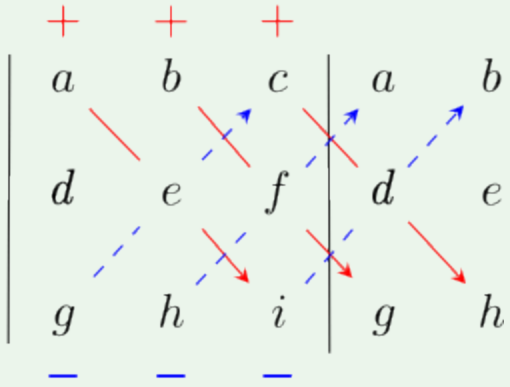
For larger matrices, the formula is more complicated. Thankfully there is a nice pattern to the next largest square matrix thanks to Pierre Sarrus.

Definition: Determinant (3×3)

The ***determinant*** of the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$$

This is easiest to memorize by copying the first two columns in order on the right side of the original matrix and adding the product along the three down-right diagonals and subtracting the product along the three up-right diagonals, as displayed in the image below:



Again, the columns of A will be *linearly dependent* when the determinant is equal to zero, and they will be *linearly independent* when the determinant is nonzero.

Example 1.2.1

Determine if the columns of each matrix below are linearly independent or linearly dependent.

(a) $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

(b) $B = \begin{bmatrix} -1 & 4 \\ 2 & -2 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & -2 & 5 \\ -1 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution

(a) $|A| = 2 \cdot 6 - 3 \cdot 4 = 12 - 12 = 0$

Since the determinant is zero, we know these columns are $\boxed{\text{linearly dependent}}$

(You can check that the second column is just two times the first)

(b) $|B| = -1 \cdot (-2) - 4 \cdot 2 = 2 - 8 = -6$

Since the determinant is non-zero, we know these columns are $\boxed{\text{linearly independent}}$

(c) $|C| = 1 \cdot (-1) \cdot (-1) + (-2) \cdot (-2) \cdot 0 + 5 \cdot (-1) \cdot 1 - 0 \cdot (-1) \cdot 5 - 1 \cdot (-2) \cdot 1 - (-1) \cdot (-1) \cdot (-2) = 1 + 0 - 5 - 0 + 2 + 2 = 0$

Since the determinant is zero, we know these columns are $\boxed{\text{linearly dependent}}$

(You can check that the third column is just three times the first column minus the second column)

(d) $|D| = 1 \cdot 1 \cdot 1 + 0 \cdot (-1) \cdot 1 + 1 \cdot 0 \cdot 1 - 1 \cdot 1 \cdot 1 - 1 \cdot (-1) \cdot 1 - 1 \cdot 0 \cdot 0 = 1 + 0 + 0 - 1 + 1 - 0 = 1$

Since the determinant is non-zero, we know these columns are $\boxed{\text{linearly independent}}$

Determinants are useful for more than just seeing how many solutions a homogeneous system has. If you have a non-homogeneous system and the coefficient matrix has non-zero determinant, you know there will be exactly one solution. We can avoid elimination and use determinants to solve for it.

Theorem 1.2.1: Cramer's Rule

If the system of equations $\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{cases}$ has a unique solution, then for each x_k we have $x_k = \dfrac{\begin{vmatrix} a_{1,1} & \cdots & b_1 & \cdots & a_{1,n} \\ a_{2,1} & \cdots & b_2 & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & b_n & \cdots & a_{n,n} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,k} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & \cdots & a_{n,n} \end{vmatrix}}$

If we denote A to be the coefficient matrix of the original system of equations, and A_k to be that matrix with the kth column replaced by the answer column vector $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, this simplifies to $x_k = \dfrac{|A_k|}{|A|}$

The proof of this Theorem uses properties that are developed in a pure linear algebra course, so we will omit it (but a curious reader can find one here)

Let's try using Cramer's Rule on one of our initial example systems: $\begin{cases} x + y = 2 \\ 3x - 3y = 6 \end{cases}$

This has coefficient matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}$ which we have seen has $|A| = -6$

If we replace the first column $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with the numbers in the answer column $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, we find

$$A_1 = \begin{bmatrix} 2 & 1 \\ 6 & -3 \end{bmatrix} \quad \text{which has} \quad |A_1| = 2 \cdot (-3) - 1 \cdot 6 = -12$$

Therefore, by Cramer's Rule, the solution will have its first variable $x = \dfrac{|A_1|}{|A|} = \dfrac{-12}{-6} = 2$

If we replace the second column $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ with the numbers in the answer column $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, we find

$$A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{which has} \quad |A_2| = 1 \cdot 6 - 3 \cdot 2 = 0$$

Therefore, by Cramer's Rule, the solution will have its second variable $y = \dfrac{|A_2|}{|A|} = \dfrac{0}{-6} = 0$

Indeed, the numbers $x = 2$ and $y = 0$ are the unique solution to the system that we saw at the beginning of this subchapter.

Note that if you tried to use Cramer's Rule on the example system $\begin{cases} x + y = 2 \\ 3x + 3y = 6 \end{cases}$, the determinant of the coefficient matrix A

$= \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ is zero. We cannot divide by zero, which lets us know that we cannot find a *unique solution* to this system. (As we have discussed previously, this system has infinitely many solutions!)

Example 1.2.2

Find the solutions to the systems of equations below by using Cramer's Rule.

(a) $\begin{cases} 4x + 3y = 5 \\ 3x + 2y = 4 \end{cases}$

(b) $\begin{cases} x + 2z = -1 \\ y - z = 1 \\ 2x + y + z = 3 \end{cases}$

Solution

(a) The coefficient matrix of this system is $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$

This has $|A| = 4 \cdot 2 - 3 \cdot 3 = -1$. Since this is non-zero, we know there will be a unique solution and we can find it using Cramer's Rule.

Replacing the first column gives $A_1 = \begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix} \implies |A_1| = 5 \cdot 2 - 3 \cdot 4 = -2$

Therefore, the first variable $x = \dfrac{|A_1|}{|A|} = \dfrac{-2}{-1} = 2$

Replacing the second column gives $A_2 = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \implies |A_2| = 4 \cdot 4 - 5 \cdot 3 = 1$

Therefore, the second variable $y = \dfrac{|A_2|}{|A|} = \dfrac{1}{-1} = -1$

We can easily see that $\boxed{x = 2 \text{ and } y = -1}$ satisfy both equations of the original system, as desired.

(b) The coefficient matrix of this system is $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$

This has $|A| = 1 + 0 + 0 - 4 - (-1) - 0 = -2$. Since this is non-zero, we know there will be a unique solution and we can find it using Cramer's Rule.

Replacing the first column gives $A_1 = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \implies |A_1| = -1 + 0 + 2 - 6 - 1 - 0 = -6$

Therefore, the first variable $x = \dfrac{|A_1|}{|A|} = \dfrac{-6}{-2} = 3$

Replacing the second column gives $A_2 = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 2 & 3 & 1 \end{bmatrix} \implies |A_2| = 1 + 2 + 0 - 4 - (-3) - 0 = 2$

Therefore, the second variable $y = \dfrac{|A_2|}{|A|} = \dfrac{2}{-2} = -1$

Replacing the third column gives $A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \implies |A_3| = 3 + 0 + 0 - (-2) - 1 - 0 = 4$

Therefore, the third variable $z = \dfrac{|A_3|}{|A|} = \dfrac{4}{-2} = -2$

We can easily see that $\boxed{x = 3,\ y = -1,\ \text{and}\ z = -2}$ satisfy all three equations of the original system, as desired.

Many of these definitions in linear algebra are recycled and used when solving certain differential equations (called *linear differential equations*). We will see precisely what that means shortly when we discuss what exactly a differential equation is and certain characteristics we look for in them.

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1.2: Differential Equation Terminology

In algebra, we are often tasked with finding a specific value of a variable in an equation. These variables start off representing numbers, like in the equation $3x + 1 = 7$, which is satisfied by the number $x = 2$. In a system of linear equations, we are looking for more than one number that satisfies more than one equation.

In differential equations, we are usually solving for an unknown function based on an equation that relates it to its derivatives. These come in many varieties, but the goal is always the same: to find the function (or functions) that satisfy the given relationship.

Just like in algebra, it is often much easier to verify that you have a solution than to find it directly.

Illustration 1.2.3

Verify that each function or functions satisfy the given differential equation.

(a) $y = \sin(x)$ and $(y')^2 + y^2 = 1$

(b) $y = e^x \cos(x)$ and $y'' - 2y' + 2y = 0$

(c) $y = Ax^2 + 6x + B$ and $x^2y'' - xy' = -6x$ where A, B are real constants.

Solution

(a) $y = \sin(x) \implies y' = \cos(x)$:

$$(y')^2 + y^2 = (\cos(x))^2 + (\sin(x))^2 = \cos^2(x) + \sin^2(x) = 1$$

This proves that $y = \sin(x)$ is one particular solution to this differential equation.

(b) $y = e^x \cos(x) \implies y' = e^x(\cos(x) - \sin(x)) \implies y'' = -2e^x \sin(x)$

$$y'' - 2y' + 2y = (-2e^x \sin(x)) - 2(e^x(\cos(x) - \sin(x))) + 2(e^x \cos(x)) = 0$$

This proves that $y = e^x \cos(x)$ is one particular solution to this differential equation.

(c) $y = Ax^2 + 6x + B \implies y' = 2Ax + 6 \implies y'' = 2A$

$$x^2y'' - xy' = x^2(2A) - x(2Ax + 6) = -6x$$

This proves that $y = Ax^2 + 6x + B$ satisfies this differential equation for all constant values of $A, B^{\text{int}}\mathbb{R}$

Also like in linear algebra, any one function that satisfies the differential equation is called a *particular solution*. As we saw above, it is also possible for there to be infinitely many solutions to a differential equation. In these cases, we say that we have a ***family of solutions***.

Definition: n-Parameter Families of Solutions

Each of the undetermined (or "free") constants in the answer is called a ***parameter***. A family of solutions with n parameters is called an *n-parameter family of solutions*.

Since $y = Ax^2 + 6x + B$ satisfied $x^2y'' - xy' = -6x$, we have found a 2-parameter family of solutions to this differential equation.

Since $y = e^x \cos(x)$ does not have any parameters, but satisfied $y'' - 2y' + 2y = 0$, we say it is a particular solution (or "0-parameter

family of solutions"). There are actually infinitely many solutions (a 2-parameter family exists) to this differential equation, we just haven't found them yet.

To help us determine how many solutions exist for a differential equation, we have a few definitions to help us classify what *type* of differential equation we are dealing with.

Definition: Ordinary and Partial Differential Equations

When the unknown function $y(x)$ depends on *only one dependent variable* x, we say that the differential equation is an ***ordinary differential equation***.

If the function y depends on more than one variable (and thus, has more than one type of derivative in its differential equation), we say it is a ***partial differential equation***.

For example:

$y'(x) = \sqrt{x^2 - 1}$ involves only one variable so it is an ordinary differential equation.

$\dfrac{\partial y}{\partial x} + \dfrac{\partial y}{\partial t} = x + t$ involves two variables, so it is a partial differential equation.

Intended for use in an introductory course, the text will focus entirely on solving ordinary differential equations. Many of the tools used in solving these equations are useful when solving partial differential equations.

Definition: Order

The ***order*** of a differential equation is the highest level of derivative present in the equation.

For example:

$(y')^2 + y^2 = 1$ is a *first-order* differential equation.

$y'' - 2y' + 2y = 0$ is a *second-order* differential equation.

$y^{(5)} \cdot y^{(4)} = y + 1$ is a *fifth-order* differential equation.

Recall that a linear system of equations is of the form

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

The reason it is *linear* is because the unknown variables $x_1, x_2, \ldots, x_n$ are multiplied by constants (not other unknowns) and added or subtracted from one another. Putting any of these variables inside of a function like $\sin(x_2)$ or e^{x_1} would not yield a system of equations that can be solved using the same methodology we have discussed so far. The same requirements go into making a *linear differential equation*.

Definition: Homogeneous and Non-homogeneous Linear Differential Equations

A differential equation of the form $f(y, y', y'', \ldots, y^{(n)}, x) = 0$ is a ***linear differential equation*** if each of $y, y', y'', \ldots y^{(n)}$ is only multiplied by constants (or functions with x's in them, but no other y's) and added or subtracted from one another. Any other such differential equation is called a *non-linear differential equation*.

When there are no functions of x without y or one of its derivatives as a coefficient in a linear differential equation, we say that equation is ***homogeneous***. Otherwise it is called a *non-homogeneous* linear differential equation.

For example:

$(y')^2 + y^2 = 1$ is a *non-linear* differential equation, since y and y' are each squared. (We do not discuss homogeneous or not in this case)

$y'' - 2y' + 2y = 0$ is a *homogeneous linear* differential equation, since each level of derivative of y is only multiplied by constants and no functions of x are "alone".

$x^2y'' - 2xy' = -6x$ is a *non-homogeneous linear* differential equation, since each level of derivative of y is only multiplied by a function of x, and $-6x$ is "alone".

Illustration 1.2.4

Classify each differential equation below.

(a) $x^2y''' - \cos(x)y = 0$

(b) $y'' \cdot y = e^{-2x}$

(c) $y' = \sqrt{x^2 - 1}$

Solution

(a) $x^2y''' - \cos(x)y = 0$ has highest level of derivative y''' so it is third order. Each of y''' and y has coefficients that do not involve another expression of y so it is linear. There is no function of x without a coefficient involving y, so it is homogeneous.

This is a third order, homogeneous linear differential equation.

(b) $y'' \cdot y = e^{-2x}$ has highest level of derivative y'' so it is second order. Since y'' is multiplied by y it is not linear. Since it is not linear, the term homogeneous does not have any meaning.

This is a second order non-linear differential equation.

(c) $y' = \sqrt{x^2 - 1}$ has highest level of derivative y' so it is first order. The y' term does not have any y coefficients so it is linear. The function $\sqrt{x^2 - 1}$ does not have any y coefficients so it is non-homogeneous.

This is a first order, non-homogeneous linear differential equation.

This section has introduced the fundamental concepts of differential equations. We have connected the task of finding unknown functions to the familiar algebraic goal of finding unknown numbers. We have also defined key classifications, such as order, linearity, and homogeneity, that are essential for organizing and solving these equations.

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1.2: Practice Set

Exercise 1.2.1a

Use elimination to solve

$$\begin{cases} 2x + 3y = 7 \\ x - y = 1 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.1b

Use elimination to solve

$$\begin{cases} 4a - b = 13 \\ 3a + 2b = 7 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.1c

Use elimination to solve

$$\begin{cases} 3x + 6y = 9 \\ x + 2y = 3 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.1d

Use elimination to solve

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x + 2y - z = 2 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.2a

Evaluate the determinant $\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix}$

Answer available at the end of this section.

Exercise 1.2.2b

Evaluate the determinant $\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix}$

Answer available at the end of this section.

Exercise 1.2.2c

Evaluate the determinant $\begin{vmatrix} 2 & 3 & -1 \\ 4 & -2 & 3 \\ -1 & 0 & 5 \end{vmatrix}$

Answer available at the end of this section.

Exercise 1.2.2d

Evaluate the determinant $\begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}$

Answer available at the end of this section.

Exercise 1.2.3

What value(s) of k will yield infinitely many solutions to the system $\begin{cases} kx + 4y = 0 \\ 9x + ky = 0 \end{cases}$

Answer available at the end of this section.

Exercise 1.2.4a

Use Cramer's Rule to solve

$$\begin{cases} 5x - 2y = 8 \\ 3x + y = 7 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.4b

Use Cramer's Rule to solve

$$\begin{cases} 4a - 7b = 13 \\ 3a + 5b = 7 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.4c

Use Cramer's Rule to solve

$$\begin{cases} 2p + 5q = 3 \\ -p + 3q = 1 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.4d

Use Cramer's Rule to solve

$$\begin{cases} x+y+z=6 \\ 2x-y+z=3 \\ x+2y-z=2 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.4e

Use Cramer's Rule to solve

$$\begin{cases} 2x-z=11 \\ x+4y=1 \\ x+2y+3z=0 \end{cases}$$

Answer available at the end of this section.

Exercise 1.2.5

Prove if $|A|=0$ for $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then columns are linearly dependent

Answer available at the end of this section.

Exercise 1.2.6a

Classify $y''+3y'=e^x$

Answer available at the end of this section.

Exercise 1.2.6b

Classify $u_t=u_{xx}+\sin(u)$

Answer available at the end of this section.

Exercise 1.2.6c

Classify $\sin(x)y'''+e^xy'=0$

Answer available at the end of this section.

Exercise 1.2.7a

Verify that $y=Ce^{-3x}+2$ is a 1-parameter family of solutions for $y'+3y=6$

Answer available at the end of this section.

Exercise $1.2.7b$

Verify that $y = \tan(x)$ is a particular solution for $(y')^2 = y^4 + 2y^2 + 1$

Answer available at the end of this section.

Exercise $1.2.7c$

Verify that $y = 0$ is a particular solution for $y''' + (y')^2 = 0$

Answer available at the end of this section.

Exercise $1.2.7d$

Verify that $y = Ce^{2x} + De^{-2x}$ is a 2-parameter family of solutions for $y'' - 4y = 0$

Answer available at the end of this section.

Exercise $1.2.8a$

Use integration to find a 2-parameter family of solutions to $y'' = \cos(3x)$

Answer available at the end of this section.

Exercise $1.2.8b$

Find all value(s) of C which make $f(x) = x^2 + Cx$ a particular solution to $xy' - y = x^2$

Answer available at the end of this section.

Exercise $1.2.8c$

Find constants a, b such that $y = a\sin(x) + b\cos(x)$ satisfies $y'' + 3y' + 2y = 14\cos(x) - 12\sin(x)$

Answer available at the end of this section.

Exercise $1.2.8d$

Find constants a, b such that $y = a\sin(x) + b\cos(x)$ satisfies $y'' - y = 6\sin(x)$

Answer available at the end of this section.

Exercise $1.2.8e$

Find constants a, b such that $y = ae^x + be^{-x}$ satisfies $y'' + y = 2e^x - 4e^{-x}$

Answer available at the end of this section.

Exercise $1.2.8f$

Find constants a, b such that $y = ae^x + be^{-x}$ satisfies $y''' + 2y' - y = 0$

Answer available at the end of this section.

Answers

Exercise $1.2.1a$

We eliminate y by adding the first equation to 3 times the second equation:

$$(2x + 3y = 7) + 3(x - y = 1) \implies 5x = 10 \implies x = 2$$

Plugging this into either of the initial equations we see $y = 1$

There is a unique solution $x = 2,\ y = 1$

Exercise $1.2.1b$

We eliminate b by adding 2 times the first equation to the second equation:

$$2(4a - b = 13) + (3a + 2b = 7) \implies 11a = 33 \implies a = 3$$

Plugging this into either of the initial equations gives $b = -1$

There is a unique solution $a = 3,\ b = -1$

Exercise $1.2.1c$

We eliminate x by subtracting the first equation from 3 times the second equation:

$$3(x + 2y = 3) - (3x + 6y = 9) \implies 0y = 0$$

This means y can be any number (and there will be infinitely many solutions.

When $y = t$, we can solve $x + 2y = 3 \implies x = 3 - 2t$

The family of solutions looks like $x = 3 - 2t,\ y = t$ for any real number t

Exercise $1.2.1d$

We eliminate y by adding the first equation to the second equation:

$$(x + y + z = 6) + (2x - y + z = 3) \implies 3x + 2z = 9$$

We eliminate y by subtracting the third equation from 2 times the first equation:

$$2(x + y + z = 6) - (x + 2y - z = 2) \implies x + 3z = 10$$

Finally we subtract 3 times this equation from our previous result:

$$(3x + 2z = 9) - 3(x + 3z = 10) \implies -7z = -21 \implies z = 3$$

Plugging this into $x + 3z = 10 \implies x = 1$

Plugging these into either of the initial equations gives $y = 2$

There is a unique solution $x = 1,\ y = 2,\ z = 3$

Exercise $1.2.2a$

$$\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} = 4 \cdot 3 - 2 \cdot (-1) = 14$$

Exercise $1.2.2b$

$$\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} = \cos(\theta) \cdot \cos(\theta) - \sin(\theta) \cdot (-\sin(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

Exercise $1.2.2c$

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & -2 & 3 \\ -1 & 0 & 5 \end{vmatrix} = -20 - 9 + 0 - (-2) - 0 - 60 = -87$$

Exercise $1.2.2d$

$$\begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = x + 0 + 1 - 0 - (-x) - 1 = 2x$$

Exercise $1.2.3$

We take the determinant of the coefficient matrix $A = \begin{bmatrix} k & 4 \\ 9 & k \end{bmatrix}$:

$$\begin{vmatrix} k & 4 \\ 9 & k \end{vmatrix} = k \cdot k - 4 \cdot 9 = k^2 - 36$$

When this determinant is zero, the system will have infinite solutions:

$$k^2 - 36 = 0 \implies k = \pm 6$$

Exercise $1.2.4a$

$$A = \begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix} = 5(1) - (-2)(3) = 11$$
$$A_1 = \begin{vmatrix} 8 & -2 \\ 7 & 1 \end{vmatrix} = 22 \implies x = \frac{22}{11} = 2$$
$$A_2 = \begin{vmatrix} 5 & 8 \\ 3 & 7 \end{vmatrix} = 11 \implies y = \frac{11}{11} = 1$$

The unique solution is $x = 2,\ y = 1$

Exercise 1.2.4*b*

$$A = \begin{vmatrix} 4 & -7 \\ 3 & 5 \end{vmatrix} = 4(5) - 3(-7) = 41$$
$$A_1 = \begin{vmatrix} 13 & -7 \\ 7 & 5 \end{vmatrix} = 114 \implies a = \frac{114}{41}$$
$$A_2 = \begin{vmatrix} 4 & 13 \\ 3 & 7 \end{vmatrix} = -11 \implies b = \frac{-11}{41}$$

The unique solution is $a = \frac{114}{41},\ b = -\frac{11}{41}$

Exercise 1.2.4*c*

$$A = \begin{vmatrix} 2 & 5 \\ -1 & 3 \end{vmatrix} = 2(3) - 5(-1) = 11$$
$$A_1 = \begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix} = 4 \implies p = \frac{4}{11}$$
$$A_2 = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5 \implies q = \frac{5}{11}$$

The unique solution is $p = \frac{4}{11},\ q = \frac{5}{11}$

Exercise 1.2.4*d*

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 7$$
$$A_1 = \begin{vmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ 2 & 2 & -1 \end{vmatrix} = 7 \implies x = \frac{7}{7} = 1$$
$$A_2 = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 14 \implies y = \frac{14}{7} = 2$$
$$A_3 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ 1 & 2 & 2 \end{vmatrix} = 21 \implies z = \frac{21}{7} = 3$$

The unique solution is $x = 1,\ y = 2,\ z = 3$

Exercise 1.2.4*e*

$$A = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 4 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 26$$
$$A_1 = \begin{vmatrix} 11 & 0 & -1 \\ 1 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} = 130 \implies x = \frac{130}{26} = 5$$
$$A_2 = \begin{vmatrix} 2 & 11 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -26 \implies y = \frac{-26}{26} = -1$$
$$A_3 = \begin{vmatrix} 2 & 0 & 11 \\ 1 & 4 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -26 \implies z = \frac{-26}{26} = -1$$

The unique solution is $x = 5,\ y = -1,\ z = -1$

Exercise 1.2.5

We are given that $|A| = ad - bc = 0 \implies ad = bc$.

Case 1: $a \neq 0$

Then $ad = bc \implies d = \frac{bc}{a} \implies \begin{bmatrix} b \\ d \end{bmatrix} = \frac{b}{a}\begin{bmatrix} a \\ c \end{bmatrix}$ are clearly scalar multiples.

Case 2: $a = 0$

Then $ad = bc \implies b = 0$ or $c = 0$

Case 2a: $a = 0$ and $b = 0$

Then $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix}$ are definitely scalar multiples.

Case 2b: $a = 0$ and $c = 0$

Then $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} b \\ d \end{bmatrix}$ are definitely scalar multiples.

In all cases, one column is a scalar multiple of the other.

Exercise 1.2.6*a*

This is an **ordinary 2nd order, linear, non-homogeneous** differential equation.

Exercise 1.2.6*b*

This is an **partial 2nd order, non-linear** differential equation.

Exercise 1.2.6*c*

This is an **ordinary 3rd order, linear, homogeneous** differential equation.

Exercise 1.2.7*a*

$y = Ce^{-3x} + 2 \implies y' = -3Ce^{-3x}$:

$$y' + 3y = -3Ce^{-3x} + 3\left(Ce^{-3x} + 2\right) = 6$$

This confirms that y satisfies the first order linear non-homogeneous equation for all $C^{\text{int}}\mathbb{R}$.

Exercise 1.2.7*b*

$y = \tan(x) \implies y' = \sec^2(x) \implies (y')^2 = \sec^4(x)$:

$$y^4 + 2y^2 + 1 = (\tan(x))^4 + 2(\tan(x))^2 + 1 = \tan^4(x) + 2\tan^2(x) + 1 = \left(\tan^2(x) + 1\right)^2 = \left(\sec^2(x)\right)^2 = \sec^4(x) = (y')^2$$

This confirms that y satisfies the first order non-linear differential equation.

Exercise 1.2.7*c*

$y = 0 \implies y' = 0 \implies y'' = 0 \implies y''' = 0$:

$$y''' + (y')^2 = 0 + (0)^2 = 0$$

This confirms that y satisfies the third order non-linear differential equation.

Exercise 1.2.7*d*

$y = Ce^{2x} + De^{-2x} \implies y' = 2Ce^{2x} - 2De^{-2x} \implies y'' = 4Ce^{2x} + 4De^{-2x}$:

$$y'' - 4y = \left(4Ce^{2x} + 4De^{-2x}\right) - 4\left(Ce^{2x} + De^{-2x}\right) = 0$$

This confirms that y satisfies the 2nd order linear homogeneous equation for all $C, D^{\text{int}}\mathbb{R}$.

Exercise 1.2.8*a*

$$y'' = \cos(3x) \implies y' = \int y''\,dx = \frac{1}{3}\sin(3x) + C$$

$$y' = \frac{1}{3}\sin(3x) + C \implies y = \int y'\,dx = -\frac{1}{9}\cos(3x) + Cx + D$$

Exercise 1.2.8*b*

$y = x^2 + Cx \implies y' = 2x + C$

Plugging these into the DE:

$$xy' - y = x(2x + C) - \left(x^2 + Cx\right) = x^2$$

We see this holds for **all possible values of** C, and we actually have a 1-parameter family of solutions.

Exercise 1.2.8*c*

$$\begin{aligned} y &= a\sin(x) + b\cos(x) \\ y' &= a\cos(x) - b\sin(x) \\ y'' &= -a\sin(x) - b\cos(x) \end{aligned}$$

We substitute these into the DE:

$$y'' + 3y' + 2y = (-a\sin(x) - b\cos(x)) + 3(a\cos(x) - b\sin(x)) + 2(a\sin(x) + b\cos(x)) = (a - 3b)\sin(x) + (3a + b)\cos(x)$$

Setting this equal to the right side of the DE gives:

$$\begin{cases} a - 3b = -12 \\ 3a + b = 14 \end{cases} \implies \begin{cases} a = 3 \\ b = 5 \end{cases}$$

We see this holds for $a = 3,\ b = 5$.

Exercise 1.2.8*d*

$$\begin{aligned} y &= a\sin(x) + b\cos(x) \\ y' &= a\cos(x) - b\sin(x) \\ y'' &= -a\sin(x) - b\cos(x) \end{aligned}$$

We substitute these into the DE:

$$y'' - y = (-a\sin(x) - b\cos(x)) - (a\sin(x) + b\cos(x)) = -2a\sin(x) - 2b\cos(x)$$

Setting this equal to the right side of the DE gives:

$$\begin{cases} -2a = 6 \\ -2b = 0 \end{cases} \implies \begin{cases} a = -3 \\ b = 0 \end{cases}$$

We see this holds for $a = -3,\ b = 0$.

Exercise 1.2.8*e*

$$y = ae^{x} + be^{-x}$$
$$y' = ae^{x} - be^{-x}$$
$$y'' = ae^{x} + be^{-x}$$

We substitute these into the DE:

$$y'' + y = \left(ae^{x} + be^{-x}\right) + \left(ae^{x} + be^{-x}\right) = 2ae^{x} + 2be^{-x}$$

Setting this equal to the right side of the DE gives:

$$\begin{cases} 2a = 2 \\ 2b = -4 \end{cases} \implies \begin{cases} a = 1 \\ b = -2 \end{cases}$$

We see this holds for $a = 1,\ b = -2$.

Exercise $1.2.8f$

$$y = ae^{x} + be^{-x}$$
$$y' = ae^{x} - be^{-x}$$
$$y'' = ae^{x} + be^{-x}$$
$$y''' = ae^{x} - be^{-x}$$

We substitute these into the DE:

$$y''' + 2y' - y = \left(ae^{x} - be^{-x}\right) + 2\left(ae^{x} - be^{-x}\right)) - \left(ae^{x} + be^{-x}\right) = 2ae^{x} - 4be^{-x}$$

Setting this equal to the right side of the DE gives:

$$\begin{cases} 2a = 0 \\ -4b = 0 \end{cases} \implies \begin{cases} a = 0 \\ b = 0 \end{cases}$$

We see this holds for $a = 0,\ b = 0$ (which is the trivial solution).

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1.3: Partial Derivatives and Exact Equations

Learning Objectives

- Review and apply the concepts of partial derivatives and partial integrals.
- Determine whether a given first-order differential equation is exact.
- Solve exact differential equations by reconstructing a potential function.
- Use integrating factors to transform non-exact equations into exact ones.

Here, we explore a specific class of first-order equations that can be solved by recognizing them as the total differential of a function. This subchapter bridges concepts from multivariable calculus, reviewing partial derivatives and integrals to develop a method for reconstructing a potential function whose level curves are the solutions.

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1.3: Partial Differentiation and Integration

While this textbook will focus on ordinary differential equations, we do make use of some of the tools found in a multivariable calculus course.

The first of these tools is the *partial derivative*. When differentiating a single variable function $y = f(x)$, we are taking the slope of the line tangent to that curve at a given point. We find this slope by evaluating the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This limit does not exist for every function at every point, but when it does we call it the derivative of that function at that point. The value of $f'(a)$ represents a multiplier of how much a small change in the input variable x (from $x = a$) will cause a change in the output function f (from $f(a)$).

For functions with more than one variable, the same small change in two different input variables can have dramatically different effects on the output f. To tell them apart, we define a different derivative for each of these inputs.

Definition: Partial Derivative

Let $f(x, y)$ be a function of two independent variables x and y.

The ***partial derivative of f with respect to x*** (written $\dfrac{\partial f}{\partial x}$ or f_x) is given by the limit:

$$f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

The ***partial derivative of f with respect to y*** (written $\dfrac{\partial f}{\partial y}$ or f_y) is given by the limit:

$$f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Although they are defined as a limit, we evaluate these derivatives in the same way we do for single variable functions. Just remember that any variable you are not differentiating with respect to acts like a constant.

Example 1.3.1

Find each partial derivative of the multivariable functions below.

(a) $f(x, y) = x^2 y - x \sin(2y)$

(b) $g(x, y) = xe^{2xy^3}$

Solution

(a)

$$f_x = \frac{\partial}{\partial x}\left(x^2 y - x\sin(2y)\right) = \frac{\partial}{\partial x}\left(x^2\right) \cdot y - \frac{\partial}{\partial x}(x) \cdot \sin(2y) = 2x \cdot y - 1 \cdot \sin(2y) = \boxed{2xy - \sin(2y)}$$

$$f_y = \frac{\partial}{\partial y}\left(x^2 y - x\sin(2y)\right) = x^2 \cdot \frac{\partial}{\partial y}(y) - x\frac{\partial}{\partial y}(\sin(2y)) = x^2 \cdot 1 - x \cdot 2\cos(2y) = \boxed{x^2 - 2x\cos(2y)}$$

(b)

$$g_x = \frac{\partial}{\partial x}\left(xe^{2xy^3}\right) = \frac{\partial}{\partial x}(x)\cdot e^{2xy^3} + x\frac{\partial}{\partial x}\left(e^{2xy^3}\right) = 1\cdot e^{2xy^3} + x\cdot\left(e^{2xy^3}\cdot 2y^3\right) = \boxed{\left(1+2xy^3\right)e^{2xy^3}}$$

$$g_y = \frac{\partial}{\partial y}\left(xe^{2xy^3}\right) = x\cdot\frac{\partial}{\partial y}\left(e^{2xy^3}\right) = x\cdot\left(e^{2xy^3}\cdot 6xy^2\right) = \boxed{6x^2y^2e^{2xy^3}}$$

Since each partial derivative of a function $f(x,y)$ is also a multivariable function, we can differentiate multiple times (just like in single variable calculus). The standard notation for this is

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \quad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

In this notation, we read the derivatives from left to right in order, so f_{xy} differentiates with respect to x first, then y second.

Example 1.3.2

Find each possible second partial derivative of the multivariable functions below.

(a) $f(x,y) = x^2y - x\sin(2y)$

(b) $g(x,y) = xe^{2xy^3}$

Solution

(a) We have already seen that $f_x = 2xy - \sin(2y)$ and $f_y = x^2 - 2x\cos(2y)$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(2xy - \sin(2y)) = 2\cdot y - 0 = \boxed{2y}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(2xy - \sin(2y)) = 2x\cdot 1 - \cos(2y)\cdot 2 = \boxed{2x - 2\cos(2y)}$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(x^2 - 2x\cos(2y)\right) = 0 - 2x\cdot(-2\sin(2y)) = \boxed{4x\sin(2y)}$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(x^2 - 2x\cos(2y)\right) = 2x - 2\cdot\cos(2y) = \boxed{2x - 2\cos(2y)}$$

(b) We have already seen that $g_x = \left(1+2xy^3\right)e^{2xy^3}$ and $g_y = 6x^2y^2e^{2xy^3}$

$$g_{xx} = \frac{\partial}{\partial x}(g_x) = \frac{\partial}{\partial x}\left(\left(1+2xy^3\right)e^{2xy^3}\right) = 2y^3\cdot e^{2xy^3} + \left(1+2xy^3\right)e^{2xy^3}\cdot 2y^3 = \boxed{\left(4y^3+4xy^6\right)e^{2xy^3}}$$

$$g_{xy} = \frac{\partial}{\partial y}(g_x) = \frac{\partial}{\partial y}\left(\left(1+2xy^3\right)e^{2xy^3}\right) = 6xy^2\cdot e^{2xy^3} + \left(1+2xy^3\right)e^{2xy^3}\cdot 6xy^2 = \boxed{\left(12xy^2+12x^2y^5\right)e^{2xy^3}}$$

$$g_{yy} = \frac{\partial}{\partial y}(g_y) = \frac{\partial}{\partial y}\left(6x^2y^2e^{2xy^3}\right) = 12x^2y\cdot e^{2xy^3} + 6x^2y^2e^{2xy^3}\cdot 6xy^2 = \boxed{\left(12x^2y+36x^3y^4\right)e^{2xy^3}}$$

$$g_{yx} = \frac{\partial}{\partial x}(g_y) = \frac{\partial}{\partial x}\left(6x^2y^2e^{2xy^3}\right) = 12xy^2\cdot e^{2xy^3} + 6x^2y^2e^{2xy^3}\cdot 2y^3 = \boxed{\left(12xy^2+12x^2y^5\right)e^{2xy^3}}$$

Notice that in each of the example functions above, we calculated the *mixed partial derivative* of the function two different ways $\frac{\partial^2}{\partial x\partial y}$ and $\frac{\partial^2}{\partial y\partial x}$ and ended up with the same result. This is not a coincidence!

Theorem 1.3.1: Clairaut's Theorem

Let $f(x,y)$ be a function of two variables such that each of f_{xy} and f_{yx} exists and is continuous.

Then $f_{xy} = f_{yx}$ must hold.

The proof of this theorem uses Green's Theorem in multivariable calculus, so the reader here may rely on the following intuitive argument.

If $f(x, y)$ represents the elevation of a hill, then f_x represents its slope in the *east-west* direction and f_y represents its slope in the *north-south* direction. Then f_{xy} represents the rate of change of this *east-west slope* as we move in the *north-south* direction, while f_{yx} represents the rate of change of the *north-south slope* as we move in the *east-west* direction. These two should be equal on a sufficiently smooth hill.

In a multivariable calculus course, this means that students have two different paths they could use to reach the same derivative. In a differential equations course, it means that we have a way to check if two different functions $M(x, y)$ and $N(x, y)$ are each a different derivative of the same function $f(x, y)$!

For example, if we are curious if there is some function $f(x, y)$ satisfying $f_x = 6x^2y^2$ and $f_y = 4x^3y$, we can check the prospective mixed partials:

$$\frac{\partial}{\partial y}\left(6x^2y^2\right) = 12x^2y \quad \text{and} \quad \frac{\partial}{\partial x}\left(4x^3y\right) = 12x^2y$$

Since these two match, there is some **potential function** $f(x, y)$ with the derivatives described. We will discuss how to find that function shortly.

Alternatively, if we are curious if there is some function $g(x, y)$ satisfying $g_x = 12x + 4y$ and $g_y = 6x - 2y$:

$$\frac{\partial}{\partial y}\left(12x + 4y\right) = 4 \quad \text{and} \quad \frac{\partial}{\partial x}\left(6x - 2y\right) = 6$$

Since these two mixed partial derivatives do not match, there is no possible function $g(x, y)$ with the partial derivatives above.

So how do we find the potential function when it exists? We use partial anti-derivatives (sometimes referred to as partial integrals).

Definition: Partial Anti-Derivative

Let $f(x, y)$ be a function of two independent variables x and y.

A **partial anti-derivative of f with respect to** x is a function $F(x, y)$ satisfying $F_x = f$:

$$F(x, y) = \int f(x, y)\, dx$$

A **partial anti-derivative of f with respect to** y is a function $F(x, y)$ satisfying $F_y = f$:

$$F(x, y) = \int f(x, y)\, dy$$

Partial anti-derivatives are not unique, they can differ by any term which differentiates to zero with respect to the variable in question!

We can evaluate a partial anti-derivative just like an indefinite integral in single variable calculus. Just remember to treat the other variable as a constant, and that your $+C$ term is not necessarily constant with respect to the *other variable*!

Returning to the example function $f(x, y)$ satisfying $f_x = 6x^2y^2$ and $f_y = 4x^3y$, we calculate each partial anti-derivative:

$$f(x, y) = \int 6x^2y^2\, dx = 6y^2 \cdot \int x^2\, dx = 6y^2 \cdot \frac{1}{3}x^3 + C(y) = 2x^3y^2 + C(y)$$
$$f(x, y) = \int 4x^3y\, dy = 4x^3 \cdot \int y\, dy = 4x^3 \cdot \frac{1}{2}y^2 + D(x) = 2x^3y^2 + D(x)$$

Note that the constants of integration $C(y)$ and $D(x)$ represent objects satisfying $\frac{\partial}{\partial x}C(y) = 0$ and $\frac{\partial}{\partial y}D(x) = 0$. We see that if $C(y)$ had any y terms, they would have appeared in the second integral. If $D(x)$ had any x terms, they would have appeared in the first integral. Because of this, we can see that $C(y) = D(x) = K$ for some constant K. Therefore, the potential functions are all of the form $f(x, y) = 2x^3y^2 + K$.

Example 1.3.3

Find the function with the given partial derivatives (if one exists):

(a) $f_x = 2xy\cos(x^2y) - y + 6x$ and $f_y = x^2\cos(x^2y) - x + 3$

(b) $g_x = \dfrac{y}{1 + x^2y^2}$ and $g_y = \dfrac{x}{1 + x^2y^2} - 8y$

(c) $h_x = \dfrac{1}{x^2 - y^2}$ and $h_y = \dfrac{1}{x^2 - y^2}$

Solution

(a) We first check if the mixed partials match:

$$\frac{\partial}{\partial y}\left(2xy\cos(x^2y) - y + 6x\right) = 2x\cos(x^2y) - 2x^3y\sin(x^2y) - 1$$
$$\frac{\partial}{\partial x}\left(x^2\cos(x^2y) - x + 3\right) = 2x\cos(x^2y) - 2x^3y\sin(x^2y) - 1$$

Since these match, we know that a potential function exists:

$$\int 2xy\cos(x^2y) - y + 6x\,dx = \sin(x^2y) - xy + 3x^2 + C(y)$$
$$\int x^2\cos(x^2y) - x + 3\,dy = \sin(x^2y) - xy + 3y + D(x)$$

All "mixed" xy terms match, and we see that $C(y) = 3y + K$ and $D(x) = 3x^2 + K$ make the rest of the terms match.

Therefore, the potential function is $\boxed{f(x, y) = \sin(x^2y) - xy + 3x^2 + 3y + K}$

(b) We first check if the mixed partials match:

$$\frac{\partial}{\partial y}\left(\frac{y}{1 + x^2y^2}\right) = \frac{(1 + x^2y^2)\cdot 1 - y\cdot(2x^2y)}{(1 + x^2y^2)^2} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2}$$
$$\frac{\partial}{\partial x}\left(\frac{x}{1 + x^2y^2} - 8y\right) = \frac{(1 + x^2y^2)\cdot 1 - x\cdot(2xy^2)}{(1 + x^2y^2)^2} - 0 = \frac{1 - x^2y^2}{(1 + x^2y^2)^2}$$

Since these match, we know that a potential function exists:

$$\int \frac{y}{1 + x^2y^2}\,dx = \tan^{-1}(xy) + C(y)$$
$$\int \frac{x}{1 + x^2y^2} - 8y\,dy = \tan^{-1}(xy) - 4y^2 + D(x)$$

All "mixed" xy terms match, and we see that $C(y) = -4y^2 + K$ and $D(x) = K$ make the rest of the terms match.

Therefore, the potential function is $\boxed{g(x, y) = \tan^{-1}(xy) - 4y^2 + K}$

(c) We first check if the mixed partials match:

$$\frac{\partial}{\partial y}\left(\frac{1}{x^2-y^2}\right) = -(x^2-y^2)^{-2}\cdot(-2y) = \frac{2y}{(x^2-y^2)^2}$$
$$\frac{\partial}{\partial x}\left(\frac{1}{x^2-y^2}\right) = -(x^2-y^2)^{-2}\cdot 2x = -\frac{2x}{(x^2-y^2)^2}$$

These do not match, so no possible potential function $h(x,y)$ can exist with the given partial derivatives.

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1.3: Exact Equations

We will make liberal use of the partial derivative notation in this textbook, even though we are focused on solving *ordinary* differential equations. This connection is most obvious when discussing implicit differentiation.

Students will recall that when a function $y(x)$ is defined implicitly, like in the equation $y^3 + y - x = 0$, we find its derivative by using the chain rule:

$$\frac{d}{dx}\left(y^3 + y - x = 0\right) \implies 3\left[y(x)\right]^2 y'(x) + y'(x) - 1 = 0 \implies y'(x) = \frac{1}{3\left[y(x)\right]^2 + 1}$$

This can be replicated by considering the partial derivatives of the function $F(x, y) = y^3 + y - x$.

$$F_x = -1 \quad F_y = 3y^2 + 1 \implies \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{-1}{3y^2 + 1} = \frac{1}{3y^2 + 1}$$

This is a direct result of the *chain rule* for multivariable functions, which says that if y is a function of x then $\frac{d}{dx}\left(F\left(x, y(x)\right)\right) = F_x + F_y \cdot \frac{dy}{dx}$. This gives us a specific type of differential equation that would be very easy to solve!

Definition: Exact Differential Equation

A first order differential equation of the form $M(x, y) + N(x, y) \cdot \frac{dy}{dx} = 0$ is an ***exact equation*** if $M_y = N_x$

We can see that taking the implicit derivative of $F\left(x, y(x)\right) = K$ gives $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{M}{N}$.

Therefore, the left side of the exact differential equation becomes $M + N \cdot \left(-\frac{M}{N}\right)$ which definitely simplifies down to zero.

For example, consider the differential equation $y + x \ \frac{dy}{dx} = 0$:

$$M = y \implies M_y = 1 \quad N = x \implies N_x = 1$$

This is an exact differential equation. We look for $F(x, y)$ satisfying $F_x = M = y$ and $F_y = N = x$:

$$\int y\, dx = xy + C(y) \quad \text{and} \quad \int x\, dy = xy + D(x)$$

We see that $F(x, y) = xy$ is one such potential function, and therefore the one-parameter solution to this differential equation is $xy = K$.

Rule: Solving Exact Differential Equations

Given an exact first order differential equation of the form $M(x, y) + N(x, y) \cdot \frac{dy}{dx} = 0$ which satisfies $M_y = N_x$, there exists some potential function F with $F_x = M$ and $F_y = N$. A one parameter family of solutions to this equation is $F(x, y) = K$

Example 1.3.4

Find the solutions to each exact differential equation below.

(a) $2xy + y^3 + (x^2 + 3xy^2)\dfrac{dy}{dx} = 0$

(b) $(y+1)\ dx = (1-x)\ dy$

(c) $\dfrac{dy}{dx} = \dfrac{4x^3 - y\sec^2(xy)}{x\sec^2(xy) + 2y}$

Solution

(a) First we confirm the partial derivatives of match:

$$\begin{aligned} M = 2xy + y^3 &\implies M_y = 2x + 3y^2 \\ N = x^2 + 3xy^2 &\implies N_x = 2x + 3y^2 \end{aligned}$$

Now we find the potential function with these partials:

$$\begin{cases} F(x,y) = \displaystyle\int M\,dx = \int 2xy + y^3\,dx = x^2y + xy^3 + C(y) \\ F(x,y) = \displaystyle\int N\,dy = \int x^2 + 3xy^2\,dy = x^2y + xy^3 + D(x) \end{cases} \implies F(x,y) = x^2y + xy^3$$

Therefore, the one-parameter family of solutions to this differential equation is $\boxed{x^2y + xy^3 = K}$

(b) We rewrite this differential equation so that we can clearly see M and N:

$$(y+1)\,dx = (1-x)\,dy \implies y + 1 = (1-x)\cdot\frac{dy}{dx} \implies (y+1) + (x-1)\frac{dy}{dx} = 0$$

Now we confirm the partial derivatives match.

$$\begin{aligned} M = y + 1 &\implies M_y = 1 \\ N = x - 1 &\implies N_x = 1 \end{aligned}$$

Now we find the potential function with these partials:

$$\begin{cases} F(x,y) = \displaystyle\int M\,dx = \int y + 1\,dx = xy + x + C(y) \\ F(x,y) = \displaystyle\int N\,dy = \int x - 1\,dy = xy - y + D(x) \end{cases} \implies F(x,y) = xy + x - y$$

Therefore, the one-parameter family of solutions to this differential equation is $\boxed{xy + x - y = K}$

(c) We rewrite this differential equation so that we can clearly see M and N:

$$\frac{dy}{dx} = \frac{4x^3 - y\sec^2(xy)}{x\sec^2(xy) + 2y} \implies 4x^3 - y\sec^2(xy) = \big(x\sec^2(xy) + 2y\big)\cdot\frac{dy}{dx} \implies 4x^3 - y\sec^2(xy) + \big(-x\sec^2(xy) - 2y\big)\cdot\frac{dy}{dx} = 0$$

Now we confirm the partial derivatives match.

$$\begin{aligned} M = 4x^3 - y\sec^2(xy) &\implies M_y = -\sec^2(xy) - 2xy\sec^2(xy)\tan(xy) \\ N = -x\sec^2(xy) - 2y &\implies N_x = -\sec^2(xy) - 2xy\sec^2(xy)\tan(xy) \end{aligned}$$

Now we find the potential function with these partials:

$$\begin{cases} F(x,y) = \displaystyle\int M\,dx = \int 4x^3 - y\sec^2(xy)\,dx = x^4 - \tan(xy) + C(y) \\ F(x,y) = \displaystyle\int N\,dy = \int -x\sec^2(xy) - 2y\,dy = -\tan(xy) - y^2 + D(x) \end{cases} \implies F(x,y) = x^4 - y^2 - \tan(xy)$$

Therefore, the one-parameter family of solutions to this differential equation is $\boxed{x^4 - y^2 - \tan(xy) = K}$

Sometimes a differential equation contains additional conditions, like a particular point the solution must contain. These are generally referred to as *initial conditions*, and they give us additional information with which to solve for which member of the family of solutions we are interested in. In layman's terms, this gives us something with which to try to solve for the constant K.

Using the example above $2xy + y^3 + (x^2 + 3xy^2)\dfrac{dy}{dx} = 0$, we showed that the solutions were of the form $x^2y + xy^3 = K$. If we want one that contains the point $(2, 3)$, then we know $2^2 \cdot 3 + 2 \cdot 3^3 = K \implies K = 66$. Therefore, the particular solution we are interested in is the curve $x^2y + xy^3 = 66$.

Example 1.3.5

Find the solution to each exact differential equation with initial condition below.

(a) $\dfrac{\ln(y)}{x^2} - \dfrac{1}{xy}\dfrac{dy}{dx} = 0$ with $y(2) = e^2$

(b) $\dfrac{dy}{dx} = \dfrac{4xy^4 - 9x^2y^2}{6x^3y - 8x^2y^3}$ with $y(-1) = 2$

Solution

(a) First we confirm the partial derivatives of match:

$$M = \frac{\ln(y)}{x^2} \implies M_y = \frac{1}{x^2y}$$
$$N = -\frac{1}{xy} \implies N_x = \frac{1}{x^2y}$$

Now we find the potential function with these partials:

$$\begin{cases} F(x,y) = \displaystyle\int M\,dx = \int \frac{\ln(y)}{x^2}\,dx = -\frac{\ln(y)}{x} + C(y) \\ F(x,y) = \displaystyle\int N\,dy = \int -\frac{1}{xy}\,dy = -\frac{\ln(y)}{x} + D(x) \end{cases} \implies F(x,y) = -\frac{\ln(y)}{x}$$

Therefore, the one-parameter family of solutions to this differential equation is $-\dfrac{\ln(y)}{x} = K$

Plugging in $x = 2$ and $y = e^2$ this becomes $-\dfrac{\ln(e^2)}{2} = K \implies K = -1$

Therefore, the solution we want is $-\dfrac{\ln(y)}{x} = -1$ which we can rewrite as $x = \ln(y)$ or $\boxed{y = e^x}$

(b) We rewrite this differential equation so that we can clearly see M and N:

$$\frac{dy}{dx} = 4xy^4 - 9x^2y^2 \xrightarrow{\frac{6x^3y - 8x^2y^3}{\implies}} 4xy^4 - 9x^2y^2 = \left(6x^3y - 8x^2y^3\right) \cdot \frac{dy}{dx} \implies 4xy^4 - 9x^2y^2 + \left(8x^2y^3 - 6x^3y\right) \cdot \frac{dy}{dx} = 0$$

Now we confirm the partial derivatives match.

$$M = 4xy^4 - 9x^2y^2 \implies M_y = 16xy^3 - 18x^2y$$
$$N = 8x^2y^3 - 6x^3y \implies N_x = 16xy^3 - 18x^2y$$

Now we find the potential function with these partials:

$$\begin{cases} F(x,y) = \int M\,dx = \int 4xy^4 - 9x^2y^2\,dx = 2x^2y^4 - 3x^3y^2 + C(y) \\ F(x,y) = \int N\,dy = \int 8x^2y^3 - 6x^3y\,dy = 2x^2y^4 - 3x^2y^2 + D(x) \end{cases} \implies F(x,y) = 2x^2y^4 - 3x^3y^2$$

Therefore, the one-parameter family of solutions to this differential equation is $2x^2y^4 - 3x^3y^2 = K$

Plugging in $x = -1$ and $y = 2$ this becomes $2 \cdot (-1)^2 \cdot 2^4 - 3 \cdot (-1)^3 \cdot 2^2 = K \implies K = 20$

Therefore, the solution we want is $\boxed{2x^2y^4 - 3x^3y^2 = 20}$

Notice in the very last example that the original differential equation $\frac{dy}{dx} = \frac{6x^3y - 8x^2y^3}{4xy^4 - 9x^2y^2}$ could have been reduced to $\frac{dy}{dx} = \frac{6x^2 - 8xy^2}{4y^3 - 9xy}$

In this form, re-writing it as $(4y^3 - 9xy) + (8xy^2 - 6x^2) \cdot \frac{dy}{dx} = 0$ is *not an exact equation*:

$$\hat{M} = 4y^3 - 9xy \implies \hat{M}_y = 12y^2 - 9x$$
$$\hat{N} = 8xy^2 - 6x^2 \implies \hat{N}_x = 8y^2 - 12x$$

So how would we approach this question if it started off in this reduced form? Consider that we reduced away a function $\mu(x,y) = xy$ from each of M and N:

$$M(x,y) = 4xy^4 - 9x^2y^2 = \mu(x,y) \cdot \hat{M}(x,y) \quad \text{and} \quad N(x,y) = 8x^2y^3 - 6x^3y = \mu(x,y) \cdot \hat{N}(x,y)$$

If we begin with $\hat{M} + \hat{N} \cdot \frac{dy}{dx} = 0$ and find that $\hat{M}_y \neq \hat{N}_x$, then we "guess" that such an $M = \mu(x,y) \cdot \hat{M}$ and $N = \mu(x,y) \cdot \hat{N}$ exist. That way we can multiply the non-exact differential equation throughout by $\mu(x,y)$ to get:

$$\mu\hat{M} + \mu\hat{N} \cdot \frac{dy}{dx} = 0$$

This new equation will be exact when $\frac{\partial}{\partial y}\left(\mu\hat{M}\right) = \frac{\partial}{\partial x}\left(\mu\hat{N}\right)$, or equivalently:

$$\mu_y\hat{M} + \mu\hat{M}_y = \mu_x\hat{N} + \mu\hat{N}_x \longleftrightarrow \hat{M}_y - \hat{N}_x = \frac{\mu_x}{\mu}\hat{N} - \frac{\mu_y}{\mu}\hat{M}$$

If we can find such a function $\mu(x,y)$, it is called an **integrating factor**. The unfortunate truth about finding an integrating factor though, is that it requires solving a partial differential equation!

Returning to our example $(4y^3 - 9xy) + (8xy^2 - 6x^2) \cdot \frac{dy}{dx} = 0$ above, again let $\hat{M} = 4y^3 - 9xy$ and $\hat{N} = 8xy^2 - 6x^2$

$$\hat{M}_y - \hat{N}_x = (12y^2 - 9x) - (8y^2 - 12x) = 4y^2 + 3x = \frac{\mu_x}{\mu} \cdot \left(8xy^2 - 6x^2\right) - \frac{\mu_y}{\mu}\left(4y^3 - 9xy\right)$$

It is easy to check that for $\mu(x,y) = xy$ we have $\frac{\mu_x}{\mu} = \frac{1}{x}$ and $\frac{\mu_y}{\mu} = \frac{1}{y}$ which satisfies the equation above, but it is not easy to find that $\mu(x,y)$ in the first place. In general **all non-exact first order differential equations can be made exact if multiplied by the proper integrating factor**. But finding this integrating factor is sometimes harder than using other approaches.

Consider the special case where $\mu(x,y)$ depends *only* on x, (so $\mu_y = 0$), then we would want:

$$\hat{M}_y - \hat{N}_x = \frac{\mu_x}{\mu} \cdot \hat{N} - 0 \cdot \hat{M} \implies \mu(x) = e^{\int \frac{\hat{M}_y - \hat{N}_x}{\hat{N}}\,dx}$$

And alternatively, if $\mu(x,y)$ depends *only* on y, (so $\mu_x = 0$), then we would want:

$$\hat{M}_y - \hat{N}_x = 0 \cdot \hat{N} - \frac{\mu_y}{\mu} \cdot \hat{M} \implies \mu(y) = e^{\int \frac{\hat{N}_x - \hat{M}_y}{\hat{M}}\, dy}$$

Not every non-exact equation will fall into one of these cases, but all of them in this part of the textbook will!

Illustration 1.3.1

Solve each non-exact differential equation by finding a suitable integrating factor.

(a) $(y+1) - x \cdot \dfrac{dy}{dx} = 0$ with $y(1) = 2$

(b) $(1+2x)y^2 + 2xy \cdot \dfrac{dy}{dx} = 0$ with $y(1) = \dfrac{1}{e}$

Solution

(a) We have $\hat{M} = y+1 \implies \hat{M}_y = 1$ and $\hat{N} = -x \implies \hat{N}_x = -1$

This can be done with either of our two types of integrating factor, so we will show both.

Consider $\displaystyle\int \frac{\hat{M}_y - \hat{N}_x}{\hat{N}}\, dx = \int \frac{2}{-x}\, dx = -2\ln|x|$

Therefore, we can use $\mu(x) = e^{-2\ln|x|} = x^{-2}$ as an integrating factor, turning the differential equation into:

$$x^{-2}(y+1) - x^{-1} \cdot \frac{dy}{dx} = 0$$

Now we integrate to find the potential function:

$$\begin{cases} F(x,y) = \displaystyle\int M\, dx = \int x^{-2}(y+1)\, dx = -\frac{y+1}{x} + C(y) \\ F(x,y) = \displaystyle\int N\, dy = \int -x^{-1}\, dy = -\frac{y}{x} + D(x) \end{cases} \implies F(x,y) = -\frac{y+1}{x}$$

Therefore, the solutions to the differential equations are of the form $-\dfrac{y+1}{x} = K$

Plugging in the initial condition $x = 1$ and $y = 2$ gives $-\dfrac{3}{1} = K \implies K = -3$

This gives a final solution of $-\dfrac{y+1}{x} = -3$ which looks nicer re-written as $\boxed{y = 3x - 1}$

If we instead found an integrating factor in terms of y, we will get the same answer.

We use $\displaystyle\int \frac{\hat{N}_x - \hat{M}_y}{\hat{M}}\, dy = \int \frac{-2}{y+1}\, dx = -2\ln|y+1|$

Therefore, we can use $\mu(y) = e^{-2\ln|y+1|} = (y+1)^{-2}$ as an integrating factor, turning the differential equation into:

$$(y+1)^{-1} - x(y+1)^{-2} \cdot \frac{dy}{dx} = 0$$

Again we integrate to find the potential function:

$$\begin{cases} F(x,y) = \displaystyle\int M\, dx = \int (y+1)^{-1}\, dx = \frac{x}{y+1} + C(y) \\ F(x,y) = \displaystyle\int N\, dy = \int -x(y+1)^{-2}\, dy = \frac{x}{y+1} + D(x) \end{cases} \implies F(x,y) = \frac{x}{y+1}$$

Therefore, the solutions to the differential equations are of the form $\dfrac{x}{y+1} = K$

Plugging in the initial condition $x = 1$ and $y = 2$ gives $\dfrac{1}{3} = K$

This gives a final solution of $\dfrac{x}{y+1} = \dfrac{1}{3}$ which is the same as before: $\boxed{y = 3x - 1}$

(b) This differential equation has $\hat{M} = (1+2x)y^2 \implies \hat{M}_y = 2y + 4xy$ and $\hat{N} = 2xy \implies \hat{N}_x = 2y$

We see that $\hat{M}_y - \hat{N}_x = 4xy$, so $\dfrac{\hat{N}_x - \hat{M}_y}{M} = \dfrac{-4x}{(1+2x)y}$ and $\dfrac{\hat{M}_y - \hat{N}_x}{N} = 2$

This tells us that we will use $\mu(x) = e^{\int 2\,dx} = e^{2x}$ as our integrating factor, turning the differential equation into:

$$(1+2x)y^2e^{2x} + 2xye^{2x} \cdot \frac{dy}{dx} = 0$$

Now we integrate to find the potential function:

$$\begin{cases} F(x,y) = \displaystyle\int M\,dx = \int (1+2x)y^2e^{2x}\,dx = xy^2e^{2x} + C(y) \\ F(x,y) = \displaystyle\int N\,dy = \int 2xye^{2x}\,dy = xy^2e^{2x} + D(x) \end{cases} \implies F(x,y) = xy^2e^{2x}$$

Therefore, the solutions to the differential equations are of the form $xy^2e^{2x} = K$

Plugging in the initial condition $x = 1$ and $y = \dfrac{1}{e}$ gives $1 \cdot \left(\dfrac{1}{e}\right)^2 e^{2\cdot 1} = K \implies K = 1$

This gives a final solution of $xy^2e^{2x} = 1$ which looks nicer re-written as $\boxed{y = \dfrac{1}{e^x\sqrt{x}}}$

Finding an integrating factor can make a seemingly impossible differential equation into a simple exact one! Not all first order differential equations can be made exact by a *single variable* integrating factor, but we will discuss other methods of solving differential equations in the coming chapters.

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1.3: Practice Set

Exercise 1.3.1a

Find both first partial derivatives of $f(x, y) = x^3y + e^x y^2$

Answer available at the end of this section.

Exercise 1.3.1b

Find both first partial derivatives of $f(x, y) = \sin(xy) + y\ln(x)$

Answer available at the end of this section.

Exercise 1.3.1c

Find both first partial derivatives of $f(x, y) = \dfrac{x^2 + y^2}{xy}$

Answer available at the end of this section.

Exercise 1.3.2a

Verify that $f_{xx} + f_{yy} > 0$ for $f(x, y) = e^{x^2+y}$

Answer available at the end of this section.

Exercise 1.3.2b

Verify that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ for $z = \ln(x^2 + y^2)$

Answer available at the end of this section.

Exercise 1.3.2c

Verify that $f_{xx} + f_{yy} = f_{xy} - f_{yx}$ for $f(x, y) = x^3y - xy^3$

Answer available at the end of this section.

Exercise 1.3.3a

Evaluate the partial integral $\displaystyle\int (3x^2y + 2y)\, dx$

Answer available at the end of this section.

Exercise 1.3.3b

Evaluate the partial integral $\int (e^x + \cos(y))\, dy$

Answer available at the end of this section.

Exercise 1.3.3c

Determine if there exists $f(x, y)$ such that $f_x = 2xy$ and $f_y = x^2 + 3y^2$

Answer available at the end of this section.

Exercise 1.3.3d

Determine if there exists $f(x, y)$ and $k^{\text{int}}\mathbb{R}$ such that $f_x = kx^2y$ and $f_y = x^3 + 2y$

Answer available at the end of this section.

Exercise 1.3.4a

Find the family of solutions to the differential equation

$$(2xy + y^3) + (x^2 + 3xy^2) \cdot \frac{dy}{dx} = 0$$

Answer available at the end of this section.

Exercise 1.3.4b

Find the family of solutions to the differential equation

$$(e^y + 2x)\, dx + (xe^y + 3y^2)\, dy = 0$$

Answer available at the end of this section.

Exercise 1.3.4c

Find the family of solutions to the differential equation

$$(y + y\cos(xy))\ dx + (x + x\cos(xy))\ dy = 0$$

Answer available at the end of this section.

Exercise 1.3.4d

Find the family of solutions to the differential equation

$$\frac{dy}{dx} = -\frac{3x^2y + y^3}{x^3 + 3xy^2 + 6y^2}$$

Answer available at the end of this section.

Exercise 1.3.4e

Find the family of solutions to the differential equation

$$\frac{y^3}{(y^2-x^2)^{\frac{3}{2}}}\,dx = \left(\frac{x^3}{(y^2-x^2)^{\frac{3}{2}}} - 1\right)\,dy$$

Answer available at the end of this section.

Exercise 1.3.4f

Find the family of solutions to the differential equation

$$\frac{dy}{dx} = \frac{1-4xy}{4x^2y^2+2x^2-2xy-2y}$$

Answer available at the end of this section.

Exercise 1.3.5a

Find the solution to the differential equation

$$3x^2+2y+(2x+6y^2)\cdot\frac{dy}{dx} = 0 \quad \text{with } y(1)=1$$

Answer available at the end of this section.

Exercise 1.3.5b

Find the solution to the differential equation

$$\left(\frac{8x}{1+x^4}+y\right)\,dx + \left(x+4\tan^2(y)\right)\,dy = 0 \quad \text{with } y(1)=0$$

Answer available at the end of this section.

Exercise 1.3.5c

Find the solution to the differential equation

$$\frac{dy}{dx} = -\frac{2x+e^y\cos(x)}{e^y\sin(x)+4y^3} \quad \text{with } y(\pi)=\sqrt{\pi}$$

Answer available at the end of this section.

Exercise 1.3.5d

Find the solution to the differential equation

$$xy - xy\tan^2\left(\frac{x^2}{4}\right) + 2\tan\left(\frac{x^2}{4}\right)\cdot\frac{dy}{dx} = 0 \quad \text{with } y\left(\sqrt{\pi}\right) = 2$$

Answer available at the end of this section.

Answers

Exercise 1.3.1*a*

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y) + \frac{\partial}{\partial x}(e^x y^2) = 3x^2y + e^x y^2$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y) + \frac{\partial}{\partial y}(e^x y^2) = x^3 + 2e^x y$$

Exercise 1.3.1*b*

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\sin(xy)) + \frac{\partial}{\partial x}(y\ln(x)) = y\cos(xy) + \frac{y}{x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\sin(xy)) + \frac{\partial}{\partial y}(y\ln(x)) = x\cos(xy) + \ln(x)$$

Exercise 1.3.1*c*

$$\frac{\partial f}{\partial x} = \frac{(2x)(xy) - (x^2+y^2)(y)}{(xy)^2} = \frac{2x^2y - x^2y - y^3}{x^2y^2} = \frac{x^2 - y^2}{x^2y}$$
$$\frac{\partial f}{\partial y} = \frac{(2y)(xy) - (x^2+y^2)(x)}{(xy)^2} = \frac{2xy^2 - x^3 - xy^2}{x^2y^2} = \frac{y^2 - x^2}{xy^2}$$

Exercise 1.3.2*a*

$$f_x = 2xe^{x^2+y}$$
$$f_{xx} = (4x^2+2)e^{x^2+y}$$
$$f_y = e^{x^2+y}$$
$$f_{yy} = e^{x^2+y}$$

Clearly $f_{xx} + f_{yy} = (4x^2+3)e^{x^2+y} > 0$

Exercise 1.3.2*b*

$$z_x = \frac{2x}{x^2+y^2}$$
$$z_{xx} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$$
$$z_y = \frac{2y}{x^2+y^2}$$
$$z_{yy} = \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2} = \frac{2x^2 - 2y^2}{(x^2+y^2)^2}$$

Clearly $z_{xx} + z_{yy} = 0$

Exercise 1.3.2*c*

$$f_x = 3x^2y - y^3 \qquad f_y = x^3 - 3xy^2$$
$$f_{xx} = 6xy \qquad f_{xy} = 3x^2 - 3y^2$$
$$f_{yx} = 3x^2 - 3y^2 \qquad f_{yy} = -6xy$$

Clearly $f_{xx} + f_{yy} = 0 = f_{xy} - f_{yx}$

Exercise 1.3.3*a*

$$\int (3x^2y + 2y)\,dx = y\int 3x^2dx + 2y\int dx = y(x^3) + 2y(x) + C(y) = x^3y + 2xy + C(y)$$

Exercise 1.3.3*b*

$$\int (e^x + \cos(y))\,dy = e^x\int dy + \int \cos(y)\,dy = e^x y + \sin(y) + C(x)$$

Exercise 1.3.3*c*

$$\frac{\partial}{\partial y}(2xy) = 2x \quad \text{and} \quad \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, such f exists:

$$f(x,y) = \int 2xy\,dx = x^2y + C(y)$$
$$f(x,y) = \int (x^2 + 3y^2)\,dy = x^2y + y^3 + D(x)$$
$$C(y) = y^3, \quad D(x) = 0$$

Therefore, the potential functions are of the form $f(x,y) = x^2y + y^3 + K$

Exercise 1.3.3*d*

$$\frac{\partial}{\partial y}(kx^2y) = kx^2 \quad \text{and} \quad \frac{\partial}{\partial x}(x^3 + 2y) = 3x^2$$

When $k = 3$, we see $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so such f exists:

$$f(x,y) = \int 3x^2y\,dx = x^3y + C(y)$$
$$f(x,y) = \int (x^3 + 2y)\,dy = x^3y + y^2 + D(x)$$
$$C(y) = y^2, \quad D(x) = 0$$

Therefore, the potential functions are of the form $f(x,y) = x^3y + y^2 + K$

Exercise 1.3.4*a*

We check for exactness:

$$\frac{\partial}{\partial y}(2xy + y^3) = 2x + 3y^2 \quad \text{and} \quad \frac{\partial}{\partial x}(x^2 + 3xy^2) = 2x + 3y^2$$

These are equal, so we find the potential function:

$$f(x,y) = \int (2xy + y^3)\,dx = x^2y + xy^3 + C(y)$$
$$f(x,y) = \int (x^2 + 3xy^2)\,dy = x^2y + xy^3 + D(x)$$
$$C(y) = 0, \quad D(x) = 0$$
$$\implies f(x,y) = x^2y + xy^3$$

Thus, the family of solutions is $x^2y + xy^3 = K$

Exercise 1.3.4*b*

Rewrite as $e^y + 2x + (xe^y + 3y^2) \cdot \frac{dy}{dx} = 0$

We check for exactness:

$$\frac{\partial}{\partial y}(e^y + 2x) = e^y \quad \text{and} \quad \frac{\partial}{\partial x}(xe^y + 3y^2) = e^y$$

These are equal, so we find the potential function:

$$f(x,y) = \int (e^y + 2x)\,dx = xe^y + x^2 + C(y)$$
$$f(x,y) = \int (xe^y + 3y^2)\,dy = xe^y + y^3 + D(x)$$
$$C(y) = y^3, \quad D(x) = x^2$$
$$\implies f(x,y) = xe^y + x^2 + y^3$$

Thus, the family of solutions is $xe^y + x^2 + y^3 = K$

Exercise 1.3.4*c*

Rewrite as $y + y\cos(xy) + (x + x\cos(xy)) \cdot \frac{dy}{dx} = 0$

We check for exactness:

$$\frac{\partial}{\partial y}(y + y\cos(xy)) = 1 + \cos(xy) - xy\sin(xy) \quad \text{and} \quad \frac{\partial}{\partial x}(x + x\cos(xy)) = 1 + \cos(xy) - xy\sin(xy)$$

These are equal, so we find the potential function:

$$f(x,y) = \int (y + y\cos(xy))\,dx = xy + \sin(xy) + C(y)$$
$$f(x,y) = \int (x + x\cos(xy))\,dy = xy + \sin(xy) + D(x)$$
$$C(y) = 0, \quad D(x) = 0$$
$$\implies f(x,y) = xy + \sin(xy)$$

Thus, the family of solutions is $xy + \sin(xy) = K$

Exercise 1.3.4*d*

Rewrite as $3x^2y + y^3 + (x^3 + 3xy^2 + 6y^2) \cdot \frac{dy}{dx} = 0$

We check for exactness:

$$\frac{\partial}{\partial y}(3x^2y + y^3) = 3x^2 + 3y^2 \quad \text{and} \quad \frac{\partial}{\partial x}(x^3 + 3xy^2 + 6y^2) = 3x^2 + 3y^2$$

These are equal, so we find the potential function:

$$f(x,y) = \int (3x^2y + y^3)\,dx = x^3y + xy^3 + C(y)$$
$$f(x,y) = \int (x^3 + 3xy^2 + 6y^2)\,dy = x^3y + xy^3 + 2y^3 + D(x)$$
$$C(y) = 2y^3, \quad D(x) = 0$$
$$\implies f(x,y) = x^3y + xy^3 + 2y^3$$

Thus, the family of solutions is $x^3y + xy^3 + 2y^3 = K$

Exercise 1.3.4*e*

Rewrite as $\dfrac{y^3}{(y^2-x^2)^{\frac{3}{2}}} + \left(-\dfrac{x^3}{(y^2-x^2)^{\frac{3}{2}}} + 1\right) \cdot \dfrac{dy}{dx} = 0$

We check for exactness:

$$\frac{\partial}{\partial y}\left(\frac{y^3}{(y^2-x^2)^{\frac{3}{2}}}\right) = -\frac{3x^2y^2}{(y^2-x^2)^{\frac{5}{2}}} \quad \text{and} \quad \frac{\partial}{\partial x}\left(-\frac{x^3}{(y^2-x^2)^{\frac{3}{2}}} + 1\right) = -\frac{3x^2y^2}{(y^2-x^2)^{\frac{5}{2}}}$$

These are equal, so we find the potential function:

$$f(x,y) = \int \frac{y^3}{(y^2-x^2)^{\frac{3}{2}}}\,dx = \frac{xy}{\sqrt{y^2-x^2}} + C(y)$$
$$f(x,y) = \int \left(-\frac{x^3}{(y^2-x^2)^{\frac{3}{2}}} + 1\right) dy = \frac{xy}{\sqrt{y^2-x^2}} + y + D(x)$$
$$C(y) = y, \quad D(x) = 0$$
$$\implies f(x,y) = \frac{xy}{\sqrt{y^2-x^2}} + y$$

Thus, the family of solutions is $\frac{xy}{\sqrt{y^2-x^2}} + y = K$

Exercise 1.3.4*f*

Rewrite as $4xy - 1 + (4x^2y^2 + 2x^2 - 2xy - 2y) \cdot \dfrac{dy}{dx} = 0$

We check for exactness:

$$\frac{\partial}{\partial y}(4xy - 1) = 4x \quad \text{and} \quad \frac{\partial}{\partial x}\left(4x^2y^2 + 2x^2 - 2xy - 2y\right) = 8xy^2 + 4x - 2y$$

These are not equal, so we look for an integrating factor:

$$\frac{\hat{M}_y - \hat{N}_x}{\hat{N}} = \frac{-8xy^2 + 2y}{4x^2y^2 + 2x^2 - 2xy - 2y} \quad \text{and} \quad \frac{\hat{N}_x - \hat{M}_y}{\hat{M}} = \frac{8xy^2 - 2y}{4xy - 1} = 2y$$

We use $\mu(y) = e^{\int 2y\,dy} = e^{y^2}$ and rewrite the differential equation as:

$$(4xy - 1)\,e^{y^2} + \left(4x^2y^2 + 2x^2 - 2xy - 2y\right)e^{y^2} \cdot \frac{dy}{dx} = 0$$

Now we find the potential function:

$$f(x,y) = \int (4xy - 1)\,e^{y^2}\,dx = \left(2x^2y - x\right)e^{y^2} + C(y)$$
$$f(x,y) = \int \left(4x^2y^2 + 2x^2 - 2xy - 2y\right)e^{y^2}\,dy = \left(2x^2y - x - 1\right)e^{y^2} + D(x)$$
$$C(y) = -e^{y^2}, \quad D(x) = 0$$
$$\implies f(x,y) = \left(2x^2y - x - 1\right)e^{y^2}$$

Thus, the family of solutions is $\left(2x^2y - x - 1\right)e^{y^2} = K$

Exercise 1.3.5*a*

We check for exactness:

$$\frac{\partial}{\partial y}(3x^2 + 2y) = 2 \quad \text{and} \quad \frac{\partial}{\partial x}(2x + 6y^2) = 2$$

These are equal, so we find the potential function:

$$f(x,y) = \int (3x^2 + 2y)\,dx = x^3 + 2xy + C(y)$$
$$f(x,y) = \int (2x + 6y^2)\,dy = 2xy + 2y^3 + D(x)$$
$$C(y) = 2y^3, \quad D(x) = x^3$$
$$\implies f(x,y) = x^3 + 2xy + 2y^3$$

Thus, the family of general solutions is $x^3 + 2xy + 2y^3 = K$

Plugging in the initial condition gives

$$1^3 + 2(1)(1) + 2(1)^3 = K \implies K = 5$$

Therefore, the unique solution is $x^3 + 2xy + 2y^3 = 5$

Exercise 1.3.5*b*

We check for exactness:

$$\frac{\partial}{\partial y}\left(\frac{8x}{1+x^4} + y\right) = 1 \quad \text{and} \quad \frac{\partial}{\partial x}\left(x + 4\tan^2(y)\right) = 1$$

These are equal, so we find the potential function:

$$f(x,y) = \int \left(\frac{8x}{1+x^4} + y\right)\, dx = 4\tan^{-1}\left(x^2\right) + xy + C(y)$$
$$f(x,y) = \int \left(x + 4\tan^2(y)\right)\, dy = xy + 4\tan(y) - 4y + D(x)$$
$$C(y) = 4\tan(y) - 4y, \quad D(x) = 4\tan^{-1}\left(x^2\right)$$
$$\implies f(x,y) = xy + 4\tan^{-1}\left(x^2\right) + 4\tan(y) - 4y$$

Thus, the family of general solutions is $xy + 4\tan^{-1}\left(x^2\right) + 4\tan(y) - 4y = K$

Plugging in the initial condition gives

$$1(0) + 4\tan^{-1}(1) + 4\tan(0) - 4(0) = K \implies K = \pi$$

Therefore, the unique solution is $xy + 4\tan^{-1}\left(x^2\right) + 4\tan(y) - 4y = \pi$

Exercise 1.3.5*c*

Rewrite as $2x + e^y\cos(x) + \left(e^y\sin(x) + 4y^3\right)\cdot\dfrac{dy}{dx} = 0$

We check for exactness:

$$\frac{\partial}{\partial y}\left(2x + e^y\cos(x)\right) = e^y\cos(x) \quad \text{and} \quad \frac{\partial}{\partial x}\left(e^y\sin(x) + 4y^3\right) = e^y\cos(x)$$

These are equal, so we find the potential function:

$$f(x,y) = \int \left(2x + e^y\cos(x)\right)\, dx = x^2 + e^y\sin(x) + C(y)$$
$$f(x,y) = \int \left(e^y\sin(x) + 4y^3\right)\, dy = e^y\sin(x) + y^4 + D(x)$$
$$C(y) = y^4, \quad D(x) = x^2$$
$$\implies f(x,y) = x^2 + e^y\sin(x) + y^4$$

Thus, the family of general solutions is $x^2 + e^y\sin(x) + y^4 = K$

Plugging in the initial condition gives

$$\pi^2 + e^{\sqrt{\pi}}\sin(\pi) + \left(\sqrt{\pi}\right)^4 = K \implies K = 2\pi^2$$

Therefore, the unique solution is $x^2 + e^y\sin(x) + y^4 = 2\pi^2$

Exercise 1.3.5*d*

We check for exactness:

$$\frac{\partial}{\partial y}\left(xy - xy\tan^2\left(\frac{x^2}{4}\right)\right) = x - x\tan^2\left(\frac{x^2}{4}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(2\tan\left(\frac{x^2}{4}\right)\right) = x\sec^2\left(\frac{x^2}{4}\right) = x + x\tan^2\left(\frac{x^2}{4}\right)$$

These are not equal, so we look for an integrating factor:

$$\frac{\hat{M}_y - \hat{N}_x}{\hat{N}} = \frac{-2x\tan^2\left(\frac{x^2}{4}\right)}{2\tan\left(\frac{x^2}{4}\right)} = -x\tan\left(\frac{x^2}{4}\right) \quad \text{and} \quad \frac{\hat{N}_x - \hat{M}_y}{\hat{M}} = \frac{2x\tan^2\left(\frac{x^2}{4}\right)}{xy - xy\tan^2\left(\frac{x^2}{4}\right)}$$

We use $\mu(x) = e^{\int -x\tan\left(\frac{x^2}{4}\right)dx} = e^{2\ln\left(\cos\left(\frac{x^2}{4}\right)\right)} = \cos^2\left(\dfrac{x^2}{4}\right)$ and rewrite the differential equation as:

$$xy\left(\cos^2\left(\frac{x^2}{4}\right) - \sin^2\left(\frac{x^2}{4}\right)\right) + 2\sin\left(\frac{x^2}{4}\right)\cos\left(\frac{x^2}{4}\right)\cdot\frac{dy}{dx} = 0$$

Now we find the potential function:

$$f(x,y) = \int xy\left(\cos^2\left(\frac{x^2}{4}\right) - \sin^2\left(\frac{x^2}{4}\right)\right)\, dx = 2y\sin\left(\frac{x^2}{4}\right)\cos\left(\frac{x^2}{4}\right) + C(y)$$
$$f(x,y) = \int 2\sin\left(\frac{x^2}{4}\right)\cos\left(\frac{x^2}{4}\right)\, dy = 2y\sin\left(\frac{x^2}{4}\right)\cos\left(\frac{x^2}{4}\right) + D(x)$$
$$C(y) = 0, \quad D(x) = 0$$
$$\implies f(x,y) = 2y\sin\left(\frac{x^2}{4}\right)\cos\left(\frac{x^2}{4}\right)$$

Thus, the family of general solutions is $2y\sin\left(\dfrac{x^2}{4}\right)\cos\left(\dfrac{x^2}{4}\right) = K$

Plugging in the initial condition gives

$$2\cdot 2\cdot\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = K \implies K = 2$$

Therefore, the unique solution is $2y\sin\left(\dfrac{x^2}{4}\right)\cos\left(\dfrac{x^2}{4}\right) = 2$

We can rewrite this as $y = \csc\left(\dfrac{x^2}{2}\right)$

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1.4: Separable Equations and Existence Theorem

Learning Objectives

- Understand the conditions that guarantee a solution to an Initial Value Problem (IVP) exists and is unique.
- Identify and solve first-order separable differential equations.
- Connect the theoretical result of Picard's Theorem to the practical process of solving separable equations, particularly regarding intervals of definition and singular solutions.

This subchapter connects a deep theoretical question—"When does a solution exist, and is it unique?"—with a practical solution technique. Picard's Theorem provides the assurance we need to proceed, while the method for separable equations allows us to solve a wide class of equations by simply separating variables and integrating.

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1.4: Picard's Existence Theorem

We now have at least one tool in our toolbox for solving differential equations (as long as they are exact or close to exact). We have also seen that sometimes a differential equation includes an initial condition of the form $y(a) = b$ that helps us narrow down a family of solutions into one unique solution. We call this combination of information an Initial Value Problem.

Definition: Initial Value Problem

A first order differential equation $\frac{dy}{dx} = f(x, y)$ combined with an initial condition $y(x_0) = y_0$ is an ***initial value problem***, or IVP.

In all of our examples thus far, we have been able to use the initial condition to solve for the unknown constant in our family of solutions. A natural question arises: Does a solution always exist, and if so, is it unique?

Consider the following IVP:

$$\frac{dy}{dx} = \frac{1}{x} \quad \text{with } y(0) = 1$$

This is a very simple type of differential equation, since we know $y' = \frac{1}{x} \implies y = \int \frac{1}{x}\, dx = \ln|x| + C$

But when we go to plug in the initial conditions $x = 0$ and $y = 1$, we get $1 = \ln|0| + C$ which does not make sense since $\ln(0)$ is *undefined*.

This initial condition does not lead to any possible solutions that fit the differential equation!

Alternatively, examine the IVP:

$$\frac{dy}{dx} = 3y^{\frac{2}{3}} \quad \text{with } y(0) = 0$$

It is not hard to check that both $f(x) = x^3$ and the constant function $g(x) = 0$ satisfy both the differential equation and initial condition.

This initial condition leads to more than one solution to the differential equation!

It is safe to say that in general, we cannot assume $\frac{dy}{dx} = f(x, y)$ combined with an initial condition $y(x_0) = y_0$ will yield one unique solution.

However, there are a few conditions which ensure that we *will* find a unique solution to a given IVP.

Theorem 1.4.1: Picard's Existence and Uniqueness Theorem

Let R be a rectangular region in the xy-plane defined by \(a

If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions on R, then the IVP $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ has a *unique solution* for all x on some interval centered at x_0.

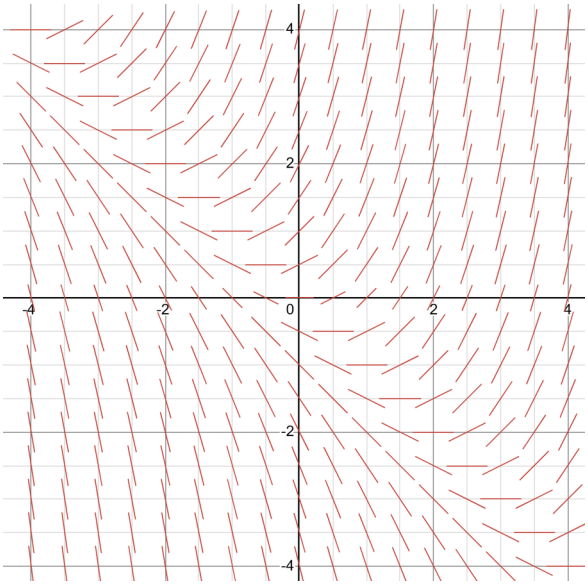
The proof of this theorem uses the Banach fixed point theorem, and some heavy duty tools from advanced algebra and analysis courses. Since these are beyond the scope of this textbook, we will instead give a hand-waving justification for each of the continuity requirements involved.

In order for $f(x, y)$ to be the slope of the solution function, it needs to exist nearby the initial point (x_0, y_0). If this function is not continuous, then it may lead to holes or asymptotes in the graph of our solution (which would clearly not go through the point in the initial condition).

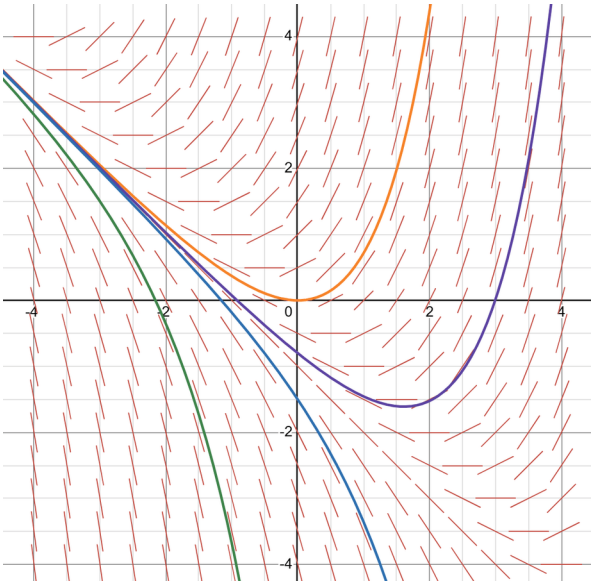
The reason for $\dfrac{\partial f}{\partial y}$ to be continuous, is that we need the slope near that point not to change too wildly as y increases or decreases.

To help us to visualize this result, we look at a few **slope fields**, which are graphs showing the output slopes of $\dfrac{dy}{dx} = f(x, y)$ throughout the xy-plane.

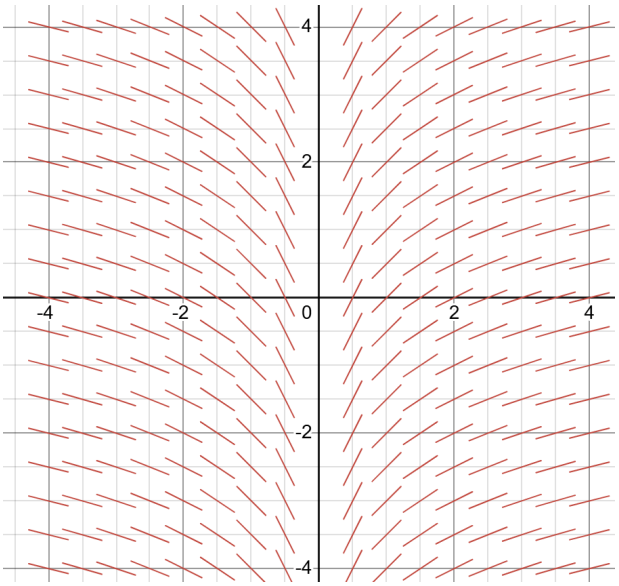
For example, consider the differential equation $\dfrac{dy}{dx} = x + y$. If we plot the slope at each point in the plane, we get the following graph:



Notice that this function $f(x, y) = x + y$ is continuous everywhere, and its partial derivative $\dfrac{\partial f}{\partial y} = 1$ is also continuous everywhere. Thus, Picard's Theorem tells us that for any initial value point, there is exactly one solution going through that point.

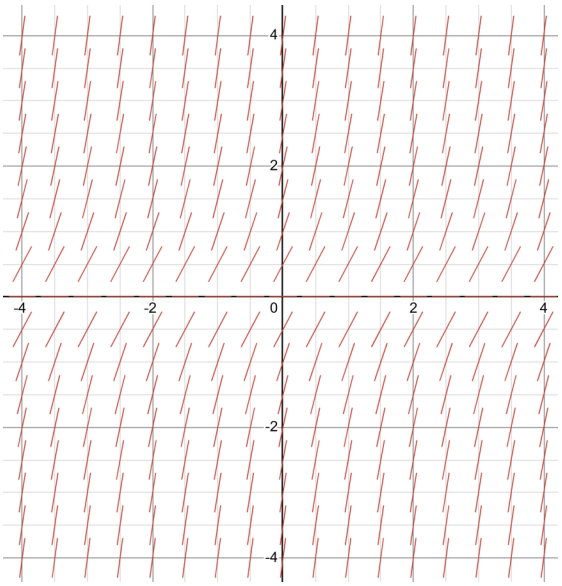


Compare that to the slope field for our first example in this subchapter: $\frac{dy}{dx} = \frac{1}{x}$



The function $f(x, y) = \frac{1}{x}$ is not defined (or continuous) at $x = 0$, which is why we cannot find a solution going through that point.

We can also look at the slope field for our second example: $\frac{dy}{dx} = 3y^{\frac{2}{3}}$



The function $f(x, y) = 3y^{\frac{2}{3}}$ is continuous everywhere, but its partial derivative $\frac{\partial f}{\partial y} = 2y^{-\frac{1}{3}}$ is not. Near $y = 0$, we see the slopes change dramatically near the x-axis. This is why we can find more than one solution passing through the origin.



Before solving any first order IVP, it is important to consider if you can expect to get a unique solution or not by looking at $f(x,y)$ and $\frac{\partial f}{\partial y}$ near the initial condition!

Illustration 1.4.1

We will determine what initial conditions on the given differential equation would yield unique solutions for each differential equation:

(a) $\frac{dy}{dx} = x^2 + y^2$

(b) $\frac{dy}{dx} = \sqrt{x} \cdot \tan\left(\frac{\pi y}{2}\right)$

(c) $\frac{dy}{dx} = \frac{\sqrt[5]{y}}{1+\cos(\pi x)}$

(a) The function $f(x,y) = x^2 + y^2$ is continuous everywhere, and $\frac{\partial f}{\partial y} = 2y$ is continuous everywhere.

All possible initial conditions would yield unique solutions to this differential equation.

(b) The function $f(x,y) = \sqrt{x} \cdot \tan\left(\frac{\pi y}{2}\right)$ is continuous when $x \geq 0$ and y is not an odd integer.

Similarly, $\frac{\partial f}{\partial y} = \sqrt{x} \cdot \frac{\pi}{2}\sec^2\left(\frac{\pi y}{2}\right)$ is continuous when $x \geq 0$ and y is not an odd integer.

There will be unique solutions to this DE when $x \geq 0$ and y is not an odd integer.

(c) The function $f(x,y) = \frac{\sqrt[5]{y}}{1+\cos(\pi x)}$ is continuous when x is not an odd integer for all possible y.

Similarly, $\frac{\partial f}{\partial y} = \frac{1}{5y^{\frac{4}{5}}(1+\cos(\pi x))}$ is continuous when x is not an odd integer and $y \neq 0$.

There will be unique solutions to this DE when x is not an odd integer and $y \neq 0$.

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1.4: Separable Differential Equations

Not all first order differential equations are the same level of difficulty. For example, the differential equation $\frac{dy}{dx} = f(x)$ is very straightforward:

$$\frac{dy}{dx} = f(x) \implies y = \int f(x)\,dx = F(x)$$

Even though this answer is easy to reach, we can get to it by re-writing it as an exact equation:

$$\frac{dy}{dx} = f(x) \leftrightarrow f(x) - 1 \cdot \frac{dy}{dx} = 0$$

Clearly $\frac{\partial}{\partial y}\left(f(x)\right) = 0$ and $\frac{\partial}{\partial x}\left(-1\right) = 0$ so we can take the partial anti-derivatives:

$$\int f(x)\,dx = F(x) + C(y)$$
$$\int -1\,dy = -y + D(x)$$

Thus, the general solution is of the form $F(x) - y = K$, which we can rewrite as $y = F(x) + K$ to get the same answer we expected.

Using this approach though, we see that the function -1 could be replaced by any function of y and still be exact.

We say a differential equation of the form $\frac{dy}{dx} = g(x) \cdot h(y)$ is **separable**.

These can always be re-written in the form $g(x) - \frac{1}{h(y)} \cdot \frac{dy}{dx} = 0$ which is exact.

$$\int g(x)\,dx = G(x) + C(y)$$
$$\int -\frac{1}{h(y)}\,dy = -H(y) + D(x)$$

Thus, the general solution is of the form $G(x) - H(y) = K$, which is equivalent to $G(x) = H(y) + K$

As a shortcut, we can instead just "cross-multiply" the original equation and then integrate both sides:

$$\frac{dy}{dx} = g(x)h(y) \implies \frac{1}{h(y)}dy = g(x)dx \implies \int \frac{1}{h(y)}\,dy = \int g(x)\,dx \implies H(y) = G(x) + K$$

Rule: Solving Separable Differential Equations

For a separable differential equation of the form $\frac{dy}{dx} = g(x) \cdot h(y)$, a one parameter family of solutions is given by

$$\int g(x)\,dx = \int \frac{1}{h(y)}\,dy$$

As long as you can evaluate the integral, you can solve a separable differential equation.

Example 1.4.1

Find the general family of solutions to each separable differential equation below.

(a) $\dfrac{dy}{dx} = \dfrac{x^2}{y^5}$

(b) $\dfrac{dy}{dx} = (-3x^2 + 4x - 4)e^{2y}$

(c) $\dfrac{dy}{dx} = 2xy + 4x - y - 2$

Solution

(a) We cross-multiply and then integrate both sides:

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2}{y^5} \\ y^5\,dy &= x^2\,dx \\ \int y^5\,dy &= \int x^2\,dx \\ \frac{1}{6}y^6 &= \frac{1}{3}x^3 + C \\ y^6 &= 2x^3 + 6C \end{aligned}$$

Notice that $6C$ is still some unknown constant, so we can relabel it as a single constant K.

Therefore, the general solution is $\boxed{y^6 = 2x^3 + K}$

(b) We cross-multiply and then integrate both sides:

$$\begin{aligned} \frac{dy}{dx} &= (-3x^2 + 4x - 4)e^{2y} \\ e^{-2y}\,dy &= (-3x^2 + 4x - 4)\,dx \\ \int e^{-2y}\,dy &= \int -3x^2 + 4x - 4\,dx \\ -\frac{1}{2}e^{-2y} &= -x^3 + 2x^2 - 4x + C \\ e^{-2y} &= 2x^3 - 4x^2 + 8x - 2C \end{aligned}$$

We can solve for y explicitly in this equation and replace $-2C$ with K to get the general solution $\boxed{y = -\frac{1}{2}\ln\left(2x^3 - 4x^2 + 8x + K\right)}$

(c) First we factor the right hand side $2xy + 4x - y - 2 = (2x - 1)(y + 2)$ to see it is separable.

We must be careful before we divide both sides by $y + 2$. There are two cases: $y = -2$ for all x, or $y \neq -2$ for some x.

We see that the constant function $y = -2$ satisfies the differential equation: $\dfrac{dy}{dx} = (2x - 1) \cdot 0 = 0$.

If we assume $y \neq -2$ for some value(s) of x, we can divide by $y + 2$ and multiply both sides by dx as before:

$$\begin{aligned} \frac{dy}{dx} &= (2x - 1)(y + 2) \\ \frac{1}{y+2}\,dy &= (2x - 1)\,dx \\ \int \frac{1}{y+2}\,dy &= \int 2x - 1\,dx \\ \ln|y + 2| &= x^2 - x + C \end{aligned}$$

We can solve for y explicitly in this equation to get the general solution:

$$\ln|y + 2| = x^2 - x + C \implies |y + 2| = e^{x^2 - x + C} \implies y + 2 = \pm e^C \cdot e^{x^2 - x}$$

Since $\pm e^C$ is just another (non-zero) unknown constant, our general solution in this case is $y = Ke^{x^2-x} - 2$

The case when $K = 0$ is the constant function $y = -2$ we began with.

Therefore, we can write all of these possible solutions in the form $\boxed{y = Ke^{x^2-x} - 2}$

Whenever we change $\frac{dy}{dx} = g(x)h(y)$ we must be extra careful when discussing values of y which make $h(y) = 0$.

Any c satisfying $h(c) = 0$ turns the differential equation into $\frac{dy}{dx} = 0$, giving us a constant solution $y = c$ to consider. As above, these often have the same form as the rest of our solutions, but we must be careful to check that each time!

When an initial condition is added into the mix, we should always begin the problem by first checking if a unique solution will exist by Picard's Theorem.

Example 1.4.2

Find the solutions to each separable IVP below.

(a) $\frac{dy}{dx} = \frac{1}{xy\ln(x)}$ with $y(e) = -2$

(b) $\frac{dy}{dx} = \sqrt{\frac{y}{x}}$ with $y(0) = 1$

Solution

(a) First we check the uniqueness conditions on $f(x, y) = \frac{1}{xy\ln(x)}$ and $\frac{\partial f}{\partial y} = -\frac{1}{xy^2\ln(x)}$

These are both continuous at $(e, -2)$, so there will be a unique solution to this IVP.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{xy\ln(x)} \\
y\,dy &= \frac{1}{x\ln(x)}\,dx \\
\int y\,dy &= \int \frac{1}{x\ln(x)}\,dx \\
\frac{1}{2}y^2 &= \ln|\ln(x)| + C \\
y^2 &= 2\ln|\ln(x)| + 2C
\end{aligned}$$

Since $y(e) = -2 < 0$, we know $y = -\sqrt{2\ln|\ln(x)| + K}$

Now we plug in the initial condition: $-2 = -\sqrt{2\ln|\ln(x)| + K} \implies K = 4$

Therefore, the unique explicit solution is $\boxed{y = -\sqrt{2\ln|\ln(x)| + 4}}$

(b) First we check the uniqueness conditions on $f(x, y) = \sqrt{\frac{y}{x}}$ and $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{xy}}$

These are both *not* continuous at $(0, 1)$, so there *may not* be a unique solution to this IVP.

We see that the constant function $y = 0$ will be a solution in general, but won't satisfy our initial condition so we may ignore it.

$$\frac{dy}{dx} = \sqrt{\frac{y}{x}}$$
$$\frac{1}{\sqrt{y}}\,dy = \frac{1}{\sqrt{x}}\,dx$$
$$\int y^{-\frac{1}{2}}\,dy = \int x^{-\frac{1}{2}}\,dx$$
$$2y^{\frac{1}{2}} = 2x^{\frac{1}{2}} + C$$
$$\sqrt{y} = \sqrt{x} + \frac{1}{2}C$$

We can solve for y explicitly in this equation and replace $\frac{1}{2}C$ with K to get the general solution $y = \left(\sqrt{x} + K\right)^2$

Plugging in the initial conditions gives us the equation $1 = \left(\sqrt{0} + K\right)^2 \implies K = \pm 1$

Therefore, we have *two solutions*: $\boxed{y = \left(\sqrt{x} + 1\right)^2}$ and $\boxed{y = \left(\sqrt{x} - 1\right)^2}$

Sometimes the integration on one side (or both sides) when solving a separable IVP can get difficult or even impossible. Alternatively, solving for the unknown constant can sometimes be tedious. If we know that the unique solution exists, we can use definite integrals to express this answer.

Rule: Solving Separable IVPs

For a separable initial value problem of the form $\frac{dy}{dx} = g(x) \cdot h(y)$ with $y(x_0) = y_0$, when there is a unique solution it is given by:

$$\int_{x_0}^{x} g(t)\,dt = \int_{y_0}^{y} \frac{1}{h(t)}\,dt$$

Example 1.4.3

Find the solution to each IVP below using definite integrals.

(a) $\frac{dy}{dx} = \frac{1}{xy\ln(x)}$ with $y(e) = -2$

(b) $\frac{dy}{dx} = 2xe^{x^2-y^2}$ with $y(0) = 1$

Solution

(a) We have already checked above that there will be a unique solution to this IVP.

$$\frac{dy}{dx} = \frac{1}{xy\ln(x)}$$
$$y\,dy = \frac{1}{x\ln(x)}\,dx$$
$$\int_{-2}^{y} t\,dt = \int_{e}^{x} \frac{1}{t\ln(t)}\,dt$$
$$\frac{1}{2}t^2\Big|_{-2}^{y} = \ln|\ln(t)|\,\Big|_{e}^{x}$$
$$\frac{1}{2}y^2 - 2 = \ln|\ln(x)| - \ln|\ln(e)|$$
$$\frac{1}{2}y^2 - 2 = \ln|\ln(x)| - 0$$
$$y^2 = 2\ln|\ln(x)| + 4$$

Since $y(e) = -2 < 0$, we know the unique explicit solution is $\boxed{y = -\sqrt{2\ln|\ln(x)| + 4}}$, as we solved previously.

(b) First we check the uniqueness conditions on $f(x, y) = 2xe^{x^2-y^2}$ and $\dfrac{\partial f}{\partial y} = -4xye^{x^2-y^2}$

These are both continuous in a region containing $(0, 1)$, so there will be a unique solution to this IVP.

$$\begin{aligned}
\frac{dy}{dx} &= 2xe^{x^2-y^2} \\
e^{y^2}\,dy &= 2xe^{x^2}\,dx \\
\int_1^y e^{t^2}\,dt &= \int_0^x 2te^{t^2}\,dt \\
\int_1^y e^{t^2}\,dt &= e^{t^2}\Big|_0^x \\
\int_1^y e^{t^2}\,dt &= e^{x^2} - 1
\end{aligned}$$

Since the integral on the left cannot be expressed in terms of elementary functions, we cannot progress beyond this expression. It does represent our solution to the IVP though, as the definite integral is well defined for all real values of y and can be approximated if necessary with calculators or computers. So our answer is $\boxed{\displaystyle\int_1^y e^{t^2}\,dt = e^{x^2} - 1}$

In this subchapter, we have studied separable differential equations, a class of first-order equations that can be solved by direct integration. These equations can often be rewritten as exact equations, providing a methodical approach to finding their solutions. While the general solution technique is straightforward, we have highlighted the importance of carefully handling constant solutions and verifying the existence of unique solutions for initial value problems. This foundational method provides the tools to solve a wide range of practical differential equations.

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1.4: Practice Set

Exercise 1.4.1a

Find all initial conditions $y(x_0) = y_0$ where the solution is guaranteed to exist and be unique to $\frac{dy}{dx} = \sqrt{y^2 - 4}$

Answer available at the end of this section.

Exercise 1.4.1b

Find all initial conditions $y(x_0) = y_0$ where the solution is guaranteed to exist and be unique to $\frac{dy}{dx} = \frac{x + y}{y^2 - 9}$

Answer available at the end of this section.

Exercise 1.4.1c

Find all initial conditions $y(x_0) = y_0$ where the solution is guaranteed to exist and be unique to $\frac{dy}{dx} = \frac{x}{x^2 - y^2}$

Answer available at the end of this section.

Exercise 1.4.2a

Find the general solution to $y' = e^{x-y}$ and determine what initial conditions will yield a unique solution.

Answer available at the end of this section.

Exercise 1.4.2b

Find the general solution to $y' = \frac{\cos^2(y)}{\sqrt{x} + x\sqrt{x}}$ and determine what initial conditions will yield a unique solution.

Answer available at the end of this section.

Exercise 1.4.2c

Find the general solution to $\frac{dy}{dx} = \frac{x\sqrt[3]{y}}{1 + x^2}$ and determine what initial conditions will yield a unique solution.

Answer available at the end of this section.

Exercise 1.4.2d

Find the general solution to $\frac{dP}{dt} = P(1 - P)$ and determine what initial conditions will yield a unique solution.

Answer available at the end of this section.

Exercise $1.4.3a$

Find the solution to the IVP $\frac{dy}{dx} = 2xe^{-x^2}$ with $y(0) = 3$

Answer available at the end of this section.

Exercise $1.4.3b$

Find the solution to the IVP $\frac{dy}{dx} = \frac{4}{x^2+4}$ with $y(2) = \pi$

Answer available at the end of this section.

Exercise $1.4.3c$

Find the solution to the IVP $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$ with $y(0) = 1$

Answer available at the end of this section.

Exercise $1.4.3d$

Find the solution to the IVP $\frac{dy}{dx} = \frac{2\tan(y)}{\sqrt{x^2-1}}$ with $y\left(\frac{5}{4}\right) = \frac{\pi}{6}$

Answer available at the end of this section.

Exercise $1.4.3e$

Find the solution to the IVP $y' = \frac{y\cos(x)}{1+2y^2}$ with $y(0) = 1$

Answer available at the end of this section.

Exercise $1.4.3f$

Find the solution to the IVP $\frac{dy}{dx} = \frac{\ln(x)\sqrt{y^2+1}}{x^2(2y^2+1)}$ with $y(1) = 0$

Answer available at the end of this section.

Exercise $1.4.3g$

Find the solution to the IVP $\frac{dy}{dx} = \frac{y^2-4}{x^2-9}$ with $y(4) = 3$

Answer available at the end of this section.

Exercise 1.4.4a

Find the curve through $(0,1)$ where the slope at any point (x,y) is $\dfrac{y}{\sqrt{4-x^2}}$

Answer available at the end of this section.

Exercise 1.4.4b

Find the curve through $(4,0)$ where the slope at any point (x,y) is $-\dfrac{x}{2y^3}$

Answer available at the end of this section.

Answers

Exercise 1.4.1a

$f(x,y)=\sqrt{y^2-4}$ is continuous when $y^2-4\geq 0$

$\dfrac{\partial f}{\partial y}=\dfrac{y}{\sqrt{y^2-4}}$ is continuous when $y^2-4>0$

Thus, a unique solution exists through any (x_0,y_0) where $|y_0|>2$

Exercise 1.4.1b

$f(x,y)=\dfrac{x+y}{y^2-9}$ is continuous when $y^2-9\neq 0$

$\dfrac{\partial f}{\partial y}=\dfrac{(y^2-9)\cdot 1-2y\cdot(x+y)}{(y^2-9)^2}$ is continuous when$y^2-9\neq 0$

Thus, a unique solution exists through any (x_0,y_0) where $|y_0|\neq 3$

Exercise 1.4.1c

$f(x,y)=\dfrac{x}{x^2-y^2}$ is continuous when $x^2-y^2\neq 0$

$\dfrac{\partial f}{\partial y}=\dfrac{2xy}{(x^2-y^2)^2}$ is continuous when $x^2-y^2\neq 0$

Thus, a unique solution exists through any (x_0,y_0) where $|x_0|\neq|y_0|$

Exercise 1.4.2a

First, we separate: $\dfrac{dy}{dx}=e^xe^{-y}\Rightarrow e^y\ dy=e^x\ dx$

Then we integrate: $\displaystyle\int e^y\,dy=\int e^x\,dx$

This gives the general implicit solution: $e^y=e^x+C$

Solving for y gives the explicit answer: $y=\ln|e^x+C|$

Since $f(x,y)=e^{x-y}$ and $\dfrac{\partial f}{\partial y}=-e^{x-y}$, this answer is uniquely defined for all initial conditions $y(x_0)=y_0$

Exercise 1.4.2b

Considering the constant cases first, we see that $y=\dfrac{(2k+1)\pi}{2}$ is a solution for all integers k.

Now we separate: $\dfrac{dy}{dx}=\dfrac{\cos^2(y)}{\sqrt{x}+x\sqrt{x}}\Rightarrow \sec^2(y)\ dy=\dfrac{1}{\sqrt{x}+x\sqrt{x}}\ dx$

Then we integrate: $\displaystyle\int\sec^2(y)\,dy=\int\frac{1}{\sqrt{x}+x\sqrt{x}}\,dx$

This gives the general implicit solution: $\tan(y)=2\tan^{-1}(\sqrt{x})+C$

Since $f(x,y)=\dfrac{\cos^2(y)}{\sqrt{x}+x\sqrt{x}}$ and $\dfrac{\partial f}{\partial y}=\dfrac{-2\cos(y)\sin(y)}{\sqrt{x}+x\sqrt{x}}$, these answers are uniquely defined for all initial conditions $y(x_0)=y_0$ with $x_0>0$

Exercise 1.4.2c

Considering the constant case first, we see that $y=0$ is a solution to the DE.

Now we separate: $\dfrac{dy}{dx}=\dfrac{x\sqrt[3]{y}}{1+x^2}\Rightarrow y^{-\frac{1}{3}}\ dy=\dfrac{x}{1+x^2}\ dx$

Then we integrate: $\displaystyle\int 2y^{-\frac{1}{3}}\,dy=\int\frac{2x}{1+x^2}\,dx$

This gives the general implicit solution: $3y^{\frac{2}{3}}=\ln\left|1+x^2\right|+C$

Solving for y gives the explicit answer: $y=\left(\dfrac{1}{3}\ln\left|1+x^2\right|+\hat{C}\right)^{\frac{3}{2}}$

Since $f(x,y)=\dfrac{x\sqrt[3]{y}}{1+x^2}$ and $\dfrac{\partial f}{\partial y}=\dfrac{x}{3y^{\frac{2}{3}}(1+x^2)}$, this answer is uniquely defined for all initial conditions $y(x_0)=y_0$ with $y_0\neq 0$.

We can see that for $y(x_0) = 0$ we will get two solutions: our constant $y = 0$ and $y = \left(\frac{1}{3}\ln|1+x^2| - \frac{1}{3}\ln|1+x_0^2|\right)^{\frac{3}{2}}$

Exercise 1.4.2*d*

Considering the constant cases first, we see that $P = 0$ and $P = 1$ are both solutions to the DE.

Next we separate: $\frac{dP}{dt} = P(1-P) \Rightarrow \frac{1}{P(1-P)}\ dP = 1\ dt$

Then we integrate: $\int \left(\frac{1}{P} + \frac{1}{1-P}\right)\ dP = \int 1\, dt$

This gives the general implicit solution: $\ln|P| - \ln|1-P| = t + C$

Solving for P gives the explicit answer:

$$\begin{aligned} \ln\left(\frac{P}{1-P}\right) &= t + C \\ \frac{P}{1-P} &= e^{t+C} \\ -1 + \frac{1}{1-P} &= \hat{C}e^t \\ \frac{1}{1-P} &= \hat{C}e^t + 1 \\ P &= 1 - \frac{1}{\hat{C}e^t + 1} \end{aligned}$$

Since $f(t,P) = P(1-P)$ and $\frac{\partial f}{\partial P} = 1 - 2P$, this answer is uniquely defined for all initial conditions $P(t_0) = P_0$

Notice that the constant solution $P = 0$ occurs when $\hat{C} = 0$, while $P = 1$ occurs for $\hat{C} = \pm\infty$

Exercise 1.4.3*a*

First, we check if our solution will be unique: $f(x,y) = 2xe^{-x^2}$ and $\frac{\partial f}{\partial y} = 0$ are both continuous at $(0,3)$.

We now separate and integrate both sides:

$$\int dy = \int 2xe^{-x^2}\, dx$$

This yields the general explicit solution $y = -e^{-x^2} + C$

Plugging in the initial condition $y(0) = 3$ gives

$$3 = -1 + C \implies C = 4$$

Thus, the unique solution is $y = -e^{-x^2} + 4$

Exercise 1.4.3*b*

First, we check if our solution will be unique: $f(x,y) = \frac{4}{x^2+4}$ and $\frac{\partial f}{\partial y} = 0$ are both continuous at $(2,\pi)$.

We now separate and integrate both sides:

$$\int dy = \int \frac{4}{x^2+4}\, dx$$

This yields the general explicit solution $y = 2\tan^{-1}\left(\frac{x}{2}\right) + C$

Plugging in the initial condition $y(2) = \pi$ gives

$$\pi = 2\tan^{-1}(1) + C \implies C = \frac{\pi}{2}$$

Thus, the unique solution is $y = 2\tan^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{2}$

Exercise 1.4.3*c*

First, we check if our solution will be unique: $f(x,y) = \frac{y^2+1}{x^2+1}$ and $\frac{\partial f}{\partial y} = \frac{2y}{x^2+1}$ are both continuous at $(0,1)$.

We now separate and integrate both sides:

$$\int \frac{1}{y^2+1}\, dy = \int \frac{1}{x^2+1}\, dx$$

This yields the general implicit solution $\tan^{-1}(y) = \tan^{-1}(x) + C$

Plugging in the initial condition $y(0) = 1$ gives

$$\tan^{-1}(1) = \tan^{-1}(0) + C \implies C = \frac{\pi}{4}$$

Implicitly, this becomes $\tan^{-1}(y) = \tan^{-1}(x) + \frac{\pi}{4}$

Solving for y explicitly, the unique solution is $y = \tan\left(\tan^{-1}(x) + \frac{\pi}{4}\right)$

Exercise 1.4.3*d*

First, we check if our solution will be unique: $f(x,y) = \frac{2\tan(y)}{\sqrt{x^2-1}}$ and $\frac{\partial f}{\partial y} = \frac{2\sec^2(y)}{\sqrt{x^2-1}}$ are both continuous at $\left(\frac{5}{4}, \frac{\pi}{6}\right)$.

There are constant solutions $y = \frac{(2k+1)\pi}{2}$, but none of them satisfy the initial condition so they may be ignored.

We now separate and integrate both sides:

$$\int \cot(y)\, dy = \int \frac{2}{\sqrt{x^2-1}}\, dx$$

This yields the general implicit solution $\ln|\sin(y)| = \ln\left|x+\sqrt{x^2-1}\right| - \ln\left|x-\sqrt{x^2-1}\right| + C$

Plugging in the initial condition $y\left(\frac{5}{4}\right) = \frac{\pi}{6}$ gives

$$\ln\left(\frac{1}{2}\right) = \ln\left(\frac{5}{4}+\frac{3}{4}\right) - \ln\left(\frac{5}{4}-\frac{3}{4}\right) + C \implies C = -3\ln(2)$$

Implicitly, this becomes $\ln|\sin(y)| = \ln\left|x+\sqrt{x^2-1}\right| - \ln\left|x-\sqrt{x^2-1}\right| - 3\ln(2)$

Solving for y explicitly, the unique solution is $y = \sin^{-1}\left(\frac{x+\sqrt{x^2-1}}{8(x-\sqrt{x^2-1})}\right)$

Exercise $1.4.3e$

First, we check if our solution will be unique: $f(x,y) = \frac{y\cos(x)}{1+2y^2}$ and $\frac{\partial f}{\partial y} = \frac{(1-2y^2)\cos(x)}{(1+2y^2)^2}$ are both continuous at $(0,1)$.

The constant solution $y=0$ satisfies the DE, but not our initial condition so we may ignore it.

We now separate and integrate both sides:

$$\int \left(y^{-1}+2y\right)\,dy = \int \cos(x)\,dx$$

This yields the general implicit solution $\ln|y| + y^2 = \sin(x) + C$

Plugging in the initial condition $y(0)=1$ gives

$$\ln(1) + 1^2 = \sin(0) + C \implies C = 1$$

Implicitly, this becomes $\ln|y| + y^2 = \sin(x) + 1$

Since $y>0$ and we cannot solve for it explicitly, we settle for $\ln(y) + y^2 = \sin(x) + 1$

Exercise $1.4.3f$

First, we check if our solution will be unique: $f(x,y) = \frac{\ln(x)\sqrt{y^2+1}}{x^2(2y^2+1)}$ and $\frac{\partial f}{\partial y} = -\frac{y(2y^2+3)\ln(x)}{x^2(2y^2+1)^2\sqrt{y^2+1}}$ are both continuous at $(1,0)$.

We now separate and integrate both sides:

$$\int \frac{2y^2+1}{\sqrt{y^2+1}}\,dy = \int \frac{\ln(x)}{x^2}\,dx$$

After a generous helping of trigonometric substitution and integration by parts, this yields the general implicit solution $y\sqrt{y^2+1} = -x^{-1}\ln(x) - x^{-1} + C$

Plugging in the initial condition $y(1)=0$ gives

$$0 = 0 - 1 + C \implies C = 1$$

Implicitly, this becomes $y\sqrt{y^2+1} = -x^{-1}\ln(x) - x^{-1} + 1$

Since we cannot solve for y explicitly, we settle for $y\sqrt{y^2+1} = 1 - \frac{\ln(x)+1}{x}$

Exercise $1.4.3g$

First, we check if our solution will be unique: $f(x,y) = \frac{y^2-4}{x^2-9}$ and $\frac{\partial f}{\partial y} = \frac{2y}{x^2-9}$ are both continuous at $(4,3)$.

The constant solutions $y=2$ and $y=-2$ satisfy the DE but not the initial condition, so they may be ignored.

We now separate and integrate both sides:

$$\int \left(\frac{3}{y-2} - \frac{3}{y+2}\right)\,dy = \int \left(\frac{2}{x-3} - \frac{2}{x+3}\right)\,dx$$

This yields the general implicit solution $3\ln|y-2| - 3\ln|y+2| = 2\ln|x-3| - 2\ln|x+3| + C$

Plugging in the initial condition $y(4)=3$ gives

$$3\ln(1) - 3\ln(5) = 2\ln(1) - 2\ln(7) + C \implies C = \ln\left(\frac{7^2}{5^3}\right)$$

Implicitly, this becomes $3\ln\left(\frac{y-2}{y+2}\right) = 2\ln\left(\frac{x-3}{x+3}\right) + \ln\left(\frac{7^2}{5^3}\right)$

Solving for y, the unique explicit solution is $y = 2\cdot\dfrac{5+\left(\frac{7(x-3)}{x+3}\right)^{\frac{2}{3}}}{5-\left(\frac{7(x-3)}{x+3}\right)^{\frac{2}{3}}}$

Exercise $1.4.4a$

We are looking for the solution to the IVP $\frac{dy}{dx} = \frac{y}{\sqrt{4-x^2}}$ with $y(0)=1$

$f(x,y) = \frac{y}{\sqrt{4-x^2}}$ and $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{4-x^2}}$ are continuous at $(0,1)$, so there will be a unique solution.

We separate and integrate both sides:

$$\int \frac{1}{y}\,dy = \int \frac{1}{\sqrt{4-x^2}}\,dx$$

This yields our general implicit solution $\ln|y| = \sin^{-1}\left(\frac{x}{2}\right) + C$

Plugging in the initial condition $y(0)=1$, we get

$$0 = 0 + C \Rightarrow C = 0$$

So our implicit solution looks like $\ln|y| = \sin^{-1}\left(\frac{x}{2}\right)$

Solving for y explicitly gives our unique solution: $y = e^{\sin^{-1}\left(\frac{x}{2}\right)}$

Exercise $1.4.4b$

We are looking for the solution to the IVP $\frac{dy}{dx} = -\frac{x}{2y^3}$ with $y(4) = 0$

$f(x,y) = -\frac{x}{2y^3}$ and $\frac{\partial f}{\partial y} = \frac{3x}{2y^4}$ are *not continuous* at $(4,0)$, so there *may not* be a unique solution.

We separate and integrate both sides:

$$\int 2y^3\,dy = \int -x\,dx$$

This yields our general implicit solution $\frac{1}{2}y^4 = -\frac{1}{2}x^2 + C$

Plugging in the initial condition $y(4) = 0$, we get

$$0 = -8 + C \Rightarrow C = 8$$

So our implicit solution looks like $\frac{1}{2}y^4 = -\frac{1}{2}x^2 + 8$

Solving for y explicitly gives us two possible solutions: $y = \pm\sqrt[4]{16 - x^2}$

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1.5: First Order Linear Differential Equations

Learning Objectives

- Identify a first-order linear differential equation in standard form.
- Derive and apply the integrating factor method to solve such equations.
- Understand the shape and structure of solutions to first order linear differential equations.

We introduce a powerful, systematic algorithm for solving any first-order linear ODE. The method of integrating factors transforms a non-exact equation into an exact one, providing a guaranteed solution path for equations that model a vast array of phenomena from circuit analysis to population dynamics.

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1.5: Solving Linear Differential Equations

Recall that we put certain differential equations into the category of **linear** when each derivative of y is only multiplied by coefficients involving functions of x instead of other y expressions. First order linear differential equations (LDEs) take the general form

$$a(x) \cdot \frac{dy}{dx} + b(x) \cdot y = c(x)$$

We take this subchapter to explore this very special type of differential equation. We can safely assume that $a(x)$ is not the constant zero function (or else this would not be a differential equation at all!), so we can divide through by it to get the more manageable:

$$\frac{dy}{dx} + p(x) \cdot y = q(x)$$

When the function $q(x)$ is the constant 0, this is a **homogeneous** first order LDE. This specific variant of the first order LDE is *always separable*.

$$\frac{dy}{dx} + p(x) \cdot y = 0 \longleftrightarrow \frac{dy}{dx} = -p(x) \cdot y$$

The constant function $y = 0$ is always a solution to the homogeneous LDE, and is referred to as a **trivial solution**. For a non-trivial solution, we separate and integrate:

$$\int \frac{1}{y}\, dy = \int -p(x)\, dx \implies \ln|y| = -\int p(x)\, dx \implies y(x) = e^{-\int p(x)\, dx}$$

When the constant of integration appears, we can pull it down to the front to get $y(x) = Ce^{-\int p(x)\, dx}$ as the one parameter family of solutions to the first order homogeneous LDE. (When $C = 0$ we get the trivial solution discussed above.)

Example 1.5.1

Find the general solution to each homogeneous LDE below.

(a) $\frac{dy}{dx} - 2y = 0$

(b) $y' + 2xy = 0$

(c) $\cos(x) \cdot \frac{dy}{dx} + 2\sin(x) \cdot y = 0$

Solution

(a) We see $p(x) = -2$ so $-\int p(x)\, dx = \int 2\, dx = 2x$

Therefore, our family of homogeneous solutions is of the form $\boxed{y(x) = Ce^{2x}}$

(b) We see $p(x) = 2x$ so $-\int p(x)\, dx = \int -2x\, dx = -x^2$

Therefore, our family of homogeneous solutions is of the form $\boxed{y(x) = Ce^{-x^2}}$

(c) Re-writing this to eliminate the coefficient of y' gives: $\frac{dy}{dx} + 2\tan(x) \cdot y = 0$

We see $p(x) = 2\tan(x)$ so $-\int p(x)\,dx = \int -2\tan(x)\,dx = 2\ln|\cos(x)|$ and $e^{2\ln|\cos(x)|} = \cos^2(x)$

Therefore, our family of homogeneous solutions is of the form $\boxed{y(x) = C\cos^2(x)}$

Next, we consider the non-homogeneous case: when $q(x)$ is not identically zero.

Pretend for a moment that we found two functions y_1 and y_2 that each satisfied the non-homogeneous LDE:

$$y_1'(x) + p(x)\cdot y_1(x) = q(x) \quad \text{and} \quad y_2'(x) + p(x)\cdot y_2(x) = q(x)$$

Then consider the function $f(x) = y_1(x) - y_2(x)$ which has derivative $f'(x) = y_1'(x) - y_2'(x)$, we see that:

$$f'(x) + p(x)\cdot f(x) = (y_1'(x) - y_2'(x)) + p(x)\cdot(y_1(x) - y_2(x)) = (y_1'(x) + p(x)\cdot y_1(x)) - (y_2'(x) + p(x)\cdot y_2(x)) = q(x) - q(x) = 0$$

This proves the following useful fact:

Fact

The difference between any two solutions to the same non-homogeneous LDEs is a solution to the homogeneous LDE.

In practice, this means that instead of trying to solve for the entire family of solutions, we can simply try to find *one particular solution* to the non-homogeneous LDE, $y_p(x)$, and then add on the family of solutions to the homogeneous LDE.

$$y'(x) + p(x)\cdot y(x) = q(x) \implies y(x) = y_p(x) + Ce^{-\int p(x)\,dx}$$

When $q(x)$ is a constant multiple of $p(x)$, this can be done by guessing a constant particular solution.

Illustration 1.5.1

We can also find the general solution to each non-homogeneous LDE below:

(a) $\dfrac{dy}{dx} - 2y = 20$

(b) $y' + 2xy = 6x$

(c) $\cos(x)\cdot\dfrac{dy}{dx} + 2\sin(x)\cdot y = \sin(x)$

(a) We already solved the homogeneous version of this LDE, and found solutions Ce^{2x}.

We can guess a constant solution to this non-homogeneous version, $y_p(x) = K$, which will have derivative zero.

This makes the LDE into $0 - 2\cdot K = 20 \implies K = -10$, so $y_p(x) = -10$

Therefore, our family of solutions is $y(x) = -10 + Ce^{2x}$

(b) We already solved the homogeneous version of this LDE, and found solutions Ce^{-x^2}.

We can guess a constant solution to this non-homogeneous version, $y_p(x) = K$, which will have derivative zero.

This makes the LDE into $0 + 2x\cdot K = 6x \implies K = 3$, so $y_p(x) = 3$

Therefore, our family of solutions is $y(x) = 3 + Ce^{-x^2}$

(c) We already solved the homogeneous version of this LDE, and found solutions $C\cos^2(x)$.

We can guess a constant solution to this non-homogeneous version, $y_p(x) = K$, which will have derivative zero.

This makes the LDE into $\cos(x) \cdot 0 + 2\sin(x) \cdot K = \sin(x) \implies K = \frac{1}{2}$, so $y_p(x) = \frac{1}{2}$

Therefore, our family of solutions is $y(x) = \frac{1}{2} + C\cos^2(x)$

Of course, not every $q(x)$ is a constant multiple of $p(x)$, so we need a more general strategy.

We can consider the non-homogeneous LDE as an almost exact equation:

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \implies (p(x) \cdot y - q(x)) + 1 \cdot \frac{dy}{dx} = 0$$

We see $\hat{M} = p(x) \cdot y - q(x) \implies \hat{M}_y = p(x)$ and $\hat{N} = 1 \implies \hat{N}_x = 0$

These are not equal, but we can find an integrating factor $\mu(x)$:

$$\frac{\hat{M}_y - \hat{N}_x}{\hat{N}} = p(x) \implies \mu(x) = e^{\int p(x)\,dx}$$

Re-writing the equation to make it exact we get:

$$(p(x) \cdot y - q(x))\, e^{\int p(x)\,dx} + e^{\int p(x)\,dx} \cdot \frac{dy}{dx} = 0$$

Now we can take the partial integrals of both terms:

$$\int (p(x) \cdot y - q(x))\, e^{\int p(x)\,dx}\,dx = ye^{\int p(x)\,dx} - \int q(x)e^{\int p(x)}\,dx + C(y)$$
$$\int e^{\int p(x)\,dx}\,dy = ye^{\int p(x)\,dx} + D(x)$$

These match when $C(y) = 0$ and $D(x) = -\int q(x)e^{\int p(x)}\,dx$ which gives us general solution

$$ye^{\int p(x)\,dx} - \int q(x)e^{\int p(x)}\,dx = K$$

Finally, we can solve for the unique explicit solution by multiplying both sides by $e^{-\int p(x)\,dx}$ and isolating y:

$$y = e^{-\int p(x)\,dx} \cdot \int q(x)e^{\int p(x)\,dx}\,dx + Ke^{-\int p(x)\,dx}$$

Notice this solution is exactly what we predicted: a particular solution (given by that nasty integral expression) plus the family of homogeneous solutions!

We simplify this solution by writing it in terms of $\mu(x)$ instead of $p(x)$:

$$y = \frac{1}{\mu(x)} \cdot \int \mu(x) \cdot q(x)\,dx$$

Rule: Solving a First Order Linear Differential Equation

In the first order LDE $y' + p(x)\ y = q(x)$, if we define $\mu(x) = e^{\int p(x)\,dx}$, the general solution is given by:

$$y = \frac{1}{\mu(x)} \cdot \int \mu(x) \cdot q(x)\,dx$$

Example 1.5.2

Find the general solution to each non-homogeneous LDE below.

(a) $\dfrac{dy}{dx} - 2y = e^{3x}$

(b) $y' + 2xy = \dfrac{4x}{e^{x^2}}$

(c) $\cos(x) \cdot \dfrac{dy}{dx} + 2\sin(x) \cdot y = \cos^4(x)$

Solution

(a) We have $p(x) = -2 \implies \mu(x) = e^{\int -2\,dx} = e^{-2x}$

$$\frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{e^{-2x}} \cdot \int e^{-2x} \cdot e^{3x}\,dx = e^{2x} \cdot \int e^{x}\,dx = e^{2x} \cdot [e^x + C] = e^{3x} + Ce^{2x}$$

Therefore, the general solution to the LDE is $\boxed{y = e^{3x} + Ce^{2x}}$

(b) We have $p(x) = 2x \implies \mu(x) = e^{\int 2x\,dx} = e^{x^2}$

$$\frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{e^{x^2}} \cdot \int e^{x^2} \cdot \frac{4x}{e^{x^2}}\,dx = e^{-x^2} \cdot \int 4x\,dx = e^{-x^2} \cdot \left[2x^2 + C\right] = 2x^2 e^{-x^2} + Ce^{-x^2}$$

Therefore, the general solution to the LDE is $\boxed{y = 2x^2 e^{-x^2} + Ce^{-x^2}}$

(c) First, we rewrite the LDE as $\dfrac{dy}{dx} + 2\tan(x) \cdot y = \cos^3(x)$

We have $p(x) = 2\tan(x) \implies \mu(x) = e^{\int 2\tan(x)\,dx} = e^{2\ln|\sec(x)|} = \sec^2(x)$

$$\frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{\sec^2(x)} \cdot \int \sec^2(x) \cdot \cos^3(x)\,dx = \cos^2(x) \cdot \int \cos(x)\,dx = \cos^2(x) \cdot [\sin(x) + C] = \sin(x)\cos^2(x) + C\cos^2(x)$$

Therefore, the general solution to the LDE is $\boxed{y = \sin(x)\cos^2(x) + C\cos^2(x)}$

Notice that a first order LDE (homogeneous or non-homogeneous) will always yield a unique solution where $p(x)$ and $q(x)$ are continuous:

$$\frac{dy}{dx} = q(x) - p(x) \cdot y \implies f(x, y) = q(x) - p(x) \cdot y \text{ and } \frac{\partial f}{\partial y} = -p(x)$$

This means the vast majority of initial conditions will yield exactly one solution for a LDE.

Example 1.5.3

Find the unique explicit solution to each IVP below.

(a) $\dfrac{dy}{dx} + 3 \cdot y = \dfrac{1}{e^{3x} + e^{2x}}$ with $y(0) = \ln(2)$

(b) $(1 + x^2) \cdot y' - y = \tan^{-1}(x)$ with $y(0) = 0$

(c) $2x \cdot \dfrac{dy}{dx} = \cos\left(\sqrt{x}\right) - y$ with $y\left(\dfrac{\pi^2}{4}\right) = \dfrac{10}{\pi}$

Solution

(a) We see $p(x) = 3 \implies \mu(x) = e^{\int 3\,dx} = e^{3x}$

$$\frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{e^{3x}} \cdot \int e^{3x} \cdot \frac{1}{e^{3x} + e^{2x}}\,dx = e^{-3x} \cdot \int \frac{e^x}{e^x + 1}\,dx = e^{-3x} \cdot [\ln(1 + e^x) + C]$$

Plugging in the initial conditions $x = 0$ and $y = \ln(2)$, this becomes $\ln(2) = 1 \cdot [\ln(1+1) + C] \implies C = 0$

Therefore, the unique solution is $\boxed{y(x) = e^{-3x} \ln(1 + e^x)}$

(b) Re-writing this we get $y' - \frac{1}{1+x^2} \cdot y = \frac{\tan^{-1}(x)}{1+x^2}$

Therefore, $p(x) = -\frac{1}{1+x^2} \implies \mu(x) = e^{\int -\frac{1}{1+x^2}\,dx} = e^{-\tan^{-1}(x)}$

$$\frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{e^{-\tan^{-1}(x)}} \cdot \int e^{-\tan^{-1}(x)} \cdot \frac{\tan^{-1}(x)}{1+x^2}\,dx = e^{\tan^{-1}(x)} \cdot \int \frac{\tan^{-1}(x) e^{-\tan^{-1}(x)}}{1+x^2}\,dx = e^{\tan^{-1}(x)} \cdot \left[-\left(1 + \tan^{-1}(x)\right) e^{-}\right.$$

Plugging in the initial conditions $x = 0$ and $y = 0$, this becomes $0 = -1 - 0 + C \cdot e^0 \implies C = 1$

Therefore, the unique solution is $\boxed{y(x) = -1 - \tan^{-1}(x) + e^{\tan^{-1}(x)}}$

(c) Re-writing this we get $\frac{dy}{dx} + \frac{1}{2x} \cdot y = \frac{1}{2x}\cos(\sqrt{x})$

Therefore, $p(x) = \frac{1}{2x} \implies \mu(x) = e^{\int \frac{1}{2x}\,dx} = e^{\frac{1}{2}\ln|x|} = \sqrt{x}$

$$\frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{\sqrt{x}} \cdot \int \sqrt{x} \cdot \frac{1}{2x}\cos(\sqrt{x})\,dx = x^{-\frac{1}{2}} \cdot \int \frac{1}{2} x^{-\frac{1}{2}} \cos(\sqrt{x})\,dx = x^{-\frac{1}{2}} \cdot \left[\sin(\sqrt{x}) + C\right]$$

Plugging in the initial conditions $x = \frac{\pi^2}{4}$ and $y = \frac{10}{\pi}$, this becomes $\frac{10}{\pi} = \frac{2}{\pi} \cdot [1 + C] \implies C = 4$

Therefore, the unique solution is $\boxed{y(x) = \frac{\sin(\sqrt{x}) + 4}{\sqrt{x}}}$

In this subchapter, we have studied first-order linear differential equations. We began by solving the simpler homogeneous case, which is always separable. For the non-homogeneous case, we developed a systematic method using an integrating factor to find the general solution, which consists of a particular solution plus the family of homogeneous solutions. This reliable technique allows us to solve a wide variety of first-order linear equations and their associated initial value problems.

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1.5: Practice Set

Exercise 1.5.1a

Find the general solution to the LDE $\frac{dy}{dx} + 3y = 15e^{2x}$

Denote which part of your answer is the particular solution, and which is the family of homogeneous solutions.

Answer available at the end of this section.

Exercise 1.5.1b

Find the general solution to the LDE $\frac{dy}{dx} - \frac{2}{x}y = x^3$

Denote which part of your answer is the particular solution, and which is the family of homogeneous solutions.

Answer available at the end of this section.

Exercise 1.5.1c

Find the general solution to the LDE $y' + y\tan(x) = \sec(x)$

Denote which part of your answer is the particular solution, and which is the family of homogeneous solutions.

Answer available at the end of this section.

Exercise 1.5.1d

Find the general solution to the LDE $xy' + 2y = \sin(x)$

Denote which part of your answer is the particular solution, and which is the family of homogeneous solutions.

Answer available at the end of this section.

Exercise 1.5.1e

Find the general solution to the LDE $y' + \frac{1}{x}y = 2\ln(x^2)$

Denote which part of your answer is the particular solution, and which is the family of homogeneous solutions.

Answer available at the end of this section.

Exercise 1.5.1f

Find the general solution to the LDE $y' + \frac{1}{x\ln(x)}y = \frac{2}{x}$

Denote which part of your answer is the particular solution, and which is the family of homogeneous solutions.

Answer available at the end of this section.

Exercise $1.5.2a$

Find the unique solution to the IVP $\frac{dy}{dx} + 2y = 6xe^{-2x}$ with $y(0) = 2$

Answer available at the end of this section.

Exercise $1.5.2b$

Find the unique solution to the IVP $\frac{dy}{dx} - y = 20\cos(2x)$ with $y(\pi) = -4$

Answer available at the end of this section.

Exercise $1.5.2c$

Find the unique solution to the IVP $\frac{dy}{dx} + \frac{2}{x^2-1}y = 3x^2 - 3$ with $y(0) = 0$

Answer available at the end of this section.

Exercise $1.5.2d$

Find the unique solution to the IVP $\frac{dy}{dx} - \sin(x)y = 3\sqrt{x}\ e^{-\cos(x)}$ with $y(0) = 1$

Answer available at the end of this section.

Exercise $1.5.2e$

Find the unique solution to the IVP $\frac{dy}{dx} - \frac{2}{x}y = \frac{1}{\sqrt{x^2-1}}$ with $y\left(\frac{5}{3}\right) = -\frac{5}{9}$

Answer available at the end of this section.

Exercise $1.5.2f$

Find the unique solution to the IVP $\frac{dy}{dx} + \frac{1}{\sqrt{x}}y = 12x$ with $y(1) = 3$

Answer available at the end of this section.

Exercise $1.5.2g$

Find the unique solution to the IVP $\frac{dy}{dx} + \frac{9x^2}{x^3+1}y = \frac{4}{x^6+2x^3+1}$ with $y(1) = -2$

Answer available at the end of this section.

Exercise $1.5.2h$

Find the unique solution to the IVP $\frac{dy}{dx} - \sec(x)y = 1$ with $y(\pi) = 2$

Answer available at the end of this section.

Exercise $1.5.3$

Assume the first order LDE $y' + p(x)\ y = q(x)$ has particular solution $y_p(x) = (\sqrt{x} + 1)^2$ and homogeneous solution $y_h(x) = x + 1$

(a) Find a solution to the IVP $y' + p(x)y = 0$, with $y(1) = -4$

(b) Find the solution to the IVP $y' + p(x)\ y = q(x)$ with $y(1) = -4$

(c) Find the original differential equation by solving for $p(x)$ and $q(x)$

Answer available at the end of this section.

Exercise $1.5.4$

Assume the first order LDE $y' + p(x)\ y = q(x)$ has solutions $y_1(x) = (x + 1)e^x$ and $y_2(x) = (x - 1)e^x$

(a) Find a nonzero function that satisfies $y' + p(x)y = 0$

(b) Find the general form of the solutions to $y' + p(x)\ y = q(x)$

(c) Find the solution to the IVP $y' + p(x)\ y = q(x)$ with $y(1) = 0$

(d) Find the original differential equation by solving for $p(x)$ and $q(x)$

Answer available at the end of this section.

Exercise $1.5.5$

Assume a first order LDE $y' + p(x)\ y = q(x)$ has solutions $y_1(x) = 3e^x - 2e^{2x}$ and $y_2(x) = 2e^x + 5e^{2x}$

(a) Find a nonzero function that satisfies $y' + p(x)y = 0$

(b) Find the general form of the solutions to $y' + p(x)\ y = q(x)$

(c) Find the solution to the IVP $y' + p(x)\ y = q(x)$ with $y(0) = 19$

(d) Find the original differential equation by solving for $p(x)$ and $q(x)$

Answer available at the end of this section.

Answers

Exercise $1.5.1a$

We see $p(x) = 3 \implies \mu(x) = e^{\int 3\,dx} = e^{3x}$

$$\int \mu(x)q(x)\,dx = \int e^{3x}\cdot 15e^{2x}\,dx = \int 15e^{5x}\,dx = 3e^{5x}+C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{e^{3x}}\left(3e^{5x}+C\right) = 3e^{2x}+Ce^{-3x}$$

Therefore, $y(x) = 3e^{2x}+Ce^{-3x}$

The **particular solution** is $y_p(x) = 3e^{2x}$

The **homogeneous solutions** are $y_h(x) = Ce^{-3x}$

Exercise 1.5.1*b*

We see $p(x) = -\frac{2}{x} \implies \mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln(x)} = x^{-2}$

$$\int \mu(x)q(x)\,dx = \int x^{-2}\cdot x^3\,dx = \int x\,dx = \frac{1}{2}x^2+C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x^{-2}}\left(\frac{1}{2}x^2+C\right) = \frac{1}{2}x^4+Cx^2$$

Therefore, $y(x) = \frac{1}{2}x^4+Cx^2$

The **particular solution** is $y_p(x) = \frac{1}{2}x^4$

The **homogeneous solutions** are $y_h(x) = Cx^2$

Exercise 1.5.1*c*

We see $p(x) = \tan(x) \implies \mu(x) = e^{\int \tan(x)\,dx} = e^{\ln(\sec(x))} = \sec(x)$

$$\int \mu(x)q(x)\,dx = \int \sec(x)\cdot\sec(x)\,dx = \int \sec^2(x)\,dx = \tan(x)+C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{\sec(x)}\left(\tan(x)+C\right) = \sin(x)+C\,\cos(x)$$

Therefore, $y(x) = \sin(x)+C\,\cos(x)$

The **particular solution** is $y_p(x) = \sin(x)$

The **homogeneous solutions** are $y_h(x) = C\ \cos(x)$

Exercise 1.5.1*d*

First we must rewrite it as $y' + \frac{2}{x}y = \frac{\sin(x)}{x}$

We see $p(x) = \frac{2}{x} \implies \mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln(x)} = x^2$

$$\int \mu(x)q(x)\,dx = \int x^2\cdot\frac{\sin(x)}{x}\,dx = \int x\sin(x)\,dx = -x\cos(x)+\sin(x)+C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x^2}\left(-x\cos(x)+\sin(x)+C\right) = \frac{\sin(x)-x\cos(x)+C}{x^2}$$

Therefore, $y(x) = \frac{\sin(x)-x\cos(x)+C}{x^2}$

The **particular solution** is $y_p(x) = -x^{-1}\cos(x)+x^{-2}\sin(x)$

The **homogeneous solutions** are $y_h(x) = Cx^{-2}$

Exercise 1.5.1*e*

We see $p(x) = \frac{1}{x} \implies \mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln(x)} = x$

$$\int \mu(x)q(x)\,dx = \int x\cdot 2\ln(x^2)\,dx = \int 4x\ln(x)\,dx = 2x^2\ln(x)-x^2+C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x}\left(2x^2\ln(x)-x^2+C\right) = 2x\ln(x)-x+Cx^{-1}$$

Therefore, $y(x) = 2x\ln(x)-x+Cx^{-1}$

The **particular solution** is $y_p(x) = 2x\ln(x)-x$

The **homogeneous solutions** are $y_h(x) = Cx^{-1}$

Exercise 1.5.1*f*

We see $p(x) = \frac{1}{x\ln(x)} \implies \mu(x) = e^{\int \frac{1}{x\ln(x)}\,dx} = e^{\ln(\ln(x))} = \ln(x)$

$$\int \mu(x)q(x)\,dx = \int \ln(x)\cdot\frac{2}{x}\,dx = \int \frac{2\ln(x)}{x}\,dx = [\ln(x)]^2+C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{\ln(x)}\left([\ln(x)]^2+C\right) = \ln(x)+\frac{C}{\ln(x)}$$

Therefore, $y(x) = \ln(x)+\frac{C}{\ln(x)}$

The **particular solution** is $y_p(x) = \ln(x)$

The **homogeneous solutions** are $y_h(x) = \frac{C}{\ln(x)}$

Exercise 1.5.2*a*

We see $p(x) = 2 \implies \mu(x) = e^{\int 2\,dx} = e^{2x}$

$$\int \mu(x)q(x)\,dx = \int e^{2x}\cdot 6xe^{-2x}\,dx = \int 6x\,dx = 3x^2 + C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{e^{2x}}\left(3x^2 + C\right) = 3x^2e^{-2x} + Ce^{-2x}$$

The general solution is therefore $y(x) = 3x^2e^{-2x} + Ce^{-2x}$

Plugging in $y(0) = 2$ gives: $2 = 0 + C\cdot 1 \implies C = 2$

Therefore, the unique solution is $y(x) = (3x^2+2)e^{-2x}$

Exercise 1.5.2*b*

We see $p(x) = -1 \implies \mu(x) = e^{\int -1\,dx} = e^{-x}$

$$\int \mu(x)q(x)\,dx = \int e^{-x}\cdot 20\cos(2x)\,dx = 8e^{-x}\sin(2x) - 4e^{-x}\cos(2x) + C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{e^{-x}}\left(8e^{-x}\sin(2x) - 4e^{-x}\cos(2x) + C\right) = 8\sin(2x) - 4\cos(2x) + Ce^{x}$$

The general solution is therefore $y(x) = 8\sin(2x) - 4\cos(2x) + Ce^{x}$

Plugging in $y(\pi) = -4$ gives: $-4 = 0 - 4 + C\cdot 1 \implies C = 0$

Therefore, the unique solution is $y(x) = 8\sin(2x) - 4\cos(2x)$

Exercise 1.5.2*c*

We see $p(x) = \dfrac{2}{x^2-1} \implies \mu(x) = e^{\int \frac{1}{x-1} - \frac{1}{x+1}\,dx} = e^{\ln|x-1| - \ln|x+1|} = \left|\dfrac{x-1}{x+1}\right| = \dfrac{1-x}{x+1}$

$$\int \mu(x)q(x)\,dx = \int \frac{1-x}{x+1}\cdot 3(x^2-1)\,dx = \int -3(1-x)^2\,dx = (1-x)^3 + C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{\frac{1-x}{x+1}}\left((1-x)^3 + C\right) = (1-x)^2(x+1) + C\,\frac{x+1}{1-x}$$

The general solution is therefore $y(x) = (1-x)^2(x+1) + C\,\dfrac{x+1}{1-x}$

Plugging in $y(0) = 0$ gives: $0 = 1 + C\cdot 1 \implies C = -1$

Therefore, the unique solution is $y(x) = (1-x)^2(x+1) - \dfrac{x+1}{1-x}$

Exercise 1.5.2*d*

We see $p(x) = -1 \implies \mu(x) = e^{\int -\sin(x)\,dx} = e^{\cos(x)}$

$$\int \mu(x)q(x)\,dx = \int e^{\cos(x)}\cdot 3\sqrt{x}e^{-\cos(x)}\,dx = \int 3x^{\frac{1}{2}} = 2x^{\frac{3}{2}} + C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{e^{\cos(x)}}\left(2x^{\frac{3}{2}} + C\right) = \left(2x^{\frac{3}{2}} + C\right)e^{-\cos(x)}$$

The general solution is therefore $y(x) = \left(2x^{\frac{3}{2}} + C\right)e^{-\cos(x)}$

Plugging in $y(0) = 1$ gives: $1 = (0 + C)\cdot e^{-1} \implies C = e$

Therefore, the unique solution is $y(x) = \left(2x^{\frac{3}{2}} + e\right)e^{-\cos(x)}$

Exercise 1.5.2*e*

We see $p(x) = -\dfrac{2}{x} \implies \mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln|x|} = x^{-2}$

$$\int \mu(x)q(x)\,dx = \int x^{-2}\cdot\frac{1}{\sqrt{x^2-1}}\,dx = \int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \frac{\sqrt{x^2-1}}{x} + C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x^{-2}}\left(\frac{\sqrt{x^2-1}}{x} + C\right) = x\sqrt{x^2-1} + C\,x^2$$

The general solution is therefore $y(x) = x\sqrt{x^2-1} + C\,x^2$

Plugging in $y\left(\dfrac{5}{3}\right) = -\dfrac{5}{9}$ gives: $-\dfrac{5}{9} = \dfrac{5}{3}\cdot\dfrac{4}{3} + C\cdot\dfrac{25}{9} \implies C = -1$

Therefore, the unique solution is $y(x) = x\sqrt{x^2-1} - x^2$

Exercise 1.5.2*f*

We see $p(x) = \dfrac{1}{\sqrt{x}} \implies \mu(x) = e^{\int x^{-\frac{1}{2}}\,dx} = e^{2\sqrt{x}}$

$$\int \mu(x)q(x)\,dx = \int e^{2\sqrt{x}}\cdot 12x\,dx = \left(12x^{\frac{3}{2}} - 18x + 18x^{\frac{1}{2}} - 9\right)e^{2\sqrt{x}} + C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{e^{2\sqrt{x}}}\left(\left(12x^{\frac{3}{2}} - 18x + 18x^{\frac{1}{2}} - 9\right)e^{2\sqrt{x}} + C\right) = 12x^{\frac{3}{2}} - 18x + 18x^{\frac{1}{2}} - 9 + Ce^{-2\sqrt{x}}$$

The general solution is therefore $y(x) = 12x^{\frac{3}{2}} - 18x + 18x^{\frac{1}{2}} - 9 + Ce^{-2\sqrt{x}}$

Plugging in $y(1) = 3$ gives: $3 = 12 - 18 + 18 - 9 + C\cdot e^{-2} \implies C = 0$

Therefore, the unique solution is $y(x) = 12x^{\frac{3}{2}} - 18x + 18x^{\frac{1}{2}} - 9$

Exercise 1.5.2*g*

We see $p(x) = \dfrac{9x^2}{x^3+1} \implies \mu(x) = e^{\int \frac{9x^2}{x^3+1}\,dx} = e^{3\ln|x^3+1|} = (x^3+1)^3$

$$\int \mu(x)q(x)\,dx = \int (x^3+1)^3 \cdot \frac{4}{(x^3+1)^2}\,dx = \int 4(x^3+1)\,dx = x^4+4x+C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{(x^3+1)^3}\left(x^4+4x+C\right) = \frac{x^4+4x+C}{(x^3+1)^3}$$

The general solution is therefore $y(x) = \dfrac{x^4+4x+C}{(x^3+1)^3}$

Plugging in $y(1) = -2$ gives: $-2 = \dfrac{1+4+C}{2^3} \implies C = -21$

Therefore, the unique solution is $y(x) = \dfrac{x^4+4x-21}{(x^3+1)^3}$

Exercise 1.5.2*h*

We see $p(x) = -\sec(x) \implies \mu(x) = e^{\int -\sec(x)\,dx} = e^{-\ln|\sec(x)+\tan(x)|} = \dfrac{1}{\sec(x)+\tan(x)}$

$$\int \mu(x)q(x)\,dx = \int \frac{1}{\sec(x)+\tan(x)} \cdot 1\,dx = \int \frac{\cos(x)}{1+\sin(x)}\,dx = \ln(1+\sin(x)) + C$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{\frac{1}{\sec(x)+\tan(x)}}\left(\ln(1+\sin(x))+C\right) = (\sec(x)+\tan(x))\left(\ln(1+\sin(x))+C\right)$$

The general solution is therefore $y(x) = (\sec(x)+\tan(x))\left(\ln(1+\sin(x))+C\right)$

Plugging in $y(\pi) = 2$ gives: $2 = (-1+0)(\ln(1)+C) \implies C = -2$

Therefore, the unique solution is $y(x) = (\sec(x)+\tan(x))\left(\ln(1+\sin(x))-2\right)$

Exercise 1.5.3

(a) All possible homogeneous solutions must be of the form $y_h(x) = C(x+1)$

Plugging in $x = 1, y = -4$ gives: $-4 = C \cdot 2 \implies C = -2$

Therefore, the unique solution is $y(x) = -2(x+1)$

(b) We know the general solution to the non-homogeneous LDE is $y(x) = y_p(x) + y_h(x) = (\sqrt{x}+1)^2 + C(x+1)$

Plugging in $x = 1, y = -4$ gives: $-4 = 4 + C \cdot 2 \implies C = -4$

Therefore, the unique solution is $y(x) = (\sqrt{x}+1)^2 - 4(x+1)$

(c) We know $\dfrac{1}{\mu(x)} = y_h(x) = x+1 \implies \mu(x) = \dfrac{1}{x+1}$

$$\mu(x) = e^{\int p(x)\,dx} \implies \int p(x)\,dx = \ln(\mu(x)) \implies p(x) = \frac{d}{dx}\left[\ln\left(\frac{1}{x+1}\right)\right] = -\frac{1}{x+1}$$

We can plug this in to solve for $q(x)$:

$$q(x) = y_p' + p(x)y_p = \frac{\sqrt{x}+1}{\sqrt{x}} - \frac{1}{x+1}\cdot\left(\sqrt{x}+1\right)^2 = \frac{1-x}{(x+1)\sqrt{x}}$$

Therefore, the original equation was $y' - \dfrac{1}{x+1}\,y = \dfrac{1-x}{(x+1)\sqrt{x}}$

Exercise 1.5.4

(a) We know that the difference between two particular solutions is a homogeneous solution:

$$y_h(x) = y_1(x) - y_2(x) = (x+1)e^x - (x-1)e^x = 2e^x$$

Therefore, we know all solutions are of the form $y_h(x) = C \cdot 2e^x$

Note this is the same as saying $y_h(x) = Ke^x$

(b) Using either particular solution, we know the general solution is of the form $y(x) = y_p(x) + y_h(x)$:

Therefore, $y(x) = (x+1)e^x + C \cdot 2e^x = xe^x + (2C+1)e^x$

Note again, this is the same as writing $y(x) = (x+K)e^x$

(c) Plugging in the initial condition $y(1) = 0$ gives $0 = e + (2C+1)e = 2(C+1)e0 \implies C = -1$

Thus, the unique solution to this IVP is $y(x) = (x-1)e^x$

(d) We know $\dfrac{1}{\mu(x)} = y_h(x) = 2e^x \implies \mu(x) = \dfrac{1}{2}e^{-x}$

$$\mu(x) = e^{\int p(x)\,dx} \implies \int p(x)\,dx = \ln(\mu(x)) \implies p(x) = \frac{d}{dx}\left[\ln\left(\frac{1}{2}e^{-x}\right)\right] = -1$$

We can plug this in to solve for $q(x)$:

$$q(x) = y_p' + p(x)y_p = \left((x+2)e^x\right) + (-1)\cdot\left((x+1)e^x\right) = e^x$$

Therefore, the original equation was $y' - y = e^x$

Exercise 1.5.5

(a) We know that the difference between two particular solutions is a homogeneous solution:

$$y_h(x) = y_1(x) - y_2(x) = \left(3e^x - 2e^{2x}\right) - \left(2e^x + 5e^{2x}\right) = e^x - 7e^{2x}$$

Therefore, we know all solutions are of the form $y_h(x) = C\left(e^x - 7e^{2x}\right)$

(b) Using either particular solution, we know $y(x) = y_p(x) + y_h(x)$:

Therefore, $y(x) = 3e^x - 2e^{2x} + C\left(e^x - 7e^{2x}\right) = (3+C)e^x - (7C+2)e^{2x}$

(c) Plugging in the initial condition $y(0) = 19$ gives $19 = (3+C)e^0 - (7C+2)e^0 = 1 - 6C \implies C = -3$

Thus, the unique solution to this IVP is $y(x) = 19e^{2x}$

(d) We know $\frac{1}{\mu(x)} = y_h(x) = e^x - 7e^{2x} \implies \mu(x) = (e^x - 7e^{2x})^{-1}$

$$\int p(x)dx = \ln(\mu(x)) \implies p(x) = \frac{d}{dx}\left[\ln\left((e^x - 7e^{2x})^{-1}\right)\right] = \frac{14e^x - 1}{1 - 7e^x}$$

We can plug this in to solve for $q(x)$:

$$q(x) = y_p' + p(x)y_p = \left(3e^x - 4e^{2x}\right) + \frac{14e^x - 1}{1 - 7e^x} \cdot \left(3e^x - 2e^{2x}\right) = \frac{19e^{2x}}{1 - 7e^x}$$

Therefore, the original equation was $y' + \frac{14e^x - 1}{1 - 7e^x}\, y = \frac{19e^{2x}}{1 - 7e^x}$

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1.6: Substitutions

Learning Objectives

- Recognize differential equation forms amenable to specific substitutions.
- Apply the substitution method to transform nonlinear equations into linear or separable forms.
- Derive and implement the general solutions for Bernoulli and Riccati equations.
- Develop problem-solving intuition for choosing effective substitutions.

Many nonlinear equations can be transformed into solvable linear or separable ones with a clever change of variable. This subchapter teaches you to recognize equation forms amenable to specific substitutions, ultimately deriving and applying the general solutions for the famous Bernoulli and Riccati equations.

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1.6: Substitution In Differential Equations

When solving an integral (the most basic form of differential equation), the method of substitution is used to help simplify the anti-derivative by hiding some part of the chain rule that occurred during differentiation. When we do this, we replace the independent variable (typically x) with a new variable (typically u) that is related to the original variable through some function or relationship, $u(x)$. We also make sure to replace the differential inside of the integral (the dx) with its equivalent in terms of the differential of the new variable du. When everything goes smoothly, the end result is a new integral that is easier to solve.

The exact same thing can be said about substitution inside of a general differential equation. We can let u be some function of x or y (or both) and try to use it to replace all of one variable (and its derivatives) inside of the differential equation, in hopes of making a new differential equation which is easier to solve.

For a super simple example, consider the separable differential equation

$$2y \cdot \frac{dy}{dx} = \frac{1+y^2}{\sqrt{3-2x}}$$

Instead of immediately separating the variables and integrating, let's try doing a substitution of $u = 1 + y^2$

Taking the derivative of both sides with respect to x, this gives $\frac{du}{dx} = 2y \cdot \frac{dy}{dx}$

Both of these are easy to find and replace in the differential equation, turning it into:

$$\frac{du}{dx} = \frac{u}{\sqrt{3-2x}}$$

This is still separable, so let's separate and integrate both sides:

$$\int \frac{1}{u}\, du = \int (3-2x)^{-\frac{1}{2}} dx \implies \ln|u| = -\sqrt{3-2x} + C$$

Since $u = 1 + y^2$, this can be re-written in terms of y as: $\ln\left|1+y^2\right| = -\sqrt{3-2x} + C$

This substitution is exactly the same one we would have done inside of the integral if we had just separated and integrated in the first place:

$$\int \frac{2y}{1+y^2}\, dy = \int (3-2x)^{-\frac{1}{2}} dx \implies \ln\left|1+y^2\right| = -\sqrt{3-2x} + C$$

In that same differential equation, we could have instead substituted $u = 3 - 2x$ so $\frac{du}{dx} = -2$ and left y alone.

Notice that our differential equation would look like $2y \cdot \frac{dy}{dx} = \frac{1+y^2}{\sqrt{u}}$

There are still three variables in its current form! **We cannot solve this until all of the x's or y's are eliminated**.

We can do this by multiplying $\frac{dy}{dx} \cdot \frac{dx}{du} = \frac{dy}{du} \leftrightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$:

$$2y \cdot \frac{dy}{du} \cdot (-2) = \frac{1+y^2}{\sqrt{u}}$$

Now it is once again separable, so we separate and integrate:

$$\int -\frac{4y}{1+y^2}\, dy = \int u^{-\frac{1}{2}}\, du \implies -2\ln\left|1+y^2\right| = 2\sqrt{u} + C$$

Substituting back in $u = 3 - 2x$ this gives us an equivalent solution: $-2\ln\left|1+y^2\right| = 2\sqrt{3-2x} + C$

Example 1.6.1

Use the given substitution to rewrite each differential equation and then find its general solution.

(a) $\dfrac{dy}{dx} = \dfrac{x^5y^3 - 4y}{2x}$ with $u = y^{-2}$

(b) $\dfrac{dy}{dx} = x^2\sqrt{6x + 3y + 1} - 2$ with $v = 6x + 3y + 1$

(c) $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{\frac{y}{x}}$ with $w = \dfrac{y}{x}$

Solution

(a) Differentiating with respect to x we see $u = y^{-2} \implies \dfrac{du}{dx} = -2y^{-3} \cdot \dfrac{dy}{dx}$

We multiply both sides of the original differential equation by $-2y^{-3}$ to make things easier to see:

$$-2y^{-3} \cdot \frac{dy}{dx} = -x^4 + \frac{4}{x}y^{-2}$$

Now we eliminate y via substitution:

$$\frac{du}{dx} = -x^4 + \frac{4}{x} \cdot u$$

This is now a linear differential equation:

$$\frac{du}{dx} - \frac{4}{x}u = -x^4$$

We use $\mu(x) = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln|x|} = x^{-4}$:

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x^{-4}}\int x^{-4} \cdot \left(-x^4\right)\,dx = x^4\int -1\,dx = x^4 \cdot (-x + C) = -x^5 + Cx^4$$

Now we back substitute u into terms of y to get our final answer: $\boxed{y^{-2} = -x^5 + Cx^4}$

(b) Differentiating with respect to$x$ we see $v = 6x + 3y + 1 \implies \dfrac{dv}{dx} = 6 + 3 \cdot \dfrac{dy}{dx}$

Solving for $\dfrac{dy}{dx}$ gives $\dfrac{dy}{dx} = \dfrac{1}{3}\dfrac{dv}{dx} - 2$, so we substitute:

$$\frac{1}{3}\frac{dv}{dx} - 2 = x^2\sqrt{v} - 2$$

This simplifies down to $\dfrac{dv}{dx} = 3x^2\sqrt{v}$, which is separable:

$$\int \frac{1}{\sqrt{v}}\,dv = \int 3x^2\,dx \implies 2\sqrt{v} = x^3 + C$$

Again, we back substitute to get our final answer: $\boxed{2\sqrt{6x + 3y + 1} = x^3 + C}$

(c) Differentiating with respect to x we see $w = \dfrac{y}{x} \implies \dfrac{dw}{dx} = \dfrac{x \cdot \frac{dy}{dx} - y \cdot 1}{x^2}$

This is unpleasant to deal with directly, so instead we rewrite our substitution *before* we differentiate:

$$w = \frac{y}{x} \implies y = wx \implies \frac{dy}{dx} = \frac{dw}{dx} \cdot x + w \cdot 1$$

This is much easier to substitute into the original differential equation:

$$\frac{dw}{dx} \cdot x + w = w - e^w$$

This simplifies to $\frac{dw}{dx} = -\frac{e^w}{x}$ which is separable:

$$\int -e^{-w}\, dw = \int \frac{1}{x}\, dx \implies e^{-w} = \ln|x| + C$$

Back substituting gives our final answer: $\boxed{e^{-\frac{y}{x}} = \ln|x| + C}$

Choosing the right substitution in a first order differential equation depends on recognizing patterns within its structure. If you see a term like $\frac{y}{x}$ repeated, try using it as your substitution. Just like in an integral though, you will need to be able to find the derivative of your chosen substitution or be able to manipulate that substitution expression sufficiently to solve back for y in order to eliminate the $\frac{dy}{dx}$ term from the equation. This comes with experience and practice!

Example 1.6.2

Find the solutions to the following first order IVPs by choosing a sufficient substitution.

(a) $\frac{dy}{dx} = (x+y)^2$ with $y(0) = 1$

(b) $\frac{dy}{dx} = \frac{2x^2 + 3y^2}{4xy}$ with $y(1) = 1$

(c) $3y^2 \cdot \frac{dy}{dx} = 2x + e^{\tan(x)} + \frac{y^3 - x^2}{\cos^2(x)}$ with $y(0) = 2$

Solution

(a) We see $x + y$ inside of another function, so let's try $u = x + y \implies \frac{du}{dx} = 1 + \frac{dy}{dx}$:

$$\frac{du}{dx} - 1 = u^2 \implies \frac{du}{dx} = 1 + u^2$$

This is separable:

$$\int \frac{1}{1+u^2}\, du = \int dx \implies \tan^{-1}(u) = x + C$$

Back-substitution gives our general solution: $\tan^{-1}(x+y) = x + C$

Plugging in the initial condition $y(0) = 1$ gives us: $\tan^{-1}(1) = 0 + C \implies C = \frac{\pi}{4}$

Therefore, the unique solution to this DE is $\tan^{-1}(x+y) = x + \frac{\pi}{4}$

We can solve for y, giving explicit solution $\boxed{y = \tan\left(x + \frac{\pi}{4}\right) - x}$

(b) This substitution is not obvious in the current form of the differential equation, so we rewrite it:

$$\frac{dy}{dx} = \frac{2 + 3\left(\frac{y}{x}\right)^2}{4\left(\frac{y}{x}\right)}$$

We can now see that $v = \frac{y}{x}$ is a good choice, which can be written as $y = vx \implies \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot 1$:

$$\frac{dv}{dx} \cdot x + v = \frac{2 + 3v^2}{4v} \implies \frac{dv}{dx} = \frac{2 - v^2}{4xv}$$

This is separable:

$$\int \frac{4v}{2 - v^2}\, dv = \int \frac{1}{x}\, dx \implies -2\ln\left(2 - v^2\right) = \ln(x) + C$$

Back-substitution gives our general solution: $-2\ln\left(2 - \left(\frac{y}{x}\right)^2\right) = \ln(x) + C$

Plugging in the initial condition $y(1) = 1$ gives us: $-2\ln\left(2 - 1^2\right) = \ln(1) + C \implies C = 0$

Therefore, the solution to this DE is $-2\ln\left(2 - \left(\frac{y}{x}\right)^2\right) = \ln(x)$

We can solve for y, giving $y = \pm\sqrt{2x^2 - x^{\frac{3}{2}}}$

Since $y(1) = 1 > 0$, we know to take the positive root for our unique solution: $\boxed{y = \sqrt{2x^2 - x^{\frac{3}{2}}}}$

(c) This looks messy, but even a small substitution might help: $w = y^3 - x^2 \implies \frac{dw}{dx} = 3y^2 \cdot \frac{dy}{dx} - 2x$

$$\frac{dw}{dx} + 2x = 2x + e^{\tan(x)} + \frac{w}{\cos^2(x)} \implies \frac{dw}{dx} - \sec^2(x)\, w = e^{\tan(x)}$$

This is now linear, and we can use $\mu(x) = e^{\int -\sec^2(x)\, dx} = e^{-\tan(x)}$:

$$w = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{e^{-\tan(x)}} \int e^{-\tan(x)} \cdot e^{\tan(x)}\, dx = e^{\tan(x)} \int 1\, dx = e^{\tan(x)}\,[x + C]$$

Back-substitution gives our general solution: $y^3 - x^2 = (x + C)e^{\tan(x)}$

Plugging in the initial condition $y(0) = 2$ gives us: $8 - 0 = (0 + C)e^0 \implies C = 8$

Therefore, the solution to this DE is $y^3 - x^2 = (x + 8)e^{\tan(x)}$

We can solve for y, giving our explicit solution $\boxed{y = \sqrt[3]{x^2 + (x + 8)e^{\tan(x)}}}$

Usually a simple substitution will help things progress to their next logical step in your differential equations. You can always do more than one substitution in the same problem (just like in integrals), so there is no need to try to make a fancy substitution all at once!

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1.6: Famous DEs and Their Substitutions

There are a few famous forms of first order differential equations that it is advantageous to memorize substitutions for, which will be discussed here.

The first of these is the Bernoulli Equation, which is *almost* a first order linear differential equation.

Definition: Bernoulli Equation

A **Bernoulli Equation** is a first order differential equation of the form

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \cdot y^n \quad \text{for some } n \neq 0, 1$$

Notice that when $n = 0$ or $n = 1$ this turns into a standard LDE, so we do not need any substitution to solve it.

However for all other choices of n, we need to do a substitution of $u = y^{1-n} \implies \frac{du}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$

We multiply the original equation through by $(1-n)y^{-n}$ to make this easier:

$$(1-n)y^{-n} \cdot \frac{dy}{dx} + (1-n)p(x) \cdot y^{1-n} = (1-n)q(x)$$

Now everything easily transitions into terms of u:

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$$

The end result is a linear differential equation, which we know how to solve.

Rule: Solving Bernoulli Equations

Given a Bernoulli Equation of the form $\frac{dy}{dx} + p(x) \cdot y = q(x) \cdot y^n$, use the substitution $u = y^{1-n}$ to create a first order linear differential equation in u.

Example 1.6.3

Solve the following first order IVPs.

(a) $y' + y = xy^2$ with $y(0) = \frac{1}{2}$

(b) $2\frac{dy}{dx} + 3y\tan(x) = 3y^{\frac{1}{3}}\sec(x)$ with $y\left(\frac{\pi}{4}\right) = 4\sqrt[4]{2}$

Solution

(a) We substitute $u = y^{1-2} = y^{-1} \implies \frac{du}{dx} = -y^{-2} \cdot \frac{dy}{dx}$:

$$-y^{-2} \cdot y' - y^{-1} = -x \implies u' - u = -x$$

We use $\mu(x) = e^{\int -1\,dx} = e^{-x}$:

$$u = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{e^{-x}} \int e^{-x} \cdot (-x)\, dx = e^{x} \left[(x+1)e^{-x} + C \right] = x + 1 + Ce^{x}$$

Back-substituting gives our general solution: $y^{-1} = x + 1 + Ce^{x}$

Plugging in the initial condition $y(0) = \frac{1}{2}$ gives: $2 = 0 + 1 + C \cdot 1 \implies C = 1$

Thus, our solution will be $y^{-1} = x + 1 + e^{x}$, which we can solve explicitly for y as $\boxed{y = \frac{1}{x + 1 + e^{x}}}$

(b) We substitute $v = y^{1-\frac{1}{3}} = y^{\frac{2}{3}} \implies \frac{dv}{dx} = \frac{2}{3} y^{-\frac{1}{3}} \cdot \frac{dy}{dx}$:

$$\frac{2}{3} y^{-\frac{1}{3}} \cdot \frac{dy}{dx} + \tan(x) \cdot y^{\frac{2}{3}} = \sec(x) \implies \frac{dv}{dx} + \tan(x)\, v = \sec(x)$$

We use $\mu(x) = e^{\int \tan(x)\, dx} = e^{\ln|\sec(x)|} = \sec(x)$:

$$v = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{\sec(x)} \int \sec(x) \cdot \sec(x)\, dx = \cos(x) \int \sec^2(x)\, dx = \cos(x) \left[\tan(x) + C \right] = \sin(x) + C\cos(x)$$

Back-substituting gives our general solution: $y^{\frac{2}{3}} = \sin(x) + C\cos(x)$

Plugging in the initial condition $y\left(\frac{\pi}{4}\right) = 2^{\frac{9}{4}}$ gives: $2^{\frac{3}{2}} = \frac{\sqrt{2}}{2} + C \cdot \frac{\sqrt{2}}{2} \implies C = 3$

Thus, our solution will be $y^{\frac{2}{3}} = \sin(x) + 3\cos(x)$, which we can solve explicitly for y as $\boxed{y = (\sin(x) + 3\cos(x))^{\frac{3}{2}}}$

Another famous type of first order differential equation is *almost* a Bernoulli equation.

Definition: Riccati Equation

A **Riccati Equation** is a first order differential equation of the form

$$\frac{dy}{dx} = P(x) + Q(x) \cdot y + R(x) \cdot y^2$$

Riccati Equations are intimidating, but once you find *one particular solution*, you can find the entire family of solutions with a simple substitution.

Assume for a moment that we are lucky enough to have guessed (or been given) a solution $y = f(x)$ to the equation $\frac{dy}{dx} = P(x) + Q(x)\ y + R(x)\ y^2$.

$$\frac{df}{dx} = P(x) + Q(x) \cdot f(x) + R(x) \cdot [f(x)]^2$$

Then, if we substitute $u = y - f$, we have $\frac{du}{dx} = \frac{dy}{dx} - \frac{df}{dx}$, which turns the equation into

$$\frac{du}{dx} + \frac{df}{dx} = P(x) + Q(x) \cdot (u + f(x)) + R(x) \cdot (u + f(x))^2$$

This simplifies to

$$\frac{du}{dx} = \left(P(x) + Q(x) \cdot f(x) + R(x) \cdot [f(x)]^2 - \frac{df}{dx} \right) + (Q(x) + 2fR(x)) \cdot u + R(x) \cdot u^2$$

Since $f(x)$ was a known solution, the big parenthesis term is equal to zero and the result is a *Bernoulli Equation* for u.

This means we can use substitution $v = u^{1-2} = u^{-1}$ to turn this into a linear equation, which will be solvable!

Rule: Solving Riccati Equations

Given a Riccati Equation of the form $\frac{dy}{dx} = P(x) + Q(x) \cdot y + R(x) \cdot y^2$ with one known solution $y = f(x)$, use the substitution $y = f + u^{-1}$ to create a first order linear differential equation in u.

Example 1.6.4

Find the general solution to each Riccati Equation by using the given solution.

(a) $\frac{dy}{dx} = \frac{1 - xy - x^2y^2}{x^2}$ given $f(x) = \frac{1}{x}$

(b) $\frac{dy}{dx} = 1 + 2\tan^2(x) - y^2$ given $f(x) = \tan(x)$

Solution

(a) We use the substitution $y = \frac{1}{x} + u^{-1} \implies \frac{dy}{dx} = -\frac{1}{x^2} - u^{-2} \cdot \frac{du}{dx}$:

$$-\frac{1}{x^2} - u^{-2} \cdot \frac{du}{dx} = \frac{1 - x\left(\frac{1}{x} + u^{-1}\right) - x^2\left(\frac{1}{x} + u^{-1}\right)^2}{x^2}$$

This simplifies down to the linear differential equation

$$\frac{du}{dx} - \frac{3}{x} \cdot u = 1$$

This gives us $\mu(x) = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln(x)} = x^{-3}$

$$u = \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{x^{-3}} \int x^{-3} \cdot 1\,dx = x^3\left[-\frac{1}{2}x^{-2} + C\right] = \frac{Kx^3 - x}{2}$$

Back-substituting this gives us our general solution $\boxed{y = \frac{1}{x} + \frac{2}{Kx^3 - x}}$

(b) We use the substitution $y = \tan(x) + v^{-1} \implies \frac{dy}{dx} = \sec^2(x) - v^{-2} \cdot \frac{dv}{dx}$:

$$\sec^2(x) - v^{-2} \cdot \frac{dv}{dx} = 1 + 2\tan^2(x) - \left(\tan(x) + v^{-1}\right)^2$$

This simplifies down to the linear differential equation

$$\frac{dv}{dx} - 2\tan(x) \cdot v = 1$$

This gives us $\mu(x) = e^{\int -2\tan(x)\,dx} = e^{2\ln|\cos(x)|} = \cos^2(x)$

$$u = \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{\cos^2(x)} \int \cos^2(x) \cdot 1\,dx = \sec^2(x) \int \frac{1 + \cos(2x)}{2}\,dx = \sec^2(x)\left[\frac{1}{2}x + \frac{1}{4}\sin(2x) + C\right] = \frac{2x + \sin(2x) +$$

Back-substituting this gives us our general solution $\boxed{y = \tan(x) + \frac{4\cos^2(x)}{2x + \sin(2x) + C}}$

Instead of giving you the particular solution outright, sometimes you only get a hint to solve for it.

Illustration 1.6.1

Find the general solution to each Riccati Equation by using the given solution.

(a) $\frac{dy}{dx} = 1 + 2x^2 - 4x^5 + (8x^4 - 2x)y - 4x^3y^2$ given there exists some linear solution

(b) $\frac{dy}{dx} = 4e^{2x} + (1 - 4e^x)\,y + y^2$ given there exists some solution of the form $y = Ce^x$

(a) If we assume $y_p = mx + b$, then $y_p' = m$ so the differential equation would be satisfied when:

$$m = 1 + 2x^2 - 4x^5 + (8x^4 - 2x)(mx + b) - 4x^3(mx + b)^2$$

This is equivalent to:

$$0 = (1 - m) - 2bx + (2 - 2m)x^2 - 4x^5 - 4b^2x^3 + 8b(1 - m)x^4 + 4m(1 - m)x^5$$

Clearly, $m = 1$ is necessary for the constant term to vanish, and $b = 0$ for the x term to vanish. Luckily those two choices for m and b satisfy the rest of the coefficients becoming zero, meaning our particular solution is $y_p = x$.

We use the substitution $y = x + u^{-1} \implies \frac{dy}{dx} = 1 - u^{-2} \cdot \frac{du}{dx}$:

$$1 - u^{-2} \cdot \frac{du}{dx} = 1 + 2x^2 - 4x^5 + (8x^4 - 2x)\left(x + u^{-1}\right) - 4x^3\left(x + u^{-1}\right)^2$$

This simplifies down to the linear differential equation

$$\frac{du}{dx} - 2x \cdot u = 4x^3$$

This gives us $\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$

$$u = \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx = \frac{1}{e^{-x^2}} \int e^{-x^2} \cdot 4x^3\,dx = e^{x^2}\left[-\left(2x^2 + 2\right)e^{-x^2} + C\right] = -2x^2 - 2 + Ce^{x^2}$$

Back-substituting this gives us our general solution $y = x + \dfrac{1}{Ce^{x^2} - 2 - 2x^2}$

(b) If we assume $y_p = Ce^x$, then $y_p' = Ce^x$ so the differential equation would be satisfied when:

$$Ce^x = 4e^{2x} + (1 - 4e^x) \cdot Ce^x + (Ce^x)^2$$

This is equivalent to:

$$0 = (C - 2)^2 e^{2x}$$

This is only satisfied when $C = 2$, so our particular solution is $y_p = 2e^x$.

We use the substitution $y = 2e^x + v^{-1} \implies \frac{dy}{dx} = 2e^x - v^{-2} \cdot \frac{dv}{dx}$:

$$2e^x - v^{-2} \cdot \frac{dv}{dx} = 4e^{2x} + (1 - 4e^x)\left(2e^x + v^{-1}\right) + \left(2e^x + v^{-1}\right)^2$$

This simplifies down to the linear differential equation

$$\frac{dv}{dx} + v = -1$$

This gives us $\mu(x) = e^{\int 1\,dx} = e^x$

$$v = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{e^x} \int e^x \cdot (-1)\, dx = e^{-x} \left[-e^x + C \right] = -1 + Ce^{-x}$$

Back-substituting this gives us our general solution $y = 2e^x + \dfrac{1}{Ce^{-x} - 1}$

There are many other famous equations named after the mathematicians that discovered the substitution that unlocks their solutions. Keep trying clever substitutions and maybe you can join them one day!

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1.6: Practice Set

Exercise 1.6.1a

Find the general explicit solution to the differential equation $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

Answer available at the end of this section.

Exercise 1.6.1b

Find the general explicit solution to the differential equation $(x^3 + y^3)dx - xy^2dy = 0$

Answer available at the end of this section.

Exercise 1.6.1c

Find the general explicit solution to the differential equation $y' = \frac{y}{x} + e^{\frac{y}{x}}$

Answer available at the end of this section.

Exercise 1.6.1d

Find the general explicit solution to the differential equation $x^2y' = y^2 + 2xy - 6x^2$

Answer available at the end of this section.

Exercise 1.6.1e

Find the general explicit solution to the differential equation $3y^2y' + \frac{2}{x} \cdot y^3 = 4x$

Answer available at the end of this section.

Exercise 1.6.1f

Find the general explicit solution to the differential equation $\frac{2y}{x}y' = \frac{y^2}{x^2} + \sec\left(\frac{y^2}{x}\right)$

Answer available at the end of this section.

Exercise 1.6.2a

Find the unique solution to the IVP $xy' + y = y^2 \ln(x)$ with $y(1) = \frac{1}{2}$

Answer available at the end of this section.

Exercise $1.6.2b$

Find the unique solution to the IVP $y' - y = 3e^x y^3$ with $y(0) = -1$

Answer available at the end of this section.

Exercise $1.6.2c$

Find the unique solution to the IVP $x^2y' + 2xy = 7y^4$ with $y(1) = -1$

Answer available at the end of this section.

Exercise $1.6.2d$

Find the unique solution to the IVP $y' + 2y\tan(x) = \sqrt{y}\sec(x)$ with $y(\pi) = 0$

Answer available at the end of this section.

Exercise $1.6.3a$

Given $y_1(x) = \dfrac{1}{x}$ satisfies $y' = y^2 - \dfrac{3y}{x} + \dfrac{1}{x^2}$, find all possible solutions.

Answer available at the end of this section.

Exercise $1.6.3b$

Given $y_1(x) = \cot(x)$ satisfies $y' = -\csc^2(x) - y\cot(x) + y^2$, find all possible solutions.

Answer available at the end of this section.

Exercise $1.6.3c$

Given $y_1(x) = e^x$ satisfies $y' = e^{2x} + (1 - 2e^x)y + y^2$, find all possible solutions.

Answer available at the end of this section.

Exercise $1.6.3d$

Given $y_1(x) = x^3$ satisfies $y' = x^6 + \dfrac{3y}{x} - y^2$, find all possible solutions.

Answer available at the end of this section.

Exercise $1.6.4a$

Find the solution to the IVP $y' = \dfrac{1}{x + y + 2}$ with $y(1) = 0$

Answer available at the end of this section.

Exercise 1.6.4b

Find the solution to the IVP $(1+3y^2y') + \dfrac{1}{x}(x+y^3) = 12x^2$ with $y(2) = 0$

Answer available at the end of this section.

Exercise 1.6.4c

Find the solution to the IVP $yy' = \sqrt{x^2+y^2} - x$ with $y(3) = 4$

Answer available at the end of this section.

Exercise 1.6.4d

Find the solution to the IVP $\cos(x+y)y' + \sin(x)\sin(x+y) + \cos(x+y) = 2xe^{\cos(x)}$ with $y\left(\dfrac{\pi}{2}\right) = \dfrac{3\pi}{2}$

Answer available at the end of this section.

Exercise 1.6.4e

Find the solution to the IVP $(6(x^4+y^4)^2 + 2x(x^4+y^4))(4x^3dx + 4y^3dy) + ((x^4+y^4)^2 + 12x^2)\ dx = 0$ with $y(1) = 1$

Answer available at the end of this section.

Answers

Exercise 1.6.1a

First, we rewrite: $\dfrac{dy}{dx} = 1 + \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$

This suggests we use $u = \dfrac{y}{x} \implies y = xu \implies y' = u + xu'$

The DE becomes:

$$u + xu' = 1 + u + u^2 \implies x \cdot \frac{du}{dx} = 1 + u^2$$

This is separable:

$$\int \frac{1}{1+u^2}\,du = \int \frac{1}{x}\,dx \implies \tan^{-1}(u) = \ln|x| + C$$

Back-substituting gives $\tan^{-1}\left(\dfrac{y}{x}\right) = \ln|x| + C$

We can solve this for y to get our explicit solutions: $y = x\tan(\ln|x| + C)$

Exercise 1.6.1b

First, we rewrite: $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right)^{-2} + \dfrac{y}{x}$

This suggests we use $u = \dfrac{y}{x} \implies y = xu \implies y' = u + xu'$

The DE becomes:

$$u + xu' = u^{-2} + u \implies x \cdot \frac{du}{dx} = u^{-2}$$

This is separable:

$$\int u^2\,du = \int \frac{1}{x}\,dx \implies \frac{1}{3}u^3 = \ln|x| + C$$

Back-substituting gives $\dfrac{y^3}{3x^3} = \ln|x| + C$

We can solve this for y to get our explicit solutions: $y = x\sqrt[3]{3\ln|x| + C}$

Exercise 1.6.1*c*

This suggests we use $u = \dfrac{y}{x} \implies y = xu \implies y' = u + xu'$

The DE becomes:

$$u + xu' = u + e^u \implies x \cdot \frac{du}{dx} = e^u$$

This is separable:

$$\int e^{-u}\,du = \int \frac{1}{x}\,dx \implies -e^{-u} = \ln|x| + C$$

Back-substituting gives $-e^{-\frac{y}{x}} = \ln|x| + C$

We can solve this for y to get our explicit solutions: $y = -x\ln(C - \ln|x|)$

Exercise 1.6.1*d*

First, we rewrite: $y' = \left(\dfrac{y}{x}\right)^2 + 2\dfrac{y}{x} - 6$

This suggests we use $u = \dfrac{y}{x} \implies y = xu \implies y' = u + xu'$

The DE becomes:

$$u + xu' = u^2 + 2u - 6 \implies x \cdot \frac{du}{dx} = u^2 + u - 6$$

This is separable:

$$\int \frac{1}{(u+3)(u-2)}\,du = \int \frac{1}{x}\,dx \implies \frac{1}{5}\ln|u-2| - \frac{1}{5}\ln|u+3| = \ln|x| + C$$

Back-substituting gives $\dfrac{1}{5}\ln\left|\dfrac{y}{x} - 2\right| - \dfrac{1}{5}\ln\left|\dfrac{y}{x} + 3\right| = \ln|x| + C$

We can solve this for y to get our explicit solutions: $y = \dfrac{5x}{1 - Cx^5} - 3x$

Exercise 1.6.1*e*

We use $u = y^3 \implies u' = 3y^2y'$

The DE becomes:

$$u' + \frac{2}{x}u = x$$

This is linear, so we use $\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln(x)} = x^2$:

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x^2}\int x^2 \cdot 4x\,dx = x^{-2}\int 4x^3\,dx = x^{-2}\left[x^4 + C\right] = x^2 + Cx^{-2}$$

Back-substituting gives $y^3 = x^2 + Cx^{-2}$

We can solve this for y to get our explicit solutions: $y = \sqrt[3]{x^2 + Cx^{-2}}$

Exercise 1.6.1*f*

We use $u = \dfrac{y^2}{x} \Rightarrow u' = \dfrac{2y}{x}y' - \dfrac{y^2}{x^2}$

The DE becomes:

$$u' = \sec(u)$$

This is separable:

$$\int \cos(u)\,du = \int 1\,dx \implies \sin(u) = x + C$$

Back-substituting gives $\sin\left(\dfrac{y^2}{x}\right) = x + C$

We can solve this for y to get our explicit solutions: $y = \pm\sqrt{x\sin^{-1}(x + C)}$

Exercise 1.6.2*a*

First, we rewrite: $y^{-2}y' + \dfrac{1}{x}y^{-1} = \dfrac{\ln(x)}{x}$

This is Bernoulli with $n = 2$, so we use the substitution $u = y^{-1} \Rightarrow u' = -y^{-2}y'$

The DE becomes linear:

$$u' - \frac{1}{x}u = -\frac{\ln x}{x}$$

We use $\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln(x)} = \dfrac{1}{x}$

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{\frac{1}{x}}\int \frac{1}{x}\cdot\left(-\frac{\ln(x)}{x}\right)dx = x\int -x^{-2}\ln(x)\,dx = x\left[\frac{\ln(x)+1}{x} + C\right] = \ln(x) + 1 + Cx$$

Back-substituting gives $y^{-1} = \ln(x) + 1 + Cx$

Plugging in the initial condition $y(1) = \dfrac{1}{2}$, we get: $2 = 0 + 1 + C \cdot 1 \implies C = 1$

Thus, our implicit solution is $y^{-1} = \ln(x) + 1 + x$

We can solve for y to get our unique explicit solution $y = \frac{1}{\ln(x)+1+x}$

Exercise 1.6.2*b*

First, we rewrite: $y^{-3}y' - y^{-2} = 3e^x$

This is Bernoulli with $n = 3$, so we use the substitution $u = y^{-2} \Rightarrow u' = -2y^{-3}y'$

The DE becomes linear:

$$u' + 2u = -6e^x$$

We use $\mu(x) = e^{\int 2\,dx} = e^{2x}$

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{e^{2x}}\int e^{2x}\cdot(-6e^x)\,dx = e^{-2x}\int -6e^{3x}\,dx = e^{-2x}\left[-2e^{3x} + C\right] = -2e^x + Ce^{-2x}$$

Back-substituting gives $y^{-2} = -2e^x + Ce^{-2x}$

Plugging in the initial condition $y(0) = -1$, we get: $1 = -2 + C\cdot 1 \implies C = 3$

Thus, our implicit solution is $y^{-2} = -2e^x + 3e^{-2x}$

We can solve for y to get our unique explicit solution $y = -\dfrac{1}{\sqrt{3e^{-2x} - 2e^x}}$

Exercise 1.6.2*c*

This is Bernoulli with $n = 4$, so we use the substitution $u = y^{-3} \Rightarrow u' = -3y^{-4}y'$

The DE becomes linear:

$$u' - \frac{6}{x}u = -\frac{21}{x^2}$$

We use $\mu(x) = e^{\int -\frac{6}{x}\,dx} = e^{-6\ln(x)} = x^{-6}$

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x^{-6}}\int x^{-6}\cdot\left(-\frac{21}{x^2}\right)\,dx = x^6\int -21x^{-8}\,dx = x^6\left[3x^{-7} + C\right] = 3x^{-1} + Cx^6$$

Back-substituting gives $y^{-3} = 3x^{-1} + Cx^6$

Plugging in the initial condition $y(1) = -1$, we get: $-1 = 3 + C\cdot 1 \implies C = -4$

Thus, our implicit solution is $y^{-3} = 3x^{-1} - 4x^6$

We can solve for y to get our unique explicit solution $y = \dfrac{1}{\sqrt[3]{3x^{-1} - 4x^6}}$

Exercise 1.6.2*d*

This is Bernoulli with $n = \dfrac{1}{2}$, so we use the substitution $u = y^{\frac{1}{2}} \Rightarrow u' = \dfrac{1}{2}y^{-\frac{1}{2}}y'$

The DE becomes linear:

$$u' + \tan(x)\cdot u = \frac{1}{2}\sec(x)$$

We use $\mu(x) = e^{\int \tan(x)\,dx} = e^{\ln(\sec(x))} = \sec(x)$

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{\sec(x)}\int \sec(x)\cdot\left(\frac{1}{2}\sec(x)\right)\,dx = \cos(x)\int \frac{1}{2}\sec^2(x)\,dx = \cos(x)\left[\frac{1}{2}\tan(x) + C\right] = \frac{1}{2}\sin(x) + C\,\cos(x)$$

Back-substituting gives $y^{\frac{1}{2}} = \dfrac{1}{2}\sin(x) + C\,\cos(x)$

Plugging in the initial condition $y(\pi) = 0$, we get: $0 = 0 + C\cdot(-1) \implies C = 0$

Thus, our implicit solution is $y^{\frac{1}{2}} = \dfrac{1}{2}\sin(x)$

We can solve for y to get our unique explicit solution $y = \dfrac{1}{4}\sin^2(x)$

Exercise 1.6.3*a*

This is Riccati, so we substitute $y = \dfrac{1}{x} + v^{-1} \Rightarrow y' = -\dfrac{1}{x^2} - v^{-2}v'$

The DE becomes:

$$-\frac{1}{x^2} - v^{-2}v' = \left(\frac{1}{x} + v^{-1}\right)^2 - \frac{3}{x}\left(\frac{1}{x} + v^{-1}\right) + \frac{1}{x^2}$$

This simplifies to become linear:

$$v' - \frac{1}{x}v = -1$$

We use $\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln(x)} = x^{-1}$

$$v = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{x^{-1}}\int x^{-1}\cdot(-1)\,dx = x\left[-\ln|x| + C\right] = -x\ln|x| + Cx$$

Back-substituting gives our general solution: $y = \dfrac{1}{x} + \dfrac{1}{-x\ln|x| + Cx}$

Exercise 1.6.3*b*

This is Riccati, so we substitute $y = \cot(x) + v^{-1} \Rightarrow y' = -\csc^2(x) - v^{-2}v'$

The DE becomes:

$$-\csc^2(x) - v^{-2}v' = -\csc^2(x) - \cot(x)\left(\cot(x) + v^{-1}\right) + \left(\cot(x) + v^{-1}\right)^2$$

This simplifies to become linear:

$$v' + \cot(x)v = -1$$

We use $\mu(x) = e^{\int \cot(x)\,dx} = e^{\ln(\sin(x))} = \sin(x)$

$$v = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{\sin(x)} \int \sin(x) \cdot (-1)\, dx = \frac{\cos(x) + C}{\sin(x)}$$

Back-substituting gives our general solution: $y = \cot(x) + \dfrac{\sin(x)}{\cos(x) + C}$

Exercise 1.6.3*c*

This is Riccati, so we substitute $y = e^x + v^{-1} \Rightarrow y' = e^x - v^{-2}v'$

The DE becomes:

$$e^x - v^{-2}v' = e^{2x} + (1 - 2e^x)\left(e^x + v^{-1}\right) + \left(e^x + v^{-1}\right)^2$$

This simplifies to become linear:

$$v' + v = -1$$

We use $\mu(x) = e^{\int 1\, dx} = e^x$

$$v = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{e^x} \int e^x \cdot (-1)\, dx = \frac{-e^x + C}{e^x}$$

Back-substituting gives our general solution: $y = e^x + \dfrac{e^x}{C - e^x}$

Exercise 1.6.3*d*

This is Riccati, so we substitute $y = x^3 + v^{-1} \Rightarrow y' = 3x^2 - v^{-2}v'$

The DE becomes:

$$3x^2 - v^{-2}v' = x^6 + \frac{3}{x}\left(x^3 + v^{-1}\right) + \left(x^3 + v^{-1}\right)^2$$

This simplifies to become linear:

$$v' + \left(\frac{3}{x} - 2x^3\right) v = 1$$

We use $\mu(x) = e^{\int \frac{3}{x} - 2x^3\, dx} = e^{3\ln(x) - \frac{1}{2}x^4} = x^3 e^{-\frac{1}{2}x^4}$

$$v = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{x^3 e^{-\frac{1}{2}x^4}} \int x^3 e^{-\frac{1}{2}x^4} \cdot 1\, dx = \frac{-\frac{1}{2} e^{-\frac{1}{2}x^4} + C}{x^3 e^{-\frac{1}{2}x^4}} = \frac{C e^{\frac{1}{2}x^4} - 1}{2x^3}$$

Back-substituting gives our general solution: $y = x^3 + \dfrac{2x^3}{Ce^{\frac{1}{2}x^4} - 1}$

Exercise 1.6.4*a*

We use the substitution $u = x + y + 2 \implies y = u - x - 2 \implies y' = u' - 1$

The DE becomes:

$$u' - 1 = \frac{1}{u} \implies \frac{du}{dx} = \frac{u + 1}{u}$$

This is separable:

$$\int \frac{u}{u+1}\, du = \int dx \implies u - \ln|u + 1| = x + C$$

Back-substituting gives our general solution: $x + y + 2 - \ln|x + y + 3| = x + C$

This simplifies to $y - \ln|x + y + 3| = C$

Plugging in the initial condition $y(1) = 0$ gives: $0 - \ln|1 + 0 + 3| = C \Rightarrow C = -\ln(4)$

Therefore, our unique solution is $y - \ln|x + y + 3| = -\ln(4)$

Exercise 1.6.4*b*

We use the substitution $u = x + y^3 \implies \dfrac{du}{dx} = 1 + 3y^2 y'$

The DE becomes:

$$\frac{du}{dx} + \frac{1}{x}u = 12x^2$$

This is linear, so we use $mu(x) = e^{\int \frac{1}{x}\, dx} = e^{\ln(x)} = x$

$$u = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{x} \int x \cdot 12x^2\, dx = x^{-1} \int 12x^3\, dx = x^{-1}\left[3x^4 + C\right] = 3x^3 + Cx^{-1}$$

Back-substituting gives our general solution: $x + y^3 = 3x^3 + Cx^{-1}$

This simplifies to $y = \sqrt[3]{3x^3 - x + Cx^{-1}}$

Plugging in the initial condition $y(2) = 0$ gives: $0 = \sqrt[3]{24 - 2 + \dfrac{C}{2}} \Rightarrow C = -44$

Therefore, our unique solution is $y = \sqrt[3]{3x^3 - x - 44x^{-1}}$

Exercise 1.6.4*c*

We use the substitution $u = x^2 + y^2 \Rightarrow u' = 2x + 2yy' \implies yy' = \dfrac{1}{2}u' - x$

The DE becomes:

$$\frac{1}{2}u' - x = \sqrt{u} - x \implies \frac{du}{dx} = 2\sqrt{u}$$

This is separable:

$$\int \frac{1}{2}u^{-\frac{1}{2}}\,du = \int dx \implies u^{\frac{1}{2}} = x + C$$

Back-substituting gives our general solution: $\sqrt{x^2+y^2} = x + C$

Plugging in the initial condition $y(3) = 4$ gives: $\sqrt{3^2+4^2} = 3 + C \Rightarrow C = 2$

Therefore, our unique solution is $\sqrt{x^2+y^2} = x + 2$

Exercise $1.6.4d$

We use the substitution $u = \sin(x+y) \Rightarrow u' = \cos(x+y)(1+y')$

The DE becomes:

$$u' + \sin(x)u = 2xe^{\cos(x)}$$

This is linear, so we use $mu(x) = e^{\int \sin(x)\,dx} = e^{-\cos(x)}$

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{e^{-\cos(x)}}\int e^{-\cos(x)}\cdot 2xe^{\cos(x)}\,dx = e^{\cos(x)}\int 2x\,dx = e^{\cos(x)}\left[x^2+C\right]$$

Back-substituting gives our general solution: $\sin(x+y) = e^{\cos(x)}\left(x^2+C\right)$

Plugging in the initial condition $y\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$ gives: $\sin\left(\frac{\pi}{2}+\frac{3\pi}{2}\right) = e^{-\cos\left(\frac{\pi}{2}\right)}\left(\left(\frac{\pi}{2}\right)^2+C\right) \Rightarrow C = -\frac{\pi^2}{4}$

Therefore, our unique solution is $\sin(x+y) = e^{\cos(x)}\left(x^2-\frac{\pi^2}{4}\right)$

Exercise $1.6.4e$

We rewrite the DE: $(6(x^4+y^4)^2+2x(x^4+y^4))\left(4x^3+4y^3\cdot\frac{dy}{dx}\right)+((x^4+y^4)^2+12x^2) = 0$

We use the substitution $u = x^4+y^4 \Rightarrow \frac{du}{dx} = 4x^3+4y^3\cdot\frac{dy}{dx}$

The DE becomes:

$$(6u^2+2ux)\cdot\frac{du}{dx}+(u^2+12x^2) = 0$$

Notice $\frac{\partial}{\partial x}(6u^2+2ux) = 2u$ and $\frac{\partial}{\partial u}(u^2+12x^2) = 2u$, so this is exact:

$$\int (6u^2+2ux)\,du = 2u^3+u^2x+C(x)$$
$$\int (u^2+12x^2)\,dx = u^2x+4x^3+D(u)$$

Using $C(x) = 4x^3$ and $D(u) = 2u^3$ gives us solution $u^2x+2u^3+4x^3 = K$

Back-substituting gives our general solution: $2(x^4+y^4)^3+(x^4+y^4)^2x+4x^3 = K$

Plugging in the initial condition $y(1) = 1$ gives: $2\cdot 2^3+2^2\cdot 1+4 = K \Rightarrow K = 24$

Therefore, our unique solution is $2(x^4+y^4)^3+(x^4+y^4)^2x+4x^3 = 24$

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2: HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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2.2: Constant Coefficient LDEs

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2.1: Independent Solutions

Learning Objectives

- Understand the fundamental theorem governing the number of linearly independent solutions to a homogeneous linear ODE.
- Construct the general solution for both homogeneous and nonhomogeneous equations from their fundamental components.
- Analyze the differences in solution sets for Initial Value Problems (IVPs) and Boundary Value Problems (BVPs).

We begin this chapter by building a theoretical framework for understanding the solutions to higher-order equations. This subchapter proves that the solution space is a vector space whose dimension equals the order of the equation and explores the different solution sets for initial value versus boundary value problems.

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2.1: Higher Order LDE Solution Sets

A linear differential equation of order n is one that can be written in the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x)$$

Here each of $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ are continuous functions on some common interval and $a_n(x)$ is not the constant zero function on this interval.

Just like for first order LDEs, when $g(x)$ is the constant zero function we say this LDE is *homogeneous*, and in any other case we say it is *non-homogeneous*.

We first explore the shape of solutions to homogeneous LDEs.

Theorem 2.1.1: Superposition Principle

Assume $y_1, y_2, \ldots, y_m$ are each solutions to the homogeneous LDE

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

Then the function $y(x) = \sum_{k=1}^{m} c_k y_k(x)$ will also be a solution for any choice of constants $c_1, \ldots, c_m^{\text{int}}\mathbb{R}$

The proof of this theorem is a straightforward calculation, since each derivative $y^{(i)}(x) = \sum_{k=1}^{m} y_k^{(i)}(x)$:

$$\begin{aligned} &a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) \\ &\quad = a_n(x)\cdot\left(\sum_{k=1}^{m} c_k y_k^{(n)}(x)\right) + \cdots + a_1(x)\cdot\left(\sum_{k=1}^{m} c_k y_k'(x)\right) + a_0(x)\cdot\left(\sum_{k=1}^{m} c_k y_k(x)\right) \\ &\quad = \sum_{k=1}^{m} c_k\left(a_n(x)y_k^{(n)}(x) + \cdots a_1(x)y_k'(x) + a_0(x)y_k(x)\right) \\ &\quad = \sum_{k=1}^{m} c_k \cdot 0 = 0 \end{aligned}$$

This means that given any number of solutions to a homogeneous linear differential equation (of any order), we can add scalar multiples of them together to create another solution. We will prove shortly that a homogeneous LDE of order n has an n-parameter family of solutions. This sounds like great news, but it means we have to be careful when describing these solutions. For example, it is easy to see that

$$y_1 = \sin(x), y_2(x) = \cos(x), \text{ and } y_3(x) = 2\sin(x) - \cos(x) \text{ all satisfy the LDE } y''' + y' = 0$$

This tells us that all functions of the form $y = A\sin(x) + B\cos(x) + C(2\sin(x) - \cos(x))$ satisfy $y''' + y' = 0$, which is a three parameter family of solutions. However, this family of solutions can be re-written as $y = (A + 2C)\sin(x) + (B_C)\cos(x) = \hat{A}\sin(x) + \hat{B}\cos(x)$, which is a two parameter family of solutions. This is because one of the solutions we found, y_3, can be made out of the other two: $y_3 = 2y_1 - y_2$. Just like vectors in linear algebra, we say these solutions are *linearly dependent*.

Definition: Linearly Dependent Solutions

We say a set of functions $\{f_1(x), \ldots, f_n(x)\}$ is ***linearly dependent*** on an interval if there exists some constants $c_1, \ldots, c_n$ which aren't all zero such that

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0$$

for all x in that interval.

In this case, at least one of the functions can be written as a combination of the others. Assume $c_j \neq 0$, then we can rewrite the combination above:

$$-c_j f_j(x) = c_1 f_1(x) + \cdots + c_{j-1} f_{j-1}(x) + c_{j+1} f_{j+1}(x) + \cdots + c_n f_n(x)$$

Then we can divide both sides by $-c_j$ (since it is non-zero):

$$f_j(x) = \left(-\frac{c_1}{c_j}\right) f_1(x) + \cdots + \left(-\frac{c_{j-1}}{c_j}\right) f_{j-1}(x) + \left(-\frac{c_{j+1}}{c_j}\right) f_{j+1}(x) + \cdots + \left(-\frac{c_n}{c_j}\right) f_n(x)$$

If we imagine these n functions are all solutions to a common homogeneous LDE, then any combination of them can be written in terms of these $n-1$ functions instead.

Alternatively, if the set of functions is *not* linearly dependent, we say it is **linearly independent**. We borrow the same tools used in linear algebra to determine if our solutions are linearly independent or dependent: determinants.

Definition: Wronskian

The ***Wronskian*** of a set of functions $\{f_1(x), f_2(x), \ldots f_n(x)\}$ with at least $n-1$ derivatives on some common interval is the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

The set of functions is linearly independent if the Wronskian is non-zero for at least one point in that interval.

Example 2.1.1

Find the Wronskian of each set of functions below.

(a) $\{x^2, x^3\}$

(b) $\{e^{2x}, e^{3x}, e^{4x}\}$

(c) $\{x\sin^2(x), x\cos^2(x), 2x\}$

Solution

(a) $f_1(x) = x^2$ and $f_2(x) = x^3$ are differentiable infinitely many times on any interval:

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^2 \cdot 3x^2 - x^3 \cdot 2x = 3x^4 - 2x^4 = \boxed{x^4}$$

Since x^4 is non-zero for some x in any interval, we know f_1 and f_2 are linearly independent.

(b) $f_1(x) = e^{2x}$, $f_2(x) = e^{3x}$, and $f_3(x) = e^{4x}$ are differentiable infinitely many times on any interval:

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} & e^{4x} \\ 2e^{2x} & 3e^{3x} & 4e^{4x} \\ 4e^{2x} & 9e^{3x} & 16e^{4x} \end{vmatrix}$$

Using the method of Sarrus, this determinant expands to:

$$e^{2x}\cdot 3e^{3x}\cdot 16e^{4x}+e^{3x}\cdot 4e^{4x}\cdot 4e^{2x}+e^{4x}\cdot 2e^{2x}\cdot 9e^{3x}-4e^{2x}\cdot 3e^{3x}\cdot e^{4x}-9e^{3x}\cdot 4e^{4x}\cdot e^{2x}-16e^{4x}\cdot 2e^{2x}\cdot e^{3x}=\boxed{2e^{9x}}$$

Since $2e^{9x}$ is non-zero for any x in any interval, we know f_1, f_2, and f_3 are linearly independent.

(c) $f_1(x)=x\sin^2(x)$, $f_2(x)=x\cos^2(x)$, and $f_3(x)=2x$ are differentiable infinitely many times on any interval:

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}=\begin{vmatrix} x\sin^2(x) & x\cos^2(x) & 2x \\ x\sin(2x)+\sin^2(x) & -x\sin(2x)+\cos^2(x) & 2 \\ 2x\cos(2x)+2\sin(2x) & -2x\cos(2x)-2\sin(2x) & 0 \end{vmatrix}$$

Using the method of Sarrus, this determinant expands to:

$$x\sin^2(x)\cdot\big(-x\sin(2x)+\cos^2(x)\big)\cdot 0+x\cos^2(x)\cdot 2\cdot(2x\cos(2x)+2\sin(2x))+2x\cdot\big(x\sin(2x)+\sin^2(x)\big)\cdot(-2x\cos(2x)-2\sin(2x))$$

Since this simplifies down to the constant $\boxed{0}$ on any interval, we know f_1, f_2, and f_3 are linearly dependent.

This means that we can make one of the three functions out of the other two. One such way is:

$$2\cdot\big(x\sin^2(x)\big)+2\cdot\big(x\cos^2(x)\big)=2x$$

It is important to pay attention to the interval on which you are comparing these functions. Consider the functions $f_1(x)=x$ and $f_2(x)=|x|$.

If we compare these two on the interval $I=(0,\infty)$, they are both differentiable with $f_1(x)=1$ and $f_2(x)=1$, so their Wronskian is:

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}=\begin{vmatrix} x & |x| \\ 1 & 1 \end{vmatrix}=x\cdot 1-|x|\cdot 1=x-|x|$$

Since x is always positive in our interval I, we have $x-|x|=0$ for all x. These two functions are linearly dependent. That should make sense, since $x=|x|$ for all x in the interval, so they are the same function.

If we instead consider the interval $I=(-\infty,0)$ they are both differentiable with $f_1(x)=1$ and $f_2(x)=-1$, so their Wronskian is:

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}=\begin{vmatrix} x & |x| \\ 1 & -1 \end{vmatrix}=x\cdot(-1)-|x|\cdot 1=-x-|x|$$

Since x is always negative in this interval, we again have $-x-|x|=0$ for all x. These two functions are linearly dependent. That again makes sense, since $x=-|x|$ for all x in the interval, so they are multiples of the same function.

But we know that x and $|x|$ are not multiples of each other in general, why is this happening? Since $y=|x|$ is not differentiable at $x=0$, the Wronskian cannot compare them on any interval containing 0. This is why we cannot "detect" the linear independence of these two functions.

Theorem 2.1.2: Existence and Uniqueness

Let $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x), g(x)$ be continuous functions on some common interval and $a_n(x)\neq 0$ for all x on this interval. Then the IVP of order n defined by $y(x_0)=b_0, y'(x_0)=b_1, \dots, y^{(n-1)}(x_0)=b_{n-1}$ and

$$a_n(x)y^{(n)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g(x)$$

has a unique solution on that interval.

This theorem is the generalized form of Picard's Existence and Uniqueness Theorem we discussed for first order LDEs. It is proved by re-writing this equation as a first order LDE for a vector based function and applying Picard's Theorem to it. The most important result of this theorem though is to confirm that we should find exactly n linearly independent solutions to the homogeneous LDE

of order n.

Assume $y_1(x), y_2(x), \dots, y_n(x)$ are each solutions to the homogeneous LDE of order n satisfying different initial conditions:

$$\begin{aligned} y_1(x_0) = 1, y_1'(x_0) = 0, \dots, y_1^{(n-1)}(x_0) = 0 \\ y_2(x_0) = 0, y_2'(x_0) = 1, \dots, y_2^{(n-1)}(x_0) = 0 \\ \vdots \\ y_n(x_0) = 0, y_n'(x_0) = 0, \dots, y_n^{(n-1)}(x_0) = 1 \end{aligned}$$

Then these n solutions are clearly linearly independent, since their Wronskian at x_0 is:

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1$$

This shows there are *at least* n linearly independent solutions to the homogeneous LDE of order n. To prove there aren't any others, consider that there must exist some unique solution with initial conditions $y(x_0) = b_0, y'(x_0) = b_1, \dots, y^{(n-1)}(x_0) = b_{n-1}$.

The function $f(x) = b_0 y_1(x) + b_1 y_2(x) + \cdots + b_{n-1} y_n(x)$ satisfies all of these conditions. That means that *any initial conditions* yield a solution that is some combination of these $y_1, \dots, y_n$. Therefore, there cannot be any other linearly independent solutions to this LDE.

Fact

The homogeneous linear differential equation of order n has exactly n linearly independent solutions.

Now that we have fully explored the homogeneous case, it is easy to discuss the non-homogeneous LDEs of order n.

Rule: Solution Set to n^{th} order Linear Differential Equations

If y_p is a particular solution to the non-homogeneous LDE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x)$$

and $y_1, \dots, y_n$ are the linearly independent solutions to the *homogeneous* LDE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0$$

then the general family of solutions to the non-homogeneous LDE must be of the form $y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$

This is proven just like it was in the first order case: If $f(x)$ and $g(x)$ are each particular solutions, then their difference $y(x) = f(x) - g(x)$ must satisfy the homogeneous LDE.

Illustration 2.1.1

Consider the functions $f(x) = e^x + 3e^{-x}, g(x) = e^x - e^{-x}$, and $h(x) = e^x + 4$ and the second order LDE

$$y'' + y' = 2e^x$$

(a) Verify that $f(x), g(x)$ and $h(x)$ are each solutions to the LDE.

(b) Find the solution set to the homogeneous LDE $y'' + y' = 0$

(c) Find the general solutions to $y'' + y' = 2e^x$

(d) Find the unique solution to the IVP $y'' + y' = 2e^x$ with $y(0) = 4, y'(0) = 3$

(a) We show function is a solution separately.

$$f(x) = e^x + 3e^{-x} \implies f'(x) = e^x - 3e^{-x} \implies f''(x) = e^x + 3e^{-x}$$
$$f'' + f' = (e^x + 3e^{-x}) + (e^x - 3e^{-x}) = 2e^x$$
$$g(x) = e^x - e^{-x} \implies g'(x) = e^x + e^{-x} \implies g''(x) = e^x - e^{-x}$$
$$g'' + g' = (e^x - e^{-x}) + (e^x + e^{-x}) = 2e^x$$
$$h(x) = e^x + 4 \implies h'(x) = e^x \implies h''(x) = e^x$$
$$h'' + h' = (e^x) + (e^x) = 2e^x$$

This shows that all three of these functions are particular solutions to the LDE.

(b) Since the difference between any two particular solutions is a homogeneous solution, we can see that

$$y_1(x) = f(x) - g(x) = 4e^{-x}, \quad \text{and} \quad y_2(x) = f(x) - h(x) = 3e^{-x} - 4, \quad \text{and} \quad y_3(x) = g(x) - h(x) = -e^{-x} - 4$$

are each solutions to the homogeneous LDE $y'' + y' = 0$.

These three are clearly linearly dependent (try taking their Wronskian, it will be the constant zero function), but any two of them are linearly independent.

Thus, we can find all possible solutions to $y'' + y' = 0$ in the form $y(x) = Ay_1(x) + By_2(x) = A \cdot 4e^{-x} + B(3e^{-x} - 4) = (4A + 3B)e^{-x} - 4B$

Renaming these constants $C = 4A + 3B$ and $D = -4B$, this looks nicer as $y(x) = Ce^{-x} + D$

(c) Picking any of our three particular solutions and adding the answer from part b will give us the general solution to the LDE:

$$y(x) = f(x) + Ce^{-x} + D = e^x + 3e^{-x} + Ce^{-x} + D = e^x + (C+3)e^{-x} + D$$
$$y(x) = g(x) + Ce^{-x} + D = e^x - e^{-x} + Ce^{-x} + D = e^x + (C-1)e^{-x} + D$$
$$y(x) = h(x) + Ce^{-x} + D = e^x + 4 + Ce^{-x} + D = e^x + Ce^{-x} + (D+4)$$

These all have the same form: $y = e^x + \hat{C}e^{-x} + \hat{D}$

(d) To find the unique solution to the IVP, we simply take our general solution from part c and solve for the appropriate constants.

$$y(x) = e^x + \hat{C}e^{-x} + \hat{D} \implies y(0) = 1 + \hat{C} \cdot 1 + \hat{D} \implies 4 = 1 + \hat{C} + \hat{D}$$
$$y'(x) = e^x - \hat{C}e^{-x} \implies y'(0) = 1 - \hat{C} \cdot 1 \implies 3 = 1 - \hat{C}$$

Solving $\begin{cases} 4 = 1 + \hat{C} + \hat{D} \\ 3 = 1 - \hat{C} \end{cases} \implies C = -2, \ D = 5$

Therefore, the unique solution is $y(x) = e^x - 2e^{-x} + 5$

All of the things here should resemble the statements we made about first order LDEs because they are the generalization of the statements made about first order linear differential equations and initial value problems. However, the introduction of more parameters allows for us to attempt to solve them with slightly different information. An initial value problem lists out all of the values of the function and its derivative at a single point, but we could alternatively describe a function that goes through multiple

different points in its domain. These questions are fundamentally different than IVPs, and called *Boundary Value Problems*.

Definition: Boundary Value Problem

Any LDE of order n along with n conditions specified at two or more different points is called a ***Boundary Value Problem***.

Unlike IVPs, boundary value problems do not necessarily have one unique solution. This is because IVPs specify the initial "state" and "velocity" of a system, leading to a unique trajectory, BVPs specify conditions at the boundaries, which may or may not be compatible with the system's dynamics.

For example, consider the second order LDE $y'' + y = 0$ which is satisfied by the general solution $y = A\cos(x) + B\sin(x)$

If we use the conditions $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 1$, we get:

$$y(0) = A \cdot 1 + B \cdot 0 \implies 0 = A \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = A \cdot 0 + B \cdot 1 \implies 1 = B$$

Therefore, there is a unique solution $y(x) = \sin(x)$

If we instead used the conditions $y(0) = 0$ and $y(\pi) = 1$, we get:

$$y(0) = A \cdot 1 + B \cdot 0 \implies 0 = A \quad \text{and} \quad y(\pi) = A \cdot (-1) + B \cdot 0 \implies 1 = -A$$

It is impossible to find any A satisfying both conditions, so there is no possible solution satisfying these conditions.

Lastly, if we used the conditions $y(0) = 0$ and $y(\pi) = 0$, we get:

$$y(0) = A \cdot 1 + B \cdot 0 \implies 0 = A \quad \text{and} \quad y(\pi) = A \cdot (-1) + B \cdot 0 \implies 0 = -A$$

This clearly gives $A = 0$, but there is no restriction on B, so there are infinitely many solutions satisfying these conditions, which all look like $y(x) = B\sin(x)$

This section has established the foundational theory for higher-order linear differential equations. We defined the crucial concepts of linear dependence and independence using the Wronskian, and proved the existence of exactly n linearly independent solutions for an nth-order homogeneous equation. The general solution to a non-homogeneous equation was shown to be the sum of a particular solution and the homogeneous solution set. Finally, we distinguished between initial value problems, which have a unique solution, and boundary value problems, where solutions may be unique, non-existent, or infinite in number. This theoretical framework is essential for solving linear differential equations of any order.

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2.1: Practice Set

Exercise 2.1.1a

If $y = e^x$ and $y = x$ are both solutions to the DE $y'' + p(x)y' + q(x)y = 0$, find a solution to the IVP $y'' + p(x)y' + q(x)y = 0$ with $y(0) = 7,\ y'(0) = 4$

Answer available at the end of this section.

Exercise 2.1.1b

If $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$ are both solutions to the DE $x^2y'' + axy' + by = 0$, find a solution to the BVP $x^2y'' + axy' + by = 0$ with $y(1) = 3,\ y(4) = 0$

Answer available at the end of this section.

Exercise 2.1.1c

If $y = Ax + B$ is the general solution to the homogeneous DE $y'' = 0$, find the solution to the IVP $y'' = -4\cos(2x)$ with $y(0) = 2$ and $y'(0) = 2$

Answer available at the end of this section.

Exercise 2.1.1d

If $y = A\sin(2x) + B\cos(2x)$ is the general solution to the homogeneous DE $y'' + 4y = 0$, find the solution to the BVP $y'' + 4y = 4x$ with $y(\pi) = 0$ and $y(2\pi) = 0$

Answer available at the end of this section.

Exercise 2.1.2

If $y = p(x)e^x$ satisfies $y''' + \alpha(x)y'' + \beta(x)y' + \gamma(x)y = 0$, for *every quadratic* $p(x)$, solve the IVP $y''' + \alpha(x)y'' + \beta(x)y' + \gamma(x)y = \gamma(x)$ with $y(0) = 1,\ y'(0) = 0$, and $y''(0) = 2$

Answer available at the end of this section.

Exercise 2.1.3a

Determine if the set $\{x, x^3\}$ is linearly independent or dependent on the interval $(-\infty, \infty)$

Answer available at the end of this section.

Exercise 2.1.3b

Determine if the set $\{3e^x, 4e^x\}$ is linearly independent or dependent on the interval $(-\infty, \infty)$

Answer available at the end of this section.

Exercise $2.1.3c$

Determine if the set $\{\tan(x), \sec(x)\}$ is linearly independent or dependent on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Answer available at the end of this section.

Exercise $2.1.3d$

Determine if the set $\{x^2, x|x|\}$ is linearly independent or dependent on the interval $[-1, 1]$

Answer available at the end of this section.

Exercise $2.1.3e$

Determine if the set $\{e^x, (2+3x)e^x, (4+5x)e^x\}$ is linearly independent or dependent on the interval $[-3, 5]$

Answer available at the end of this section.

Exercise $2.1.3f$

Determine if the set $\{e^x, xe^x, x^2e^x\}$ is linearly independent or dependent on the interval $(-\infty, \infty)$

Answer available at the end of this section.

Exercise $2.1.3g$

Determine if the set $\{\ln(x), \ln(x^2), \ln(x^3)\}$ is linearly independent or dependent on the interval $(1, \infty)$

Answer available at the end of this section.

Exercise $2.1.4a$

Let $f(x), g(x)$, and $h(x)$ be three linearly independent functions. Is it possible for f, g, and h to all be solutions to a homogeneous second order linear differential equation?

Answer available at the end of this section.

Exercise $2.1.4b$

Let $f(x), g(x)$, and $h(x)$ be three linearly independent functions. Is it possible for f, g, and h to all be solutions to a second order linear differential equation?

Answer available at the end of this section.

Answers

Exercise 2.1.1*a*

We have two homogeneous solutions, so we know all solutions are of the form $y(x) = Ae^x + Bx$

Plugging in $y(0) = 7$ gives $7 = A \cdot 1 + B \cdot 0 \implies A = 7$

We can differentiate to get $y'(x) = Ae^x + B$

Plugging in $y'(0) = 4$ gives $4 = A \cdot 1 + B \implies B = 4 - A = -3$

Therefore, the unique solution to this IVP is $y = 7e^x - 3x$

Exercise 2.1.1*b*

We have two homogeneous solutions, so we know all solutions are of the form $y(x) = A\sqrt{x} + \dfrac{B}{\sqrt{x}}$

Plugging in $y(1) = 3$ gives $3 = A \cdot 1 + \dfrac{B}{1}$

Plugging in $y(4) = 0$ gives $0 = A \cdot 2 + \dfrac{B}{2}$

Together these form a linear system of equations:

$$\begin{cases} A + B = 3 \\ 2A + \frac{B}{2} = 0 \end{cases} \longrightarrow \begin{cases} A = -1 \\ B = 4 \end{cases}$$

Therefore, the unique solution to this IVP is $y = -\sqrt{x} + \dfrac{4}{\sqrt{x}}$

Exercise 2.1.1*c*

We have the general form of the homogeneous solutions: $y_h(x) = Ax + B$

We must find a particular solution to $y'' = -4\cos(2x)$

Hopefully one can see that $y_p(x) = \cos(2x)$ is one such solution.

Therefore, the general solutions are of the form $y(x) = \cos(2x) + Ax + B$

Plugging in $y(0) = 2$ gives $2 = 1 + A \cdot 0 + B \implies B = 1$

We can differentiate to get $y'(x) = -2\sin(2x) + A$

Plugging in $y'(0) = 2$ gives $2 = 0 + A \implies A = 2$

Therefore, the unique solution to this IVP is $y = \cos(2x) + 2x + 1$

Exercise 2.1.1*d*

We have the general form of the homogeneous solutions: $y_h(x) = A\sin(2x) + B\cos(2x)$

We must find a particular solution to $y'' + 4y = 4x$

Hopefully one can see that $y_p(x) = x$ is one such solution.

Therefore, the general solutions are of the form $y(x) = x + A\sin(2x) + B\cos(2x)$

Plugging in $y(\pi) = 0$ gives $0 = \pi + A \cdot 0 + B \cdot 1 \implies B = -\pi$

Plugging in $y(2\pi) = 0$ gives $0 = 2\pi + A \cdot 0 + B \cdot 1 \implies B = -2\pi$

These two equations cannot both be true at the same time. There is no solution to this BVP.

Exercise 2.1.2

We have the general form of the homogeneous solutions: $y_h(x) = (Ax^2 + Bx + C)e^x$

We must find a particular solution to $y''' + \alpha(x)y'' + \beta(x)y' + \gamma(x)y = \gamma(x)$

Hopefully one can see that $y_p(x) = 1$ is one such solution.

Therefore, the general solutions are of the form $y(x) = 1 + (Ax^2 + Bx + C)e^x$

Plugging in $y(0) = 1$ gives $1 = 1 + (A \cdot 0 + B \cdot 0 + C) \cdot 1 \implies C = 0$

We can differentiate to get $y'(x) = (Ax^2 + Bx + C + 2Ax + B)e^x$

Plugging in $y'(0) = 0$ gives $0 = (A \cdot 0 + B \cdot 0 + C + 2A \cdot 0 + B) \implies B = 0$

We can differentiate to get $y''(x) = (Ax^2 + Bx + C + 4Ax + 2B + 2A)e^x$

Plugging in $y''(0) = 2$ gives $2 = (A \cdot 0 + B \cdot 0 + C + 4A \cdot 0 + 2B + 2A) \implies A = 1$

Therefore, the unique solution to this IVP is $y = 1 + x^2e^x$

Exercise 2.1.3*a*

We find the Wronskian:

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3$$

Since this is not identically zero, we know that the set is **linearly independent**.

Exercise 2.1.3*b*

We find the Wronskian:

$$W = \begin{vmatrix} 3e^x & 4e^x \\ 3e^x & 4e^x \end{vmatrix} = 12e^{2x} - 12e^{2x} = 0$$

Since this is identically zero, and we can see the functions are scalar multiples of each other, we know that this set is **linearly dependent**.

Exercise 2.1.3c

We find the Wronskian:

$$W = \begin{vmatrix} \tan(x) & \sec(x) \\ \sec^2(x) & \tan(x)\sec(x) \end{vmatrix} = \tan^2(x)\sec(x) - \sec^3(x)$$

Since this is not identically zero, we know that the set is **linearly independent**.

Exercise 2.1.3d

Notice that for $f(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$ we have $f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases} = 2|x|$

We find the Wronskian:

$$W = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2x^2|x| - 2x^2|x| = 0$$

This is identically zero, but the independence or dependence depends on the interval in question:

For any interval containing only non-negative numbers, we know $|x| = x$, so the set of functions would be $\{x^2, x^2\}$, which would be linearly dependent.

For any interval containing only non-positive numbers, we know $|x| = -x$, so the set of functions would be $\{x^2, -x^2\}$, which would be linearly dependent.

But these two functions are *not scalar multiples* of each other on any interval containing both positive and negative numbers, like the one provided ($[-1, 1]$), so they are **linearly independent**.

Notice: This is the famous example showing that the Wronskian being zero does not imply linear dependence, and that independence/dependence is based on the specific interval in question.

Exercise 2.1.3e

We find the Wronskian:

$$W = \begin{vmatrix} e^x & (2+3x)e^x & (4+5x)e^x \\ e^x & (5+3x)e^x & (9+5x)e^x \\ e^x & (8+3x)e^x & (14+5x)e^x \end{vmatrix} = 0$$

This is identically zero, and we can see that $e^x = -\frac{5}{2} \cdot (2+3x)e^x + \frac{3}{2} \cdot (4+5x)e^x$

Therefore, this set is **linearly dependent**.

Exercise 2.1.3f

We find the Wronskian:

$$W = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & (1+x)e^x & (2x+x^2)e^x \\ e^x & (2+x)e^x & (2+4x+x^2)e^x \end{vmatrix} = 2e^{3x}$$

Since this is not identically zero, we know that the set is **linearly independent**.

Exercise 2.1.3g

We find the Wronskian:

$$W = \begin{vmatrix} \ln(x) & \ln(x^2) & \ln(x^3) \\ \frac{1}{x} & \frac{2}{x} & \frac{3}{x} \\ -\frac{1}{x^2} & -\frac{2}{x^2} & -\frac{3}{x^2} \end{vmatrix} = 0$$

This is identically zero, and we can see that $2 \cdot \ln(x) = \ln(x^2)$

Therefore, this set is **linearly dependent**.

Exercise 2.1.4a

We know that a second order homogeneous linear differential equation has general solution $y(x) = Ay_1(x) + By_2(x)$ where $y_1(x)$ and $y_2(x)$ are linearly independent functions.

Thus, $f(x) = A_f y_1(x) + B_f y_2(x)$, $g(x) = A_g y_1(x) + B_g y_2(x)$, and $h(x) = A_h y_1(x) + B_h y_2(x)$

If all three A_f, A_g, A_h are zero, then these are clearly dependent. Thus, at least one of these must be nonzero, similarly at least one of B_f, B_g and B_h is nonzero. Even then, we will show these three cannot be linearly independent, since:

$$A_g f(x) - A_f g(x) = A_g\left(A_f y_1(x) + B_f y_2(x)\right) - A_f\left(A_g y_1(x) + B_g y_2(x)\right) = \left(A_g B_f - A_f B_g\right) y_2(x)$$
$$A_h f(x) - A_f h(x) = A_h\left(A_f y_1(x) + B_f y_2(x)\right) - A_f\left(A_h y_1(x) + B_h y_2(x)\right) = \left(A_h B_f - A_f B_h\right) y_2(x)$$

This shows that $\left(A_h B_f - A_f B_h\right)\left(A_g f(x) - A_f g(x)\right) = \left(A_g B_f - A_f B_g\right)\left(A_h f(x) - A_f h(x)\right)$

Clearly this means one of the functions can be solved in terms of the other two. Which would be impossible if f, g, and h were linearly independent.

Therefore, it is **impossible**.

Exercise 2.1.4b

In Exercise 4a above, we showed that this is impossible for a homogeneous LDE.

But for a nonhomogeneous equation, it is entirely possible. One such example:

$y'' - y = -1$ has solutions $f(x) = 1, g(x) = 1 + e^x, h(x) = 1 + e^{-x}$

These three are clearly linearly independent since

$$W = \begin{vmatrix} 1 & 1+e^x & 1+e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2 \neq 0$$

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2.2: Constant Coefficient and Cauchy Euler LDEs

Learning Objectives

- Solve homogeneous constant coefficient linear ODEs by finding the roots of the characteristic equation.
- Solve homogeneous Cauchy-Euler equations using a characteristic equation derived from an assumed power solution.
- Apply Euler's Formula to express complex roots in terms of real-valued functions.
- Construct general solutions for equations of order 2, 3, and 4.
- Solve Initial Value Problems (IVPs) and Boundary Value Problems (BVPs) for both equation types.

This subchapter provides the primary tools for finding solutions to homogeneous higher-order equations. We develop the characteristic equation method for constant coefficient equations and the analogous power solution for Cauchy-Euler equations, demonstrating how their roots dictate the general solution's form.

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2.2: Constant Coefficient LDEs

One particularly tractable family of linear differential equations is the case where all of the coefficient functions are constants with respect to x:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = g(x)$$

We call this the constant coefficient linear differential equation (CC LDE).

Solving the homogeneous version of these differential equations is easily done by "guessing" the form of the solution to be $y = e^{rx}$:

$$y(x) = e^{rx} \implies y'(x) = re^{rx} \implies y''(x) = r^2 e^{rx} \implies \cdots \implies y^{(k)}(x) = r^k e^{rx} \text{ for all integers } k$$

Plugging these into the homogeneous CC LDE gives us:

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} = 0$$

Multiplying both sides by the nonzero term e^{-rx} simplifies this to:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

This is a polynomial equation in terms of the parameter r.

Definition: Characteristic Polynomial of a CC LDE

For any linear differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = g(x)$$

the ***characteristic equation*** of the differential equation is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Notice that for any value of r that satisfies the characteristic equation, we know that $y(x) = e^{rx}$ is a solution to the corresponding CC LDE.

For a simple example, consider the first order CC LDE

$$y' - 2y = 0$$

The characteristic equation would then be:

$$r - 2 = 0$$

This clearly has only one solution, and that is $r = 2$. This tells us that $y = e^{2x}$ is a solution to the original LDE. Since that LDE is separable, we can confirm this easily by solving that equation using the methods of Chapter 1:

$$\frac{dy}{dx} - 2y = 0 \implies \int \frac{1}{y}\, dy = \int 2\, dx \implies \ln|y| = 2x + C \implies y = Ce^{2x}$$

Of course, the real value of the characteristic equation is that it helps us to solve higher order CC LDEs.

Rule: Solutions to Homogeneous CC LDE

For any homogeneous linear differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0$$

If r is a root of the characteristic equation, then $y = e^{rx}$ is a solution to the CC LDE.

Example 2.2.1

Find the general solution to each homogeneous CC LDE below.

(a) $y'' - 9y = 0$

(b) $2y'' + 3y' = 0$

(c) $y''' - y'' - 4y' + 4y = 0$

Solution

(a) The characteristic equation for this CC LDE is $r^2 - 9 = 0$

This factors to $(r-1)(r+1) = 0$, so we have two distinct solutions: $r_1 = 3$ and $r_2 = -3$

Each of these yields a solution to the original homogeneous LDE: $y_1 = e^{3x}$ and $y_2 = e^{-3x}$

These are linearly independent, since

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

Therefore, we know all solutions to this second order LDE are of the form $y(x) = Ae^{3x} + Be^{-3x}$

(b) The characteristic equation for this CC LDE is $2r^2 + 3r = 0$

This factors to $r(2r+3) = 0$, so we have two distinct solutions: $r_1 = 0$ and $r_2 = -\dfrac{3}{2}$

Each of these yields a solution to the original homogeneous LDE: $y_1 = e^{0x} = 1$ and $y_2 = e^{-\frac{3}{2}x}$

These are linearly independent, since

$$W = \begin{vmatrix} 1 & e^{-\frac{3}{2}x} \\ 0 & -\dfrac{3}{2}e^{-\frac{3}{2}x} \end{vmatrix} = -\frac{3}{2}e^{-\frac{3}{2}x} \neq 0$$

Therefore, we know all solutions to this second order LDE are of the form $y(x) = A + Be^{-\frac{3}{2}x}$

(c) The characteristic equation for this CC LDE is $r^3 - r^2 - 4r + 4 = 0$

This factors to $(r-1)(r-2)(r+2) = 0$, so we have three distinct solutions: $r_1 = 1,\ r_2 = 2$ and $r_3 = -2$

Each of these yields a solution to the original homogeneous LDE: $y_1 = e^{1x} = e^x,\ y_2 = e^{2x}$ and $y_3 = e^{-2x}$

These are linearly independent, since

$$W = \begin{vmatrix} e^x & e^{2x} & e^{-2x} \\ e^x & 2e^{2x} & -2e^{-2x} \\ e^x & 4e^{2x} & 4e^{-2x} \end{vmatrix} = 12e^x \neq 0$$

Therefore, we know all solutions to this third order LDE are of the form $y(x) = Ae^x + Be^{2x} + Ce^{-2x}$

Whenever the characteristic equation of any CC LDE of order n has n distinct roots, they will always yield n linearly independent solutions to the homogeneous version of that LDE. This is because the Wronskian calculation will always look like:

$$W = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} & \cdots & e^{r_n x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & \cdots & r_2 e^{r_n x} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n--1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \cdots & r_n^{n-1} e^{r_n x} \end{vmatrix} = e^{(r_1+r_2+\cdots r_n)x} \cdot \prod_{1 \leq i \leq j \leq n} (r_j - r_i)$$

Since each $r_i \neq r_j$ in the product (which was discovered by Vandermonde) is nonzero, the total Wronskian is nonzero.

This means that *as long as you can factor the characteristic equation into distinct linear terms, you can solve the homogeneous CC LDE.*

Sadly, this will not always go as smoothly as it did in the examples above. Consider the second order LDE $y'' + 2y' - 4y = 0$, which has characteristic equation $r^2 + 2r - 4 = 0$. This quadratic equation does not factor at all, so we use the quadratic formula:

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1} = -1 \pm \sqrt{5}$$

These are still two distinct roots, so we can say the solutions to the LDE are of the form $y = Ae^{\left(-1+\sqrt{5}\right)x} + Be^{\left(-1-\sqrt{5}\right)x}$. These aren't the prettiest solutions, but they are still legitimate.

Consider another possible roadblock: complex roots. The LDE $y'' + y = 0$ has characteristic polynomial $r^2 + 1 = 0$, which has roots $r = \pm i$. Writing our solution in the form $y = Ae^{ix} + Be^{-ix}$ seems mathematically abstract (and inherently complex).

To help with this, we recall Euler's Identity.

Definition: Euler's Identity

$$e^{ix} = \cos(x) + i\sin(x)$$

This identity is often used in precalculus courses when discussing complex numbers and calculations in the complex plane (like taking roots of unity). It is typically justified by manipulating Taylor expansions for each of the involved functions:

$$\begin{aligned} e^{ix} = \sum_{n=0}^{\infty} \frac{1}{n!}(ix)^n &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}(ix)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(ix)^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!}x^{2k} + i\sum_{k=0}^{\infty} \frac{i^{2k}}{(2k+1)!}x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} + i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} \\ &= \cos(x) + i\sin(x) \end{aligned}$$

This identity allows us to rewrite our solution as:

$$\begin{aligned} y &= Ae^{ix} + Be^{-ix} \\ &= A\left(\cos(x) + i\sin(x)\right) + B\left(\cos(-x) + i\sin(-x)\right) \\ &= A\left(\cos(x) + i\sin(x)\right) + B\left(\cos(x) - i\sin(x)\right) \\ &= (A+B)\cos(x) + (Ai - Bi)\sin(x) \\ &= C\cos(x) + D\sin(x) \end{aligned}$$

Here we use $C = A + B$ and $D = Ai - Bi$, which seems problematic, but notice that if A and B are complex conjugates, then we will have C and D both become real numbers. Therefore, this version of our solution will make physical sense.

This calculation occurs anytime you have two complex conjugate roots to your characteristic equation, so we use the following shortcut:

Note

$$C_1 e^{(a+bi)x} + C_2 e^{(a-bi)x} = D_1 e^{ax}\cos(bx) + D_2 e^{ax}\sin(bx) \quad \text{for some constants } D_1, D_2$$

The functions $y_1 = e^{ax}\cos(bx)$ and $y_2 = e^{ax}\sin(bx)$ are always linearly independent when $b \neq 0$:

$$W = \begin{vmatrix} e^{ax}\cos(bx) & e^{ax}\sin(bx) \\ e^{ax}(a\cos(bx) - b\sin(bx)) & e^{ax}(a\sin(bx) + b\cos(bx)) \end{vmatrix} = be^{2ax}$$

This means that even when we have complex roots to a characteristic equation, we can find the general solution to a homogeneous CC LDE.

Rule: Complex Roots of Characteristic Polynomials in Homogeneous CC LDEs

For any homogeneous linear differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0$$

If $r = a \pm bi$ are complex roots of the characteristic equation, then $y = e^{ax}\cos(bx)$ and $y = e^{ax}\sin(bx)$ are two linearly independent solutions to the CC LDE.

Example 2.2.2

Find the general solution to each homogeneous CC LDE below.

(a) $y'' + 9y = 0$

(b) $2y''' + 2y'' + 5y' = 0$

Solution

(a) The characteristic equation for this CC LDE is $r^2 + 9 = 0$

This has roots $r = \pm 3i$, so our general solution is $y = Ae^{0x}\cos(3x) + Be^{0x}\sin(3x)$

This simplifies to $y = A\cos(3x) + B\sin(3x)$

(b) The characteristic equation for this CC LDE is $2r^3 + 2r^2 + 5r = 0$

This factors to $r(2r^2 + 2r + 5) = 0$, which has roots $r = 0$ and $r = -\frac{1}{2} \pm \frac{3i}{2}$

Thus, our general solution is $y = A + Be^{-\frac{1}{2}x}\cos\left(\frac{3}{2}x\right) + Ce^{-\frac{1}{2}x}\sin\left(\frac{3}{2}x\right)$

As before, we can include initial conditions to these types of questions to find unique solutions, or make them into boundary value problems for excitement.

Example 2.2.3

Find the solution(s) to each IVP or BVP below.

(a) $y'' + \pi^2 y = 0$ with $y(1) = 3,\ y(3) = 1$

(b) $y''' - 4y'' + 6y' - 4y = 0$ with $y(0) = 1,\ y'(0) = 2,\ y''(0) = 2$

(c) $y^{(4)} + 5y'' + 4y = 0$ with $y(0) = 0,\ y(\pi) = 0,\ y'(0) = 0,\ y'(\pi) = -8$

Solution

(a) The characteristic equation for this CC LDE is $r^2 + \pi^2 = 0$

This has roots $r = \pm\pi i$, so our general solution is $y = A\cos(\pi x) + B\sin(\pi x)$

Plugging in the boundary conditions gives:

$$\begin{aligned} y(1) = A\cdot(-1) + B\cdot 0 &\implies 3 = -A \\ y(3) = A\cdot(-1) + B\cdot 0 &\implies 1 = -A \end{aligned}$$

Clearly $A = -3$ and $A = -1$ cannot both be satisfied at once, so there is no solution to this BVP.

(b) The characteristic equation for this CC LDE is $r^3 - 4r^2 + 6r - 4 = 0$

This factors to $(r-2)(r^2 - 2r + 2) = 0$, which has roots $r = 2$ and $r = 1 \pm i$

Thus, our general solution is $y = Ae^{2x} + Be^x\cos(x) + Ce^x\sin(x)$

This has derivatives $y' = 2Ae^{2x} + Be^x(\cos(x) - \sin(x)) + Ce^x(\sin(x) + \cos(x))$ and $y'' = 4Ae^{2x} - 2Be^x\sin(x) + 2Ce^x\cos(x)$

Plugging in the initial conditions gives:

$$\begin{aligned} y(0) = A\cdot 1 + B\cdot 1 + C\cdot 0 &\implies 1 = A + B \\ y'(0) = 2A\cdot 1 + B\cdot 1 + C\cdot 1 &\implies 2 = 2A + B + C \\ y''(0) = 4A\cdot 1 - 2B\cdot 0 + 2C\cdot 1 &\implies 2 = 4A + 2C \end{aligned}$$

Putting these all together we can solve the system:

$$\begin{cases} A + B = 1 \\ 2A + B + C = 2 \\ 4A + 2C = 2 \end{cases} \implies \begin{cases} A = 0 \\ B = 1 \\ C = 1 \end{cases}$$

This tells us the unique solution is $y = e^x\cos(x) + e^x\sin(x)$

(c) The characteristic equation for this CC LDE is $r^4 + 5r^2 + 4 = 0$

This factors to $(r^2 + 4)(r^2 + 1) = 0$, which has roots $r = \pm 2i, \pm i$

Thus, our general solution is $y = A\cos(2x) + B\sin(2x) + C\cos(x) + D\sin(x)$

This has derivative $y' = -2A\sin(2x) + 2B\cos(2x) - C\sin(x) + D\cos(x)$

Plugging in each of our boundary conditions this gives:

$$\begin{aligned} y(0) = A\cdot 1 + B\cdot 0 + C\cdot 1 + D\cdot 0 &\implies 0 = A + C \\ y(\pi) = A\cdot 1 + B\cdot 0 + C\cdot(-1) + D\cdot 0 &\implies 0 = A - C \\ y'(0) = -2A\cdot 0 + 2B\cdot 1 - C\cdot 0 + D\cdot 1 &\implies 0 = 2B + D \\ y'(\pi) = -2A\cdot 0 + 2B\cdot 1 - C\cdot 0 + D\cdot(-1) &\implies -8 = 2B - D \end{aligned}$$

Putting these all together we can solve the system:

$$\begin{cases} A + C = 0 \\ A - C = 0 \\ 2B + D = 0 \\ 2B - D = -8 \end{cases} \Longrightarrow \begin{cases} A = 0 \\ B = -2 \\ C = 0 \\ D = 4 \end{cases}$$

This tells us the unique solution is $y = -2\sin(2x) + 4\sin(x)$

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2.2: Cauchy Euler LDEs

We next consider the case where all of the coefficient functions are power functions whose exponents match the order of the derivatives of y they are multiplied by:

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y(x) = g(x)$$

We call this the Cauchy-Euler linear differential equation (CE LDE).

Solving the homogeneous version of these differential equations is easily done by "guessing" the form of the solution to be $y = x^m$:

$$y(x) = x^m \implies y'(x) = m x^{m-1} \implies y''(x) = m(m-1)x^{m-2} \implies \cdots \implies y^{(k)}(x) = \frac{m!}{(m-k)!} x^{m-k} \text{ for all integers } k$$

Plugging these into the homogeneous CE LDE gives us:

$$a_n x^n \left(\frac{m!}{(m-n)!} x^{m-n} \right) + a_{n-1} x^{n-1} \left(\frac{m!}{(m-(n-1))!} x^{m-(n-1)} \right) + \cdots + a_1 x \left(m x^{m-1} \right) + a_0 x^m = 0$$

This simplifies to:

$$\left(a_n \frac{m!}{(m-n)!} + a_{n-1} \frac{m!}{(m-(n-1))!} + \cdots + a_1 m + a_0 \right) x^m = 0$$

Since we want this to hold on some interval, we can assume x^m is not identically zero, giving us a polynomial equation in terms of the parameter m.

Definition: Characteristic Polynomial (CE LDE)

For any linear differential equation with constant coefficients

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y(x) = g(x)$$

The ***characteristic equation*** of the differential equation is

$$a_n \frac{m!}{(m-n)!} + a_{n-1} \frac{m!}{(m-(n-1))!} + \cdots + a_1 m + a_0 = 0$$

Notice that for any value of m that satisfies the characteristic equation, we know that $y(x) = x^m$ is a solution to the corresponding CE LDE.

Once again for a simple example, consider the first order CE LDE

$$x y' - 3y = 0$$

The characteristic equation would then be:

$$m - 3 = 0$$

This clearly has only one solution, and that is $m = 3$. This tells us that $y = x^3$ is a solution to the original LDE. Since that LDE is separable, we can confirm this easily by solving that equation using the methods of Chapter 1:

$$x \frac{dy}{dx} - 3y = 0 \implies \int \frac{1}{y}\, dy = \int \frac{3}{x}\, dx \implies \ln|y| = 3\ln|x| + C \implies y = Cx^3$$

Once more, the real value of the characteristic equation is that it helps us to solve higher order CE LDEs.

Rule: Solutions to Homogeneous CE LDE

For any homogeneous linear differential equation with constant coefficients

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y(x) = 0$$

If m is a root of the characteristic equation, then $y = x^m$ is a solution to the CE LDE.

Example 2.2.4

Find the general solution to each homogeneous CE LDE below.

(a) $x^2 y'' - 12y = 0$

(b) $2x^2 y'' + 5xy' = 0$

(c) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$

Solution

(a) The characteristic equation for this CE LDE is $m(m-1) - 12 = 0 \implies m^2 - m - 12 = 0$

This factors to $(m-4)(m+3) = 0$, so we have two distinct solutions: $m_1 = 4$ and $m_2 = -3$

Each of these yields a solution to the original homogeneous LDE: $y_1 = x^4$ and $y_2 = x^{-3}$

These are linearly independent, since

$$W = \begin{vmatrix} x^4 & x^{-3} \\ 4x^3 & -3x^{-4} \end{vmatrix} = -7x \neq 0$$

Therefore, we know all solutions to this second order LDE are of the form $y(x) = Ax^4 + Bx^{-3}$

(b) The characteristic equation for this CE LDE is $2m(m-1) + 5m = 0 \implies 2m^2 + 3m = 0$

This factors to $m(2m+3) = 0$, so we have two distinct solutions: $m_1 = 0$ and $m_2 = -\dfrac{3}{2}$

Each of these yields a solution to the original homogeneous LDE: $y_1 = x^0 = 1$ and $y_2 = x^{-\frac{3}{2}}$

These are linearly independent, since

$$W = \begin{vmatrix} 1 & x^{-\frac{3}{2}} \\ 0 & -\dfrac{3}{2}x^{-\frac{5}{2}} \end{vmatrix} = -\frac{3}{2}x^{-\frac{5}{2}} \neq 0$$

Therefore, we know all solutions to this second order LDE are of the form $y(x) = A + Bx^{-\frac{3}{2}}$

(c) The characteristic equation for this CE LDE is $m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0 \implies m^3 - 6m^2 + 11m - 6 = 0$

This factors to $(m-1)(m-2)(m-3) = 0$, so we have three distinct solutions: $m_1 = 1,\ m_2 = 2$ and $m_3 = 3$

Each of these yields a solution to the original homogeneous LDE: $y_1 = x^1 = x,\ y_2 = x^2$ and $y_3 = x^3$

These are linearly independent, since

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^2 \neq 0$$

Therefore, we know all solutions to this third order LDE are of the form $y(x) = Ax + Bx^2 + Cx^3$

Whenever the characteristic equation of any CE LDE of order n has n distinct roots, they will always yield n linearly independent solutions to the homogeneous version of that LDE. Instead of using the Wronskian directly, we can realize this by using the analogous result from our CC LDEs. Consider the substitution $x = e^t$:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot e^t = x\frac{dy}{dx} \\
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(e^t \frac{dy}{dx}\right) = e^t \cdot \frac{dy}{dx} + e^t \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right) = e^t \cdot \frac{dy}{dx} + e^t \cdot e^t \frac{d^2y}{dx^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} \\
\frac{d^3y}{dt^3} &= \frac{d}{dt}\left(e^t \frac{dy}{dx} + e^{2t}\frac{d^2y}{dx^2}\right) = e^t \cdot \frac{dy}{dx} + e^t \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right) + 2e^{2t} \cdot \frac{d^2y}{dx^2} + e^{2t} \cdot \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) \\
&= e^t \cdot \frac{dy}{dx} + e^t \cdot e^t \frac{d^2y}{dx^2} + 2e^{2t} \cdot \frac{d^2y}{dx^2} + e^{2t} \cdot e^t \frac{d^3y}{dx^3} = x\frac{dy}{dx} + 3x^2\frac{d^2y}{dx^2} + x^3\frac{d^3y}{dx^3} \\
&\vdots
\end{aligned}$$

If we continue in this way and solve each expression for $\frac{d^ny}{dx^n}$ in terms of the derivatives with respect to t this becomes:

$$\begin{aligned}
x\frac{dy}{dx} &= \frac{dy}{dt} \\
x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dt^2} - \frac{dy}{dt} \\
x^3\frac{d^3y}{dx^3} &= \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} \\
&\vdots
\end{aligned}$$

Substituting this into a Cauchy-Euler LDE we see:

$$\begin{aligned}
&a_n x^n \frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0 y = b_n\frac{d^ny}{dt^n} + b_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + b_1\frac{dy}{dt} + b_0 y = 0 \\
&\text{where } a_n\frac{m!}{(m-n)!} + a_{n-1}\frac{m!}{(m-(n-1))!} + \cdots + a_1 m + a_0 = b_n m^n + b_{n-1}m^{n-1} + \cdots + b_1 m + b_0
\end{aligned}$$

In other words, the roots of the characteristic equation of a CE LDE with respect to x are exactly the roots of the corresponding CC LDE with respect to $t = \ln(x)$.

This is why our solutions to the CE LDE are $y = e^{mt} = e^{m\cdot\ln(x)} = x^m$. Since we know that distinct roots of a characteristic equation from a CC LDE yield linearly independent solutions, now we know that distinct roots of the characteristic equation from a CE LDE must also yield linearly independent solutions.

We can also take the results of Euler's Identity we used for CC LDEs to determine the solutions that correspond to complex roots for a CE LDE.

$$x^{a+bi} = e^{(a+bi)\ln(x)} = e^{a\ln(x)}\cos(b\ln(x)) + ie^{a\ln(x)}\sin(b\ln(x)) = x^a(\cos(b\ln(x)) + i\sin(b\ln(x)))$$

Therefore, we get an analogous rule about complex roots in CE LDEs:

Rule: Complex Roots of Characteristic Polynomials in Homogeneous CE LDEs

For any homogeneous linear differential equation with constant coefficients

$$a_n x^n y^{(n)}(x) + a_{n-1}x^{n-1}y^{(n-1)}(x) + \cdots + a_1 xy'(x) + a_0 y(x) = 0$$

If $m = a \pm bi$ are complex roots of the characteristic equation, then $y = x^a \cos(b\ln(x))$ and $y = x^a \sin(b\ln(x))$ are two linearly independent solutions to the CE LDE.

It is worth quickly mentioning that our solutions generally apply for $x > 0$ thanks to the logarithmic terms involved.

Example 2.2.5

Find the general solution to each homogeneous CE LDE below.

(a) $x^2y'' - xy' + 5y = 0$

(b) $x^3y''' + 12x^2y'' + 35xy' + 25y = 0$

Solution

(a) The characteristic equation for this CE LDE is $m(m-1) - m + 5 = 0 \implies m^2 - 2m + 5 = 0$

This has roots $m = 1 \pm 2i$, so our general solution is $y = Ax^1 \cos(2\ln(x)) + Bx^1 \sin(2\ln(x))$

This simplifies to $y = Ax\cos(2\ln(x)) + Bx\sin(2\ln(x))$

(b) The characteristic equation for this CE LDE is $m(m-1)(m-2) + 12m(m-1) + 35m + 25 = 0 \implies m^3 + 9m^2 + 25m + 25 = 0$

This factors to $(m+5)(r^2+2r+5) = 0$, which has roots $m = -5$ and $m = -2 \pm i$

Thus, our general solution is $y = Ax^{-5} + Bx^{-2}\cos(\ln(x)) + Cx^{-2}\sin(\ln(x))$

Once again, we can apply initial conditions to these types of problems to find unique solutions, or make them into boundary value problems for excitement.

Example 2.2.6

Find the solution(s) to each IVP or BVP below.

(a) $x^2y'' + xy' - 4y = 0$ with $y(1) = 0,\ y(2) = 15$

(b) $x^3y''' + 3x^2y'' + 17xy' = 0$ with $y(1) = 1,\ y'(1) = 4,\ y''(1) = -4$

(c) $x^3y''' - 3x^2y'' + 3xy' = 0$ with $y(1) = 0,\ y(-1) = 0,\ y'(1) = 0$

Solution

(a) The characteristic equation for this CE LDE is $m(m-1) + m - 4 = 0 \implies m^2 - 4 = 0$

This has roots $r = \pm 2$, so our general solution is $y = Ax^2 + Bx^{-2}$

Plugging in the boundary conditions gives:

$$y(1) = A \cdot 1 + B \cdot 1 \implies 0 = A + B$$
$$y(2) = A \cdot 4 + B \cdot \frac{1}{4} \implies 15 = 4A + \frac{1}{4}B$$

Putting these all together we can solve the system:

$$\begin{cases} A+B=0 \\ 4A+\frac{1}{4}B=15 \end{cases} \implies \begin{cases} A=4 \\ B=-4 \end{cases}$$

This tells us the unique solution is $y=4x^2-4x^{-2}$

(b) The characteristic equation for this CE LDE is $m(m-1)(m-2)+3m(m-1)+17m=0 \implies m^3+16m=0$

This factors to $m(m^2+16)=0$, which has roots $m=0$ and $m=\pm 4i$

Thus, our general solution is $y=A+B\cos(4\ln(x))+C\sin(4\ln(x))$

This has derivative $y'=-\frac{4B}{x}\sin(4\ln(x))+\frac{4C}{x}\cos(4\ln(x))$

The second derivative is $y''=\frac{4B}{x^2}\sin(4\ln(x))-\frac{16B}{x^2}\cos(4\ln(x))-\frac{4C}{x^2}\cos(4\ln(x))-\frac{16C}{x^2}\sin(4\ln(x))$

Plugging in the initial conditions gives:

$$\begin{aligned} y(1)=A+B\cdot 1+C\cdot 0 &\implies 1=A+B \\ y'(1)=-4B\cdot 0+4C\cdot 1 &\implies 4=4C \\ y''(1)=4B\cdot 0-16B\cdot 1-4C\cdot 1-16C\cdot 0 &\implies -4=-16B-4C \end{aligned}$$

Putting these all together we can solve the system:

$$\begin{cases} A+B=1 \\ 4C=4 \\ -16B-4C=-4 \end{cases} \implies \begin{cases} A=1 \\ B=0 \\ C=1 \end{cases}$$

This tells us the unique solution is $y=1+\sin(4\ln(x))$

(c) The characteristic equation for this CE LDE is $m(m-1)(m-2)-3m(m-1)+3m=0 \implies m^3-6m^2+8m=0$

This factors to $m(m-2)(m-4)=0$, which has roots $m=0,2,4$

Thus, our general solution is $y=A+Bx^2+Cx^4$

This has derivative $y'=2Bx+4Cx^3$

Plugging in each of our boundary conditions this gives:

$$\begin{aligned} y(1)=A+B\cdot 1+C\cdot 1 &\implies 0=A+B+C \\ y(-1)=A+B\cdot 1+C\cdot 1 &\implies 0=A+B+C \\ y'(1)=2B\cdot 1+4C\cdot 1 &\implies 0=2B+4C \end{aligned}$$

Putting these all together we can solve the system:

$$\begin{cases} A+B+C=0 \\ A+B+C=0 \\ 2B+4C=0 \end{cases} \implies \begin{cases} A=C \\ B=-2C \\ C=\text{anything} \end{cases}$$

This tells us there are infinitely many solutions of the form $y=-C+2Cx^2+Cx^4$, which simplifies to $y=C(x^2-1)^2$

In this subchapter, we have focused on two fundamental and highly solvable classes of higher-order linear differential equations: constant coefficient and Cauchy-Euler equations. By assuming an exponential form for the solution, we reduced each differential equation to an algebraic characteristic equation. Solving this polynomial allowed us to construct the complete set of linearly independent solutions for the homogeneous case. This method provides a direct and powerful technique for finding general solutions, forming the cornerstone for analyzing the behavior of linear systems described by these specific, yet widely applicable, equation types.

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2.2: Practice Set

Exercise 2.2.1a

Find the general solution to $2y'' + y' - 3y = 0$

Answer available at the end of this section.

Exercise 2.2.1b

Find the general solution to $x^2y'' - 12y = 0$

Answer available at the end of this section.

Exercise 2.2.1c

Find the general solution to $6y''' - 5y'' - 4y' = 0$

Answer available at the end of this section.

Exercise 2.2.1d

Find the general solution to $2\frac{d^2y}{dw^2} + 18y = 0$

Answer available at the end of this section.

Exercise 2.2.1e

Find the general solution to $4x^3y''' + 8x^2y'' + 2xy' = 0$

Answer available at the end of this section.

Exercise 2.2.1f

Find the general solution to $y^{(5)}(x) - y'(x) = 0$

Answer available at the end of this section.

Exercise 2.2.1g

Find the general solution to $4u^2p''(u) - 3p(u) = 0$

Answer available at the end of this section.

Exercise $2.2.1h$

Find the general solution to $x^2y'' + xy' + 4y = 0$

Answer available at the end of this section.

Exercise $2.2.1i$

Find the general solution to $2y''' + 13y'' + 13y' - 10y = 0$

Answer available at the end of this section.

Exercise $2.2.1j$

Find the general solution to $t^4y^{(4)} + 4t^3y^{(3)}(t) = 0$

Answer available at the end of this section.

Exercise $2.2.1k$

Find the general solution to $\frac{d^4y}{dz^4} + 2\frac{d^3y}{dz^3} - 2\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = 0$

Answer available at the end of this section.

Exercise $2.2.2a$

Find the solution to $s''(t) = 4s'(t) - 5s(t)$ with $s(0) = 1,\ s'(0) = 4$

Answer available at the end of this section.

Exercise $2.2.2b$

Find the solution to $w^3y''' - 2w^2y'' + 2wy' = 0$ with $y(-1) = 1,\ y'(-1) = 3,\ y''(-1) = 0$

Answer available at the end of this section.

Exercise $2.2.2c$

Find the solution to $y'' + y = 0$ with $y(\pi) = 3,\ y'(\pi) = -2$

Answer available at the end of this section.

Exercise $2.2.2d$

Find the solution to $p^{(4)}(t) - 20p''(t) + 64p(t) = 0$ with $p(0) = 0,\ p(\ln(2)) = 0,\ p(\ln(3)) = 0,\ p(\ln(4)) = 0$

Answer available at the end of this section.

Exercise 2.2.2e

Find the solution to $x^2y'' - 12y = 0$ with $y(1) = 0,\ y'(1) = 14$

Answer available at the end of this section.

Exercise 2.2.2f

Find the solution to $y'''(x) - 4y'(x) = 0$ with $y(0) = 9,\ y(\ln(2)) = 0,\ y(-\ln(2)) = 0$

Answer available at the end of this section.

Exercise 2.2.2g

Find the solution to $x^2y'' - 3xy' = 0$ with $y(1) = 2,\ y(-1) = 2$

Answer available at the end of this section.

Answers

Exercise 2.2.1a

This is a CC LDE, so we use $y = e^{rx}$:

Thus, the characteristic equation is $2r^2 + r - 3 = 0 \implies (r-1)(2r+3) = 0$

The roots are $r = 1, -\frac{3}{2}$

This gives the general solution $y(x) = Ae^{x} + Be^{-\frac{3}{2}x}$

Exercise 2.2.1b

This is a CE LDE, so we use $y = x^m$:

Thus, the characteristic equation is $m(m-1) - 12 = 0 \implies (m+3)(m-4) = 0$

The roots are $m = -3, 4$

This gives the general solution $y(x) = Ax^{-3} + Bx^{4}$

Exercise 2.2.1c

This is a CC LDE, so we use $y = e^{rx}$:

Thus, the characteristic equation is $6r^3 - 5r^2 - 4r = 0 \implies r(2r+1)(3r-4) = 0$

The roots are $r = 0, -\frac{1}{2}, \frac{4}{3}$

This gives the general solution $y(x) = A + Be^{-\frac{1}{2}x} + Ce^{\frac{4}{3}x}$

Exercise 2.2.1d

This is a CC LDE, so we use $y = e^{rw}$:

Thus, the characteristic equation is $2r^2 + 18 = 0 \implies 2(r^2+9) = 0$

The roots are $r = \pm 3i$

This gives the general solution $y(w) = A\cos(3w) + B\sin(3w)$

Exercise 2.2.1e

This is a CE LDE, so we use $y = x^m$:

Thus, the characteristic equation is $4m(m-1)(m-2) + 8m(m-1) + 2m = 0 \implies 2m(2m^2 - 2m + 1) = 0$

The roots are $m = 0, \frac{1}{2} \pm \frac{i}{2}$

This gives the general solution $y(x) = A + Bx^{\frac{1}{2}}\cos\left(\frac{1}{2}\ln(x)\right) + Cx^{\frac{1}{2}}\sin\left(\frac{1}{2}\ln(x)\right)$

Exercise 2.2.1f

This is a CC LDE, so we use $y = e^{rx}$:

Thus, the characteristic equation is $r^5 - r = 0 \implies r(r-1)(r+1)(r^2+1) = 0$

The roots are $r = 0, 1, -1, \pm i$

This gives the general solution $y(x) = A + Be^{x} + Ce^{-x} + D\sin(x) + E\cos(x)$

Exercise 2.2.1g

This is a CE LDE, so we use $p = u^m$:

Thus, the characteristic equation is $4m(m-1) - 3 = 0 \implies (2m-3)(2m+1) = 0$

The roots are $m = \frac{3}{2}, -\frac{1}{2}$

This gives the general solution $p(u) = Au^{\frac{3}{2}} + Bu^{-\frac{1}{2}}$

Exercise 2.2.1*h*

This is a CE LDE, so we use $y = x^m$:

Thus, the characteristic equation is $m(m-1) + m + 4 = 0 \implies m^2 + 4 = 0$

The roots are $m = \pm 2i$

This gives the general solution $y(x) = A\cos(2\ln(x)) + B\sin(2\ln(x))$

Exercise 2.2.1*i*

This is a CC LDE, so we use $y = e^{rx}$:

Thus, the characteristic equation is $2r^3 + 13r^2 + 13r - 10 = 0 \implies (r+5)(r+2)(2r-1) = 0$

The roots are $r = -5, -2, \frac{1}{2}$

This gives the general solution $y(x) = Ae^{-5x} + Be^{-2x} + Ce^{\frac{1}{2}x}$

Exercise 2.2.1*j*

This is a CE LDE, so we use $y = t^m$:

Thus, the characteristic equation is $m(m-1)(m-2)(m-3) + 4m(m-1)(m-2) = 0 \implies m(m-1)(m-2)(m+1) = 0$

The roots are $m = 0, 1, 2, -1$

This gives the general solution $y(t) = A + Bt + Ct^2 + Dt^{-1}$

Exercise 2.2.1*k*

This is a CC LDE, so we use $y = e^{rx}$:

Thus, the characteristic equation is $r^4 + 2r^3 - 2r^2 - 2r + 1 = 0 \implies (r-1)(r+1)(r^2+2r-1) = 0$

The roots are $r = 1, -1, -1 \pm \sqrt{2}$

This gives the general solution $y = Ae^z + Be^{-z} + Ce^{(-1+\sqrt{2})z} + De^{(-1-\sqrt{2})z}$

Exercise 2.2.2*a*

This is a CC LDE, so we use $s = e^{rt}$:

Thus, the characteristic equation is $r^2 = 4r - 5 \implies r^2 - 4r + 5 = 0$

The roots are $r = 2 \pm i$ which gives the general solution $s = Ae^{2t}\cos(t) + Be^{2t}\sin(t)$

Plugging in the initial conditions:

$$\begin{aligned} s(t) &= Ae^{2t}\cos(t) + Be^{2t}\sin(t) \\ s(0) = 1 &\implies 1 = A \cdot 1 + B \cdot 0 \implies A = 1 \\ s'(t) &= 2Ae^{2t}\cos(t) - Ae^{2t}\sin(t) + 2Be^{2t}\sin(t) + Be^{2t}\cos(t) \\ s'(0) = 4 &\implies 4 = 2A \cdot 1 - A \cdot 0 + 2B \cdot 0 + B \cdot 1 \implies B = 2 \end{aligned}$$

The unique solution is $s(t) = e^{2t}\cos(t) + 2e^{2t}\sin(t)$

Exercise 2.2.2*b*

This is a CE LDE, so we use $y = x^m$:

Thus, the characteristic equation is $m(m-1)(m-2) - 2m(m-1) + 2m = 0 \implies m(m-2)(m-3) = 0$

The roots are $m = 0, 2, 3$ which gives the general solution $y = A + Bw^2 + Cw^3$

Plugging in the initial conditions:

$$\begin{aligned} y(w) &= A + Bw^2 + Cw^3 \\ y(-1) = 1 &\implies 1 = A + B \cdot 1 + C \cdot (-1) \implies A + B - C = 1 \\ y'(w) &= 2Bw + 3Cw^2 \\ y'(-1) = 3 &\implies 3 = 2B \cdot (-1) + 3C \cdot 1 \implies -2B + 3C = 3 \\ y''(w) &= 2B + 6Cw \\ y''(-1) = 0 &\implies 0 = 2B + 6C \cdot (-1) \implies 2B - 6C = 0 \end{aligned}$$

$$\begin{cases} A + B - C = 1 \\ -2B + 3C = 3 \\ 2B - 6C = 0 \end{cases} \implies \begin{cases} A = 3 \\ B = -3 \\ C = -1 \end{cases}$$

The unique solution is $y = 3 - 3w^2 - w^3$

Exercise 2.2.2*c*

This is a CC LDE, so we use $y = e^{rx}$:

Thus, the characteristic equation is $r^2 + 1 = 0$

The roots are $r = \pm i$ which gives the general solution $y = A\cos(x) + B\sin(x)$

Plugging in the initial conditions:

$$\begin{aligned} y(x) &= A\cos(x) + B\sin(x) \\ y(\pi) = 3 &\implies 3 = A \cdot (-1) + B \cdot 0 \implies A = -3 \\ y'(x) &= -A\sin(x) + B\cos(x) \\ y'(\pi) = -2 &\implies -2 = -A \cdot 0 + B \cdot (-1) \implies B = 2 \end{aligned}$$

The unique solution is $y = -3\cos(x) + 2\sin(x)$

Exercise 2.2.2*d*

This is a CC LDE, so we use $p = e^{rt}$:

Thus, the characteristic equation is $r^4 - 20r^2 + 64 = 0 \implies (r-2)(r+2)(r-4)(r+4) = 0$

The roots are $r = 2, -2, 4, -4$ which gives the general solution $p = Ae^{2t} + Be^{-2t} + Ce^{4t} + De^{-4t}$

Plugging in the boundary conditions:

$$\begin{aligned} p(t) &= Ae^{2t} + Be^{-2t} + Ce^{4t} + De^{-4t} \\ p(0) = 0 &\implies A + B + C + D = 0 \\ p(\ln(2)) = 0 &\implies 4A + \frac{1}{4}B + 16C + \frac{1}{16}D = 0 \\ p(\ln(3)) = 0 &\implies 9A + \frac{1}{9}B + 81C + \frac{1}{81}D = 0 \\ p(\ln(4)) = 0 &\implies 16A + \frac{1}{16}B + 256C + \frac{1}{256}D = 0 \end{aligned}$$

$$\begin{cases} A + B + C + D = 0 \\ 4A + \frac{1}{4}B + 16C + \frac{1}{16}D = 0 \\ 9A + \frac{1}{9}B + 81C + \frac{1}{81}D = 0 \\ 16A + \frac{1}{16}B + 256C + \frac{1}{256}D = 0 \end{cases} \implies \begin{cases} A = 0 \\ B = 0 \\ C = 0 \\ D = 0 \end{cases}$$

The unique solution is $y = 0$ (the trivial solution)

Exercise $2.2.2e$

This is a CE LDE, so we use $y = x^m$:

Thus, the characteristic equation is $m(m-1) - 12 = 0 \implies (m-4)(m+3) = 0$

The roots are $m = 4, -3$ which gives the general solution $y = Ax^4 + Bx^{-3}$

Plugging in the initial conditions:

$$\begin{aligned} y(x) &= Ax^4 + Bx^{-3} \\ y(1) = 0 &\implies 0 = A \cdot 1 + B \cdot 1 \implies A + B = 0 \\ y'(x) &= 4Ax^3 - 3Bx^{-4} \\ y'(1) = 14 &\implies 14 = 4A \cdot 1 - 3B \cdot 1 \implies 4A - 3B = 14 \end{aligned}$$

$$\begin{cases} A + B = 0 \\ 4A - 3B = 14 \\ A + \frac{1}{4}B + 4C = 0 \end{cases} \implies \begin{cases} A = 2 \\ B = -2 \end{cases}$$

So the unique solution is $y = 2x^4 - 2x^{-3}$

Exercise $2.2.2f$

This is a CC LDE, so we use $y = e^{rx}$:

Thus, the characteristic equation is $r^3 - 4r = 0 \implies r(r-2)(r+2) = 0$

The roots are $r = 0, 2, -2$ which gives the general solution $y = A + Be^{2x} + Ce^{-2x}$

Plugging in the boundary conditions:

$$\begin{aligned} y(x) &= A + Be^{2x} + Ce^{-2x} \\ y(0) = 9 &\implies 9 = A + B \cdot 1 + C \cdot 1 \implies A + B + C = 9 \\ y(\ln(2)) = 0 &\implies 0 = A + B \cdot 4 + C \cdot \frac{1}{4} \implies A + 4B + \frac{1}{4}C = 0 \\ y(-\ln(2)) = 0 &\implies 0 = A + B \cdot \frac{1}{4} + C \cdot 4 \implies A + \frac{1}{4}B + 4C = 0 \end{aligned}$$

$$\begin{cases} A + B + C = 9 \\ A + 4B + \frac{1}{4}C = 0 \\ A + \frac{1}{4}B + 4C = 0 \end{cases} \implies \begin{cases} A = 9 \\ B = 0 \\ C = 0 \end{cases}$$

So the unique solution is $y = 9$

Exercise $2.2.2g$

This is a CE LDE, so we use $y = x^m$:

Thus, the characteristic equation is $m(m-1) - 3m = 0 \implies m(m-4) = 0$

The roots are $m = 0, 4$ which gives the general solution $y = A + Bx^4$

Plugging in the boundary conditions:

$$\begin{aligned} y(x) &= A + Bx^4 \\ y(1) = 2 &\implies 2 = A + B \cdot 1 \implies A + B = 2 \\ y'(-1) = 2 &\implies 2 = A + B \cdot 1 \implies A + B = 2 \end{aligned}$$

$$\begin{cases} A + B = 2 \\ A + B = 2 \end{cases} \implies \begin{cases} A = 2 - B \\ B = \text{anything} \end{cases}$$

So we have infinitely many solutions of the form $y = 2 + B(x^4 - 1)$

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2.3: Reduction of Order

Learning Objectives

- Apply the reduction of order method to find a second linearly independent solution when one solution is known.
- Use the method for constant coefficient equations with repeated roots.
- Use the method for Cauchy-Euler equations with repeated roots.
- Apply the method to other linear ODEs where one solution can be guessed or is given.

When one solution to a second-order homogeneous equation is known, Reduction of Order provides a method to find a second, linearly independent solution. This technique is essential for handling constant coefficient equations with repeated roots and Cauchy-Euler equations, and is a valuable tool for more general problems.

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2.3: Reduction of Order in 2nd Order LDEs

We now have the tools to solve *most* CC and CE linear differential equations, by finding the roots of their characteristic equation. However there is one case we have not covered: repeated roots. Consider the CC LDE $y'' = 0$, which has characteristic equation:

$$r^2 = 0 \implies r = 0$$

The double root $r = 0$ yields only one solution: $y = e^{0x} = 1$. We know there are supposed to be two linearly independent solutions to this second order LDE, but have no way to find the second one. This subsection focuses on exactly that scenario.

Assume that we know $y = f(x)$ is a non-trivial solution to the general second order LDE of the form

$$y'' + p(x)y' + q(x)y = 0$$

We will use a substitution to solve for the general solution, setting $y(x) = f(x) \cdot u(x)$ and trying to solve for the unknown $u(x)$.

$$y = f \cdot u \implies y' = f' \cdot u + f \cdot u' \implies y'' = f'' \cdot u + 2f' \cdot u' + f \cdot u''$$

Plugging these into the LDE we see:

$$\begin{gathered}(f'' \cdot u + 2f' \cdot u' + f \cdot u'') + p \cdot (f' \cdot u + f \cdot u') + q \cdot (f \cdot u) = 0 \\ \Downarrow \\ \underbrace{(f'' + p \cdot f' + q \cdot f)}_{\text{equals } 0 \text{ since } f \text{ is a solution}} \cdot u + (2f' + pf) \cdot u' + f \cdot u'' = 0 \\ \Downarrow \\ (2f' + pf) \cdot u'(x) + f \cdot u''(x) = 0\end{gathered}$$

This can be separated and integrated with respect to x:

$$\int \frac{u''(x)}{u'(x)}\,dx = \int -\frac{2f'(x)}{f(x)} - p(x)\,dx \implies \ln|u'(x)| = -2\ln|f(x)| - \int p(x)\,dx$$

Solving this for $u'(x)$ we get:

$$u'(x) = \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx}$$

Since we know $f(x)$ and $p(x)$, we can integrate the expression on the right to solve for $u(x)$. Notice that it will be of the form $u(x) = Cg(x) + D$ for some constants C and D (one from each integral). Plugging this back into our original substitution gives our general solutions:

$$y(x) = f(x) \cdot u(x) = f(x) \cdot (Cg(x) + D) = Cf(x)g(x) + Df(x)$$

This will give our two linearly independent solutions, as expected. This substitution is known as **Reduction of Order** (since it takes a second order LDE and turns it into a first order differential equation instead).

Rule: Reduction of Order

Let $y = f(x)$ be a non-trivial solution to the homogeneous second order LDE of the form

$$y'' + p(x)y' + q(x)y = 0$$

For any function $u(x)$ satisfying $u'(x) = \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx}$, we know that $y = u(x)f(x)$ is also a solution to the LDE.

For our inspirational example $y'' = 0$, we have $p(x) = 0$ and known solution $f(x) = 1$, so the expression for u' becomes:

$$u'(x) = \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx} = \frac{1}{[1]^2} e^{-\int 0\,dx} = 1 \cdot e^C = C$$

Integrating this gives $u(x)$:

$$u(x) = \int u'(x)\,dx = \int C\,dx = Cx + D$$

Therefore, our solutions are $y(x) = u(x) \cdot f(x) = (Cx + D) \cdot 1 = Cx + D$. This general solution is easily verifiable to satisfy $y'' = 0$ and the functions x and 1 are clearly linearly independent, as desired.

Example 2.3.1

Use reduction of order to turn the given solution to the provided LDE into the general solution.

(a) $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$ with known solution $f(x) = x$

(b) $xy'' + (2x+2)\,y' + 2y = 0$ with known solution $f(x) = \frac{1}{x}$

Solution

(a) We can see that $p(x) = -\frac{x}{x-1}$, so we calculate

$$-\int p(x)\,dx = \int \frac{x}{x-1}\,dx = \int 1 + \frac{1}{x-1}\,dx = x + \ln|x-1| + C$$

This means $u'(x) = \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx} = \frac{1}{[x]^2} e^{x+\ln|x-1|+C} = \frac{1}{x^2} \cdot C(x-1)e^x = \frac{C(x-1)}{x^2} e^x$

Now we integrate to solve for $u(x)$:

$$u(x) = \int u'(x)\,dx = C\int \left(x^{-1} - x^{-2}\right) e^x\,dx = C\left(\int x^{-1}e^x\,dx - \int x^{-2}e^x\,dx\right)$$

Use integration by parts on the second integral expression to get:

$$\int x^{-2}e^x\,dx = -x^{-1}e^x + \int x^{-1}e^x\,dx$$

Therefore, the original $u(x)$ integral expression simplifies to:

$$u(x) = C\left(\int x^{-1}e^x\,dx - \left(-x^{-1}e^x + \int x^{-1}e^x\,dx\right)\right) = Cx^{-1}e^x + D$$

Plugging this into the substitution expression for $y(x)$ gives our final answer:

$$y(x) = u(x) \cdot f(x) = \left(Cx^{-1}e^x + D\right) \cdot x = Ce^x + Dx$$

Thus, our general solution to the LDE is $\boxed{y = Ce^x + Dx}$

(b) We must first rewrite the LDE so that we can identify $p(x)$ correctly:

$$y'' + \left(2 + \frac{2}{x}\right) y' + \frac{2}{x}y = 0 \implies p(x) = 2 + \frac{2}{x}$$

Now we can calculate:

$$-\int p(x)\,dx = \int -2 - \frac{2}{x}\,dx = -2x - 2\ln|x| + C$$

This means $u'(x) = \dfrac{1}{[f(x)]^2} e^{-\int p(x)\,dx} = \dfrac{1}{\left[\frac{1}{x}\right]^2} e^{-2x-2\ln|x|+C} = x^2 \cdot Cx^{-2}e^{-2x} = Ce^{-2x}$

We can integrate to solve for $u(x)$:

$$u(x) = \int u'(x)\,dx = \int Ce^{-2x}\,dx = -\frac{C}{2}e^{-2x} + D$$

Plugging this into the substitution expression for $y(x)$ gives our final answer:

$$y(x) = u(x) \cdot f(x) = \left(-\frac{C}{2}e^{-2x} + D\right) \cdot \frac{1}{x} = \hat{C}\frac{e^{-2x}}{x} + \frac{D}{x}$$

Thus, our general solution to the LDE is $\boxed{y = \hat{C}x^{-1}e^{-2x} + Dx^{-1}}$

We now turn to using reduction of order on the specific case of a second order CC LDE with repeated roots.

$$y'' - 2ay' + a^2y = 0$$

This has characteristic equation $r^2 - 2ar + a^2 = 0 \implies (r-a)^2 = 0$

We easily see that since $r = a$ we must have the solution $f(x) = e^{ax}$, and that $p(x) = -2a$:

$$-\int p(x)\,dx = \int 2a\,dx = 2ax + C$$

Therefore, $u'(x) = \dfrac{1}{[f(x)]^2} e^{-\int p(x)\,dx} = \dfrac{1}{[e^{ax}]^2} e^{2ax+C} = e^{-2ax} \cdot Ce^{2ax} = C$

$$u(x) = \int u'(x)\,dx = \int C\,dx = Cx + D$$

Therefore, our general solution is $y(x) = u(x) \cdot f(x) = (Cx + D)\,e^{ax} = Cxe^{ax} + De^{ax}$.

We already knew that $y_1(x) = e^{ax}$ was a solution, but let's check why $y_2(x) = xe^{ax}$ satisfies the equation:

$$y_2(x) = xe^{ax} \implies y_2'(x) = (ax+1)e^{ax} \implies y_2''(x) = (a^2x + 2a)e^{ax}$$

Plugging this into our CC LDE gives:

$$\begin{gathered}\left(a^2x + 2a\right)e^{ax} + p(x) \cdot (ax+1)\,e^{ax} + q(x) \cdot xe^{ax} \\ \Downarrow \\ \left(a^2 + p(x) \cdot a + q(x) \cdot 1\right)xe^{ax} + \left(2a + p(x) \cdot 1\right)e^{ax}\end{gathered}$$

When $p(x) = -2a$ and $q(x) = a^2$, we see the coefficients of xe^{ax} and e^{ax} become zero. If you think of the characteristic polynomial $C(r) = r^2 - 2ar + a^2$, we see the coefficient of xe^{ax} looks just like it (with our derivatives of xe^{ax} plugged in as "powers" of r). Notice that $C'(r) = 2r - 2a$, and we see the coefficient of (e^{ax}\) looks just like that polynomial (with our derivatives of e^{ax} plugged in as "powers" of r. Since $r = a$ is a double root of $C(r)$, it is also a root of $C'(r)$, and multiplying the function e^{ax} by x yielded the product rule above, with both coefficients becoming zero because of that reason.

This tells us that not only does this happen for all double roots, but we can generalize it for any multiplicity of root for a CC LDE.

Rule: Repeated Roots of a Homogeneous CC LDE

If a CC LDE has characteristic equation $C(r) = 0$ and $r = a$ is a root of this equation with multiplicity k, then

$$y_1(x) = e^{ax}, \quad y_2(x) = xe^{ax}, \ldots, \quad y_k(x) = x^{k-1}e^{ax}$$

are k linearly independent solutions to the homogeneous CC LDE.

The trick to remember is that for repeated roots, you simply increase the power of x next to e^{rx} until you get the necessary number of solutions.

Example 2.3.2

Find the general solution to each CC LDE below.

(a) $9y'' + 12y' + 4y = 0$

(b) $4y^{(6)}(x) - 4y^{(5)}(x) + y^{(4)}(x) = 0$

Solution

(a) The characteristic equation is $9r^2 + 12r + 4 = 0$

This factors to $(3r + 2)^2 = 0$, which has roots $r = -\dfrac{2}{3}$ (double)

Therefore, the general solution is $\boxed{y(x) = Ae^{-\frac{2}{3}x} + Bxe^{-\frac{2}{3}x}}$

(b) The characteristic equation is $4r^6 - 4r^5 + r^4 = 0$

This factors to $r^4(2r - 1)^2 = 0$, which has roots $r = \dfrac{1}{2}$ (double), 0 (quadruple)

Therefore, the general solution is $\boxed{y(x) = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x} + C + Dx + Ex^2 + Fx^3}$

We can use the substitution connection between CC LDEs of x and CE LDEs of t, $x = \ln(t)$, to get the repeated roots rule for Cauchy Euler equations.

Rule: Repeated Roots of a Homogeneous CE LDE

If a CE LDE has characteristic equation $C(m) = 0$ and $m = a$ is a root of this equation with multiplicity k, then

$$y_1(x) = x^a, \quad y_2(x) = x^a \ln(x), \ \ldots, \quad y_k(x) = x^a \left[\ln(x)\right]^{k-1}$$

are k linearly independent solutions to the homogeneous CE LDE.

So for repeated roots in the Cauchy Euler case, you simply increase the power of $\ln(x)$ next to x^m until you get the necessary number of solutions.

Example 2.3.3

Find the general solution to each CE LDE below.

(a) $x^3y''' - x^2y'' + xy' = 0$

(b) $x^3y''' + 6x^2y'' + 7xy' + y = 0$

Solution

(a) The characteristic equation is $m(m-1)(m-2) - m(m-1) + m = 0$

This factors to $m(m-2)^2 = 0$, which has roots $m = 0, \ 2$ (double)

Therefore, the general solution is $\boxed{y(x) = A + Bx^2 + Cx^2\ln(x)}$

(b) The characteristic equation is $m(m-1)(m-2) + 6m(m_1) + 7m + 1 = 0$

This factors to $(m+1)^3 = 0$, which has roots $m = -1$ (triple)

Therefore, the general solution is $\boxed{y(x) = Ax^{-1} + Bx^{-1}\ln(x) + Cx^{-1}[\ln(x)]^2}$

As usual, we can also include initial or boundary conditions in hopes of finding unique solutions.

Illustration 2.3.1

We will find the general solution to each IVP or BVP below:

(a) $x^3y''' - 5x^2y'' + 13xy' - 16y = 0$ with $y(1) = -1,\ y(e) = 0$, and $y'(e) = -e^3$

(b) $y'''' - 4y'' + 8y'' - 8y' + 4y = 0$ with $y(0) = 0,\ y'(0) = 0,\ y''(0) = 0$, and $y'''(0) = -2$

(a) This is a CE LDE with characteristic equation $m(m-1)(m-2) - 5m(m-1) + 13m - 16 = 0$

This factors to $(m-2)^2(m-4) = 0$, which has roots $m = 4,\ 2$ (double)

Therefore, the general solution is $y(x) + Ax^2 + Bx^2\ln(x) + Cx^4$

Plugging in the boundary conditions gives:

$$\begin{aligned}
y(1) &= A\cdot 1 + B\cdot 0 + C\cdot 1 \implies A + C = e^2 - e^4\\
y(e) &= A\cdot e^2 + B\cdot e^2 + C\cdot e^4 \implies Ae^2 + Be^2 + Ce^4 = 0\\
y' &= 2Ax + 2Bx\ln(x) + Bx + 4Cx^3\\
y'(e) &= 2A\cdot e + 3B\cdot e + 4Ce^3 \implies 2Ae + 3Be + 4Ce^3 = -e^3
\end{aligned}$$

Solving this system of equations gives:

$$\begin{cases} A + C = -1\ Ae^2 + Be^2 + Ce^4 = 0\\ 2Ae + 3Be + 4Ce^3 = -e^3 \end{cases} \implies \begin{cases} A = 0\\ B = e^2\\ C = -e^4 \end{cases}$$

This gives us the unique solution $y(x) = e^2x^2\ln(x) - x^4$

(b) This is a CC LDE with characteristic equation $r^4 - 4r^3 + 8r^2 - 8r + 4 = 0$

This factors to $(r^2 - 2r + 2)^2 = 0$, which has roots $r = 1 \pm i$ (double)

Therefore, the general solution is $y(x) = Ae^x\cos(x) + Be^x\sin(x) + Cxe^x\cos(x) + Dxe^x\sin(x)$

Plugging in the initial conditions gives:

$$\begin{aligned}
y(0) &= A\cdot 1 + B\cdot 0 + C\cdot 0 + D|cdot0 \implies A = 0\\
y' &= (A + B + C)\,e^x\cos(x) + (B - A + D)e^x\sin(x) + (C + D)xe^x\cos(x) + (D - C)xe^x\sin(x)\\
y'(0) &= (A + B + C)\cdot 1 + (B - A + D)\cdot 0 + (C + D)\cdot 0 + (D - C)\cdot 0 \implies A + B + C = 0\\
y'' &= (2B + 2C + 2D)\,e^x\cos(x) + (-2A - 2C + 2D)e^x\sin(x) + 2Dxe^x\cos(x) = 2Cxe^x\sin(x)\\
y''(0) &= (2B + 2C + 2D)\cdot 1 + (-2A - 2C + 2D)\cdot 0 + 2D\cdot 0 - 2C\cdot 0 \implies 2B + 2C + 2D = 0\\
y''' &= (-2A + 2B + 6D)\,e^x\cos(x) + (-2A - 2B - 6C)e^x\sin(x) + (2D - 2C)xe^x\cos(x) - (2C + 2D)xe^x\sin(x)\\
y'''(0) &= (-2A + 2B + 6D)\cdot 1 + (-2A - 2B - 6C)\cdot 0 + (2D - 2C)\cdot 0 - (2C + 2D)\cdot 0 \implies -2A + 2B + 6D = -2
\end{aligned}$$

Solving this system of equations gives:

$$\begin{cases} A = 0 \\ A + B + C = 0 \\ 2B + 2C + 2D = 0 \\ -2A + 2B + 6D = -2 \end{cases} \Longrightarrow \begin{cases} A = 0 \\ B = -1 \\ C = 1 \\ D = 0 \end{cases}$$

This gives us the unique solution $y(x) = xe^x \cos(x) - e^x \sin(x)$

This subchapter introduced the reduction of order method, a powerful technique for solving second-order linear differential equations when one solution is already known. This method allows us to find a complete fundamental set of solutions from a single known solution, whether for homogeneous equations with non-constant coefficients or for finding a particular solution to non-homogeneous equations. Reduction of order thus extends our ability to solve complex second-order problems by building directly from known information.

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2.3: Practice Set

Exercise 2.3.1a

Find the general solution to the LDE $y'' - 16y' + 64y = 0$

Answer available at the end of this section.

Exercise 2.3.1b

Find the general solution to the LDE $x^2y'' + 5xy' + 4y = 0$

Answer available at the end of this section.

Exercise 2.3.1c

Find the general solution to the LDE $2y''' - 3y'' = 0$

Answer available at the end of this section.

Exercise 2.3.1d

Find the general solution to the LDE $y^{(4)}(x) - 3y^{(3)}(x) + 4y'(x) = 0$

Answer available at the end of this section.

Exercise 2.3.1e

Find the general solution to the LDE $t^3\dfrac{d^3z}{dt^3} + t\dfrac{dz}{dt} = z$

Answer available at the end of this section.

Exercise 2.3.1f

Find the general solution to the LDE $x^4y^{(4)}(x) + 6x^3y^{(3)}(x) + 15x^2y''(x) + 9xy'(x) + 16y(x) = 0$

Answer available at the end of this section.

Exercise 2.3.1g

Find the general solution to the LDE $y^{(4)}(t) + 2y''(t) + y(t) = 0$

Answer available at the end of this section.

Exercise $2.3.1h$

Find the general solution to the LDE $x^4y^{(4)}(x) + 8x^3y^{(3)}(x) + 15x^2y''(x) + 5xy'(x) = 0$

Answer available at the end of this section.

Exercise $2.3.2a$

Find the general solution to the BVP $4y'' - 4y' + y = 0$ with $y(2\ln(2)) = 4,\ y(2\ln(3)) = 6$

Answer available at the end of this section.

Exercise $2.3.2b$

Find the general solution to the BVP $2u^2w''(u) - 10uw'(u) + 18w(u) = 0$ with $w(1) = 0,\ w(2) = 8$

Answer available at the end of this section.

Exercise $2.3.2c$

Find the general solution to the IVP $y''' + 6y'' + 12y' + 8y = 0$ with $y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$

Answer available at the end of this section.

Exercise $2.3.2d$

Find the general solution to the BVP $x^3y''' + x^2y'' = 0$ with $y(1) = e - e^2,\ y(e) = 0, y(e^2) = e^3 - e^2$

Answer available at the end of this section.

Exercise $2.3.2e$

Find the general solution to the BVP $\dfrac{d^4p}{dt^4} + \dfrac{d^2p}{dt^2} = 0$ with $p(0) = 1,\ p(\pi) = 1,\ p(2\pi) = 1,\ p(3\pi) = 0$

Answer available at the end of this section.

Exercise $2.3.2f$

Find the general solution to the IVP $x^3y''' - 47xy' - 81y = 0$ with $y(1) = 0,\ y'(1) = 0,\ y''(1) = 144$

Answer available at the end of this section.

Exercise $2.3.3$

Given $f(x) = \sin(x^2)$ satisfies $y'' + 4x^2y = \dfrac{1}{x}y'$, find all solutions.

Answer available at the end of this section.

Exercise 2.3.4

Given $f(x) = e^{2x}$ satisfies $xy'' - (4x+4)y' + (4x+8)y = 0$, find all solutions.

Answer available at the end of this section.

Exercise 2.3.5

Given $f(x) = \sec(x)$ satisfies $y'' - \tan(x)y' - \sec^2(x)y = 0$, find the solution with $y(0) = 1,\ y'(0) = 1$

Answer available at the end of this section.

Exercise 2.3.6

Given $f(x) = e^{\sqrt{x}}$ satisfies $4x^{\frac{3}{2}}y'' - 2xy' + y = 0$, find the solution with $y(1) = 2e^2 - 3e,\ y(4) = 0$

Answer available at the end of this section.

Answers

Exercise 2.3.1*a*

This is a CC LDE, so we use $y = e^{rx}$:
The characteristic equation is: $r^2 - 16r + 64 = 0$
This factors into: $(r-8)^2 = 0$
The roots are: $r = 8$ (double)
Therefore, we have general solution $y(x) = Ae^{8x} + Bxe^{8x}$

Exercise 2.3.1*b*

This is a CE LDE, so we use $y = x^m$:
The characteristic equation is: $m(m-1) + 5m + 4 = 0$
This factors into: $(m+2)^2 = 0$
The roots are: $m = -2$ (double)
Therefore, we have general solution $y(x) = Ax^{-2} + Bx^{-2}\ln(x)$

Exercise 2.3.1*c*

This is a CC LDE, so we use $y = e^{rx}$:
The characteristic equation is: $2r^3 - 3r^2 = 0$
This factors into: $r^2(2r-3) = 0$
The roots are: $r = 0$ (double), $\frac{3}{2}$
Therefore, we have general solution $y(x) = A + Bx + Ce^{\frac{3}{2}x}$

Exercise 2.3.1*d*

This is a CC LDE, so we use $y = e^{rx}$:
The characteristic equation is: $r^4 - 3r^3 + 4r = 0$
This factors into: $r(r+1)(r-2)^2 = 0$
The roots are: $r = 0, -1, 2$ (double)
Therefore, we have general solution $y(x) = A + Be^{-x} + Ce^{2x} + Dxe^{2x}$

Exercise 2.3.1*e*

This is a CE LDE, so we use $z = t^m$:
The characteristic equation is: $m(m-1)(m-2) + m = 1$
This factors into: $(m-1)^2 = 0$
The roots are: $m = 1$(triple)
Therefore, we have general solution $z(t) = At + Bt\ln(t) + Ct(\ln(t))^2$

Exercise 2.3.1*f*

This is a CE LDE, so we use $y = x^m$:
The characteristic equation is: $m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) + 15m(m-1) + 9m + 16 = 0$
This factors into: $(m^2+4)^2 = 0$
The roots are: $m = \pm 2i$ (double)

Therefore, we have general solution $y(x) = A\cos(2\ln(x)) + B\cos(2\ln(x))\ln(x) + C\sin(2\ln(x)) + D\sin(2\ln(x))\ln(x)$

Exercise 2.3.1*g*

This is a CC LDE, so we use $y = e^{rt}$:

The characteristic equation is: $r^4 + 2r^2 + 1 = 0$

This factors into: $(r^2+1)^2 = 0$

The roots are: $r = \pm i$ (double)

Therefore, we have general solution $y(x) = A\cos(x) + Bx\cos(x) + C\sin(x) + Dx\sin(x)$

Exercise 2.3.1*h*

This is a CE LDE, so we use $y = x^m$:

The characteristic equation is: $m(m-1)(m-2)(m-3) + 8m(m-1)(m-2) + 15m(m-1) + 5m = 0$

This factors into: $m^2(m^2+2m+2) = 0$

The roots are: $m = 0$ (double), $-1 \pm i$

Therefore, we have general solution $y(x) = A + B\ln(x) + Cx^{-1}\cos(\ln(x)) + Dx^{-1}\sin(\ln(x))$

Exercise 2.3.2*a*

This is a CC LDE, so we use $y = e^{rx}$:

The characteristic equation is: $4r^2 - 4r + 1 = 0$

This factors into: $(2r-1)^2 = 0$

The roots are: $r = \dfrac{1}{2}$ (double)

Therefore, we have general solution $y = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$

Plugging in the boundary conditions:

$$\begin{gathered} y(x) = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x} \\ y(2\ln(2)) = 4 \implies 4 = A\cdot 2 + B\cdot 4\ln(2) \implies 2A + 4\ln(2)\,B = 4 \\ y(2\ln(3)) = 6 \implies 6 = A\cdot 3 + B\cdot 9\ln(3) \implies 3A + 9\ln(3)\,B = 8 \\ \begin{cases} 2A + 4\ln(2)\,B = 4 \\ 3A + 9\ln(3)\,B = 6 \end{cases} \implies \begin{cases} A = 2 \\ B = 0 \end{cases} \end{gathered}$$

So, the unique solution is $y = 2e^{\frac{1}{2}x}$

Exercise 2.3.2*b*

This is a CE LDE, so we use $w = u^m$:

The characteristic equation is: $2m(m-1) - 10m + 18 = 0$

This factors into: $2(m-3)^2 = 0$

The roots are: $m = 3$ (double)

Therefore, we have general solution $w = Au^3 + Bu^3\ln(u)$

Plugging in the boundary conditions:

$$\begin{gathered} w(u) = Au^3 + Bu^3\ln(u) \\ w(1) = 0 \implies 0 = A\cdot 1 + B\cdot 0 \implies A = 0 \\ w(2) = 1 \implies 8 = A\cdot 8 + B\cdot 8\ln(2) \implies B = \frac{1}{\ln(2)} \end{gathered}$$

So, the unique solution is $w(u) = \dfrac{1}{\ln(2)}u^3\ln(u)$

Exercise 2.3.2*c*

This is a CC LDE, so we use $y = e^{rx}$:

The characteristic equation is: $r^3 + 6r^2 + 12r + 8 = 0$

This factors into: $(r+2)^3 = 0$

The roots are: $r = -2$ (triple)

Therefore, we have general solution $y = (A + Bx + Cx^2)e^{-2x}$

Plugging in the initial conditions:

$$\begin{gathered} y(x) = (A + Bx + Cx^2)e^{-2x} \\ y(0) = 1 \implies 1 = A\cdot 1 \implies A = 1 \\ y'(x) = -2(A + Bx + Cx^2)e^{-2x} + (B + 2Cx)e^{-2x} \\ y'(0) = 0 \implies 0 = -2A\cdot 1 + B\cdot 1 \implies B = 2 \\ y''(x) = 4(A + Bx + Cx^2)e^{-2x} + 2 - 4(B + 2Cx)e^{-2x} + 2Ce^{-2x} \\ y''(0) = 0 \implies 0 = 4A\cdot 1 - 4B\cdot 1 + 2C\cdot 1 \implies C = 2 \end{gathered}$$

So, the unique solution is $y(x) = (1 + 2x + 2x^2)e^{-2x}$

Exercise 2.3.2*d*

This is a CE LDE, so we use $y = x^m$:

The characteristic equation is: $m(m-1)(m-2) + m(m-1) = 0$

This factors into: $m(m-1)^2 = 0$

The roots are: $m = 0, 1$ (double)

Therefore, we have general solution $y = A + Bx + Cx\ln(x)$

Plugging in the boundary conditions:

$$
\begin{aligned}
y(x) &= A + Bx + Cx\ln(x)\\
y(1) = e - e^2 &\implies A + B = e - e^2\\
y(e) = 0 &\implies A + eB + eC = 0\\
y(e^2) = e^3 - e^2 &\implies A + e^2B + 2e^2C = e^3 - e^2\\
\begin{cases} A + B = e - e^2 \\ A + eB + eC = 0 \\ A + e^2B + 2e^2C = e^3 - e^2 \end{cases} &\implies \begin{cases} A = -e^2 \\ B = e \\ C = 0 \end{cases}
\end{aligned}
$$

So, the unique solution is $y(x) = -e^2 + ex$

Exercise 2.3.2*e*

This is a CC LDE, so we use $p = e^{rt}$:

The characteristic equation is: $r^4 + r^2 = 0$

This factors into: $r^2(r^2 + 1) = 0$

The roots are: $r = 0\ (\text{double}), \pm i$

Therefore, we have general solution $p = A + Bt + C\cos(t) + D\sin(t)$

Plugging in the boundary conditions:

$$
\begin{aligned}
p(t) &= A + Bt + C\cos(t) + D\sin(t)\\
p(0) = 1 &\implies 1 = A + B\cdot 0 + C\cdot 1 + D\cdot 0 \implies A + C = 1\\
p(\pi) = 1 &\implies 1 = A + B\cdot \pi + C\cdot(-1) + D\cdot 0 \implies A + \pi B - C = 1\\
p(2\pi) = 1 &\implies 1 = A + B\cdot 2\pi + C\cdot 1 + D\cdot 0 \implies A + 2\pi B + C = 1\\
p(3\pi) = 0 &\implies 0 = A + B\cdot 3\pi + C\cdot(-1) + D\cdot 0 \implies A + 3\pi B - C = 0\\
&\begin{cases} A + C = 1 \\ A + \pi B - C = 1 \\ A + 2\pi B + C = 1 \\ A + 3\pi B - C = 0 \end{cases} \implies \emptyset\ (\text{impossible})
\end{aligned}
$$

So, **no such solution exists.**

Exercise 2.3.2*f*

This is a CE LDE, so we use $y = x^m$:

The characteristic equation is: $m(m-1)(m-2) - 47m - 81 = 0$

This factors into: $(m-9)(m+3)^2 = 0$

The roots are: $m = 9, -3\ (\text{double})$

Therefore, we have general solution $y = Ax^9 + Bx^{-3} + Cx^{-3}\ln(x)$

Plugging in the initial conditions:

$$
\begin{aligned}
y(x) &= Ax^9 + Bx^{-3} + Cx^{-3}\ln(x)\\
y(1) = 0 &\implies A + B = 0\\
y'(x) &= 9Ax^8 - 3Bx^{-4} - 3Cx^{-4}\ln(x) + Cx^{-4}\\
y'(1) = 0 &\implies 9A - 3B + C = 0\\
y''(x) &= 72Ax^7 + 12Bx^{-5} + 12Cx^{-5}\ln(x) - 7Cx^{-5}\\
y''(1) = 144 &\implies 72A + 12B - 7C = 144\\
\begin{cases} A + B = 0 \\ 9A - 3B + C = 0 \\ 72A + 12B - 7C = 144 \end{cases} &\implies \begin{cases} A = 1 \\ B = -1 \\ C = -12 \end{cases}
\end{aligned}
$$

So, the unique solution is $y(x) = x^9 - x^{-3} - 12x^{-3}\ln(x)$

Exercise 2.3.3

We are given one solution $f(x) = \sin(x^2)$:

Rewriting this DE: $y'' - \dfrac{1}{x}y' + 4x^2y = 0$, we see $p(x) = -\dfrac{1}{x}$

This gives: $-\displaystyle\int p(x) = \int \frac{1}{x}\,dx = \ln|x| + C$

So, $u'(x) = \dfrac{e^{-\int p(x)\,dx}}{[f(x)]^2} = \dfrac{e^{\ln|x|+C}}{[\sin(x^2)]^2} = \dfrac{Cx}{\sin^2(x^2)}$

$$u(x) = \int \frac{Cx}{\sin^2(x^2)}\,dx = C\int x\csc^2(x^2)\,dx = -\frac{C}{2}\cot(x^2) + D$$

Therefore, our function $y(x) = u(x)f(x) = \left(-\dfrac{C}{2}\cot(x^2) + D\right)\sin(x^2)$

This simplifies to $y(x) = \hat{C}\cos(x^2) + D\sin(x^2)$

Exercise 2.3.4

We are given one solution $f(x) = e^{2x}$:

Rewriting this DE: $y'' - \left(4 + \dfrac{4}{x}\right)y' + \left(4 + \dfrac{8}{x}\right)y = 0$, we see $p(x) = -4 - \dfrac{4}{x}$

This gives: $-\displaystyle\int p(x) = \int \left(4 + \frac{4}{x}\right)dx = 4x + 4\ln|x| + C$

So, $u'(x) = \dfrac{e^{-\int p(x)\,dx}}{[f(x)]^2} = \dfrac{e^{4x+4\ln|x|+C}}{[e^{2x}]^2} = \dfrac{Cx^4e^{4x}}{e^{4x}} = Cx^4$

$$u(x) = \int Cx^4\,dx = \frac{C}{5}x^5 + D$$

Therefore, our function $y(x) = u(x)f(x) = \left(\frac{C}{5}x^5 + D\right)e^{2x}$

This simplifies to $y(x) = \hat{C}x^5e^{2x} + De^{2x}$

Exercise 2.3.5

We are given one solution $f(x) = \sec(x)$:

We see $p(x) = -\tan(x)$

This gives: $-\int p(x) = \int \tan(x)\,dx = \ln|\sec(x)| + C$

So, $u'(x) = \dfrac{e^{-\int p(x)\,dx}}{[f(x)]^2} = \dfrac{e^{\ln|\sec(x)|+C}}{[\sec(x)]^2} = \dfrac{C\sec(x)}{\sec^2(x)} = C\cos(x)$

$$u(x) = \int C\cos(x)\,dx = C\sin(x) + D$$

Therefore, our function $y(x) = u(x)f(x) = (C\sin(x) + D)\sec(x) = C\tan(x) + D\sec(x)$

Plugging in the initial conditions:

$$\begin{aligned} y(0) = 1 &\implies 1 = C\cdot 0 + D\cdot 1 \implies D = 1 \\ y'(0) = 1 &\implies 1 = C\cdot 1 + D\cdot 0 \implies C = 1 \end{aligned}$$
$$y' = C\sec^2(x) + D\sec(x)\tan(x)$$

So, the unique solution is $y(x) = \tan(x) + \sec(x)$

Exercise 2.3.6

We are given one solution $f(x) = e^{\sqrt{x}}$:

Rewriting this DE: $y'' - \dfrac{1}{2\sqrt{x}}y' + \dfrac{1}{4x^{\frac{3}{2}}}y = 0$, we see $p(x) = -\dfrac{1}{2\sqrt{x}}$

This gives: $-\int p(x) = \int \dfrac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$

So, $u'(x) = \dfrac{e^{-\int p(x)\,dx}}{[f(x)]^2} = \dfrac{e^{\sqrt{x}}+C}{[e^{\sqrt{x}}]^2} = \dfrac{Ce^{\sqrt{x}}}{e^{2\sqrt{x}}} = Ce^{-\sqrt{x}}$

$$u(x) = \int Ce^{-\sqrt{x}}\,dx = -2Ce^{-\sqrt{x}}(\sqrt{x}+1) + D$$

Therefore, our function $y(x) = u(x)f(x) = \left(-2Ce^{-\sqrt{x}}(\sqrt{x}+1) + D\right)e^{\sqrt{x}} = \hat{C}(\sqrt{x}+1) + De^{\sqrt{x}}$

Plugging in the boundary conditions:

$$\begin{aligned} y(1) = 1 &\implies 1 = \hat{C}\cdot 2 + D\cdot e \implies 2\hat{C} + eD = 1 \\ y(4) = 0 &\implies 0 = \hat{C}\cdot 3 + D\cdot e^2 \implies 3\hat{C} + e^2D = 0 \end{aligned}$$
$$\begin{cases} 2\hat{C} + eD = 2e^2 - 3e \\ 3\hat{C} + e^2D = 0 \end{cases} \implies \begin{cases} \hat{C} = e^2 \\ D = -3 \end{cases}$$

So, the unique solution is $y(x) = e^2(\sqrt{x}+1) - 3e^{\sqrt{x}}$

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2.4: Undetermined Coefficients

Learning Objectives

- Understand the structure of particular solutions for common forcing functions.
- Apply the Method of Undetermined Coefficients to find particular solutions for constant coefficient nonhomogeneous ODEs.
- Apply the Modification Rule when the trial solution overlaps with the homogeneous solution.
- Solve complete initial value problems using the method.

To solve nonhomogeneous equations, we need a method to find a particular solution. This subchapter presents the "educated guess" method of Undetermined Coefficients, teaching you to construct trial solutions based on the form of the nonhomogeneous term and modify them when they conflict with the homogeneous solution.

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2.4: Guessing Solutions

Now, we move to solving the non-homogeneous version of the higher order LDEs:

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x) = g(x)$$

We recall the shape of the solution set for any n^{th} order LDE:

$$y(x) = y_p(x) + C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$$

where $y_p(x)$ is *any* particular solution to the non-homogeneous LDE, and each $y_1, y_2, \ldots, y_n$ is a linearly independent solution to the homogeneous LDE.

Since we have found the way to find these homogeneous solutions for CC and CE LDEs, all that remains is a method to find the particular solution.

The **Method of Undetermined Coefficients** is a simple "guess and check" method, where we make an educated guess as to the form of y_p based on the function $g(x)$ (and the homogeneous solutions we have already found). Then we substitute this guess into the differential equation and solve for the coefficients in our guess to make it the precise particular solution we need.

For a simple example, consider the LDE $y'' - 3y' = 10e^{2x}$.

The characteristic equation $r^2 - 3r = 0$ has roots $r = 0, 3$, so we know the homogeneous solutions are $y_1(x) = 1$ and $y_2(x) = e^{3x}$.

If we want a term involving e^{2x} to appear, we presume that the particular solution involves some multiple of e^{2x} in it. Therefore, we guess $y_p(x) = Ce^{2x}$ for some constant C.

$$y_p = Ce^{2x} \implies y_p' = 2Ce^{2x} \implies y_p'' = 4Ce^{2x}$$
$$\Downarrow$$
$$y_p'' - 3y_p' = 4Ce^{2x} - 3 \cdot 2Ce^{2x} = -2Ce^{2x}$$

In order for $-2Ce^{2x}$ to match $g(x) = 10e^{2x}$, we must have $-2C = 10 \implies C = -5$. This tells us that one possible particular solution is $y_p(x) = -5e^{2x}$.

Now we know that the general solution to this non-homogeneous LDE is $y(x) = -5e^{2x} + A + Be^{3x}$.

Just like with partial fraction decomposition, this "guess and check" method turns questions into simple algebra *as long as you make the correct shape of guess.*

Illustration 2.4.1

Find the general solution to each CC LDE below.

(a) $y'' - 2y' - 3y = 6x - 5$

(b) $y'' + y = 4x^2$

(c) $y''' - 2y'' + y' = 50\sin(2x)$

(a) First we find the homogeneous solutions by using the characteristic equation:

$$r^2 - 2r - 3 = 0 \implies (r-3)(r+1) = 0 \implies r = 3, -1 \implies y_1 = e^{3x},\ y_2 = e^{-x}$$

Now we guess the shape of our particular solution. Notice that if y_p is a polynomial, then every derivative $y_p^{(k)}$ will have smaller degree than the original function y_p. This tells us that if we start with a linear polynomial, the right side of the equation will stay linear (or smaller).

This leads us to guess $y_p = Cx + D \implies y_p' = C \implies y_p'' = 0$:

$$y_p'' - 2y_p' - 3y_p = 0 - 2(C) - 3(Cx + D) = -3Cx + (-2C - 3D)$$

Setting this equal to the function $g(x) = 6x - 5$, we solve the system

$$\begin{cases} -3C = 6 \\ -2C - 3D = -5 \end{cases} \implies \begin{cases} C = -2 \\ D = 3 \end{cases}$$

We have found $y_p(x) = -2x + 3$ is a particular solution to the non-homogeneous LDE.

Therefore, our general solution is $y(x) = -2x + 3 + Ae^{3x} + Be^{-x}$

(b) First we find the homogeneous solutions by using the characteristic equation:

$$r^2 + 1 = 0 \implies r = \pm i \implies y_1 = \cos(x),\ y_2 = \sin(x)$$

Now we guess the shape of our particular solution. This time, if we start with a quadratic polynomial, the right side of the equation will stay quadratic (or smaller).

This leads us to guess $y_p = Cx^2 + Dx + E \implies y_p' = 2Cx + D \implies y_p'' = 2C$:

$$y_p'' + y_p = 2C + \left(Cx^2 + Dx + E\right) = Cx^2 + Dx + (E + 2C)$$

Setting this equal to the function $g(x) = 4x^2$, we solve the system

$$\begin{cases} C = 4 \\ D = 0 \\ E + 2C = 0 \end{cases} \implies \begin{cases} C = 4 \\ D = 0 \\ E = -8 \end{cases}$$

We have found $y_p(x) = 4x^2 - 8$ is a particular solution to the non-homogeneous LDE.

Therefore, our general solution is $y(x) = 4x^2 - 8 + A\cos(x) + B\sin(x)$

(c) First we find the homogeneous solutions by using the characteristic equation:

$$r^3 - 2r^2 + r = 0 \implies r(r-1)^2 = 0 \implies r = 0, 1 \text{ (double)} \implies y_1 = 1,\ y_2 = e^x,\ y_3 = xe^x$$

Now we guess the shape of our particular solution. Notice that if y_p is a sine *or* cosine function, then every derivative $y_p^{(k)}$ will contain sines and/or cosines. This tells us that if we start with either sines or cosines, the right side of the equation will contain sines and/or cosines. We do not know which one it would take to make our exact $g(x)$, so to be safe we use both sine and cosine: $y_p = A\cos(2x) + B\sin(2x)$.

This has derivatives $y_p' = -2A\sin(2x) + 2B\cos(2x),\ y_p'' = -4A\cos(2x) - 4B\sin(2x),\ y_p''' = 8A\sin(2x) - 8B\cos(2x)$:

$$y_p''' - 2y_p'' + y_p' = (8A\sin(2x) - 8B\cos(2x)) - 2(-4A\cos(2x) - 4B\sin(2x)) + (-2A\sin(2x) + 2B\cos(2x)) = (6A + 8B)\sin(2x) + (8$$

Setting this equal to the function $g(x) = 50\sin(2x)$, we solve the system

$$\begin{cases} 6A + 8B = 50 \\ 8A - 6B = 0 \end{cases} \implies \begin{cases} A = 3 \\ B = 4 \end{cases}$$

We have found $y_p(x) = 3\cos(2x) + 4\sin(2x)$ is a particular solution to the non-homogeneous LDE.

Therefore, our general solution is $y(x) = 3\cos(2x) + 4\sin(2x) + C + De^x + Exe^x$

We see that when the coefficients are all constants, we can use logic to deduce the right "guess" for the shape of y_p. The table below summarizes what guess we should use based on the shape of $g(x)$ in a CC LDE.

$g(x)$	General guess for y_p
1 (or any constant)	$y_p = A$
x (or any linear function)	$y_p = Ax + B$
x^2 (or any quadratic function)	$y_p = Ax^2 + Bx + C$
x^n (or any polynomial of degree n)	$y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
e^{kx}	$y_p = Ae^{kx}$
xe^{kx}	$y_p = (Ax + B)e^{kx}$
x^2e^{kx}	$y_p = (Ax^2 + Bx + C)e^{kx}$
$cos\,(kx)$	$y_p = A\cos(kx) + B\sin(kx)$
$sin(kx)$	$y_p = A\cos(kx) + B\sin(kx)$
$x\cos(kx)$	$y_p = (A_1x + A_0)\cos(kx) + (B_1x + B_0)\sin(kx)$
$x\sin(kx)$	$y_p = (A_1x + A_0)\cos(kx) + (B_1x + B_0)\sin(kx)$
$x^2\cos(kx)$	$y_p = (A_2x^2 + A_1x + A_0)\cos(kx) + (B_2x^2 + B_1x + B_0)\sin(kx)$
$x^2\sin(kx)$	$y_p = (A_2x^2 + A_1x + A_0)\cos(kx) + (B_2x^2 + B_1x + B_0)\sin(kx)$
$e^{jx}\cos(kx)$	$y_p = Ae^{jx}\cos(kx) + Be^{jx}\sin(kx)$
$e^{jx}\sin(kx)$	$y_p = Ae^{jx}\cos(kx) + Be^{jx}\sin(kx)$
$xe^{jx}\cos(kx)$	$y_p = (A_1x + A_0)\,e^{jx}\cos(kx) + (B_1x + B_0)\,e^{jx}\sin(kx)$
$xe^{jx}\sin(kx)$	$y_p = (A_1x + A_0)\,e^{jx}\cos(kx) + (B_1x + B_0)\,e^{jx}\sin(kx)$
$x^2e^{jx}\cos(kx)$	$y_p = (A_2x^2 + A_1x + A_0)\,e^{jx}\cos(kx) + (B_2x^2 + B_1x + B_0)\,e^{jx}\sin(kx)$
$x^2e^{jx}\sin(kx)$	$y_p = (A_2x^2 + A_1x + A_0)\,e^{jx}\cos(kx) + (B_2x^2 + B_1x + B_0)\,e^{jx}\sin(kx)$

Students should not try to memorize this exact table, but rather the pattern inside it:

1. The particular guess whenever a polynomial is involved is a general polynomial of the same degree.
2. The particular guess whenever an exponential times a polynomial is involved is the general polynomial of the same degree multiplied by the exponential.
3. The particular guess whenever sine or cosine are involved uses *both* sine and cosine.
4. The particular guess whenever sine or cosine are multiplied by some coefficient function uses the standard guess for that coefficient function times *both* sine and cosine.

Example 2.4.1

Find the unique solution to the IVP $y''' + y'' = 4x^2e^x - 90\cos(3x)$ with $y(0) = 0,\ y'(0) = 0,\ y''(0) = -26$

Solution

First we find the homogeneous solutions by using the characteristic equation:

$$r^3 + r^2 = 0 \implies r^2(r+1) = 0 \implies r = -1, 0\ (\text{double}) \implies y_1 = e^{-x},\ y_2 = 1,\ y_3 = x$$

Now we guess the shape of our particular solution. We break $g(x) = 4x^2e^x - 90\cos(3x)$ into two pieces.

To get the $4x^2e^x$ term, we know to guess $y_p = (Ax^2 + Bx + C)e^x$

$$y_p''' + y_p'' = \left(Ax^2 + (6A + B)x + (6A + 3B + C)\right)e^x + \left(Ax^2 + (4A + B)x + (2A + 2B + C)\right)e^x = \left(2Ax^2 + (10A + 2B)x + (8A + 5B$$

Setting this equal to the function $g_2(x) = 4x^2e^x$, we solve the system

$$\begin{cases} 2A = 4 \\ 10A + 2B = 0 \\ 8A + 5B + 2C = 0 \end{cases} \implies \begin{cases} A = 2 \\ B = -10 \\ C = 17 \end{cases}$$

We have found $y_{1,p}(x) = \left(2x^2 - 10x + 17\right)e^x$ is the first half of our particular solution.

To get the $-90\cos(3x)$ term, we know to guess $y_p = D\cos(3x) + E\sin(3x)$

$$y_p''' + y_p'' = (27D\sin(3x) - 27E\cos(3x)) + (-9D\cos(3x) - 9E\sin(3x)) = (-9D - 27E)\cos(3x) + (27D - 9E)\sin(3x)$$

Setting this equal to the function $g_1(x) = -90\cos(3x)$, we solve the system

$$\begin{cases} -9D - 27E = -90 \\ 27D - 9E = 0 \end{cases} \implies \begin{cases} D = 1 \\ E = 3 \end{cases}$$

We have found $y_{2,p}(x) = \cos(3x) + 3\sin(3x)$ is the second half of our particular solution.

Therefore, our particular solution is $y_p(x) = \left(2x^2 - 10x + 17\right)e^x + \cos(3x) + 3\sin(3x)$

We know our general solution must be $y(x) = \left(2x^2 - 10x + 17\right)e^x + \cos(3x) + 3\sin(3x) + Fe^{-x} + G + Hx$

Finally, we can plug in our initial conditions:

$$\begin{aligned} y(0) &= 17 + 1 + 0 + F \cdot 1 + G + H \cdot 0 \implies F + G = -18 \\ y' &= \left(2x^2 - 6x + 7\right)e^x - 3\sin(3x) + 9\cos(3x) - Fe^{-x} + H \\ y'(0) &= 7 - 0 + 9 - F \cdot 1 + H \implies -F + H = -16 \\ y'' &= \left(2x^2 - 2x + 1\right)e^x - 9\cos(3x) - 27\sin(3x) + Fe^{-x} \\ y''(0) &= 1 - 9 + 0 + F \cdot 1 \implies F = -18 \end{aligned}$$

This gives us the system

$$\begin{cases} F + G = -18 \\ -F + H = -16 \\ F = -18 \end{cases} \implies \begin{cases} F = -18 \\ G = 0 \\ H = -34 \end{cases}$$

Thus, the unique solution to this IVP is $\boxed{y(x) = \left(2x^2 - 10x + 17\right)e^x + \cos(3x) + 3\sin(3x) - 18e^{-x} - 34x}$

There is one wrinkle in the Undetermined Coefficient method, and that is when your "guess" for the particular solution matches one of your homogeneous solutions.

Consider the first order CC LDE $y' - y = 8e^x$.

If we attempt this problem using the methods in this subsection, we see the characteristic equation is $r - 1 = 0 \implies r = 1$.

Therefore, the homogeneous solutions are all multiples of $y_1(x) = e^x$.

But the function $g(x) = 8e^x$ would usually make us guess $y_p(x) = Ae^x$ for some A. When we plug this into the LDE:

$$y_p' - y_p = (Ae^x) - (Ae^x) = 0$$

The coefficient A disappears, and we cannot find any such constant that will yield $8e^x$ on the right side. This will happen *any time* that your guess for the particular solution contains components that are homogeneous solutions. If we use the methods of Chapter 1 to solve this LDE we see $p(x) = -1 \implies \mu(x) = e^{-x}$:

$$y = \frac{1}{\mu(x)} \int \mu(x) \cdot q(x)\, dx = \frac{1}{e^{-x}} \int e^{-x} \cdot 8e^{x}\, dx = e^{x} \int 8\, dx = e^{x}\left[8x + C\right] = 8xe^{x} + Ce^{x}$$

The particular solution is of the form $y_p = Axe^x$ instead of our original guess $y_p = Ae^x$. This is because the particular solution for a non-homogeneous LDE along with the homogeneous solutions must *always form a linearly independent set*.

Rule: Avoiding Homogeneous Duplication

If your guess for $y_p(x)$ contains one or more of your homogeneous solution functions, multiply it by x^k where k is the multiplicity of the root that yielded that homogeneous function.

Since $r = 1$ was a single root in the LDE $y' - y = 8e^x$, our "usual guess" of $y_p = Ae^x$ should be multiplied by x^1, giving $y_p(x) = Axe^x$.

For another example, consider the LDE $y''' = 48x$. This has characteristic equation $r^3 = 0$ so the homogeneous solutions are $y_1 = 1,\ y_2 = x,\ y_3 = x^2$.

The "usual guess" when $g(x) = 48x$ would be $y_p = Ax + B$, but instead we will multiply by x^3 (since $r = 0$ is a triple root) and use $y_p = Ax^4 + Bx^3$.

$$y_p''' = 24Ax + 6B \implies \begin{cases} 24A = 48 \\ 6B = 0 \end{cases} \implies \begin{cases} A = 2 \\ B = 0 \end{cases}$$

Therefore, we see that the particular solution is $y_p = 2x^4$ and our general solution will be $y(x) = 2x^4 + A + Bx + Cx^2$.

Example 2.4.2

Find the general solution to each LDE below.

(a) $y'' - 8y' + 16y = 12e^{4x}$

(b) $y'' + y = 4\sin(x)$

Solution

(a) The characteristic equation for this CC LDE is $r^2 - 8r + 16 = (r-4)^2 = 0$ which has double root $r = 4$

This gives us the homogeneous solution functions $y_1 = e^{4x}$ and $y_2 = xe^{4x}$

The function $g(x) = 12e^{4x}$ usually uses the guess $y_p = Ae^{4x}$, but since $r = 4$ is a double root we use $y_p = Ax^2e^{4x}$:

$$y_p'' - 8y_p' + 16y_p = \left(16Ax^2 + 16Ax + 2A\right)e^{4x} - 8\left(4x^2 + 2Ax\right)e^{4x} + 16\left(Ax^2\right)e^{4x} = 2Ae^{4x}$$

Setting this equal to $g(x)$ gives us $2A = 12 \implies A = 6$ so our particular solution is $y_p = 6x^2e^{4x}$

The general solution is therefore $\boxed{y(x) = 6x^2e^{4x} + Be^{4x} + Cxe^{4x}}$

(b) The characteristic equation for this CC LDE is $r^2 + 1 = 0$ which has roots $r = \pm i$.

This gives us the homogeneous solution functions $y_1 = \cos(x)$ and $y_2 = \sin(x)$

The function $g(x) = 4\sin(x)$ usually uses the guess $y_p = A\cos(x) + B\sin(x)$, but instead we use $y_p = Ax\cos(x) + Bx\sin(x)$:

$$y_p'' + y_p = \left[(2B - Ax)\cos(x) + (-2A - Bx)\sin(x)\right] + \left[Ax\cos(x) + Bx\sin(x)\right] = 2B\cos(x) - 2A\sin(x)$$

Setting this equal to $g(x)$ gives us $2B = 0 \implies B = 0$ and $-2A = 4 \implies A = -2$, so our particular solution is $y_p = -2x\cos(x)$

The general solution is therefore $\boxed{y(x) = -2x\cos(x) + C\cos(x) + D\sin(x)}$

The number of undetermined coefficients grows quickly as the function $g(x)$ increases in complexity, but we can use linearity to simplify these calculations. If $g(x) = g_1(x) + g_2(x)$, we can find the particular solution that yields $g_1(x)$ and add it to the particular solution that yields $g_2(x)$ to get one total particular solution that works in our LDE.

Illustration 2.4.2

We will find the general solution to the LDE $y''' - y'' - y' + y = 24xe^x - 24x^2e^{-x}$

The characteristic equation for this CC LDE is $r^3 - r^2 - r + 1 = (r-1)^2(r+1) = 0$, which has roots $r = -1,\ 1\ (\text{double})$

This gives us the homogeneous solution functions $y_1 = e^{-x},\ y_2 = e^x$, and $y_3 = xe^x$

The function $g(x) = 24xe^x - 24x^2e^{-x}$ can be split into two pieces: $g_1 = 12xe^x$ and $g_2 = -x^2e^{-x}$

The usual guess for $g_1 = 24xe^x$ would be $y_p = (Ax + B)e^x$, but since $r = 1$ is a double root we guess $y_p = (Ax^3 + Bx^2)e^x$:

$$\begin{aligned} y_p &= \left(Ax^3 + Bx^2\right)e^x \\ y_p' &= \left(Ax^3 + (3A+B)x^2 + 2Bx\right)e^x \\ y_p'' &= \left(Ax^3 + (6A+B)x^2 + (6A+4B)x + 2B\right)e^x \\ y_p''' &= \left(Ax^3 + (9A+B)x^2 + (18A+6B)x + 6A + 6B\right)e^x \\ \implies y_p''' - y_p'' - y_p' + y_p &= (12Ax + 6A + 4B)e^x \end{aligned}$$

Setting this equal to $g_1(x)$ gives $\begin{cases} 12A = 24 \\ 6A + 4B = 0 \end{cases} \implies \begin{cases} A = 2 \\ B = -3 \end{cases}$

Therefore, our first particular solution is $y_{1,p}(x) = \left(2x^3 - 3x^2\right)e^x$

The usual guess for $y_2 = -x^2e^{-x}$ is $y_p = \left(Cx^2 + Dx + E\right)e^{-x}$, but since $r = -1$ is a root we guess $y_p = \left(Cx^3 + Dx^2 + Ex\right)e^{-x}$:

$$\begin{aligned} y_p &= \left(Cx^3 + Dx^2 + Ex\right)e^{-x} \\ y_p' &= \left(-Cx^3 + (3C-D)x^2 + (2D-E)x + E\right)e^{-x} \\ y_p'' &= \left(Cx^3 - (6C-D)x^2 + (6C-4D+E)x + 2D - 2E\right)e^{-x} \\ y_p''' &= \left(-Cx^3 + (9C-D)x^2 - (18C-6D+E)x + 6C - 6D + 3E\right)e^{-x} \\ \implies y_p''' - y_p'' - y_p' + y_p &= \left(12Cx^2 + (-24C+8D)x + (6C-8D+4E)\right)e^{-x} \end{aligned}$$

Setting this equal to $g_2(x)$ gives $\begin{cases} 12C = 24 \\ -24C + 8D = 0 \\ 6C - 8D + 4E = 0 \end{cases} \implies \begin{cases} C = 2 \\ D = 6 \\ E = 9 \end{cases}$

Therefore, our second particular solution is $y_{2,p}(x) = \left(2x^3 + 6x^2 + 9x\right)e^{-x}$

Putting this together gives our general solution $y(x) = \left(2x^3 - 3x^2\right)e^x + \left(2x^3 + 6x^2 + 9x\right)e^{-x} + Fe^x + Gxe^x + He^{-x}$

We observe that the method of undetermined coefficients, while systematic, relies fundamentally on a process of explicit guesswork and algebraic verification to construct particular solutions to nonhomogeneous constant-coefficient linear differential equations.

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2.4: Practice Set

Exercise 2.4.1a

Find the general solution to $y'' - 5y' + 6y = 12$

Answer available at the end of this section.

Exercise 2.4.1b

Find the general solution to $2y'' + y' - y = 10\cos(x)$

Answer available at the end of this section.

Exercise 2.4.1c

Find the general solution to $y'' + 9y = 100e^{-x}$

Answer available at the end of this section.

Exercise 2.4.1d

Find the general solution to $y'' - 8y' + 16y = 16x + 56$

Answer available at the end of this section.

Exercise 2.4.1e

Find the general solution to $y'' + 4y' + 13y = 9e^{-2x} - 13$

Answer available at the end of this section.

Exercise 2.4.1f

Find the general solution to $y'' + y = 20e^{x}\sin(x)$

Answer available at the end of this section.

Exercise 2.4.2a

Find the general solution to $y'' + 3y' = 18x$

Answer available at the end of this section.

Exercise 2.4.2b

Find the general solution to $y'' - 4y = -16e^{2x}$

Answer available at the end of this section.

Exercise 2.4.2c

Find the general solution to $y'' + 16y = 8\cos(4x)$

Answer available at the end of this section.

Exercise 2.4.2d

Find the general solution to $y'' + 4y' + 4y = 12x(x+1)e^{-2x}$

Answer available at the end of this section.

Exercise 2.4.2e

Find the general solution to $y'' + 2y' + 2y = 10e^{-x}\cos(x)$

Answer available at the end of this section.

Exercise 2.4.2f

Find the general solution to $y^{(4)} - y = 16xe^{x}$

Answer available at the end of this section.

Exercise 2.4.2g

Find the general solution to $y''' - 3y'' + 3y' - y = 24e^{x}$

Answer available at the end of this section.

Exercise 2.4.3a

Find the general solution to the IVP $y'' + 2y' + 5y = 10$ with $y(0) = 1,\ y'(0) = 1$

Answer available at the end of this section.

Exercise 2.4.3b

Find the general solution to the IVP $y'' - 6y' + 8y = 32x$ with $y(0) = 0,\ y'(0) = 0$

Answer available at the end of this section.

Exercise $2.4.3c$

Find the general solution to the IVP $y'' + 4y' + 4y = 8e^{-2x}$ with $y(0) = 3,\ y'(0) = 1$

Answer available at the end of this section.

Exercise $2.4.3d$

Find the general solution to the IVP $y''' + y' = 4x$ with $y(0) = 0,\ y'(0) = 0,\ y''(0) = -1$

Answer available at the end of this section.

Exercise $2.4.3e$

Find the general solution to the BVP $y'' + 4y' = 20\sin(2x)$ with $y(0) = 0,\ y(\pi) = 0$

Answer available at the end of this section.

Exercise $2.4.3f$

Find the general solution to the IVP $y'' - y' - 2y = 4x^2 + 130\sin(3x)$ with $y(0) = 0,\ y'(0) = 2$

Answer available at the end of this section.

Answers

Exercise $2.4.1a$

The characteristic equation is $r^2 - 5r + 6 = (r-2)(r-3) = 0$ which has roots $r = 2, 3$

Therefore, our homogeneous solution is of the form $y_h = Ae^{2x} + Be^{3x}$

We will use particular guess $y_p = C \implies y_p' = 0, \quad y_p'' = 0$:

$$y_p'' - 5y_p' + 6y_p = 0 - 0 + 6C = 6C$$

Setting this equal to $g(x) = 12$ gives $6C = 12 \implies C = 2$

Therefore, $y_p = 2$ and our general solution is $y(x) = 2 + Ae^{2x} + Be^{3x}$

Exercise $2.4.1b$

The characteristic equation is $2r^2 + r - 1 = (2r-1)(r+1) = 0$ which has roots $r = \frac{1}{2}, -1$

Therefore, our homogeneous solution is of the form $y_h = Ae^{\frac{1}{2}x} + Be^{-x}$

We will use particular guess $y_p = C\cos(x) + D\sin(x) \implies y_p' = -C\sin(x) + D\cos(x), \quad y_p'' = -C\cos(x) - D\sin(x)$:

$$2y_p'' + y_p' - y_p = 2\left(-C\cos(x) - D\sin(x)\right) + \left(-C\sin(x) + D\cos(x)\right) - \left(C\cos(x) + D\sin(x)\right) = (-3C + D)\cos(x) + (-C - 3D)\sin(x)$$

Setting this equal to $g(x) = 10\cos(x) + 0\sin(x)$ gives $\begin{cases} -3C + D = 10 \\ -C - 3D = 0 \end{cases} \implies \begin{cases} C = -3 \\ D = 1 \end{cases}$

Therefore, $y_p = -3\cos(x) + \sin(x)$ and our general solution is $y(x) = -3\cos(x) + \sin(x) + Ae^{\frac{1}{2}x} + Be^{-x}$

Exercise $2.4.1c$

The characteristic equation is $r^2 + 9 = 0$ which has roots $r = \pm 3i$

Therefore, our homogeneous solution is of the form $y_h = A\cos(3x) + B\sin(3x)$

We will use particular guess $y_p = Ce^{-x} \implies y_p' = -Ce^{-x}, \quad y_p'' = Ce^{-x}$:

$$y_p'' + 9y_p = Ce^{-x} + 9(Ce^{-x}) = 10Ce^{-x}$$

Setting this equal to $g(x) = 100e^{-x}$ gives $10C = 100 \implies C = 10$

Therefore, $y_p = 10e^{-x}$ and our general solution is $y(x) = 10e^{-x} + A\cos(3x) + B\sin(3x)$

Exercise $2.4.1d$

The characteristic equation is $r^2 - 8r + 16 = (r-4)^2 = 0$ which has roots $r = 4$ (double)

Therefore, our homogeneous solution is of the form $y_h = Ae^{4x} + Bxe^{4x}$

We will use particular guess $y_p = Cx + D \implies y_p' = C, \quad y_p'' = 0$:

$$y_p'' - 8y_p' + 16y_p = 0 - 8C + 16(Cx + D) = 16Cx + (-8C + 16D)$$

Setting this equal to $g(x) = 16x + 56$ gives $\begin{cases} 16C = 16 \\ -8C + 16D = 56 \end{cases} \implies \begin{cases} C = 1 \\ D = 4 \end{cases}$

Therefore, $y_p = x + 4$ and our general solution is $y(x) = x + 4 + Ae^{4x} + Bxe^{4x}$

Exercise 2.4.1*e*

The characteristic equation is $r^2 + 4r + 13 = 0$ which has roots $r = -2 \pm 3i$

Therefore, our homogeneous solution is of the form $y_h = Ae^{-2x}\cos(3x) + Be^{-2x}\sin(3x)$

We will use particular guess $y_p = Ce^{-2x} + D \implies y_p' = -2Ce^{-2x}, \quad y_p'' = 4Ce^{-2x}$:

$$y_p'' + 4y_p' + 13y_p = 4Ce^{-2x} + 4(-2Ce^{-2x}) + 13(Ce^{-2x} + D) = 9Ce^{-2x} + 13D$$

Setting this equal to $g(x) = 9e^{-2x} - 13$ gives $\begin{cases} 9C = 9 \\ 13D = -13 \end{cases} \implies \begin{cases} C = 1 \\ D = -1 \end{cases}$

Therefore, $y_p = e^{-2x} - 1$ and our general solution is $y(x) = e^{-2x} - 1 + Ae^{-2x}\cos(3x) + Be^{-2x}\sin(3x)$

Exercise 2.4.1*f*

The characteristic equation is $r^2 + 1 = 0$ which has roots $r = \pm i$

Therefore, our homogeneous solution is of the form $y_h = A\cos(x) + B\sin(x)$

We will use particular guess $y_p = Ce^x\cos(x) + De^x\sin(x)$

This has derivatives $y_p' = (C + D)e^x\cos(x) + (D - C)e^x\sin(x), \quad y_p'' = 2De^x\cos(x) - 2Ce^x\sin(x)$:

$$y_p'' + y_p = (2De^x\cos(x) - 2Ce^x\sin(x)) + (Ce^x\cos(x) + De^x\sin(x)) = (C + 2D)e^x\cos(x) + (D - 2C)e^x\sin(x)$$

Setting this equal to $g(x) = 20e^x\sin(x)$ gives $\begin{cases} C + 2D = 0 \\ D - 2C = 20 \end{cases} \implies \begin{cases} C = -8 \\ D = 4 \end{cases}$

Therefore, $y_p = -8e^x\cos(x) + 4e^x\sin(x)$ and our general solution is $y(x) = (A - 8e^x)\cos(x) + (B + 4e^x)\sin(x)$

Exercise 2.4.2*a*

The characteristic equation is $r^2 + 3r = r(r + 3) = 0$ which has roots $r = 0, -3$

Therefore, our homogeneous solution is of the form $y_h = A + Be^{-3x}$

We would normally guess $y_p = Cx + D$, but instead use particular guess $y_p = Cx^2 + Dx$

This has derivatives $y_p' = 2Cx + D, \quad y_p'' = 2C$:

$$y_p'' + 3y_p' = 2C + 3(2Cx + D) = 6Cx + (2C + 3D)$$

Setting this equal to $g(x) = 18x$ gives $\begin{cases} 6C = 18 \\ 2C + 3D = 0 \end{cases} \implies \begin{cases} C = 3 \\ D = -2 \end{cases}$

Therefore, $y_p = 3x^2 - 2x$ and our general solution is $y(x) = 3x^2 - 2x + A + Be^{-3x}$

Exercise 2.4.2*b*

The characteristic equation is $r^2 - 4 = (r - 2)(r + 2) = 0$ which has roots $r = 2, -2$

Therefore, our homogeneous solution is of the form $y_h = Ae^{2x} + Be^{-2x}$

We would normally guess $y_p = Ce^{2x}$, but instead use particular guess $y_p = Cxe^{2x}$

This has derivatives $y_p' = C(1 + 2x)e^{2x}, \quad y_p'' = C(4 + 4x)e^{2x}$:

$$y_p'' - 4y_p = C(4 + 4x)e^{2x} - 4Cxe^{2x} = 4Ce^{2x}$$

Setting this equal to $g(x) = -16e^{2x}$ gives $4C = -16 \implies C = -4$

Therefore, $y_p = -4xe^{2x}$ and our general solution is $y(x) = -4xe^{2x} + Ae^{2x} + Be^{-2x}$

Exercise 2.4.2*c*

The characteristic equation is $r^2 + 16 = 0$ which has roots $r = \pm 4i$

Therefore, our homogeneous solution is of the form $y_h = A\cos(4x) + B\sin(4x)$

We would normally guess $y_p = C\cos(4x) + D\sin(4x)$, but instead use particular guess $y_p = Cx\cos(4x) + Dx\sin(4x)$

This has derivatives $y_p' = (C + 4Dx)\cos(4x) + (D - 4Cx)\sin(4x), \quad y_p'' = (-8C - 16Dx)\sin(4x) + (8D - 16Cx)\cos(4x)$:

$$y_p'' + 16y_p = (-8C - 16Dx)\sin(4x) + (8D - 16Cx)\cos(4x) + 16\left(Cx\cos(4x) + Dx\sin(4x)\right) = -8C\sin(4x) + 8D\cos(4x)$$

Setting this equal to $g(x) = 8\cos(4x) + 0\sin(4x)$ gives $\begin{cases} -8C = 0 \\ 8D = 8 \end{cases} \implies \begin{cases} C = 0 \\ D = 1 \end{cases}$

Therefore, $y_p = x\sin(4x)$ and our general solution is $y(x) = x\sin(4x) + A\cos(4x) + B\sin(4x)$

Exercise 2.4.2*d*

The characteristic equation is $r^2 + 4r + 4 = (r + 2)^2 = 0$ which has roots $r = -2$ (double)

Therefore, our homogeneous solution is of the form $y_h = (A + Bx)e^{-2x}$

We would normally guess $y_p = (Cx^2 + Dx + E)e^{-2x}$, but instead use particular guess $y_p = (Cx^4 + Dx^3 + Ex^2)e^{-2x}$ which has derivatives

$$y_p' = \left(-2Cx^4 + (4C - 2D)x^3 + (3D - 2E)x^2 + 2Ex\right)e^{-2x}$$
$$y_p'' = \left(4Cx^4 + (-16C + 4D)x^3 + (12C - 12D + 4E)x^2 + 2(3D - 6E)x + 2E\right)e^{-2x}$$
$$y_p'' + 4y_p' + 4y_p = \left(12Cx^2 + (6D - 4E)x + 2E\right)e^{-2x}$$

Setting this equal to $g(x) = (12x^2 + 12x + 0)e^{-2x}$ gives $\begin{cases} 12C = 12 \\ 6D - 4E = 12 \\ 2E = 0 \end{cases} \implies \begin{cases} C = 1 \\ D = 2 \\ E = 0 \end{cases}$

Therefore, $y_p = (x^4 + 2x^3)e^{-2x}$ and our general solution is $y(x) = \left(A + Bx + 2x^3 + x^4\right)e^{-2x}$

Exercise 2.4.2*e*

The characteristic equation is $r^2+2r+2=0$ which has roots $r=-1\pm i$

Therefore, our homogeneous solution is of the form $y_h=Ae^{-x}\cos(x)+Be^{-x}\sin(x)$

We would normally guess $y_p=Ce^{-x}\cos(x)+De^{-x}\sin(x)$, but instead use particular guess $y_p=Cxe^{-x}\cos(x)+Dxe^{-x}\sin(x)$

This has derivatives $y_p'=(-Cx+Dx+C)e^{-x}\cos(x)+(-Cx-Dx+C)e^{-x}\sin(x),\quad y_p''=2(-Dx+D-C)e^{-x}\cos(x)+2(Cx-C-D)e^{-x}\sin(x)$:

$$y_p''+2y_p'+2y_p=-2Ce^{-x}\sin(x)+2De^{-x}\cos(x)$$

Setting this equal to $g(x)=0e^{-x}\sin(x)+10e^{-x}\cos(x)$ gives $\begin{cases}-2C=0\\2D=10\end{cases}\implies\begin{cases}C=0\\D=5\end{cases}$

Therefore, $y_p=5xe^{-x}\sin x$ and our general solution is $y(x)=Ae^{-x}\cos(x)+(B+5x)e^{-x}\sin(x)$

Exercise 2.4.2*f*

The characteristic equation is $r^4-1=(r-1)(r+1)(r^2+1)=0$ which has roots $r=1,-1,\pm i$

Therefore, our homogeneous solution is of the form $y_h=Ae^{x}+Be^{-x}+C\cos(x)+D\sin(x)$

We would normally guess $y_p=(Ex+F)e^x$, but instead use particular guess $y_p=(Ex^2+Fx)e^x$ which has derivatives:

$$\begin{aligned}y_p'&=\left(Ex^2+(2E+F)x+F\right)e^x\\y_p''&=\left(Ex^2+(4E+F)x+(2E+2F)\right)e^x\\y_p'''&=\left(Ex^2+(6E+F)x+(6E+3F)\right)e^x\\y_p^{(4)}&=\left(Ex^2+(8E+F)x+(12E+4F)\right)e^x\end{aligned}$$

$$y_p^{(4)}-y_p=\left(Ex^2+(8E+F)x+(12E+4F)\right)e^x-(Ex^2+Fx)e^x=\left(8Ex+(12E+4F)\right)e^x$$

Setting this equal to $g(x)=16xe^x$ gives $\begin{cases}8E=16\\12E+4F=0\end{cases}\implies\begin{cases}E=2\\F=-6\end{cases}$

Therefore, $y_p=(2x^2-6x)e^x$ and our general solution is $y(x)=(2x^2-6x+A)e^x+Be^{-x}+C\cos(x)+D\sin(x)$

Exercise 2.4.2*g*

The characteristic equation is $r^3-3r^2+3r-1=(r-1)^3=0$ which has roots $r=1$ (triple)

Therefore, our homogeneous solution is of the form $y_h=(A+Bx+Cx^2)e^x$

We would normally guess $y_p=De^x$, but instead use particular guess $y_p=Dx^3e^x$

This has derivatives $y_p'=D(3x^2+x^3)e^x,\ y_p''=D(6x+6x^2+x^3)e^x,\ y_p'''=D(6+18x+9x^2+x^3)e^x$:

$$y_p'''-3y_p''+3y_p'-y_p=D(6+18x+9x^2+x^3)e^x-3D(6x+6x^2+x^3)e^x+3D(3x^2+x^3)e^x-Dx^3e^x=6De^x$$

Setting this equal to $g(x)=24e^x$ gives $6D=24\implies D=4$

Therefore, $y_p=4x^3e^x$ and our general solution is $y(x)=(A+Bx+Cx^2+4x^3)e^x$

Exercise 2.4.3*a*

The characteristic equation is $r^2+2r+5=0$ which has roots $r=-1\pm 2i$

Therefore, our homogeneous solution is of the form $y_h=Ae^{-x}\cos(2x)+Be^{-x}\sin(2x)$

We will use particular guess $y_p=C\implies y_p'=0,\ y_p''=0$:

$$y_p''+2y_p'+5y_p=0+0+5C$$

Setting this equal to $g(x)=10$ gives $5C=10\implies C=2$

Therefore, $y_p=2$ and our general solution is $y(x)=2+Ae^{-x}\cos(2x)+Be^{-x}\sin(2x)$

Plugging in the initial conditions:

$$\begin{aligned}y(x)&=2+Ae^{-x}\cos(2x)+Be^{-x}\sin(2x)\\y(0)=1&\implies 1=2+A\cdot1+B\cdot0\implies A=-1\\y'(x)&=-Ae^{-x}\cos(2x)-2Ae^{-x}\sin(2x)-Be^{-x}\sin(2x)+2Be^{-x}\cos(2x)\\y'(0)=1&\implies 1=-A\cdot1-2A\cdot0-B\cdot0+2B\cdot1\implies -A+2B=1\implies B=0\end{aligned}$$

Therefore, our unique solution is $y(x)=2-e^{-x}\cos(2x)$

Exercise 2.4.3*b*

The characteristic equation is $r^2-6r+8=(r-2)(r-4)=0$ which has roots $r=2,4$

Therefore, our homogeneous solution is of the form $y_h=Ae^{2x}+Be^{4x}$

We will use particular guess $y_p=Cx+D\implies y_p'=C,\ y_p''=0$:

$$y''-6y'+8y=0-6C+8(Cx+D)=8Cx+(8D-6C)$$

Setting this equal to $g(x)=32x+0$ gives $\begin{cases}8C=32\\8D-6C=0\end{cases}\implies\begin{cases}C=4\\D=3\end{cases}$

Therefore, $y_p=4x+3$ and our general solution is $y(x)=4x+3+Ae^{2x}+Be^{4x}$

Plugging in the initial conditions:

$$\begin{aligned}y(x)&=4x+3+Ae^{2x}+Be^{4x}\\y(0)=0&\implies 0=3+A\cdot1+B\cdot1\implies A+B=-3\\y'(x)&=4+2Ae^{2x}+4Be^{4x}\\y'(0)=0&\implies 0=4+2A\cdot1+4B\cdot1\implies 2A+4B=-4\\&\begin{cases}A+B=-3\\2A+4B=-4\end{cases}\implies\begin{cases}A=-4\\B=1\end{cases}\end{aligned}$$

Therefore, our unique solution is $y(x)=4x+3-4e^{2x}+e^{4x}$

Exercise 2.4.3*c*

The characteristic equation is $r^2+4r+4=(r+2)^2=0$ which has roots $r=-2$ (double)

Therefore, our homogeneous solution is of the form $y_h=(A+Bx)e^{-2x}$

We will use particular guess $y_p=Cx^2e^{-2x} \implies y_p'=2C(x-x^2)e^{-2x},\ y_p''=2C(1-4x+2x^2)e^{-2x}$:

$$y_p''+4y_p'+4y_p=2Ce^{-2x}$$

Setting this equal to $g(x)=8e^{-2x}$ gives $2C=8 \implies C=4$

Therefore, $y_p=4x^2e^{-2x}$ and our general solution is $y(x)=(A+Bx+4x^2)e^{-2x}$

Plugging in the initial conditions:

$$\begin{aligned} y(x)&=(A+Bx+4x^2)e^{-2x}\\ y(0)=3 &\implies 3=(A+B\cdot 0+0)\cdot 1 \implies A=3\\ y'(x)&=-2(A+Bx+4x^2)e^{-2x}+(B+8x)e^{-2x}\\ y'(0)=1 &\implies 1=-2A\cdot 1+B\cdot 1 \implies B=7 \end{aligned}$$

Therefore, our unique solution is $y(x)=(3+7x+4x^2)e^{-2x}$

Exercise 2.4.3*d*

The characteristic equation is $r^3+r=r(r^2+1)=0$ which has roots $r=0,\pm i$

Therefore, our homogeneous solution is of the form $y_h=A+B\cos(x)+C\sin(x)$

We will use particular guess $y_p=y_p=Dx^2+Ex$

This has derivatives $y_p'=2Dx+E,\ y_p''=2D,\ y_p'''=0$:

$$y_p'''+y_p'=0+(2Ax+B)=2Ax+B$$

Setting this equal to $g(x)=4x+0$ gives $\begin{cases} 2A=4\\ B=0 \end{cases} \implies \begin{cases} A=2\\ B=0 \end{cases}$

Therefore, $y_p=2x^2$ and our general solution is $y(x)=2x^2+A+B\cos(x)+C\sin(x)$

Plugging in the initial conditions:

$$\begin{aligned} y(x)&=2x^2+A+B\cos(x)+C\sin(x)\\ y(0)=0 &\implies 0=0+A+B\cdot 1+C\cdot 0 \implies A+B=0\\ y'(x)&=4x-B\sin(x)+C\cos(x)\\ y'(0)=0 &\implies 0=0-B\cdot 0+C\cdot 1 \implies C=0\\ y''(x)&=4-B\cos(x)-C\sin(x)\\ y''(0)=-1 &\implies -1=4-B\cdot 1-C\cdot 0 \implies B=5\\ A+B=0 &\implies A=-5 \end{aligned}$$

Therefore, our unique solution is $y(x)=2x^2-5+5\cos(x)$

Exercise 2.4.3*e*

The characteristic equation is $r^2+4r=r(r+4)=0$ which has roots $r=0,-4$

Therefore, our homogeneous solution is of the form $y_h=A+Be^{-4x}$

We will use particular guess $y_p=y_p=C\cos(2x)+D\sin(2x)$

This has derivatives $y_p'=-2C\sin(2x)+2D\cos(2x),\quad y_p''=-4C\cos(2x)-4D\sin(2x)$:

$$y_p''+4y_p'=(-4C+8D)\cos(2x)+(-4D-8C)\sin(2x)$$

Setting this equal to $g(x)=0\cos(2x)+20\sin(2x)$ gives $\begin{cases} -4C+8D=0\\ -4D-8C=20 \end{cases} \implies \begin{cases} C=-2\\ D=-1 \end{cases}$

Therefore, $y_p=-2\cos(2x)-\sin(2x)$ and our general solution is $y(x)=-2\cos(2x)-\sin(2x)+A+Be^{-4x}$

Plugging in the boundary conditions:

$$\begin{aligned} y(x)&=-2\cos(2x)-\sin(2x)+A+Be^{-4x}\\ y(0)=0 &\implies 0=-2+A+B\cdot 1 \implies A+B=2\\ y(\pi)=0 &\implies 0=-2+A+B\cdot e^{-4\pi} \implies A+e^{-4\pi}B=-2\\ \begin{cases} A+B=2\\ A+e^{\pi}B=2 \end{cases} &\implies \begin{cases} A=2\\ B=0 \end{cases} \end{aligned}$$

Therefore, our unique solution is $y(x)=-2\cos(2x)-\sin(2x)+2$

Exercise 2.4.3*f*

The characteristic equation is $r^2-r-2=(r-2)(r+1)=0$ which has roots $r=2,-1$

Therefore, our homogeneous solution is of the form $y_h=Ae^{2x}+Be^{-x}$

We will use particular guess $y_p=Cx^2+Dx+E+F\cos(3x)+G\sin(3x)$

This has derivatives $y_p'=2Cx+D-3F\sin(3x)+3G\cos(3x),\quad y_p''=2C-9F\cos(3x)-9G\sin(3x)$:

$$y_p''-y_p'-2y_p=-2Cx^2+(-2D-2C)x+(-2E-D+2C)+(-11F-3G)\cos(3x)+(-11G+3F)\sin(3x)$$

Setting this equal to $g(x)=4x^2+0x+0+0\cos(3x)+130\sin(3x)$ gives $\begin{cases} -2C=4\\ -2D-2C=0\\ -2E-D+2C=0\\ -11F-3G=0\\ -11G+3F=130 \end{cases} \implies \begin{cases} C=-2\\ D=2\\ E=-3\\ F=3\\ G=-11 \end{cases}$

Therefore, $y_p=-2x^2+2x+1+3\cos(3x)-11\sin(3x)$ and our general solution is $y(x)=-2x^2+2x-3+3\cos(3x)-11\sin(3x)+Ae^{2x}+Be^{-x}$

Plugging in the initial conditions:

$$
\begin{aligned}
y(x) &= -2x^2 + 2x - 3 + 3\cos(3x) - 11\sin(3x) + Ae^{2x} + Be^{-x} \\
y(0) = 4 \implies & \quad 4 = 4 + A \cdot 1 + B \cdot 1 \implies A + B = 0 \\
y'(x) &= -4x + 2 - 9\sin(3x) - 33\cos(3x) + 2Ae^{2x} - Be^{-x} \\
y' &= -4x + 2 - 9\sin(3x) - 33\cos(3x) + 2Ae^{2x} - Be^{-x} \\
y'(0) = 2 \implies & \quad 2 = -31 + 2A \cdot 1 - B \cdot 1 \implies 2A - B = 33 \\
& \begin{cases} A + B = 0 \\ 2A - B = 33 \end{cases} \implies \begin{cases} A = 11 \\ B = -11 \end{cases}
\end{aligned}
$$

Therefore, our unique solution is $y(x) = -2x^2 + 2x - 3 + 3\cos(3x) - 11\sin(3x) + 11e^{2x} - 11e^{-x}$

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2.5: Variation of Parameters

Learning Objectives

- Derive the Variation of Parameters formulas for a second-order linear ODE.
- Apply the method to solve nonhomogeneous constant coefficient equations.
- Apply the method to solve nonhomogeneous Cauchy-Euler equations.
- Understand the advantages and limitations of Variation of Parameters compared to Undetermined Coefficients.

For equations where Undetermined Coefficients is difficult or inapplicable, Variation of Parameters offers a more general, formula-based approach. This method works for any linear equation where the homogeneous solution is known, deriving a particular solution by "varying the constants" in the homogeneous solution.

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2.5: Variation of Parameters for 2nd Order LDEs

It is not always apparent what "guess" you should start with in the Method of Undetermined Coefficients when solving a non-homogeneous second order LDE.

$$y'' + p(x)\,y' + q(x)y = g(x)$$

This can be because of a particularly unreasonable function $g(x)$, or because the coefficient functions $p(x)$ and $q(x)$ are non-constant (like in a CE LDE).

For this reason, we derive a second method for solving these LDEs. Assuming we know the two linearly independent homogeneous solutions $y_1(x)$ and $y_2(x)$, we define a substitution similar to the reduction of order:

$$y(x) = u_1(x) \cdot y_1(x) + u_2(x) \cdot y_2(x)$$

We can see by inspection that our general solution will have u_1 and u_2 each have some $+C_1$ and $+C_2$ respectively (this will yield the homogeneous form of our solution set). The goal is to find the rest of these functions, which will be the components that combine to make our particular solution. Thus, by *varying* these coefficient functions from constants to functions, we will be able to recover y_p.

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ y' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ y'' &= u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2'' \end{aligned}$$

Plugging these into the LDE $y'' + py' + qy = g$gives:

$$\begin{gathered} [u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''] + p\,[u_1'y_1 + u_1y_1' + u_2'y_2 + u^2y_2'] + q\,[u_1y_1 + u_2y_2] = g \\ \Downarrow \\ u_1 \cdot \underbrace{[y_1'' + py_1' + qy_1]}_{y_1 \text{ homogeneous solution} \implies 0} + u_2 \cdot \underbrace{[y_2'' + py_2' + qy_2]}_{y_2 \text{ homogeneous solution} \implies 0} + u_1''y_1 + 2u_1'y_1' + u_2''y_2 + 2u_2'y_2' + p\,[u_1'y_1 + u_2'y_2] = g \\ \Downarrow \\ u_1''y_1 + 2u_1'y_1' + u_2''y_2 + 2u_2'y_2' + p\,[u_1'y_1 + u_2'y_2] = g \\ \Downarrow \\ (u_1'y_1 + u_2'y_2)' + u_1'y_1' + u_2'y_2' + p\,[u_1'y_1 + u_2'y_2] = g \end{gathered}$$

Since there are two unknown functions u_1 and u_2 and only one equation, there will be infinitely many such pairs of solution functions to this equation. We can implement an additional restriction to narrow it down to a unique solution, and it seems a good choice would be to set the term $u_1'y_1 + u_2'y_2 = 0$ to simplify the resulting equation above greatly:

$$\begin{gathered} \underbrace{(u_1'y_1 + u_2'y_2)'}_{=0} + u_1'y_1' + u_2'y_2' + p\,\underbrace{[u_1'y_1 + u_2'y_2]}_{=0} = g \\ \Downarrow \\ u_1'y_1' + u_2'y_2' = g \end{gathered}$$

Thus, the resulting system of equations that u_1' and u_2' must satisfy is

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g \end{cases}$$

We can solve this by using Cramer's Rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

We see the denominator is the *Wronskian* of y_1 and y_2, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, which will be non-zero.

We denote the corresponding numerators as $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}$ and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$.

Now that we have expressions for u_1' and u_2' we can integrate them to solve for our desired functions u_1 and u_2:

$$u_1(x) = \int \frac{W_1}{W}\,dx \quad \text{and} \quad u_2(x) = \int \frac{W_2}{W}\,dx$$

Then, we can substitute them into our expression $y = u_1y_1 + u_2y_2$ to find our general solution.

Rule: Variation of Parameters

If y_1 and y_2 are linearly independent solutions to the homogeneous second order LDE

$$y'' + p(x)\,y' + q(x)y = 0$$

Then, the function $y = u_1y_1 + u_2y_2$ is the general solution to the second order LDE

$$y'' + p(x)\,y' + q(x)y = g(x)$$

Where

$$u_1(x) = \int \frac{W_1}{W}\,dx \quad \text{and} \quad u_2(x) = \int \frac{W_2}{W}\,dx$$

Let's demonstrate this method in action with a problem we know how to solve using the Method of Undetermined Coefficients:

Illustration 2.5.1

We use the method of Variation of Parameters to solve the CC LDE:

$$y'' + 4y = 8$$

The characteristic equation is $r^2 + 4 = 0$, which has roots $r = \pm 2i$

This means our two linearly independent homogeneous solutions are $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$

We can see that the function $g(x) = 8$, so we find each of the determinants necessary:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = \cos(2x)\cdot 2\cos(2x) - \sin(2x)\cdot(-2\sin(2x)) = 2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin(2x) \\ 8 & 2\cos(2x) \end{vmatrix} = 0\cdot 2\cos(2x) - \sin(2x)\cdot 8 = -8\sin(2x)$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} \cos(2x) & 0 \\ -2\sin(2x) & 8 \end{vmatrix} = \cos(2x)\cdot 8 - 0\cdot(-2\sin(2x)) = 8\cos(2x)$$

Now we set up and solve each integral to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{-8\sin(2x)}{2}\,dx = \int -4\sin(2x)\,dx = 2\cos(2x) + C$$

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{8\cos(2x)}{2}\,dx = \int 4\cos(2x)\,dx = 2\sin(2x) + D$$

This gives us our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (2\cos(2x) + C)\cos(2x) + (2\sin(2x) + D)\sin(2x) \\ &= 2\cos^2(x) + C\cos(2x) + 2\sin^2(2x) + D\sin(2x) \\ &= \left(2\cos^2(2x) + 2\sin^2(2x)\right) + C\cos(2x) + D\sin(2x) \\ &= 2 + C\cos(2x) + D\sin(2x) \end{aligned}$$

Notice that this is exactly the same solution we would have gotten using the method of our previous subchapter:

$$\begin{gathered} y_p = A \implies y_p' = 0,\ y_p'' = 0 \\ y_p'' + 4y_p = 0 + 4 \cdot C = 4C \implies 4C = 8 \implies C = 2 \implies y_p = 2 \\ y(x) = 2 + C\cos(2x) + D\sin(2x) \end{gathered}$$

The specific LDE in Illustration $2.5.1$ seems easier to solve using Undetermined Coefficients, but some problems will be easier to solve using Variation of Parameters. It is best to have both tools available in our tool-belt when solving second order LDEs.

Example 2.5.1

Use Variation of Parameters to solve each non-homogeneous LDE below.

(a) $y'' - 2y' + 10y = 9e^x \sec(3x)$

(b) $x^2y'' + xy' = 2x$

(c) $x^2y'' - 2y = x^3\sin(x)$

Solution

(a) This is a CC LDE with characteristic equation $r^2 - 2r + 10 = 0 \implies r = 1 \pm 3i$

Therefore, the two linearly independent homogeneous solution functions are $y_1 = e^x\cos(3x)$ and $y_2 = e^x\sin(3x)$

We can see that $g(x) = 9e^x\sec(3x)$, so we calculate:

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x\cos(3x) & e^x\sin(3x) \\ e^x(\cos(3x) - 3\sin(3x)) & e^x(\sin(3x) + 3\cos(3x)) \end{vmatrix} = 3e^{2x} \\ W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^x\sin(3x) \\ 9e^x\sec(3x) & e^x(\sin(3x) + 3\cos(3x)) \end{vmatrix} = -9e^{2x}\tan(3x) \\ W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^x\cos(3x) & 0 \\ e^x(\cos(3x) - 3\sin(3x)) & 9e^x\sec(3x) \end{vmatrix} = 9e^{2x} \end{aligned}$$

Now we set up and solve each integral to find u_1 and u_2:

$$\begin{aligned} u_1 &= \int \frac{W_1}{W}\,dx = \int \frac{-9e^{2x}\tan(3x)}{3e^{2x}}\,dx = \int -3\tan(3x)\,dx = \ln|\cos(3x)| + C \\ u_2 &= \int \frac{W_2}{W}\,dx = \int \frac{9e^{2x}}{3e^{2x}}\,dx = \int 3\,dx = 3x + D \end{aligned}$$

This gives us our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (\ln|\cos(3x)| + C)\,e^x\cos(3x) + (3x + D)\,e^x\sin(3x) \\ &= e^x\cos(3x)\ln|\cos(3x)| + Ce^x\cos(3x) + 3xe^x\sin(3x) + De^x\sin(3x) \\ &= e^x\cos(3x)\ln|\cos(3x)| + 3xe^x\sin(3x) + Ce^x\cos(3x) + De^x\sin(3x) \end{aligned}$$

Therefore, we have $\boxed{y(x) = e^x\cos(3x)\ln|\cos(3x)| + 3xe^x\sin(3x) + Ce^x\cos(3x) + De^x\sin(3x)}$

(b) This is a CE LDE with characteristic equation $m(m-1) + m = 0 \implies m = 0$ (double)

Therefore, the two linearly independent homogeneous solution functions are $y_1 = 1$ and $y_2 = \ln(x)$

We rewrite the LDE as $y'' + \frac{1}{x}y' = \frac{2}{x}$ to see that $g(x) = \frac{2}{x}$, so we calculate:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x}$$
$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \ln(x) \\ \frac{2}{x} & \frac{1}{x} \end{vmatrix} = -\frac{2}{x}\ln(x)$$
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{2}{x} \end{vmatrix} = \frac{2}{x}$$

Now we set up and solve each integral to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{-\frac{2}{x}\ln(x)}{\frac{1}{x}}\,dx = \int -2\ln(x)\,dx = 2x - 2x\ln(x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{\frac{2}{x}}{\frac{1}{x}}\,dx = \int 2\,dx = 2x + D$$

This gives us our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (2x - 2x\ln(x) + C)\cdot 1 + (2x + D)\ln(x) \\ &= 2x - 2x\ln(x) + C + 2x\ln(x) + D\ln(x) \\ &= 2x + C + D\ln(x) \end{aligned}$$

Therefore, we have $\boxed{y(x) = 2x + C + D\ln(x)}$

(c) This is a CE LDE with characteristic equation $m(m-1) - 2 = 0 \implies m = -1, 2$

Therefore, the two linearly independent homogeneous solution functions are $y_1 = x^{-1}$ and $y_2 = x^2$

We rewrite the LDE as $y'' - \frac{2}{x^2}y = x\sin(x)$ to see that $g(x) = x\sin(x)$, so we calculate:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3$$
$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^2 \\ x\sin(x) & 2x \end{vmatrix} = -x^3\sin(x)$$
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & x\sin(x) \end{vmatrix} = \sin(x)$$

Now we set up and solve each integral to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{-x^3\sin(x)}{3}\,dx = \int -\frac{1}{3}x^3\sin(x)\,dx = \left(\frac{1}{3}x^3 - 2x\right)\cos(x) + (2 - x^2)\sin(x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{\sin(x)}{3}\,dx = \int \frac{1}{3}\sin(x)\,dx = -\frac{1}{3}\cos(x) + D$$

This gives us our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(\left(\frac{1}{3}x^3 - 2x\right)\cos(x) + (2 - x^2)\sin(x) + C\right)x^{-1} + \left(-\frac{1}{3}\cos(x) + D\right)x^2 \\ &= \left(\frac{1}{3}x^2 - 2\right)\cos(x) + (2x^{-1} - x)\sin(x) + Cx^{-1} - \frac{1}{3}x^2\cos(x) + Dx^2 \\ &= -2\cos(x) + (2x^{-1} - x)\sin(x) + Cx^{-1} + Dx^2 \end{aligned}$$

Therefore, we have $\boxed{y(x) = -2\cos(x) + (2x^{-1} - x)\sin(x) + Cx^{-1} + Dx^2}$

As always, we can also include initial or boundary conditions in these questions to help us determine unique solutions when they exist.

Example 2.5.2

Use Variation of Parameters to solve each IVP below.

(a) $y'' - 4y' + 4y = \dfrac{1}{xe^{-2x}}$ with $y(1) = e^2,\ \ y'(1) = 0$

(b) $x^2y'' - 2xy' + 2y = x^3e^x$ with $y(1) = 0,\ \ y'(1) = 2e$

Solution

(a) This is a CC LDE with characteristic equation $r^2 - 4r + 4 = 0 \implies r = 2$ (double)

Therefore, the two linearly independent homogeneous solution functions are $y_1 = e^{2x}$ and $y_2 = xe^{2x}$.

We can see that $g(x) = x^{-1}e^{2x}$, so we calculate:

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix} = e^{4x} \\ W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{2x} \\ x^{-1}e^{2x} & (2x+1)e^{2x} \end{vmatrix} = -e^{4x} \\ W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & x^{-1}e^{2x} \end{vmatrix} = x^{-1}e^{4x} \end{aligned}$$

Now we set up and solve each integral to find u_1 and u_2:

$$\begin{aligned} u_1 &= \int \frac{W_1}{W}\,dx = \int \frac{-e^{4x}}{e^{4x}}\,dx = \int -1\,dx = -x + C \\ u_2 &= \int \frac{W_2}{W}\,dx = \int \frac{x^{-1}e^{4x}}{e^{4x}}\,dx = \int x^{-1}\,dx = \ln|x| + D \end{aligned}$$

This gives us our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (-x + C)\,e^{2x} + (\ln|x| + D)\,xe^{2x} \\ &= -xe^{2x} + Ce^{2x} + xe^{2x}\ln|x| + Dxe^{2x} \\ &= xe^{2x}\ln|x| + Ce^{2x} + (D-1)xe^{2x} \\ &= xe^{2x}\ln|x| + Ce^{2x} + \hat{D}xe^{2x} \end{aligned}$$

We plug in our initial conditions to solve for C and $\hat{D}$:

$$
\begin{aligned}
y &= xe^{2x}\ln(x) + Ce^{2x} + \hat{D}xe^{2x} \\
y(1) &= 0 + C\cdot e^2 + \hat{D}\cdot e^2 \implies e^2C + e^2\hat{D} = e^2 \implies C + \hat{D} = 1 \\
y' &= e^{2x}\ln(x) + 2xe^{2x}\ln(x) + e^{2x} + 2Ce^{2x} + \hat{D}(2x+1)e^{2x} \\
y'(1) &= 0 + 0 + e^2 + 2C\cdot e^2 + \hat{D}\cdot 3e^2 \implies 2e^2C + 3e^2\hat{D} = -e^2 \implies 2C + 3\hat{D} = -1 \\
&\begin{cases} C + \hat{D} = 1 \\ 2C + 3\hat{D} = -1 \end{cases} \implies \begin{cases} C = 4 \\ \hat{D} = -3 \end{cases}
\end{aligned}
$$

Therefore, our unique solution is $\boxed{y(x) = xe^{2x}\ln(x) + 4e^{2x} - 3xe^{2x}}$

(b) This is a CE LDE with characteristic equation $m(m-1) - 2m + 2 = 0 \implies m = 1, 2$

Therefore, the two linearly independent homogeneous solution functions are $y_1 = x$ and $y_2 = x^2$.

We rewrite the LDE as $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = xe^x$ to see that $g(x) = xe^x$, so we calculate:

$$
\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \\
W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^2 \\ xe^x & 2x \end{vmatrix} = -x^3e^x \\
W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} x & 0 \\ 1 & xe^x \end{vmatrix} = x^2e^x
\end{aligned}
$$

Now we set up and solve each integral to find u_1 and u_2:

$$
\begin{aligned}
u_1 &= \int \frac{W_1}{W}\,dx = \int \frac{-x^3e^x}{x^2}\,dx = \int -xe^x\,dx = (1-x)e^x + C \\
u_2 &= \int \frac{W_2}{W}\,dx = \int \frac{x^2e^x}{x^2}\,dx = \int e^x\,dx = e^x + D
\end{aligned}
$$

This gives us our general solution:

$$
\begin{aligned}
y &= u_1y_1 + u_2y_2 \\
&= ((1-x)e^x + C)\,x + (e^x + D)\,x^2 \\
&= xe^x - x^2e^x + Cx + x^2e^x + Dx^2 \\
&= xe^x + Cx + Dx^2
\end{aligned}
$$

We plug in our initial conditions to solve for C and D:

$$
\begin{aligned}
y &= xe^x + Cx + Dx^2 \\
y(1) &= e + C\cdot 1 + D\cdot 1 \implies C + D = -e \\
y' &= (x+1)e^x + C + 2Dx \\
y'(1) &= 2e + C + 2D\cdot 1 \implies C + 2D = 0 \\
&\begin{cases} C + D = -e \\ C + 2D = 0 \end{cases} \implies \begin{cases} C = -2e \\ D = e \end{cases}
\end{aligned}
$$

Therefore, our unique solution is $\boxed{y(x) = xe^x - 2ex + ex^2}$

Remember that not all second order LDEs are CC or CE. As long as you have a way to get both homogeneous solutions, you can solve the non-homogeneous LDE.

Illustration 2.5.2

Consider the $xy'' - (2x+1)y' + (x+1)y = 2xe^x$

(a) Show that $y = e^x$ is a solution to $xy'' - (2x+1)y' + (x+1)y = 0$

(b) Find the general solution to $xy'' - (2x+1)y' + (x+1)y = 0$

(c) Find the general solution to $xy'' - (2x+1)y' + (x+1)y = 2xe^x$

(d) Find the solution to $xy'' - (2x+1)y' + (x+1)y = xe^x$ with $y(1) = e,\ y'(1) = 4e$

(a) We see that $y = e^x \implies y' = e^x,\ y'' = e^x$, so:

$$xy'' - (2x+1)y' + (x+1)y = xe^x - (2x+1)e^x + (x+1)e^x = (x - (2x+1) + (x+1))\,e^x = 0e^x = 0$$

(b) Since we know $f(x) = e^x$ is one homogeneous solution, we can use reduction of order to find the second.

Rewriting the LDE as $y'' + \left(-2 - \frac{1}{x}\right)y' + \left(1 + \frac{1}{x}\right)y = 0$ we see $p(x) = -2 - \frac{1}{x}$.

$$\begin{aligned} -\int p(x)\,dx &= \int 2 + \frac{1}{x}\,dx = 2x + \ln|x| + C \\ u' = \frac{1}{[f(x)]^2}e^{-\int p(x)\,dx} &= \frac{1}{[e^x]^2}e^{2x+\ln|x|+C} = e^{-2x}\cdot Cxe^{2x} = Cx \\ u &= \int u'\,dx = \int Cx\,dx = \frac{C}{2}x^2 + D \\ y = u\cdot f &= \left(\frac{C}{2}x^2 + D\right)e^x = \hat{C}x^2e^x + De^x \end{aligned}$$

This gives us general solution $y(x) = Cx^2e^x + De^x$

(c) Now that we have two linearly independent solutions to the homogeneous version of this LDE, $y_1 = e^x$ and $y_2 = x^2e^x$, we can use variation of parameters to solve the non-homogeneous version. Again, we rewrite it first as $y'' + \left(-2 - \frac{1}{x}\right)y' + \left(1 + \frac{1}{x}\right)y = 2e^x$ to see that $g(x) = 2e^x$.

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x^2e^x \\ e^x & (x^2+2x)e^x \end{vmatrix} = 2xe^{2x} \\ W_1 &= \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^2e^x \\ 2e^x & (x^2+2x)e^x \end{vmatrix} = -2x^2e^{2x} \\ W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} e^x & 0 \\ e^x & 2e^x \end{vmatrix} = 2e^{2x} \end{aligned}$$

Now we set up and solve each integral to find u_1 and u_2:

$$\begin{aligned} u_1 &= \int \frac{W_1}{W}\,dx = \int \frac{-2x^2e^{2x}}{2xe^{2x}}\,dx = \int -x\,dx = -\frac{1}{2}x^2 + C \\ u_2 &= \int \frac{W_2}{W}\,dx = \int \frac{2e^{2x}}{2xe^{2x}}\,dx = \int \frac{1}{dx}\,dx = \ln|x| + D \end{aligned}$$

This gives us our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= \left(-\frac{1}{2}x^2 + C\right)e^x + (\ln|x| + D)\,x^2e^x \\ &= -\frac{1}{2}x^2e^x + Ce^x + x^2e^x\ln|x| + Dx^2e^x \\ &= x^2e^x\ln(x) + Ce^x + \left(D - \frac{1}{2}\right)x^2e^x \\ &= x^2e^x\ln(x) + Ce^x + \hat{D}x^2e^x \end{aligned}$$

Therefore, we have $y(x) = x^2e^x\ln(x) + Ce^x + Dx^2e^x$

(d) Plugging in these initial conditions allows us to solve for C and D in the general solution from part (c):

$$\begin{aligned} y &= x^2e^x\ln(x) + Ce^x + Dx^2e^x \\ y(1) &= 0 + C\cdot e + D\cdot e \implies eC + eD = e \implies C + D = 1 \\ y' &= 2xe^x\ln(x) + x^2e^x\ln(x) + xe^x + Ce^x + D(x^2+2x)e^x \\ y'(1) &= 0 + 0 + e + C\cdot e + D\cdot 3e \implies eC + 3eD = 3e \implies C + 3D = 3 \end{aligned}$$

$$\begin{cases} C + D = 1 \\ C + 3D = 3 \end{cases} \implies \begin{cases} C = 0 \\ D = 1 \end{cases}$$

Therefore, our unique solution is $y(x) = x^2e^x\ln(x) + x^2e^x$

Variation of Parameters is a straightforward tool for solving any second order non-homogeneous linear differential equation, provided you know both of the homogeneous solutions already.

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2.5: Practice Set

Exercise 2.5.1a

Find the general solution to $y'' + 4y = 4\tan(2x)$ for \(-\dfrac{\pi}{4}

Answer available at the end of this section.

Exercise 2.5.1b

Find the general solution to $y'' + 3y' + 2y = \sin(e^x)$

Answer available at the end of this section.

Exercise 2.5.1c

Find the general solution to $x^2y'' - 2xy' + 2y = x^3\cos(x)$

Answer available at the end of this section.

Exercise 2.5.1d

Find the general solution to $y'' - 2y' + y = \dfrac{e^x}{x}$ for $x > 0$

Answer available at the end of this section.

Exercise 2.5.1e

Find the general solution to $y'' + y = 2\csc^3(x)$ for $0 < x < \pi$

Answer available at the end of this section.

Exercise 2.5.1f

Find the general solution to $x^2y'' - 3xy' + 4y = \dfrac{x^2}{\ln(x)}$ for $x > 1$

Answer available at the end of this section.

Exercise 2.5.1g

Find the general solution to $y'' - y = \dfrac{2}{e^x + 1}$

Answer available at the end of this section.

Exercise 2.5.1h

Find the general solution to $y'' + 2y' + y = 4e^{-x}\ln(x)$

Answer available at the end of this section.

Exercise 2.5.1i

Find the general solution to $x^2y'' + xy' - y = \dfrac{2x}{\sqrt{x^2-1}}$ for $x > 1$

Answer available at the end of this section.

Exercise 2.5.1j

Find the general solution to $y'' + 4y' + 4y = \dfrac{2e^{-2x}}{x^2+1}$

Answer available at the end of this section.

Exercise 2.5.2a

Find the general solution to the IVP $y'' - 2y' = 4e^x\sin(x)$ with $y(0) = 1,\ y'(0) = 0$

Answer available at the end of this section.

Exercise 2.5.2b

Find the general solution to the IVP $2x^2y'' + xy' = xe^{\sqrt{x}}$ with $y(1) = 0,\ y'(1) = 0$

Answer available at the end of this section.

Exercise 2.5.2c

Find the general solution to the BVP $x^2y'' + xy' + y = 6\sin(\ln(x))\cos(\ln(x))$ with $y(1) = 1,\ y(e^{\pi}) = 0$

Answer available at the end of this section.

Exercise 2.5.2d

Find the general solution to the BVP $y'' + y = 2\sec(x)$ with $y(0) = -2\ln(2),\ y\left(\frac{\pi}{3}\right) = 0$

Answer available at the end of this section.

Exercise 2.5.3

Use the fact that $y = \sec(x)$ satisfies the homogeneous LDE $y'' - \tan(x)y' - \sec^2(x)y = 0$
to find the solution to the IVP $y'' - \tan(x)y' - \sec^2(x)y = 1$ with $y(0) = 0,\ y'(0) = 0$

Answer available at the end of this section.

Exercise 2.5.4

Use the fact that $y = \sqrt{1+e^x}$ satisfies the homogeneous LDE $(2+2e^x)^2 y'' - (2+2e^x)^2 y' + e^{2x} y = 0$ to find the general solution to LDE $(2+2e^x)^2 y'' - (2+2e^x)^2 y' + e^{2x} y = 8e^{2x}\sqrt{1+e^x}$

Answer available at the end of this section.

Answers

Exercise 2.5.1*a*

This is a CC LDE with characteristic equation $r^2 + 4 = 0$

This has roots $r = \pm 2i$, so our two linearly independent homogeneous solutions are $y_1 = \cos(2x),\ y_2 = \sin(2x)$

We see $g(x) = 4\tan(2x)$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2$$
$$W_1 = \begin{vmatrix} 0 & \sin(2x) \\ 4\tan(2x) & 2\cos(2x) \end{vmatrix} = -\sin(2x)\cdot 4\tan(2x) = 4\cos(2x) - 4\sec(2x)\tan(2x)$$
$$W_2 = \begin{vmatrix} \cos(2x) & 0 \\ -2\sin(2x) & 4\tan(2x) \end{vmatrix} = \cos(2x)\cdot 4\tan(2x) = 4\sin(2x)$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int (2\cos(2x) - 2\sec(2x)\tan(2x))\,dx = \sin(2x) - \ln|\sec(2x)+\tan(2x)| + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int 2\sin(2x)\,dx = -\cos(2x) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (\sin(2x) - \ln|\sec(2x)+\tan(2x)| + C)\cos(2x) + (-\cos(2x) + D)\sin(2x) \end{aligned}$$

Thus, our solutions look like $y(x) = -\cos(2x)\ln|\sec(2x)+\tan(2x)| + C\cos(2x) + D\sin(2x)$

Exercise 2.5.1*b*

This is a CC LDE with characteristic equation $r^2 + 3r + 2 = (r+1)(r+2) = 0$

This has roots $r = -1, -2$, so our two linearly independent homogeneous solutions are $y_1 = e^{-x},\ y_2 = e^{-2x}$

We see $g(x) = \sin(e^x)$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$
$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin(e^x) & -2e^{-2x} \end{vmatrix} = -2e^{-2x}\sin(e^x)$$
$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin(e^x) \end{vmatrix} = e^{-x}\sin(e^x)$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = e^x\sin(e^x)\,dx = -\cos(e^x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int -e^{2x}\sin(e^x)\,dx = e^x\cos(e^x) - \sin(e^x) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (-\cos(e^x) + C)\,e^{-x} + (e^x\cos(e^x) - \sin(e^x) + D)\,e^{-2x} \end{aligned}$$

Thus, our solutions look like $y(x) = -e^{-2x}\sin(e^x) + Ce^{-x} + De^{-2x}$.

Exercise 2.5.1*c*

This is a CE LDE with characteristic equation $m(m-1) - 2m + 2 = (m-1)(m-2) = 0$

This has roots $m = 1, 2$, so our two linearly independent homogeneous solutions are $y_1 = x,\ y_2 = x^2$

Rewriting the LDE to isolate y'': $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x\cos(x)$

We see $g(x) = x\cos(x)$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x \cdot 2x - 1 \cdot x^2 = x^2$$
$$W_1 = \begin{vmatrix} 0 & x^2 \\ x\cos(x) & 2x \end{vmatrix} = -x^2 \cdot x\cos(x) = -x^3\cos(x)$$
$$W_2 = \begin{vmatrix} x & 0 \\ 1 & x\cos(x) \end{vmatrix} = x \cdot x\cos(x) = x^2\cos(x)$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -x\cos(x)\,dx = -x\sin(x) - \cos(x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \cos(x)\,dx = \sin(x) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= (-x\sin(x) - \cos(x) + C)\,x + (\sin(x) + D)x^2 \end{aligned}$$

Thus, our solutions look like $y(x) = -x\cos(x) + Cx + Dx^2$

Exercise 2.5.1*d*

This is a CC LDE with characteristic equation $r^2 - 2r + 1 = (r-1)^2 = 0$

This has roots $r = 1$ (double), so our two linearly independent homogeneous solutions are $y_1 = e^x,\ y_2 = xe^x$

We see $g(x) = \dfrac{e^x}{x}$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^x \cdot (1+x)e^x - xe^x \cdot e^x = e^{2x}$$
$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & (1+x)e^x \end{vmatrix} = -xe^x \cdot \frac{e^x}{x} = -e^{2x}$$
$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix} = e^x \cdot \frac{e^x}{x} = \frac{e^{2x}}{x}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -1\,dx = -x + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{1}{x}\,dx = \ln(x) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= (-x + C)\,e^x + (\ln(x) + D)\,xe^x \end{aligned}$$

Thus, our solutions look like $y(x) = xe^x\ln(x) + Ce^x + \hat{D}xe^x$

Exercise 2.5.1*e*

This is a CC LDE with characteristic equation $r^2 + 1 = 0$

This has roots $r = \pm i$, so our two linearly independent homogeneous solutions are $y_1 = \cos(x),\ y_2 = \sin(x)$

We see $g(x) = 2\csc^3(x)$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1$$
$$W_1 = \begin{vmatrix} 0 & \sin(x) \\ 2\csc^3(x) & \cos(x) \end{vmatrix} = -\sin(x) \cdot 2\csc^3(x) = -2\csc^2(x)$$
$$W_2 = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & 2\csc^3(x) \end{vmatrix} = \cos(x) \cdot 2\csc^3(x) = \frac{2\cos(x)}{\sin^3(x)}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -2\csc^2(x)\,dx = 2\cot(x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{2\cos(x)}{\sin^3(x)}\,dx = -\csc^2(x) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= (2\cot(x) + C)\cos(x) + \left(-\csc^2(x) + D\right)\sin(x) \end{aligned}$$

Thus, our solutions look like $y(x) = 2\cot(x)\sin(x) - \csc(x) + C\cos(x) + D\sin(x)$

Exercise 2.5.1*f*

This is a CE LDE with characteristic equation $m(m-1) - 3m + 4 = (m-2)^2 = 0$

This has roots $m = 2$ (double), so our two linearly independent homogeneous solutions are $y_1 = x^2,\ y_2 = x^2\ln(x)$

Rewriting the LDE to isolate y'': $y'' - \dfrac{3}{x}y' + \dfrac{4}{x^2}y = \dfrac{1}{\ln(x)}$

We see $g(x) = \dfrac{1}{\ln(x)}$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} x^2 & x^2\ln(x) \\ 2x & (2\ln(x)+1)x \end{vmatrix} = x^2\cdot(2\ln(x)+1)x - x^2\ln(x)\cdot 2x = x^3$$
$$W_1 = \begin{vmatrix} 0 & x^2\ln(x) \\ \frac{1}{\ln(x)} & (2\ln(x)+1)x \end{vmatrix} = -x^2\ln(x)\cdot\frac{1}{\ln(x)} = -x^2$$
$$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & \frac{1}{\ln(x)} \end{vmatrix} = x^2\cdot\frac{1}{\ln(x)} = \frac{x^2}{\ln(x)}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -\frac{1}{x}\,dx = -\ln(x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{1}{x\ln(x)}\,dx = \ln(\ln(x)) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= (-\ln(x)+C)\,x^2 + (\ln(\ln(x))+D)\,x^2\ln(x) \end{aligned}$$

Thus, our solutions look like $y(x) = x^2\ln(x)\ln(\ln(x)) + Cx^2 + \hat{D}x^2\ln(x)$

Exercise 2.5.1*g*

This is a CC LDE with characteristic equation $r^2 - 1 = (r+1)(r-1) = 0$

This has roots $r = \pm 1$, so our two linearly independent homogeneous solutions are $y_1 = e^x,\ y_2 = e^{-x}$

We see $g(x) = \dfrac{2}{e^x+1}$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$
$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{2}{e^x+1} & -e^{-x} \end{vmatrix} = -e^{-x}\cdot\frac{2}{e^x+1} = -\frac{2}{e^x(e^x+1)}$$
$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{2}{e^x+1} \end{vmatrix} = e^x\cdot\frac{2}{e^x+1} = \frac{2e^x}{e^x+1}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{1}{e^x(e^x+1)}\,dx = -e^{-x} - x + \ln(e^x+1) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int -\frac{e^x}{e^x+1}\,dx = -\ln(e^x+1) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= (-e^{-x} - x + \ln(e^x+1) + C)\,e^x + (-\ln(e^x+1)+D)\,e^{-x} \end{aligned}$$

Thus, our solutions look like $y(x) = -1 + (e^x - e^{-x})\ln(e^x+1) + (C-x)e^x + De^{-x}$

Exercise 2.5.1*h*

This is a CC LDE with characteristic equation $r^2 + 2r + 1 = (r+1)^2 = 0$

This has roots $r = -1$ (double), so our two linearly independent homogeneous solutions are $y_1 = e^{-x},\ y_2 = xe^{-x}$

We see $g(x) = 4e^{-x}\ln(x)$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-x}(1-x)e^{-x} - xe^{-x}(-e^{-x}) = e^{-2x}$$
$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ 4e^{-x}\ln(x) & (1-x)e^{-x} \end{vmatrix} = -xe^{-x}\cdot e^{-x}\ln(x) = -4xe^{-2x}\ln(x)$$
$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & 4e^{-x}\ln(x) \end{vmatrix} = e^{-x}\cdot 4e^{-x}\ln(x) = 4e^{-2x}\ln(x)$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -4x\ln(x)\,dx = x^2 - 2x^2\ln(x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int 4\ln(x)\,dx = 4x\ln(x) - 4x + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= (x^2 - 2x^2\ln(x) + C)\,e^{-x} + (4x\ln(x) - 4x + D)\,xe^{-x} \end{aligned}$$

Thus, our solutions look like $y(x) = (2x^2\ln(x) - 3x^2 + C + Dx)\,e^{-x}$

Exercise 2.5.1*i*

This is a CE LDE with characteristic equation $m(m-1) + m - 1 = (m-1)(m+1) = 0$

This has roots $m = 1, -1$, so our two linearly independent homogeneous solutions are $y_1 = x,\ y_2 = x^{-1}$

Rewriting the LDE to isolate y'': $y'' + \dfrac{1}{x}y' - \dfrac{1}{x^2}y = \dfrac{2}{x\sqrt{x^2-1}}$

We see $g(x) = \dfrac{2}{x\sqrt{x^2-1}}$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = x \cdot (-x^{-2}) - 1 \cdot x^{-1} = -2x^{-1}$$
$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ \frac{2}{x\sqrt{x^2-1}} & -x^{-2} \end{vmatrix} = -x^{-1} \cdot \frac{2}{x\sqrt{x^2-1}} = -\frac{2}{x^2\sqrt{x^2-1}}$$
$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x\sqrt{x^2-1}} \end{vmatrix} = x \cdot \frac{2}{x\sqrt{x^2-1}} = \frac{2}{\sqrt{x^2-1}}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1} + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(\sec^{-1}(x) + C\right) x + (\sqrt{x^2-1} + D)x^{-1} \end{aligned}$$

Thus, our solutions look like $y(x) = x\sec^{-1}(x) + \dfrac{\sqrt{x^2-1}}{x} + Cx + Dx^{-1}$

Exercise 2.5.1*j*

This is a CC LDE with characteristic equation $r^2 + 4r + 4 = (r+2)^2 = 0$

This has roots $r = -2$ (double), so our two linearly independent homogeneous solutions are $y_1 = e^{-2x},\ y_2 = xe^{-2x}$

We see $g(x) = \dfrac{2e^{-2x}}{x^2+1}$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-4x}$$
$$W_1 = \begin{vmatrix} 0 & xe^{-2x} \\ \frac{2e^{-2x}}{x^2+1} & (1-2x)e^{-2x} \end{vmatrix} = -\frac{2xe^{-4x}}{x^2+1}$$
$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{2e^{-2x}}{x^2+1} \end{vmatrix} = \frac{2e^{-4x}}{x^2+1}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -\frac{2x}{x^2+1} \implies u_1 = -\ln(x^2+1) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{2}{x^2+1}\,dx = 2\tan^{-1}(x) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(-\ln(x^2+1) + C\right) e^{-2x} + \left(2\tan^{-1}(x) + D\right) xe^{-2x} \end{aligned}$$

Thus, our solutions look like $y(x) = \left(2x\tan^{-1}(x) - \ln(x^2+1) + C + Dx\right) e^{-2x}$

Exercise 2.5.2*a*

This is a CC LDE with characteristic equation $r^2 - 2r = r(r-2) = 0$

This has roots $r = 0, 2$, so our two linearly independent homogeneous solutions are $y_1 = 1,\ y_2 = e^{2x}$

We see $g(x) = 4e^x \sin(x)$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$
$$W_1 = \begin{vmatrix} 0 & e^{2x} \\ 4e^x \sin(x) & 2e^{2x} \end{vmatrix} = -e^{2x} \cdot 4e^x \sin(x) = -4e^{3x}\sin(x)$$
$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 4e^x \sin(x) \end{vmatrix} = 1 \cdot 4e^x \sin(x) = 4e^x \sin(x)$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -2e^x \sin(x)\,dx = -e^x(\cos(x) - \sin(x)) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int 2e^{-x}\sin(x)\,dx = -e^{-x}(\sin(x) + \cos(x)) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (-e^x(\cos(x) - \sin(x)) + C) \cdot 1 + (-e^{-x}(\sin(x) + \cos(x)) + D) \cdot e^{2x} \end{aligned}$$

Thus, our solutions look like $y(x) = -2e^x \sin(x) + C + De^{2x}$.

Plugging in our initial conditions:

$$\begin{aligned} y(x) &= -2e^x \sin(x) + C + De^{2x} \\ y(0) = 1 &\implies 0 = 0 + C + D \cdot 1 \implies C + D = 0 \\ y'(x) &= -2e^x(\cos(x) - \sin(x)) + 2De^{2x} \\ y'(0) = 0 &\implies 0 = -2 + 2D \cdot 1 \implies D = 1 \\ C + D = 0 &\implies C = -1 \end{aligned}$$

So, our unique solution is $y(x) = -2e^x \sin(x) - 1 + e^{2x}$

Exercise 2.5.2*b*

This is a CE LDE with characteristic equation $2m(m-1) + m = m(2m-1) = 0$

This has roots $m = 0, \frac{1}{2}$, so our two linearly independent homogeneous solutions are $y_1 = 1,\ y_2 = \sqrt{x}$

Rewriting the LDE to isolate y'': $y'' + \frac{1}{2x}y' = \frac{e^{\sqrt{x}}}{2x}$

We see $g(x) = \frac{e^{\sqrt{x}}}{2x}$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} 1 & \sqrt{x} \\ 0 & \frac{1}{2}x^{-\frac{1}{2}} \end{vmatrix} = 1 \cdot \frac{1}{2}x^{-\frac{1}{2}} - 0 \cdot \sqrt{x} = \frac{1}{2\sqrt{x}}$$
$$W_1 = \begin{vmatrix} 0 & \sqrt{x} \\ \frac{e^{\sqrt{x}}}{2x} & \frac{1}{2}x^{-\frac{1}{2}} \end{vmatrix} = -\sqrt{x} \cdot \frac{e^{\sqrt{x}}}{2x} = -\frac{e^{\sqrt{x}}}{2x^{\frac{1}{2}}}$$
$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & \frac{e^{\sqrt{x}}}{2x} \end{vmatrix} = 1 \cdot \frac{e^{\sqrt{x}}}{2x} = \frac{e^{\sqrt{x}}}{2x}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int -e^{\sqrt{x}}\,dx = 2(1-\sqrt{x})e^{\sqrt{x}} + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(2(1-\sqrt{x})e^{\sqrt{x}} + C\right)1 + \left(2e^{\sqrt{x}} + D\right)\sqrt{x} \end{aligned}$$

Thus, our solutions look like $y(x) = 2e^{\sqrt{x}} + C + D\sqrt{x}$

Plugging in our initial conditions:

$$\begin{aligned} y(x) &= 2e^{\sqrt{x}} + C + D\sqrt{x} \\ y(1) = 0 &\implies 0 = 2e + C + D \cdot 1 \implies C + D = -2e \\ y'(x) &= \frac{e^{\sqrt{x}}}{\sqrt{x}} + \frac{D}{2\sqrt{x}} \\ y'(1) = 0 &\implies 0 = e + \frac{D}{2} \implies D = -2e \\ C + D = -2e &\implies C = 0 \end{aligned}$$

So, our unique solution is $y(x) = 2e^{\sqrt{x}} - 2e\sqrt{x}$

Exercise 2.5.2*c*

This is a CE LDE with characteristic equation $m(m-1) + m + 1 = m^2 + 1 = 0$

This has roots $m = \pm i$, so our two linearly independent homogeneous solutions are $y_1 = \sin(\ln(x)),\ y_2 = \cos(\ln(x))$

Rewriting the LDE to isolate y'': $y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{6\sin(\ln(x))\cos(\ln(x))}{x^2}$

We see $g(x) = \frac{6}{x^2}\sin(\ln(x))\cos(\ln(x))$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} \sin(\ln(x)) & \cos(\ln(x)) \\ \frac{1}{x}\cos(\ln(x)) & -\frac{1}{x}\sin(\ln(x)) \end{vmatrix} = -\frac{1}{x}\sin^2(\ln(x)) - \frac{1}{x}\cos^2(\ln(x)) = -\frac{1}{x}$$
$$W_1 = \begin{vmatrix} 0 & \cos(\ln(x)) \\ \frac{6}{x^2}\sin(\ln(x))\cos(\ln(x)) & -\frac{1}{x}\sin(\ln(x)) \end{vmatrix} = -\frac{6}{x^2}\cos^2(\ln(x))\sin(\ln(x))$$
$$W_2 = \begin{vmatrix} \sin(\ln(x)) & 0 \\ \frac{1}{x}\cos(\ln(x)) & \frac{6}{x^2}\sin(\ln(x))\cos(\ln(x)) \end{vmatrix} = \frac{6}{x^2}\sin^2(\ln(x))\cos(\ln(x))$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{6}{x}\cos^2(\ln(x))\sin(\ln(x))\,dx = -2\cos^3(\ln(x)) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int -\frac{6}{x}\sin^2(\ln(x))\cos(\ln(x))\,dx = -2\sin^3(\ln(x)) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(-2\cos^3(\ln(x)) + C\right)\sin(\ln(x)) + \left(-2\sin^3(\ln(x)) + D\right)\cos(\ln(x)) \end{aligned}$$

Thus, our solutions look like $y(x) = -2\sin(\ln(x))\cos(\ln(x)) + C\sin(\ln(x)) + D\cos(\ln(x))$

Plugging in our boundary conditions:

$$\begin{aligned} y(x) &= -2\sin(\ln(x))\cos(\ln(x)) + C\sin(\ln(x)) + D\cos(\ln(x)) \\ y(1) = 1 &\implies 1 = 0 + C \cdot 0 + D \cdot 1 \implies D = 1 \\ y(e^\pi) = 0 &\implies 0 = 0 = 0 + C \cdot 0 + D \cdot (-1) \implies D = 0 \end{aligned}$$

So, **no such solution exists**.

Exercise 2.5.2*d*

This is a CC LDE with characteristic equation $r^2 + 1 = 0$

This has roots $r = \pm i$, so our two linearly independent homogeneous solutions are $y_1 = \cos(x),\ y_2 = \sin(x)$
We see $g(x) = 2\sec(x)$ and calculate the values of $W,\ W_1,\ W_2$:

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1$$
$$W_1 = \begin{vmatrix} 0 & \sin(x) \\ 2\sec(x) & \cos(x) \end{vmatrix} = -\sin(x) \cdot 2\sec(x) = -2\tan(x)$$
$$W_2 = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & 2\sec(x) \end{vmatrix} = \cos(x) \cdot 2\sec(x) = 2$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int 2\tan(x)\,dx = 2\ln|\sec(x)| + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int 2\,dx = 2x + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (2\ln|\sec(x)| + C)\cos(x) + (2x + D)\sin(x) \end{aligned}$$

Thus, our solutions look like $y(x) = 2\cos(x)\ln|\sec(x)| + 2x\sin(x) + C\cos(x) + D\sin(x)$
Plugging in our boundary conditions:

$$\begin{aligned} y(x) &= 2\cos(x)\ln|\sec(x)| + 2x\sin(x) + C\cos(x) + D\sin(x) \\ y(0) = -2\ln(2) &\implies -2\ln(2) = 0 + 0 + C\cdot 1 + D\cdot 0 \implies C = -2\ln(2) \\ y\left(\frac{\pi}{3}\right) = 0 &\implies 0 = \ln(2) + \frac{\pi}{\sqrt{3}} + C\cdot\frac{1}{2} + D\cdot\frac{\sqrt{3}}{2} \\ C + D\sqrt{3} &= -2\ln(2) - \frac{2\pi}{\sqrt{3}} \implies D = -\frac{2\pi}{3} \end{aligned}$$

So, our unique solution is $y(x) = 2\cos(x)\ln\left|\frac{\sec(x)}{2}\right| + 2\left(x - \frac{\pi}{3}\right)\sin(x)$

Exercise 2.5.3

We are given one homogeneous solution, $f(x) = \sec(x)$, so by reduction of order we can find the second:

$$\begin{aligned} p(x) = -\tan(x) &\implies -\int p(x)\,dx = \int \tan(x)\,dx = \ln|\sec(x)| + C \\ u'(x) = \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx} &= \frac{1}{[\sec(x)]^2}\cdot e^{\ln|\sec(x)| + C} = \cos^2(x)\cdot C\sec(x) = C\cos(x) \\ u(x) &= \int u'(x)\,dx = \int C\cos(x)\,dx = C\sin(x) + D \end{aligned}$$

This means our homogeneous solutions are of the form $y_h(x) = u(x)f(x) = (C\sin(x) + D)\sec(x) = C\tan(x) + D\sec(x)$.
Now we use variation of parameters with $y_1 = \tan(x),\ y_2 = \sec(x)$, and $g(x) = 1$:

$$W = \begin{vmatrix} \tan(x) & \sec(x) \\ \sec^2(x) & \sec(x)\tan(x) \end{vmatrix} = \sec(x)\tan^2(x) - \sec^3(x) = -\sec(x)$$
$$W_1 = \begin{vmatrix} 0 & \sec(x) \\ 1 & \sec(x)\tan(x) \end{vmatrix} = -\sec(x)$$
$$W_2 = \begin{vmatrix} \tan(x) & 0 \\ \sec^2(x) & 1 \end{vmatrix} = \tan(x)$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int 1\,dx = x + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int -\sin(x)\,dx = \cos(x) + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (x + C)\tan(x) + (\cos(x) + D)\sec(x) \end{aligned}$$

Thus, our solutions look like $y(x) = x\tan(x) + 1 + C\tan(x) + D\sec(x)$
Plugging in our initial conditions:

$$\begin{aligned} y(x) &= x\tan(x) + 1 + C\tan(x) + D\sec(x) \\ y(0) = 0 &\implies 0 = 0 + 1 + C\cdot 0 + D\cdot 1 \implies D = -1 \\ y'(x) &= \tan(x) + x\sec^2(x) + C\sec^2(x) + D\sec(x)\tan(x) \\ y'(0) = 0 &\implies 0 = 0 + 0 + C\cdot 1 + D\cdot 0 \implies C = 0 \end{aligned}$$

So, our unique solution is $y(x) = x\tan(x) + 1 - \sec(x)$

Exercise 2.5.4

First, rewrite the LDE as $y'' - y' + \frac{e^{2x}}{4(1+e^x)^2}y = \frac{2e^{2x}}{(1+e^x)^{\frac{3}{2}}}$
We are given one homogeneous solution, $f(x) = \sqrt{1+e^x}$, so by reduction of order we can find the second:

$$p(x) = -1 \implies -\int p(x)\,dx = \int 1\,dx = x + C$$
$$u'(x) = \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx} = \frac{1}{\left[\sqrt{1+e^x}\right]^2} \cdot e^{x+C} = \frac{1}{1+e^x} \cdot Ce^x = \frac{Ce^x}{1+e^x}$$
$$u(x) = \int u'(x)\,dx = \int \frac{Ce^x}{1+e^x}\,dx = C\ln\left(1+e^x\right) + D$$

This means our homogeneous solutions are of the form $y_h(x) = u(x)f(x) = \left(C\ln\left(1+e^x\right)+D\right)\sqrt{1+e^x} = C\ln\left(1+e^x\right)\sqrt{1+e^x} + D\sqrt{1+e^x}$

Now we use variation of parameters with $y_1 = \ln\left(1+e^x\right)\sqrt{1+e^x},\ \ y_2 = \sqrt{1+e^x}$, and $g(x) = \dfrac{2e^{2x}}{\left(1+e^x\right)^{\frac{3}{2}}}$:

$$W = \begin{vmatrix} \ln\left(1+e^x\right)\sqrt{1+e^x} & \sqrt{1+e^x} \\ \frac{e^x}{2\sqrt{1+e^x}}\left(2+\ln\left(1+e^x\right)\right) & \frac{e^x}{2\sqrt{1+e^x}} \end{vmatrix} = -e^x$$
$$W_1 = \begin{vmatrix} 0 & \sqrt{1+e^x} \\ \frac{2e^{2x}}{\left(1+e^x\right)^{\frac{3}{2}}} & \frac{e^x}{2\sqrt{1+e^x}} \end{vmatrix} = -\frac{2e^{2x}}{1+e^x}$$
$$W_2 = \begin{vmatrix} \ln\left(1+e^x\right)\sqrt{1+e^x} & 0 \\ \frac{e^x}{2\sqrt{1+e^x}}\left(2+\ln\left(1+e^x\right)\right) & \frac{2e^{2x}}{\left(1+e^x\right)^{\frac{3}{2}}} \end{vmatrix} = \frac{2e^{2x}\ln\left(1+e^x\right)}{1+e^x}$$

Now we integrate to find u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{2e^x}{1+e^x}\,dx = 2\ln\left(1+e^x\right) + C$$
$$u_2 = \int \frac{W_2}{W}\,dx = \int -\frac{2e^x\ln\left(1+e^x\right)}{1+e^x}\,dx = -\left[\ln\left(1+e^x\right)\right]^2 + D$$

We plug this in to find our general solution:

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= \left(2\ln\left(1+e^x\right)+C\right)\ln\left(1+e^x\right)\sqrt{1+e^x} + \left(-\left[\ln\left(1+e^x\right)\right]^2 + D\right)\sqrt{1+e^x} \end{aligned}$$

Thus, our solutions look like $y(x) = \left[\ln\left(1+e^x\right)\right]^2\sqrt{1+e^x} + C\ln\left(1+e^x\right)\sqrt{1+e^x} + D\sqrt{1+e^x}$

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2.6: Systems of LDEs

Learning Objectives

- Solve systems of two first-order linear differential equations by systematically eliminating one variable.
- Recognize when the resulting higher-order equation is constant coefficient or Cauchy-Euler.
- Solve the resulting higher-order equation and back-substitute to find the complete system solution.
- Apply initial conditions to solve systems of Initial Value Problems.

Many real-world systems involve multiple interdependent quantities. This subchapter shows how to solve systems of linear differential equations by systematically eliminating variables to reduce the system to a single, higher-order equation that we already know how to solve.

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2.6: Solving Linear Systems of LDEs

In the real world, there are sometimes two or more interacting quantities in a system which affect each other's derivatives. This leads to a system of differential equations. For example, a system of two first order LDEs looks like:

$$\begin{cases} a_1(t)x'(t) + b_1(t)y'(t) + c_1(t)x(t) + d_1(t)y(t) = g_1(t) \\ a_2(t)x'(t) + b_2(t)y'(t) + c_2(t)x(t) + d_2(t)y(t) = g_2(t) \end{cases}$$

Solving such a system involves attempting to find the two functions $x(t)$ and $y(t)$ that satisfy both equations. This can vary in difficulty, so we begin with a simple homogeneous example:

$$\begin{cases} x'(t) - x(t) = 0 \\ y'(t) + y(t) - 4x(t) = 0 \end{cases}$$

This system of differential equations is "nice" because one of the functions, $x(t)$, is already isolated in its own equation.
We can solve this first order LDE via separation:

$$\frac{dx}{dt} = x \implies \int \frac{1}{x}\,dx = \int dt \implies \ln|x| = t + C \implies x(t) = Ce^t$$

Now we can substitute this answer into the second equation in our system to get

$$y'(t) + y(t) - 4Ce^t = 0 \implies y'(t) + y(t) = 4Ce^t$$

This is a first order LDE so we use $\mu(t) = e^{\int p(t)\,dt} = e^{\int 1\,dt} = e^t$:

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)\cdot q(t)\,dt = \frac{1}{e^t}\int e^t \cdot 4Ce^t\,dt = e^{-t}\left[2Ce^{2t} + D\right] = 2Ce^t + De^{-t}$$

Therefore, our solution to this system is the pair of functions $x(t) = Ce^t,\ \ y(t) = 2Ce^t + De^{-t}$
Notice that this constant C must be the same in both solutions in order for the second equation to be satisfied.

For many systems of LDEs, like the one above, we can use the solution for one of the functions in order to obtain the solution for the other. The best case scenario, would be if we did not have to solve a second differential equation in order to solve for the second function. For instance, if we had somehow solved the system above for $y(t)$ first, we could have easily rewritten the second equation as $x(t) = \frac{1}{4}y'(t) + \frac{1}{4}y(t)$

Solving for $y(t)$ first would require a little more work at the front end of the problem: we would need to eliminate $x(t)$ first. We could do this via substitution:

$$x(t) = \frac{1}{4}y'(t) + \frac{1}{4}y(t) \implies x'(t) = \frac{1}{4}y''(t) + \frac{1}{4}y''(t)$$

Plugging these into the first equation we get:

$$x'(t) - x(t) = 0 \implies \left(\frac{1}{4}y''(t) + \frac{1}{4}y''(t)\right) - \left(\frac{1}{4}y'(t) + \frac{1}{4}y(t)\right) = 0$$

This simplifies down to the second order LDE $y''(t) - y(t) = 0$

This is a CC LDE with characteristic equation $r^2 - 1 = 0$ which has roots $r = 1, -1$. This tells us the general solution is $y(t) = Ae^t + Be^{-t}$

Now that we have $y(t)$, we substitute it back in to solve for $x(t)$:

$$x(t) = \frac{1}{4}y'(t) + \frac{1}{4}y(t) = \frac{1}{4}\left(Ae^t - Be^{-t}\right) + \frac{1}{4}\left(Ae^t + Be^{-t}\right) = \frac{1}{2}Ae^t$$

This gives us solutions $x(t) = \frac{1}{2}Ae^t,\ y(t) = Ae^t + Be^{-t}$. Notice that if $A = 2C$ and $B = D$ these answers are identical!

Since not all systems will begin with one variable isolated, this method of substitution is the default method for solving a system.

Consider a slightly tougher homogeneous example:

Illustration 2.6.1

We will find the general solutions to the system of LDEs:

$$\begin{cases} x'(t) + x(t) - y(t) = 0 \\ x(t) + y(t) + y'(t) = 0 \end{cases}$$

First, note that if we solved for $x(t)$ first, we could use the top equation to easily recover $y(t)$. If we instead solved for $y(t)$ first, we could use the bottom equation to easily recover $x(t)$. This means it does not matter which function we eliminate first (which is always worth checking before doing any actual eliminating or substituting).

Solving for $y(t)$ in the first equation gives $y(t) = x'(t) + x(t)$. Differentiating makes this $y'(t) = x''(t) + x'(t)$. Now we plug this into the second equation:

$$x(t) + y(t) + y'(t) = 0 \implies x(t) + (x'(t) + x(t)) + (x''(t) + x'(t)) = 0 \implies x''(t) + 2x'(t) + 2x(t) = 0$$

This is a second order CC LDE with characteristic equation $r^2 + 2r + r = 0$
The roots are $r = -1 \pm i$, which makes the general solution $x(t) = Ae^{-t}\cos(t) + Be^{-t}\sin(t)$

Plugging these back into $y(t) = x'(t) + x(t)$ gives our second function:

$$\begin{aligned} y(t) &= x'(t) + x(t) \\ &= \left(-Ae^{-t}\cos(t) - Ae^{-t}\sin(t) - Be^{-t}\sin(t) + Be^{-t}\cos(t)\right) + \left(Ae^{-t}\cos(t) + Be^{-t}\sin(t)\right) \\ &= -Ae^{-t}\sin(t) + Be^{-t}\cos(t) \end{aligned}$$

Therefore, our solution is $x(t) = Ae^{-t}\cos(t) + Be^{-t}\sin(t),\ y(t) = Be^{-t}\cos(t) - Ae^{-t}\sin(t)$

Example 2.6.1

Find the solution to each homogeneous system of linear equations below.

(a) $\begin{cases} x'(t) + y(t) = 0 \\ 2x''(t) + x(t) + y'(t) = 0 \end{cases}$

(b) $\begin{cases} tx'(t) - 5x(t) + 6y(t) = 0 \\ ty'(t) - x(t) = 0 \end{cases}$

Solution

(a) We see that if we first solve for $x(t)$, the top equation allows us to recover $y(t)$, but if we solved for $y(t)$ first, we would still have to solve a differential equation to find $x(t)$. This means we will try to eliminate $y(t)$ first by using $y(t) = -x'(t) \implies y'(t) = -x''(t)$:

$$2x''(t) + x(t) + y'(t) = 0 \implies 2x''(t) + x(t) + (-x''(t)) = 0 \implies x''(t) + x(t) = 0$$

This is a second order CC LDE with characteristic equation $r^2 + 1 = 0$, which has roots $r = \pm i$
Therefore, the general solution is $x(t) = A\cos(t) + B\sin(t)$. We substitute this into $y(t) = -x'(t)$ to get:

$$y(t) = -x'(t) = -\left(-A\sin(t) + B\cos(t)\right) = A\sin(t) - B\cos(t)$$

Thus, our solution to the system is $\boxed{x(t) = A\cos(t) + B\sin(t),\ y(t) = A\sin(t) - B\cos(t)}$

(b) We see that if we first solve for $x(t)$, we can use the top equation to recover $y(t)$. If we instead first solved for $y(t)$, we could use the bottom equation to recover $x(t)$. This means we can eliminate whichever is easiest, so we will use $x(t) = ty'(t) \implies x'(t) = y'(t) + ty''(t)$:

$$tx'(t) - 5x(t) + 6y(t) = 0 \implies t\left(y'(t) + ty''(t)\right) - 5\left(ty'(t)\right) + 6y(t) = 0 \implies t^2y''(t) - 4ty'(t) + 6y(t) = 0$$

This is a second order CE LDE with characteristic equation $m(m-1) - 4m + 6 = (m-2)(m-3) = 0$, which has roots $m = 2, 3$ Therefore, the general solution is $y(t) = At^2 + Bt^3$. We substitute this into $x(t) = ty'(t)$ to get:

$$x(t) = ty'(t) = t \cdot \left(2At + 3Bt^2\right) = 2At^2 + 3Bt^3$$

Thus, our solution to the system is $\boxed{x(t) = 2At^2 + 3Bt^3,\ y(t) = At^2 + Bt^3}$

We can also use elimination instead of substitution to create an LDE with only one function in it. To make this elimination easier, we use the differential operator notation:

$$Df = \frac{d}{dt}f$$

This allows us to rewrite our system of differential equations in a way that makes it more obvious how to eliminate one of the functions.

For example, consider the system from Illustration 2.6.1:

$$\begin{cases} x'(t) + x(t) - y(t) = 0 \\ x(t) + y(t) + y'(t) = 0 \end{cases} \implies \begin{cases} Dx + x - y = 0 \\ x + y + Dy = 0 \end{cases} \implies \begin{cases} (D+1)x - y = 0 \\ x + (D+1)y = 0 \end{cases}$$

We can "multiply" the top equation by $D+1$ and then add the two equations together to eliminate y:

$$(D+1)\left[(D+1)x - y = 0\right] + \left[x + (D+1)y = 0\right] \implies \left((D+1)^2 + 1\right)x = 0$$

This multiplies out to $\left(D^2 + 2D + 2\right)x = 0$ which is the same second order CC LDE we got through substitution: $x'' + 2x' + 2x = 0$. From this point, we follow the same steps we did above to solve for $x(t)$ and then substitute it in to solve for $y(t)$.

Example 2.6.2

Find the solution to the homogeneous system of linear equations $\begin{cases} x'(t) = 3x(t) - 4y(t) \\ y'(t) = z(t) \\ z'(t) = x(t) \end{cases}$

Solution

We rewrite this system in terms of the operator D and move all terms to the left side:

$$\begin{cases} x'(t) = 3x(t) - 4y(t) \\ y'(t) = z(t) \\ z'(t) = x(t) \end{cases} \implies \begin{cases} Dx = 3x - 4y \\ Dy = z \\ Dz = x \end{cases} \implies \begin{cases} (D-3)x + 4y = 0 \\ Dy - z = 0 \\ -x + Dz = 0 \end{cases}$$

Now we can decide which functions to eliminate. Knowing x makes it easy to find y, knowing y makes it easy to find z, and knowing z makes it easy to find x. This means it does not matter which function we eliminate first, and we will do whatever is simplest.

We substitute the second equation $z = Dy$ into the third equation $x = Dz$ to get $x = D^2y$.
Then, we substitute this result into the first equation to get $(D-3)D^2y + 4y = 0 \implies \left(D^3 - 3D^2 + 4\right)y = 0$

This can be rewritten as a CC LDE:

$$y''' - 3y'' + 4y = 0$$

This has characteristic equation $r^3 - 3r^2 + 4 = (r-2)^2(r+1) = 0$, which has roots $r = -1,\ 2\ (\text{double})$.

Thus, the general solution is $y(t) = Ae^{-t} + Be^{2t} + Cte^{2t}$.

We plug this into the equation $z(t) = y'(t) = -Ae^{-t} + 2Be^{2t} + C\left(2t+1\right)e^{2t} = -Ae^{-t} + (2B+C)e^{2t} + 2Cte^{2t}$.

Lastly, we plug this into the equation $x(t) = z'(t) = Ae^{-t} + 2(2B+C)e^{2t} + 2C(2t+1)e^{2t} = Ae^{-t} + (4B+4C)e^{2t} + 4Cte^{2t}$.

Our solution to this first order system of three differential equations is: $\boxed{\begin{cases} x(t) = Ae^{-t} + (4B+4C)e^{2t} + 4Cte^{2t} \\ y(t) = Ae^{-t} + Be^{2t} + Cte^{2t} \\ z(t) = -Ae^{-t} + (2B+C)e^{2t} + 2Cte^{2t} \end{cases}}$

The method for solving non-homogeneous systems is the same as we have seen above, but students should use caution. When applying the operator D to a non-zero function, we must take its derivative (not "multiply" in the standard sense). Similarly, when applying $D - a$ to a function $g(x)$, we see:

$$(D-a)g(x) = Dg(x) - ag(x) = g'(x) - ag(x)$$

We will demonstrate this by solving a non-homogeneous example:

Illustration 2.6.2

We will solve the system of LDEs:

$$\begin{cases} 2tx'(t) = x(t) + 2y(t) + 3t \\ ty'(t) = x(t) + 2y(t) - 6t^2 \end{cases}$$

First, we rewrite it in terms of the operator D and move all x and y to one side: $\begin{cases} (2tD-1)x - 2y = t^2 \\ -x + (tD-2)y = t^2 - e^{2t} \end{cases}$

We can eliminate x by "multiplying" the second equation by $2tD-1$ and then adding them together:

$$\left[(2tD-1)x - 2y = t^2\right] + (2tD-1)\left[-x + (tD-2)y = t^2 - e^{2t}\right] \implies -2y + (2tD-1)(tD-2)y = 3t + (2tD-1)\left(-6t^2\right)$$

Here we carefully calculate $(2tD-1)(tD-2)y(t)$:

$$\begin{aligned} \left(2t\cdot\frac{d}{dt} - 1\right)\left(t\cdot\frac{d}{dt} - 2\right)y(t) &= \left(2t\cdot\frac{d}{dt} - 1\right)\left(ty'(t) - 2y(t)\right) \\ &= 2t\left(y'(t) + ty''(t) - 2y'(t)\right) - 1\left(ty'(t) - 2y(t)\right) = 2t^2y''(t) - 3ty'(t) + 2y(t) \end{aligned}$$

Therefore, the expression $-2y + (2tD-1)(tD-2)y$ becomes $2t^2y''(t) - 3ty'(t)$.

We also carefully calculate $3t + (2tD-1)\left(-6t^2\right) = 3t + 2t\cdot\frac{d}{dt}\left(-6t^2\right) - 1\left(-6t^2\right) = 3t + 2t\cdot(-12t) + 6t^2 = 3t - 18t^2$.

Putting these together, we end up with the second order non-homogeneous LDE for $y(t)$:

$$2t^2y''(t) - 3ty'(t) = 3t - 18t^2$$

This is a CE LDE with characteristic equation $2m(m-1) - 3m = m(2m-5) = 0$, which has roots $m = 0,\ \frac{5}{2}$

Thus, the homogeneous solutions are of the form $y(t) = A + Bt^{\frac{5}{2}}$
We use the method of undetermined coefficients to find the particular solution, guessing $y_p = Ct^2 + Dt$:

$$2t^2y_p'' - 3ty_p' = 2t^2\cdot 2C - 3t\cdot(2Ct + D) = -2Ct^2 - 3Dt$$

Setting this equal to $g(t) = -18t^2 + 3t$ we see $-2C = 18 \implies C = 9$ and $-3D = 3 \implies D = -1$

Thus, the general solution for y is $y(t) = A + Bt^{\frac{5}{2}} + 9t^2 - t$. We can substitute this into the second equation to solve for $x(t)$:

$$x(t) = ty'(t) - 2y(t) + 6t^2 = t\left(\frac{5}{2}Bt^{\frac{3}{2}} + 18t - 1\right) - 2\left(A + Bt^{\frac{5}{2}} + 9t^2 - t\right) + 6t^2 = -2A + \frac{1}{2}Bt^{\frac{5}{2}} + 6t^2 + t$$

Therefore, our final solutions are $\begin{cases} x(t) = -2A + \hat{B}t^{\frac{5}{2}} + 6t^2 + t \\ y(t) = A + 2\hat{B}t^{\frac{5}{2}} + 9t^2 - t \end{cases}$

Example 2.6.3

Find the general solution to each non-homogeneous system of LDEs below.

(a) $\begin{cases} x'(t) = -9x(t) - 5y(t) + 11e^{2t} \\ y'(t) = 6x(t) + 8y(t) - 6e^{2t} \end{cases}$

(b) $\begin{cases} x'(t) = \frac{2}{t}x(t) - \frac{1}{t}y(t) \\ y'(t) = \frac{3}{t}x(t) - \frac{2}{t}y(t) - te^t \end{cases}$

Solution

(a) First, we rewrite in terms of the operator D and move all x and y to one side: $\begin{cases} (D+9)x + 5y = 11e^{2t} \\ -6x + (D-8)y = -6e^{2t} \end{cases}$

We can eliminate x by "multiplying" the first equation by 6 and the second equation by $D+9$ and then adding them together:

$$6\left[(D+9)x + 5y = 11e^{2t}\right] + (D+9)\left[-6x + (D-8)y = -6e^{2t}\right] \implies 30y + (D+9)(D-8)y = 66e^{2t} + (D+9)\left(-6e^{2t}\right)$$

Here we see $30y + (D+9)(D-8)y = (D^2 + D - 42)\,y(t) = y''(t) + y'(t) - 42y(t)$.

Similarly, $66e^{2t} + (D+9)\left(-6e^{2t}\right) = 66e^{2t} - 12e^{2t} - 54e^{2t} = 0$.

Putting these together, we end up with the second order homogeneous LDE for $y(t)$:

$$y''(t) + y'(t) - 42y(t) = 0$$

This is a CC LDE with characteristic equation $r^2 + r - 42 = (r-6)(r+7) = 0$, which has roots $r = 6,\ -7$.

Thus, the general solution for y is $y(t) = Ae^{6t} + Be^{-7t}$. We can substitute this into the second equation to solve for $x(t)$:

$$x(t) = \frac{1}{6}y'(t) - \frac{4}{3}y(t) + e^{2t} = \frac{1}{6}\left(6Ae^{6t} - 7Be^{-7t}\right) - \frac{4}{3}\left(Ae^{6t} + Be^{-7t}\right) + e^{2t} = -\frac{1}{3}Ae^{6t} - \frac{5}{2}Be^{-7t} + e^{2t}$$

Therefore, our final solutions are $\boxed{\begin{cases} x(t) = -\hat{A}e^{6t} - 5\hat{B}e^{-7y} + e^{2t} \\ y(t) = 3\hat{A}e^{6t} + 2\hat{B}e^{-7t} \end{cases}}$

(b) First, we rewrite it in terms of the operator D and move all x and y to one side: $\begin{cases} (tD-2)\,x + y = 0 \\ -3x + (tD+2)\,y = -t^2e^t \end{cases}$

We can eliminate y by "multiplying" the first equation by $tD+2$ and subtracting the second equation:

$$(tD+2)\left[(tD-2)\,x + y = 0\right] - \left[-3x + (tD+2)\,y = -t^2e^t\right] \implies (tD+2)\,(tD-2)\,x + 3x = t^2e^t$$

Here we carefully calculate $(tD+2)(tD-2)x(t)$:

$$\begin{aligned}\left(t\frac{d}{dt}+2\right)\left(t\frac{d}{dt}-2\right)x(t) &= \left(t\frac{d}{dt}+2\right)(tx'(t)-2x(t)) \\ &= t(x'(t)+tx''(t)-2x'(t))+2(tx'(t)-2x(t)) \\ &= t^2x''(t)+tx'(t)-4x(t)\end{aligned}$$

Therefore, the expression $(tD+2)(tD-2)x+3x$ becomes $t^2x''(t)+tx'(t)-x(t)$.

Now, we end up with the second order non-homogeneous LDE for $x(t)$:

$$t^2x''(t)+tx'(t)-x(t)=t^2e^t$$

This is a CE LDE with characteristic equation $m(m-1)-m-1=(m-1)(m+1)=0$, which has roots $m=-1,\ 1$.

Thus, the homogeneous solutions are of the form $x(t)=At^{-1}+Bt$.
We use the method of variation of parameters to find the particular solution, with $x_1=t^{-1},\ x_2=t,\ g(t)=e^t$:

$$W=\begin{vmatrix}t^{-1} & t\\ -t^{-2} & 1\end{vmatrix}=2t^{-1},\quad W_1=\begin{vmatrix}0 & t\\ e^t & 1\end{vmatrix}=-te^t,\quad W_2=\begin{vmatrix}t^{-1} & 0\\ -t^{-2} & e^t\end{vmatrix}=t^{-1}e^t$$

Now we integrate to find u_1 and u_2:

$$\begin{aligned}u_1 &= \int\frac{W_1}{W}\,dt=\int-\frac{1}{2}t^2e^t\,dt=\left(-\frac{1}{2}t^2+t-1\right)e^t+C\\ u_2 &= \int\frac{W_2}{W}\,dt=\int\frac{1}{2}e^t\,dt=\frac{1}{2}e^t+D\end{aligned}$$

We plug this in to find our general solution:

$$\begin{aligned}x(t) &= u_1x_1+u_2x_2\\ &= \left(\left(-\frac{1}{2}t^2+t-1\right)e^t+C\right)t^{-1}+\left(\frac{1}{2}e^t+D\right)t\\ &= \left(1-t^{-1}\right)e^t+Ct^{-1}+Dt\end{aligned}$$

We can substitute this into the first equation to solve for $y(t)$:

$$y(t)=2x(t)-tx'(t)=2\left(\left(1-t^{-1}\right)e^t+Ct^{-1}+Dt\right)-t\left(\left(1-t^{-1}+t^{-2}\right)e^t-Ct^{-2}+D\right)=\left(3-t-3t^{-1}\right)e^t+3Ct^{-1}+Dt$$

Therefore, our final solutions are $\boxed{\begin{cases}x(t)=\left(1-t^{-1}\right)e^t+Ct^{-1}+Dt\\ y(t)=\left(3-t-3t^{-1}\right)e^t+3Ct^{-1}+Dt\end{cases}}$

As always, the inclusion of initial conditions can allow for us to solve for the unique coefficients in the general solutions to these systems.

Example 2.6.4

Find the solution to the non-homogeneous system $\begin{cases}x'(t)=2x(t)+y(t)+e^{2t}\\ y'(t)=x(t)+2y(t)\end{cases}$ with $x(0)=1,\ y(0)=0$

Solution

First, we rewrite in terms of the operator D and move all x and y to one side: $\begin{cases}(D-2)x-y=e^{2t}\\ -x+(D-2)y=0\end{cases}$

We can eliminate x by "multiplying" the second equation by $D-2$ and adding it to first equation:

$$(D-2)\left[-x+(D-2)y=0\right]+\left[(D-2)x-y=e^{2t}\right] \implies (D-2)(D-2)y-y=e^{2t}$$

Here we see $(D-2)(D-2)y-y=\left(D^2-4D+3\right)y(t)=y''(t)-4y'(t)+3y(t)$.

So, we end up with the second order non-homogeneous LDE for $y(t)$:

$$y''(t)-4y'(t)+3y(t)=e^{2t}$$

This is a CC LDE with characteristic equation $r^2-4r+3=(r-3)(r-1)=0$, which has roots $r=1,\ 3$.

Thus, the homogeneous solution for y is $y_h(t)=Ae^t+Be^{3t}$.

We can use the method of undetermined coefficients to find the particular solution, guessing $y_p=Ce^{2t}$:

$$y_p''-4y_p'+3y_p=4Ce^{2t}-4\left(2Ce^{2t}\right)+3\left(Ce^{2t}\right)=-Ce^{2t}$$

Setting this equal to $g(t)=e^{2t}$, we get $-C=1 \implies C=-1$. Thus, $y_p=-e^{2t}$.

This means our general solution for y is $y(t)=Ae^t+Be^{3t}-e^{2t}$.

We can substitute this into the second equation to solve for $x(t)$:

$$x(t)=y'(t)-2y(t)=\left(Ae^t+3Be^{3t}-2e^{2t}\right)-2\left(Ae^t+Be^{3t}-e^{2t}\right)=-Ae^t+Be^{3t}$$

Therefore, our general solutions are $\begin{cases} x(t)=-Ae^t+Be^{3t} \\ y(t)=Ae^t+Be^{3t}-e^{2t} \end{cases}$

Plugging in the initial conditions gives:

$$\begin{aligned} x(0)=-A\cdot 1+B\cdot 1 &\implies -A+B=1 \\ y(0)=A\cdot 1+B\cdot 1-1 &\implies A+B=1 \\ \begin{cases} -A+B=1 \\ A+B=1 \end{cases} &\implies \begin{cases} A=0 \\ B=1 \end{cases} \end{aligned}$$

Therefore, the unique solution to this system is $\boxed{\begin{cases} x(t)=e^{3t} \\ y(t)=e^{3t}-e^{2t} \end{cases}}$

This subchapter has introduced the methods for solving systems of linear differential equations. By employing techniques such as substitution, elimination, and differential operator notation, we can reduce a system to a single higher-order equation in one unknown function. Once that equation is solved, we can systematically find the remaining functions. This process applies to both homogeneous and non-homogeneous systems and can be extended to initial value problems to find unique solutions. Solving these systems is crucial for modeling interconnected dynamic processes in fields like physics, engineering, and biology.

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2.6: Practice Set

Exercise 2.6.1a

Find the general solution to the system $\begin{cases} y_1'(t) = 3y_1(t) \\ y_2'(t) = y_1(t) - 2y_2(t) \end{cases}$

Answer available at the end of this section.

Exercise 2.6.1b

Find the general solution to the system $\begin{cases} y_1'(t) = y_1(t) + 2y_2(t) \\ ty_2'(t) = 2y_2(t) \end{cases}$

Answer available at the end of this section.

Exercise 2.6.1c

Find the general solution to the system $\begin{cases} x'(t) = y(t) - \dfrac{2}{t}x(t) \\ y''(t) = y(t) \end{cases}$

Answer available at the end of this section.

Exercise 2.6.1d

Find the general solution to the system $\begin{cases} x'(t) = 2\cos(t) + x(t) \\ y''(t) = x(t) + \cos(t) - \sin(t) + y(t) \end{cases}$

Answer available at the end of this section.

Exercise 2.6.1e

Find the general solution to the system $\begin{cases} u'(x) = u(x)v(x) \\ x^2v''(x) = 12x + 6v(x) \end{cases}$

Answer available at the end of this section.

Exercise 2.6.2a

Find the general solution to the system $\begin{cases} y_1'(t) = y_1(t) + y_2(t) \\ y_2'(t) = -y_1(t) - y_2(t) \end{cases}$

Answer available at the end of this section.

Exercise 2.6.2b

Find the general solution to the system $\begin{cases} u'(x) = 36x + 3v(x) \\ v'(x) = 2u(x) + v(x) \end{cases}$

Answer available at the end of this section.

Exercise 2.6.2c

Find the general solution to the system $\begin{cases} tx' = 2x - y \\ ty' = -x + 2y \end{cases}$

Answer available at the end of this section.

Exercise 2.6.2d

Find the general solution to the system $\begin{cases} ty_1' = y_1 - y_2 \\ ty_2' = -y_1 + y_2 - \dfrac{2t^2}{\sqrt{t^2-1}} \end{cases}$

Answer available at the end of this section.

Exercise 2.6.2e

Find the general solution to the system $\begin{cases} x' = 9x - 35y + (37t^4 - 24t^3)e^{3t} \\ y' = x - 3y + 6t^4 e^{3t} \end{cases}$

Answer available at the end of this section.

Exercise 2.6.3a

Find the general solution to the system $\begin{cases} x'(t) = -7x(t) + y(t) + t^2 \\ y'(t) = z(t) - 2t \\ z'(t) = 36x(t) \end{cases}$ with $x(0) = 0,\ y(0) = 5,\ z(0) = 30$

Answer available at the end of this section.

Exercise 2.6.3b

Find the general solution to the system $\begin{cases} v'(z) = 4v(z) + 2w(z) \\ w'(z) = 3v(z) + 3w(z) \end{cases}$ with $v(0) = 5,\ w(0) = 0$

Answer available at the end of this section.

Exercise 2.6.3c

Find the general solution to the system $\begin{cases} x'(t) = x(t) + 3y(t) + 6e^t \\ y'(t) = 2x(t) + 2y(t) - 6e^t \end{cases}$ with $x(0) = 3,\ y(0) = -3$

Answer available at the end of this section.

Exercise 2.6.3d

Find the general solution to the system $\begin{cases} ty_1' = 3y_1 - 2y_2 \\ ty_2' = -4y_1 + 5y_2 \end{cases}$ with $y_1(1) = 0,\ y_2(1) = 6$

Answer available at the end of this section.

Exercise 2.6.3e

Find the general solution to the system $\begin{cases} tx' = 2x - y + 6t^2 \\ ty' = -x + 2y - 6t^2 \end{cases}$ with $x(1) = 4,\ y(1) = 6$

Answer available at the end of this section.

Exercise 2.6.3f

Find the general solution to the system $\begin{cases} u'(x) = 3u(x) - 3v(x) - 12 \\ v''(x) = 2u(x) - 2v(x) - 10 \end{cases}$ with $u(0) = 6,\ v(0) = 3,\ v'(0) = 1$

Answer available at the end of this section.

Exercise 2.6.3g

Find the general solution to the system $\begin{cases} x'(t) = 3x(t) + y(t) + e^{-t} \\ y'(t) = z(t) \\ z'(t) = -4x(t) + 8e^{-t} \end{cases}$ with $x(0) = 1,\ y(0) = -1,\ z(0) = -6$

Answer available at the end of this section.

Answers

Exercise 2.6.1a

The top equation already has $y_2(t)$ eliminated and is a CC LDE:

$y_1' - 3y_1 = 0$ has characteristic equation $r - 3 = 0 \implies r = 3$

The general solution for y_1 is therefore $y_1(t) = Ae^{3t}$

Plugging this into the bottom equation gives: $y_2' = Ae^{3t} - 2y_2 \implies y_2' + 2y_2 = Ae^{3t}$

This is linear, with $\mu(t) = e^{\int 2\,dt} = e^{2t}$:

$$y_2(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\,dt = \frac{1}{e^{2t}} \int e^{2t} \cdot Ae^{3t}\,dt = e^{-2t} \int Ae^{5t}\,dt = e^{-2t}\left[\frac{A}{5}e^{5t} + C\right]$$

Therefore, we have $y_2(t) = \frac{A}{5}e^{3t} + Ce^{-2t}$

We write $A = 5\hat{A}$, and the general solution is $\begin{cases} y_1(t) = 5\hat{A}e^{3t} \\ y_2(t) = \hat{A}e^{3t} + Ce^{-2t} \end{cases}$

Exercise 2.6.1b

The bottom equation already has $y_1(t)$ eliminated and is a CE LDE:

$ty_2' - 2y_2 = 0$ has characteristic equation $m - 2 = 0 \implies m = 2$

The general solution for y_2 is therefore $y_2(t) = At^2$

Plugging this into the top equation gives: $y_1' = y_1 + 2 \cdot At^2 \implies y_1' - y_1 = 2At^2$

This is linear, with $\mu(t) = e^{\int -1\,dt} = e^{-t}$:

$$y_1(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\,dt = \frac{1}{e^{-t}} \int e^{-t} \cdot 2At^2\,dt = e^{t}\left[-2A\left(t^2 + 2t + 2\right)e^{-t} + C\right]$$

Therefore, we have $y_1(t) = -2A\left(t^2+2t+2\right) + Ce^t$

The general solution is $\begin{cases} y_1(t) = -2A\left(t^2+2t+2\right) + Ce^t \\ y_2(t) = At^2 \end{cases}$

Exercise 2.6.1*c*

The bottom equation already has $x(t)$ eliminated and is a CC LDE:

$y'' - y = 0$ has characteristic equation $r^2 - 1 = 0 \implies r = \pm 1$

The general solution for y is therefore $y(t) = Ae^t + Be^{-t}$

Plugging this into the top equation gives: $x' = Ae^t + Be^{-t} - \dfrac{2}{t}x \implies x' + \dfrac{2}{t}x = Ae^t + Be^{-t}$

This is linear, with $\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln(t)} = t^2$:

$$x(t) = \frac{1}{\mu(t)}\int \mu(t)q(t)\,dt = \frac{1}{t^2}\int t^2\cdot\left(Ae^t + Be^{-t}\right)\,dt = t^{-2}\left[A(t^2-2t+2)e^t - B(t^2+2t+2)e^{-t} + C\right]$$

Therefore, we have $x(t) = A(1-2t^{-1}+2t^{-2})e^t - B(1+2t^{-1}+2t^{-2})e^{-t} + Ct^{-2}$

The general solution is $\begin{cases} x(t) = A(1-2t^{-1}+2t^{-2})e^t - B(1+2t^{-1}+2t^{-2})e^{-t} + Ct^{-2} \\ y(t) = Ae^t + Be^{-t} \end{cases}$

Exercise 2.6.1*d*

The top equation already has $y(t)$ eliminated and is a linear first order LDE:

$x' - x = 2\cos(t)$ so we use $\mu(t) = e^{\int -1\,dt} = e^{-t}$:

$$x(t)\frac{1}{\mu(t)}\int \mu(t)q(t)\,dt = \frac{1}{e^{-t}}\int e^{-t}\cdot 2\cos(t)\,dt = e^t\left[e^{-t}\sin(t) - e^{-t}\cos(t) + C\right]$$

The general solution for x is therefore $x(t) = \sin(t) - \cos(t) + Ce^t$.

Plugging this into the bottom equation gives: $y'' - y = Ce^t$.

This is a second order CC LDE with characteristic equation $r^2 - 1 = 0 \implies r = \pm 1$

Thus, the homogeneous solution is $y_h(t) = Ae^t + Be^{-t}$, and we guess $y_p(t) = Dte^t$:

$$y_p'' - y_p = D(t+2)e^t - Dte^t = 2De^t$$

Setting this equal to $g(t) = Ce^t$ gives $C = 2D$

The general solution is therefore $\begin{cases} x(t) = \sin(t) - \cos(t) + 2De^t \\ y(t) = (A+Dt)e^t + Be^{-t} \end{cases}$

Exercise 2.6.1*e*

The bottom equation already has $u(x)$ eliminated and is a CE LDE:

$x^2v'' - 6y = 0$ has characteristic equation $m(m-1) - 6 = (m-3)(m+2) = 0 \implies m = 3, -2$

The homogeneous solution for v is therefore $v_h(x) = Ax^3 + Bx^{-2}$

We guess the particular solution is $v_p(x) = Cx$

$$v_p'' - 6v_p = 0 - 6\cdot Cx = -6Cx$$

Setting this equal to $g(x) = 12x$ gives $-6C = 12 \implies C = -2$

The general solution for v is thus $v(x) = Ax^3 + Bx^{-2} - 2x$

Plugging this into the top equation gives: $u'(x) = u(x)\cdot\left(Ax^3 + Bx^{-2} - 2x\right)$

This is separable:

$$\int \frac{u'(x)}{u(x)}\,dx = \int \left(Ax^3 + Bx^{-2} - 2x\right)\,dx \implies \ln(u(x)) = \frac{A}{4}x^4 - Bx^{-1} - x^2 + D$$

Therefore, we have $u(x) = e^{\frac{A}{4}x^4 - Bx^{-1} - x^2 + D} = De^{\frac{A}{4}x^4 - Bx^{-1} - x^2}$

Writing $\hat{A} = \dfrac{A}{4}$, the general solution is $\begin{cases} u(x) = De^{\hat{A}x^4 - Bx^{-1} - x^2} \\ v(x) = 4\hat{A}x^3 + Bx^{-2} - 2x \end{cases}$

Exercise 2.6.2*a*

First, we rewrite the system using the D notation:

$$\begin{cases} Dy_1 = y_1 + y_2 \\ Dy_2 = -y_1 - y_2 \end{cases} \implies \begin{cases} (D-1)y_1 - y_2 = 0 \\ y_1 + (D+1)y_2 = 0 \end{cases}$$

We add $(D+1)$ "times" the top equation to the bottom equation to get:

$$\left((D+1)(D-1) + 1\right)y_1 = 0 \implies y_1'' = 0$$

This is a CC LDE with characteristic equation $r^2 = 0 \implies r = 0$ (double)

Thus, we have general solution $y_1(t) = At + B$

Substituting this into the top equation gives:

$$y_2 = y_1' - y_1 = A - (At + B) = -At + (A - B)$$

The general solution is $\begin{cases} y_1(t) = At + B \\ y_2(t) = -At + (A-B) \end{cases}$

Exercise 2.6.2*b*

First, we rewrite the system using the D notation:

$$\begin{cases} Du = 36x + 3v \\ Dv = 2u + v \end{cases} \implies \begin{cases} Du - 3v = 36x \\ -2u + (D-1)v = 0 \end{cases}$$

We add 2 times the top equation to D "times" the bottom equation to get:

$$(-6+D(D-1))\,v = 72x \implies v''-v'-6v=0$$

This is a CC LDE with characteristic equation $r^2-r-6=0 \implies r=3,-2$

The homogeneous solution is therefore $v_h = Ae^{3x}+Be^{-2x}$

We guess a particular solution of the form $v_p = Cx+D$:

$$v_p''-v_p'-6v_p = 0-C-6(Cx+D) = -6Ct-(C+6D)$$

Setting this equal to $g(x)=72x$ gives $\begin{cases} -6C=72 \\ -C-6D=0 \end{cases} \implies \begin{cases} C=-12 \\ D=2 \end{cases}$

This gives general solution $v(x) = Ae^{3x}+Be^{-2x}-12x+2$

Substituting this into the bottom equation gives:

$$u = \frac{1}{2}v'-\frac{1}{2}v = \frac{1}{2}\left(3Ae^{3x}-2Be^{-2x}-12\right)-\frac{1}{2}\left(Ae^{3x}+Be^{-2x}-12x+2\right)$$

Writing $B=2\hat{B}$, the general solution is $\begin{cases} u(x) = Ae^{3x}-3\hat{B}e^{-2x}+6x-7 \\ v(x) = Ae^{3x}+2\hat{B}e^{-2x}-12x+2 \end{cases}$

Exercise $2.6.2c$

First, we rewrite the system using the D notation:

$$\begin{cases} tDx = 2x-y \\ tDy = -x+2y \end{cases} \implies \begin{cases} (tD-2)\,x+y=0 \\ x+(tD-2)\,y=0 \end{cases}$$

We subtract the bottom equation from $(tD-2)$ "times" the top equation to get:

$$\left((tD-2)^2-1\right)x = 0 \implies t^2x''-3tx'+3x=0$$

This is a CE LDE with characteristic equation $m(m-1)-3m+3=(m-1)(m-3)=0 \implies m=1,3$

The general solution is therefore $x(t) = At+Bt^3$

Substituting this into the top equation gives:

$$y = 2x-tx' = 2\left(At+Bt^3\right)-t\left(A+3Bt^2\right) = At-Bt^3$$

The general solution is $\begin{cases} x(t) = At+Bt^3 \\ y(t) = At-Bt^3 \end{cases}$

Exercise $2.6.2d$

First, we rewrite the system using the D notation:

$$\begin{cases} tDy_1 = y_1-y_2 \\ tDy_2 = -y_1+y_2-\dfrac{2t^2}{\sqrt{t^2-1}} \end{cases} \implies \begin{cases} (tD-1)\,y_1+y_2=0 \\ y_1+(tD-1)\,y_2 = -\dfrac{2t^2}{\sqrt{t^2-1}} \end{cases}$$

We subtract the bottom equation from $(tD-1)$ "times" the top equation to get:

$$\left((tD-1)^2-1\right)y_1 = \frac{2t^2}{\sqrt{t^2-1}} \implies t^2y_1''-ty_1' = \frac{2t^2}{\sqrt{t^2-1}}$$

This is a CE LDE with characteristic equation $m(m-1)-m=m(m-2)=0 \implies m=0,2$

The homogeneous solution is therefore $y_{1,h}(t) = A+Bt^2$

To find the particular solution, we use variation of parameters:

$$W = \begin{vmatrix} 1 & t^2 \\ 0 & 2t \end{vmatrix} = 2t, \quad W_1 = \begin{vmatrix} 0 & t^2 \\ \frac{2}{\sqrt{t^2-1}} & 2t \end{vmatrix} = -\frac{2}{\sqrt{t^2-1}}, \quad W_2 = \begin{vmatrix} 1 & 0 \\ 0 & \frac{2}{\sqrt{t^2-1}} \end{vmatrix} = \frac{2t^2}{\sqrt{t^2-1}}$$

Integrating gives us u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dt = \int -\frac{t}{\sqrt{t^2-1}}\,dt = -\sqrt{t^2-1}+A$$
$$u_2 = \int \frac{W_2}{W}\,dt = \int \frac{1}{t\sqrt{t^2-1}}\,dt = \sec^{-1}(t)+B$$

Therefore, the general solution is: $y_1(t) = -\sqrt{t^2-1}+t^2\sec^{-1}(t)+A+Bt^2$

Substituting this into the top equation gives:

$$y_2 = y_1-ty_1' = \left(-\sqrt{t^2-1}+t^2\sec^{-1}(t)+A+Bt^2\right)-t\left(2t\sec^{-1}(t)+2Bt\right) = -\sqrt{t^2-1}-t^2\sec^{-1}(t)+A-Bt^2$$

The general solution is $\begin{cases} y_1(t) = -\sqrt{t^2-1}+t^2\sec^{-1}(t)+A+Bt^2 \\ y_2(t) = -\sqrt{t^2-1}-t^2\sec^{-1}(t)+A-Bt^2 \end{cases}$

Exercise $2.6.2e$

First, we rewrite the system using the D notation:

$$\begin{cases} Dx = 9x-35y+(37t^4-24t^3)e^{3t} \\ Dy = x-3y+6t^4e^{3t} \end{cases} \implies \begin{cases} (D-9)\,x+35y = (37t^4-24t^3)e^{3t} \\ -x+(D+3)\,y = 6t^4e^{3t} \end{cases}$$

We add the $(D-9)$ "times" the bottom equation to the top equation to get:

$$((D-9)(D+3)+35)\,y = (37t^4-24t^3)e^{3t}+(D-9)6t^4e^{3t} \implies y''-6y'+8y = t^4e^{3t}$$

This is a CC LDE with characteristic equation $r^2-6r+8=(r-2)(r-4)=0 \implies r=2,4$

The homogeneous solution is therefore $y_h(t) = Ae^{2t} + Be^{4t}$

To find the particular solution, we use variation of parameters:

$$W = \begin{vmatrix} e^{2t} & e^{4t} \\ 2e^{2t} & 4e^{4t} \end{vmatrix} = 2e^{6t}, \quad W_1 = \begin{vmatrix} 0 & e^{4t} \\ t^4e^{3t} & 4e^{4t} \end{vmatrix} = t^4e^{7t}, \quad W_2 = \begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & t^4e^{3t} \end{vmatrix} = -t^4e^{5t}$$

Integrating gives us u_1 and u_2:

$$u_1 = \int \frac{W_1}{W}\,dt = \int \frac{1}{2}t^4e^t\,dt = \left(\frac{1}{2}t^4 - 2t^3 + 6t^2 - 12t + 12\right)e^t + A$$
$$u_2 = \int \frac{W_2}{W}\,dt = \int -\frac{1}{2}t^4e^{-t}\,dt = \left(\frac{1}{2}t^4 + 2t^3 + 6t^2 + 12t + 12\right)e^{-t} + B$$

Therefore, the general solution is: $y(t) = \left(t^4 + 12t^2 + 24\right)e^{3t} + Ae^{2t} + Be^{4t}$

Substituting this into the bottom equation gives:

$$x = y' + 3y - t^4e^{3t} = \left(\left(3t^4 + 4t^3 + 36t^2 + 24t + 72\right)e^{3t} + 2Ae^{2t} + 4Be^{4t}\right) + 3\left(\left(t^4 + 12t^2 + 24\right)e^{3t} + Ae^{2t} + Be^{4t}\right) - 6t^4e^{3t}$$

This simplifies to $x(t) = \left(4t^3 + 72t^2 + 24t + 144\right)e^{3t} + 5Ae^{2t} + 7Be^{4t}$

The general solution is $\begin{cases} x(t) = \left(4t^3 + 72t^2 + 24t + 144\right)e^{3t} + 5Ae^{2t} + 7Be^{4t} \\ y(t) = \left(t^4 + 12t^2 + 24\right)e^{3t} + Ae^{2t} + Be^{4t} \end{cases}$

Exercise 2.6.3*a*

First, we rewrite the system using the D notation:

$$\begin{cases} Dx = -7x + y + t^2 \\ Dy = z - 2t \\ Dz = 36x \end{cases} \implies \begin{cases} (D+7)x - y = t^2 \\ Dy - z = -2t \\ -36x + Dz = 0 \end{cases}$$

We add D^2 "times" the top equation to D times the middle equation to the bottom equation to get:

$$\left(D^2(D+7) - 36\right)x = D^2(t^2) + D(-2t) + 0 \implies x''' + 7x'' - 36x = 0$$

This is a homogeneous CC LDE with characteristic equation: $r^3 + 7r^2 - 36 = (r-2)(r+3)(r+6) = 0 \implies r = 2, -3, -6$

Therefore, $x(t) = Ae^{2t} + Be^{-3t} + Ce^{-6t}$

Substituting this into the top equation gives:

$$y = x' + 7x - t^2 = \left(2Ae^{2t} - 3Be^{-3t} - 6Ce^{-6t}\right) + 7\left(Ae^{2t} + Be^{-3t} + Ce^{-6t}\right) - t^2 = 9Ae^{2t} + 4Be^{-3t} + Ce^{-6t} - t^2$$

Substituting this into the middle equation gives:

$$z = y' + 2t = \left(18Ae^{2t} - 12Be^{-3t} - 6Ce^{-6t} - 2t\right) + 2t = 18Ae^{2t} - 12Be^{-3t} - 6Ce^{-6t}$$

Applying the initial conditions:

$$\begin{aligned} x(t) &= Ae^{2t} + Be^{-3t} + Ce^{-6t} \\ x(0) = 0 &\implies A + B + C = 0 \\ y(t) &= 9Ae^{2t} + 4Be^{-3t} + Ce^{-6t} - t^2 \\ y(0) = 5 &\implies 9A + 4B + C = 5 \\ z(t) &= 18Ae^{2t} - 12Be^{-3t} - 6Ce^{-6t} \\ z(0) = 30 &\implies 18A - 12B - 6C = 30 \end{aligned}$$

$$\begin{cases} A + B + C = 0 \\ 9A + 4B + C = 5 \\ 18A - 12B - 6C = 30 \end{cases} \implies \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

The solution is $\begin{cases} x(t) = e^{2t} - e^{-3t} \\ y(t) = 9e^{2t} - 4e^{-3t} - t^2 \\ z(t) = 18e^{2t} + 12e^{-3t} \end{cases}$

Exercise 2.6.3*b*

First, we rewrite the system using the D notation:

$$\begin{cases} Dv = 4v + 2w \\ Dw = 3v + 3w \end{cases} \implies \begin{cases} (D-4)v - 2w = 0 \\ -3v + (D-3)w = 0 \end{cases}$$

We "multiply" the first equation by $(D-3)$ and add 2 times the second equation to eliminate y_2:

$$\left((D-3)(D-4) - 6\right)v = 0 \implies v'' - 7v' + 6v = 0$$

This is a CC LDE with characteristic equation $r^2 - 7r + 6 = (r-1)(r-6) = 0 \implies r = 1, 6$

This has general solution $v(z) = Ae^z + Be^{6z}$

Substituting this into the first original equation gives:

$$w = \frac{1}{2}v' - 2v = \frac{1}{2}\left(Ae^z + 6Be^{6z}\right) - 2\left(Ae^z + Be^{6z}\right) = -\frac{3}{2}Ae^z + Be^{6z}$$

Applying the initial conditions:

$$v(z) = Ae^z + Be^{6z}$$
$$v(0) = 5 \implies A + B = 5$$
$$w(z) = -\frac{3}{2}Ae^z + Be^{6z}$$
$$w(0) = 0 \implies -\frac{3}{2}A + B = 0$$
$$\begin{cases} A + B = 5 \\ -\frac{3}{2}A + B = 0 \end{cases} \implies \begin{cases} A = 2 \\ B = 3 \end{cases}$$

The solution is $\begin{cases} v(z) = 2e^z + 3e^{6z} \\ w(z) = -3e^z + 3e^{6z} \end{cases}$

Exercise 2.6.3*c*

First, we rewrite the system using the D notation:

$$\begin{cases} Dx = x + 3y + 6e^t \\ Dy = 2x + 2y - 6e^t \end{cases} \implies \begin{cases} (D-1)x - 3y = 6e^t \\ -2x + (D-2)y = -6e^t \end{cases}$$

Adding $(D-2)$ "times" the top equation to 3 times the bottom gives:

$$((D-2)(D-1) - 6)\,x = (D-2)6e^t + 3(-6e^t) \implies x'' - 3x' - 4x = -24e^t$$

This is a CC LDE with characteristic equation $r^2 - 3r - 4 = (r+1)(r-4) = 0 \implies r = -1, 4$

This gives us the homogeneous solution $x_h(t) = Ae^{-t} + Be^{4t}$

We guess a particular solution of the form $x_p = Ce^t$:

$$x_p'' - 3x_p' - 4x_p = Ce^t - 3Ce^t - 4Ce^t = -6Ce^t$$

Setting this equal to $g(t) = -24e^t$ gives $-6C = -24 \implies C = 4$

Therefore, we have $x(t) = 4e^t + Ae^{-t} + Be^{4t}$

Substituting this into the first equation gives:

$$y = \frac{1}{3}x' - \frac{1}{3}x - 2e^t = \frac{1}{3}\left(4e^t - Ae^{-t} + 4Be^{4t}\right) - \frac{1}{3}\left(4e^t + Ae^{-t} + Be^{4t}\right) - 2e^t = -2e^t - \frac{2}{3}Ae^{-t} + Be^{4t}$$

Applying the initial conditions:

$$x(t) = 4e^t + Ae^{-t} + Be^{4t}$$
$$x(0) = 3 \implies 4 + A + B = 3 \implies A + B = -1$$
$$y(t) = -2e^t - \frac{2}{3}Ae^{-t} + Be^{4t}$$
$$y(0) = -3 \implies -2 - \frac{2}{3}A + B = -3 \implies -\frac{2}{3}A + B = -1$$
$$\begin{cases} A + B = -1 \\ -\frac{2}{3}A + B = -1 \end{cases} \implies \begin{cases} A = 0 \\ B = -1 \end{cases}$$

The solution is $\begin{cases} x(t) = 4e^t - e^{4t} \\ y(t) = -2e^t - e^{4t} \end{cases}$

Exercise 2.6.3*d*

First, we rewrite the system using the D notation:

$$\begin{cases} tDy_1 = 3y_1 - 2y_2 \\ tDy_2 = -4y_1 + 5y_2 \end{cases} \implies \begin{cases} (tD-3)y_1 + 2y_2 = 0 \\ 4y_1 + (tD-5)y_2 = 0 \end{cases}$$

Subtracting 2 times the bottom equation from $(tD-5)$ "times" the top equation gives:

$$((tD-5)(tD-3) - 8)\,y_1 = 0 \implies t^2y_1'' - 7ty_1' + 7y_1 = 0$$

This is a CE LDE with characteristic equation $m(m-1) - 7m + 8 = (m-1)(m-7) = 0 \implies m = 1, 7$

The general solution is therefore $y_1(t) = At + Bt^7$

Substituting this into the original top equation gives:

$$y_2 = \frac{3}{2}y_1 - \frac{1}{2}ty_1' = \frac{3}{2}(At + Bt^7) - \frac{1}{2}t(A + 7Bt^6) = At - 2Bt^7$$

Applying the initial conditions gives:

$$y_1(t) = At + Bt^7$$
$$y_1(1) = 0 \implies A + B = 0$$
$$y_2(t) = At - 2Bt^7$$
$$y_2(1) = 6 \implies A - 2B = 6$$
$$\begin{cases} A + B = 0 \\ A - 2B = 6 \end{cases} \implies \begin{cases} A = 2 \\ B = -2 \end{cases}$$

The solution is $\begin{cases} y_1(t) = 2t - 2t^7 \\ y_2(t) = 2t + 4t^7 \end{cases}$

Exercise 2.6.3*e*

First, we rewrite the system using the D notation:

$$\begin{cases} tDx = 2x - y + 6t^2 \\ tDy = -x + 2y - 6t^2 \end{cases} \implies \begin{cases} (tD-2)x + y = 6t^2 \\ x + (tD-2)y = -6t^2 \end{cases}$$

Subtracting the bottom equation from $(tD-2)$ "times" the top equation gives:

$$\left((tD-2)^2-1\right)x=(tD-2)6t^2-(-6t^2)\implies t^2x''-3tx'+3x=6t^2$$

This is a CE LDE with characteristic equation $m(m-1)-3m+3=(m-1)(m-3)=0\implies m=1,3$

This has homogeneous solution $x_h(t)=At+Bt^3$

To find the particular solution, we guess $x_p(t)=Ct^2$:

$$t^2x_p''-3tx_p'+3x_p=t^2\left(2C\right)-3t\left(2Ct\right)+3\left(Ct^2\right)=-Ct^2$$

Using $g(t)=6t^2$ we see $-C=6\implies C=-6$

Thus, the general solution is $x(t)=At+Bt^3-6t^2$

Substituting this into the top equation gives:

$$y=2x-tx'+6t^2=2\left(At+Bt^3-6t^2\right)-t\left(A+3Bt^2-12t\right)+6t^2=At-Bt^3+6t^2$$

Applying the initial conditions gives:

$$\begin{aligned}x(t)&=At+Bt^3-6t^2\\ x(1)=4&\implies A+B-6=4\implies A+B=10\\ y(t)&=At-Bt^3+6t^2\\ y(1)=6&\implies A-B+6=6\implies A-B=0\\ \begin{cases}A+B=10\\A-B=0\end{cases}&\implies\begin{cases}A=5\\B=5\end{cases}\end{aligned}$$

The solution is $\begin{cases}x(t)=5t+5t^3-6t^2\\y(t)=5t-5t^3+6t^2\end{cases}$

Exercise 2.6.3f

First, we rewrite the system using the D notation:

$$\begin{cases}Du=3u-3v-12\\D^2v=2u-2v-10\end{cases}\implies\begin{cases}(D-3)u+3v=-12\\-2u+(D^2+2)v=-10\end{cases}$$

We add 2 times the top equation to $(D-3)$ "times" the bottom equation to get:

$$\left(6+(D^2+2)(D-3)\right)v=2(-12)+(D-3)(-10)\implies v'''-3v''+2v'=6$$

This is a CC LDE with characteristic equation $r^3-3r^2+2r=r(r-1)(r-2)=0\implies r=0,1,2$

The homogeneous solution is therefore $v_h(x)=A+Be^x+Ce^{2x}$

Since $1=e^{0x}$ is a homogeneous solution, we guess the particular solution is of the form $v_p=Dx$:

$$v_p'''-3v_p''+2v_p'=0-3\cdot0+2\cdot D=2D$$

Setting this equal to $g(x)=6$ we see $2D=6\implies D=3$

Thus, the general solution is $v(x)=3x+A+Be^x+Ce^{2x}$

Substituting this into the bottom equation gives:

$$u=\frac{1}{2}v''+v+5=\frac{1}{2}\left(Be^x+4Ce^{2x}\right)+\left(3x+A+Be^x+Ce^{2x}\right)+5=3x+A+5+\frac{3}{2}Be^x+3Ce^{2x}$$

Applying the initial conditions gives:

$$\begin{aligned}u(x)&=3x+A+5+\frac{3}{2}Be^x+3Ce^{2x}\\ u(0)=6&\implies A+5+\frac{3}{2}B+3C=6\implies A+\frac{3}{2}B+3C=1\\ v(x)&=3x+A+Be^x+Ce^{2x}\\ v(0)=3&\implies A+B+C=3\\ v'(x)&=3+Be^x+2Ce^{2x}\\ v'(0)=1&\implies 3+B+2C=1\implies B+2C=-2\\ \begin{cases}A+\frac{3}{2}B+3C=1\\A+B+C=1\\B+2C=-2\end{cases}&\implies\begin{cases}A=4\\B=0\\C=-1\end{cases}\end{aligned}$$

The solution is $\begin{cases}u(x)=3x+9-3e^{2x}\\v(x)=3x+4-e^{2x}\end{cases}$

Exercise 2.6.3g

First, we rewrite the system using the D notation:

$$\begin{cases}Dx=3x+y+e^{-t}\\Dy=z\\Dz=-4x+8e^{-t}\end{cases}\implies\begin{cases}(D-3)x-y=e^{-t}\\Dy-z=0\\4x+Dz=8e^{-t}\end{cases}$$

We add D^2 "times" the top equation to D "times" the middle equation to the bottom equation:

$$\left(D^2(D-3)+4\right)x=D^2(e^{-t})+D(0)+8e^{-t}\implies x'''-3x''+4x=9e^{-t}$$

This is a CC LDE with characteristic equation $r^3-4r^2+4=(r-2)^2(r+1)=0\implies r=2\text{ (double)},-1$

Therefore, we have homogeneous solution $x_h(t)=(A+Bt)e^{2t}+Ce^{-t}$

Since e^{-t} is a homogeneous solution, we guess the particular solution is of the form $x_p=Dte^{-t}$:

$$x_p'''-3x_p''+4x_p=D(3-t)e^{-t}-3\cdot D(t-2)e^{-t}+4Dte^{-t}=9De^{-t}$$

Using $g(t) = 9e^{-t}$, we have $9D = 9 \implies D = 1$

Therefore, the general solution is $x(t) = (A + Bt)e^{2t} + (C + t)e^{-t}$

Substituting this into the top equation gives:

$$\begin{aligned} y &= x' - 3x + e^{-t} \\ &= \left((B + 2A + 2Bt)e^{2t} + (1 - C - t)e^{-t}\right) - 3\left((A + Bt)e^{2t} + (C + t)e^{-t}\right) - e^{-t} \\ &= (B - A - Bt)e^{2t} + (-4C - 4t)e^{-t} \end{aligned}$$

Substituting this into the middle equation gives:

$$z = y' = (B - 2A - 2Bt)e^{2t} + (-4 + 4C + 4t)e^{-t}$$

Applying the initial conditions gives:

$$\begin{aligned} & x(t) = (A + Bt)e^{2t} + (C + t)e^{-t} \\ & x(0) = 1 \implies A + C = 1 \\ & y(t) = (B - A - Bt)e^{2t} + (-4C - 4t)e^{-t} \\ & y(0) = -1 \implies B - A - 4C = 1 \implies -A + B - 4C = -1 \\ & z(t) = (B - 2A - 2Bt)e^{2t} + (-4 + 4C + 4t)e^{-t} \\ & z(0) = -6 \implies B - 2A - 4 + 4C = -6 \implies -2A + B + 4C = -6 \end{aligned}$$

$$\begin{cases} A + C = 1 \\ -A + B - 4C = -1 \\ -2A + B + 4C = -2 \end{cases} \implies \begin{cases} A = 1 \\ B = 0 \\ C = 0 \end{cases}$$

The solution is $\begin{cases} x(t) = e^{2t} + te^{-t} \\ y(t) = -e^{2t} - 4te^{-t} \\ z(t) = -2e^{2t} + (4t - 4)e^{-t} \end{cases}$

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3: LAPLACE TRANSFORMATIONS

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3.2: Differentiation of Inputs of Laplace

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3.1: The Laplace Transformation

Learning Objectives

- Understand the definition of the Laplace transform as an integral operator.
- Calculate the Laplace transform of basic functions from the definition.
- Derive and recognize common Laplace transforms that form the foundation for the method.
- Understand the concept of the inverse Laplace transform and apply it to simple rational functions.
- Use partial fraction decomposition to find inverse Laplace transforms.

We introduce the Laplace transform as a powerful integral operator that converts calculus problems into algebra problems. This foundation covers the definition, basic transforms, and the concept of the inverse transform, using partial fractions to recover functions from their transformed state.

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3.1: Laplace Transformation Basics

The culmination of the second calculus course for most students centers upon power series and Taylor Series. That is, we take an infinitely differentiable function $f(x)$ at $x = a$, and we find a power series $\hat{f}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ that converges to $f(x)$ whenever it converges.

It is interesting that we can get the full "essence" of a smooth function just by knowing the values of a sequence $f(a), f'(a), f''(a), \ldots$ (so that we can plug them into the Taylor series function $\hat{f}(x)$).

This is not the only way of transforming one object (a function) into another object (a sequence) that we can use to study a function. This chapter will focus entirely on a different (but inherently similar) such transformation.

Definition: Laplace Transformation

Let $f(t) : [0, \infty) \to \mathbb{R}$ be a real valued function. We define the ***Laplace Transformation*** of $f(t)$ to be the function $\mathscr{L}(f(t))$ defined by the integral

$$\mathscr{L}(f(t)) = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

for all s that make the integral converge.

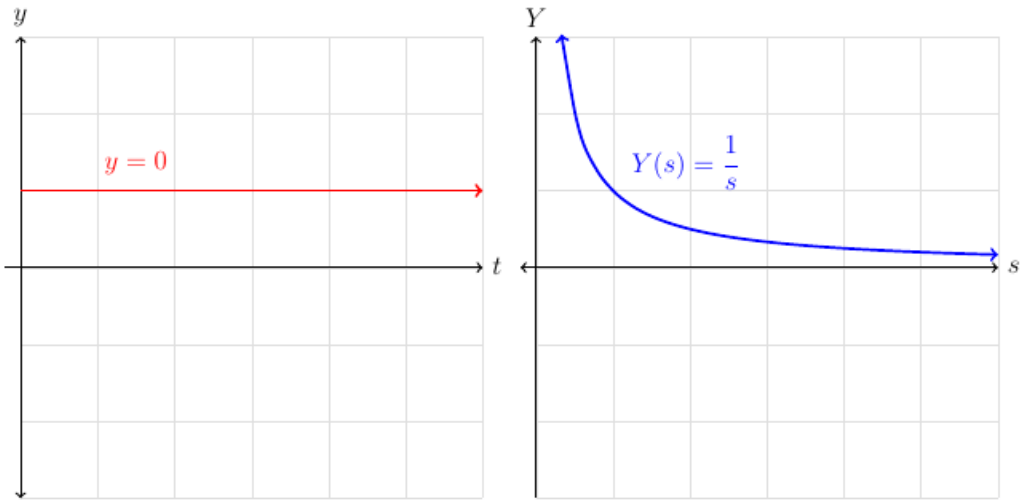
The Laplace Transformation turns one function, $f(t)$, into another one $F(s)$. Fans of probability theory may recognize a close relative of the *generating function of a distribution*. The Laplace Transformation gives us a way to "decode" the function itself (how it acts on a variable t) into an expression of how it integrates against various rates of growth or decay (s).

In practice, the Laplace Transformation is fairly easy to calculate for many popular functions.

Illustration 3.1.1:

Consider the constant unit function $f(t) = 1$. The Laplace Transformation of this constant function is given by

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \cdot 1\, dt \\ &= \lim_{T\to\infty} \int_0^{T} e^{-st}\, dt \\ &= \lim_{T\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^{T} \\ &= \lim_{T\to\infty} \frac{e^{-sT} - 1}{-s} \\ &= \frac{0-1}{-s} \\ &= \frac{1}{s} \end{aligned}$$



This gives us a baseline for what the most boring function's Laplace Transformation will look like. Calculating it for exponential functions is relatively straightforward.

Example 3.1.1

Find $\mathscr{L}\left(e^{at}\right)$ where $a^{\text{int}}\mathbb{R}$ is a fixed constant.

Solution

We calculate the integral directly:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\cdot e^{at}\,dt \\ &= \lim_{T\to\infty}\int_0^T e^{(a-s)t}\,dt \\ &= \lim_{T\to\infty}\left.\frac{e^{(a-s)t}}{a-s}\right|_0^T \\ &= \lim_{T\to\infty}\frac{e^{(a-s)T}-1}{a-s} \\ &= \frac{0-1}{a-s} \\ &= \frac{1}{s-a} \end{aligned}$$

This integral converges whenever $s > a$, so $\boxed{\mathscr{L}\left(e^{at}\right) = \frac{1}{s-a}}$

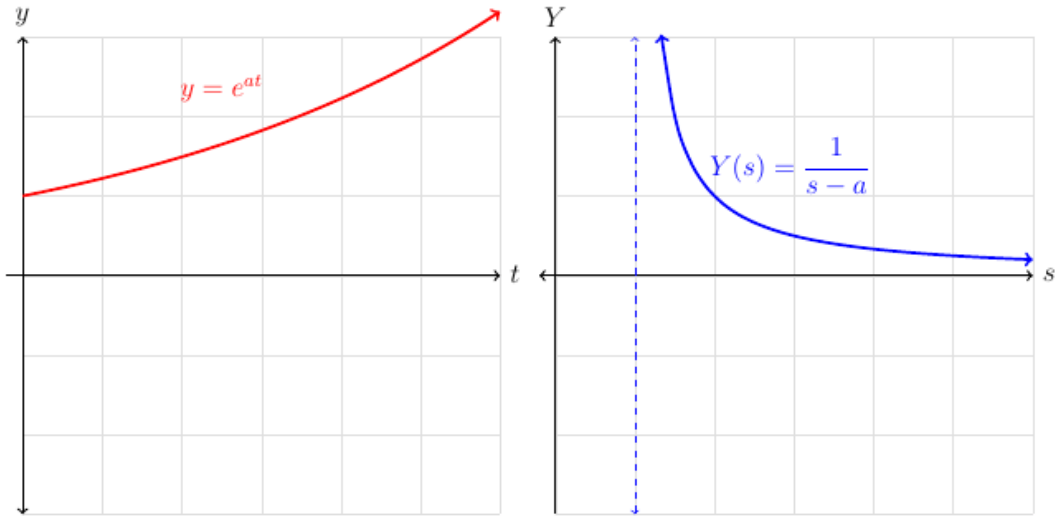


Illustration 3.1.2:

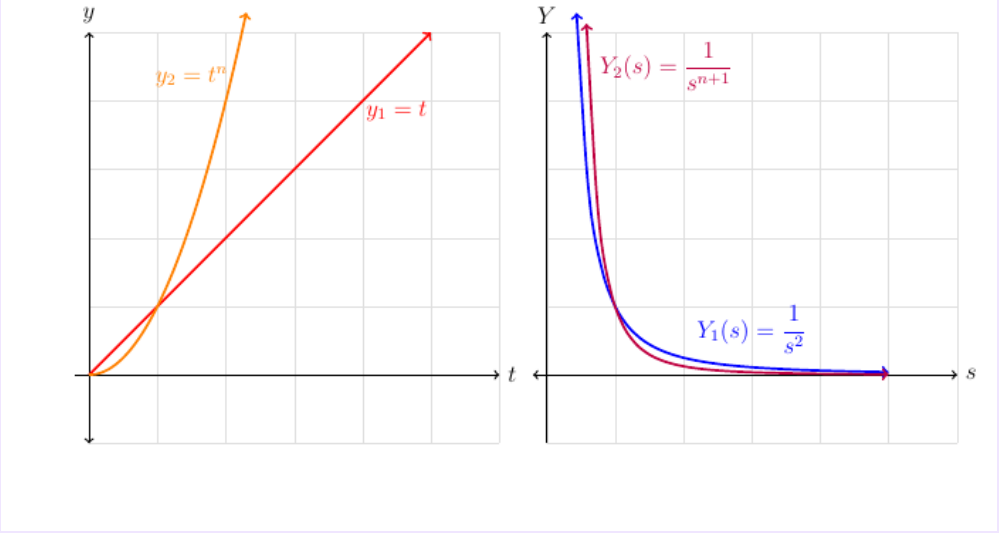
Consider the family of power functions $f(t) = t^b$. Whenever $b = n$ is a non-negative integer, we recognize a very solvable integral:

$$\mathscr{L}(t^n) = \int_0^\infty e^{-st} \cdot t^n \, dt = \lim_{T\to\infty} \int_0^T t^n e^{-st} \, dt$$

This integral is solvable by the tabular method (but can be unwieldy for large n). Instead, we use just one integration by parts:

$$\mathscr{L}(t^n) = \lim_{T\to\infty} \int_0^T t^n e^{-st} \, dt = \lim_{T\to\infty} t^n \frac{e^{-st}}{-s}\Big|_0^T - \int_0^T nt^{n-1} \frac{e^{-st}}{-s} \, dt = \frac{n}{s} \lim_{T\to\infty} \int_0^T t^{n-1} e^{-st} \, dt = \frac{n}{s} \mathscr{L}(t^{n-1})$$

Chaining these together gives $\mathscr{L}(t^n) = \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{1}{s} \cdot \mathscr{L}(1) = \frac{n!}{s^{n+1}}$ for $n \geq 0$



Note: Domain of the Laplace Transformation

Not every function has a convergent Laplace Transformation. Consider the function $z(t) = e^{t^2}$:

$$Z(t) = \int_0^\infty e^{-st} \cdot e^{t^2}\, dt = \int_0^\infty e^{(t-\frac{s}{2})^2 + \frac{s^2}{4}}\, dt$$

This diverges for all possible values of s. In order for the Laplace Transformation to exist, the function must be able to "be decayed" by an exponential function. This means we will only consider functions satisfying $|f(t)| \leq Me^{ct}$ for some constants $M, c \geq 0$.

All of this work with Laplace Transformations should remind you of when we first learned to differentiate. We began with a rather cumbersome formula (involving a limit) so that we could fully understand how to differentiate. Then, we noticed patterns like the power rule. We now have our equivalent of the power rule with $\mathscr{L}(t^n) = \dfrac{n!}{s^{n+1}}$.

After the power rule, you move on to differentiating polynomials by using the linearity of differentiation. Luckily, the Laplace Transformation is *also linear:*

Theorem 3.1.1: Linearity of Laplace Transformation

Assume $f(t)$ and $g(t)$ are functions satisfying $F(s) = \mathscr{L}(f(t))$ and $G(s) = \mathscr{L}(g(t))$

(i) The function $h(t) = f(t) + g(t)$ has Laplace Transformation $H(s) = F(s) + G(s)$

(ii) The function $h(t) = r \cdot f(t)$ has Laplace Transformation $H(s) = r \cdot F(s)$

Proof

It is easy to prove (i):

$$\begin{aligned} H(s) &= \int_0^\infty e^{-st} \cdot h(t)\, dt \\ &= \int_0^\infty e^{-st} \cdot [f(t) + g(t)]\, dt \\ &= \int_0^\infty e^{-st} \cdot f(t)\, dt + \int_0^\infty e^{-st} \cdot g(t)\, dt \\ &= F(s) + G(s) \end{aligned}$$

And similar to prove (ii):

$$\begin{aligned} H(s) &= \int_0^\infty e^{-st} \cdot h(t)\, dt \\ &= \int_0^\infty e^{-st} \cdot [r \cdot f(t)]\, dt \\ &= r \int_0^\infty e^{-st} \cdot f(t)\, dt \\ &= r \cdot F(s) \end{aligned}$$

This means that Laplace Transforms distribute across addition and scale with multiplication.

Example 3.1.2

Evaluate the Laplace Transformation for each function below.

(a) $f(t) = 2 - e^{3t}$

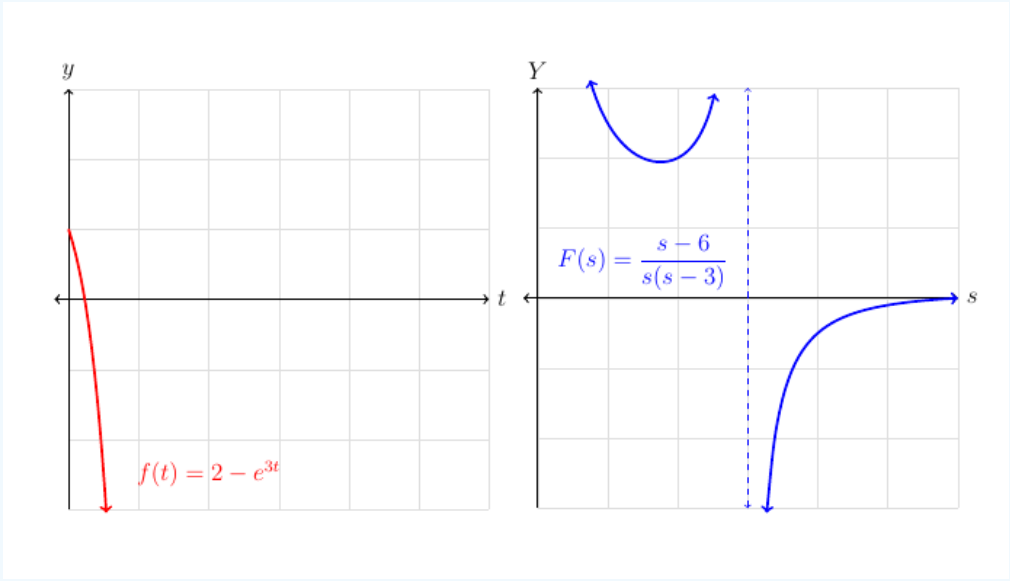
(b) $g(t) = 4t + 2t^2$

(c) $h(t) = (t+1)^3$

Solution

(a) We use linearity:

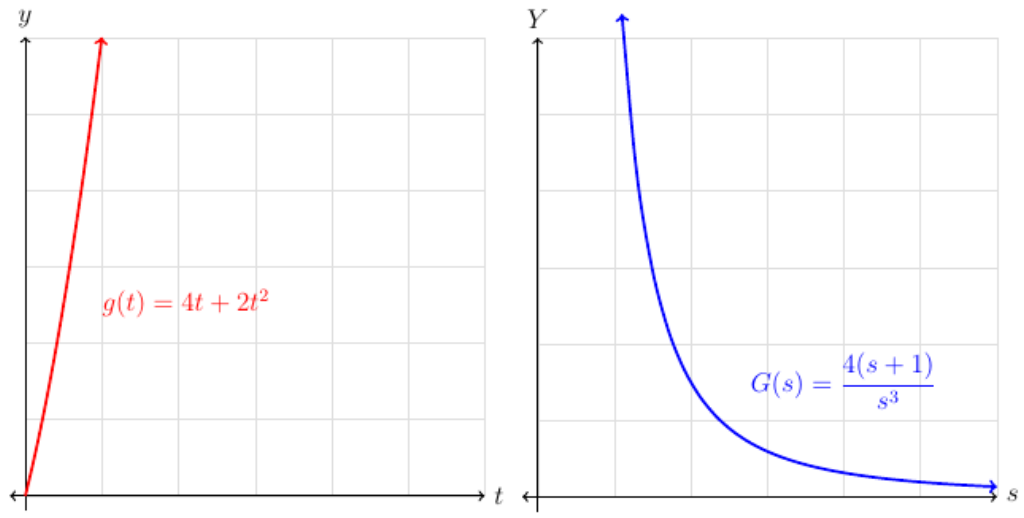
$$\mathscr{L}\left(2 - e^{3t}\right) = 2 \cdot \mathscr{L}(1) - \mathscr{L}\left(e^{3t}\right) = 2 \cdot \frac{1}{s} - \frac{1}{s-3} = \frac{s-6}{s(s-3)}$$



Thus, $\boxed{F(s) = \frac{s-6}{s(s-3)}}$

(b) We use linearity:

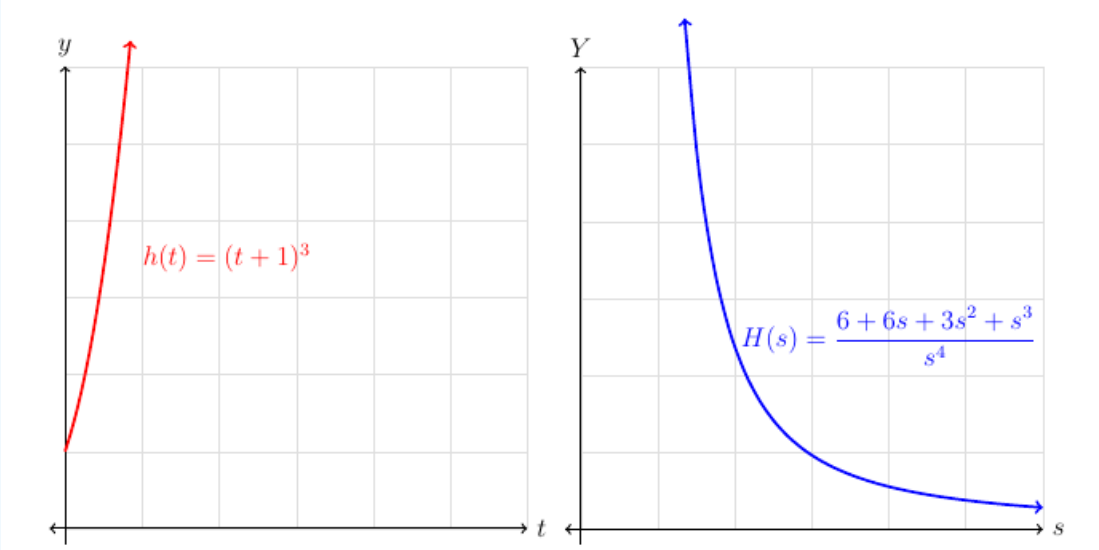
$$\mathscr{L}\left(4t + 2t^2\right) = 4 \cdot \mathscr{L}(t) + 2\mathscr{L}\left(t^2\right) = 4 \cdot \frac{1!}{s^{1+1}} + 2 \cdot \frac{2!}{s^{2+1}} = \frac{4(s+1)}{s^3}$$



Thus, $\boxed{G(s) = \frac{4(s+1)}{s^3}}$

(c) First we rewrite $h(t) = t^3 + 3t^2 + 3t + 1$, then use linearity:

$$\mathscr{L}\left(t^3 + 3t^2 + 3t + 1\right) = \mathscr{L}\left(t^3\right) + 3 \cdot \mathscr{L}\left(t^2\right) + 3 \cdot \mathscr{L}\left(t\right) + \mathscr{L}\left(1\right) = \frac{3!}{s^4} + 3 \cdot \frac{2!}{s^3} + 3 \cdot \frac{1!}{s^2} + \frac{1}{s} = \frac{6 + 6s + 3s^2 + s^3}{s^4}$$



Thus, $\boxed{H(s) = \frac{6 + 6s + 3s^2 + s^3}{s^4}}$

This linearity property immediately turns the exponential formula into its two hyperbolic components as well:

Illustration 3.1.3:

The hyperbolic sine function $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ has Laplace Transformation

$$\mathscr{L}(\sinh(kx)) = \frac{1}{2}\mathscr{L}\left(e^{kx}\right) - \frac{1}{2}\mathscr{L}\left(e^{-kx}\right) = \frac{1}{2}\cdot\frac{1}{s-k} - \frac{1}{2}\cdot\frac{1}{s+k} = \frac{k}{s^2-k^2}$$

The hyperbolic cosine function $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ has Laplace Transformation

$$\mathscr{L}(\cosh(kx)) = \frac{1}{2}\mathscr{L}\left(e^{kx}\right) + \frac{1}{2}\mathscr{L}\left(e^{-kx}\right) = \frac{1}{2}\cdot\frac{1}{s-k} + \frac{1}{2}\cdot\frac{1}{s+k} = \frac{s}{s^2-k^2}$$

This leads us to the natural question of the Laplace Transformation of the circular sine and cosine functions.

These periodic functions lend themselves nicely to this transformation.

Theorem 3.1.2

If $f(t) : [0,\infty)^{\text{int}}\mathbb{R}$ is periodic with $f(t) = f(t+T)$, then $F(s) = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-sx} f(x)\,dx$

This is a simple application of the geometric sum formula:

$$\begin{aligned}
\mathscr{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \sum_{k=0}^{\infty}\int_{kT}^{(k+1)T} e^{-st} f(t)\,dt \\
&= \sum_{k=0}^{\infty}\int_0^T e^{-s(x+kT)} f(x+kT)\,dx \\
&= \sum_{k=0}^{\infty} e^{-skT}\int_0^T e^{-sx} f(x)\,dx \\
&= \frac{1}{1-e^{-sT}}\int_0^T e^{-sx} f(x)\,dx
\end{aligned}$$

We can solve each of sine and cosine using this formula with period $T = 2\pi$.

Illustration 3.1.4

We will find $\mathscr{L}(\sin(t))$ and $\mathscr{L}(\cos(t))$

Consider the formula from Theorem 3.1.2 above for each of sine and cosine:

$$\mathscr{L}(\sin(t)) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-sx}\sin(x)\,dx \quad \mathscr{L}(\cos(t)) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-sx}\cos(x)$$

Each of these integrals will be solved with integration by parts, so we define $I = \displaystyle\int_0^{2\pi} e^{-sx}\sin(x)\,dx$ and $J = \displaystyle\int_0^{2\pi} e^{-sx}\cos(x)\,dx$:

$$I = \int_0^{2\pi} e^{-sx}\sin(x)\,dx = \sin(x)\cdot\frac{e^{-sx}}{-s}\bigg|_0^{2\pi} - \int_0^{2\pi}\frac{e^{-sx}}{-s}\cdot\cos(x)\,dx = \frac{1}{s}J$$

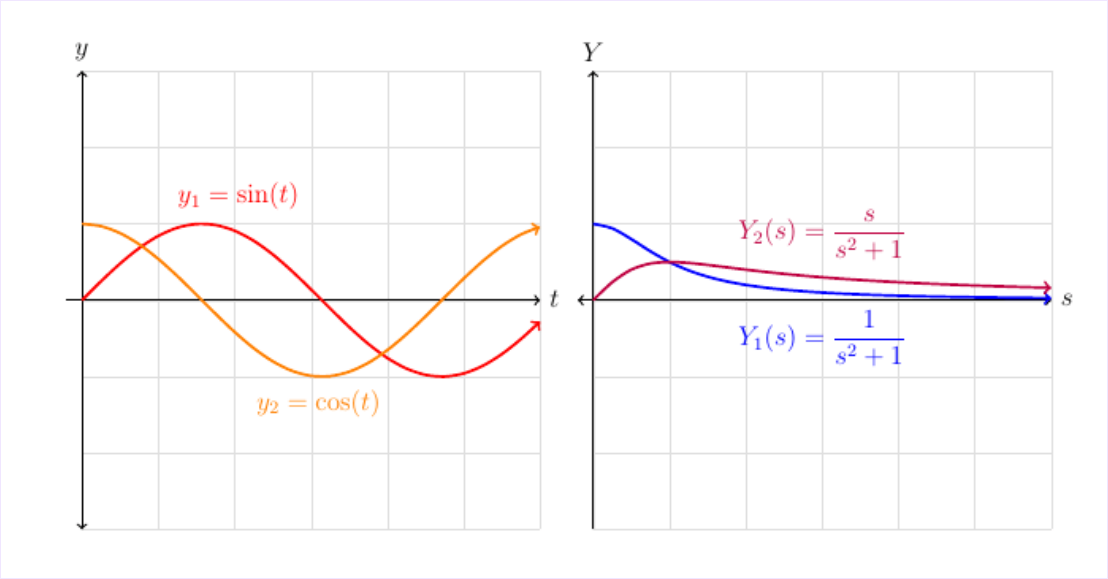
$$J = \int_0^{2\pi} e^{-sx}\cos(x)\,dx = \cos(x)\cdot\frac{e^{-sx}}{-s}\bigg|_0^{2\pi} - \int_0^{2\pi}\frac{e^{-sx}}{-s}\cdot(-\sin(x))\,dx = \frac{1-e^{-2\pi s}}{s} + \frac{1}{s}I$$

We put these together to get:

$$\begin{cases} I = \dfrac{1}{s}J \\ J = \dfrac{1-e^{-2\pi s}}{s} + \dfrac{1}{s}I \end{cases} \implies \begin{cases} I = \dfrac{1-e^{-2\pi s}}{s^2+1} \\ J = \dfrac{\left(1-e^{-2\pi s}\right)s}{s^2+1} \end{cases}$$

Substituting this into the original equations gives our final answers:

$$\mathscr{L}\left(\sin(t)\right) = \frac{1}{s^2+1} \quad \text{and} \quad \mathscr{L}\left(\cos(t)\right) = \frac{s}{s^2+1}$$



Of course, we may have seen this coming! Look at their hyperbolic counterparts:

$$\mathscr{L}\left(\sinh(t)\right) = \frac{1}{s^2-1} \longleftrightarrow \mathscr{L}\left(\sin(t)\right) = \frac{1}{s^2+1} \quad \text{and} \quad \mathscr{L}\left(\cosh(t)\right) = \frac{s}{s^2-1} \longleftrightarrow \mathscr{L}\left(\cos(t)\right) = \frac{s}{s^2+1}$$

The big difference between the circular and hyperbolic trig functions are their Pythagorean Identities:

$$\cosh^2(x) - \sinh^2(x) = 1 \quad \text{and} \quad \cos^2(x) + \sin^2(x) = 1$$

To a skeptic, this may still need further verification. We lean on Euler:

$$\begin{cases} e^{i\theta} = \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} = \cos(\theta) - i\sin(\theta) \end{cases} \implies \begin{cases} \cos(\theta) = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \cosh(i\theta) \\ \sin(\theta) = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = -i\sinh(i\theta) \end{cases}$$

Thus, we know $\mathscr{L}\left(\cos(kt)\right) = \mathscr{L}\left(\cosh(ikt)\right) = \dfrac{s}{s^2-(-ik)^2} = \dfrac{s}{s^2+k^2}$

Similarly, $\mathscr{L}\left(\sin(kt)\right) = \mathscr{L}\left(-i\sinh(ikt)\right) = -i\mathscr{L}\left(\sinh(ikt)\right) = -i\cdot\dfrac{ik}{s^2-(-ik)^2} = \dfrac{k}{s^2+k^2}$

We put these results together in a table, which will be worth becoming familiar with:

$f(t)$	$\mathscr{L}\left(f(t)\right)$
1	$\frac{1}{s}$

t^m	$\frac{m!}{s^{m+1}}$
e^{at}	$\frac{1}{s-a}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$

While not *all* of our favorite functions are on this list, a large percentage of the building blocks of functions we use are here.

For most other functions, we simply calculate the integral directly.

Example 3.1.3

Find the Laplace Transformation for each function below.

(a) $f(t) = te^t \sin(t)$

(b) $g(t) = \begin{cases} 0 & 0 \le t < 3 \\ t^2 - 6t + 9 & t \ge 3 \end{cases}$

(c) $h(t) = \begin{cases} e^{t-2k} & 2k \le t < 2k+1 \\ e^{2k+2-t} & 2k+1 \le t < 2k+2 \end{cases}$

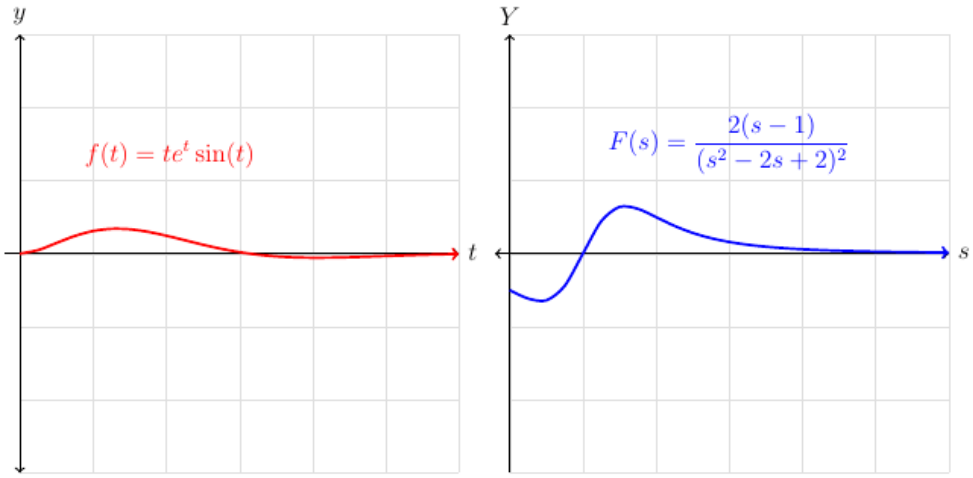
Solution

(a) We evaluate the integral directly:

$$F(s) = \int_0^\infty e^{-st} \cdot te^t \sin(t)\, dt = \lim_{T\to\infty} \int_0^T te^{-(s-1)t} \sin(t)\, dt$$

We use integration by parts with $u = t \implies u' = dt$ and $v' = e^{-(s-1)t}\sin(t) \implies v = \frac{e^{-(s-1)t}}{s^2-2s+2}\left[(1-s)\sin(t) - \cos(t)\right]$:

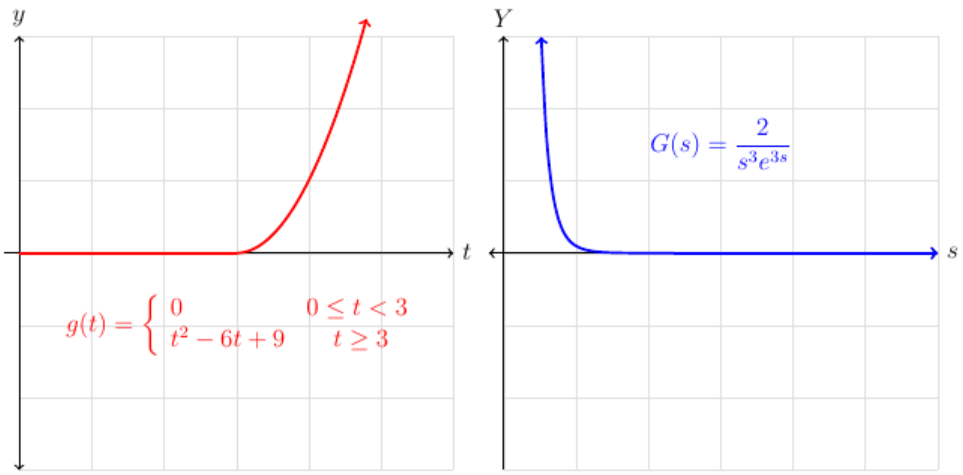
$$\begin{aligned}
F(s) &= \lim_{T\to\infty} t\frac{e^{-(s-1)t}}{s^2-2s+2}\left[(1-s)\sin(t)-\cos(t)\right]\Big|_0^T - \int_0^T \frac{e^{-(s-1)t}}{s^2-2s+2}\left[(1-s)\sin(t)-\cos(t)\right]\, dt \\
&= \lim_{T\to\infty} -T\frac{e^{-(s-1)T}}{s^2-2s+2}\left[(1-s)\sin(T)-\cos(T)\right] - \int_0^T \frac{e^{-(s-1)t}}{s^2-2s+2}\left[(1-s)\sin(t)-\cos(t)\right]\, dt \\
&= \lim_{T\to\infty} -\frac{1}{s^2-2s+2}\left[(1-s)\int_0^T e^{-(s-1)t}\sin(t)\, dt - \int_0^T e^{-(s-1)t}\cos(t)\, dt\right] \\
&= \frac{s-1}{s^2-2s+2}\cdot\int_0^\infty e^{-(s-1)t}\sin(t)\, dt + \frac{1}{s^2-2s+2}\cdot\int_0^\infty e^{-(s-1)t}\cos(t)\, dt \\
&= \frac{s-1}{s^2-2s+2}\cdot\frac{1}{(s-1)^2+1} + \frac{1}{s^2-2s+2}\cdot\frac{s-1}{(s-1)^2+1} \\
&= \frac{2(s-1)}{(s^2-2s+2)^2}
\end{aligned}$$



Notice that we used $\mathscr{L}(\sin(t)) = \dfrac{1}{s^2+1}$ and $\mathscr{L}(\cos(t)) = \dfrac{s}{s^2+1}$ to evaluate the resulting integrals. We will find easier ways to reach it in the future, but the answer will still be $\boxed{F(s) = \dfrac{2(s-1)}{(s^2-2s+2)^2}}$

(b) We evaluate again directly:

$$\begin{aligned}
G(s) &= \int_0^\infty e^{-st} \cdot g(t)\, dt \\
&= \lim_{T\to\infty} \int_0^3 e^{-st} \cdot 0\, dt + \int_3^T e^{-st} \cdot (t^2 - 6t + 9)\, dt \\
&= \lim_{T\to\infty} \int_3^T e^{-st}(t-3)^2\, dt \\
&= \lim_{T\to\infty} -\left(\frac{(t-3)^2}{s} + \frac{2(t-3)}{s^2} + \frac{2}{s^3}\right) e^{-st}\Big|_3^T \\
&= \lim_{T\to\infty} -\left(\frac{(T-3)^2}{s} + \frac{2(T-3)}{s^2} + \frac{2}{s^3}\right) e^{-sT} + \frac{2}{s^3}e^{-3s} \\
&= \frac{2}{s^3}e^{-3s}
\end{aligned}$$



Thus, we have found $\boxed{G(s) = \frac{2}{s^3 e^{3s}}}$

(c) This function looks terrible, but it is periodic with period $T = 2$, we can use the formula:

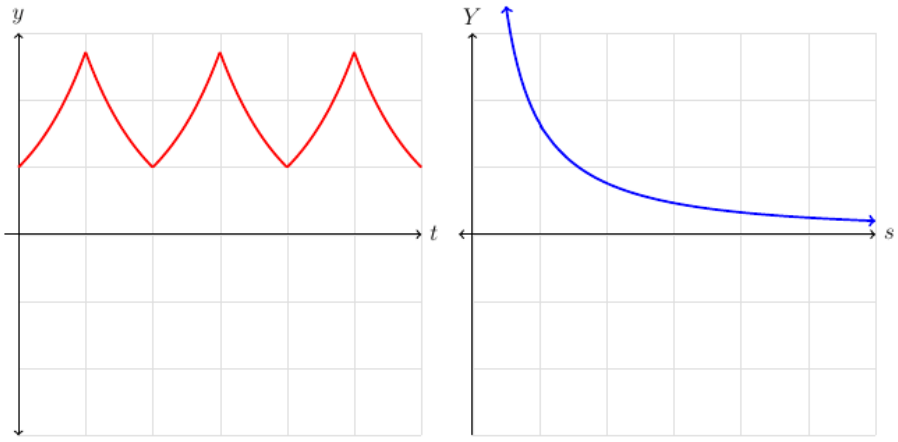
$$H(s) = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} h(t)\, dt$$

Now we simply calculate

$$\begin{aligned} \int_0^2 e^{-st} h(t)\, dt &= \int_0^1 e^{-st} \cdot e^{t-0}\, dt + \int_1^2 e^{-st} \cdot e^{2-t}\, dt \\ &= \int_0^1 e^{(1-s)t}\, dt + \int_1^2 e^{2-(1+s)t}\, dt \\ &= -\frac{1}{s-1} e^{(1-s)t}\Big|_0^1 - \frac{1}{1+s} e^{2-(1+s)t}\Big|_1^2 \\ &= -\frac{1}{s-1} e^{1-s} + \frac{1}{s-1} - \frac{1}{1+s} e^{-2s} + \frac{1}{1+s} e^{1-s} \end{aligned}$$

Plugging this into the periodic formula gives: $\boxed{H(s) = \frac{1}{1 - e^{-2s}} \left[\frac{2e^{1-s}}{1-s^2} - \frac{1}{1-s} - \frac{e^{-2s}}{1+s} \right]}$

$$h(t) = \begin{cases} e^{t-2k} & 2k \le t < 2k+1 \\ e^{2k+2-t} & 2k+1 \le t < 2k+2 \end{cases} \qquad H(s) = \frac{1}{1-e^{-2s}}\left[\frac{2e^{1-s}}{1-s^2} - \frac{1}{1-s} - \frac{e^{-2s}}{1+s}\right]$$



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3.1: The Inverse Laplace Transformation

One of the best properties of the Laplace Transformation is that it is one-to-one:

$$\mathscr{L}(f(t)) = \mathscr{L}(g(t)) \implies \int_0^\infty e^{-st}(f(t) - g(t))\,dt = 0 \text{ for all } s \implies f(t) = g(t)$$

This means that there is an *inverse function* to the Laplace Transformation that sends objects in its range ($F(s)$) back to the original element in the domain ($f(t)$) that satisfies $\mathscr{L}(f(t)) = F(s)$. There is an explicit formula for this Inverse Laplace Transformation, but we do not ever need to acknowledge it in order to make use of it.

Definition: Inverse Laplace Transformation

For any function $f(t)$ with Laplace Transformation $F(s) = \mathscr{L}(f(t))$, we define $\mathscr{L}^{-1}(F(s)) = f(t)$

The transformation $\mathscr{L}^{-1}$ is known as the ***Inverse Laplace Transformation***.
As the inverse of a linear transformation, it is an invertible linear transformation.

This means that we can "recover" a function $f(t)$ based only on the Laplace Transformation $F(s)$. To continue the analogy of a calculus course, we saw how to do the transformation (derivative/Laplace), and now we learn how to do the reverse (anti-derivatives/Inverse Laplace) by simply reversing the previous knowledge.

To make this process easier, we will painfully calculate out the general formula for the function $f(t) = t^n e^{at}$:

$$\begin{aligned}
\mathscr{L}(t^n e^{at}) &= \int_0^\infty e^{-st} \cdot t^n e^{at}\,dt \\
&= \lim_{T\to\infty} \int_0^T e^{(a-s)t} \cdot t^n\,dt \\
&= t^n \cdot \frac{e^{(a-s)t}}{a-s}\bigg|_0^T - \int_0^T nt^{n-1}\frac{e^{(a-s)t}}{a-s}\,dt \\
&= \lim_{T\to\infty} \frac{T^n}{a-s}e^{(a-s)T} + \frac{n}{s-a}\int_0^T e^{(a-s)t}t^{n-1}\,dt \\
&= 0 + \frac{n}{s-a}\mathscr{L}\left(t^{n-1}e^{at}\right)
\end{aligned}$$

Iterating this formula n times gives us the general Laplace Formula shortcut

$$\mathscr{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

This is particularly useful since it extends to the cases where n or a is zero, giving us two shortcuts in one! The real power of this result occurs when we write this statement in terms of the Inverse Laplace Transform:

Rule: Inverse Laplace Transformation Shortcut

$$\mathscr{L}^{-1}\left(\frac{1}{(s-a)^{n+1}}\right) = \frac{1}{n!}t^n e^{at} \text{ for all integers } n \geq 0$$

Using this formula and linearity, we can evaluate the Inverse Laplace Transformation of many different functions.

Example 3.1.4

Find a function with Laplace Transformation equal to:

(a) $F(s) = \dfrac{3}{s^2} - \dfrac{1}{s-4}$

(b) $G(s) = \dfrac{12}{(s+1)^4} - \dfrac{2}{s}$

(c) $H(s) = \dfrac{4}{s^4 - 2s^2 + 1}$

Solution

(a) We see that

$$f(t) = \mathscr{L}^{-1}\left(F(s)\right) = 3\mathscr{L}^{-1}\left(\frac{1}{s^2}\right) - \mathscr{L}^{-1}\left(\frac{1}{s-4}\right) = 3 \cdot \frac{1}{1!}t^1e^{0t} - \frac{1}{0!}t^0e^{4t}$$

This simplifies down to $\boxed{f(t) = 3t - e^{4t}}$

(b) We see that

$$g(t) = \mathscr{L}^{-1}\left(G(s)\right) = 12\mathscr{L}^{-1}\left(\frac{1}{(s+1)^4}\right) - 2\mathscr{L}^{-1}\left(\frac{1}{s}\right) = 12 \cdot \frac{1}{3!}t^3e^{-t} - 2 \cdot \frac{1}{0!}t^0e^{0t}$$

This simplifies down to $\boxed{g(t) = 2t^3e^{-t} - 2}$

(c) Partial fraction decomposition gives us

$$h(t) = \mathscr{L}^{-1}\left(H(s)\right) = \mathscr{L}^{-1}\left(\frac{1}{s+1}\right) + \mathscr{L}^{-1}\left(\frac{1}{(s+1)^2}\right) - \mathscr{L}^{-1}\left(\frac{1}{s-1}\right) + \mathscr{L}^{-1}\left(\frac{1}{(s-1)^2}\right) = \frac{1}{0!}t^0e^{-t} + \frac{1}{1!}t^1e^{-t} - \frac{1}{0!}t^0e^{t} + \frac{1}{1!}t^1$$

This simplifies down to $\boxed{h(t) = (t+1)e^{-t} + (t-1)e^{t}}$

We can add in the other famous Laplace Transformations we recently discovered to get our Inverse Laplace Transformation formula table:

$F(s)$	$f(t)$
$\dfrac{1}{(s-a)^{n+1}}$	$\dfrac{1}{n!}t^ne^{at}$
$\dfrac{s}{s^2+k^2}$	$\cos(kt)$
$\dfrac{1}{s^2+k^2}$	$\dfrac{1}{k}\sin(kt)$
$\dfrac{s}{s^2-k^2}$	$\cosh(kt)$
$\dfrac{1}{s^2-k^2}$	$\dfrac{1}{k}\sinh(kt)$

With these shortcuts and basic partial fraction decomposition, we can apply the Inverse Laplace Transformation to a wide array of functions.

Example 3.1.5

Evaluate each function below.

(a) $\mathscr{L}^{-1}\left(\dfrac{8s-8}{4s^2-1}\right)$

(b) $\mathscr{L}^{-1}\left(\dfrac{3s^2+2s+3}{s^4-2s^3+2s^2-2s+1}\right)$

Solution

(a) We can approach this example two different ways. The more direct method is to use partial fraction decomposition to rewrite:

$$\frac{8s-8}{(2s-1)(2s+1)} = \frac{6}{2s+1} - \frac{2}{2s-1} = \frac{3}{s+\frac{1}{2}} - \frac{1}{s-\frac{1}{2}}$$

$$\mathscr{L}^{-1}\left(\frac{8s-8}{4s^2-1}\right) = 3\mathscr{L}^{-1}\left(\frac{1}{s+\frac{1}{2}}\right) - \mathscr{L}^{-1}\left(\frac{1}{s-\frac{1}{2}}\right) = 3e^{-\frac{1}{2}t} - e^{\frac{1}{2}t}$$

Thus, we have $\boxed{\mathscr{L}^{-1}\left(\dfrac{8s-8}{4s^2-1}\right) = 3e^{-\frac{1}{2}t} - e^{\frac{1}{2}t}}$

If we preferred, we could have reduced $\dfrac{8s-8}{4s^2-1} = \dfrac{2s-2}{s^2-\frac{1}{4}}$, and used hyperbolics:

$$\mathscr{L}^{-1}\left(\frac{8s-8}{4s^2-1}\right) = 2\mathscr{L}^{-1}\left(\frac{s}{s^2-\frac{1}{4}}\right) - 2\mathscr{L}^{-1}\left(\frac{1}{s^2-\frac{1}{4}}\right) = 2\cosh\left(\frac{1}{2}t\right) - 2\cdot\frac{1}{\frac{1}{2}}\sinh\left(\frac{1}{2}t\right)$$

This simplifies down to the same answer:

$$f(t) = 2\cosh\left(\frac{1}{2}t\right) - 4\sinh\left(\frac{1}{2}t\right) = 2\left(\frac{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}{2}\right) - 4\left(\frac{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}}{2}\right) = -e^{\frac{1}{2}t} + 3e^{-\frac{1}{2}t}$$

(b) This example is just a good old-fashioned four variable partial fraction decomposition.

$$\frac{3s^2+2s+3}{s^4-2s^3+2s^2-2s+1} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1} \implies \begin{cases} A=0 \\ B=4 \\ C=0 \\ D=-1 \end{cases}$$

Now taking the Inverse Laplace Transformation is straightforward.

$$\mathscr{L}^{-1}\left(\frac{3s^2+2s+3}{s^4-2s^3+2s^2-2s+1}\right) = 4\mathscr{L}^{-1}\left(\frac{1}{(s-1)^2}\right) - \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) = 4\cdot\frac{1}{1!}t^1e^{1t} - \frac{1}{1}\sin(1t)$$

This simplifies down to our final answer $\boxed{\mathscr{L}^{-1}\left(\dfrac{3s^2+2s+3}{s^4-2s^3+2s^2-2s+1}\right) = 4te^t - \sin(t)}$

While seemingly dry, this provides a foundation of the basic tools you will need to calculate both Laplace and Inverse Laplace Transformations. We will soon see that these transformations do more than just change some of our favorite functions into new ones, they also change their relationships to each other in ways we will appreciate.

Illustration 3.1.5:

For just one example, consider the integral $\int_0^\infty \frac{\cos(et)-\cos(t)}{t}\,dt$

That integral does not invite any conventional approach, but consider the Laplace Transformation of the function $h(t)=\frac{f(t)}{t}$:

$$H(s)=\mathscr{L}\left(h(t)\right)=\int_0^\infty e^{-st}h(t)\,dt=\int_0^\infty e^{-st}\frac{f(t)}{t}\,dt$$
$$\Downarrow$$
$$H'(s)=\int_0^\infty\left(-te^{-st}\right)\frac{f(t)}{t}\,dt=-\int_0^\infty e^{-st}f(t)\,dt=-F(s)$$

We have discovered that $H'(s)=-F(s)$, which gives us a simple differential equation to solve.

Back to our example, we have $f(t)=\cos(et)-\cos(t)$, so $F(s)=\mathscr{L}\left(\cos(et)-\cos(t)\right)=\frac{s}{s^2+e^2}-\frac{s}{s^2+1^2}$

Now we can try to solve for $H(s)=\int -F(s)\,ds$:

$$H(s)=\int\frac{s}{s^2+1}-\frac{s}{s^2+e^2}\,ds=\frac{1}{2}\ln\left|s^2+1\right|-\frac{1}{2}\ln\left|s^2+e^2\right|=\frac{1}{2}\ln\left(\frac{s^2+1}{s^2+e^2}\right)+C$$

Here we notice that $H(s)$ itself is a Laplace Transform, so any constant but $C=0$ would be troublesome as a result. This is because the limiting behavior (as $s\to+\infty$) of a Laplace Transformation forces the decay to happen very quickly. Thus, the function value is almost entirely dependent on its behavior near $t=0$. Since all functions involved are sufficiently smooth, we can see that $C=0$ must hold for $\lim_{s\to\infty}H(s)=0$ to hold.

Finally, we can see that our helpful function in this problem is

$$H(s)=\frac{1}{2}\ln\left(\frac{s^2+1}{s^2+e^2}\right)\implies H(0)=\int_0^\infty e^{-0t}h(t)\,dt=\frac{1}{2}\ln\left(\frac{1}{e^2}\right)=-1$$

We have officially shown that $\int_0^\infty\frac{\cos(et)-\cos(t)}{t}\,dt=-1$

Laplace Transformations clearly have some connection between derivatives and integrals that help us in unexpected ways!

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3.1: Practice Set

Exercise 3.1.1a

Find the Laplace Transformation of $a(t) = 5e^{-2t} + 3$

Answer available at the end of this section.

Exercise 3.1.1b

Find the Laplace Transformation of $b(t) = 4t^3 - t^2 + 6t$

Answer available at the end of this section.

Exercise 3.1.1c

Find the Laplace Transformation of $c(t) = (1 - e^{-t})^2$

Answer available at the end of this section.

Exercise 3.1.1d

Find the Laplace Transformation of $d(t) = 4\sin\left(3t + \frac{\pi}{2}\right)$

Answer available at the end of this section.

Exercise 3.1.1e

Find the Laplace Transformation of $e(t) = 8\sin^2(t)$

Answer available at the end of this section.

Exercise 3.1.2a

Use the integral definition to find the Laplace Transformation of $h(t) = e^t \sin(3t)$

Answer available at the end of this section.

Exercise 3.1.2b

Use the integral definition to find the Laplace Transformation of $q(t) = 2t\cosh(2t)$

Answer available at the end of this section.

Exercise 3.1.2c

Use the integral definition to find the Laplace Transformation of $w(t) = \begin{cases} 1 & 0 \le t < 2 \\ t & t \ge 2 \end{cases}$

Answer available at the end of this section.

Exercise 3.1.2d

Use the integral definition to find the Laplace Transformation of $u(t) = e^{-4t}\left(t^2 + \cos(t)\right)$

Answer available at the end of this section.

Exercise 3.1.2e

Find the Laplace Transformation of $z(t) = \begin{cases} \pi & 2k \le t < 2k+1 \text{ for some integer } k \\ 0 & 2k+1 \le t < 2k+2 \text{ for some integer } k \end{cases}$

Answer available at the end of this section.

Exercise 3.1.3a

Find the Inverse Laplace Transformation of $A(s) = \dfrac{5}{s-7} + \dfrac{2s}{s^2+9}$

Answer available at the end of this section.

Exercise 3.1.3b

Find the Inverse Laplace Transform of $B(s) = \dfrac{12}{(s-7)^4}$

Answer available at the end of this section.

Exercise 3.1.3c

Find the Inverse Laplace Transform of $C(s) = \dfrac{720}{(s-1)^6} - \dfrac{2}{(s+8)^3}$

Answer available at the end of this section.

Exercise 3.1.3d

Find the Inverse Laplace Transformation of $D(s) = \dfrac{35}{s^2 - 3s - 4}$

Answer available at the end of this section.

Exercise 3.1.3e

Find the Inverse Laplace Transformation of $E(s) = \dfrac{12}{(s^2+4)(s^2+1)}$

Answer available at the end of this section.

Exercise 3.1.3f

Find the Inverse Laplace Transformation of $F(s) = \dfrac{s+1}{s^2(s^2+1)}$

Answer available at the end of this section.

Exercise 3.1.3g

Find the Inverse Laplace Transformation of $G(s) = \dfrac{4s^3 - s^2 + 8}{s^4 - s^3 - 2s^2}$

Answer available at the end of this section.

Exercise 3.1.3h

Find the Inverse Laplace Transformation of $H(s) = \dfrac{16s^4 + 9s^3 - 12s^2 + 16s - 192}{s^5 + 16s^3}$

Answer available at the end of this section.

Exercise 3.1.4

Find the value of the integral $\displaystyle\int_0^\infty \frac{\sin(t)}{t}\,dt$

Answer available at the end of this section.

Answers

Exercise 3.1.1a

We use the linearity of $\mathscr{L}$ to write:

$$\mathscr{L}(a(t)) = 5\mathscr{L}\left(e^{-2t}\right) + 3\mathscr{L}(1) = 5\cdot\frac{1}{s+2} + 3\cdot\frac{1}{s}$$

Therefore, $A(s) = \dfrac{5}{s+2} + \dfrac{3}{s}$

Exercise 3.1.1b

We use the linearity of $\mathscr{L}$ to write:

$$\mathscr{L}(b(t)) = 4\mathscr{L}\left(t^3\right) - \mathscr{L}\left(t^2\right) + 6\mathscr{L}(t) = 4\cdot\frac{3!}{s^4} - \frac{2!}{s^3} + 6\cdot\frac{1!}{s^2}$$

Therefore, $B(s) = \dfrac{24}{s^4} - \dfrac{2}{s^3} + \dfrac{6}{s^2}$

Exercise 3.1.1c

We use the linearity of $\mathscr{L}$ and expand $c(t) = 1 - 2e^{-t} + e^{-2t}$ to write:

$$\mathscr{L}(c(t)) = \mathscr{L}(1) - 2\mathscr{L}\left(e^{-t}\right) + \mathscr{L}\left(e^{-2t}\right) = \frac{1}{s} - 2\cdot\frac{1}{s+1} + \frac{1}{s+2}$$

Therefore, $C(s) = \dfrac{1}{s} - \dfrac{2}{s+1} + \dfrac{1}{s+2}$

Exercise 3.1.1*d*

Recall the sum of angle identity, $\sin\left(\theta+\frac{\pi}{2}\right)=\cos(\theta)$

That means $d(t)=4\cos(3t)$, and we use the linearity of $\mathscr{L}$ to write:

$$\mathscr{L}\left(d(t)\right)=4\mathscr{L}\left(\cos(3t)\right)=4\cdot\frac{s}{s^2+9}$$

Therefore, $D(s)=\frac{4s}{s^2+9}$

Exercise 3.1.1*e*

Recall the half-angle identity, $\sin^2(t)=\frac{1}{2}\left(1-\cos(2t)\right)$

We have $e(t)=4-4\cos(2t)$, and we use the linearity of $\mathscr{L}$ to write:

$$\mathscr{L}\left(e(t)\right)=4\cdot\mathscr{L}\left(1\right)-4\mathscr{L}\left(\cos(2t)\right)=4\cdot\frac{1}{s}-4\cdot\frac{s}{s^2+4}$$

Therefore, we have $E(s)=\frac{4}{s}-\frac{4s}{s^2+4}$

Exercise 3.1.2*a*

$$\begin{aligned}\mathscr{L}\left(h(t)\right)&=\int_0^\infty e^{-st}e^t\sin(3t)dt\\&=\lim_{T\to\infty}\int_0^T e^{(1-s)t}\sin(3t)\,dt\\&=\lim_{T\to\infty}\left[\frac{e^{(1-s)t}((1-s)\sin(3t)-3\cos(3t))}{(1-s)^2+9}\right]_0^T\\&=\lim_{T\to\infty}\frac{e^{(1-s)T}((1-s)\sin(3T)-3\cos(3T))}{(1-s)^2+9}-\frac{-3}{(1-s)^2+9}\\&=0+\frac{3}{(s-1)^2+9}\end{aligned}$$

Therefore, $H(s)=\dfrac{3}{(s-1)^2+9}$

Exercise 3.1.2*b*

Recall the definition of $\cosh(2t)=\dfrac{e^{2t}+e^{-2t}}{2}$

We have $q(t)=te^{2t}+te^{-2t}$. We calculate using the integral definition directly:

$$\begin{aligned}\mathscr{L}\left(q(t)\right)&=\mathscr{L}\left(te^{2t}\right)+\mathscr{L}\left(te^{-2t}\right)\\&=\lim_{T\to\infty}\int_0^T te^{(2-s)t}\,dt+\int_0^T te^{(-2-s)t}\,dt\\&=\lim_{T\to\infty}\left[\left(\frac{t}{2-s}-\frac{1}{(2-s)^2}\right)e^{(2-s)t}\right]_0^T+\left[\left(\frac{t}{-2-s}-\frac{1}{(-2-s)^2}\right)e^{(-2-s)t}\right]_0^T\\&=\lim_{T\to\infty}\left(\frac{T}{2-s}-\frac{1}{(2-s)^2}\right)e^{(2-s)T}-\left(-\frac{1}{(2-s)^2}\right)+\left(\frac{T}{-2-s}-\frac{1}{(-2-s)^2}\right)e^{(-2-s)T}-\left(-\frac{1}{(-2-s)^2}\right)\\&=0+\frac{1}{(s-2)^2}+0+\frac{1}{(s+2)^2}\end{aligned}$$

Therefore, $Q(s)=\frac{1}{(s-2)^2}+\frac{1}{(s+2)^2}$

Exercise 3.1.2*c*

$$\begin{aligned}\mathscr{L}\left(w(t)\right)&=\int_0^2 e^{-st}\cdot 1\,dt+\int_2^\infty e^{-st}\cdot t\,dt\\&=\int_0^2 e^{-st}\,dt+\lim_{T\to\infty}\int_2^T te^{-st}\,dt\\&=\left[-\frac{e^{-st}}{s}\right]_0^2+\lim_{T\to\infty}\left[\left(-\frac{t}{s}-\frac{1}{s^2}\right)e^{-st}\right]_2^T\\&=\left(-\frac{e^{-2s}}{s}+\frac{1}{s}\right)+\lim_{T\to\infty}\left(-\frac{T}{s}-\frac{1}{s^2}\right)e^{-sT}-\left(-\frac{2}{s}-\frac{1}{s^2}\right)e^{-2s}\\&=\left(-\frac{e^{-2s}}{s}+\frac{1}{s}\right)+\left(0-\left(-\frac{2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}\right)\right)\end{aligned}$$

Therefore, $W(s)=\frac{1}{s}+\frac{e^{-2s}}{s}+\frac{e^{-2s}}{s^2}$

Exercise 3.1.2*d*

$$\begin{aligned}\mathscr{L}\left(u(t)\right)&=\mathscr{L}\left(t^2e^{-4t}\right)+\mathscr{L}\left(e^{-4t}\cos(t)\right)\\&=\int_0^\infty e^{-st}\cdot t^2e^{-4t}\,dt+\int_0^\infty e^{-st}\cdot e^{-4t}\cos(t)\,dt\\&=\lim_{T\to\infty}\int_0^T t^2e^{-(s+4)t}\,dt+\int_0^T e^{-(s+4)t}\cos(t)\,dt\\&=\lim_{T\to\infty}\left[\left(-\frac{t^2}{s+4}-\frac{2t}{(s+4)^2}-\frac{2}{(s+4)^3}\right)e^{-(s+4)t}+\frac{1}{1+(s+4)^2}\left(\sin(t)-(s+4)\cos(t)\right)e^{-(s+4)t}\right]_0^T\\&=\lim_{T\to\infty}\left(-\frac{T^2}{s+4}-\frac{2T}{(s+4)^2}-\frac{2}{(s+4)^3}\right)e^{-(s+4)T}+\frac{1}{1+(s+4)^2}\left(\sin(T)-(s+4)\cos(T)\right)e^{-(s+4)T}-\left(-\frac{2}{(s+4)^3}-\frac{s+4}{(s+4)^2+1}\right)\\&=\frac{2}{(s+4)^3}+\frac{s+4}{(s+4)^2+1}\end{aligned}$$

Therefore, $U(s) = \frac{2}{(s+4)^3} + \frac{s+4}{(s+4)^2+1}$

Exercise 3.1.2*e*

We see that $z(t+2) = z(t)$ so this function is periodic with $T = 2$:

$$\begin{aligned}\mathscr{L}(z(t)) &= \frac{1}{1-e^{-s\cdot 2}}\int_0^2 e^{-st}z(t)\,dt \\ &= \frac{1}{1-e^{-2s}}\int_0^1 \pi e^{-st}\,dt \\ &= \frac{1}{1-e^{-2s}}\left[-\frac{\pi}{s}e^{-st}\right]_0^1 \\ &= \frac{1}{1-e^{-2s}}\left(-\frac{\pi}{s}e^{-s}+\frac{\pi}{s}\right) \\ &= \frac{\pi(1-e^{-s})}{s(1-e^{-2s})}\end{aligned}$$

Therefore, $Z(s) = \frac{\pi(1-e^{-s})}{s(1-e^{-2s})}$

Exercise 3.1.3*a*

We use the linearity of $\mathscr{L}^{-1}$ to get:

$$\mathscr{L}^{-1}(A(s)) = 5\mathscr{L}^{-1}\left(\frac{1}{s-7}\right) + 2\mathscr{L}^{-1}\left(\frac{s}{s^2+3^2}\right) = 5\cdot e^{7t} + 2\cdot\cos(3t)$$

Therefore, $a(t) = 5e^{7t} + 2\cos(3t)$

Exercise 3.1.3*b*

We know that

$$\mathscr{L}^{-1}(B(s)) = 12\cdot\frac{1}{3!}t^3\cdot e^{7t}$$

Therefore, $b(t) = 2t^3e^{7t}$

Exercise 3.1.3*c*

We use linearity of $\mathscr{L}^{-1}$ to get:

$$\begin{aligned}\mathscr{L}^{-1}(B(s)) &= 720\mathscr{L}^{-1}\left(\frac{1}{(s-1)^6}\right) - 2\mathscr{L}^{-1}\left(\frac{1}{(s+8)^3}\right) \\ &= 720\cdot\frac{1}{5!}t^5\cdot e^t - 2\cdot\frac{1}{2!}t^2\cdot e^{-8t}\end{aligned}$$

Therefore, $c(t) = 6t^5e^t - t^2e^{-8t}$

Exercise 3.1.3*d*

First, we use partial fractions to rewrite $D(s) = \dfrac{7}{s-4} - \dfrac{7}{s+1}$

We use the linearity of $\mathscr{L}^{-1}$ to get:

$$\mathscr{L}^{-1}(D(s)) = 7\mathscr{L}^{-1}\left(\frac{1}{s-4}\right) - 7\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = 7\cdot e^{4t} - 7\cdot e^{-1t}$$

Therefore, $d(t) = 7e^{4t} - 7e^{-t}$

Exercise 3.1.3*e*

First, we use partial fractions to rewrite $E(s) = \dfrac{4}{s^2+1} - \dfrac{4}{s^2+4}$.

We use the linearity of $\mathscr{L}^{-1}$ to get:

$$\mathscr{L}^{-1}(E(s)) = 4\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) - 4\mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right) = 4\cdot\frac{1}{1}\sin(t) - 4\cdot\frac{1}{2}\sin(2t)$$

Therefore, $e(t) = 4\sin(t) - 2\sin(2t)$

Exercise 3.1.3*f*

First, we use partial fractions to rewrite $F(s) = \dfrac{1}{s} + \dfrac{1}{s^2} - \dfrac{1}{s^2+1} - \dfrac{s}{s^2+1}$.

We use the linearity of $\mathscr{L}^{-1}$ to get:

$$\mathscr{L}^{-1}(F(s)) = \mathscr{L}^{-1}\left(\frac{1}{s}\right) + \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) - \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) - \mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right) = 1 + t - 1\sin(t) - 1\cos(t)$$

Therefore, $f(t) = 1 + t - \sin(t) - \cos(t)$

Exercise 3.1.3*g*

First, we use partial fractions to rewrite $G(s) = \dfrac{2}{s} - \dfrac{4}{s^2} + \dfrac{3}{s-2} - \dfrac{1}{s+1}$.

We use the linearity of $\mathscr{L}^{-1}$ to get:

$$\mathscr{L}^{-1}(G(s)) = 2\mathscr{L}^{-1}\left(\frac{1}{s}\right) - 4\mathscr{L}^{-1}\left(\frac{1}{s^2}\right) + 3\mathscr{L}^{-1}\left(\frac{1}{s-2}\right) - \mathscr{L}\left(\frac{1}{s+1}\right) = 2\cdot 1 - 4\cdot t + 3\cdot e^{2t} - 1\cdot e^{-1t}$$

Therefore, $g(t) = 2 - 4t + 3e^{2t} - e^{-t}$

Exercise 3.1.3*h*

First, we use partial fractions to rewrite $H(s) = \dfrac{12}{s^2} - \dfrac{12}{s^3} + \dfrac{16s}{s^2+16} + \dfrac{8}{s^2+16}$.

We use the linearity of $\mathscr{L}^{-1}$ to get:

$$\mathscr{L}^{-1}\left(H(s)\right) = 12\mathscr{L}^{-1}\left(\frac{1}{s^2}\right) - 12\mathscr{L}^{-1}\left(\frac{1}{s^3}\right) + 16\mathscr{L}^{-1}\left(\frac{s}{s^2+16}\right) + 8\mathscr{L}\left(\frac{1}{s^2+16}\right) = 12\cdot t - 12\cdot\frac{1}{2!}t^2 + 16\cdot\cos(4t) + 8\cdot\frac{1}{4}\sin(4t)$$

Therefore, $h(t) = 12t - 6t^2 + 16\cos(4t) + 2\sin(4t)$

Exercise 3.1.4

For $h(t) = \dfrac{\sin(t)}{t}$ and $f(t) = \sin(t)$, we have $H'(s) = -F(s)$.

$$F(s) = \mathscr{L}\left(f(t)\right) = \mathscr{L}\left(\sin(t)\right) = \frac{1}{s^2+1}$$

Therefore, we know $H'(s) = -\dfrac{1}{s^2+1}$, and taking an anti-derivative we get $H(s) = -\tan^{-1}(s) + C$.

To find C, remember that we must have $\lim\limits_{s\to\infty} H(s) = 0 \implies C = \dfrac{\pi}{2}$.

Now we know that $H(s) = \dfrac{\pi}{2} - \tan^{-1}(s)$

Finally, plugging in $s = 0$ gives $H(0) = \dfrac{\pi}{2}$.

This can be written as the equivalent integral $H(0) = \displaystyle\int_0^\infty e^{-0\cdot t}\frac{\sin(t)}{t}\,dt = \int_0^\infty \frac{\sin(t)}{t}\,dt$.

Thus, we have proven that $\displaystyle\int_0^\infty \frac{\sin(t)}{t}\,dt = \frac{\pi}{2}$

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3.2: Laplace Derivative Properties

Learning Objectives

- Apply the derivative properties of the Laplace transform to convert derivatives into algebraic expressions.
- Use the first translation theorem to find transforms of functions multiplied by exponentials.
- Solve initial value problems for linear ODEs using the Laplace transform method.
- Understand the complete workflow of the Laplace transform solution method.

The true power of the Laplace transform is unlocked here. We derive the properties that transform derivatives into algebraic expressions and introduce the first translation theorem, culminating in a revolutionary method for solving initial value problems by converting them into simple algebraic equations.

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3.2: Differentiation of Outputs of Laplace

The Laplace Transformation has been shown to be reasonable to calculate for almost all of the functions we have seen involved in our homogeneous LDE solution sets.

This naturally leads us to ask if the relationships between two functions and their Laplace Transformations look different.

For example, we have already proven that if $h(t) = af(t) + bg(t)$, then $H(s) = aF(s) + bG(s)$.

Something else of interest is that multiplication by an exponential function in the t-space is equivalent to a shift on the s-axis.

Theorem 3.2.1: First Laplace Translation Theorem

If $f(t)$ is a function with $F(s) = \mathscr{L}(f(t))$, and $g(t) = e^{at} \cdot f(t)$, then $G(s) = F(s-a)$

This Theorem is the result of a direct calculation:

$$G(s) = \int_0^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$$

This explains our formula $\mathscr{L}(t^n e^{at}) = \dfrac{n!}{(s-a)^{n+1}}$ in a new light:

$$\mathscr{L}(t^n e^{at}) = [\mathscr{L}(t^n)]_{s\to s-a} = \left[\frac{n!}{s^{n+1}}\right]_{s\to s-a} = \frac{n!}{(s-a)^{n+1}}$$

It is simply the First Translation Theorem applied to our Laplace power rule!

This is particularly useful when dealing with mixed exponential and trigonometric terms.

Example 3.2.1

Find the Laplace Transformation of each function below.

(a) $f(t) = e^{2t}\cos(3t)$

(b) $g(t) = e^{-t}\sin(2t) + e^{-t}\cos(2t)$

Solution

(a) We use the First Translation Theorem:

$$\mathscr{L}(e^{2t}\cos(3t)) = [\mathscr{L}(\cos(3t))]_{s\to s-2} = \left[\frac{s}{s^2+3^2}\right]_{s\to s-2} = \frac{s-2}{(s-2)^2+9}$$

Therefore, $\boxed{F(s) = \dfrac{s-2}{(s-2)^2+9}}$

(b) Once again, the First Translation Theorem gives

$$\begin{aligned} G(s) &= \mathscr{L}\left(e^{-t}\sin(2t)\right) + \mathscr{L}\left(e^{-t}\cos(2t)\right) \\ &= \left[\mathscr{L}\left(\sin(2t)\right)\right]_{s\to s+1} + \left[\mathscr{L}\left(\cos(2t)\right)\right]_{s\to s+1} \\ &= \left[\frac{2}{s^2+2^2}\right]_{s\to s+1} + \left[\frac{s}{s^2+2^2}\right]_{s\to s+1} \\ &= \frac{2}{(s+1)^2+4} + \frac{s+1}{(s+1)^2+4} \end{aligned}$$

This gives us $\boxed{G(s) = \dfrac{s+3}{(s+1)^2+4}}$

This Theorem connects multiplication by an exponential function in the t-world as some shift in the domain of the s-world of Laplace functions. We can turn this statement into one about inverse Laplace Transformations as well.

Rule: Reverse First Translation Theorem

If $f(t)$ is a function with $\mathscr{L}\left(f(t)\right) = F(s)$, then for any $a \in \mathbb{R}$, we have

$$\mathscr{L}^{-1}\left(F(s)\right) = e^{at}\cdot\mathscr{L}^{-1}\left(F(s+a)\right)$$

This allows us to much more easily take the Inverse Laplace Transformation of non-factorable quadratic denominators by completing the square.

Example 3.2.2

Evaluate each Inverse Laplace Transformation below.

(a) $\mathscr{L}^{-1}\left(\dfrac{1}{s^2+10s+34}\right)$

(b) $\mathscr{L}^{-1}\left(\dfrac{s}{s^2-4s+5}\right)$

Solution

(a) We see the denominator $s^2+10s+34$ does not factor, so we complete the square:

$$\mathscr{L}^{-1}\left(\frac{1}{(s+5)^2+9}\right) = e^{-5t}\cdot\mathscr{L}^{-1}\left(\frac{1}{s^2+9}\right) = e^{-5t}\cdot\frac{1}{3}\sin(3t)$$

Therefore, $\boxed{\mathscr{L}^{-1}\left(\dfrac{1}{s^2+10s+34}\right) = \dfrac{1}{3}e^{-5t}\sin(3t)}$

(b) Once again, the denominator s^2-4s+5 does not factor, so we complete the square.

$$\mathscr{L}^{-1}\left(\frac{s}{(s-2)^2+1}\right) = e^{2t}\cdot\mathscr{L}^{-1}\left(\frac{s+2}{s^2+1}\right) = e^{2t}\left[\cos(t) + 2\cdot\frac{1}{1}\sin(t)\right]$$

Therefore, $\boxed{\mathscr{L}^{-1}\left(\dfrac{s}{s^2-4s+5}\right) = e^{2t}\left(\cos(t)+2\sin(t)\right)}$

The First Laplace Translation Theorem is a powerful one for filling in the gaps in the types of functions we know to expect in our solution sets for LDEs.

The Laplace Transformation also has a special relationship with differentiation, both with respect to t and to s.

When differentiating $F'(s)$, we can see that

$$F'(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right] = \int_0^\infty \frac{d}{ds}\left[e^{-st}\right] f(t)\,dt = \int_0^\infty -te^{-st} f(t)\,dt = \mathscr{L}\left(-t \cdot f(t)\right)$$

This means that "a derivative in s" is equivalent to "multiplication by $-t$ in t".

This gives us yet another way to consider our favorite Laplace Transformation formula:

$$\mathscr{L}\left(t^n e^{at}\right) = (-1)^n \mathscr{L}\left((-t)^n e^{at}\right) = (-1)^n \frac{d^n}{ds^n}\mathscr{L}\left(e^{at}\right) = (-1)^n \frac{d^n}{ds^n}\left[\frac{1}{s-a}\right] = \frac{n!}{(s-a)^{n+1}}$$

In practice, it gives us another tool for evaluating the Laplace Transformation of tricky combinations of the functions we may see.

Example 3.2.3

Evaluate the Laplace Transformation of each function below.

(a) $f(t) = t\cos(3t)$

(b) $g(t) = te^t \sin(t)$

(c) $h(t) = t\sin(2t) \cdot t\cos(2t)$

Solution

(a) We trade the t for $-\frac{d}{ds}$:

$$\begin{aligned} F(s) &= \mathscr{L}\left(t\cos(3t)\right) \\ &= -\frac{d}{ds}\mathscr{L}\left(\cos(3t)\right) \\ &= -\frac{d}{ds}\left[\frac{s}{s^2+9}\right] \\ &= \frac{s^2-9}{(s^2+9)^2} \end{aligned}$$

Therefore, $\boxed{\mathscr{L}\left(t\cos(3t)\right) = \frac{s^2-9}{(s^2+9)^2}}$

(b) We trade the t for $-\frac{d}{ds}$, and the e^t for a shift $s \to s-1$:

$$\begin{aligned} G(s) &= \mathscr{L}\left(te^t\sin(t)\right) \\ &= -\frac{d}{ds}\mathscr{L}\left(e^t\sin(t)\right) \\ &= -\frac{d}{ds}\left[\mathscr{L}\left(\sin(t)\right)\right]_{s\to s-1} \\ &= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]_{s\to s-1} \\ &= \left[\frac{2s}{(s^2+1)^2}\right]_{s\to s-1} \\ &= \frac{2(s-1)}{((s-1)^2+1)^2} \end{aligned}$$

Therefore, $\boxed{\mathscr{L}\left(te^t\sin(t)\right) = \frac{2(s-1)}{((s-1)^2+1)^2}}$

(c) We use the identity $\sin(4t) = 2\sin(2t)\cos(2t)$ and trade the t^2 for two $-\frac{d}{ds}$:

$$\begin{aligned} H(s) &= \mathscr{L}\left(t^2 \cdot \frac{1}{2}\sin(4t)\right) \\ &= (-1)^2 \frac{d^2}{ds^2}\mathscr{L}\left(\frac{1}{2}\sin(4t)\right) \\ &= \frac{d^2}{ds^2}\left[\frac{2}{s^2+16}\right] \\ &= \frac{d}{ds}\left[\frac{-4s}{(s^2+16)^2}\right] \\ &= \frac{12s^2-64}{(s^2+16)^3} \end{aligned}$$

Therefore, $\boxed{\mathscr{L}\left(t\sin(2t)\cdot t\cos(2t)\right) = \frac{12s^2-64}{(s^2+16)^3}}$

Once more, there is an equivalent rule you can use for this in terms of the Inverse Laplace Transformation.

$$\mathscr{L}\left(tf(t)\right) = -F'(s) \implies -t\cdot\mathscr{L}^{-1}\left(F(s)\right) = \mathscr{L}^{-1}\left(F'(s)\right)$$

This formula can be used for some of the stranger inverses, like $\mathscr{L}^{-1}\left(\tan^{-1}\left(\frac{\pi}{s}\right)\right)$:

Illustration 3.2.1

We will evaluate $f(t) = \mathscr{L}^{-1}\left(\tan^{-1}\left(\frac{\pi}{s}\right)\right)$:

Notice that this function looks a lot better after differentiation:

$$F(s) = \tan^{-1}\left(\frac{\pi}{s}\right) \implies F'(s) = \frac{-\pi}{s^2+\pi^2}$$

Therefore, $-t\cdot f(t) = \mathscr{L}^{-1}\left(-\frac{\pi}{s^2+\pi^2}\right)$

We recognize that as $-\sin(\pi t)$, so we can solve $-tf(t) = -\sin(\pi t) \implies f(t) = \frac{\sin(\pi t)}{t}$

Example 3.2.4

Evaluate $\mathscr{L}^{-1}\left(\ln\left(1+s^{-2}\right)\right)$

Solution

If we let $F(s) = \ln\left(1+s^{-2}\right)$, then $F'(s) = \frac{-2s^{-3}}{1+s^{-2}} = \frac{-2}{s(s^2+1)}$, so:

$$-tf(t) = \mathscr{L}^{-1}\left(\frac{-2}{s(s^2+1)}\right) = \mathscr{L}^{-1}\left(\frac{2s}{s^2+1} - \frac{2}{s}\right) = 2\cos(t) - 2$$

Dividing both sides by $-t$ gives our answer, $\boxed{f(t) = \frac{2-2\cos(t)}{t}}$

While not an exhaustive list, we can see that many connections between functions based on Laplace Transforms are tied together in different ways in the t and s spaces. Next, we tackle the most important of these properties for a differential equations class: "how do derivatives in t" affect s?

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3.2: Differentiation of Inputs of Laplace

We can use integration by parts to evaluate the Laplace Transformation of $f'(t)$:

$$\mathscr{L}\left(f'(t)\right) = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st}\cdot f(t)\Big|_0^\infty - \int_0^\infty f(t)\cdot\left(-se^{-st}\right)\,dt = -f(0) + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\mathscr{L}\left(f(t)\right)$$

This is a good formula to remember: $\mathscr{L}\left(f'(t)\right) = s\mathscr{L}\left(f(t)\right) - f(0)$

We can even use it twice:

$$\begin{aligned}\mathscr{L}\left(f''(t)\right) &= s\mathscr{L}\left(f'(t)\right) - f'(0)\\ &= s\left[s\mathscr{L}\left(f(t)\right) - f(0)\right] - f'(0)\\ &= s^2\mathscr{L}\left(f(t)\right) - sf(0) - f'(0)\end{aligned}$$

If we use it n consecutive times, we get a formula for the nth derivative:

Rule: Laplace Transformation of a Derivative

For any function $f(t)$ such that $\mathcal{L}\left(f(t)\right)$ converges, and integer $n \geq 1$:

$$\mathscr{L}\left(f^{(n)}(t)\right) = s^n\cdot\mathscr{L}\left(f(t)\right) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$$

This is the final ingredient for our goal of using Laplace Transformations to solve differential equations. We can now apply the Laplace Transformation across an entire differential equation in the t-space and turn it into some other type of equation in the s-space.

Illustration 3.2.2:

Consider the simple first order IVP $y'(t) + y(t) = e^{-t}$ with $y(0) = 1$

If we apply the Laplace Transformation to all three objects we get:

$$\begin{aligned}\mathscr{L}\left(y'(t)\right) &= sY(s) - y(0) = sY(s) - 1\\ \mathscr{L}\left(y(t)\right) &= Y(s)\\ \mathscr{L}\left(e^{-t}\right) &= \frac{1}{s+1}\end{aligned}$$

Now our differential equation becomes $sY(s) - 1 + Y(s) = \dfrac{1}{s+1}$

We can solve this for $Y(s)$, getting $Y(s) = \dfrac{1}{(s+1)^2} + \dfrac{1}{s+1}$

Our last step is to recover $y(t) = \mathscr{L}^{-1}\left(Y(s)\right) = \mathscr{L}^{-1}\left(\dfrac{1}{(s+1)^2} + \dfrac{1}{s+1}\right) = te^{-t} + e^{-t}$

This gives us the unique solution to this linear IVP: $y(t) = (t+1)e^{-t}$

This is the blueprint for solving any IVPs via Laplace Transformations:

1. We take the Laplace Transformation of everything in the equation
2. We algebraically solve for the Laplace Transformation of the function we want to know.
3. We use the Inverse Laplace Transformation to recover the solution function.

The ability of this transformation to turn differential equations in t-world into simple rational equations in s is very powerful.

Example 3.2.4

Solve each IVP below by using Laplace Transformations.

(a) $y'' = 24t^2$ with $y(0) = 0,\ y'(0) = 6$

(b) $y'' + 4y = 8t$ with $y(0) = 0,\ y'(0) = 0$

(c) $y''' + 2y'' + 2y' = 6$ with $y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$

Solution

(a) We take the Laplace Transformation of both sides:

$$\mathscr{L}\left(y''\right) = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 6$$
$$\mathscr{L}\left(24t^2\right) = 24 \cdot \frac{2!}{s^3} = \frac{48}{s^3}$$

Therefore, the equation becomes $s^2Y(s) - 6 = \dfrac{48}{s^3} \implies Y(s) = \dfrac{48}{s^5} + \dfrac{6}{s^2}$

We can take the Inverse Laplace Transformation to get:

$$y(t) = \mathscr{L}^{-1}\left(Y(s)\right) = 48 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^5}\right) + 6 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) = 48 \cdot \frac{1}{4!}t^4 + 6 \cdot \frac{1}{1!}t^1$$

Thus, our unique solution is the function $\boxed{y(t) = 2t^4 + 6t}$

(b) We take the Laplace Transformation of all involved objects:

$$\mathscr{L}\left(y''\right) = s^2Y(s) - y'(0) - sy(0) = s^2Y(s)$$
$$\mathscr{L}\left(y\right) = Y(s)$$
$$\mathscr{L}\left(8t\right) = 8 \cdot \frac{1!}{s^2} = \frac{8}{s^2}$$

Therefore, the equation becomes $s^2Y(s) + 4 \cdot Y(s) = \dfrac{8}{s^2} \implies Y(s) = \dfrac{8}{s^2(s^2+4)}$

We do a *little bit* of partial fraction decomposition:

$$\frac{8}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} \implies \begin{cases} A = 0 \\ B = 2 \\ C = 0 \\ D = -2 \end{cases}$$

We can take the Inverse Laplace Transformation to get:

$$y(t) = \mathscr{L}^{-1}\left(Y(s)\right) = 2 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) - 2 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right) = 2 \cdot \frac{1}{1!}t^1 - 2 \cdot \frac{1}{2}\sin\left(2t\right)$$

The solution to this differential equation is therefore $\boxed{y(t) = 2t - \sin(2t)}$

(c) We take the Laplace Transformation of all involved objects:

$$\mathscr{L}(y''') = s^3Y(s) - y''(0) - sy'(0) - s^2y(0) = s^3Y(s) - s$$
$$\mathscr{L}(y'') = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 1$$
$$\mathscr{L}(y') = sY(s) - y(0) = sY(s)$$
$$\mathscr{L}(6) = \frac{6}{s}$$

Therefore, the equation becomes $s^3Y(s) - s + 2\left(s^2Y(s) - 1\right) + 2\left(sY(s)\right) = \frac{6}{s} \implies Y(s) = \frac{s^2 + 2s + 6}{s^2(s^2 + 2s + 2)}$

We do a *little bit* of partial fraction decomposition:

$$\frac{s^2 + 2s + 6}{s^2(s^2 + 2s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s+1) + D}{(s+1)^2 + 1} \implies \begin{cases} A = -2 \\ B = 3 \\ C = 2 \\ D = 0 \end{cases}$$

We can take the Inverse Laplace Transformation to get:

$$y(t) = \mathscr{L}^{-1}\left(Y(s)\right) = -2 \cdot \mathscr{L}^{-1}\left(\frac{1}{s}\right) + 3 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) + 2\mathscr{L}^{-1}\left(\frac{s+1}{(s+1)^2 + 1}\right) = -2 \cdot 1 + 3 \cdot \frac{1}{1!}t^1 + 2 \cdot e^{-t}\cos\left(1t\right)$$

The solution to this differential equation is therefore $\boxed{y(t) = 3t - 2 + 2e^{-t}\cos(t)}$

We will find that this method (Laplace, solve, Inverse Laplace) is a reliable tool for solving some of our old favorite CC LDEs, but only if you are given the initial conditions. Finding general solutions is skipped when you apply this method, but the particular solution is calculated at the same time as your specific homogeneous constants (all in one ugly calculation).

For now, this does not seem impressive. We already have methods to solve these types of differential equations. The true power comes with a few more rules and relationships between $f(t)$ and $F(s)$ that help us to expand our pool of solvable LDEs.

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3.2: Practice Set

Exercise 3.2.1a

Find the Laplace Transformation of $a(t) = t^2 e^{3t}$

Answer available at the end of this section.

Exercise 3.2.1b

Find the Laplace Transformation of $b(t) = e^{7t}\cos(2t)$

Answer available at the end of this section.

Exercise 3.2.1c

Find the Laplace Transformation of $c(t) = t\sin(11t)$

Answer available at the end of this section.

Exercise 3.2.1d

Find the Laplace Transformation of $d(t) = \cosh(t)\sin(\pi t)$

Answer available at the end of this section.

Exercise 3.2.1e

Find the Laplace Transformation of $e(t) = t\cos^2\left(\frac{t}{4}\right)$

Answer available at the end of this section.

Exercise 3.2.1f

Find the Laplace Transformation of $f(t) = te^{-9t}\cos(4t)$

Answer available at the end of this section.

Exercise 3.2.1g

Find the Laplace Transformation of $g(t) = t^2 e^{2t}\sin(3t)$

Answer available at the end of this section.

Exercise $3.2.2a$

Find the Inverse Laplace Transformation of $A(s) = \dfrac{s}{s^2 + 8s + 20}$

Answer available at the end of this section.

Exercise $3.2.2b$

Find the Inverse Laplace Transformation of $B(s) = \dfrac{12s}{(s^2 + 9)^2}$

Answer available at the end of this section.

Exercise $3.2.2c$

Find the Inverse Laplace Transformation of $C(s) = \dfrac{2s + 2}{s^4 + 2s^3 + 2s^2}$

Answer available at the end of this section.

Exercise $3.2.2d$

Find the Inverse Laplace Transformation of $D(s) = \dfrac{4}{(s^2 - 1)^2}$

Answer available at the end of this section.

Exercise $3.2.2e$

Find the Inverse Laplace Transformation of $E(s) = \dfrac{7s^3 + 42s^2 + 90s + 56}{s^4 + 6s^3 + 12s^2 + 8s}$

Answer available at the end of this section.

Exercise $3.2.3a$

Use Laplace Transformations to solve the IVP $y'' + 4y' + 4y = 36t^2e^{-2t}$ with $y(0) = 0,\ \ y'(0) = 0$

Answer available at the end of this section.

Exercise $3.2.3b$

Use Laplace Transformations to solve the IVP $y'' - 4y' + 5y = 4e^{2t}\cos(t)$ with $y(0) = 0,\ \ y'(0) = 0$

Answer available at the end of this section.

Exercise $3.2.3c$

Use Laplace Transformations to solve the IVP $y'' - 2y' - 3y = 10e^{t}\cos(t)$ with $y(0) = 1,\ \ y'(0) = 3$

Answer available at the end of this section.

Exercise $3.2.3d$

Use Laplace Transformations to solve the IVP $y''' - y = 2t^3 - 12$ with $y(0) = 4,\ y'(0) = 4,\ y''(0) = 4$

Answer available at the end of this section.

Answers

Exercise $3.2.1a$

We use the First Translation Theorem:

$$\mathscr{L}\left(a(t)\right) = \mathscr{L}\left(t^2 e^{3t}\right) = \mathscr{L}\left(t^2\right)\Big|_{s\to s-3} = \frac{2!}{s^3}\Big|_{s\to s-3} = \frac{2}{(s-3)^3}$$

Thus, $A(s) = \dfrac{2}{(s-3)^3}$

Exercise $3.2.1b$

We use the First Translation Theorem:

$$\mathscr{L}\left(b(t)\right) = \mathscr{L}\left(e^{7t}\cos(2t)\right) = \mathscr{L}\left(\cos(2t)\right)\Big|_{s\to s-7} = \frac{s}{s^2+2^2}\Big|_{s\to s-7} = \frac{s-7}{(s-7)^2+4}$$

Thus, $B(s) = \dfrac{s-7}{s^2-14s+53}$

Exercise $3.2.1c$

We use the differentiation property:

$$\mathscr{L}\left(c(t)\right) = \mathscr{L}\left(t\sin(11t)\right) = -\frac{d}{ds}\left[\mathscr{L}\left(\sin(11t)\right)\right] = -\frac{d}{ds}\left[\frac{11}{s^2+11^2}\right] = \frac{22s}{(s^2+121)^2}$$

Thus, $C(s) = \dfrac{22s}{(s^2+121)^2}$

Exercise $3.2.1d$

We use the First Translation Theorem:

$$\begin{aligned}
\mathscr{L}\left(d(t)\right) &= \mathscr{L}\left(\left(\frac{e^t+e^{-t}}{2}\right)\sin(\pi t)\right)\\
&= \frac{1}{2}\mathscr{L}\left(e^t\sin(\pi t)\right) + \frac{1}{2}\mathscr{L}\left(e^{-t}\sin(\pi t)\right)\\
&= \frac{1}{2}\mathscr{L}\left(\sin(\pi t)\right)\Big|_{s\to s-1} + \frac{1}{2}\mathscr{L}\left(\sin(\pi t)\right)\Big|_{s\to s+1}\\
&= \frac{1}{2}\cdot\frac{\pi}{s^2+\pi^2}\Big|_{s\to s-1} + \frac{1}{2}\cdot\frac{\pi}{s^2+\pi^2}\Big|_{s\to s+1}\\
&= \frac{\pi}{2(s-1)^2+2\pi^2} + \frac{\pi}{2(s+1)^2+2\pi^2}
\end{aligned}$$

Thus, $D(s) = \dfrac{\pi}{2(s-1)^2+2\pi^2} + \dfrac{\pi}{2(s+1)^2+2\pi^2}$

Exercise $3.2.1e$

We use the differentiation property:

$$\begin{aligned}
\mathscr{L}\left(e(t)\right) &= \mathscr{L}\left(t\cdot\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{t}{2}\right)\right)\right)\\
&= -\frac{d}{ds}\left[\mathscr{L}\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{t}{2}\right)\right)\right]\\
&= -\frac{d}{ds}\left[\frac{1}{2}\mathscr{L}(1) + \frac{1}{2}\mathscr{L}\left(\cos\left(\frac{t}{2}\right)\right)\right]\\
&= -\frac{d}{ds}\left[\frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+\left(\frac{1}{2}\right)^2}\right]\\
&= -\frac{d}{ds}\left[\frac{1}{2s} + \frac{2s}{4s^2+1}\right]\\
&= \frac{1}{2s^2} + \frac{8s^2-2}{(4s^2+1)^2}
\end{aligned}$$

Thus, $E(s) = \frac{1}{2s^2} + \frac{8s^2-2}{(4s^2+1)^2}$

Exercise $3.2.1f$

We use the differentiation property and First Translation Theorem:

$$\begin{aligned}\mathscr{L}(f(t)) &= \mathscr{L}\left(te^{-9t}\cos(4t)\right)\\ &= -\frac{d}{ds}\left[\mathscr{L}\left(e^{-9t}\cos(4t)\right)\right]\\ &= -\frac{d}{ds}\left[\mathscr{L}(\cos(4t))\right]\Big|_{s\to s+9}\\ &= -\frac{d}{ds}\left[\frac{s}{s^2+4^2}\right]\Big|_{s\to s+9}\\ &= \frac{s^2-16}{(s^2+16)^2}\Big|_{s\to s+9}\\ &= \frac{(s+9)^2-16}{((s+9)^2+16)^2}\end{aligned}$$

Thus, $F(s) = \dfrac{(s+9)^2-16}{((s+9)^2+16)^2}$

Exercise $3.2.1g$

We use the differentiation property (twice) and First Translation Theorem:

$$\begin{aligned}\mathscr{L}(g(t)) &= \mathscr{L}\left(t^2e^{2t}\sin(3t)\right)\\ &= (-1)^2\frac{d^2}{ds^2}\left[\mathscr{L}\left(e^{2t}\sin(3t)\right)\right]\\ &= \frac{d^2}{ds^2}\left[\mathscr{L}(\sin(3t))\right]\Big|_{s\to s-2}\\ &= \frac{d^2}{ds^2}\left[\frac{3}{s^2+3^2}\right]\Big|_{s\to s-2}\\ &= \frac{d}{ds}\left[-\frac{6s}{(s^2+9)^2}\right]\Big|_{s\to s-2}\\ &= \left[\frac{18s^2-54}{(s^2+9)^3}\right]\Big|_{s\to s-2}\\ &\frac{18(s-2)^2-54}{((s-2)^2+9)^3}\end{aligned}$$

Thus, $G(s) = \dfrac{18(s-2)^2-54}{((s-2)^2+9)^3}$

Exercise $3.2.2a$

We use the First Translation Theorem:

$$\begin{aligned}\mathscr{L}^{-1}(A(s)) &= \mathscr{L}^{-1}\left(\frac{s}{s^2+8s+20}\right) = \mathscr{L}^{-1}\left(\frac{s}{(s+4)^2+4}\right)\\ &= e^{-4t}\mathscr{L}^{-1}\left(\frac{s-4}{s^2+4}\right)\\ &= e^{-4t}\mathscr{L}^{-1}\left(\frac{s}{s^2+4}\right) - 4e^{-4t}\mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right)\\ &= e^{-4t}\cdot\cos(2t) - 4e^{-4t}\cdot\frac{1}{2}\sin(2t)\end{aligned}$$

Thus, $a(t) = (\cos(2t) - 2\sin(2t))\,e^{-4t}$

Exercise $3.2.2b$

We use the differentiation property:

$$\begin{aligned}\mathscr{L}^{-1}(B(s)) &= 12\mathscr{L}^{-1}\left(\frac{s}{(s^2+9)^2}\right)\\ &= -12t\mathscr{L}^{-1}\left(\int\frac{s}{(s^2+9)^2}ds\right)\\ &= -12t\mathscr{L}^{-1}\left(-\frac{1}{2(s^2+9)}\right)\\ &= 6t\mathscr{L}^{-1}\left(\frac{1}{s^2+9}\right)\\ &= 6t\cdot\frac{1}{3}\sin(3t)\end{aligned}$$

Thus, $b(t) = 2t\sin(3t)$

Exercise $3.2.2c$

We use partial fraction decomposition and the First Translation Theorem:

$$\begin{aligned}\mathscr{L}^{-1}(C(s)) &= \mathscr{L}^{-1}\left(\frac{2s+2}{s^4+2s^3+2s^2}\right)\\ &= \mathscr{L}^{-1}\left(\frac{1}{s^2} - \frac{1}{(s+1)^2+1}\right)\\ &= \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) - \mathscr{L}\left(\frac{1}{(s+1)^2+1}\right)\\ &= t - e^{-t}\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)\\ &= t - e^{-t}\sin(t)\end{aligned}$$

Thus, $c(t) = t - e^{-t}\sin(t)$

Exercise $3.2.2d$

We use partial fraction decomposition and the First Translation Theorem:

$$\begin{aligned}\mathscr{L}^{-1}\left(D(s)\right) &= \mathscr{L}^{-1}\left(\frac{4}{(s^2-1)^2}\right)\\ &= \mathscr{L}^{-1}\left(\frac{1}{s+1}+\frac{1}{(s+1)^2}-\frac{1}{s-1}+\frac{1}{(s-1)^2}\right)\\ &= \mathscr{L}^{-1}\left(\frac{1}{s+1}\right)+\mathscr{L}^{-1}\left(\frac{1}{(s+1)^2}\right)-\mathscr{L}^{-1}\left(\frac{1}{s-1}\right)+\mathscr{L}^{-1}\left(\frac{1}{(s-1)^2}\right)\\ &= e^{-t}\mathscr{L}^{-1}\left(\frac{1}{s}\right)+e^{-t}\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)-e^{t}\mathscr{L}^{-1}\left(\frac{1}{s}\right)+e^{t}\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)\\ &= e^{-t}\cdot 1+e^{-t}\cdot t-e^{t}\cdot 1+e^{t}\cdot t\end{aligned}$$

Thus, $d(t)=(t+1)e^{-t}+(t-1)e^{t}$

Exercise 3.2.2*e*

We use partial fraction decomposition and the First Translation Theorem:

$$\begin{aligned}\mathscr{L}^{-1}\left(E(s)\right) &= \mathscr{L}^{-1}\left(\frac{7s^3+42s^2+90s+56}{s(s+2)^3}\right)\\ &= \mathscr{L}^{-1}\left(\frac{7}{s}+\frac{6}{(s+2)^3}\right)\\ &= 7\mathscr{L}^{-1}\left(\frac{1}{s}\right)+6\mathscr{L}^{-1}\left(\frac{1}{(s+2)^3}\right)\\ &= 7\cdot 1+6e^{-2t}\mathscr{L}^{-1}\left(\frac{1}{s^3}\right)\\ &= 7+6e^{-2t}\cdot\frac{1}{2!}t^2\end{aligned}$$

Thus, $e(t)=7+3t^2e^{-2t}$

Exercise 3.2.3*a*

We take the Laplace Transformation of each term:

$$\begin{aligned}\mathscr{L}\left(y''\right) &= -y'(0)-sy(0)+s^2Y(s)=s^2Y(s)\\ \mathscr{L}\left(4y'\right) &= 4\left(-y(0)+sY(s)\right)=4sY(s)\\ \mathscr{L}\left(4y\right) &= 4Y(s)\\ \mathscr{L}\left(36t^2e^{-2t}\right) &= 36\mathscr{L}t^2)\Big|_{s\to s+2}=36\cdot\frac{2!}{s^3}\Big|_{s\to s+2}=\frac{72}{(s+2)^3}\end{aligned}$$

Therefore, the differential equation becomes:

$$\begin{gathered}s^2Y(s)+4sY(s)+4Y(s)=\frac{72}{(s+2)^3}\\ \Downarrow\\ Y(s)=\frac{72}{(s+2)^5}\end{gathered}$$

Taking the inverse Laplace Transformation gives:

$$y(t)=\mathscr{L}^{-1}\left(Y(s)\right)=\mathscr{L}^{-1}\left(\frac{72}{(s+2)^5}\right)=72e^{-2t}\mathscr{L}^{-1}\left(\frac{1}{s^5}\right)=72e^{-2t}\cdot\frac{t^4}{4!}$$

Therefore, the unique solution is $y(t)=3t^4e^{-2t}$

Exercise 3.2.3*b*

We take the Laplace Transformation of each term:

$$\begin{aligned}\mathscr{L}\left(y''\right) &= -y'(0)-sy(0)+s^2Y(s)=s^2Y(s)\\ \mathscr{L}\left(4y'\right) &= 4\left(-y(0)+sY(s)\right)=4sY(s)\\ \mathscr{L}\left(5y\right) &= 5Y(s)\\ \mathscr{L}\left(4e^{t}\cos(t)\right) &= 4\mathscr{L}(\cos(t))\Big|_{s\to s-2}=4\cdot\frac{s}{s^2+1}\Big|_{s\to s-2}=\frac{4(s-2)}{(s-2)^2+1}\end{aligned}$$

Therefore, the differential equation becomes:

$$\begin{gathered}s^2Y(s)-4sY(s)+5Y(s)=\frac{4(s-2)}{(s-2)^2+1}\\ \Downarrow\\ Y(s)=\frac{4(s-2)}{\left((s-2)^2+1\right)^2}\end{gathered}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\ &= \mathscr{L}^{-1}\left(\frac{4(s-2)}{\left((s-2)^2+1\right)^2}\right) \\ &= 4e^{2t}\mathscr{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) \\ &= 4e^{2t}\cdot\left(-t\mathscr{L}^{-1}\left(\int\frac{s}{(s^2+1)^2}\,ds\right)\right) \\ &= -4te^{2t}\mathscr{L}^{-1}\left(-\frac{1}{2(s^2+1)}\right) \\ &= 2te^{2t}\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) \\ &= 2te^{2t}\sin(t) \end{aligned}$$

Therefore, the unique solution is $y(t) = 2te^{2t}\sin(t)$

Exercise $3.2.3c$

We take the Laplace Transformation of each term:

$$\begin{aligned} \mathscr{L}\left(y''\right) &= -y'(0) - sy(0) + s^2Y(s) = -3 - s + s^2Y(s) \\ \mathscr{L}\left(2y'\right) &= 2\left(-y(0) + sY(s)\right) = -2 + 2sY(s) \\ \mathscr{L}\left(3y\right) &= 3Y(s) \\ \mathscr{L}\left(10e^t\cos(t)\right) &= 10\mathscr{L}(\cos(t))\Big|_{s\to s-1} = 10\cdot\frac{s}{s^2+1}\Big|_{s\to s-1} = \frac{10(s-1)}{(s-1)^2+1} \end{aligned}$$

Therefore, the differential equation becomes:

$$\begin{gathered} \left(-3 - s + s^2Y(s)\right) - \left(-2 + 2sY(s)\right) - 3Y(s) = \frac{10(s-1)}{(s-1)^2+1} \\ \Downarrow \\ (s^2 - 2s - 3)Y(s) = \frac{10s-10}{(s-1)^2+1} + 1 + s \\ \Downarrow \\ Y(s) = \frac{10s-10}{\left((s-1)^2+1\right)(s-3)(s+1)} + \frac{1}{s-3} \end{gathered}$$

Using partial fraction decomposition we can rewrite this as:

$$Y(s) = \frac{-2s+2}{(s-1)^2+1} + \frac{1}{s+1} + \frac{2}{s-3}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\ &= \mathscr{L}^{-1}\left(\frac{-2s+2}{(s-1)^2+1} + \frac{1}{s+1} + \frac{2}{s-3}\right) \\ &= -2\mathscr{L}^{-1}\left(\frac{s-1}{(s-1)^2+1}\right) + \mathscr{L}^{-1}\left(\frac{1}{s+1}\right) + 2\mathscr{L}^{-1}\left(\frac{1}{s-3}\right) \\ &= -2e^t\mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right) + e^{-t} + 2\cdot e^{3t} \\ &= -2e^t\cos(t) + e^{-t} + 2e^{3t} \end{aligned}$$

Therefore, the unique solution is $y(t) = -2e^t\cos(t) + e^{-t} + 2e^{3t}$

Exercise $3.2.3d$

We take the Laplace Transformation of each term:

$$\begin{aligned} \mathscr{L}\left(y'''\right) &= -y''(0) - sy'(0) - s^2y(0) + s^3Y(s) = -4 - 4s - 4s^2 + s^3Y(s) \\ \mathscr{L}\left(y\right) &= Y(s) \\ \mathscr{L}\left(2t^3 - 12\right) &= \frac{12}{s^4} - \frac{12}{s} = \frac{12 - 12s^3}{s} \end{aligned}$$

Therefore, the differential equation becomes:

$$\begin{gathered} \left(-4 - 4s - 4s^2 + s^3Y(s)\right) - Y(s) = \frac{12-12s^3}{s^4} \\ \Downarrow \\ (s^3 - 1)Y(s) = \frac{12-12s^3}{s^4} + 4 + 4s + 4s^2 \\ \Downarrow \\ Y(s) = \frac{12-12s^3}{s^4(s^3-1)} + \frac{4+4s+4s^2}{s^3-1} \end{gathered}$$

Factoring $(s^3 - 1) = (s-1)(s^2+s+1)$, we see this reduces to:

$$Y(s) = -\frac{12}{s^4} + \frac{4}{s-1}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\ &= \mathscr{L}^{-1}\left(\frac{4}{s-1} - \frac{12}{s^4}\right) \\ &= 4\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) - 12\mathscr{L}^{-1}\left(\frac{1}{s}\right) \\ &= 4 \cdot e^t - 12 \cdot \frac{1}{3!}t^3 \end{aligned}$$

Therefore, the unique solution is $y(t) = 4e^t - 2t^3$

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3.3: Piecewise Differential Equations

Learning Objectives

- Define and graph the unit step (Heaviside) function.
- Express piecewise-defined functions using unit step functions.
- Apply the second translation theorem to find Laplace transforms of shifted functions.
- Solve differential equations with piecewise continuous forcing functions using Laplace transforms.

This subchapter tackles differential equation problems involving forces that turn on and off, like a switch in a circuit. We learn to express piecewise functions using the unit step function and apply the second translation theorem to solve differential equations with these discontinuous inputs.

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3.3: Piecewise Functions and the Unit Step Function

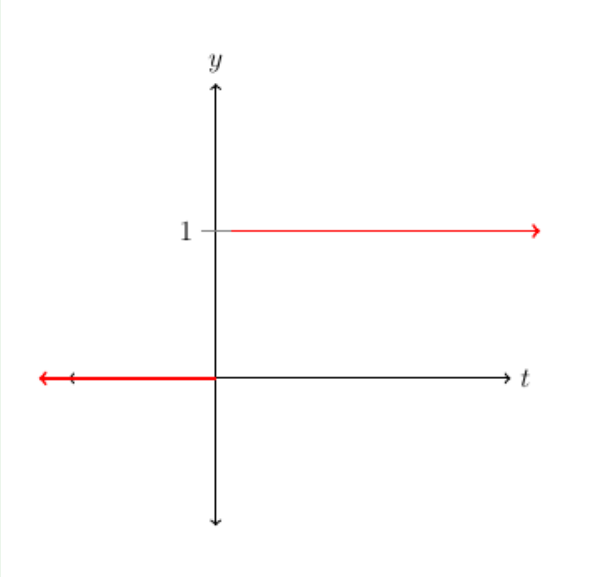
First, we will define an object we can use to express all possible piecewise functions more easily.

Definition: Unit Step Function

The ***unit step function*** is defined as the piecewise function

$$\mathcal{U}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

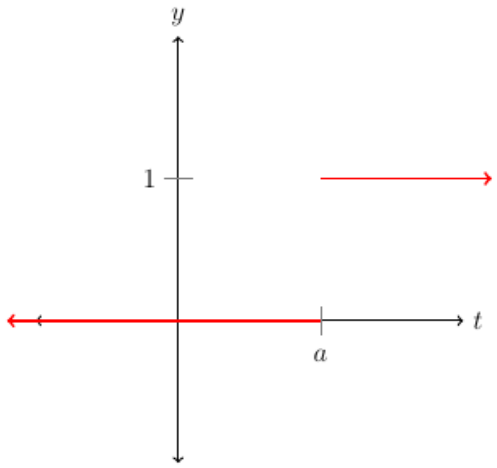
This is the "on/off" switch that turns from off (when $t < 0$) to on (when $t \geq 0$)



In a chapter about Laplace Transformations, where every function has domain $0 \leq t < \infty$, this function looks completely useless.

Indeed $f(t) \cdot \mathcal{U}(t) = f(t)$ for all functions $f(t)$ with this domain. However, we can use shifts in the $\mathcal{U}(t)$ function to move where a step occurs.

For any $a > 0$, we see \(\mathcal{U}(t-a)=\left\{ \begin{array} {lr} 0 & t



We can use this function to rewrite any piecewise functions in terms of various shifts of $\mathcal{U}$.

Example 3.3.1

Write each piecewise function in terms of unit step functions.

(a) $a(t) = \begin{cases} 0 & t < 2 \\ t & t \geq 2 \end{cases}$

(b) $b(t) = \begin{cases} t^2 & t < 3 \\ 6t - 9 & t \geq 3 \end{cases}$

(c) $c(t) = \begin{cases} 2t & t < 1 \\ 0 & 1 \leq t < \pi \\ \sin(t) & t \geq \pi \end{cases}$

Solution

(a) We rewrite this function as:

$$a(t) = \begin{cases} 0 & t < 2 \\ t & t \geq 2 \end{cases} = t \cdot \begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases} = t \cdot \mathcal{U}(t-2)$$

Therefore, we have written $\boxed{a(t) = t \cdot \mathcal{U}(t-2)}$

(b) We rewrite this function as:

$$b(t) = \begin{cases} t^2 & t < 3 \\ 6t - 9 & t \geq 3 \end{cases} = t^2 + (6t - 9 - t^2) \cdot \begin{cases} 0 & t < 3 \\ 1 & t \geq 3 \end{cases} = t^2 + (6t - 9 - t^2) \cdot \mathcal{U}(t-3)$$

Therefore, we have written $\boxed{b(t) = t^2 + (6t - 9 - t^2) \cdot \mathcal{U}(t-3)}$

(c) We rewrite this function as:

$$c(t) = \begin{cases} 2t & t < 1 \\ 0 & 1 \le t < \pi \\ \sin(t) & t \ge \pi \end{cases} = 2t - 2t \cdot \begin{cases} 0 & t < 1 \\ 1 & t \ge 1 \end{cases} + \sin(t) \cdot \begin{cases} 0 & t < \pi \\ 1 & t \ge \pi \end{cases} = 2t - 2t \cdot \mathcal{U}(t-1) + \sin(t) \cdot \mathcal{U}(t-\pi)$$

Therefore, we have written $\boxed{c(t) = 2t - 2t \cdot \mathcal{U}(t-1) + \sin(t) \cdot \mathcal{U}(t-\pi)}$

Notice that the Laplace Transformation has an interesting relationship with the shifted unit step function.

Theorem 3.3.1: Second Laplace Translation Theorem

If $\mathscr{L}(f(t)) = F(s)$, then $\mathscr{L}(f(t-a) \cdot \mathcal{U}(t-a)) = e^{-as}F(s)$

This is another straightforward proof by direct calculation:

$$\begin{aligned} \mathscr{L}(f(t-a) \cdot \mathcal{U}(t-a)) &= \int_0^\infty e^{-st} \cdot f(t-a) \cdot \mathcal{U}(t-a)\, dt \\ &= \int_0^a e^{-st} \cdot f(t-a) \cdot 0\, dt + \int_a^\infty e^{-st} \cdot f(t-a) \cdot 1\, dt \\ &= \int_a^\infty e^{-st} \cdot f(t-a)\, dt \\ &= \int_0^\infty e^{-s(w+a)} \cdot f(w)\, dw \\ &= e^{-as} \int_0^\infty e^{-sw} f(w)\, dw \\ &= e^{-as} \cdot \mathscr{L}(f(t)) \\ &= e^{-as} F(s) \end{aligned}$$

We see that a piecewise "shift" in the t-axis turns into exponential decay in the s-space. Substituting in $t + a$ everywhere there is a t gives a more tractable version of this theorem:

Rule: Revised Second Laplace Translation Theorem

$$\mathscr{L}(f(t)\mathcal{U}(t-a)) = e^{-as}\mathscr{L}(f(t+a))$$

We can use this to find the Laplace Transformation of many piecewise defined functions.

Example 3.3.2

Find the Laplace Transformation of each function below.

(a) $a(t) = \begin{cases} 0 & t < 2 \\ t & t \ge 2 \end{cases}$

(b) $b(t) = \begin{cases} t^2 & t < 3 \\ 6t - 9 & t \ge 3 \end{cases}$

(c) $c(t) = \begin{cases} 2t & t < 1 \\ 0 & 1 \le t < \pi \\ \sin(t) & t \ge \pi \end{cases}$

Solution

(a) We use the previous result $a(t) = t \cdot \mathcal{U}(t-2)$ and the Second Laplace Translation Theorem:

$$\begin{aligned} A(s) &= \mathscr{L}\left(a(t)\right) \\ &= \mathscr{L}\left(t \cdot \mathcal{U}(t-2)\right) \\ &= e^{-2s}\mathscr{L}\left(t+2\right) \\ &= e^{-2s}\left[\frac{1}{s^2}+\frac{2}{s}\right] \\ &= \frac{1+2s}{s^2}e^{-2s} \end{aligned}$$

Therefore, we have $\boxed{A(s) = \frac{1+2s}{s^2e^{2s}}}$

(b) We use the previous result $b(t) = t^2 + (6t - 9 - t^2) \cdot \mathcal{U}(t-3)$ and the Second Laplace Translation Theorem:

$$\begin{aligned} B(s) &= \mathscr{L}\left(b(t)\right) \\ &= \mathscr{L}\left(t^2 + (6t-9-t^2)\cdot\mathcal{U}(t-3)\right) \\ &= \mathscr{L}\left(t^2\right) + \mathscr{L}\left((6t-9-t^2)\cdot\mathcal{U}(t-3)\right) \\ &= \frac{2}{s^3} + e^{-3s}\mathscr{L}\left(6(t+3)-9-(t+3)^2\right) \\ &= \frac{2}{s^3} + e^{-3s}\mathscr{L}\left(-t^2\right) \\ &= \frac{2}{s^3} + e^{-3s}\left[-\frac{2}{s^3}\right] \\ &= \frac{2}{s^3}\left(1-e^{-3s}\right) \end{aligned}$$

Therefore, we have $\boxed{B(s) = \frac{2}{s^3}\left(1-e^{-3s}\right)}$

(c) We use the previous result $c(t) = 2t - 2t \cdot \mathcal{U}(t-1) + \sin(t) \cdot \mathcal{U}(t-\pi)$ and the Second Laplace Translation Theorem:

$$\begin{aligned} C(s) &= \mathscr{L}\left(c(t)\right) \\ &= \mathscr{L}\left(2t - 2t\cdot\mathcal{U}(t-1) + \sin(t)\cdot\mathcal{U}(t-\pi)\right) \\ &= \mathscr{L}\left(2t\right) - \mathscr{L}\left(2t\cdot\mathcal{U}(t-1)\right) + \mathscr{L}\left(\sin(t)\cdot\mathcal{U}(t-\pi)\right) \\ &= \frac{2}{s^2} - e^{-s}\mathscr{L}\left(2(t+1)\right) + e^{-\pi s}\mathscr{L}\left(\sin(t+\pi)\right) \\ &= \frac{2}{s^2} - e^{-s}\mathscr{L}\left(2t+2\right) + e^{-\pi s}\mathscr{L}\left(-\sin(t)\right) \\ &= \frac{2}{s^2} - e^{-s}\left[\frac{2}{s^2}+\frac{2}{s}\right] + e^{-\pi s}\left[-\frac{1}{s^2+1}\right] \\ &= \frac{2-(2+2s)e^{-s}}{s^2} - \frac{e^{-\pi s}}{s^2+1} \end{aligned}$$

Therefore, we have $\boxed{C(s) = \frac{2-(2+2s)e^{-s}}{s^2} - \frac{e^{-\pi s}}{s^2+1}}$

Once more, the Second Laplace Translation Theorem also works in the opposite direction. In terms of the Inverse Laplace Transformation, we have

Rule: Reverse Second Laplace Translation Theorem

$$\mathscr{L}^{-1}\left(e^{-as}F(s)\right) = \left[\mathscr{L}^{-1}\left(F(s)\right)\right]_{t\to t-a}\cdot\mathcal{U}(t-a)$$

Example 3.3.3

Find the Inverse Laplace Transformation of each function below.

(a) $F(s) = \dfrac{s - 4e^{-3s}}{s^3}$

(b) $G(s) = \dfrac{2e^{-s}}{s^2 + 4s + 5}$

(c) $H(s) = \dfrac{e^{-\pi s} + e^{-2\pi s}}{s^2 + 1}$

Solution

(a) We see $F(s) = \dfrac{s - 4e^{-3s}}{s^3} = \dfrac{1}{s^2} - \dfrac{4}{s^3}e^{-3s}$

$$\mathscr{L}^{-1}\left(\frac{1}{s^2}\right) = t$$
$$\mathscr{L}^{-1}\left(\frac{4}{s^3}e^{-3s}\right) = \left[\mathscr{L}^{-1}\left(\frac{4}{s^3}\right)\right]_{t\to t-3} \cdot \mathcal{U}(t-3) = \left[2t^2\right]_{t\to t-3} \cdot \mathcal{U}(t-3) = 2(t-3)^2\mathcal{U}(t-3)$$

Therefore, $\boxed{f(t) = t - 2(t-3)^2\mathcal{U}(t-3) = \begin{cases} t & 0 \le t < 3 \\ t - 2(t-3)^2 & t \ge 3 \end{cases}}$

(b) Once again, we rewrite $G(s) = \dfrac{2}{(s+2)^2 + 1}e^{-s}$:

$$\mathscr{L}^{-1}\left(\frac{2}{(s+2)^2+1}e^{-s}\right) = \left[\mathscr{L}^{-1}\left(\frac{2}{(s+2)^2+1}\right)\right]_{t\to t-1} \cdot \mathcal{U}(t-1) = \left[2e^{-2t}\sin(t)\right]_{t\to t-1} \cdot \mathcal{U}(t-1) = 2e^{2-2t}\sin(t-1)\mathcal{U}(t-1)$$

Therefore, $\boxed{g(t) = 2e^{2-2t}\sin(t-1)\mathcal{U}(t-1) = \begin{cases} 0 & 0 \le t < 1 \\ 2e^{2-2t}\sin(t-1) & t \ge 1 \end{cases}}$

(c) We split $H(s) = \dfrac{1}{s^2+1}e^{-\pi s} + \dfrac{1}{s^2+1}e^{-2\pi s}$:

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+1}e^{-\pi s}\right) = \left[\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)\right]_{t\to t-\pi} \cdot \mathcal{U}(t-\pi) = \left[\sin(t)\right]_{t\to t-\pi} \cdot \mathcal{U}(t-\pi) = -\sin(t)\mathcal{U}(t-\pi)$$
$$\mathscr{L}^{-1}\left(\frac{1}{s^2+1}e^{-2\pi s}\right) = \left[\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)\right]_{t\to t-2\pi} \cdot \mathcal{U}(t-2\pi) = \left[\sin(t)\right]_{t\to t-2\pi} \cdot \mathcal{U}(t-2\pi) = \sin(t)\mathcal{U}(t-2\pi)$$

Thus, $\boxed{h(t) = -\sin(t)\mathcal{U}(t-\pi) + \sin(t)\mathcal{U}(t-2\pi) = \begin{cases} 0 & 0 \le t < \pi \\ -\sin(t) & \pi \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}}$

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3.3: Laplace Transformations of Piecewise Functions

Now that we are comfortable using the Second Translation Theorem ($\mathscr{L}\{f(t-a)\cdot\mathcal{U}(t-a)\}=e^{-as}F(s)$) in each direction, we can use these skills to solve CC LDE IVPs with piecewise forcing functions. There is no realistic assumption that the external force your system is subjected to is always continuous.

To tackle these problems, we simply write all piecewise functions down in terms of the unit step function $\mathcal{U}(t-a)$, then take the Laplace Transformation of the entire equation, solve the resulting algebraic equation, and use the Inverse Laplace Transformation to get back to the solution.

Illustration 3.3.1

We will use Laplace Transformations to find the unique solution to $y'+y=\begin{cases} t+1 & 0\le t<2 \\ t^2-t+1 & t\ge 2 \end{cases}$ with $y(0)=0$

First, we rewrite the piecewise function:

$$g(t)=\begin{cases} t+1 & 0\le t<2 \\ t^2-t+1 & t\ge 2 \end{cases} = (t+1)+(t^2-2t)\mathcal{U}(t-2)$$

Now, we take the Laplace Transformation of the objects in the IVP:

$$\begin{aligned} \mathscr{L}(y') &= sY(s)-y(0)=sY(s) \\ \mathscr{L}(y) &= Y(s) \\ \mathscr{L}(t+1) &= \frac{1}{s^2}+\frac{1}{s}=\frac{s+1}{s^2} \\ \mathscr{L}\left((t^2-2t)\mathcal{U}(t-2)\right) &= \mathscr{L}\left((t+2)^2-2(t+2)\right)e^{-2s} = \mathscr{L}\left(t^2+2t\right)e^{-2s} = \left[\frac{2}{s^3}+\frac{2}{s^2}\right]e^{-2s} = \frac{2(s+1)}{s^3}e^{-2s} \end{aligned}$$

This turns our differential equation into the equation:

$$sY(s)+Y(s)=\frac{s+1}{s^2}+\frac{2(s+1)}{s^3}e^{-2s} \implies Y(s)=\frac{1}{s^2}+\frac{2}{s^3}e^{-2s}$$

We can now take the Inverse Laplace Transformation of each piece:

$$\begin{aligned} \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) &= t \\ \mathscr{L}^{-1}\left(\frac{2}{s^3}e^{-2s}\right) &= \left[\mathscr{L}^{-1}\left(\frac{2}{s^3}\right)\right]_{t\to t-2}\cdot\mathcal{U}(t-2) = \left[t^2\right]_{t\to t-2}\cdot\mathcal{U}(t-2) = (t-2)^2\mathcal{U}(t-2) \end{aligned}$$

This gives us final solution $y(t)=t+(t-2)^2\mathcal{U}(t-2)$, which we write as: $\boxed{y(t)=\begin{cases} t & 0\le t<2 \\ t^2-3t+4 & t\ge 2 \end{cases}}$

These can increase in difficulty when you have *more pieces* or *more derivatives*.

Example 3.3.4

Find the unique solution to $y''-y=\begin{cases} t^2-2 & 0\le t<2 \\ -t^2+8t-6 & 2\le t<4 \\ 8 & t\ge 4 \end{cases}$ with $y(0)=0,\ y'(0)=0$

Solution

First, we rewrite the piecewise function:

$$g(t)=\begin{cases} t^2-2 & 0\le t<2 \\ -t^2+8t-6 & 2\le t<4 \\ 8 & t\ge 4 \end{cases} = t^2-2+\left(-2t^2+8t-4\right)\mathcal{U}(t-2)+(t^2-8t+14)\mathcal{U}(t-4)$$

Now, we take the Laplace Transformation of the objects in the IVP:

$$\begin{aligned} \mathscr{L}\left(y''\right) &= s^2Y(s)-sy(0)-y'(0)=s^2Y(s) \\ \mathscr{L}\left(y\right) &= Y(s) \\ \mathscr{L}\left(t^2-2\right) &= \frac{2!}{s^3}-\frac{2}{s}=\frac{2-2s^2}{s^3} \end{aligned}$$

$$\mathscr{L}\left((-2t^2+8t-4)\mathcal{U}(t-2)\right)=\mathscr{L}\left(-2(t+2)^2+8(t+2)-4\right)e^{-2s}=\mathscr{L}\left(-2t^2+4\right)e^{-2s}=\left[-2\cdot\frac{2!}{s^3}+4\cdot\frac{1}{s}\right]e^{-2s}=\frac{-4+4s^2}{s^3}e^{-2s}$$

$$\mathscr{L}\left((t^2-8t+14)\mathcal{U}(t-4)\right)=\mathscr{L}\left((t+4)^2-8(t+4)+14\right)e^{-4s}=\mathscr{L}\left(t^2-2\right)e^{-4s}=\left[\frac{2}{s^3}-\frac{2}{s}\right]e^{-4s}=\frac{2-2s^2}{s^3}e^{-4s}$$

This turns our differential equation into the equation:

$$s^2Y(s)-Y(s)=\frac{2-2s^2}{s^3}+\frac{4-4s^2}{s^3}e^{-2s}+\frac{2-2s^2}{s^3}e^{-4s} \implies Y(s)=-\frac{2}{s^3}+\frac{4}{s^3}e^{-2s}-\frac{2}{s^3}e^{-4s}$$

We can now take the Inverse Laplace Transformation of each piece:

$$\begin{aligned} \mathscr{L}^{-1}\left(\frac{2}{s^3}\right) &= 2\cdot\frac{1}{2!}t^2=t^2 \\ \mathscr{L}^{-1}\left(\frac{4}{s^3}e^{-2s}\right) &= \left[\mathscr{L}^{-1}\left(\frac{4}{s^3}\right)\right]_{t\to t-2}\cdot\mathcal{U}(t-2)=\left[2t^2\right]_{t\to t-2}\cdot\mathcal{U}(t-2)=2(t-2)^2\mathcal{U}(t-2) \\ \mathscr{L}^{-1}\left(\frac{2}{s^3}e^{-4s}\right) &= \left[\mathscr{L}^{-1}\left(\frac{2}{s^3}\right)\right]_{t\to t-4}\cdot\mathcal{U}(t-4)=\left[t^2\right]_{t\to t-4}\cdot\mathcal{U}(t-4)=(t-4)^2\mathcal{U}(t-4) \end{aligned}$$

This gives us final solution $y(t)=-t^2+2(t-2)^2\mathcal{U}(t-2)-(t-4)^2\mathcal{U}(t-4)$, which we write as:

$$\boxed{y(t)=\begin{cases} -t^2 & 0\le t<2 \\ t^2-8t+8 & 2\le t<4 \\ -8 & t\ge 4 \end{cases}}$$

Example 3.3.5

Find the solution to $y^{(4)}(t)-4y^{(3)}(t)+6y''(t)-4y'(t)+y(t)=\begin{cases} 24e^t & 0\le t<1 \\ 0 & 1\le t<3 \\ (24t-72)e^t & t\ge 3 \end{cases}$ with $y(0)=0,\ y'(0)=0,\ y''(0)=0,$ $y'''(0)=0$

Solution

First, we rewrite the piecewise function:

$$g(t)=\begin{cases} 24e^t & 0\le t<1 \\ 0 & 1\le t<3 \\ (24t-72)e^t & t\ge 3 \end{cases} = 24e^t-24e^t\mathcal{U}(t-1)+(24t-72)e^t\mathcal{U}(t-3)$$

Now, we take the Laplace Transformation of the objects in the IVP:

$$\begin{aligned}
\mathscr{L}\left(y^{(4)}\right) &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y(s)\\
\mathscr{L}\left(y^{(3)}\right) &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) = s^3Y(s)\\
\mathscr{L}\left(y''\right) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s)\\
\mathscr{L}\left(y'\right) &= sY(s) - y(0) = sY(s)\\
\mathscr{L}\left(y\right) &= Y(s)\\
\mathscr{L}\left(24e^t\right) &= \frac{24}{s-1}
\end{aligned}$$

$$\mathscr{L}\left(-24e^t\mathcal{U}(t-1)\right) = \mathscr{L}\left(-24e^{t+1}\right)e^{-s} = \mathscr{L}\left(-24e^t\right)e^{1-s} = -\frac{24}{s-1}e^{1-s}$$

$$\mathscr{L}\left((24t-72)e^t\mathcal{U}(t-3)\right) = \mathscr{L}\left((24(t+3)-72)\,e^{t+3}\right)e^{-3s} = \mathscr{L}\left(24te^t\right)e^{3-3s} = \left[24\cdot\frac{1}{(s-1)^2}\right]e^{3-3s} = \frac{24}{(s-1)^2}e^{3-3s}$$

This turns our differential equation into the equation:

$$s^4Y(s) - 4s^3Y(s) + 6s^2Y(s) - 4sY(s) + Y(s) = \frac{24}{s-1} - \frac{24}{s-1}e^{1-s} + \frac{24}{(s-1)^2}e^{3-3s} \implies Y(s) = \frac{24}{(s-1)^5} - \frac{24}{(s-1)^5}e^{1-s} + \frac{24}{(s-$$

We can now take the Inverse Laplace Transformation of each piece:

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{24}{(s-1)^5}\right) &= 24\cdot\frac{1}{4!}t^4e^t = t^4e^t\\
\mathscr{L}^{-1}\left(\frac{24}{(s-1)^5}e^{1-s}\right) &= 24e\left[\mathscr{L}^{-1}\left(\frac{1}{(s-1)^5}\right)\right]_{t\to t-1}\cdot\mathcal{U}(t-1) = 24e\left[\frac{1}{4!}t^4e^t\right]_{t\to t-1}\cdot\mathcal{U}(t-1) = (t-1)^4e^t\mathcal{U}(t-1)\\
\mathscr{L}^{-1}\left(\frac{24}{(s-1)^6}e^{3-3s}\right) &= 24e^3\left[\mathscr{L}^{-1}\left(\frac{1}{(s-1)^6}\right)\right]_{t\to t-3}\cdot\mathcal{U}(t-3) = 24e^3\left[\frac{1}{5!}t^5e^t\right]_{t\to t-3}\cdot\mathcal{U}(t-3) = \frac{1}{5}(t-3)^5e^t\mathcal{U}(t-3)
\end{aligned}$$

This gives us final solution $y(t) = t^4e^t - (t-1)^4e^t\mathcal{U}(t-1) + \frac{1}{5}(t-3)^5e^t\mathcal{U}(t-3)$, which we write as:

$$y(t) = \begin{cases} t^4e^t & 0 \le t < 1\\ \left(t^4-(t-1)^4\right)e^t & 1 \le t < 3\\ \left(t^4-(t-1)^4+\frac{1}{5}(t-3)^5\right)e^t & t \ge 3 \end{cases}$$

The Laplace Transformation gives us a method to solve many CC LDEs that have piecewise functions involved.

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3.3: Practice Set

Exercise 3.3.1a

Find the Laplace Transformation of the function $a(t) = \begin{cases} 0 & 0 \le t < 3 \\ 8 & t \ge 3 \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1b

Find the Laplace Transformation of the function $b(t) = \begin{cases} 1 & 0 \le t < 1 \\ t & t \ge 1 \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1c

Find the Laplace Transformation of the function $c(t) = \begin{cases} e^t & 0 \le t < 2 \\ 0 & t \ge 2 \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1d

Find the Laplace Transformation of the function $d(t) = \begin{cases} \sin(t) & 0 \le t < 2\pi \\ 2\sin(t) & t \ge 2\pi \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1e

Find the Laplace Transformation of the function $e(t) = \begin{cases} 3\cos(2t) & 0 \le t < \pi \\ 4\sin(2t) & t \ge \pi \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1f

Find the Laplace Transformation of the function $f(t) = \begin{cases} t^2 & 0 \le t < 4 \\ 8t & t \ge 4 \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1g

Find the Laplace Transformation of the function $g(t) = \begin{cases} t^3 e^{5t} & 0 \le t < 2 \\ (6t^2 - 12t + 8)e^{5t} & t \ge 2 \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1h

Find the Laplace Transformation of the function $h(t) = \begin{cases} 0 & 0 \le t < 2 \\ 3t & 2 \le t < 4 \\ 0 & t \ge 4 \end{cases}$

Answer available at the end of this section.

Exercise 3.3.1i

Find the Laplace Transformation of the function $i(t) = \begin{cases} 2 & 0 \le t < 1 \\ te^{2t} & 1 \le t < 3 \\ (t^2 - 5t + 9)e^{2t} & t \ge 3 \end{cases}$

Answer available at the end of this section.

Exercise 3.3.2a

Find the Inverse Laplace Transformation of the function $A(s) = 10e^{-2s}s^{-3}$

Answer available at the end of this section.

Exercise 3.3.2b

Find the Inverse Laplace Transformation of the function $B(s) = \dfrac{1}{s^2 e^{4s} + e^{4s}}$

Answer available at the end of this section.

Exercise 3.3.2c

Find the Inverse Laplace Transformation of the function $C(s) = \dfrac{4 - 2se^{-5s}}{s^3}$

Answer available at the end of this section.

Exercise 3.3.2d

Find the Inverse Laplace Transformation of the function $D(s) = \dfrac{18 + 6e^{-s}}{(s+1)^4}$

Answer available at the end of this section.

Exercise 3.3.2e

Find the Inverse Laplace Transformation of the function $E(s) = \dfrac{3se^{-\pi s}}{s^2 - 8s + 25}$

Answer available at the end of this section.

Exercise 3.3.2f

Find the Inverse Laplace Transformation of the function $F(s) = \dfrac{e^{-4s} - 4se^{-s}}{s^2(s-1))}$

Answer available at the end of this section.

Exercise 3.3.2g

Find the Inverse Laplace Transformation of the function $G(s) = \dfrac{-2e^{-\frac{\pi}{2}s} + e^{2\pi - \frac{3\pi}{2}s}}{s^2 - 4s + 5}$

Answer available at the end of this section.

Exercise 3.3.3a

Use Laplace Transformations to solve the IVP $y' - 2y = \begin{cases} -4t^2 & 0 \le t < 1 \\ 4(t^2 - 4t + 2) & t \ge 1 \end{cases}$ with $y(0) = 1$

Answer available at the end of this section.

Exercise 3.3.3b

Use Laplace Transformations to solve the IVP $y'' - y = (t-1) \cdot \mathcal{U}(t-2)$ with $y(0) = 2,\ y'(0) = -2$

Answer available at the end of this section.

Exercise 3.3.3c

Use Laplace Transformations to solve the IVP $y''' = \begin{cases} 0 & 0 \le t < 4 \\ 72(t-4) & 4 \le t < 7 \\ 24(t+2) & t \ge 7 \end{cases}$ with $y(0) = -5,\ y'(0) = 0,\ y''(0) = 2$

Answer available at the end of this section.

Exercise 3.3.3d

Use Laplace Transformations to solve the IVP $4y'' + y = \begin{cases} 0 & 0 \le t < 4\pi \\ 4\cos\left(\dfrac{t}{2}\right) & t \ge 4\pi \end{cases}$ with $y(0) = 0,\ y'(0) = 2\pi$

Answer available at the end of this section.

Exercise 3.3.3e

Use Laplace Transformations to solve the IVP $y' = \begin{cases} 1 & 2k \le t < 2k+1 \text{ for some integer } k \\ -1 & 2k+1 \le t < 2k+2 \text{ for some integer } k \end{cases}$ with $y(0) = 0$

Answer available at the end of this section.

Answers

Exercise 3.3.1a

We re-write $a(t) = 8 \cdot \mathcal{U}(t-3)$

$$\begin{aligned} \mathscr{L}(a(t)) &= 8\mathscr{L}(\mathcal{U}(t-3)) \\ &= 8e^{-3s} \cdot \mathscr{L}(1) \\ &= 8e^{-3s} \cdot \frac{1}{s} \end{aligned}$$

Thus, $A(s) = \dfrac{8e^{-3s}}{s}$

Exercise 3.3.1b

We re-write $b(t) = 1 + (t-1) \cdot \mathcal{U}(t-1)$

$$\begin{aligned} \mathscr{L}(b(t)) &= \mathscr{L}(1 + (t-1) \cdot \mathcal{U}(t-1)) \\ &= \mathscr{L}(1) + \mathscr{L}((t-1) \cdot \mathcal{U}(t-1)) \\ &= \frac{1}{s} + e^{-s} \cdot \mathscr{L}(t) \\ &= \frac{1}{s} + e^{-s} \cdot \frac{1}{s^2} \end{aligned}$$

Thus, $B(s) = \dfrac{s + e^{-s}}{s^2}$

Exercise 3.3.1c

We re-write $c(t) = e^t - e^t \cdot \mathcal{U}(t-2)$

$$\begin{aligned} \mathscr{L}(c(t)) &= \mathscr{L}(e^t - e^t \cdot \mathcal{U}(t-2)) \\ &= \mathscr{L}(e^t) - \mathscr{L}(e^t \cdot \mathcal{U}(t-2)) \\ &= \frac{1}{s-1} - e^{-2s} \cdot \mathscr{L}\left(e^{t+2}\right) \\ &= \frac{1}{s-1} - e^{-2s} \cdot e^2 \mathscr{L}(e^t) \\ &= \frac{1}{s-1} - e^{2-2s} \cdot \frac{1}{s-1} \end{aligned}$$

Thus, $C(s) = \dfrac{1 - e^{2-2s}}{s-1}$

Exercise 3.3.1d

We re-write $d(t) = \sin(t) + \sin(t) \cdot \mathcal{U}(t-2\pi)$

$$\begin{aligned} \mathscr{L}(d(t)) &= \mathscr{L}(\sin(t) + \sin(t) \cdot \mathcal{U}(t-2\pi)) \\ &= \mathscr{L}(\sin(t)) + \mathscr{L}(\sin(t) \cdot \mathcal{U}(t-2\pi)) \\ &= \frac{1}{s^2+1} + e^{-2\pi s} \cdot \mathscr{L}(\sin(t+2\pi)) \\ &= \frac{1}{s^2+1} + e^{-2\pi s} \cdot \mathscr{L}(\sin(t)) \\ &= \frac{1}{s^2+1} + e^{-2\pi s} \cdot \frac{1}{s^2+1} \end{aligned}$$

Thus, $D(s) = \dfrac{1 + e^{-2\pi s}}{s^2+1}$

Exercise 3.3.1e

We re-write $e(t) = 3\cos(2t) + (4\sin(2t) - 3\cos(2t)) \cdot \mathcal{U}(t-\pi)$

$$\begin{aligned} \mathscr{L}(e(t)) &= \mathscr{L}(3\cos(2t) + (4\sin(2t) - 3\cos(2t)) \cdot \mathcal{U}(t-\pi)) \\ &= 3\mathscr{L}(\cos(t)) + 4\mathscr{L}(\sin(2t) \cdot \mathcal{U}(t-\pi)) - 3\mathscr{L}(\cos(2t) \cdot \mathcal{U}(t-\pi)) \\ &= 3 \cdot \frac{s}{s^2+4} + 4e^{-\pi s} \cdot \mathscr{L}(\sin(2t+2\pi)) - 3e^{-\pi s} \cdot \mathscr{L}(\cos(2t+2\pi)) \\ &= \frac{3s}{s^2+4} + 4e^{-\pi s} \cdot \mathscr{L}(\sin(2t)) - 3e^{-\pi s} \cdot \mathscr{L}(\cos(2t)) \\ &= \frac{3s}{s^2+4} + 4e^{-\pi s} \cdot \frac{2}{s^2+4} - 3e^{-\pi s} \cdot \frac{s}{s^2+4} \end{aligned}$$

Thus, $E(s) = \dfrac{3s + (8-3s)e^{-\pi s}}{s^2+4}$

Exercise 3.3.1f

We re-write $f(t) = t^2 + (8t - t^2) \cdot \mathcal{U}(t-4)$

$$\begin{aligned}
\mathscr{L}(f(t)) &= \mathscr{L}\left(t^2 + (8t - t^2) \cdot \mathcal{U}(t-4)\right) \\
&= \mathscr{L}\left(t^2\right) + \mathscr{L}\left((8t - t^2) \cdot \mathcal{U}(t-4)\right) \\
&= \frac{2}{s^3} + e^{-4s} \cdot \mathscr{L}\left(8(t+4) - (t+4)^2\right) \\
&= \frac{2}{s^3} + e^{-4s} \cdot \mathscr{L}\left(16 - t^2\right) \\
&= \frac{2}{s^3} + e^{-4s} \cdot \left(\frac{16}{s} - \frac{2}{s^3}\right)
\end{aligned}$$

Thus, $F(s) = \dfrac{2 + (16s^2 - 2)e^{-4s}}{s^3}$

Exercise 3.3.1*g*

We re-write $g(t) = t^3 e^{5t} + ((6t^2 - 12t + 8 - t^3)e^{5t}) \cdot \mathcal{U}(t-2)$

$$\begin{aligned}
\mathscr{L}(g(t)) &= \mathscr{L}\left(t^3 e^{5t} + ((6t^2 - 12t + 8 - t^3)e^{5t}) \cdot \mathcal{U}(t-2)\right) \\
&= \mathscr{L}\left(t^3 e^{5t}\right) + \mathscr{L}\left((6t^2 - 12t + 8 - t^3)e^{5t} \cdot \mathcal{U}(t-2)\right) \\
&= \mathscr{L}\left(t^3\right)\Big|_{s \to s-5} + e^{-2s} \cdot \mathscr{L}\left((6(t+2)^2 - 12(t+2) + 8 - (t+2)^3)e^{5t}\right) \\
&= \frac{6}{s^4}\Big|_{s \to s-5} + e^{-2s} \cdot \mathscr{L}\left(-t^3 e^{5t}\right) \\
&= \frac{6}{(s-5)^4} - e^{-2s} \cdot \left[\mathscr{L}\left(t^3\right)\right]\Big|_{s \to s-5} \\
&= \frac{6}{(s-5)^4} - e^{-2s} \cdot \left[\frac{6}{s^4}\right]\Big|_{s \to s-5} \\
&= \frac{6\left(1 - e^{-2s}\right)}{(s-5)^4}
\end{aligned}$$

Thus, $G(s) = \dfrac{6\left(1 - e^{-2s}\right)}{(s-5)^4}$

Exercise 3.3.1*h*

We re-write $h(t) = 3t \cdot \mathcal{U}(t-2) - 3t \cdot \mathcal{U}(t-4)$

$$\begin{aligned}
\mathscr{L}(h(t)) &= \mathscr{L}\left(3t \cdot \mathcal{U}(t-2) - 3t \cdot \mathcal{U}(t-4)\right) \\
&= 3\mathscr{L}\left(t \cdot \mathcal{U}(t-2)\right) - 3\mathscr{L}\left(t \cdot \mathcal{U}(t-4)\right) \\
&= 3e^{-2s} \cdot \mathscr{L}\left(t+2\right) - 3e^{-4s} \cdot \mathscr{L}\left(t+4\right) \\
&= 3e^{-2s} \cdot \left(\frac{1}{s^2} + \frac{2}{s}\right) - 3e^{-4s} \cdot \left(\frac{1}{s^2} + \frac{4}{s}\right)
\end{aligned}$$

Thus, $H(s) = \dfrac{(3 + 6s)e^{-2s} + (3 + 12s)e^{-4s}}{s^2}$

Exercise 3.3.1*i*

We re-write $i(t) = 2 + (te^{2t} - 2) \cdot \mathcal{U}(t-1) + ((t^2 - 6t + 9)e^{2t}) \cdot \mathcal{U}(t-3)$

$$\begin{aligned}
\mathscr{L}(i(t)) &= \mathscr{L}\left(2 + \left(te^{2t} - 2\right) \cdot \mathcal{U}(t-1) + \left((t^2 - 6t + 9)e^{2t}\right) \cdot \mathcal{U}(t-3)\right) \\
&= 2\mathscr{L}\left(1\right) + \mathscr{L}\left(\left(te^{2t} - 2\right) \cdot \mathcal{U}(t-1)\right) + \mathscr{L}\left((t-3)^2 e^{2t} \cdot \mathcal{U}(t-3)\right) \\
&= 2 \cdot \frac{1}{s} + e^{-s} \cdot \mathscr{L}\left((t+1)e^{2t+2} - 2\right) + e^{-3s} \cdot \mathscr{L}\left(t^2 e^{2t+6}\right) \\
&= \frac{2}{s} + e^{-s} \cdot e^2 \cdot \mathscr{L}\left((t+1)e^{2t}\right) - 2e^{-s} \cdot \mathscr{L}(1) + e^{-3s} \cdot e^6 \cdot \mathscr{L}\left(t^2 e^{2t}\right) \\
&= \frac{2}{s} + e^{2-s} \cdot \mathscr{L}\left(t+1\right)\Big|_{s \to s-2} - 2e^{-s} \cdot \frac{1}{s} + e^{6-3s} \cdot \mathscr{L}\left(t^2\right)\Big|_{s \to s-2} \\
&= \frac{2}{s} + e^{2-s} \cdot \left[\frac{1}{s^2} + \frac{1}{s}\right]\Big|_{s \to s-2} - 2e^{-s} \cdot \frac{1}{s} + e^{6-3s} \cdot \left[\frac{2}{s^3}\right]\Big|_{s \to s-2} \\
&= \frac{2}{s} + e^{2-s} \cdot \left[\frac{1}{(s-2)^2} + \frac{1}{s-2}\right] - 2e^{-s} \cdot \frac{1}{s} + e^{6-3s} \cdot \left[\frac{2}{(s-2)^3}\right]
\end{aligned}$$

Thus, $I(s) = \dfrac{2 - 2e^{-s}}{s} + \dfrac{(s-2)(s-1)e^{2-s} + 2e^{6-3s}}{(s-2)^3}$

Exercise 3.3.2*a*

$$\begin{aligned}
\mathscr{L}^{-1}(A(s)) &= 10\mathscr{L}^{-1}\left(e^{-2s}s^{-3}\right) \\
&= 10 \cdot \mathscr{L}^{-1}\left(s^{-3}\right)\Big|_{t \to t-2} \cdot \mathcal{U}(t-2) \\
&= 10 \cdot \left[\frac{1}{2!}t^2\right]\Big|_{t \to t-2} \cdot \mathcal{U}(t-2) \\
&= 5 \cdot (t-2)^2 \cdot \mathcal{U}(t-2)
\end{aligned}$$

Therefore, $\boxed{a(t) = \begin{cases} 0 & 0 \le t < 2 \\ 5(t-2)^2 & t \ge 2 \end{cases}}$

Exercise 3.3.2*b*

$$\begin{aligned}\mathscr{L}^{-1}\left(B(s)\right) &= \mathscr{L}^{-1}\left(\frac{1}{s^2+1}e^{-4s}\right)\\ &= \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)\Big|_{t\to t-4}\cdot\mathcal{U}(t-4)\\ &= \left[\sin(t)\right]\Big|_{t\to t-4}\cdot\mathcal{U}(t-4)\\ &= \sin(t-4)\cdot\mathcal{U}(t-4)\end{aligned}$$

Therefore, $b(t)=\begin{cases}0 & 0\le t<4\\ \sin(t-4) & t\ge 4\end{cases}$

Exercise 3.3.2*c*

$$\begin{aligned}\mathscr{L}^{-1}\left(C(s)\right) &= 4\mathscr{L}^{-1}\left(\frac{1}{s^3}\right)-2\mathscr{L}^{-1}\left(\frac{1}{s^2}e^{-5s}\right)\\ &= 4\cdot\frac{1}{2!}t^2-2\mathscr{L}^{-1}\left(\frac{1}{s}\right)\Big|_{t\to t-5}\cdot\mathcal{U}(t-5)\\ &= 2t^2-2\cdot\left[t\right]\Big|_{t\to t-5}\cdot\mathcal{U}(t-5)\\ &= 2t^2-2\cdot(t-5)\cdot\mathcal{U}(t-5)\end{aligned}$$

Therefore, $c(t)=\begin{cases}2t^2 & 0\le t<5\\ 2t^2-2t+10 & t\ge 5\end{cases}$

Exercise 3.3.2*d*

$$\begin{aligned}\mathscr{L}^{-1}\left(D(s)\right) &= 18\mathscr{L}^{-1}\left(\frac{1}{(s+1)^4}\right)+6\mathscr{L}^{-1}\left(\frac{1}{(s+1)^4e^{-s}}\right)\\ &= 18e^{-t}\cdot\mathscr{L}^{-1}\left(\frac{1}{s^4}\right)+6\cdot\mathscr{L}^{-1}\left(\frac{1}{(s+1)^4}\right)\Big|_{t\to t-1}\cdot\mathcal{U}(t-1)\\ &= 18e^{-t}\cdot\frac{1}{3!}t^3+6\cdot\left[e^{-t}\cdot\frac{1}{3!}t^3\right]\Big|_{t\to t-1}\cdot\mathcal{U}(t-1)\\ &= 3t^3e^{-t}+(t-1)^3e^{1-t}\cdot\mathcal{U}(t-1)\end{aligned}$$

Therefore, $d(t)=\begin{cases}3t^3e^{-t} & 0\le t<1\\ 3t^3e^{-t}+(t-1)^3e^{1-t} & t\ge 1\end{cases}$

Exercise 3.3.2*e*

$$\begin{aligned}\mathscr{L}^{-1}\left(E(s)\right) &= 3\cdot\mathscr{L}^{-1}\left(\frac{s}{s^2-8s+25}e^{-\pi s}\right)\\ &= 3\cdot\mathscr{L}^{-1}\left(\frac{s}{(s-4)^2+9}\right)\Big|_{t\to t-\pi}\cdot\mathcal{U}(t-\pi)\\ &= 3\cdot\left[e^{4t}\mathscr{L}^{-1}\left(\frac{s+4}{s^2+9}\right)\right]\Big|_{t\to t-\pi}\cdot\mathcal{U}(t-\pi)\\ &= 3\cdot\left[e^{4t}\left(\cos(3t)+4\cdot\frac{1}{3}\sin(3t)\right)\right]\Big|_{t\to t-\pi}\cdot\mathcal{U}(t-\pi)\\ &= \left(3\cdot e^{4t-4\pi}\cos\left(3t-3\pi\right)+4\cdot e^{4\pi-4t}\sin(3t-3\pi)\right)\cdot\mathcal{U}(t-\pi)\\ &= \left(-3\cdot e^{4t-4\pi}\cos\left(3t\right)-4\cdot e^{4\pi-4t}\sin(3t)\right)\cdot\mathcal{U}(t-\pi)\end{aligned}$$

Therefore, $e(t)=\begin{cases}0 & 0\le t<\pi\\ -(4\sin(3t)+3\cos(3t))e^{4t-4\pi} & t\ge\pi\end{cases}$

Exercise 3.3.2*f*

$$\begin{aligned}\mathscr{L}^{-1}\left(F(s)\right) &= \mathscr{L}^{-1}\left(\frac{1}{s^2(s-1)}e^{-4s}\right)-4\mathscr{L}^{-1}\left(\frac{1}{s(s-1)}e^{-s}\right)\\ &= \mathscr{L}^{-1}\left(\frac{1}{s-1}-\frac{1}{s}-\frac{1}{s^2}\right)\Big|_{t\to t-4}\cdot\mathcal{U}(t-4)-4\cdot\mathscr{L}^{-1}\left(\frac{1}{s-1}-\frac{1}{s}\right)\Big|_{t\to t-1}\cdot\mathcal{U}(t-1)\\ &= \left[e^t-1-t\right]\Big|_{t\to t-4}\cdot\mathcal{U}(t-4)-4\cdot\left[e^t-1\right]\Big|_{t\to t-1}\cdot\mathcal{U}(t-1)\\ &= \left(e^{t-4}-t+3\right)\cdot\mathcal{U}(t-4)-4\cdot\left(e^{t-1}-1\right)\cdot\mathcal{U}(t-1)\end{aligned}$$

Therefore, $f(t)=\begin{cases}0 & 0\le t<1\\ 4-4e^{t-1} & 1\le t<4\\ 7-t-4e^{t-1}+e^{t-4} & t\ge 4\end{cases}$

Exercise 3.3.2*g*

$$\begin{aligned}\mathscr{L}^{-1}\left(G(s)\right) &= -2\cdot\mathscr{L}^{-1}\left(\frac{1}{s^2-4s+5}e^{-\frac{\pi}{2}s}\right)+e^{2\pi}\cdot\mathscr{L}^{-1}\left(\frac{1}{s^2-4s+5}e^{-\frac{3\pi}{2}s}\right)\\ &= -2\cdot\mathscr{L}^{-1}\left(\frac{1}{(s-2)^2+1}\right)\Big|_{t\to t-\frac{\pi}{2}}\cdot\mathcal{U}\left(t-\frac{\pi}{2}\right)+e^{2\pi}\cdot\mathscr{L}^{-1}\left(\frac{1}{(s-2)^2+1}\right)\Big|_{t\to t-\frac{3\pi}{2}}\cdot\mathcal{U}\left(t-\frac{3\pi}{2}\right)\\ &= -2\cdot\left[e^{2t}\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)\right]\Big|_{t\to t-\frac{\pi}{2}}\cdot\mathcal{U}\left(t-\frac{\pi}{2}\right)+e^{2\pi}\cdot\left[e^{2t}\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)\right]\Big|_{t\to t-\frac{3\pi}{2}}\cdot\mathcal{U}\left(t-\frac{3\pi}{2}\right)\\ &= -2\cdot\left[e^{2t}\sin(t)\right]\Big|_{t\to t-\frac{\pi}{2}}\cdot\mathcal{U}\left(t-\frac{\pi}{2}\right)+e^{2\pi}\cdot\left[e^{2t}\sin(t)\right]\Big|_{t\to t-\frac{3\pi}{2}}\cdot\mathcal{U}\left(t-\frac{3\pi}{2}\right)\\ &= -2e^{2t-\pi}\sin\left(t-\frac{\pi}{2}\right)\cdot\mathcal{U}\left(t-\frac{\pi}{2}\right)+e^{2\pi}\cdot e^{2t-3\pi}\sin\left(t-\frac{3\pi}{2}\right)\cdot\mathcal{U}\left(t-\frac{3\pi}{2}\right)\\ &= 2e^{2t-\pi}\cos(t)\cdot\mathcal{U}\left(t-\frac{\pi}{2}\right)+e^{2t-\pi}\cos(t)\cdot\mathcal{U}\left(t-\frac{3\pi}{2}\right)\end{aligned}$$

Therefore, $g(t) = \begin{cases} 0 & 0 \le t < \frac{\pi}{2} \\ 2e^{2t-\pi}\cos(t) & \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ 3e^{2t-\pi}\cos(t) & t \ge \frac{3\pi}{2} \end{cases}$

Exercise $3.3.3a$

We take the Laplace Transformation of each term:

$$\mathscr{L}(y') = -y(0) + sY(s) = 1 + sY(s)$$
$$\mathscr{L}(y) = Y(s)$$
$$\mathscr{L}\left(-4t^2 + (8t^2 - 16t + 8)\cdot\mathcal{U}(t-1)\right) = -4\cdot\frac{2}{s^3} + 8e^{-s}\cdot\mathscr{L}(t^2) = -\frac{8}{s^3} + 8e^{-s}\cdot\frac{2}{s^3}$$

Therefore, the differential equation becomes:

$$1 + sY(s) - 2Y(s) = -\frac{8}{s^3} + 8e^{-s}\cdot\frac{2}{s^3}$$
$$\Downarrow$$
$$Y(s) = -\frac{8}{s^3(s-2)} + \frac{16}{s^3(s-2)}e^{-s} + \frac{1}{s-2}$$

Using partial fraction decomposition, we write this as:

$$Y(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{4}{s^3} + \left(\frac{2}{s-2} - \frac{8}{s^3} - \frac{4}{s^2} - \frac{2}{s}\right)e^{-s}\$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}(Y(s)) \\ &= \mathscr{L}^{-1}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{4}{s^3}\right) + \mathscr{L}^{-1}\left(\left(\frac{2}{s-2} - \frac{8}{s^3} - \frac{4}{s^2} - \frac{2}{s}\right)e^{-s}\right) \\ &= \left(1 + 2t + 2t^2\right) + \left[\mathscr{L}^{-1}\left(\frac{2}{s-2} - \frac{8}{s^3} - \frac{4}{s^2} - \frac{2}{s}\right)\right]\Big|_{t\to t-1}\cdot\mathcal{U}(t-1) \\ &= \left(1 + 2t + 2t^2\right) + \left[2e^{2t} - 4t^2 - 4t - 2\right]\Big|_{t\to t-1}\cdot\mathcal{U}(t-1) \\ &= \left(1 + 2t + 2t^2\right) + \left(2e^{2t-2} - 4t^2 + 4t - 2\right)\cdot\mathcal{U}(t-1) \end{aligned}$$

Therefore, the unique solution is $y(t) = \begin{cases} 1 + 2t + 2t^2 & 0 \le t < 1 \\ -2t^2 + 6t - 1 + 2e^{2t-2} & t \ge 1 \end{cases}$

Exercise $3.3.3b$

We take the Laplace Transformation of each term:

$$\mathscr{L}(y'') = -y'(0) - sy(0) + s^2Y(s) = 2 - 2s + s^2Y(s)$$
$$\mathscr{L}(y) = Y(s)$$
$$\mathscr{L}\left((t-1)\cdot\mathcal{U}(t-2)\right) = e^{-2s}\cdot\mathscr{L}(t+1) = e^{-2s}\cdot\left(\frac{1}{s^2} + \frac{1}{s}\right) = \frac{1+s}{s^2}e^{-2s}$$

Therefore, the differential equation becomes:

$$2 - 2s + s^2Y(s) - Y(s) = \frac{1+s}{s^2}e^{-2s}$$
$$\Downarrow$$
$$Y(s) = \frac{1}{s^2(s-1)}e^{-2s} + \frac{2}{s+1}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}(Y(s)) \\ &= \mathscr{L}^{-1}\left(\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right)e^{-2s}\right) + 2\cdot\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) \\ &= \mathscr{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right)\Big|_{t\to t-2}\cdot\mathcal{U}(t-2) + 2\cdot e^{-t} \\ &= \left[e^t - 1 - t\right]\Big|_{t\to t-2}\cdot\mathcal{U}(t-2) + 2\cdot e^{-t} \\ &= \left(e^{t-2} + 1 - t\right)\cdot\mathcal{U}(t-2) + 2\cdot e^{-t} \end{aligned}$$

Therefore, the unique solution is $y(t) = \begin{cases} 2e^{-t} & 0 \le t < 2 \\ e^{t-2} + 1 - t + 2e^{-t} & t \ge 2 \end{cases}$

Exercise $3.3.3c$

We take the Laplace Transformation of each term:

$$\mathscr{L}(y''') = -y''(0) - sy'(0) - s^2y(0) + s^3Y(s) = -2 + 5s^2 + s^3Y(s)$$
$$\mathscr{L}\left(72(t-4)\cdot\mathcal{U}(t-4) - 48(t-7)\cdot\mathcal{U}(t-7)\right) = \frac{72}{s^2}e^{-4s} - \frac{48}{s^2}e^{-7s}$$

Therefore, the differential equation becomes:

$$-2 + 5s^2 + s^3Y(s) = \frac{72}{s^2}e^{-4s} - \frac{48}{s^2}e^{-7s}$$
$$\Downarrow$$
$$Y(s) = \frac{72}{s^5}e^{-4s} - \frac{48}{s^5}e^{-7s} + \frac{2-5s^2}{s^3}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\
&= \mathscr{L}^{-1}\left(\frac{72}{s^5}e^{-4s}\right) - \mathscr{L}^{-1}\left(\frac{48}{s^5}e^{-7s}\right) + \mathscr{L}^{-1}\left(\frac{2}{s^3} - \frac{5}{s}\right) \\
&= \mathscr{L}^{-1}\left(\frac{72}{s^5}\right)\Big|_{t\to t-4} \cdot \mathcal{U}(t-4) - \mathscr{L}^{-1}\left(\frac{48}{s^5}\right)\Big|_{t\to t-7} \cdot \mathcal{U}(t-7) + \left(t^2-5\right) \\
&= \left[3t^4\right]\Big|_{t\to t-4} \cdot \mathcal{U}(t-4) - \left[2t^4\right]\Big|_{t\to t-7} \cdot \mathcal{U}(t-7) + t^2 - 5 \\
&= 3(t-4)^4 \cdot \mathcal{U}(t-4) - 2(t-7)^4 \cdot \mathcal{U}(t-7) + t^2 - 5
\end{aligned}$$

Therefore, the unique solution is $y(t) = \begin{cases} t^2-5 & 0 \le t < 4 \\ t^2-5+3(t-4)^4 & 4 \le t < 7 \\ t^2-5+3(t-4)^4-2(t-7)^4 & t \ge 7 \end{cases}$

Exercise 3.3.3*d*

We take the Laplace Transformation of each term:

$$\begin{aligned}
&\mathscr{L}\left(4y''\right) = 4\left(-y'(0) - sy(0) + s^2Y(s)\right) = -8\pi + 4s^2Y(s) \\
&\mathscr{L}\left(y\right) = Y(s) \\
&\mathscr{L}\left(4\cos\left(\frac{1}{2}t\right)\cdot\mathcal{U}(t-4\pi)\right) = 4\cdot\frac{s}{s^2+\frac{1}{4}}\cdot e^{-4\pi s} = \frac{16s}{4s^2+1}e^{-4\pi s}
\end{aligned}$$

Therefore, the differential equation becomes:

$$\begin{gathered}
-8\pi + 4s^2Y(s) + Y(s) = \frac{16s}{4s^2+1}e^{-4\pi s} \\
\Downarrow \\
Y(s) = \frac{16s}{(4s^2+1)^2}e^{-4\pi s} + \frac{8\pi}{4s^2+1}
\end{gathered}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\
&= 16\cdot\mathscr{L}^{-1}\left(\frac{s}{(4s^2+1)^2}e^{-4\pi s}\right) + 2\pi\cdot\mathscr{L}^{-1}\left(\frac{1}{s^2+\frac{1}{4}}\right) \\
&= 16\cdot\mathscr{L}^{-1}\left(\frac{s}{(4s^2+1)^2}\right)\Big|_{t\to t-4\pi}\cdot\mathcal{U}(t-4\pi) + 2\pi\cdot 2\sin\left(\frac{1}{2}t\right) \\
&= 16\cdot\left[-t\cdot\mathscr{L}^{-1}\left(\int\frac{s}{(4s^2+1)^2}\,ds\right)\right]\Big|_{t\to t-4\pi}\cdot\mathcal{U}(t-4\pi) + 4\pi\sin\left(\frac{1}{2}t\right) \\
&= \left[-16t\cdot\mathscr{L}^{-1}\left(-\frac{\frac{1}{8}}{4s^2+1}\right)\right]\Big|_{t\to t-4\pi}\cdot\mathcal{U}(t-4\pi) + 4\pi\sin\left(\frac{1}{2}t\right) \\
&= \left[\frac{1}{2}t\cdot\mathscr{L}^{-1}\left(\frac{1}{s^2+\frac{1}{4}}\right)\right]\Big|_{t\to t-4\pi}\cdot\mathcal{U}(t-4\pi) + 4\pi\sin\left(\frac{1}{2}t\right) \\
&= \left[t\sin\left(\frac{1}{2}t\right)\right]\Big|_{t\to t-4\pi}\cdot\mathcal{U}(t-4\pi) + 4\pi\sin\left(\frac{1}{2}t\right) \\
&= \left[(t-4\pi)\sin\left(\frac{1}{2}t-2\pi\right)\right]\cdot\mathcal{U}(t-4\pi) + 4\pi\sin\left(\frac{1}{2}t\right) \\
&= \left[(t-4\pi)\sin\left(\frac{1}{2}t\right)\right]\cdot\mathcal{U}(t-4\pi) + 4\pi\sin\left(\frac{1}{2}t\right)
\end{aligned}$$

Therefore, the unique solution is $y(t) = \begin{cases} 4\pi\sin\left(\frac{t}{2}\right) & 0 \le t < 4\pi \\ t\sin\left(\frac{t}{2}\right) & t \ge 4\pi \end{cases}$

Exercise 3.3.3*e*

The piecewise function on the right can be written as an infinite sum: $\begin{cases} 1 & 2k \le t < 2k+1 \text{ for some integer } k \\ -1 & 2k+1 \le t < 2k+2 \text{ for some integer } k \end{cases} = 1 + 2\sum_{n=1}^{\infty}(-1)^n\,\mathcal{U}(t-n)$

We take the Laplace Transformation of each term:

$$\begin{aligned}
&\mathscr{L}\left(y'\right) = -y(0) + sY(s) = sY(s) \\
&\mathscr{L}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n\,\mathcal{U}(t-n)\right) = \frac{1}{s} + 2\sum_{n=1}^{\infty}(-1)^n\frac{1}{s}e^{-sn}
\end{aligned}$$

Therefore, the differential equation becomes:

$$\begin{gathered}
sY(s) = \frac{1}{s} + 2\sum_{n=1}^{\infty}(-1)^n\frac{1}{s}e^{-sn} \\
\Downarrow \\
Y(s) = \frac{1}{s^2} + 2\sum_{n=1}^{\infty}(-1)^n\frac{1}{s^2}e^{-sn}
\end{gathered}$$

Taking the inverse Laplace Transformation gives:

$$
\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\
&= \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) + 2\sum_{n=1}^{\infty}(-1)^n \mathscr{L}^{-1}\left(\frac{1}{s^2}e^{-sn}\right) \\
&= t + 2\sum_{n=1}^{\infty}(-1)^n \left[\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)\right]\Big|_{t\to t-n} \cdot \mathcal{U}(t-n) \\
&= t + 2\sum_{n=1}^{\infty}(-1)^n \cdot t\Big|_{t\to t-n} \cdot \mathcal{U}(t-n) \\
&= t + 2\sum_{n=1}^{\infty}(-1)^n \cdot (t-n) \cdot \mathcal{U}(t-n) \\
&= t - 2(t-1)\mathcal{U}(t-1) + 2(t-2)\mathcal{U}(t-2) - 2(t-3)\mathcal{U}(t-3) + \cdots \\
&= \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ t-2 & 2 \le t < 3 \\ 4-t & 3 \le t < 4 \\ t-4 & 4 \le t < 5 \\ \vdots & \end{cases}
\end{aligned}
$$

Therefore, our unique solution is $y(t) = \begin{cases} t-2k & 2k \le t < 2k+1 \text{ for some integer } k \\ 2k+2-t & 2k+1 \le t < 2k+2 \text{ for some integer } k \end{cases}$

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3.4: Integro-Differential Equations

Learning Objectives

- Define the convolution integral and understand its properties.
- Prove and apply the convolution theorem for Laplace transforms.
- Use convolution to find inverse Laplace transforms of products.
- Solve integro-differential equations using Laplace transforms.

We explore the concept of convolution, a mathematical operation that represents "blending" two functions. The Convolution Theorem simplifies finding inverse transforms of products and provides the key to solving integro-differential equations, which model systems with memory effects.

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3.4: Convolutions and Laplace

In calculus we study a myriad of ways to make new functions out of old ones: addition, subtraction, multiplication, division, composition. When you have two functions with common domain $[0, \infty)$, we define another way to combine them: *convolution*.

Definition: Convolution

For any two functions $f(t)$ and $g(t)$ with domain $[0, \infty)$, we define the ***convolution*** of f and g to be the function

$$(f \star g)(t) = \int_0^t f(x) \cdot g(t-x)\, dx$$

The convolution of two functions is a specific way of "blending" or "averaging" $f(t)$ and $g(t)$ over the interval $[0, t]$.

Example 3.4.1

Find the convolution of each pair of functions below.

(a) $f(t) = t$ and $g(t) = e^t$

(b) $f(t) = 2\sin(t)$ and $g(t) = e^{-t}$

Solution

(a) We set up the convolution integral and evaluate:

$$\begin{aligned}(f \star g)(t) &= \int_0^t f(x) \cdot g(t-x)\, dx \\ &= \int_0^t x \cdot e^{t-x}\, dx \\ &= e^t \int_0^t x e^{-x}\, dx \\ &= e^t \cdot (-x-1)e^{-x}\Big|_0^t \\ &= e^t \cdot ((-t-1)e^{-t} + 1) \\ &= e^t - 1 - t\end{aligned}$$

Therefore, we have $\boxed{(f \star g)(t) = e^t - 1 - t}$

(b) We set up the convolution integral and evaluate:

$$\begin{aligned}(f \star g)(t) &= \int_0^t f(x) \cdot g(t-x)\, dx \\ &= \int_0^t 2\sin(x) \cdot e^{-(t-x)}\, dx \\ &= e^{-t} \int_0^t 2e^x \sin(x)\, dx \\ &= e^{-t} \cdot (\sin(x) - \cos(x))\, e^x\Big|_0^t \\ &= e^{-t} \cdot ((\sin(t) - \cos(t))\, e^t + 1) \\ &= \sin(t) - \cos(t) + e^{-t}\end{aligned}$$

Therefore, we have $\boxed{(f \star g)(t) = \sin(t) - \cos(t) + e^{-t}}$

There are many properties that the convolution function inherits from the integral definition.

Properties of Convolutions

For all functions $f(t), g(t)$ and $h(t)$ defined on $[0, \infty)$, we must have:

1. $f \star g = g \star f$
2. $f \star (g \star h) = (f \star g) \star h$
3. $f \star (g + h) = (f \star g) + (f \star h)$
4. $f \star 0 = 0$

The proof of the properties above (all properties follow by the linearity of integrals) is left as an exercise to the reader.

The reason we care about convolutions, is because of how they interact with the Laplace Transformation.

Theorem 3.4.1: Laplace Transformation Convolution Theorem

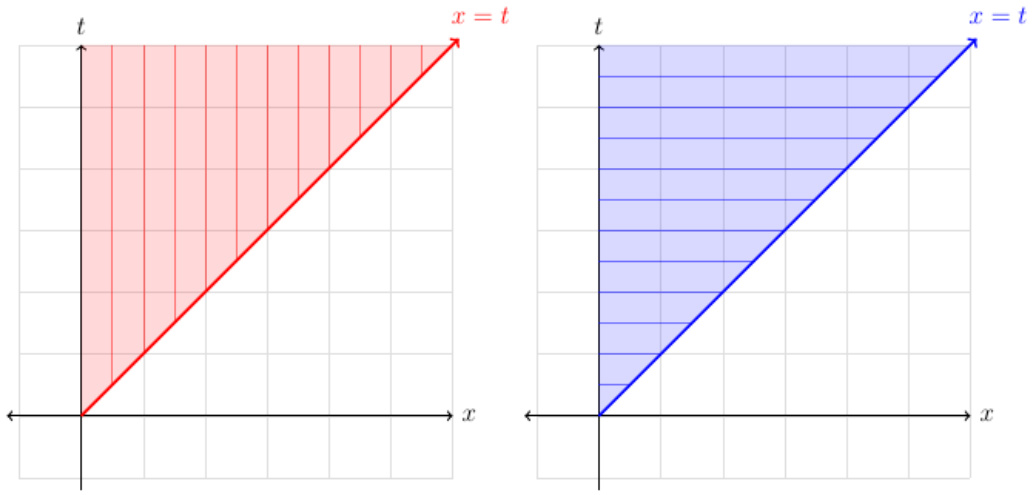
If $f(t)$ and $g(t)$ are functions defined on $[0, \infty)$ with convergent Laplace Transformations, then

$$\mathscr{L}\left((f \star g)(t)\right) = \mathscr{L}\left(f(t)\right) \cdot \mathscr{L}\left(g(t)\right)$$

To prove this statement, we will use a little bit of knowledge from multivariable integration.

$$\begin{aligned}
\mathscr{L}\left(f(t)\right) \cdot \mathscr{L}\left(g(t)\right) &= \int_0^\infty e^{-sx} f(x)\, dx \cdot \int_0^\infty e^{-sy} g(y)\, dy \\
&= \int_0^\infty \int_0^\infty e^{-s(x+y)} f(x) g(y)\, dy\, dx \\
&= \int_0^\infty \int_x^\infty e^{-st} f(x) g(t-x)\, dt\, dx
\end{aligned}$$

Above, we used the substitution $t = x + y$ to change the inner integral. Notice that the region we are now integrating on in the xt -plane looks like:



Instead of describing this region as $\{(x,t) \mid 0 \le x < \infty,\, x \le t < \infty\}$ (in red above, each fixed value of x corresponds to t values between x and $+\infty$), we can say $\{(x,t) \mid 0 \le t < \infty,\, 0 \le x \le t\}$ (in blue above, each fixed value of t corresponds to x values between 0 and t).

This means when we switch the order of integration in the integral above, it becomes:

$$\begin{aligned}\int_0^\infty \int_x^\infty e^{-sw} f(x) g(w-x)\, dw\, dx &= \int_0^\infty \int_0^w e^{-sw} f(x) g(w-x)\, dx\, dw \\ &= \int_0^\infty \left(\int_0^w e^{-sw} f(x) g(w-x)\, dx \right) dw \\ &= \int_0^\infty e^{-sw} (f \star g)(w)\, dw \\ &= \mathscr{L}\{f \star g\}\end{aligned}$$

This officially proves that the Laplace Transformation of a convolution of two functions will be equal to the product of their individual Laplace Transforms.

Example 3.4.2

Evaluate the Laplace Transformation of each function below using the Convolution Theorem.

(a) $a(t) = e^{2t} \star e^{-t}$

(b) $b(t) = \cos(t) \star \sin(t)$

Solution

(a) According to the Convolution Theorem, we have

$$\mathscr{L}\left\{e^{2t} \star e^{-t}\right\} = \mathscr{L}\left\{e^{2t}\right\} \cdot \mathscr{L}\left\{e^{-t}\right\} = \frac{1}{s-2} \cdot \frac{1}{s+1} = \frac{1}{(s-2)(s+1)}$$

Clearly, we have $\boxed{A(s) = \dfrac{1}{(s-2)(s+1)}}$. We can check this answer by manually calculating $e^{2t} \star e^{-t}$:

$$a(t) = \int_0^t e^{2x} \cdot e^{-(t-x)}\, dx = e^{-t} \int_0^t e^{3x}\, dx = e^{-t} \cdot \frac{1}{3} e^{3x} \Big|_0^t = e^{-t} \cdot \left(\frac{1}{3} e^{3t} - \frac{1}{3} \right) = \frac{1}{3} e^{2t} - \frac{1}{3} e^{-t}$$

Notice that we have $A(s) = \mathscr{L}\left(\frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}\right) = \frac{1}{3}\cdot\frac{1}{s-2} - \frac{1}{3}\cdot\frac{1}{s+1} = \frac{1}{(s-2)(s+1)}$

(b) According to the Convolution Theorem, we have

$$\mathscr{L}\left(\cos(t)\star\sin(t)\right) = \mathscr{L}\left(\cos(t)\right)\cdot\mathscr{L}\left(\sin(t)\right) = \frac{s}{s^2+1}\cdot\frac{1}{s^2+1} = \frac{s}{(s^2+1)^2}$$

Clearly, we have $\boxed{B(s) = \frac{s}{(s^2+1)^2}}$. We can check this answer by manually calculating $\cos(t)\star\sin(t)$:

$$b(t) = \int_0^t \cos(x)\cdot\sin(t-x)\,dx = \int_0^t \cos^2(x)\sin(t) - \cos(t)\sin(x)\cos(x)\,dx = \left[\sin(t)\cdot\left(\frac{1}{2}x + \frac{1}{4}\sin(2x)\right) - \cos(t)\cdot\frac{1}{2}\sin^2(x)\right]\Bigg|_0^t =$$

Notice that we have $B(s) = \mathscr{L}\left(\frac{1}{2}t\sin(t)\right) = -\frac{d}{ds}\left(\frac{1}{2}\mathscr{L}\left(\sin(t)\right)\right) = -\frac{1}{2}\cdot\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{1}{2}\cdot\frac{2s}{(s^2+1)^2} = \frac{s}{(s^2+1)^2}$

As always, we can use the Convolution Theorem to help with Inverse Laplace Transformations as well.

Example 3.4.3

Evaluate the Inverse Laplace Transformation of each function below.

(a) $A(s) = \dfrac{1}{(s^2+4)^2}$

(b) $B(s) = \dfrac{\tan^{-1}\left(\frac{1}{s}\right)}{s}$

(c) $C(s) = \dfrac{108}{s(s-2)^2(s+1)^3}$

Solution

(a) We write $A(s) = \frac{1}{s^2+4}\cdot\frac{1}{s^2+4}$ and solve:

$$\begin{aligned}
\mathscr{L}^{-1}\left(A(s)\right) &= \mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right)\star\mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right)\\
&= \frac{1}{2}\sin(2t)\star\frac{1}{2}\sin(2t)\\
&= \frac{1}{4}\int_0^t \sin(2x)\cdot\sin(2t-2x)\,dx\\
&= \frac{1}{4}\int_0^t \sin(2x)\sin(2t)\cos(2x) - \sin^2(2x)\cos(2t)\,dx\\
&= \frac{1}{4}\left[\sin(2t)\cdot\frac{1}{4}\sin^2(2x) - \cos(2t)\cdot\left(\frac{1}{2}x - \frac{1}{8}\sin(4x)\right)\right]\Bigg|_0^t\\
&= \frac{1}{16}\sin(2t) - \frac{1}{8}t\cos(2t)
\end{aligned}$$

Thus, $\boxed{a(t) = \frac{1}{16}\sin(2t) - \frac{1}{8}t\cos(2t)}$

(b) We write $B(s) = \tan^{-1}\left(\frac{1}{s}\right)\cdot\frac{1}{s}$. Taking $\mathscr{L}^{-1}\left(\frac{1}{s}\right) = 1$ is straightforward.

Notice that for $D(s) = \tan^{-1}\left(\frac{1}{s}\right)$ we have $D'(s) = \frac{1}{1+s^{-2}}\cdot\left(-\frac{1}{s^2}\right) = -\frac{1}{s^2+1} = \mathscr{L}\left(-\sin(t)\right)$.

This means that $-td(t) = -\sin(t) \implies d(t) = \dfrac{\sin(t)}{t}$. Now we can take their convolution:

$$\mathscr{L}^{-1}\left(B(s)\right) = \mathscr{L}^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right) \star \mathscr{L}^{-1}\left(\frac{1}{s}\right) = \frac{\sin(t)}{t} \star 1 = \int_0^t \frac{\sin(x)}{x}\,dx$$

This is a famously unsolvable integral, called the *sine integral function,* so our answer is $\boxed{b(t) = \displaystyle\int_0^t \frac{\sin(x)}{x}\,dx}$

(c) While partial fractions is a viable option here, it will require us to solve a system of six unknowns.

Instead we write $C(s) = \dfrac{108}{s} \cdot \dfrac{1}{(s-2)^2} \cdot \dfrac{1}{(s+1)^3}$ and use a triple convolution.

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{(s-2)^2} \cdot \frac{1}{(s+1)^3}\right) &= \mathscr{L}^{-1}\left(\frac{1}{(s-2)^2}\right) \star \mathscr{L}^{-1}\left(\frac{1}{(s+1)^3}\right) \\
&= te^{2t} \star \frac{1}{2}t^2e^{-t} \\
&= \frac{1}{2}\int_0^t xe^{2x} \cdot (t-x)^2 e^{-(t-x)}\,dx \\
&= \frac{1}{2}e^{-t}\int_0^t (t^2x - 2tx^2 + x^3)e^{3x}\,dx \\
&= \frac{1}{2}e^{-t} \cdot \left(\frac{1}{3}t^2x - \frac{1}{9}t^2 - \frac{2}{3}tx^2 + \frac{4}{3}tx - \frac{4}{27}t + 9x^3 - 9x^2 + 6x - 2\right)e^{3x}\Big|_0^t \\
&= \frac{1}{54}\left((3t^2 + 4t + 2)e^{-t} + (2t-2)e^{2t}\right)
\end{aligned}$$

This gives us:

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{108}{s} \cdot \frac{1}{(s-2)^2(s+1)^3}\right) &= \mathscr{L}^{-1}\left(\frac{108}{s}\right) \star \mathscr{L}^{-1}\left(\frac{1}{(s-2)^2(s+1)^3}\right) \\
&= 108 \star \frac{1}{54}\left((3t^2 + 4t + 2)e^{-t} + (2t-2)e^{2t}\right) \\
&= \int_0^t (6x^2 + 8x + 4)e^{-x} + (4x-4)e^{2x}\,dx = \left[(-6x^2 - 20x - 24)e^{-x} + (2x-3)e^{2x}\right]\Big|_0^t \\
&= 27 - 3e^{2t} + 2te^{2t} - 24e^{-t} - 20te^{-t} - 6t^2e^{-t}
\end{aligned}$$

Our final answer is therefore $\boxed{c(t) = 27 - 3e^{2t} + 2te^{2t} - 24e^{-t} - 20te^{-t} - 6t^2e^{-t}}$

This is the exact result you would have gotten using partial fraction decomposition, since:

$$\frac{108}{s(s-2)^2(s+1)^3} = \frac{27}{s} - \frac{3}{s-2} + \frac{2}{(s-2)^2} - \frac{24}{s+1} - \frac{20}{(s+1)^2} - \frac{12}{(s+1)^3}$$

While this text has focused mainly on ordinary differential equations, we now have the tools to solve equations involving *integrals* of an unknown function as well. We treat this situation just like we would for a differential equation: Laplace everything, solve for the unknown function algebraically, and then Inverse Laplace.

Illustration 3.4.1

A simple example is the **integro-differential equation** defined by $y(t) = t - \int_0^t y(x)e^{-x}\,dx$. Since the integral term can be written as a convolution, we see:

$$\mathscr{L}\left(y(t)\right) = \mathscr{L}(t) - \mathscr{L}\left(y(t) \star e^{-t}\right)$$
$$\Downarrow$$
$$Y(s) = \frac{1}{s^2} - Y(s) \cdot \frac{1}{s+1}$$
$$\Downarrow$$
$$Y(s) = \frac{1}{s(s+1)}$$

Taking the inverse Laplace Transformation of this gives the solution to this equation:

$$y(t) = \mathscr{L}^{-1}\left(\frac{1}{s(s+1)}\right) = \mathscr{L}^{-1}\left(\frac{1}{s}\right) - \mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = 1 - e^{-t}$$

Whenever you see an integral inside of your differential equation, think convolutions!

Example 3.4.4

Solve the following Integro-Differential Equations by using the Laplace Transformation.

(a) $y(t) + \int_0^t (4t - 4x)y(x)\,dx = 4t$

(b) $y'(t) + 2y(t) + \int_0^t y(x)\,dx = \mathcal{U}(t-1)$, with $y(0) = 1$

Solution

(a) We recognize the integral $\int_0^t (4t - 4x)y(x)\,dx = 4t \star y(t)$ and take the Laplace Transformation of the entire equation:

$$\mathscr{L}\left(y(t)\right) + \mathscr{L}\left(4t \star y(t)\right) = \mathscr{L}(4t)$$
$$\Downarrow$$
$$Y(s) + \frac{4}{s^2} \cdot Y(s) = \frac{4}{s^2}$$
$$\Downarrow$$
$$Y(s) = \frac{4}{s^2+4}$$

Taking the Inverse Laplace Transformation gives us $y(t) = \mathscr{L}^{-1}\left(\frac{4}{s^2+4}\right) = 2\sin(2t)$

Therefore, the unique solution to this equation is $\boxed{y(t) = 2\sin(2t)}$

(b) We see the integral $\int_0^t y(x)\,dx = y(t) \star 1$ and take the Laplace Transformation of the entire equation:

$$\mathscr{L}\left(y'(t)\right) + 2\mathscr{L}\left(y(t)\right) + \mathscr{L}\left(y(t) \star 1\right) = \mathscr{L}\left(\mathcal{U}(t-1)\right)$$
$$\Downarrow$$
$$sY(s) - y(0) + 2 \cdot Y(s) + Y(s) \cdot \frac{1}{s} = \frac{1}{s}e^{-s}$$
$$\Downarrow$$
$$\left(s + 2 + \frac{1}{s}\right) Y(s) = 1 + \frac{1}{s}e^{-s}$$
$$\Downarrow$$
$$Y(s) = \frac{s}{(s+1)^2} + \frac{1}{(s+1)^2}e^{-s}$$

Using partial fraction decomposition, we can rewrite $\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$, giving us:

$$
\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\
&= \mathscr{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right) + \mathscr{L}^{-1}\left(\left(\frac{1}{(s+1)^2}\right)e^{-s}\right) \\
&= e^{-t} - te^{-t} + \left[\mathscr{L}^{-1}\left(\frac{1}{(s+1)^2}\right)\right]_{t\to t-1} \cdot \mathcal{U}(t-1) \\
&= (1-t)e^{-t} + \left[te^{-t}\right]_{t\to t-1} \cdot \mathcal{U}(t-1) \\
&= (1-t)e^{-t} + (t-1)e^{-(t-1)}\mathcal{U}(t-1)
\end{aligned}
$$

Putting this into piecewise form gives our unique solution $\boxed{y(t) = \begin{cases} (1-t)e^{-t} & 0 \le t < 1 \\ (1-e)(1-t)e^{-t} & t \ge 1 \end{cases}}$

This subchapter demonstrates how convolution provides a powerful tool for handling products of transforms and for solving equations that involve both derivatives and integrals, which are a common occurrence in applications involving memory effects or cumulative processes.

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3.4: Practice Set

Exercise 3.4.1a

Find the Laplace Transformation of the function $a(t) = \int_0^t (t-x)^3 e^{-4x}\,dx$

Answer available at the end of this section.

Exercise 3.4.1b

Find the Laplace Transformation of the function $b(t) = \int_0^t (t-x)e^{2x}\sin(3x)\cos(t-x)\,dx$

Answer available at the end of this section.

Exercise 3.4.1c

Find the Laplace Transformation of the function $c(t) = \int_0^t (t-x)^2 e^{t-x}\sin(2x)\,dx$

Answer available at the end of this section.

Exercise 3.4.1d

Find the Laplace Transformation of the function $d(t) = \int_0^t (t-x)e^{t-2x}\,dx$

Answer available at the end of this section.

Exercise 3.4.1e

Find the Laplace Transformation of the function $e(t) = \int_0^t \left((t-x)^7 - 42(t-x)^5\right)\,dx$

Answer available at the end of this section.

Exercise 3.4.1f

Find the Laplace Transformation of the function $f(t) = \int_0^t x^2 \sin\left(x\sqrt{2}\right)\cos^2(t-x)\,dx$

Answer available at the end of this section.

Exercise 3.4.2a

Use the properties of the Laplace Transformation to exactly evaluate the integral $\int_0^t 2e^x \sin(t-x)\,dx$

Answer available at the end of this section.

Exercise $3.4.2b$

Use the properties of the Laplace Transformation to exactly evaluate the integral $\int_0^t \cos(2x)\sin(2t-2x)\,dx$

Answer available at the end of this section.

Exercise $3.4.2c$

Use the properties of the Laplace Transformation to exactly evaluate the integral $\int_0^t (t-x)^n \cdot \mathcal{U}(t-a)\,dx$

Answer available at the end of this section.

Exercise $3.4.2d$

Use the properties of the Laplace Transformation to exactly evaluate the integral $\int_0^t \mathcal{U}(t-x-a)\cdot\mathcal{U}(x-b)\,dx$

Answer available at the end of this section.

Exercise $3.4.2e$

Use the properties of the Laplace Transformation to exactly evaluate the integral $\int_0^1 x^n(1-x)^m\,dx$

Answer available at the end of this section.

Exercise $3.4.3a$

Find the Inverse Laplace Transformation of $A(s) = \dfrac{10}{(s-1)(s+4)}$

Answer available at the end of this section.

Exercise $3.4.3b$

Find the Inverse Laplace Transformation of $B(s) = \dfrac{16}{s^3(s-2)^2}$

Answer available at the end of this section.

Exercise $3.4.3c$

Find the Inverse Laplace Transformation of $C(s) = \dfrac{2ks}{(s^2+k^2)^2}$

Answer available at the end of this section.

Exercise 3.4.3d

Find the Inverse Laplace Transformation of $D(s) = \dfrac{2k^3}{(s^2+k^2)^2}$

Answer available at the end of this section.

Exercise 3.4.4a

Find the solution to the integro-differential equation $y' + \int_0^t e^{2t-2x} y(x)\,dx = e^{2t}$ with $y(0) = 0$

Answer available at the end of this section.

Exercise 3.4.4b

Find the solution to the integro-differential equation $\int_0^t y(t)y(t-x)\,dx = 6t^3$

Answer available at the end of this section.

Exercise 3.4.4c

Find the solution to the integro-differential equation $y'' - \int_0^t 2\sin(t)y(t-x)\,dx = 2t^2 + 2$ with $y(0) = 0,\ y'(0) = 0$

Answer available at the end of this section.

Exercise 3.4.4d

Find the solution to the integro-differential equation $\int_0^t y'(x)e^{t-x}\,dx - 2\int_0^t y(t-x)\,dx = 2$ with $y(0) = 2$

Answer available at the end of this section.

Exercise 3.4.4e

Find the solution to the integro-differential equation $y''(t) + 5y'(t) + 20 = 2e^{3t} + 6y(t) - \int_0^t 2y(x)e^{3t-3x}\,dx$ with $y(0) = 3,\ y'(0) = 0$

Answer available at the end of this section.

Answers

Exercise 3.4.1a

We have

$$\begin{aligned}\mathscr{L}\left(a(t)\right) &= \mathscr{L}\left(t^3 \star e^{-4t}\right)\\ &= \mathscr{L}\left(t^3\right)\cdot\mathscr{L}\left(e^{-4t}\right)\\ &= \frac{3!}{s^4}\cdot\frac{1}{s+4}\end{aligned}$$

Therefore, $A(s) = \dfrac{6}{(s^4(s+4)}$

Exercise 3.4.1*b*

We have

$$\begin{aligned}
\mathscr{L}\left(b(t)\right) &= \mathscr{L}\left(e^{2t}\sin(3t) \star t\cos(t)\right) \\
&= \mathscr{L}\left(e^{2t}\sin(3t)\right) \cdot \mathscr{L}\left(t\cos(t)\right) \\
&= \mathscr{L}\left(\sin(3t)\right)\Big|_{s\to s-2} \cdot \left(-\frac{d}{ds}\left[\mathscr{L}\left(\cos(t)\right)\right]\right) \\
&= \frac{3}{s^2+9}\Big|_{s\to s-2} \cdot \left(-\frac{d}{ds}\left[\frac{s}{s^2+1}\right]\right) \\
&= \frac{3}{(s-2)^2+9} \cdot \frac{s^2-1}{(s^2+1)^2}
\end{aligned}$$

Therefore, $B(s) = \dfrac{3s^2-3}{(s^2-4s+13)(s^2+1)^2}$

Exercise 3.4.1*c*

We have

$$\begin{aligned}
\mathscr{L}\left(c(t)\right) &= \mathscr{L}\left(t^2e^t \star \sin(2t)\right) \\
&= \mathscr{L}\left(t^2e^t\right) \cdot \mathscr{L}\left(\sin(2t)\right) \\
&= \mathscr{L}\left(t^2\right)\Big|_{s\to s-1} \cdot \frac{2}{s^2+4} \\
&= \frac{2}{s^3}\Big|_{s\to s-1} \cdot \frac{2}{s^2+4} \\
&= \frac{2}{(s-1)^3} \cdot \frac{2}{s^2+4}
\end{aligned}$$

Therefore, $C(s) = \dfrac{4}{(s-1)^3(s^2+4)}$

Exercise 3.4.1*d*

Notice that $(t-x)e^{t-2x} = (t-x)e^{t-x} \cdot e^{-x}$

$$\begin{aligned}
\mathscr{L}\left(d(t)\right) &= \mathscr{L}\left(te^t \star e^{-t}\right) \\
&= \mathscr{L}\left(te^t\right) \cdot \mathscr{L}\left(e^{-t}\right) \\
&= \mathscr{L}\left(t\right)\Big|_{s\to s-1} \cdot \frac{1}{s+1} \\
&= \frac{1}{s^2}\Big|_{s\to s-1} \cdot \frac{1}{s+1} \\
&= \frac{1}{(s-1)^2} \cdot \frac{1}{s+1}
\end{aligned}$$

Therefore, $D(s) = \dfrac{1}{(s-1)^2(s+1)}$

Exercise 3.4.1*e*

We have

$$\begin{aligned}
\mathscr{L}\left(e(t)\right) &= \mathscr{L}\left(\left(t^7-42t^5\right) \star 1\right) \\
&= \mathscr{L}\left(t^7-42t^5\right) \cdot \mathscr{L}\left(1\right) \\
&= \left(\frac{7!}{s^8} - 42 \cdot \frac{5!}{s^7}\right) \cdot \frac{1}{s} \\
&= \frac{7!(1-s^2)}{s^8} \cdot \frac{1}{s}
\end{aligned}$$

Therefore, $E(s) = \dfrac{7!(1-s^2)}{s^9}$

Exercise 3.4.1*f*

We have

$$\begin{aligned}
\mathscr{L}\left(f(t)\right) &= \mathscr{L}\left(t^2\sin\left(t\sqrt{2}\right) \star \left[\frac{1}{2} + \frac{1}{2}\cos(2t)\right]\right) \\
&= \mathscr{L}\left(t^2\sin\left(t\sqrt{2}\right)\right) \cdot \frac{1}{2}\mathscr{L}\left(1+\cos(2t)\right) \\
&= \frac{d^2}{ds^2}\left[\mathscr{L}\left(\sin\left(t\sqrt{2}\right)\right)\right] \cdot \frac{1}{2} \cdot \left(\frac{1}{s} + \frac{s}{s^2+4}\right) \\
&= \frac{d^2}{ds^2}\left[\frac{\sqrt{2}}{s^2+2}\right] \cdot \frac{s^2+2}{s^2+4} \\
&= \frac{2\sqrt{2}(3s^2-2)}{(s^2+2)^3} \cdot \frac{s^2+2}{s^2+4}
\end{aligned}$$

Therefore, $F(s) = \dfrac{2\sqrt{2}(3s^2-2)}{(s^2+2)^2(s^2+4)}$

Exercise 3.4.2*a*

First, we take the Laplace Transformation of $f(t) = \displaystyle\int_0^t 2e^x\sin(t-x)\,dx$:

$$\begin{aligned} F(s) &= \mathscr{L}\left(f(t)\right) \\ &= \mathscr{L}\left(2e^t \star \sin(t)\right) \\ &= \mathscr{L}\left(2e^t\right) \cdot \mathscr{L}\left(\sin(t)\right) \\ &= \frac{2}{s-1} \cdot \frac{1}{s^2+1} \\ &= \frac{2}{(s-1)(s^2+1)} \\ &= \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1} \end{aligned}$$

Taking the Inverse Laplace Transformation gives us back $f(t)$:

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left(F(s)\right) \\ &= \mathscr{L}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right) \\ &= \mathscr{L}\left(\frac{1}{s-1}\right) - \mathscr{L}\left(\frac{s}{s^2+1}\right) - \mathscr{L}\left(\frac{1}{s^2+1}\right) \\ &= e^t - \cos(t) - \sin(t) \end{aligned}$$

Therefore, the integral is $\displaystyle\int_0^t 2e^x \sin(t-x)\,dx = e^t - \cos(t) - \sin(t)$

Exercise 3.4.2*b*

First, we take the Laplace Transformation of $f(t) = \displaystyle\int_0^t \cos(2x)\sin(2t-2x)\,dx$:

$$\begin{aligned} F(s) &= \mathscr{L}\left(f(t)\right) \\ &= \mathscr{L}\left(\cos(2t) \star \sin(2t)\right) \\ &= \mathscr{L}\left(\cos(2t)\right) \cdot \mathscr{L}\left(\sin(2t)\right) \\ &= \frac{s}{s^2+4} \cdot \frac{2}{s^2+4} \\ &= \frac{2s}{(s^2+4)^2} \end{aligned}$$

Taking the Inverse Laplace Transformation gives us back $f(t)$:

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left(F(s)\right) \\ &= \mathscr{L}\left(\frac{2s}{(s^2+4)^2}\right) \\ &= -t \cdot \mathscr{L}\left(\int \frac{2s}{(s^2+4)^2}\,ds\right) \\ &= -t \cdot \mathscr{L}\left(-\frac{1}{s^2+4}\right) \\ &= t \cdot \frac{1}{2}\sin(2t) \end{aligned}$$

Therefore, the integral is $\displaystyle\int_0^t \cos(2x)\sin(2t-2x)\,dx = \frac{1}{2}t\sin(2t)$

Exercise 3.4.2*c*

First, we take the Laplace Transformation of $f(t) = \displaystyle\int_0^t (t-x)^n \cdot \mathcal{U}(t-a)\,dx$:

$$\begin{aligned} F(s) &= \mathscr{L}\left(f(t)\right) \\ &= \mathscr{L}\left(t^n \star \mathcal{U}(t-a)\right) \\ &= \mathscr{L}\left(t^n\right) \cdot \mathscr{L}\left(\mathcal{U}(t-a)\right) \\ &= \left(\frac{n!}{s^{n+1}}\right) \cdot \frac{1}{s} e^{-as} \\ &= \frac{n!}{s^{n+2}} e^{-as} \end{aligned}$$

Taking the Inverse Laplace Transformation gives us back $$f(t)$:

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left(F(s)\right) \\ &= n! \cdot \mathscr{L}\left(\frac{1}{s^{n+2}} e^{-as}\right) \\ &= n! \cdot \mathscr{L}\left(\frac{1}{s^{n+2}}\right)\Big|_{t\to t-a} \cdot \mathcal{U}(t-a) \\ &= n! \cdot \frac{t^{n+1}}{(n+1)!}\Big|_{t\to t-a} \cdot \mathcal{U}(t-a) \\ &= \frac{1}{n+1}(t-a)^{n+1} \cdot \mathcal{U}(t-a) \end{aligned}$$

Therefore, the integral is: $\displaystyle\int_0^t (t-x)^n \cdot \mathcal{U}(t-a)\,dx = \begin{cases} 0 & 0 \le t < a \\ \frac{1}{n+1}(t-a)^{n+1} & t \ge a \end{cases}$

Exercise 3.4.2*d*

First, we take the Laplace Transformation of $f(t) = \displaystyle\int_0^t \mathcal{U}(t-x-a) \cdot \mathcal{U}(x-b)\,dx$:

$$\begin{aligned} F(s) &= \mathscr{L}\left(f(t)\right) \\ &= \mathscr{L}\left(\mathcal{U}(t-a) \star \mathcal{U}(t-b)\right) \\ &= \mathscr{L}\left(\mathcal{U}(t-a)\right) \cdot \mathscr{L}\left(\mathcal{U}(t-b)\right) \\ &= \frac{1}{s} e^{-as} \cdot \frac{1}{s} e^{-bs} \\ &= \frac{1}{s^2} e^{-(a+b)s} \end{aligned}$$

Taking the Inverse Laplace Transformation gives us back $f(t)$:

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left(F(s)\right) \\ &= \mathscr{L}\left(\frac{1}{s^2} e^{-(a+b)s}\right) \\ &= \mathscr{L}\left(\frac{1}{s^2}\right)\Big|_{t \to t-(a+b)} \cdot \mathcal{U}(t-(a+b)) \\ &= t\Big|_{t \to t-a-b} \cdot \mathcal{U}(t-a-b) \\ &= (t-a-b) \cdot \mathcal{U}(t-a-b) \end{aligned}$$

Therefore, the integral is $\displaystyle\int_0^t \mathcal{U}(t-x-a) \cdot \mathcal{U}(x-b)\, dx = \begin{cases} 0 & 0 \le t < a+b \\ t-a-b & t \ge a+b \end{cases}$

Exercise 3.4.2*e*

We want $f(1)$ for $f(t) = \displaystyle\int_0^t x^n (t-x)^m\, dx$

$$\begin{aligned} \mathscr{L}\left(f(t)\right) &= \mathscr{L}\left(t^n \star t^m\right) \\ &= \mathscr{L}\left(t^n\right) \cdot \mathscr{L}\left(t^m\right) \\ &= \frac{n!}{s^{n+1}} \cdot \frac{m!}{s^{m+1}} \\ &= \frac{n! \cdot m!}{s^{n+m+2}} \end{aligned}$$

Taking the Inverse Laplace Transformation gives us back $f(t)$:

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left(F(s)\right) \\ &= n! \cdot m! \cdot \mathscr{L}\left(\frac{1}{s^{n+m+2}}\right) \\ &= n! \cdot m! \cdot \frac{t^{n+m+1}}{(n+m+1)!} \end{aligned}$$

Therefore, we know $f(t) = \dfrac{n! \cdot m!}{(n+m+1)!} t^{n+m+1}$

Plugging in $t = 1$ gives $f(1) = \dfrac{n! \cdot m!}{(n+m+1)!}$

So, our desired integral is $\displaystyle\int_0^1 x^n (1-x)^m\, dx = \frac{n! \cdot m!}{(n+m+1)!}$. (*This is known as the Beta Integral.*)

Exercise 3.4.3*a*

We apply the Inverse Laplace Transformation:

$$\begin{aligned} \mathscr{L}^{-1}\left(A(s)\right) &= \mathscr{L}^{-1}\left(\frac{10}{s-1} \cdot \frac{1}{s+4}\right) \\ &= \mathscr{L}^{-1}\left(\frac{10}{s-1}\right) \star \mathscr{L}^{-1}\left(\frac{1}{s+4}\right) \\ &= 10e^t \star e^{-4t} \\ &= \int_0^t 10e^x \cdot e^{-4(t-x)}\, dx \\ &= 10e^{-4t} \int_0^t e^{5x}\, dx \\ &= 10e^{-4t} \cdot \frac{1}{5} e^{5x}\Big|_0^t \\ &= 2e^{-4t} \cdot \left(e^{5t} - 1\right) \\ &= 2e^t - 2e^{-4t} \end{aligned}$$

This lines up with the answer we would get by using partial fraction decomposition, it is not difficult to calculate:

$$\frac{10}{(s-1)(s+4)} = \frac{2}{s-1} - \frac{2}{s+4}$$

Exercise 3.4.3*b*

We apply the Inverse Laplace Transformation:

$$\begin{aligned}
\mathscr{L}^{-1}(B(s)) &= \mathscr{L}^{-1}\left(\frac{16}{s^3}\cdot\frac{1}{(s-2)^2}\right)\\
&= \mathscr{L}^{-1}\left(\frac{16}{s^3}\right)\star\mathscr{L}^{-1}\left(\frac{1}{(s-2)^2}\right)\\
&= 16\cdot\frac{t^2}{2!}\star e^{2t}\cdot\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)\\
&= 8t^2\star e^{2t}\cdot t\\
&= 8t^2\star te^{2t}\\
&= \int_0^t 8x^2\cdot(t-x)e^{2(t-x)}\,dx\\
&= 8e^{2t}\int_0^t\left(tx^2-x^3\right)e^{-2x}\,dx\\
&= 8e^{2t}\cdot\left(\frac{1}{2}x^3-\frac{1}{2}tx^2+\frac{3}{4}x^2-\frac{1}{2}tx+\frac{3}{4}x-\frac{1}{4}t+\frac{3}{8}\right)e^{-2x}\Big|_0^t\\
&= 8e^{2t}\cdot\left[\left(\frac{1}{4}t^2+\frac{1}{2}t+\frac{3}{8}\right)e^{-2t}+\frac{1}{4}t-\frac{3}{8}\right]\\
&= 2t^2+4t+3+2te^{2t}-3e^{2t}
\end{aligned}$$

This lines up with the answer we would get by using partial fraction decomposition. It is possible to calculate:

$$\frac{16}{s^3(s-2)^2}=\frac{4}{s^3}+\frac{4}{s^2}+\frac{3}{s}-\frac{3}{s-2}+\frac{2}{(s-2)^2}$$

Exercise 3.4.3*c*

We apply the Inverse Laplace Transformation:

$$\begin{aligned}
\mathscr{L}^{-1}(C(s)) &= \mathscr{L}^{-1}\left(\frac{2k}{s^2+k^2}\cdot\frac{s}{s^2+k^2}\right)\\
&= \mathscr{L}^{-1}\left(\frac{2k}{s^2+k^2}\right)\star\mathscr{L}^{-1}\left(\frac{s}{s^2+k^2}\right)\\
&= 2\sin(kt)\star\cos(kt)\\
&= \int_0^t 2\sin(kx)\cdot\cos(kt-kx)\,dx\\
&= 2\int_0^t\sin(kx)\cdot(\cos(kt)\cos(kx)+\sin(kt)\sin(kx))\,dx\\
&= 2\cos(kt)\int_0^t\sin(kx)\cos(kx)\,dx+2\sin(kt)\int_0^t\sin^2(kx)\,dx\\
&= 2\cos(kt)\,\frac{1}{2k}\sin^2(kx)\Big|_0^t+2\sin(kt)\left[\frac{1}{2}x-\frac{1}{4k}\sin(2kx)\right]\Big|_0^t\\
&= \frac{1}{2k}\cos(kt)\cdot\sin^2(kt)+2\sin(kt)\cdot\left(\frac{1}{2}t-\frac{1}{4k}\sin(2kt)\right)\\
&= \frac{1}{2k}\sin^2(kt)\cos(kt)+\sin(kt)\cdot\left(t-\frac{1}{2k}\sin(kt)\cos(kt)\right)\\
&= t\sin(kt)
\end{aligned}$$

This lines up with the answer we would get by using the anti-derivative property:

$$\mathscr{L}^{-1}\left(\frac{2ks}{(s^2+k^2)^2}\right)=-t\cdot\mathscr{L}^{-1}\left(\int\frac{2ks}{(s^2+k^2)^2}\,ds\right)=-t\cdot\mathscr{L}^{-1}\left(-\frac{k}{s^2+k^2}\right)=t\sin(kt)$$

Exercise 3.4.3*d*

We apply the Inverse Laplace Transformation:

$$\begin{aligned}
\mathscr{L}^{-1}(D(s)) &= \mathscr{L}^{-1}\left(\frac{k}{s^2+k^2}\cdot\frac{k}{s^2+k^2}\right)\\
&= \mathscr{L}^{-1}\left(\frac{2k^2}{s^2+k^2}\right)\star\mathscr{L}^{-1}\left(\frac{k}{s^2+k^2}\right)\\
&= 2k\sin(kt)\star\sin(kt)\\
&= \int_0^t 2k\sin(kx)\cdot\sin(kt-kx)\,dx\\
&= 2k\int_0^t\sin(kx)\cdot(\sin(kt)\cos(kx)-\cos(kt)\sin(kx))\,dx\\
&= 2k\sin(kt)\int_0^t\sin(kx)\cos(kx)\,dx-2k\cos(kt)\int_0^t\sin^2(kx)\,dx\\
&= 2k\sin(kt)\,\frac{1}{2k}\sin^2(kx)\Big|_0^t-2k\cos(kt)\left[\frac{1}{2}x-\frac{1}{4k}\sin(2kx)\right]\Big|_0^t\\
&= \sin(kt)\cdot\sin^2(kt)-\cos(kt)\cdot\left(kt-\frac{1}{2}\sin(2kt)\right)\\
&= \sin^3(kt)-\cos(kt)\cdot(kt-\sin(kt)\cos(kt))\\
&= \sin(kt)\left(1-\cos^2(kt)\right)-kt\cos(kt)+\sin(kt)\cos^2(kt)\\
&= \sin(kt)-kt\cos(kt)
\end{aligned}$$

This Inverse Laplace Transformation is very difficult to solve without using convolutions.

Exercise 3.4.4*a*

Note that $\displaystyle\int_0^t e^{2t-2x}y(x)\,dx=e^t\star y(t)$.

We take the Laplace Transformation of each term:

$$\mathscr{L}(y') = -y(0) + sY(s) = sY(s)$$
$$\mathscr{L}\left(e^{2t} \star y(t)\right) = \frac{1}{s-2} \cdot Y(s)$$
$$\mathscr{L}\left(e^{2t}\right) = \frac{1}{s-2}$$

Therefore, the differential equation becomes:

$$sY(s) + \frac{1}{s-2} \cdot Y(s) = \frac{1}{s-2}$$
$$\Downarrow$$
$$Y(s) = \frac{1}{(s-1)^2}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}(Y(s)) \\ &= \mathscr{L}^{-1}\left(\frac{1}{(s-1)^2}\right) \\ &= e^t \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) \\ &= e^t \cdot t \end{aligned}$$

So, our unique solution is $y(t) = te^t$

Exercise 3.4.4*b*

Note that $\int_0^t y(t)y(t-x)\,dx = y(t) \star y(t)$.

We take the Laplace Transformation of each term:

$$\mathscr{L}(y(t) \star y(t)) = Y(s) \cdot Y(s) = [Y(s)]^2$$
$$\mathscr{L}\left(6t^3\right) = 6 \cdot \frac{3!}{s^4} = \frac{36}{s^4}$$

Therefore, the equation becomes:

$$[Y(s)]^2 = \frac{36}{s^4}$$
$$\Downarrow$$
$$Y(s) = \pm\frac{6}{s^2}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}(Y(s)) \\ &= \pm 6 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) \\ &= \pm 6 \cdot t \end{aligned}$$

So, there are two solutions: $y(t) = 6t$ and $y(t) = -6t$

Exercise 3.4.4*c*

Note that $\int_0^t 2\sin(t)y(t-x)\,dx = 2\sin(t) \star y(t)$.

We take the Laplace Transformation of each term:

$$\mathscr{L}(y'') = -y'(0) - sy(0) + s^2Y(s) = s^2Y(s)$$
$$\mathscr{L}(2\sin(t) \star y(t)) = \frac{2}{s^2+1} \cdot Y(s)$$
$$\mathscr{L}\left(2t^2 + 2\right) = \frac{4}{s^3} + \frac{2}{s} = \frac{4+2s^2}{s^3}$$

Therefore, the differential equation becomes:

$$s^2Y(s) - 2\frac{1}{s^2+1} \cdot Y(s) = \frac{4+2s^2}{s^3}$$
$$\Downarrow$$
$$Y(s) = \frac{2(s^2+1)}{s^3(s-1)(s+1)}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}(Y(s)) \\ &= \mathscr{L}^{-1}\left(\frac{2}{s+1} + \frac{2}{s-1} - \frac{2}{s^3} - \frac{4}{s}\right) \\ &= 2\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) + 2\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) - 2\mathscr{L}^{-1}\left(\frac{1}{s^3}\right) - 4\mathscr{L}^{-1}\left(\frac{1}{s}\right) \\ &= 2 \cdot e^{-t} + 2 \cdot e^t - 2 \cdot \frac{1}{2!}t^2 - 4 \cdot 1 \end{aligned}$$

So, the unique solution is $y(t) = 2e^{-t} + 2e^t - t^2 - 4$

Exercise 3.4.4*d*

Note that $\int_0^t y'(x)e^{t-x}\,dx = y'(t) \star e^t$ and $\int_0^t y(t-x)\,dx = 1 \star y(t)$.

We take the Laplace Transformation of each term:

$$\mathscr{L}\left(y'(t) \star e^t\right) = (-y(0) + sY(s)) \cdot \frac{1}{s-1} = \frac{s}{s-1}Y(s) - \frac{2}{s-1}$$
$$\mathscr{L}\left(1 \star y(t)\right) = \frac{1}{s} \cdot Y(s) = \frac{1}{s}Y(s)$$
$$\mathscr{L}\left(2\right) = \frac{2}{s}$$

Therefore, the differential equation becomes:

$$\frac{s}{s-1}Y(s) - \frac{2}{s-1} - 2 \cdot \frac{1}{s}Y(s) = \frac{2}{s}$$
$$\Downarrow$$
$$Y(s) = \frac{4s-2}{s^2-2s+2}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\ &= \mathscr{L}^{-1}\left(\frac{4s-2}{s^2-2s+2}\right) \\ &= e^t \cdot \mathscr{L}^{-1}\left(\frac{4s+2}{s^2+1}\right) \\ &= e^t \cdot \left(4\cos(t) + 2\sin(t)\right) \end{aligned}$$

Thus, our unique solution is $y(t) = 4e^t\cos(t) + 2e^t\sin(t)$

Exercise $3.4.4e$

Note that $\int_0^t 2y(x)e^{3t-3x}\,dx = y(t) \star 2e^{3t}$.

We take the Laplace Transformation of each term:

$$\mathscr{L}\left(y''(t)\right) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3s$$
$$\mathscr{L}\left(y'(t)\right) = sY(s) - y(0) = sY(s) - 3$$
$$\mathscr{L}\left(20\right) = \frac{20}{s}$$
$$\mathscr{L}\left(2e^{3t}\right) = \frac{2}{s-3}$$
$$\mathscr{L}\left(y(t) \star 2e^{3t}\right) = Y(s) \cdot \frac{2}{s-3} = \frac{2}{s-3}Y(s)$$

Therefore, the differential equation becomes:

$$s^2Y(s) - 3s + 5\left(sY(s) - 3\right) + \frac{20}{s} = \frac{2}{s-3} + 6Y(s) - \frac{2}{s-3}Y(s)$$
$$\Downarrow$$
$$Y(s) = \frac{3}{s}$$

Taking the inverse Laplace Transformation gives:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\ &= \mathscr{L}^{-1}\left(\frac{3}{s}\right) \\ &= 3 \end{aligned}$$

Thus, our unique solution is $y(t) = 3$

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3.5: Dirac Delta

Learning Objectives

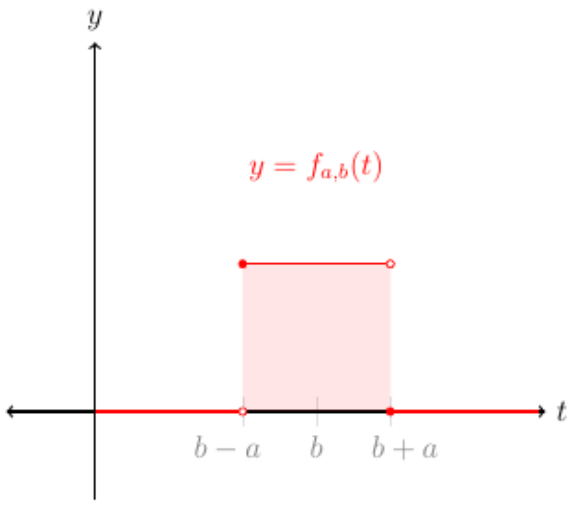
- Define the unit impulse function as a model for instantaneous impulses.
- Define the Dirac Delta function and its Laplace Transformation.
- Solve differential and integro-differential equations involving the Dirac Delta function.

The Laplace transform techniques developed so far are powerful, but they rely on forcing functions that are piecewise continuous and of exponential order. How do we model a force that acts with tremendous magnitude over an infinitesimally short time interval: hammer strike, a voltage spike, or an instantaneous injection? Such idealized *impulses* are mathematically represented by the **Dirac Delta function** $\delta_0(t)$. Although we will find it is not a *function* in the classical sense, it is defined precisely by how it behaves inside an integral. In this subchapter, we will explore the defining properties of the delta function, learn how to incorporate it into the Laplace Transform framework, and use it to solve initial value problems where the forcing term or solution involves an instantaneous impulse.

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3.5: The Dirac Delta Function

Consider two positive real numbers $b > a > 0$ and define the function \(f_{a,b}(t)=\left\{ \begin{array} {lr} 0 & 0\le t



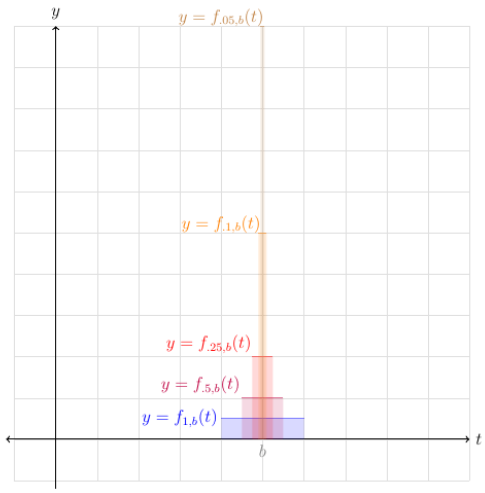
For any choice of $b > a > 0$, we see that the area under $f_{a,b}(t)$ is the same:

$$\int_0^\infty f_{a,b}(t)\,dt = \int_0^{b-a} f_{a,b}(t)\,dt + \int_{b-a}^{b+a} f_{a,b}(t)\,dt + \int_{b+a}^\infty f_{a,b}(t)\,dt = \int_{b-a}^{b+a} \frac{1}{2a}\,dt = \frac{1}{2a}\Big|_{b-a}^{b+a} = 1$$

In physics, this integral can be used to represent a force of size $\frac{1}{2a}$ applied over an interval of time $[b-a, b+a]$. The product of force and time is referred to as *impulse*. Thus, all of these functions represent the same impulse of 1 spread over different length intervals.

If we take the limit as $a \to 0^+$ of this function, we define a new "function":

$$\delta_b(t) = \lim_{a\to 0^+} f_{a,b}(t) = \begin{cases} +\infty & t = b \\ 0 & t \neq b \end{cases}$$



It is important to note that $\delta_b(t)$ above is not actually a function, since we do not consider $+\infty$ to be a valid output for functions in this text.
Instead, we avoid this thorny issue by re-defining $\delta_b(t)$ as the function satisfying the following equation:

$$\int_0^\infty g(t)\delta_b(t)\,dt = g(b) \text{ for all continuous functions } g(t)$$

This is well defined, and known as the **unit impulse function** at b.

Notice that $\lim_{a\to 0^+}\int_0^\infty g(t)f_{a,b}(t)\,dt = g(b)$ for all continuous functions $g(t)$ as well. This means we can define this $\delta_b(t)$ as the limit of $f_{a,b}(t)$ even though we cannot find its equivalent expression in explicit form.

This function helps us to model situations with very large forces applied on very short time intervals, like a hammer striking a nail.

Theorem 3.5.1

For any $b > 0$, the function $\delta_b(t) = \lim_{a\to 0^+} f_{a,b}(t)$ satisfies $\mathscr{L}(\delta_b(t)) = e^{-bs}$

This formula is a direct calculation of the Laplace Transformation of δ_b:
First we will write $f_{a,b}(t) = \frac{1}{2a}\mathcal{U}(t-(b-a)) - \frac{1}{2a}\mathcal{U}(t-(b+a))$, and take its Laplace Transformation:

$$\mathscr{L}(f_{a,b}(t)) = \int_0^\infty e^{-st}\cdot f_{a,b}(t)\,dt = \int_{b-a}^{b+a} e^{-st}\cdot\frac{1}{2a}\,dt = -\frac{1}{2as}e^{-st}\Big|_{b-a}^{b+a} = \frac{e^{-sb+sa}-e^{-sb-sa}}{2as}$$

Now if you believe in the interchanging of limits (a non-trivial assumption), we will say $\lim_{a\to 0^+}\mathscr{L}(f_{a,b}(t)) = \mathscr{L}\left(\lim_{a\to 0^+} f_{a,b}(t)\right)$. Therefore,

$$\mathscr{L}(\delta_b(t)) = \lim_{a\to 0^+}\frac{e^{-sb+sa}-e^{-sb-sa}}{2as} \overset{L'H}{=} \lim_{a\to 0^+}\frac{se^{-sb+sa}+se^{-sb-sa}}{2s} = e^{-bs}$$

This finishes the proof of our Theorem above.

Alternatively, you can use $g(t) = e^{-st}$ in the well-defined version of $\delta_b(t)$ to say that $\mathscr{L}(\delta_b(t)) = \int_0^\infty e^{-st}\delta_b(t) = e^{-sb}$

Illustration 3.5.1

Solve the IVP $y'' + y = \delta_{3\pi}(t)$ with $y(0) = 0,\ y'(0) = 1$

Solution

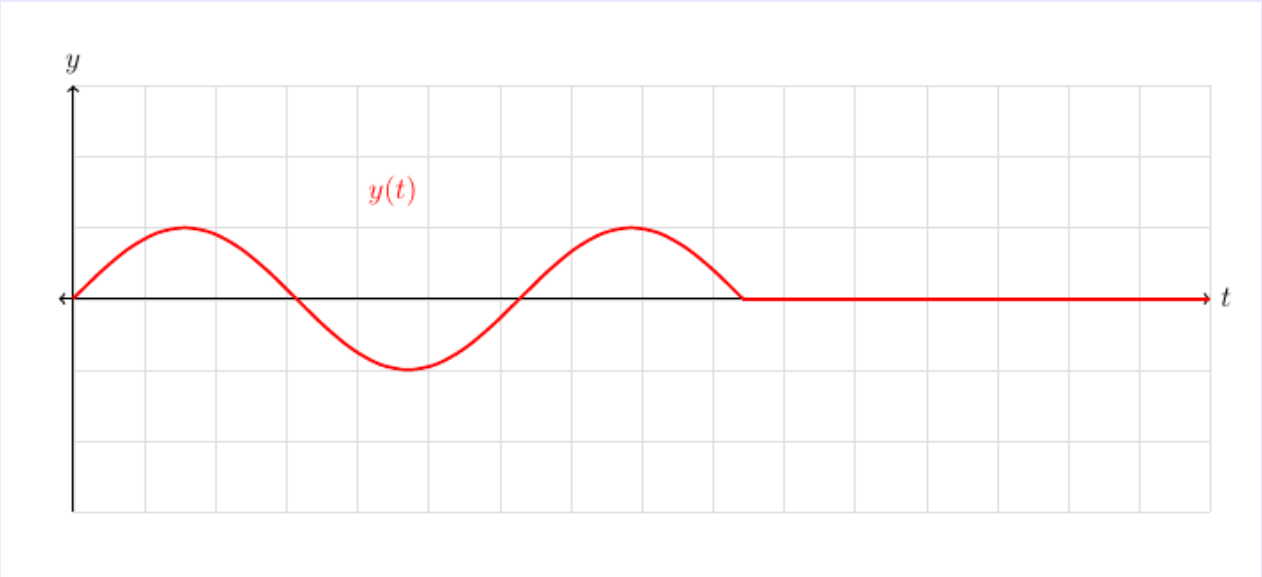
We take the Laplace Transformation of both sides of the equation:

$$\begin{gathered}\mathscr{L}\left(y''(t)\right) + \mathscr{L}\left(y(t)\right) = \mathscr{L}\left(\delta_{3\pi}(t)\right) \\ \Downarrow \\ s^2Y(s) - 1 + Y(s) = e^{-3\pi s} \\ \Downarrow \\ (s^2+1)Y(s) = 1 + e^{-3\pi s} \\ \Downarrow \\ Y(s) = \frac{1}{s^2+1} + \frac{1}{s^2+1}e^{-3\pi s}\end{gathered}$$

Now, we take the Inverse Laplace Transformation to recover $y(t)$:

$$\begin{aligned}\mathscr{L}^{-1}\left(Y(s)\right) &= \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathscr{L}^{-1}\left(\frac{1}{s^2+1}e^{-3\pi s}\right) \\ &= \sin(t) + [\sin(t)]_{t\to t-3\pi}\cdot\mathcal{U}(t-3\pi) \\ &= \sin(t) + \sin(t-3\pi)\cdot\mathcal{U}(t-3\pi) \\ &= \sin(t) - \sin(t)\mathcal{U}(t-3\pi)\end{aligned}$$

Rewriting this as a piecewise function, we see our solution is $\begin{cases}\sin(t) & 0 \le t < 3\pi \\ 0 & t \ge 3\pi\end{cases}$



This system represents an oscillating object that is "hit" with an impulse at time $t = 3\pi$ that perfectly stops its motion. (Think of it as a person jumping up and down on a trampoline and a very strong person suddenly grabs them just as they hit the trampoline and stops their motion entirely.)

For any $b > 0$, we can see that $\lim_{s\to\infty}\mathscr{L}\left(\delta_b(t)\right) = 0$ still holds.
This is because the functions are still continuous and bounded near $t = 0$.

If we allow for $b = 0$, we define a new function, known as the **Dirac delta** function.

Definition: Dirac Delta Function

The function $\delta_0(t) = \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases}$ is the ***Dirac delta function***.

This function represents the unit impulse at time $t = 0$.

We can define this function as the limit of \(\tilde{f}_a(t) =\left\{ \begin{array} {lr} \dfrac{1}{a} & 0\le t

If we go through all of the same steps above, we see that $\mathscr{L}(\delta_0(t)) = e^{-0s} = 1$

Recall that all functions $y(t)$ we have dealt with thus far have satisfied $\lim_{s\to\infty} Y(s) = 0$. This is a direct result of their continuity at and around $t = 0$. Notice that this discontinuity at $t = 0$ causes the Laplace Transformation *not to tend to zero* as $s \to \infty$. This is just one of the many fantastical properties of the Dirac delta function that caused historical mathematicians to question its validity. Thankfully, we can simply extend the well-definition of the unit impulse functions:

$$\int_0^\infty g(t)\delta_0(t)\,dt = g(0) \text{ for all continuous functions } g(t)$$

This definition aligns with our previous one, and allows us to confidently say that $\mathscr{L}(\delta_0(t)) = e^{-0s} = 1$

More excitingly, it means that we now have a way to take the Inverse Laplace Transformation of constant functions:

$$\mathscr{L}^{-1}(1) = \delta_0(t)$$

As always, we can combine the Dirac delta function with our previously learned Laplace skills as well.

Illustration 3.5.2

Are there any functions defined on $[0, \infty)$ which are equal to their convolution with themselves?

This is an integro-differential equation: $y(t) \star y(t) = y(t)$

We Laplace both sides to get:

$$\begin{gathered} \mathscr{L}(y(t) \star y(t)) = \mathscr{L}(y(t)) \\ \Downarrow \\ Y(s) \cdot Y(s) = Y(s) \\ \Downarrow \\ Y(s) = 0 \text{ or } 1 \end{gathered}$$

Using $Y(s) = 0$ gives us $y(t) = \mathscr{L}^{-1}(0) = 0$, the trivial solution.

Using $Y(s) = 1$ gives us $y(t) = \mathscr{L}^{-1}(1) = \delta_0(t)$, the Dirac delta function.

Thus, there are exactly two such functions: $y(t) = 0$ or $y(t) = \delta_0(t)$

One thing to be wary of, is that the Dirac delta function causes strange things to occur with initial conditions.

Example 3.5.1

Solve the IVP: $y' - y = \delta_0(t)$ with $y(0) = 0$ (we assume the domain of $y(t)$ is $[0, \infty)$)

Solution

We take the Laplace Transformation of both sides of the equation:

$$\mathscr{L}(y'(t)) - \mathscr{L}(y(t)) = \mathscr{L}(\delta_0(t))$$
$$\Downarrow$$
$$sY(s) - Y(s) = 1$$

$$\Downarrow$$
$$(s-1)Y(s) = 1$$

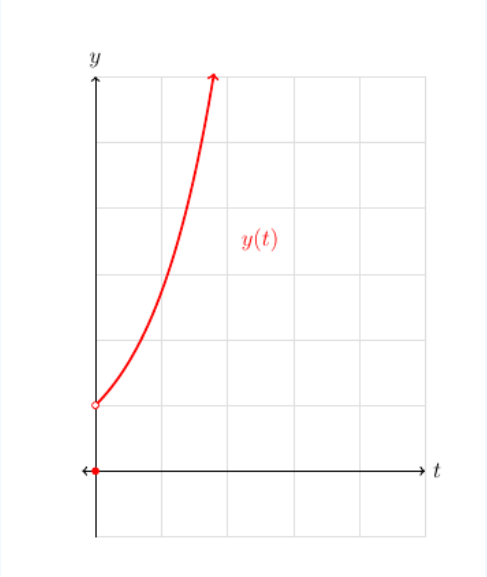
$$\Downarrow$$
$$Y(s) = \frac{1}{s-1}$$

Now, we take the Inverse Laplace Transformation to recover $y(t)$:

$$\mathscr{L}^{-1}(Y(s)) = \mathscr{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$$

Notice however, that $y(t) = e^t \implies y(0) = 1$, not $y(0) = 0$ like the problem requires!

To fix this, we realize that $y(t)$ must have a discontinuity at $t = 0$, and write $\boxed{y(t) = \begin{cases} 0 & t = 0 \\ e^t & t > 0 \end{cases}}$



It is important to remember that *whenever your original equation or solution involves the Dirac delta function, you need to double check your initial conditions.*

Example 3.5.2

Solve the integro-differential equation: $\int_0^t y'(x)e^{t-x}\,dx - y(t) = \begin{cases} te^t & 0 \le t < 2 \\ (t - e^{-2})\,e^t & t \ge 2 \end{cases}$ with $y(0) = 0$ (you may assume $t \ge 0$ is the domain of $y(t)$)

Solution

We can rewrite the piecewise forcing function on the right as $g(t) = te^t - e^{t-2}\mathcal{U}(t-2)$

We take the Laplace Transformation of both sides of the equation:

$$\mathscr{L}\left(y'(t) \star e^t\right) - \mathscr{L}\left(y(t)\right) = \mathscr{L}\left(te^t\right) - \mathscr{L}\left(e^{t-2}\mathcal{U}(t-2)\right)$$
$$\Downarrow$$
$$sY(s) \cdot \frac{1}{s-1} - Y(s) = \frac{1}{(s-1)^2} - \frac{1}{s-1}e^{-2s}$$
$$\Downarrow$$
$$\frac{1}{s-1}Y(s) = \frac{1-(s-1)e^{-2s}}{(s-1)^2}$$
$$\Downarrow$$
$$Y(s) = \frac{1}{s-1} - e^{-2s}$$

Now, we take the Inverse Laplace Transformation to recover $y(t)$:

$$\mathscr{L}^{-1}\left(Y(s)\right) = \mathscr{L}^{-1}\left(\frac{1}{s-1} - e^{-2s}\right) = e^t - \delta_2(t)$$

We check $y(t) = e^t - \delta_2(t) \implies y(0) = 1$, not $y(0) = 0$ again! We fix this by introducing a discontinuity at $t = 0$

Our solution is thus \(\boxed{y(t)=\left\{\begin{array} {lc} 0 & t=0 \ e^t & 0 2 \end{array} \right. }\)

In the example above, it is tempting to write $y(2) = -\infty$ to match our informal definition of the Dirac delta function, but that would not make $y(t)$ *well-defined*. If your answer ever includes any $\delta_b(t)$ term for $b > 0$, just write it out as we did above!

The Dirac delta function and unit impulse functions provide a powerful mathematical model for instantaneous impulses occurring at specific times. With these tools, we can extend the reach of the Laplace Transformation to include impulsive phenomena. This brings mathematical precision to the study of sudden, impactful events in physical systems.

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3.5: Practice Set

Exercise 3.5.1a

Find the Laplace Transformation of the function $a(t) = t^2 + \delta_2(t)$

Answer available at the end of this section.

Exercise 3.5.1b

Find the Laplace Transformation of the function $b(t) = \int_0^t \delta_a(x) \cdot \delta_b(t-x)\, dx$

Answer available at the end of this section.

Exercise 3.5.1c

Find the Laplace Transformation of the function $c(t) = t\delta_b(t)$

Answer available at the end of this section.

Exercise 3.5.1d

Consider the function $w(x) = \sum_{k=0}^{\infty} c_k x^k$ with positive radius of convergence.

Find the Laplace Transformation of the function $d(t) = \sum_{k=0}^{\infty} c_k \delta_k(t)$

Answer available at the end of this section.

Exercise 3.5.2a

Find the Inverse Laplace Transformation of $A(s) = \dfrac{e^{-s} - se^{-3s}}{s}$

Answer available at the end of this section.

Exercise 3.5.2b

Find the Inverse Laplace Transformation of $B(s) = \dfrac{2s^2 + 2s - 6}{s^2 + 2s - 8}$

Answer available at the end of this section.

Exercise $3.5.2c$

Find the Inverse Laplace Transformation of $C(s) = \dfrac{s^2(e^s - e^{4s})}{s^2e^{4s} + e^{4s}}$

Answer available at the end of this section.

Exercise $3.5.2d$

Find the Inverse Laplace Transformation of $D(s) = \dfrac{s^4 - 4s^3 + 10s^2 - 10s + 8}{(s-1)^2(s^2+4)}$.

Answer available at the end of this section.

Exercise $3.5.2e$

Find the Inverse Laplace Transformation of $E(s) = \ln(1 + e^{-s})$.
(Hint: consider Exercise 1d above)

Answer available at the end of this section.

Exercise $3.5.3a$

Find the solution to $y'' + 4y = 2\delta_\pi(t)$ with $y(0) = 0,\ y'(0) = 0$

Answer available at the end of this section.

Exercise $3.5.3b$

Find the solution to $y' + 3y = 9t - \delta_2(t)$ with $y(0) = 1$

Answer available at the end of this section.

Exercise $3.5.3c$

Find solution to $y'' + y = \delta_{\frac{\pi}{2}}(t) + \delta_\pi(t)$ with $y(0) = 1,\ y'(0) = 0$

Answer available at the end of this section.

Exercise $3.5.3d$

Find solution to $y' - y = te^t\left[\delta_1(t) - \delta_2(t)\right]$ with $y(0) = 1$

Answer available at the end of this section.

Exercise $3.5.3e$

Find solution to $y(t) - \int_0^t 9(t-w)y(w)\ dw = 2\delta_7(t)$

Answer available at the end of this section.

Exercise $3.5.3f$

Find solution to $y'(t) + \int_0^t y(w)e^{2t-2w}\, dw = \delta_1(t)$ with $y(0) = 0$

Answer available at the end of this section.

Exercise $3.5.3g$

Find solution to $\int_0^t (y'(w) - y(w))\,(t-w)e^{t-w}\, dw = \mathcal{U}(t-a)$ with $y(0) = 0$

Answer available at the end of this section.

Exercise $3.5.3h$

Find solution to $\int_0^t y''(w) \cdot y(t-w)\, dw = \delta_0(t)$ with $y(0) = 0,\ y'(0) = 0$

Answer available at the end of this section.

Answers

Exercise $3.5.1a$

We use the linearity of the Laplace Transformation to get:

$$\mathscr{L}(a(t)) = \mathscr{L}(t^2) + \mathscr{L}(\delta_2(t)) = \frac{2!}{s^3} + e^{-2s}$$

Thus, $A(s) = \frac{2}{s^3} + e^{-2s}$

Exercise $3.5.1b$

Notice that $b(t) = \delta_a(t) \star \delta_b(t)$, so:

$$\mathscr{L}(b(t)) = \mathscr{L}(\delta_a(t) \star \delta_b(t)) = \mathscr{L}(\delta_a(t)) \cdot \mathscr{L}(\delta_b(t)) = e^{-as} \cdot e^{-bs}$$

Thus, $B(s) = e^{-(a+b)s}$

Notice that if we take the Inverse Laplace Transformation of $B(s)$, we get $b(t) = \delta_{a+b}(t)$ which is equivalent to $b(t) = \delta_a(t) \star \delta_b(t)$

Exercise $3.5.1c$

We use the differentiation property of Laplace Transformations:

$$\mathscr{L}(c(t)) = \mathscr{L}(t \cdot \delta_b(t)) = -\frac{d}{ds}\left[\mathscr{L}(\delta_b(t))\right] = -\frac{d}{ds}\left[e^{-bs}\right] = be^{-bs}$$

Thus, $C(s) = be^{-bs}$

Notice that if we take the Inverse Laplace Transformation of $C(s)$, we get $c(t) = b\delta_b(t)$ which is equivalent to $c(t) = t\delta_b(t)$

Exercise $3.5.1d$

We use the linearity of the Laplace Transformation to get:

$$\mathscr{L}(d(t)) = \mathscr{L}\left(\sum_{k=0}^{\infty} c_k\delta_k(t)\right) = \sum_{k=0}^{\infty} c_k\mathscr{L}(\delta_k(t)) = \sum_{k=0}^{\infty} c_k e^{-ks} = \sum_{k=0}^{\infty} c_k\left(e^{-s}\right)^k = w\left(e^{-s}\right)$$

Thus, $D(s) = w\left(e^{-s}\right)$

Exercise $3.5.2a$

We rewrite this as $A(s) = \frac{1}{s}e^{-s} - e^{-3s}$ and use the linearity of the Inverse Laplace Transformation:

$$\mathscr{L}^{-1}\left(A(s)\right) = \mathscr{L}^{-1}\left(\frac{1}{s}e^{-s}\right) - \mathscr{L}^{-1}\left(e^{-3s}\right) = 1 \cdot \mathcal{U}(t-1) - \delta_3(t)$$

Thus, $a(t) = \mathcal{U}(t-1) - \delta_3(t)$

Exercise 3.5.2*b*

We use long division and partial fraction decomposition to write this as $b(t) = 2 + \frac{1}{s-2} - \frac{3}{s+4}$. So,

$$\mathscr{L}^{-1}\left(B(s)\right) = \mathscr{L}^{-1}\left(2 + \frac{1}{s-2} - \frac{3}{s+4}\right) = 2 \cdot \delta_0(t) + e^{2t} - 3e^{-4t}$$

Thus, $b(t) = 2\delta_0(t) + e^{2t} - 3e^{-4t}$

Exercise 3.5.2*c*

We rewrite this as $C(s) = \frac{s^2}{s^2+1} \cdot \frac{e^s - e^{4s}}{e^{4s}} = \left(1 - \frac{1}{s^2+1}\right)\left(e^{-3s} - 1\right)$:

$$\mathscr{L}^{-1}\left(C(s)\right) = \mathscr{L}^{-1}\left(\left(1 - \frac{1}{s^2+1}\right)e^{-3s} - \left(1 - \frac{1}{s^2+1}\right)\right) = \delta_3(t) - \sin(t-3)\mathcal{U}(t-3) - \delta_0(t) + \sin(t)$$

Thus, $c(t) = \sin(t) - \delta_0(t) + \delta_3(t) - \sin(t-3)\mathcal{U}(t-3)$

Exercise 3.5.2*d*

We use long division and partial fraction decomposition to write this as $b(t) = 1 + \frac{1}{(s-1)^2} - \frac{2s}{s^2+4}$. So,

$$\mathscr{L}^{-1}\left(D(s)\right) = \mathscr{L}^{-1}\left(1 + \frac{1}{(s-1)^2} - \frac{2s}{s^2+4}\right) = \delta_0(t) + te^t - 2\cos(2t)$$

Thus, $d(t) = \delta_0(t) + te^t - 2\cos(2t)$

Exercise 3.5.2*e*

Considering $w(x) = \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$, we have $E(s) = w\left(e^{-s}\right)$.

By the result in Exercise 1d, we know that $e(t) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\delta_k(t)$

Exercise 3.5.3*a*

We take the Laplace Transformation on both sides:

$$\begin{gathered}\mathscr{L}\left(y''(t)\right) + 4\mathscr{L}\left(y(t)\right) = 2\mathscr{L}\left(\delta_\pi(t)\right) \\ \Downarrow \\ s^2Y(s) + 4Y(s) = 2e^{-\pi s} \\ \Downarrow \\ Y(s) = \frac{2}{s^2+4}e^{-\pi s}\end{gathered}$$

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\mathscr{L}^{-1}\left(Y(s)\right) = \mathscr{L}^{-1}\left(\frac{2}{s^2+4}e^{-\pi s}\right) = \left[\sin(2t)\right]_{t\to t-\pi} \cdot \mathcal{U}(t-\pi) = \sin(2t)\mathcal{U}(t-\pi)$$

Therefore, the solution is $y(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin(2t) & t \ge \pi \end{cases}$

Exercise 3.5.3*b*

We take the Laplace Transformation on both sides:

$$\begin{gathered}\mathscr{L}\left(y'(t)\right) + 3\mathscr{L}\left(y(t)\right) = 9\mathscr{L}\left(t\right) - \mathscr{L}\left(\delta_2(t)\right) \\ \Downarrow \\ sY(s) - 1 + 3Y(s) = 9 \cdot \frac{1}{s^2} - e^{-2s} \\ \Downarrow \\ Y(s) = \frac{1 - e^{-2s}}{s+3} + \frac{6}{s^2(s+3)}\end{gathered}$$

Using partial fraction decomposition, we can write this as $Y(s) = \frac{3}{s^2} - \frac{1}{s} + \frac{2}{s+3} - \frac{1}{s+3}e^{-2s}$.

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\begin{aligned}\mathscr{L}^{-1}\left(Y(s)\right) &= \mathscr{L}^{-1}\left(\frac{3}{s^2} - \frac{1}{s} + \frac{2}{s+3} - \frac{1}{s+3}e^{-2s}\right) \\ &= 3t - 1 + 2e^{-3t} - \left[e^{-3t}\right]_{t\to t-2} \cdot \mathcal{U}(t-2) \\ &= 3t - 1 + 2e^{-3t} - e^{6-3t} \cdot \mathcal{U}(t-2)\end{aligned}$$

Therefore, the solution is $y(t) = \begin{cases} 3t - 1 + 2e^{-3t} & 0 \le t < 2 \\ 3t - 1 + 2e^{-3t} - e^{6-3t} & t \ge 2 \end{cases}$

Exercise 3.5.3*c*

We take the Laplace Transformation on both sides:

$$\mathscr{L}\left(y''(t)\right)+\mathscr{L}\left(y(t)\right)=\mathscr{L}\left(\delta_{\frac{\pi}{2}}(t)\right)+\mathscr{L}\left(\delta_{\pi}(t)\right)$$
$$\Downarrow$$
$$s^2Y(s)-s+Y(s)=e^{-\frac{\pi}{2}s}+e^{-\pi s}$$
$$\Downarrow$$
$$Y(s)=\frac{s}{s^2+1}+\frac{1}{s^2+1}e^{-\frac{\pi}{2}s}+\frac{1}{s^2+1}e^{-\pi s}$$

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\begin{aligned}\mathscr{L}^{-1}\left(Y(s)\right)&=\mathscr{L}^{-1}\left(\frac{s}{s^2+1}+\frac{1}{s^2+1}e^{-\frac{\pi}{2}s}+\frac{1}{s^2+1}e^{-\pi s}\right)\\&=\cos(t)+\left[\sin(t)\right]_{t\to t-\frac{\pi}{2}}\cdot\mathcal{U}\left(t-\frac{\pi}{2}\right)+\left[\sin(t)\right]_{t\to t-\pi}\cdot\mathcal{U}(t-\pi)\\&=\cos(t)-\cos(t)\cdot\mathcal{U}\left(t-\frac{\pi}{2}\right)-\sin(t)\mathcal{U}(t-\pi)\end{aligned}$$

Therefore, the solution is $y(t)=\begin{cases}\cos(t) & 0\le t<\frac{\pi}{2}\\ 0 & \frac{\pi}{2}\le t<\pi\\ -\sin(t) & t\ge\pi\end{cases}$

Exercise $3.5.3d$

We take the Laplace Transformation on both sides:

$$\mathscr{L}\left(y'(t)\right)-\mathscr{L}\left(y(t)\right)=\mathscr{L}\left(te^t\delta_1(t)\right)-\mathscr{L}\left(te^t\delta_2(t)\right)$$
$$\Downarrow$$
$$sY(s)-1-Y(s)=e\cdot e^{-s}-2e^2\cdot e^{-2s}$$
$$\Downarrow$$
$$Y(s)=\frac{1}{s-1}+\frac{e}{s-1}e^{-s}-\frac{2e^2}{s-1}e^{-2s}$$

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\begin{aligned}\mathscr{L}^{-1}\left(Y(s)\right)&=\mathscr{L}^{-1}\left(\frac{1}{s-1}+\frac{e}{s-1}e^{-s}-\frac{e^2}{s-1}e^{-2s}\right)\\&=e^t+e\cdot\left[e^t\right]_{t\to t-1}\cdot\mathcal{U}\left(t-1\right)-2e^2\left[e^t\right]_{t\to 2}\cdot\mathcal{U}(t-2)\\&=e^t+e^t\cdot\mathcal{U}\left(t-1\right)-2e^t\mathcal{U}(t-2)\end{aligned}$$

Therefore, the solution is $y(t)=\begin{cases}e^t & 0\le t<1\\ 2e^t & 1\le t<2\\ 0 & t\ge 2\end{cases}$

Exercise $3.5.3e$

We take the Laplace Transformation on both sides:

$$\mathscr{L}\left(y(t)\right)-\mathscr{L}\left(y(t)\star 9t\right)=\mathscr{L}\left(2\delta_7(t)\right)$$
$$\Downarrow$$
$$Y(s)-Y(s)\cdot\frac{9}{s^2}=2e^{-7s}$$
$$\Downarrow$$
$$Y(s)=\left(2+\frac{3}{s-3}-\frac{3}{s+3}\right)e^{-7s}$$

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\begin{aligned}\mathscr{L}^{-1}\left(Y(s)\right)&=\mathscr{L}^{-1}\left(\left(2+\frac{3}{s-3}-\frac{3}{s+3}\right)e^{-7s}\right)\\&=2\delta_7(t)+\left[3e^{3t}-3e^{-3t}\right]_{t\to t-7}\cdot\mathcal{U}\left(t-7\right)\\&=2\delta_7(t)+3\left(e^{3(t-7)}-e^{-3(t-7)}\right)\cdot\mathcal{U}\left(t-7\right)\end{aligned}$$

Therefore, the solution is $y(t)=\begin{cases}0 & 0\le t<7\\ 2\delta_7(t) & t=7\\ 3e^{3t-21}-3e^{21-3t} & t>7\end{cases}$

Exercise $3.5.3f$

We take the Laplace Transformation on both sides:

$$\mathscr{L}\left(y'(t)\right)+\mathscr{L}\left(y(t)\star e^{2t}\right)=\mathscr{L}\left(\delta_1(t)\right)$$
$$\Downarrow$$
$$sY(s)+Y(s)\cdot\frac{1}{s-2}=e^{-s}$$
$$\Downarrow$$
$$Y(s)=\left(\frac{1}{s-1}-\frac{1}{(s-1)^2}\right)e^{-s}$$

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\begin{aligned}\mathscr{L}^{-1}\left(Y(s)\right)&=\mathscr{L}^{-1}\left(\left(\frac{1}{s-1}-\frac{1}{(s-1)^2}\right)e^{-s}\right)\\&=\left[e^t-te^t\right]_{t\to t-1}\cdot\mathcal{U}\left(t-1\right)\\&=(2-t)e^{t-1}\cdot\mathcal{U}\left(t-1\right)\end{aligned}$$

Therefore, the solution is $y(t)=\begin{cases}0 & 0\le t<1\\ (2-t)e^{t-1} & t\ge 1\end{cases}$

Exercise $3.5.3g$

We take the Laplace Transformation on both sides:

$$\begin{gathered}\mathscr{L}\left(y'(t)\star te^t\right)-\mathscr{L}\left(y(t)\star te^t\right)=\mathscr{L}\left(\mathcal{U}(t-a)\right)\\ \Downarrow\\ sY(s)\cdot\frac{1}{(s-1)^2}-Y(s)\cdot\frac{1}{(s-1)^2}=\frac{1}{s}e^{-as}\\ \Downarrow\\ Y(s)=e^{-as}-\frac{1}{s}e^{-as}\end{gathered}$$

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\mathscr{L}^{-1}\left(Y(s)\right)=\mathscr{L}^{-1}\left(e^{-as}-\frac{1}{s}e^{-as}\right)=\delta_a(t)-1\cdot\mathcal{U}(t-a)$$

Therefore, the solution is \({ y(t) = \left\{ \begin{array} {lr} 0 & 0\le ta \end{array} \right. }\)

Exercise $3.5.3h$

We take the Laplace Transformation on both sides:

$$\begin{gathered}\mathscr{L}\left(y''(t)\star y(t)\right)=\mathscr{L}\left(\delta_0(t)\right)\\ \Downarrow\\ s^2Y(s)\cdot Y(s)=1\\ \Downarrow\\ Y(s)=\pm\frac{1}{s}\end{gathered}$$

Now we take the Inverse Laplace Transformation to recover $y(t)$:

$$\mathscr{L}^{-1}\left(Y(s)\right)=\mathscr{L}^{-1}\left(\pm\frac{1}{s}\right)=\pm1$$

This yields two distinct solutions, $y_1(t)=1$ and $y_2(t)=-1$, neither of which satisfy $y(0)=0$

We introduce discontinuities at $t=0$ to get our final solutions: $y_1(t)=\begin{cases}0 & t=0\\ 1 & t>0\end{cases}$ and $y_2(t)=\begin{cases}0 & t=0\\ -1 & t>0\end{cases}$

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3.6: Systems of Differential Equations Revisited

Learning Objectives

- Apply Laplace transforms to solve systems of linear differential equations.
- Handle systems with piecewise continuous forcing functions.
- Solve systems containing integral terms (integro-differential systems).
- Use the method to solve coupled mechanical and electrical systems.

The Laplace transform method elegantly extends to systems of differential equations. This section demonstrates how to transform a system of ODEs into a system of algebraic equations, solve for the transformed variables, and then invert them to find the solution to the original coupled system.

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3.6: Solving Systems using Laplace

When we are solving systems of linear differential equations, we can often use Laplace Transformations to solve them. However, it is important that the system begins with initial conditions at $t = 0$, or else students should use the standard D operator method we discussed in Chapter 2.

Illustration 3.6.1

Compare these methods for the system $\begin{cases} x' = 2x + y \\ y' = x + 2y \end{cases}$ with $x(0) = 2,\ y(0) = 0$

Using the D notation, we would normally rewrite this as $\begin{cases} (D-2)x - y = 0 \\ -x + (D-2)y = 0 \end{cases}$

Then we could eliminate y to get $(D-2)^2 - 1)\, x = 0 \implies x''(t) - 4x'(t) + 3x(t) = 0$

This CC LDE has general solution $x(t) = Ae^t + Be^{3t}$, and substituting this into the original first equation gives:

$$y(t) = x'(t) - 2x(t) = \left(Ae^t + 3Be^{3t}\right) - 2\left(Ae^t + Be^{3t}\right) = -Ae^t + Be^{3t}$$

With the general solution in hand, we plug in the initial conditions:

$$\begin{aligned} x(t) = Ae^t + Be^{3t} &\implies x(0) = A + B \implies A + B = 2 \\ y(t) = -Ae^t + Be^{3t} &\implies y(0) = -A + B \implies -A + B = 0 \end{aligned}$$

It is straightforward to solve that $A = 1,\ B = 1$ yields our final solution $\begin{cases} x(t) = e^t + e^{3t} \\ y(t) = -e^t + e^{3t} \end{cases}$

Now, compare this to solving the same original system using Laplace Transformations:

$$\begin{aligned} \mathscr{L}(x') &= sX(s) - x(0) = sX(s) - 2 \\ \mathscr{L}(y') &= sY(s) - y(0) = sY(s) \end{aligned}$$

This turns our system of equations into:

$$\begin{cases} \mathscr{L}(x') = 2\mathscr{L}(x) + \mathscr{L}(y) \\ \mathscr{L}(y') = \mathscr{L}(x) + 2\mathscr{L}(y) \end{cases} \implies \begin{cases} sX(s) - 2 = 2X(s) + Y(s) \\ sY(s) = X(s) + 2Y(s) \end{cases} \implies \begin{cases} (s-2)X(s) - Y(s) = 2 \\ -X(s) + (s-2)Y(s) = 0 \end{cases}$$

We use substitution (or elimination) to remove $Y(s)$ and get the equation:

$$(s-2)^2 - 1)\, X(s) = 2(s-2) \implies X(s) = \frac{2s-4}{s^2 - 4s + 3} = \frac{1}{s-1} + \frac{1}{s-3}$$

Taking the Inverse Laplace Transformation gives the same first function: $x(t) = \mathscr{L}^{-1}\left(\frac{1}{s-1} + \frac{1}{s-3}\right) = e^t + e^{3t}$

At this point, we can do the same substitution and solve for $y(t)$ using $x' = 2x + y$, or we can solve for $Y(s)$ in the Laplace equation and then take its Inverse Laplace Transformation to solve for $y(t)$. Both of these methods will yield the same solution ($y(t) = -e^t + e^{3t}$), so we just do whichever is easier in a given problem.

In general, if you have a system of linear IVPs with initial conditions at zero, the steps to solve it are:

1. Take the Laplace Transformation of all terms.
2. Substitute or eliminate to solve for one of the functions (of s) in the system.
3. Take the Inverse Laplace Transformation of that function.
4. Use this to solve for the remaining functions using any desired method.

Example 3.6.1

Solve the system $\begin{cases} x' = 5x + 3y + 15e^{4t} \\ y' = 2x + 4y \\ z' = 2x + z \end{cases}$ with $x(0) = 0,\ y(0) = 0,\ z(0) = 0$

Solution

First, we take the Laplace Transformation of all (non-obvious) terms:

$$\begin{aligned} \mathscr{L}\left(x'(t)\right) &= sX(s) - x(0) = sX(s) \\ \mathscr{L}\left(15e^{4t}\right) &= \frac{15}{s-4} \\ \mathscr{L}\left(y'(t)\right) &= sY(s) - y(0) = sY(s) \\ \mathscr{L}\left(z'(t)\right) &= sZ(s) - z(0) = sZ(s) \end{aligned}$$

This turns our system into:

$$\begin{cases} sX(s) = 5X(s) + 3Y(s) + \frac{15}{s-4} \\ sY(s) = 2X(s) + 4Y(s) \\ sZ(s) = 2X(s) + Z(s) \end{cases} \implies \begin{cases} (s-5)X(s) - 3Y(s) = \frac{15}{s-4} \\ -2X(s) + (s-4)Y(s) = 0 \\ -2X(s) + (s-1)Z(s) = 0 \end{cases}$$

The middle equation can be written $Y(s) = \frac{2}{s-4}X(s)$. Substituting this into the top equation gives us:

$$(s-5)X(s) - 3 \cdot \frac{2}{s-4}X(s) = \frac{15}{s-4} \implies X(s) = \frac{15}{(s-2)(s-7)} = \frac{3}{s-7} - \frac{3}{s-2}$$

We take the Inverse Laplace of this to find $x(t) = \mathscr{L}^{-1}\left(\frac{3}{s-7} - \frac{3}{s-2}\right) = 3e^{7t} - 3e^{2t}$

Now we can use this function in the original top equation to solve for y:

$$y(t) = \frac{1}{3}x'(t) - \frac{5}{3}x(t) - 5e^{4t} = \frac{1}{3}\left(21e^{7t} - 6e^{2t}\right) - \frac{5}{3}\left(3e^{7t} - 3e^{2t}\right) - 5e^{4t} = 2e^{7t} + 3e^{2t} - 5e^{4t}$$

Unfortunately substituting into the bottom equation does not immediately give us $z(t)$, just a differential equation. Instead we solve for $Z(s)$ using $-2X(s) + (s-1)Z(s) = 0$:

$$Z(s) = \frac{2}{s-1}X(s) = \frac{30}{(s-1)(s-2)(s-7)} = \frac{1}{s-7} - \frac{6}{s-2} + \frac{5}{s-1}$$

Taking the Inverse Laplace Transformation gives us:

$$z(t) = \mathscr{L}^{-1}\left(\frac{1}{s-7} - \frac{6}{s-2} + \frac{5}{s-1}\right) = e^{7t} - 6e^{2t} + 5e^{t}$$

This gives us the unique solution to the system of IVPs is $\boxed{\begin{cases} x(t) = 3e^{7t} - 3e^{2t} \\ y(t) = 2e^{7t} + 3e^{2t} - 5e^{4t} \\ z(t) = e^{7t} - 6e^{2t} + 5e^{t} \end{cases}}$

Since we know how Laplace Transformations interact with piecewise functions, this allows us to solve more exciting systems.

Example 3.6.2

Solve the system $\begin{cases} x' = y + 4\mathcal{U}(t-1) \\ y' = -4x + 4 \end{cases}$ with $x(0) = 4,\ y(0) = 8$

Solution

First, we take the Laplace Transformation of all (non-obvious) terms:

$$\begin{aligned}\mathscr{L}\left(x'(t)\right) &= sX(s) - x(0) = sX(s) - 4\\ \mathscr{L}\left(4\mathcal{U}(t-1)\right) &= \frac{4}{s}e^{-s}\\ \mathscr{L}\left(y'(t)\right) &= sY(s) - y(0) = sY(s) - 8\\ \mathscr{L}\left(4\right) &= \frac{4}{s}\end{aligned}$$

This turns our system into:

$$\begin{cases} sX(s) - 4 = Y(s) + \frac{4}{s}e^{-s} \\ sY(s) - 8 = -4X(s) + \frac{4}{s} \end{cases} \implies \begin{cases} sX(s) - Y(s) = 4 + \frac{4}{s}e^{-s} \\ 4X(s) + sY(s) = 8 + \frac{4}{s} \end{cases}$$

In the top equation, we see that $Y(s) = 4 + \frac{4}{s}e^{-s} - sX(s)$, so we substitute this into the bottom equation:

$$4X(s) + s\left(sX(s) - 4 - \frac{4}{s}e^{-s}\right) = 8 + \frac{4}{s} \implies (s^2+4)X(s) = \frac{4s+8}{s^2+4} + \frac{4}{s(s^2+4)} + \frac{4}{s^2+4}e^{-s}$$

Using partial fraction decomposition, we can rewrite this as $X(s) = \frac{1}{s} + \frac{3s}{s^2+4} + \frac{8}{s^2+4} + \frac{4}{s^2+4}e^{-s}$ Thus,

$$\begin{aligned}x(t) &= \mathscr{L}^{-1}\left(X(s)\right)\\ &= \mathscr{L}^{-1}\left(\frac{1}{s} + \frac{3s}{s^2+4} + \frac{8}{s^2+4} + \frac{4}{s^2+4}e^{-s}\right)\\ &= 1 + 3\cos(2t) + 8\cdot\frac{1}{2}\sin(2t) + \left[4\cdot\frac{1}{2}\sin(2t)\right]_{t\to t-1}\cdot\mathcal{U}(t-1)\\ &= 1 + 3\cos(2t) + 4\sin(2t) + 2\sin(2t-2)\mathcal{U}(t-1)\end{aligned}$$

Substituting this into the top equation gives us $y(t)$:

$$\begin{aligned}y(t) &= x'(t) - 4\mathcal{U}(t-1)\\ &= \left(-6\sin(2t) + 8\cos(2t) + 4\cos(2t-2)\mathcal{U}(t-1)\right) - 4\mathcal{U}(t-1)\\ &= -6\sin(2t) + 8\cos(2t) + \left(4\cos(2t-2) - 4\right)\mathcal{U}(t-1)\end{aligned}$$

Therefore, the unique solution to this system is $\boxed{\begin{cases} x(t) = 1 + 3\cos(2t) + 4\sin(2t) + 2\sin(2t-2)\mathcal{U}(t-1) \\ y(t) = -6\sin(2t) + 8\cos(2t) + \left(4\cos(2t-2) - 4\right)\mathcal{U}(t-1) \end{cases}}$

Laplace also allows us to solve systems involving convolution integrals (which is common for systems with "memory").

Example 3.6.3

Solve the system $\begin{cases} x' = \int_0^t y(w)\,dw + 3t^2 \\ y' = \int_0^t x(t-w)\,dw + 2 \end{cases}$ with $x(0) = 0,\ y(0) = 0$

Solution

First, we take the Laplace Transformation of all (non-obvious) terms:

$$\mathscr{L}(x'(t)) = sX(s) - x(0) = sX(s)$$
$$\mathscr{L}(y(t) \star 1) = Y(s) \cdot \frac{1}{s}$$
$$\mathscr{L}(3t^2) = \frac{6}{s^3}$$
$$\mathscr{L}(y'(t)) = sY(s) - y(0) = sY(s)$$
$$\mathscr{L}(x(t) \star 1) = X(s) \cdot \frac{1}{s}$$
$$\mathscr{L}(2) = \frac{2}{s}$$

This turns our system into:

$$\begin{cases} sX(s) = \frac{1}{s}Y(s) + \frac{6}{s^3} \\ sY(s) = \frac{1}{s}X(s) + \frac{2}{s} \end{cases} \implies \begin{cases} s^2X(s) - Y(s) = \frac{6}{s^2} \\ -X(s) + s^2Y(s) = 2 \end{cases}$$

We can multiply the top equation by s^2 and then add it to the bottom equation to eliminate $Y(s)$:

$$(s^4 - 1)X(s) = 8 \implies X(s) = \frac{8}{s^4 - 1} = \frac{4}{s^2 - 1} - \frac{4}{s^2 + 1}$$

Taking the Inverse Laplace Transformation gives us $x(t) = \mathscr{L}^{-1}\left(\frac{4}{s^2 - 1} - \frac{4}{s^2 + 1}\right) = 4\sinh(t) - 4\sin(t)$

There is no nice equation to recover $y(t)$, so we solve for $Y(s)$ instead by substituting in $X(s) = \frac{8}{s^4 - 1}$ into either equation:

$$-X(s) + s^2Y(s) = 2 \implies \frac{8}{s^4 - 1} + s^2Y(s) = 2 \implies Y(s) = \frac{2s^4 - 10}{s^2(s^4 - 1)} = \frac{4}{s^2 - 1} + \frac{2}{s^2 + 1} - \frac{6}{s^2}$$

Taking the Inverse Laplace Transformation gives us $y(t) = \mathscr{L}^{-1}\left(\frac{4}{s^2 - 1} + \frac{2}{s^2 + 1} - \frac{6}{s^2}\right) = 4\sinh(t) + 2\sin(t) - 6t$

Therefore, our unique solution is $\boxed{\begin{cases} x(t) = 4\sinh(t) - 4\sin(t) \\ y(t) = 4\sinh(t) + 2\sin(t) - 6t \end{cases}}$

While not all (non-linear) systems are solvable with this method, there are still some that it can be helpful for.

Example 3.6.4

Solve the system $\begin{cases} x' = y + \int_0^t x(w) \cdot x(t - w)\, dw \\ y' = -x \end{cases}$ with $x(0) = 1,\ y(0) = 0$.

Solution

Note that $\int_0^t x(w) \cdot x(t - w)\, dw = x(t) \star x(t)$. We take the Laplace Transformation of all terms:

$$\mathscr{L}(x'(t)) = sX(s) - x(0) = sX(s) - 1$$
$$\mathscr{L}(x(t) \star x(t)) = X(s) \cdot X(s) = [X(s)]^2$$
$$\mathscr{L}(y'(t)) = sY(s) - y(0) = sY(s)$$

This turns our system into: $\begin{cases} sX(s) - 1 = Y(s) + [X(s)]^2 \\ sY(s) = -X(s) \end{cases}$

Substituting $X(s) = -sY(s)$ into the top equation gives:

$$-s^2Y(s) - 1 = Y(s) + s^2\left[Y(s)\right]^2 \implies s^2\left[Y(s)\right]^2 + (s^2+1)Y(s) + 1 = 0$$

Using the quadratic formula, we see $Y(s) = \dfrac{-(s^2+1) \pm \sqrt{(s^2+1)^2 - 4s^2}}{2s^2} = \dfrac{-(s^2+1) \pm (s^2-1)}{2s^2}$

Simplifying these two roots, we have $Y(s) = -\dfrac{1}{s^2}$ and $Y(s) = -1$

Taking the Inverse Laplace Transformation gives us $y_1(t) = \mathscr{L}^{-1}\left(-\dfrac{1}{s^2}\right) = -t$ and $y_2(t) = \mathscr{L}^{-1}(-1) = -\delta_0(t)$.

Note that $y(0) = 0$ would be impossible to satisfy using the Dirac delta function, so we ignore $y_2(t) = -\delta_0(t)$ and use $y(t) = -t$

The bottom equation gives $x(t) = -y'(t) = 1$

Therefore, our solution to this system is $\boxed{\begin{cases} x(t) = 1 \\ y(t) = -t \end{cases}}$

This subchapter demonstrates that the Laplace Transformation method provides a comprehensive, systematic approach for solving complex systems of differential equations that arise in interconnected mechanical, electrical, and biological systems. The ability to handle piecewise inputs and integral terms makes it particularly valuable for modeling real-world systems with switching behavior and memory effects.

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3.6: Practice Set

Exercise 3.6.1

Find the solution to $\begin{cases} x' = x - 4y \\ y' = x + y \end{cases}$ with $x(0) = 2,\ y(0) = 0$

Answer available at the end of this section.

Exercise 3.6.2

Find the solution to $\begin{cases} x' = y + 2x + 10 - 10e^{-t} \\ y' = -y + 4x + 20 \end{cases}$ with $x(0) = 0,\ y(0) = 0$

Answer available at the end of this section.

Exercise 3.6.3

Find the solution to $\begin{cases} x' = y + \sin(t) \\ y' = -x + \cos(t) \end{cases}$ with $x(0) = 1,\ y(0) = 0$

Answer available at the end of this section.

Exercise 3.6.4

Find the solution to $\begin{cases} x' = 2y + 2te^{t} \\ y' = 3x + y - 6te^{t} \end{cases}$ with $x(0) = -1,\ y(0) = 2$

Answer available at the end of this section.

Exercise 3.6.5

Find the solution to $\begin{cases} x' = 2x + y - te^{3t} \\ y' = -x + 4y + (3 - t)e^{3t} \end{cases}$ with $x(0) = 0,\ y(0) = -3$

Answer available at the end of this section.

Exercise 3.6.6

Find the solution to $\begin{cases} x' = 2x - y + 1 \\ y'' = -2x + 4t^2 + 2 \end{cases}$ with $x(0) = -3,\ y(0) = -5,\ y'(0) = -4$

Answer available at the end of this section.

Exercise 3.6.7

Find the solution to $\begin{cases} x'' + x' = y \\ y' = 2x + 2e^{-t} \end{cases}$ with $x(0) = 0,\ x'(0) = 2,\ y(0) = 2$

Answer available at the end of this section.

Exercise 3.6.8

Find the solution to $\begin{cases} x' = y - t\,\mathcal{U}(t-1) \\ y' = x - t + t\,\mathcal{U}(t-1) \end{cases}$ with $x(0) = 0,\ y(0) = 1$

Answer available at the end of this section.

Exercise 3.6.9

Find the solution to $\begin{cases} \displaystyle\int_0^t x(w) \cdot e^{2t-2w}\,dw = y(t) \\ \displaystyle\int_0^t 3y(w)\,dw = x(t) + 2 - 4e^t \end{cases}$

Answer available at the end of this section.

Exercise 3.6.10

Find the solution to $\begin{cases} \displaystyle\int_0^t x(w) \cdot (t-w)\,dw = 2x(t) + y(t) \\ y'(t) = x(t) + 3 \end{cases}$ with $y(0) = 0$

Answer available at the end of this section.

Exercise 3.6.11

Find the solution to $\begin{cases} x' = y - 2x + 4\delta_2(t) \\ y' = x - 2y \end{cases}$ with $x(0) = 0,\ y(0) = 0$

Answer available at the end of this section.

Exercise 3.6.12

Find the solution to $\begin{cases} x'' = 4y - 5x + 6\delta_\pi(t) \\ y'' = 4x - 5y \end{cases}$ with $x(0) = 0,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = 0$

Answer available at the end of this section.

Exercise 3.6.13

Find the solution to $\begin{cases} x' = y - t \\ y' = z + 3 - t^3 \\ z' = 3x - 6y + 36 \end{cases}$ with $x(0) = 0,\ y(0) = 6,\ z(0) = 0$

Answer available at the end of this section.

Exercise 3.6.14

Find the solution to $\begin{cases} x' = y - x \\ y' = 2z \\ z' = x - z - 2te^{-t} \end{cases}$ with $x(0) = 0,\ y(0) = 2,\ z(0) = -1$

Answer available at the end of this section.

Exercise 3.6.15

Find the solution to $\begin{cases} x'(t) = z'(t) \\ y'(t) = x(t) - z(t) \\ \displaystyle\int_0^t x(w) + z(w)\,dw = 2\mathcal{U}(t-2) \end{cases}$ with $x(0) = -1,\ y(0) = 0,\ z(0) = 1$

Answer available at the end of this section.

Answers

Exercise 3.6.1

We take the Laplace Transformation of all interesting terms:

$$\mathscr{L}(x') = -x(0) + sX(s) = -2 + sX(s)$$
$$\mathscr{L}(y') = -y(0) + sY(s) = sY(s)$$

The system of equations becomes:

$$\begin{cases} -2 + sX(s) = X(s) - 4Y(s) \\ sY(s) = X(s) + Y(s) \end{cases} \Longrightarrow \begin{cases} (s-1)X(s) + 4Y(s) = 2 \\ X(s) = (s-1)Y(s) \end{cases}$$

Substituting in $X(s)$ from the second equation into the first gives:

$$(s-1)\left[(s-1)Y(s)\right] + 4Y(s) = 2$$
$$\Downarrow$$
$$\left((s-1)^2 + 4\right)Y(s) = 2$$
$$\Downarrow$$
$$Y(s) = \frac{2}{(s-1)^2 + 4}$$

Taking the Inverse Laplace Transformation gives:

$$y(t) = \mathscr{L}^{-1}\left(\frac{2}{(s-1)^2+4}\right) = e^t \cdot \mathscr{L}^{-1}\left(\frac{2}{s^2+4}\right) = e^t \sin(2t)$$

Substituting this into the second equation gives:

$$x(t) = y'(t) - y(t) = \left[e^t\left(2\cos(2t) + \sin(2t)\right)\right] - e^t \sin(2t) = 2e^t \cos(2t)$$

Therefore, the solution is $\begin{cases} x(t) = 2e^t \cos(2t) \\ y(t) = e^t \sin(2t) \end{cases}$

Exercise 3.6.2

We take the Laplace Transformation of all interesting terms:

$$\mathscr{L}(x') = -x(0) + sX(s) = sX(s)$$
$$\mathscr{L}(10 - 10e^{-t}) = \frac{10}{s} - \frac{10}{s+1} = \frac{10}{s(s+1)}$$
$$\mathscr{L}(y') = -y(0) + sY(s) = sY(s)$$
$$\mathscr{L}(20) = \frac{20}{s}$$

The system of equations becomes:

$$\begin{cases} sX(s) = Y(s) + 2X(s) + \dfrac{10}{s(s+1)} \\ sY(s) = -Y(s) + 4X(s) + \dfrac{20}{s} \end{cases} \Longrightarrow \begin{cases} Y(s) = (s-2)X(s) - \dfrac{10}{s(s+1)} \\ (s+1)Y(s) - 4X(s) = \dfrac{20}{s} \end{cases}$$

Substituting in $Y(s)$ from the first equation into the second gives:

$$(s+1)\left[(s-2)X(s) - \frac{10}{s(s+1)}\right] - 4X(s) = \frac{20}{s}$$
$$\Downarrow$$
$$\left(s^2 - s - 6\right)X(s) = \frac{30}{s}$$
$$\Downarrow$$
$$X(s) = \frac{30}{s(s-3)(s+2)} = \frac{3}{s+2} + \frac{2}{s-3} - \frac{5}{s}$$

Taking the Inverse Laplace Transformation gives:

$$x(t) = \mathscr{L}^{-1}\left(\frac{3}{s+2} + \frac{2}{s-3} - \frac{5}{s}\right) = 3e^{-2t} + 2e^{3t} - 5$$

Substituting this into the first equation gives:

$$y(t) = x'(t) - 2x(t) - 10 + 10e^{-t} = \left[-6e^{-2t} + 6e^{3t}\right] - 2\left[3e^{-2t} + 2e^{3t} - 5\right] - 10 + 10e^{-t} = -12e^{-2t} + 10e^{-t} + 2e^{3t}$$

Therefore, the solution is $\begin{cases} x(t) = 3e^{-2t} + 2e^{3t} - 5 \\ y(t) = -12e^{-2t} + 2e^{3t} + 10e^{-t} \end{cases}$

Exercise 3.6.3

We take the Laplace Transformation of all interesting terms:

$$\mathscr{L}(x') = -x(0) + sX(s) = -1 + sX(s)$$
$$\mathscr{L}(\sin(t)) = \frac{1}{s^2+1}$$
$$\mathscr{L}(y') = -y(0) + sY(s) = sY(s)$$
$$\mathscr{L}(\cos(t)) = \frac{s}{s^2+1}$$

The system of equations becomes:

$$\begin{cases} -1 + sX(s) = Y(s) + \dfrac{1}{s^2+1} \\ sY(s) = -X(s) + \dfrac{s}{s^2+1} \end{cases} \Longrightarrow \begin{cases} sX(s) - Y(s) = \dfrac{s^2+2}{s^2+1} \\ X(s) = \dfrac{s}{s^2+1} - sY(s) \end{cases}$$

Substituting in $X(s)$ from the second equation into the first gives:

$$s\left[\frac{s}{s^2+1} - sY(s)\right] - Y(s) = \frac{s^2+2}{s^2+1}$$
$$\Downarrow$$
$$\left(-s^2 - 1\right)Y(s) = \frac{2}{s^2+1}$$
$$\Downarrow$$
$$X(s) = -\frac{2}{(s^2+1)^2}$$

Taking the Inverse Laplace Transformation gives:

$$y(t) = -2\mathscr{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = -2\sin(t) \star \sin(t) = t\cos(t) - \sin(t)$$

Substituting this into the second equation gives:

$$x(t) = -y'(t) + \cos(t) = -\left[-t\sin(t)\right] + \cos(t) = t\sin(t) + \cos(t)$$

Therefore, the solution is $\begin{cases} x(t) = t\sin(t) + \cos(t) \\ y(t) = t\cos(t) - \sin(t) \end{cases}$

Exercise 3.6.4

We take the Laplace Transformation of all interesting terms:

$$\mathscr{L}(x') = -x(0) + sX(s) = 1 + sX(s)$$
$$\mathscr{L}(2te^{t}) = \frac{2}{(s-1)^2}$$
$$\mathscr{L}(y') = -y(0) + sY(s) = -2 + sY(s)$$
$$\mathscr{L}(6te^{t}) = \frac{6}{(s-1)^2}$$

The system of equations becomes:

$$\begin{cases} 1+sX(s)=2Y(s)+\dfrac{2}{(s-1)^2} \\ -2+sY(s)=3X(s)+Y(s)-\dfrac{6}{(s-1)^2} \end{cases} \Longrightarrow \begin{cases} X(s)=\dfrac{2}{s}Y(s)-\dfrac{1}{s}+\dfrac{2}{s(s-1)^2} \\ 3X(s)+(1-s)Y(s)=\dfrac{6}{(s-1)^2}-2 \end{cases}$$

Substituting in $X(s)$ from the first equation into the second gives:

$$\begin{gathered} 3\left[\frac{2}{s}Y(s)-\frac{1}{s}+\frac{2}{s(s-1)^2}\right]+(1-s)Y(s)=\frac{6}{(s-1)^2}-2 \\ \Downarrow \\ \left(\frac{6}{s}+1-s\right)Y(s)=\frac{2s^2+s+3}{s(s-1)} \\ \Downarrow \\ Y(s)=\frac{2s+1}{(s-1)(s+2)}=\frac{1}{s-1}+\frac{1}{s+2} \end{gathered}$$

Taking the Inverse Laplace Transformation gives:

$$y(t)=\mathscr{L}^{-1}\left(\frac{1}{s-1}+\frac{1}{s+2}\right)=e^t+e^{-2t}$$

Substituting this into the second equation gives:

$$x(t)=\frac{1}{3}y'(t)-\frac{1}{3}y(t)+2te^t=\frac{1}{3}\left[e^t-2e^{-2t}\right]-\frac{1}{3}\left[e^t+e^{-2t}\right]+2te^t=2te^t-e^{-2t}$$

Therefore, the solution is $\begin{cases} x(t)=2te^t-e^{-2t} \\ y(t)=e^t+e^{-2t} \end{cases}$

Exercise 3.6.5

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned} \mathscr{L}\left(x'\right)&=-x(0)+sX(s)=sX(s) \\ \mathscr{L}\left(te^{3t}\right)&=\frac{1}{(s-3)^2} \\ \mathscr{L}\left(y'\right)&=-y(0)+sY(s)=3+sY(s) \\ \mathscr{L}\left((3-t)e^{3t}\right)&=\frac{3}{s-3}-\frac{1}{(s-3)^2}=\frac{3s-10}{(s-3)^2} \end{aligned}$$

The system of equations becomes:

$$\begin{cases} sX(s)=2X(s)+Y(s)-\dfrac{1}{(s-3)^2} \\ 3+sY(s)=-X(s)+4Y(s)+\dfrac{3s-10}{(s-3)^2} \end{cases} \Longrightarrow \begin{cases} X(s)=\dfrac{1}{s-2}Y(s)-\dfrac{1}{(s-2)(s-3)^2} \\ X(s)+(s-4)Y(s)=\dfrac{3s-10}{(s-3)^2}-3 \end{cases}$$

Substituting in $X(s)$ from the first equation into the second gives:

$$\begin{gathered} \left[\frac{1}{s-2}Y(s)-\frac{1}{(s-2)(s-3)^2}\right]+(s-4)Y(s)=\frac{3s-10}{(s-3)^2}-3 \\ \Downarrow \\ \left(\frac{1}{s-2}+s-4\right)Y(s)=\frac{-3s^2+18s-25}{(s-3)(s-2)} \\ \Downarrow \\ Y(s)=\frac{-3s^2+18s-25}{(s-3)^3}=\frac{2}{(s-3)^3}-\frac{3}{s-3} \end{gathered}$$

Taking the Inverse Laplace Transformation gives:

$$y(t)=\mathscr{L}^{-1}\left(\frac{2}{(s-3)^3}-\frac{3}{s-3}\right)=t^2e^{3t}-3e^{3t}=(t^2-3)e^{3t}$$

Substituting this into the second equation gives:

$$x(t)=-y'(t)+4y(t)+(3-t)e^{3t}=-\left[(3t^2+2t-9)e^{3t}\right]+4\left[(t^2-3)e^{3t}\right]+(3-t)e^{3t}=(t^2-3t)e^{3t}$$

Therefore, the solution is $\begin{cases} x(t)=(t^2-3t)e^{3t} \\ y(t)=(t^2-3)e^{3t} \end{cases}$

Exercise 3.6.6

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned} \mathscr{L}\left(x'\right)&=-x(0)+sX(s)=3+sX(s) \\ \mathscr{L}\left(1\right)&=\frac{1}{s} \\ \mathscr{L}\left(y''\right)&=-y'(0)-sy(0)+s^2Y(s)=4+5s+s^2Y(s) \\ \mathscr{L}\left(4t^2+2\right)&=\frac{8}{s^3}+\frac{2}{s}=\frac{8+2s^2}{s^3} \end{aligned}$$

The system of equations becomes:

$$\begin{cases} 3+sX(s)=2X(s)-Y(s)+\dfrac{1}{s} \\ 4+5s+s^2Y(s)=-2X(s)+\dfrac{8+2s^2}{s^3} \end{cases} \Longrightarrow \begin{cases} Y(s)=(2-s)X(s)+\dfrac{1}{s}-3 \\ 2X(s)+s^2Y(s)=\dfrac{8+2s^2}{s^3}-4-5s \end{cases}$$

Substituting in $Y(s)$ from the first equation into the second gives:

$$2X(s)+s^2\left[(2-s)X(s)+\frac{1}{s}-3\right]=\frac{8+2s^2}{s^3}-4-5s$$
$$\Downarrow$$
$$\left(2+s^2(2-s)\right)X(s)=\frac{3s^5-6s^4-4s^3+2s^2+8}{s^5}$$
$$\Downarrow$$
$$X(s)=-\frac{3s^5-6s^4-4s^3+2s^2+8}{s^5(s^3-2s^2-2)}$$

Polynomial long division simplifies this greatly:

$$X(s)=\frac{4-3s^2}{s^3}=\frac{4}{s^3}-\frac{3}{s}$$

Taking the Inverse Laplace Transformation gives:

$$x(t)=\mathscr{L}^{-1}\left(\frac{4}{s^3}-\frac{3}{s}\right)=2t^2-3$$

Substituting this into the first equation gives:

$$y(t)=2x(t)-x'(t)+1=2\left[2t^2-3\right]-[4t]+1=4t^2-4t-5$$

Therefore, the solution is $\begin{cases} x(t)=2t^2-3 \\ y(t)=4t^2-4t-5 \end{cases}$

Exercise 3.6.7

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned}\mathscr{L}\left(x''\right)&=-x'(0)-sx(0)+s^2X(s)=-2+s^2X(s)\\ \mathscr{L}\left(x'\right)&=-x(0)+sX(s)=sX(s)\\ \mathscr{L}\left(y'\right)&=-y(0)+sY(s)=-2+sY(s)\\ \mathscr{L}\left(2e^{-t}\right)&=\frac{2}{s+1}\end{aligned}$$

The system of equations becomes:

$$\begin{cases} -2+s^2X(s)+sX(s)=Y(s) \\ -2+sY(s)=2X(s)+\dfrac{2}{s+1} \end{cases} \implies \begin{cases} Y(s)=(s^2+s)X(s)-2 \\ -2X(s)+sY(s)=\dfrac{2}{s+1}+2 \end{cases}$$

Substituting in $Y(s)$ from the first equation into the second gives:

$$-2X(s)+s\left[(s^2+s)X(s)-2\right]=\frac{2}{s+1}+2$$
$$\Downarrow$$
$$\left(-2+s(s^2+s)\right)X(s)=\frac{2s^2+4s+4}{s+1}$$
$$\Downarrow$$
$$X(s)=\frac{2s^2+4s+4}{(s+1)(s^3+s^2-2)}=\frac{2(s^2+2s+2)}{(s+1)(s-1)(s^2+2s+2)}=\frac{2}{(s+1)(s-1)}=\frac{1}{s-1}-\frac{1}{s+1}$$

Taking the Inverse Laplace Transformation gives:

$$x(t)=\mathscr{L}^{-1}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)=e^t-e^{-t}$$

Substituting this into the first equation gives:

$$y(t)=x''(t)+x'(t)=\left[e^t-e^{-t}\right]+\left[e^t+e^{-t}\right]=2e^t$$

Therefore, the solution is $\begin{cases} x(t)=e^t-e^{-t} \\ y(t)=2e^t \end{cases}$

Exercise 3.6.8

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned}\mathscr{L}\left(x'\right)&=-x(0)+sX(s)=sX(s)\\ \mathscr{L}\left(t\cdot\mathcal{U}(t-1)\right)&=\mathscr{L}\left(t+1\right)\cdot e^{-s}=\left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s}=\frac{1+s}{s^2}e^{-s}\\ \mathscr{L}\left(y'\right)&=-y(0)+sY(s)=-1+sY(s)\\ \mathscr{L}\left(-t\right)&=\frac{1}{s^2}\end{aligned}$$

The system of equations becomes:

$$\begin{cases} sX(s)=Y(s)-\dfrac{1+s}{s^2}e^{-s} \\ -1+sY(s)=X(s)-\dfrac{1}{s^2}+\dfrac{1+s}{s^2}e^{-s} \end{cases} \implies \begin{cases} Y(s)=sX(s)+\dfrac{1+s}{s^2}e^{-s} \\ -X(s)+sY(s)=\dfrac{1+s}{s^2}e^{-s}-\dfrac{1}{s^2}+1 \end{cases}$$

Substituting in $Y(s)$ from the first equation into the second gives:

$$-X(s)+s\left[sX(s)+\frac{1+s}{s^2}e^{-s}\right]=\frac{1+s}{s^2}e^{-s}-\frac{1}{s^2}+1$$
$$\Downarrow$$
$$\left(-1+s^2\right)X(s)=\frac{1-s^2}{s^2}e^{-s}+\frac{s^2-1}{s^2}$$
$$\Downarrow$$
$$X(s)=-\frac{1}{s^2}e^{-s}+\frac{1}{s^2}$$

Taking the Inverse Laplace Transformation gives:

$$x(t)=\mathscr{L}^{-1}\left(-\frac{1}{s^2}e^{-s}+\frac{1}{s^2}\right)=[-t]\Big|_{t\to t-1}\cdot\mathcal{U}(t-1)+t=t-(t-1)\cdot\mathcal{U}(t-1)$$

Substituting this into the first equation gives:

$$y(t)=x'(t)+t\,\mathcal{U}(t-1)=[1-1\cdot\mathcal{U}(t-1)]+t\,\mathcal{U}(t-1)=1+(t-1)\cdot\mathcal{U}(t-1)$$

Therefore, the solution is $\begin{cases} x(t)=t-(t-1)\cdot\mathcal{U}(t-1) \\ y(t)=1+(t-1)\cdot\mathcal{U}(t-1) \end{cases}$

Exercise 3.6.9

We take the Laplace Transformation of all interesting terms:

$$\mathscr{L}\left(x(t)\star e^{2t}\right)=X(s)\cdot\frac{1}{s-2}$$
$$\mathscr{L}\left(y(t)\star 3\right)=Y(s)\cdot\frac{3}{s}$$
$$\mathscr{L}\left(2-4e^{t}\right)=\frac{2}{s}-\frac{4}{s-1}$$

The system of equations becomes:

$$\begin{cases} X(s)\cdot\dfrac{1}{s-2}=Y(s) \\ Y(s)\cdot\dfrac{3}{s}=X(s)+\dfrac{2}{s}-\dfrac{4}{s-1} \end{cases}\quad\Longrightarrow\quad\begin{cases} X(s)=(s-2)Y(s) \\ Y(s)\cdot\dfrac{3}{s}-X(s)=\dfrac{-2s-2}{s(s-1)} \end{cases}$$

Substituting in $X(s)$ from the first equation into the second gives:

$$Y(s)\cdot\frac{3}{s}-(s-2)Y(s)=\frac{-2s-2}{s(s-1)}$$
$$\Downarrow$$
$$\left(\frac{3}{s}-s+2\right)Y(s)=\frac{-2(s+1)}{s(s-1)}$$
$$\Downarrow$$
$$Y(s)=\frac{2}{(s-1)(s-3)}=\frac{1}{s-3}-\frac{1}{s-1}$$

Taking the Inverse Laplace Transformation gives:

$$y(t)=\mathscr{L}^{-1}\left(\frac{1}{s-3}-\frac{1}{s-1}\right)=e^{3t}-e^{t}$$

Substituting this into the second equation gives

$$x(t)=\int_0^t 3y(w)\,dw-2+4e^{t}=\int_0^t\left(3e^{3w}-3e^{w}\right)\,dw-2+4e^{t}=\left[e^{3w}-3e^{w}\right]\Big|_0^t-2+4e^{t}=e^{3t}+e^{t}$$

Therefore, the solution is $\begin{cases} x(t)=e^{3t}+e^{t} \\ y(t)=e^{3t}-e^{t} \end{cases}$

Exercise 3.6.10

We take the Laplace Transformation of all interesting terms:

$$\mathscr{L}\left(x(t)\star t\right)=X(s)\cdot\frac{1}{s^2}$$
$$\mathscr{L}\left(y'(t)\right)=-y(0)+sY(s)=sY(s)$$
$$\mathscr{L}\left(3\right)=\frac{3}{s}$$

The system of equations becomes:

$$\begin{cases} X(s)\cdot\dfrac{1}{s^2}=2X(s)+Y(s) \\ sY(s)=2X(s)+\dfrac{3}{s} \end{cases}\quad\Longrightarrow\quad\begin{cases} Y(s)=\left(\dfrac{1}{s^2}-2\right)X(s) \\ sY(s)-2X(s)=\dfrac{3}{s} \end{cases}$$

Substituting in $Y(s)$ from the first equation into the second gives:

$$s\cdot\left(\frac{1}{s^2}-2\right)X(s)-X(s)=\frac{3}{s}$$
$$\Downarrow$$
$$\left(\frac{1}{s}-2s-1\right)X(s)=\frac{3}{s}$$
$$\Downarrow$$
$$X(s)=-\frac{3}{(s+1)(2s-1)}=\frac{1}{s+1}-\frac{1}{s-\frac{1}{2}}$$

Taking the Inverse Laplace Transformation gives:

$$x(t) = \mathscr{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s-\frac{1}{2}}\right) = e^{-t} - e^{\frac{1}{2}t}$$

We can use the first equation (in terms of s) to get:

$$Y(s) = \left(\frac{1}{s^2} - 2\right)X(s) = \frac{6s^2-3}{s^2(s+1)(2s-1)} = \frac{3}{s^2} + \frac{3}{s} - \frac{1}{s+1} - \frac{2}{s-\frac{1}{2}}$$

Taking the Inverse Laplace Transformation of this gives:

$$y(t) = \mathscr{L}^{-1}\left(\frac{3}{s^2} + \frac{3}{s} - \frac{1}{s+1} - \frac{2}{s-\frac{1}{2}}\right) = 3t - 3 - e^{-t} - 2e^{\frac{1}{2}t}$$

Therefore, the solution is $\begin{cases} x(t) = e^{-t} - e^{\frac{1}{2}t} \\ y(t) = 3t - 3 - e^{-t} - 2e^{\frac{1}{2}t} \end{cases}$

Exercise 3.6.11

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned} \mathscr{L}(x') &= -x(0) + sX(s) = sX(s) \\ \mathscr{L}(4\delta_2(t)) &= 4e^{-2s} \\ \mathscr{L}(y') &= -y(0) + sY(s) = sY(s) \end{aligned}$$

The system of equations becomes:

$$\begin{cases} sX(s) = Y(s) - 2X(s) + 4e^{-2s} \\ sY(s) = X(s) - 2Y(s) \end{cases} \implies \begin{cases} (s+2)X(s) - Y(s) = 4e^{-2s} \\ X(s) = (s+2)Y(s) \end{cases}$$

Substituting in $X(s)$ from the second equation into the first gives:

$$(s+2)\cdot(s+2)Y(s) - Y(s) = 4e^{-2s} \implies Y(s) = \frac{4}{(s+1)(s+3)}e^{-2s} = \left(\frac{2}{s+1} - \frac{2}{s+3}\right)e^{-2s}$$

Taking the Inverse Laplace Transformation gives:

$$y(t) = \mathscr{L}^{-1}\left(\left(\frac{2}{s+1} - \frac{2}{s+3}\right)e^{-2s}\right) = \left[2e^{-t} - 2e^{-3t}\right]_{t\to t-2}\cdot\mathcal{U}(t-2) = \left(2e^{2-t} - 2e^{6-3t}\right)\mathcal{U}(t-2)$$

Substituting this into the second equation gives:

$$x(t) = y'(t) + 2y(t) = \left[\left(-2e^{2-t} + 6e^{6-3t}\right)\mathcal{U}(t-2)\right] + 2\cdot\left(2e^{2-t} - 2e^{6-3t}\right)\mathcal{U}(t-2) = \left(2e^{2-t} + 2e^{6-3t}\right)\mathcal{U}(t-2)$$

Therefore, the solution is $\begin{cases} x(t) = \left(2e^{2-t} + 2e^{6-3t}\right)\mathcal{U}(t-2) \\ y(t) = \left(2e^{2-t} - 2e^{6-3t}\right)\mathcal{U}(t-2) \end{cases}$

Exercise 3.6.12

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned} \mathscr{L}(x'') &= -x'(0) - sx(0) + s^2X(s) = s^2X(s) \\ \mathscr{L}(6\delta_\pi(t)) &= 6e^{-\pi s} \\ \mathscr{L}(y'') &= -y'(0) - sy(0) + s^2Y(s) = s^2Y(s) \end{aligned}$$

The system of equations becomes:

$$\begin{cases} s^2X(s) = 4Y(s) - 5X(s) + 6e^{-\pi s} \\ s^2Y(s) = 4X(s) - 5Y(s) \end{cases} \implies \begin{cases} (s^2+5)X(s) - 4Y(s) = 6e^{-\pi s} \\ -4X(s) + (s^2+5)Y(s) = 0 \end{cases}$$

We multiply the bottom equation by s^2+5 and add it to 4 times the top equation to eliminate $X(s)$:

$$\left((s^2+5)^2 - 16\right)Y(s) = 24e^{-\pi s} \implies Y(s) = \frac{24}{(s^2+1)(s^2+9)}e^{-\pi s} = \left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right)e^{-\pi s}$$

Taking the Inverse Laplace Transformation gives:

$$y(t) = \mathscr{L}^{-1}\left(\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right)e^{-\pi s}\right) = \left[3\sin(t) - \sin(3t)\right]_{t\to t-\pi}\cdot\mathcal{U}(t-\pi) = \left(-3\sin(t) + \sin(3t)\right)\mathcal{U}(t-\pi)$$

Using $Y(s) = \dfrac{24}{(s^2+1)(s^2+9)}e^{-\pi s}$, we can substitute this into either equation to find $X(s)$:

$$-4X(s) + (s^2+5)\cdot\frac{24}{(s^2+1)(s^2+9)}e^{-\pi s} = 0 \implies X(s) = \frac{6s^2+30}{(s^2+1)(s^2+9)}e^{-\pi s} = \left(\frac{3}{s^2+1} + \frac{3}{s^2+9}\right)e^{-\pi s}$$

Taking the Inverse Laplace Transformation gives:

$$x(t) = \mathscr{L}^{-1}\left(\left(\frac{3}{s^2+1} + \frac{3}{s^2+9}\right)e^{-\pi s}\right) = \left[3\sin(t) + \sin(3t)\right]_{t\to t-\pi}\cdot\mathcal{U}(t-\pi) = \left(-3\sin(t) - \sin(3t)\right)\mathcal{U}(t-\pi)$$

Therefore, the solution is $\begin{cases} x(t) = \left(-3\sin(t) - \sin(3t)\right)\mathcal{U}(t-\pi) \\ y(t) = \left(-3\sin(t) + \sin(3t)\right)\mathcal{U}(t-\pi) \end{cases}$

Exercise 3.6.13

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned} \mathscr{L}(x') &= -x(0) + sX(s) = sX(s) & \mathscr{L}(t) &= \frac{1}{s^2} \\ \mathscr{L}(y') &= -y(0) + sY(s) = -6 + sY(s) & \mathscr{L}(3-t^3) &= \frac{3}{s} - \frac{6}{s^4} \\ \mathscr{L}(z') &= -z(0) + sZ(s) = sZ(s) & \mathscr{L}(36) &= \frac{36}{s} \end{aligned}$$

The system of equations becomes:

$$\begin{cases} sX(s) = Y(s) - \frac{1}{s^2} \\ -6 + sY(s) = Z(s) + \frac{3}{s} - \frac{6}{s^4} \\ sZ(s) = 3X(s) - 6Y(s) + \frac{36}{s} \end{cases} \implies \begin{cases} Y(s) = sX(s) + \frac{1}{s^2} \\ Z(s) = sY(s) + \frac{6}{s^4} - \frac{3}{s} - 6 \\ -3X(s) + 6Y(s) + sZ(s) = \frac{36}{s} \end{cases}$$

Substituting in $Y(s)$ from the first equation into the second gives:

$$Z(s) = s\left[sX(s) + \frac{1}{s^2}\right] + \frac{6}{s^4} - \frac{3}{s} - 6 = s^2X(s) + \frac{6}{s^4} - \frac{2}{s} - 6$$

Substituting each of these into the third equation gives:

$$s\left[s^2X(s) + \frac{6}{s^4} - \frac{2}{s} - 6\right] = 3X(s) - 6\left[sX(s) + \frac{1}{s^2}\right] + \frac{36}{s}$$
$$\Downarrow$$
$$s^3X(s) + \frac{6}{s^3} - 2 - 6s = 3X(s) - 6sX(s) - \frac{6}{s^2} + \frac{36}{s}$$
$$\Downarrow$$
$$X(s) = \frac{6s^4 + 2s^3 + 36s^2 - 6s - 6}{s^3(s^3 + 6s - 3)}$$

Some polynomial long division helps to factor this into:

$$X(s) = \frac{2(3s+1)(s^3 + 6s - 3)}{s^3(s^3 + 6s - 3)} = \frac{6s+2}{s^3} = \frac{6}{s^2} + \frac{2}{s^3}$$

Taking the Inverse Laplace Transformation gives:

$$x(t) = \mathscr{L}^{-1}\left(\frac{6}{s^2} + \frac{2}{s^3}\right) = 6t + t^2$$

Substituting this into the first equation gives:

$$y(t) = x'(t) + t = [6 + 2t] + t = 6 + 3t$$

Substituting this into the second equation gives:

$$z(t) = y'(t) - 3 + t^3 = [3] - 3 + t^3 = t^3$$

Therefore, the solution is $\begin{cases} x(t) = t^3 + 6t \\ y(t) = 3t + 6 \\ z(t) = t^3 \end{cases}$

Exercise 3.6.14

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned} \mathscr{L}(x') &= -x(0) + sX(s) = sX(s) \\ \mathscr{L}(y') &= -y(0) + sY(s) = -2 + sY(s) \\ \mathscr{L}(z') &= -z(0) + sZ(s) = 1 + sZ(s) \\ \mathscr{L}(2te^{-t}) &= \frac{2}{(s+1)^2} \end{aligned}$$

The system of equations becomes:

$$\begin{cases} sX(s) = Y(s) - X(s) \\ -2 + sY(s) = 2Z(s) \\ 1 + sZ(s) = X(s) - Z(s) - \frac{2}{(s+1)^2} \end{cases} \implies \begin{cases} Y(s) = (s+1)X(s) \\ Y(s) = \frac{2}{s}Z(s) + \frac{2}{s} \\ (s+1)Z(s) - X(s) = -\frac{2}{(s+1)^2} - 1 \end{cases}$$

Substituting in $Y(s)$ from the top equation into the middle equation gives:

$$(s+1)X(s) = \frac{2}{s}Z(s) + \frac{2}{s} \implies X(s) = \frac{2}{s(s+1)}Z(s) + \frac{2}{s(s+1)}$$

Substituting these both into the third equation gives:

$$(s+1)Z(s) - \left[\frac{2}{s(s+1)}Z(s) + \frac{2}{s(s+1)}\right] = -\frac{2}{(s+1)^2} - 1$$
$$\Downarrow$$
$$\left((s+1) - \frac{2}{s(s+1)}\right)Z(s) = -\frac{2}{(s+1)^2} + \frac{2}{s(s+1)} - 1$$
$$\Downarrow$$
$$\frac{s^3 + 2s^2 + s - 2}{s(s+1)} \cdot Z(s) = -\frac{s^3 + 2s^2 + s - 2}{s(s+1)^2} \implies Z(s) = -\frac{1}{s+1}$$

Taking the Inverse Laplace Transformation gives:

$$z(t) = \mathscr{L}^{-1}\left(-\frac{1}{s+1}\right) = -e^{-t}$$

Substituting this into the third equation gives:

$$x(t) = z'(t) + z(t) + 2te^{-t} = [e^{-t}] + [-e^{-t}] + 2te^{-t} = 2te^{-t}$$

Substituting this into the first equation gives:

$$y(t) = x'(t) + x(t) = [(2 - 2t)e^{-t}] + [2te^{-t}] = 2e^{-t}$$

Therefore, the solution is $\begin{cases} x(t) = 2te^{-t} \\ y(t) = 2e^{-t} \\ z(t) = -e^{-t} \end{cases}$

Exercise 3.6.15

We take the Laplace Transformation of all interesting terms:

$$\begin{aligned} \mathscr{L}(x') &= -x(0) + sX(s) = 1 + sX(s) \\ \mathscr{L}(z') &= -z(0) + sZ(s) = -1 + sZ(s) \\ \mathscr{L}(y') &= -y(0) + sY(s) = sY(s) \\ \mathscr{L}((x+z) \star 1) &= (X(s) + Z(s)) \cdot \frac{1}{s} = \frac{1}{s}X(s) + \frac{1}{s}Z(s) \\ \mathscr{L}(2\mathcal{U}(t-2)) &= \frac{2}{s}e^{-2s} \end{aligned}$$

The system of equations becomes:

$$\begin{cases} 1 + sX(s) = -1 + sZ(s) \\ sY(s) = X(s) - Z(s) \\ \frac{1}{s}X(s) + \frac{1}{s}Z(s) = \frac{2}{s}e^{-2s} \end{cases} \implies \begin{cases} X(s) = Z(s) - \frac{2}{s} \\ Y(s) = \frac{1}{s}X(s) - \frac{1}{s}Z(s) \\ X(s) + Z(s) = 2e^{-2s} \end{cases}$$

Substituting in $X(s)$ from the top equation into the bottom equation gives:

$$Z(s) - \frac{2}{s} + Z(s) = 2e^{-2s} \implies Z(s) = e^{-2s} + \frac{1}{s}$$

Substituting this into each of the first and second equations gives us $X(s)$ and $Y(s)$ respectively:

$$X(s) = Z(s) - \frac{2}{s} = e^{-2s} - \frac{1}{s} \quad Y(s) = \frac{1}{s}X(s) - \frac{1}{s}Z(s) = -\frac{2}{s^2}$$

Taking the Inverse Laplace Transformation of each of these three gives:

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}(X(s)) = \mathscr{L}^{-1}\left(e^{-2s} - \frac{1}{s}\right) = \delta_2(t) - 1 \\ y(t) &= \mathscr{L}^{-1}(Y(s)) = \mathscr{L}^{-1}\left(-\frac{2}{s^2}\right) = -2t \\ z(t) &= \mathscr{L}^{-1}(Z(s)) = \mathscr{L}^{-1}\left(e^{-2s} + \frac{1}{s}\right) = \delta_2(t) + 1 \end{aligned}$$

Therefore, the solution is $\begin{cases} x(t) = \delta_2(t) - 1 \\ y(t) = -2t \\ z(t) = \delta_2(t) + 1 \end{cases}$

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4: POWER SERIES METHODS AND APPROXIMATIONS

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4.1: Power Series

Learning Objectives

- Review power series notation, convergence, and basic operations.
- Master the techniques of index shifting and differentiation of power series.
- Understand the concept of analytic functions and their importance in solving ODEs.
- Develop proficiency in manipulating power series for substitution into differential equations.

Before solving equations with series, we build the essential toolkit. This subchapter reviews power series operations, with a heavy emphasis on the crucial technique of index shifting, and introduces the concept of analytic functions, which guarantees the existence of series solutions.

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4.1: Power Series Review

In calculus, students learn that sufficiently smooth functions can be approximated by their tangent lines, and later that they can be approximated by any degree polynomial (called a *Taylor Polynomial*). As the degree of the Taylor polynomial increases, the approximation generally gets "closer" to the actual function $f(x)$.

If $f(x)$ is "very smooth" at $x = a$, we extend this process to find an "infinite degree" Taylor polynomial, known as the *Taylor Series* of $f(x)$ centered at $x = a$.

$$\hat{f}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This new function $\hat{f}(x)$ is a *power series* that (we hope) represents $f(x)$ when x is sufficiently close to a.

Definition: Power Series

Given any sequence of real numbers c_n for $n = 0, 1, \ldots$ and $a^{\text{int}}\mathbb{R}$, the function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is a ***power series centered at*** $x = a$.

The domain of a power series contains all $x^{\text{int}}\mathbb{R}$ such that the sum that defines $f(x)$ converges. We call this domain the ***interval of convergence*** of $f(x)$.
(We denote the *radius of convergence* as the distance from either endpoint of the interval of convergence to a.)

Determining the interval and radius of convergence for a given power series is generally done via the Ratio Test or Root Test.

Illustration 4.1.1

We will find the interval and radius of convergence of each power series below:

(a) $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$

(b) $g(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(x+5)^n$

(c) $h(x) = \sum_{n=9}^{\infty} (2n)!(x-7)^n$

(a) The coefficients inside of $f(x)$ are $c_n = \frac{1}{n}$. We can find the radius of convergence R with the root test:

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{n}} = \frac{1}{1} \implies R = 1$$

Since this power series is centered at $a = 2$, we know the interval of convergence must contain the open interval $(a - R, a + R) = (1, 3)$

We manually check each endpoint to determine convergence.

At $x=1$, we have $f(1)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges (due to the *Alternating Series Test*).

At $x=3$, we have $f(3)=\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges (due to the p-*test* or *integral test*).

Therefore, the interval of convergence of this power series is $[1,3)$

(b) The coefficients inside of $g(x)$ are $c_n=\frac{1}{n!}$. We can find the radius of convergence R with the ratio test:

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=\lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}=\lim_{n\to\infty}\frac{1}{n+1}=0\implies R=+\infty$$

Since the radius of convergence is infinite, we know the interval of convergence must contain all real numbers: $(-\infty,\infty)$

(c) The coefficients inside of $h(x)$ are $c_n=(2n)!$. We can find the radius of convergence R with the ratio test:

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=\lim_{n\to\infty}\frac{(2n+2)!}{(2n)!}=\lim_{n\to\infty}(2n+1)(2n+2)=+\infty\implies R=0$$

Since this power series is centered at $a=7$, we know the interval of convergence must contain only the point $x=7$

(This is not much of a function! It is only defined at one point, $h(7)=0$)

Functions defined as power series can be combined and manipulated in the same manner that any other functions can in calculus.

Theorem 4.1.1

Let $f(x)=\sum_{n=0}^{\infty}f_n(x-a)^n$ and $g(x)=\sum_{n=0}^{\infty}g_n(x-a)^n$ be power series with radius of convergence $R_f>0$ and $R_g>0$. Then,

$$h(x)=(f\pm g)(x)\implies h(x)=\sum_{n=0}^{\infty}(f_n\pm g_n)(x-a)^n \text{ for } |x-a|<\min(R_f,R_g)$$

This theorem says that the sum or difference of two power series centered at $x=a$ is a function which is also a power series centered at $x=a$, and that the addition or subtraction of these series is done *term-by-term*. For instance, if we know that $e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ and $e^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n$, then:

$$e^x+e^{-x}=\sum_{n=0}^{\infty}\frac{1}{n!}x^n+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n=\sum_{n=0}^{\infty}\frac{1+(-1)^n}{n!}x^n=\sum_{k=0}^{\infty}\frac{2}{(2k)!}x^{2k}$$
$$e^x-e^{-x}=\sum_{n=0}^{\infty}\frac{1}{n!}x^n-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n=\sum_{n=0}^{\infty}\frac{1-(-1)^n}{n!}x^n=\sum_{k=0}^{\infty}\frac{2}{(2k+1)!}x^{2k+1}$$

Note that this gives us the power series representations of $2\cosh(x)$ and $2\sinh(x)$ respectively.

Theorem 4.1.2

Let $f(x)=\sum_{n=0}^{\infty}f_n(x-a)^n$ be a power series with radius of convergence $R>0$. Then,

\[h(x)=f'(x) \implies h(x)=\sum_{n=0}^\infty nf_n (x-a)^{n-1} \text{ for } |x-a|

This theorem says that the derivative of a power series function centered at $x=a$ always exists and is also a power series centered at $x=a$. We see that terms in the derivative function are simply the *term-by-term* derivatives of the original function $f(x)$. We can use $f(x)=\sinh(x)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}$ to see:

$$f'(x)=\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}\right)'=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(x^{2n+1}\right)'=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left((2n+1)x^{2n}\right)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}$$

This gives the power series representation of $\cosh(x)$, which is exactly equal to the derivative of $f(x)=\sinh(x)$

Theorem 4.1.3

Let $f(x)=\sum_{n=0}^{\infty}f_n(x-a)^n$ be a power series with radius of convergence $R>0$. Then,

\[h(x)=\int_a^x f(t)\,dt \implies h(x)=\sum_{n=0}^\infty \dfrac{f_n}{n+1} (x-a)^{n+1} \text{ for } |x-a|

This theorem says that the anti-derivative of a power series function centered at $x=a$ always exists and is also a power series centered at $x=a$. We see that terms in the anti-derivative function are simply the *term-by-term* anti-derivatives of the original function $f(x)$. This should come as no surprise, and it is a direct result of the previous theorem.

The number n in each series above is called the *index* of the sum, and since the derivative and anti-derivative change the power of x we see in the power series, it is often advantageous to rewrite a power series using a different index. This is just like substitution in an integral: we define how our new index relates to the old one, and then make sure to change the limits of our sum to match the new index.

For example, consider the function $f(x)=\sum_{n=0}^{\infty}(n-1)^2x^{n+1}$

If we use $m=n-1\leftrightarrow n=m+1$, we see that $n=0\leftrightarrow m=-1,\ \ n=\infty\leftrightarrow m=\infty$ so:

$$f(x)=\sum_{m=-1}^{\infty}\left((m+1)-1\right)^2x^{(m+1)+1}=\sum_{m=-1}^{\infty}m^2x^{m+2}$$

Alternatively, if we use $j=n+1\leftrightarrow n=j-1$, we see that $n=0\leftrightarrow j=1,\ \ n=\infty\leftrightarrow j=\infty$ so:

$$f(x)=\sum_{j=1}^{\infty}\left((j-1)-1\right)^2x^{(j-1)+1}=\sum_{j=1}^{\infty}(j-2)^2x^j$$

Each of these substitutions shifts the index of the power series to make it appear different, but still represents the same function. This operation (referred to as an **index shift**) allows us to more easily add or subtract power series together.

Example 4.1.1

Consider the function $f(x)=\sum_{n=0}^{\infty}f_n(x-a)^n$ and simplify each expression below.

(a) $f(x)+f'(x)$

(b) $f''(x)-4\int_a^x f(t)\,dt$

Solution

(a) First, we find the expression for $f'(x)$ by differentiating term-by-term: $f'(x) = \sum_{n=0}^{\infty} n f_n (x-a)^{n-1}$

Since the powers of x do not match, we cannot immediately add this to $f(x) = \sum_{n=0}^{\infty} f_n (x-a)^n$

We must do an index shift, and the most obvious choice is to make the exponent of $(x-a)$ as simple as possible. Therefore, we set $m = n-1$ and adjust $f'(x)$:

$$f'(x) = \sum_{n=0}^{\infty} n f_n (x-a)^{n-1} = \sum_{m=-1}^{\infty} (m+1) f_{m+1} (x-a)^m$$

Just to make the letters match, we write $m = n$ in $f(x)$: $f(x) = \sum_{m=0}^{\infty} f_m (x-a)^m$

Now, when we try to add these series together we see that they do not have the same index limits (f starts at $m=0$ while f' starts at $m=-1$). To fix this, we always manually write out the "extra terms" from the series which starts at the lower number.

$$\sum_{m=-1}^{\infty} (m+1) f_{m+1} (x-a)^m = \underbrace{0 f_0 (x-a)^{-1}}_{m=-1} + \sum_{m=0}^{\infty} (m+1) f_{m+1} (x-a)^m = \sum_{m=0}^{\infty} (m+1) f_{m+1} (x-a)^m$$

Finally, we can add these together:

$$f(x) + f'(x) = \sum_{m=0}^{\infty} f_m (x-a)^m + \sum_{m=0}^{\infty} (m+1) f_{m+1} (x-a)^m = \sum_{m=0}^{\infty} \left(f_m + (m+1) f_{m+1}\right) (x-a)^m$$

Therefore, our final answer is $\boxed{f(x) + f'(x) = \sum_{m=0}^{\infty} \left(f_m + (m+1) f_{m+1}\right) (x-a)^m}$

(b) First, we find the expression for $f''(x)$ by differentiating twice term-by-term: $f''(x) = \sum_{n=0}^{\infty} n(n-1) f_n (x-a)^{n-2}$

Next, we integrate term-by-term to find $\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{f_n}{n+1} (x-a)^{n+1}$

We index shift each of these so the exponent of $(x-a)$ is exactly m:

$$m = n-2 \implies f''(x) = \sum_{m=-2}^{\infty} (m+2)(m+1) f_{m+2} (x-a)^m$$
$$m = n+1 \implies \int_0^x f(t)\,dt = \sum_{m=1}^{\infty} \frac{f_{m-1}}{m} (x-a)^m$$

Once again, these do not start at the same initial value of m, so we manually write out the terms they *do not* have in common:

$$\sum_{m=-2}^{\infty} (m+2)(m+1) f_{m+2} (x-a)^m = \underbrace{0 f_0 (x-a)^{-2}}_{m=-2} + \underbrace{0 f_1 (x-a)^{-1}}_{m=-1} + \underbrace{2 f_2 (x-a)^0}_{m=0} + \sum_{m=1}^{\infty} (m+2)(m+1) f_{m+2} (x-a)^m = 2f_2 + \sum_{m=1}^{\infty} (m$$

Finally, we can combine the series:

$$\begin{aligned} f''(x) - 4\int_0^x f(t)\,dt &= 2f_2 + \sum_{m=1}^{\infty} (m+2)(m+1) f_{m+2} (x-a)^m - 4\sum_{m=1}^{\infty} \frac{f_{m-1}}{m} (x-a)^m \\ &= 2f_2 + \sum_{m=1}^{\infty} \left((m+2)(m+1) f_{m+2} - \frac{4}{m} f_{m-1} \right) (x-a)^m \end{aligned}$$

Therefore, our final answer is $\boxed{f''(x) - 4\int_0^x f(t)\,dt = 2f_2 + \sum_{m=1}^{\infty}\left((m+2)(m+1)f_{m+2} - \frac{4}{m}f_{m-1}\right)(x-a)^m}$

For simple functions $g(x)$, such as linear functions or power functions, we can find the power series representation of composition as well.

Theorem 4.1.4

Let $f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n$ be a power series with radius of convergence $R > 0$. Then,

\[h(x)=f\left(g(x)\right) \implies h(x)=\sum_{n=0}^\infty f_n \left(g(x)-a\right)^n \text{ for } |g(x)-a|

This allows us to see that since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$, we must have

$$e^{2x} = \sum_{n=0}^{\infty}\frac{1}{n!}(2x)^n = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n \quad \text{and} \quad e^{x^2} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(x^2\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^{2n}$$

Note that this is not particularly useful for functions like $e^{\sin(x)}$, since $\sum_{n=0}^{\infty}\frac{1}{n!}(\sin(x))^n$ is not a power series.

Multiplication of two power series is also possible, but not pretty.

Illustration 4.1.2

Consider the example where $f(x) = g(x) = e^x$:

$$f(x) = g(x) = \sum_{n=0}^{\infty}\frac{1}{n!}x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots$$

When we multiply $f(x)\cdot g(x)$ we see:

$$f(x)\cdot g(x) = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots\right)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots\right)$$

Notice that this is just the infinitely long version of FOIL. When we collect like powers of x from this product we see:

$$f(x)\cdot g(x) = (1\cdot 1) + (1\cdot x + x\cdot 1) + \left(1\cdot\frac{1}{2}x^2 + x\cdot x + \frac{1}{2}x^2\cdot 1\right) + \left(1\cdot\frac{1}{3!}x^3 + x\cdot\frac{1}{2}x^2 + \frac{1}{2}x^2\cdot x + \frac{1}{3!}x^3\cdot 1\right) + \ldots$$

Theorem 4.1.5: Cauchy's Product Formula

Let $f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} g_n(x-a)^n$ be power series with radius of convergence $R_f > 0$ and $R_g > 0$. Then,

$$h(x) = f(x)\cdot g(x) \implies h(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} f_k g_{n-k}\right)(x-a)^n \text{ for } |x-a| < \min(R_f, R_g)$$

Using this product formula, we can more easily simplify our product in Illustration 4.1.2 above. Since $f_n = \frac{1}{n!}$ and $g_n = \frac{1}{n!}$ we get:

$$\sum_{k=0}^{n} f_k g_{n-k} = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{1}{(n-k)!} = \frac{2^n}{n!}$$

(This sum is just the binomial expansion formula for $(1+1)^n$ scaled by $n!$, but the reader does not need to worry if they do not recognize it!)

Finally, plugging this into Cauchy's Product Formula, we see the product becomes:

$$e^x \cdot e^x = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} (2x)^n = e^{2x}$$

Notice that we may not always be able to find an explicit way to simplify $\sum_{k=0}^{n} f_k g_{n-k}$, but the formula still holds.

Example 4.1.2

Find the first four nonzero terms of the power series representing $\cosh(x^2) \cdot \sinh(2x)$

Solution

To take this product, we first need to find the individual power series that represent $f(x) = \cosh(x^2)$ and $g(x) = \sinh(2x)$

We have seen that $\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$, so

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{4n} = 1 + \frac{1}{2}x^4 + \frac{1}{24}x^8 + \ldots$$

We also know that $\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$, so

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1} = 2x + \frac{4}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{315}x^7 + \ldots$$

We take the product of these two power series to get:

$$\begin{aligned} f(x) \cdot g(x) &= (1 \cdot 2x) + \left(1 \cdot \frac{4}{3}x^3\right) + \left(1 \cdot \frac{4}{15}x^5 + \frac{1}{2}x^4 \cdot 2x\right) + \left(1 \cdot \frac{8}{315}x^7 + \frac{1}{2}x^4 \cdot \frac{4}{3}x^3\right) + \ldots \\ &= 2x + \frac{4}{3}x^3 + \frac{19}{15}x^5 + \frac{218}{315}x^7 + \ldots \end{aligned}$$

Therefore, the power series we are looking for begins with $\boxed{h(x) = 2x + \frac{4}{3}x^3 + \frac{19}{15}x^5 + \frac{218}{315}x^7 + \ldots}$

Just like when multiplying two polynomials, if one of the functions involved in the product is a power function (monomial), multiplication is much easier.
For instance, the function $f(x) = x^r e^x$ can be written:

$$f(x) = x^3 \cdot e^x = x^r \cdot \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+r}$$

Notice that when r is not a natural number, the resulting function is *not a power series* since the powers of x are not all non-negative integers. Not all functions *can* be written as a power series centered around a given point. Remember that Taylor's formula tells us the formula for finding a power series representation of $f(x)$ centered at $x = a$ requires the values of $f^{(n)}(a)$ to be finite for all n. One way to refer to this property is to say that $f(x)$ is *infinitely smooth (or infinitely differentiable) at $x = a$.*

Definition: Analytic Function

A function $f(x)$ is called ***analytic*** at $x = a$ if there exists some power series $\hat{f}(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ such that $f(x) = \hat{f}(x)$ for some interval containing $x = a$.

Any function which is analytic at $x = a$ must have finite values of $f^{(n)}(a)$ for each $n = 0, 1, ...$

In the context of differential equations, it is important to be able to tell if a given function is analytic or not at a given point. Let's examine a few familiar functions.

Illustration 4.1.3

We shall find all values of $a^{\text{int}}\mathbb{R}$ such that the given function is analytic at $x = a$ for each function below.

(a) $p(x) = x^2 + 3$

(b) $f(x) = \dfrac{1}{1-x}$

(c) $g(x) = x^{\frac{5}{2}}$

(a) The function $p(x) = x^2 + 3$ is a polynomial, which is technically a power series already.

To see it centered at $x = a$, we rewrite: $p(x) = (x-a)^2 + 2a(x-a) + (a^2+3)$

Clearly, this function is analytic everywhere.

(b) The function $f(x) = \dfrac{1}{1-x}$ has domain $(-\infty, 1) \cup (1, \infty)$

Since $f(1)$ is not defined, it is impossible for $f(x)$ to be analytic at $a = 1$

However, for any $a \neq 1$, we can see that $f^{(n)}(a) = n!(1-a)^{-n-1}$, and $\hat{f}(x) = \sum_{n=0}^{\infty} \dfrac{(x-a)^n}{(1-a)^{n+1}}$

This power series has radius of convergence $R = |1-a| > 0$, and $f(x) = \hat{f}(x)$ for $|x-a| < |1-a|$

Therefore, this function is analytic for all $a \neq 1$

(c) The function $g(x) = x^{\frac{5}{2}}$ is defined and continuous on its domain $D = [0, \infty)$

The derivative $g'(x) = \dfrac{5}{2}x^{\frac{3}{2}}$ and second derivative $g''(x) = \dfrac{15}{4}x^{\frac{1}{2}}$ are also defined and continuous on D.

However, $g'''(x) = \dfrac{15}{8}x^{-\frac{1}{2}}$ is not defined at $a = 0$, so $g(x)$ cannot be analytic at $a = 0$

For all positive values of a it is possible to find the power series representation of $g(x)$ centered at a:

$$\hat{g}(x) = \sum_{n=0}^{\infty} \frac{5 \cdot 3 \cdot 1 \cdot (-1) \cdots (7-2n) \cdot a^{\frac{5}{2}-n}}{n! \cdot 2^n}(x-a)^n = a^{\frac{5}{2}} + \frac{5}{2}a^{\frac{3}{2}}(x-a) + \frac{15}{8}a^{\frac{1}{2}}(x-a)^2 - 15a^{\frac{5}{2}} \cdot \sum_{n=3}^{\infty} \frac{(2n-7)!!}{n! \cdot (-2a)^n}(x-a)^n$$

We know that $\hat{g}(x)$ has a positive radius of convergence, so that it equals $g(x)$ on some interval.

Therefore, this function is analytic for all $a > 0$

Note

Note that in the example above, we used the notation $(2n-7)!! = (2n-7)\cdot(2n-9)\cdot(2n-11)\cdots 3\cdot 1$, which is a *double factorial*.
In general, the number of factorials listed after an expression tells you how much to incrementally decrease the initial value by in the calculation.

$$7!! = 7\cdot 5\cdot 3\cdot 1 = 105 \quad 7!!! = 7\cdot 4\cdot 1 = 28 \quad 7!!!! = 7\cdot 3 = 21$$

It is always implied that once you reach or pass the number 1, the factorial stops.

Thankfully, we do know that the sum, difference, or product of two analytic functions is always analytic.
Below are a few famous power series (centered at $a=0$) of analytic functions that are worth memorizing.

Explicit Function Definition	Power Series Representation around $a=0$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$ for $\lvert x\rvert < 1$
$(1+x)^{\alpha}$	$\left(\sum_{n=0}^{\infty} \alpha \atop nx^n\right)$ for all $\lvert x\rvert < 1$
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for all x
$\cosh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$ for all x
$\sinh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$ for all x
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ for all x
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ for all x

The last operation we will discuss is *division* by a power series. As we have seen, multiplication of two power series yields the Cauchy product formula which is sometimes difficult to simplify. This should hint to us that division will also be unpleasant.

Theorem 4.1.6

Let $f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} g_n(x-a)^n$ be power series with radius of convergence $R_f > 0$ and $R_g > 0$. Then,

$$h(x) = \frac{f(x)}{g(x)} \implies h(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ for } \lvert x-a\rvert < \min(R_f, R_g)$$

where c_n satisfies $\sum_{k=0}^{n} c_k g_{n-k} = f_n$ for each $n = 0, 1, \ldots$

This theorem simply states that if $g(x)\cdot h(x) = f(x)$, then $h(x) = \frac{f(x)}{g(x)}$. In practice, it gives us an infinitely long system of equations to solve for $c_0, c_1, \ldots$

Illustration 4.1.4

For simplicity, consider the function $h(x) = \dfrac{1}{1-x-x^2}$ which is clearly equal to $\dfrac{f(x)}{g(x)}$ for the functions $f(x) = 1$ and $g(x) = 1 - x - x^2$. These are each polynomials, but we will consider them power series for this calculation. Thus, $f_0 = 1$ and $f_n = 0$ for $n \geq 1$, while $g_0 = 1,\ g_1 = -1,\ g_2 = -1$, and $g_n = 0$ for $n \geq 3$.

The first equation we solve is:

$$\sum_{k=0}^{0} c_k g_{0-k} = f_0 \leftrightarrow c_0 g_0 = f_0 \implies c_0 \cdot 1 = 1 \implies c_0 = 1$$

Next, we use $n = 1$:

$$\sum_{k=0}^{1} c_k g_{1-k} = f_1 \leftrightarrow c_0 g_1 + c_1 g_0 = f_1 \implies c_0 \cdot (-1) + c_1 \cdot 1 = 0 \implies c_1 = 1$$

Now for $n = 2$:

$$\sum_{k=0}^{2} c_k g_{2-k} = f_2 \leftrightarrow c_0 g_2 + c_1 g_1 + c_2 g_0 = f_2 \implies c_0 \cdot (-1) + c_1 \cdot (-1) + c_2 \cdot 1 = 0 \implies c_2 = c_1 + c_0 = 2$$

This is where the pattern starts to emerge, looking at $n = 3$:

$$\sum_{k=0}^{3} c_k g_{3-k} = f_3 \leftrightarrow c_0 g_3 + c_1 g_2 + c_2 g_1 + c_3 g_0 = f_3 \implies c_0 \cdot 0 + c_1 \cdot (-1) + c_2 \cdot (-1) + c_3 \cdot 1 = 0 \implies c_3 = c_2 + c_1 = 3$$

We see that for any $n \geq 3$, we will have:

$$\sum_{k=0}^{n} c_k g_{n-k} = f_n \leftrightarrow \sum_{k=0}^{n} c_k g_{n-k} = 0 \implies \sum_{k=0}^{n-3} c_k \cdot 0 + c_{n-2} \cdot (-1) + c_{n-1} \cdot (-1) + c_n \cdot 1 = 0 \implies c_n = c_{n-1} + c_{n-2}$$

Thus, the coefficients c_n must obey the recursive relationship $c_n = c_{n-1} + c_{n-2}$ with $c_0 = 1,\ c_1 = 1$. This is the famous sequence known as the *Fibonacci numbers*. Thus, we see that $h(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \cdots$

In general, finding the pattern inside of the quotient of a power series division problem is difficult, so instead we manually calculate as many terms as deemed necessary.

Example 4.1.4

Find the coefficients c_0, c_1, c_2, c_3, c_4 of the power series centered at $a = 0$ that represents each function below.

(a) $A(x) = \sec(x)$

(b) $B(x) = \dfrac{e^x - 1}{\sin(x)}$

Solution

(a) We write $A(x) = \dfrac{1}{\cos(x)} = \dfrac{f(x)}{g(x)}$.

This means that $f(x) = 1$ has $f_0 = 1$ and $f_n = 0$ for $n \geq 1$.
Also, $g(x) = \cos(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ has $g_{2k} = \dfrac{(-1)^k}{(2k)!}$ and $g_{2k+1} = 0$ for all integers $k \geq 0$.

The first equation we solve is:

$$\sum_{k=0}^{0} c_k g_{0-k} = f_0 \longleftrightarrow c_0 g_0 = f_0$$
$$c_0 \cdot 1 = 1 \implies c_0 = 1$$

For $n = 1$:

$$\sum_{k=0}^{1} c_k g_{1-k} = f_1 \longleftrightarrow c_0 g_1 + c_1 g_0 = f_1$$
$$c_0 \cdot 0 + c_1 \cdot 1 = 0 \implies c_1 = 0$$

For $n = 2$:

$$\sum_{k=0}^{2} c_k g_{2-k} = f_2 \longleftrightarrow c_0 g_2 + c_1 g_1 + c_2 g_0 = f_2$$
$$c_0 \cdot \left(-\frac{1}{2}\right) + c_1 \cdot 0 + c_2 \cdot 1 = 0 \implies c_2 = \frac{1}{2} c_0 = \frac{1}{2}$$

For $n = 3$:

$$\sum_{k=0}^{3} c_k g_{3-k} = f_3 \longleftrightarrow c_0 g_3 + c_1 g_2 + c_2 g_1 + c_3 g_0 = f_3$$
$$c_0 \cdot 0 + c_1 \cdot \left(-\frac{1}{2}\right) + c_2 \cdot 0 + c_3 \cdot 1 = 0 \implies c_3 = \frac{1}{2} c_1 = 0$$

For $n = 4$:

$$\sum_{k=0}^{4} c_k g_{4-k} = f_4 \longleftrightarrow c_0 g_4 + c_1 g_3 + c_2 g_2 + c_3 g_1 + c_4 g_0 = f_4$$
$$c_0 \cdot \frac{1}{24} + c_1 \cdot 0 + c_2 \cdot \left(-\frac{1}{2}\right) + c_3 \cdot 0 + c_4 \cdot 1 = 0 \implies c_4 = \frac{1}{2} c_2 - \frac{1}{24} c_0 = \frac{5}{24}$$

This tells us that the first few terms are $\boxed{A(x) = 1 + 0x + \frac{1}{2}x^2 + 0x^3 + \frac{5}{24}x^4 + \cdots}$

(b) We write $B(x) = \dfrac{e^x - 1}{\sin(x)} = \dfrac{f(x)}{g(x)}$.

This means that $f(x) = e^x - 1$ has $f_0 = 0$ and $f_n = \dfrac{1}{n!}$ for $n \geq 1$.

Also, $g(x) = \sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ has $g_{2k} = 0$ and $g_{2k+1} = \dfrac{(-1)^k}{(2k+1)!}$ for all integers $k \geq 0$.

The first equation we solve is:

$$\sum_{k=0}^{0} c_k g_{0-k} = f_0 \longleftrightarrow c_0 g_0 = f_0$$
$$c_0 \cdot 0 = 0 \implies \text{nothing useful}$$

For $n = 1$:

$$\sum_{k=0}^{1} c_k g_{1-k} = f_1 \longleftrightarrow c_0 g_1 + c_1 g_0 = f_1$$
$$c_0 \cdot 1 + c_1 \cdot 0 = 1 \implies c_0 = 1$$

For $n = 2$:

$$\sum_{k=0}^{2} c_k g_{2-k} = f_2 \longleftrightarrow c_0 g_2 + c_1 g_1 + c_2 g_0 = f_2$$
$$c_0 \cdot 0 + c_1 \cdot 1 + c_2 \cdot 0 = \frac{1}{2} \implies c_1 = \frac{1}{2}$$

For $n = 3$:

$$\sum_{k=0}^{3} c_k g_{3-k} = f_3 \longleftrightarrow c_0 g_3 + c_1 g_2 + c_2 g_1 + c_3 g_0 = f_3$$
$$c_0 \cdot \left(-\frac{1}{6}\right) + c_1 \cdot 0 + c_2 \cdot 1 + c_3 \cdot 0 = \frac{1}{6} \implies c_2 = \frac{1}{6}(c_0 + 1) = \frac{1}{3}$$

For $n = 4$:

$$\sum_{k=0}^{4} c_k g_{4-k} = f_4 \longleftrightarrow c_0 g_4 + c_1 g_3 + c_2 g_2 + c_3 g_1 + c_4 g_0 = f_4$$
$$c_0 \cdot 0 + c_1 \cdot \left(-\frac{1}{6}\right) + c_2 \cdot 0 + c_3 \cdot 1 + c_4 \cdot 0 = \frac{1}{24} \implies c_3 = \frac{1}{24}(4c_1 + 1) = \frac{1}{8}$$

For $n = 5$:

$$\sum_{k=0}^{5} c_k g_{5-k} = f_5 \longleftrightarrow c_0 g_5 + c_1 g_4 + c_2 g_3 + c_3 g_2 + c_4 g_1 + c_5 g_0 = f_5$$
$$c_0 \cdot \frac{1}{120} + c_1 \cdot 0 + c_2 \cdot \left(-\frac{1}{6}\right) + c_3 \cdot 0 + c_4 \cdot 1 + c_5 \cdot 0 = \frac{1}{120} \implies c_4 = \frac{1}{120}(20c_2 - c_0 + 1) = \frac{1}{18}$$

This tells us that the first few terms are $\boxed{B(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{8}x^3 + \frac{1}{18}x^4 + \cdots}$

This foundation in power series manipulation and the concept of analyticity provides the essential toolkit for the power series method of solving differential equations, which will be developed in the next subchapter. The ability to skillfully shift indices and recognize analytic functions is crucial for success with series solutions.

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4.1: Practice Set

Exercise 4.1.1a

Find all points at which the function $f(x) = \dfrac{1}{x^2+4}$ is analytic.

Answer available at the end of this section.

Exercise 4.1.1b

Find all points at which the function $f(x) = \dfrac{10x}{(x-2)(x+3)}$ is analytic.

Answer available at the end of this section.

Exercise 4.1.1c

Find all points at which the function $f(x) = |x|$ is analytic.

Answer available at the end of this section.

Exercise 4.1.2a

Find the power series representation of the function $f(x) = \sin(x^2)$ centered at $a = 0$ and its interval of convergence.

Answer available at the end of this section.

Exercise 4.1.2b

Find the power series representation of the function $f(x) = \dfrac{1}{1+3x}$ centered at $a = 0$ and its interval of convergence.

Answer available at the end of this section.

Exercise 4.1.2c

Find the power series representation of the function $f(x) = \ln(1+x)$ centered at $a = 0$ and its interval of convergence.

Answer available at the end of this section.

Exercise 4.1.2d

Find the power series representation of the function $f(x) = \dfrac{1}{(2-x^2)^2}$ centered at $a = 0$ and its interval of convergence.

Answer available at the end of this section.

Exercise 4.1.2e

Given $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ and $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^n$, find $h(x) = f(x) - g(x)$ and its interval of convergence.

Answer available at the end of this section.

Exercise 4.1.3a

Write $\sum_{n=2}^{\infty} n(n-1)x^{n-2} + \sum_{n=0}^{\infty} x^{n+1}$ as a single power series.

Answer available at the end of this section.

Exercise 4.1.3b

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, write $x^2 f''(x) - 2x f'(x) + f(x)$ as a single power series.

Answer available at the end of this section.

Exercise 4.1.3c

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, write $f''(x) + x f'(x)$ as a single power series.

Answer available at the end of this section.

Exercise 4.1.3d

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, write $(x-1)f'(x) + \int_0^x t f(t)\, dt$ as a single power series.

Answer available at the end of this section.

Exercise 4.1.4a

Find the first four terms of the power series representation of $f(x) = \sin(x)\cos(x)$ centered at $a = 0$

Answer available at the end of this section.

Exercise 4.1.4b

Find the first four terms of the power series representation of $f(x) = e^x \ln(1+x)$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.1.4c$

Find the first four terms of the power series representation of $f(x) = \tan(x)$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.1.4d$

Find the first four terms of the power series representation of $f(x) = \dfrac{x}{e^x - 1}$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.1.5a$

Verify that the power series for e^x satisfies $y' - y = 0$

Answer available at the end of this section.

Exercise $4.1.5b$

Verify that the power series for $\cosh(x),\ \sinh(x)$, and e^x each satisfies $y'' - y = 0$

Answer available at the end of this section.

Answers

Exercise $4.1.1a$

The domain of $f(x)$ is $(-\infty, \infty)$, and it is continuous everywhere on its domain.

We see it is infinitely differentiable, with:

$$\begin{aligned} f(x) &= \frac{1}{x^2+4} \\ f'(x) &= \frac{-2x}{(x^2+4)^2} \\ f''(x) &= \frac{6x^2-8}{(x^2+4)^3} \\ &\vdots \\ f^{(n)}(x) &= \frac{p_n(x)}{(x^2+4)^{n+1}} \text{ for some polynomial } p_n(x) \end{aligned}$$

Therefore, it is possible to find the Taylor Series representing $f(x)$ centered at any value $a \in \mathbb{R}$, and it will have a positive radius of convergence. This tells us \{f(x)\} is analytic everywhere.

Exercise $4.1.1b$

We can easily rewrite this function as $f(x) = \dfrac{4}{x-2} + \dfrac{6}{x+3}$.

It is clear that $\dfrac{4}{x-2}$ is analytic everywhere but $x = 2$.

Similarly, we know that $\dfrac{6}{x+3}$ is analytic everywhere but $x = -3$.

Since the sum of analytic functions is analytic, we know $f(x)$ is analytic at any point in $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$

Exercise $4.1.1c$

We know $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ is continuous everywhere.

However, it is not differentiable at $x = 0$, and since $f'(0)$ does not exist it cannot be analytic at $x = 0$.

When $x > 0$, we have $f(x) = x$ which is a polynomial and therefore analytic everywhere.

When $x < 0$, we have $f(x) = -x$ which is a polynomial and therefore analytic everywhere.

Thus, $f(x)$ is analytic for all points in $(-\infty, 0) \cup (0, \infty)$

Exercise $4.1.2a$

We know that $g(x) = \sin(x)$ is equal to the series $\hat{g}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all x.

Since $f(x) = \sin(x^2) = g(x^2)$ we know $\hat{f}(x) = \hat{g}(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$

The interval of convergence is easily calculated to be $(-\infty, \infty)$

Therefore, $\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$ for all $x^{\text{int}}(-\infty, \infty)$

Exercise 4.1.2*b*

We know that $g(x) = \frac{1}{1-x}$ is equal to the series $\hat{g}(x) = \sum_{n=0}^{\infty} x^n$ for all $|x| < 1$.

Since $f(x) = \frac{1}{1+3x} = g(-3x)$ we know $\hat{f}(x) = \hat{g}(-3x) = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-3)^n x^n$

The interval of convergence is easily calculated to be $\left(-\frac{1}{3}, \frac{1}{3}\right)$

Therefore, $\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3)^n x^n$ for all $x^{\text{int}} \left(-\frac{1}{3}, \frac{1}{3}\right)$

Exercise 4.1.2*c*

We know that $g(x) = \frac{1}{1-x}$ is equal to the series $\hat{g}(x) = \sum_{n=0}^{\infty} x^n$ for all $|x| < 1$.

Since $f(x) = \ln(1+x) = \int_0^x \frac{1}{1+t}\, dt = \int_0^x g(-t)\, dt$ we know:

$$\hat{f}(x) = \int_0^x \hat{g}(-t)\, dt = \int_0^x \sum_{n=0}^{\infty} (-t)^n\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} x^m$$

The interval of convergence is easily calculated to be $[-1, 1)$

Therefore, $\ln(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} x^m$ for all $x^{\text{int}}\, [-1, 1)$

Exercise 4.1.2*d*

We know that $g(x) = \frac{1}{1-x}$ is equal to the series $\hat{g}(x) = \sum_{n=0}^{\infty} x^n$ for all $|x| < 1$.

Differentiating this gives $g'(x) = \frac{1}{(1-x)^2}$ equal to $\hat{g}'(x) = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{m=-1}^{\infty} (m+1) x^m = 0 + \sum_{m=0}^{\infty} (m+1) x^m$.

Since $f(x) = \frac{1}{(2-x^2)^2} = \frac{1}{4} \cdot \frac{1}{\left(1 - \frac{x^2}{2}\right)^2} = \frac{1}{4} g'\left(\frac{x^2}{2}\right)$ we know $\hat{f}(x) = \frac{1}{4} \hat{g}'\left(\frac{x^2}{2}\right) = \sum_{m=0}^{\infty} (m+1) \left(\frac{x^2}{2}\right)^m = \sum_{m=0}^{\infty} \frac{m+1}{2^m} x^{2m}$

The interval of convergence is easily calculated to be $\left(-\sqrt{2}, \sqrt{2}\right)$

Therefore, $\frac{1}{(2-x^2)^2} = \sum_{m=0}^{\infty} \frac{m+1}{2^m} x^{2m}$ for all $x^{\text{int}} \left(-\sqrt{2}, \sqrt{2}\right)$

Exercise 4.1.2*e*

We add the two power series together term-by-term:

$$f(x) - g(x) = \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^2}\right] x^n = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2} x^{2k+1}$$

The interval of convergence is easily calculated to be $[-1, 1]$

Therefore, $h(x) = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2} x^{2k+1}$ for all $x^{\text{int}} [-1, 1]$

Exercise 4.1.3*a*

We shift the first sum using $m = n - 2$, giving $\sum_{n=2}^{\infty} n(n-1) x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) x^m$

We shift the second sum using $m = n + 1$, giving $\sum_{n=0}^{\infty} x^{n+1} = \sum_{m=1}^{\infty} x^m$

To make both sums start at $m = 1$, we manually write the $m = 0$ term from the first sum then collect them together:

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) x^{n-2} + \sum_{n=0}^{\infty} x^{n+1} &= \sum_{m=0}^{\infty} (m+2)(m+1) x^m + \sum_{m=1}^{\infty} x^m \\ &= 2x^0 + \sum_{m=1}^{\infty} (m+2)(m+1) x^m + \sum_{m=1}^{\infty} x^m \\ &= 2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1) + 1\right] x^m \end{aligned}$$

Exercise 4.1.3*b*

We see $f(x) = \sum_{n=0}^{\infty} c_n x^n \implies f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \implies f''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$

Note that $x^2 f''(x) = x^2 \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} n(n-1) c_n x^n$

Also, we have $2xf'(x) = 2x\sum_{n=0}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty} 2nc_n x^n$

Since all sums start at $n = 0$, we collect them together:

$$\begin{aligned} x^2f''(x) - 2xf'(x) + f(x) &= \sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} \left[n(n-1)c_n - 2nc_n + c_n\right] x^n \\ &= \sum_{n=0}^{\infty} (n^2 - 3n + 1)c_n x^n \end{aligned}$$

Exercise 4.1.3*c*

We see $f(x) = \sum_{n=0}^{\infty} c_n x^n \implies f'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1} \implies f''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$

We shift $f''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$ using $m = n-2$ to get $f''(x) = \sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2}x^m$

We shift $xf'(x) = x\sum_{n=0}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty} nc_n x^n$ using $m = n$ to get $xf'(x) = \sum_{m=0}^{\infty} mc_m x^m$

To make both sums start at $m = 0$, we manually write the $m = -2, -1$ terms from the first sum then collect them together:

$$\begin{aligned} f''(x) + xf'(x) &= \sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2}x^m + \sum_{m=0}^{\infty} mc_m x^m \\ &= 0x^{-2} + 0x^{-1} + \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2}x^m + \sum_{m=0}^{\infty} mc_m x^m \\ &= \sum_{m=0}^{\infty} \left[(m+2)(m+1)c_{m+2} + mc_m\right] x^m \end{aligned}$$

Exercise 4.1.3*d*

We see $f(x) = \sum_{n=0}^{\infty} c_n x^n \implies f'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1}$

Distributing $(x-1)f'(x) = xf'(x) - f'(x)$, so we consider each piece separately.

We shift $xf'(x) = x\sum_{n=0}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty} nc_n x^n$ using $m = n$ to get $xf'(x) = \sum_{m=0}^{\infty} mc_m x^m$

We shift $f'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1}$ using $m = n-1$ to get $f'(x) = \sum_{m=-1}^{\infty} (m+1)c_{m+1}x^m$

Lastly, we take $\int_0^x tf(t)\,dt = \int_0^x t\sum_{n=0}^{\infty} c_n t^n\,dt = \sum_{n=0}^{\infty} c_n \int_0^x t^{n+1}\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+2}x^{n+2}$ and shift using $m = n+2$ to get $\int_0^x tf(t)\,dt = \sum_{m=2}^{\infty} \frac{c_{m-2}}{m}x^m$

To make all sums start at $m = 2$, we manually write the $m = -1, 0, 1$ terms from the first two sums then collect them together:

$$\begin{aligned} (x-1)f'(x) + \int_0^x tf(t)\,dt &= \sum_{m=0}^{\infty} mc_m x^m - \sum_{m=-1}^{\infty} (m+1)c_{m+1}x^m + \sum_{m=2}^{\infty} \frac{c_{m-2}}{m}x^m \\ &= 0c_0x^0 + 1c_1x^1 + \sum_{m=2}^{\infty} mc_m x^m - 0c_0x^{-1} - 1c_1x^0 - 2c_2x^1 - \sum_{m=2}^{\infty}(m+1)c_{m+1}x^m + \sum_{m=2}^{\infty} \frac{c_{m-2}}{m}x^m \\ &= -c_1 + (c_1 - 2c_2)x + \sum_{m=2}^{\infty} \left[mc_m - (m+1)c_{m+1} + \frac{c_{m-2}}{m}\right] x^m \end{aligned}$$

Exercise 4.1.4*a*

We want $f(x) = g(x)\cdot h(x)$ for $g(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ and $h(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

We will use $f(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} g_k h_{n-k}\right) x^n$:

$$\begin{aligned} f_0 &= g_0h_0 = 0\cdot 1 = 0 \\ f_1 &= g_0h_1 + g_1h_0 = 0\cdot 0 + 1\cdot 1 = 1 \\ f_2 &= g_0h_2 + g_1h_1 + g_2h_0 = 0\cdot\left(-\frac{1}{2}\right) + 1\cdot 0 + 0\cdot 1 = 0 \\ f_3 &= g_0h_3 + g_1h_2 + g_2h_1 + g_3h_0 = 0\cdot 0 + 1\cdot\left(-\frac{1}{2}\right) + 0\cdot 0 + \left(-\frac{1}{6}\right)\cdot 1 = -\frac{2}{3} \end{aligned}$$

Therefore, $f(x) = 0 + 1x + 0x^2 - \frac{2}{3}x^3 + \ldots$

Exercise 4.1.4*b*

We want $f(x) = g(x)\cdot h(x)$ for $g(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ and $h(x) = \ln(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}x^m$ (from Exercise 2c)

We will use $f(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} g_k h_{n-k}\right) x^n$:

$$f_0 = g_0h_0 = 1 \cdot 0 = 0$$
$$f_1 = g_0h_1 + g_1h_0 = 1 \cdot 1 + 1 \cdot 0 = 1$$
$$f_2 = g_0h_2 + g_1h_1 + g_2h_0 = 1 \cdot \left(-\frac{1}{2}\right) + 1 \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$$
$$f_3 = g_0h_3 + g_1h_2 + g_2h_1 + g_3h_0 = 1 \cdot \frac{1}{3} + 1 \cdot \left(-\frac{1}{2}\right) + \frac{1}{2} \cdot 1 + \frac{1}{6} \cdot 0 = \frac{1}{3}$$

Therefore, $f(x) = 0 + 1x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$

Exercise 4.1.4*c*

We want $f(x) = \frac{g(x)}{h(x)}$ for $g(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ and $h(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

We will use $g_n = \sum_{k=0}^{n} f_k h_{n-k}$:

$$g_0 = f_0h_0 = f_0 \cdot 1$$
$$\implies f_0 = 0$$
$$g_1 = f_0h_1 + f_1h_0 = f_0 \cdot 0 + f_1 \cdot 1$$
$$\implies f_1 = 1$$
$$g_2 = f_0h_2 + f_1h_1 + f_2h_0 = f_0 \cdot \left(-\frac{1}{2}\right) + f_1 \cdot 0 + f_2 \cdot 1$$
$$\implies -\frac{1}{2}f_0 + f_2 = 0 \implies f_2 = 0$$
$$g_3 = f_0h_3 + f_1h_2 + f_2h_1 + f_3h_0 = f_0 \cdot 0 + f_1 \cdot \left(-\frac{1}{2}\right) + f_2 \cdot 0 + f_3 \cdot 1$$
$$\implies -\frac{1}{2}f_1 + f_3 = -\frac{1}{6} \implies f_3 = \frac{1}{3}$$

Therefore, $f(x) = 0 + 1x + 0x^2 + \frac{1}{3}x^3 + \ldots$

Exercise 4.1.4*d*

We want $f(x) = \frac{g(x)}{h(x)}$ for $g(x) = x = 0 + x + \sum_{n=2}^{\infty} 0x^n$ and $h(x) = e^x - 1 = \sum_{n=1}^{\infty} \frac{1}{n!}x^n$

We will use $g_n = \sum_{k=0}^{n} f_k h_{n-k}$:

$$g_0 = f_0h_0 = f_0 \cdot 0$$
$$\implies 0 = 0 \implies \text{no information}$$
$$g_1 = f_0h_1 + f_1h_0 = f_0 \cdot 1 + f_1 \cdot 0$$
$$\implies f_0 = 1$$
$$g_2 = f_0h_2 + f_1h_1 + f_2h_0 = f_0 \cdot \frac{1}{2} + f_1 \cdot 1 + f_2 \cdot 0$$
$$\implies \frac{1}{2}f_0 + f_1 = 0 \implies f_1 = -\frac{1}{2}$$
$$g_3 = f_0h_3 + f_1h_2 + f_2h_1 + f_3h_0 = f_0 \cdot \frac{1}{6} + f_1 \cdot \frac{1}{2} + f_2 \cdot 1 + f_3 \cdot 0$$
$$\implies \frac{1}{6}f_0 + \frac{1}{2}f_1 + f_2 = 0 \implies f_2 = \frac{1}{12}$$
$$g_4 = f_0h_4 + f_1h_3 + f_2h_2 + f_3h_1 + f_4h_0 = f_0 \cdot \frac{1}{24} + f_1 \cdot \frac{1}{6} + f_2 \cdot \frac{1}{2} + f_3 \cdot 1 + f_4 \cdot 0$$
$$\implies \frac{1}{24}f_0 + \frac{1}{6}f_1 + \frac{1}{2}f_2 + f_3 = 0 \implies f_3 = 0$$

Therefore, $f(x) = 1 - \frac{1}{2}x + \frac{1}{12}x^2 + 0x^3 + \ldots$

Exercise 4.1.5*a*

Let $y = e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$

Differentiating term-by-term gives: $y' = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot nx^{n-1} = 0 + \sum_{n=1}^{\infty} \frac{1}{n!} \cdot nx^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^{n-1}$

We shift using $m = n - 1$, which makes $y' = \sum_{m=0}^{\infty} \frac{1}{m!}x^m$

Now we have $y' - y = \sum_{m=0}^{\infty} \frac{1}{m!}x^m - \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 0$

This proves that the power series for e^x satisfies the differential equation $y' - y = 0$

Exercise 4.1.5*b*

Let $y = \cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$

Differentiating term-by-term twice gives: $y'' = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \cdot 2n(2n-1)x^{2n-2} = 0 + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \cdot 2n(2n-1)x^{2n-2} = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!}x^{2n-2}$

We shift using $m = n - 1$, which makes $y'' = \sum_{m=0}^{\infty} \frac{1}{(2m)!}x^{2m}$

Now we have $y'' - y = \sum_{m=0}^{\infty} \frac{1}{(2m)!}x^{2m} - \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} = 0$

This proves that the power series for $\cosh(x)$ satisfies the differential equation $y'' - y = 0$

Let $y = \sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$

Differentiating term-by-term twice gives: $y'' = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \cdot (2n+1)2nx^{2n-1} = 0 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \cdot (2n+1)2nx^{2n-1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} x^{2n-1}$

We shift using $m = n - 1$, which makes $y'' = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{2m+1}$

Now we have $y'' - y = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{2m+1} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = 0$

This proves that the power series for $\sinh(x)$ satisfies the differential equation $y'' - y = 0$

Let $y = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Differentiating term-by-term twice gives: $y'' = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot n(n-1)x^{n-2} = 0 + 0 + \sum_{n=2}^{\infty} \frac{1}{n!} \cdot n(n-1)x^{n-2} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} x^{n-2}$

We shift using $m = n - 2$, which makes $y'' = \sum_{m=0}^{\infty} \frac{1}{m!} x^m$

Now we have $y'' - y = \sum_{m=0}^{\infty} \frac{1}{m!} x^m - \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 0$

This proves that the power series for e^x satisfies the differential equation $y'' - y = 0$

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4.2: First order LDE Power Series solutions

Learning Objectives

- Apply the power series method to solve first-order linear ODEs.
- Determine recurrence relations for the series coefficients.
- Recognize when the resulting series represents a known elementary function.
- Handle simple nonhomogeneous equations and equations requiring division by series.

We apply the power series method to first-order equations, assuming a solution in the form of an infinite series. The process yields recurrence relations for the coefficients, and we often discover that the resulting series represents a familiar elementary function.

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4.2: Power Series Solutions

With our minds refreshed on what a power series is, and how to manipulate functions defined as power series, we can verify whether or not a given power series satisfies a linear differential equation. Better still, if we assume that the solution to some LDE is analytic, we can try to *find* a power series that satisfies it.

Illustration 4.2.1

We turn to a simple example that we already know the solution to: $y' - y = 0$

If we assume that $y = \sum_{n=0}^{\infty} c_n x^n$, then we must have $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$

We plug these into the differential equation and shift all series as necessary to combine them into a single power series:

$$\begin{aligned} y' - y &= 0 \\ \underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{m=n} &= 0 \\ \sum_{m=-1}^{\infty} (m+1) c_{m+1} x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\ \left[0 c_0 x^{-1} + \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m \right] - \sum_{m=0}^{\infty} c_m x^m &= 0 \\ \sum_{m=0}^{\infty} \left[(m+1) c_{m+1} - c_m \right] x^m &= 0 \end{aligned}$$

We see that the resulting power series must be exactly equal to zero for all x. The only way this occurs is when each coefficient is equal to zero.

$$\begin{aligned} m = 0: &\quad 1c_1 - c_0 = 0 &&\implies c_1 = c_0 \\ m = 1: &\quad 2c_2 - c_1 = 0 &&\implies c_2 = \frac{1}{2} c_1 = \frac{1}{2!} c_0 \\ m = 2: &\quad 3c_3 - c_2 = 0 &&\implies c_3 = \frac{1}{3} c_2 = \frac{1}{3!} c_0 \\ m = 3: &\quad 4c_4 - c_3 = 0 &&\implies c_4 = \frac{1}{4} c_3 = \frac{1}{4!} c_0 \\ &\quad \vdots &&\quad \vdots \end{aligned}$$

We see a pattern begins to emerge, giving us $c_n = \frac{1}{n!} c_0$ for all $n \geq 1$

Plugging this into our assumption $y = \sum_{n=0}^{\infty} c_n x^n$, we see our power series solution looks like:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=1}^{\infty} c_n x^n = c_0 + \sum_{n=1}^{\infty} \frac{1}{n!} c_0 x^n = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n \right)$$

This series happens to be identical to $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, which we know is our solution to the LDE.

While not every power series solution will be recognizable as a function like e^x, the method we used to reach this solution will always work to find any analytic solutions to any first-order LDE.

Example 4.2.1

Find the power series solution to the LDE

(a) $y' + 2xy = 0$

(b) $y' + 2xy = 1$

Solution

(a) We assume that $y = \sum_{n=0}^{\infty} c_n x^n$, so that $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$

We plug these into the differential equation and shift all series as necessary to combine them into a single power series:

$$\begin{aligned}
y' + 2xy &= 0 \\
\sum_{n=0}^{\infty} n c_n x^{n-1} + 2x \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} 2 c_n x^{n+1}}_{m=n+1} &= 0 \\
\sum_{m=-1}^{\infty} (m+1) c_{m+1} x^m + \sum_{m=1}^{\infty} 2 c_{m-1} x^m &= 0 \\
\left[0 c_0 x^{-1} + 1 c_1 x^0 + \sum_{m=1}^{\infty} (m+1) c_{m+1} x^m \right] + \sum_{m=1}^{\infty} 2 c_{m-1} x^m &= 0 \\
c_1 + \sum_{m=0}^{\infty} \left[(m+1) c_{m+1} + 2 c_{m-1} \right] x^m &= 0
\end{aligned}$$

The resulting power series must be exactly equal to zero for all x. The only way this occurs is when each coefficient is equal to zero. Since c_1 is the only constant term (coefficient of x^0), we must have $c_1 = 0$. For each coefficient inside of the bigger sum, we solve:

$$\begin{aligned}
m = 1: &\quad 2c_2 + 2c_0 = 0 &&\implies c_2 = -c_0 \\
m = 2: &\quad 3c_3 + 2c_1 = 0 &&\implies c_3 = -\frac{2}{3} c_1 = 0 \\
m = 3: &\quad 4c_4 + 2c_2 = 0 &&\implies c_4 = -\frac{1}{2} c_2 = \frac{(-1)^2}{2!} c_0 \\
m = 4: &\quad 5c_5 + 2c_3 = 0 &&\implies c_5 = -\frac{2}{5} c_3 = 0 \\
m = 5: &\quad 6c_6 + 2c_4 = 0 &&\implies c_6 = -\frac{1}{3} c_4 = \frac{(-1)^3}{3!} c_0 \\
&\quad \vdots &&\quad \vdots
\end{aligned}$$

We see two patterns emerge. All odd coefficients must equal zero, so $c_{2k+1} = 0$ for all $k \geq 0$

The even coefficients satisfy $c_{2k} = \frac{(-1)^k}{k!} c_0$ for all $k \geq 1$

Plugging this into our assumption $y = \sum_{n=0}^{\infty} c_n x^n$, we see our power series solution looks like:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} + \sum_{k=1}^{\infty} c_{2k} x^{2k} = c_0 + \sum_{k=0}^{\infty} 0 x^{2k+1} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} c_0 x^{2k} = \boxed{c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} x^{2k} \right)}$$

This series happens to be identical to $e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$, which is the solution we would have gotten using the methods of Chapter 1.

(b) The only change for this LDE compared to the one in part (a) is that the resulting power series must add up to 1 instead of 0.

If we go through the same steps as above, we end up at:

$$c_1 + \sum_{m=0}^{\infty} \left[(m+1)c_{m+1} + 2c_{m-1}\right] x^m = 1$$

The constant term c_1 must equal the constant term on the right, so $c_1 = 1$. All other coefficients in the above sum must be equal to zero:

$$\begin{aligned}
m = 1: &\quad 2c_2 + 2c_0 = 0 &&\implies c_2 = -c_0 \\
m = 2: &\quad 3c_3 + 2c_1 = 0 &&\implies c_3 = -\frac{2}{3}c_1 = -\frac{2}{3} = \frac{(-2)^1}{3!!} \\
m = 3: &\quad 4c_4 + 2c_2 = 0 &&\implies c_4 = -\frac{1}{2}c_2 = \frac{(-1)^2}{2!}c_0 \\
m = 4: &\quad 5c_5 + 2c_3 = 0 &&\implies c_5 = -\frac{2}{5}c_3 = \frac{(-2)^2}{5!!} \\
m = 5: &\quad 6c_6 + 2c_4 = 0 &&\implies c_6 = -\frac{1}{3}c_4 = \frac{(-1)^3}{3!}c_0 \\
m = 6: &\quad 7c_7 + 2c_5 = 0 &&\implies c_7 = -\frac{2}{7}c_5 = \frac{(-2)^3}{7!!} \\
&\quad \vdots &&\quad \vdots
\end{aligned}$$

Once more, two different patterns emerge. The odd coefficients satisfy $c_{2k+1} = \dfrac{(-2)^k}{(2k+1)!!}$ for all $k \geq 1$

The even coefficients satisfy $c_{2k} = \dfrac{(-1)^k}{k!}c_0$ for all $k \geq 1$

Plugging this into our assumption $y = \sum_{n=0}^{\infty} c_n x^n$, we see our power series solution looks like:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k} x^{2k} + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \\
&= c_0 + 1x + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} c_0 x^{2k} + \sum_{k=1}^{\infty} \frac{(-2)^k}{(2k+1)!!} x^{2k+1} \\
&= \boxed{\left(x + \sum_{k=1}^{\infty} \frac{(-2)^k}{(2k+1)!!} x^{2k+1}\right) + c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} x^{2k}\right)}
\end{aligned}$$

This solution is of the form we expect: a particular solution (that we don't recognize as a famous elementary function, but can still be expressed as a power series) and all multiples of a homogeneous solution (which we saw was equivalent to e^{-x^2} in part (a)).

This method exchanges integration (when using our standard $y(x) = \dfrac{1}{\mu(x)} \displaystyle\int \mu(x) q(x)\, dx$ formula) for series manipulation and pattern recognition.

Example 4.2.2

Find the power series solution to the LDE $(1+x)y' - y = 0$,

(a) centered at $a = 0$

(b) centered at $a = -1$

Solution

(a) We assume that $y = \sum_{n=0}^{\infty} c_n x^n$, so that $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$

We plug these into the differential equation and shift all series as necessary to combine them into a single power series:

$$\begin{aligned}
(1+x)y' - y &= 0 \\
\sum_{n=0}^{\infty} n c_n x^{n-1} + x\sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} n c_n x^{n}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n}}_{m=n} &= 0 \\
\sum_{m=-1}^{\infty} (m+1)c_{m+1}x^m + \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\
\left[0c_0 x^{-1} + \sum_{m=0}^{\infty}(m+1)c_{m+1}x^m\right] + \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[(m+1)c_{m+1} + m c_m - c_m\right] x^m &= 0
\end{aligned}$$

The resulting power series must be exactly equal to zero for all x. The only way this occurs is when each coefficient is equal to zero:

$$\begin{aligned}
m = 0: &\quad 1c_1 + 0c_0 - c_0 = 0 &&\implies c_1 = c_0 \\
m = 1: &\quad 2c_2 + 1c_1 - c_1 = 0 &&\implies c_2 = 0 \\
m = 2: &\quad 3c_3 + 2c_2 - c_2 = 0 &&\implies c_3 = -\frac{1}{3}c_2 = 0 \\
m = 3: &\quad 4c_4 + 3c_3 - c_3 = 0 &&\implies c_4 = -\frac{1}{2}c_3 = 0 \\
m = 4: &\quad 5c_5 + 4c_4 - c_4 = 0 &&\implies c_5 = -\frac{3}{5}c_3 = 0 \\
&\quad \vdots &&\quad \vdots
\end{aligned}$$

We see a simple pattern emerge: $c_n = 0$ for all $n \geq 2$

Plugging this into our assumption $y = \sum_{n=0}^{\infty} c_n x^n$, we see our power series solution looks like:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n = c_0 + c_0 x + \sum_{n=2}^{\infty} 0 x^n = \boxed{c_0\,(1+x)}$$

This series solution is actually a polynomial, which is easy to see satisfies the homogeneous LDE.

(b) We will repeat the same steps as in part (a), but under the assumption that $y = \sum_{n=0}^{\infty} c_n (x+1)^n$, so that $y' = \sum_{n=0}^{\infty} n c_n (x+1)^{n-1}$

We plug these into the differential equation and shift all series as necessary to combine them into a single power series:

$$
\begin{aligned}
(1+x)y' - y &= 0 \\
(1+x)\sum_{n=0}^{\infty} nc_n(x+1)^{n-1} - \sum_{n=0}^{\infty} c_n(x+1)^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} nc_n(x+1)^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} c_n(x+1)^n}_{m=n} &= 0 \\
\sum_{m=0}^{\infty} mc_m(x+1)^m - \sum_{m=0}^{\infty} c_m(x+1)^m &= 0 \\
\sum_{m=0}^{\infty} (m-1)c_m(x+1)^m &= 0
\end{aligned}
$$

The resulting power series must be exactly equal to zero for all x. The only way this occurs is when each coefficient is equal to zero:

$$
\begin{array}{lll}
m=0: & -1c_0 = 0 & \Longrightarrow c_0 = 0 \\
m=1: & 0c_1 = 0 & \Longrightarrow c_1 = \text{ anything} \\
m=2: & 1c_2 = 0 & \Longrightarrow c_2 = 0 \\
m=3: & 2c_3 = 0 & \Longrightarrow c_3 = 0 \\
m=4: & 3c_4 = 0 & \Longrightarrow c_4 = 0 \\
 & \vdots & \vdots
\end{array}
$$

We see that: $c_n = 0$ for all $n \neq 1$

Plugging this into our assumption $y = \sum_{n=0}^{\infty} c_n(x+1)^n$, we see our power series solution looks like:

$$
y = \sum_{n=0}^{\infty} c_n(x+1)^n = c_0 + c_1(x+1) + \sum_{n=2}^{\infty} c_n(x+1)^n = 0 + c_1(x+1) + \sum_{n=2}^{\infty} 0(x+1)^n = \boxed{c_1\,(x+1)}
$$

This solution is the same as the one we found in part (a), but the work along the way was less complicated.

In Chapter 1, the introduction of an initial condition meant we had to solve for specific values of the parameters that appear in our general solution. When dealing with power series, this step can seem daunting (especially when you cannot recognize the function represented by your power series solution).

Consider a function $f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n$, we see that$f(a) = \sum_{n=0}^{\infty} f_n(a-a)^n = f_0 + \sum_{n=1}^{\infty} f_n \cdot 0 = f_0$.

This means that when your solution is centered at $x = a$, the initial condition $y(a) = b$ is always equal to c_0. Thus, if we are given a choice, we can use the initial condition to determine *where the series should be centered*. Then, we can immediately know the first coefficient, which may simplify the pattern recognition.

Example 4.2.3

Find the power series solution to $(x^2 - 2x + 2)y' - xy = 2$ with $y(1) = -1$

Solution

Since our initial condition is at $x = 1$, we will center our guess at $a = 1$

We assume that $y = \sum_{n=0}^{\infty} c_n(x-1)^n$, so that $y' = \sum_{n=0}^{\infty} nc_n(x-1)^{n-1}$

We plug these into the differential equation and shift all series as necessary to combine them into a single power series:

$$
\begin{aligned}
(x^2-2x+2)y'-xy &= 2\\
\left((x-1)^2+1\right)y'-\left((x-1)+1\right)y &= 2\\
(x-1)^2\sum_{n=0}^{\infty}nc_n(x-1)^{n-1}+\sum_{n=0}^{\infty}nc_n(x-1)^{n-1}-(x-1)\sum_{n=0}^{\infty}c_n(x-1)^n-\sum_{n=0}^{\infty}c_n(x-1)^n &= 2\\
\underbrace{\sum_{n=0}^{\infty}nc_n(x-1)^{n+1}}_{m=n+1}+\underbrace{\sum_{n=0}^{\infty}nc_n(x-1)^{n-1}}_{m=n-1}-\underbrace{\sum_{n=0}^{\infty}c_n(x-1)^{n+1}}_{m=n+1}-\underbrace{\sum_{n=0}^{\infty}c_n(x-1)^n}_{m=n} &= 2\\
\sum_{m=1}^{\infty}(m-1)c_{m-1}(x-1)^m+\sum_{m=-1}^{\infty}(m+1)c_{m+1}(x-1)^m-\sum_{m=1}^{\infty}c_{m-1}(x-1)^m-\sum_{m=0}^{\infty}c_m(x-1)^m &= 2\\
\sum_{m=1}^{\infty}(m-1)c_{m-1}(x-1)^m+\left[0c_0+1c_1+\sum_{m=1}^{\infty}(m+1)c_{m+1}(x-1)^m\right]-\sum_{m=1}^{\infty}c_{m-1}(x-1)^m-\left[c_0+\sum_{m=1}^{\infty}c_m(x-1)^m\right] &= 2\\
c_1-c_0+\sum_{m=1}^{\infty}\left[(m-1)c_{m-1}+(m+1)c_{m+1}-c_{m-1}-c_m\right](x-1)^m &= 2
\end{aligned}
$$

The resulting power series must be exactly equal to the constant 2 for all x. The only way this occurs is when $c_1-c_0=2$, so we get $c_1=c_0+2$. This seems like a good time to remember that because we picked $y(x)$ as a power series centered at $a=1$, we have $y(1)=c_0$. This is a great starting point, we have $c_0=-1$ and so $c_1=1$.

Next, the coefficients of the collected series must all equal zero, so we solve for the largest indexed term as a function of the others:

$$
(m-1)c_{m-1}+(m+1)c_{m+1}-c_{m-1}-c_m=0 \implies c_{m+1}=\frac{c_m-(m-2)c_{m-1}}{m+1} \quad \text{for } m\geq 1
$$

Plugging in a few quick values of m, we search for a pattern.

$$
\begin{aligned}
m=1: &\quad c_2=\frac{c_1+c_0}{2}=\frac{1+(-1)}{2}=0\\
m=2: &\quad c_3=\frac{c_2+0c_1}{3}=\frac{0+0\cdot 1}{3}=0\\
m=3: &\quad c_4=\frac{c_3-c_2}{4}=\frac{0-0}{4}=0\\
m=4: &\quad c_5=\frac{c_4-2c_3}{5}=\frac{0-2\cdot 0}{5}=0\\
&\quad\vdots
\end{aligned}
$$

We see that $c_n=0$ for all $n\geq 2$, so our unique solution to this differential equation is:

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty}c_n(x-1)^n\\
&= c_0+c_1(x-1)+\sum_{n=2}^{\infty}c_n(x-1)^n\\
&= -1+1\cdot(x-1)+\sum_{n=2}^{\infty}0\cdot(x-1)^n\\
&= \boxed{x-2}
\end{aligned}
$$

Unfortunately, power series are not a magic bullet for all first-order LDEs. One of the limitations of "guessing" that $y(x)$ is a power series is that you exclude possible answers which are *non-analytic*.

Illustration 4.2.2

Find the power series solutions to $xy'+y=0$

(a) centered at $a\neq 0$

(b) centered at $a = 0$

(a) We want a solution of the form $y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \implies y'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$:

$$\begin{aligned}
xy'(x) + y(x) &= 0 \\
[(x-a)+a]\, y'(x) + y(x) &= 0 \\
(x-a)\sum_{n=0}^{\infty} n c_n (x-a)^{n-1} + a\sum_{n=0}^{\infty} n c_n (x-a)^{n-1} + \sum_{n=0}^{\infty} c_n (x-a)^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n c_n (x-a)^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} a n c_n (x-a)^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n (x-a)^n}_{m=n} &= 0 \\
\sum_{m=0}^{\infty} m c_m (x-a)^m + \sum_{m=-1}^{\infty} a(m+1) c_{m+1} (x-a)^m + \sum_{m=0}^{\infty} c_m (x-a)^m &= 0 \\
\sum_{m=0}^{\infty} m c_m (x-a)^m + \left[a \cdot 0 c_0 + \sum_{m=0}^{\infty} a(m+1) c_{m+1} (x-a)^m \right] + \sum_{m=0}^{\infty} c_m (x-a)^m &= 0 \\
\sum_{m=0}^{\infty} \left[m c_m + a(m+1) c_{m+1} + c_m \right] (x-a)^m &= 0
\end{aligned}$$

Setting each term equal to zero gives us the recurrence relation:

$$m c_m + a(m+1) c_{m+1} + c_m = 0 \implies c_{m+1} = -\frac{c_m}{a} \quad \text{for } m \geq 0$$

This pattern is pretty easy to spot:

$$\begin{aligned}
m = 0: &\quad c_1 = -\frac{c_0}{a} = \frac{c_0}{-a} \\
m = 1: &\quad c_2 = -\frac{c_1}{a} = \frac{c_0}{(-a)^2} \\
m = 2: &\quad c_3 = -\frac{c_2}{a} = \frac{c_0}{(-a)^3} \\
m = 3: &\quad c_4 = -\frac{c_3}{a} = \frac{c_0}{(-a)^4} \\
&\vdots
\end{aligned}$$

We have $c_n = \dfrac{c_0}{(-a)^n}$ for all $n \geq 1$. Plugging this into our power series format gives our final answer:

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\
&= c_0 + \sum_{n=1}^{\infty} c_n (x-a)^n \\
&= c_0 + \sum_{n=1}^{\infty} \frac{c_0}{(-a)^n} (x-a)^n \\
&= c_0 \left(1 + \sum_{n=1}^{\infty} \frac{(x-a)^n}{(-a)^n} \right)
\end{aligned}$$

(b) We want a solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n \implies y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$:

$$xy'(x) + y(x) = 0$$
$$x\sum_{n=0}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=0}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=0}^{\infty} (n+1)c_n x^n = 0$$

Setting each term equal to zero gives us the recurrence relation:

$$(n+1)c_n = 0 \implies c_n = 0 \quad \text{for } n \geq 0$$

This means we immediately have all $c_n = 0$, which is the trivial solution $y(x) = 0$ (the constant zero function).

This is definitely not the solution we expected to see! This is a tiny CE LDE, with characteristic equation $m + 1 = 0$

We know this means we have solutions of the form $y(x) = \dfrac{K}{x}$
Why didn't we find this function as a power series? Because it is not analytic at $a = 0$

This means $y(x)$ cannot be written as a power series centered at $a = 0$, even though a solution exists.

The second limitation of the power series method is that it can sometimes lead to power series products and division. Because of this, when the pattern is not easily recognizable, we often settle for the first "few" coefficients (which is useful for approximations).

Example 4.2.4

Consider the IVP: $y' + \cos(x)y = e^x$ with $y(0) = 0$

Find the first six terms in the power series solution centered around $a = 0$

Solution

We want to find solutions of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n \implies y'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1}$:

$$\begin{aligned} y(x) \cdot \cos(x) &= \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right) \\ &= \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right) \\ &= c_0 + c_1 x + \left[c_2 - \frac{c_0}{2}\right]x^2 + \left[c_3 - \frac{c_1}{2}\right]x^3 + \left[c_4 - \frac{c_2}{2} + \frac{c_0}{24}\right]x^4 + \ldots \end{aligned}$$

We use this truncated sum to solve for the first few coefficients that satisfy the LDE:

$$\begin{aligned} y' + \cos(x) \cdot y &= \sum_{n=0}^{\infty} nc_n x^{n-1} + \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right) \\ &= \left[c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \ldots\right] + \left[c_0 + c_1 x + \left[c_2 - \frac{c_0}{2}\right]x^2 + \left[c_3 - \frac{c_1}{2}\right]x^3 + \left[c_4 - \frac{c_2}{2} + \frac{c_0}{24}\right]x^4 + \ldots\right] \\ &= (c_0 + c_1) + (c_1 + 2c_2)\,x + \left(c_2 - \frac{c_0}{2} + 3c_3\right)x^2 + \left(c_3 - \frac{c_1}{2} + 4c_4\right)x^3 + \left(5c_5 + c_4 - \frac{c_2}{2} + \frac{c_0}{24}\right)x^4 + \ldots \end{aligned}$$

We are given $y(0) = c_0 = 0$, and this power series must equal e^x for at least the first few coefficients:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

Now we can solve for the first few coefficients:

$$
\begin{aligned}
x^0: &\quad c_0 + c_1 = 1 \implies 0 + c_1 = 1 \implies c_1 = 1 \\
x^1: &\quad c_1 + 2c_2 = 1 \implies 1 + 2c_2 = 1 \implies c_2 = 0 \\
x^2: &\quad c_2 - \frac{c_0}{2} + 3c_3 = \frac{1}{2} \implies 0 - \frac{0}{2} + 3c_3 = \frac{1}{2} \implies c_3 = \frac{1}{6} \\
x^3: &\quad c_3 - \frac{c_1}{2} + 4c_4 = \frac{1}{6} \implies \frac{1}{6} - \frac{1}{2} + 4c_4 = \frac{1}{6} \implies c_4 = \frac{1}{8} \\
x^4: &\quad c_4 - \frac{c_2}{2} + \frac{c_0}{24} + 5c_5 = \frac{1}{24} \implies \frac{1}{8} - \frac{0}{2} + \frac{0}{24} + 5c_5 = \frac{1}{24} \implies c_5 = -\frac{1}{60}
\end{aligned}
$$

Therefore, we have a solution of the form $\boxed{y(x) = 0 + 1x + 0x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{60}x^5 + \ldots}$

This subchapter demonstrates that the power series method not only provides solutions to ODEs but often reveals that these solutions are familiar elementary functions, building intuition for the relationship between differential equations and their solutions.

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4.2: Practice Set

Exercise 4.2.1a

Find the explicit pattern of the sequence of numbers satisfying $a_{n+1} = \dfrac{a_n}{n+3}$ with $a_0 = 1$

Answer available at the end of this section.

Exercise 4.2.1b

Find the explicit pattern of the sequence of numbers satisfying $b_{n+1} = -\dfrac{b_n}{3n+1}$

Answer available at the end of this section.

Exercise 4.2.1c

Find the explicit pattern of the sequence of numbers satisfying $c_{n+2} = \dfrac{3c_n}{(n+4)(n+2)}$ for $n \geq 0$

Answer available at the end of this section.

Exercise 4.2.2a

Show that the power series $y = \displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n$ satisfies the LDE $y' + 2y = 0$

Answer available at the end of this section.

Exercise 4.2.2b

Show that the power series $y = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$ satisfies the LDE $xy'' - 2x^2y' = 0$

Answer available at the end of this section.

Exercise 4.2.3a

Find the general power series solution to the LDE $y' + 3x^2y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise 4.2.3b

Find the general power series solution to the LDE $(1+x)y' - y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.2.3c$

Find the general power series solution to the LDE $2xy' + y = e^x$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.2.3d$

Find the general power series solution to the LDE $(1 - 2x)y' + y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.2.3e$

Find the general power series solution to the LDE $(x - 1)y' - 3y = 0$ centered at $a = 1$

Answer available at the end of this section.

Exercise $4.2.3f$

Find the general power series solution to the LDE $(x^2 + 4x)y' - (x + 2)y = 0$ centered at $a = -2$

Answer available at the end of this section.

Exercise $4.2.4a$

Find the unique power series solution to the IVP $y' - xy = 1$ with $y(0) = 2$

Answer available at the end of this section.

Exercise $4.2.4b$

Find the unique power series solution to the IVP $y' + xy = 1 + 2x$ with $y(0) = 2$

Answer available at the end of this section.

Exercise $4.2.4c$

Find the unique power series solution to the IVP $xy' + y = 0$ with $y(1) = 1$

Answer available at the end of this section.

Exercise $4.2.4d$

Find the unique power series solution to the IVP $(x - 3)^2y' - y = 6$ with $y(3) = -6$

Answer available at the end of this section.

Exercise $4.2.4e$

Find the unique power series solution to the IVP $(x+2)y' + 2(x+2)^2 y = 0$ with $y(-2) = -2$

Answer available at the end of this section.

Answers

Exercise $4.2.1a$

Plugging in a few $n \geq 0$, we can see the pattern quickly:

$$\begin{aligned} a_1 &= \frac{a_0}{3} = \frac{1}{3} \\ a_2 &= \frac{a_1}{4} = \frac{1}{3 \cdot 4} \\ a_3 &= \frac{a_2}{5} = \frac{1}{3 \cdot 4 \cdot 5} \\ &\vdots \\ a_n &= \frac{2}{(n+2)!} \end{aligned}$$

Therefore, we have $a_n = \dfrac{2}{(n+2)!}$ for all $n \geq 1$

(Technically this explicit formula holds for $n = 0$ too, but we only looked for the pattern after $n = 1$)

Exercise $4.2.1b$

Plugging in a few $n \geq 0$, we can see the pattern eventually:

$$\begin{aligned} b_1 &= -\frac{b_0}{1} \\ b_2 &= -\frac{b_1}{4} = \frac{(-1)^2 b_0}{4 \cdot 1} \\ b_3 &= -\frac{b_2}{7} = \frac{(-1)^3 b_0}{7 \cdot 4 \cdot 1} \\ b_4 &= -\frac{b_3}{10} = \frac{(-1)^4 b_0}{10 \cdot 7 \cdot 4 \cdot 1} \\ &\vdots \\ b_n &= \frac{(-1)^n}{(3n-2)!!!} b_0 \end{aligned}$$

Therefore, the sequence is explicitly defined by $b_n = \dfrac{(-1)^n}{(3n-2)!!!} b_0$

Exercise $4.2.1c$

Plugging in a few $n \geq 0$, we can see the pattern is different for even and odd terms:

$$\begin{aligned} c_2 &= \frac{3c_0}{4 \cdot 2} = \frac{3^1}{4!! \cdot 2!!} \cdot 2c_0 \\ c_3 &= \frac{3c_1}{5 \cdot 3} = \frac{3^1}{5!! \cdot 3!!} \cdot 3c_1 \\ c_4 &= \frac{3c_2}{6 \cdot 4} = \frac{3^2 c_0}{6 \cdot 4 \cdot 4 \cdot 2} = \frac{3^2}{6!! \cdot 4!!} \cdot 2c_0 \\ c_5 &= \frac{3c_3}{7 \cdot 5} = \frac{3^2 c_1}{7 \cdot 5 \cdot 5 \cdot 3} = \frac{3^2}{7!! \cdot 5!!} \cdot 3c_1 \\ &\vdots \\ c_{2k} &= \frac{3^k}{(2k+2)!!(2k)!!} \cdot 2c_0 \\ c_{2k+1} &= \frac{3^k}{(2k+3)!!(2k+1)!!} \cdot 3c_1 \end{aligned}$$

Therefore, the sequence is explicitly defined by $c_n = \begin{cases} \dfrac{3^k}{(2k+2)!!(2k)!!} \cdot 2c_0 & n = 2k \\ \dfrac{3^k}{(2k+3)!!(2k+1)!!} \cdot 3c_1 & n = 2k+1 \end{cases}$

Exercise $4.2.2a$

We see $y = \displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n \implies y' = \sum_{n=0}^{\infty} \frac{(-2)^n n}{n!} x^{n-1}$

Therefore, the LDE becomes:

$$\begin{aligned} y' + 2y &= \sum_{n=0}^{\infty} \frac{(-2)^n n}{n!} x^{n-1} + 2 \cdot \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n \\ &= \sum_{m=-1}^{\infty} \frac{(-2)^{m+1}(m+1)}{(m+1)!} x^m + \sum_{m=0}^{\infty} \frac{2 \cdot (-2)^m}{m!} x^m \\ &= 0 c_0 x^{-1} + \sum_{m=0}^{\infty} \frac{(-2)^{m+1}}{m!} x^m + \sum_{m=0}^{\infty} \frac{2 \cdot (-2)^m}{m!} x^m \\ &= \sum_{m=0}^{\infty} \left(\frac{(-2)^{m+1}}{m!} + 2\frac{(-2)^m}{m!} \right) x^m \\ &= \sum_{m=0}^{\infty} 0 = 0 \end{aligned}$$

This proves that the given power series satisfies the LDE.

Exercise $4.2.2b$

We see $y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} \implies y' = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \implies y'' = \sum_{n=0}^{\infty} \frac{2n x^{2n-1}}{n!}$

Therefore, the LDE becomes:

$$\begin{aligned} xy'' - 2x^2 y' &= x \cdot \sum_{n=0}^{\infty} \frac{2n x^{2n-1}}{n!} - 2x^2 \cdot \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{2n x^{2n}}{n!} - \sum_{n=0}^{\infty} \frac{2x^{2n+2}}{n!} \\ &= \left(0 \cdot \frac{x^0}{0!} + \sum_{n=1}^{\infty} \frac{2x^{2n}}{(n-1)!} \right) - \underbrace{\sum_{n=0}^{\infty} \frac{2x^{2n+2}}{n!}}_{m=n+1} \\ &= \sum_{n=1}^{\infty} \frac{2x^{2n}}{(n-1)!} - \sum_{m=1}^{\infty} \frac{2x^{2m}}{(m-1)!} \\ &= \sum_{m=1}^{\infty} 0 = 0 \end{aligned}$$

This proves that the given power series satisfies the LDE.

Exercise $4.2.3a$

We assume $y = \sum_{n=0}^{\infty} c_n x^n$, so $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$:

$$\begin{aligned} y' + 3x^2 y &= 0 \\ \sum_{n=0}^{\infty} n c_n x^{n-1} + 3x^2 \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} 3 c_n x^{n+2}}_{m=n+2} &= 0 \\ \sum_{m=-1}^{\infty} (m+1) c_{m+1} x^m + \sum_{m=2}^{\infty} 3 c_{m-2} x^m &= 0 \\ \left(0 c_0 x^{-1} + 1 c_1 x^0 + 2 c_2 x^1 + \sum_{m=2}^{\infty} (m+1) c_{m+1} x^m \right) + \sum_{m=2}^{\infty} 3 c_{m-2} x^m &= 0 \\ c_1 + 2c_2 x + \sum_{m=2}^{\infty} \left[(m+1) c_{m+1} + 3 c_{m-2} \right] x^m &= 0 \end{aligned}$$

Setting the coefficients of 1 and x equal to zero gives us $c_1 = 0$ and $c_2 = 0$ respectively, and for larger c_n we see:

$$(m+1) c_{m+1} + 3 c_{m-2} \implies c_{m+1} = -\frac{3}{m+1} c_{m-2} \text{ for all } m \geq 2$$

Plugging these in we see more than one pattern emerge:

$$\begin{aligned} m = 2: &\quad c_3 = -\frac{3}{3} c_0 = -c_0 \\ m = 3: &\quad c_4 = -\frac{3}{4} c_1 = 0 \\ m = 4: &\quad c_5 = -\frac{3}{5} c_2 = 0 \\ m = 5: &\quad c_6 = -\frac{3}{6} c_3 = \frac{(-1)^2}{2} c_0 \\ m = 6: &\quad c_7 = -\frac{3}{7} c_4 = 0 \\ m = 7: &\quad c_8 = -\frac{3}{8} c_5 = 0 \\ m = 8: &\quad c_9 = -\frac{3}{9} c_6 = \frac{(-1)^3}{3!} c_0 \\ m = 9: &\quad c_{10} = -\frac{3}{10} c_7 = 0 \\ m = 10: &\quad c_{11} = -\frac{3}{11} c_8 = 0 \\ &\vdots \end{aligned}$$

We see that $c_{3k+1} = 0$ and $c_{3k+2} = 0$ for all k, and that $c_{3k} = \dfrac{(-1)^k}{k!} c_0$ for $k \geq 1$

Plugging this into our assumed power series form gives:

$$\begin{aligned} y &= c_0 + \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} c_{3k+2} x^{3k+2} + \sum_{k=1}^{\infty} c_{3k} x^{3k} \\ &= c_0 + \sum_{k=0}^{\infty} 0 \cdot x^{3k+1} + \sum_{k=0}^{\infty} 0 \cdot x^{3k+2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} x^{3k} \\ &= c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k}}{k!} \right) \end{aligned}$$

Thus, the general solution is $y(x) = c_0 \left(1 + \sum_{k=1}^{\infty} \dfrac{(-1)^k x^{3k}}{k!} \right)$

It is worth noticing that this series is equivalent to the solution $y = c_0 e^{-x^3}$ for all x.

Exercise $4.2.3b$

We assume $y = \sum_{n=0}^{\infty} c_n x^n$ so $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$:

$$\begin{aligned} y' + xy' - y &= 0 \\ \sum_{n=0}^{\infty} n c_n x^{n-1} + x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} n c_n x^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{m=n} &= 0 \\ \sum_{m=-1}^{\infty} (m+1) c_{m+1} x^n + \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\ \left(0 c_0 x^{-1} + \sum_{m=0}^{\infty} (m+1) c_{m+1} x^n \right) + \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\ \sum_{m=0}^{\infty} \left[(m+1) c_{m+1} + m c_m - c_m \right] x^m &= 0 \end{aligned}$$

Setting each coefficient equal to zero gives:

$$(m+1) c_{m+1} + m c_m - c_m = 0 \implies c_{m+1} = \frac{(1-m) c_m}{m+1} \text{ for all } m \geq 0$$

Plugging in a few values of m, we see that:

$$\begin{aligned} m = 0: &\quad c_1 = \frac{1 c_0}{1} = c_0 \\ m = 1: &\quad c_2 = \frac{0 c_1}{2} = 0 \\ m = 2: &\quad c_3 = \frac{-1 c_2}{3} = 0 \\ m = 3: &\quad c_4 = \frac{-2 c_3}{4} = 0 \\ &\vdots \end{aligned}$$

Therefore, we have $c_0 = c_1$ and $c_n = 0$ for all $n \geq 2$. Plugging this into our assumed power series form gives:

$$\begin{aligned} y &= c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n \\ &= c_0 + c_0 x + \sum_{n=2}^{\infty} 0 x^n \\ &= c_0 (1 + x) \end{aligned}$$

Thus, the general solution is $y = c_0(1+x)$

Exercise $4.2.3c$

We assume $y = \sum_{n=0}^{\infty} c_n x^n$ so $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$:

$$\begin{aligned} 2xy' + y &= e^x \\ 2x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ \sum_{n=0}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ \sum_{n=0}^{\infty} (2n+1) c_n x^n &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \end{aligned}$$

This will be satisfied when $(2n+1) c_n = \dfrac{1}{n!}$, thus we must have $c_n = \dfrac{1}{n!(2n+1)}$ for all $n \geq 0$

Plugging this into our assumed power series form gives:

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{(2n+1) n!} x^n$$

Exercise $4.2.3d$

We assume $y = \sum_{n=0}^{\infty} c_n x^n$ so $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$:

$$\begin{aligned}
y' - 2xy' + y &= 0 \\
\sum_{n=0}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty} 2n c_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{m=n} &= 0 \\
\sum_{m=-1}^{\infty} (m+1) c_{m+1} x^m - \sum_{m=0}^{\infty} 2m c_m x^m + \sum_{m=0}^{\infty} c_m x^m &= 0 \\
\left(0 c_0 x^{-1} + \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m \right) - \sum_{m=0}^{\infty} 2m c_m x^m + \sum_{m=0}^{\infty} c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left((m+1) c_{m+1} - 2m c_m + c_m \right) x^m &= 0
\end{aligned}$$

Setting this equal to zero gives us:

$$(m+1)c_{m+1} - 2mc_m + c_m = 0 \implies c_{m+1} = \frac{(2m-1)c_m}{m+1} \text{ for all } m \geq 0$$

Plugging in a few values of m we see:

$$\begin{aligned}
m = 0: &\quad c_1 = \frac{-1c_0}{1} = -c_0 \\
m = 1: &\quad c_2 = \frac{1c_1}{2} = -\frac{1}{2} c_0 \\
m = 2: &\quad c_3 = \frac{3c_2}{3} = -\frac{3 \cdot 1}{3 \cdot 2} c_0 \\
m = 3: &\quad c_4 = \frac{5c_2}{4} = -\frac{5 \cdot 3 \cdot 1}{4 \cdot 3 \cdot 2} c_0 \\
m = 4: &\quad c_5 = \frac{7c_2}{5} = -\frac{7!!}{5!} c_0 \\
&\quad \vdots
\end{aligned}$$

The pattern emerges after $c_1 = -c_0$, which we see as $c_n = -\frac{(2n-3)!!}{n!} c_0$ for $n \geq 2$

Plugging this into our assumed power series form gives:

$$y = c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n = c_0 - c_0 x - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{n!} c_0 x^n = c_0 \left(1 - x - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{n!} x^n \right)$$

Exercise $4.2.3e$

We assume $y = \sum_{n=0}^{\infty} c_n (x-1)^n$ so $y' = \sum_{n=0}^{\infty} n c_n (x-1)^{n-1}$:

$$\begin{aligned}
(x-1)y' - 3y &= 0 \\
(x-1) \cdot \sum_{n=0}^{\infty} n c_n (x-1)^{n-1} - 3 \sum_{n=0}^{\infty} c_n (x-1)^n &= 0 \\
\sum_{n=0}^{\infty} n c_n (x-1)^n - \sum_{n=0}^{\infty} 3 c_n (x-1)^n &= 0 \\
\sum_{n=0}^{\infty} (n-3) c_n x^n &= 0
\end{aligned}$$

This will be satisfied when $(n-3)c_n = 0$. For $n \neq 3$, this forces $c_n = 0$

When $n = 3$, the equation becomes $0 \cdot c_3 = 0$, so c_3 can be any number.

Plugging this into our assumed power series form gives:

$$y = \sum_{n=0}^{\infty} c_n (x-1)^n = 0 + 0(x-1) + 0(x-1)^2 + c_3 (x-1)^3 + 0(x-1)^4 + \cdots = c_3 (x-1)^3$$

Exercise $4.2.3f$

We assume $y = \sum_{n=0}^{\infty} c_n (x+2)^n$ so $y' = \sum_{n=0}^{\infty} n c_n (x+2)^{n-1}$:

$$\left((x+2)^2-4\right)y'-(x+2)y=0$$
$$(x+2)^2\cdot\sum_{n=0}^{\infty}nc_n(x+2)^{n-1}-4\cdot\sum_{n=0}^{\infty}nc_n(x+2)^{n-1}-(x+2)\cdot\sum_{n=0}^{\infty}c_n(x+2)^n=0$$
$$\underbrace{\sum_{n=0}^{\infty}nc_n(x+2)^{n+1}}_{m=n+1}-\underbrace{\sum_{n=0}^{\infty}4nc_n(x+2)^{n-1}}_{m=n-1}-\underbrace{\sum_{n=0}^{\infty}c_n(x+2)^{n+1}}_{m=n+1}=0$$
$$\sum_{m=1}^{\infty}(m-1)c_{m-1}(x+2)^m-\sum_{m=-1}^{\infty}4(m+1)c_{m+1}(x+2)^m-\sum_{m=1}^{\infty}c_{m-1}(x+2)^m=0$$
$$\sum_{m=1}^{\infty}(m-1)c_{m-1}(x+2)^m-\left(0c_0(x+2)^{-1}+4c_1+\sum_{m=1}^{\infty}4(m+1)c_{m+1}(x+2)^m\right)-\sum_{m=1}^{\infty}c_{m-1}(x+2)^m=0$$
$$-4c_1+\sum_{m=1}^{\infty}\left[(m-1)c_{m-1}-4(m+1)c_{m+1}-c_{m-1}\right](x+2)^m=0$$

The constant term $-4c_1$ must match the constant zero on the right, so $-4c_1=0 \implies c_1=0$

Then for $m\geq 1$, we get: $(m-1)c_{m-1}-4(m+1)c_{m+1}-c_{m-1}=0 \implies c_{m+1}=\dfrac{(m-2)}{4(m+1)}c_{m-1}$

Plugging in a few values of m, we see:

$$\begin{aligned}
m=1: &\quad c_2=\frac{-1}{4\cdot 2}c_0\\
m=2: &\quad c_3=\frac{0}{4\cdot 3}c_1=0\\
m=3: &\quad c_4=\frac{1}{4\cdot 4}c_2=-\frac{1}{4^2\cdot 4!!}c_0\\
m=4: &\quad c_5=\frac{2}{4\cdot 5}c_3=0\\
m=5: &\quad c_6=\frac{3}{4\cdot 6}c_4=\frac{3!!}{4^3\cdot 6!!}c_0\\
&\quad\vdots
\end{aligned}$$

The pattern emerges as $c_{2k+1}=0$ for all $k\geq 0$, and $c_{2k}=-\dfrac{(2k-3)!!}{4^k\cdot(2k)!!}c_0$ for $k\geq 2$

Plugging this into our assumed power series form gives:

$$\begin{aligned}
y&=\sum_{n=0}^{\infty}c_n(x+2)^n\\
&=c_0+c_2(x+2)^2+\sum_{k=2}^{\infty}c_{2k}(x+2)^{2k}+\sum_{k=0}^{\infty}c_{2k+1}(x+2)^{2k+1}\\
&=c_0-\frac{1}{8}c_0(x+2)^2-\sum_{k=2}^{\infty}\frac{(2k-3)!!}{4^k(2k)!!}c_0(x+2)^{2k}+\sum_{k=0}^{\infty}0(x+2)^{2k+1}\\
&=c_0\left(1-\frac{1}{8}(x+2)^2-\sum_{k=2}^{\infty}\frac{(2k-3)!!}{4^k(2k)!!}(x+2)^{2k}\right)
\end{aligned}$$

Exercise 4.2.4*a*

We assume $y=\sum_{n=0}^{\infty}c_nx^n$ so $y'=\sum_{n=0}^{\infty}nc_nx^{n-1}$:

$$y'-xy=1$$
$$\sum_{n=0}^{\infty}nc_nx^{n-1}-x\sum_{n=0}^{\infty}c_nx^n=1$$
$$\underbrace{\sum_{n=0}^{\infty}nc_nx^{n-1}}_{m=n-1}-\underbrace{\sum_{n=0}^{\infty}c_nx^{n+1}}_{m=n+1}=1$$
$$\sum_{m=-1}^{\infty}(m+1)c_{m+1}x^m-\sum_{m=1}^{\infty}c_{m-1}x^m=1$$
$$\left(0c_0x^{-1}+1c_1x^0\sum_{m=1}^{\infty}(m+1)c_{m+1}x^m\right)-\sum_{m=1}^{\infty}c_{m-1}x^m=1$$
$$c_1+\sum_{m=1}^{\infty}\left((m+1)c_{m+1}-c_{m-1}\right)x^m=1$$

We are given $y(0)=2$, which tells us that $c_0=2$. Looking above, we see that for the constant to match on both sides, we must have $c_1=1$

To make the remaining coefficients equal zero, we must have:

$$(m+1)c_{m+1}-c_{m-1}=0 \implies c_{m+1}=\frac{1}{m+1}c_{m-1} \text{ for all } m\geq 1$$

Plugging in a few values of m, we see the pattern is different for even and odd coefficients:

$$\begin{aligned}
m=1: &\quad c_2=\frac{1}{2}c_0=\frac{2}{2!!}\\
m=2: &\quad c_3=\frac{1}{3}c_1=\frac{1}{3!!}\\
m=3: &\quad c_4=\frac{1}{4}c_2=\frac{1}{4\cdot 2}c_0=\frac{2}{4!!}\\
m=4: &\quad c_5=\frac{1}{5}c_3=\frac{1}{5\cdot 3}c_1=\frac{1}{5!!}\\
m=5: &\quad c_6=\frac{1}{6}c_4=\frac{1}{6!!}c_0=\frac{2}{6!!}\\
m=6: &\quad c_7=\frac{1}{7}c_5=\frac{1}{7!!}c_1=\frac{1}{7!!}\\
&\quad\vdots
\end{aligned}$$

We see that for even coefficients, we have $c_{2k}=\dfrac{2}{(2k)!!}$ and for odd coefficients we use $c_{2k+1}=\dfrac{1}{(2k+1)!!}$

Plugging this into our assumed power series form gives:

$$y=\sum_{n=0}^{\infty}c_nx^n=\sum_{k=0}^{\infty}c_{2k}x^{2k}+\sum_{k=0}^{\infty}c_{2k+1}x^{2k+1}=\sum_{k=0}^{\infty}\frac{2}{(2k)!!}x^{2k}+\sum_{k=0}^{\infty}\frac{1}{(2k+1)!!}x^{2k+1}$$

Exercise 4.2.4*b*

We assume $y=\sum_{n=0}^{\infty}c_nx^n$, so $y'=\sum_{n=0}^{\infty}nc_nx^{n-1}$:

$$\begin{aligned}
y'+xy&=1+2x\\
\sum_{n=0}^{\infty}nc_nx^{n-1}+x\cdot\sum_{n=0}^{\infty}c_nx^n&=1+2x\\
\underbrace{\sum_{n=0}^{\infty}nc_nx^{n-1}}_{m=n-1}+\underbrace{\sum_{n=0}^{\infty}c_nx^{n+1}}_{m=n+1}&=1+2x\\
\sum_{m=-1}^{\infty}(m+1)c_{m+1}x^m+\sum_{m=1}^{\infty}c_{m-1}x^m&=1+2x\\
\left(0c_0x^{-1}+1c_1x^0+\sum_{m=1}^{\infty}(m+1)c_{m+1}\right)+\sum_{m=1}^{\infty}c_{m-1}x^m&=1+2x\\
c_1+\sum_{m=1}^{\infty}\left((m+1)c_{m+1}+c_{m-1}\right)x^m&=1+2x
\end{aligned}$$

We are given $c_0=y(0)=2$, and must have $c_1=1$ to satisfy the line above.

When we use $m=1$, we get $2c_2+c_0=2$, and thus $c_2=0$

The remaining coefficients must all equal zero, giving us:

$$(m+1)c_{m+1}+c_{m-1}=0\implies c_{m+1}=-\frac{1}{m+1}c_{m-1}\text{ for all }m\geq 2$$

Plugging in a few values of m gives us the patterns:

$$\begin{aligned}
m=2: &\quad c_3=-\frac{1}{3}c_1=\frac{-1}{3}\\
m=3: &\quad c_4=-\frac{1}{4}c_2=0\\
m=4: &\quad c_5=-\frac{1}{5}c_3=\frac{(-1)^2}{5!!}\\
m=5: &\quad c_6=-\frac{1}{6}c_4=0\\
&\quad\vdots
\end{aligned}$$

We see that $c_{2k}=0$ for all $k\geq 1$, and that $c_{2k+1}=\dfrac{(-1)^k}{(2k+1)!!}$ for all $k\geq 1$

Plugging this into our assumed power series form gives:

$$y=c_0+c_1x+\sum_{k=1}^{\infty}c_{2k}x^{2k}+\sum_{k=1}^{\infty}c_{2k+1}x^{2k+1}=2+1x+\sum_{k=1}^{\infty}0x^{2k}+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!!}x^{2k+1}=2+x+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!!}x^{2k+1}$$

Exercise 4.2.4*c*

We assume $y=\sum_{n=0}^{\infty}c_n(x-1)^n$, so $y'=\sum_{n=0}^{\infty}nc_n(x-1)^{n-1}$:

$$\begin{aligned}
((x-1)+1)\,y' + y &= 0 \\
(x-1)\cdot\sum_{n=0}^{\infty} nc_n(x-1)^{n-1} + \sum_{n=0}^{\infty} nc_n(x-1)^{n-1} + \sum_{n=0}^{\infty} c_n(x-1)^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} nc_n(x-1)^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} nc_n(x-1)^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n(x-1)^n}_{m=n} &= 0 \\
\sum_{m=0}^{\infty} mc_m(x-1)^m + \sum_{m=-1}^{\infty} (m+1)c_{m+1}(x-1)^m + \sum_{m=0}^{\infty} c_m(x-1)^m &= 0 \\
\sum_{m=0}^{\infty} mc_m(x-1)^m + \left(0c_0 + \sum_{m=0}^{\infty}(m+1)c_{m+1}(x-1)^m\right) + \sum_{m=0}^{\infty} c_m(x-1)^m &= 0 \\
\sum_{m=0}^{\infty} \left(mc_m + (m+1)c_{m+1} + c_m\right)(x-1)^m &= 0
\end{aligned}$$

We are given $c_0 = y(1) = 1$, and setting the coefficients above all equal to zero gives us:

$$mc_m + (m+1)c_{m+1} + c_m = 0 \implies c_{m+1} = -c_m \text{ for all } m \geq 1$$

Clearly, we have $c_n = (-1)^n c_0 = (-1)^n$ for all n. Plugging this into our assumed power series form gives:

$$y = \sum_{n=0}^{\infty} c_n(x-1)^n = \sum_{n=0}^{\infty} (-1)^n(x-1)^n = \sum_{n=0}^{\infty} (1-x)^n$$

Exercise 4.2.4*d*

We assume $y = \sum_{n=0}^{\infty} c_n(x-3)^n$, so $y' = \sum_{n=0}^{\infty} nc_n(x-3)^{n-1}$:

$$\begin{aligned}
(x-3)^2 y' - y &= 6 \\
(x-3)^2 \cdot \sum_{n=0}^{\infty} nc_n(x-3)^{n-1} - \sum_{n=0}^{\infty} c_n(x-3)^n &= 6 \\
\underbrace{\sum_{n=0}^{\infty} nc_n(x-3)^{n+1}}_{m=n+1} - \underbrace{\sum_{n=0}^{\infty} c_n(x-3)^n}_{m=n} &= 6 \\
\sum_{m=1}^{\infty} (m-1)c_{m-1}(x-3)^m - \sum_{m=0}^{\infty} c_m(x-3)^m &= 6 \\
\sum_{m=1}^{\infty} (m-1)c_{m-1}(x-3)^m - \left(c_0 + \sum_{m=1}^{\infty} c_m(x-3)^m\right) &= 6 \\
-c_0 + \sum_{m=1}^{\infty} \left((m-1)c_{m-1} - c_m\right)(x-3)^m &= 6
\end{aligned}$$

We are given $c_0 = y(3) = -6$, which sets the constant term equal to 6 on both sides of the above equation.
The remaining coefficients above all equal to zero, which gives us:

$$(m-1)c_{m-1} - c_m = 0 \implies c_m = (m-1)c_{m-1} \text{ for all } m \geq 1$$

When $m = 1$, we get $c_1 = 0$, and then all subsequent $c_n = 0$ as well.
Plugging this into our assumed power series form gives:

$$y = c_0 + \sum_{n=1}^{\infty} c_n(x-3)^n = -6 + \sum_{n=1}^{\infty} 0 \cdot (x-3)^n = -6$$

Exercise 4.2.4*e*

We assume $y = \sum_{n=0}^{\infty} c_n(x+2)^n$, so $y' = \sum_{n=0}^{\infty} nc_n(x+2)^{n-1}$:

$$\begin{aligned}
(x+2)y' + 2(x+2)^2 y &= 0 \\
(x+2)\cdot\sum_{n=0}^{\infty} nc_n(x+2)^{n-1} + 2(x+2)^2 \cdot \sum_{n=0}^{\infty} c_n(x+2)^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} nc_n(x+2)^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 2c_n(x+2)^{n+2}}_{m=n+2} &= 0 \\
\sum_{m=0}^{\infty} mc_m(x+2)^m + \sum_{m=2}^{\infty} 2c_{m-2}(x+2)^m &= 0 \\
\left(0c_0 + 1c_1(x+2)^1 + \sum_{m=2}^{\infty} mc_m(x+2)^m\right) + \sum_{m=2}^{\infty} 2c_{m-2}(x+2)^m &= 0 \\
c_1(x+2)^1 + \sum_{m=2}^{\infty} \left(mc_m + 2c_{m-2}\right)(x+2)^m &= 0
\end{aligned}$$

We are given $c_0 = y(-2) = -2$, and must have $c_1 = 0$ for the $(x+2)^1$ terms to match.
Setting the coefficients above all equal to zero gives us:

$$mc_m + 2c_{m-2} = 0 \implies c_m = -\frac{2}{m}c_{m-2} \text{ for all } m \geq 2$$

Plugging in a few values for m, we can see the pattern:

$$
\begin{aligned}
m=2: &\quad c_2=-\frac{2}{2}c_0=1\\
m=3: &\quad c_3=-\frac{2}{3}c_1=0\\
m=4: &\quad c_4=-\frac{2}{4}c_2=\frac{(-1)^2}{2!}\\
m=5: &\quad c_5=-\frac{2}{5}c_3=0\\
m=6: &\quad c_6=-\frac{2}{6}c_4=\frac{(-1)^3}{3!}\\
&\quad\vdots
\end{aligned}
$$

We have $c_{2k+1}=0$ for all $k\geq 0$, and $c_{2k}=\dfrac{(-1)^k}{k!}$ for $k\geq 1$

Plugging this into our assumed power series form gives:

$$
\begin{aligned}
y&=\sum_{n=0}^{\infty}c_n(x+2)^n\\
&=c_0+\sum_{k=0}^{\infty}c_{2k+1}(x+2)^{2k+1}+\sum_{k=1}^{\infty}c_{2k}(x+2)^{2k}\\
&=-2+\sum_{k=0}^{\infty}0(x+2)^{2k+1}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}(x+2)^{2k}\\
&=-2+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}(x+2)^{2k}
\end{aligned}
$$

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4.3: Second Order Power Series Solutions

Learning Objectives

- Apply the power series method to solve second-order linear ODEs.
- Solve for coefficients using recurrence relations with two arbitrary constants.
- Recognize when series solutions correspond to elementary functions.
- Apply the method to constant coefficient and Cauchy-Euler equations to verify known solutions.
- Solve initial value problems with series centered at various points.

The method extends to second-order equations, where the solution structure involves two arbitrary constants. We solve several important equations (like Airy's and Hermite's), often recognizing special functions, and apply the method to both constant coefficient and Cauchy-Euler types to verify known solutions.

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4.3: Second Order Power Series Solutions

Consider the second order LDE of the form

$$y''(x) + P(x)\,y'(x) + Q(x)\,y(x) = 0$$

If both $P(x)$ and $Q(x)$ are analytic at $a = 0$, and if we assume that the solutions to this differential equation are analytic at the point $a = 0$, we can write:

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty} n c_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) = 0$$

If we examine the first few terms of each series we see:

$$\begin{aligned}
\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} &= 0c_0 + 0c_1 + 2c_2 + 6c_3 x + 12c_4 x^2 + \ldots \\
\left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty} n c_n x^{n-1}\right) &= \left(p_0 + p_1 x + p_2 x^2 + \ldots\right)\left(0c_0 + 1c_1 + 2c_2 x + 3c_3 x^2 + \ldots\right) \\
&= p_0 c_1 + \left(p_0 2c_2 + p_1 c_1\right) x + \left(p_0 3c_3 + p_1 2c_2 + p_2 c_1\right) x^2 + \ldots \\
\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) &= \left(q_0 + q_1 x + q_2 x^2 + \ldots\right)\left(c_0 + c_1 x + c_2 x^2 + \ldots\right) \\
&= q_0 c_0 + \left(q_0 c_1 + q_1 c_0\right) x + \left(q_0 c_2 + q_1 c_1 + q_2 c_0\right) x^2 + \ldots
\end{aligned}$$

So when we add them up, and set it equal to zero we must have each coefficient equal zero:

$$\begin{aligned}
x^0: &\quad 2c_2 + p_0 c_1 + q_0 c_0 = 0 && \Longrightarrow c_2 = -\frac{p_0}{2}c_1 - \frac{q_0}{2}c_0 \\
x^1: &\quad 6c_3 + p_0 2c_2 + p_1 c_1 + q_0 c_1 + q_1 c_0 = 0 && \Longrightarrow c_3 = \text{ some function of } (c_0, c_1) \\
x^2: &\quad 12c_4 + p_0 3c_3 + p_1 2c_2 + p_2 c_1 + q_0 c_2 + q_1 c_1 + q_2 c_0 = 0 && \Longrightarrow c_4 = \text{ some function of } (c_0, c_1) \\
&\quad \vdots && \quad \vdots
\end{aligned}$$

This is a constructive proof of the existence of a family of solutions of two parameters (written in their power series forms).

Theorem 4.3.1: Existence of Analytic Solutions

If both $P(x)$ and $Q(x)$ are analytic at $a = 0$, then any solution $y(x)$ to the differential equation

$$y''(x) + P(x)\,y'(x) + Q(x)\,y(x) = 0$$

is analytic at $a = 0$. Moreover, the general solution can be written as $y(x) = c_0 y_1(x) + c_1 y_2(x)$ where y_1 and y_2 are linearly independent analytic functions whose power series can be computed using the recurrence relations derived from the LDE.

Illustration 4.3.1: Airy's Equation

Consider the LDE $y'' - xy = 0$

The coefficient functions involved are not CC or CE, but they are all analytic at zero, so we look for power series solutions around zero.

Assume $y = \sum_{n=0}^{\infty} c_n x^n$, which gives $y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$:

$$
\begin{aligned}
y'' - xy &= 0 \\
\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - x \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{m=n+1} &= 0 \\
\sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2} x^m - \sum_{m=1}^{\infty} c_{m-1} x^m &= 0 \\
\left(0c_0 + 0c_1 + 2c_2 x^0 + \sum_{m=1}^{\infty} (m+2)(m+1)c_{m+2} x^m\right) - \sum_{m=1}^{\infty} c_{m-1} x^m &= 0 \\
2c_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1)c_{m+2} - c_{m-1}\right] x^m &= 0
\end{aligned}
$$

We immediately see that $2c_2 = 0 \implies c_2 = 0$ must hold. Setting the remaining coefficients equal to zero gives:

$$(m+2)(m+1)c_{m+2} - c_{m-1} = 0 \implies c_{m+2} = \frac{1}{(m+2)(m+1)} c_{m-1} \text{ for all } m \geq 1$$

Plugging in a few values of m, we see:

$$
\begin{aligned}
m = 1: &\quad c_3 = \frac{1}{3 \cdot 2} c_0 \\
m = 2: &\quad c_4 = \frac{1}{4 \cdot 3} c_1 \\
m = 3: &\quad c_5 = \frac{1}{5 \cdot 4} c_2 = 0 \\
m = 4: &\quad c_6 = \frac{1}{6 \cdot 5} c_3 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} c_0 = \frac{4!!!}{6!} c_0 \\
m = 5: &\quad c_7 = \frac{1}{7 \cdot 6} c_4 = \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} c_1 = \frac{5!!!}{7!} c_1 \\
m = 6: &\quad c_8 = \frac{1}{8 \cdot 7} c_5 = 0 \\
m = 7: &\quad c_9 = \frac{1}{9 \cdot 8} c_6 = \frac{7!!!}{9!} c_0 \\
m = 8: &\quad c_{10} = \frac{1}{10 \cdot 9} c_7 = \frac{8!!!}{10!} c_1 \\
m = 9: &\quad c_{11} = \frac{1}{11 \cdot 10} c_8 = 0 \\
&\quad \vdots
\end{aligned}
$$

The patterns that emerge are $c_{3k} = \frac{(3k-2)!!!}{(3k)!} c_0$ for $k \geq 1$, and $c_{3k+1} = \frac{(3k-1)!!!}{(3k+1)!} c_1$ for $k \geq 1$, with $c_{3k+2} = 0$ for all $k \geq 0$

Putting this into our general power series form gives:

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + \sum_{k=0}^{\infty} c_{3k+2} x^{3k+2} + \sum_{k=1}^{\infty} c_{3k} x^{3k} + \sum_{k=1}^{\infty} c_{3k+1} x^{3k+1} \\
&= c_0 + c_1 x + \sum_{k=0}^{\infty} 0 x^{3k+2} + \sum_{k=1}^{\infty} \frac{(3k-2)!!!}{(3k)!} c_0 x^{3k} + \sum_{k=1}^{\infty} \frac{(3k-1)!!!}{(3k+1)!} c_1 x^{3k+1} \\
&= c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(3k-2)!!!}{(3k)!} x^{3k}\right) + c_1 \left(x + \sum_{k=1}^{\infty} \frac{(3k-1)!!!}{(3k+1)!} x^{3k+1}\right)
\end{aligned}
$$

These two solutions are known as the *Airy Functions*. This equation appears in situations which involve transitions between wave-like and particle-like behaviors in physics and engineering. They are even involved in the background of why we see rainbows!

This method is generally more time intensive than our others for the constant coefficient case, but it is still applicable.

Illustration 4.3.2: Constant Coefficients

Consider the simple LDE $y'' - y = 0$

This is a CC LDE, which can be solved using the methods of Chapter 2, but we will attempt to find power series solutions around zero to compare.

If we assume $y = \sum_{n=0}^{\infty} c_n x^n$, which gives $y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$, this becomes:

$$\begin{aligned} y'' - y &= 0 \\ \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{m=n} &= 0 \\ \sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2}x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\ \left(0c_0 + 0c_1 + \sum_{m=0}^{\infty}(m+2)(m+1)c_{m+2}x^m\right) - \sum_{m=0}^{\infty} c_m x^m &= 0 \\ \sum_{m=0}^{\infty} \left[(m+2)(m+1)c_{m+2} - c_m\right] x^m &= 0 \end{aligned}$$

This gives us $(m+2)(m+1)c_{m+2} - c_m = 0$, which we write as $c_{m+2} = \dfrac{1}{(m+2)(m+1)}c_m$ for all $m \geq 0$:

$$\begin{aligned} m = 0: &\quad c_2 = \frac{1}{2\cdot 1}c_0 \\ m = 1: &\quad c_3 = \frac{1}{3\cdot 2}c_1 \\ m = 2: &\quad c_4 = \frac{1}{4\cdot 3}c_2 = \frac{1}{4!}c_0 \\ m = 3: &\quad c_5 = \frac{1}{5\cdot 4}c_3 = \frac{1}{5!}c_1 \\ m = 4: &\quad c_6 = \frac{1}{6\cdot 5}c_4 = \frac{1}{6!}c_0 \\ &\vdots \end{aligned}$$

This gives two patterns, with $c_{2k} = \dfrac{1}{(2k)!}c_0$ and $c_{2k+1} = \dfrac{1}{(2k+1)!}c_1$ for all $k \geq 1$

Plugging these into our general power series solution, we see:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k}x^{2k} + \sum_{k=1}^{\infty} c_{2k+1}x^{2k+1} \\ &= c_0 + c_1 x + \sum_{k=1}^{\infty} \frac{1}{(2k)!}c_0 x^{2k} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!}c_1 x^{2k+1} \\ &= c_0\left(1 + \sum_{k=1}^{\infty}\frac{1}{(2k)!}x^{2k}\right) + c_1\left(x + \sum_{k=1}^{\infty}\frac{1}{(2k+1)!}x^{2k+1}\right) \end{aligned}$$

These happen to be exactly the power series representing $\cosh(x)$ and $\sinh(x)$ respectively.

Illustration 4.3.3: Hermite's Equation

Consider the LDE $y'' - 2xy' + 2y = 0$

The coefficient functions involved are not CC or CE, but they are all analytic at zero, so we look for power series solutions around zero.

Assume $y = \sum_{n=0}^{\infty} c_n x^n$, so $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$:

$$\begin{aligned}
y'' - 2xy' + 2y &= 0 \\
\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 2x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 2n c_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^n}_{m=n} &= 0 \\
\sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2} x^m - \sum_{m=0}^{\infty} 2m c_m x^m + \sum_{m=0}^{\infty} 2c_m x^m &= 0 \\
\left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2} x^m\right) - \sum_{m=0}^{\infty} 2m c_m x^m + \sum_{m=0}^{\infty} 2c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[(m+2)(m+1)c_{m+2} - (2m-2)c_m\right] x^m &= 0
\end{aligned}$$

Setting the coefficients equal to zero gives:

$$(m+2)(m+1)c_{m+2} - (2m-2)c_m = 0 \implies c_{m+2} = \frac{2(m-1)}{(m+2)(m+1)} c_m \text{ for all } m \geq 0$$

Plugging in a few values of m, we see:

$$\begin{aligned}
m = 0: &\quad c_2 = \frac{-2}{2 \cdot 1} c_0 = -c_0 \\
m = 1: &\quad c_3 = \frac{0}{3 \cdot 2} c_1 = 0 \\
m = 2: &\quad c_4 = \frac{2}{4 \cdot 3} c_2 = -\frac{2!!!!}{4!} c_0 \\
m = 3: &\quad c_5 = \frac{4}{5 \cdot 4} c_3 = 0 \\
m = 4: &\quad c_6 = \frac{6}{6 \cdot 5} c_4 = -\frac{6!!!!}{6!} c_0 \\
m = 5: &\quad c_7 = \frac{8}{7 \cdot 6} c_5 = 0 \\
m = 6: &\quad c_8 = \frac{10}{8 \cdot 7} c_6 = -\frac{10!!!!}{8!} c_0 \\
m = 7: &\quad c_9 = \frac{12}{9 \cdot 8} c_7 = 0 \\
m = 8: &\quad c_{10} = \frac{14}{10 \cdot 9} c_8 = \frac{14!!!!}{10!} c_0 \\
&\vdots
\end{aligned}$$

The patterns that emerge are $c_{2k} = -\frac{(4k-6)!!!}{(2k)!} c_0$ for $k \geq 2$, and $c_{2k+1} = 0$ for $k \geq 1$

Putting this into our general power series form gives:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+2} + \sum_{k=2}^{\infty} c_{2k} x^{2k} \\ &= c_0 + c_1 x - c_0 x^2 + \sum_{k=1}^{\infty} 0 x^{2k+1} - \sum_{k=2}^{\infty} \frac{(4k-6)!!!!}{(2k)!} c_0 x^{2k} \\ &= c_0 \left(1 - x^2 - \sum_{k=2}^{\infty} \frac{(4k-6)!!!!}{(2k)!} x^{2k}\right) + c_1 x \end{aligned}$$

This Hermite equation appears in many fundamental physical phenomena (like quantum vibrations) and is applicable in the fields of computational science and data analysis.

When solving a first order LDE, we saw that $y(a) = c_0$ when the power series is centered at $x = a$. This was because any function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \implies f(a) = c_0 + c_1 \cdot 0 + c_2 \cdot 0^2 + \ldots$$

If we differentiate term by term before we plug in a, we can see that:

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1} = 0c_0 + 1c_1(x-a)^0 + 2c_2(x-a)^1 + \cdots \implies f(a) = 0 + c_1 \cdot 1 + 2c_2 \cdot 0^1 + \ldots$$

Thus, $c_1 = f'(a)$. The general formula for a higher order LDE is simply $c_n = n! f^{(n)}(a)$

(*Look familiar? Taylor's Theorem says that this series must have coefficients* $c_n = \dfrac{f^{(n)}(a)}{n!}$)

Example 4.3.1

Find the unique solution to $(x^2+1)y'' + xy' - 4y = 0$ with $y(0) = 1,\ y'(0) = 0$

Solution

We assume $y = \sum_{n=0}^{\infty} c_n x^n$ so $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$:

$$\begin{aligned} (x^2+1)y'' + xy' - 4y &= 0 \\ x^2 \cdot \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} - 4 \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} + \underbrace{\sum_{n=0}^{\infty} n c_n x^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 4c_n x^n}_{m=n} &= 0 \\ \sum_{m=0}^{\infty} m(m-1)c_m x^m + \sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2} x^m + \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} 4c_m x^m &= 0 \\ \sum_{m=0}^{\infty} m(m-1)c_m x^m + \left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2} x^m\right) + \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} 4c_m x^m &= 0 \\ \sum_{m=0}^{\infty} \left[m(m-1)c_m + (m+2)(m+1)c_{m+2} + m c_m - 4c_m\right] x^m &= 0 \end{aligned}$$

Setting each coefficient equal to zero gives us:

$$m(m-1)c_m + (m+2)(m+1)c_{m+2} + mc_m - 4c_m = 0 \implies c_{m+2} = \frac{2-m}{m+1} c_m \text{ for all } m \geq 0$$

The initial conditions give $y(0) = c_0 = 1$ and $y'(0) = c_1 = 0$, so we continue for $m \geq 0$:

$$\begin{aligned} m = 0: &\quad c_2 = \frac{2}{1} c_0 = 2 \\ m = 1: &\quad c_3 = \frac{1}{2} c_1 = 0 \\ m = 2: &\quad c_4 = \frac{0}{3} c_2 = 0 \\ m = 3: &\quad c_5 = \frac{-1}{4} c_3 = 0 \\ m = 4: &\quad c_6 = \frac{-2}{5} c_4 = 0 \\ &\vdots \end{aligned}$$

The only nonzero terms were $c_0 = 1$ and $c_2 = 2$

Therefore, the unique solution is $\boxed{y(x) = 1 + 2x^2}$

Remember to center the power series around the initial condition whenever possible.

Example 4.3.2

Find the unique solution to $xy'' + (x-1)y' - 3y = 0$ with $y(1) = 4,\ y'(1) = 9$

Solution

We write $xy'' + (x-1)y' - 3y = 0 \longleftrightarrow ((x-1)+1)\, y'' + (x-1)y' - 3y = 0$

If we assume $y = \sum_{n=0}^{\infty} c_n (x-1)^n$, then $y' = \sum_{n=0}^{\infty} n c_n (x-1)^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) c_n (x-1)^{n-2}$:

$$((x-1)+1)\, y'' + (x-1)y' - 3y = 0$$

$$(x-1) \cdot \sum_{n=0}^{\infty} n(n-1) c_n (x-1)^{n-2} + \sum_{n=0}^{\infty} n(n-1) c_n (x-1)^{n-2} + (x-1) \cdot \sum_{n=0}^{\infty} n c_n (x-1)^{n-1} - 3 \sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

$$\underbrace{\sum_{n=0}^{\infty} n(n-1) c_n (x-1)^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} n(n-1) c_n (x-1)^{n-2}}_{m=n-2} + \underbrace{\sum_{n=0}^{\infty} n c_n (x-1)^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 3 c_n (x-1)^n}_{m=n} = 0$$

$$\sum_{m=-1}^{\infty} (m+1) m c_{m+1} (x-1)^m + \sum_{m=-2}^{\infty} (m+2)(m+1) c_{m+2} (x-1)^m + \sum_{m=0}^{\infty} m c_m (x-1)^m - \sum_{m=0}^{\infty} 3 c_m (x-1)^m = 0$$

$$\left(0c_0 + \sum_{m=0}^{\infty} (m+1) m c_{m+1} (x-1)^m\right) + \left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} (x-1)^m\right) + \sum_{m=0}^{\infty} m c_m (x-1)^m - \sum_{m=0}^{\infty} 3 c_m (x-1)^m = 0$$

$$\sum_{m=0}^{\infty} \left[(m+1) m c_{m+1} + (m+2)(m+1) c_{m+2} + m c_m - 3 c_m\right] (x-1)^m = 0$$

Setting each of these coefficients equal to zero gives us:

$$(m+1) m c_{m+1} + (m+2)(m+1) c_{m+2} + m c_m - 3 c_m = 0$$

$$\Downarrow$$

$$c_{m+2} = -\frac{1}{(m+2)(m+1)} \left[m(m+1) c_{m+1} + (m-3)\, c_m\right] \text{ for all } m \geq 0$$

We know that $y(1) = c_0 = 4$ and $y'(1) = c_1 = 9$. Next, we plug in a few values of m to see if we can recognize a pattern:

$$\begin{aligned} m=0: &\quad c_2=-\frac{1}{2\cdot 1}\left[0c_1-3c_0\right]=6\\ m=1: &\quad c_3=-\frac{1}{3\cdot 2}\left[2c_2-2c_1\right]=1\\ m=2: &\quad c_4=-\frac{1}{4\cdot 3}\left[6c_3-1c_2\right]=0\\ m=3: &\quad c_5=-\frac{1}{5\cdot 4}\left[12c_4+0c_3\right]=0\\ &\vdots \end{aligned}$$

Thankfully, we have $c_n=0$ for all $n\geq 4$, and can see $c_0=4,\ \ c_1=9,\ \ c_2=6,\ \ c_3=1$

Therefore, the unique solution is the polynomial $\boxed{y(x)=4+9(x-1)+6(x-1)^2+1(x-1)^3=x^3+3x^2-2}$

Example 4.3.3

Find the unique solution to $2y''+(x+\pi)y'+y=0$ with $y(-\pi)=0,\ \ y'(-\pi)=2$

Solution

If we assume $y=\sum_{n=0}^{\infty}c_n(x+\pi)^n$, then $y'=\sum_{n=0}^{\infty}nc_n(x+\pi)^{n-1}$ and $y''=\sum_{n=0}^{\infty}n(n-1)c_n(x+\pi)^{n-2}$:

$$\begin{aligned} 2y''+(x+\pi)y'+y&=0\\ 2\cdot\sum_{n=0}^{\infty}n(n-1)c_n(x+\pi)^{n-2}+(x+\pi)\cdot\sum_{n=0}^{\infty}nc_n(x+\pi)^{n-1}+\sum_{n=0}^{\infty}c_n(x+\pi)^n&=0\\ \underbrace{\sum_{n=0}^{\infty}2n(n-1)c_n(x+\pi)^{n-2}}_{m=n-2}+\underbrace{\sum_{n=0}^{\infty}nc_n(x+\pi)^{n}}_{m=n}+\underbrace{\sum_{n=0}^{\infty}c_n(x+\pi)^{n}}_{m=n}&=0\\ \sum_{m=-2}^{\infty}2(m+2)(m+1)c_{m+2}(x+\pi)^m+\sum_{m=0}^{\infty}mc_m(x+\pi)^m+\sum_{m=0}^{\infty}c_m(x+\pi)^m&=0\\ \left(0c_0+0c_1+\sum_{m=0}^{\infty}2(m+2)(m+1)c_{m+2}(x+\pi)^m\right)+\sum_{m=0}^{\infty}mc_m(x+\pi)^m+\sum_{m=0}^{\infty}c_m(x+\pi)^m&=0\\ \sum_{m=0}^{\infty}\left[2(m+2)(m+1)c_{m+2}+mc_m+c_m\right](x+\pi)^m&=0 \end{aligned}$$

Setting each of these coefficients equal to zero gives us:

$$\begin{gathered} 2(m+2)(m+1)c_{m+2}+mc_m+c_m=0\\ \Downarrow\\ c_{m+2}=-\frac{1}{2(m+2)}c_m \text{ for all } m\geq 0 \end{gathered}$$

We know that $y(-\pi)=c_0=0$ and $y'(-\pi)=c_1=2$. Next, we plug in a few values of m to see if we can recognize a pattern:

$$\begin{aligned} m=0: &\quad c_2=-\frac{1}{2\cdot 2}c_0=0\\ m=1: &\quad c_3=-\frac{1}{2\cdot 3}c_1=\frac{2}{(-2)^1\cdot 3!!}\\ m=2: &\quad c_4=-\frac{1}{2\cdot 4}c_2=0\\ m=3: &\quad c_5=-\frac{1}{2\cdot 5}c_3=\frac{2}{(-2)^2\cdot 5!!}\\ &\vdots \end{aligned}$$

Thankfully, we have $c_{2k}=0$ for all $k\geq 0$, and $c_{2k+1}=\dfrac{2}{(-2)^k\cdot(2k+1)!!}$ for $k\geq 0$

We plug this into the general power series formula:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n (x+\pi)^n \\ &= \sum_{k=0}^{\infty} c_{2k}(x+\pi)^{2k} + \sum_{k=0}^{\infty} c_{2k+1}(x+\pi)^{2k+1} \\ &= \sum_{k=0}^{\infty} 0(x+\pi)^{2k} + \sum_{k=0}^{\infty} \frac{2}{(-2)^k \cdot (2k+1)!!}(x+\pi)^{2k+1} \\ &\boxed{= \sum_{k=0}^{\infty} \frac{2}{(-2)^k \cdot (2k+1)!!}(x+\pi)^{2k+1}} \end{aligned}$$

As always, if the pattern is difficult to notice we settle for the first handful of terms.

Example 4.3.4

Find the first two nonzero coefficients of the power series solution centered at $a = 0$ for the IVP $y'' + e^x y = \sin(x)$ with $y(0) = 0$ and $y'(0) = 0$.

Solution

If we assume $y = \sum_{n=0}^{\infty} c_n x^n$, then $y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$.

We know $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ and $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$:

$$\begin{aligned} & & y'' + e^x \cdot y &= \sin(x) \\ \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} &+ \left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \end{aligned}$$

We write out the first few terms of each sum:

$$\begin{aligned} y'' &= \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} \\ &= 0c_0 x^{-2} + 0c_1 x^{-1} + \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2}x^m \\ &= 2c_2 x^0 + 6c_3 x^1 + 12c_4 x^2 + 20c_5 x^3 + \ldots \\ \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \\ &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots \\ &= 0 + 1x + 0x^2 - \frac{1}{6}x^3 + 0x^4 + \frac{1}{120}x^5 + \ldots \\ e^x \cdot y &= \left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) \\ &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots\right)\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\right) \\ &= c_0 + (c_0 + c_1)\,x + \left(\frac{1}{2}c_0 + c_1 + c_2\right)x^2 + \left(\frac{1}{6}c_0 + \frac{1}{2}c_1 + c_2 + c_3\right)x^3 + \ldots \end{aligned}$$

We are given $y(0) = c_0 = 0$ and $y'(0) = c_1 = 0$, and collecting like terms gives us an equation for each power of x:

$$
\begin{aligned}
x^0: &\quad 2c_2 + c_0 = 0 \implies c_2 = 0 \\
x^1: &\quad 6c_3 + c_0 + c_1 = 1 \implies c_3 = \frac{1}{6} \\
x^2: &\quad 12c_4 + \frac{1}{2}c_0 + c_1 + c_2 = 0 \implies c_4 = 0 \\
x^3: &\quad 20c_5 + \frac{1}{6}c_0 + \frac{1}{2}c_1 + c_2 + c_3 = -\frac{1}{6} \implies c_5 = -\frac{1}{60} \\
&\quad \vdots
\end{aligned}
$$

Therefore, the power series around $a = 0$ satisfying this IVP begins with $\boxed{y(x) = \frac{1}{6}x^3 - \frac{1}{60}x^5 + \ldots}$

We finish by once more discussing the limitations of this approach. We began all problems *assuming* that an analytic solution existed to the LDE. We have seen that it is entirely possible to have non-analytic solutions to some LDEs (like a Cauchy-Euler LDE with negative roots).

If the coefficient functions in

$$y''(x) + P(x)\,y'(x) + Q(x)\,y(x) = 0$$

are not analytic, then there is no promise that there will be any analytic solutions for $y(x)$. There is still a method we can use for special cases of "almost analytic" functions, which will be discussed in the next subchapter. This subchapter demonstrates that the power series method provides a systematic approach to solving second-order LDEs and often reveals the underlying special functions that appear throughout physics and engineering.

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4.3: Practice Set

Exercise $4.3.1a$

Find the general series solution centered at $a = 0$ for the LDE $(x^2 + 1)y'' - 2y = 0$

Answer available at the end of this section.

Exercise $4.3.1b$

Find the general series solution centered at $a = 0$ for the LDE $y'' + xy = 0$

Answer available at the end of this section.

Exercise $4.3.1c$

Find the general series solution centered at $a = 0$ for the LDE $(x^2 - 2)y'' + 3xy' = 0$

Answer available at the end of this section.

Exercise $4.3.1d$

Find the general series solution centered at $a = 0$ for the LDE $y'' + xy' + y = 0$

Answer available at the end of this section.

Exercise $4.3.1e$

Find the general series solution centered at $a = 0$ for the LDE $3y'' - 2xy' - 4y = 0$

Answer available at the end of this section.

Exercise $4.3.1f$

Find the general series solution centered at $a = 0$ for the LDE $y'' - 4x^2y' - 12xy = 0$

Answer available at the end of this section.

Exercise $4.3.1g$

Find the general series solution centered at $a = 0$ for the LDE $(x^3 - 1)y'' - 6xy = 0$

Answer available at the end of this section.

Exercise $4.3.1h$

Find the general series solution centered at $a = 0$ for the LDE $(x^2 + 1)y'' + 4xy' + 2y = 0$

Answer available at the end of this section.

Exercise $4.3.2a$

Find the unique power series solution to $(1 - 2x^2)y'' + xy' = 0$ with $y(0) = 11,\ y'(0) = 2$

Answer available at the end of this section.

Exercise $4.3.2b$

Find the unique power series solution to $3y'' + xy' + 2y = 0$ with $y(0) = 0,\ y'(0) = 1$

Answer available at the end of this section.

Exercise $4.3.2c$

Find the unique power series solution to $y'' - 4x^2y = 0$ with $y(0) = 1,\ y'(0) = 0$

Answer available at the end of this section.

Exercise $4.3.2d$

Find the unique power series solution to $(x^2 + 4x)y'' + (4x + 8)y' + 2y = 0$ with $y(-2) = 4,\ y'(-2) = 1$

Answer available at the end of this section.

Exercise $4.3.3a$

Find the first 4 nonzero terms of the power series solution to $y'' + xy' + xy = 0$ with $y(0) = 360, y'(0) = 0$

Answer available at the end of this section.

Exercise $4.3.3b$

Find the first 5 nonzero terms of the power series solution to $2y'' - x^2y' + 2xy = 0$ with $y(0) = 720, y'(0) = -48$

Answer available at the end of this section.

Exercise $4.3.3c$

Find the first 6 nonzero terms of the power series solution to $(x + 1)y'' + (x - 1)y' + y = 0$ with $y(0) = 10, y'(0) = 30$

Answer available at the end of this section.

Exercise $4.3.3d$

Find the first 6 nonzero terms of the power series solution to $y'' - 2(x-1)y' + y = 0$ with $y(1) = 1, y'(1) = 1$

Answer available at the end of this section.

Answers

Exercise $4.3.1a$

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$\begin{aligned}
(x^2+1)y'' - 2y &= 0 \\
x^2 \cdot \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 2\left(\sum_{n=0}^{\infty} c_n x^n\right) &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 2c_n x^n}_{m=n} &= 0 \\
\sum_{m=0}^{\infty} m(m-1)c_m x^m + \sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2}x^m - \sum_{m=0}^{\infty} 2c_m x^m &= 0 \\
\sum_{m=0}^{\infty} m(m-1)c_m x^m + \left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2}x^m\right) - \sum_{m=0}^{\infty} 2c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[m(m-1)c_m + (m+2)(m+1)c_{m+2} - 2c_m\right] x^m &= 0
\end{aligned}$$

Setting each coefficient in the sum equal to zero gives:

$$(m-2)(m+1)c_m + (m+2)(m+1)c_{m+2} = 0 \implies c_{m+2} = \frac{2-m}{m+2}c_m \text{ for all } m \geq 0$$

Plugging in a few values of m, we see:

$$\begin{aligned}
m = 0: &\quad c_2 = \frac{2}{2}c_0 = c_0 \\
m = 1: &\quad c_3 = \frac{1}{3}c_1 \\
m = 2: &\quad c_4 = \frac{0}{4}c_2 = 0 \\
m = 3: &\quad c_5 = \frac{-1}{5}c_3 = \frac{-1}{5!!}c_1 \\
m = 4: &\quad c_6 = \frac{-2}{6}c_4 = 0 \\
m = 5: &\quad c_7 = \frac{-3}{7}c_5 = \frac{(-1)^2 3!!}{7!!}c_1 \\
&\vdots
\end{aligned}$$

The pattern here is that $c_{2k} = 0$ for $k \geq 2$, and $c_{2k+1} = \dfrac{(-1)^{k-1}(2k-3)!!}{(2k+1)!!}c_1 = \dfrac{(-1)^{k-1}}{(2k+1)(2k-1)}c_1$ for $k \geq 2$.

Putting this into the general power series formula, we get:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \sum_{k=2}^{\infty} c_{2k} x^{2k} + \sum_{k=2}^{\infty} c_{2k+1} x^{2k+1} \\
&= c_0 + c_1 x + c_0 x^2 + \frac{1}{3}c_1 x^3 + \sum_{k=2}^{\infty} 0 x^{2k} + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k+1)(2k-1)} c_1 x^{2k+1} \\
&= c_0\left(1 + x^2\right) + c_1\left(x + \frac{1}{3}x^3 + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k+1)(2k-1)} x^{2k+1}\right)
\end{aligned}$$

Exercise $4.3.1b$

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$
\begin{aligned}
y'' + xy &= 0 \\
\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + x \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^{n+1}}_{m=n+1} &= 0 \\
\sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2}x^m + \sum_{m=1}^{\infty} c_{m-1}x^m &= 0 \\
\left(0c_0 + 0c_1 + 2c_2 x^0 + \sum_{m=1}^{\infty}(m+2)(m+1)c_{m+2}x^m\right) + \sum_{m=1}^{\infty} c_{m-1}x^m &= 0 \\
2c_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1)c_{m+2} + c_{m-1}\right] x^m &= 0
\end{aligned}
$$

Setting the x^0 term to zero gives $2c_2 = 0 \implies c_2 = 0$

Setting each coefficient in the sum equal to zero gives:

$$(m+2)(m+1)c_{m+2} + c_{m-1} = 0 \implies c_{m+2} = -\frac{1}{(m+2)(m+1)}c_{m-1} \text{ for all } m \geq 1$$

Plugging in a few values of m, we see:

$$
\begin{aligned}
m = 1: &\quad c_3 = -\frac{1}{3 \cdot 2}c_0 \\
m = 2: &\quad c_4 = -\frac{1}{4 \cdot 3}c_1 \\
m = 3: &\quad c_5 = -\frac{1}{5 \cdot 4}c_2 = 0 \\
m = 4: &\quad c_6 = -\frac{1}{6 \cdot 5}c_3 = \frac{(-1)^2 \cdot 4!!}{6!}c_0 \\
m = 5: &\quad c_7 = -\frac{1}{7 \cdot 6}c_4 = \frac{(-1)^2 \cdot 5!!}{7!}c_1 \\
m = 6: &\quad c_8 = -\frac{1}{8 \cdot 7}c_5 = 0 \\
&\vdots
\end{aligned}
$$

The patterns that emerge are $c_{3k} = \dfrac{(-1)^k \cdot (3k-2)!!!}{(3k)!}c_0$ and $c_{3k+1} = \dfrac{(-1)^k(3k-1)!!!}{(3k+1)!}c_1$ for $k \geq 1$ (and $c_{3k+2} = 0$ for all k).

Putting this into the general power series formula, we get:

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + c_2 x^2 + \sum_{k=1}^{\infty} c_{3k}x^{3k} + \sum_{k=1}^{\infty} c_{3k+1}x^{3k+1} + \sum_{k=1}^{\infty} c_{3k+2}x^{3k+2} \\
&= c_0 + c_1 x + 0x^2 + \sum_{k=1}^{\infty} \frac{(-1)^k \cdot (3k-2)!!!}{(3k)!}c_0 x^{3k} + \sum_{k=1}^{\infty} \frac{(-1)^k(3k-1)!!!}{(3k+1)!}c_1 x^{3k+1} + \sum_{k=1}^{\infty} 0x^{3k+2} \\
&= c_0\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k(3k-2)!!}{(3k)!}x^{3k}\right) + c_1\left(x + \sum_{k=1}^{\infty} \frac{(-1)^k(3k-1)!!}{(3k+1)!}x^{3k+1}\right)
\end{aligned}
$$

Exercise $4.3.1c$

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$
\begin{aligned}
(x^2-2)y'' + 3xy' &= 0 \\
x^2 \cdot \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 2 \cdot \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + 3x \cdot \sum_{n=0}^{\infty} nc_n x^{n-1} &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 2n(n-1)c_n x^{n-2}}_{m=n-2} + \underbrace{\sum_{n=0}^{\infty} 3nc_n x^n}_{m=n} &= 0 \\
\sum_{m=0}^{\infty} m(m-1)c_m x^m - \sum_{m=-2}^{\infty} 2(m+2)(m+1)c_{m+2}x^m + \sum_{n=0}^{\infty} 3mc_m x^m &= 0 \\
\sum_{m=0}^{\infty} m(m-1)c_m x^m - \left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} 2(m+2)(m+1)c_{m+2}x^m\right) + \sum_{n=0}^{\infty} 3mc_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[m(m-1)c_m - 2(m+2)(m+1)c_{m+2} + 3mc_m\right] x^m &= 0
\end{aligned}
$$

Setting each coefficient in the sum equal to zero gives:

$$m(m+2)c_m - 2(m+2)(m+1)c_{m+2} = 0 \implies c_{m+2} = \frac{m}{2(m+1)}c_m \text{ for all } m \geq 0$$

Plugging in a few values of m, we see:

$$\begin{aligned} m=0: &\quad c_2=\frac{0}{2\cdot 1}c_0=0\\ m=1: &\quad c_3=\frac{1}{2\cdot 2}c_1\\ m=2: &\quad c_4=\frac{2}{2\cdot 3}c_2=0\\ m=3: &\quad c_5=\frac{3}{2\cdot 4}c_3=\frac{3!!}{2^2\cdot 4!!}c_1\\ &\vdots \end{aligned}$$

The patterns that emerge are $c_{2k}=0$ for $k\geq 0$ and $c_{2k+1}=\dfrac{(2k-1)!!}{2^k\cdot(2k)!!}c_1$ for $k\geq 1$

Putting this into the general power series formula, we get:

$$\begin{aligned} y&=\sum_{n=0}^{\infty}c_nx^n\\ &=c_0+c_1x+\sum_{k=1}^{\infty}c_{2k}x^{2k}+\sum_{k=1}^{\infty}c_{2k+1}x^{2k+1}\\ &=c_0+c_1x+\sum_{k=1}^{\infty}0x^{2k}+\sum_{k=1}^{\infty}\frac{(2k-1)!!}{2^k\cdot(2k)!!}c_1x^{2k+1}\\ &=c_0+c_1\left(x+\sum_{k=1}^{\infty}\frac{(2k-1)!!}{2^k\cdot(2k)!!}x^{2k+1}\right) \end{aligned}$$

Exercise 4.3.1*d*

We assume $y=\sum_{n=0}^{\infty}c_nx^n$:

$$\begin{aligned} y''+xy'-y&=0\\ \sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}+x\cdot\sum_{n=0}^{\infty}nc_nx^{n-1}+\sum_{n=0}^{\infty}c_nx^n&=0\\ \underbrace{\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}}_{m=n-2}+\underbrace{\sum_{n=0}^{\infty}nc_nx^n}_{m=n}+\underbrace{\sum_{n=0}^{\infty}c_nx^n}_{m=n}&=0\\ \sum_{m=-2}^{\infty}(m+2)(m+1)c_{m+2}x^m+\sum_{m=0}^{\infty}mc_mx^m+\sum_{m=0}^{\infty}c_mx^m&=0\\ \left(0c_0+0c_1+\sum_{m=0}^{\infty}(m+2)(m+1)c_{m+2}x^m\right)+\sum_{m=0}^{\infty}mc_mx^m+\sum_{m=0}^{\infty}c_mx^m&=0\\ \sum_{m=0}^{\infty}\left[(m+2)(m+1)c_{m+2}+mc_m+c_m\right]x^m&=0 \end{aligned}$$

Setting each coefficient in the sum equal to zero gives:

$$(m+2)(m+1)c_{m+2}+(m+1)c_m=0\implies c_{m+2}=-\frac{1}{m+2}c_m \text{ for all } m\geq 0$$

Plugging in a few values of m, we see:

$$\begin{aligned} m=0: &\quad c_2=-\frac{1}{2}c_0\\ m=1: &\quad c_3=-\frac{1}{3}c_1\\ m=2: &\quad c_4=-\frac{1}{4}c_2=\frac{(-1)^2}{4!!}c_0\\ m=3: &\quad c_5=-\frac{1}{5}c_3=\frac{(-1)^2}{5!!}c_1\\ &\vdots \end{aligned}$$

The patterns that emerge are $c_{2k}=\dfrac{(-1)^k}{(2k)!!}c_0$ for $k\geq 1$ and $c_{2k+1}=\dfrac{(-1)^k}{(2k+1)!!}c_1$ for $k\geq 1$

Putting this into the general power series formula, we get:

$$\begin{aligned} y&=\sum_{n=0}^{\infty}c_nx^n\\ &=c_0+c_1x+\sum_{k=1}^{\infty}c_{2k}x^{2k}+\sum_{k=1}^{\infty}c_{2k+1}x^{2k+1}\\ &=c_0+c_1x+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!!}c_0x^{2k}+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!!}c_1x^{2k+1}\\ &=c_0\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!!}x^{2k}\right)+c_1\left(x+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!!}x^{2k+1}\right) \end{aligned}$$

Exercise 4.3.1*e*

We assume $y=\sum_{n=0}^{\infty}c_nx^n$:

$$
\begin{aligned}
3y'' - 2xy' - 4y &= 0 \\
3 \cdot \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 2x \cdot \sum_{n=0}^{\infty} nc_n x^{n-1} - 4 \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} 3n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 2nc_n x^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 4c_n x^n}_{m=n} &= 0 \\
\sum_{m=-2}^{\infty} 3(m+2)(m+1)c_{m+2} x^m - \sum_{m=0}^{\infty} 2mc_m x^m - \sum_{m=0}^{\infty} 4c_m x^m &= 0 \\
\left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} 3(m+2)(m+1)c_{m+2} x^m\right) - \sum_{m=0}^{\infty} 2mc_m x^m - \sum_{m=0}^{\infty} 4c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[3(m+2)(m+1)c_{m+2} - 2mc_m - 4c_m\right] x^m &= 0
\end{aligned}
$$

Setting each coefficient in the sum equal to zero gives:

$$3(m+2)(m+1)c_{m+2} - 2(m+2)c_m = 0 \implies c_{m+2} = \frac{2}{3(m+1)} c_m \text{ for all } m \geq 0$$

Plugging in a few values of m, we see:

$$
\begin{aligned}
m = 0: &\quad c_2 = \frac{2}{3 \cdot 1} c_0 \\
m = 1: &\quad c_3 = \frac{2}{3 \cdot 2} c_1 \\
m = 2: &\quad c_4 = \frac{2}{3 \cdot 3} c_2 = \frac{2^2}{3^2 \cdot 3!!} c_0 \\
m = 3: &\quad c_5 = \frac{2}{3 \cdot 4} c_3 = \frac{2^2}{3^2 \cdot 4!!} c_1 \\
&\vdots
\end{aligned}
$$

The patterns that emerge are $c_{2k} = \dfrac{2^k}{3^k \cdot (2k-1)!!} c_0$ for $k \geq 1$ and $c_{2k+1} = \dfrac{2^k}{3^k \cdot (2k)!!} c_1$ for $k \geq 1$

Putting this into the general power series formula, we get:

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k} x^{2k} + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \\
&= c_0 + c_1 x + \sum_{k=1}^{\infty} \frac{2^k}{3^k \cdot (2k-1)!!} c_0 x^{2k} + \sum_{k=1}^{\infty} \frac{2^k}{3^k \cdot (2k)!!} c_1 x^{2k+1} \\
&= c_0 \left(1 + \sum_{k=1}^{\infty} \frac{2^k}{3^k \cdot (2k-1)!!} x^{2k}\right) + c_1 \left(x + \sum_{k=1}^{\infty} \frac{2^k}{3^k \cdot (2k)!!} x^{2k+1}\right)
\end{aligned}
$$

Exercise 4.3.1f

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$
\begin{aligned}
y'' - 4x^2 - 12xy &= 0 \\
\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 4x^2 \cdot \sum_{n=0}^{\infty} nc_n x^n - 12x \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 4nc_n x^{n+1}}_{m=n+1} - \underbrace{\sum_{n=0}^{\infty} 12c_n x^{n+1}}_{m=n+1} &= 0 \\
\sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2} x^m - \sum_{m=1}^{\infty} 4(m-1)c_{m-1} x^m - \sum_{m=1}^{\infty} 12c_{m-1} x^m &= 0 \\
\left(0c_0 + 0c_1 + 2c_2 x^0 + \sum_{m=1}^{\infty} (m+2)(m+1)c_{m+2} x^m\right) - \sum_{m=1}^{\infty} 4(m-1)c_{m-1} x^m - \sum_{m=1}^{\infty} 12c_{m-1} x^m &= 0 \\
2c_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1)c_{m+2} - 4(m-1)c_{m-1} - 12c_{m-1}\right] x^m &= 0
\end{aligned}
$$

Setting the x^0 term to zero gives $2c_2 = 0 \implies c_2 = 0$

Setting each remaining coefficient in the above sum equal to zero gives:

$$(m+2)(m+1)c_{m+2} - 4(m+2)c_{m-1} = 0 \implies c_{m+2} = \frac{4}{m+1} c_{m-1} \text{ for all } m \geq 1$$

Plugging in a few values of m, we see:

$$
\begin{aligned}
m=1: &\quad c_3=\frac{4}{2}c_0\\
m=2: &\quad c_4=\frac{4}{3}c_1\\
m=3: &\quad c_5=\frac{4}{4}c_2=0\\
m=4: &\quad c_6=\frac{4}{5}c_3=\frac{4^2}{5!!!}c_0\\
m=5: &\quad c_7=\frac{4}{6}c_4=\frac{4^2}{6!!!}c_1\\
&\quad\vdots
\end{aligned}
$$

The patterns that emerge are $c_{3k}=\dfrac{4^k}{(3k-1)!!!}c_0$ for $k\geq 1$ and $c_{3k+1}=\dfrac{4^k}{(3k)!!!}c_1$ for $k\geq 1$, with $c_{3k+2}=0$ for all $k\geq 0$

Putting this into the general power series formula, we get:

$$
\begin{aligned}
y&=\sum_{n=0}^{\infty}c_nx^n\\
&=c_0+c_1x+\sum_{k=0}^{\infty}c_{3k+2}x^{3k+2}+\sum_{k=1}^{\infty}c_{3k}x^{3k}+\sum_{k=1}^{\infty}c_{3k+1}x^{3k+1}\\
&=c_0+c_1x+\sum_{k=0}^{\infty}0x^{3k+2}+\sum_{k=1}^{\infty}\frac{4^k}{(3k-1)!!!}c_0x^{3k}+\sum_{k=1}^{\infty}\frac{4^k}{(3k)!!!}c_1x^{3k+1}\\
&=c_0\left(1+\sum_{k=1}^{\infty}\frac{4^k}{(3k-1)!!!}x^{3k}\right)+c_1\left(x+\sum_{k=1}^{\infty}\frac{4^k}{(3k)!!!}x^{3k+1}\right)
\end{aligned}
$$

Exercise $4.3.1g$

We assume $y=\sum_{n=0}^{\infty}c_nx^n$:

$$
\begin{aligned}
(x^3-1)y''-xy&=0\\
x^3\cdot\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}-\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}-6x\cdot\sum_{n=0}^{\infty}c_nx^n&=0\\
\underbrace{\sum_{n=0}^{\infty}n(n-1)c_nx^{n+1}}_{m=n+1}-\underbrace{\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}}_{m=n-2}-\underbrace{\sum_{n=0}^{\infty}6c_nx^{n+1}}_{m=n+1}&=0\\
\sum_{m=1}^{\infty}(m-1)(m-2)c_{m-1}x^m-\sum_{m=-2}^{\infty}(m+2)(m+1)c_{m+2}x^m-\sum_{m=1}^{\infty}6c_{m-1}x^m&=0\\
\sum_{m=1}^{\infty}(m-1)(m-2)c_{m-1}x^m-\left(0c_0+0c_1+2c_2x^0+\sum_{m=-2}^{\infty}(m+2)(m+1)c_{m+2}x^m\right)-\sum_{m=1}^{\infty}6c_{m-1}x^m&=0\\
-2c_2+\sum_{m=1}^{\infty}\left[(m-1)(m-2)c_{m-1}-(m+2)(m+1)c_{m+2}-6c_{m-1}\right]x^m&=0
\end{aligned}
$$

Setting the x^0 term to zero gives $-2c_2=0\implies c_2=0$

Setting each coefficient in the sum equal to zero gives:

$$(m+1)(m-4)c_{m-1}-(m+2)(m+1)c_{m+2}=0\implies c_{m+2}=\frac{m-4}{m+2}c_m \text{ for all } m\geq 1$$

Plugging in a few values of m, we see:

$$
\begin{aligned}
m=1: &\quad c_3=\frac{-3}{3}c_0\\
m=2: &\quad c_4=\frac{-2}{4}c_1\\
m=3: &\quad c_5=\frac{-1}{5}c_2=0\\
m=4: &\quad c_6=\frac{0}{6}c_3=0\\
m=5: &\quad c_7=\frac{1}{7}c_4=\frac{-2\cdot 1!!!}{7!!!}c_1\\
m=6: &\quad c_8=\frac{2}{8}c_5=0\\
m=7: &\quad c_9=\frac{3}{9}c_6=0\\
m=8: &\quad c_{10}=\frac{4}{10}c_7=\frac{-2\cdot 4!!!}{10!!!}c_1\\
&\quad\vdots
\end{aligned}
$$

The patterns that emerge are $c_{3k+1}=\dfrac{-2\cdot(3k-5)!!!}{(3k+1)!!!}c_1$ for $k\geq 2$, with $c_{3k+2}=0$ for all $k\geq 0$ and $c_{3k}=0$ for $k\geq 2$.

Putting this into the general power series formula, we get:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \sum_{k=1}^{\infty} c_{3k+2} x^{3k+2} + \sum_{k=2}^{\infty} c_{3k} x^{3k} + \sum_{k=2}^{\infty} c_{3k+1} x^{3k+1} \\
&= c_0 + c_1 x + 0x^2 - c_0 x^3 - \frac{1}{2} c_1 x^4 + \sum_{k=1}^{\infty} 0 x^{3k+2} + \sum_{k=2}^{\infty} 0 x^{3k} + \sum_{k=2}^{\infty} \frac{-2 \cdot (3k-5)!!!}{(3k+1)!!!} c_1 x^{3k+1} \\
&= c_0 \left(1 - x^3\right) + c_1 \left(x - \frac{1}{2} x^3 - 2 \sum_{k=2}^{\infty} \frac{(3k-5)!!!}{(3k+1)!!!} x^{3k+1} \right)
\end{aligned}$$

Exercise 4.3.1*h*

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$\begin{aligned}
(x^2+1)y'' + 4xy' + 2y &= 0 \\
x^2 \cdot \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + 4x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} + 2 \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1) c_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}}_{m=n-2} + \underbrace{\sum_{n=0}^{\infty} 4 n c_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 2 c_n x^n}_{m=n} &= 0 \\
\sum_{m=0}^{\infty} m(m-1) c_m x^m + \sum_{m=-2}^{\infty} (m+2)(m+1) c_{m+2} x^m + \sum_{m=0}^{\infty} 4 m c_m x^m + \sum_{m=0}^{\infty} 2 c_m x^m &= 0 \\
\sum_{m=0}^{\infty} m(m-1) c_m x^m + \left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m \right) + \sum_{m=0}^{\infty} 4 m c_m x^m + \sum_{m=0}^{\infty} 2 c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[m(m-1) c_m + (m+2)(m+1) c_{m+2} + 4 m c_m + 2 c_m \right] x^m &= 0
\end{aligned}$$

Setting each coefficient in the sum equal to zero gives:

$$(m+2)(m+1)c_m + (m+2)(m+1)c_{m+2} = 0 \implies c_{m+2} = -c_m \text{ for all} m \geq 0$$

This pattern is easy to see after a few values of m:

$$\begin{aligned}
m = 0: &\quad c_2 = -c_0 \\
m = 1: &\quad c_3 = -c_1 \\
m = 2: &\quad c_4 = -c_2 = (-1)^2 c_0 \\
m = 3: &\quad c_5 = -c_3 = (-1)^2 c_1 \\
&\vdots
\end{aligned}$$

We see that $c_{2k} = (-1)^k c_0$ and $c_{2k+1} = (-1)^k c_1$ for all $k \geq 1$.
Putting this into the general power series formula, we get:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k} x^{2k} + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \\
&= c_0 + c_1 x + \sum_{k=1}^{\infty} (-1)^k c_0 x^{2k} + \sum_{k=1}^{\infty} (-1)^k c_1 x^{2k+1} \\
&= c_0 \left(1 + \sum_{k=1}^{\infty} (-1)^k x^{2k} \right) + c_1 \left(x + \sum_{k=2}^{\infty} (-1)^k x^{2k+1} \right)
\end{aligned}$$

Exercise 4.3.2*a*

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$\begin{aligned}
(1-2x^2)y'' + y &= 0 \\
\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - x^2 \cdot \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - 2x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 2n(n-1) c_n x^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 2 n c_n x^n}_{m=n} &= 0 \\
\sum_{m=-2}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{n=0}^{\infty} 2m(m-1) c_m x^m - \sum_{m=0}^{\infty} 2 m c_m x^m &= 0 \\
\left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m \right) - \sum_{n=0}^{\infty} 2m(m-1) c_m x^m - \sum_{m=0}^{\infty} 2 m c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[(m+2)(m+1) c_{m+2} - 2m(m-1) c_m - 2 m c_m \right] x^m &= 0
\end{aligned}$$

Setting each coefficient in the sum equal to zero gives:

$$(m+2)(m+1)c_{m+2} - 2m^2 c_m = 0 \implies c_{m+2} = \frac{2m^2 c_m}{(m+2)(m+1)} \text{ for all } m \geq 0$$

By the initial conditions, we have $y(0) = c_0 = 11$ and $y'(0) = c_1 = 2$. We plug in m values to find the remaining pattern:

$$
\begin{aligned}
m=0: \quad c_2 &= \frac{2\cdot 0^2\cdot c_0}{2\cdot 1} = 0 \\
m=1: \quad c_3 &= \frac{2\cdot 1^2\cdot c_1}{3\cdot 2} = \frac{2^2\cdot 1^2}{3!} \\
m=2: \quad c_4 &= \frac{2\cdot 2^2\cdot c_2}{4\cdot 3} = 0 \\
m=3: \quad c_5 &= \frac{2\cdot 3^2\cdot c_3}{5\cdot 4} = \frac{2^3\,(3!!)^2}{5!} \\
m=4: \quad c_6 &= \frac{2\cdot 4^2\cdot c_4}{6\cdot 5} = 0 \\
m=5: \quad c_7 &= \frac{2\cdot 5^2\cdot c_5}{7\cdot 6} = \frac{2^4\,(5!!)^2}{7!} \\
&\vdots
\end{aligned}
$$

This gives $c_{2k}=0$ for $k\geq 1$ and $c_{2k+1}=\dfrac{((2k-1)!!)^2\cdot 2^{k+1}}{(2k+1)!}$ for $k\geq 1$

Putting this into the general power series formula, we get:

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k} x^{2k} + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \\
&= 11 + 2x + \sum_{k=1}^{\infty} 0x^{2k} + \sum_{k=1}^{\infty} \frac{((2k-1)!!)^2\cdot 2^{k+1}}{(2k+1)!} x^{2k+1} \\
&= 11 + 2x + \sum_{k=1}^{\infty} \frac{((2k-1)!!)^2\cdot 2^{k+1}}{(2k+1)!} x^{2k+1}
\end{aligned}
$$

Exercise 4.3.2*b*

We assume $y=\sum_{n=0}^{\infty} c_n x^n$:

$$
\begin{aligned}
3y'' + xy' + 2y &= 0 \\
3\cdot\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + x\cdot\sum_{n=0}^{\infty} nc_n x^{n-1} + 2\cdot\sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} 3n(n-1)c_n x^{n-2}}_{m=n-2} + \underbrace{\sum_{n=0}^{\infty} nc_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^n}_{m=n} &= 0 \\
\sum_{m=-2}^{\infty} 3(m+2)(m+1)c_{m+2}x^m + \sum_{n=0}^{\infty} mc_m x^m + \sum_{m=0}^{\infty} 2c_m x^m &= 0 \\
\left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} 3(m+2)(m+1)c_{m+2}x^m\right) + \sum_{n=0}^{\infty} mc_m x^m + \sum_{m=0}^{\infty} 2c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[3(m+2)(m+1)c_{m+2} + 2mc_m + c_m\right] x^m &= 0
\end{aligned}
$$

Setting each coefficient in the sum equal to zero gives:

$$
3(m+2)(m+1)c_{m+2} + (m+2)c_m = 0 \implies c_{m+2} = -\frac{c_m}{3(m+1)} \text{ for all } m\geq 0
$$

By the initial conditions, we have $y(0)=c_0=0$ and $y'(0)=c_1=1$. We plug in m values to find the remaining pattern:

$$
\begin{aligned}
m=0: \quad c_2 &= -\frac{c_0}{3\cdot 1} = 0 \\
m=1: \quad c_3 &= -\frac{c_1}{3\cdot 2} = -\frac{1}{3\cdot 2} \\
m=2: \quad c_4 &= -\frac{c_2}{3\cdot 3} = 0 \\
m=3: \quad c_5 &= -\frac{c_3}{3\cdot 4} = \frac{(-1)^2}{3^2\cdot 4!!} \\
m=4: \quad c_6 &= -\frac{c_4}{3\cdot 5} = 0 \\
m=5: \quad c_7 &= -\frac{c_5}{3\cdot 6} = \frac{(-1)^3}{3^3\cdot 6!!} \\
&\vdots
\end{aligned}
$$

We see that $c_{2k}=0$ for $k\geq 1$ and $c_{2k+1}=\dfrac{(-1)^k}{3^k\cdot(2k)!!}$ for $k\geq 1$

Putting this into the general power series formula, we get:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k} x^{2k} + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \\ &= 0 + 1x + \sum_{k=1}^{\infty} 0 x^{2k} + \sum_{k=1}^{\infty} \frac{(-1)^k}{3^k \cdot (2k)!!} x^{2k+1} \\ &= x + \sum_{k=1}^{\infty} \frac{(-1)^k}{3^k \cdot (2k)!!} x^{2k+1} \end{aligned}$$

Exercise 4.3.2*c*

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$\begin{aligned} y'' - 4x^2 y &= 0 \\ \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - 4x^2 \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \underbrace{\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 4 c_n x^{n+2}}_{m=n+2} &= 0 \\ \sum_{m=-2}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{m=2}^{\infty} 4 c_{m-2} x^m &= 0 \\ \left(0c_0 + 0c_1 + 2c_2 x^0 + 6c_3 x^1 + \sum_{m=2}^{\infty} (m+2)(m+1) c_{m+2} x^m \right) - \sum_{m=2}^{\infty} 4 c_{m-2} x^m &= 0 \\ 2c_2 + 6c_3 x + \sum_{m=2}^{\infty} \left[(m+2)(m+1) c_{m+2} - 4 c_{m-2} \right] x^m &= 0 \end{aligned}$$

Setting the coefficients of x^0 and x^1 equal to zero gives $c_2 = 0, \;\; c_3 = 0$

Setting each coefficient in the sum equal to zero gives:

$$(m+2)(m+1) c_{m+2} - 4 c_{m-2} = 0 \implies c_{m+2} = \frac{4 c_{m-2}}{(m+2)(m+1)} \text{ for all } m \geq 2$$

By the initial conditions, we have $y(0) = c_0 = 1$ and $y'(0) = c_1 = 0$. We plug in m values to find the remaining pattern:

$$\begin{aligned} m = 2: &\quad c_4 = \frac{4c_0}{4 \cdot 3} = \frac{1}{3!!!! \cdot 1!} \\ m = 3: &\quad c_5 = \frac{4c_1}{5 \cdot 4} = 0 \\ m = 4: &\quad c_6 = \frac{4c_2}{6 \cdot 5} = 0 \\ m = 5: &\quad c_7 = \frac{4c_3}{7 \cdot 6} = 0 \\ m = 6: &\quad c_8 = \frac{4c_2}{8 \cdot 7} = \frac{1}{7!!!! \cdot 2!} \\ m = 7: &\quad c_9 = \frac{4c_5}{9 \cdot 8} = 0 \\ m = 8: &\quad c_{10} = \frac{4c_6}{10 \cdot 9} = 0 \\ m = 9: &\quad c_{11} = \frac{4c_7}{11 \cdot 10} = 0 \\ &\vdots \end{aligned}$$

We see that $c_{4k} = \frac{1}{(4k-1)!!!! \cdot k!}$ for $k \geq 1$, and $c_{4k+1} = c_{4k+2} = c_{4k+3} = 0$ for $k \geq 0$

Putting this into the general power series formula, we get:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + \sum_{k=0}^{\infty} c_{4k+1} x^{4k+1} + \sum_{k=0}^{\infty} c_{4k+2} x^{4k+2} + \sum_{k=0}^{\infty} c_{4k+3} x^{4k+3} + \sum_{k=1}^{\infty} c_{4k} x^{4k} \\ &= 1 + \sum_{k=0}^{\infty} 0 x^{4k+1} + \sum_{k=0}^{\infty} 0 x^{4k+2} + \sum_{k=0}^{\infty} 0 x^{4k+3} + \sum_{k=1}^{\infty} \frac{1}{(4k-1)!!!! \cdot k!} x^{4k} \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{(4k-1)!!!! \cdot k!} x^{4k} \end{aligned}$$

Exercise 4.3.2*d*

We assume $y = \sum_{n=0}^{\infty} c_n (x+2)^n$:

$$(x^2+4x)y''+(4x+8)y'+2y=0$$
$$(x+2)^2y''-4y''+4(x+2)y'+2y=0$$
$$(x+2)^2\cdot\sum_{n=0}^{\infty}n(n-1)c_n(x+2)^{n-2}-4\cdot\sum_{n=0}^{\infty}n(n-1)c_n(x+2)^{n-2}+4(x+2)\cdot\sum_{n=0}^{\infty}nc_n(x+2)^{n-1}+2\cdot\sum_{n=0}^{\infty}c_n(x+2)^n=0$$
$$\underbrace{\sum_{n=0}^{\infty}n(n-1)c_n(x+2)^n}_{m=n}-\underbrace{\sum_{n=0}^{\infty}4n(n-1)c_n(x+2)^{n-2}}_{m=n-2}+\underbrace{\sum_{n=0}^{\infty}4nc_n(x+2)^n}_{m=n}+\underbrace{\sum_{n=0}^{\infty}2c_n(x+2)^n}_{m=n}=0$$
$$\sum_{m=0}^{\infty}m(m-1)c_m(x+2)^m-\sum_{m=-2}^{\infty}4(m+2)(m+1)c_{m+2}(x+2)^m+\sum_{m=0}^{\infty}4mc_m(x+2)^m+\sum_{m=0}^{\infty}2c_m(x+2)^m=0$$
$$\sum_{m=0}^{\infty}m(m-1)c_m(x+2)^m-\left(0c_0+0c_1+\sum_{m=0}^{\infty}4(m+2)(m+1)c_{m+2}(x+2)^m\right)+\sum_{m=0}^{\infty}4mc_m(x+2)^m+\sum_{m=0}^{\infty}2c_m(x+2)^m=0$$
$$\sum_{m=0}^{\infty}\left[m(m-1)c_m-4(m+2)(m+1)c_{m+2}+4mc_m+2c_m\right](x+2)^m=0$$

Setting each coefficient in the sum equal to zero gives:

$$(m+1)(m+2)c_m-4(m+2)(m+1)c_{m+2}=0\implies c_{m+2}=\frac{c_m}{4}\text{ for all }m\geq 0$$

By the initial conditions, we have $y(-2)=c_0=4$ and $y'(-2)=c_1=1$. We plug in m values to find the remaining pattern:

$$\begin{aligned}
m=0:&\quad c_2=\frac{c_0}{4}=4^0\\
m=1:&\quad c_3=\frac{c_1}{4}=4^{-1}\\
m=2:&\quad c_4=\frac{c_2}{4}=4^{-1}\\
m=3:&\quad c_5=\frac{c_3}{4}=4^{-2}\\
m=4:&\quad c_6=\frac{c_4}{4}=4^{-2}\\
m=5:&\quad c_7=\frac{c_5}{4}=4^{-3}\\
&\vdots
\end{aligned}$$

We see that $c_{2k}=4^{1-k}$ and $c_{2k+1}=4^{-k}$ for $k\geq 1$

Putting this into the general power series formula, we get:

$$\begin{aligned}
y&=\sum_{n=0}^{\infty}c_n(x+2)^n\\
&=c_0+c_1x+\sum_{k=1}^{\infty}c_{2k}(x+2)^{2k}+\sum_{k=1}^{\infty}c_{2k+1}(x+2)^{2k+1}\\
&=4+1(x+2)+\sum_{k=1}^{\infty}4^{1-k}(x+2)^{2k}+\sum_{k=1}^{\infty}4^{-k}(x+2)^{2k+1}\\
&=4+(x+2)+\sum_{k=1}^{\infty}4^{1-k}(x+2)^{2k}+\sum_{k=1}^{\infty}4^{-k}(x+2)^{2k+1}
\end{aligned}$$

Exercise $4.3.3a$

We assume $y=\sum_{n=0}^{\infty}c_nx^n$:

$$y''+xy'+xy=0$$
$$\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}+x\cdot\sum_{n=0}^{\infty}nc_nx^{n-1}+x\cdot\sum_{n=0}^{\infty}c_nx^n=0$$
$$\underbrace{\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}}_{m=n-2}+\underbrace{\sum_{n=0}^{\infty}nc_nx^n}_{m=n}+\underbrace{\sum_{n=0}^{\infty}c_nx^{n+1}}_{m=n+1}=0$$
$$\sum_{m=-2}^{\infty}(m+2)(m+1)c_{m+2}x^m+\sum_{m=0}^{\infty}mc_mx^m+\sum_{m=1}^{\infty}c_{m-1}x^m=0$$
$$\left(0c_0+0c_1+2c_2x^0+\sum_{m=1}^{\infty}(m+2)(m+1)c_{m+2}x^m\right)+\left(0c_0+\sum_{m=1}^{\infty}mc_mx^m\right)+\sum_{m=1}^{\infty}c_{m-1}x^m=0$$
$$2c_2+\sum_{m=1}^{\infty}\left[(m+2)(m+1)c_{m+2}+mc_m+c_{m-1}\right]x^m=0$$

Setting the x^0 term equal to zero gives $c_2=0$, and setting the remaining coefficients equal to zero gives:

$$(m+2)(m+1)c_{m+2}+mc_m+c_{m-1}=0\implies c_{m+2}=-\frac{mc_m+c_{m-1}}{(m+2)(m+1)}\text{ for all }m\geq 1$$

By the initial conditions, we have $y(0)=c_0=360$ and $y'(0)=c_1=0$

We plug in a few values of m until we reach four nonzero coefficients:

$$
\begin{aligned}
c_3 &= -\frac{1\cdot c_1 + c_0}{3\cdot 2} = -\frac{0+360}{6} = -60\\
c_4 &= -\frac{2\cdot c_2 + c_1}{4\cdot 3} = -\frac{0+0}{12} = 0\\
c_5 &= -\frac{3\cdot c_3 + c_2}{5\cdot 4} = -\frac{-180+0}{20} = 9\\
c_6 &= -\frac{4\cdot c_4 + c_3}{6\cdot 5} = -\frac{0-60}{30} = 2\\
&\vdots
\end{aligned}
$$

Therefore, the series solution starts off as $y = 360 + 0x + 0x^2 - 60x^3 + 0x^4 + 9x^5 + 2x^6 + \cdots$

Exercise 4.3.3*b*

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$
\begin{aligned}
2y'' - x^2y' + 2xy &= 0\\
2\cdot\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - x^2\cdot\sum_{n=0}^{\infty} nc_n x^{n-1} + 2x\cdot\sum_{n=0}^{\infty} c_n x^n &= 0\\
\underbrace{\sum_{n=0}^{\infty} 2n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} nc_n x^{n+1}}_{m=n+1} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^{n+1}}_{m=n+1} &= 0\\
\sum_{m=-2}^{\infty} 2(m+2)(m+1)c_{m+2}x^m + \sum_{m=1}^{\infty}(m-1)c_{m-1}x^m + \sum_{m=1}^{\infty} 2c_{m-1}x^m &= 0\\
\left(0c_0 + 0c_1 + 4c_2x^0 + \sum_{m=1}^{\infty} 2(m+2)(m+1)c_{m+2}x^m\right) + \sum_{m=1}^{\infty}(m-1)c_{m-1}x^m + \sum_{m=1}^{\infty} 2c_{m-1}x^m &= 0\\
2c_2 + \sum_{m=1}^{\infty}\left[2(m+2)(m+1)c_{m+2} - (m-1)c_{m-1} + 2c_{m-1}\right]x^m &= 0
\end{aligned}
$$

Setting the x^0 term equal to zero gives $c_2 = 0$, and setting the remaining coefficients equal to zero gives:

$$(m+2)(m+1)c_{m+2} - (m-3)c_{m-1} = 0 \implies c_{m+2} = \frac{m-3}{2(m+2)(m+1)}c_{m-1} \text{ for all } m \geq 1$$

By the initial conditions, we have $y(0) = c_0 = 720$ and $y'(0) = c_1 = 1$.

We plug in a few values of m until we reach five nonzero coefficients:

$$
\begin{aligned}
c_3 &= \frac{-2}{2\cdot 3\cdot 2}c_0 = -120\\
c_4 &= \frac{-1}{2\cdot 4\cdot 3}c_1 = 2\\
c_5 &= \frac{0}{2\cdot 5\cdot 4}c_2 = 0\\
c_6 &= \frac{1}{2\cdot 6\cdot 5}c_3 = -2\\
&\vdots
\end{aligned}
$$

Therefore, the series solution starts off as $y = 720 - 48x + 0x^2 - 120x^3 + 2x^4 + 0x^5 - 2x^6 + \cdots$

Exercise 4.3.3*c*

We assume $y = \sum_{n=0}^{\infty} c_n x^n$:

$$
\begin{aligned}
(x+1)y'' + (x-1)y' + y &= 0\\
x\cdot\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + x\cdot\sum_{n=0}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= 0\\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} + \underbrace{\sum_{n=0}^{\infty} nc_n x^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} nc_n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{m=n} &= 0\\
\sum_{m=-1}^{\infty}(m+1)mc_{m+1}x^m + \sum_{m=-2}^{\infty}(m+2)(m+1)c_{m-2}x^m + \sum_{m=0}^{\infty} mc_m x^m - \sum_{m=-1}^{\infty}(m+1)c_{m+1}x^m + \sum_{m=0}^{\infty} c_m x^m &= 0\\
\sum_{m=0}^{\infty}\left[(m+1)mc_{m+1} + (m+2)(m+1)c_{m+2} + mc_m - (m+1)c_{m+1} + c_m\right]x^m &= 0
\end{aligned}
$$

Setting each coefficient equal to zero gives:

$$(m+2)(m+1)c_{m+2} + (m+1)(m-1)c_{m+1} + (m+1)c_m = 0 \implies c_{m+2} = -\frac{(m-1)c_{m+1} + c_m}{m+2} \text{ for all } m \geq 0$$

By the initial conditions, we have $y(0) = c_0 = 10$ and $y'(0) = c_1 = 30$.

We plug in a few values of m until we reach six nonzero coefficients:

$$
\begin{aligned}
c_2 &= -\frac{-1\cdot c_1 + c_0}{2} = -\frac{-30+10}{2} = 10\\
c_3 &= -\frac{0\cdot c_2 + c_1}{3} = -\frac{0+30}{3} = -10\\
c_4 &= -\frac{1\cdot c_3 + c_2}{4} = -\frac{-10+10}{4} = 0\\
c_5 &= -\frac{2\cdot c_4 + c_3}{5} = -\frac{0-10}{5} = 2\\
c_6 &= -\frac{3\cdot c_5 + c_4}{6} = -\frac{6+0}{6} = -1\\
&\vdots
\end{aligned}
$$

Therefore, the series solution starts off as $y = 10 + 30x + 10x^2 - 10x^3 + 0x^4 + 2x^5 - x^6 + \cdots$

Exercise $4.3.3d$

We assume $y = \sum_{n=0}^{\infty} c_n (x-1)^n$:

$$
\begin{aligned}
y'' - 2(x-1)y' + y &= 0\\
\sum_{n=0}^{\infty} n(n-1)c_n(x-1)^{n-2} - 2(x-1)\cdot\sum_{n=0}^{\infty} nc_n(x-1)^{n-1} + \sum_{n=0}^{\infty} c_n(x-1)^n &= 0\\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n(x-1)^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 2nc_n(x-1)^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} c_n(x-1)^n}_{m=n} &= 0\\
\sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2}(x-1)^m - \sum_{m=0}^{\infty} 2mc_m(x-1)^m + \sum_{m=0}^{\infty} c_m(x-1)^m &= 0\\
\left(0c_0 + 0c_1 + \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2}(x-1)^m\right) - \sum_{m=0}^{\infty} 2mc_m(x-1)^m + \sum_{m=0}^{\infty} c_m(x-1)^m &= 0\\
\sum_{m=0}^{\infty} \left[(m+2)(m+1)c_{m+2} - 2mc_m + c_m\right](x-1)^m &= 0
\end{aligned}
$$

Setting each coefficient equal to zero gives:

$$(m+2)(m+1)c_{m+2} - (2m-1)c_m = 0 \implies c_{m+2} = \frac{2m-1}{(m+2)(m+1)}c_m \text{ for all } m \geq 0$$

By the initial conditions, we have $y(1) = c_0 = 240$ and $y'(1) = c_1 = 48$.

We plug in a few values of m until we reach six nonzero coefficients:

$$
\begin{aligned}
c_2 &= \frac{-1}{2\cdot 1}c_0 = -120\\
c_3 &= \frac{1}{3\cdot 2}c_1 = 8\\
c_4 &= \frac{3}{4\cdot 3}c_2 = -30\\
c_5 &= \frac{5}{5\cdot 4}c_3 = 2
\end{aligned}
$$

Therefore, the series solution starts off as $y = 240 + 48(x-1) - 120(x-1)^2 + 8(x-1)^3 - 30(x-1)^4 + 2(x-1)^5 + \cdots$

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4.4: Power Series Solutions at Singular Points

Learning Objectives

- Identify regular singular points of linear ODEs.
- Apply the Frobenius method to find series solutions about regular singular points.
- Solve the indicial equation and handle cases with distinct roots and repeated roots.
- Find second solutions using reduction of order when roots differ by an integer.
- Recognize when Frobenius solutions correspond to elementary functions.

When the standard power series method fails at a singular point, the Frobenius method comes to the rescue. This subchapter teaches you to identify regular singular points and find series solutions of the form $x^r \cdot \sum_{n=0}^{\infty} a_n x^n$, handling cases with distinct, repeated, and integer-differing roots.

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4.4: The Method of Frobenius and Indicial Equation

As we discussed earlier in this chapter, one of the major limitations using power series to solve an LDE is that it only finds the *analytic* solutions.

For instance, we know that $y = x^{-2}$ satisfies $xy' + 2y = 0$, but if we tried to find a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$, we see:

$$\begin{aligned} xy' + 2y &= 0 \\ x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} + 2 \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+2) c_n x^n &= 0 \end{aligned}$$

Since $n + 2 \neq 0$, we must have $c_n = 0$ for all $n \geq 0$, and thus we get only $y = 0$ (the trivial solution).

To make the method of power series solutions more robust, we expand our search from strictly analytic functions to "almost analytic" functions. We search for functions $y(x) = x^r \cdot \sum_{n=0}^{\infty} c_n x^n$, where r is some constant of our choosing.

Illustration 4.4.1

In the example $xy' + 2y = 0$, if we guess that $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, we see:

$$\begin{aligned} xy' + 2y &= 0 \\ x \cdot \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + 2 \cdot \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} 2 c_n x^{n+r} &= 0 \\ x^r \left[\sum_{n=0}^{\infty} (n+r+2) c_n x^n \right] &= 0 \end{aligned}$$

Dividing both sides by x^r, we can set each coefficient equal to zero to get:

$$\begin{aligned} (r+2)c_0 &= 0 \\ (r+3)c_1 &= 0 \\ (r+4)c_2 &= 0 \\ &\vdots \end{aligned}$$

If we pick $r = -2$, there is exactly one equation of the form $0c_0 = 0$, which forces all other coefficients to be $c_n = 0$ when $n \geq 1$

Thus, our solution would be $y = c_0 x^{0-2} = c_0 x^{-2}$, which is exactly what we expected.

If we had picked $r = -6$ instead, the equation would have been $0c_4 = 0$, and the solution would have been $y = c_4 x^{4-6} = c_4 x^{-2}$ We see that we could choose any integer smaller than -2 for r, and our solution would still be equivalent to the one formed when $r = -2$

This trick of "multiplying by a clever power of x" can often make the solution analytic, and thus lead us to finding the power series representation we desire. This trick is commonly referred to as the **Method of Frobenius**.

Example 4.4.1

Find the non-analytic solutions to the LDE $x^2y' + 4xy = 0$ centered at $a = 0$

Solution

We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, so:

$$\begin{aligned} x^2y' + 4xy &= 0 \\ x^2 \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + 4x \cdot \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty}(n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty} 4c_n x^{n+r+1} &= 0 \\ x^r\left[\underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+1}}_{m=n+1} + \underbrace{\sum_{n=0}^{\infty} 4c_n x^{n+1}}_{m=n+1}\right] &= 0 \\ \sum_{m=1}^{\infty}(m-1+r)c_{m-1}x^m + \sum_{m=1}^{\infty} 4c_{m-1}x^m &= 0 \\ \sum_{m=1}^{\infty}(m+3+r)c_{m-1}x^m &= 0 \end{aligned}$$

Setting each coefficient equal to zero gives us: $(m+3+r)c_{m-1} = 0$ for all $m \geq 1$

This means that for all $m \neq -3-r$, we have $c_{m-1} = 0$. If we chose r such that $m+3+r = 0$ for some $m \geq 1$, we could find a solution.

Let $r = -4$, so that when $m = 1$ we have $0c_0 = 0$, and c_0 is free to be any constant.

Therefore, we have solution $y = c_0x^{0+r} = \boxed{c_0x^{-4}}$

It is important to notice that we do not *always need* to include the $+r$ in our exponents and work when solving an LDE. This is only necessary when we expect there are non-analytic solutions centered around the point of interest.

Definition: Singular Point

Consider the second order homogeneous LDE defined by

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

If all coefficient functions $P(x)$ and $Q(x)$ are analytic at $x = a$, then $x = a$ is an ***ordinary point***.
The methods of the previous subsection will provide us with two linearly independent analytic solutions centered at $x = a$

If $x = a$ is not a regular point, then it is a ***singular point*** of the differential equation.

When $\hat{P}(x) = (x-a)P(x)$ and $\hat{Q}(x) = (x-a)^2Q(x)$ are each analytic at $x = a$, then $x = a$ is a ***regular singular point***.
We will be able to use the *Method of Frobenius* to attempt to find solutions.

If either of $\hat{P}$ or $\hat{Q}$ is still not analytic at $x = a$, then it is an ***irregular singular point.***
Don't let anyone see you cry, just move onto the next problem and hope someone else figures out a solution.

Before we dive into the second order LDE examples, we will run over the proof as to why this method will work.

If we write the general form of the LDE

$$\begin{gathered} y'' + P(x)y' + Q(x)y = 0 \\ \Downarrow \\ (x-a)^2 \cdot y'' + (x-a)P(x) \cdot (x-a)y' + (x-a)^2 Q(x) \cdot y = 0 \\ \Downarrow \\ (x-a)^2 y'' + \hat{P}(x) \cdot (x-a)y' + \hat{Q}(x) \cdot y = 0 \end{gathered}$$

Assume that $y = \sum_{n=0}^{\infty} c_n (x-a)^{n+r}$:

$$\begin{aligned} (x-a)^2 y'' + \hat{P}(x) \cdot (x-a)y' + \hat{Q}(x) \cdot y &= 0 \\ (x-a)^2 \cdot \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n (x-a)^{n+r-2} + \hat{P}(x) \left((x-a) \cdot \sum_{n=0}^{\infty} (n+r)c_n (x-a)^{n+r-1} \right) + \hat{Q}(x) \cdot \left(\sum_{n=0}^{\infty} c_n (x-a)^{n+r} \right) &= 0 \\ x^r \left[\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n (x-a)^{n} + \hat{P}(x) \cdot \left(\sum_{n=0}^{\infty} (n+r)c_n (x-a)^{n} \right) + \hat{Q}(x) \cdot \left(\sum_{n=0}^{\infty} c_n (x-a)^{n} \right) \right] &= 0 \\ \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n (x-a)^{n} + \left(\sum_{n=0}^{\infty} p_n (x-a)^{n} \right) \cdot \left(\sum_{n=0}^{\infty} (n+r)c_n (x-a)^{n} \right) + \left(\sum_{n=0}^{\infty} q_n (x-a)^{n} \right) \cdot \left(\sum_{n=0}^{\infty} c_n (x-a)^{n} \right) &= 0 \end{aligned}$$

We see that setting the first coefficient (when $n = 0$) equal to zero gives us:

$$r(r-1)c_0 + rc_0 p_0 + c_0 q_0 = 0 \implies r^2 + (p_0 - 1)r + q_0 = 0$$

This is known as the ***indicial equation*** of the LDE, and it tells us which value(s) of r will yield non-trivial solutions.

Once the roots of the indicial equation are known, we plug them into the recurrence relationship based on the remaining coefficients being set equal to zero. If all goes well, we should end up with two linearly independent solutions expressed in terms of a power series centered around $x = a$.

Theorem 4.4.1: Method of Frobenius

If $x = a$ is a regular singular point for the LDE $y'' + p(x)y' + q(x)y = 0$, then there is at least one non-trivial solution of the form

$$y(x) = (x-a)^r \sum_{n=0}^{\infty} c_n (x-a)^n \text{ with } c_0 \neq 0$$

where the exponent r is a root of the indicial equation $r(r-1) + \hat{p}(a)r + \hat{q}(a) = 0$

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4.4: Distinct Power Series Solutions using Frobenius Method

If we are using the Method of Frobenius to solve a second order LDE, the lowest power of x will always yield the *indicial equation* as a quadratic equation.

As such, the roots of the indicial equation are either:

1. Two distinct real roots: these two roots give us two solutions $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ and $y_2 = \sum_{n=0}^{\infty} d_n x^{n+r_2}$
2. A double real root: approached using reduction of order (we will see examples of this in the next subchapter).
3. Complex conjugates: these problems are regrettable and if we have good karma we will not run into any of them in our lifetime.

We will focus on the first case here, where we can generate two power series solutions to the LDE. Under the right conditions, these solutions should be linearly independent.

Example 4.4.2

Find the general solution to $2x^2y'' + (3x - x^2)y' - 6y = 0$ centered at $a = 0$

Solution

If we write this LDE as $y'' + \dfrac{3x - x^2}{2x^2}y' - \dfrac{3}{x^2}y = 0$.

We see that $P(x) = \dfrac{3x - x^2}{2x^2}$ is not analytic at $a = 0$, but $\hat{P}(x) = x \cdot P(x) = \dfrac{3x^2 - x^3}{2x^2} = \dfrac{3 - x}{2}$ is analytic at $a = 0$.

Note that $Q(x) = -\dfrac{3}{x^2}$ is not analytic at $a = 0$, but $\hat{Q}(x) = x^2 \cdot Q(x) = -3$ is analytic at $a = 0$.

This tells us that $a = 0$ is a regular singular point, and the Method of Frobenius will help in this problem.

We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$2x^2y'' + (3x - x^2)y' - 6y = 0$$

$$2x^2 \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 3x \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - x^2 \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - 6 \cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r+1}}_{m=n+1} - \underbrace{\sum_{n=0}^{\infty} 6c_n x^{n+r}}_{m=n} = 0$$

$$x^r\left[\sum_{m=0}^{\infty} 2(m+r)(m+r-1)c_m x^m + \sum_{m=0}^{\infty} 3(m+r)c_m x^m - \sum_{m=1}^{\infty}(m+r-1)c_{m-1}x^m - \sum_{m=0}^{\infty} 6c_m x^m\right] = 0$$

$$\left(2r(r-1)c_0x^0 + \sum_{m=1}^{\infty} 2(m+r)(m+r-1)c_m x^m\right) + \left(3rc_0x^0 + \sum_{m=1}^{\infty} 3(m+r)c_m x^m\right) - \sum_{m=1}^{\infty}(m+r-1)c_{m-1}x^m - \left(6c_0x^0 + \sum_{m=1}^{\infty} 6c_m x^m\right) = 0$$

$$(2r(r-1) + 3r - 6)\,c_0x^0 + \sum_{m=1}^{\infty}\left[2(m+r)(m+r-1)c_m + 3(m+r)c_m - (m+r-1)c_{m-1} - 6c_m\right]x^m = 0$$

We see the indicial equation is $2r(r-1) + 3r - 6 = 0 \implies (r+2)(2r-3) = 0 \implies r_1 = -2,\ r_2 = \dfrac{3}{2}$

Notice that each value of r makes the recurrence relationship look different. When $r = -2$:

$$2(m+r)(m+r-1)c_m + 3(m+r)c_m - (m+r-1)c_{m-1} - 6c_m = 0 \implies c_m = \frac{m-3}{m(2m-7)}c_{m-1} \text{ for all } m \geq 1$$

We plug in a few values of m:

$$
\begin{aligned}
m=1: &\quad c_1=\frac{-2}{1\cdot(-5)}c_0=\frac{2}{5}c_0\\
m=2: &\quad c_2=\frac{-1}{2\cdot(-3)}c_1=\frac{1}{15}c_0\\
m=3: &\quad c_3=\frac{0}{3\cdot(-1)}c_2=0\\
m=4: &\quad c_4=\frac{1}{4\cdot 1}c_3=0\\
&\vdots
\end{aligned}
$$

The coefficients in this pattern all become zero after c_2, thus

$$y_1(x)=\sum_{n=0}^{\infty}c_nx^{n+r}=x^r\left[c_0+c_1x+c_2x^2+\sum_{n=3}^{\infty}0x^n\right]=x^{-2}\left[c_0+\frac{2}{5}c_0x+\frac{1}{15}c_0x^2\right]=\frac{c_0}{15}\left(15x^{-2}+6x^{-1}+1\right)$$

Now we look at the other value of r from the indicial equation, $r=\frac{3}{2}$:

$$2(m+r)(m+r-1)d_m+3(m+r)d_m-(m+r-1)d_{m-1}-6d_m=0\implies d_m=\frac{2m+1}{2m(2m+7)}d_{m-1}\text{ for all }m\geq 1$$

We plug in a few values of m:

$$
\begin{aligned}
m=1: &\quad d_1=\frac{3}{2\cdot 9}d_0=\frac{3!!\cdot 7!!}{2!!\cdot 9!!}d_0\\
m=2: &\quad d_2=\frac{5}{4\cdot 11}d_1=\frac{5!!\cdot 7!!}{4!!\cdot 11!!}d_0\\
m=3: &\quad d_3=\frac{7}{6\cdot 13}d_2=\frac{7!!\cdot 7!!}{6!!\cdot 13!!}d_0\\
&\vdots
\end{aligned}
$$

The coefficients in this pattern all fit $d_n=\frac{(2n+1)!!\cdot 7!!}{(2n)!!\cdot(2n+7)!!}d_0$ for $n\geq 1$. Therefore, we have

$$y_2(x)=\sum_{n=0}^{\infty}d_nx^{n+r}=x^r\left[d_0+\sum_{n=1}^{\infty}\frac{(2n+1)!!\cdot 7!!}{(2n)!!\cdot(2n+7)!!}d_0x^n\right]=d_0x^{\frac{3}{2}}\left[1+\sum_{n=1}^{\infty}\frac{(2n+1)!!\cdot 7!!}{(2n)!!\cdot(2n+7)!!}x^n\right]$$

These two solutions are very clearly linearly independent, so we know the general solution to this LDE is:

$$\boxed{y(x)=\frac{c_0}{15}\left(15x^{-2}+6x^{-1}+1\right)+d_0x^{\frac{3}{2}}\left(1+\sum_{n=1}^{\infty}\frac{(2n+1)!!\cdot 7!!}{(2n)!!\cdot(2n+7)!!}x^n\right)}$$

Example 4.4.3

Find the general solution to $(2x^2-2)y''+(5x-3)y'+y=0$ centered at $a=1$

Solution

We rewrite this as $(2x^2-2)y''+(5x-3)y'+y=0\leftrightarrow y''+\frac{5x-3}{2(x-1)(x+1)}y'+\frac{1}{2(x-1)(x+1)}y=0$

We see that $P(x)=\frac{5x-3}{2(x-1)(x+1)}$ is not analytic at $a=1$, but $\hat{P}(x)=(x-1)\cdot P(x)$ is.

Similarly, $Q(x)=\frac{1}{2(x-1)(x+1)}$ is not analytic at $a=1$, but $\hat{Q}(x)=(x-1)^2\cdot Q(x)$ is.

This tells us we can use the method of Frobenius, and should finally rewrite the LDE as:

$$(2(x-1)^2+4(x-1))y''+(5(x-1)+2)y'+y=0$$

Now we assume $y=\sum_{n=0}^{\infty}c_n(x-1)^{n+r}$:

$$(2(x-1)^2+4(x-1))y''+(5(x-1)+2)y'+y=0$$
$$2(x-1)^2\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(x-1)^{n+r-2}+4(x-1)\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(x-1)^{n+r-2}+5(x-1)\cdot\sum_{n=0}^{\infty}(n+r)c_n(x-1)^{n+r-1}+2\cdot\sum_{n=0}^{\infty}(n+r)c_n(x-1)^{n+r-1}+\sum_{n=0}^{\infty}c_n(x-1)^{n+r}=0$$
$$\underbrace{\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n(x-1)^{n+r}}_{m=n}+\underbrace{\sum_{n=0}^{\infty}4(n+r)(n+r-1)c_n(x-1)^{n+r-1}}_{m=n-1}+\underbrace{\sum_{n=0}^{\infty}5(n+r)c_n(x-1)^{n+r}}_{m=n}+\underbrace{\sum_{n=0}^{\infty}2(n+r)c_n(x-1)^{n+r-1}}_{m=n-1}+\underbrace{\sum_{n=0}^{\infty}c_n(x-1)^{n+r}}_{m=n}=0$$
$$(x-1)^r\left[\sum_{m=0}^{\infty}2(m+r)(m+r-1)c_m(x-1)^m+\sum_{m=-1}^{\infty}4(m+r+1)(m+r)c_{m+1}(x-1)^m+\sum_{m=0}^{\infty}5(m+r)c_m(x-1)^m+\sum_{m=-1}^{\infty}2(m+r+1)c_{m+1}(x-1)^m+\sum_{m=0}^{\infty}c_m(x-1)^m\right]=0$$
$$\sum_{m=0}^{\infty}2(m+r)(m+r-1)c_m(x-1)^m+\left(4r(r-1)c_0(x-1)^{-1}+\sum_{m=0}^{\infty}4(m+r+1)(m+r)c_{m+1}(x-1)^m\right)+\sum_{m=0}^{\infty}5(m+r)c_m(x-1)^m+\left(2rc_0(x-1)^{-1}+\sum_{m=0}^{\infty}2(m+r+1)c_{m+1}(x-1)^m\right)+\sum_{m=0}^{\infty}c_m(x-1)^m=0$$
$$(4r(r-1)+2r)\,c_0(x-1)^{-1}+\sum_{m=0}^{\infty}\left[(2(m+r)(m+r-1)+5(m+r)+1)\,c_m+(4(m+r)+2)(m+r+1)c_{m+1}\right](x-1)^m=0$$

The indicial equation is $4r(r-1)+2r=0 \implies 2r(2r-1)=0 \implies r_1=0,\ r_2=\frac{1}{2}$

When $r=0$:

$$(2(m+r)(m+r-1)+5(m+r)+1)\,c_m+(4(m+r)+2)(m+r+1)c_{m+1}=0 \implies c_{m+1}=-\frac{1}{2}c_m \text{ for all } m\geq 0$$

To find the first pattern, we plug in a few values of m:

$$\begin{aligned}m=0:&\quad c_1=-\frac{1}{2}c_0\\ m=1:&\quad c_2=-\frac{1}{2}c_1=\left(-\frac{1}{2}\right)^2c_0\\ m=2:&\quad c_3=-\frac{1}{2}c_2=\left(-\frac{1}{2}\right)^3c_0\\ &\vdots\end{aligned}$$

The pattern is $c_n=\left(-\frac{1}{2}\right)^n c_0$ for $n\geq 1$, so we have first solution:

$$y_1(x)=\sum_{n=0}^{\infty}c_n(x-1)^{n+r}=(x-1)^r\left[c_0+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n c_0(x-1)^n\right]=c_0(x-1)^0\left(1+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n(x-1)^n\right)=\frac{c_0}{1-\left(-\frac{1}{2}(x-1)\right)}=$$

When $r=\frac{1}{2}$:

$$(2(m+r)(m+r-1)+5(m+r)+1)\,d_m+(4(m+r)+2)(m+r+1)d_{m+1}=0 \implies d_{m+1}=-\frac{1}{2}d_m \text{ for all } m\geq 0$$

To find the first pattern, we plug in a few values of m:

$$\begin{aligned}m=0:&\quad d_1=-\frac{1}{2}d_0\\ m=1:&\quad d_2=-\frac{1}{2}d_1=\left(-\frac{1}{2}\right)^2d_0\\ m=2:&\quad d_3=-\frac{1}{2}d_2=\left(-\frac{1}{2}\right)^3d_0\\ &\vdots\end{aligned}$$

The pattern is $d_n=\left(-\frac{1}{2}\right)^n c_0$ for $n\geq 1$, so we have first solution:

$$y_2(x)=\sum_{n=0}^{\infty}d_n(x-1)^{n+r}=(x-1)^r\left[d_0+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n d_0(x-1)^n\right]=d_0(x-1)^{\frac{1}{2}}\left(1+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n(x-1)^n\right)=\frac{d_0\sqrt{x-1}}{1-\left(-\frac{1}{2}(x-1)\right)}$$

These are two linearly independent solutions, (whether or not you recognize them in their explicit form).

Therefore, our general solution is $\boxed{y(x) = \dfrac{A + B\sqrt{x-1}}{x+1}}$

It is always good to practice the ability to recognize the famous power series when they appear in our answers!

Example 4.4.4

Find all solutions to $4xy'' + 2y' + y = 0$ centered at $a = 0$

Solution

We see that $y'' + \dfrac{1}{2x}y' + \dfrac{1}{4x}y = 0$ does not have analytic coefficient functions at $a = 0$, but this singular point is regular so the Method of Frobenius will be useful.

We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$\begin{aligned}
4xy'' + 2y' + y &= 0 \\
4x \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 2 \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\underbrace{\sum_{n=0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} &= 0 \\
x^r\left[\sum_{m=-1}^{\infty} 4(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=-1}^{\infty} 2(m+r+1)c_{m+1}x^m + \sum_{m=0}^{\infty} c_m x^m\right] &= 0 \\
\left(4r(r-1)c_0 x^{-1} + \sum_{m=0}^{\infty} 4(m+r+1)(m+r)c_{m+1}x^m\right) + \left(2rc_0 x^{-1} + \sum_{m=0}^{\infty} 2(m+r+1)c_{m+1}x^m\right) + \sum_{m=0}^{\infty} c_m x^m &= 0 \\
(4r(r-1) + 2r)\,c_0 x^{-1} + \sum_{m=0}^{\infty}\left[4(m+r+1)(m+r)c_{m+1} + 2(m+r+1)c_{m+1} + c_m\right]x^m &= 0
\end{aligned}$$

The indicial equation is $4r(r-1) + 2r = 0 \implies 4r^2 - 2r = 0 \implies r_1 = 0,\ r_2 = \dfrac{1}{2}$

When $r = 0$:

$$4(m+r+1)(m+r)c_{m+1} + 2(m+r+1)c_{m+1} + c_m = 0 \implies c_{m+1} = -\frac{1}{2(2m+1)(m+1)}c_m \text{ for all } m \geq 0$$

Plugging in a few m, we see:

$$\begin{aligned}
m = 0: &\quad c_1 = -\frac{1}{2\cdot 1\cdot 1}c_0 \\
m = 1: &\quad c_2 = -\frac{1}{2\cdot 3\cdot 2}c_1 = \frac{(-1)^2}{4!}c_0 \\
m = 2: &\quad c_3 = -\frac{1}{2\cdot 5\cdot 3}c_2 = \frac{(-1)^3}{6!}c_0 \\
&\quad \vdots
\end{aligned}$$

The pattern is $c_n = \dfrac{(-1)^n}{(2n)!}c_0$ for $n \geq 1$, so we have first solution:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r} = x^r\left[c_0 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}c_0 x^n\right] = c_0 x^0\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^n\right) = c_0\cos\left(\sqrt{x}\right)$$

When $r=\frac{1}{2}$:

$$4(m+r+1)(m+r)d_{m+1}+2(m+r+1)d_{m+1}+d_m=0 \implies d_{m+1}=-\frac{1}{2(2m+3)(m+1)}d_m \text{ for all } m\geq 0$$

Plugging in a few m, we see:

$$\begin{aligned} m=0: &\quad d_1=-\frac{1}{2\cdot 3\cdot 1}d_0 \\ m=1: &\quad d_2=-\frac{1}{2\cdot 5\cdot 2}d_1=\frac{(-1)^2}{5!}d_0 \\ m=2: &\quad d_3=-\frac{1}{2\cdot 7\cdot 3}d_2=\frac{(-1)^3}{7!}d_0 \\ &\vdots \end{aligned}$$

The pattern is $d_n=\frac{(-1)^n}{(2n+1)!}d_0$ for $n\geq 1$, so we have second solution:

$$y_2(x)=\sum_{n=0}^{\infty}d_nx^{n+r}=x^r\left[d_0+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}d_0x^n\right]=d_0x^{\frac{1}{2}}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}x^n\right)=d_0\sin\left(\sqrt{x}\right)$$

These are linearly independent, so we have general solution

$$\boxed{y(x)=c_0\cos\left(\sqrt{x}\right)+d_0\sin\left(\sqrt{x}\right)}$$

The method of Frobenius works on any second order LDE, but there is no promise that the two series we find will be linearly independent.

Illustration 4.4.2

We will now use the method of Frobenius to find solutions to $x^2y''+xy'+(x^2-1)y=0$ centered at $a=0$

We assume $y=\sum_{n=0}^{\infty}c_nx^{n+r}$:

$$\begin{aligned} x^2y''+xy'+(x^2-1)y&=0 \\ x^2\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}+x\cdot\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}+x^2\cdot\sum_{n=0}^{\infty}c_nx^{n+r}-\sum_{n=0}^{\infty}c_nx^{n+r}&=0 \\ \underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r}}_{m=n}+\underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}}_{m=n}+\underbrace{\sum_{n=0}^{\infty}c_nx^{n+r+2}}_{m=n+2}-\underbrace{\sum_{n=0}^{\infty}c_nx^{n+r}}_{m=n}&=0 \\ x^r\left[\sum_{m=0}^{\infty}(m+r)(m+r-1)c_mx^m+\sum_{m=0}^{\infty}(m+r)c_mx^m+\sum_{m=2}^{\infty}c_{m-2}x^m-\sum_{m=0}^{\infty}c_mx^m\right]&=0 \\ \left(r(r-1)c_0x^0+(r+1)rc_1x^1+\sum_{m=2}^{\infty}(m+r)(m+r-1)c_mx^m\right)+\left(rc_0x^0+(r+1)c_1x^1+\sum_{m=2}^{\infty}(m+r)c_mx^m\right)+\sum_{m=2}^{\infty}c_{m-2}x^m-\left(c_0x^0+c_1x^1+\sum_{m=2}^{\infty}c_mx^m\right)&=0 \\ (r(r-1)+r-1)\,c_0x^0+((r+1)r+(r+1)-1)\,c_1x^1+\sum_{m=2}^{\infty}\left[(m+r)(m+r-1)c_m+(m+r)c_m+c_{m-2}-c_m\right]x^m&=0 \end{aligned}$$

We see the indicial equation is $r(r-1)+r-1=0 \implies (r+1)(r-1)=0 \implies r_1=-1,\ r_2=1$

Again, each value of r makes the recurrence relationship look different. When $r=-1$:

$$(m+r)(m+r-1)c_m+(m+r)c_m+c_{m-2}-c_m=0 \implies m(m-2)c_m=-c_{m-2} \text{ for all } m\geq 2$$

Note that the coefficient of x^1 tells us that $c_1=0$. To find the rest, we plug in a few values of m:

$$
\begin{aligned}
m=2: &\quad 0c_2=-c_0 \implies c_0=0\\
m=3: &\quad 3c_3=-c_1 \implies c_3=0\\
m=4: &\quad c_4=-\frac{1}{4\cdot 2}c_2\\
m=5: &\quad c_5=-\frac{1}{5\cdot 3}c_3=0\\
m=6: &\quad c_6=-\frac{1}{6\cdot 4}c_4=\frac{(-1)^2}{6!!\cdot 4!!}c_2\\
m=7: &\quad c_7=-\frac{1}{7\cdot 5}c_5=0\\
m=8: &\quad c_8=-\frac{1}{8\cdot 6}c_6=\frac{(-1)^3}{8!!\cdot 6!!}c_2\\
&\quad\vdots
\end{aligned}
$$

The pattern is $c_{2k}=\dfrac{(-1)^{k-1}}{(2k)!!\cdot(2k-2)!!}c_2$ for $k\geq 2$, so we have first solution:

$$
\begin{aligned}
y_1(x)&=\sum_{n=0}^{\infty}c_nx^{n+r}\\
&=x^r\left[c_0+c_1x+c_2x^2+\sum_{k=1}^{\infty}c_{2k+1}x^{2k+1}+\sum_{k=2}^{\infty}c_{2k}x^{2k}\right]\\
&=x^{-1}\left[0+0x+c_2x^2+\sum_{k=1}^{\infty}0x^{2k+1}+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k)!!\cdot(2k-2)!!}c_2x^{2k}\right]\\
&=c_2\left(x+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k)!!\cdot(2k-2)!!}x^{2k-1}\right)\\
&=c_2\left(x+\sum_{m=1}^{\infty}\frac{(-1)^{m}}{(2m+2)!!\cdot(2m)!!}x^{2m+1}\right)
\end{aligned}
$$

This time, we compare to what we get from $r=1$:

$$
(m+r)(m+r-1)d_m+(m+r)d_m+d_{m-2}-d_m=0 \implies d_m=-\frac{1}{m(m+2)}d_{m-2} \text{ for all } m\geq 2
$$

Note that the coefficient of x^1 tells us that $d_1=0$ for the same reasons as the $r=-1$ case. To find the rest, we plug in a few values of m:

$$
\begin{aligned}
m=2: &\quad d_2=-\frac{1}{2\cdot 4}d_0\\
m=3: &\quad d_3=-\frac{1}{3\cdot 5}d_1=0\\
m=4: &\quad d_4=-\frac{1}{4\cdot 6}d_2=\frac{(-1)^2}{4!!\cdot 6!!}d_0\\
m=5: &\quad d_5=-\frac{1}{5\cdot 7}d_3=0\\
&\quad\vdots
\end{aligned}
$$

The pattern is $d_{2k}=\dfrac{(-1)^{k}}{(2k+2)!!\cdot(2k)!!}d_0$ for $k\geq 1$, so we have second solution:

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} d_n x^{n+r} \\ &= x^r \left[d_0 + d_1 x + \sum_{k=1}^{\infty} d_{2k} x^{2k} + \sum_{k=1}^{\infty} d_{2k+1} x^{2k+1} \right] \\ &= x^1 \left[d_0 + 0x + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+2)!! \cdot (2k)!!} d_0 x^{2k} + \sum_{k=1}^{\infty} 0 x^{2k+1} \right] \\ &= d_0 \left(x + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+2)!! \cdot (2k)!!} x^{2k+1} \right) \end{aligned}$$

The solutions y_1 and y_2 are identical! These are clearly *not linearly independent, they are multiples of each other.*
Thus, the Method of Frobenius has only given us one of the two linearly independent solutions to this differential equation.

It is entirely possible (and entirely unpleasant) to find the second solution in Illustration $4.4.2$, and we will discuss that further in the next subchapter.

The reason why these two solutions were identical is because the two roots of the indicial equation differed by an integer. Whenever this occurs, the two recurrence relationships that are created will always be an index shifted copy of one another.

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4.4: Frobenius Method (bad cases)

We have seen that in the very specific case that the roots of the indicial equation are distinct real numbers that do not differ by an integer, the Method of Frobenius yields the two linearly independent solutions we would expect to find of a second order homogeneous LDE.

Recall Illustration 4.4.2, in which we solved $x^2y'' + xy' + (x^2-1)y = 0$ centered at $a = 0$:

Since $y'' + \frac{1}{x}y' + \frac{x^2-1}{x^2}y = 0$ has a regular singular point at $a = 0$, we used the Method of Frobenius. However, since the roots of the indicial equation $r = -1$ and $r = 1$ differed by an integer, we ended up with two copies of the same solution: $y_1(x) = x + 2\sum_{k=1}^{\infty} \frac{(-1)^k \cdot (2k-1)!!}{(2k+2)!! \cdot (2k)!!} x^{2k+1}$

Instead, we use the method of reduction of order to find the second solution to this LDE.

Illustration 4.4.2 Continued

Recall that $p(x) = \frac{1}{x}$ and $f(x) = y_1(x)$ is the known solution, so $y = u \cdot f$ for :

$$u(x) = \int \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx}\,dx = \int \frac{1}{[y_1(x)]^2} e^{-\ln(x)} = \int \frac{C}{x\,[y_1(x)]^2}\,dx$$

We use long division to find:

$$\frac{1}{x\left[x - \frac{1}{8}x^3 + \frac{1}{64}x^5 + \cdots\right]^2} = \frac{1}{x^3 - \frac{1}{4}x^5 + \frac{3}{64}x^7 + \cdots} = x^{-3} + \frac{1}{4}x^{-1} + \frac{1}{64}x^1 + \cdots$$

Thus, we can integrate term-by-term to find:

$$u(x) = \int \left(x^{-3} + \frac{1}{4}x^{-1} + \frac{1}{64}x^1 + \cdots\right)\,dx = D - \frac{1}{2}x^{-2} + \frac{1}{4}\ln|x| + \frac{1}{128}x^2 + \cdots$$

And thus, our general solution looks like:

$$y(x) = u(x) \cdot f(x) = \left(D - \frac{1}{2}x^{-2} + \frac{1}{4}\ln|x| + \frac{1}{128}x^2 + \cdots\right) \cdot C\left(x + 2\sum_{k=1}^{\infty} \frac{(-1)^k \cdot (2k-1)!!}{(2k+2)!! \cdot (2k)!!} x^{2k+1}\right)$$

This unfortunate calculation is the plan of attack whenever we are in the less desirable situation involving real roots. (Either a double root, or roots separated by an integer).

We will always be able to find *one solution via Frobenius* and then find the *second solution via reduction of order*.

Neither of these promises to be a completely pleasant experience.

Example 4.4.5: Bessel's Equation

Find the solutions to the LDE $xy'' + y' + xy = 0$ centered at $a = 0$

Solution

Writing this as $y'' + \frac{1}{x}y' + y = 0$, we see that $P(x) = \frac{1}{x}$ is not analytic at $a = 0$, but $\hat{P}$ is, so we will use the Method of Frobenius.

We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$xy'' + y' + xy = 0$$
$$x \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + x \cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$
$$\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+1}}_{m=n+1} = 0$$
$$x^r\left[\sum_{m=-1}^{\infty}(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=-1}^{\infty}(m+r+1)c_{m+1}x^m + \sum_{m=1}^{\infty} c_{m-1}x^m\right] = 0$$
$$\left(r(r-1)c_0x^{-1} + (r+1)rc_1x^0 + \sum_{m=1}^{\infty}(m+r+1)(m+r)c_{m+1}x^m\right) + \left(rc_0x^{-1} + (r+1)c_1x^0 + \sum_{m=1}^{\infty}(m+r+1)c_{m+1}x^m\right) + \sum_{m=1}^{\infty} c_{m-1}x^m = 0$$
$$r^2c_0x^{-1} + (r+1)^2c_1x^0 + \sum_{m=1}^{\infty}\left[(m+r+1)(m+r)c_{m+1} + (m+r+1)c_{m+1} + c_{m-1}\right]x^m = 0$$

The indicial equation is $r^2 = 0$, which has double roots at $r = 0$. We can see that this causes $c_1 = 0$ to hold, and then we set the remaining coefficients equal to zero:

$$(m+r+1)(m+r)c_{m+1} + (m+r+1)c_{m+1} + c_{m-1} = 0 \implies c_{m+1} = -\frac{1}{(m+1)^2}c_{m-1} \text{ for all } m \geq 1$$

Plugging in a few values of m, we see the pattern:

$$\begin{aligned}
m = 1: &\quad c_2 = -\frac{1}{2^2}c_0 \\
m = 2: &\quad c_3 = -\frac{1}{3^2}c_1 = 0 \\
m = 3: &\quad c_4 = -\frac{1}{4^2}c_2 = \frac{(-1)^2}{[4!!]^2} \\
m = 4: &\quad c_5 = -\frac{1}{5^2}c_3 = 0 \\
m = 5: &\quad c_6 = -\frac{1}{6^2}c_4 = \frac{(-1)^3}{[6!!]^2} \\
&\vdots
\end{aligned}$$

It is clear that $c_{2k} = \frac{(-1)^k}{[(2k)!!]^2}c_0$ for $k \geq 1$ and $c_{2k+1} = 0$ for all k.

Plugging this into our general form gives:

$$\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= x^r\left[c_0 + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1} + \sum_{k=1}^{\infty} c_{2k}x^{2k}\right] \\
&= x^0\left[c_0 + \sum_{k=0}^{\infty} 0x^{2k+1} + \sum_{k=1}^{\infty}\frac{(-1)^k}{[(2k)!!]^2}c_0x^{2k}\right) \\
&= c_0\left(1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{[(2k)!!]^2}x^{2k}\right)
\end{aligned}$$

Now we use reduction of order, since $p(x) = \frac{1}{x}$, we have $u' = \frac{1}{[y_1(x)]^2}e^{-\int p(x)\,dx} = \frac{1}{x\,[y_1(x)]^2}$

We write the first few terms:

$$\begin{aligned}[y_1(x)]^2 &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{[(2n)!!]^2} x^{2n}\right)^2 \\ &= \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots\right)^2 \\ &= 1 - \frac{1}{2}x^2 + \frac{3}{32}x^4 + \cdots\end{aligned}$$

Therefore, we can do some long division with these few terms:

$$\frac{1}{[y(x)]^2} = \frac{1}{1 - \frac{1}{2}x^2 + \frac{3}{32}x^4 + \cdots} = 1 + \frac{1}{2}x^2 + \frac{5}{32}x^4 + \cdots$$

Finally, we have:

$$u = \int u'\,dx = \int \frac{1}{x} \cdot \frac{1}{[y(x)]^2}\,dx = \int x^{-1} + \frac{1}{2}x^1 + \frac{5}{32}x^3 + \cdots\,dx = D + \ln|x| + \frac{1}{4}x^2 + \frac{5}{96}x^4 + \cdots$$

This gives our general solution as

$$\boxed{y(x) = \left(D + \ln|x| + \frac{1}{4}x^2 + \frac{5}{96}x^4 + \cdots\right) \cdot c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{[(2k)!!]^2} x^{2k}\right)}$$

While that covers all of our possible cases with real roots, we must still consider the case where the roots of the indicial equation are complex numbers.

Illustration 4.4.3

Consider the example LDE $x^2y'' + xy' + (x+1)y = 0$, which we solve using the Method of Frobenius:

$$\begin{aligned}x^2y'' + xy' + (x+1)y &= 0 \\ x^2 \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + x \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + x \cdot \sum_{n=0}^{\infty} c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\ \underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+1}}_{m=n+1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} &= 0 \\ x^r\left[\sum_{m=0}^{\infty}(m+r)(m+r-1)c_m x^m + \sum_{m=0}^{\infty}(m+r)c_m x^m + \sum_{m=1}^{\infty} c_{m-1}x^m + \sum_{m=0}^{\infty} c_m x^m\right] &= 0 \\ \left(r(r-1)c_0x^0 + \sum_{m=1}^{\infty}(m+r)(m+r-1)c_m x^m\right) + \left(rc_0x^0 + \sum_{m=1}^{\infty}(m+r)c_m x^m\right) + \sum_{m=1}^{\infty} c_{m-1}x^m + \left(c_0x^0 + \sum_{m=1}^{\infty} c_m x^m\right) &= 0 \\ (r(r-1) + r + 1)\,c_0x^0 + \sum_{m=1}^{\infty}\left[(m+r)(m+r-1)c_m + (m+r)c_m + c_{m-1} + c_m\right]x^m &= 0\end{aligned}$$

Thus, the indicial equation is $r(r-1) + r = 1 = 0 \implies r^2 + 1 = 0 \implies r = \pm i$

$$(m+r)(m+r-1)c_m + (m+r)c_m + c_{m-1} + c_m = 0 \implies c_m = -\frac{1}{m(m \pm 2i)}c_{m-1} \text{ for all } m \geq 1$$

These complex roots make for difficult to express patterns. For each c_m, we know the conjugate value of r will yield the conjugate $\bar{c}_m$, so we check $r = i$:

$$
\begin{aligned}
m=1: \quad & c_1 = -\frac{1}{1\cdot(1+2i)}c_0 \\
m=2: \quad & c_2 = -\frac{1}{2\cdot(2+2i)}c_1 = \frac{(-1)^2}{2!(1+2i)(2+2i)}c_0 \\
m=3: \quad & c_3 = -\frac{1}{3(3+2i)}c_2 = \frac{(-1)^3}{3!(1+2i)(2+2i)(3+2i)}c_0 \\
& \vdots
\end{aligned}
$$

The pattern here is $c_n = \dfrac{(-1)^n}{n!(1+2i)\cdots(n+2i)}c_0$ for $n \geq 1$. This clearly has complex coefficients, but consider the conjugate $r = -i$ with its own coefficients $d_n = \dfrac{(-1)^n}{n!(1-2i)\cdots(n-2i)}d_0$ for $n \geq 1$. Each of these patterns works together to give:

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= x^r\left[c_0 + \sum_{n=1}^{\infty} c_n x^n\right] \\
&= x^i\left[c_0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1+2i)\cdots(n+2i)}c_0 x^n\right] \\
&= c_0 x^i\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1+2i)\cdots(n+2i)}x^n\right) \\
y_2(x) &= \sum_{n=0}^{\infty} d_n x^{n+r} \\
&= x^r\left[d_0 + \sum_{n=1}^{\infty} d_n x^n\right] \\
&= x^{-i}\left[d_0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1-2i)\cdots(n-2i)}d_0 x^n\right] \\
&= d_0 x^{-i}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1-2i)\cdots(n-2i)}x^n\right)
\end{aligned}
$$

While each of these functions has complex coefficient terms, we can take conjugate multiples of each of them and cancel out all complex terms:

$$
\begin{aligned}
y(x) &= (\alpha+i\beta)y_1(x) + (\alpha-i\beta)y_2(x) \\
&= (\alpha+i\beta)x^i\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1+2i)\cdots(n+2i)}x^n\right) + (\alpha-i\beta)x^{-i}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1-2i)\cdots(n-2i)}x^n\right) \\
&= \left((\alpha+i\beta)x^i + (\alpha-i\beta)x^{-i}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left[\frac{\alpha+i\beta}{(1+2i)\cdots(n+2i)}x^i + \frac{\alpha-i\beta)}{(1-2i)(2-2i)\cdots(n-2i)}x^{-i}\right]x^n \\
&= 2a\cos(\ln(x)) - 2b\sin(\ln(x)) + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot \text{ some function of } \cos(\ln(x)) \text{ and } \sin(\ln(x))}{n!}x^n
\end{aligned}
$$

Therefore, our solution is actually of the form $y(x) = f(\cos(\ln(x)), \sin(\ln(x)))$

In general, complex roots will lead to a similar calculation as this (which ultimately ends with "there is some pattern here that would be very difficult to write out explicitly"), and we will be satisfied to know the general shape of our solution (oscillatory in $\ln(x)$, decaying, etc.)

For these reasons, students will almost always interact entirely with the "nice case" of Frobenius, where the indicial equation has two real roots and they are separated by a non-integer. For an extra challenge, or when the pattern is recognizable as a famous function, repeated roots and the reduction of order go hand-in-hand. No reasonable instructor should expect you to do any examples with complex roots.

The Method of Frobenius extends the power series approach to handle equations with regular singular points, revealing a rich

family of special functions that are fundamental in mathematical physics and engineering applications.

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4.4: Practice Set

Exercise 4.4.1a

Use the Method of Frobenius to solve the first order LDE $4xy' + (1 - x)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise 4.4.1b

Use the Method of Frobenius to solve the first order LDE $3xy' + (2 + 3x)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise 4.4.1c

Use the Method of Frobenius to solve the first order LDE $5xy' - (5x^2 + 1)y = 0$

Answer available at the end of this section.

Exercise 4.4.2a

Find the indicial equation and its roots for the LDE $2x^2y'' + 6xy' + (7x + 2)y = 0$

Answer available at the end of this section.

Exercise 4.4.2b

Find the indicial equation and its roots for the LDE $x^2y'' + (2x^3 + x)y' + (3x^2 - 4)y = 0$

Answer available at the end of this section.

Exercise 4.4.2c

Find the indicial equation and its roots for the LDE $xy'' + (6x + 11)y' - x^2y = 0$

Answer available at the end of this section.

Exercise 4.4.2d

Find the indicial equation and its roots for the LDE $(2x^2 - 3x)y'' + (4x + 9)y' + 7xy = 0$

Answer available at the end of this section.

Exercise $4.4.2e$

Find the indicial equation and its roots for the LDE $(4x^3 + x^2)y'' + (x^2 - x)y' + (6x^3 + 5)y = 0$

Answer available at the end of this section.

Exercise $4.4.2f$

Find the indicial equation and its roots for the LDE $2x^2y'' + 2xy' + (x - 6)y = 0$

Answer available at the end of this section.

Exercise $4.4.2g$

Find the indicial equation and its roots for the LDE $x^2y'' + \sin(x)y' + e^xy = 0$

Answer available at the end of this section.

Exercise $4.4.2h$

Find the indicial equation and its roots for the LDE $(1 - \cos(x))\, y'' - 2\ln(1 + x)\ y' + 3\tan^{-1}(x)y = 0$

Answer available at the end of this section.

Exercise $4.4.3a$

Use the Method of Frobenius to find the general solution to the LDE $2xy'' + y' - 2y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3b$

Use the Method of Frobenius to find the general solution to the LDE $3xy'' - 2y' + y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3c$

Use the Method of Frobenius to find the general solution to the LDE $2xy'' + 3y' - 3xy = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3d$

Use the Method of Frobenius to find the general solution to the LDE $2x^2y'' + (2x^2 - x)y' - (3x + 5)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3e$

Use the Method of Frobenius to find the general solution to the LDE $6x^2y'' + xy' + (1 - 4x^2)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3f$

Use the Method of Frobenius to find the general solution to the LDE $3xy'' + (3x + 1)y' - 2y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3g$

Use the Method of Frobenius to find the general solution to the LDE $3x^2y'' - (6x^2 + 5x)y' - (8x + 3)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3h$

Use the Method of Frobenius to find the general solution to the LDE $2x^2y'' + (2x^3 + x)y' + (2x^2 - 1)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.3i$

Use the Method of Frobenius to find the general solution to the LDE $3x^2y'' + (3x^4 - x)y' + (5x^3 + 1)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.4a$

Find one nontrivial solution to the LDE $2xy'' + 2y' - y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.4b$

Find one nontrivial solution to the LDE $x^2y'' + 3xy' + (1 - 4x^2)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.4c$

Find one nontrivial solution to the LDE $x^2y'' + xy' + (2x - 9)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.4d$

Find one nontrivial solution to the LDE $xy'' + (x^2+1)y' + 2xy = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.4e$

Find one nontrivial solution to the LDE $4xy'' + (8x^3+4)y' + (20x^2-1)y = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.5a$

Use the Method of Frobenius to find the general solution to the LDE $xy'' + 2y' - xy = 0$ centered at $a = 0$

Answer available at the end of this section.

Exercise $4.4.5b$

Use the Method of Frobenius to find the general solution to the LDE $(x^2-x)y'' + (x+1)y' - y = 0$ centered at $a = 0$

Answer available at the end of this section.

Answers

Exercise $4.4.1a$

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ so $y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$:

$$\begin{aligned}
4xy' + y - xy &= 0 \\
4x \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} - x \cdot \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\underbrace{\sum_{n=0}^{\infty} 4(n+r)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+1}}_{m=n+1} &= 0 \\
x^r \left[\sum_{m=0}^{\infty} 4(m+r)c_m x^m + \sum_{m=0}^{\infty} c_m x^m - \sum_{m=1}^{\infty} c_{m-1} x^m \right] &= 0 \\
\left(4rc_0 x^0 + \sum_{m=1}^{\infty} 4(m+r)c_m x^m \right) + \left(c_0 x^0 + \sum_{m=1}^{\infty} c_m x^m \right) - \sum_{m=1}^{\infty} c_{m-1} x^m &= 0 \\
(4r+1)c_0 x^0 + \sum_{m=1}^{\infty} \left[4(m+r+1)c_m - c_{m-1} \right] x^m &= 0
\end{aligned}$$

We see that $(4r+1)c_0 x^0 = 0 \implies r = -\frac{1}{4}$. Since each coefficient of the sum must be zero we get:

$$4\left(m + \frac{3}{4}\right)c_m - c_{m-1} = 0 \implies c_m = \frac{c_{m-1}}{4m+3} \text{ for all} m \geq 1$$

We plug in a few values of m:

$$\begin{aligned}
m = 1: &\quad c_1 = \frac{c_0}{7} = \frac{1}{7!!!!} \cdot 3c_0 \\
m = 2: &\quad c_2 = \frac{c_1}{11} = \frac{1}{11!!!!} \cdot 3c_0 \\
m = 3: &\quad c_3 = \frac{c_2}{15} = \frac{1}{15!!!!} \cdot 3c_0 \\
&\vdots
\end{aligned}$$

The pattern emerges as $c_n = \frac{1}{(4n+3)!!!!} \cdot 3c_0$ for all $n \geq 1$.

We put this into the general power series form:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n^{n+r} \\ &= x^r \left[c_0 + \sum_{n=1}^{\infty} c_n x^n \right] \\ &= c_0 x^{-\frac{1}{4}} \left(1 + 3 \sum_{n=1}^{\infty} \frac{1}{(4n+3)!!!!} x^n \right) \end{aligned}$$

Exercise 4.4.1*b*

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ so $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$:

$$\begin{aligned} 3xy' + 2y + 3xy &= 0 \\ 3x \cdot \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + 2 \cdot \sum_{n=0}^{\infty} c_n x^{n+r} + 3x \cdot \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\ \underbrace{\sum_{n=0}^{\infty} 3(n+r) c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 3c_n x^{n+r+1}}_{m=n+1} &= 0 \\ x^r \left[\sum_{m=0}^{\infty} 3(m+r) c_m x^m + \sum_{m=0}^{\infty} 2c_m x^m + \sum_{m=1}^{\infty} 3c_{m-1} x^m \right] &= 0 \\ \left(3rc_0 x^0 + \sum_{m=1}^{\infty} 3(m+r) c_m x^m \right) + \left(2c_0 x^0 + \sum_{m=1}^{\infty} 2c_m x^m \right) + \sum_{m=1}^{\infty} 3c_{m-1} x^m &= 0 \\ (3r+2) c_0 x^0 + \sum_{m=1}^{\infty} \left[(3m+3r+2) c_m + 3c_{m-1} \right] x^m &= 0 \end{aligned}$$

Wee see that $(3r+2)c_0 x^0 = 0 \implies r = -\frac{2}{3}$. Since each coefficient of the sum must be zero we get:

$$3mc_m + 3c_{m-1} = 0 \implies c_m = -\frac{c_{m-1}}{m} \text{ for all } m \geq 1$$

We plug in a few values of m:

$$\begin{aligned} m = 1: &\quad c_1 = -\frac{c_0}{1} \\ m = 2: &\quad c_2 = -\frac{c_1}{2} = \frac{(-1)^2}{2!} c_0 \\ m = 3: &\quad c_3 = -\frac{c_2}{3} = \frac{(-1)^3}{3!} c_0 \\ &\vdots \end{aligned}$$

The pattern is $c_n = \frac{(-1)^n}{n!}$ for all $n \geq 1$.

We put this into the general power series form:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n^{n+r} \\ &= x^r \left[c_0 + \sum_{n=1}^{\infty} c_n x^n \right] \\ &= c_0 x^{-\frac{2}{3}} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n \right) \\ &= c_0 x^{-\frac{2}{3}} e^{-x} \end{aligned}$$

Exercise 4.4.1*c*

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ so $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$:

$$\begin{aligned} 5xy' - 5x^2 y - y &= 0 \\ 5x \cdot \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - 5x^2 \cdot \sum_{n=0}^{\infty} c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\ \underbrace{\sum_{n=0}^{\infty} 5(n+r) c_n x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 5c_n x^{n+r+2}}_{m=n+2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} &= 0 \\ x^r \left[\sum_{m=0}^{\infty} 5(m+r) c_m x^m - \sum_{m=2}^{\infty} 5c_{m-2} x^m - \sum_{m=0}^{\infty} c_m x^m \right] &= 0 \\ \left(5rc_0 x^0 + 5(r+1) c_1 x^1 + \sum_{m=2}^{\infty} 5(m+r) c_m x^m \right) - \sum_{m=2}^{\infty} 5c_{m-2} x^m - \left(c_0 + c_1 x + \sum_{m=2}^{\infty} c_m x^m \right) &= 0 \\ (5r-1) c_0 x^0 + (5r-6) c_1 x^1 + \sum_{m=2}^{\infty} \left[(5m+5r-1) c_m - 5c_{m-2} \right] x^m &= 0 \end{aligned}$$

We see that $(5r-1)c_0 x^0 = 0 \implies r = \frac{1}{5}$. since $(5r-2)c_1 x^1 = 0$ we must have $c_1 = 0$.

Since each coefficient of the sum must be zero we get:

$$5mc_m - 5c_{m-2} = 0 \implies c_m = \frac{c_{m-2}}{m} \text{ for all } m \geq 2$$

We plug in a few values of m:

$$\begin{aligned} m=2: &\quad c_2=\frac{c_0}{2} \\ m=3: &\quad c_3=\frac{c_1}{3}=0 \\ m=4: &\quad c_4=\frac{c_2}{4}=\frac{1}{4!!}c_0 \\ m=5: &\quad c_5=\frac{c_3}{5}=0 \\ m=6: &\quad c_6=\frac{c_4}{6}=\frac{1}{6!!}c_0 \\ m=7: &\quad c_7=\frac{c_5}{7}=0 \\ &\vdots \end{aligned}$$

The pattern is $c_{2k}=\dfrac{1}{(2k)!!}$ for $k\geq 1$ and $c_{2k+1}=0$ for $k\geq 0$.

We put this into the general power series form:

$$\begin{aligned} y&=\sum_{n=0}^{\infty}c_n^{n+r} \\ &=x^r\left[c_0+\sum_{k=0}^{\infty}c_{2k+1}x^{2k+1}+\sum_{k=1}^{\infty}c_{2k}x^{2k}\right] \\ &=c_0x^{\frac{1}{2}}\left(1+\sum_{n=1}^{\infty}\frac{1}{(2k)!!}x^{2k}\right) \end{aligned}$$

Exercise $4.4.2a$

The indicial equation stems from lowest power terms:

$$\begin{aligned} 2x^2y''&=2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}=\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_nx^{n+r} \\ 6xy'&=6x\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}=\sum_{n=0}^{\infty}6(n+r)c_nx^{n+r} \\ 2y&=2\sum_{n=0}^{\infty}c_nx^{n+r}=\sum_{n=0}^{\infty}2c_nx^{n+r} \end{aligned}$$

Adding lowest power (x^{n+r}) terms together gives:

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)+6(n+r)+2\right]c_nx^{n+r}$$

Thus, the indicial equation is

$$2r(r-1)+6r+2=0 \implies 2r^2+4r+2=0 \implies r=-1 \text{ (double)}$$

Exercise $4.4.2b$

The indicial equation stems from lowest power terms:

$$\begin{aligned} x^2y''&=x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r} \\ xy'&=x\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r} \\ -4y&=-4\sum_{n=0}^{\infty}c_nx^{n+r}=\sum_{n=0}^{\infty}-4c_nx^{n+r} \end{aligned}$$

Adding lowest power (x^{n+r}) terms together gives:

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+(n+r)-4\right]c_nx^{n+r}$$

Thus, the indicial equation is

$$r(r-1)+r-4=0 \implies r^2-4=0 \implies r=\pm 2$$

Exercise $4.4.2c$

The indicial equation stems from lowest power terms:

$$\begin{aligned} xy''&=x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1} \\ 11y'&=5\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}=\sum_{n=0}^{\infty}11(n+r)c_nx^{n+r-1} \\ -x^2y&=-x^2\sum_{n=0}^{\infty}c_nx^{n+r}=-\sum_{n=0}^{\infty}c_nx^{n+r+2} \end{aligned}$$

Adding lowest power (x^{n+r-1}) terms together gives:

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+11(n+r)\right]c_nx^{n+r-1}$$

Thus, the indicial equation is

$$r(r-1)+11r=0 \implies r^2+10r=0 \implies r=0,-10$$

Exercise $4.4.2d$

The indicial equation stems from lowest power terms:

$$-3xy'' = -3x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} = -\sum_{n=0}^{\infty}3(n+r)(n+r-1)c_n x^{n+r-1}$$
$$9y' = 9\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} = \sum_{n=0}^{\infty}9(n+r)c_n x^{n+r-1}$$
$$7xy = 7x\sum_{n=0}^{\infty}c_n x^{n+r} = \sum_{n=0}^{\infty}7c_n x^{n+r+1}$$

Adding lowest power (x^{n+r-1}) terms together gives:

$$\sum_{n=0}^{\infty}\left[-3(n+r)(n+r-1)+9(n+r)\right]c_n x^{n+r-1}$$

Thus, the indicial equation is

$$-3r(r-1)+9r=0 \implies -3r^2+12r=0 \implies r=0,4$$

Exercise $4.4.2e$

The indicial equation stems from lowest power terms:

$$x^2y'' = x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}$$
$$-xy' = -x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} = -\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}$$
$$5y = 5\sum_{n=0}^{\infty}c_n x^{n+r} = \sum_{n=0}^{\infty}5c_n x^{n+r}$$

Adding lowest power (x^{n+r}) terms together gives:

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-(n+r)+5\right]c_n x^{n+r}$$

Thus, the indicial equation is

$$r(r-1)-r+5=0 \implies r^2-2r+5=0 \implies r=1\pm 2i$$

Exercise $4.4.2f$

The indicial equation stems from lowest power terms:

$$2x^2y'' = 2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} = \sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n+r}$$
$$2xy' = 2x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} = \sum_{n=0}^{\infty}2(n+r)c_n x^{n+r}$$
$$-6y = -6\sum_{n=0}^{\infty}c_n x^{n+r} = \sum_{n=0}^{\infty}-6c_n x^{n+r}$$

Adding lowest power (x^{n+r}) terms together gives:

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)+2(n+r)-6\right]c_n x^{n+r}$$

Thus, the indicial equation is

$$2r(r-1)+2r-6=0 \implies 2r^2-6=0 \implies r=\pm\sqrt{3}$$

Exercise $4.4.2g$

he indicial equation stems from lowest power terms:

$$x^2y'' = x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}$$
$$\sin(x)y' = \left(x-\frac{1}{3}x^3+\cdots\right)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r}+\cdots$$
$$e^x y = (1+x+\cdots)\sum_{n=0}^{\infty}c_n x^{n+r} = \sum_{n=0}^{\infty}c_n x^{n+r}+\cdots$$

Adding lowest power (x^{n+r}) terms together gives:

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+(n+r)+1\right]c_n x^{n+r}$$

Thus, the indicial equation is

$$r(r-1)+r+1=0 \implies r^2+1=0 \implies r=\pm i$$

Exercise $4.4.2h$

The indicial equation stems from lowest power terms:

$$(1-\cos(x))\,y'' = \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} = \sum_{n=0}^{\infty}\frac{1}{2}(n+r)(n+r-1)c_n x^{n+r} + \cdots$$

$$2\ln(1+x)y' = 2\left(x - \frac{x^2}{2} + \cdots\right)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} = \sum_{n=0}^{\infty}2(n+r)c_n x^{n+r} + \cdots$$

$$3\tan^{-1}(x)y = (3x - x^3 + \cdots)\sum_{n=0}^{\infty}c_n x^{n+r} = \sum_{n=0}^{\infty}3c_n x^{n+r+1} + \cdots$$

Adding lowest power (x^{n+r}) terms together gives:

$$\sum_{n=0}^{\infty}\left[\frac{1}{2}(n+r)(n+r-1) + 2(n+r)\right]c_n x^{n+r}$$

Thus, the indicial equation is

$$\frac{1}{2}r(r-1) + 2r = 0 \implies \frac{1}{2}r^2 + \frac{3}{2}r = 0 \implies r = 0, -3$$

Exercise 4.4.3*a*

Assume $y = \sum_{n=0}^{\infty}c_n x^{n+r}$:

$$\begin{aligned}
2xy'' + y' - 2y &= 0\\
2x\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - 2\cdot\sum_{n=0}^{\infty}c_n x^{n+r} &= 0\\
\underbrace{\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty}2c_n x^{n+r}}_{m=n} &= 0\\
x^r\left[\sum_{m=-1}^{\infty}2(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=-1}^{\infty}(m+r+1)c_{m+1}x^m - \sum_{m=0}^{\infty}2c_m x^m\right] &= 0\\
(2r(r-1)+r)\,c_0 x^{-1} + \sum_{m=0}^{\infty}\left[2(m+r+1)(m+r)c_{m+1} + (m+r+1)c_{m+1} - 2c_m\right]x^m &= 0
\end{aligned}$$

This gives us indicial equation $2r(r-1)+r=0 \implies r=0,\ r=\frac{1}{2}$

For $r=0$, setting each coefficient in the sum equal to zero gives:

$$(m+1)(2m+1)c_{m+1} - 2c_m = 0 \implies c_{m+1} = \frac{2c_m}{(m+1)(2m+1)} \text{ for all } m \geq 0$$

$$\begin{aligned}
c_1 &= \frac{2}{1\cdot 1}c_0\\
c_2 &= \frac{2}{2\cdot 3}c_1 = \frac{2^2}{2!\cdot 3!!}c_0\\
c_3 &= \frac{2}{3\cdot 5}c_2 = \frac{2^3}{3!\cdot 5!!}c_0\\
&\vdots
\end{aligned}$$

We see the pattern is $c_n = \frac{2^n}{n!\cdot(2n-1)!!}c_0$ for all $n \geq 1$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + \sum_{n=1}^{\infty}c_n x^n\right) = c_0 x^0\left(1 + \sum_{n=1}^{\infty}\frac{2^n}{n!\cdot(2n-1)!!}x^n\right)$$

For $r=\frac{1}{2}$, setting each coefficient in the sum equal to zero gives:

$$(2m+3)\,(m+1)d_{m+1} - 2d_m = 0 \implies d_{m+1} = \frac{2d_m}{(m+1)(2m+3)} \text{ for all } m \geq 0$$

$$\begin{aligned}
d_1 &= \frac{2}{1\cdot 3}d_0\\
d_2 &= \frac{2}{2\cdot 5}d_1 = \frac{2^2}{2!\cdot 5!!}d_0\\
d_3 &= \frac{2}{3\cdot 7}d_2 = \frac{2^3}{3!\cdot 7!!}d_0\\
&\vdots
\end{aligned}$$

We see the pattern is $d_n = \frac{2^n}{n!\cdot(2n+1)!!}d_0$ for all $n \geq 1$. The second series solution must be:

$$y_2 = x^r\left(d_0 + \sum_{n=1}^{\infty}d_n x^n\right) = d_0 x^{\frac{1}{2}}\left(1 + \sum_{n=1}^{\infty}\frac{2^n}{n!\cdot(2n+1)!!}x^n\right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0 x^0\left(1 + \sum_{n=1}^{\infty}\frac{2^n}{n!\cdot(2n-1)!!}x^n\right) + d_0 x^{\frac{1}{2}}\left(1 + \sum_{n=1}^{\infty}\frac{2^n}{n!\cdot(2n+1)!!}x^n\right)$$

Exercise 4.4.3*b*

Assume $y = \sum_{n=0}^{\infty}c_n x^{n+r}$:

$$3xy'' - 2y' + y = 0$$
$$3x \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - 2 \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$
$$\underbrace{\sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} = 0$$
$$x^r \left[\sum_{m=-1}^{\infty} 3(m+r+1)(m+r)c_{m+1}x^m - \sum_{m=-1}^{\infty} 2(m+r+1)c_{m+1}x^m + \sum_{m=0}^{\infty} c_m x^m \right] = 0$$
$$(3r(r-1) - 2r)\, c_0 x^{-1} + \sum_{m=0}^{\infty} \left[3(m+r+1)(m+r)c_{m+1} - 2(m+r+1)c_{m+1} + c_m\right] x^m = 0$$

This gives us indicial equation $3r(r-1) - 2r = 0 \implies r = 0,\ r = \dfrac{5}{3}$

For $r = 0$, setting each coefficient in the sum equal to zero gives:

$$(m+1)(3m-2)c_{m+1} + c_m = 0 \implies c_{m+1} = -\frac{c_m}{(m+1)(3m-2)} \text{ for all } m \geq 0$$
$$c_1 = -\frac{1}{1 \cdot (-2)} c_0 = \frac{c_0}{2}$$
$$c_2 = -\frac{1}{2 \cdot 1} c_1 = -\frac{1}{2! \cdot 1} \frac{c_0}{2}$$
$$c_3 = -\frac{1}{3 \cdot 4} c_2 = \frac{1}{3! \cdot 4!!!} \frac{c_0}{2}$$
$$c_4 = -\frac{1}{4 \cdot 7} c_3 = -\frac{1}{4! \cdot 7!!!} \frac{c_0}{2}$$
$$\vdots$$

The pattern is $c_n = \dfrac{(-1)^{n-1}}{n! \cdot (3n-5)!!!} \dfrac{c_0}{2}$$ for all $n \geq 2$. The first series solution is therefore:

$$y_1 = x^r \left(c_0 + c_1 x + \sum_{n=1}^{\infty} c_n x^n \right) = c_0 x^0 \left(1 + \frac{1}{2}x + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n! \cdot (3n-5)!!!} x^n \right)$$

For $r = \dfrac{5}{3}$, setting each coefficient in the sum equal to zero gives:

$$(3m+8)\,(m+1)d_{m+1} - 2d_m = 0 \implies d_{m+1} = \frac{2d_m}{(m+1)(3m+8)} \text{ for all } m \geq 0$$
$$d_1 = \frac{2}{1 \cdot 8} d_0 = \frac{2}{1! \cdot 8!!!} \cdot 10 d_0$$
$$d_2 = \frac{2}{2 \cdot 11} d_1 = \frac{2^2}{2! \cdot 11!!!} \cdot 10 d_0$$
$$d_3 = \frac{2}{3 \cdot 14} d_2 = \frac{2^3}{3! \cdot 14!!!} \cdot 10 d_0$$
$$\vdots$$

The pattern is $d_n = \dfrac{2^n}{n! \cdot (3n+5)!!!} \cdot 10 d_0$ for $n \geq 1$. The second series solution is therefore:

$$y_2 = x^r \left(d_0 + \sum_{n=1}^{\infty} d_n x^n \right) d_0 x^{\frac{5}{3}} \left(1 + 10 \sum_{n=1}^{\infty} \frac{2^n}{n! \cdot (3n+5)!!!} x^n \right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0 x^0 \left(1 + \frac{1}{2}x + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n! \cdot (3n-5)!!!} x^n \right) + d_0 x^{\frac{5}{3}} \left(1 + 10 \sum_{n=1}^{\infty} \frac{2^n}{n! \cdot (3n+5)!!!} x^n \right)$$

Exercise $4.4.3c$

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$2xy'' + 3y' - 3xy = 0$$
$$2x \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 3 \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - 3x \cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$
$$\underbrace{\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty} 3c_n x^{n+r+1}}_{m=n+1} = 0$$
$$x^r \left[\sum_{m=-1}^{\infty} 2(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=-1}^{\infty} 3(m+r+1)c_{m+1}x^m - \sum_{m=1}^{\infty} 3c_{m-1}x^m \right] = 0$$
$$(2r(r-1) + 3r)\, c_0 x^{-1} + (2(r+1)r + 3(r+1))\, c_1 x^0$$
$$+ \sum_{m=1}^{\infty} \left[2(m+r+1)(m+r)c_{m+1} + 3(m+r+1)c_{m+1} - 2c_{m-1}\right] x^m = 0$$

The indicial equation is $2r(r-1) + 3r = 0 \implies r = 0,\ r = -\dfrac{1}{2}$

For $r = 0$, setting each coefficient in the sum equal to zero gives:

$$(m+1)(2m+3)c_{m+1} - 2c_{m-1} = 0 \implies c_{m+1} = \frac{c_{m-1}}{(m+1)(2m+3)} \text{ for all } m \geq 1$$

$$
\begin{aligned}
c_2 &= \frac{2}{2\cdot 5}c_0 = \frac{1}{5}c_0\\
c_3 &= \frac{2}{3\cdot 7}c_1 = 0\\
c_4 &= \frac{2}{4\cdot 9}c_2 = \frac{1}{2!\cdot 9!!!!}c_0\\
c_5 &= \frac{2}{5\cdot 11}c_3 = 0\\
c_6 &= \frac{2}{4\cdot 13}c_4 = \frac{1}{3!\cdot 13!!!!}c_0\\
c_7 &= \frac{2}{5\cdot 15}c_5 = 0\\
&\vdots
\end{aligned}
$$

The pattern is $c_{2k} = \dfrac{1}{k!\cdot(4k+1)!!!!}c_0$ for $k \geq 1$ and $c_{2k+1} = 0$ for $k \geq 1$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + c_1x + \sum_{k=1}^{\infty} c_{2k}x^{2k} + \sum_{k=1}^{\infty} c_{2k+1}x^{2k+1}\right) = c_0x^0\left(1 + \sum_{k=1}^{\infty}\frac{1}{k!\cdot(4k+1)!!!!}x^{2k}\right)$$

For $r = -\dfrac{1}{2}$, setting each coefficient in the sum equal to zero gives:

$$(2m+1)(m+1)d_{m+1} - 2d_{m-1} = 0 \implies d_{m+1} = \frac{2d_{m-1}}{(m+1)(2m+1)} \text{ for all } m \geq 1$$

$$
\begin{aligned}
d_2 &= \frac{2}{2\cdot 3}d_0 = \frac{1}{3}d_0\\
d_3 &= \frac{2}{3\cdot 5}d_1 = 0\\
d_4 &= \frac{2}{4\cdot 7}d_2 = \frac{1}{2!\cdot 7!!!!}d_0\\
d_5 &= \frac{2}{5\cdot 9}d_3 = 0\\
d_6 &= \frac{2}{4\cdot 11}d_4 = \frac{1}{3!\cdot 11!!!!}d_0\\
d_7 &= \frac{2}{5\cdot 13}d_5 = 0\\
&\vdots
\end{aligned}
$$

The pattern is $d_{2k} = \dfrac{1}{k!\cdot(4k-1)!!!!}d_0$ for $k \geq 1$ and $d_{2k+1} = 0$ for $k \geq 1$. The second series solution is therefore:

$$y_2 = x^r\left(d_0 + d_1x + \sum_{k=1}^{\infty} d_{2k}x^{2k} + \sum_{k=1}^{\infty} d_{2k+1}x^{2k+1}\right) = d_0x^{-\frac{1}{2}}\left(1 + \sum_{k=1}^{\infty}\frac{1}{k!\cdot(4k-1)!!!!}x^{2k}\right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0x^0\left(1 + \sum_{k=1}^{\infty}\frac{1}{k!\cdot(4k+1)!!!!}x^{2k}\right) + d_0x^{-\frac{1}{2}}\left(1 + \sum_{k=1}^{\infty}\frac{1}{k!\cdot(4k-1)!!!!}x^{2k}\right)$$

Exercise 4.4.3*d*

Assume $y = \sum_{n=0}^{\infty} c_nx^{n+r}$:

$$
\begin{aligned}
2x^2y'' + (2x^2 - x)y' - (3x+5)y &= 0\\
2x^2\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2} + 2x^2\cdot\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} - x\cdot\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} - 3x\cdot\sum_{n=0}^{\infty}c_nx^{n+r} - 5\cdot\sum_{n=0}^{\infty}c_nx^{n+r} &= 0\\
\underbrace{\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_nx^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty}2(n+r)c_nx^{n+r+1}}_{m=n+1} - \underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty}3c_nx^{n+r+1}}_{m=n+1} - \underbrace{\sum_{n=0}^{\infty}5c_nx^{n+r}}_{m=n} &= 0\\
x^r\left[\sum_{m=0}^{\infty}2(m+r)(m+r-1)c_mx^m + \sum_{m=1}^{\infty}2(m+r-1)c_{m-1}x^m - \sum_{m=0}^{\infty}(m+r)c_mx^m - \sum_{m=1}^{\infty}3c_{m-1}x^m - \sum_{m=0}^{\infty}5c_mx^m\right] &= 0\\
(2r(r-1) - r - 5)\,c_0x^0 + \sum_{m=1}^{\infty}\left[2(m+r)(m+r-1)c_m + 2(m+r-1)c_{m-1} - (m+r)c_m - 3c_{m-1} - 5c_m\right]x^m &= 0
\end{aligned}
$$

The indicial equation is $2r(r-1) - r - 5 = 0 \implies r = -1,\ r = \dfrac{5}{2}$

For $r = -1$, setting each coefficient in the sum equal to zero gives:

$$m(2m-7)c_m + (2m-7)c_{m-1} = 0 \implies c_m = -\frac{c_{m-1}}{m} \text{ for all } m \geq 1$$

$$
\begin{aligned}
c_1 &= -\frac{1}{1}c_0\\
c_2 &= -\frac{1}{2}c_1 = \frac{(-1)^2}{2!}c_0\\
c_3 &= -\frac{1}{3}c_2 = \frac{(-1)^3}{3!}c_0\\
&\vdots
\end{aligned}
$$

The pattern is $c_n = \dfrac{(-1)^n}{n!}c_0$ for all $n \geq 1$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + \sum_{n=1}^{\infty} c_n x^n\right) = c_0 x^{-1}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n\right) = c_0 x^{-1} e^{-x}$$

For $r = \frac{5}{2}$, setting each coefficient in the sum equal to zero gives:

$$m(2m+7)d_{m+1} + 2md_{m-1} = 0 \implies d_m = -\frac{2d_{m-1}}{2m+7} \text{ for all } m \geq 1$$

$$\begin{aligned} d_1 &= -\frac{2}{9}d_0 = \frac{-2}{9!!} \cdot 105 d_0 \\ d_2 &= -\frac{2}{11}d_1 = \frac{(-2)^2}{11!!} \cdot 105 d_0 \\ d_3 &= -\frac{2}{13}d_2 = \frac{(-2)^3}{13!!} \cdot 105 d_0 \\ &\vdots \end{aligned}$$

The pattern is $d_n = \frac{(-2)^n}{(2n+7)!!} \cdot 105 d_0$ for $n \geq 1$. The second series solution is therefore:

$$y_2 = x^r\left(d_0 + \sum_{n=1}^{\infty} d_n x^n\right) = d_0 x^{\frac{5}{2}}\left(1 + 105\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+7)!!} x^n\right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0 x^{-1} e^{-x} + d_0 x^{\frac{5}{2}}\left(1 + 105\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+7)!!} x^n\right)$$

Exercise 4.4.3e

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$\begin{aligned} 6x^2y'' + xy' + (1-4x^2)y &= 0 \\ 6x^2 \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + x \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} - 4x^2 \cdot \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\ \underbrace{\sum_{n=0}^{\infty} 6(n+r)(n+r-1)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 4c_n x^{n+r+2}}_{m=n+2} &= 0 \\ x^r\left[\sum_{m=0}^{\infty} 6(m+r)(m+r-1)c_m x^m + \sum_{m=1}^{\infty}(m+r)c_m x^m + \sum_{m=0}^{\infty} c_m x^m - \sum_{m=2}^{\infty} 4c_{m-2}x^m\right] &= 0 \\ (6r(r-1)+r+1)\,c_0 x^0 + (6(r+1)r+(r+1)+1)\,c_1 x^1 + \sum_{m=2}^{\infty}\left[6(m+r)(m+r-1)c_m + (m+r)c_m + c_m - 4c_{m-2}\right]x^m &= 0 \end{aligned}$$

The indicial equation is $6r(r-1)+r+1 \implies r = \frac{1}{3},\ r = \frac{1}{2}$

For $r = \frac{1}{3}$, setting each coefficient in the sum equal to zero gives:

$$(m(6m-1)c_m - 4c_{m-2} = 0 \implies c_m = \frac{4c_{m-2}}{m(6m-1)} \text{ for all } m \geq 2$$

$$\begin{aligned} c_2 &= \frac{4}{2 \cdot 11}c_0 = \frac{2}{11!!!!!!} \cdot 5c_0 \\ c_3 &= \frac{4}{3 \cdot 17}c_1 = 0 \\ c_4 &= \frac{4}{4 \cdot 23}c_2 = \frac{2^2}{2!23!!!!!!} \cdot 5c_0 \\ c_5 &= \frac{4}{5 \cdot 29}c_3 = 0 \\ c_6 &= \frac{4}{6 \cdot 35}c_4 = \frac{2^3}{3!35!!!!!!} \cdot 5c_0 \\ c_7 &= \frac{4}{7 \cdot 41}c_5 = 0 \\ &\vdots \end{aligned}$$

The pattern is $c_{2k} = \frac{2^k}{k!(12k-1)!!!!!!}c_0$ for $k \geq 1$ and $c_{2k+1} = 0$ for $k \geq 0$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k}x^{2k} + \sum_{k=1}^{\infty} c_{2k+1}x^{2k+1}\right) = c_0 x^{\frac{1}{3}}\left(1 + \sum_{k=1}^{\infty} \frac{2^k}{k!(12k-1)!!!!!!} x^{2k}\right)$$

For $r = \frac{1}{2}$, setting each coefficient in the sum equal to zero gives:

$$m(6m+1)d_m - 4d_{m-2} = 0 \implies d_m = \frac{4d_{m-2}}{m(6m+1)} \text{ for all } m \geq 2$$

$$
\begin{aligned}
d_2 &= \frac{4}{2\cdot 13}d_0 = \frac{2}{13!!!!!!}\cdot 7d_0\\
d_3 &= \frac{4}{3\cdot 19}d_1 = 0\\
d_4 &= \frac{4}{4\cdot 25}d_2 = \frac{2^2}{2!25!!!!!!}\cdot 7d_0\\
d_5 &= \frac{4}{5\cdot 31}d_3 = 0\\
d_6 &= \frac{4}{6\cdot 37}d_4 = \frac{2^3}{3!37!!!!!!}\cdot 7d_0\\
d_7 &= \frac{4}{7\cdot 43}d_5 = 0\\
&\vdots
\end{aligned}
$$

The pattern is $d_{2k} = \dfrac{2^k}{k!(12k+1)!!!!!!}d_0$ for $k \geq 1$ and $d_{2k+1} = 0$ for $k \geq 0$. The second series solution is therefore:

$$y_2 = x^r\left(d_0 + d_1x + \sum_{k=1}^{\infty} d_{2k}x^{2k} + \sum_{k=1}^{\infty} d_{2k+1}x^{2k+1}\right) = d_0x^{\frac{1}{2}}\left(1 + \sum_{k=1}^{\infty}\frac{2^k}{k!(12k+1)!!!!!!}x^{2k}\right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0x^{\frac{1}{3}}\left(1 + \sum_{k=1}^{\infty}\frac{2^k}{k!(12k-1)!!!!!!}x^{2k}\right) + d_0x^{\frac{1}{2}}\left(1 + \sum_{k=1}^{\infty}\frac{2^k}{k!(12k+1)!!!!!!}x^{2k}\right)$$

Exercise $4.4.3f$

Assume $y = \sum_{n=0}^{\infty} c_nx^{n+r}$:

$$
\begin{aligned}
3xy'' + (3x+1)y' - 2y &= 0\\
3x\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2} + 3x\cdot\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} - 2\cdot\sum_{n=0}^{\infty}c_nx^{n+r} &= 0\\
\underbrace{\sum_{n=0}^{\infty}3(n+r)(n+r-1)c_nx^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty}3(n+r)c_nx^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty}2c_nx^{n+r}}_{m=n} &= 0\\
x^r\left[\sum_{m=-1}^{\infty}3(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=1}^{\infty}3(m+r)c_mx^m + \sum_{m=-1}^{\infty}(m+r+1)c_{m+1}x^m - \sum_{m=1}^{\infty}2c_mx^m\right] &= 0\\
(3r(r-1)+r)\,c_0x^{-1} + \sum_{m=0}^{\infty}\left[3(m+r+1)(m+r)c_{m+1} + 3(m+r)c_m + (m+r+1)c_{m+1} - 2c_m\right]x^m &= 0
\end{aligned}
$$

The indicial equation is $3r(r-1)+r = 0 \implies r = 0,\ r = \dfrac{2}{3}$

For $r = 0$, setting each coefficient in the sum equal to zero gives:

$$(3m+1)(m+1)c_{m+1} + (3m-2)c_m = 0 \implies c_m = -\frac{(3m-2)c_m}{(3m+1)(m+1)} \text{ for all } m \geq 1$$

$$
\begin{aligned}
c_1 &= -\frac{1}{4\cdot 2}c_0 = \frac{-1}{4\cdot 2!}c_0\\
c_2 &= -\frac{4}{7\cdot 3}c_1 = \frac{(-1)^2}{7\cdot 3!}c_0\\
c_3 &= -\frac{7}{10\cdot 4}c_2 = \frac{(-1)^3}{10\cdot 4!}c_0\\
&\vdots
\end{aligned}
$$

The pattern is $c_n = \dfrac{(-1)^n}{(3n+1)\cdot n!}c_0$ for $n \geq 1$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + \sum_{n=1}^{\infty}c_nx^n\right) = c_0x^0\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(3n+1)\cdot n!}x^n\right)$$

For $r = \dfrac{2}{3}$, setting each coefficient in the sum equal to zero gives:

$$(3m+1)(m+5)d_{m+1} + 3md_m = 0 \implies d_m = -\frac{3md_{m-1}}{(3m+1)(m+5)} \text{ for all } m \geq 1$$

$$
\begin{aligned}
d_1 &= -\frac{3}{4\cdot 6}d_0 = \frac{-3}{4!!!\cdot 6!}\cdot 120d_0\\
d_2 &= -\frac{6}{7\cdot 7}d_1 = \frac{(-3)^2\cdot 2!}{7!!!\cdot 7!}\cdot 120d_0\\
d_3 &= -\frac{9}{10\cdot 8}d_2 = \frac{(-3)^3\cdot 3!}{10!!!\cdot 8!}\cdot 120d_0\\
&\vdots
\end{aligned}
$$

The pattern is $d_n = \dfrac{(-3)^n\cdot n!}{(3n+1)!!!\cdot(n+5)!}\cdot 120d_0$ for all $n \geq 1$. The second series solution is therefore:

$$y_2 = x^r\left(d_0 + \sum_{n=1}^{\infty}d_nx^n\right) = d_0x^{\frac{2}{3}}\left(1 + 120\sum_{n=1}^{\infty}\frac{(-3)^n\cdot n!}{(3n+1)!!!\cdot(n+5)!}x^n\right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0 x^0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+1)\cdot n!} x^n\right) + d_0 x^{\frac{2}{3}} \left(1 + 120 \sum_{n=1}^{\infty} \frac{(-3)^n \cdot n!}{(3n+1)!!! \cdot (n+5)!} x^n\right)$$

Exercise 4.4.3*g*

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$3x^2 y'' - (6x^2 + 5x)y' - (8x+3)y = 0$$

$$3x^2 \cdot \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} - (6x^2+5x) \cdot \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} - (8x+3) \cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 6(n+r)c_n x^{n+r+1}}_{m=n+1} - \underbrace{\sum_{n=0}^{\infty} 5(n+r)c_n x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 8c_n x^{n+r+1}}_{m=n+1} - \underbrace{\sum_{n=0}^{\infty} 3c_n x^{n+r}}_{m=n} = 0$$

$$x^r \left[\sum_{m=0}^{\infty} 3(m+r)(m+r-1)c_m x^m - \sum_{m=1}^{\infty} 6(m+r-1)c_{m-1}x^m - \sum_{m=0}^{\infty} 5(m+r)c_m x^m - \sum_{m=1}^{\infty} 8c_{m-1}x^m - \sum_{m=0}^{\infty} 3c_m x^m\right] = 0$$

$$(3r(r-1) - 5r - 3)\, c_0 x^0 + \sum_{m=1}^{\infty} \left[3(m+r)(m+r-1)c_m - 6(m+r-1)c_{m-1} - 5(m+r)c_m - 8c_{m-1} - 3c_m\right] x^m = 0$$

The indicial equation is $3r(r-1) - 5r - 3 = 0 \implies r = 3,\ r = -\frac{1}{3}$

For $r = 3$, setting each coefficient in the sum equal to zero gives:

$$m(3m+10)c_m - 2(3m+10)c_{m-1} = 0 \implies c_m = \frac{2c_{m-1}}{m} \text{ for all } m \geq 1$$

$$c_1 = \frac{2}{1}c_0 = \frac{2}{1!}c_0$$
$$c_2 = \frac{2}{2}c_1 = \frac{2^2}{2!}c_0$$
$$c_3 = \frac{2}{3}c_2 = \frac{2^3}{3!}c_0$$
$$\vdots$$

The pattern is $c_n = \frac{2^n}{n!}c_0$ for $n \geq 1$. The first series solution is therefore:

$$y_1 = x^r \left(c_0 + \sum_{n=1}^{\infty} c_n x^n\right) = c_0 x^3 \left(1 + \sum_{n=1}^{\infty} \frac{2^n}{n!} x^n\right) = c_0 x^3 e^{2x}$$

For $r = -\frac{1}{3}$, setting each coefficient in the sum equal to zero gives:

$$m(3m-10)d_m - 6md_{m-1} = 0 \implies d_m = \frac{6d_{m-1}}{3m-10} \text{ for all } m \geq 1$$

$$d_1 = \frac{6}{-7}d_0 = -\frac{6}{7}d_0$$
$$d_2 = \frac{6}{-4}d_1 = \frac{9}{7}d_0$$
$$d_3 = \frac{6}{-1}d_2 = -\frac{54}{7}d_0$$
$$d_4 = \frac{6}{2}d_3 = \frac{6}{2!!!} \cdot \frac{54}{7}d_0$$
$$d_5 = \frac{6}{5}d_4 = \frac{6^2}{5!!!} \cdot \frac{54}{7}d_0$$
$$\vdots$$

The pattern is $d_n = \frac{6^{n-3}}{(3n-10)!!!} \cdot \frac{54}{7}d_0$ for $n \geq 4$. The second series solution is therefore:

$$y_2 = x^r \left(d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \sum_{n=4}^{\infty} d_n x^n\right) = d_0 x^{-\frac{1}{3}} \left(1 - \frac{6}{7}x + \frac{9}{7}x^2 - \frac{54}{7}x^3 + \frac{54}{7}\sum_{n=4}^{\infty} \frac{6^{n-3}}{(3n-10)!!!} x^n\right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0 x^3 e^{2x} + d_0 x^{-\frac{1}{3}} \left(1 - \frac{6}{7}x + \frac{9}{7}x^2 - \frac{54}{7}x^3 + \frac{54}{7}\sum_{n=4}^{\infty} \frac{6^{n-3}}{(3n-10)!!!} x^n\right)$$

Exercise 4.4.3*h*

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$2x^2y'' + (2x^3 + x)y' + (2x^2 - 1)y = 0$$

$$2x^2 \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (2x^3 + x) \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + (2x^2 - 1) \cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r+2}}_{m=n+2} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^{n+r+2}}_{m=n+2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} = 0$$

$$x^r \left[\sum_{m=0}^{\infty} 2(m+r)(m+r-1)c_m x^m + \sum_{m=2}^{\infty} 2(m+r-2)c_{m-2}x^m + \sum_{m=0}^{\infty}(m+r)c_m x^m + \sum_{m=2}^{\infty} 2c_{m-2}x^m - \sum_{m=0}^{\infty} c_m x^m\right] = 0$$

$$(2r(r-1) + r - 1)\, c_0 x^0 + (2(r+1)r + (r+1) - 1)\, c_1 x^1 + \sum_{m=2}^{\infty} \left[2(m+r)(m+r-1)c_m + 2(m+r-2)c_{m-2} + (m+r)c_m + 2c_{m-2} - c_m\right] x^m = 0$$

The indicial equation $2r(r-1) + r - 1 = 0 \implies r = 1,\ r = -\dfrac{1}{2}$

For $r = 1$, setting each coefficient in the sum equal to zero gives:

$$m(2m+3)c_m + 2mc_{m-2} = 0 \implies c_m = -\frac{2c_{m-2}}{2m+3} \text{ for all } m \geq 2$$

$$\begin{aligned}
c_2 &= -\frac{2}{7}c_0 = \frac{-2}{7!!!!} \cdot 3c_0 \\
c_3 &= -\frac{2}{9}c_1 = 0 \\
c_4 &= -\frac{2}{11}c_2 = \frac{(-2)^2}{11!!!!} \cdot 3c_0 \\
c_5 &= -\frac{2}{13}c_3 = 0 \\
c_6 &= -\frac{2}{15}c_4 = \frac{(-2)^3}{15!!!!} \cdot 3c_0 \\
c_7 &= -\frac{2}{17}c_5 = 0 \\
&\vdots
\end{aligned}$$

The pattern is $c_{2k} = \dfrac{(-2)^k}{(4k+3)!!!!} \cdot 3c_0$ for $k \geq 1$ and $c_{2k+1} = 0$ for $k \geq 0$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k}x^{2k} + \sum_{k=1}^{\infty} c_{2k+1}x^{2k+1}\right) = c_0 x^1\left(1 + 3\sum_{k=1}^{\infty}\frac{(-2)^k}{(4k+3)!!!!}x^{2k}\right)$$

For $r = -\dfrac{1}{2}$, setting each coefficient in the sum equal to zero gives:

$$m(2m-3)d_m + (2m-3)d_{m-1} = 0 \implies d_m = -\frac{d_{m-2}}{m} \text{ for all } m \geq 2$$

$$\begin{aligned}
d_2 &= -\frac{1}{2}d_0 = \frac{-1}{2!!}d_0 \\
d_3 &= -\frac{2}{3}d_1 = 0 \\
d_4 &= -\frac{1}{4}d_2 = \frac{(-1)^2}{4!!}d_0 \\
d_5 &= -\frac{2}{5}d_3 = 0 \\
d_6 &= -\frac{1}{6}d_4 = \frac{(-1)^3}{6!!}d_0 \\
d_7 &= -\frac{2}{7}d_5 = 0 \\
&\vdots
\end{aligned}$$

The pattern is $d_{2k} = \dfrac{(-1)^k}{(2k)!!}d_0$ for $k \geq 1$ and $d_{2k+1} = 0$ for $k \geq 0$. The second series solution is therefore:

$$y_2 = x^r\left(d_0 + d_1 x + \sum_{k=1}^{\infty} d_{2k}x^{2k} + \sum_{k=1}^{\infty} d_{2k+1}x^{2k+1}\right) = d_0 x^{-\frac{1}{2}}\left(1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!!}x^{2k}\right) = d_0 x^{-\frac{1}{2}}e^{-\frac{x^2}{2}}$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0 x^1\left(1 + 3\sum_{k=1}^{\infty}\frac{(-2)^k}{(4k+3)!!!!}x^{2k}\right) + d_0 x^{-\frac{1}{2}}e^{-\frac{x^2}{2}}$$

Exercise 4.4.3*i*

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$3x^2y'' + (3x^4 - x)y' + (5x^3 + 1)y = 0$$

$$3x^2 \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (3x^4 - x) \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + (5x^3+1) \cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r+3}}_{m=n+3} - \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 5c_n x^{n+r+3}}_{m=n+3} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} = 0$$

$$x^r\left[\sum_{m=0}^{\infty} 3(m+r)(m+r-1)c_m x^m + \sum_{m=3}^{\infty} 3(m+r-3)c_{m-2}x^m - \sum_{m=0}^{\infty}(m+r)c_m x^m + \sum_{m=3}^{\infty} 5c_{m-3}x^m + \sum_{m=0}^{\infty} c_m x^m\right] = 0$$

$$(3r(r-1) - r + 1)\,c_0 x^0 + (3(r+1)r - (r+1) + 1)\,c_1 x^1 + (3(r+2)(r+1) - (r+2) + 1)\,c_2 x^2 +$$
$$\sum_{m=2}^{\infty}\left[3(m+r)(m+r-1)c_m + 3(m+r-2)c_{m-3} - (m+r)c_m + 5c_{m-3} + c_m\right]x^m = 0$$

The indicial equation $3r(r-1) - r + 1 = 0 \implies r = 1,\ r = \dfrac{1}{3}$

For $r = 1$, setting each coefficient in the sum equal to zero gives:

$$m(3m+2)c_m + (3m+2)c_{m-3} = 0 \implies c_m = -\frac{c_{m-3}}{m} \text{ for all } m \geq 3$$

$$c_3 = -\frac{1}{3}c_0 = \frac{-1}{3!!!} \cdot c_0$$
$$c_4 = -\frac{1}{4}c_1 = 0$$
$$c_5 = -\frac{1}{5}c_2 = 0$$
$$c_6 = -\frac{1}{6}c_3 = \frac{(-1)^2}{6!!!!}c_0$$
$$c_7 = -\frac{1}{7}c_4 = 0$$
$$c_8 = -\frac{1}{8}c_5 = 0$$
$$\vdots$$

The pattern is $c_{3k} = \dfrac{(-1)^k}{(3k)!!!}c_0$ for $k \geq 1$, and $c_{3k+1} = c_{3k+2} = 0$ for $k \geq 0$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + c_1x + c_2x^2 + \sum_{k=1}^{\infty} c_{3k}x^{3k} + \sum_{k=1}^{\infty} c_{3k+1}x^{3k+1} + \sum_{k=1}^{\infty} c_{3k+2}x^{3k+2}\right) = c_0x^1\left(1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(3k)!!!}x^{3k}\right)$$

For $r = \dfrac{1}{3}$, setting each coefficient in the sum equal to zero gives:

$$m(3m-2)d_m + 3md_{m-1} = 0 \implies d_m = -\frac{3d_{m-2}}{3m-2} \text{ for all } m \geq 3$$

$$d_3 = -\frac{1}{7}d_0 = \frac{-1}{7!!!!!!!!!} \cdot d_0$$
$$d_4 = -\frac{1}{10}d_1 = 0$$
$$d_5 = -\frac{1}{13}d_2 = 0$$
$$d_6 = -\frac{1}{16}d_3 = \frac{(-1)^2}{16!!!!!!!!!}d_0$$
$$d_7 = -\frac{1}{19}d_4 = 0$$
$$d_8 = -\frac{1}{22}d_5 = 0$$
$$\vdots$$

The pattern is $d_{3k} = \dfrac{(-1)^k}{(9k-2)!!!!!!!!!}d_0$ for $k \geq 1$, and $d_{3k+1} = d_{3k+2} = 0$ for $k \geq 0$. The second series solution is therefore:

$$y_2 = x^r\left(d_0 + d_1x + d_2x^2 + \sum_{k=1}^{\infty} d_{3k}x^{3k} + \sum_{k=1}^{\infty} d_{3k+1}x^{3k+1} + \sum_{k=1}^{\infty} d_{3k+2}x^{3k+2}\right) = d_0x^{\frac{1}{3}}\left(1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(9k-2)!!!!!!!!!}x^{3k}\right)$$

Putting these together, we see the general solution to this LDE is:

$$y = c_0x^1\left(1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(3k)!!!}x^{3k}\right) + d_0x^{\frac{1}{3}}\left(1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(9k-2)!!!!!!!!!}x^{3k}\right)$$

Exercise 4.4.4*a*

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$2xy'' + 2y' - y = 0$$

$$2x \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 2 \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} = 0$$

$$x^r\left[\sum_{m=-1}^{\infty} 2(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=-1}^{\infty} 2(m+r+1)c_{m+1}x^m - \sum_{m=0}^{\infty} c_m x^m\right] = 0$$

$$(2r(r-1)+2r)\,c_0 x^{-1} + \sum_{m=0}^{\infty}\left[2(m+r+1)(m+r)c_{m+1} + 2(m+r+1)c_{m+1} - c_m\right]x^m = 0$$

The indicial equation is $2r(r-1)+2r=0 \implies r=0$ (double)

For $r=0$, setting each coefficient in the sum equal to zero gives:

$$2(m+1)^2 c_{m+1} - c_m = 0 \implies c_{m+1} = \frac{c_m}{(m+1)^2} \text{ for all } m \geq 0$$

$$\begin{aligned} c_1 &= \frac{1}{2\cdot 1^2}c_0 \\ c_2 &= \frac{1}{2\cdot 2^2}c_1 = \frac{1}{2^2\cdot(2!)^2}c_0 \\ c_3 &= \frac{1}{2\cdot 3^2}c_2 = \frac{1}{2^3\cdot(3!)^2}c_0 \\ &\vdots \end{aligned}$$

The pattern is $c_n = \dfrac{1}{2^n\cdot(n!)^2}c_0$ for $n \geq 1$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + \sum_{n=1}^{\infty} c_n x^n\right) = c_0 x^0\left(1 + \sum_{n=1}^{\infty}\frac{1}{2^n\cdot(n!)^2}x^n\right)$$

Thankfully, this question does not ask for us to find the second solution via reduction of order.

Exercise 4.4.4*b*

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$x^2y'' + 3xy' + (1-4x^2)y = 0$$

$$x^2 \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 3x \cdot \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + (1-4x^2)\cdot\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 4c_n x^{n+r+2}}_{m=n+2} = 0$$

$$x^r\left[\sum_{m=0}^{\infty}(m+r)(m+r-1)c_m x^m + \sum_{m=0}^{\infty} 3(m+r)c_m x^m + \sum_{m=0}^{\infty} c_m x^m - \sum_{m=2}^{\infty} 4c_{m-2}x^m\right] = 0$$

$$(r(r-1)+3r+1)\,c_0 x^0 + ((r+1)r+3(r+1)+1)\,c_1 x^1$$

$$+\sum_{m=2}^{\infty}\left[(m+r)(m+r-1)c_m + 3(m+r)c_m + c_m - 4c_{m-2}\right]x^m = 0$$

The indicial equation is $r(r-1)+3r+1=0 \implies r=-1$ (double).

We see that we must have $c_1 = 0$, and setting each other coefficient in the sum equal to zero gives:

$$m^2 c_m - 4c_{m-2} = 0 \implies c_m = \frac{4c_{m-2}}{m^2} \text{ for all } m \geq 2$$

$$\begin{aligned} c_2 &= \frac{4}{2^2}c_0 = \frac{1}{(1!)^2}c_0 \\ c_3 &= \frac{1}{3^2}c_1 = 0 \\ c_4 &= \frac{4}{4^2}c_2 = \frac{1}{(2!)^2}c_0 \\ c_5 &= \frac{1}{5^2}c_3 = 0 \\ c_6 &= \frac{4}{6^2}c_4 = \frac{1}{(3!)^2}c_0 \\ c_7 &= \frac{1}{7^2}c_5 = 0 \\ &\vdots \end{aligned}$$

The pattern is $c_{2k} = \dfrac{1}{(k!)^2}c_0$ for $k \geq 1$ and $c_{2k+1} = 0$ for all k. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + c_1 x + \sum_{k=1}^{\infty} c_{2k}x^{2k} + \sum_{k=1}^{\infty} c_{2k+1}x^{2k+1}\right) = c_0 x^{-1}\left(1 + \sum_{k=1}^{\infty}\frac{1}{(k!)^2}x^{2k}\right)$$

Thankfully, this question does not ask for us to find the second solution via reduction of order.

Exercise 4.4.4*c*

Assume $y=\sum_{n=0}^{\infty} c_n x^{n+r}$:

$$x^2y''+xy'+(2x-9)y=0$$
$$x^2\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}+x\cdot\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}+(2x-9)\cdot\sum_{n=0}^{\infty}c_nx^{n+r}=0$$
$$\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r}}_{m=n}+\underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}}_{m=n}+\underbrace{\sum_{n=0}^{\infty}2c_nx^{n+r+1}}_{m=n+1}-\underbrace{\sum_{n=0}^{\infty}9c_nx^{n+r}}_{m=n}=0$$
$$x^r\left[\sum_{m=0}^{\infty}(m+r)(m+r-1)c_mx^m+\sum_{m=0}^{\infty}(m+r)c_mx^m+\sum_{m=1}^{\infty}2c_{m-1}x^m-\sum_{m=0}^{\infty}9c_{m-2}x^m\right]=0$$
$$(r(r-1)+r-9)\,c_0x^0+\sum_{m=1}^{\infty}\left[(m+r)(m+r-1)c_m+(m+r)c_m+2c_{m-1}-9c_m\right]x^m=0$$

The indicial equation is $r(r-1)+r-9=0 \implies r=\pm 3$. Using $r=3$, we set each coefficient in the sum equal to zero giving:

$$m(m+6)c_m+2c_{m-1}=0 \implies c_m=-\frac{2c_{m-1}}{m(m+6)} \text{ for all } m\geq 1$$
$$c_1=-\frac{2}{1\cdot 7}c_0=\frac{(-2)^1}{1!\cdot 7!}\cdot 720c_0$$
$$c_2=-\frac{2}{2\cdot 8}c_1=\frac{(-2)^2}{2!\cdot 8!}\cdot 720c_0$$
$$c_3=-\frac{2}{3\cdot 9}c_2=\frac{(-2)^3}{3!\cdot 9!}\cdot 720c_0$$
$$\vdots$$

The pattern is $c_n=\frac{(-2)^n}{n!\cdot(n+6)!}\cdot 720c_0$ for $n\geq 1$. The first series solution is therefore:

$$y_1=x^r\left(c_0+\sum_{n=1}^{\infty}c_nx^n\right)=c_0x^3\left(1+720\sum_{n=1}^{\infty}\frac{(-2)^n}{n!(n+6)!}x^n\right)$$

Exercise 4.4.4*d*

Assume $y=\sum_{n=0}^{\infty} c_n x^{n+r}$:

$$xy''+(x^2+1)y'+2xy=0$$
$$x\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}+(x^2+1)\cdot\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}+2x\cdot\sum_{n=0}^{\infty}c_nx^{n+r}=0$$
$$\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}}_{m=n-1}+\underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r+1}}_{m=n+1}+\underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}}_{m=n-1}+\underbrace{\sum_{n=0}^{\infty}2c_nx^{n+r+1}}_{m=n+1}=0$$
$$x^r\left[\sum_{m=-1}^{\infty}(m+r+1)(m+r)c_{m+1}x^m+\sum_{m=1}^{\infty}(m+r-1)c_{m-1}x^m+\sum_{m=-1}^{\infty}(m+r+1)c_{m+1}x^m+\sum_{m=1}^{\infty}2c_{m-1}x^m\right]=0$$
$$(r(r-1)+r)\,c_0x^{-1}+((r+1)r+(r+1))\,c_1x^0$$
$$+\sum_{m=1}^{\infty}\left[(m+r+1)(m+r+1)c_{m+1}+((m+r-1)+2)\,c_{m-1}\right]x^m=0$$

The indicial equation is $r(r-1)+r=0 \implies r=0$ (double). With $r=0$ and setting each coefficient in the sum equal to zero:

$$(m+1)^2c_{m+1}+(m+1)c_{m-1}=0 \implies c_{m+1}=-\frac{c_{m-1}}{m+1} \text{ for all } m\geq 1$$
$$c_2=-\frac{1}{2}c_0$$
$$c_3=-\frac{1}{3}c_1=0$$
$$c_4=-\frac{1}{4}c_2=\frac{(-1)^2}{4!!}c_0$$
$$c_5=\frac{1}{5}c_3=0$$
$$c_6=-\frac{1}{6}c_4=\frac{(-1)^3}{6!!}c_0$$
$$c_7=\frac{1}{7}c_5=0$$
$$\vdots$$

The pattern is $c_{2k}=\frac{(-1)^k}{(2k)\,!!}c_0$ for $k\geq 1$ and $c_{2k+1}=0$ for all k. The first series solution is therefore:

$$y_1=x^r\left(c_0+c_1x+\sum_{k=1}^{\infty}c_{2k}x^{2k}+\sum_{k=1}^{\infty}c_{2k+1}x^{2k+1}\right)=c_0x^0\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!!}x^{2k}\right)$$

This can be rewritten, using $(2k)!!=2^k\cdot k!$, so

$$y_1=c_0x^0\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!!}x^{2k}\right)=c_0\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{x^2}{2}\right)^k\right)=c_0e^{-\frac{x^2}{2}}$$

This means reduction of order might be reasonable! Consider:

$$u' = \frac{1}{[f(x)]^2} e^{-\int p(x)\,dx} = \frac{1}{\left[c_0 e^{-\frac{x^2}{2}}\right]^2} e^{-\int \frac{x^2+1}{x}\,dx} = \frac{e^{x^2}}{c_0^2} e^{-\frac{1}{2}x^2 - \ln(x)} = Cx^{-1}e^{\frac{x^2}{2}}$$

Thus, $u = \int u'\,dx = \int Cx^{-1}e^{\frac{x^2}{2}}\,dx$. This is not an integral we can express with a nice form, but it explains why our power series answer would look so terrible if we were asked for the second solution.

Exercise $4.4.4e$

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$4xy'' + (8x^3+4)y' + (20x^2-1)y = 0$$
$$4x \cdot \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (8x^3+4)\cdot\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + (20x^2-1)\cdot\sum_{n=0}^{\infty}c_n x^{n+r} = 0$$
$$\underbrace{\sum_{n=0}^{\infty}4(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty}8(n+r)c_n x^{n+r+2}}_{m=n+2} + \underbrace{\sum_{n=0}^{\infty}4(n+r)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty}2c_n x^{n+r+2}}_{m=n+2} - \underbrace{\sum_{n=0}^{\infty}20c_n x^{n+r}}_{m=n} = 0$$
$$x^r\left[\sum_{m=-1}^{\infty}4(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=1}^{\infty}8(m+r-2)c_{m-2}x^m + \sum_{m=-1}^{\infty}4(m+r+1)c_{m+1}x^m + \sum_{m=2}^{\infty}20c_{m-2}x^m - \sum_{m=0}^{\infty}c_m x^m\right] = 0$$
$$(4r(r-1)+4r-1)\,c_0x^{-1} + (4(r+1)r+4(r+1)-1)\,c_1x^0 + + (4(r+2)r+4(r+2)-1)\,c_2x^1$$
$$+\sum_{m=1}^{\infty}\left[(4(m+r+1)(m+r+1)-1)c_{m+1} + (8(m+r-2)+20)\,c_{m-1}\right]x^m = 0$$

The indicial equation is $4r(r-1)+4r-1=0 \implies r = \pm\frac{1}{2}$. Using $r=\frac{1}{2}$, we set each coefficient in the sum equal to zero:

$$4(m+1)(m+2)c_{m+1} + 8(m+1)c_{m-1} = 0 \implies c_{m+1} = -\frac{2c_{m-2}}{(m+2)} \text{ for all } m \geq 0$$

$$\begin{aligned}
c_3 &= -\frac{2}{4}c_0\\
c_4 &= -\frac{2}{5}c_1 = 0\\
c_5 &= -\frac{2}{6}c_2 = 0\\
c_6 &= -\frac{2}{7}c_3 = \frac{(-2)^2}{7!!!}c_0\\
c_7 &= -\frac{2}{8}c_3 = 0\\
c_8 &= -\frac{2}{9}c_5 = 0\\
c_9 &= -\frac{2}{10}c_6 = \frac{(-2)^3}{10!!!}c_0\\
c_{10} &= \frac{2}{11}c_7 = 0\\
c_{11} &= -\frac{2}{12}c_8 = 0\\
&\vdots
\end{aligned}$$

The pattern is $c_{3k} = \frac{(-2)^k}{(3k+1)\,!!!}c_0$ for $k \geq 1$, and $c_{3k+1} = c_{3k+2} = 0$ for all k. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + c_1x + c_2x^2 + \sum_{k=1}^{\infty}c_{3k}x^{3k} + \sum_{k=1}^{\infty}c_{3k+1}x^{3k+1} + \sum_{k=1}^{\infty}c_{3k+2}x^{3k+2}\right) = c_0x^{\frac{1}{2}}\left(1 + \sum_{k=1}^{\infty}\frac{(-2)^k}{(3k+1)!!!}x^{3k}\right)$$

Thankfully, this question does not ask for us to find the second solution via reduction of order.

Exercise $4.4.5a$

Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$xy'' + 2y' - xy = 0$$
$$x\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 2\cdot\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - x\cdot\sum_{n=0}^{\infty}c_n x^{n+r} = 0$$
$$\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty}2(n+r)c_n x^{n+r-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty}c_n x^{n+r+1}}_{m=n+1} = 0$$
$$x^r\left[\sum_{m=-1}^{\infty}(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=-1}^{\infty}2(m+r+1)c_{m+1}x^m - \sum_{m=1}^{\infty}c_{m-1}x^m\right] = 0$$
$$(r(r-1)+2r)\,c_0x^{-1} + ((r+1)r+2(r+1))\,c_1x^0 + \sum_{m=1}^{\infty}\left[(m+r)(m+r+1)c_{m+1} + 2(m+r+1)c_{m+1} - c_{m-1}\right]x^m = 0$$

The indicial equation is $r(r-1)+2r=0 \implies r=-1,\ r=0$

For $r=-1$, setting each coefficient in the sum equal to zero gives:

$$(m+r)(m+r+1)c_{m+1} + 2(m+r+1)c_{m+1} - c_{m-1} = 0 \implies c_{m+1} = \frac{1}{m(m+1)}c_{m-1} \text{ for all } m \geq 1$$

Clearly, we must have $c_1 = 0$, and if we plug in a few more values of m:

$$
\begin{aligned}
c_2 &= \frac{1}{1\cdot 2}c_0 = \frac{1}{2!}c_0\\
c_3 &= \frac{1}{2\cdot 3}c_1 = 0\\
c_4 &= \frac{1}{3\cdot 4}c_2 = \frac{1}{4!}c_0\\
c_5 &= \frac{1}{4\cdot 5}c_3 = 0\\
c_6 &= \frac{1}{5\cdot 6}c_4 = \frac{1}{6!}c_0\\
&\vdots
\end{aligned}
$$

The pattern is $c_{2k} = \dfrac{1}{(2k)!}c_0$ for $k \geq 1$ and $c_{2k+1} = 0$ for all k. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1} + \sum_{k=1}^{\infty} c_{2k}x^{2k}\right) = c_0x^{-1}\left(1 + \sum_{k=1}^{\infty}\frac{1}{(2k)!}x^{2k}\right) = c_0x^{-1}\cosh(x)$$

Since the second root is off by an integer, we know it will yield the same solution. Instead, we use reduction of order.
The original equation can be written as $y'' + \dfrac{2}{x}y' - y = 0$, so $p(x) = \dfrac{2}{x}$:

$$u' = \frac{1}{[f(x)]^2}e^{-\int p(x)\,dx} = \frac{1}{[c_0x^{-1}\cosh(x)]^2}e^{-\int \frac{2}{x}\,dx} = \frac{x^2}{c_0^2\cosh^2(x)}e^{-2\ln(x)} = C\operatorname{sech}^2(x)$$

Thus, $u = \displaystyle\int u'\,dx = C\tanh(x) + D$. We plug this into $y = u\cdot f$:

$$y(x) = (C\tanh(x) + D)\cdot x^{-1}\cosh(x) = Cx^{-1}\sinh(x) + Dx^{-1}\cosh(x)$$

Exercise 4.4.5*b*

Assume $y = \displaystyle\sum_{n=0}^{\infty} c_nx^{n+r}$:

$$
\begin{aligned}
(x^2-x)y'' + (x+1)y' - y &= 0\\
(x^2-x)\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2} + (x+1)\cdot\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} - \sum_{n=0}^{\infty}c_nx^{n+r} &= 0\\
\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty}c_nx^{n+r}}_{m=n} &= 0\\
x^r\left[\sum_{m=0}^{\infty}(m+r)(m+r-1)c_mx^m - \sum_{m=-1}^{\infty}(m+r+1)(m+r)c_{m+1}x^m + \sum_{m=0}^{\infty}(m+r)c_mx^m + \sum_{m=-1}^{\infty}(m+r+1)c_{m+1}x^m - \sum_{m=0}^{\infty}c_mx^m\right] &= 0\\
(-r(r-1)+r)\,c_0x^{-1} + \sum_{m=0}^{\infty}\left[\left((m+r)^2-1\right)c_m + \left(1-(m+r)^2\right)c_{m+1}\right]x^m &= 0
\end{aligned}
$$

The indicial equation is $-r(r-1)+r = 0 \implies r = 0, 2$. Using $r = 2$, we set each coefficient in the sum equal to zero:

$$(m+1)(m+3)c_m - (m+1)(m+3)c_{m+1} = 0 \implies c_{m+1} = c_m \text{ for all } m \geq 0$$

We know $c_0 = c_1 = c_2 = \cdots \implies c_n = c_0$ for all $n \geq 1$. The first series solution is therefore:

$$y_1 = x^r\left(c_0 + \sum_{n=1}^{\infty}c_nx^n\right) = c_0x^2\left(1 + \sum_{n=1}^{\infty}x^n\right) = \frac{c_0x^2}{1-x}$$

The original equation can be written as $y'' + \dfrac{x+1}{x(x-1)}y' + \dfrac{1}{x(x-1)}y = 0$, so $p(x) = \dfrac{x+1}{x(x-1)} = \dfrac{2}{x-1} - \dfrac{1}{x}$. Therefore

$$u' = \frac{1}{[f(x)]^2}e^{-\int p(x)\,dx} = \frac{1}{\left(\frac{c_0x^2}{1-x}\right)^2}e^{\int \frac{1}{x} - \frac{2}{x-1}\,dx} = \frac{(1-x)^2}{c_0^2x^4}e^{\ln(x)-2\ln(x-1)} = \frac{(1-x)^2}{c_0^2x^4}\cdot\frac{x}{(x-1)^2} = \frac{1}{c_0^2x^3}$$

So, the reduction of order formula says that: $u = \displaystyle\int u'\,dx = \frac{C}{x^2} + D$, and our final answer is:

$$y = u\cdot f = \left(\frac{C}{x^2} + D\right)\cdot\frac{x^2}{1-x} = \frac{C + Dx2}{1-x}$$

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4.5: Approximating Solutions

Learning Objectives

- Understand when and why numerical methods are necessary for solving differential equations.
- Implement Euler's method and analyze its error behavior.
- Apply the Runge-Kutta method (particularly RK4) for improved accuracy.
- Estimate errors and understand convergence properties of numerical methods.
- Apply these methods to systems of differential equations.

For equations that cannot be solved analytically, we turn to numerical approximation. This subchapter introduces fundamental algorithms like Euler's method and the highly accurate fourth-order Runge-Kutta method (RK4), providing the tools to estimate solutions for virtually any initial value problem.

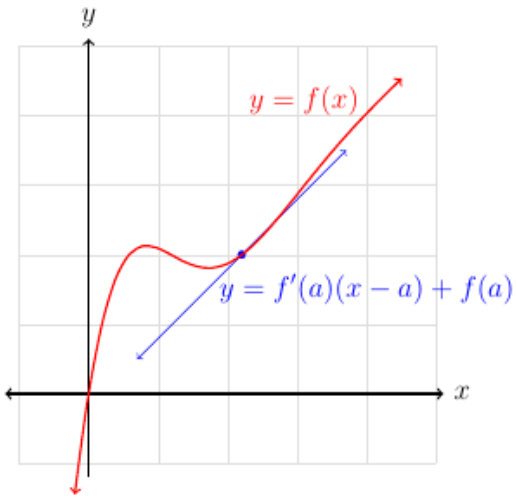
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4.5: Euler's Method

As we have seen, there are many differential equations which have solutions we cannot express in closed form (in terms of nice elementary functions).

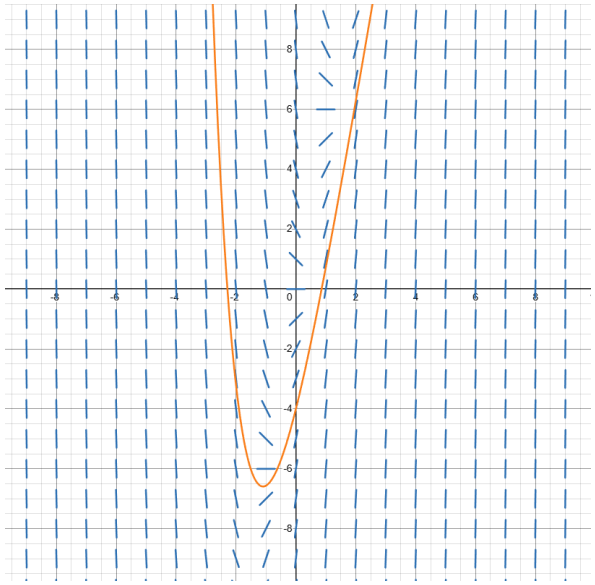
Finding the power series representation of such a solution function gives a sequence of polynomials which converge to the solution.
In the most basic sense, this means that we can approximate our solution using its Taylor polynomial.

As in a calculus course, the most straightforward example of this is in linearization. A single variable function $f(x)$ "acts like" its tangent line for values of x near to $(a, f(a))$ where it is tangent.



If we simply treat every solution as a line, we have a very crude (brute force) method of approximating these solutions.

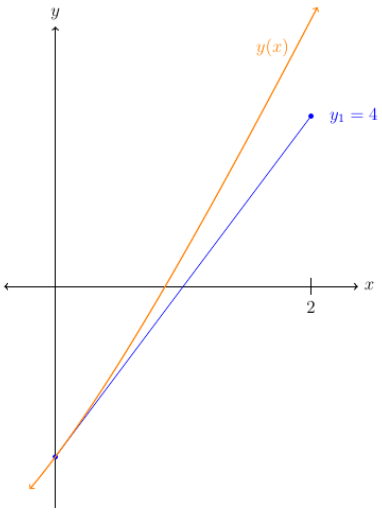
Consider the example $y' = 6x - y$ with $y(0) = -4$. If we pretend that we cannot solve this LDE, then we can visualize its slope field and possible solution:



As nice as this picture is, we cannot recover an explicit formula from it. What if we need to know the value of $y(2)$?

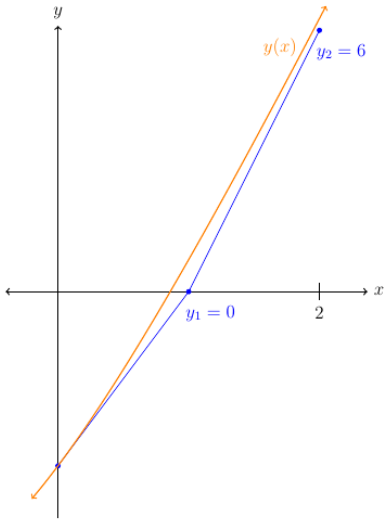
If we use straight lines, then the real question is just how lazy do we want to be?

Case 1: (One line) the best approximation of y starts at $(0,-4)$ with slope $y' = 6 \cdot 0 - (-4) = 4$, so it is the line $y = 4x - 4$



Thus, $y(2) \approx 4 \cdot 2 - 4 = 4$

Case 2: (Two lines) we go halfway (to $x = 1$), then re-assess with a new line for the second half.
The first line is still $y_1 = 4x - 4$, and at $x = 1$, we have $y_1(1) = 0$. At that point the slope is $y' = 6 \cdot 1 - 0 = 6$, so it is the line $y = 6x - 6$



This second line ends at $y(2) \approx 6 \cdot 2 - 6 = 6$

Case 3: (n lines) we split the interval $[a,b]$ into n pieces of equal length.
Just like with integration, we split the interval into n subintervals $[x_{k-1}, x_k]$ each of width $h = \dfrac{b-a}{n}$

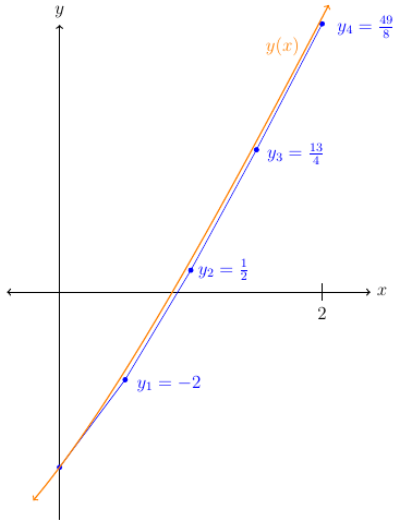
We start at the point $(x_0, y_0) = (0,-4)$ and use the first line $y_1(x) = (6x_0 - y_0)(x - x_0) + y_0$

This gives us the y value of the second point $y_1 = (6x_0 - y_0)h + y_0$

Now we are at the point (x_1, y_1) and use the new line $y_2(x) = (6x_1 - y_1)(x - x_1) + y_1$
This gives us the y value of the third point $y_2 = (6x_1 - y_1)h + y_1$

We continue this process n times until we reach x_n and y_n (our estimated point.)

For reference, here is the sequence of points for $y' = 6x - y$ with $y(0) = -4$ using $n = 4$ steps:



We know the solution to this IVP is $y = 6x - 6 + 2e^{-x}$, so $y(2) = 6 + 2e^{-2} \approx 6.271\ldots$

This process of iterated approximations via lines is called **Euler's Method**.

Definition: Euler's Method

To approximate the solution to $y' = f(x, y)$ with $y(x_0) = y_0$ at $y(b)$ using n steps, define $h = \dfrac{b - x_0}{n}$, and set

$$y_{k+1} = y_k + f(x_k, y_k) \cdot h \text{ for } k = 0, 1, 2, \ldots, n - 1$$

Illustration 4.5.1

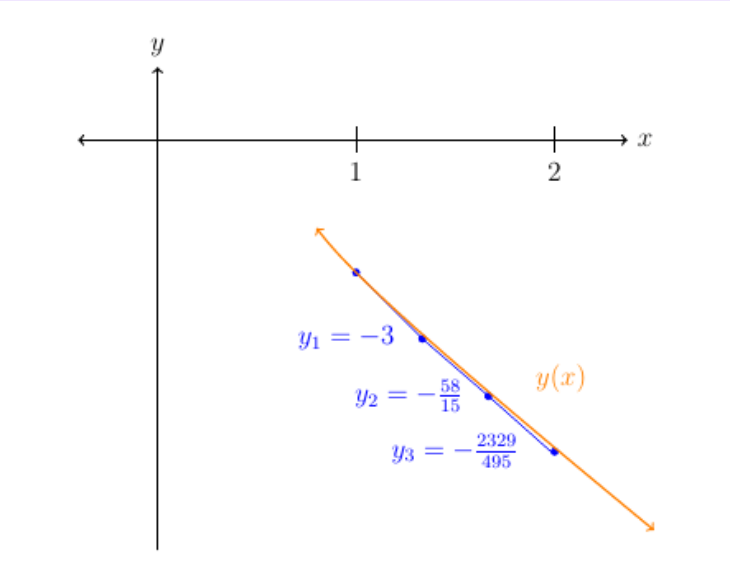
We will use 3 steps to approximate the value at $x = 2$ of the solution to $y' = \dfrac{x - y}{x + y}$ through $(1, -2)$

We have $f(x, y) = \dfrac{x - y}{x + y}$, and start at $x_0 = 1, y_0 = -2$.

The value of $n = 3$ steps tells us that $h = \dfrac{b - a}{n} = \dfrac{2 - 1}{3} = \dfrac{1}{3}$.

$$(x_0, y_0) = (1, -2) \implies f(x_0, y_0) = -3$$
$$y_1 = y_0 + f(x_0, y_0) \cdot h = -2 - 3 \cdot \frac{1}{3} = -3$$
$$(x_1, y_1) = \left(\frac{4}{3}, -3\right) \implies f(x_1, y_1) = -\frac{13}{5}$$
$$y_2 = y_1 + f(x_1, y_1) \cdot h = -3 - \frac{13}{5} \cdot \frac{1}{3} = -\frac{58}{15}$$
$$(x_2, y_2) = \left(\frac{5}{3}, -\frac{58}{15}\right) \implies f(x_2, y_2) = -\frac{83}{33}$$
$$y_3 = y_2 + f(x_2, y_2) \cdot h = -\frac{58}{15} - \frac{83}{33} \cdot \frac{1}{3} = -\frac{2329}{495}$$

We end at $(x_3, y_3) = \left(2, -\frac{2329}{495}\right) \approx (2, -4.705\ldots)$



This is relatively close to the real solution, which is $y = -x - \sqrt{2x^2 - 1} \implies y(2) = -2 - \sqrt{7} = -4.64575$

This process is straightforward, and easy to program a computer to do. Unfortunately, it is not incredibly accurate if your differential equation has large changes in slope over relatively small intervals of x. This also means that if you want to approximate "far away" from the initial value provided, you would need to take a very large number of steps. We can definitely do better.

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4.5: Runge-Kutta Approximation

To gain a more accurate approximation, it is tempting to look into using a higher order Taylor Polynomial to approximate the unknown function in Euler's Method. This is intuitive, but computationally intensive. Instead, we can split each interval up into smaller subintervals to increase our accuracy.

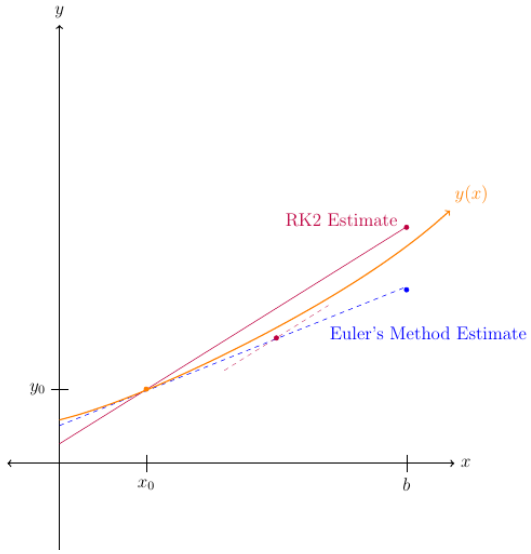
Definition: Runge-Kutta 2 Algorithm (Midpoint Method)

Assume $y(x)$ is the solution to the first order differential equation $y' = f(x, y)$ with $y(x_0) = y_0$

To approximate $y(b)$ using n steps, set $h = \dfrac{b - x_0}{n}$, and on the jth iteration use:

$$y_{j+1} = y_j + h \cdot f\left(x_j + \frac{h}{2}, y_j + \frac{h}{2} f\left(x_j, y_j\right)\right) \text{ for each } j = 0, 1, \ldots, n-1$$

This takes each subinterval and replaces the straight line from Euler's Method (which has slope determined by the left endpoint of the subinterval), with the line whose slope determined by the "predicted average" location of the point.



In practice, this gives us a slightly more accurate approximation than Euler's Method (when using the same number of steps).

To keep things organized, it is common to call the first slope $k_1 = f(x_j, y_j)$ and the second slope $k_2 = f\left(x_j + \dfrac{h}{2}, y_j + \dfrac{h}{2} k_1\right)$

Thus, we estimate $y_{j+1} = y_j + hk_2$

Example 4.5.1: RK2

Consider the example $y' = 6x - y$ with $y(0) = -4$. Find the RK2 approximation of $y(2)$ using

(a) one subinterval

(b) two subintervals

(c) four subintervals

Solution

(a) $n = 1 \implies h = \dfrac{b-a}{n} = \dfrac{2-0}{1} = 2$:

$$(x_0, y_0) = (0, -4) \implies k_1 = f(x_0, y_0) = 6 \cdot 0 - (-4) = 4$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + 1, -4 + 1 \cdot 4\right) = f(1, 0) = 6 \cdot 1 - 0 = 6$$
$$y_1 = y_0 + h \cdot k_2 = -4 + 2 \cdot 6 = 8$$

Thus, we approximate $\boxed{y(2) \approx 8}$

(b) $n = 2 \implies h = \dfrac{b-a}{n} = \dfrac{2-0}{2} = 1$:

$$(x_0, y_0) = (0, -4) \implies k_1 = f(x_0, y_0) = 6 \cdot 0 - (-4) = 4$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + \frac{1}{2}, -4 + \frac{1}{2} \cdot 4\right) = f\left(\frac{1}{2}, -2\right) = 6 \cdot \frac{1}{2} - (-2) = 5$$
$$y_1 = y_0 + h \cdot k_2 = -4 + 1 \cdot 5 = 1$$
$$(x_1, y_1) = (1, 1) \implies k_1 = f(x_1, y_1) = 6 \cdot 1 - 1 = 5$$
$$f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_1\right) = f\left(1 + \frac{1}{2}, 1 + \frac{1}{2} \cdot 5\right) = f\left(\frac{3}{2}, \frac{7}{2}\right) = 6 \cdot \frac{3}{2} - \frac{7}{2} = \frac{11}{2}$$
$$y_2 = y_1 + h \cdot k_2 = 1 + 1 \cdot \frac{11}{2} = \frac{13}{2}$$

Thus, we approximate $\boxed{y(2) \approx \dfrac{13}{2}}$

(c) $n = 4 \implies h = \dfrac{b-a}{n} = \dfrac{2-0}{4} = \dfrac{1}{2}$:

$$(x_0, y_0) = (0, -4) \implies k_1 = f(x_0, y_0) = 6 \cdot 0 - (-4) = 4$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + \frac{1}{4}, -4 + \frac{1}{4} \cdot 4\right) = f\left(\frac{1}{4}, -3\right) = 6 \cdot \frac{1}{4} - (-3) = \frac{9}{2}$$
$$y_1 = y_0 + h \cdot k_2 = -4 + \frac{1}{2} \cdot \frac{9}{2} = -\frac{7}{4}$$
$$(x_1, y_1) = \left(\frac{1}{2}, -\frac{7}{4}\right) \implies k_1 = f(x_1, y_1) = 6 \cdot \frac{1}{2} - \left(-\frac{7}{4}\right) = \frac{19}{4}$$
$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_1\right) = f\left(\frac{1}{2} + \frac{1}{4}, -\frac{7}{4} + \frac{1}{4} \cdot \frac{19}{4}\right) = f\left(\frac{3}{4}, -\frac{9}{16}\right) = 6 \cdot \frac{3}{4} - \left(-\frac{9}{16}\right) = \frac{81}{16}$$
$$y_2 = y_1 + h \cdot k_2 = -\frac{7}{4} + \frac{1}{2} \cdot \frac{81}{16} = \frac{25}{32}$$
$$(x_2, y_2) = \left(1, \frac{25}{32}\right) \implies k_1 = f(x_2, y_2) = 6 \cdot 1 - \frac{25}{32} = \frac{167}{32}$$
$$k_2 = f\left(x_2 + \frac{h}{2}, y_2 + \frac{h}{2} \cdot k_1\right) = f\left(1 + \frac{1}{4}, \frac{25}{32} + \frac{1}{4} \cdot \frac{167}{32}\right) = f\left(\frac{5}{4}, \frac{267}{128}\right) = 6 \cdot \frac{5}{4} - \frac{267}{128} = \frac{693}{128}$$
$$y_3 = y_2 + h \cdot k_2 = \frac{25}{32} + \frac{1}{2} \cdot \frac{693}{128} = \frac{893}{256}$$
$$(x_3, y_3) = \left(\frac{3}{2}, \frac{893}{256}\right) \implies k_1 = f(x_3, y_3) = 6 \cdot \frac{3}{2} - \frac{893}{256} = \frac{1411}{256}$$
$$k_2 = f\left(x_3 + \frac{h}{2}, y_3 + \frac{h}{2} \cdot k_1\right) = f\left(\frac{3}{2} + \frac{1}{4}, \frac{893}{256} + \frac{1}{4} \cdot \frac{1411}{256}\right) = f\left(\frac{7}{4}, \frac{4983}{1024}\right) = 6 \cdot \frac{7}{4} - \frac{4983}{1024} = \frac{5769}{1024}$$
$$y_4 = y_3 + h \cdot k_2 = \frac{893}{256} + \frac{1}{2} \cdot \frac{5769}{1024} = \frac{12913}{2048}$$

Thus, we approximate $\boxed{y(2) \approx \dfrac{12913}{2048} = 6.305}$

We can compare the results above to the Euler Method approximations using the same values of n to see that they are each indeed closer to the real value of

$$y(2) = 6 + 2e^{-2} \approx 6.271\ldots$$

As always, if you are willing to do more work you can get more accuracy. In the next algorithm, we will divide each subinterval up into four pieces instead of two.

Definition: Runge-Kutta 4 Algorithm

Assume $y(x)$ is the solution to the first order differential equation $y' = f(x, y)$ with $y(x_0) = y_0$

To approximate $y(b)$ using n steps, set $h = \dfrac{b - x_0}{n}$, and on the jth iteration use:

$$\begin{aligned} k_1 &= f(x_j, y_j) \\ k_2 &= f\left(x_j + \frac{h}{2}, y_j + \frac{h}{2} \cdot k_1\right) \\ k_3 &= f\left(x_j + \frac{h}{2}, y_j + \frac{h}{2} \cdot k_2\right) \\ k_4 &= f\left(x_j + h, y_j + h \cdot k_3\right) \\ y_{j+1} &= y_j + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \end{aligned}$$

This is the more complicated way to "sample" where the expected average slope should be during the interval $[x_j, x_{j+1}]$. It greatly improves accuracy, but obviously there is much more work done during each step.

Example 4.5.2: RK4

Consider the example $y' = 6x - y$ with $y(0) = -4$. Find the RK4 approximation of $y(2)$ using

(a) one subinterval

(b) two subintervals

Solution

(a) $n = 1 \implies h = \dfrac{b-a}{n} = \dfrac{2-0}{1} = 2$:

$$\begin{aligned} k_1 &= f(x_0, y_0) = f(0, -4) = 6 \cdot 0 - (-4) = 4 \\ k_2 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f(0 + 1, -4 + 4) = 6 \cdot 1 - 0 = 6 \\ k_3 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) = f(0 + 1, -4 + 6) = 6 \cdot 1 - 2 = 4 \\ k_4 &= f(x_0 + h, y_0 + h \cdot k_3) = f(0 + 2, -4 + 8) = 6 \cdot 2 - 4 = 8 \\ y_1 &= y_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = -4 + \frac{1}{3}\left(4 + 2 \cdot 6 + 2 \cdot 4 + 8\right) = -4 + \frac{32}{3} = \frac{20}{3} \end{aligned}$$

Thus, we approximate $\boxed{y(2) \approx \dfrac{20}{3} = 6.666\ldots}$

(b) $n = 2 \implies h = \dfrac{b-a}{n} = \dfrac{2-0}{2} = 1$:

$$k_1 = f(x_0, y_0) = f(0, -4) = 6 \cdot 0 - (-4) = 4$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + \frac{1}{2}, -4 + 2\right) = 6 \cdot \frac{1}{2} - (-2) = 5$$
$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) = f\left(0 + \frac{1}{2}, -4 + \frac{5}{2}\right) = 6 \cdot \frac{1}{2} - \left(-\frac{3}{2}\right) = \frac{9}{2}$$
$$k_4 = f(x_0 + h, y_0 + h \cdot k_3) = f\left(0 + 1, -4 + \frac{9}{2}\right) = 6 \cdot 1 - \frac{1}{2} = \frac{11}{2}$$
$$y_1 = y_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = -4 + \frac{1}{6}\left(4 + 2 \cdot 5 + 2 \cdot \frac{9}{2} + \frac{11}{2}\right) = -4 + \frac{19}{4} = \frac{3}{4}$$
$$k_1 = f(x_1, y_1) = f\left(1, \frac{3}{4}\right) = 6 \cdot 1 - \frac{3}{4} = \frac{21}{4}$$
$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_1\right) = f\left(1 + \frac{1}{2}, \frac{3}{4} + \frac{21}{8}\right) = 6 \cdot \frac{3}{2} - \frac{27}{8} = \frac{45}{8}$$
$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_2\right) = f\left(1 + \frac{1}{2}, \frac{3}{4} + \frac{45}{16}\right) = 6 \cdot \frac{3}{2} - \frac{57}{16} = \frac{87}{16}$$
$$k_4 = f(x_1 + h, y_1 + h \cdot k_3) = f\left(1 + 1, \frac{3}{4} + \frac{87}{16}\right) = 6 \cdot 2 - \frac{99}{16} = \frac{93}{16}$$
$$y_2 = y_1 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \frac{3}{4} + \frac{1}{6}\left(\frac{21}{4} + 2 \cdot \frac{45}{8} + 2 \cdot \frac{87}{16} + \frac{93}{16}\right) = \frac{3}{4} + \frac{177}{32} = \frac{201}{32}$$

Thus, we approximate $\boxed{y(2) \approx \frac{201}{32} = 6.2812\ldots}$

As predicted, this is definitely much closer (and time intensive) than Euler's Method or RK2!

So how do we know which method to use? That depends on what you are trying to estimate, and how accurate you need to be. If you need a simple quick estimate, use Euler's Method. (just realize that it is the least accurate approximation method we have). If you need a lot of accuracy, use RK 4 (the computer does all of the work anyway for these types of situations). If you want medium accuracy, RK2 is much easier to iterate than RK4. Remember in all of these methods, you can increase accuracy by increasing the number of steps (decreasing h).

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4.5: Power Series Approximation

We have seen the analytic solutions to higher order LDEs can be found via power series manipulation. Since power series can be considered the limit of a sequence of functions, we should be able to estimate arbitrarily close to the exact value of $y(b)$ if we use "enough terms". This gives us a fantastic jumping-off point for estimating solutions to these higher order LDEs.

Illustration 4.5.2

We demonstrate this method on a simple first order example by estimating $y(.1)$ where y is the solution to the IVP $(1-x)y' - y = 0$ with $y(0) = 1$

We solve for the powers series $y = \sum_{n=0}^{\infty} c_n x^n$ satisfying this LDE:

$$\begin{aligned}
(1-x)y' - y &= 0 \\
\sum_{n=0}^{\infty} n c_n x^{n-1} - x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty} n c_n x^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{m=n} &= 0 \\
\sum_{m=-1}^{\infty} (m+1) c_{m+1} x^m - \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\
\left(0 c_0 x^{-1} + \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m \right) - \sum_{m=0}^{\infty} m c_m x^m - \sum_{m=0}^{\infty} c_m x^m &= 0 \\
\sum_{m=0}^{\infty} \left[(m+1) c_{m+1} - m c_m - c_m \right] x^m &= 0
\end{aligned}$$

Setting each coefficient equal to zero gives us

$$(m+1)c_{m+1} - m c_m - c_m = 0 \implies c_{m+1} = c_m \text{ for all } m \geq 0$$

We are given $c_0 = y(0) = 1$, so this gives us the pattern as $c_n = 1$ for all $n \geq 0$ and general solution

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n$$

Pretending that we do not recognize this function as $\dfrac{1}{1-x}$, we can use this sum to estimate $y(.1)$:

$$y(.1) = \sum_{n=0}^{\infty} (.1)^n \approx .1 + (.1)^2 + (.1)^3 + \cdots + (.1)^N \text{ for large values of } N$$

Depending on N (the highest power of the terms in the polynomial you use to approximate the series), this estimate looks different:

$$\begin{aligned}
\text{Exact Value}: \quad & y(.1) = \frac{1}{1-.1} = \frac{10}{9} = 1.111111\ldots \\
N = 0: \quad & y(.1) \approx 1 \\
N = 1: \quad & y(.1) \approx 1.1 \\
N = 2: \quad & y(.1) \approx 1.11 \\
N = 3: \quad & y(.1) \approx 1.111 \\
& \vdots
\end{aligned}$$

We can see that if you wanted an estimate accurate to m digits in this example, you would need to use $N = m$ as your largest power. This is because the terms with higher powers of $(.1)^n$ will only change digits after the mth digit.

In general, we will be given an IVP and want to estimate $y(b)$ within ϵ (which is some tiny positive number that is an acceptable margin of error) by using a power series centered at $x = a$. Advanced analysis and Taylor's Theorem tell us that a good rule of "when to stop" is to find when $\left|c_{N+1}(b-a)^{N+1}\right| < \epsilon$, since that ensures that the very next term in your power series will not change your estimate by more than ϵ. Please note this is not a strict rule, as it is possible that the next two, three, or ten terms add up to change your estimate by more than ϵ, so it is a good idea to check a few extra terms to be safe in practice.

Example 4.5.3

Estimate $y(-.01)$ within one billionth, where y satisfies the IVP $y'' - 4xy' + (4x^2 - 2)y = 0$ with $y(0) = 0,\ y'(0) = 1$

Solution

First, we assume $y = \sum_{n=0}^{\infty} c_n x^n$ and solve the LDE:

$$\begin{aligned}
y'' - 4xy' + (4x^2 - 2)y &= 0 \\
\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 4x \cdot \sum_{n=0}^{\infty} n c_n x^{n-1} + 4x^2 \cdot \sum_{n=0}^{\infty} c_n x^n - 2 \cdot \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} 4n c_n x^n}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 4c_n x^{n+2}}_{m=n+2} - \underbrace{\sum_{n=0}^{\infty} 2c_n x^n}_{m=n} &= 0 \\
\sum_{m=-2}^{\infty} (m+2)(m+1)c_{m+2}x^m - \sum_{m=0}^{\infty} 4mc_m x^m + \sum_{m=2}^{\infty} 4c_{m-2}x^m - \sum_{m=0}^{\infty} 2c_m x^m &= 0 \\
\left(0c_0x^{-2} + 0c_1x^{-1} + 2c_2x^0 + 6c_3x^1 + \sum_{m=2}^{\infty}(m+2)(m+1)c_{m+2}x^m\right) - \left(0c_0x^0 + 4c_1x^1 + \sum_{m=2}^{\infty} 4mc_m x^m\right) & \\
+ \sum_{m=2}^{\infty} 4c_{m-2}x^m - \left(2c_0x^0 + 2c_1x^1 + \sum_{m=2}^{\infty} 2c_m x^m\right) &= 0 \\
(2c_2 - 2c_0) + (6c_3 - 6c_1)x + \sum_{m=2}^{\infty} \left[(m+2)(m+1)c_{m+2} - 4mc_m + 4c_{m-2} - 2c_m\right] x^m &= 0
\end{aligned}$$

The initial conditions tell us that $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$

Setting the coefficient of x^0 equal to zero gives $2c_2 - 2c_0 = 0 \implies c_2 = c_0 = 0$

Setting the coefficient of x^1 equal to zero gives $6c_3 - 6c_1 = 0 \implies c_3 = c_1 = 1$

Setting the remaining coefficients in the sum equal to zero gives:

$$(m+2)(m+1)c_{m+2} - 4mc_m + 4c_{m-2} - 2c_m = 0 \implies c_{m+2} = \frac{4m+2}{(m+2)(m+1)}c_m - \frac{4}{(m+2)(m+1)}c_{m-2}$$

Plugging in a few more values of m we see:

$$\begin{aligned}
c_4 &= \frac{10}{4 \cdot 3}c_2 - \frac{4}{4 \cdot 3}c_0 = 0 \\
c_5 &= \frac{14}{5 \cdot 4}c_3 - \frac{4}{5 \cdot 4}c_1 = \frac{1}{2} \\
&\vdots
\end{aligned}$$

Since $(-.01)^6$ is one trillionth, it is safe to say that we will not need more than these coefficients for our estimate.

$$y(x) \approx 0 + 1x + 0x^2 + 1x^3 + 0x^4 + \frac{1}{2}x^5$$
$$y(-.01) \approx 1(-.01)^1 + 1(-.01)^3 + \frac{1}{2}(-.01)^5 = \boxed{-0.01000100005}$$

We can also estimate values further from zero, as long as we can find the power series solution centered nearby.

Example 4.5.4

Estimate $y(4.2)$ within one thousandth, where y satisfies the IVP $y'' - xy' - 2y = 0$ and $y(4) = 9, y'(4) = 0$

Solution

First, we assume $y = \sum_{n=0}^{\infty} c_n(x-4)^n$ and solve the LDE:

$$y'' - (x-4)y' - 4y' - 2y = 0$$
$$\sum_{n=0}^{\infty} n(n-1)c_n(x-4)^{n-2} - (x-4)\cdot\sum_{n=0}^{\infty} nc_n(x-4)^{n-1} - 4\cdot\sum_{n=0}^{\infty} nc_n(x-4)^n - 2\cdot\sum_{n=0}^{\infty} c_n(x-4)^n = 0$$
$$\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n(x-4)^{n-2}}_{m=n-2} - \underbrace{\sum_{n=0}^{\infty} nc_n(x-4)^n}_{m=n} - \underbrace{\sum_{n=0}^{\infty} 4nc_n(x-4)^{n-1}}_{m=n-1} - \underbrace{\sum_{n=0}^{\infty} 2c_n(x-4)^n}_{m=n} = 0$$
$$\sum_{m=-2}^{\infty}(m+2)(m+1)c_{m+2}(x-4)^m - \sum_{m=0}^{\infty} mc_m(x-4)^m - \sum_{m=-1}^{\infty} 4(m+1)c_{m+1}(x-4)^m - \sum_{m=0}^{\infty} 2c_m(x-4)^m = 0$$
$$\left(0c_0(x-4)^{-2} + 0c_1(x-4)^{-1} + \sum_{m=0}^{\infty}(m+2)(m+1)c_{m+2}(x-4)^m\right) - \sum_{m=0}^{\infty} mc_m(x-4)^m$$
$$-\left(0c_0(x-4)^{-1} + \sum_{m=0}^{\infty} 4(m+1)c_{m+1}(x-4)^m\right) - \sum_{m=0}^{\infty} 2c_m(x-4)^m = 0$$
$$\sum_{m=0}^{\infty}\left[(m+2)(m+1)c_{m+2} - mc_m - 4(m+1)c_{m+1} - 2c_m\right](x-4)^m = 0$$

The initial conditions tell us that $c_0 = y(4) = 9$ and $c_1 = y'(4) = 0$

Setting the coefficients in the sum equal to zero gives:

$$(m+2)(m+1)c_{m+2} - mc_m - 4(m+1)c_{m+1} - 2c_m = 0 \implies c_{m+2} = \frac{4}{m+2}c_{m+1} + \frac{1}{m+1}c_m$$

Plugging in a few values of m we see:

$$c_2 = \frac{4}{2}c_1 + \frac{1}{1}c_0 = 9$$
$$c_3 = \frac{4}{3}c_2 + \frac{1}{2}c_1 = 12$$
$$c_4 = \frac{4}{4}c_3 + \frac{1}{3}c_2 = 15$$
$$c_5 = \frac{4}{5}c_4 + \frac{1}{4}c_3 = 15$$
$$c_6 = \frac{4}{6}c_5 + \frac{1}{5}c_4 = 13$$
$$\vdots$$

Since $(.2)^6 = .000064$, it is safe to say that we will not need more than these coefficients for our estimate.

$$y(x) \approx 9 + 0(x-4) + 9(x-4)^2 + 12(x-4)^3 + 15(x-4)^4 + 15(x-4)^5 + 13(x-4)^6$$
$$y(4.2) \approx 9 + 9(.2)^2 + 12(.2)^3 + 15(.2)^4 + 15(.2)^5 + 13(.2)^6 = \boxed{9.485632}$$

This subchapter provides the foundation for practical computation with differential equations, bridging the gap between theoretical analysis and real-world application. Where power series approximations give us exact but often impractical infinite representations, numerical methods like Runge-Kutta provide finite, computable solutions. These numerical techniques are essential for problems that resist analytical treatment—nonlinear equations, irregular singular points, and complicated forcing functions. The Runge-Kutta methods, particularly RK4, remain the workhorse for most scientific and engineering applications due to their excellent balance of accuracy, stability, and computational efficiency.

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4.5: Practice Set

Exercise 4.5.1

Consider the differential equation $y' = \dfrac{1}{x+y}$ with $y(1) = 0$. Approximate $y(2)$ using

(a) 1 step and the Euler Method

(b) 1 step and the RK2 Method

(c) 1 step and the RK4 Method

Answer available at the end of this section.

Exercise 4.5.2

Consider the differential equation $y' = y^x$ with $y(1) = 1$. Approximate $y(3)$ using

(a) 1 step and the Euler Method

(b) 1 step and the RK2 Method

(c) 1 step and the RK4 Method

Answer available at the end of this section.

Exercise 4.5.3

Consider the differential equation $y' = 4x + 2y$ with $y(0) = 3$. Approximate $y(2)$ using

(a) 1 step and the Euler Method

(b) 1 step and the RK2 Method

(c) 1 step and the RK4 Method

(d) What is the exact value of $y(2)$?

Answer available at the end of this section.

Exercise 4.5.4

Consider the differential equation $y' = y$ with $y(0) = 1$. Approximate $y(b)$ using

(a) 1 step and the Euler Method

(b) 1 step and the RK2 Method

(c) 1 step and the RK4 Method

(d) What is the exact value of $y(b)$?

Answer available at the end of this section.

Exercise 4.5.5

Consider the differential equation $y' = 5y - y^2$ with $y(0) = 2$. Approximate $y(2)$ using

(a) 2 steps and the Euler Method

(b) 2 steps and the RK2 Method

(c) 2 steps and the RK4 Method

(d) What is the exact value of $y(2)$?

Answer available at the end of this section.

Exercise 4.5.6

Consider the differential equation $y' = \dfrac{2y + x^3}{x}$ with $y(1) = 2$. Approximate $y(3)$ using

(a) 2 steps and the Euler Method

(b) 2 steps and the RK2 Method

(c) 2 steps and the RK4 Method

(d) What is the exact value of $y(3)$?

Answer available at the end of this section.

Exercise 4.5.7

Consider the differential equation $y' = x^y$ with $y(1) = -1$. Approximate $y(2)$ using

(a) 2 steps and the Euler Method

(b) 2 steps and the RK2 Method

(c) 2 steps and the RK4 Method

Answer available at the end of this section.

Exercise 4.5.8

Consider the differential equation $y' = \dfrac{ye^{xy} + 2x}{1 - xe^{xy}}$ with $y(0) = 1$. Approximate $y(-1)$ using

(a) 3 steps and the Euler Method

(b) 3 steps and the RK2 Method

(c) 3 steps and the RK4 Method

(d) What is the exact value of $y(-1)$?

Answer available at the end of this section.

Exercise $4.5.9a$

Estimate $y(.1)$ within one millionth for the solution to the IVP $y'' - 3xy = 0$ with $y(0) = 1,\ y'(0) = 0$

Answer available at the end of this section.

Exercise $4.5.9b$

Estimate $y(0.8)$ within one ten thousandth for the solution to the IVP $y'' - (x-1)y' - 2y = 0$ with $y(1) = 0,\ y'(1) = -1$

Answer available at the end of this section.

Exercise $4.5.9c$

Estimate $y(-1.01)$ within one millionth for the solution to $y''' + 3y = 0$ with $y(-1) = 2,\ y'(-1) = 0,\ y''(-1) = 10$

Answer available at the end of this section.

Answers

Exercise $4.5.1$

(a) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-1}{1} = 1$:

$$\begin{aligned}(x_0, y_0) = (1, 0) \implies f(x_0, y_0) = 1\\ y_1 = y_0 + f(x_0, y_0) \cdot h = 0 + 1 \cdot 1 = 1\end{aligned}$$

Thus, the Euler estimate is $y(2) \approx 1$

(b) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-1}{1} = 1$:

$$\begin{aligned}(x_0, y_0) = (1, 0) \implies k_1 = f(x_0, y_0) = 1\\ k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(1 + \frac{1}{2}, 0 + \frac{1}{2} \cdot 1\right) = f\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{1}{2}\\ y_1 = y_0 + h \cdot k_2 = 0 + 1 \cdot \frac{1}{2} = \frac{1}{2}\end{aligned}$$

Thus, the RK 2 estimate is $y(2) \approx \dfrac{1}{2}$

(c) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-1}{1} = 1$:

$$\begin{aligned}(x_0, y_0) = (1, 0) \implies k_1 = f(x_0, y_0) = f(1, 0) = 1\\ k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(1 + \frac{1}{2}, 0 + \frac{1}{2}\right) = \frac{1}{2}\\ k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) = f\left(1 + \frac{1}{2}, 0 + \frac{1}{4}\right) = \frac{4}{7}\\ k_4 = f(x_0 + h, y_0 + h \cdot k_3) = f\left(1 + 1, 0 + \frac{4}{7}\right) = \frac{7}{18}\\ y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0 + \frac{1}{6}\left(1 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{4}{7} + \frac{7}{18}\right) = \frac{445}{756}\end{aligned}$$

Thus, the RK 4 estimate is $y(2) \approx \dfrac{445}{756} = .5886$

Exercise $4.5.2$

(a) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{3-1}{1} = 2$:

$$\begin{aligned}(x_0, y_0) = (1, 1) \implies f(x_0, y_0) = 1\\ y_1 = y_0 + f(x_0, y_0) \cdot h = 1 + 1 \cdot 2 = 3\end{aligned}$$

Thus, the Euler estimate is $y(3) \approx 3$

(b) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{3-1}{1} = 2$:

$$(x_0, y_0) = (1,1) \implies k_1 = f(x_0, y_0) = 1$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f(1+1, 1+1) = f(2,2) = 4$$
$$y_1 = y_0 + h \cdot k_2 = 1 + 2 \cdot 4 = 9$$

Thus, the RK 2 estimate is $y(3) \approx 9$

(c) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{3-1}{1} = 2$:

$$(x_0, y_0) = (1,1) \implies k_1 = f(x_0, y_0) = f(1,1) = 1$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f(1+1, 1+1) = 4$$
$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) = f(1+1, 1+4) = 25$$
$$k_4 = f(x_0 + h, y_0 + h \cdot k_3) = f(1+2, 1+50) = 265302$$
$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{3}(1 + 2 \cdot 4 + 2 \cdot 25 + 132651) = \frac{132713}{3}$$

Thus, the RK 4 estimate is $y(3) \approx \dfrac{132713}{3} = 44237.6$

Exercise 4.5.3

(a) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-0}{1} = 2$:

$$(x_0, y_0) = (0,3) \implies f(x_0, y_0) = 6$$
$$y_1 = y_0 + f(x_0, y_0) \cdot h = 3 + 6 \cdot 2 = 15$$

Thus, the Euler estimate is $y(2) \approx 15$

(b) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-0}{1} = 2$:

$$(x_0, y_0) = (0,3) \implies k_1 = f(x_0, y_0) = 6$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f(0+1, 3+1 \cdot 6) = f(1,9) = 22$$
$$y_1 = y_0 + h \cdot k_2 = 3 + 2 \cdot 22 = 47$$

Thus, the RK 2 estimate is $y(2) \approx 47$

(c) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-0}{1} = 2$:

$$(x_0, y_0) = (0,3) \implies k_1 = f(x_0, y_0) = f(0,3) = 6$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f(0+1, 3+6) = 22$$
$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) = f(0+1, 3+22) = 54$$
$$k_4 = f(x_0 + h, y_0 + h \cdot k_3) = f(0+2, 3+108) = 230$$
$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 3 + \frac{1}{3}(6 + 2 \cdot 22 + 2 \cdot 54 + 230) = \frac{397}{3}$$

Thus, the RK 4 estimate is $y(2) \approx \dfrac{397}{3} = 132.33$

(d) This equation is linear:

$$y = \frac{1}{\mu(x)} \int \mu(x) q(x)\, dx = \frac{1}{e^{-2x}} \int e^{-2x} \cdot 4x\, dx = e^{2x}\left(-(2x+1)e^{-2x} + C\right)$$

We can rewrite this as $y = -2x - 1 + Ce^{2x}$ for some constant C, and plug in $y(0) = 3$ to get:

$$y(0) = -1 + Ce^0 \longleftrightarrow C = 4$$

Therefore, the explicit solution to this IVP is $y(x) = -2x - 1 + 4e^{2x}$, which gives us exact value $y(2) = 4e^4 - 5 \approx 213.3926$

Exercise 4.5.4

(a) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{b-0}{1} = b$:

$$(x_0, y_0) = (0,1) \implies f(x_0, y_0) = 1$$
$$y_1 = y_0 + f(x_0, y_0) \cdot h = 1 + 1 \cdot b = 1 + b$$

Thus, the Euler estimate is $y(b) \approx 1 + b$

(b) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{b-0}{1} = b$:

$$(x_0, y_0) = (0,1) \implies k_1 = f(x_0, y_0) = 1$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + \frac{b}{2}, 1 + \frac{b}{2} \cdot 1\right) = f\left(\frac{b}{2}, \frac{b+2}{2}\right) = \frac{b+2}{2}$$
$$y_1 = y_0 + h \cdot k_2 = 1 + b \cdot \frac{b+2}{2} = 1 + b + \frac{1}{2}b^2$$

Thus, the RK 2 estimate is $y(b) \approx 1 + b + \dfrac{1}{2}b^2$

(c) The value of $n = 1$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{b-0}{1} = b$:

$$
\begin{aligned}
(x_0, y_0) = (0,1) \implies k_1 &= f(x_0, y_0) = f(0,1) = 1 \\
k_2 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + \frac{b}{2}, 1 + \frac{b}{2}\right) = \frac{b+2}{2} \\
k_3 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) = f\left(0 + \frac{b}{2}, 1 + \frac{b^2+2b}{4}\right) = \frac{b^2+2b+4}{4} \\
k_4 &= f(x_0 + h, y_0 + h \cdot k_3) = f\left(0 + b, 1 + \frac{b^3+2b^2+4b}{4}\right) = \frac{b^3+2b^2+4b+4}{4} \\
y_1 &= y_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\
&= 1 + \frac{b}{6}\left(1 + 2 \cdot \frac{b+2}{2} + 2 \cdot \frac{b^2+2b+4}{4} + \frac{b^3+2b^2+4b+4}{4}\right) \\
&= 1 + \frac{b^4+4b^3+12b^2+24b}{24} = 1 + b + \frac{1}{2}b^2 + \frac{1}{6}b^3 + \frac{1}{24}b^4
\end{aligned}
$$

Thus, the RK 4 estimate is $y(b) \approx 1 + b + \frac{1}{2}b^2 + \frac{1}{6}b^3 + \frac{1}{24}b^4$

(d) This equation is separable:

$$
\frac{dy}{dx} = y \implies \int \frac{1}{y}\, dy = \int dx \implies \ln|y| = x + C
$$

We can rewrite this as $y = Ce^x$ for some constant C, and plug in $y(0) = 1$ to get:

$$
y(0) = Ce^0 \longleftrightarrow C = 1
$$

Therefore, the explicit solution to this IVP is $y(x) = e^x$, which gives us exact value $y(b) = e^b \approx 1 + b + \frac{1}{2}b^2 + \frac{1}{6}b^3 + \frac{1}{24}b^4 + \cdots$

Exercise 4.5.5

(a) The value of $n = 2$ steps tells us that $h = \frac{b-a}{n} = \frac{2-0}{2} = 1$:

$$
\begin{aligned}
(x_0, y_0) = (0,2) &\implies f(x_0, y_0) = 6 \\
y_1 &= y_0 + f(x_0, y_0) \cdot h = 2 + 6 \cdot 1 = 8 \\
(x_1, y_1) = (1,8) &\implies f(x_1, y_1) = -24 \\
y_2 &= y_1 + f(x_1, y_1) \cdot h = 8 - 24 \cdot 1 = -16
\end{aligned}
$$

Thus, the Euler estimate is $y(2) \approx -16$

(b) The value of $n = 2$ steps tells us that $h = \frac{b-a}{n} = \frac{2-0}{2} = 1$:

$$
\begin{aligned}
(x_0, y_0) = (0,2) \implies k_1 &= f(x_0, y_0) = 6 \\
k_2 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + \frac{1}{2}, 2 + \frac{1}{2} \cdot 6\right) = f\left(\frac{1}{2}, 5\right) = 0 \\
y_1 &= y_0 + h \cdot k_2 = 2 + 1 \cdot 0 = 2 \\
(x_1, y_1) = (1,2) \implies k_1 &= f(x_1, y_1) = 6 \\
k_2 &= f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_1\right) = f\left(1 + \frac{1}{2}, 2 + \frac{1}{2} \cdot 6\right) = f\left(\frac{3}{2}, 5\right) = 0 \\
y_2 &= y_1 + h \cdot k_2 = 2 + 1 \cdot 0 = 2
\end{aligned}
$$

Thus, the RK 2 estimate is $y(2) \approx 2$

(c) The value of $n = 2$ steps tells us that $h = \frac{b-a}{n} = \frac{2-0}{2} = 1$:

$$
\begin{aligned}
(x_0, y_0) = (0,2) \implies k_1 &= f(x_0, y_0) = f(0,2) = 6 \\
k_2 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) = f\left(0 + \frac{1}{2}, 2 + 3\right) = 0 \\
k_3 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) = f\left(0 + \frac{1}{2}, 2 + 0\right) = 6 \\
k_4 &= f(x_0 + h, y_0 + h \cdot k_3) = f\left(0 + 1, 2 + 6\right) = -24 \\
y_1 &= y_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 2 + \frac{1}{6}\left(6 + 2 \cdot 0 + 2 \cdot 6 + (-24)\right) = 2 - 1 = 1 \\
(x_1, y_1) = (1,1) \implies k_1 &= f(x_1, y_1) = f\left(1, 1\right) = 4 \\
k_2 &= f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_1\right) = f\left(1 + \frac{1}{2}, 1 + 2\right) = 6 \\
k_3 &= f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_2\right) = f\left(1 + \frac{1}{2}, 1 + 3\right) = 4 \\
k_4 &= f(x_1 + h, y_1 + h \cdot k_3) = f\left(1 + 1, 1 + 4\right) = 0 \\
y_2 &= y_1 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1 + \frac{1}{6}\left(4 + 2 \cdot 6 + 2 \cdot 4 + 0\right) = 1 + 4 = 5
\end{aligned}
$$

Thus, the RK 4 estimate is $y(2) \approx 5$

(d) This equation is separable:

$$
\frac{dy}{dx} = 5y - y^2 \implies \int \frac{1}{y(5-y)}\, dy = \int dx \implies \ln\left(\frac{y}{5-y}\right) = 5x + C
$$

We can rewrite this as $y = \frac{5}{De^{-5x} + 1}$ for some constant D, and plug in $y(0) = 2$ to get:

$$
y(0) = \frac{5}{D \cdot 1 + 1} \longleftrightarrow D = \frac{3}{2}
$$

Therefore, the explicit solution to this IVP is $y(x) = \frac{5}{\frac{3}{2}e^{-5x} + 1}$, which gives us exact value $y(2) = \frac{5}{\frac{3}{2}e^{-10} + 1} \approx 5$

Exercise 4.5.6

(a) The value of $n = 2$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{3-1}{2} = 1$:

$$\begin{aligned}(x_0, y_0) = (1, 2) &\implies f(x_0, y_0) = 5\\ y_1 = y_0 + f(x_0, y_0) \cdot h &= 2 + 5 \cdot 1 = 7\\ (x_1, y_1) = (2, 7) &\implies f(x_1, y_1) = 11\\ y_2 = y_1 + f(x_1, y_1) \cdot h &= 7 + 11 \cdot 1 = 18\end{aligned}$$

Thus, the Euler estimate is $y(3) \approx 18$

(b) The value of $n = 2$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{3-1}{2} = 1$:

$$\begin{aligned}(x_0, y_0) = (1, 2) &\implies k_1 = f(x_0, y_0) = 5\\ k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) &= f\left(1 + \frac{1}{2}, 2 + \frac{1}{2} \cdot 5\right) = f\left(\frac{1}{2}, \frac{9}{2}\right) = \frac{33}{4}\\ y_1 = y_0 + h \cdot k_2 &= 2 + 1 \cdot \frac{33}{4} = \frac{41}{4}\\ (x_1, y_1) = \left(2, \frac{41}{4}\right) &\implies k_1 = f(x_1, y_1) = \frac{57}{4}\\ k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_1\right) &= f\left(2 + \frac{1}{2}, \frac{41}{4} + \frac{1}{2} \cdot \frac{57}{4}\right) = f\left(\frac{5}{2}, \frac{139}{8}\right) = \frac{403}{20}\\ y_2 = y_1 + h \cdot k_2 &= \frac{41}{4} + 1 \cdot \frac{403}{20} = \frac{152}{5}\end{aligned}$$

Thus, the RK 2 estimate is $y(3) \approx \dfrac{152}{5} = 30.4$

(c) The value of $n = 2$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{3-1}{2} = 1$:

$$\begin{aligned}(x_0, y_0) = (1, 2) &\implies k_1 = f(x_0, y_0) = f(1, 2) = 5\\ k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_1\right) &= f\left(1 + \frac{1}{2}, 2 + \frac{5}{2}\right) = \frac{33}{4}\\ k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot k_2\right) &= f\left(1 + \frac{1}{2}, 2 + \frac{33}{8}\right) = \frac{125}{12}\\ k_4 = f(x_0 + h, y_0 + h \cdot k_3) &= f\left(1 + 1, 2 + \frac{125}{12}\right) = \frac{197}{12}\\ y_1 = y_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) &= 2 + \frac{1}{6}\left(5 + 2 \cdot \frac{33}{4} + 2 \cdot \frac{125}{12} + \frac{197}{12}\right) = 2 + \frac{235}{24} = \frac{283}{24}\\ (x_1, y_1) = \left(2, \frac{283}{24}\right) &\implies k_1 = f(x_1, y_1) = 1 \cdot f\left(2, \frac{283}{24}\right) = \frac{379}{24}\\ k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_1\right) &= f\left(2 + \frac{1}{2}, \frac{283}{24} + \frac{379}{48}\right) = 22\\ k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot k_2\right) &= f\left(2 + \frac{1}{2}, \frac{283}{24} + 11\right) = \frac{1469}{60}\\ k_4 = f(x_1 + h, y_1 + h \cdot k_3) &= f\left(2 + 1, \frac{283}{24} + \frac{1469}{60}\right) = \frac{1451}{40}\\ y_2 = y_1 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) &= \frac{283}{24} + \frac{1}{6}\left(\frac{379}{24} + 2 \cdot 22 + 2 \cdot \frac{1469}{50} + \frac{1991}{60}\right) = \frac{283}{24} + \frac{17033}{120} = \frac{25523}{720}\end{aligned}$$

Thus, the RK 4 estimate is $y(3) \approx \dfrac{25523}{720} = 35.4486\ldots$

(d) This equation is linear:

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \implies y = \frac{1}{\mu(x)}\int \mu(x) q(x)\, dx = \frac{1}{x^{-2}}\int x^{-2} \cdot x^2\, dx = x^2 \int 1\, dx = x^2(x + C) = x^3 + Cx$$

We plug in $y(1) = 2$ to get:

$$y(1) = 1^3 + C \cdot 1 \implies C = 1$$

Therefore, the explicit solution to this IVP is $y(x) = x^3 + x^2$, which gives us exact value $y(3) = 36$

Exercise 4.5.7

(a) The value of $n = 2$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-1}{2} = \dfrac{1}{2}$:

$$\begin{aligned}(x_0, y_0) = (1, -1) &\implies f(x_0, y_0) = 1\\ y_1 = y_0 + f(x_0, y_0) \cdot h &= -1 + 1 \cdot \frac{1}{2} = -\frac{1}{2}\\ (x_1, y_1) = \left(\frac{3}{2}, -\frac{1}{2}\right) &\implies f(x_1, y_1) = 0.816\\ y_2 = y_1 + f(x_1, y_1) \cdot h &= -\frac{1}{2} + 0.816 \cdot \frac{1}{2} = -0.092\end{aligned}$$

Thus, the Euler estimate is $y(2) \approx -0.092$

(b) The value of $n = 2$ steps tells us that $h = \dfrac{b-a}{n} = \dfrac{2-1}{2} = \dfrac{1}{2}$:

$$(x_0,y_0)=(1,-1) \implies k_1=f(x_0,y_0)=1$$
$$k_2=f\left(x_0+\frac{h}{2},y_0+\frac{h}{2}\cdot k_1\right)=f\left(1+\frac{1}{4},-1+\frac{1}{4}\cdot 1\right)=f\left(\frac{5}{4},-\frac{3}{4}\right)=0.846$$
$$y_1=y_0+h\cdot k_2=-1+\frac{1}{2}\cdot 0.846=-0.577$$
$$(x_1,y_1)=\left(\frac{3}{2},-0.577\right) \implies k_1=f(x_1,y_1)=0.792$$
$$k_2=f\left(x_1+\frac{h}{2},y_1+\frac{h}{2}\cdot k_1\right)=f\left(\frac{3}{2}+\frac{1}{4},-0.577+\frac{1}{4}\cdot 0.792\right)=f\left(\frac{7}{4},-0.379\right)=0.809$$
$$y_2=y_1+h\cdot k_2=-0.577+\frac{1}{2}\cdot 0.809=-0.173$$

Thus, the RK 2 estimate is $y(2)\approx -0.173$

(c) The value of $n=2$ steps tells us that $h=\dfrac{b-a}{n}=\dfrac{2-1}{2}=\dfrac{1}{2}$:

$$(x_0,y_0)=(1,-1) \implies k_1=f(x_0,y_0)=f(1,-1)=1$$
$$k_2=f\left(x_0+\frac{h}{2},y_0+\frac{h}{2}\cdot k_1\right)=f\left(1+\frac{1}{4},-1+\frac{1}{4}\right)=0.846$$
$$k_3=f\left(x_0+\frac{h}{2},y_0+\frac{h}{2}\cdot k_2\right)=f\left(1+\frac{1}{4},-1+\frac{1}{4}\cdot 0.846\right)=0.838$$
$$k_4=f(x_0+h,y_0+h\cdot k_3)=f\left(1+\frac{1}{2},-1+\frac{1}{2}\cdot 0.838\right)=0.790$$
$$y_1=y_0+\frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)=-1+\frac{1}{12}\left(1+2\cdot 0.846+2\cdot 0.838+0.790\right)=-1+0.430=-0.570$$
$$(x_1,y_1)=\left(\frac{3}{2},-0.570\right) \implies k_1=f(x_1,y_1)=f\left(\frac{3}{2},-0.570\right)=0.774$$
$$k_2=f\left(x_1+\frac{h}{2},y_1+\frac{h}{2}\cdot k_1\right)=f\left(\frac{3}{2}+\frac{1}{4},-0.570+\frac{1}{4}\cdot 0.774\right)=0.812$$
$$k_3=f\left(x_1+\frac{h}{2},y_1+\frac{h}{2}\cdot k_2\right)=f\left(\frac{3}{2}+\frac{1}{4},-0.570+\frac{1}{2}\cdot 0.812\right)=0.814$$
$$k_4=f(x_1+h,y_1+h\cdot k_3)=f\left(\frac{3}{2}+\frac{1}{2},-0.570+\frac{1}{2}\cdot 0.814\right)=0.892$$
$$y_2=y_1+\frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)=-0.570+\frac{1}{12}\left(0.774+2\cdot 0.812+2\cdot 0.814+0.892\right)=-0.570+0.412=-0.158$$

Thus, the RK 4 estimate is $y(2)\approx -0.158$

Exercise 4.5.8

(a) The value of $n=3$ steps tells us that $h=\dfrac{b-a}{n}=\dfrac{0-1}{3}=-\dfrac{1}{3}$:

$$(x_0,y_0)=(0,1) \implies f(x_0,y_0)=1$$
$$y_1=y_0+f(x_0,y_0)\cdot h=1-1\cdot\frac{1}{3}=\frac{2}{3}$$
$$(x_1,y_1)=\left(-\frac{1}{3},\frac{2}{3}\right) \implies f(x_1,y_1)=-.105$$
$$y_2=y_1+f(x_1,y_1)\cdot h=\frac{2}{3}+1.05\cdot\frac{1}{3}=0.702$$
$$(x_2,y_2)=\left(-\frac{2}{3},0.702\right) \implies f(x_2,y_2)=-0.630$$
$$y_3=y_2+f(x_2,y_2)\cdot h=0.702+.630\cdot\frac{1}{3}=0.912$$

Thus, the Euler estimate is $y(-11)\approx 0.912$

(b) The value of $n=3$ steps tells us that $h=\dfrac{b-a}{n}=\dfrac{0-1}{3}=-\dfrac{1}{3}$:

$$(x_0,y_0)=(0,1) \implies k_1=f(x_0,y_0)=1$$
$$k_2=f\left(x_0+\frac{h}{2},y_0+\frac{h}{2}\cdot k_1\right)=f\left(0-\frac{1}{6},1-\frac{1}{6}\cdot 1\right)=f\left(\frac{1}{6},\frac{5}{6}\right)=0.342$$
$$y_1=y_0+h\cdot k_2=1-\frac{1}{3}\cdot 0.342=0.886$$
$$(x_1,y_1)=\left(-\frac{1}{3},0.886\right) \implies k_1=f(x_1,y_1)=-0.006$$
$$k_2=f\left(x_1+\frac{h}{2},y_1+\frac{h}{2}\cdot k_1\right)=f\left(-\frac{1}{3}-\frac{1}{6},0.886+\frac{1}{6}\cdot 0.006\right)=f\left(-\frac{1}{2},0.887\right)=-0.326$$
$$y_2=y_1+h\cdot k_2=0.886-\frac{1}{3}\cdot(-0.326)=0.995$$
$$(x_2,y_2)=\left(-\frac{2}{3},0.995\right) \implies k_1=f(x_2,y_2)=-0.611$$
$$k_2=f\left(x_2+\frac{h}{2},y_2+\frac{h}{2}\cdot k_1\right)=f\left(-\frac{2}{3}-\frac{1}{6},0.995+\frac{1}{6}\cdot 0.611\right)=f\left(-\frac{5}{6},1.096\right)=-0.919$$
$$y_3=y_2+h\cdot k_2=0.995+\frac{1}{3}\cdot 0.919=1.301$$

Thus, the RK 2 estimate is $y(-1)\approx 1.301$

(c) The value of $n=3$ steps tells us that $h=\dfrac{b-a}{n}=\dfrac{0-1}{3}=-\dfrac{1}{3}$:

$$(x_0,y_0)=(0,1)\implies k_1=f(x_0,y_0)=f(0,1)=1$$
$$k_2=f\left(x_0+\frac{h}{2},y_0+\frac{h}{2}\cdot k_1\right)=f\left(0-\frac{1}{6},1-\frac{1}{6}\cdot 1\right)=0.342$$
$$k_3=f\left(x_0+\frac{h}{2},y_0+\frac{h}{2}\cdot k_2\right)=f\left(0-\frac{1}{6},1-\frac{1}{6}\cdot 0.342\right)=0.414$$
$$k_4=f(x_0+h,y_0+h\cdot k_3)=f\left(0-\frac{1}{3},1-\frac{1}{3}\cdot 0.414\right)=-0.015$$
$$y_1=y_0+\frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)=1-\frac{1}{18}\left(1+2\cdot 0.342+2\cdot 0.414-0.015\right)=1-0.139=0.861$$
$$(x_1,y_1)=\left(-\frac{1}{3},0.861\right)\implies k_1=f(x_1,y_1)=f\left(-\frac{1}{3},0.861\right)=-0.015$$
$$k_2=f\left(x_1+\frac{h}{2},y_1+\frac{h}{2}\cdot k_1\right)=f\left(-\frac{1}{3}-\frac{1}{6},0.861-\frac{1}{6}\cdot(-0.015)\right)=-0.330$$
$$k_3=f\left(x_1+\frac{h}{2},y_1+\frac{h}{2}\cdot k_2\right)=f\left(-\frac{1}{3}-\frac{1}{6},0.861-\frac{1}{6}\cdot(-0.330)\right)=-0.318$$
$$k_4=f(x_1+h,y_1+h\cdot k_3)=f\left(-\frac{1}{3}-\frac{1}{3},0.861-\frac{1}{3}(-0.318)\right)=-0.612$$
$$y_2=y_1+\frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)=0.861-\frac{1}{18}\left(-0.015+2\cdot(-0.330)+2\cdot(-0.318)+(-0.612)\right)=0.861+.107=0.968$$
$$(x_2,y_2)=\left(-\frac{2}{3},0.968\right)\implies k_1=f(x_2,y_2)=f\left(-\frac{2}{3},0.968\right)=-0.612$$
$$k_2=f\left(x_2+\frac{h}{2},y_2+\frac{h}{2}\cdot k_1\right)=f\left(-\frac{2}{3}-\frac{1}{6},0.968-\frac{1}{6}\cdot(-0.612)\right)=-0.915$$
$$k_3=f\left(x_2+\frac{h}{2},y_2+\frac{h}{2}\cdot k_2\right)=f\left(-\frac{2}{3}-\frac{1}{6},0.968-\frac{1}{6}\cdot(-0.915)\right)=-0.927$$
$$k_4=f(x_2+h,y_2+h\cdot k_3)=-\frac{1}{3}\cdot f\left(-\frac{2}{3}-\frac{1}{3},0.968-\frac{1}{3}\cdot(-0.927)\right)=-1.284$$
$$y_3=y_2+\frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)=0.968-\frac{1}{18}\left(-0.612+2\cdot(-0.915)+2\cdot(-0.924)+(-1.284)\right)=0.9685+0.3097=1.2782$$

Thus, the RK 4 estimate is $y(-1)\approx 1.2782\ldots$

(d) This equation is exact:

$$y'=\frac{ye^{xy}+2x}{1-xe^{xy}}\implies (ye^{xy}+2x)+(xe^{xy}-1)y'=0\implies e^{xy}+x^2-y=K$$

We plug in $y(0)=1$ to get:

$$e^0+0^2-1=K\implies K=-1$$

Therefore, the implicit solution to this IVP is $y=e^{xy}+x^2+1$, and if we plug in $x=-1$, we get: $y=e^{-y}+2$

These two functions clearly intersect at exactly one point, and using the intermediate value theorem, we can narrow it down to approximately $y=1.27846$

Exercise $4.5.9a$

We let $y=\sum_{n=0}^{\infty}c_nx^n$, and plug it into the LDE:

$$\begin{aligned}
y''-xy&=0\\
\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}-3x\cdot\sum_{n=0}^{\infty}c_nx^n&=0\\
\underbrace{\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}}_{m=n-2}-\underbrace{\sum_{n=0}^{\infty}3c_nx^{n+1}}_{m=n+1}&=0\\
\sum_{m=-2}^{\infty}(m+2)(m+1)c_{m+2}x^m-\sum_{m=1}^{\infty}3c_{m-1}x^m&=0\\
0c_0x^{-2}+0c_1x^{-1}+2c_2x^0+\sum_{m=1}^{\infty}\left[(m+2)(m+1)c_{m+2}-3c_{m-1}\right]x^m&=0
\end{aligned}$$

Setting each coefficient equal to zero gives $c_2=0$, and $c_{m+2}=\frac{1}{(m+2)(m+1)}c_{m-1}$ for $m\geq 1$

The initial conditions give $c_0=y(0)=1,\ \ c_1=y'(0)=0$. We plug in a few m to find the next terms:

$$\begin{aligned}
c_3&=\frac{3}{3\cdot 2}c_0=\frac{1}{1!\cdot 2!!!}\\
c_4&=\frac{3}{4\cdot 3}c_1=0\\
c_5&=\frac{3}{5\cdot 4}c_2=0\\
c_6&=\frac{3}{6\cdot 5}c_3=\frac{1}{2!\cdot 5!!!}\\
&\vdots
\end{aligned}$$

The pattern here is $c_{3k}=\frac{1}{k!\cdot(3k-1)!!!}$ for $k\geq 1$ and $c_{3k+1}=c_{3k+2}=0$ for all k.

Thus, $y=1+\sum_{k=1}^{\infty}\frac{1}{k!\cdot(3k-1)!!!}x^{3k}=1+\frac{1}{2}x^3+\frac{1}{20}x^6+\ldots$

$$y(.1)=1+\frac{1}{2}(.1)^3+\frac{1}{20}(.1)^6+\cdots=1+.0005+.00000005+\cdots\approx 1.00050005$$

Exercise $4.5.9b$

We let $y=\sum_{n=0}^{\infty} c_n(x-1)^n$, and plug it into the LDE:

$$\begin{aligned}
y''-(x-1)y'-2y&=0\\
\sum_{n=0}^{\infty} n(n-1)c_n(x-1)^{n-2}-(x-1)\cdot\sum_{n=0}^{\infty} nc_n(x-1)^{n-1}-2\sum_{n=0}^{\infty} c_n(x-1)^n&=0\\
\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n(x-1)^{n-2}}_{m=n-2}-\underbrace{\sum_{n=0}^{\infty} nc_n(x-1)^{n}}_{m=n}-\underbrace{\sum_{n=0}^{\infty} 2c_n(x-1)^{n}}_{m=n}&=0\\
\sum_{m=-2}^{\infty}(m+2)(m+1)c_{m+2}(x-1)^m-\sum_{m=0}^{\infty} mc_m(x-1)^m-\sum_{m=0}^{\infty} 2c_m(x-1)^m&=0\\
0c_0(x-1)^{-2}+0c_1(x-1)^{-1}+\sum_{m=0}^{\infty}\left[(m+2)(m+1)c_{m+2}-(m+2)c_m\right]x^m&
\end{aligned}$$

Setting each coefficient equal to zero gives $c_{m+2}=\dfrac{1}{m+1}c_m$ for $m\geq 0$

The initial conditions give $c_0=y(1)=0,\ \ c_1=y'(1)=-1$. We plug in a few m to find the next terms:

$$\begin{aligned}
c_2&=\frac{1}{1}c_0=0\\
c_3&=\frac{1}{2}c_1=-\frac{1}{2!!}\\
c_4&=\frac{1}{3}c_2=0\\
c_5&=\frac{1}{4}c_3=-\frac{1}{4!!}\\
&\vdots
\end{aligned}$$

The pattern here is $c_{2k+1}=-\dfrac{1}{(2k)!!}$ for $k\geq 1$ and $c_{2k}=0$ for all k.

Thus, $y=-(x-1)-\sum_{k=1}^{\infty}\dfrac{1}{(2k)!!}(x-1)^{2k+1}=(1-x)+\dfrac{1}{2}(1-x)^3+\dfrac{1}{8}(1-x)^5+\ldots$

$$y(.8)=.2+\frac{1}{2}(.2)^3+\frac{1}{8}(.2)^5+\cdots=.2+.004+.00004+\cdots\approx 0.20404$$

Exercise $4.5.9c$

We let $y=\sum_{n=0}^{\infty} c_n(x+1)^n$, and plug it into the LDE:

$$\begin{aligned}
y'''+3y&=0\\
\sum_{n=0}^{\infty} n(n-1)(n-2)c_n(x+1)^{n-3}+3\sum_{n=0}^{\infty} c_n(x+1)^n&=0\\
\underbrace{\sum_{n=0}^{\infty} n(n-1)(n-2)c_n(x+1)^{n-3}}_{m=n-3}+\underbrace{\sum_{n=0}^{\infty} 3c_n(x+1)^{n}}_{m=n}&=0\\
\sum_{m=-3}^{\infty}(m+3)(m+2)(m+1)c_{m+3}(x+1)^m+\sum_{m=0}^{\infty} 3c_m(x+1)^m&=0\\
0c_0(x+1)^{-3}+0c_1(x+1)^{-2}+0c_2(x+1)^{-1}+\sum_{m=0}^{\infty}\left[(m+3)(m+2)(m+1)c_{m+3}+3c_m\right](x+1)^m&=0
\end{aligned}$$

Setting each coefficient equal to zero gives $c_{m+3}=-\dfrac{3c_m}{(m+3)(m+2)(m+1)}$ for $m\geq 0$

The initial conditions give $c_0=y(-1)=2,\ \ c_1=y'(-1)=0,\ \ c_2=\dfrac{y''(-1)}{2}=5$. We plug in a few m to find the next terms:

$$\begin{aligned}
c_3&=-\frac{3c_0}{3\cdot 2\cdot 1}=-1\\
c_4&=-\frac{3c_1}{4\cdot 3\cdot 2}=0\\
c_5&=-\frac{3c_2}{5\cdot 4\cdot 3}=-\frac{1}{4}\\
c_6&=-\frac{3c_3}{6\cdot 5\cdot 4}=\frac{1}{40}\\
&\vdots
\end{aligned}$$

The pattern here is $c_{3k}=\dfrac{2(-3)^k}{(3k)!}$, $c_{3k+1}=0$, and $c_{3k+2}=\dfrac{10(-3)^k}{(3k+2)!}$ for all k.

Thus, $y=\sum_{k=0}^{\infty}\dfrac{2(-3)^k}{(3k)!}(x+1)^{3k}+\sum_{k=0}^{\infty}\dfrac{10(-3)^k}{(3k+2)!}(x+1)^{3k+2}=2+5(x+1)^2-1(x+1)^3+\ldots$

$$y(-1.01)=2+5(-.01)^2-1(-.01)^3+\cdots=2+.0005+.000001+\cdots\approx 2.000501$$

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5: APPLICATIONS

5.1: Applications of First Order Differential Equations

5.1: Separable Models

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5.1: Applications of First Order Differential Equations

Learning Objectives

- Model real-world phenomena using first-order differential equations.
- Solve and interpret population growth models, including the logistic equation.
- Apply Newton's Law of Cooling to temperature change problems.

This subchapter demonstrates the remarkable power of first-order differential equations to model diverse real-world phenomena across biology, physics, engineering, and social sciences. The ability to translate a verbal description of a process into a differential equation and extract meaningful predictions is one of the most valuable skills in applied mathematics.

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5.1: Separable Models

First order differential equations are used to model many real life situations. We will explore a few famous ones here that result in separable DEs.

The classic unbounded population growth model assumes that a population with unlimited resources and space will expand at a rate proportional to its current population. Intuitively this is because in a population of P people (or animals or bacteria) we expect a fixed percentage of them to reproduce during each unit of time.

$$\frac{dP}{dt} = r \cdot P \quad \text{for some } r > 0$$

This differential equation is separable and straightforward to solve:

$$\int \frac{1}{P}\, dP = \int r\, dt \implies \ln|P| = rt + C$$

If necessary, we can solve for this solution explicitly: $P(t) = Ke^{rt}$

This formula may look familiar, it is known as the **exponential growth equation**.

Rule: Exponential Growth Equation

A population which satisfies the differential equation $\frac{dP}{dt} = r \cdot P$ has general solution $P(t) = Ke^{rt}$

Exponential growth serves as the most common application problem for students practicing logarithm properties in a precalculus and/or calculus course.

Illustration 5.1.1

The population of California in 1900 was 1.5 million people, and by 1920 it grew to 3.5 million people.

(a) Use the differential equation $\frac{dP}{dt} = rP$ to model the population (in millions) t decades after 1900.

(b) Use the result from part a to approximate the population of California in 2020.

(c) If the population in 2020 is actually around 40 million people, why does this not agree with the answer in part b?

(a) We have seen that solving the separable DE $\frac{dP}{dt} = rP$ gives the solution $P(t) = Ke^{rt}$.

The population in 1900 (which is when $t = 0$) must satisfy $P(0) = K \cdot e^{r \cdot 0} \implies 1.5 = K$

The population in 1920 (which is when $t = 2$) must satisfy $P(2) = K \cdot e^{r \cdot 2} \implies 3.5 = Ke^{2r}$

We can substitute in $K = 1.5$ to re-write the second equation as $3.5 = 1.5 \cdot e^{2r} \implies r = \frac{1}{2}\ln\left(\frac{3.5}{1.5}\right)$

Thus, the population should satisfy the function $P(t) = 1.5e^{\frac{1}{2}t\ln(\frac{7}{3})}$

(b) The year 2020 is 12 decades after 1900, so $t = 12$:

$$P(12) = 1.5e^{\frac{1}{2}\cdot 12\cdot \ln\left(\frac{7}{3}\right)} = 1.5e^{\ln\left(\left(\frac{7}{3}\right)^6\right)} = 1.5\cdot\left(\frac{7}{3}\right)^6 \approx 231.5$$

Therefore, the population is estimated to be 231.5 million people.

(c) This model assumes *unlimited resources and space*, and any California resident knows these are not realistic assumptions for the population! In fact, the exponential solution overestimates California's actual population numbers almost immediately after 1930.

Year	$P(t)$	Actual Population (in millions)
1900	1.5	1.5
1920	3.5	3.5
1940	8.2	6.9
1960	19	15.7
1980	44.5	23.6
2000	103.7	33.8

While easy to solve, this type of model is not realistic for long term predictions (or even medium term predictions) in real life.

We see that the growth rate of a population actually decreases as that population approaches the carrying capacity of the environment. For example, the number of fish in a pond cannot grow exponentially forever since they will not have enough resources or space to support a very large population. To refine our population model, we adjust the differential equation used:

$$\frac{dP}{dt} = rP(M-P) \quad \text{for some } r, M > 0$$

Here r is still determined by the type of animal that we are modeling (rabbits have more babies per year than elephants do), and M is the carrying capacity of the environment (the maximum amount of population that it can support). This differential equation is still separable:

$$\int \frac{1}{P(M-P}\,dP = \int r\,dt \implies \frac{1}{M}\ln|P| - \frac{1}{M}\ln|M-P| = rt + C$$

We can again solve for P explicitly to get $P(t) = \dfrac{M}{1+Ke^{-Mrt}}$

This solution is known as the **logistic growth equation**.

Rule: Logistic Growth Equation

A population which satisfies the differential equation $\dfrac{dP}{dt} = r\cdot P(M-P)$ has general solution $P(t) = \dfrac{M}{1+Ke^{-Mrt}}$

The logistic growth equation is often used to model populations in an environment with a fixed carrying capacity.

Illustration 5.1.2

The population of California in 1900 was 1.5 million people, and by 1920 it grew to 3.5 million people.

(a) Use the differential equation $\dfrac{dP}{dt} = rP(40-P)$ to model the population (in millions) t decades after 1900.

(b) Use the result from part a to approximate the population of California in 1940, 1960, 1980, 2000, and 2020.

(a) We have seen above that the solution to this differential equation is the logistic growth equation $P(t) = \dfrac{40}{1+Ke^{-rt}}$

Plugging in the population in 1900 ($t = 0$) gives $1.5 = \dfrac{40}{1+Ke^{-Mr\cdot 0}} \implies 1.5 = \dfrac{40}{1+K} \implies K = \dfrac{77}{3}$

Plugging in the population in 1920 ($t = 2$) gives $3.5 = \dfrac{40}{1+Ke^{-Mr\cdot 2}} \implies 3.5 = \dfrac{40}{1+Ke^{-2Mr}}$

Substituting in $K = \dfrac{77}{3}$ makes this second equation into:

$$3.5 = \frac{40}{1+\frac{77}{3}e^{-2Mr}} \implies e^{-2Mr} = \frac{73\cdot 3}{7\cdot 77} \implies -Mr = \frac{1}{2}\ln\left(\frac{219}{539}\right) \approx .45$$

Therefore, the population is modeled by the function $P(t) = \dfrac{40}{1+\frac{77}{3}e^{\frac{1}{2}t\ln\left(\frac{219}{539}\right)}}$

(b) Plugging in $t = 4, 6, 8, 10,$ and 12 into the function found in part a gives us the following populations (in millions).

Year	$P(t)$	Actual Population (in millions)
1900	1.5	1.5
1920	3.5	3.5
1940	7.6	6.9
1960	14.7	15.7
1980	23.5	23.6
2000	31.1	33.8

We see these numbers are indeed much closer to the actual populations of California during those decades!

In many physical, biological, and chemical contexts, we observe a different pattern: *decay*. From radioactive materials losing their potency to a cup of coffee cooling on a counter, these processes are often modeled by first-order equations where the rate of decay is proportional to the amount of substance remaining.

We model the decay of an amount of substance using a very similar differential equation:

$$\frac{dA}{dt} = -r\cdot A \quad \text{for some } r > 0$$

This has the same family of solutions as the exponential growth function: $A(t) = Ke^{-rt}$

This formula is often referred to as the **exponential decay equation**.

Rule: Exponential Decay Equation

A quantity which satisfies the differential equation $\dfrac{dA}{dt} = -r\cdot A$ has general solution $A(t) = Ke^{-rt}$

Example 5.1.1

A sample of a radioactive isotope is decaying at a rate proportional to the amount present. After 8 days, the sample is measured to have a mass of 80 mg and is decaying at an instantaneous rate of 20 mg per day. How much of the isotope was initially present?

Solution

Above, we found the solution to differential equation $\frac{dA}{dt} = -rA$ to be $A(t) = Ke^{-rt}$

We are given the amount after 8 days, so $A(8) = 80$:

$$A(8) = Ke^{-r\cdot 8} \implies 80 = Ke^{-8r}$$

We are also given the rate of decay after 8 days, so $A'(8) = -20$:

$$A'(t) = -rKe^{-rt} \implies A'(8) = -rKe^{-r\cdot 8} \implies -20 = -rKe^{-8r}$$

Substituting the first equation into the second one gives us $-20 = -r \cdot 80 \implies r = \frac{1}{4}$

Plugging this into the first equation gives us $80 = Ke^{-2} \implies K = 80e^2$

Therefore, our function is $A(t) = 80e^2 \cdot e^{-\frac{1}{4}t} = 80e^{2-\frac{1}{4}t}$

The initial amount of the sample must be $\boxed{A(0) = 80e^2 \approx 591 \text{ mg}}$

Sometimes this decay is described in terms of the **half-life** of the material involved. This is defined as the amount of time needed for an initial sample to decay to half of its original size. If a sample has a half-life of h, then:

$$A(h) = \frac{1}{2}A(0) \implies e^{-rh} = \frac{1}{2} \implies r = \frac{1}{h}\ln(2)$$

This is just a subtle way to give you the value of r in a given problem.

Rule: Half-Life Formula

A quantity with half-life h satisfies $A(t) = Ke^{-rt}$ for $r = \frac{1}{h}\ln(2)$

Example 5.1.2

An archaeologist discovers a bone in his backyard and discovers that the amount of Carbon-14 in it has decayed to 1% of the original amount. If Carbon-14 has a half life of 5730 years, how old is this bone?

Solution

Since the total carbon has decayed to 1% of the initial amount, if the bone is T years old we must have

$$A(T) = .01 \cdot A(0) \implies Ke^{-rT} = .01 \cdot K \implies e^{-rT} = .01$$

We are given the half life $h = 5730$, so we know $r = \frac{1}{5730}\ln(2)$.

We plug this into the above equation and solve for T:

$$e^{-\frac{1}{5730}\ln(2)\cdot T} = .01 \implies T = -\frac{5730\ln(.01)}{\ln(2)} \approx 38069$$

Therefore, the bone is approximately $\boxed{38069 \text{ years old}}$

Just like a radioactive isotope decays to zero, the difference in temperature between an object and its environment decays as well. This is referred to as **Newton's Law of Cooling**.

If the temperature of the environment is denoted T_e, this means that the temperature of an object obeys the differential equation

$$\frac{dT}{dt} = -r\,(T - T_e) \qquad \text{for some } r > 0$$

This is once more a separable differential equation:

$$\int \frac{1}{T - T_e}\,dT = \int -r\,dt \implies \ln|T - T_e| = -rt + C$$

We can solve for T explicitly to get our family of solutions: $T(t) = T_e + Ke^{-rt}$

Rule: Newton's Law of Cooling Equation

The temperature of an object with an ambient temperature T_e satisfies the differential equation $\frac{dT}{dt} = -r \cdot (T - T_e)$ and has general solution $T(t) = T_e + Ke^{-rt}$

Notice that for $K > 0$ we will have $T(t) > T_e$ for all t, while for $K < 0$ we will have \(T(t)

Example 5.1.3

The police discover a body at 3 PM and find it has temperature 86° F. If it was found inside of an air conditioned room that is exactly 68° F, and the body was measured to be exactly 80° F at 4PM, what was the approximate time of death? (A living body maintains a temperature of 98.6° F.)

Solution

We use the solution to the differential equation defined by Newton's Law of Cooling: $T(t) = T_e + Ke^{-rt}$

We are given the temperature of the room is $T_e = 68$, so this function looks like $T(t) + 68 + Ke^{-rt}$

The temperature at time $t = 3$ is 86°, meaning $T(3) = 86$:

$$T(3) = 68 + Ke^{-r\cdot 3} \implies 86 = 68 + Ke^{-3r} \implies 18 = Ke^{-3r}$$

The temperature at time $t = 4$ is 80°, meaning $T(4) = 80$:

$$T(4) = 68 + Ke^{-r\cdot 4} \implies 80 = 68 + Ke^{-4r} \implies 12 = Ke^{-4r}$$

Dividing these two equations by each other gives

$$\frac{18}{12} = \frac{Ke^{-3r}}{Ke^{-4r}} \implies \frac{3}{2} = e^r \implies r = \ln\left(\frac{3}{2}\right)$$

Plugging this into either equation and solving for K gives us $K = \frac{243}{4}$

Therefore, the temperature of the body t hours after noon satisfies the equation $T(t) = 68 + \frac{243}{4}e^{-t\ln\left(\frac{3}{2}\right)}$

We want to know when $T(t) = 98.6$:

$$98.6 = 68 + \frac{243}{4} e^{-t \ln\left(\frac{3}{2}\right)} \implies t = -\frac{\ln\left(30.6 \cdot \frac{4}{243}\right)}{\ln\left(\frac{3}{2}\right)} \approx 1.7$$

Thus, the time of death should be approximately 1.7 hours after noon, or $\boxed{1:41 \text{ PM}}$

This subchapter has shown how separable first-order differential equations model key real-world processes. The exponential growth and more realistic logistic equations describe populations, while analogous equations describe radioactive decay and cooling. These applications illustrate the broad utility of separable differential equations.

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5.1: Practice Set

Exercise $5.1.1a$

You find 1875 lbs of uranium in your back yard. Three days later, it only weighs 1500 lbs. How much will it weigh 9 days after you found it?

Answer available at the end of this section.

Exercise $5.1.1b$

A colony of ants is measured by its weight. On May 1st, they weighed 405 kg. On May 3rd, they grew to 540 kg. If they continue to grow exponentially, what date will they weigh 1280 kg?

Answer available at the end of this section.

Exercise $5.1.1c$

A zombie virus has been spreading exponentially throughout the world. On day 10 of the epidemic, there were 63 zombies and then on day 25 there were 1701 zombies. How many people were infected on day 5?

Answer available at the end of this section.

Exercise $5.1.2a$

A thermometer is taken from a warm room to the outside where the air temperature is 5°. After 4 minutes the thermometer reads 55° and after 6 minutes the reading is 30°. What was the initial temperature of the room?

Answer available at the end of this section.

Exercise $5.1.2b$

A human being (with body temperature 98.6°) gets into a jacuzzi at 7 pm with water temperature 105°. Fifteen minutes later, the body temperature of this human has risen to 100.2°. If a temperature of 102.3 can be fatal, by what time should this person exit the jacuzzi?

Answer available at the end of this section.

Exercise $5.1.2c$

A big pot of soup at 250° is put into the walk in freezer. After one hour, the soup is only 130°. After three hours it is 40°. What is the temperature inside of the freezer?

Answer available at the end of this section.

Exercise $5.1.3a$

Suppose some species of fish in a lake is modeled by a logistic population model $\frac{dP}{dt} = P(a - bP)$ with relative growth rate of $a = .3$ per year and a carrying capacity of $K = 100$ fish. If 25 fish are initially in the lake, how long will it take for this population to grow to 80 fish?

Answer available at the end of this section.

Exercise $5.1.3b$

Assume $a > b > 0$, what are the inflection points of the solution to the logistic equation $\frac{dP}{dt} = P(a - bP)$ with $P(0) = \frac{a}{2b}$?

Answer available at the end of this section.

Exercise $5.1.3c$

A population of frogs in a pond obeys the logistic equation $\frac{dP}{dt} = P(a - bP)$. Scientists initially count that there are 28 frogs, then after 1 month there are 52 frogs, and after 2 months there are 91 frogs. What is the carrying capacity of the frogs in this pond?

Answer available at the end of this section.

Exercise $5.1.3d$

A population of sick squirrels satisfies an almost logistic differential equation: $\frac{dA}{dt} = A\,(1 + tA)$ (where t is measured in years and A is measured in hundreds of squirrels). If after 2 months there are 600 sick squirrels, how many were sick initially?

Answer available at the end of this section.

Exercise $5.1.4a$

Two chemicals A and B are combined to form a chemical C. Initially there are 40 grams of A and 50 grams of B, and for each gram of B, 2 grams of A are used. The rate or velocity of the reactions is proportional to the product of the instantaneous amounts of A and B that are not converted to chemical C, so $\frac{dX}{dt} = k\left(40 - \frac{2}{3}X\right)\left(50 - \frac{1}{3}X\right)$. It is observed that 15 grams of C are formed in 1 minute.

(i) How much is formed in 2 minutes?
(ii) What is the limiting amount of C after a long time?
(iii) How much of chemical A and B remains after a long time?

Answer available at the end of this section.

Exercise 5.1.4b

A compound C is formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each 2 grams of A, 3 grams of B are used. The rate of the reaction is proportional to the amounts of A and B remaining, so $\frac{dX}{dt} = k\left(40 - \frac{2}{5}X\right)\left(60 - \frac{3}{5}X\right)$. It is observed that 25 grams of the compound C are formed in 4 minutes. Determine the amount of C at any time if initially there are 40 grams of A and 60 grams of B. How much of the compound C is present after 12 minutes?

Answer available at the end of this section.

Exercise 5.1.5

The population of alien droids t days after the invasion began is governed by the differential equation $\frac{dA}{dt} = \frac{A(t)}{t} - ke^{\frac{A(t)}{t}}$. If there were 800 droids spotted on day 4 of the invasion, and only 200 droids on day 8, how many will be expected on the 16th day?

Answer available at the end of this section.

Exercise 5.1.6

The amount of money you earn on your paycheck p is determined by an implicit relationship with your effort level ℓ:

$$\frac{dp}{d\ell} = \frac{28 - p - 3p^2\ell^2}{2p\ell^3 + \ell + 7}$$

If $\ell = 2$ corresponds to a paycheck of $p = 4$, what is the paycheck for $\ell = 1$?

Answer available at the end of this section.

Answers

Exercise $5.1.1a$

First, we solve the separable DE:

$$\int \frac{1}{A}\,dA = \int k\,dt \implies \ln|A| = kt + C$$

The initial amount is $A = 1875$ on day $t = 0$:

$$\ln|1875| = k \cdot 0 + C \implies C = \ln(1875)$$

On day $t = 3$ the amount is $A = 1500$:

$$\ln|1500| = k \cdot 3 + \ln(1875) \implies 3k = \ln\left(\frac{1500}{1875}\right) \implies k = \frac{1}{3}\ln\left(\frac{4}{5}\right)$$

Thus, the implicit solution is $\ln(A) = \frac{1}{3}t\ln\left(\frac{4}{5}\right) + \ln(1875)$

To solve for A when $t = 9$, we get the equation:

$$\ln(A) = \frac{1}{3} \cdot 9\ln\left(\frac{4}{5}\right) + \ln(1875) \implies A = 960$$

Therefore, there will be 960 lbs on day 9.

Exercise $5.1.1b$

First, we solve the separable DE:

$$\int \frac{1}{A}\,dA = \int k\,dt \implies \ln|A| = kt + C$$

On day $t = 1$, the amount is $A = 405$:

$$\ln|405| = k \cdot 1 + C \implies k + C = \ln(405)$$

On day $t = 3$ the amount is $A = 540$:

$$\ln|540| = k \cdot 3 + C \implies 3k + C = \ln(540)$$

This gives us a system of equations to solve:

$$\begin{cases} k + C = \ln(405) \\ 3k + C = \ln(540) \end{cases} \implies \begin{cases} k = \frac{1}{2}\ln\left(\frac{4}{3}\right) \\ C = \ln(405) - \frac{1}{2}\ln\left(\frac{4}{3}\right) \end{cases}$$

Thus, the implicit solution is $\ln(A) = \frac{1}{2}(t-1)\ln\left(\frac{4}{3}\right) + \ln(405)$

To solve for t when $A = 1280$, we get the equation:

$$\ln(1280) = \frac{1}{2}(t-1)\ln\left(\frac{4}{3}\right) + \ln(405) \implies t = 9$$

Therefore, there will be 1280 kg of ants on May 9th.

Exercise $5.1.1c$

First, we solve the separable DE:

$$\int \frac{1}{A}\,dA = \int k\,dt \implies \ln|A| = kt + C$$

On day $t = 10$, the amount is $A = 63$:

$$\ln|63| = k \cdot 10 + C \implies 10k + C = \ln(63)$$

On day $t = 25$ the amount is $A = 1701$:

$$\ln|1701| = k \cdot 25 + C \implies 25k + C = \ln(1701)$$

This gives us a system of equations to solve:

$$\begin{cases} 10k + C = \ln(63) \\ 25k + C = \ln(1701) \end{cases} \implies \begin{cases} k = \frac{1}{5}\ln(3) \\ C = \ln(7) \end{cases}$$

Thus, the implicit solution is $\ln(A) = \frac{1}{5}t\ln(3) + \ln(7)$

To solve for A when $t = 5$, we get the equation:

$$\ln(A) = \frac{1}{5} \cdot 5\ln(3) + \ln(7) \implies A = 21$$

Therefore, on day 5 there were 21 zombies.

Exercise $5.1.2a$

First, we solve the separable DE:

$$\int \frac{1}{T - T_m}\,dT = \int k\,dt \implies \ln|T - T_m| = kt + C$$

The temperature of the air outside is $T_m = 5$

The temperature at time $t = 4$ is $T = 55$:

$$\ln|55 - 5| = k \cdot 4 + C \implies 4k + C = \ln(50)$$

The temperature at time $t = 6$ is $T = 30$:

$$\ln|30 - 5| = k \cdot 6 + C \implies 6k + C = \ln(25)$$

This gives us a system of equations to solve:

$$\begin{cases} 4k + C = \ln(50) \\ 6k + C = \ln(25) \end{cases} \implies \begin{cases} k = -\frac{1}{2}\ln(2) \\ C = \ln(200) \end{cases}$$

Thus, the implicit solution is $\ln(T-5) = -\frac{1}{2}t\ln(2) + \ln(200)$

To solve for T in the room, we input $t = 0$ to get:

$$\ln(T-5) = -\frac{1}{2} \cdot 0 \cdot \ln(2) + \ln(200) \implies T = 205$$

Therefore the room has temperature 205°. (Quite warm indeed!)

Exercise $5.1.2b$

First, we solve the separable DE:

$$\int \frac{1}{T - T_m}\,dT = \int k\,dt \implies \ln|T - T_m| = kt + C$$

The temperature of the jacuzzi is $T_m = 105$

The temperature at time $t = 0$ minutes after 7pm is $T = 98.6$:

$$\ln|98.6 - 105| = k \cdot 0 + C \implies C = \ln(6.4)$$

The temperature at time $t = 15$ minutes after 7pm is $T = 100.2$:

$$\ln|100.2 - 105| = k \cdot 15 + \ln(6.4) \implies 15k = \ln\left(\frac{4.8}{6.4}\right) \implies k = \frac{1}{15}\ln\left(\frac{3}{4}\right)$$

Thus, the implicit solution is $\ln(105 - T) = \frac{1}{15}t\ln\left(\frac{3}{4}\right) + \ln(6.4)$

To solve for when the fatal temperature is reached, we input $T = 102.3$ to get:

$$\ln(105 - 102.3) = \frac{1}{15}t\ln\left(\frac{3}{4}\right) + \ln(6.4) \implies t = 45$$

Therefore, the human being should exit before 7:45 pm.

Exercise $5.1.2c$

First, we solve the separable DE:

$$\int \frac{1}{T - T_m}\,dT = \int k\,dt \implies \ln|T - T_m| = kt + C$$

The temperature of the freezer is T_m.

The initial temperature of the soup (at time $t = 0$ hours) is $T = 250$:

$$\ln|250 - T_m| = k \cdot 0 + C \implies C = \ln(250 - T_m)$$

The temperature at time $t = 1$ hours is $T = 130$:

$$\ln|130 - T_m| = k \cdot 1 + \ln(250 - T_m) \implies k = \ln\left(\frac{130 - T_m}{250 - T_m}\right)$$

The temperature at time $t = 3$ hours is $T = 40$:

$$\ln|40 - T_m| = k \cdot 3 + \ln(250 - T_m) \implies k = \frac{1}{3}\ln\left(\frac{40 - T_m}{250 - T_m}\right)$$

Setting these two expressions for k equal to each other gives:

$$\begin{aligned}
&\frac{1}{3}\ln\left(\frac{40 - T_m}{250 - T_m}\right) = \ln\left(\frac{130 - T_m}{250 - T_m}\right) \\
\implies &\left(\frac{40 - T_m}{250 - T_m}\right) = \left(\frac{130 - T_m}{250 - T_m}\right)^3 \\
\implies &(40 - T_m)(250 - T_m)^2 = (130 - T_m)^3 \\
\implies &150T_m^2 - 31800T_m + 303000 = 0 \\
\implies &T_m = 10 \text{ or } 202
\end{aligned}$$

Therefore, the temperature inside of the freezer is 10°.

Exercise $5.1.3a$

First, we plug in $a = .3$ and rewrite the DE:

$$\frac{dP}{dt} = P(.3 - bP) \implies \frac{dP}{dt} = \frac{3}{10}P\left(1 - \hat{b}P\right) \implies \frac{dP}{P(1 - \hat{b}P)} = \frac{3}{10}\,dt$$

We solve the separable DE:

$$\int \left(\frac{1}{P} - \frac{\hat{b}}{\hat{b}P - 1}\right) dP = \int \frac{10}{3}\,dt \implies \ln\left|\frac{P}{\hat{b}P - 1}\right| = \frac{10}{3}t + C$$

The carrying capacity occurs as $t \to \infty$:

$$t \to +\infty \implies \ln\left|\frac{P}{\hat{b}P - 1}\right| \to +\infty \implies \hat{b}P - 1 \to 0 \implies P \to \frac{1}{\hat{b}}$$

This population has $\lim_{t\to\infty} P(t) = 100$, so $\hat{b} = \frac{1}{100}$

Initially (at time $t = 0$), there are $P = 25$ fish:

$$\ln\left|\frac{25}{\frac{1}{100} \cdot 25 - 1}\right| = \frac{10}{3} \cdot 0 + C \implies C = \ln\left(\frac{100}{3}\right)$$

Thus, the implicit solution is $\ln\left(\frac{P}{1 - \frac{1}{100}P}\right) = \frac{10}{3}t + \ln\left(\frac{100}{3}\right)$

To solve for the time t such that gives a population of 80, we input $P = 80$ to get:

$$\ln\left(\frac{80}{1 - \frac{1}{100} \cdot 80}\right) = \frac{10}{3}t + \ln\left(\frac{100}{3}\right) \implies t = \frac{10}{3}\ln(12)$$

Therefore, the population will reach 80 fish after $\frac{10}{3}\ln(12) \approx 8.318$ years.

Exercise $5.1.3b$

We solve the separable DE:

$$\int \left(\frac{1}{P} + \frac{b}{a - bP}\right) dP = \int a\,dt \implies \ln\left|\frac{P}{a - bP}\right| = at + C$$

We are given $P(0) = \frac{a}{2b}$:

$$\ln\left|\frac{\frac{a}{2b}}{a - b \cdot \frac{a}{2b}}\right| = a \cdot 0 + C \implies C = \ln\left(\frac{1}{b}\right)$$

Thus, the implicit solution is:

$$\ln\left(\frac{P}{a - bP}\right) = at + \ln\left(\frac{1}{b}\right) \implies t = \ln\left(\frac{bP}{a - bP}\right)$$

Recall inflection points are where $P''(t) = 0$

So we take the implicit derivative: $\frac{d}{dt}\left(\frac{dP}{dt} = P(a-bP)\right)$:

$$\begin{aligned}\frac{d^2P}{dt^2} &= \frac{dP}{dt}\cdot(a-bP) + P\cdot\left(-b\frac{dP}{dt}\right)\\ &= P(a-bP)\cdot(a-bP) - bP\cdot P(a-bP)\\ &= P(a-bP)\left[(a-bP)-bP\right]\\ &= P(a-bP)(a-2bP)\end{aligned}$$

Setting this equal to zero we see that the roots are $P=0,\ \frac{a}{2b}$, and $\frac{a}{b}$

For $P=0$ and $P=\frac{a}{b}$ there is no solution for t.

For $P=\frac{a}{2b}$, we get $t=\ln\left(\frac{b\cdot\frac{a}{2b}}{a-b\cdot\frac{a}{2b}}\right)=\ln(1)=0$

Therefore, the inflection point for this solution is at the initial point $\left(0,\frac{a}{2b}\right)$

Exercise $5.1.3c$

We solve the separable DE:

$$\int\left(\frac{1}{P}+\frac{b}{a-bP}\right)dP = \int a\,dt \implies \ln\left|\frac{P}{a-bP}\right| = at+C \implies P(t)=\frac{\frac{a}{b}}{1+\hat{C}e^{-at}}$$

We want to know $M=\lim_{t\to\infty}P(t)=\frac{a}{b}$, so we rewrite it as $P(t)=\frac{M}{1+C\cdot\alpha^t}$

Initially there are 28 frogs, so when $t=0$ we have $P=28$:

$$28=\frac{M}{1+\hat{C}\cdot\alpha^0} \implies 28(1+\hat{C})=M$$

After 1 month there are 52 frogs, so when $t=1$ we have $P=52$:

$$52=\frac{M}{1+\hat{C}\cdot\alpha^1} \implies 52(1+\hat{C}\alpha)=M \implies \hat{C}\alpha=\frac{28(1+\hat{C})}{52}-1$$

After 2 months there are 91 frogs, so when $t=2$ we have $P=91$:

$$91=\frac{M}{1+\hat{C}\cdot\alpha^2} \implies 91(1+\hat{C}\alpha^2)=M \implies \hat{C}\alpha^2=\frac{28(1+\hat{C})}{91}-1$$

Dividing these last two equations gives:

$$\frac{\hat{C}\alpha^2}{\hat{C}\alpha}=\frac{\frac{4}{13}(1+\hat{C})-1}{\frac{7}{13}(1+\hat{C})-1} \implies \alpha=\frac{4\hat{C}-9}{7\hat{C}-6}$$

Now we substitute this α into the $t=1$ equation:

$$\begin{aligned}&\hat{C}\left(\frac{4\hat{C}-9}{7\hat{C}-6}\right)=\frac{7}{13}(1+\hat{C})-1\\ \implies &\hat{C}^2-11\hat{C}-12=0\\ \implies &C=-1,12\end{aligned}$$

Since $\hat{C}=-1$ is impossible, we have $\hat{C}=12$. This gives $M=28(1+12)=28\cdot13=364$

Therefore, the carrying capacity of this pond is 364 frogs.

Exercise $5.1.3d$

We expand out the DE: $A'-A=tA^2$

This is Bernoulli, so we substitute $u=A^{1-2}=A^{-1}\leftrightarrow\frac{du}{dt}=-A^{-2}\cdot\frac{dA}{dt}$:

$$-\frac{du}{dt}-u=t \implies \frac{du}{dt}+1\cdot u=-t$$

This is linear, with $\mu(t)=e^{\int 1\,dt}=e^t$:

$$u(t)=\frac{1}{\mu(t)}\int\mu(t)\cdot q(t)\,dt=\frac{1}{e^t}\int e^t\cdot(-t)\,dt=e^{-t}\left((1-t)\,e^t+D\right)=1-t+De^{-t}$$

Thus, $u=A^{-1} \implies A=u^{-1}=\frac{1}{1-t+De^{-t}}$

Plugging in the given information at 2 months, $A\left(\frac{1}{6}\right)=6$, we get:

$$6=\frac{1}{\frac{5}{6}+De^{-\frac{1}{6}}} \implies 5+6De^{-\frac{1}{6}}=1 \implies D=\frac{2}{3}e^{\frac{1}{6}}$$

We want to find $A(0)=\frac{1}{1+D}=\frac{1}{1+\frac{2}{3}e^{\frac{1}{6}}}$.

Therefore, there were initially $\frac{100}{1+\frac{2}{3}e^{\frac{1}{6}}}\approx471$ sick squirrels.

Exercise $5.1.4a$

We let $X(t)$ be the grams of C at time t, which is made out of $\frac{2}{3}X$ from A and $\frac{1}{3}X$ is made out of B:

$$\frac{dX}{dt}=k\left(40-\frac{2}{3}X\right)\left(50-\frac{1}{3}X\right) \implies \frac{dX}{dt}=k(60-X)(150-X)$$

This is separable

$$\int \frac{1}{(60-X)(150-X)}\,dX = \int k\,dt$$
$$\int \frac{\frac{1}{90}}{60-X} - \frac{\frac{1}{90}}{150-X}\,dX = kt + C$$
$$\frac{1}{90}\ln\left|\frac{150-X}{60-X}\right| = kt + C$$
$$\left|\frac{150-X}{60-X}\right| = e^{90kt+C}$$
$$1 + \frac{90}{60-X} = Ce^{90kt}$$
$$X(t) = 60 - \frac{90}{Ce^{90kt}-1}$$

The initial condition (starting with zero grams of C) is $X(0) = 0$:

$$0 = 60 - \frac{90}{C-1} \implies C = \frac{5}{2}$$

We are given that $X(1) = 15$:

$$15 = 60 - \frac{90}{\frac{5}{2}e^{90k}-1} \implies e^{90k} = \frac{6}{5}$$

Thus, $X(t) = 60 - \dfrac{180}{5\left(\frac{6}{5}\right)^t - 2}$ for $t \geq 0$

(i) At time $t = 2$ minutes there is $X(2) = 60 - \dfrac{180}{5\cdot\frac{36}{25} - 2} = 60 - \dfrac{450}{13} = \dfrac{330}{13} \approx 25.4$ grams.

(ii) The limiting amount of C is $\lim\limits_{\to\infty} X(t) = \lim\limits_{t\to\infty} 60 - \dfrac{180}{5\left(\frac{6}{5}\right)^t - 2} = 60 - 0 = 60$

Therefore, the limiting amount is 60 grams.

(iii) This makes sense since that would "use up" all 40 grams of material A, and use 20 grams of material B, leaving exactly 30 grams of B.

Exercise $5.1.4b$

We let $X(t)$ be the grams of C at time t, which is made out of $\frac{2}{5}A$ and $\frac{3}{5}B$:

$$\frac{dX}{dt} = k\left(40 - \frac{2}{5}X\right)\left(60 - \frac{3}{5}X\right) \implies \frac{dX}{dt} = k(100-X)^2$$

This is separable:

$$\int \frac{1}{(100-X)^2}\,dX = \int k\,dt \implies \frac{1}{100-X} = kt + C \implies X(t) = 100 - \frac{1}{kt+C}$$

The initial condition (starting with zero grams of C) is $X(0) = 0$:

$$0 = 100 - \frac{1}{k\cdot 0 + C} \implies C = \frac{1}{100}$$

We are given that $X(4) = 25$:

$$25 = 100 - \frac{1}{k\cdot 4 + \frac{1}{100}} \implies k = \frac{1}{1200}$$

Therefore, $X(t) = 100 - \dfrac{1}{\frac{1}{1200}t + \frac{1}{100}} = 100 - \dfrac{1200}{t+12}$ for $t \geq 0$

At time $t = 12$ minutes there is $X(12) = 100 - \dfrac{1200}{12+12} = 50$ grams of compound C.

Exercise $5.1.5$

We do substitution to solve the DE, $u = \dfrac{A}{t} \longleftrightarrow t\cdot u' + 1\cdot u = A'$:

$$t\cdot\frac{du}{dt} + 1\cdot u = u - ke^u$$
$$\int e^{-u}\,du = \int -\frac{k}{t}dt$$
$$-e^{-u} = -k\ln(t) + C$$
$$u = -\ln\left(k\ln(t) + C\right)$$
$$A(t) = -t\ln\left(k\ln(t) + C\right)$$

Plugging in $A(4) = 800$ we get:

$$800 = -4\ln\left(k\ln(4) + C\right) \implies e^{-200} = 2k\ln(2) + C$$

On the other hand $A(8) = 200$ gives us:

$$200 = -8\ln(k\ln(8) + C) \implies e^{-25} = 3k\ln(2) + C$$

Subtracting these gives:

$$k\ln(2) = e^{-25} - e^{-200} \implies k = \frac{e^{-25} - e^{-200}}{\ln(2)}$$

Plugging this into either of the initial condition equations gives $C = 3e^{-200} - 2e^{-25}$

Thus, the function is $A(t) = -t\ln\left(\frac{e^{-25}-e^{-200}}{\ln(2)}\ln(t) + 3e^{-200} - 2e^{-25}\right)$

On day 16 there will be $A(16) = -16\ln\left((e^{-25} - e^{-200}) \cdot \dfrac{\ln(16)}{\ln(2)} + 3e^{-200} - 2e^{-25}\right) = -16\left(2e^{-25} - e^{-200}\right)$

Therefore, there will be $-16\ln\left(2e^{-25} - e^{-200}\right) \approx 389$ droids on day 16.

Exercise $5.1.6$

This re-writes into: $(2p\ell^3 + \ell + 7)dp + (3p^2\ell^2 + p - 28)d\ell = 0$

This is an exact DE:

$$\frac{\partial}{\partial \ell}(2p\ell^3 + \ell + 7) = 6p\ell^2 + 1 \quad \text{and} \quad \frac{\partial}{\partial p}(3p^2\ell^2 + p - 28) = 6p\ell^2 + 1$$

We integrate:

$$\int (2p\ell^3 + \ell + 7)\, dp = p^2\ell^3 + \ell p + 7p + C(\ell)$$
$$\int (3p^2\ell^2 + p - 28)\, d\ell = p^2\ell^3 + \ell p - 28\ell + D(p)$$

Thus, the general implicit solution is $p^2\ell^3 + \ell p + 7p - 28\ell = K$

Plugging in the initial condition $p = 4, \ell = 2 \implies K = 108$

Finally, we solve for p when $\ell = 1$:

$$p^2 + 8p - 28 = 108 \implies p = -4 \pm 2\sqrt{38}$$

Since $p = -4 - 2\sqrt{38}$ doesn't make sense, an effort level of $\ell = 1$ yields a paycheck of $p = 2\sqrt{38} - 4 \approx 8.329$

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5.2: Orthogonal Curves

Learning Objectives

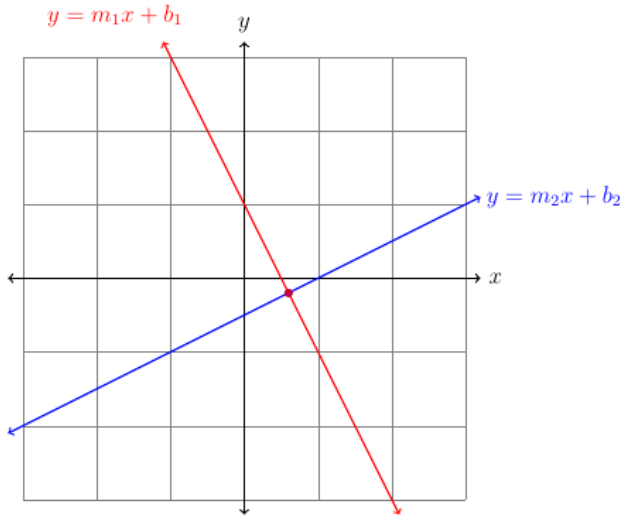
- Determine if two curves are orthogonal.
- Find orthogonal trajectories of families of curves.

This subchapter focuses on finding curves that meet at a right angle at each point of intersection. We define when two curves are considered orthogonal, and determine how to find an orthogonal family of curves to a given curve or curves.

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5.2: Orthogonal Trajectories

In algebra, we say that two lines are *perpendicular* when they intersect at a right angle.



We check for this by examining the product of their slopes:

$$y = m_1x + b_1 \text{ is perpendicular to } y = m_2x + b_2 \quad \text{ when } m_1m_2 = -1$$

This shortcut is actually the direct result of the formula for the angle between two vectors: $\cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$. The angle θ must be 90° when the dot product of two vectors is zero. In this case we say the vectors are **orthogonal**. (The word orthogonal in this context ultimately means the same thing as "meet at a right angle".)

There are other curves besides lines that also meet at a right angle whenever they intersect, and we call these orthogonal curves. If they are not lines, this means that their instantaneous slopes will have product -1.

Definition: Orthogonal Curves

Two curves y_1 and y_2 are ***orthogonal curves*** if $\dfrac{dy_1}{dx} \cdot \dfrac{dy_2}{dx} = -1$ at every point of intersection.

If we know one of these curves, then solving for the second is simply a differential equation!

Illustration 5.2.1

What curve is orthogonal to $y = x^3$ at every intersection point?

Solution

We see that the original function $y = x^3$ has $\dfrac{dy}{dx} = 3x^2$

Since we cannot predict "where" the intersection will occur before we find the second curve, the safest move is to ensure orthogonality at any possible point (x, y) by solving the differential equation:

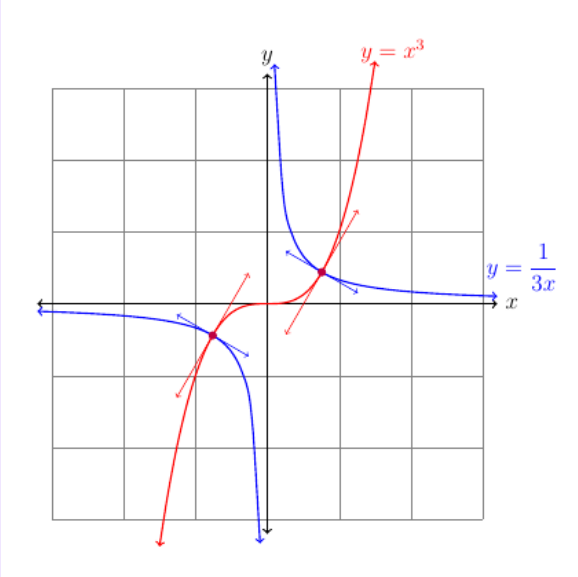
$$3x^2 \cdot \frac{dy}{dx} = -1$$

This is separable, so we can separate and integrate:

$$\int dy = \int -\frac{1}{3x^2}\,dx \implies y = \frac{1}{3x} + C$$

Thus, any curve of the form $y = \frac{1}{3x} + C$ will be orthogonal to $y = x^3$ whenever they intersect.

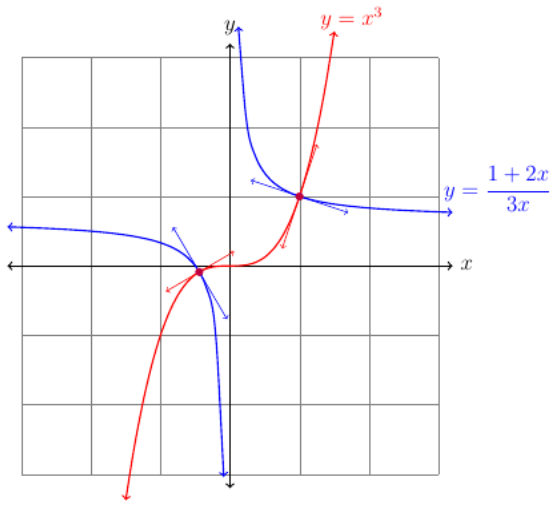
Consider the curve when $C = 0$, so $y = \frac{1}{3x}$. These curves meet when $x^3 = \frac{1}{3x} \implies x = \pm 3^{-\frac{1}{4}}$, so their two points of intersection are $\left(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{27}}\right)$ and $\left(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{27}}\right)$.



At both of these points, the slope of $y = x^3$ is $\sqrt{3}$ and the slope of $y = \frac{1}{3x}$ is $-\frac{1}{\sqrt{3}}$

For another solution, consider the curve when $C = \frac{2}{3}$, so $y = \frac{1+2x}{3x}$. These curves meet when $x^3 = \frac{1+2x}{3x} \implies (x-1)(3x^3 + 3x^2 + 3x + 1) = 0$. This yields two real roots, $x = 1$ and $x = -\frac{1}{1+2^{\frac{1}{3}}}$.

Thus the two intersection points are at $\left(1, \frac{1}{3}\right)$ and $\left(-\frac{1}{1+2^{\frac{1}{3}}}, \frac{1-2^{\frac{1}{3}}}{3}\right)$



At $x = 1$, the slope of $y = x^3$ is 3 and the slope of $y = \dfrac{1+2x}{3x}$ is $-\dfrac{1}{3}$

At $x = -\dfrac{1}{1+2^{\frac{1}{3}}}$, the slope of $y = x^3$ is $2^{\frac{2}{3}} - 1$ and the slope of $y = \dfrac{1+2x}{3x}$ is $\dfrac{1}{1-2^{\frac{1}{3}}}$

In the above example we found infinitely many different curves that are *all* orthogonal to the curve $y = x^3$ at each of their intersection points. We say the original function $y = x^3$ is an *orthogonal trajectory* to the family $y = \dfrac{1}{3x} + C$ since it is orthogonal to all members of that family.

Definition: Orthogonal Trajectories

When all the curves in one family of curves $G(x, y, c_1) = 0$ intersect orthogonally with all of the curves of another family of curves $H(x, y, c_2) = 0$, the families are called ***orthogonal trajectories*** of each other.

Whenever we want to find the orthogonal trajectories to a given family of curves, we first find the differential equation they satisfy: $\dfrac{dy}{dx} = f(x, y)$

Rule: Finding Orthogonal Trajectories

The family containing its orthogonal trajectories to the solution set of $\dfrac{dy}{dx} = f(x, y)$ will then be the solution set to the differential equation $\dfrac{dy}{dx} = -\dfrac{1}{f(x, y)}$

Illustration 5.2.2

We will find the orthogonal trajectories to the family of curves $y = \dfrac{c}{x}$:

First we find the differential equation that yields $y = \dfrac{c}{x}$ as its solution set.

It is easy to see the derivative of any member of the family is $\dfrac{dy}{dx} = -\dfrac{c}{x^2}$, but this is not a differential equation!

Since c can be any constant, if we tried to solve $\frac{dy}{dx} = -\frac{c}{x^2}$ via separation we would get:

$$\int dy = \int -cx^{-2}\,dx \implies y = \frac{c}{x} + D$$

Clearly this is an entirely different family of functions. To get the correct $f(x, y)$, we need to eliminate c.

This can be done by solving for c in the original equation $y = \frac{c}{x} \longleftrightarrow c = xy$

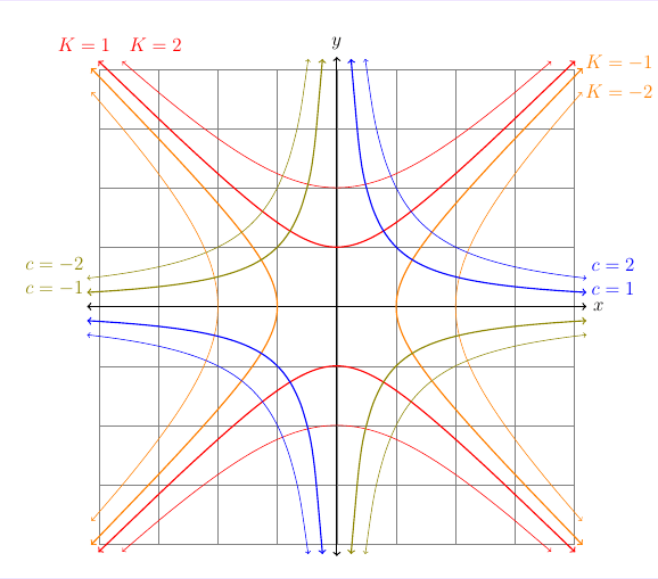
Plugging this into our derivative gives $\frac{dy}{dx} = -\frac{xy}{x^2} = -\frac{y}{x}$. This is the differential equation that yields our original family as a solution set:

$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx \implies \ln|y| = -\ln|x| + C \implies y = \frac{C}{x}$$

Thus the correct function is $f(x, y) = -\frac{y}{x}$. Now we can set up the differential equation to solve for the second family: $\frac{dy}{dx} = -\frac{1}{f(x, y)} = \frac{x}{y}$. This is also separable:

$$\int y\,dy = \int x\,dx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + C \implies y^2 - x^2 = K$$

Thus the family of hyperbolas $y^2 - x^2 = K$ is orthogonal to the family of functions $y = \frac{c}{x}$ at every point of intersection, as we can see below for various c and K:



Example 5.2.1

Find the orthogonal trajectories to the family of functions $y = \frac{cx}{2 + 3x}$

Solution

First, we find the differential equation that yields $y = \frac{cx}{2 + 3x}$ as its solution set.

We take its derivative $\frac{dy}{dx} = \frac{2c}{(2 + 3x)^2}$.

Solving for c in the original equation gives $c = \dfrac{y(2+3x)}{x}$, so we substitute it to find

$$f(x,y) = \frac{2 \cdot \frac{y(2+3x)}{x}}{(2+3x)^2} = \frac{2y}{x(2+3x)} = \frac{2y}{2x+3x^2}$$

Therefore, we need to solve the differential equation $\dfrac{dy}{dx} = -\dfrac{1}{f(x,y)} = -\dfrac{2x+3x^2}{2y}$, which is separable:

$$\int -2y\,dy = \int 2x + 3x^2\,dx \implies -y^2 = x^2 + x^3 + K$$

We have found that the family of orthogonal trajectories consists of the curves of the form $\boxed{y^2 = K - x^2 - x^3}$

Beyond their geometric elegance, orthogonal trajectories serve as a vital tool in physics and engineering. They allow us to map physical phenomena where forces and potentials interact: in *electrostatics*, electric field lines are orthogonal to equipotential curves; in *thermodynamics*, heat flow lines are orthogonal to isotherms; and in *fluid dynamics*, streamlines are orthogonal to velocity potentials. Mastering these techniques provides the mathematical language necessary to model how energy and matter navigate physical space.

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5.2: Practice Set

Exercise 5.2.1

Find the orthogonal trajectories to the family of curves $y = cx$

Answer available at the end of this section.

Exercise 5.2.2

Find the orthogonal trajectories to the family of curves $y = cx^2$

Answer available at the end of this section.

Exercise 5.2.3

Find the orthogonal trajectories to the family of curves $xy = c$

Answer available at the end of this section.

Exercise 5.2.4

Find the orthogonal trajectories to the family of curves $y = ce^{-x}$

Answer available at the end of this section.

Exercise 5.2.5

Find the orthogonal trajectories to the family of curves $\sin(y) = ce^x$

Answer available at the end of this section.

Exercise 5.2.6

Find the orthogonal trajectories to the family of curves $x^2 + y^2 = 2cx$

Answer available at the end of this section.

Exercise 5.2.7

Find the orthogonal trajectories to the family of curves $y = \dfrac{x}{1 + cx}$

Answer available at the end of this section.

Exercise 5.2.8

Find the orthogonal trajectories to the family of curves $x^2 + cy^2 = 1$

Answer available at the end of this section.

Exercise 5.2.9

Find the orthogonal trajectories to the family of curves $y^2 = cx^3$

Answer available at the end of this section.

Exercise 5.2.10

Find the orthogonal trajectories to the family of curves $x + y = 1 + ce^{-y}$

Answer available at the end of this section.

Answers

Exercise 5.2.1

First, we find the differential equation that yields $y = cx$ as its solution set. We take its derivative:

$$\frac{dy}{dx} = c$$

Solving for c in the original equation gives $c = \frac{y}{x}$, so we substitute it to find $\frac{dy}{dx} = f(x, y)$:

$$f(x, y) = \frac{y}{x}$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} = -\frac{x}{y}$$

This is separable:

$$\int y\,dy = \int -x\,dx \implies \frac{1}{2}y^2 = -\frac{1}{2}x^2 + K \implies x^2 + y^2 = K$$

We have found that the family of orthogonal trajectories consists of the curves of the form $x^2 + y^2 = K$

Exercise 5.2.2

First, we find the differential equation that yields $y = cx^2$ as its solution set. We take its derivative:

$$\frac{dy}{dx} = 2cx$$

Solving for c in the original equation gives $c = \frac{y}{x^2}$, so we substitute it to find $\frac{dy}{dx} = f(x, y)$:

$$f(x, y) = 2 \cdot \frac{y}{x^2} \cdot x = \frac{2y}{x}$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} = -\frac{x}{2y}$$

This is separable:

$$\int 2y\,dy = \int -x\,dx \implies y^2 = -\frac{1}{2}x^2 + K \implies x^2 + 2y^2 = K$$

We have found that the family of orthogonal trajectories consists of the curves of the form $x^2 + 2y^2 = K$

Exercise 5.2.3

First, we find the differential equation that yields $xy = c$ as its solution set. We take its derivative:

$$y + x\frac{dy}{dx} = 0 \implies = -\frac{y}{x}$$

Since there is no c present, we see that $\frac{dy}{dx} = f(x, y) = -\frac{y}{x}$.

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = \frac{x}{y}$$

This is separable:

$$\int y\,dy = \int x\,dx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + K \implies y^2 - x^2 = K$$

We have found that the family of orthogonal trajectories consists of the curves of the form $y^2 - x^2 = K$

Exercise 5.2.4

First, we find the differential equation that yields $y = ce^{-x}$ as its solution set. We take its derivative:

$$\frac{dy}{dx} = -ce^{-x}$$

Solving for c in the original equation gives $c = ye^{x}$, so we substitute it to find $\frac{dy}{dx} = f(x,y)$:

$$f(x,y) = -ye^{x} \cdot e^{-x} = -y$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = \frac{1}{y}$$

This is separable:

$$\int y\,dy = \int 1\,dx \implies \frac{1}{2}y^2 = x + K \implies y^2 = 2x + K$$

We have found that the family of orthogonal trajectories consists of the curves of the form $y^2 = 2x + K$

Exercise 5.2.5

First, we find the differential equation that yields $\sin(y) = ce^{x}$ as its solution set. We take its derivative:

$$\cos(y) \cdot \frac{dy}{dx} = ce^{x} \implies \frac{dy}{dx} = ce^{x}\sec(y)$$

Solving for c in the original equation gives $c = e^{-x}\sin(y)$, so we substitute it to find $\frac{dy}{dx} = f(x,y)$:

$$f(x,y) = e^{-x}\sin(y) \cdot e^{x}\sec(y) = \tan(y)$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = -\cot(y)$$

This is separable:

$$\int \tan(y)\,dy = \int -1\,dx \implies \ln|\sec(y)| = -x + K \implies y = \sec^{-1}\left(Ke^{-x}\right)$$

We have found that the family of orthogonal trajectories consists of the curves of the form $y = \sec^{-1}\left(Ke^{-x}\right)$

Exercise 5.2.6

First, we find the differential equation that yields $x^2 + y^2 = 2cx$ as its solution set. We take its derivative:

$$2x + 2y \cdot \frac{dy}{dx} = 2c \implies \frac{dy}{dx} = \frac{c - x}{y}$$

Solving for c in the original equation gives $c = \dfrac{x^2 + y^2}{2x}$, so we substitute it to find $\frac{dy}{dx} = f(x,y)$:

$$f(x,y) = \frac{\left(\frac{x^2+y^2}{2x}\right) - x}{y} = \frac{y^2 - x^2}{2xy}$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = -\frac{2xy}{y^2 - x^2}$$

This non-linear DE looks much nicer after substituting $u = \dfrac{y}{x}$:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0 \implies \left(u^2x^2 - x^2\right) \cdot \left(u + x \cdot \frac{du}{dx}\right) + 2x \cdot ux = 0$$

Now it is separable:

$$\int \frac{1 - u^2}{u(1 + u^2)}\,du = \int \frac{1}{x}\,dx \implies \ln\left|\frac{u}{1 + u^2}\right| = \ln|x| + K$$

We can substitute back $u = \dfrac{y}{x}$ and rewrite this relationship as: $\dfrac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2} = Kx$

We have found that the family of orthogonal trajectories consists of the curves of the form $x^2 + y^2 = Ky$

Exercise 5.2.7

First, we find the differential equation that yields $y = \dfrac{x}{1 + cx}$ as its solution set. We take its derivative:

$$\frac{dy}{dx} = \frac{(1 + cx) - x \cdot c}{(1 + cx)^2} = \frac{1}{(1 + cx)^2}$$

Solving for c in the original equation gives $c = \dfrac{1}{y} - \dfrac{1}{x}$, so we substitute it to find $\frac{dy}{dx} = f(x,y)$:

$$f(x,y) = \frac{1}{\left(1+\left(\frac{1}{y}-\frac{1}{x}\right)x\right)^2} = \frac{y^2}{x^2}$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = -\frac{x^2}{y^2}$$

This is separable:

$$\int y^2\,dy = \int -x^2\,dx \implies \frac{1}{3}y^3 = -\frac{1}{3}x^3 + K$$

We have found that the family of orthogonal trajectories consists of the curves of the form $x^3 + y^3 = K$

Exercise 5.2.8

First, we find the differential equation that yields $x^2 + cy^2 = 1$ as its solution set. We take its derivative:

$$2x + 2cy \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{cy}$$

Solving for c in the original equation gives $c = \dfrac{1-x^2}{y^2}$, so we substitute it to find $\dfrac{dy}{dx} = f(x,y)$:

$$f(x,y) = \frac{x}{\left(\frac{1-x^2}{y^2}\right)y} = \frac{xy}{1-x^2}$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = \frac{x^2-1}{xy}$$

This non-linear DE looks much nicer after substituting $u = y^2$:

$$xy \cdot \frac{dy}{dx} = x^2 - 1 \implies x \cdot \frac{1}{2}\frac{du}{dx} = x^2 - 1$$

This is separable:

$$\int 1\,du = \int 2x - \frac{2}{x}\,dx \implies u = x^2 - 2\ln|x| + K$$

We can substitute back $u = y^2$ and rewrite this relationship as: $y^2 = x^2 - 2x\ln|x| + K$

We have found that the family of orthogonal trajectories consists of the curves of the form $y^2 = x^2 - 2x\ln|x| + K$

Exercise 5.2.9

First, we find the differential equation that yields $y^2 = cx^3$ as its solution set. We take its derivative:

$$2y \cdot \frac{dy}{dx} = 3cx^2 \implies \frac{dy}{dx} = \frac{3cx^2}{2y}$$

Solving for c in the original equation gives $c = \dfrac{y^2}{x^3}$, so we substitute it to find $\dfrac{dy}{dx} = f(x,y)$:

$$f(x,y) = \frac{3\left(\frac{y^2}{x^3}\right)x^2}{2y} = \frac{3y}{2x}$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = -\frac{2x}{3y}$$

This is separable:

$$\int 3y\,dy = \int -2x\,dx \implies \frac{3}{2}y^2 = -x^2 + K$$

We have found that the family of orthogonal trajectories consists of the curves of the form $2x^2 + 3y^2 = K$

Exercise 5.2.10

First, we find the differential equation that yields $y = x\left(\ln|x| + c\right)$ as its solution set. We take its derivative:

$$1 + \frac{dy}{dx} = -ce^{-y}\frac{dy}{dx} \implies \frac{dy}{dx} = -\frac{1}{ce^{-y}+1}$$

Solving for c in the original equation gives $c = (x+y-1)e^y$, so we substitute it to find $\dfrac{dy}{dx} = f(x,y)$:

$$f(x,y) = -\frac{1}{(x+y-1)e^y \cdot e^{-y} + 1} = -\frac{1}{x+y}$$

Therefore, we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = x + y$$

This is linear:

$$\frac{dy}{dx} - y = x \implies y = Ke^x - x - 1$$

We have found that the family of orthogonal trajectories consists of the curves of the form $y = Ke^x - x - 1$

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5.3: Mixtures

Learning Objectives

- Recognize and solve basic single tank mixing problems
- Set up and solve multi-tank mixing problems using systems of linear differential equations.

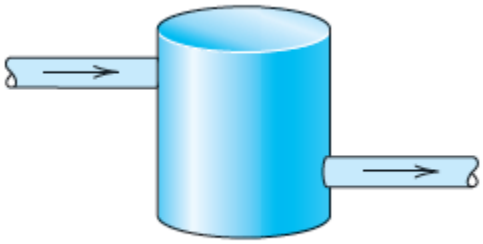
Mixing problems provide a tangible application of first-order differential equations, modeling how substances blend and concentrations change in a dynamic system. Whether tracking salt dissolved in a brine tank, pollutants in a lake, or chemicals in a reactor, these scenarios connect directly to real-world systems by balancing rates of inflow and outflow.

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5.3: Mixture Problems

Differential equations often model scenarios where the rate of change depends on both an "inward" and "outward" flow.

Imagine a tank of water with liquid being pumped in, sufficiently mixed around, and then pumped out at the same time.



If we wish to track how much of a particular substance will be inside of the tank at a given time, we can use the differential equation:

$$\frac{dA}{dt} = \delta_{in} \cdot R_{in} - \delta_{out} \cdot R_{out}$$

Here δ_{in} and δ_{out} represent the density of that substance being pumped in and out of the tank respectively, while R_{in} and R_{out} represent the rate at which the liquids are being pumped in and out of the tank. Generally, $\delta_{in}, R_{in},$ and R_{out} are given in the problem, and δ_{out} depends on the density of the mixed liquids inside of the tank. If the tank has volume $V(t)$ at time t, then this density must be $\delta_{out} = \dfrac{A(t)}{V(t)}$

Thus, the differential equation takes the following form:

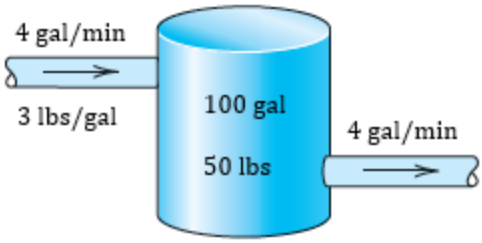
Rule: Differential Equation for Mixture Problems

$$\frac{dA}{dt} = \delta_{in} \cdot R_{in} - \frac{A}{V(t)} \cdot R_{out}$$

This is a linear differential equation for $A(t)$, which we can solve to answer any question about the mixture that we are asked.

Example 5.3.1

An aquarium is filled with 100 gallons of water containing 50 lbs of salt in it. Brine containing 3 lbs of salt per gallon is pumped into the tank at 4 gallons per minute, and the well mixed solution is pumped out of the other side at 4 gallons per minute. How long will it take for there to be exactly 200 lbs of salt inside of the tank?



Solution

We are given the constants $\delta_{in} = 3$ lbs per gallon, $R_{in} = 4$ gallons per minute, and $R_{out} = 4$ gallons per minute.

Since the water pumped in and out is equal at all times, the volume remains constant: $V(t) = 100$

Therefore, the differential equation modeling the lbs of salt in the tank is

$$\frac{dA}{dt} = 3 \cdot 4 - \frac{A}{100} \cdot 4$$

Writing this in our standard linear form gives $\frac{dA}{dt} + \frac{1}{25}A = 12$

Therefore, we use $\mu(x) = e^{\int \frac{1}{25}\,dt} = e^{\frac{1}{25}t}$:

$$A(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\,dt = \frac{1}{e^{\frac{1}{25}t}} \int e^{\frac{1}{25}} \cdot 12\,dt = e^{-\frac{1}{25}t} \cdot \left[300e^{\frac{1}{25}t} + C\right] = 300 + Ce^{-\frac{1}{25}t}$$

We know the initial amount of salt in the tank was $A(0) = 50$, so we can solve for C:

$$A(0) = 300 + Ce^{-\frac{1}{25}\cdot 0} \implies 50 = 300 + C \cdot 1 \implies C = -250$$

Thus, our solution function is $A(t) = 300 - 250e^{-\frac{1}{25}t}$.

To find out when the tank contains exactly 200 lbs of salt, we solve for t:

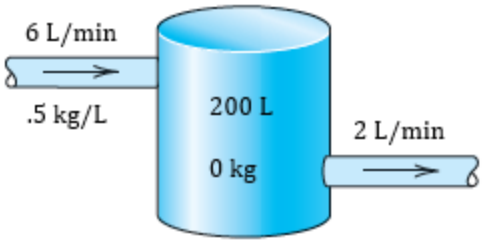
$$200 = 300 - 250e^{-\frac{1}{25}t} \implies t = 25 \ln\left(\frac{5}{2}\right) \approx 22.9$$

We have shown that there will be 200 lbs of salt in the tank after approximately $\boxed{22.9 \text{ minutes}}$

The volume of the tank does not have to remain constant for us to solve these types of questions.

Example 5.3.2

A pool is filled with 200 Liters of pure water. Water containing $\frac{1}{2}$ kg of chlorine per Liter is pumped into the pool at 6 Liters per minute, and the well mixed solution leaks out of a crack in the side at 2 Liters per minute. What is the density of the chlorine in the pool after 2.5 hours?



Solution

We are given the constants $\delta_{in} = \frac{1}{2}$ kg per Liter, $R_{in} = 6$ Liters per minute, and $R_{out} = 2$ Liters per minute.

Since the water pumped in and out is not equal, the volume changes: $V(t) = 200 + (R_{in} - R_{out}) \cdot t = 200 + 4t$ Liters

Therefore, the differential equation modeling the lbs of salt in the tank is

$$\frac{dA}{dt} = \frac{1}{2} \cdot 6 - \frac{A}{200 + 4t} \cdot 2$$

Writing this in our standard linear form gives $\frac{dA}{dt} + \frac{1}{100 + 2t} A = 3$

Therefore, we use $\mu(x) = e^{\int \frac{1}{100+2t}\, dt} = e^{\frac{1}{2}\ln|100+2t|} = \sqrt{100 + 2t}$:

$$A(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\, dt = \frac{1}{\sqrt{100 + 2t}} \int \sqrt{100 + 2t} \cdot 3\, dt = (100 + 2t)^{-\frac{1}{2}} \cdot \left[(100 + 2t)^{\frac{3}{2}} + C\right] = 100 + 2t + \frac{C}{\sqrt{100 + 2t}}$$

We know the initial amount of salt in the tank was $A(0) = 0$, so we can solve for C:

$$A(0) = 100 + 2 \cdot 0 + \frac{C}{\sqrt{100 + 2 \cdot 0}} \implies 0 = 100 + \frac{C}{10} \implies C = -1000$$

Thus, our solution function is $A(t) = 100 + 2t - \frac{1000}{\sqrt{100 + 2t}}$

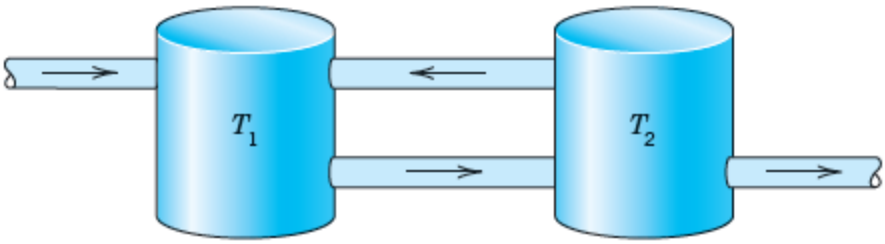
We plug in $t = 150$ to find the total amount of chlorine after 2.5 hours:

$$A(150) = 100 + 2 \cdot 150 - \frac{1000}{\sqrt{100 + 2 \cdot 150}} = 400 - \frac{1000}{\sqrt{400}} = 400 - 50 = 350 \text{ kg}$$

The volume at this time is $V(150) = 200 + 4 \cdot 150 = 800$ Liters. Therefore the density is $\boxed{\frac{350}{800} = \frac{7}{16} \text{ kg/L}}$

We can also handle more than one tank by setting up systems of LDEs.

For example, if there were two tanks containing an amount $A_1(t)$ and $A_2(t)$ of a specific chemical, and they pump water between themselves:

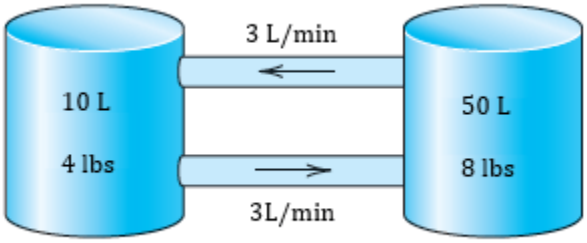


Then, for each tank, we have $\frac{dA_i}{dt} = \delta_{in} \cdot R_{in} - \delta_{out} \cdot R_{out}$

This gives us a system of equations to solve.

Example 5.3.3

Two fish tanks are attached via pumps so that they each pump in 3 L/min and pump out 3 L/min, maintaining their respective volumes. If Tank A initially contains 4 lbs of salt in 10 L of water, and Tank B initially contains 8 lbs of salt in 50 L of water, how much of the salt is in Tank A after 2 hours?



Solution

We are given the constants $R_{in} = R_{out} = 3$ Liters per minute for both tanks.

Since the water pumped in and out is equal, the volumes stay constant: $V_A(t) = 10$ and $V_B(t) = 50$.

Therefore, the differential equations modeling the lbs of salt in the tanks are

$$\begin{cases} \frac{dA}{dt} = \frac{B}{50} \cdot 3 - \frac{A}{10} \cdot 3 \\ \frac{dB}{dt} = \frac{A}{10} \cdot 3 - \frac{B}{50} \cdot 3 \end{cases} \implies \begin{cases} \frac{dA}{dt} = -0.3A + 0.06B \\ \frac{dB}{dt} = 0.3A - 0.06B \end{cases}$$

Since we know $A(0) = 4$ and $B(0) = 8$, we can use Laplace Transformations to solve this system:

$$\begin{cases} s\hat{A}(s) - 4 = -0.3\hat{A}(s) + 0.06\hat{B}(s) \\ s\hat{B}(s) - 8 = 0.3\hat{A}(s) - 0.06\hat{B}(s) \end{cases} \implies \begin{cases} (s+0.3)\hat{A}(s) - 0.06\hat{B}(s) = 4 \\ -0.3\hat{A}(s) + (s+0.06)\hat{B}(s) = 8 \end{cases}$$

Elimination or substitution turns this into $\hat{A}(s) = \frac{2}{s} + \frac{2}{s+0.36}$

Taking the Inverse Laplace Transformation turns this into: $A(t) = 2 + 2e^{-0.36t}$

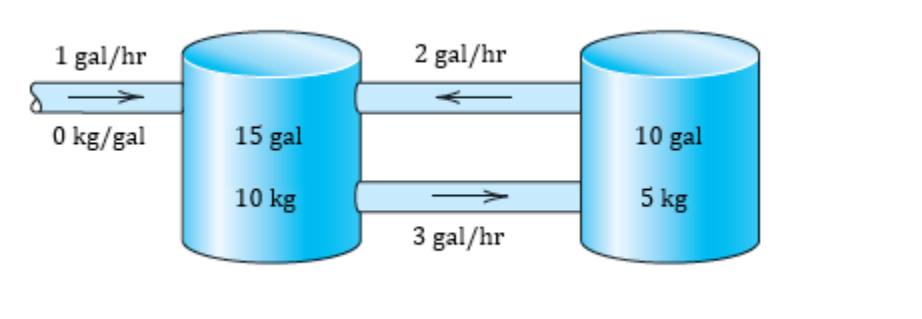
Finally, we can plug in $t = 120$ minutes (2 hours) to get $A(120) = 2 + 2e^{-43.2}$

Therefore, the salt in the first tank is $\boxed{A(120) \approx 2}$

This strategy works when there are more tanks, as well as non-constant volumes.

Illustration 5.3.1

Two tanks are connected together such that Tank A pumps 3 gallons per hour into Tank B and is supplied by a fresh water source of 1 gallon per hour, Tank B pumps 2 gallons an hour into Tank A. Tank A begins with a volume of 15 gallons containing 10 kg of microplastics, and Tank B begins with a volume of 10 gallons containing 5 kg of microplastics. What is the equation of the total amount of microplastics in Tank A after t hours?



We are given the constants $R_{A\to B} = 3, R_{B\to A} = 2, R_{A,in} = 1$ gallons per hour for all tanks.

Since the water pumped in and out is not equal, the volumes are: $V_A(t) = 15 + (2 + 1 - 3)t = 15$ and $V_B(t) = 10 + (3 - 2)t = 10 + t$

Therefore, the differential equations modeling the kg of microplastics in the tanks are:

$$\begin{cases} \dfrac{dA}{dt} = \dfrac{B}{10+t} \cdot 2 - \dfrac{A}{15} \cdot 3 \\ \dfrac{dB}{dt} = \dfrac{A}{15} \cdot 3 - \dfrac{B}{10+t} \cdot 2 \end{cases} \implies \begin{cases} \dfrac{dA}{dt} = -\dfrac{1}{5}A + \dfrac{2}{10+t}B \\ \dfrac{dB}{dt} = \dfrac{1}{5}A - \dfrac{2}{10+t}B \end{cases}$$

This system cannot be solved via Laplace Transformations, so we will use the D operator to solve:

$$\begin{cases} DA = -\dfrac{1}{5}A + \dfrac{2}{10+t}B \\ DB = \dfrac{1}{5}A - \dfrac{2}{10+t}B \end{cases} \implies \begin{cases} \left(D + \dfrac{1}{5}\right)A - \dfrac{2}{10+t}B = 0 \\ -\dfrac{1}{5}A + \left(D - \dfrac{2}{10+t}\right)B = 0 \end{cases}$$

To eliminate B, we will add $D - \dfrac{2}{10+t}$ "times" the top equation to $\dfrac{2}{10+t}$ times the bottom equation, giving:

$$\frac{dA}{dt} + \left(\frac{1}{5} + \frac{2}{10+t}\right)A = \frac{30}{10+t}$$

This is linear, so we can solve it using $\mu(t) = e^{\int \frac{1}{5} + \frac{2}{10+t}\,dt} = e^{\frac{1}{5}t + 2\ln(10+t)} = (10+t)^2 e^{\frac{1}{5}t}$:

$$A(t) = \frac{1}{\mu(t)}\int \mu(t)q(t)\,dt = \frac{1}{(10+t)^2 e^{\frac{1}{5}t}}\int (10+t)^2 e^{\frac{1}{5}t} \cdot \frac{30}{10+t}\,dt = \frac{30}{(10+t)^2 e^{\frac{1}{5}t}}\int (10+t)e^{\frac{1}{5}t}\,dt = \frac{30}{(10+t)^2 e^{\frac{1}{5}t}} \cdot \left((25+5t)e^{\frac{1}{5}t}\right.$$

Therefore, the microplastic in the first tank is $A(t) = \dfrac{150(5+t) + 250e^{-\frac{t}{5}}}{(10+t)^2}$ kg

Beyond simple tank problems, the principles of mixture modeling form the foundation of *compartmental analysis* in modern

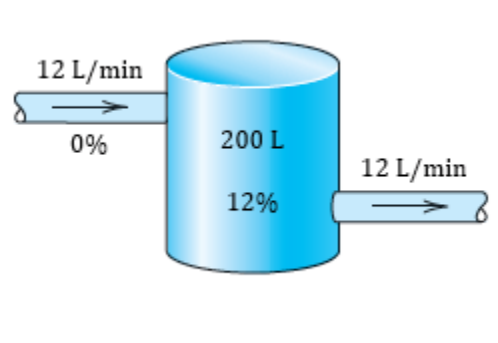
medicine and ecology. Whether calculating how a drug circulates between organs in *pharmacokinetics* or tracking the dispersion of contaminants through a watershed, the "rate in minus rate out" logic remains constant. These models demonstrate that even in systems with fluctuating volumes and complex interconnections, differential equations provide a predictable path toward understanding how substances achieve dynamic equilibrium within their environment.

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5.3: Practice Set

Exercise $5.3.1a$

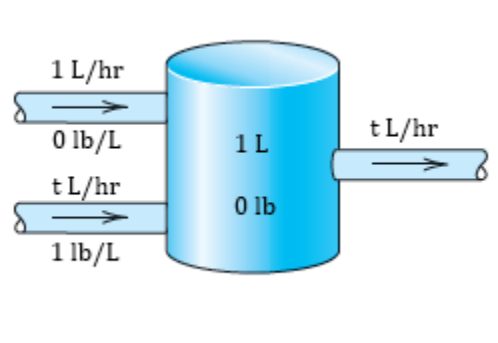
A brewery is experimenting with selling pre-made mixed drinks. They have a large 200 liter vat of one such mixed drink that is tested to be 12% alcohol. They water this down by pumping alcohol-free juice into the vat at 12 liters per minute, and pumping out the well mixed result at the same rate. How many minutes will it take for the mixed drink to reach 10% alcohol?



Answer available at the end of this section.

Exercise $5.3.1b$

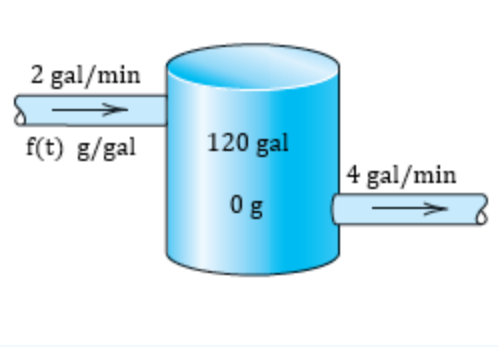
A large fish tank has a fresh water source flowing into it at 1 L/hr. To simulate a rising tide, it has a 1 lb/L salt water flowing into it at t L/hour while exactly t L/hr of the mixed tank water is drained out of it at time t hours. If the tank starts with 1 L of fresh water, approximately how many lbs of salt will be in the tank after a very long time?



Answer available at the end of this section.

Exercise $5.3.1c$

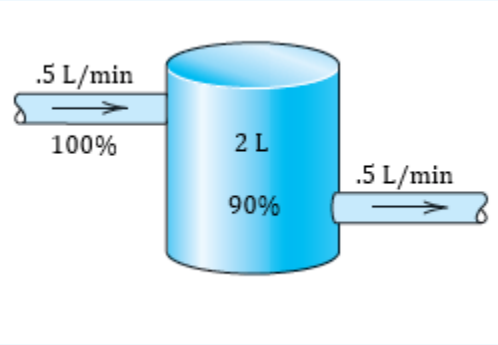
A large pond initially holds 120 gallons of clean water. It is fed by a polluted stream flowing 2 gallons per minute with $f(t) = 3600t - 120t^2 + t^3$ grams of arsenic per gallon in it after t minutes. The thoroughly mixed pond water is pumped out of the bottom at constant rate 4 gallons per minute. When will the maximum amount of arsenic be inside of the pond?



Answer available at the end of this section.

Exercise $5.3.1d$

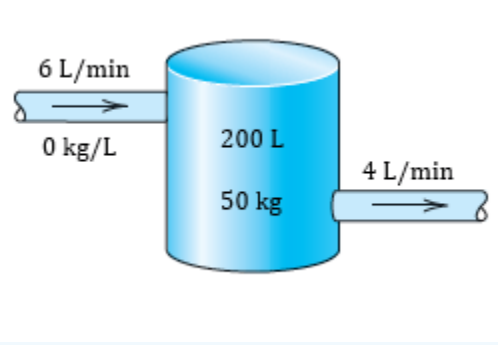
A coffee machine holds 2 L that is initially 90% coffee and 10% creamer. To strengthen it, pure coffee is pumped in at $.5$ L/min while the well-stirred mixture is drained out at $.5$ L/min. How long will it take for the mixture to become 95% coffee?



Answer available at the end of this section.

Exercise $5.3.1e$

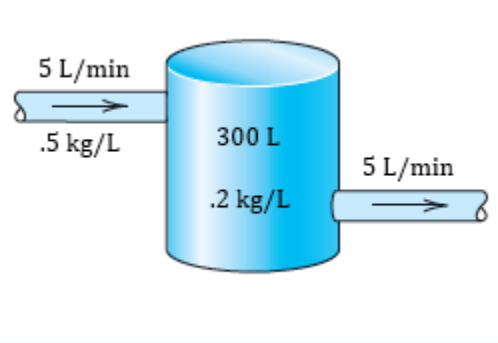
A large tank contains 200 L of polluted water, containing 50 kg of pollutant. Clean water flows in at 6 L/min and the mixture flows out at 4 L/min. How long will it take for 42 kg of pollutant to be removed from the tank?



Answer available at the end of this section.

Exercise $5.3.1f$

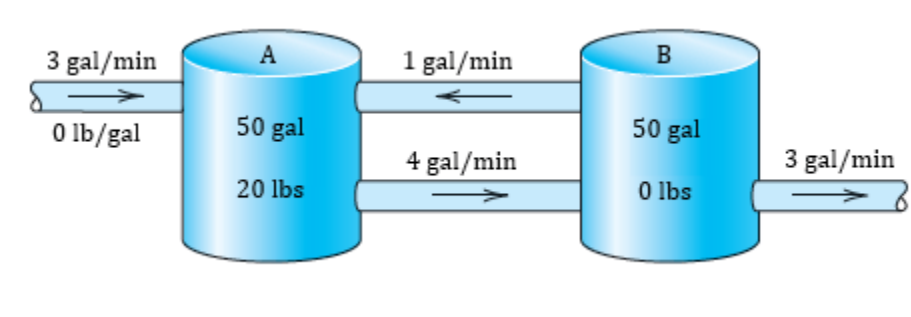
A vat contains 300 L of a sugar solution with 0.2 kg/L sugar. To sweeten it, a solution with 0.5 kg/L sugar is pumped in at 5 L/min and the mixture is drained out at 5 L/min. What will the concentration inside of the vat be after 1 hour?



Answer available at the end of this section.

Exercise $5.3.2a$

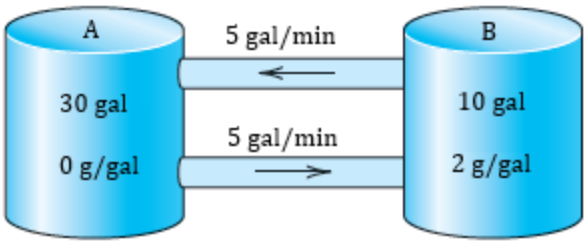
Tank A and Tank B are two 50 gallon tanks full of water. Fresh water flows into Tank A at 3 gal/min, and the liquid from Tank B is pumped into Tank A at a rate of 1 gal/min. At the same time, liquid is pumped from Tank A to Tank B at a rate of 4 gal/min, and 3 gal/min of the mixed liquid in Tank B flows over the top and down the drain. If there are initially 20 lbs of salt in Tank A and none in Tank B, find the amount of salt in each tank after 25 minutes.



Answer available at the end of this section.

Exercise $5.3.2b$

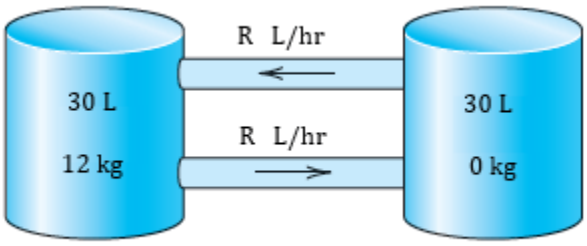
Tank A is a 30 gallon tank filled with pure water and Tank B is a 10 gallon tank filled with water that contains 2 grams of arsenic per gallon. The liquid from Tank B is pumped into Tank A at a rate of 5 gal/min. At the same time, liquid is pumped from Tank A to Tank B at a rate of 5 gal/min. When will the two tanks contain the same amount of arsenic?



Answer available at the end of this section.

Exercise $5.3.2c$

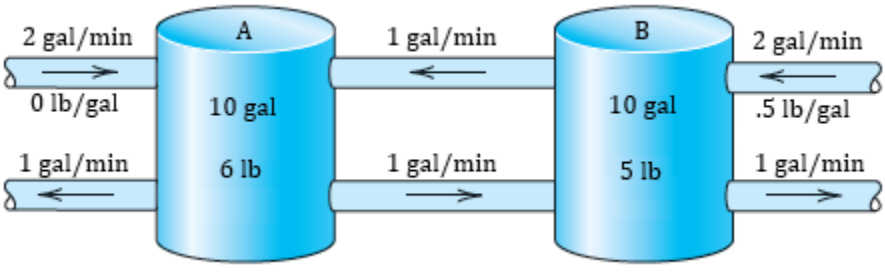
A tank containing 30 L of brine containing a total of 12 kg of salt is connected to a tank containing 30 L of pure water. The tanks exchange mixture at a rate R L/hr in each direction to preserve their volumes. If the original tank contains 8 kg of salt after exactly $5\ln(3)$ hours, what is the value of R?



Answer available at the end of this section.

Exercise $5.3.2d$

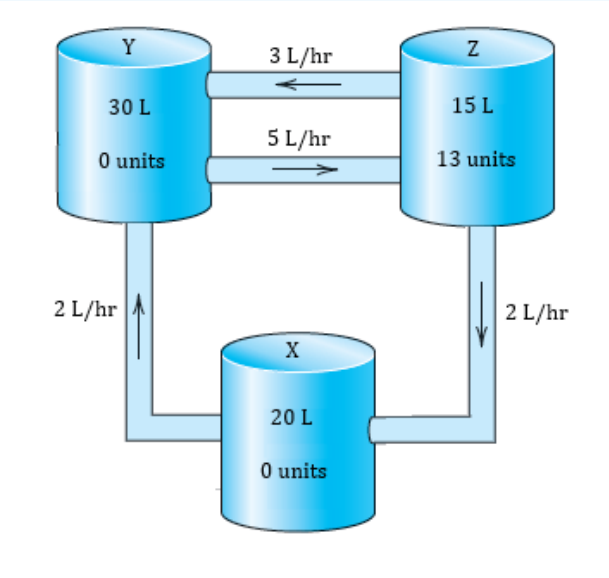
Two lakes, Lake A and Lake B, each initially contain 10 gallons of water. A freshwater stream flows into Lake A at 2 gallons per minute. A saltwater stream (concentration 0.5 lb/gal) flows into Lake B at 2 gallons per minute. Water is pumped from Lake A to Lake B at 1 gal/min, and from Lake B to Lake A at 1 gal/min. Each lake also loses water to a drain at 1 gal/min. Initially, Lake A contains 6 lb of salt and Lake B contains 5 lb of salt.
How long will it take until these two lakes have an equal amount of salt?



Answer available at the end of this section.

Exercise $5.3.2e$

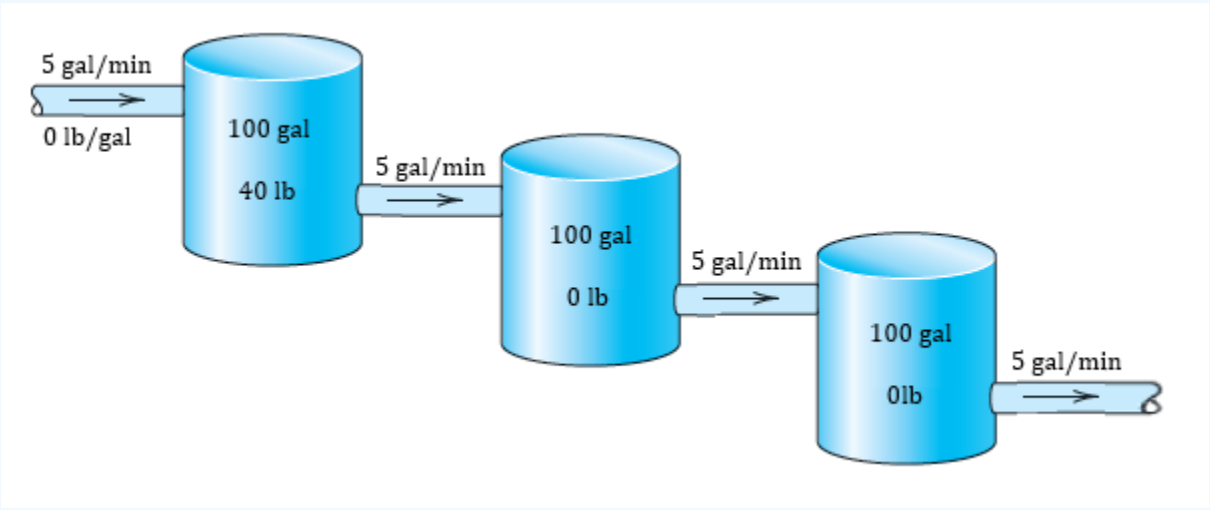
Tank X is a 20 L tank filled with pure water, Tank Y is a 30 L tank filled with pure water, and Tank Z is a 15 L tank filled with water that contains 13 units of ink. The liquid from Tank Z is pumped into Tank Y at a rate of 3 L/hr, from Tank X to Tank Y at a rate of 2 L/min, from Tank Y into Tank Z at 5 L/hr, and from Tank Z into Tank X at 2 L/hr. How much ink will be in each tank after a long time?



Answer available at the end of this section.

Exercise $5.3.2f$

Three 100 gallon tanks are filled with water and set up in a cascade, then 40 lbs of salt is mixed into top tank. Fresh water is pumped into the top tank at 5 gal/min, which causes it to push its mixed solution into the second tank at 5 gal/min. The second tank is mixed well and pumps into the bottom tank at 5 gal/min, which causes the bottom tank to overflow down the drain at 5 gal/min. How much salt is in the bottom tank after 20 minutes?



Answer available at the end of this section.

Answers

Exercise 5.3.1*a*

We find the terms involved in the mixture DE: $R_{in} = 12,\ R_{out} = 12,\ \delta_{in} = 0,\ V(t) = 200 + (R_{in} - R_{out}) \cdot t = 200$

Therefore, $\frac{dA}{dt} = 12 \cdot 0 - 12 \cdot \frac{A}{200}$ with $A(0) = .12 \cdot 200 = 24$

We solve the separable DE:

$$\int \frac{1}{A}\, dA = \int -\frac{3}{50}\, dt \ implies \ln|A| = -\frac{3}{50}t + C$$

Plugging in $A(0) = 24$ we get:

$$\ln(24) = -\frac{3}{50} \cdot 0 + C \implies C = \ln(24)$$

Therefore, the implicit solution is $\ln(A) = -\frac{3}{50}t + \ln(24)$

In order to reach 10% alcohol, we need $A = V(t) \cdot .10 = 200 \cdot .10 = 20$:

$$\ln(20) = -\frac{3}{50}t + \ln(24) \implies t = -\frac{50}{3}\ln\left(\frac{5}{6}\right)$$

Therefore, it will take $-\frac{50}{3}\ln\left(\frac{5}{6}\right)$ minutes.

Exercise 5.3.1*b*

We find the terms involved in the mixture DE: $R_{in} = t + 1,\ R_{out} = t$ are easy to find, but we must manually find the density in:

$$\delta_{in} = \frac{(0\frac{\text{lb}}{\text{L}}) \cdot (1\frac{\text{L}}{\text{hr}}) + (1\frac{\text{lb}}{\text{L}}) \cdot (t\frac{\text{L}}{\text{hr}})}{(t+1)\frac{\text{L}}{\text{min}}} = \frac{t}{t+1}\frac{\text{lb}}{\text{L}}$$

The volume at time t is given by $V(t) = V(0) + \int_0^t (R_{in}(x) - R_{out}(x))\, dx = 1 + t$

Therefore, the LDE giving the amount of salt is:

$$\frac{dA}{dt} = \frac{t}{t+1} - \frac{tA}{t+1} \implies \frac{dA}{dt} + \frac{t}{t+1}A = \frac{t}{t+1}$$

We find $\int p(t)\, dt = \int \frac{t}{t+1}\, dt = t - \ln|t+1|$, so $\mu(t) = e^{\int p(t)\, dt} = e^{t - \ln|t+1|} = (t+1)^{-1} e^t$:

$$A(t) = \frac{1}{\mu(t)} \int \mu(t) \cdot q(t)\, dt = \frac{1}{(t+1)^{-1} e^t} \int (t+1)^{-1} e^t \cdot \frac{t}{t+1}\, dt = (t+1)e^{-t} \int t(t+1)^{-2} e^t\, dt = (t+1)e^{-t}\left[(t+1)^{-1}e^t + C\right] = 1 + C(t+1)e^{-t}$$

Plugging in the initial condition $A(0) = 0$ we get $C = -1$, so $A(t) = 1 - (t+1)e^{-t}$.

Finally, we take the limit as $t \to \infty$ to simulate "a very long time":

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 1 - (t+1)e^{-t} = 1 - 0 = 1$$

So, the tank will hold approximately 1 lb of salt after a very long time.

Exercise 5.3.1*c*

We find the terms involved in the mixture DE: $R_{in} = 2,\ \delta_{in} = 3600t - 120t^2 + t^3 = t(60-t)^2,\ R_{out} = 4$

The volume at time t is given by $V(t) = V(0) + \int_0^t (R_{in}(x) - R_{out}(x))\, dx = 120 - 2t$

Therefore, the LDE giving the amount of salt is:

$$\frac{dA}{dt} = 2 \cdot t(60-t)^2 - \frac{A \cdot 4}{120 - 2t} \implies \frac{dA}{dt} + \frac{2}{60-t}A = 2t(60-t)^2$$

We find $\int p(t)\, dt = \int \frac{2}{60-t}\, dt = -2\ln|60-t|$, so $\mu(t) = e^{\int p(t)\, dt} = e^{-2\ln|60-t|} = (60-t)^{-2}$:

$$A(t) = \frac{1}{\mu(t)} \int \mu(t) \cdot q(t)\, dt = \frac{1}{(60-t)^{-2}} \int (60-t)^{-2} \cdot 2t(60-t)^2\, dt = (60-t)^2 \int 2t\, dt = (60-t)^2\left[t^2 + C\right]$$

Plugging in the initial condition $A(0) = 0$ gives $C = 0$, so $A(t) = t^2(60-t)^2$

To find its maximum, we take its derivative:

$$A'(t) = 2t \cdot (60-t)^2 + t^2 \cdot 2(t-60) = 4t(t-60)(t-30)$$

Therefore, the critical points are $t = 0, 30$ and 60, we can clearly see that $A(0) = 0$ and $A(60) = 0$, so the maximum must occur at the interior critical point. So, the maximum amount of arsenic will be present after 30 minutes.

Exercise 5.3.1*d*

We find the terms involved in the mixture DE: $R_{in} = .5,\ R_{out} = .5,\ \delta_{in} = 1,\ V(t) = 2 + (R_{in} - R_{out}) \cdot t = 2$

Therefore, $\frac{dA}{dt} = .5 \cdot 1 - .5 \cdot \frac{A}{2}$ with $A(0) = .90 \cdot 2 = 1.8$

We solve the separable DE:

$$\int \frac{4}{2-A}\, dA = \int 1\, dt \implies -4\ln|2-A| = t + C$$

Plugging in $A(0) = 1.8$ we get:

$$-4\ln(0.2) = 0 + C \implies C = -4\ln(0.2)$$

Therefore, the implicit solution is $-4\ln(2-A) = t - 4\ln(0.2)$

In order to reach 95% coffee, we need $A = V(t) \cdot .95 = 2 \cdot .95 = 1.9$:

$$-4\ln(0.1) = t - 4\ln(0.2) \implies t = 4\ln\left(\frac{0.2}{0.1}\right) = 4\ln(2)$$

Therefore, it will take $4\ln(2) \approx 2.77$ minutes.

Exercise $5.3.1e$

We find the terms involved in the mixture DE: $R_{in} = 6,\ R_{out} = 4,\ \delta_{in} = 0,\ V(t) = 200 + (R_{in} - R_{out})\,t = 200 + 2t$

Therefore, the LDE giving the amount of salt is:

$$\frac{dA}{dt} = 6 \cdot 0 - 4 \cdot \frac{A}{200 + 2t} \implies \frac{dA}{dt} = -\frac{2}{t + 100}A$$

We solve the separable DE:

$$\int \frac{1}{A}\,dA = \int -\frac{2}{t + 100}\,dt \implies \ln|A| = -2\ln|t + 100| + C \implies A(t) = \frac{K}{(t + 100)^2}$$

Plugging in the initial condition $A(0) = 50$ we get:

$$50 = \frac{K}{(0 + 100)^2} \implies K = 500000$$

Thus, the amount of pollutant is defined by the function $A(t) = \dfrac{500000}{(t + 100)^2}$.

When 42 kg of pollutant is removed, there will be exactly $A(t) = 8$ kg remaining:

$$\frac{500000}{(t + 100)^2} = 8 \implies t = 150$$

So, 42 kg of pollutant is removed after 150 minutes.

Exercise $5.3.1f$

We find the terms involved in the mixture DE: $R_{in} = 5, R_{out} = 5,\ \delta_{in} = 0.5,\ V(t) = 300 + (R_{in} - R_{out})\,t = 300$

Therefore, the LDE giving the amount of sugar is:

$$\frac{dA}{dt} = 5 \cdot 0.5 - 5 \cdot \frac{A}{300} \implies \frac{dA}{dt} + \frac{1}{60}A = \frac{5}{2}$$

We find $\displaystyle\int p(t)\,dt = \int \frac{1}{60}\,dt = \frac{1}{60}t$, so $\mu(t) = e^{\int p(t)\,dt} = e^{\frac{1}{60}t}$:

$$A(t) = \frac{1}{\mu(t)}\int \mu(t) \cdot q(t)\,dt = \frac{1}{e^{\frac{1}{60}t}}\int e^{\frac{1}{60}t} \cdot \frac{5}{2}\,dt = e^{-\frac{1}{60}t}\left[150e^{\frac{1}{60}t} + C\right] = 150 + Ce^{-\frac{1}{60}t}$$

Plugging in the initial condition $A(0) = 0.2 \cdot 300 = 60$ gives $C = -90$, so $A(t) = 150 - 90e^{-\frac{1}{60}t}$

After $t = 60$ minutes, this gives us a total of $A(60) = 150 - 90e^{-1}$ kg of sugar in the vat.

Thus, the concentration after 1 hour is $\dfrac{150 - 90e^{-1}}{300} \approx 0.390$ kg/L.

Exercise $5.3.2a$

We denote $a(t)$ and $b(t)$ to be the number of lbs of salt in each tank.

In Tank A, we have $R_{in} = 3 + 1 = 4$ gal/min, and $R_{out} = 4$ gal/min, so the volume in Tank A is constant $V_A = 50$

In Tank B, we have $R_{in} = 4$ gal/min, and $R_{out} = 1 + 3 = 4$ gal/min, so the volume in Tank B is also constant $V_B = 50$

For the salt flowing in and out of Tank A we have:

$$\frac{da}{dt} = 3 \cdot 0 + 1 \cdot \frac{b(t)}{V_B} - 4 \cdot \frac{a(t)}{V_A} = \frac{1}{50}b(t) - \frac{2}{25}a(t)$$

For the salt flowing in and out of Tank B we have:

$$\frac{db}{dt} = 4 \cdot \frac{a(t)}{V_A} - 4 \cdot \frac{b(t)}{V_B} = \frac{2}{25}a(t) - \frac{2}{25}b(t)$$

This gives us a system of LDEs:

$$\begin{cases} a' = -\dfrac{2}{25}a + \dfrac{1}{50}b \\ b' = \dfrac{2}{25}a - \dfrac{2}{25}b \end{cases} \implies \begin{cases} (50D + 4)a - b = 0 \\ -2a + (25D + 2)b = 0 \end{cases}$$

"Multiplying" the top equation by $(25D + 2)$ and adding it to the bottom equation gives:

$$((50D + 4)(25D + 2) - 2)\,a = 0 \implies 1250a'' + 200a' + 6a = 0$$

This is a CC LDE with characteristic equation $1250r^2 + 200r + 6 = 0 \implies r = -\dfrac{1}{25},\ -\dfrac{3}{25}$

Thus, $a(t) = Ce^{-\frac{1}{25}t} + De^{-\frac{3}{25}t}$. The top equation can be written as $b(t) = 50a'(t) + 4a(t)$:

$$b(t) = 50\left[-\frac{C}{25}e^{-\frac{1}{25}t} - \frac{3D}{25}e^{-\frac{3}{25}t}\right] + 4\left[Ce^{-\frac{1}{25}t} + De^{-\frac{3}{25}t}\right] = 2Ce^{-\frac{1}{25}t} - 2De^{-\frac{3}{25}t}$$

The initial amount of salt in Tank A is $a(0) = 20$ and in Tank B is $b(0) = 0$:

$$a(t) = Ce^{-\frac{1}{25}t} + De^{-\frac{3}{25}t}$$
$$a(0) = C \cdot 1 + D \cdot 1 \implies C + D = 20$$
$$b(t) = 2Ce^{-\frac{1}{25}t} - 2De^{-\frac{3}{25}t}$$
$$b(0) = 2C \cdot 1 - 2D \cdot 1 = 0 \implies 2C - 2D = 0$$
$$\begin{cases} C + D = 20 \\ 2C - 2D = 0 \end{cases} \implies \begin{cases} C = 10 \\ D = 10 \end{cases}$$

So the solutions are: $\begin{cases} a(t) = 10e^{-\frac{1}{25}t} + 10e^{-\frac{3}{25}t} \\ b(t) = 20e^{-\frac{1}{25}t} - 20e^{-\frac{3}{25}t} \end{cases}$

Plugging in $t = 25$, we get $a(25) = 10e^{-1} + 10e^{-3}$ lbs and $b(25) = 20e^{-1} - 20e^{-3}$ lbs.

Exercise $5.3.2b$

We denote $a(t)$ and $b(t)$ to be the number of grams of arsenic in each tank.

In Tank A, we have $R_{in} = 5$ gal/min, and $R_{out} = 5$ gal/min, so the volume in Tank A is constant $V_A = 30$

In Tank B, we have $R_{in} = 5$ gal/min, and $R_{out} = 5$ gal/min, so the volume in Tank B is also constant $V_B = 10$

For the arsenic flowing in and out of tank A we have:

$$\frac{da}{dt} = 5 \cdot \frac{b(t)}{V_B} - 5 \cdot \frac{a(t)}{V_A} = \frac{1}{2}b(t) - \frac{1}{6}a(t)$$

For the arsenic flowing in and out of tank B we have:

$$\frac{db}{dt} = 5 \cdot \frac{a(t)}{V_A} - 5 \cdot \frac{b(t)}{V_B} = \frac{1}{6}a(t) - \frac{1}{2}b(t)$$

This gives us the system of LDEs:

$$\begin{cases} a' = -\frac{1}{6}a + \frac{1}{2}b \\ b' = \frac{1}{6}a - \frac{1}{2}b \end{cases} \implies \begin{cases} (6D+1)a - 3b = 0 \\ -a + (6D+3)b = 0 \end{cases}$$

"Multiplying" the top equation by $(2D + 1)$ and adding it to the bottom equation gives:

$$\left((2D+1)(6D+1) - 1\right)a = 0 \implies 12a'' + 8a' = 0$$

This is a CC LDE with characteristic equation $12r^2 + 8r = 0 \implies r = 0, \ -\frac{2}{3}$

Thus, $a(t) = C + De^{-\frac{2}{3}t}$, and the top equation can be written as $b(t) = 2a'(t) + \frac{1}{3}a(t)$:

$$b(t) = 2\left[-\frac{2}{3}De^{-\frac{2}{3}t}\right] + \frac{1}{3}\left[C + De^{-\frac{2}{3}t}\right] = \frac{1}{3}C - De^{-\frac{2}{3}t}$$

The initial amount of arsenic in Tank A is $a(0) = 0$ and in Tank B is $b(0) = 2 \cdot V_B = 20$:

$$a(t) = C + De^{-\frac{2}{3}t}$$
$$a(0) = C + D \cdot 1 \implies C + D = 0$$
$$b(t) = \frac{1}{3}C - De^{-\frac{2}{3}t}$$
$$b(0) = \frac{1}{3}C \cdot 1 - D \cdot 1 = 0 \implies \frac{1}{3}C - D = 20$$
$$\begin{cases} C + D = 0 \\ \frac{1}{3}C - D = 20 \end{cases} \implies \begin{cases} C = 15 \\ D = -15 \end{cases}$$

Thus, the amount of arsenic in each tank is $\begin{cases} a(t) = 15 - 15e^{-\frac{2}{3}t} \\ b(t) = 5 + 15e^{-\frac{2}{3}t} \end{cases}$

Setting these equal to each other gives:

$$15 - 15e^{-\frac{2}{3}t} = 5 + 15e^{-\frac{2}{3}t} \implies e^{-\frac{2}{3}t} = \frac{1}{3} \implies t = -\frac{3}{2}\ln\left(\frac{1}{3}\right)$$

Therefore, the total amount will match after $-\frac{3}{2}\ln\left(\frac{1}{3}\right) \approx 1.648$ minutes.

Exercise $5.3.2c$

Denote the first tank as Tank X and the second as Tank Y. We denote $x(t)$ and $y(t)$ to be the number of lbs of salt in each tank.

In both Tank X and Y, we have $R_{in} = R_{out} = R$ L/hour, so the volume each tank is constant: $V_X = V_Y = 30$

For the salt flowing in and out of Tank X we have:

$$\frac{dx}{dt} = \frac{y(t)}{V_Y} \cdot R - \frac{x(t)}{V_X} \cdot R = \frac{R}{30}y(t) - \frac{R}{30}x(t)$$

For the salt flowing in and out of Tank Y we have:

$$\frac{dy}{dt} = \frac{x(t)}{V_X} \cdot R - \frac{y(t)}{V_Y} \cdot R = \frac{R}{30}x(t) - \frac{R}{30}y(t)$$

This gives us a system of LDEs:

$$\begin{cases} x' = \frac{R}{30}y - \frac{R}{30}x \\ y' = \frac{R}{30}x - \frac{R}{30}y \end{cases} \implies \begin{cases} \left(D + \frac{R}{30}\right)x - \frac{R}{30}y = 0 \\ -\frac{R}{30}x + \left(D + \frac{R}{30}\right)y = 0 \end{cases}$$

"Multiplying" the top equation by $\left(D+\frac{R}{30}\right)$ and adding it to the bottom equation times $\frac{R}{30}$ gives:

$$\left(\left(D+\frac{R}{30}\right)^2-\frac{R^2}{900}\right)x=0 \implies x''+\frac{R}{15}x'=0$$

This is a CC LDE with characteristic equation $r^2+\frac{R}{15}r=0 \implies r=0,-\frac{R}{15}$

Thus, $x(t)=A+Be^{-\frac{R}{15}t}$. The top equation can be written as $y(t)=\frac{30}{R}x'(t)+x(t)$:

$$y(t)=\frac{30}{R}\left[-\frac{R}{15}Be^{-\frac{R}{15}t}\right]+\left[A+Be^{-\frac{R}{15}t}\right]=A-Be^{-\frac{R}{15}t}$$

The initial amount of salt in Tank X is $x(0)=12$ and it Tank Y is $y(0)=0$:

$$\begin{gathered}x(t)=A+Be^{-\frac{R}{15}t}\\ x(0)=A+B\cdot 1 \implies A+B=12\\ y(t)=A-Be^{-\frac{R}{15}t}\\ y(0)=A-B\cdot 1=0 \implies A-B=0\\ \begin{cases}A+B=12\\A-B=0\end{cases}\implies\begin{cases}A=6\\B=6\end{cases}\end{gathered}$$

So the solutions are: $\begin{cases}x(t)=6+6e^{-\frac{R}{15}t}\\y(t)=6-6e^{-\frac{R}{15}t}\end{cases}$

We know that $x\left(5\ln(3)\right)=8$, so we solve:

$$6+6e^{-\frac{R}{15}\cdot 5\ln(3)}=8 \implies e^{-\frac{R\ln(3)}{3}}=\frac{1}{3} \implies R=3$$

Therefore, they are exchanging liquid at 3 L/hr in each direction.

Exercise $5.3.2d$

We denote $a(t)$ and $b(t)$ to be the number of lbs of salt in each lake.

In Lake A, we have $R_{in}=2+1=3$ gal/min, and $R_{out}=1+1=2$ gal/min, so the volume in Lake A is $V_A=10+t$

In Lake B, we have $R_{in}=2+1=3$ gal/min, and $R_{out}=1+1=2$ gal/min, so the volume in Lake B is $V_B=10+t$

For the salt flowing in and out of Lake A we have:

$$\frac{da}{dt}=2\cdot 0+1\cdot\frac{b(t)}{V_B}-2\cdot\frac{a(t)}{V_A}=\frac{1}{10+t}b(t)-\frac{2}{10+t}a(t)$$

For the salt flowing in and out of Lake B we have:

$$\frac{db}{dt}=2\cdot 0.5+1\cdot\frac{a(t)}{V_A}-2\cdot\frac{b(t)}{V_B}=1+\frac{1}{10+t}a(t)-\frac{2}{10+t}b(t)$$

This gives us the system of LDEs:

$$\begin{cases}a'=\frac{1}{10+t}b-\frac{2}{10+t}a\\ b'=1+\frac{1}{10+t}a-\frac{2}{10+t}b\end{cases}\implies\begin{cases}\left((t+10)D+2\right)a-b=0\\ -a+\left((t+10)D+2\right)b=t+10\end{cases}$$

"Multiplying" the top equation by $\left((t+10)D+2\right)$ and adding it to the bottom equation gives:

$$\left(\left((t+10)D+2\right)^2-1\right)a=0 \implies (t+10)^2a''+5(t+10)a'+3a=t+10$$

This is similar to a CE LDE, but we instead assume $a(t)=(t+10)^m$, so that our characteristic equation is $m(m-1)+5m+3=0 \implies m=-1,-3$

Thus, $a_h(t)=A(t+10)^{-1}+B(t+10)^{-3}$, and we can see that $a_p(t)=\frac{1}{8}(t+10)$

Thus, we have found $a(t)=\frac{A}{t+10}+\frac{B}{(t+10)^3}+\frac{t+10}{8}$

The top equation can be written as $b(t)=(t+10)a'(t)+2a(t)$:

$$b(t)=(t+10)\left[-\frac{A}{(t+10)^2}-\frac{3B}{(t+10)^4}+\frac{1}{8}\right]+2\left[\frac{A}{t+10}+\frac{B}{(t+10)^3}+\frac{t+10}{8}\right]=\frac{A}{t+10}-\frac{B}{(t+10)^3}+\frac{3(t+10)}{8}$$

The initial amount of salt in Lake A is $a(0)=6$ and in Lake B is $b(0)=5$:

$$\begin{gathered}a(t)=\frac{A}{t+10}+\frac{B}{(t+10)^3}+\frac{t+10}{8}\\ a(0)=\frac{A}{10}+\frac{B}{1000}+\frac{5}{4} \implies \frac{A}{10}+\frac{B}{1000}=\frac{15}{4}\\ b(t)=\frac{A}{t+10}-\frac{B}{(t+10)^3}+\frac{3(t+10)}{8}\\ b(0)=\frac{A}{10}-\frac{B}{1000}+\frac{15}{4} \implies \frac{A}{10}-\frac{B}{1000}=\frac{9}{4}\\ \begin{cases}\frac{A}{10}+\frac{B}{1000}=\frac{15}{4}\\ \frac{A}{10}-\frac{B}{1000}=\frac{9}{4}\end{cases}\implies\begin{cases}A=30\\B=1750\end{cases}\end{gathered}$$

Thus, the amount of salt in each lake is $\begin{cases}a(t)=\frac{30}{t+10}+\frac{1750}{(t+10)^3}+\frac{t+10}{8}\\ b(t)=\frac{30}{t+10}-\frac{1750}{(t+10)^3}+\frac{3(t+10)}{8}\end{cases}$

Setting these equal to each other gives:

$$\begin{aligned}\frac{30}{t+10}+\frac{1750}{(t+10)^3}+\frac{t+10}{8}&=\frac{30}{t+10}-\frac{1750}{(t+10)^3}+\frac{3(t+10)}{8}\\ \frac{3500}{(t+10)^3}&=\frac{t+10}{4}\\ (t+10)^4&=14000\\ t&=\sqrt[4]{14000}-10\end{aligned}$$

Therefore, the total amount of salt in each lake will match after $\sqrt[4]{14000}-10\approx 0.878$ minutes.

Exercise $5.3.2e$

We denote $x(t),\ y(t)$, and $z(t)$ to be the units of ink in each tank.

In Tank X, we have $R_{in}=2$ L/hr, and $R_{out}=2$ L/hr, so the volume in Tank X is constant $V_X=20$

In Tank Y, we have $R_{in}=3+2=5$ L/hr, and $R_{out}=5$ L/hr, so the volume in Tank Y is constant $V_Y=30$

In Tank Z, we have $R_{in}=5$ L/hr, and $R_{out}=2+3=5$ L/hr, so the volume in Tank Z is constant $V_Z=15$

For the ink flowing in and out of each tank we have:

$$\begin{aligned}\frac{dx}{dt}&=2\cdot\frac{z(t)}{V_Z}-2\cdot\frac{x(t)}{V_X}=\frac{2}{15}z(t)-\frac{1}{10}x(t)\\ \frac{dy}{dt}&=2\cdot\frac{x(t)}{V_X}+3\cdot\frac{z(t)}{V_Z}-5\cdot\frac{y(t)}{V_Y}=\frac{1}{10}x(t)+\frac{1}{5}z(t)-\frac{1}{6}y(t)\\ \frac{dz}{dt}&=5\cdot\frac{y(t)}{V_Y}-5\cdot\frac{z(t)}{V_Z}=\frac{1}{6}y(t)-\frac{1}{3}z(t)\end{aligned}$$

This gives us a system of LDEs:

$$\begin{cases}x'=-\frac{1}{10}x+\frac{2}{15}z\\ y'=\frac{1}{10}x-\frac{1}{6}y+\frac{1}{5}z\\ z'=\frac{1}{6}y-\frac{1}{3}z\end{cases}\quad\text{with } x(0)=0,\ y(0)=0,\ z(0)=13$$

We will solve this using Laplace Transformations:

$$\begin{cases}sX(s)-0=-\frac{1}{10}X(s)+\frac{2}{15}Z(s)\\ sY(s)-0=\frac{1}{10}X(s)-\frac{1}{6}Y(s)+\frac{1}{5}Z(s)\\ sZ(s)-13=\frac{1}{6}Y(s)-\frac{1}{3}Z(s)\end{cases}\implies\begin{cases}(30s+3)X(s)-4Z(s)=0\\ -3X(s)+(30s+5)Y(s)-6Z(s)=0\\ -Y(s)+(6s+2)Z(s)=78\end{cases}$$

The top equation becomes $X(s)=\dfrac{4}{3(10s+1)}Z(s)$ and the bottom equation becomes $Y(s)=2(3s+1)Z(s)-78$

Substituting these into the middle equation gives:

$$\begin{aligned}-3\cdot\frac{4}{3(10s+1)}Z(s)+(30s+5)\left(2(3s+1)Z(s)-78\right)-6Z(s)&=0\\ \left[-\frac{4}{10s+1}+10(6s+1)(3s+1)-6\right]Z(s)&=78(30s+5)\\ \frac{10s(6s+1)(30s+13)}{10s+1}Z(s)&=390(6s+1)\\ Z(s)&=\frac{39(10s+1)}{s(30s+13)}\end{aligned}$$

Partial fraction decomposition lets us write this as $Z(s)=\dfrac{3}{s}+\dfrac{10}{s+\frac{13}{30}}$

Taking inverse Laplace Transformation, we see

$$z(t)=\mathscr{L}^{-1}\left(\frac{3}{s}+\frac{10}{s+\frac{13}{30}}\right)=3+10e^{-\frac{13}{30}t}$$

We know $y(t)=6z'(t)+2z(t)$:

$$y(t)=-26e^{-\frac{13}{30}t}+6+20e^{-\frac{13}{30}t}=6-6e^{-\frac{13}{30}t}$$

We also know $x(t)=10y'(t)+\dfrac{5}{3}y(t)-2z(t)$:

$$x(t)=-26e^{-\frac{13}{30}t}+10-10e^{-\frac{13}{30}t}-6-20e^{-\frac{13}{30}t}=4-4e^{-\frac{13}{30}t}$$

This gives our solution to the system of differential equations as: $\begin{cases}x(t)=4-4e^{-\frac{13}{30}t}\\ y(t)=6-6e^{-\frac{13}{30}t}\\ z(t)=3+10e^{-\frac{13}{30}t}\end{cases}$

As $t\to\infty$, we see that $x(t)\to 4,\ y(t)\to 6,\ z(t)\to 3$

Therefore, after a long time there will be 4 units in X, 6 units in Y, and 3 units in Z.

Exercise $5.3.2f$

We denote $x(t),\ y(t)$, and $z(t)$ to be the lbs of salt in the top, middle, and bottom tank respectively.

In each tank, we have $R_{in}=R_{out}=5$ gal/min, so the volumes remain constant $V_X=V_Y=V_Z=100$

For the salt flowing in and out of each tank we have:

$$\frac{dx}{dt} = 5 \cdot 0 - 5 \cdot \frac{x(t)}{V_X} = -\frac{1}{20}x(t)$$
$$\frac{dy}{dt} = 5 \cdot \frac{x(t)}{V_X} - 5 \cdot \frac{y(t)}{V_Y} = \frac{1}{20}x(t) - \frac{1}{20}y(t)$$
$$\frac{dz}{dt} = 5 \cdot \frac{y(t)}{V_Y} - 5 \cdot \frac{z(t)}{V_Z} = \frac{1}{20}y(t) - \frac{1}{20}z(t)$$

This gives us a system of LDEs:

$$\begin{cases} x' = -\frac{1}{20}x \\ y' = \frac{1}{20}x - \frac{1}{20}y \quad \text{with } x(0) = 40,\ y(0) = 0, z(0) = 0 \\ z' = \frac{1}{20}y - \frac{1}{20}z \end{cases}$$

We will solve for $x(t)$ since it is uncoupled, satisfying $x' = -\frac{1}{20}x$. This is separable:

$$\frac{dx}{dt} = -\frac{x}{20} \implies \int \frac{1}{x}\,dx = \int -\frac{1}{20}\,dt \implies \ln|x| = -\frac{1}{20}t + C \implies x(t) = Ce^{-\frac{1}{20}t}$$

We are given $x(0) = 40$, so we easily find $C = 40$

Plugging this into the second equation, we get $y' = \frac{1}{20} \cdot 40e^{-\frac{1}{20}t} - \frac{1}{20}y$

We write this as a first order LDE: $y' + \frac{1}{20}y = 2e^{-\frac{1}{20}t}$

This is a CC LDE with characteristic equation $r + \frac{1}{20} = 0 \implies r = -\frac{1}{20}t$

Thus, the homogeneous solution is of the form $y_h(t) = De^{-\frac{1}{20}t}$

One particular solution we can use is $y_p(t) = 2te^{-\frac{1}{20}t}$, giving us general solution $y(t) = (2t + D)e^{-\frac{1}{20}t}$.

We know that $y(0) = 0$, so we must have $D = 0$

Plugging this into the third equation, we get $z' = \frac{1}{20} \cdot 2te^{-\frac{1}{20}t} - \frac{1}{20}z$

This is a CC LDE with characteristic equation $r + \frac{1}{20} = 0 \implies r = -\frac{1}{20}t$

Thus, the homogeneous solution is of the form $z_h(t) = Ke^{-\frac{1}{20}t}$

One particular solution we can use is $z_p(t) = \frac{1}{20}t^2e^{-\frac{1}{20}t}$, giving us general solution $z(t) = \left(\frac{1}{20}t^2 + K\right)e^{-\frac{1}{20}t}$

We know that $z(0) = 0$, so we must have $K = 0$

This tells us that $z(t) = \frac{1}{20}t^2e^{-\frac{1}{20}t}$, and thus, $z(20) = \frac{1}{20} \cdot 20^2 \cdot e^{-\frac{1}{20} \cdot 20} = 20e^{-1}$

Therefore, after 20 minutes there will be $20e^{-1} \approx 7.357$ lbs of salt in the bottom tank.

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5.4: Hooke's Law

Learning Objectives

- Derive the second-order differential equation for a mass-spring system using Hooke's Law and Newton's Second Law, both with and without damping or external forcing.
- Classify spring-mass systems as undamped, underdamped, critically damped, or overdamped based on their characteristic equations and physical parameters.
- Model and analyze coupled spring-mass systems by setting up and solving a system of two second-order differential equations.

Springs model oscillations everywhere: in engineering, physics, and nature. By combining Hooke’s Law with Newton’s Second Law, we derive a simple yet powerful differential equation that predicts motion—from ideal harmonic oscillation to damped or forced vibrations. This section shows how to model single and coupled spring systems, introducing key concepts like natural frequency, damping, and resonance.

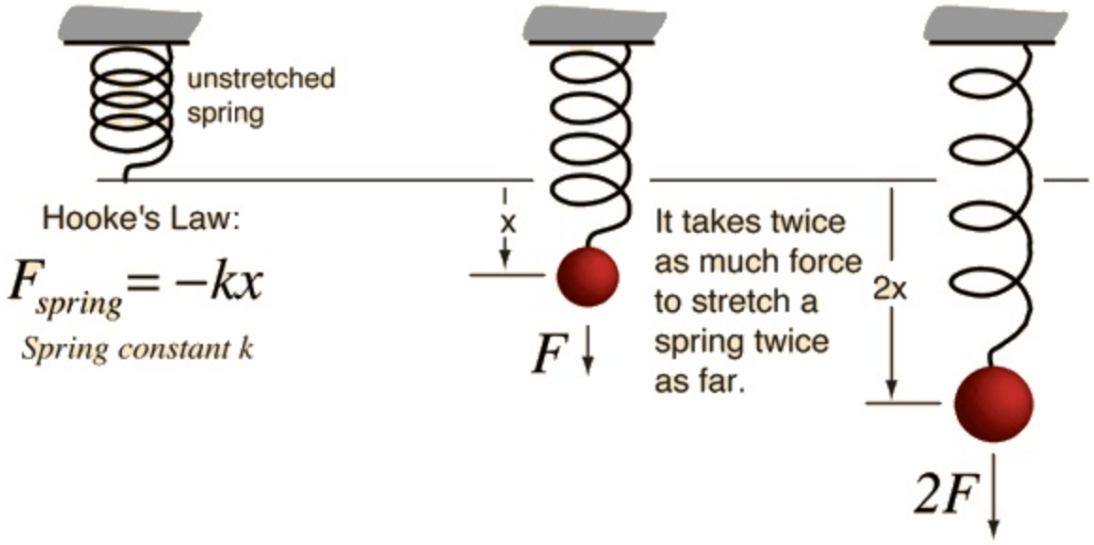
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5.4: Springs

We begin with the simplest and most fundamental force system: the mass-spring oscillator. Spring problems, including those with multiple springs, and their solutions have applications throughout engineering and physics.

Imagine a skater on an ice rink attached to a large spring fastened to one wall. At the center of the rink, the skater feels no pushing or pulling from the spring. As the skater pulls or pushes away from the center, the spring pulls or pushes in the opposite direction.

Hooke's Law states that if x is the displacement from the equilibrium point, the spring applies a force in the opposing direction that is directly proportional to x.



Let $x(t)$ be the position function (relative to equilibrium) of an object, then $F_{spring} = -k \cdot x(t)$ for some constant $k > 0$. This constant varies based on the material the spring might be made of, how many coils it uses, and the units involved in the question. However, for any given spring, it stays constant during calculation.



Newton says that force is equal to mass times acceleration, so this will have to equal $m \cdot x''(t)$ where $m > 0$ is the mass of the object.

$$mx''(t) = -kx(t)$$

Rule: Motion for Frictionless Springs

When an object with mass m is attached to a spring with coefficient k, its motion satisfies the LDE:

$$mx''(t) + kx(t) = 0$$

This is a second order CC LDE with characteristic equation $mr^2 + k = 0 \implies r = \pm i\sqrt{\frac{k}{m}}$

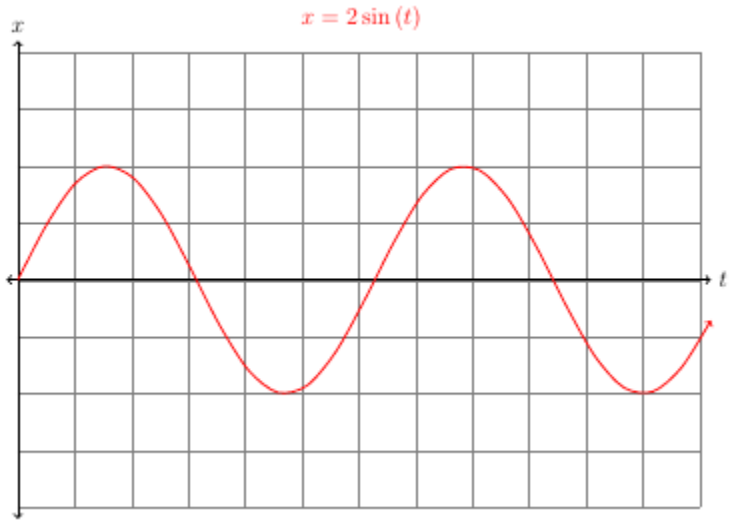
This means we have general solution $x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

Since it is a linear combination of $sin(wt)$ and $\cos(wt)$, we can re-write it using an identity:

$$\begin{aligned} \text{If } \sin(\theta) = \frac{A}{\sqrt{A^2+B^2}} \text{ and } \cos(\theta) &= \frac{B}{\sqrt{A^2+B^2}} \\ A\cos(wt) + B\sin(wt) &= \sqrt{A^2+B^2}\,[\sin(\theta)\cos(wt) + \cos(\theta)\sin(wt)] \\ &= \sqrt{A^2+B^2}\,\sin(\theta + wt) \end{aligned}$$

Therefore, this simple frictionless spring system always has a harmonic solution:

$$x(t) = C\sin(wt + \alpha) \text{ for some constants, } C,\ w \text{ and } \alpha$$



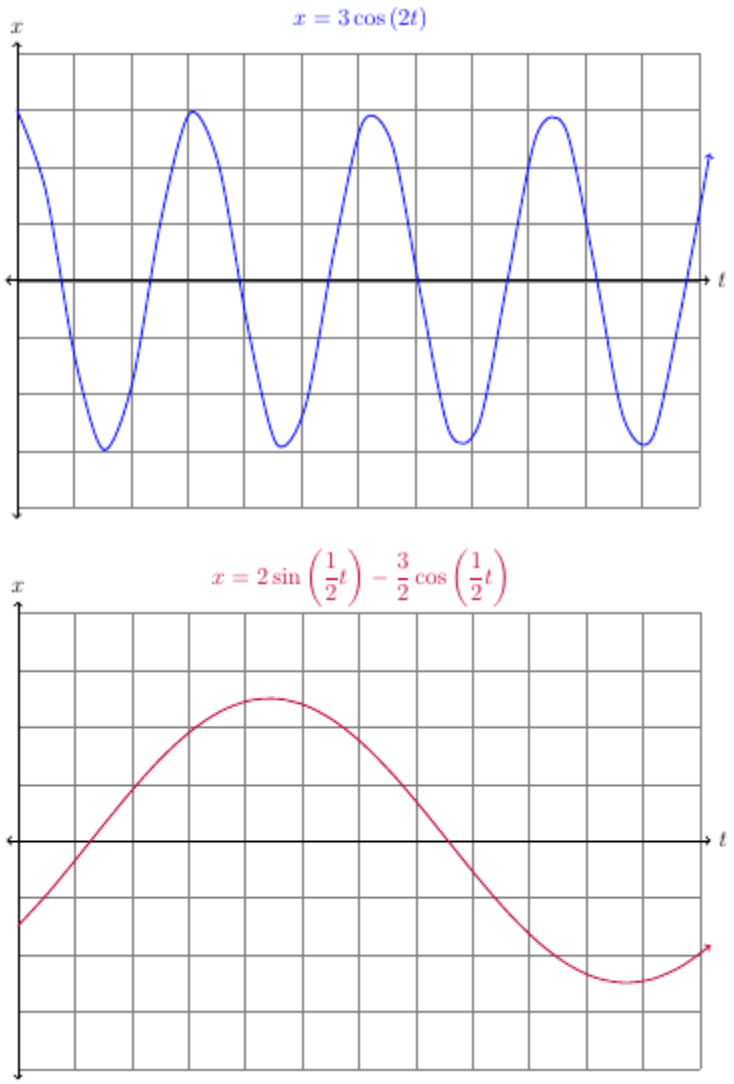


Illustration 5.4.1

A large spring hangs from the ceiling of a factory. A 16 pound weight is attached to the spring, which stretches it by two feet. This weight is then (a) pulled down by two feet and released from rest. (b) pushed up by one foot and then released with initial upward speed of $4\sqrt{3}$ feet per second.

The initial set up of this question tells us how to find m and k.

Since we are told it is a 16 pound weight, and $W = mg$ where $g = 32$ feet per second squared (assuming this is on Earth, and we orient "downward" with positive values). This gives us $m = \dfrac{16}{32} = \dfrac{1}{2}$ (measured in *slugs*). and thus k is measured in units $\dfrac{\text{lb}}{\text{ft}}$

Attaching the weight caused the spring to stretch by two feet, so $k \cdot 2 = m \cdot g \implies k = \dfrac{1}{2} \cdot 32 \cdot \dfrac{1}{2} = 8$ lb/foot.

Now we have the values of k and m, so we know the differential equation must be:

$$mx''(t) + kx(t) = 0 \implies \frac{1}{2}x'' + 8x = 0$$

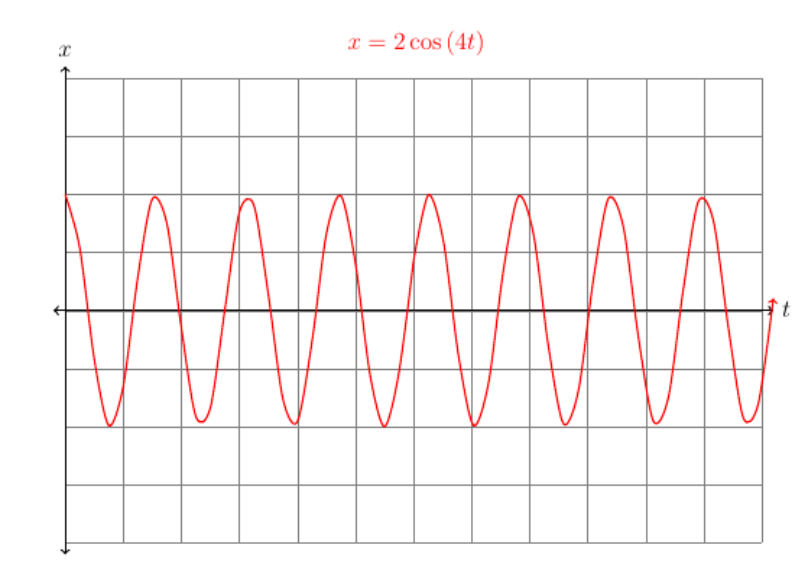
This homogeneous CC LDE is found to have general solutions $x(t) = A\cos(4t) + B\sin(4t)$

(a) For initial conditions "pulled down by two feet and then released from rest", gives us $x(0) = 2$ and $x'(0) = 0$.

Plugging these into the general solution gives:

$$\begin{aligned} x &= A\cos(4t) + B\sin(4t) \\ x(0) &= A\cdot 1 + B\cdot 0 \implies A = 2 \\ x' &= -4A\sin(4t) + 4B\cos(4t) \\ x'(0) &= -4A\cdot 0 + 4B\cdot 1 \implies 4B = 0 \implies B = 0 \end{aligned}$$

Therefore, the position (feet below equilibrium) of the weight after t seconds is given by $x(t) = 2\cos(4t)$



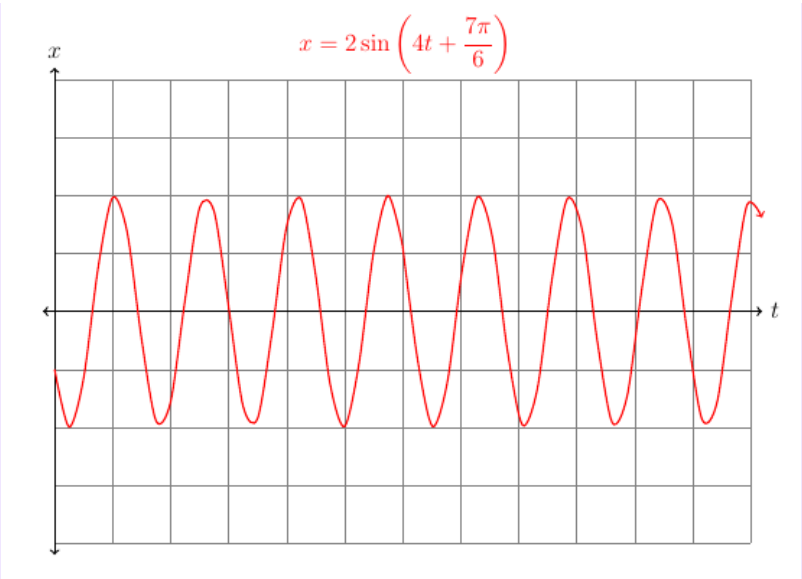
(b) For initial conditions "pushed up by one foot and then released with initial upward speed of $4\sqrt{3}$ feet per second", $x(0) = -1$ and $x'(0) = -4\sqrt{3}$

Plugging these into the general solution gives:

$$\begin{aligned} x &= A\cos(4t) + B\sin(4t) \\ x(0) &= A\cdot 1 + B\cdot 0 \implies A = -1 \\ x' &= -4A\sin(4t) + 4B\cos(4t) \\ x'(0) &= -4A\cdot 0 + 4B\cdot 1 \implies 4B = -4\sqrt{3} \implies B = -\sqrt{3} \end{aligned}$$

Therefore, the position (feet below equilibrium) of the weight after t seconds is given by $x(t) = -\cos(4t) - \sqrt{3}\sin(4t)$

This is equivalent to $x(t) = 2\sin\left(4t + \dfrac{7\pi}{6}\right)$:



If we take our ice skater attached to a spring and remove the ice itself to replace it with molasses, things dramatically change in our solutions. The introduction of friction introduces resistance to the motion in the system. In this case, we say that friction is directly proportional to the velocity of an object: $\vec{F}_{damping} = -cx'(t)$

Rule: Motion for Springs with Friction

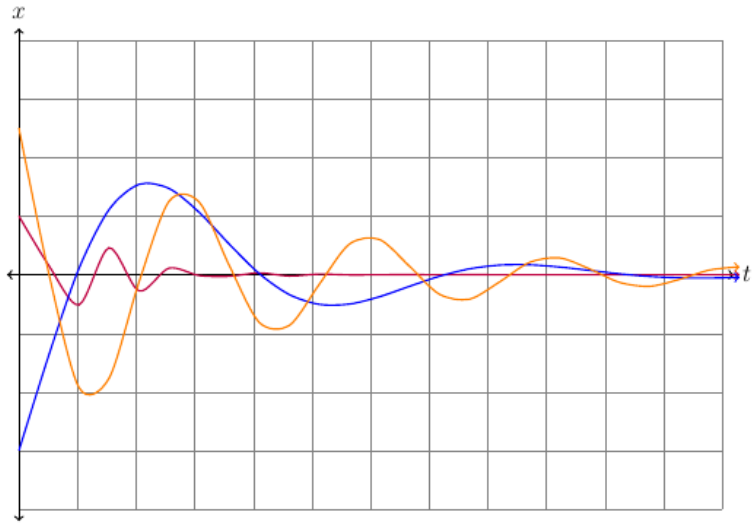
When an object with mass m is attached to a spring with coefficient k and experiences friction resisting motion with coefficient c, its motion satisfies the LDE:

$$mx''(t) + cx'(t) + kx(t) = 0$$

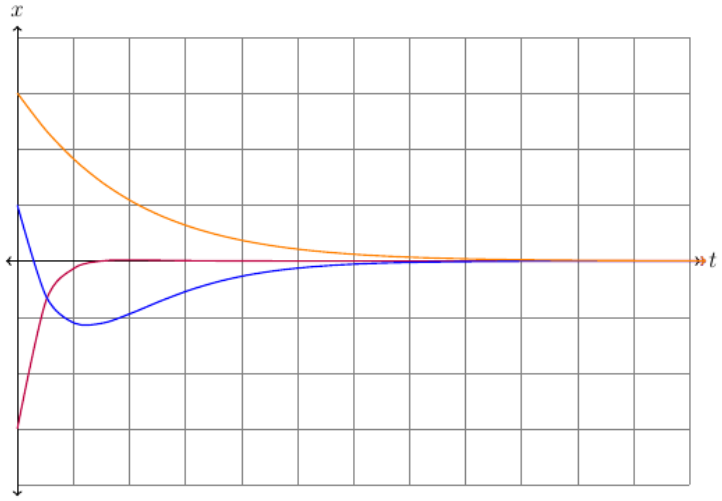
This is still a CC LDE, with characteristic equation $mr^2 + cr + k = 0 \implies r = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$

Thus, the shape of a solution depends on the value of c relative to k and m, specifically: $\Delta = c^2 - 4km$

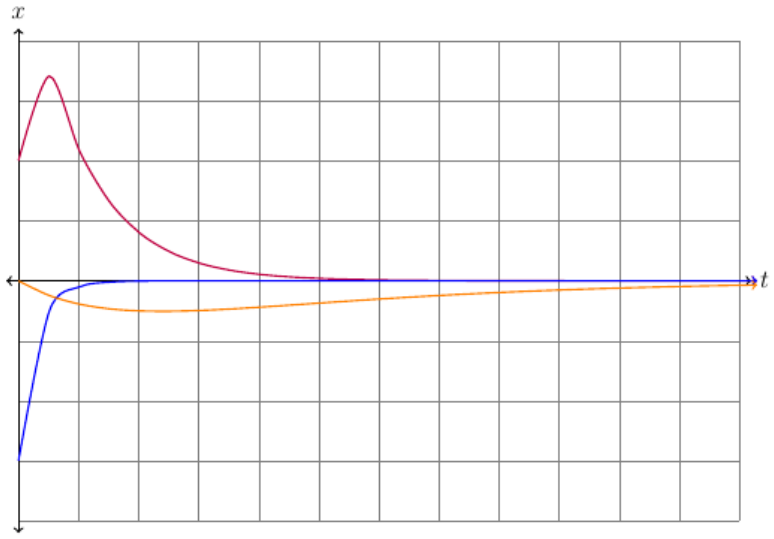
When $\Delta < 0$, there are complex roots $r = \alpha \pm \beta i$ with $\alpha < 0$, giving $x(t) = Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t)$



When $\Delta = 0$, there is a double root $r = r_1 < 0$, giving $x(t) = (A + Bt)e^{r_1 t}$



When $\Delta > 0$, there are two distinct real roots $r = r_1, r_2 < 0$, giving $x(t) = Ae^{r_1 t} + Be^{r_2 t}$



Essentially, when the friction constant is increased from zero, the system goes from free oscillation to damped oscillation (complex roots). As friction further increases, it reaches a critical point where there is no oscillation, just exponential decay to equilibrium.

Illustration 5.4.2

A 24 foot long spring hangs from the ceiling of a large room filled with a liquid that provides friction which is numerically equal to c times the instantaneous velocity. An object weighing $64lbs$ is attached to the spring, stretching it to a length of 32 feet. Find the function determining the distance between the ceiling and the object after t seconds when...

(a) $c = 4\sqrt{3}$ and the object is moved to a position 38 feet below the ceiling and released from rest.

(b) $c = 8$ and the object is released from equilibrium with an upward speed of 2 feet per second.

(c) $c = 10$ and the object is released from a position 34 feet below the ceiling with downward speed of 1 feet per second.

We start by finding m and k based on the initial information. The weight is 64 lbs, and $W = mg$ so $m = \dfrac{64}{32} = 2$ slugs.

Since the spring is extended by $32 - 24 = 8$ feet, we have $k \cdot 8 = mg \implies k = 2 \cdot 32 \cdot \dfrac{1}{8} = 8$ lbs per foot.

Therefore, the CC LDE describing the distance of this object from the ceiling will be:

$$mx'' + cx' + k(x - 32) = 0 \implies 2x'' + cx' + 8(x - 32) = 0$$

Regardless of what value c is, we can see there is a constant particular solution: $x_p(t) = 32$

(a) If $c = 4\sqrt{3}$, the characteristic equation is $2r^2 + 4\sqrt{3}\ r + 8 = 0 \implies r = -\sqrt{3} \pm i$

This gives (*underdamped*) homogeneous solutions of the form $x_h(t) = Ae^{-t\sqrt{3}}\cos(t) + Be^{-t\sqrt{3}}\sin(t)$

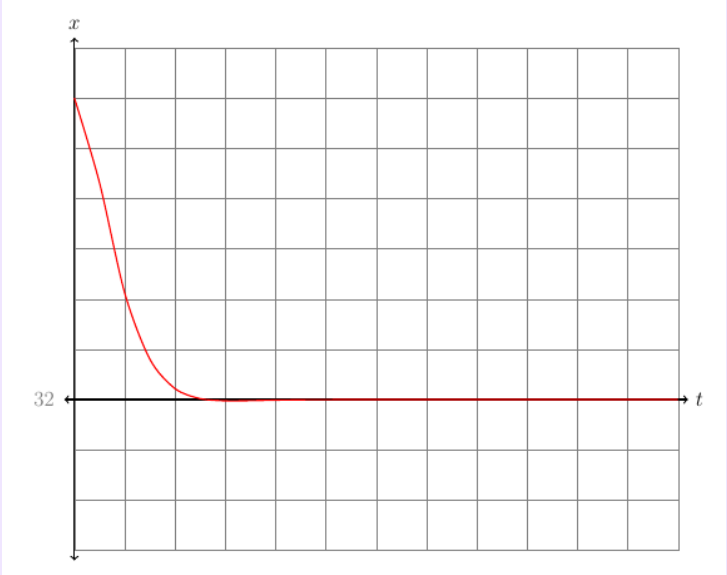
Therefore, we have general solution $x(t) = 32 + Ae^{-t\sqrt{3}}\cos(t) + Be^{-t\sqrt{3}}\sin(t)$

Plugging in the initial conditions $x(0) = 38,\ x'(0) = 0$ gives:

$$\begin{aligned} x &= 32 + Ae^{-t\sqrt{3}}\cos(t) + Be^{-t\sqrt{3}}\sin(t) \\ x(0) &= 32 + A \cdot 1 + B \cdot 0 \implies A = 6 \\ x' &= A\left(-\sqrt{3}e^{-t\sqrt{3}}\cos(t) - e^{-t\sqrt{3}}\sin(t)\right) + B\left(-\sqrt{3}e^{-t\sqrt{3}}\sin(t) + e^{-t\sqrt{3}}\cos(t)\right) \\ x'(0) &= A \cdot \left(-\sqrt{3} - 0\right) + B\,(0 + 1) \implies -A\sqrt{3} + B = 0 \implies B = 6\sqrt{3} \end{aligned}$$

This gives us final solution $x(t) = 32 + 6e^{-t\sqrt{3}}\cos(t) + 6\sqrt{3}e^{-t\sqrt{3}}\sin(t)$

This can be re-written as $x(t) = 32 + 12e^{-t\sqrt{3}}\sin\left(t + \dfrac{\pi}{6}\right)$



(b) If $c = 8$, the characteristic equation is $2r^2 + 8r + 8 = 2(r + 2)^2 = 0 \implies r = -2\ (\text{double})$

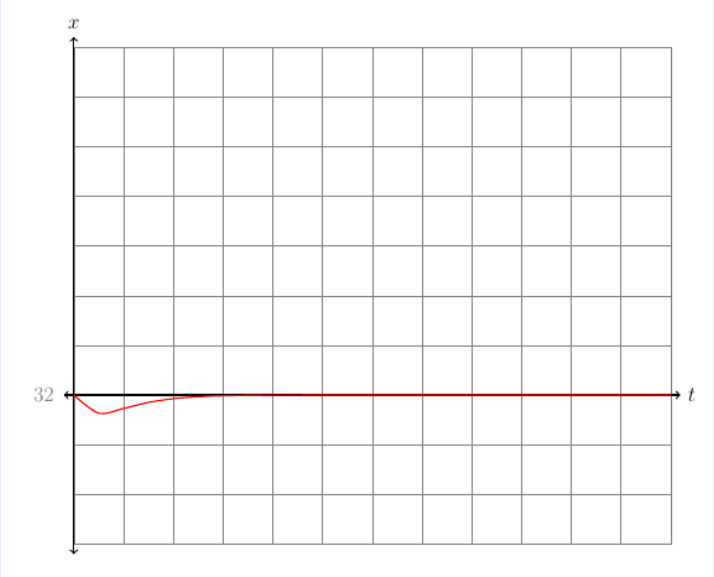
This gives (*critically damped*) homogeneous solutions of the form $x_h(t) = Ae^{-2t} + Bte^{-2t}$

Therefore, we have general solution $x(t) = 32 + Ae^{-2t} + Bte^{-2t}$

Plugging in the initial conditions $x(0) = 32,\ x'(0) = -2$ gives:

$$\begin{aligned} x &= 32 + Ae^{-2t} + Bte^{-2t} \\ x(0) &= 32 + A \cdot 1 + B \cdot 0 \implies A = 0 \\ x' &= -2Ae^{-2t} + B(-2t+1)e^{-2t} \\ x'(0) &= -2A \cdot 1 + B \cdot 1 \implies A + B = -2 \implies B = -2 \end{aligned}$$

This gives us final solution $x(t) = 32 - 2te^{-2t}$



(c) If $c = 10$, the characteristic equation is $2r^2 + 10r + 8 = 2(r+1)(r+4) \implies r = -1, -4$

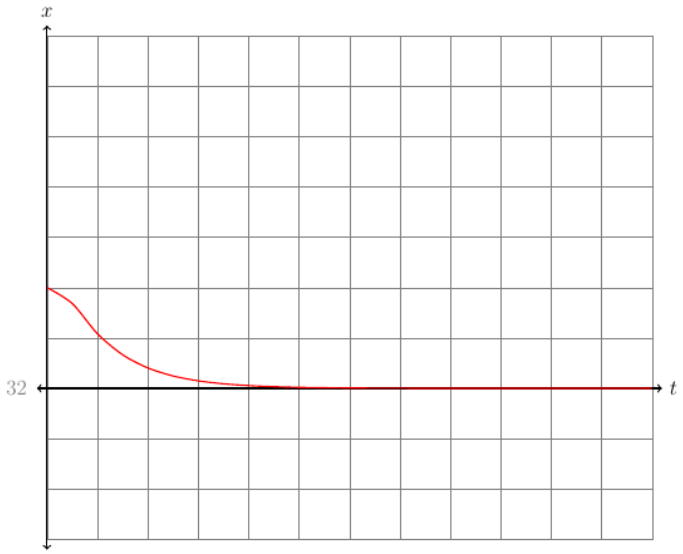
This gives (*overdamped*) homogeneous solutions of the form $x_h(t) = Ae^{-t} + Be^{-4t}$

Therefore, we have general solution $x(t) = 32 + Ae^{-t} + Be^{-4t}$

Plugging in the initial conditions $x(0) = 34,\ x'(0) = 1$ gives:

$$\begin{aligned} x &= 32 + Ae^{-t} + Be^{-4t} \\ x(0) &= 32 + A \cdot 1 + B \cdot 1 \implies A + B = 2 \\ x' &= -Ae^{-t} - 4Be^{-4t} \\ x'(0) &= -A \cdot 1 - 4B \cdot 1 \implies -A - 4B = 1 \\ \begin{cases} A + B = 2 \\ -A - 4B = 1 \end{cases} &\implies \begin{cases} A = 3 \\ B = -1 \end{cases} \end{aligned}$$

This gives us final solution $x(t) = 32 + 3e^{-t} - e^{-4t}$



When problems involve more than one mass and spring, we can use systems of CC LDEs to model their motion.

If we have a spring with constant k_1 attached to an object with $m = m_1$ slugs, and another spring with constant k_2 attached to that object and another object of mass $m = m_2$, then it is clear that the position of each object affects the force applied by each spring.

Assume that the complete system (both springs and masses) has an equilibrium point where the first spring is extended to length L_1 and the second spring is extended to length L_2. We define $x(t)$ to be the current position of the first object (and thus length of the first spring), and $y(t)$ to be the current position of the second object (so that $y(t) - x(t)$ is the length of the second spring). Then by Hooke's Law, the force applied by the second spring is $-k_2 \cdot (y(t) - x(t) - L_2)$, so we have $m_2 y''(t) = -k_2 (y(t) - x(t) - L_2)$. This second spring pushes or pulls on the second object in exactly the opposite direction that it pushes or pulls on the first object. If we include the force applied by the first spring (which is $-k_1 (x(t) - L_1)$, we get a second LDE: $m_1 x''(t) = -k_1 (x(t) - L_1) + k_2 (y(t) - x(t) - L_2)$.

Thus, the system of differential equations defining the motion of these two objects is given by:

Rule: Systems Equations for Multiple Springs without Friction

$$\begin{cases} m_1 x'' = -k_1 (x - L_1) + k_2 (y - x - L_2) \\ m_2 y'' = -k_2 (y - x - L_2) \end{cases}$$

Illustration 5.4.3

Two springs are hung vertically in series from the ceiling. The 24 foot top spring has spring constant $k_1 = 36$ lb/ft. A 192-lb ball ($m_1 = 6$ slugs) is attached to this spring, and another 24 foot spring with spring constant $k_2 = 6$ lb/ft is attached to that ball. Finally, a 32-lb ball ($m_2 = 1$ slug) hangs from the bottom spring.

(a) What are the equilibrium positions for each ball relative to the ceiling?

(b) If both balls are pulled down 5 feet from equilibrium and released from rest, what are their individual position functions?

(a) When the first spring has two masses total hanging from it, with total weight $W = 192 + 32 = 224$ lbs (or equivalently 7 slugs), it stretches to:

$$k_1 \cdot (L_1 - 24) = (m_1 + m_2)g \implies 36 \cdot (L_1 - 24) = 224 \implies L_1 = 24 + \frac{56}{9} = \frac{272}{9}$$

Thus, the equilibrium position for the first ball is $L_1 = \frac{272}{9} \approx 30.222$ feet below the ceiling.

The second spring is naturally 24 feet long, and has one ball (with mass $m_2 = 1$) attached to it. It stretches to:

$$k_2 \cdot (L_2 - 24) = m_2 g \implies 6 \cdot (L_2 - 24) = 32 \implies L_2 = 24 + \frac{16}{3} = \frac{88}{3}$$

Thus, the equilibrium length of the second spring is $L_2 = \frac{88}{3} \approx 29.333$ feet, putting the second ball at equilibrium position $\frac{272}{9} + \frac{88}{3} = \frac{536}{9} \approx 59.555$ feet below the ceiling.

The equilibrium positions are $\frac{272}{9}$ feet and $\frac{536}{9}$ below the ceiling.

(b) The system of equations that defines the motion of these balls on springs is:

$$\begin{cases} 6x'' = -36\left(x - \frac{272}{9}\right) + 6\left(y - x - \frac{88}{3}\right) \\ 1y'' = -6\left(y - x - \frac{88}{3}\right) \end{cases}$$

We write this using D notation as: $\begin{cases} (D^2+7)x - y = 152 \\ -6x + (D^2+6)y = 176 \end{cases}$

We eliminate y by "multiplying" the top equation by $D^2 + 6$ and adding the bottom equation:

$$(D^2+6)\left[(D^2+7)x - y = 152\right] + \left[-6x + (D^2+6)y = 176\right] \implies \left((D^2+7)(D^2+6) - 6\right)x = 448$$

This gives us a fourth order CC LDE: $x^{(4)}(t) + 13x''(t) + 36x(t) = 1088$

The characteristic equation is $r^4 + 13r^2 + 36 = (r^2+4)(r^2+9) = 0 \implies r = \pm 2i,\ \pm 3i$

This gives homogeneous solutions of the form $x_h(t) = A\cos(2t) + B\sin(2t) + C\cos(3t) + D\sin(3t)$.

Guessing a particular solution of the form $x_p = K$ gives $36K = 1088 \implies K = \frac{272}{9}$ (not surprising!)

This gives the general solution for x: $x(t) = \frac{272}{9} + A\cos(2t) + B\sin(2t) + C\cos(3t) + D\sin(3t)$

Substituting this into the first equation $(D^2+7)x - y = 152 \implies y = x'' + 7x - 152$:

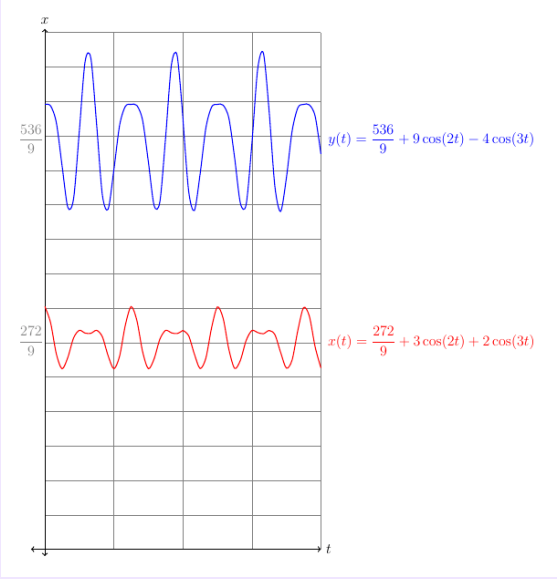
$$\begin{aligned} y(t) &= x''(t) + 7x(t) - 152 \\ &= (-4A\cos(2t) - 4B\sin(2t) - 9C\cos(3t) - 9D\sin(3t)) + 7\left(\frac{272}{9} + A\cos(2t) + B\sin(2t) + C\cos(3t) + D\sin(3t)\right) - 152 \\ &= \frac{536}{9} + 3A\cos(2t) + 3B\sin(2t) - 2C\cos(3t) - 2D\sin(3t) \end{aligned}$$

We are given both balls are pulled down 5 feet from equilibrium, so $x(0) = 5 + \frac{272}{9}$ and $y(0) = 5 + \frac{536}{9}$.
They are also released from rest, so we know $x'(0) = 0$ and $y'(0) = 0$.
Finally, we can plug in these initial conditions to find the parameters A, B, C, D:

$$x = \frac{272}{9} + A\cos(2t) + B\sin(2t) + C\cos(3t) + D\sin(3t)$$
$$x(0) = \frac{272}{9} + A\cdot 1 + B\cdot 0 + C\cdot 1 + D\cdot 0 \implies A + C = 2$$
$$x' = -2A\sin(2t) + 2B\cos(2t) - 3C\sin(3t) + 3D\cos(3t)$$
$$x'(0) = -2A\cdot 0 + 2B\cdot 1 - 3C\cdot 0 + 3D\cdot 1 \implies 2B + 3D = 0$$
$$y = \frac{536}{9} + 3A\cos(2t) + 3B\sin(2t) - 2C\cos(3t) - 2D\sin(3t)$$
$$y(0) = \frac{536}{9} + 3A\cdot 1 + 3B\cdot 0 - 2C\cdot 1 - 2D\cdot 0 \implies 3A - 2C = 2$$
$$y' = -6A\sin(2t) + 6B\cos(2t) + 6C\sin(3t) - 6D\cos(3t)$$
$$y'(0) = -6A\cdot 0 + 6B\cdot 1 + 6C\cdot 0 - 6D\cdot 1 \implies 6B - 6D = 0$$
$$\begin{cases} A + C = 5 \\ 2B + 3D = 0 \\ 3A - 2C = 5 \\ 6B - 6D = 0 \end{cases} \implies \begin{cases} A = 3 \\ B = 0 \\ C = 2 \\ D = 0 \end{cases}$$

Therefore, the position functions are $\begin{cases} x(t) = \frac{272}{9} + 3\cos(2t) + 2\cos(3t) \\ y(t) = \frac{536}{9} + 9\cos(2t) - 4\cos(3t) \end{cases}$

Plotting these two solutions together gives us a rough (upside down) picture of how the balls oscillate:



If the entire system is subjected to friction, then each object is slowed by a factor proportional to its current speed. This changes the system of LDEs into:

Rule: Systems Equations for Multiple Springs with Friction

$$\begin{cases} m_1 x'' = -k_1(x - L_1) + k_2(y - x - L_2) - c_1 x' \\ m_2 y'' = -k_2(y - x - L_2) - c_2 y' \end{cases}$$

Example 5.4.1

There are two identical balls that weigh 32 lbs each, and two identical springs each of length 30 feet. The first spring is attached to the ceiling on one end, and when the first ball is attached to it, and the second spring is then attached so that has one side attached to each ball. The system comes to rest with the lowest ball hanging exactly 84 feet from the ceiling. The top ball moves in a viscous fluid that provides a damping force proportional to its velocity with coefficient 4, and the bottom ball moves without friction. The top ball is moved to a height of \(46+\sqrt{3}\) from the ceiling while the bottom ball is held 84 feet from the ceiling. If both balls are released from rest, find the formula for the *distance between the balls* after t seconds.

Solution

We are given $m_1 = m_2 = \dfrac{32}{32} = 1$ slug.

The top spring stretches by $\dfrac{64}{k}$ feet when both of the balls are attached, and the bottom spring stretches by $\dfrac{32}{k}$ feet.

$$30 + \frac{32}{k} + 30 + \frac{64}{k} = 84 \implies \frac{96}{k} = 24 \implies k = 4$$

This gives us $L_1 = 30 + 16 = 46$ and $L_2 = 30 + 8 = 38$.

Lastly, we are also given the friction coefficient for the liquid $c_1 = 4$, and no friction on the bottom gives $c_2 = 0$.

Putting these all together gives our system of LDEs:

$$\begin{cases} 1x'' = -4\,(x - 46) + 4\,(y - x - 38) - 4x' \\ 1y'' = -4\,(y - x - 38) - 0y' \end{cases}$$

We write this as $\begin{cases} \left(D^2 + 4D + 8\right) x - 4y = 32 \\ -4x + \left(D^2 + 4\right) y = 152 \end{cases}$

To eliminate y we "multiply" the top equation by $D^2 + 4$ and add it to 4 times the bottom equation:

$$\left(D^2 + 4\right)\left[\left(D^2 + 4D + 8\right) x - 4y = 32\right] + 4\left[-4x + \left(D^2 + 4\right) y = 152\right] \implies \left(\left(D^2 + 4\right)\left(D^2 + 4D + 8\right) - 16\right) x = 736$$

If we multiply this out, we get the fourth order CC LDE: $x^{(4)} + 4x''' + 12x'' + 16x' + 16x = 736$

This has characteristic equation $r^4 + 4r^3 + 12r^2 + 16r + 16 = (r^2 + 2r + 4)^2 = 0 \implies r = -1 \pm i\sqrt{3}$ (double)

Thus, the homogeneous solutions are of the form $x_h(t) = (A + Bt)e^{-t}\cos\left(t\sqrt{3}\right) + (C + Dt)e^{-t}\sin\left(t\sqrt{3}\right)$.

We guess a particular solution of the form $x_p(t) = K \implies x_p' = 0,\ x_p'' = 0,\ x_p''' = 0,\ x_p^{(4)} = 0$:

$$x_p^{(4)} + 4x_p''' + 12x_p'' + 16x_p' + 16x_p = 16C \implies 16C = 736 \implies C = 46$$

We have found the general solution for x is $x(t) = 46 + (A + Bt)e^{-t}\cos\left(t\sqrt{3}\right) + (C + Dt)e^{-t}\sin\left(t\sqrt{3}\right)$.

Substituting this into the top equation gives us y:

$$\begin{aligned} y(t) &= \frac{1}{4}x''(t) + x'(t) + 2x(t) - 8 \\ &= 84 + \frac{1}{2}\left(A + B + C\sqrt{3} + D\sqrt{3} + Bt + Dt\sqrt{3}\right)e^{-t}\cos\left(t\sqrt{3}\right) + \frac{1}{2}\left(C + D - A\sqrt{3} - B\sqrt{3} + Dt - Bt\sqrt{3}\right)e^{-t}\sin\left(t\sqrt{3}\right) \end{aligned}$$

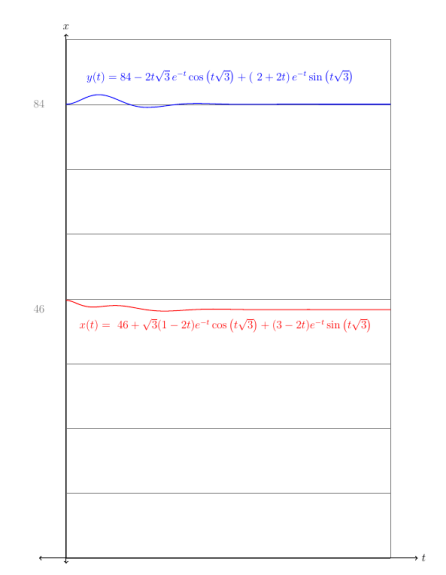
Now we plug in the initial conditions $x(0) = 46 + \sqrt{3}, x'(0) = 0, y(0) = 84, y'(0) = 0$ to get:

$$
\begin{aligned}
x &= 46 + (A+Bt)e^{-t}\cos\left(t\sqrt{3}\right) + (C+Dt)e^{-t}\sin\left(t\sqrt{3}\right) \\
x(0) &= 46 + A \\
&\implies A = \sqrt{3} \\
x' &= \left(B - A + C\sqrt{3} + Dt\sqrt{3} - Bt\right)e^{-t}\cos\left(t\sqrt{3}\right) + \left(D - C - A\sqrt{3} - Bt\sqrt{3} - Dt\right)e^{-t}\sin\left(t\sqrt{3}\right) \\
x'(0) &= B - A + C\sqrt{3} \\
&\implies B - A + C\sqrt{3} = 0 \\
y &= 84 + \frac{1}{2}\left(A + B + C\sqrt{3} + D\sqrt{3} + Bt + Dt\sqrt{3}\right)e^{-t}\cos\left(t\sqrt{3}\right) + \frac{1}{2}\left(C + D - A\sqrt{3} - B\sqrt{3} + Dt - Bt\sqrt{3}\right)e^{-t}\sin\left(t\sqrt{3}\right) \\
y(0) &= A + B + C\sqrt{3} + D\sqrt{3} \\
&\implies A + B + C\sqrt{3} + D\sqrt{3} = 0 \\
y' &= \left(-4A - 3B + D\sqrt{3} - 4Bt\right)e^{-t}\cos\left(t\sqrt{3}\right) + \left(-B\sqrt{3} - C - D - 4Dt\right)e^{-t}\sin\left(t\sqrt{3}\right) \\
y'(0) &= -4A - 3B + D\sqrt{3} \\
&\implies -4A - 3B + D\sqrt{3} = 0
\end{aligned}
$$

$$
\begin{cases} A = \sqrt{3} \\ B - A + C\sqrt{3} = 0 \\ A + B + C\sqrt{3} + D\sqrt{3} = 0 \\ -4A - 3B + D\sqrt{3} = 0 \end{cases} \implies \begin{cases} A = \sqrt{3} \\ B = -2\sqrt{3} \\ C = 3 \\ D = -2 \end{cases}
$$

Therefore, the balls satisfy equations $\begin{cases} x(t) = 46 + \sqrt{3}(1-2t)e^{-t}\cos\left(t\sqrt{3}\right) + (3-2t)e^{-t}\sin\left(t\sqrt{3}\right) \\ y(t) = 84 - 2t\sqrt{3}\,e^{-t}\cos\left(t\sqrt{3}\right) + (2+2t)\,e^{-t}\sin\left(t\sqrt{3}\right) \end{cases}$

The difference between them must be $\boxed{y(t) - x(t) = 38 - \sqrt{3}\,e^{-t}\cos\left(t\sqrt{3}\right) + (4t-1)\,e^{-t}\sin\left(t\sqrt{3}\right)}$



The mass-spring oscillator is more than just a classroom example; it is the fundamental building block for modeling vibration and resonance in the physical world. From the suspension systems in vehicles that utilize damping to prevent endless bouncing, to the structural engineering of skyscrapers designed to withstand the rhythmic swaying of wind or seismic activity, these second-order equations allow us to predict and control stability. By understanding the balance between mass, stiffness, and friction, engineers can ensure that systems return safely to equilibrium rather than succumbing to catastrophic oscillations.

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5.4: Practice Set

Exercise $5.4.1a$

A 25 lb weight is attached to a spring, extending it by 2 feet. The weight is pushed downward from this equilibrium position at an initial speed of 2 inches per second. What is the equation of motion for the weight?

Answer available at the end of this section.

Exercise $5.4.1b$

A 999 lb weight is attached to a spring, extending it by 32 feet. The weight is pulled downward to a position 3 feet below equilibrium and released from rest. What is the equation of motion for the weight?

Answer available at the end of this section.

Exercise $5.4.1c$

A 32 lb weight is attached to a spring, extending it by 3 inches. The weight is pulled 4 inches above this equilibrium position and then pushed further upward at an initial speed of 2 inches per second. What is the equation of motion for the weight?

Answer available at the end of this section.

Exercise $5.4.1d$

A $\frac{32}{9}$ lb weight is attached to a spring with spring constant 1 lb/ft. If the weight is pulled downward 4 feet from its equilibrium position and pushed back upward at an initial speed of 9 feet per second, what is maximum displacement from equilibrium of this weight?

Answer available at the end of this section.

Exercise $5.4.1e$

A 12 lb weight is attached to a spring with spring constant 1.5 lb/ft. The weight is pushed downward from a point 6 feet above its equilibrium position at an initial speed of 5 feet per second. What is the maximum speed of the weight?

Answer available at the end of this section.

Exercise $5.4.1f$

A 16 lb weight is attached to a spring that has a constant of $\frac{1}{8}$ lb/ft. If the weight is released from rest at a point 4 inches above its equilibrium position. How long does it take to return to equilibrium for the first time?

Answer available at the end of this section.

Exercise $5.4.1g$

A spring is suspended from the ceiling. When a mass weighing $2\pi^2$ lbs is attached, the spring is stretched by $\frac{1}{2}$ of a foot. The mass is removed and a person grabs the end of the stretched spring. If they take two seconds to return back to their starting position, how much does this person weigh?

Answer available at the end of this section.

Exercise $5.4.2a$

A 32 lb weight is attached to a spring, extending it by 4 inches. The medium through which the weight moves offers a resistance numerically equal to 2 times the instantaneous velocity (in feet per second). If the weight is released from rest at a point which is 1 foot below equilibrium position, what is the equation of motion for the weight?

Answer available at the end of this section.

Exercise $5.4.2b$

An 8 lb weight is attached to a spring, extending it by 6 inches. The medium through which the weight moves offers a resistance numerically equal to the instantaneous velocity (in feet per second). If the weight is pushed downward from this equilibrium position at an initial speed of $2\sqrt{3}$ feet per second, what is the equation of motion for the weight?

Answer available at the end of this section.

Exercise $5.4.2c$

A 1 lb weight is attached to a spring, extending it by 9 inches. The medium through which the weight moves offers a resistance numerically equal to $\frac{1}{\sqrt{6}}$ times the instantaneous velocity (in feet per second). If the weight released from a position 6 inches below equilibrium with an upward velocity of $\sqrt{6}$ feet per second, what is the equation of motion for the weight?

Answer available at the end of this section.

Exercise $5.4.2d$

A weight is attached to a spring with a spring constant of $k = 2$. The medium through which the weight moves offers a resistance numerically equal to the instantaneous velocity (in feet per second). If the result is critically damped, how heavy is the weight?

Answer available at the end of this section.

Exercise $5.4.2e$

A 64 lb weight is attached to a spring with a spring constant of $k = 2.5$ lb/ft. The medium through which the weight moves offers a resistance equal to 2 times the instantaneous velocity (in feet per second). If the weight is released from rest at a point 4 feet above the equilibrium position, what is its lowest position below equilibrium?

Answer available at the end of this section.

Exercise $5.4.2f$

A 24 lb weight is attached to a spring, extending it by 4 feet. The medium through which the weight moves offers a resistance numerically equal to 3 times the instantaneous velocity (in feet per second). If the weight is released from rest at a point 1 foot below its equilibrium position, what is the furthest it will ever be above equilibrium?

Answer available at the end of this section.

Exercise $5.4.3a$

A 32 lb hook is attached to a $\frac{17}{3}$ ft weightless spring hanging from the ceiling with spring constant $k_1 = 3$ lb/ft. A second 10 ft weightless spring with spring constant $k_2 = 2$ lb/ft is attached to the hook, and a 32 lb weight is attached to its end. If the hook begins 30 feet below the ceiling at rest, and the weight begins 54 feet below the ceiling at rest, what is the equation of motion for the weight relative to the ceiling?

Answer available at the end of this section.

Exercise $5.4.3b$

A 32 lb brick is attached to a 14.4 ft weightless spring hanging from the ceiling, extending it by 6.4 ft. A second 12 ft weightless spring is attached to the brick, and then a 96 lb weight is attached to its end. The entire system settles at an equilibrium point where the weight is exactly 60 feet below the ceiling. If the brick begins at equilibrium with upward velocity of 19 feet per second, and the weight begins at equilibrium with downward velocity 19 feet per sec, what is the equation of motion for the brick relative to the ceiling?

Answer available at the end of this section.

Exercise $5.4.3c$

A 32 lb bob is attached to a $\frac{35}{3}$ ft weightless spring (with spring constant $k_1 = 3$ ft/lb) hanging from bottom of a ship. A 100 ft weightless spring (with spring constant $k_1 = 2$ ft/lb.) is attached to the bob, and then a 32 lb sensor is attached to its end. If the water provides resistance numerically equivalent to 2 times the velocity of an object (in feet per second), and the weight begins at equilibrium at rest, and the sensor begins at rest 5 feet below equilibrium, what is the equation of motion for the sensor relative to the ship?

Answer available at the end of this section.

Answers

Exercise $5.4.1a$

First, we solve for the spring constant k:

$$25 = k \cdot 2 \implies k = \frac{25}{2} \text{ lb/ft}$$

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \frac{W}{g} = \frac{25}{32}$. Using the undamped DE gives:

$$mx''(t) + kx(t) = 0 \implies x''(t) + 16x(t) = 0$$

This CC LDE has characteristic equation $r^2 + 16 = 0 \implies r = \pm 4i$

Therefore, the solution is of the form $x(t) = A\cos(4t) + B\sin(4t)$

The initial condition that it is released from equilibrium means $x(0) = 0$. Since it is released with initial downward speed, we know that $x'(0) = \frac{1}{6}$:

$$x(t) = x(t) = A\cos(4t) + B\sin(4t)$$
$$x(0) = A \cdot 1 + B \cdot 0 \implies A = 0$$
$$x'(t) = -4A\sin(4t) + 4B\cos(4t)$$
$$x'(0) = -4A \cdot 0 + 4B \cdot 1 \implies B = \frac{1}{24}$$

Therefore, we have solution $x(t) = \frac{1}{24}\sin(4t)$

Exercise 5.4.1*b*

First, we solve for the spring constant k:

$$999 = k \cdot 32 \implies k = \frac{999}{32} \text{ lb/ft}$$

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \frac{W}{g} = \frac{999}{32}$. Using the undamped DE gives:

$$mx''(t) + kx(t) = 0 \implies x''(t) + x(t) = 0$$

This CC LDE has characteristic equation $r^2 + 1 = 0 \implies r = \pm i$

Therefore, the solution is of the form $x(t) = A\cos(t) + B\sin(t)$

The initial condition that it is released from below equilibrium means $x(0) = 0$. Since it is released from rest, we know that $x'(0) = 0$:

$$x(t) = A\cos(t) + B\sin(t)$$
$$x(0) = A \cdot 1 + B \cdot 0 \implies A = 3$$
$$x'(t) = -A\sin(t) + B\cos(t)$$
$$x'(0) = -A \cdot 0 + B \cdot 1 \implies B = 0$$

Therefore, we have solution $x(t) = 3\cos(t)$

Exercise 5.4.1*c*

First, we solve for the spring constant k:

$$32 = k \cdot \frac{1}{4} \implies k = 128 \text{ lb/ft}$$

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \frac{W}{g} = 1$. Using the undamped DE gives:

$$mx''(t) + kx(t) = 0 \implies x''(t) + 128x(t) = 0$$

This CC LDE has characteristic equation $r^2 + 128 = 0 \implies r = \pm 8i\sqrt{2}$

Therefore, the solution is of the form $x(t) = A\cos\left(8t\sqrt{2}\right) + B\sin\left(8t\sqrt{2}\right)$

The initial condition that it is released from above equilibrium means $x(0) = -\frac{1}{3}$. Since it is released with initial upward speed, we know that $x'(0) = -\frac{1}{6}$:

$$x(t) = A\cos\left(8t\sqrt{2}\right) + B\sin\left(8t\sqrt{2}\right)$$
$$x(0) = A \cdot 1 + B \cdot 0 \implies A = -\frac{1}{3}$$
$$x'(t) = -8\sqrt{2}\,A\sin(8t\sqrt{2}) + 8\sqrt{2}\,B\cos(8t\sqrt{2})$$
$$x'(0) = -8\sqrt{2}\,A \cdot 0 + 8\sqrt{2}\,B \cdot 1 \implies B = -\frac{1}{48\sqrt{2}}$$

Therefore, we have solution $x(t) = -\frac{1}{3}\cos(8t\sqrt{2}) - \frac{1}{48\sqrt{2}}\sin(8t\sqrt{2})$

Exercise 5.4.1*d*

We are given the spring constant $k = 1$ lb/ft.

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \frac{W}{g} = \frac{1}{9}$. Using the undamped DE gives:

$$mx''(t) + kx(t) = 0 \implies x''(t) + 9x(t) = 0$$

This CC LDE has characteristic equation $r^2 + 9 = 0 \implies r = \pm 3i$

Therefore, the solution is of the form $x(t) = A\cos(3t) + B\sin(3t)$

The initial condition that it is released from below equilibrium means $x(0) = 4$. Since it is released with initial upward speed, we know that $x'(0) = -9$:

$$x(t) = A\cos(3t) + B\sin(3t)$$
$$x(0) = A \cdot 1 + B \cdot 0 \implies A = 4$$
$$x'(t) = -3A\sin(3t) + 3B\cos(3t)$$
$$x'(0) = -4A \cdot 0 + 3B \cdot 1 \implies B = -3$$

Therefore, we have solution $x(t) = 4\cos(3t) - 3\sin(3t) = 5\sin(3t + \theta_0)$

We see that the maximum displacement (in either direction) is exactly 5 feet.

Exercise 5.4.1*e*

We are given the spring constant $k = \frac{3}{2}$ lb/ft.

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \frac{W}{g} = \frac{3}{8}$. Using the undamped DE gives:

$$mx''(t) + kx(t) = 0 \implies x''(t) + 4x(t) = 0$$

This CC LDE has characteristic equation $r^2 + 4 = 0 \implies r = \pm 2i$

Therefore, the solution is of the form $x(t) = A\cos(2t) + B\sin(2t)$

The initial condition that it is released from above equilibrium means $x(0) = -6$. Since it is released with initial downward speed, we know that $x'(0) = 5$:

$$x(t) = A\cos(2t) + B\sin(2t)$$
$$x(0) = A \cdot 1 + B \cdot 0 \implies A = -6$$
$$x'(t) = -2A\sin(2t) + 2B\cos(2t)$$
$$x'(0) = -2A \cdot 0 + 2B \cdot 1 \implies B = \frac{5}{2}$$

Therefore, we have solution $x(t) = -6\cos(2t) + \frac{5}{2}\sin(2t)$

The velocity must then be $x'(t) = 12\sin(2t) + 5\cos(2t) = 13\sin(2t + \theta_0)$

We see that the maximum speed is exactly 13 feet/sec.

Exercise $5.4.1f$

We are given that $k = \frac{1}{8}$ ft/lb and $W = 16$ lbs.

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$. Using the undamped DE gives:

$$mx''(t) + kx(t) = 0 \implies x''(t) + \frac{1}{4}x(t) = 0$$

This CC LDE has characteristic equation $r^2 + \frac{1}{4} = 0 \implies r = \pm\frac{1}{2}i$

Therefore, the solution is of the form $x(t) = A\cos\left(\frac{1}{2}t\right) + B\sin\left(\frac{1}{2}t\right)$

The initial condition that it is released 4 inches above equilibrium means $x(0) = -\frac{1}{3}$. Since it is released from rest, we know that $x'(0) = 0$:

$$x(t) = A\cos\left(\frac{1}{2}t\right) + B\sin\left(\frac{1}{2}t\right)$$
$$x(0) = A \cdot 1 + B \cdot 0 \implies A = -\frac{1}{3}$$
$$x'(t) = -2A\sin\left(\frac{1}{2}t\right) + 2B\cos\left(\frac{1}{2}t\right)$$
$$x'(0) = -2A \cdot 0 + 2B \cdot 1 \implies B = 0$$

Therefore, we have solution $x(t) = -\frac{1}{3}\cos\left(\frac{1}{2}t\right)$

To find when it reaches equilibrium position, we solve $x(t) = 0$. There are infinitely many solutions: $t = \ldots, -3\pi, -\pi, \pi, 3\pi, 5\pi, \ldots$

Since we want the "first time" it returns, we take the smallest positive such t.

It returns to the equilibrium position after π seconds.

Exercise $5.4.1g$

We use the initial weight to solve for the spring constant k:

$$2\pi^2 = k \cdot \frac{1}{2} \implies k = 4\pi^2 \text{ lbs/ft}$$

We do not know the mass of the person, so assume it is m. Using the undamped DE gives:

$$mx''(t) + kx(t) = 0 \implies x'' + \frac{4\pi^2}{m}x = 0$$

This CC LDE has characteristic equation $r^2 + \frac{4\pi^2}{m} = 0 \implies r = \pm\frac{2\pi i}{\sqrt{m}}$

Therefore, the solution is of the form $x(t) = A\cos\left(\frac{2\pi t}{\sqrt{m}}\right) + B\sin\left(\frac{2\pi t}{\sqrt{m}}\right) = C\sin\left(\frac{2\pi t}{\sqrt{m}} + D\right)$

For this to have a period of $t = 2$ second, we need $\frac{2\pi \cdot 2}{\sqrt{m}} = 2\pi \implies m = 4$

Remember that weight is just mass times gravity, and $g = 32$ ft/sec^2, so $W = mg = 4 \cdot 32 = 128$

Therefore, this person weighs 128 lbs.

Exercise $5.4.2a$

We are given $\beta = 2$, we solve for the spring constant k:

$$32 = k \cdot \frac{1}{3} \implies k = 96 \text{ lb/ft}$$

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \frac{W}{g} = 1$. Using the damped DE:

$$mx''(t) + \beta x'(t) + kx(t) = 0 \implies x''(t) + 2x'(t) + 96x(t) = 0$$

This CC LDE has characteristic equation $r^2 + 2r + 96 = 0 \implies r = -1 \pm i\sqrt{95}$

Therefore, the solution is of the form $x(t) = Ae^{-t}\cos\left(t\sqrt{95}\right) + Be^{-t}\sin\left(t\sqrt{95}\right)$

The initial condition that it is released from below equilibrium means $x(0) = 1$. Since it is released from rest, we know that $x'(0) = 0$:

$$x(t) = Ae^{-t}\cos\left(t\sqrt{95}\right) + Be^{-t}\sin\left(t\sqrt{95}\right)$$
$$x(0) = A \cdot 1 + B \cdot 0 \implies A = 1$$
$$x'(t) = -Ae^{-t}\cos(t\sqrt{95}) - A\sqrt{95}e^{-t}\sin(t\sqrt{95}) - Be^{-t}\sin(t\sqrt{95}) + B\sqrt{95}e^{-t}\cos(t\sqrt{95})$$
$$x'(0) = -A \cdot 1 - A\sqrt{95} \cdot 0 - B \cdot 0 + B\sqrt{95} \cdot 1 \implies B\sqrt{95} - A = 0 \implies B = \frac{1}{\sqrt{95}}$$

Therefore, we have solution $x(t) = e^{-t}\left(\cos(t\sqrt{95}) + \frac{1}{\sqrt{95}}\sin(t\sqrt{95})\right)$

Exercise $5.4.2b$

We are given $\beta = 1$, we solve for the spring constant k:

$$\frac{1}{2} = k \cdot 2 \implies k = \frac{1}{4} \text{ lb/ft}$$

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \dfrac{W}{g} = \dfrac{1}{4}$. Using the damped DE:

$$mx''(t) + \beta x'(t) + kx(t) = 0 \implies \frac{1}{4}x''(t) + x'(t) + \frac{1}{4}x(t) = 0$$

This CC LDE has characteristic equation $\dfrac{1}{4}r^2 + r + \dfrac{1}{4} = 0 \implies r = -2 \pm \sqrt{3}$

Therefore, the solution is of the form $x(t) = Ae^{(-2+\sqrt{3})t} + Be^{(-2-\sqrt{3})t}$

The initial condition that it is released from equilibrium means $x(0) = 0$. Since it is released with downward velocity, we know that $x'(0) = 5$:

$$\begin{aligned} x(t) &= Ae^{(-2+\sqrt{3})t} + Be^{(-2-\sqrt{3})t} \\ x(0) &= A \cdot 1 + B \cdot 1 \implies A + B = 0 \\ x'(t) &= (-2+\sqrt{3})Ae^{(-2+\sqrt{3})t} + (-2-\sqrt{3})Be^{(-2-\sqrt{3})t} \\ x'(0) &= (-2+\sqrt{3})A \cdot 1 + (-2-\sqrt{3})B \cdot 1 = \sqrt{3} \\ &\begin{cases} A + B = 0 \\ (-2+\sqrt{3})A + (-2-\sqrt{3})B = \sqrt{3} \end{cases} \implies \begin{cases} A = 1 \\ B = -1 \end{cases} \end{aligned}$$

Therefore, we have solution $x(t) = e^{(-2+\sqrt{3})t} - e^{(-2-\sqrt{3})t}$

Exercise $5.4.2c$

We are given $\beta = \dfrac{1}{\sqrt{6}}$, we solve for the spring constant k:

$$1 = k \cdot \frac{3}{4} \implies k = \frac{4}{3} \text{ lb/ft}$$

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \dfrac{W}{g} = \dfrac{1}{32}$. Using the damped DE:

$$mx''(t) + \beta x'(t) + kx(t) = 0 \implies \frac{1}{32}x''(t) + \frac{1}{\sqrt{6}}x'(t) + \frac{4}{3}x(t) = 0$$

This CC LDE has characteristic equation $\dfrac{1}{32}r^2 + \dfrac{1}{\sqrt{6}}r + \dfrac{4}{3} = 0 \implies r = -\dfrac{16}{\sqrt{6}}$ (double).

Therefore, the solution is of the form $x(t) = (A + Bt)e^{-\frac{16t}{\sqrt{6}}}$

The initial condition that it is released from below equilibrium means $x(0) = \dfrac{1}{2}$. Since it is released with upward velocity, we know that $x'(0) = -\sqrt{6}$:

$$\begin{aligned} x(t) &= (A + Bt)e^{-\frac{16t}{\sqrt{6}}} \\ x(0) &= (A + B \cdot 0) \cdot 1 \implies A = \frac{1}{2} \\ x'(t) &= \left(B - \frac{16}{\sqrt{6}}A - \frac{16}{\sqrt{6}}Bt\right)e^{-\frac{16t}{\sqrt{6}}} \\ x'(0) &= \left(B - \frac{16}{\sqrt{6}}A - \frac{16}{\sqrt{6}}B \cdot 0\right) \cdot 1 = -\sqrt{6} \implies B = \frac{2}{\sqrt{6}} \end{aligned}$$

Therefore, we have solution $x(t) = \left(\dfrac{1}{2} + \dfrac{2}{\sqrt{6}}t\right)e^{-\frac{16t}{\sqrt{6}}}$

Exercise $5.4.2d$

We are given $k = 2$ and $\beta = 1$, but want to find W. Using the damped DE:

$$mx''(t) + \beta x'(t) + kx(t) = 0 \implies mx''(t) + x'(t) + 2x(t) = 0$$

This CC LDE has characteristic equation $mr^2 + r + 2 = 0 \implies r = \dfrac{-1 \pm \sqrt{1 - 8m}}{2m}$

To be critically damped this needs to have a double root, so we must have $m = \dfrac{1}{8}$

This gives us $W = mg = \dfrac{1}{8} \cdot 32 = 4$, so the weight must be 4 lbs.

Exercise $5.4.2e$

We are given that $k = \dfrac{5}{2}$ and $\beta = 2$.

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \dfrac{W}{g} = 2$. Using the damped DE:

$$mx''(t) + \beta x'(t) + kx(t) = 0 \implies 2x''(t) + 2x'(t) + \frac{5}{2}x(t) = 0$$

This CC LDE has characteristic equation $2r^2 + 2r + \dfrac{5}{2} = 0 \implies r = -\dfrac{1}{2} \pm i$

Therefore, the solution is of the form $x(t) = Ae^{-\frac{t}{2}}\cos(t) + Be^{-\frac{t}{2}}\sin(t)$

The initial condition that it is released from above equilibrium means $x(0) = -4$. Since it is released from rest, we know that $x'(0) = 0$:

$$\begin{aligned} x(t) &= Ae^{-\frac{t}{2}}\cos(t) + Be^{-\frac{t}{2}}\sin(t) \\ x(0) &= A \cdot 1 + B \cdot 0 \implies A = -4 \\ x'(t) &= -\frac{1}{2}Ae^{-\frac{t}{2}}\cos(t) - Ae^{-\frac{t}{2}}\sin(t) - \frac{1}{2}Be^{-\frac{t}{2}}\sin(t) + Be^{-\frac{t}{2}}\cos(t) \\ x'(0) &= -\frac{1}{2}A \cdot 1 - A \cdot 0 - \frac{1}{2}B \cdot 0 + B \cdot 1 \implies B - \frac{1}{2}A = 0 \implies B = -2 \end{aligned}$$

Therefore, we have solution $x(t) = -e^{-\frac{t}{2}}\left(4\cos(t) + 2\sin(t)\right)$

To find when it reaches its lowest point (minimum) so we find the critical points:

$$x'(t) = 5e^{-\frac{t}{2}}\sin(t) = 0 \implies t = \ldots, -\pi, 0, \pi, 2\pi, \ldots$$

Therefore, the first positive critical point (which is the lowest position it will reach) occurs at time $t = \pi$, giving $x(\pi) = 4e^{-\frac{\pi}{2}}$

The lowest position the weight reaches is $4e^{-\frac{\pi}{2}}$ feet below equilibrium.

Exercise $5.4.2f$

We are given $\beta = 3$, we solve for the spring constant k:

$$24 = k \cdot 4 \implies k = 6 \text{ lb/ft}$$

Since $W = mg$ and $g = 32$ ft/sec^2, we solve for $m = \dfrac{W}{g} = \dfrac{3}{4}$. Using the damped DE:

$$mx''(t) + \beta x'(t) + kx(t) = 0 \implies \frac{3}{4}x''(t) + 3x'(t) + 6x(t) = 0$$

This CC LDE has characteristic equation $\dfrac{3}{4}\left(r^2 + 4r + 8\right) = 0 \implies r = -2 \pm 2i$

Therefore, the solution is of the form $x(t) = Ae^{-2t}\cos(2t) + Be^{-2t}\sin(2t)$

The initial condition that it is released from below equilibrium means $x(0) = 1$. Since it is released from rest, we know that $x'(0) = 0$:

$$\begin{aligned} x(t) &= Ae^{-2t}\cos(2t) + Be^{-2t}\sin(2t) \\ x(0) &= A \cdot 1 + B \cdot 0 \implies A = 1 \\ x'(t) &= -2Ae^{-2t}\cos(2t) - 2Ae^{-2t}\sin(2t) - 2Be^{-2t}\sin(2t) + 2Be^{-2t}\cos(2t) \\ x'(0) &= -2A \cdot 1 - 2A \cdot 0 - 2B \cdot 0 + 2B \cdot 1 \implies 2B - 2A = 0 \implies B = 1 \end{aligned}$$

Therefore, we have solution $x(t) = e^{-2t}\left(\cos(2t) + \sin(2t)\right)$

To find the extrema, we find the critical points:

$$x'(t) = -4e^{-2t}\sin(2t) = 0 \implies t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$$

Since this oscillates and decays, we know the largest negative value occurs at $x\left(\dfrac{\pi}{2}\right) = e^{-\pi}$.

Therefore the furthest it will be above equilibrium is $e^{-\pi}$ feet.

Exercise $5.4.3a$

We denote the downward displacement of the hook from the ceiling as $x(t)$ and the displacement of the weight from the ceiling as $y(t)$, so that the initial conditions are $x(0) = 30,\ x'(0) = 0,\ y(0) = 54,\ y'(0) = 0$.

We see that the masses of each object are $m_1 = m_2 = \dfrac{W}{g} = 1$

We first find the static equilibrium position for the entire system:

On the top spring $W_1 + W_2 = 64$ lbs, so $64 = 3\left(x_e - \dfrac{17}{3}\right) \implies x_e = 27$

On the bottom spring $W_2 = 32$, so $32 = 2(y_e - 10 - x_e) \implies y_e = 53$

Now we can more easily find the net force applied on each object.

We see at equilibrium, the bottom spring is stretched to $y_e - x_e = 26$ feet, so the net downward force on the weight is:

$$F_2 = -k_2(y(t) - x(t) - 26) = 52 - 2y(t) + 2x(t)$$

Since $F_2 = m_2 a$, we get $y''(t) = 52 - 2y(t) + 2x(t)$

The top spring at equilibrium is stretched to $x_e = 27$ feet, so the net downward force on the hook will be:

$$F_1 = -k_1\left(x(t) - 27\right) - F_2 = -3x(t) + 81 - (52 - 2y(t) + 2x(t)) = 29 - 5x(t) + 2y(t)$$

Since $F_1 = m_1 a$, we get $x''(t) = 29 - 5x(t) + 2y(t)$

Together these give us a system of differential equations:

$$\begin{cases} x'' = 29 - 5x + 2y \\ y'' = 52 - 2y + 2x \end{cases} \quad \text{with } x(0) = 30,\ x'(0) = 0,\ y(0) = 54,\ y'(0) = 0$$

We use Laplace Transformations to solve this system:

$$\begin{cases} s^2X(s) - 30s = \dfrac{29}{s} - 5X(s) + 2Y(s) \\ s^2Y(s) - 54s = \dfrac{52}{s} - 2Y(s) + 2X(s) \end{cases} \implies \begin{cases} (s^2+5)X(s) - 2Y(s) = 30s + \dfrac{29}{s} \\ -2X(s) + (s^2+2)Y(s) = 54s + \dfrac{52}{s} \end{cases}$$

Adding twice the top equation to $(s^2 + 5)$ times the bottom gives:

$$\begin{aligned} \left[(s^2+5)(s^2+2) - 4\right]Y(s) &= (s^2+5)54s + (s^2+5)\frac{52}{s} + 60s + \frac{58}{s} \\ \$(s^4 + 7s^2 + 6)Y(s) &= \frac{54s^4 + 382s^2 + 318}{s} \\ \$Y(s) &= \frac{54s^4 + 382s^2 + 318}{s(s^2+1)(s^2+6)} \end{aligned}$$

Using partial fraction decomposition, we can write: $Y(s) = \dfrac{53}{s} + \dfrac{2s}{s^2+1} - \dfrac{s}{s^2+6}$

Now we take the inverse Laplace Transformation:

$$y(t) = 53 \cdot \mathscr{L}^{-1}\left(\frac{1}{s}\right) + 2 \cdot \mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right) - \mathscr{L}^{-1}\left(\frac{s}{s^2+6}\right) = 53 \cdot 1 + 2 \cdot \cos(t) - \cos(t\sqrt{6})$$

Therefore, the equation of motion for the weight is given by $y(t) = 53 + 2\cos(t) - \cos(t\sqrt{6})$

Exercise 5.4.3*b*

We denote the downward displacement of the brick from the ceiling as $x(t)$ and the displacement of the weight from the ceiling as $y(t)$, so that the initial conditions are $x(0) = 40,\ x'(0) = -19,\ y(0) = 60,\ y'(0) = 19$

The mass of the brick is $m_1 = \dfrac{32}{32} = 1$ and of the weight is $m_2 = \dfrac{96}{32} = 3$

We first find the spring constants for each spring:

The top spring is stretched by 6.4 feet by the brick, so $32 = k_1 \cdot 6.4 \implies k_1 = 5$

When the weight is added, this further stretches the top spring to $x_e = 14.4 + \dfrac{96+32}{5} = 40$ feet.

Since the entire system settles with the weight exactly 60 ft below the ceiling, we know the second spring has extended to $60 - 40 = 20$ feet, so

$$96 = k_2(20 - 12) \implies k_2 = 12$$

Now we can find the net downward force applied on each object. On the weight this is:

$$F_2 = -12(y(t) - x(t) - 20) = 240 - 12y(t) + 12x(t)$$

Since $F_2 = m_2 a$, we get $y''(t) = 80 - 4y(t) + 4x(t)$

The top spring at equilibrium is stretched to $x_e = 40$ feet, so the net downward force on the hook will be:

$$F_1 = -5\,(x(t) - 40) - F_2 = -5x(t) + 200 - (240 - 12y(t) + 12x(t)) = -40 - 17x(t) + 12y(t)$$

Since $F_1 = m_1 a$, we get $x''(t) = -40 - 17x(t) + 12y(t)$

Together these give us a system of differential equations:

$$\begin{cases} x'' = -40 - 17x + 12y \\ y'' = 80 - 4y + 4x \end{cases} \quad \text{with } x(0) = 40,\ x'(0) = -19,\ y(0) = 60,\ y'(0) = 19$$

We use Laplace Transformations to solve this system:

$$\begin{cases} s^2X(s) - 40s + 19 = -\dfrac{40}{s} - 17X(s) + 12Y(s) \\ s^2Y(s) - 60s - 19 = \dfrac{80}{s} - 4Y(s) + 4X(s) \end{cases} \implies \begin{cases} (s^2+17)X(s) - 12Y(s) = 40s - 19 - \dfrac{40}{s} \\ -4X(s) + (s^2+4)Y(s) = 60s + 19 + \dfrac{80}{s} \end{cases}$$

Adding (s^2+4) times the top equation to 12 times the bottom gives:

$$\begin{aligned} \left[(s^2+17)(s^2+4) - 48\right] X(s) &= (s^2+4)\left(40s - 19 - \frac{40}{s}\right) + 12\left(60s + 19 + \frac{80}{s}\right) \\ (s^4 + 21s^2 + 20)X(s) &= \frac{40s^4 - 19s^3 + 840s^2 + 152s + 800}{s} \\ X(s) &= \frac{40s^4 - 19s^3 + 840s^2 + 152s + 800}{s(s^2+1)(s^2+20)} \end{aligned}$$

Using partial fraction decomposition, we can write: $X(s) = \dfrac{40}{s} + \dfrac{9}{s^2+1} - \dfrac{28}{s^2+20}$

Now we take the inverse Laplace Transformation:

$$x(t) = 40 \cdot \mathscr{L}^{-1}\left(\frac{1}{s}\right) + 9 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) - 28 \cdot \mathscr{L}^{-1}\left(\frac{1}{s^2+20}\right) = 40 \cdot 1 + 9 \cdot \sin(t) - 28 \cdot \frac{1}{2\sqrt{5}} \sin\left(t\sqrt{20}\right)$$

Therefore, the equation of motion for the brick is given by $x(t) = 40 + 9\sin(t) - \dfrac{14}{\sqrt{5}} \sin\left(t\sqrt{20}\right)$

Exercise 5.4.3*c*

We denote the downward displacement of the bob from the ship as $x(t)$ and the displacement of the sensor from the ship as $y(t)$, so that the initial conditions are $x(0) = 33,\ x'(0) = 0,\ y(0) = y_e + 5,\ y'(0) = 0$

The mass of the bob and sensor are each $m = \dfrac{32}{32} = 1$, and spring constants are $k_1 = 2, k_2 = 3$ ft/lb.

We first find the equilibrium positions for each object:

The top spring is stretched by $\dfrac{64}{k_1} = \dfrac{64}{3}$ feet by the bob and sensor combined, so the equilibrium position is at $x_e = \dfrac{35}{3} + \dfrac{64}{3} = 33$ feet.

The bottom spring is stretched by $\dfrac{32}{k_2} = 16$ feet by the sensor, so its equilibrium position is at $y_e = 33 + 100 + 16 = 149$ feet.

If we temporarily ignore resistance, we can find the net downward force applied on each object. On the sensor this is:

$$F_2 = -2(y(t) - x(t) - (149 - 33)) = 232 - 2y(t) + 2x(t)$$

Since $F_2 = ma$ and $\beta = 2$, we get $y''(t) = 232 - 2y(t) + 2x(t) - 2y'(t)$

The top spring at equilibrium is stretched to $x_e = 33$ feet, so the net downward force on the bob will be:

$$F_1 = -3\,(x(t) - 33) - F_2 = -3x(t) + 99 - (232 - 2y(t) + 2x(t)) = -133 - 5x(t) + 2y(t)$$

Since $F_1 = ma$ and $\beta = 2$, we get $x''(t) = -133 - 5x(t) + 2y(t) - 2x'(t)$

Together these give us a system of differential equations:

$$\begin{cases} x'' = -133 - 5x + 2y - 2x' \\ y'' = 232 - 2y + 2x - 2y' \end{cases} \quad \text{with } x(0) = 33,\ x'(0) = 0,\ y(0) = 154,\ y'(0) = 0$$

We use Laplace Transformations to solve this system:

$$\begin{cases} s^2X(s) - 33s = -\dfrac{133}{s} - 5X(s) + 2Y(s) - 2\,(sX(s) - 33) \\ s^2Y(s) - 154s = \dfrac{232}{s} - 2Y(s) + 2X(s) - 2\,(sY(s) - 154) \end{cases} \implies \begin{cases} (s^2+2s+5)X(s) - 2Y(s) = 33s + 66 - \dfrac{133}{s} \\ -2X(s) + (s^2+2s+2)Y(s) = 154s + 308 + \dfrac{232}{s} \end{cases}$$

Adding (s^2+2s+5) times the bottom equation to 2 times the top equation gives:

$$\left[(s^2+2s+5)(s^2+2s+2)-4\right]Y(s) = (s^2+2s+5)\left(154s+308+\frac{232}{s}\right)+2\left(33s+66-\frac{133}{s}\right)$$

$$(s^4+4s^3+11s^2+14s+6)Y(s) = \frac{154s^4+616s^3+1684s^2+2136s+894}{s}$$

$$Y(s) = \frac{154s^4+616s^3+1684s^2+2136s+894}{s(s+1)^2(s^2+2s+6)}$$

Using partial fraction decomposition, we can write: $Y(s) = \frac{149}{s}+\frac{4}{s+1}+\frac{4}{(s+1)^2}-\frac{s+2}{s^2+2s+6}$

Now we take the inverse Laplace Transformation:

$$y(t) = 149\cdot\mathscr{L}^{-1}\left(\frac{1}{s}\right)+4\cdot\mathscr{L}^{-1}\left(\frac{1}{s+1}\right)+4\cdot\mathscr{L}^{-1}\left(\frac{1}{(s+1)^2}\right)-\mathscr{L}^{-1}\left(\frac{s+2}{(s+1)^2+5}\right) = 149\cdot 1+4\cdot e^{-t}-4te^{-t}+e^{-t}\left[\cos\left(t\sqrt{5}\right)+\frac{1}{\sqrt{5}}\sin\left(t\sqrt{5}\right)\right]$$

Therefore, the equation of motion for the sensor is given by $y(t) = 149+e^{-t}\left[4(1+t)+\cos\left(t\sqrt{5}\right)+\frac{1}{\sqrt{5}}\sin\left(t\sqrt{5}\right)\right]$

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5.5: Circuits

Learning Objectives

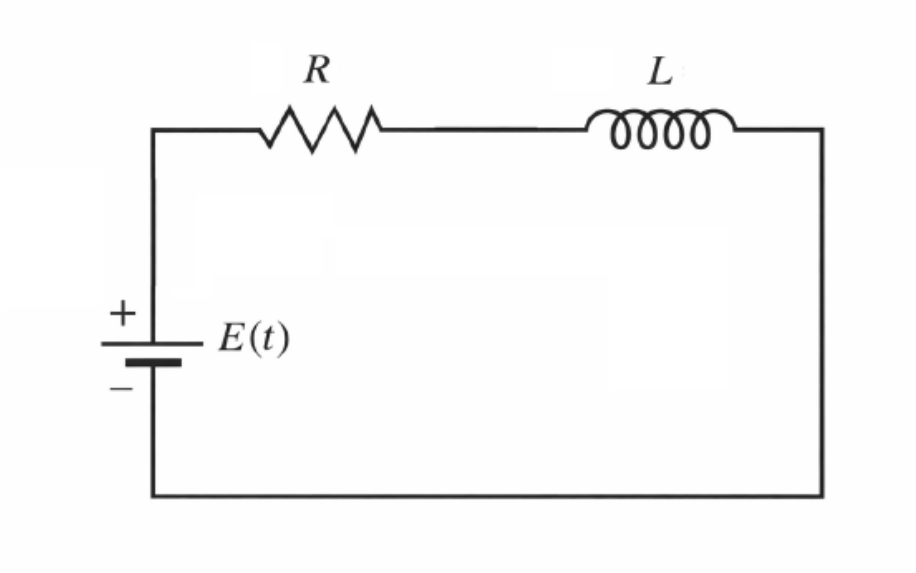
- Use Kirchhoff's laws to derive differential equations that model current and voltage in simple RL, RC, and RLC circuits.
- Solve first-order differential equations for RL and RC circuits and interpret solutions in terms of charging and discharging behavior.
- Solve second-order differential equations for RLC and interpret solutions in terms of charging and discharging behavior.
- Analyze the transient and steady-state behavior of circuit currents and voltages from the solutions of these differential equations.

Electrical circuits with resistors, capacitors, and inductors are naturally modeled by differential equations. An RC circuit leads to a first-order equation describing exponential charging or discharging, while an RLC circuit produces a second-order equation that mirrors the mathematics of spring-mass systems—complete with damping and oscillation. In this subchapter, we use Kirchhoff's laws to derive these equations and explore how circuit components shape the behavior of current and voltage over time.

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5.5: RLC Circuits and External Forces

We can model the flow of current $i(t)$ in a simple resistor-inductor (RL) circuit using a first order LDE.



We do so by using **Kirchhoff's Voltage Law:** *the sum of the voltage drops around any closed loop equals the applied voltage.*

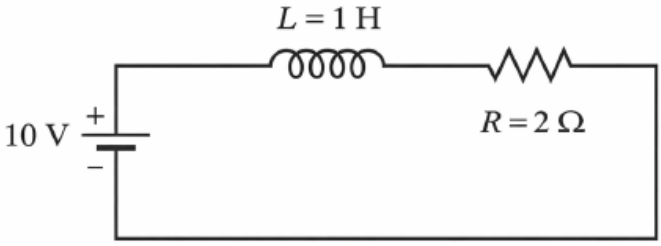
Rule: Kirchhoff's Voltage Law (RL circuit)

$$L \cdot \frac{di}{dt} + R \cdot i = E(t)$$

Here the term L (measured in *henrys*, which are *volt seconds per ampere*) is the inductance of the circuit, and R (measured in *ohms*, which are *volts per ampere*) represents the resistance of the circuit, while $E(t)$ (measured in *volts*) is the voltage source. These units yield a function $i(t)$ measured in *amperes*.

Example 5.5.1

A series circuit has a 1 henry inductor and a 2 ohm resistor connected to a 10 volt DC power supply. Initially, there is no current in the circuit. How long until the current reaches 1 ampere?



Solution

We plug in the constants $L = 1, R = 2$ and the constant voltage source $E(t) = 10$ to get our differential equation:

$$1 \cdot \frac{di}{dt} + 2 \cdot i = 10$$

This is already in standard linear form, so we use $\mu(t) = e^{\int 2\,dt} = e^{2t}$:

$$i(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\,dt = \frac{1}{e^{2t}} \int e^{2t} \cdot 10\,dt = e^{-2t}\left[5e^{2t} + C\right] = 5 + Ce^{-2t}$$

The initial condition of the current being zero gives $i(0) = 0$, so we can solve for C:

$$i(0) = 5 + Ce^{-2 \cdot 0} \implies 0 = 5 + C \cdot 1 \implies C = -5$$

Therefore the current is given by $i(t) = 5 - 5e^{-2t}$

We solve for the t which makes this current equal to 1:

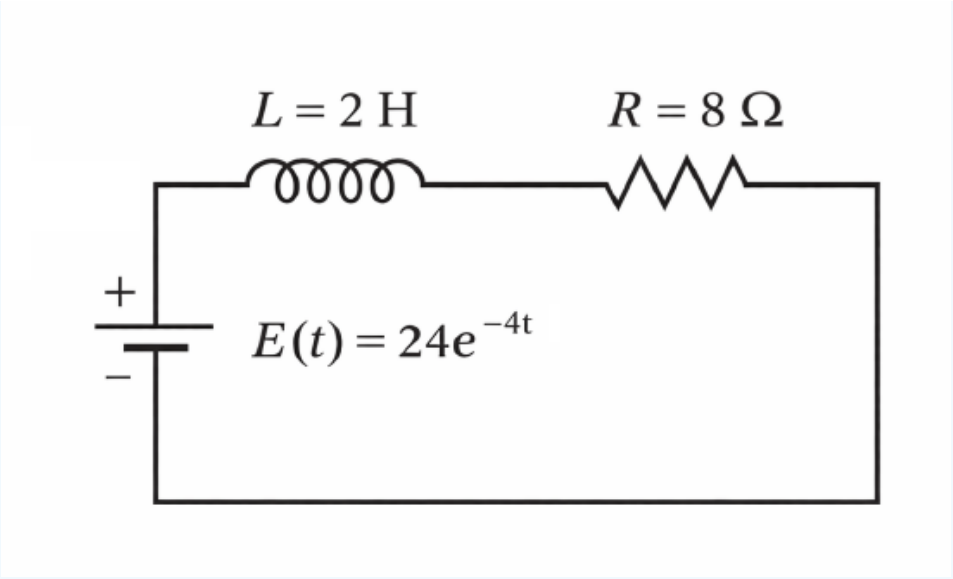
$$5 - 5e^{-2t} = 1 \implies t = -\frac{1}{2}\ln\left(\frac{4}{5}\right) \approx .111$$

Thus the current will be exactly 1 after $\boxed{.111 \text{ seconds}}$

The voltage source does not have to remain constant for us to solve these types of questions.

Example 5.5.2

A 2 H inductor and a $8\ \Omega$ resistor are connected in series to a voltage source $E(t) = 24e^{-4t}$ volts. If the initial current is 1 A, find the maximum current in the circuit and the time at which it occurs.



Solution

We plug in the constants $L = 2, R = 8$ and the voltage source $E(t) = 24e^{-4t}$ to get our differential equation:

$$2 \cdot \frac{di}{dt} + 8 \cdot i = 24e^{-4t}$$

We divide through by 2 to get this into standard linear form:

$$\frac{di}{dt} + 4i = 12e^{-4t}$$

We use $\mu(t) = e^{\int 4\,dt} = e^{4t}$:

$$i(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\,dt = \frac{1}{e^{4t}} \int e^{4t} \cdot 12e^{-4t}\,dt = e^{-4t} \int 12\,dt = e^{-4t}\,[12t + C]$$

The initial condition of the current being 1 gives $i(0) = 1$, so we can solve for C:

$$i(0) = (12 \cdot 0 + C)\,e^{-4 \cdot 0} \implies 1 = C \cdot 1 \implies C = 1$$

Therefore the current is given by $i(t) = (12t + 1)\,e^{-4t}$

We want the maximum of this function, so we take its derivative to find the critical points: $i'(t) = (-48t + 8)\,e^{-4t}$

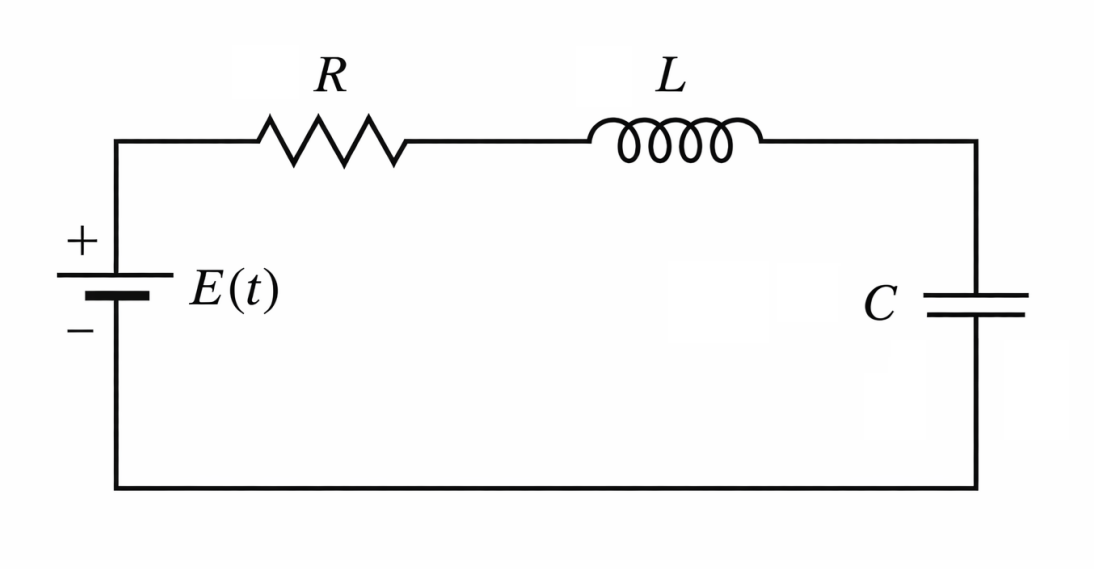
Setting this equal to zero gives $-48t + 8 = 0 \implies t = \frac{1}{6}$

Plugging in this critical point gives us $i\left(\frac{1}{6}\right) = 3e^{-\frac{2}{3}}$

Therefore the maximum current is $\boxed{3e^{-\frac{2}{3}} \text{ amperes}}$

When we introduce the capacitor to the circuit, this gives us the differential equation:

$$L \cdot i'(t) + R \cdot i(t) + \frac{1}{C} \cdot q(t) = E(t)$$



This is referred to as an RLC circuit. Once again R stands for the resistance, which you can think of as similar to the "friction" we experienced with damping springs. The constant L stands for the inductance, which is similar to the "mass" of an object in a spring problem. Lastly, C is the capacitance of the circuit, which gives the same information as the "spring constant" in a spring problem. In the equation above, the function $q(t)$ represents the charge on the capacitor, (meaning $q'(t) = i(t)$).

We can re-write this as a differential equation for q:

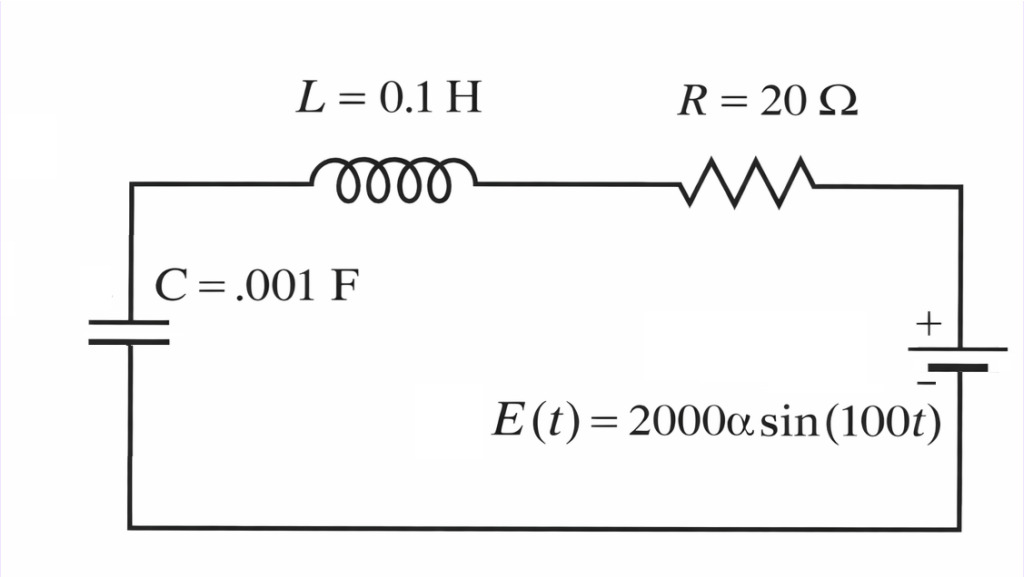
Rule: Kirchhoff's Voltage Law (RLC circuit)

$$L \cdot q''(t) + R \cdot q'(t) + \frac{1}{C} \cdot q(t) = E(t)$$

Therefore, solving these problems is a simple exercise in second order CC LDEs.

Illustration 5.5.1

Consider a circuit with $L = .1$ henrys, $R = 20\Omega$, $C = .001$ farads. If a voltage source $E(t) = 2000\alpha \sin(100t)$ is attached to the charge free circuit at time $t = 0$, find the functions representing charge and current in the circuit at time t seconds.



Plugging in the given values of $L,\ R,\ C$, and $E(t)$ gives the CC LDE:

$$.1q'' + 20q' + \frac{1}{.001}q = 2000\alpha\sin(100t)$$
$$\Downarrow$$
$$q'' + 200q' + 10000 = 20000\alpha\sin(100t)$$

This has characteristic equation $r^2 + 200r + 10000 = (r+100)^2 = 0 \implies r = -100$ (double).

The homogeneous solution must then be of the form $q_h(t) = (A + Bt)e^{-100t}$.

To find the particular solution, we guess $q_p = C\cos(100t) + D\sin(100t)$:

$$\begin{aligned} q_p'' + 200q_p' + 10000q_p &= \left(-100^2 C\cos(Bt) - 100^2 D\sin(100t)\right) + 200\left(-100C\sin(100t) + 100D\cos(100t)\right) + 10000\left(C\cos(100t) + D\sin(100t)\right) \\ &= 20000D\cos(100t) - 20000C\sin(Bt) \end{aligned}$$

Setting this equal to $g(t) = 20000\alpha\sin(100t)$ gives us system of equations:

$$\begin{cases} 20000D = 0 \\ -20000C = 20000\alpha \end{cases} \implies \begin{cases} C = -\alpha \\ D = 0 \end{cases}$$

Thus, our particular solution is $q_p(t) = -\alpha\cos(100t)$, and our general solution must be $q(t) = -\alpha\cos(100t) + (A + Bt)e^{-100t}$.

We are told the circuit is charge free at time $t = 0$, so we assume $q(0) = 0,\ q'(0) = i(0) = 0$:

$$\begin{aligned} q(t) &= -\alpha\cos(100t) + (A + Bt)e^{-100t} \\ q(0) &= -\alpha\cdot 1 + (A + B\cdot 0)\cdot 1 \implies A = \alpha \\ q'(t) &= 100\alpha\sin(100t) + (B - 100A - 100Bt)e^{-100t} \\ q'(0) &= 100\alpha\cdot 0 + (B - 100A - 100B\cdot 0)\cdot 1 \implies B - 100A = 0 \implies B = 100\alpha \end{aligned}$$

This gives us the charge function: $q(t) = \alpha\left[-\cos(100t) + (1 + 100t)\,e^{-100t}\right]$

Differentiating gives us the current function: $i(t) = 100\alpha\left[\sin(100t) - 100te^{-100t}\right]$

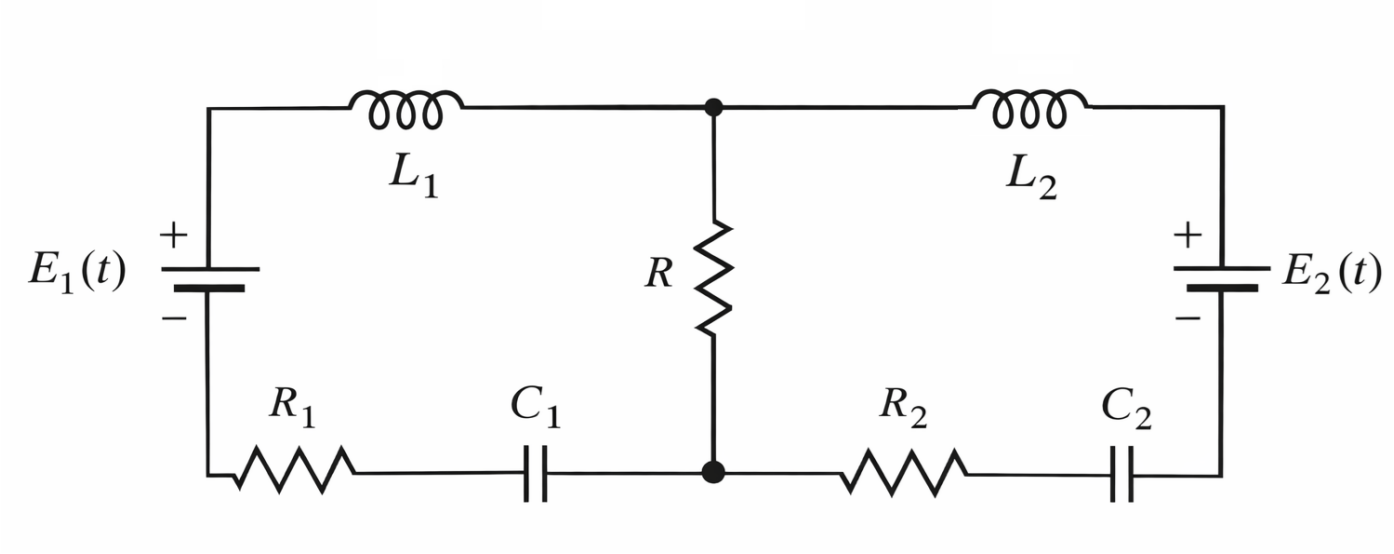
Because all of the LDEs created for RLC questions are second order, their solutions tend to look exactly like those we encountered with springs:

- Two negative real roots r_1, r_2 (*overdamped*), yielding $q(t) = Ae^{r_1 t} + Be^{r_2 t}$

- A double root at some negative r (*critically damped*), yielding $q(t) = (A + Bt)e^{rt}$

- Two complex roots $r = a + \pm bi$ (*underdamped*), yielding $q(t) = Ae^{at}\cos(bt) + Be^{at}\sin(bt)$

More complicated circuits can also be modeled in a similar way. Assume there are two independent loops of the circuit that share a common resistor of resistance R and individual resistors of resistance R_1 and R_2, with current $i_1(t)$ flowing through the first loop (with inductance L_1, capacitance C_1, and voltage source $E_1(t)$), and $i_2(t)$ flowing through the second loop (with inductance L_2, capacitance C_2, and voltage source $E_2(t)$).



Then, Kirchhoff's law gives:

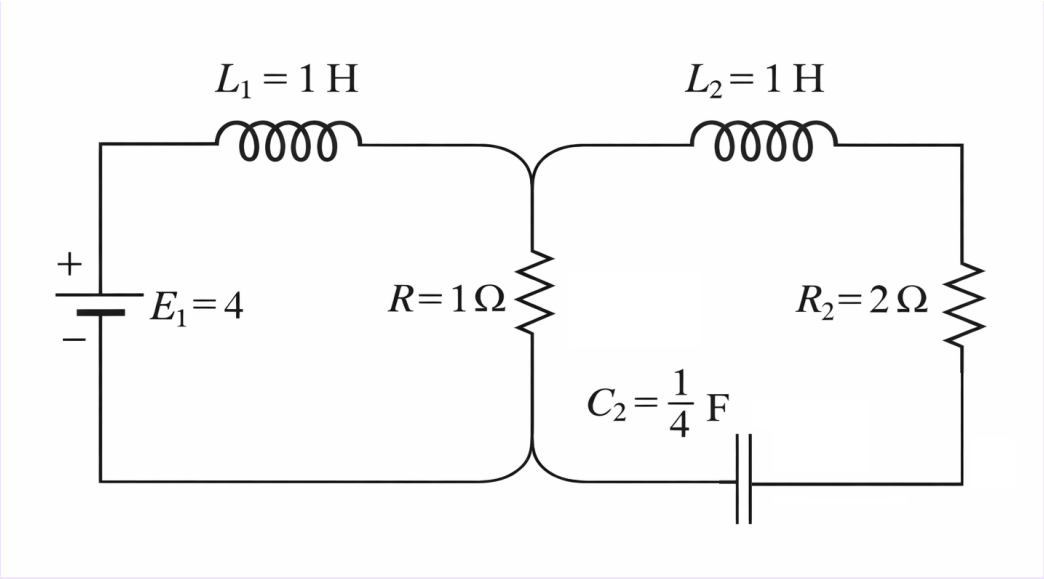
Rule: Kirchhoff's Voltage Law (RLC circuit with two loops and shared resistor)

$$\begin{cases} L_1 \cdot i_1'(t) + R \cdot (i_1(t) - i_2(t)) + R_1 \cdot i_1(t) + \dfrac{1}{C_1} \cdot q_1(t) = E_1(t) \\ L_2 \cdot i_2'(t) + R \cdot (i_2(t) - i_1(t)) + R_2 \cdot i_2(t) + \dfrac{1}{C_2} \cdot q_2(t) = E_2(t) \end{cases}$$

Intuitively, the shared resistor branch prevents whichever current is "winning" the current war, giving a little help to the underdog. We solve these by turning them into systems of LDEs for $q_1,\ q_2$.

Illustration 5.5.2

Consider a circuit with $L_1 = L_2 = 1$ henry, a resistor with $R = 1\ \Omega$ in the shared branch, another resistor with $R_2 = 2\ \Omega$ in the second loop, and a $C_2 = \dfrac{1}{4}$ farad capacitor in the second loop (no capacitor in the first loop). If a constant 4 volt source is attached to the first loop, find the current in each loop after t seconds.



We are given all the necessary ingredients to set up our system of CC LDEs:

$$\begin{cases} 1q_1'' + 1(q_1' - q_2') + 0q_1 = 4 \\ 1q_2'' + 1(q_2' - q_1') + 2q_2' + \frac{1}{\frac{1}{4}}q_2 = 0 \end{cases}$$

Writing this in differential operator notation gives $\begin{cases} (D^2 + D)q_1 - Dq_2 = 4 \\ -Dq_1 + (D^2 + 3D + 4)\,q_2 = 0 \end{cases}$

We eliminate q_1 by adding $D + 1$ "times" the bottom equation to the top equation:

$$(D+1)\left[-Dq_1 + \left(D^2 + 3D + 4\right)q_2 = 0\right] + \left[(D^2 + D)q_1 - Dq_2 = 4\right] \implies \left((D+1)\left(D^2 + 3D + 4\right) - D\right)q_1 = 4$$

This simplifies down to $q_2''' + 4q_2'' + 6q_2' + 4q_2 = 4$, which has characteristic equation:

$$r^3 + 4r^2 + 6r + 4 = (r+2)(r^2 + 2r + 2) = 0 \implies r = -2,\ -1 \pm i$$

We now see the homogeneous solution is of the form $q_{2,h}(t) = Ae^{-2t} + Be^{-t}\cos(t) + Ce^{-t}\sin(t)$.

We guess the particular solution is constant, so $q_p = \alpha$:

$$q_p''' + 4q_p'' + 6q_p' + 4q_p = 0 + 4 \cdot 0 + 6 \cdot 0 + 4 \cdot \alpha = 4\alpha$$

Setting this equal to $g(t) = 4$ gives $4\alpha = 4 \implies \alpha = 1$

Now we have our general solution for $q_2(t) = 1 + Ae^{-2t} + Be^{-t}\cos(t) + Ce^{-t}\sin(t)$

Substituting this into the second equation gives us another differential equation:

$$q_2'' + 3q_2' - q_1' + 4q_2 = 0$$
$$\Downarrow$$
$$\left(4Ae^{-2t} + 2Be^{-t}\sin(t) - 2Ce^{-t}\cos(t)\right) + 3\left(-2Ae^{-2t} + Be^{-t}\left(-\cos(t) - \sin(t)\right) + Ce^{-t}\left(-\sin(t) + \cos(t)\right)\right) - q_1' + 4\left(1 + Ae^{-2t} + Be$$
$$\Downarrow$$
$$4 + (4A - 6A + 4A)\,e^{-2t} + (-2C - 3B + 3C + 4B)\,e^{-t}\cos(t) + (2B - 3B - 3C + 4C)\,e^{-t}\sin(t) = q_1'$$
$$\Downarrow$$
$$4 + 2Ae^{-2t} + (B + C)\,e^{-t}\cos(t) + (C - B)\,e^{-t}\sin(t) = q_1'$$

We integrate both sides to get:

$$\begin{aligned} q_1(t) &= \int q_1'(t) \\ &= \int 4 + 2Ae^{-2t} - (B+C)e^{-t}\cos(t) + (C-B)e^{-t}\sin(t)\,dt \\ &= -Ae^{-2t} + 4t - Ce^{-t}\cos(t) + Be^{-t}\sin(t) + D \end{aligned}$$

Finally, we can input the initial conditions (assumed to be $q_1(0) = q_2(0) = q_1'(0) = q_2'(0) = 0$:

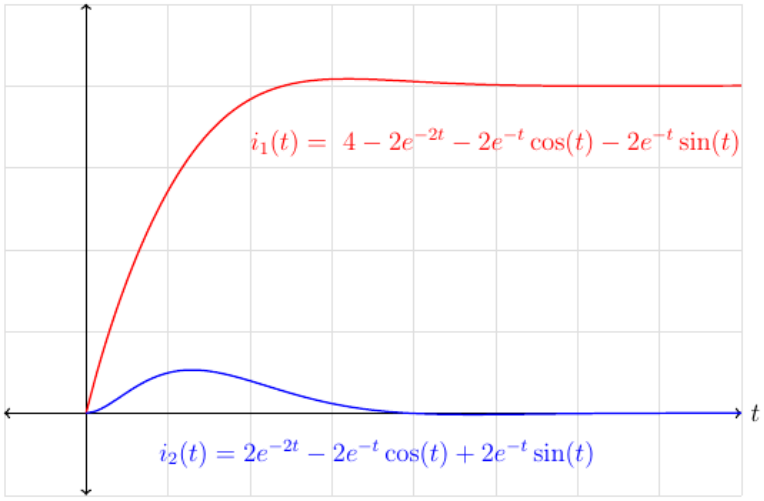
$$\begin{aligned} q_1 &= -Ae^{-2t} + 4t - Ce^{-t}\cos(t) + Be^{-t}\sin(t) + D \\ q_1(0) &= -A\cdot 1 + 0 - C\cdot 1 + B\cdot 0 + D \implies -A - C + D = 0 \\ q_1' &= 2Ae^{-2t} + 4 - (B-C)e^{-t}\cos(t) + (-C-B)e^{-t}\sin(t) \\ q_1'(0) &= 2A\cdot 1 - 4 + (B-C)\cdot 1 + (-C-B)\cdot 0 \implies 2A + B - C = 4 \end{aligned}$$

$$\begin{cases} A + B = -1 \\ -2A - B - C = 0 \\ -A - C + D = 0 \\ 2A + B - C = 4 \end{cases} \implies \begin{cases} A = -1 \\ B = 0 \\ C = -2 \\ D = -3 \end{cases}$$

Finally, we have our unique solutions: $\begin{cases} q_1(t) = 4t - 3 + e^{-2t} + 2e^{-t}\cos(t) \\ q_2(t) = 1 - e^{-2t} - 2e^{-t}\sin(t) \end{cases}$

The functions we wanted were the currents, $i_1(t) = q_1'(t)$ and $i_2(t) = q_2'(t)$, which we find by differentiation.

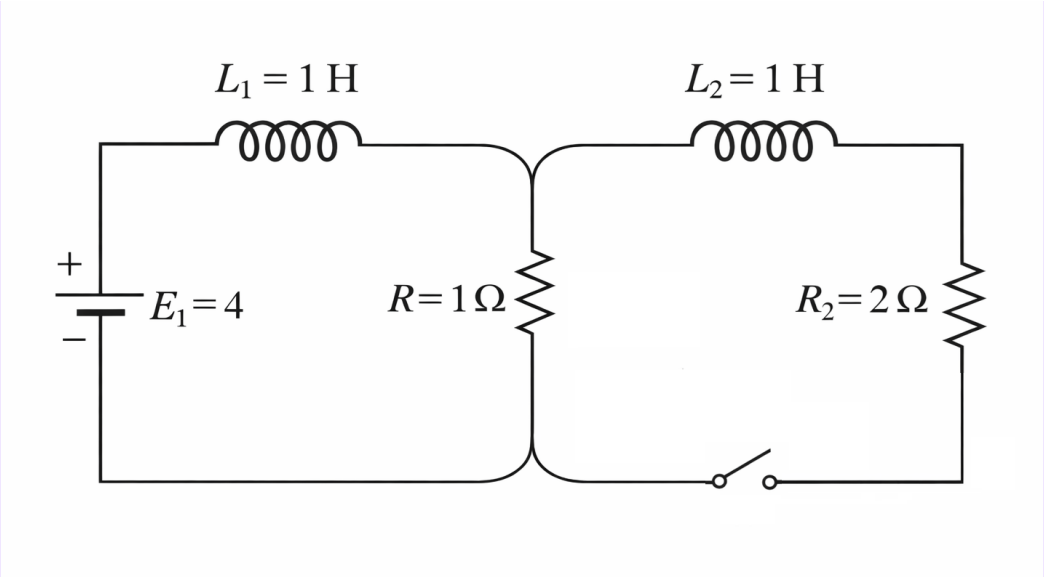
Thus, our final answers are the functions $\begin{cases} i_1(t) = 4 - 2e^{-2t} - 2e^{-t}\cos(t) - 2e^{-t}\sin(t) \\ i_2(t) = 2e^{-2t} - 2e^{-t}\cos(t) + 2e^{-t}\sin(t) \end{cases}$



Students with a background in the field may recognize the results of this graph as a steady state. This is generally calculated as $I = \dfrac{V}{R} = \dfrac{\text{voltage}}{\text{resistance}}$

The current in the first loop begins at zero and because the total resistance is 1 in this loop, approaches the steady state value of 4 amperes.

The current in the second loop begins at zero and has a capacitor, for which common practice is to assume "ultimately acts like a break in the wire".



Thus, the current approaches a steady state of zero.

The reason the solutions to the RLC equations look and act similar to the solutions to the springs is because they are both examples of a more general dynamical system. When a system is subjected to some external force (like a battery or oscillating voltage supply in an RLC circuit, or wind blowing on an object attached to a spring), we model it using some linear CC LDE:

$$\alpha x'' + \beta x' + \gamma x = g(t)$$

In the case of springs, α relates to the mass, β relates to the friction, and γ relates to the spring constant. $g(t)$ is the force being applied to the object outside of what is caused by the spring.

In the case of RLC, α relates to the inductance, β relates to resistance, and γ relates to capacitance. $g(t)$ is the voltage source applied to the circuit.

Note

In either case, we follow the same first steps. First, we solve the characteristic equation:

$$\alpha r^2 + \beta r + \gamma = 0 \implies r = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

We have seen that for positive $\alpha,\ \beta,\ \gamma$, this can lead to three types of functions:

Case 1: There are two real roots. $r = -\frac{\beta}{2\alpha} \pm \sqrt{\left(\frac{\beta}{2\alpha}\right)^2 - \frac{\gamma}{\alpha}}$

When all of $\alpha,\ \beta,\ \gamma$ are positive, it is clear that both roots for r must be negative.

Therefore, the homogeneous solutions are of the form $x_h(t) = Ae^{-r_1 t} + Be^{-r_2 t}$ (*decaying quickly, not oscillating*)

Case 2: There is a double root. $r = -\frac{\beta}{2\alpha} < 0$

In this case, the homogeneous solutions take slightly longer to decay: $x_h(t) = (A + Bt)e^{-\frac{\beta}{2\alpha}t}$ (*critical phase between oscillating and decaying*)

Case 3: There are complex roots. $r = -\frac{\beta}{2\alpha} \pm i\sqrt{\frac{\gamma}{\alpha} - \left(\frac{\beta}{2\alpha}\right)^2} = \frac{\beta}{2\alpha} \pm i\omega$

Here we see homogeneous functions of the form $x_h(t) = (A\cos(\omega t) + B\sin(\omega t))\, e^{-\frac{\beta}{2\alpha}t}$ (*oscillating quickly, decaying slowly*)

This is what leads to so many real life scenarios exhibiting behavior of these types in closed systems. It is also why we often use spring problems as interchangeable with many other problems we want to model.

If the reader has ever jumped on a trampoline for an extended period of time, they may be able to imagine that the graph of their height during that time would look somewhat sinusoidal, with regular peaks and troughs. What happens though, if someone else applies extra force? Maybe someone pushes down or up on the jumper just before they hit the trampoline, or just after? The results vary dramatically depending on that external force, $g(t)$.

Illustration 5.5.3

Let $g(t)$ be an exponential function, $g(t) = Ke^{at}$. Examine the potential behavior of solutions to

$$\alpha x''(t) + \beta x'(t) + \gamma x(t) = g(t) \quad \text{for } \alpha, \beta, \gamma > 0$$

(a) when $a > -\frac{\beta}{2\alpha}$

(b) when $a = -\frac{\beta}{2\alpha}$

(c) when $a < -\frac{\beta}{2\alpha}$

(a) In the case that $a > -\frac{\beta}{2\alpha}$, depending on the value of a and which case of homogeneous roots we have, we may pick a different particular solution.

In Case 1 with two real roots, it depends if a matches one of the homogeneous roots.
When it doesn't, we simply find particular solution $x_p(t) = Ce^{at}$
This tells us the general solution will look like

$$x(t) = Ce^{at} + Ae^{-r_1 t} + Be^{-r_2 t}$$

This describes a function decaying quickly to act like $x_p(t)$
Alternatively in Case 1, if $a = -r_1$, then we would find a particular solution $x_p(t) = Cte^{at}$
This gives the general solution of the form:

$$x(t) = (A + Ct)e^{at} + Be^{-r_2 t}$$

This still decays to essentially act like $x_p(t)$ again.

In Case 2 with the double root \(r=-\dfrac{\beta}{2\alpha} Thus, the solution will look

$$x(t) = Ce^{at} + (A + Bt)e^{-\frac{\beta}{2\alpha}t}$$

This again decays relatively quickly to act like the dominant function $x_p(t) = Ce^{at}$

In Case 3 with complex roots $r = -\frac{\beta}{2\alpha} \pm i\omega$, again would find the particular solution of the form $x_p(t) = Ce^{at}$
This gives general solution

$$x(t) = Ce^{at} + (A\cos(\omega t) + B\sin(\omega t))\, e^{-\frac{\beta}{2\alpha}t}$$

This decays (and oscillates) to act like the dominant function $x_p(t) = Ce^{at}$

In all three cases, the exponential forcing function $g(t) = Ke^{at}$ directly correlates to the long-term behavior of the solution function (with various methods of decaying or oscillating into this long-term path).

(b) In the case that $a = -\dfrac{\beta}{2\alpha}$, it would vary what choice we use for particular solutions by case.

In Case 1 with two real roots, we know $r = -\dfrac{\beta}{2\alpha} \pm \sqrt{\left(\dfrac{\beta}{2\alpha}\right)^2 - \dfrac{k}{m}}$ so a is not a homogeneous root. This means $x_p(t) = Ce^{at}$
The general solution would look like

$$x(t) = Ce^{at} + Ae^{-r_1 t} + Be^{-r_2 t}$$

All of these powers are negative, so it decays to zero.

In Case 2 with the double root $r = -\dfrac{\beta}{2\alpha} = a$, we would find a particular solution of the form $x_p(t) = Ct^2 e^{at}$
The general solution would look like

$$x(t) = (A + Bt + Ct^2)e^{-\frac{\beta}{2\alpha}t}$$

This ultimately decays to zero, but grows for a little bit first.

In Case 3 with complex roots $r = a \pm i\omega$, we would find particular solution $x_p(t) = Ce^{at}$
The general solution would look like

$$x(t) = Ce^{at} + Ae^{at}\cos(\omega t) + Be^{at}\sin(\omega t) = \left(C + A\cos(\omega t) + B\sin(\omega t)\right)e^{-\frac{\beta}{2\alpha}t}$$

This decays to zero, but with noticeable oscillations (underdamped).

In all three cases, the exponentially decaying forcing function leads all solutions to decay to zero (with various amounts of decaying and oscillation).

(c) In the case that $a < -\dfrac{\beta}{2\alpha}$, depending on the value of a and which case of homogeneous roots we have, we may pick a different particular solution.

In Case 1 with two real roots, it depends if a matches one of the homogeneous roots.
When it doesn't, we simply find particular solution $x_p(t) = Ce^{at}$
This tells us the general solution will look like

$$x(t) = Ce^{at} + Ae^{-r_1 t} + Be^{-r_2 t}$$

This describes a function decaying quickly to zero (since all exponents are negative).
Alternatively in Case 1, if $a = -r_2$, then we would find particular solution $x_p(t) = Cte^{at}$
This gives the general solution of the form

$$x(t) = (B + Ct)e^{at} + Ae^{-r_1 t}$$

This still decays to zero because of the negative exponents.

In Case 2 with the double root \(r=-\dfrac{\beta}{2\alpha} Thus, the solution will look

$$x(t) = Ce^{at} + (A + Bt)e^{-\frac{\beta}{2\alpha}t}$$

This again decays to zero.

In Case 3 with complex roots $r = -\frac{\beta}{2\alpha} \pm i\omega$, again would find the particular solution of the form $x_p(t) = Ce^{at}$
This gives general solution

$$x(t) = Ce^{at} + (A\cos(\omega t) + B\sin(\omega t))\, e^{-\frac{\beta}{2\alpha}t}$$

This decays quickly to act like the underdamped oscillation to zero.

In all three cases, the exponentially decaying forcing function $g(t) = Ke^{at}$ leads to fairly little effect, and all results decay as usual to zero.

Illustration 5.5.4

Let $g(t)$ be a sinusoidal function, $g(t) = K_1\sin(at) + K_2\cos(at)$. Examine the potential behavior of solutions to

$$\alpha x''(t) + \beta x'(t) + \gamma x(t) = g(t) \quad \text{for } \alpha, \beta, \gamma > 0$$

(a) when $a \neq \sqrt{\frac{\gamma}{\alpha} - \left(\frac{\beta}{2\alpha}\right)^2}$

(b) when $a = \sqrt{\frac{\gamma}{\alpha} - \left(\frac{\beta}{2\alpha}\right)^2}$

(c) when $\beta = 0$ and $a = \sqrt{\frac{\gamma}{\alpha}}$

(a) In the case of $a \neq \sqrt{\frac{\gamma}{\alpha} - \left(\frac{\beta}{2\alpha}\right)^2}$, we will always find the same particular solution: $x_p(t) = C\cos(at) + D\sin(at)$

In Case 1 with two real roots, this gives general solution

$$x(t) = C\cos(at) + D\sin(at) + Ae^{-r_1 t} + Be^{-r_2 t}$$

This function decays quickly into its periodic particular solution based on the forcing function.

In Case 2 with a double root, this gives general solution

$$x(t) = C\cos(at) + D\sin(at) + (A + Bt)e^{-\frac{\beta}{2\alpha}t}$$

This again decays exponentially down into its periodic particular solution.

In Case 3 with complex roots, this gives general solution

$$x(t) = C\cos(at) + D\sin(at) + (A\cos(\omega t) + B\sin(\omega t))\, e^{-\frac{\beta}{2\alpha}t}$$

The exponential decay eventually means this will act like the periodic particular solution in the long term (but it may oscillate a bit on its way there)

In all three cases, the long term behavior imparted by a periodic forcing function will unsurprisingly lead the model to decay toward some periodic equilibrium position.

(b) In the case of $a = \sqrt{\frac{\gamma}{\alpha} - \left(\frac{\beta}{2\alpha}\right)^2}$, we will again always find the same particular solution: $x_p(t) = C\cos(at) + D\sin(at)$

In Case 1 with two real roots, this gives general solution

$$x(t) = C\cos(at) + D\sin(at) + Ae^{-r_1 t} + Be^{-r_2 t}$$

This function decays quickly into its periodic particular solution based on the forcing function.

In Case 2 with a double root, this gives general solution

$$x(t) = C\cos(at) + D\sin(at) + (A + Bt)e^{-\frac{\beta}{2\alpha}t}$$

This again decays exponentially down into its periodic particular solution.

In Case 3 with complex roots, we see $a = \omega$, this gives general solution

$$x(t) = C\cos(at) + D\sin(at) + (A\cos(at) + B\sin(at))\, e^{-\frac{\beta}{2\alpha}t}$$

The exponential decay eventually means this will act like the periodic particular solution in the long term (but may potentially oscillate a lot on its way there)

In all three cases, the long term behavior imparted by a periodic forcing function will unsurprisingly lead the model to decay toward some periodic equilibrium position.

(c) In the case where $\beta = 0$ and $a = \sqrt{\frac{\gamma}{\alpha}}$, this means there is no Case 1 or Case 2.

The only possibility is Case 3, which is $\alpha r^2 + \gamma = 0 \implies r = \pm i\sqrt{\frac{\gamma}{\alpha}} = \pm ia$

Therefore, our homogeneous solutions are $x_h(t) = A\cos(at) + B\sin(at)$, and our particular solutions will be $x_p(t) + Ct\cos(at) + Dt\sin(at)$

This gives us general solution of the form

$$x(t) = (A + Ct)\cos(at) + (B + Dt)\sin(at)$$

This solution *oscillates and grows larger* than even the exterior forcing function $g(t) = K_1\sin(at) + K_2\cos(at)$

This is known as a resonance frequency.

Consider when you are on that trampoline and someone gives the "double jump", where they press the trampoline down in a way to make your next jump go much higher. When a washing machine is filled just right and the tumbler rotates at just the right speed for the suspension springs to oscillate faster than they can decay, the machine jumps and moves around! This is the result of an exterior force interacting just right (or just wrong) with the spring system or circuit it is modeling or equivalent to.

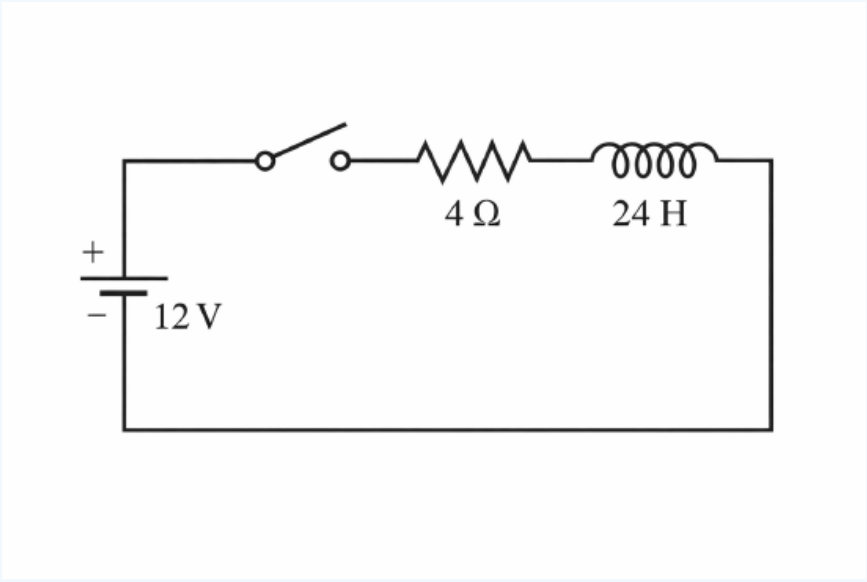
The striking mathematical similarity between mass-spring systems and RLC circuits is a cornerstone of *analogue modeling*. Because the underlying differential equations are identical, engineers can simulate complex mechanical vibrations—like the stress on an airplane wing—using small, inexpensive electrical circuits. By swapping a stiffer spring for a capacitor with lower capacitance or a heavier mass for a larger inductor, we can predict the behavior of massive physical structures through the behavior of electrons. This "universality" of second-order linear equations ensures that once you master the math of one physical domain, you have effectively mastered them all.

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5.5: Practice Set

Exercise $5.5.1a$

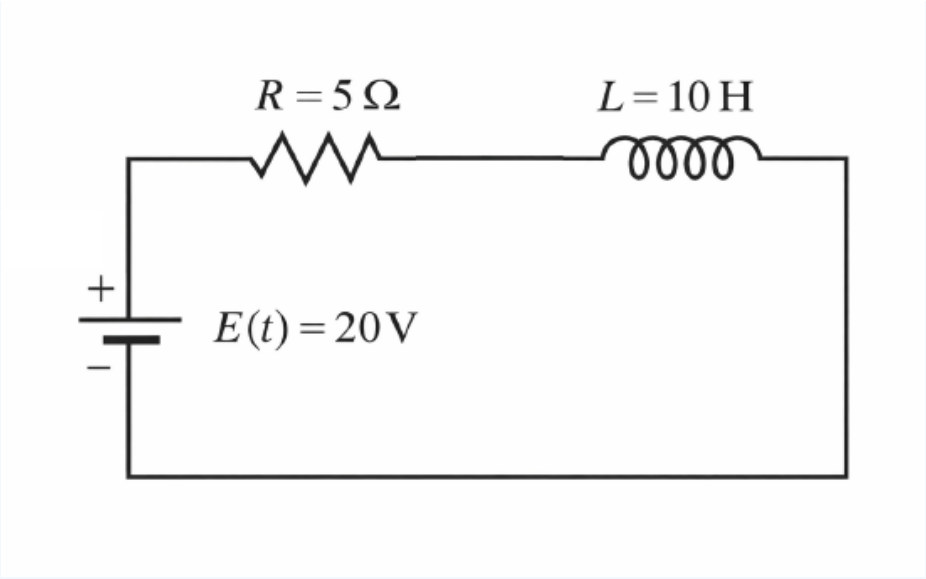
A circuit consists of a 12-V battery, a switch, a $4\ \Omega$ resistor, and a 24-H inductor in series. Initially, the switch is open and no current flows. At time $t = 0$, the switch is closed. What is the current after 6 seconds?



Answer available at the end of this section.

Exercise $5.5.1b$

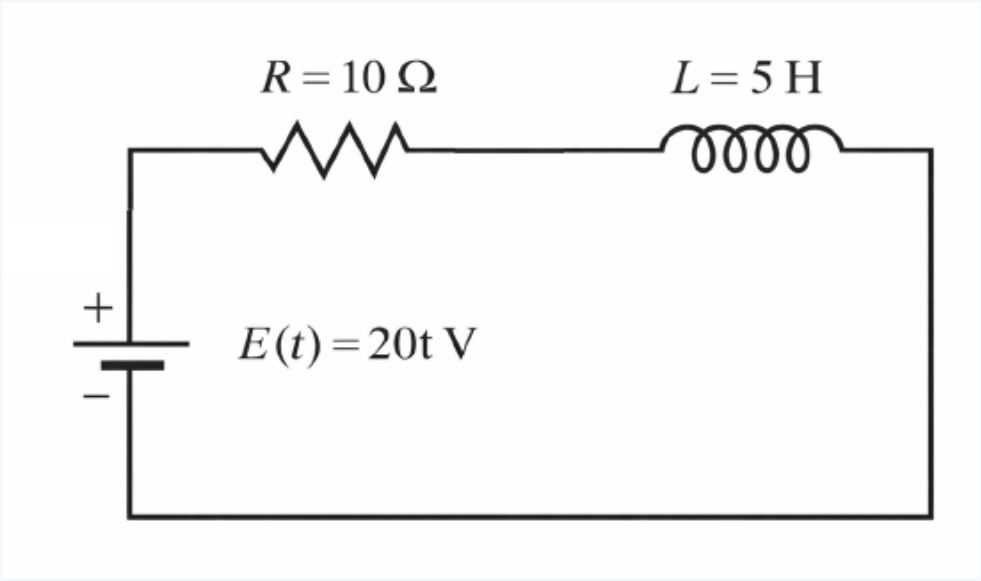
An RL circuit has $R = 5\ \Omega$ and $L = 10$ H. Initially, there is no current. At $t = 0$, a voltage source $E(t) = 20$ V is applied. What is the maximum possible current and when is it essentially reached (within 1% of max)?



Answer available at the end of this section.

Exercise 5.5.1c

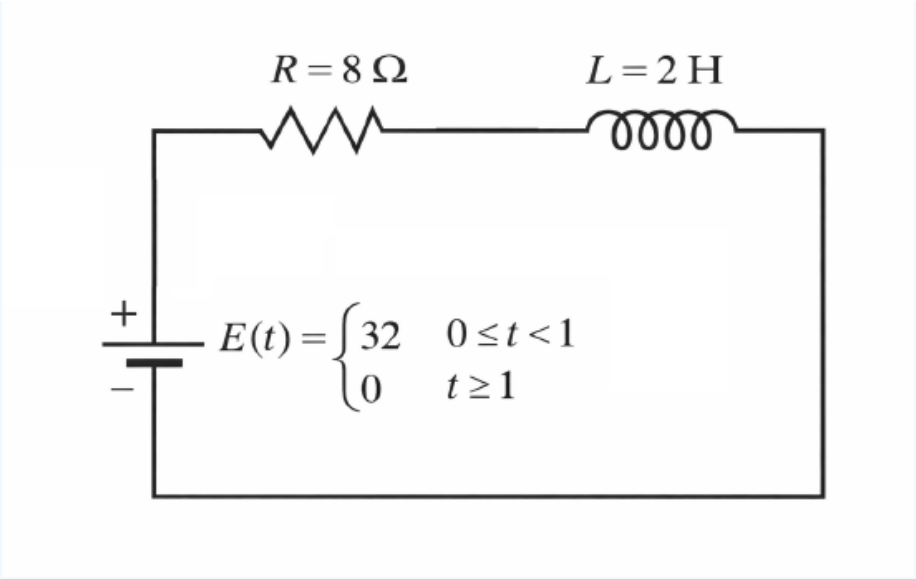
A circuit has $R = 10\ \Omega$, $L = 5$ H, and a voltage source $E(t) = 20t$ volts. Initial current is zero. Find the current after two seconds.



Answer available at the end of this section.

Exercise 5.5.1d

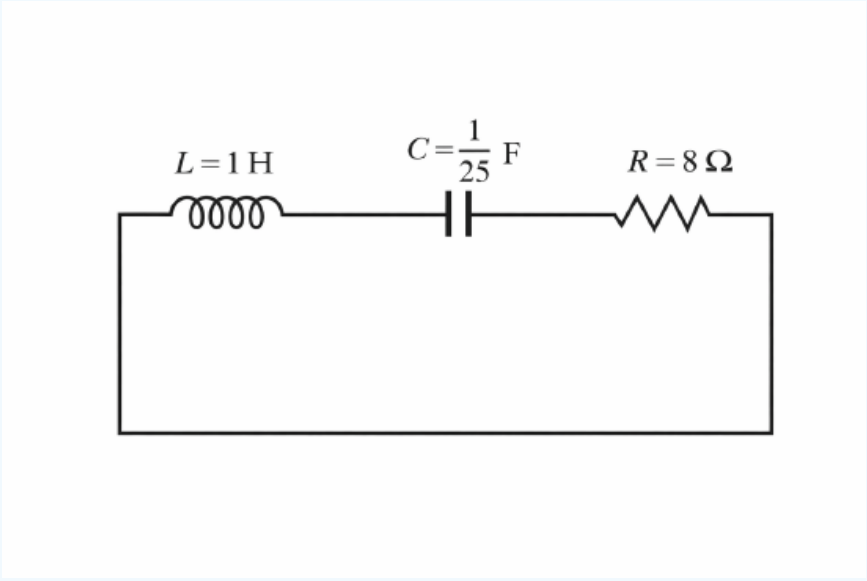
An RL circuit has $R = 8\ \Omega$ and $L = 2$ H. The voltage source is $E(t) = \begin{cases} 32 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$. If the initial current is zero, find $i(t)$.



Answer available at the end of this section.

Exercise $5.5.2a$

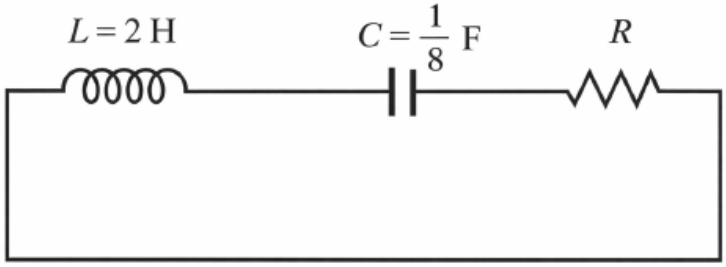
An RLC circuit has $L = 1$ H, $C = \frac{1}{25}$ F, and $R = 8\ \Omega$. The initial charge on the capacitor is $q(0) = 3$ and initial current is $i(0) = 0$. When does the charge first reach zero?



Answer available at the end of this section.

Exercise $5.5.2b$

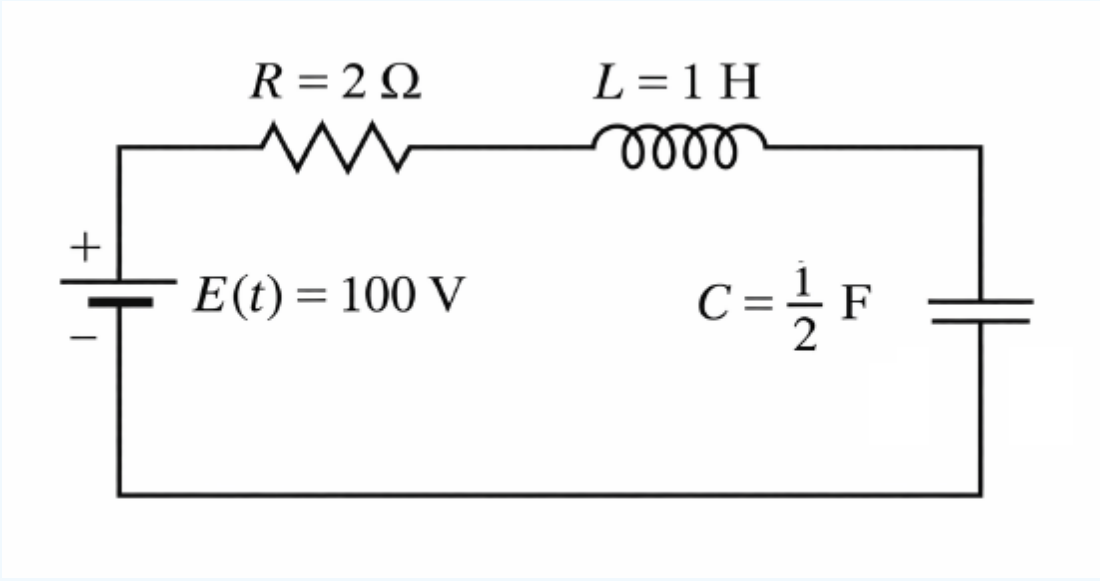
A series RLC circuit has $L = 2$ H, $C = \frac{1}{8}$ F, and R is chosen for critical damping. Initially $q(0) = 8$, and $i(0) = -20$. Find the value of R and solve for the maximum magnitude of the current.



Answer available at the end of this section.

Exercise $5.5.2c$

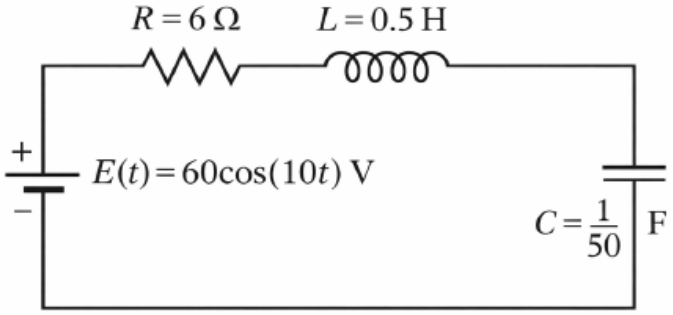
A circuit has $R = 2\ \Omega$, $L = 1$ H, $C = \frac{1}{2}$ F, and $E(t) = 100$ V. If there is initially no charge or current, what is the maximum possible current in the circuit.



Answer available at the end of this section.

Exercise $5.5.2d$

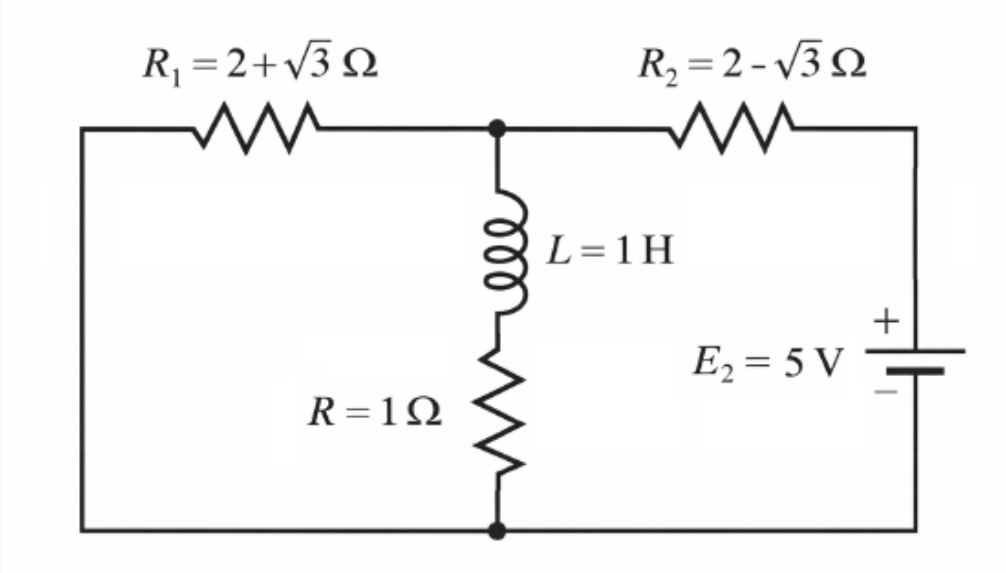
An RLC circuit with $L = 0.5$ H, $C = \frac{1}{50}$ F, $R = 6\ \Omega$ is driven by $E(t) = 60\cos(10t)$ V. If the initial conditions are $q(0) = 1$ and $i(0) = 4$, find $q(t)$.



Answer available at the end of this section.

Exercise 5.5.3a

A circuit has two loops with resistor $R_1 = 2 + \sqrt{3}\ \Omega$ in the first loop, and battery $E_2 = 5$ V and resistor $R_2 = 2 - \sqrt{3}\ \Omega$ in the second loop. On the middle (shared) branch there is a resistor $R = 1\ \Omega$ and inductor $L = 1$ H. If there is initially no current in either loop, find $i_1(t)$ and $i_2(t)$.

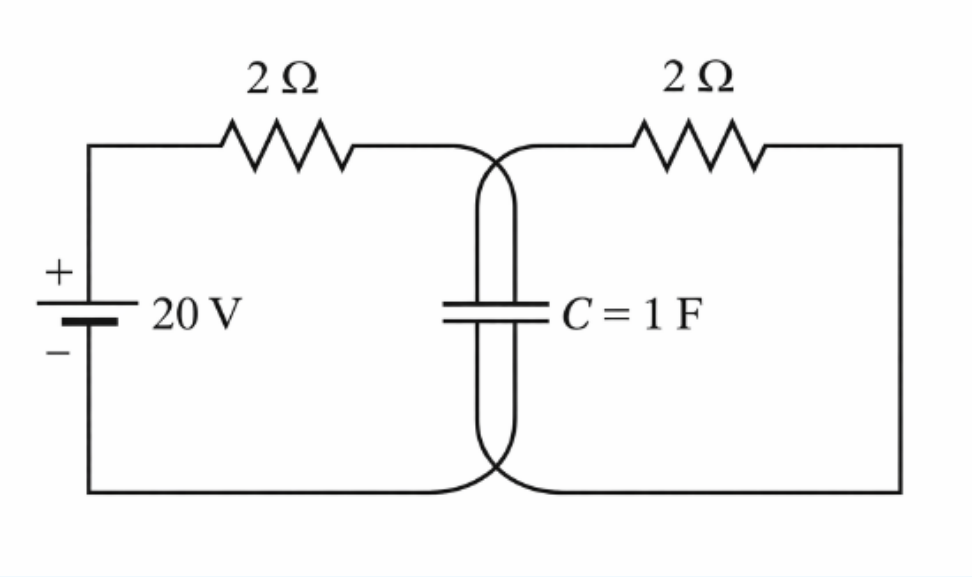


Hint: When an inductor is shared, use the system of LDES: $\begin{cases} L \cdot i_1'(t) + R \cdot (i_1(t) - i_2(t)) + R_1 \cdot i_1(t) = E_1(t) \\ L \cdot i_2'(t) + R \cdot (i_2(t) - i_1(t)) + R_2 \cdot i_2(t) = E_2(t) \end{cases}$

Answer available at the end of this section.

Exercise $5.5.3b$

Two loops share a capacitor $C = 1\,\text{F}$ in the common branch. Loop 1 consists of a 20 V battery and a $2\ \Omega$ resistor; and loop 2 consists of a $2\ \Omega$ resistor. If the capacitor begins with no charge, when does the capacitor reach 90% of its final charge?

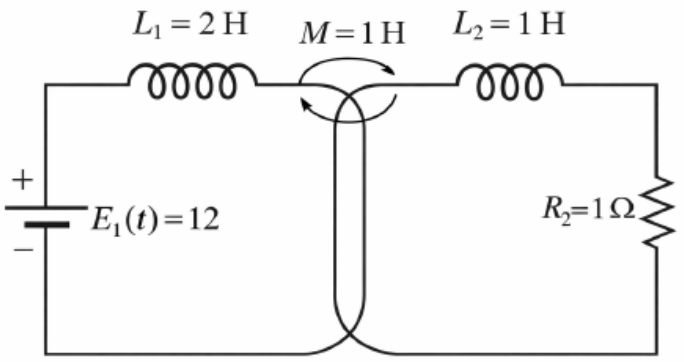


Hint: When a capacitor is shared, use the system of LDEs: $\begin{cases} L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) + \dfrac{1}{C} q(t) = E_1(t) \\ L_2 \cdot i_2'(t) + R_2 \cdot i_2(t) - \dfrac{1}{C} \cdot q(t) = E_2(t) \\ q'(t) = i_1(t) - i_2(t) \end{cases}$

Answer available at the end of this section.

Exercise $5.5.3c$

Two loops share mutual inductance $M = 1$ H in the same direction, with self-inductances $L_1 = 2$ H, and $L_2 = 1\,\text{H}$, a resistor in the second loop $R_2 = 1\ \Omega$. All currents begin at zero, then a voltage source with $E_1(t) = 12$ is attached to the first loop. What is the maximum current in the second loop?

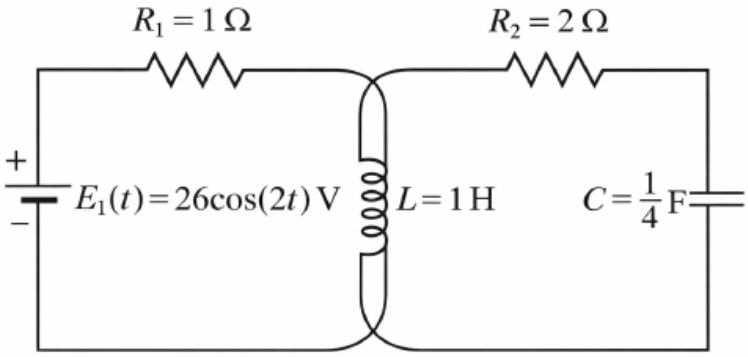


Hint: When a mutual inductance is shared in the same direction, use the system of LDEs: $\begin{cases} L_1 \cdot i_1'(t) + M \cdot i_2'(t) + R_1 \cdot i_1(t) = E_1(t) \\ L_2 \cdot i_2'(t) + M \cdot i_1'(t) + R_2 \cdot i_2(t) = E_2(t) \end{cases}$

Answer available at the end of this section.

Exercise $5.5.3d$

A circuit is made of two loops that share an inductor $L = 1$ H. Loop 1 has $E_1(t) = 26\cos(2t)$ V and $R_1 = 1\ \Omega$, while loop 2 has $R_2 = 2\ \Omega$ and $C = \frac{1}{4}$ F. If the initial current is zero, and the capacitor is uncharged, find the respective currents $i_1(t)$ and $i_2(t)$.

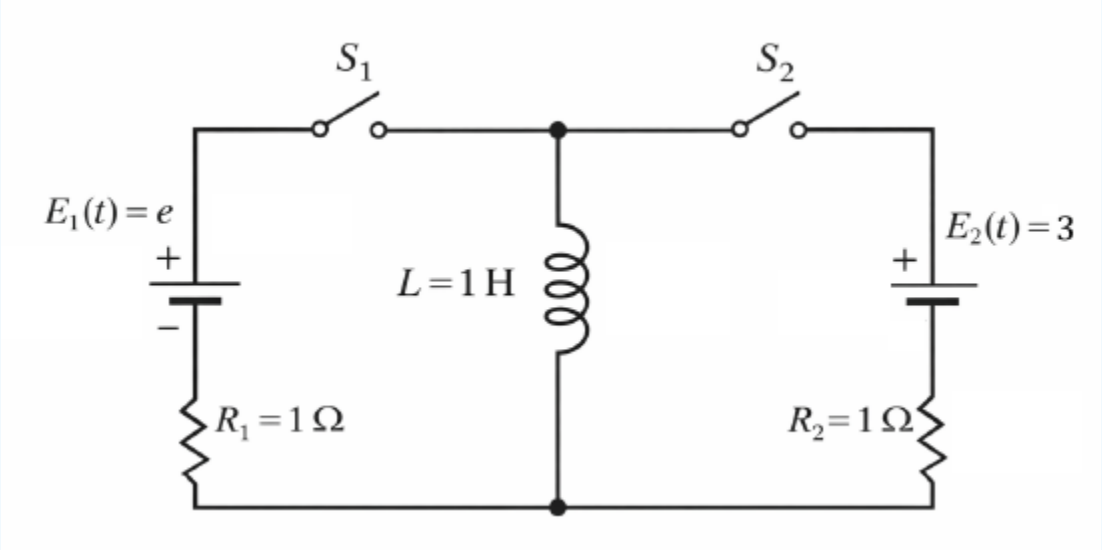


Hint: When an inductor is shared and only one loop has a capacitor, use the system of LDEs: $\begin{cases} L \cdot (i_1'(t) - q_2''(t)) + L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) = E_1(t) \\ L \cdot (q_2''(t) - i_1'(t)) + L_2 \cdot q_2''(t) + R_2 \cdot q_2'(t) + \frac{1}{C_2} q_2(t) = E_2(t) \end{cases}$

Answer available at the end of this section.

Exercise $5.5.4a$

A circuit is made of two loops that share an inductor $L = 1$ H. Loop 1 has a switch S_1, $E_1(t) = e$ V, and $R_1 = 1\ \Omega$, while loop 2 has a switch S_2, $E_2(t) = 3$ V and $R_2 = 1\ \Omega$. Both switches begin in the open position. At time $t = 0$, switch S_1 is closed. At time $t = 1$, the switch S_2 is closed. Find the current in loop 2 after an additional two seconds.

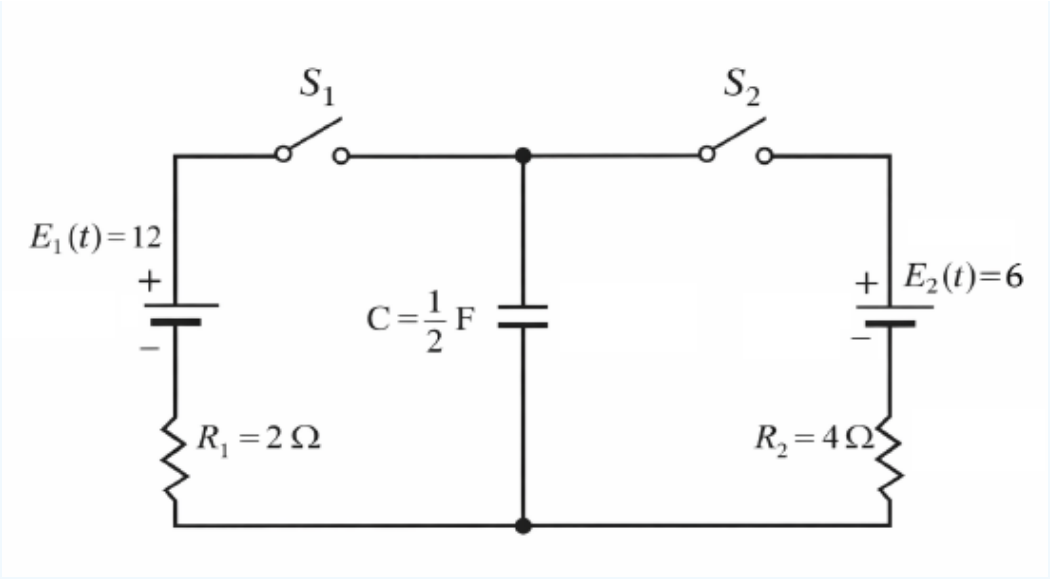


Hint: When an inductor is shared and there are no capacitors, use the system of LDEs:
$$\begin{cases} L \cdot (i_1'(t) - i_2'(t)) + L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) = E_1(t) \\ L \cdot (i_2'(t) - i_1'(t)) + L_2 \cdot i_2'(t) + R_2 \cdot i_2(t) = E_2(t) \end{cases}$$

Answer available at the end of this section.

Exercise $5.5.4b$

Consider a circuit with two loops sharing a common branch containing a capacitor $C = \frac{1}{2}$ F. Loop 1 has a switch S_1, $E_1(t) = 12$ V, and $R_1 = 2\ \Omega$, while loop 2 has a switch S_2, $E_2(t) = 6$ V and $R_2 = 4\ \Omega$. Both switches begin in the open position. At time $t = 0$, switch S_1 is closed. At time $t = 2$, the switch S_2 is closed. Find when the capacitor reaches its maximum charge.



Hint: When a capacitor is shared, use the system of LDEs: $\begin{cases} L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) + \dfrac{1}{C} q(t) = E_1(t) \\ L_2 \cdot i_2'(t) + R_2 \cdot i_2(t) - \dfrac{1}{C} \cdot q(t) = E_2(t) \\ q'(t) = i_1(t) - i_2(t) \end{cases}$

Answer available at the end of this section.

Answers

Exercise $5.5.1a$

We are given $L = 24,\ \ R = 4$, and $E(t) = 12$. Thus, the LDE we want to solve is:

$$L \cdot i'(t) + R \cdot i(t) = E(t) \implies 24i' + 4i = 12$$

This is a CC LDE with characteristic equation $24r + 4 = 0 \implies r = -\dfrac{1}{6}$

Thus, the homogeneous solutions are $i_h(t) = Ce^{-\frac{1}{6}t}$

One particular solution is $i_p(t) = 3$, so our general solution is $i(t) = 3 + Ce^{-\frac{1}{6}t}$

The switch ensures there is no current before time zero, so we have initial condition $i(0) = 0$:

$$i(t) = 3 + Ce^{-\frac{1}{6}t}$$
$$i(0) = 3 + C \cdot 1 \implies C = -3$$

This gives us current $i(t) = 3 - 3e^{-\frac{1}{6}t}$. Plugging in $t = 6$, we see $i(6) = 3 - 3e^{-1}$

Therefore, the current after 6 seconds is $3 - 3e^{-1} \approx 1.896$ amperes.

Exercise $5.5.1b$

We are given $L = 10,\ \ R = 5$, and $E(t) = 20$. Thus, the LDE we want to solve is:

$$L \cdot i'(t) + R \cdot i(t) = E(t) \implies 10i' + 5i = 20$$

This is a CC LDE with characteristic equation $10r + 5 = 0 \implies r = -\dfrac{1}{2}$

Thus, the homogeneous solutions are $i_h(t) = Ce^{-\frac{1}{2}t}$

One particular solution is $i_p(t) = 4$, so our general solution is $i(t) = 4 + Ce^{-\frac{1}{2}t}$

We are told there is no current at time zero, so we have initial condition $i(0) = 0$:

$$i(t) = 4 + Ce^{-\frac{1}{2}t}$$
$$i(0) = 4 + C \cdot 1 \implies C = -4$$

This gives us current $i(t) = 4 - 4e^{-\frac{1}{2}t}$. This is an increasing function, so the maximum possible current is the limit:

$$\lim_{t \to \infty} i(t) = 4 - 4 \cdot 0 = 4 \text{ amperes}$$

To be within 1% of this maximum current means $i(t) \geq 4 \cdot (0.99) = 3.96$:

$$4 - 4e^{-\frac{1}{2}t} \geq 3.96 \implies t \geq 2\ln(100)$$

Therefore, the current *essentially* reaches its maximum of 4 amperes after $2\ln(100) \approx 9.21$ seconds.

Exercise $5.5.1c$

We are given $L = 5,\ R = 10$, and $E(t) = 20t$. Thus, the LDE we want to solve is:

$$L \cdot i'(t) + R \cdot i(t) = E(t) \implies 5i' + 10i = 20t$$

This is a CC LDE with characteristic equation $5r + 10 = 0 \implies r = -2$

Thus, the homogeneous solutions are $i_h(t) = Ce^{-2t}$

One particular solution is $i_p(t) = 2t - 1$, so our general solution is $i(t) = 2t - 1 + Ce^{-2t}$

We know there is no current at time zero, so we have initial condition $i(0) = 0$:

$$\begin{aligned} i(t) &= 2t - 1 + Ce^{-2t} \\ i(0) &= 0 - 1 + C \cdot 1 \implies C = 1 \end{aligned}$$

This gives us current $i(t) = 2t - 1 + e^{-2t}$, and plugging in $t = 2$ we get $i(2) = 4 - 1 + e^{-4} = 3 + e^{-4}$

Therefore, the current after 2 seconds is $3 + e^{-4} \approx 3.018$ amperes.

Exercise $5.5.1d$

We are given $L = 2,\ R = 8$, and $E(t) = \begin{cases} 32 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$. Thus, the LDE we want to solve is:

$$L \cdot i'(t) + R \cdot i(t) = E(t) \implies 2i' + 8i = 32 - 32\mathcal{U}(t-1)$$

This is best to solve via Laplace Transformation:

$$\begin{aligned} 2i' + 8i &= 32 - 32\mathcal{U}(t-1) \\ 2\mathscr{L}(i'(t)) + 8\mathscr{L}(i(t)) &= 32\mathscr{L}(1) - 32\mathscr{L}(\mathcal{U}(t-1)) \\ 2(sI(s) - 0) + 8I(s) &= \frac{32}{s} - \frac{32}{s}e^{-s} \\ (2s+8)I(s) &= \frac{32}{s}(1 - e^{-s}) \\ I(s) &= \frac{16}{s(s+4)}(1 - e^{-s}) \end{aligned}$$

Using partial fraction decomposition, we write $I(s) = \left(\frac{4}{s} - \frac{4}{s+4}\right)(1 - e^{-s})$

We use the Inverse Laplace Transformation to solve for $i(t)$:

$$\begin{aligned} i(t) &= \mathscr{L}^{-1}\left(\frac{4}{s} - \frac{4}{s+4}\right) - \mathscr{L}^{-1}\left(\left(\frac{4}{s} - \frac{4}{s+4}\right)e^{-s}\right) \\ &= 4 - 4e^{-4t} - \left[4 - 4e^{-4t}\right]_{t \to t-1} \cdot \mathcal{U}(t-1) \\ &= 4 - 4e^{-4t} - \left(4 - 4e^{4-4t}\right)\mathcal{U}(t-1) \end{aligned}$$

Therefore, the current function is $i(t) = \begin{cases} 4 - 4e^{-4t} & 0 \le t < 1 \\ 4(e^4 - 1)e^{-4t} & t \ge 1 \end{cases}$

Exercise $5.5.2a$

We are given $L = 1,\ R = 8, C = \frac{1}{25}$, and $E(t) = 0$. Thus, the LDE we want to solve is:

$$L \cdot q''(t) + R \cdot q'(t) + \frac{1}{C} \cdot q(t) = E(t) \implies q'' + 8q' + 25q = 0$$

This is a CC LDE with characteristic equation $r^2 + 8r + 25 = 0 \implies r = -4 \pm 3i$

Thus, the general solutions are $q(t) = Ae^{-4t}\cos(3t) + Be^{-4t}\sin(3t)$

We are given the initial conditions $q(0) = 3,\ i(0) = q'(0) = 0$:

$$\begin{aligned} q(t) &= Ae^{-4t}\cos(3t) + Be^{-4t}\sin(3t) \\ q(0) &= A \cdot 1 + B \cdot 0 \implies A = 3 \\ q'(t) &= -4Ae^{-4t}\cos(3t) - 3Ae^{-4t}\sin(3t) - 4Be^{-4t}\sin(3t) + 3Be^{-4t}\cos(3t) \\ q'(0) &= -4A \cdot 1 - 3A \cdot 0 - 4B \cdot 0 + 3B \cdot 1 \implies 3B - 4A = 0 \implies B = 4 \end{aligned}$$

This gives us charge function $q(t) = 3e^{-4t}\cos(3t) - 4e^{-4t}\sin(3t)$, and setting this equal to zero gives:

$$3e^{-4t}\cos(3t) - 4e^{-4t}\sin(3t) = 0 \implies \tan(3t) = \frac{3}{4}$$

Therefore, the charge first equals after $\frac{1}{3}\tan^{-1}\left(\frac{3}{4}\right) \approx 0.214$ seconds.

Exercise $5.5.2b$

We are given $L = 2,\ C = \frac{1}{8},\ E(t) = 0$, and R unknown constant. Thus, the LDE we want to solve is:

$$L \cdot q''(t) + R \cdot q'(t) + \frac{1}{C} \cdot q(t) = E(t) \implies 2q'' + Rq' + 8q = 0$$

This is a CC LDE with characteristic equation $2r^2 + Rr + 8 = 0 \implies r = \dfrac{-R \pm \sqrt{R^2 - 64}}{4}$

To be critically damped, this must be a double root, so $R^2 - 64 = 0 \implies R = 8$. This gives us root $r = 2$ (double)

Thus, the general solutions are $q(t) = Ae^{-2t} + Bte^{-2t}$

We are given the initial conditions $q(0) = 8,\ i(0) = q'(0) = -20$:

$$\begin{aligned} q(t) &= Ae^{-2t} + Bte^{-2t} \\ q(0) &= A \cdot 1 + B \cdot 0 \implies A = 8 \\ q'(t) &= -2Ae^{-2t} + B(-2t + 1)e^{-2t} \\ q'(0) &= -2A \cdot 1 + B \cdot 1 \implies B - 2A = -20 \implies B = -4 \end{aligned}$$

This gives us charge function $q(t) = 8e^{-2t} - 4te^{-2t}$, so the current is $i(t) = q'(t) = (8t - 20)e^{-2t}$

We set the derivative of the current equal to zero to find the critical points:

$$i'(t) = (48 - 16t)e^{-2t} = 0 \implies t = 3$$

Plugging this into $i(t)$ gives our extreme value of $i(3) = (8 \cdot 3 - 20)\, e^{-2 \cdot 3} = \dfrac{4}{e^6}$

Therefore, the maximum magnitude of the current is $\dfrac{4}{e^6} \approx 0.00992$ amperes.

Exercise $5.5.2c$

We are given $L = 1,\ C = \dfrac{1}{2},\ R = 2$, and $E(t) = 100$. Thus, the LDE we want to solve is:

$$L \cdot q''(t) + R \cdot q'(t) + \frac{1}{C} \cdot q(t) = E(t) \implies q'' + 2q' + 2q = 100$$

This is a CC LDE with characteristic equation $r^2 + 2r + 2 = 0 \implies r = -1 \pm i$

Thus, the homogeneous solutions are $q_h(t) = Ae^{-t}\cos(t) + Be^{-t}\sin(t)$

One particular solution is $q_p(t) = 50$, so our general solution is $q(t) = 50 + Ae^{-t}\cos(t) + Be^{-t}\sin(t)$

We are given the initial conditions $q(0) = 0,\ i(0) = q'(0) = 0$:

$$\begin{aligned} q(t) &= 50 + Ae^{-t}\cos(t) + Be^{-t}\sin(t) \\ q(0) &= 50 + A \cdot 1 + B \cdot 0 \implies A = -50 \\ q'(t) &= -Ae^{-t}\cos(t) - Ae^{-t}\sin(t) - Be^{-t}\sin(t) + Be^{-t}\cos(t) \\ q'(0) &= -A \cdot 1 - A \cdot 0 - B \cdot 0 + B \cdot 1 \implies B - A = 0 \implies B = -50 \end{aligned}$$

This gives us charge function $q(t) = 50 - 50e^{-t}(\cos(t) + \sin(t))$, so the current is $i(t) = q'(t) = 100e^{-t}\sin(t)$

We set the derivative of the current equal to zero to find the critical points:

$$i'(t) = 100e^{-t}(\cos(t) - \sin(t)) = 0 \implies \tan(t) = 1 \implies t = \frac{\pi}{4}$$

Plugging this into $i(t)$ gives our extreme value of $i\left(\dfrac{\pi}{4}\right) = 50\sqrt{2}\ e^{-\frac{\pi}{4}}$

Therefore, the maximum magnitude of the current is $50\sqrt{2}\ e^{-\frac{\pi}{4}} \approx 32.24$ amperes.

Exercise $5.5.2d$

We are given $L = \dfrac{1}{2},\ C = \dfrac{1}{50},\ R = 6$, and $E(t) = 50\cos(10t)$. Thus, the LDE we want to solve is:

$$L \cdot q''(t) + R \cdot q'(t) + \frac{1}{C} \cdot q(t) = E(t) \implies \frac{1}{2}q'' + 6q' + 50q = 60\cos(10t)$$

This is a CC LDE with characteristic equation $\dfrac{1}{2}r^2 + 8r + 50 = 0 \implies r = -6 \pm 8i$

Thus, the homogeneous solutions are $q_h(t) = Ae^{-6t}\cos(8t) + Be^{-6t}\sin(8t)$

One particular solution is $q_p(t) = \sin(10t)$, so our general solution is $q(t) = Ae^{-6t}\cos(8t) + Be^{-6t}\sin(8t) + \sin(10t)$

We are given the initial conditions $q(0) = 1,\ i(0) = q'(0) = 4$:

$$\begin{aligned} q(t) &= Ae^{-6t}\cos(8t) + Be^{-6t}\sin(8t) + \sin(10t) \\ q(0) &= A \cdot 1 + B \cdot 0 + 0 \implies A = 1 \\ q'(t) &= -6Ae^{-6t}\cos(8t) - 8Ae^{-6t}\sin(8t) - 6Be^{-6t}\sin(8t) + 8Be^{-6t}\cos(8t) + 10\cos(10t) \\ q'(0) &= -6A \cdot 1 - 8A \cdot 0 - 6B \cdot 0 + 8B \cdot 1 + 10 \implies 8B - 6A = -6 \implies B = 0 \end{aligned}$$

Therefore, the charge function is $q(t) = e^{-6t}\cos(8t) + \sin(10t)$

Exercise $5.5.3a$

We are given $L = 1,\ R_1 = 2 + \sqrt{3},\ R_2 = 2 - \sqrt{3},\ R = 1$ and $E_1(t) = 0,\ E_2(t) = 20$. Thus, the system of LDEs we want to solve is:

$$\begin{cases} L \cdot i_1'(t) + R \cdot (i_1(t) - i_2(t)) + R_1 \cdot i_1(t) = E_1(t) \\ L \cdot i_2'(t) + R \cdot (i_2(t) - i_1(t)) + R_2 \cdot i_2(t) = E_2(t) \end{cases} \implies \begin{cases} i_1' + (i_1 - i_2) + (2 + \sqrt{3})i_1 = 0 \\ i_2' + (i_2 - i_1) + (2 - \sqrt{3})i_2 = 20 \end{cases}$$

We write this system in terms of our D notation: $\begin{cases} (D + 3 + \sqrt{3})i_1 - i_2 = 0 \\ -i_1 + (D + 3 - \sqrt{3})i_2 = 20 \end{cases}$

If we "multiply" the top equation by $D + 3 - \sqrt{3}$ and add it to the bottom equation this becomes:

$$\left((D + 3 - \sqrt{3})(D + 3 + \sqrt{3}) - 1\right) i_1 = 20 \implies i_1'' + 6i_1' + 5i_1 = 20$$

This is a CC LDE with characteristic equation $r^2 + 6r + 5 = 0 \implies r = -1, -5$

Thus, the homogeneous solutions are $i_{1,h}(t) = Ae^{-t} + Be^{-5t}$

One particular solution is $i_{1,p}(t) = 4$, so our general solution is $i_1(t) = Ae^{-t} + Be^{-5t} + 4$

We substitute this into the original top equation to get:

$$i_2 = i_1' + (3 + \sqrt{3})i_1 = \left(-Ae^{-t} - 5Be^{-t}\right) + (3 + \sqrt{3})\left(Ae^{-t} + Be^{-5t} + 4\right) = A(2 + \sqrt{3})e^{-t} + B(-2 + \sqrt{3})e^{-5t} + 12 + 4\sqrt{3}$$

We are given the initial conditions $i_1(0) = 0,\ i_2(0) = 0$:

$$\begin{aligned} i_1(t) &= Ae^{-t} + Be^{-5t} + 4 \\ i_1(0) &= A \cdot 1 + B \cdot 1 + 4 \implies A + B = -4 \\ i_2(t) &= A(2 + \sqrt{3})e^{-t} + B(-2 + \sqrt{3})e^{-5t} + 12 + 4\sqrt{3} \\ i_2(0) &= A(2 + \sqrt{3}) \cdot 1 + B(-2 + \sqrt{3}) \cdot 1 + 12 + 4\sqrt{3} \implies (2 + \sqrt{3})A + (-2 + \sqrt{3})B = -12 - 4\sqrt{3} \end{aligned}$$

$$\begin{cases} A + B = -4 \\ (2 + \sqrt{3})A + (-2 + \sqrt{3})B = -12 - 4\sqrt{3} \end{cases} \implies \begin{cases} A = -5 \\ B = 1 \end{cases}$$

Therefore, the current functions are $\begin{cases} i_1(t) = -5e^{-t} + e^{-5t} + 4 \\ i_2(t) = (-10 - 5\sqrt{3})e^{-t} + (-2 + \sqrt{3})e^{-5t} + 12 + 4\sqrt{3} \end{cases}$

Exercise $5.5.3b$

We are given $L_1 = L_2 = 0,\ R_1 = R_2 = 2, C = 1$ and $E_1(t) = 20,\ E_2(t) = 0$. Thus, the system of LDEs we want to solve is:

$$\begin{cases} L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) + \dfrac{1}{C} q(t) = E_1(t) \\ L_2 \cdot i_2'(t) + R_2 \cdot i_2(t) - \dfrac{1}{C} \cdot q(t) = E_2(t) \\ q'(t) = i_1(t) - i_2(t) \end{cases} \implies \begin{cases} 2i_1(t) + q(t) = 20 \\ 2i_2(t) - q(t) = 0 \\ q'(t) = i_1(t) - i_2(t) \end{cases}$$

The top equation gives $i_1(t) = 10 - \dfrac{1}{2}q$ and the second equation gives $i_2 = \dfrac{1}{2}q$. Substituting these into the third equation gives:

$$q' = \left(10 - \frac{1}{2}q\right) - \frac{1}{2}q \implies q' + q = 10$$

This is a CC LDE with characteristic equation $r + 1 = 0 \implies r = -1$

Thus, the homogeneous solutions are $q_h(t) = Ae^{-t}$

One particular solution is $q_p(t) = 10$, so our general solution is $q(t) = 10 + Ae^{-t}$

We are given the initial condition $q_(0) = 0$, so it is clear that $A = -10$

The total charge on the capacitor is therefore $q(t) = 10 - 10e^{-t}$ which clearly has a maximum of 10.

Reaching 90% of this charge would be would $q(t) = 9$:

$$10 - 10e^{-t} = 9 \implies e^{-t} = \frac{1}{10} \implies t = \ln(10)$$

Therefore, the capacitor reaches 90% of its charge after $t = \ln(10) \approx 2.30$ seconds.

Exercise $5.5.3c$

We are given $M = 1,\ L_1 = 2,\ L_2 = 1,\ R_1 = 0,\ R_2 = 1$, and $E_1(t) = 12,\ E_2(t) = 0$. Thus, the system of LDEs we want to solve is:

$$\begin{cases} L_1 \cdot i_1'(t) + M \cdot i_2'(t) + R_1 \cdot i_1(t) = E_1(t) \\ L_2 \cdot i_2'(t) + M \cdot i_1'(t) + R_2 \cdot i_2(t) = E_2(t) \end{cases} \implies \begin{cases} 2i_1' + i_2' = 12 \\ i_2' + i_1' + i_2 = 0 \end{cases}$$

We write this system in terms of our D notation: $\begin{cases} 2Di_1 + Di_2 = 12 \\ Di_1 + (D+1)i_2 = 0 \end{cases}$

If we subtract twice the bottom equation from the top equation this becomes:

$$(D - 2(D+1))\, i_2 = 12 \implies i_2' + 2i_2 = -12$$

This is a CC LDE with characteristic equation $r + 2 = 0 \implies r = -2$

Thus, the homogeneous solutions are $i_{2,h}(t) = Ae^{-2t}$

One particular solution is $i_{2,p}(t) = -6$, so our general solution is $i_2(t) = Ae^{-2t} - 6$

We are given $i_2(0) = 0$, so we can easily solve for $A = 6$ without finding the function $i_1(t)$.

Thus, $i_2(t) = 6e^{-2t} - 6$, which has maximum magnitude of 6 amperes.

Exercise $5.5.3d$

We are given $L = 1,\ L_1 = 0,\ L_2 = 0,\ R_1 = 1,\ R_2 = 2, C_2 = \dfrac{1}{4}$, and $E_1(t) = 26\cos(2t),\ E_2(t) = 0$. Thus, the system of LDEs we want to solve is:

$$\begin{cases} L \cdot (i_1'(t) - q_2''(t)) + L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) = E_1(t) \\ L \cdot (q_2''(t) - i_1'(t)) + L_2 \cdot q_2''(t) + R_2 \cdot q_2'(t) + \dfrac{1}{C_2} q_2(t) = E_2(t) \end{cases} \implies \begin{cases} i_1' - q_2'' + i_1 = 26\cos(2t) \\ q_2'' - i_1' + 2q_2' + 4q_2 = 0 \end{cases}$$

We write this system in terms of our D notation: $\begin{cases} (D^2 + D)q_1 - D^2 q_2 = 26\cos(2t) \\ -D^2 q_1 + (D^2 + 2D + 4)q_2 = 0 \end{cases}$

If we add D "times" the top equation to $D + 1$ "times" the bottom equation this becomes:

$$\left(-D^3 + (D+1)(D^2 + 2D + 4)\right) q_2 = D\left(26\cos(2t)\right) \implies 3q_2'' + 6q_2' + 4q_2 = -52\sin(2t)$$

This is a CC LDE with characteristic equation $3r^2 + 6r + 4 = 0 \implies r = -1 \pm \dfrac{i}{\sqrt{3}}$

Thus, the homogeneous solutions are $q_{2,h}(t) = Ae^{-t}\cos\left(\dfrac{t}{\sqrt{3}}\right) + Be^{-t}\sin\left(\dfrac{t}{\sqrt{3}}\right)$

One particular solution is $q_{2,p}(t) = 3cos(2t) + 2\sin(2t)$, so our general solution is $q_2(t) = Ae^{-t}\cos\left(\dfrac{t}{\sqrt{3}}\right) + Be^{-t}\sin\left(\dfrac{t}{\sqrt{3}}\right) + 3\cos(2t) + 2\sin(2t)$

Taking the derivative gives us $i_2(t)$:

$$i_2(t) = q_2'(t) = \left(-A + \frac{1}{\sqrt{3}}B\right)e^{-t}\cos\left(\frac{t}{\sqrt{3}}\right) + \left(-\frac{1}{\sqrt{3}}A - B\right)e^{-t}\sin\left(\frac{t}{\sqrt{3}}\right) + 4\cos(2t) - 6\sin(2t)$$

Notice that if you add the two original equations together, you get $q_1' + 2q_2' + 4q_2 = 26\cos(2t)$

Therefore, we can find $i_1(t) = q_1'(t)$ using:

$$i_1(t) = 26\cos(2t) - 4q_2(t) - 2q_2'(t) = \left(-2A - \frac{2}{\sqrt{3}}B\right)e^{-t}\cos\left(\frac{t}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{3}}A - 2B\right)e^{-t}\sin\left(\frac{t}{\sqrt{3}}\right) + 6\cos(2t) + 4\sin(2t)$$

The initial conditions were $i_1(0) = 0,\ i_2(0) = 0$:

$$i_1(t) = \left(-2A - \frac{2}{\sqrt{3}}B\right)e^{-t}\cos\left(\frac{t}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{3}}A - 2B\right)e^{-t}\sin\left(\frac{t}{\sqrt{3}}\right) + 6\cos(2t) + 4\sin(2t)$$

$$i_1(0) = \left(-2A - \frac{2}{\sqrt{3}}B\right) + 6 \implies -2A - \frac{2}{\sqrt{3}}B = -6$$

$$i_2(t) = \left(-A + \frac{1}{\sqrt{3}}B\right)e^{-t}\cos\left(\frac{t}{\sqrt{3}}\right) + \left(-\frac{1}{\sqrt{3}}A - B\right)e^{-t}\sin\left(\frac{t}{\sqrt{3}}\right) + 4\cos(2t) - 6\sin(2t)$$

$$i_2(0) = \left(-A + \frac{1}{\sqrt{3}}B\right)\cdot 1 + 4 \implies -A + \frac{1}{\sqrt{3}}B = -4$$

$$\begin{cases} -2A - \frac{2}{\sqrt{3}}B = -6 \\ -A + \frac{1}{\sqrt{3}}B = -4 \end{cases} \implies \begin{cases} A = \frac{7}{2} \\ B = -\frac{\sqrt{3}}{2} \end{cases}$$

Thus, the currents are $\begin{cases} i_1(t) = -6e^{-t}\cos\left(\frac{t}{\sqrt{3}}\right) + \frac{10}{\sqrt{3}}e^{-t}\sin\left(\frac{t}{\sqrt{3}}\right) + 6\cos(2t) + 4\sin(2t) \\ i_2(t) = -4e^{-t}\cos\left(\frac{t}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}e^{-t}\sin\left(\frac{t}{\sqrt{3}}\right) + 4\cos(2t) - 6\sin(2t) \end{cases}$

Exercise 5.5.4*a*

This problem has two phases, first for $0 \le t \le 1$ we have only one loop.

This loop has $L = 1,\ R = 1,\ E(t) = e$. Thus, the LDE for this phase is:

$$L \cdot i_1'(t) + R \cdot i_1(t) = E(t) \implies i_1' + i_1 = e$$

This is a CC LDE with characteristic equation $r + 1 = 0 \implies r = -1$

Thus, the homogeneous solution is of the form $i_{1,h}(t) = Ae^{-t}$

One possible particular solution is $i_{1,p}(t) = e$, so the general solution is $i_1(t) = Ae^{-t} + e$

Since we know $i_1(0) = 0$, we can see that $A = -e$. Thus, we have $i_1(t) = e - e^{1-t}$ for $0 \le t \le 1$

Since the second loop is open, we must have $i_2(t) = 0$ for all $0 \le t \le 1$

These two functions tell us that by time $t = 1$, we will have $i_1 = e - 1$

After this point, the second loop is closed.

In the second phase of this problem, we are given $L = 1,\ R_1 = 1,\ R_2 = 1$, and $E_1(t) = e,\ E_2(t) = 3$

Thus, the system of LDEs we want to solve is:

$$\begin{cases} L \cdot (i_1'(t) - i_2'(t)) + L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) = E_1(t) \\ L \cdot (i_2'(t) - i_1'(t)) + L_2 \cdot i_2'(t) + R_2 \cdot i_2(t) = E_2(t) \end{cases} \implies \begin{cases} i_1' - i_2' + i_1 = e \\ i_2' - i_1' + i_2 = 3 \end{cases}$$

We write this system in terms of our D notation: $\begin{cases} (D+1)i_1 - Di_2 = e \\ -Di_1 + (D+1)i_2 = 3 \end{cases}$

If we add D "times" the top equation to $D + 1$ "times" the bottom equation this becomes:

$$\left(-D^2 + (D+1)^2\right)i_2 = D\,(e) + (D+1)\,(3) \implies 2i_2' + i_2 = 3$$

This is a CC LDE with characteristic equation $2r + 1 = 0 \implies r = -\frac{1}{2}$

Thus, the homogeneous solutions are $i_{2,h}(t) = Be^{-\frac{1}{2}t}$

One particular solution is $i_{2,p}(t) = 3$, so our general solution is $i_2(t) = Be^{-\frac{1}{2}t} + 3$

Notice that if you add the two original equations together, you get $i_1 + i_2 = 3 + e$

Therefore, we can find $i_1(t) = 3 + e - i_2(t) = 3 + e - \left(Be^{-\frac{1}{2}t} + 3\right) = e - Be^{-\frac{1}{2}t}$

The initial conditions for this phase are a little tricky. We know the current in the inductor must be continuous (since it is differentiable), but each individual current function *does not need to be continuous* at $t = 1$. To get the initial currents after the second switch is closed, consider that before it closed we had $i_1 - i_2 = e - 1$ current in the inductor. Once it closed, we know $i_1 + i_2 = 3 + e$

$$\begin{cases} i_1 - i_2 = e - 1 \\ i_1 + i_2 = 3 + e \end{cases} \implies \begin{cases} i_1 = e + 1 \\ i_2 = 2 \end{cases}$$

So, we will use $i_1(1) = e + 1$ and $i_2(1) = 2$ as our new initial conditions.

$$i_1(t) = e - Be^{-\frac{1}{2}t}$$

$$i_1(1) = e - Be^{-\frac{1}{2}} \implies B = -e^{\frac{1}{2}}$$

Thus, the current in the second loop must be $i_2(t) = \begin{cases} 0 & 0 \le t < 1 \\ 3 - e^{-\frac{1}{2}(t-1)} & t \ge 1 \end{cases}$

Two additional seconds puts us at $t = 3$, and we have $i_2(3) = 3 - e^{-1}$

Therefore, the current in the second loop after two additional seconds is $3 - e^{-1} \approx 2.63$ amperes.

Exercise 5.5.4*b*

This problem has two phases, first for $0 \le t \le 2$ we have only one loop.

This loop has $C = \frac{1}{2},\ R = 2,\ E(t) = 12$. Thus, the LDE for the charge on the capacitor during this phase is:

$$L \cdot q''(t) + R \cdot q'(t) + \frac{1}{C}q(t) = E_1(t) \implies 2q' + 2q = 12$$

This is a CC LDE with characteristic equation $2r + 2 = 0 \implies r = -1$

Thus, the homogeneous solution is of the form $q_h(t) = Ae^{-t}$

One possible particular solution is $q_p(t) = 6$, so the general solution is $q_(t) = Ae^{-t} + 6$

Since we know $q(0) = 0$, we can see that $A = -6$. Thus, we have $q(t) = 6 - 6e^{-t}$ for $0 \le t \le 2$

This tell us that by time $t = 2$, we will have $q(2) = 6 - 6e^{-2}$

After this point, the second loop is closed.

In the second phase of this problem, we are given $L_1 = L_2 = 0,\ C = \frac{1}{2},\ R_1 = 2,\ R_2 = 4$, and $E_1(t) = 12,\ E_2(t) = 6$

Thus, the system of LDEs we want to solve is:

$$\begin{cases} L_1 \cdot i_1'(t) + R_1 \cdot i_1(t) + \frac{1}{C} q(t) = E_1(t) \\ L_2 \cdot i_2'(t) + R_2 \cdot i_2(t) - \frac{1}{C} q(t) = E_2(t) \\ q'(t) = i_1(t) - i_2(t) \end{cases} \implies \begin{cases} 2i_1(t) + 2q(t) = 12 \\ 4i_2(t) - 2q(t) = 6 \\ q'(t) = i_1(t) - i_2(t) \end{cases}$$

The first two equations allow us to solve for $i_1(t) = 6 - q(t)$ and $i_2(t) = \frac{3}{2} + \frac{1}{2} q(t)$, which we plug into the bottom equation:

$$q'(t) = (6 - q(t)) - \left(\frac{3}{2} + \frac{1}{2} q(t) \right) \implies q' + \frac{3}{2} q = \frac{9}{2}$$

This is a CC LDE with characteristic equation $r + \frac{3}{2} = 0 \implies r = -\frac{3}{2}$

Thus, the homogeneous solution is of the form $q_h(t) = Be^{-\frac{3}{2}t}$

One possible particular solution is $q_p(t) = 3$, so the general solution is $q_(t) = Be^{-\frac{3}{2}t} + 3$

Since we know that $q(2) = 6 - 6e^{-2}$, we can solve for B:

$$\begin{aligned} q_(t) &= Be^{-\frac{3}{2}t} + 3 \\ q(2) = Be^{-3} + 3 &\implies B = 3e^3 - 6e \end{aligned}$$

Thus, the charge on the capacitor is $q(t) = \begin{cases} 6 - 6e^{-t} & 0 \le t < 2 \\ 3 + (3e^3 - 6e)e^{-\frac{3}{2}t} & t \ge 2 \end{cases}$

Notice this is increasing for $t < 2$ and decreasing for $t > 2$, so $t = 2$ seconds yields our maximum charge.

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