

LEBESGUE INTEGRATION ON TIME SCALES



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Chapter 1: Outer Measure on Time Scales

Suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ , respectively.

Denote by $\mathcal{F}$ the σ -algebra generated by the family of all left closed and right open intervals of $\mathbb{T}$ of the form

$$\begin{aligned}[a, b)_{\mathbb{T}} &= [a, b) \cap \mathbb{T} \\ &= \{t \in \mathbb{T} : a \leq t < b\}\end{aligned}$$

with $a, b \in \mathbb{T}$, $a \leq b$. The interval $[a, a)$ is understood as the empty set.

Theorem

$\mathcal{F}$ is a semiring of subsets of $\mathbb{T}$.

Proof

We will check all axioms for a semiring of sets. 1. The empty set $[a, a)$ belongs to $\mathcal{F}$.

2. Let $c_1, c_2, d_1, d_2 \in \mathbb{T}$ be such that $c_1 \leq d_1$ and $c_2 \leq d_2$. Consider the intervals $[c_1, d_1)_{\mathbb{T}}$ and $[c_2, d_2)_{\mathbb{T}}$. We have the following cases. (a) Assume that $[c_1, d_1)_{\mathbb{T}} \cap [c_2, d_2)_{\mathbb{T}} = \emptyset$. Then $[c_1, d_1)_{\mathbb{T}} \cap [c_2, d_2)_{\mathbb{T}} \in \mathcal{F}$.

(b) Assume that $[c_1, d_1)_{\mathbb{T}} \cap [c_2, d_2)_{\mathbb{T}} \neq \emptyset$. Then we have the following subcases. (b1) Let $c_1 \leq c_2$ and $d_2 \leq d_1$. Then

$$\begin{aligned}[c_1, d_1)_{\mathbb{T}} \cap [c_2, d_2)_{\mathbb{T}} &= [c_2, d_2)_{\mathbb{T}} \\ &\in \mathcal{F}.\end{aligned}$$

(b2) Let $c_2 \leq c_1$ and $d_1 \leq d_2$. Then

$$\begin{aligned}[c_1, d_1)_{\mathbb{T}} \cap [c_2, d_2)_{\mathbb{T}} &= [c_1, d_1)_{\mathbb{T}} \\ &\in \mathcal{F}.\end{aligned}$$

(b3) Let $c_1 \leq c_2 \leq d_1 \leq d_2$. Then

$$\begin{aligned}[c_1, d_1)_{\mathbb{T}} \cap [c_2, d_2)_{\mathbb{T}} &= [c_2, d_1)_{\mathbb{T}} \\ &\in \mathcal{F}.\end{aligned}$$

(b4) Let $c_2 \leq c_1 \leq d_2 \leq d_1$. Then

$$\begin{aligned}[c_1, d_1)_{\mathbb{T}} \cap [c_2, d_2)_{\mathbb{T}} &= [c_1, d_2)_{\mathbb{T}} \\ &\in \mathcal{F}.\end{aligned}$$

3. Let $c_1, c_2, d_1, d_2 \in \mathbb{T}$ be such that $c_1 \leq c_2 \leq d_2 \leq d_1$. Then $[c_2, d_2)_{\mathbb{T}} \subset [c_1, d_1)_{\mathbb{T}}$ and

$$[c_1, d_1)_{\mathbb{T}} = [c_1, c_2)_{\mathbb{T}} \cup [c_2, d_2)_{\mathbb{T}} \cup [d_2, d_1)_{\mathbb{T}}$$

and $[c_1, c_2)_{\mathbb{T}}, [c_2, d_2)_{\mathbb{T}}, [d_2, d_1)_{\mathbb{T}} \in \mathcal{F}$.

Therefore $\mathcal{F}$ satisfies all axioms for a semiring of sets. This completes the proof.

Definition

Let $m : \mathcal{F} \rightarrow [0, \infty]$ be the set function on $\mathcal{F}$, whose values belong to the extended real half-line $[0, \infty]$, that assigns to each interval $[a, b)_{\mathbb{T}}$ its length $b - a$, i.e.,

$$m([a, b)_{\mathbb{T}}) = b - a.$$

Theorem

m is a measure.

Proof

We will check all axioms for a measure.

1. We have

$$\begin{aligned} m(\emptyset) &= m([a, a]_{\mathbb{T}}) \\ &= a - a \\ &= 0. \end{aligned}$$

2. For any interval $[a, b]_{\mathbb{T}}$, one has

$$\begin{aligned} m([a, b]_{\mathbb{T}}) &= b - a \\ &\geq 0. \end{aligned}$$

3. Let $c_j, d_j \in \mathbb{T}$, $j \in \{1, \dots, n\}$, be such that $c_j \leq d_j$, $j \in \{1, \dots, n\}$, and $d_{j-1} \leq c_j$, $j \in \{2, \dots, n\}$. Then

$$[c_j, d_j]_{\mathbb{T}} \cap [c_k, d_k]_{\mathbb{T}} = \emptyset, \quad j, k \in \{1, \dots, n\}, \quad j \neq k.$$

Hence,

$$\begin{aligned} m\left(\bigcup_{j=1}^n [c_j, d_j]_{\mathbb{T}}\right) &= d_1 - c_1 + \dots + d_n - c_n \\ &= m([c_1, d_1]_{\mathbb{T}}) + \dots + m([c_n, d_n]_{\mathbb{T}}) \\ &= \sum_{j=1}^n m([c_j, d_j]_{\mathbb{T}}). \end{aligned}$$

Thus, m satisfies all axioms for a measure. This completes the proof.

Definition

For the pair $(\mathcal{F}, m)$ we generate an outer measure m^* on the family of all subsets of $\mathbb{T}$ as follows. Let E be any subset of $\mathbb{T}$. With $\mathcal{C}(E)$ we will denote the set of all coverings $V = \{V_j\}$ of E by a finite or countable system of intervals $V_j \in \mathcal{F}$, i.e., $E \subset \bigcup_j V_j$. Then we put

$$m^*(E) = \inf_{V \in \mathcal{C}(E)} \sum_j m(V_j)$$

if there exists at least one covering of E and the infimum exists. If there is no such covering of E or the infimum does not exist, we put

$$m^*(E) = \infty.$$

Now, we will deduct some of the properties of the outer measure.

Theorem

The outer measure m^* satisfies the following. 1. $m^*(\emptyset) = 0$.

2. Nonnegativity. For any $A \subset \mathbb{T}$, one has $m^*(A) \geq 0$.

3. Monotonicity. IF $A \subset B \subset I$, then $m^*(A) \leq m^*(B)$.

4. Subadditivity. For any countable collection of pairwise disjoint sets $E_j \subset \mathbb{T}$, we have

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j).$$

5. If $A, B \subset \mathbb{T}$ and $m^*(A) = 0$, then

$$m^*(A \cup B) = m^*(B).$$

6. If $A, B \subset \mathbb{T}$ and

$$m^*(A \triangle B) = 0,$$

then

$$m^*(A) = m^*(B).$$

Proof

1. Since $m(\emptyset) = 0$, we get $m^*(\emptyset) = 0$.
2. Since for any interval V_j of the covering $\{V_j\}$ of the set A we have $m(V_j) \geq 0$, we get $m^*(A) \geq 0$.
3. Since $A \subset B$, we have that $C(B) \subset C(A)$. Hence,

$$\begin{aligned} m^*(B) &= \inf_{V \in C(B)} \sum_j m(V_j) \\ &\geq \inf_{U \in C(A)} \sum_j m(U_j) \\ &= m^*(A). \end{aligned}$$

4. If $\sum_{j=1}^{\infty} m^*(E_j) = \infty$, we are done. Otherwise, if $\sum_{j=1}^{\infty} m^*(E_j) < \infty$, then $m^*(E_j) < \infty$, $j \in \mathbb{N}$. Fix $\epsilon > 0$ arbitrarily. By the definition of the outer measure and characterisation of infimum, for each $j \in \mathbb{N}$ there exists a countable covering $\{I_j^k\}_{k \in \mathbb{N}}$ of E_j such that

$$\begin{aligned} m^*(E_j) &\leq \sum_{k=1}^{\infty} m(I_j^k) \\ &< m^*(E_j) + \frac{\epsilon}{2^j}. \end{aligned}$$

Since $\bigcup_{j=1}^{\infty} \{I_j^k\}_{k \in \mathbb{N}}$ forms a countable covering of $\bigcup_{j=1}^{\infty} E_j$, by the definition of the outer measure, we get

$$\begin{aligned} m^*\left(\bigcup_{j=1}^{\infty} E_j\right) &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m(I_j^k) \\ &< \sum_{j=1}^{\infty} \left(m^*(E_j) + \frac{\epsilon}{2^j}\right) \\ &= \sum_{j=1}^{\infty} m^*(E_j) + \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} \\ &= \sum_{j=1}^{\infty} m^*(E_j) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrarily chosen, the last chain of inequalities yields

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j).$$

5. By the previous point, we get

$$\begin{aligned} m^*(A \cup B) &\leq m^*(A) + m^*(B) \\ &= m^*(B). \end{aligned}$$

By the monotonicity of the outer measure, we find

$$m^*(A \cup B) \geq m^*(B).$$

Hence, sandwiching, we get the desired result.

6. Since

$$A \Delta B = (A \setminus B) \cup (B \setminus A),$$

using the monotonicity of the outer measure, we arrive at

$$\begin{aligned} 0 &\leq m^*(A \setminus B) \\ &\leq m^*(A \Delta B) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\leq m^*(B \setminus A) \\ &\leq m^*(A \Delta B) \\ &= 0. \end{aligned}$$

Thus,

$$m^*(A \setminus B) = 0 \quad \text{and} \quad m^*(B \setminus A) = 0.$$

Next, note that

$$\begin{aligned} A &= (A \cap B) \cup (A \setminus B) \\ &\subset B \cup (A \setminus B) \end{aligned}$$

and as above, we have

$$B \subset A \cup (B \setminus A).$$

Hence, using the monotonicity of the outer measure, we arrive at

$$\begin{aligned} m^*(A) &\leq m^*(B \cup (A \setminus B)) \\ &\leq m^*(B) + m^*(A \setminus B) \\ &= m^*(B) \end{aligned}$$

and

$$\begin{aligned} m^*(B) &\leq m^*(A \cup (B \setminus A)) \\ &\leq m^*(A) + m^*(B \setminus A) \\ &= m^*(A). \end{aligned}$$

By sandwiching, we obtain

$$m^*(A) = m^*(B).$$

This completes the proof.

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Chapter 2: Lebesgue Delta Measure on Time Scales

Definition

A subset $A \in \mathcal{F}$ is said to be m^* -measurable, or Δ -measurable, or delta measurable, if for any $B \in \mathcal{F}$ the following holds

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c).$$

The set of all delta measurable subsets of $\mathbb{T}$ is denoted by $\mathbb{M}(m^*)$.

Theorem

$\mathbb{M}(m^*)$ is a σ -algebra.

Proof

We will check the axioms for σ -algebra. 1. Let $A \in \mathcal{F}$ be arbitrarily chosen. We have

$$\begin{aligned} A \cap \emptyset &= \emptyset, \\ A \cap \emptyset^c &= A \cap \mathbb{T} \\ &= A. \end{aligned}$$

Hence,

$$\begin{aligned} m^*(A \cap \emptyset) + m^*(A \cap \emptyset^c) &= m^*(\emptyset) + m^*(A) \\ &= m^*(A). \end{aligned}$$

Therefore $\emptyset \in \mathbb{M}(m^*)$. Next, note that

$$\begin{aligned} A \cap \mathbb{T} &= A, \\ A \cap \mathbb{T}^c &= A \cap \emptyset \\ &= \emptyset. \end{aligned}$$

Then

$$\begin{aligned} m^*(A \cap \mathbb{T}) + m^*(A \cap \mathbb{T}^c) &= m^*(A) + m^*(\emptyset) \\ &= m^*(A). \end{aligned}$$

Consequently $\mathbb{T} \in \mathbb{M}(m^*)$.

2. Let $A, B \in \mathbb{M}(m^*)$ be arbitrarily chosen. We will prove that $A \setminus B \in \mathbb{M}(m^*)$. First observe that

$$A \setminus B = A \cap B^c.$$

Thus, we will prove that

$$A \cap B^c \in \mathbb{M}(m^*) \tag{Chapter 2.1}$$

. Let

$$G \in \mathcal{F} \tag{Chapter 2.2}$$

be arbitrarily chosen. We will prove that

$$m^*(G) = m^*(G \cap A \cap B^c) + m^*(G \cap (A^c \cup B)). \tag{Chapter 2.3}$$

Since $A, B \in \mathbb{M}(m^*)$, for any $F \in \mathcal{F}$, one has

$$m^*(F) = m^*(F \cap A) + m^*(F \cap A^c) \tag{Chapter 2.4}$$

and

$$m^*(F) = m^*(F \cap B) + m^*(F \cap B^c). \tag{Chapter 2.5}$$

Substituting $F = G \cap A$ into (Chapter 2.5), we find

$$m^*(G \cap A) = m^*(G \cap A \cap B) + m^*(G \cap A \cap B^c).$$

Substituting $F = G$ into (Chapter 2.4) gives us

$$\begin{aligned} m^*(G) &= m^*(G \cap A) + m^*(G \cap A^c) \\ &= m^*(G \cap A \cap B) + m^*(G \cap A \cap B^c) + m^*(G \cap A^c). \end{aligned} \quad (\text{Chapter 2.6})$$

Now, we consider (Chapter 2.4) with $F = G \cap (A^c \cup B)$. Then

$$\begin{aligned} m(G \cap (A^c \cup B)) &= m^*(G \cap (A^c \cup B) \cap A) + m^*(G \cap (A^c \cup B) \cap A^c) \\ &= m^*(G \cap A \cap B) + m^*(G \cap (A^c \cup (B \cap A^c))) \\ &= m^*(G \cap A \cap B) + m^*(G \cap A^c), \end{aligned}$$

where we have used the distributivity of the set operations. We substitute this in (Chapter 2.6) and we get

$$m^*(G) = m^*(G \cap (A^c \cup B)) + m^*(G \cap (A \cap B^c)).$$

Therefore $A \cap B^c \in \mathbb{M}(m^*)$, or $A \setminus B \in \mathbb{M}(m^*)$.

3. Now, we will prove that $\mathbb{M}(m^*)$ is closed under countable union. Let $\{E_j\}_{j \in \mathbb{N}}$ be countable many sets in $\mathbb{M}(m^*)$. Without loss of generality, assume that they are all pairwise disjoint. First, we will prove that $\bigcup_{j=1}^n E_j \in \mathbb{M}(m^*)$ for any $n \in \mathbb{N}$. For this aim, we will use induction. (a) For $n = 2$, we have

$$E_1 \cup E_2 = T \setminus ((T \setminus E_1) \setminus E_2).$$

Since $E_1, E_2 \in \mathbb{M}(m^*)$, by the previous point we get the implications

$$T \setminus E_1 \in \mathbb{M}(m^*) \Rightarrow (T \setminus E_1) \setminus E_2 \in \mathbb{M}(m^*) \Rightarrow T \setminus ((T \setminus E_1) \setminus E_2) \in \mathbb{M}(m^*).$$

Thus, $E_1 \cup E_2 \in \mathbb{M}(m^*)$.

(b) Assume that $\bigcup_{j=1}^k E_j \in \mathbb{M}(m^*)$ for some $k \in \mathbb{N}$.

(c) We will prove that $\bigcup_{j=1}^{k+1} E_j \in \mathbb{M}(m^*)$. Really, we have that

$$\bigcup_{j=1}^{k+1} E_j = \bigcup_{j=1}^k E_j \cup E_{k+1}.$$

Since $\bigcup_{j=1}^k E_j, E_{k+1} \in \mathbb{M}(m^*)$, we get the following implications

$$\begin{aligned} T \setminus \bigcup_{j=1}^k E_j \in \mathbb{M}(m^*) &\Rightarrow \left(T \setminus \bigcup_{j=1}^k E_j \right) \setminus E_{k+1} \in \mathbb{M}(m^*) \Rightarrow \\ &T \setminus \left(\left(T \setminus \bigcup_{j=1}^k E_j \right) \setminus E_{k+1} \right) \in \mathbb{M}(m^*). \end{aligned}$$

By induction, we conclude that $\bigcup_{j=1}^n E_j \in \mathbb{M}(m^*)$ for any $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Then, for any $F \in \mathcal{F}$, one has

$$m^*(F) = m^* \left(F \cap \left(\bigcup_{j=1}^n E_j \right) \right) + m^* \left(F \cap \left(\bigcup_{j=1}^n E_j \right)^c \right). \quad (\text{Chapter 2.7})$$

On the other hand, since $E_n \in \mathbb{M}(m^*)$ and $F \cap \left(\bigcup_{j=1}^n E_j \right) \in \mathcal{F}$, one has

$$\begin{aligned}
 m^* \left(F \cap \left(\bigcup_{j=1}^n E_j \right) \right) &= m^* \left(\left(F \cap \left(\bigcup_{j=1}^n E_j \right) \right) \cap E_n \right) + m^* \left(\left(F \cap \left(\bigcup_{j=1}^n E_j \right) \right) \cap E_n^c \right) \\
 &= m^*(F \cap E_n) + m^* \left(F \cap \left(\bigcup_{j=1}^{n-1} E_j \right) \right) \\
 &= m^*(F \cap E_n) + m^* \left(F \cap \left(\bigcup_{j=1}^{n-2} E_j \right) \right) + m^*(F \cap E_{n-1}) \\
 &\vdots \\
 &= \sum_{j=1}^n m^*(F \cap E_j).
 \end{aligned}$$

Substituting this into in (Chapter 2.7), we get

$$m^*(F) = \sum_{j=1}^n m^*(F \cap E_j) + m^* \left(F \cap \left(\bigcup_{j=1}^n E_j \right)^c \right). \quad (\text{Chapter 2.8})$$

Now, using that

$$\bigcup_{j=1}^n E_j \subset \bigcup_{j=1}^{\infty} E_j,$$

we find

$$\left(\bigcup_{j=1}^{\infty} E_j \right)^c \subset \left(\bigcup_{j=1}^n E_j \right)^c.$$

Hence, using the monotonicity of the outer measure and using (Chapter 2.8), we arrive at

$$m^*(F) \geq \sum_{j=1}^n m^*(F \cap E_j) + m^* \left(F \cap \left(\bigcup_{j=1}^{\infty} E_j \right)^c \right).$$

The last inequality is true for any $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ and using the subadditivity of m^* , we then get

$$\begin{aligned}
 m^*(F) &\geq \sum_{j=1}^{\infty} m^*(F \cap E_j) + m^* \left(F \cap \left(\bigcup_{j=1}^{\infty} E_j \right)^c \right) \\
 &\geq m^* \left(\bigcup_{j=1}^{\infty} (F \cap E_j) \right) + m^* \left(F \cap \left(\bigcup_{j=1}^{\infty} E_j \right)^c \right) \\
 &= m^* \left(F \cap \left(\bigcup_{j=1}^{\infty} E_j \right) \right) + m^* \left(F \cap \left(\bigcup_{j=1}^{\infty} E_j \right)^c \right) \\
 &\geq m^*(F),
 \end{aligned} \quad (\text{Chapter 2.9})$$

where the final inequality is true by subadditivity of m^* and

$$F = \left(F \cap \left(\bigcup_{j=1}^{\infty} E_j \right) \right) \cup \left(F \cap \left(\bigcup_{j=1}^{\infty} E_j \right)^c \right).$$

So, all the inequalities in (Chapter 2.9) are in fact equalities. This implies that $\bigcup_{j=1}^{\infty} E_j \in \mathbf{M}(m^*)$. Consequently $\mathbf{M}(m^*)$ is a σ -algebra. This completes the proof.

Definition

The restriction of m^* to $\mathbf{M}(m^*)$ is said to be the Lebesgue Δ -measure and it is denoted by μ_{Δ} .

Theorem

The Lebesgue Δ -measure μ_Δ is a measure.

Proof

We will check all axioms for a measure.

1. Since $m^*(\emptyset) = 0$ and $\emptyset \in \mathbb{M}(m^*)$, we conclude that $\mu_\Delta(\emptyset) = 0$.
2. Since m^* is nonnegative, we get that μ_Δ is nonnegative.
3. Consider a countably many disjoint subsets $\{E_j\}_{j=1}^\infty$ of $\mathbb{M}(m^*)$. We have that $\mathbb{M}(m^*)$ is a σ -algebra and then $\bigcup_{j=1}^\infty E_j \in \mathbb{M}(m^*)$. Hence,

$$\begin{aligned}\mu_\Delta(F) &= \mu_\Delta\left(F \cap \left(\bigcup_{j=1}^\infty E_j\right)\right) + \mu_\Delta\left(F \cap \left(\bigcup_{j=1}^\infty E_j\right)^c\right) \\ &= \mu_\Delta\left(F \cap \left(\bigcup_{j=1}^\infty E_j\right)\right) + \mu_\Delta\left(F \cap \left(\bigcup_{j=1}^\infty E_j^c\right)\right) \\ &= \sum_{j=1}^\infty \mu_\Delta(F \cap E_j) + \mu_\Delta\left(F \cap \left(\bigcup_{j=1}^\infty E_j^c\right)\right)\end{aligned}$$

for any $F \in \mathcal{F}$, where we have used that $\{E_j\}_{j=1}^\infty$ are disjoint and the σ -additivity of m^* . By setting $F = \bigcup_{k=1}^\infty E_k$, we arrive at

$$\begin{aligned}\mu_\Delta\left(\bigcup_{j=1}^\infty E_j\right) &= \sum_{j=1}^\infty \mu_\Delta\left(\left(\bigcup_{k=1}^\infty E_k\right) \cap E_j\right) + \mu_\Delta\left(\left(\bigcup_{k=1}^\infty E_k\right) \cap \left(\bigcap_{j=1}^\infty E_j^c\right)\right) \\ &= \sum_{j=1}^\infty \mu_\Delta(E_j).\end{aligned}$$

Thus, μ_Δ satisfies all axioms for a norm. This completes the proof.

By the definition for a Lebesgue delta measurable set it follows that any interval in $\mathbb{T}$ and $\mathbb{T}$ are Lebesgue delta measurable sets.

Theorem

Let $\{E_j\}_{j \in \mathbb{N}}$ be a countable collection of sets of $\mathcal{F}$ such that $E_j \subset E_{j+1}$, $j \in \mathbb{N}$. Then

$$\mu_\Delta\left(\bigcup_{j=1}^\infty E_j\right) = \lim_{n \rightarrow \infty} \mu(E_n).$$

Proof

Define the sets

$$F_1 = E_1,$$

$$F_j = E_j \setminus E_{j-1}, \quad j \in \mathbb{N}, \quad j \geq 2.$$

Then the sets $\{F_j\}_{j \in \mathbb{N}}$ are pairwise disjoint. Moreover, for any $n \in \mathbb{N}$, one has

$$E_n = \bigcup_{j=1}^n F_j.$$

Hence,

$$\bigcup_{j=1}^\infty E_j = \bigcup_{j=1}^\infty F_j.$$

Since the sets $\{F_j\}_{j \in \mathbb{N}}$ are pairwise disjoint, we have the equality

$$\begin{aligned}\mu_{\Delta}(E_n) &= \mu_{\Delta}\left(\bigcup_{j=1}^n F_j\right) \\ &= \sum_{j=1}^n \mu_{\Delta}(F_j).\end{aligned}$$

By the σ -additivity of μ_{Δ} , we find

$$\begin{aligned}\mu_{\Delta}\left(\bigcup_{j=1}^{\infty} E_j\right) &= \mu_{\Delta}\left(\bigcup_{j=1}^{\infty} F_j\right) \\ &= \sum_{j=1}^{\infty} \mu_{\Delta}(F_j) \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu_{\Delta}(F_j) \\ &= \lim_{n \rightarrow \infty} \mu_{\Delta}(E_n).\end{aligned}$$

This completes the proof.

Theorem

Let $\{E_j\}_{j \in \mathbb{N}}$ be a countable collection of sets of $\mathcal{F}$ such that $E_{j+1} \subset E_j$, $j \in \mathbb{N}$. Then

$$\mu_{\Delta}\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{n \rightarrow \infty} \mu_{\Delta}(E_n). \quad (\text{Chapter 2.10})$$

Proof

For each $j \in \mathbb{N}$, define the sets

$$F_j = E_1 \setminus E_j.$$

Then $\{F_j\}_{j \in \mathbb{N}}$ is a sequence of sets of $\mathcal{F}$ such that $F_j \subset F_{j+1}$, $j \in \mathbb{N}$. Note that

$$\bigcup_{j=1}^{\infty} F_j = E_1 \setminus \bigcap_{j=1}^{\infty} E_j.$$

Then, we get

$$\begin{aligned}\lim_{n \rightarrow \infty} \mu_{\Delta}(F_n) &= \mu_{\Delta}\left(\bigcup_{j=1}^{\infty} F_j\right) \\ &= \mu_{\Delta}\left(E_1 \setminus \bigcap_{j=1}^{\infty} E_j\right).\end{aligned} \quad (\text{Chapter 2.11})$$

Let

$$Y = E_1 \setminus \bigcap_{j=1}^{\infty} E_j.$$

Then

$$E_1 = Y \cup \bigcap_{j=1}^{\infty} E_j.$$

Note that Y and $\bigcap_{j=1}^{\infty} E_j$ are disjoint. Then

$$\mu_{\Delta}(E_1) = \mu_{\Delta}(Y) + \mu_{\Delta}\left(\bigcap_{j=1}^{\infty} E_j\right),$$

whereupon

$$\mu_{\Delta} \left(E_1 \setminus \bigcap_{j=1}^{\infty} E_j \right) = \mu_{\Delta}(E_1) - \mu_{\Delta} \left(\bigcap_{j=1}^{\infty} E_j \right). \quad (\text{Chapter 2.12})$$

On the other hand, we have

$$E_1 = E_n \cup F_n, \quad n \in \mathbb{N},$$

and E_n and F_n are disjoint, $n \in \mathbb{N}$. Therefore

$$\begin{aligned} \mu_{\Delta}(E_1) &= \mu_{\Delta}(E_n \cup F_n) \\ &= \mu_{\Delta}(E_n) + \mu_{\Delta}(F_n), \quad n \in \mathbb{N}, \end{aligned}$$

and

$$\mu_{\Delta}(F_n) = \mu_{\Delta}(E_1) - \mu_{\Delta}(E_n), \quad n \in \mathbb{N}.$$

Hence and (Chapter 2.11), applying the last equality and (Chapter 2.12), we obtain

$$\mu_{\Delta}(E_1) - \lim_{n \rightarrow \infty} \mu_{\Delta}(E_n) = \mu_{\Delta}(E_1) - \mu_{\Delta} \left(\bigcap_{j=1}^{\infty} E_j \right),$$

from where we arrive at (Chapter 2.10). This completes the proof.

Theorem

Suppose that $\mathbf{T}$ is a bounded time scale with a finite maximum $\mathbf{Tau}_0$. Then the singleton-point set $\{\mathbf{Tau}_0\}$ and also any Lebesgue Δ -measurable subset of $\mathbf{T}$ containing $\mathbf{Tau}_0$ have Lebesgue Δ -measure infinity.

Proof

Note that the set $X = \mathbf{T} \setminus \{\mathbf{Tau}_0\}$ can be represented as a finite or countable union of intervals of the family $\mathcal{F}$. Therefore the set X is Lebesgue Δ -measurable. Since

$$\{\mathbf{Tau}_0\} = \mathbf{T} \setminus X,$$

we conclude that $\{t_0\}$ is Lebesgue Δ -measurable as the difference of the Lebesgue Δ -measurable sets $\mathbf{T}$ and X . Note that $\{\mathbf{Tau}_0\}$ does not have a finite or countable covering by intervals of $\mathcal{F}$. Therefore the singleton-point set $\{\mathbf{Tau}_0\}$ and also any Lebesgue Δ -measurable subset of $\mathbf{T}$ containing $\mathbf{Tau}_0$ have Lebesgue Δ -measure infinity. This completes the proof.

Theorem

For any $t_0 \in \mathbf{T}^*$, one has

$$\mu_{\Delta}(\{t_0\}) = \sigma(t_0) - t_0.$$

Proof

We will consider the following cases. 1. Let t_0 be right-scattered. Then

$$\{t_0\} = [t_0, \sigma(t_0)).$$

Hence,

$$\begin{aligned} \mu_{\Delta}(\{t_0\}) &= m([t_0, \sigma(t_0))) \\ &= \sigma(t_0) - t_0. \end{aligned}$$

2. Now, suppose that t_0 is right-dense. Then there is a decreasing sequence $(t_k)_{k \in \mathbb{N}}$ of elements of $\mathbf{T}$ such that $t_k \rightarrow t_0$ as $k \rightarrow \infty$ and $t_k > t_0$, $k \in \mathbb{N}$. We have

$$[t_0, t_1) \supset [t_0, t_2) \supset \dots \supset [t_0, t_k) \supset \dots$$

and

$$\{t_0\} = \bigcap_{k=1}^{\infty} [t_0, t_k).$$

Hence, $\{t_0\}$ is Lebesgue Δ -measurable as a countable intersection of Lebesgue Δ -measurable sets. Now, we get

$$\begin{aligned}\mu_{\Delta}(\{t_0\}) &= \mu_{\Delta}\left(\bigcap_{k=1}^{\infty} [t_0, t_k)\right) \\ &= \lim_{k \rightarrow \infty} \mu_{\Delta}([t_0, t_k)) \\ &= \lim_{k \rightarrow \infty} (t_k - t_0) \\ &= 0.\end{aligned}$$

This completes the proof.

Theorem

Let $a, b \in \mathbb{T}^*$. Then we have the following. 1. $\mu_{\Delta}([a, b]_{\mathbb{T}}) = b - a$. 2. $\mu_{\Delta}(a, b)_{\mathbb{T}} = b - \sigma(a)$. 3. $\mu_{\Delta}((a, b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$. 4. $\mu_{\Delta}([a, b]_{\mathbb{T}}) = \sigma(b) - a$.

Proof

1. The first formula is obvious because

$$\begin{aligned}\mu_{\Delta}([a, b]_{\mathbb{T}}) &= m([a, b]_{\mathbb{T}}) \\ &= b - a.\end{aligned}$$

2. Note that

$$[a, b]_{\mathbb{T}} = \{a\} \cup (a, b)_{\mathbb{T}}$$

and $\{a\}$ and $(a, b)_{\mathbb{T}}$ are disjoint. Then

$$\begin{aligned}b - a &= \mu_{\Delta}([a, b]_{\mathbb{T}}) \\ &= \mu_{\Delta}(\{a\} \cup (a, b)_{\mathbb{T}}) \\ &= \mu_{\Delta}(\{a\}) + \mu_{\Delta}((a, b)_{\mathbb{T}}) \\ &= \sigma(a) - a + \mu_{\Delta}((a, b)_{\mathbb{T}}),\end{aligned}$$

whereupon

$$\mu_{\Delta}((a, b)_{\mathbb{T}}) = b - \sigma(a).$$

3. Note that

$$(a, b]_{\mathbb{T}} = (a, b)_{\mathbb{T}} \cup \{b\}$$

and $(a, b)_{\mathbb{T}}$ and $\{b\}$ are disjoint. Then

$$\begin{aligned}\mu_{\Delta}((a, b]_{\mathbb{T}}) &= \mu_{\Delta}((a, b)_{\mathbb{T}} \cup \{b\}) \\ &= \mu_{\Delta}((a, b)_{\mathbb{T}}) + \mu_{\Delta}(\{b\}) \\ &= b - \sigma(a) + \sigma(b) - b \\ &= \sigma(b) - \sigma(a).\end{aligned}$$

4. Note that

$$[a, b]_{\mathbb{T}} = [a, b)_{\mathbb{T}} \cup \{b\}$$

and $[a, b)_{\mathbb{T}}$ and $\{b\}$ are disjoint. Then

$$\begin{aligned}\mu_{\Delta}([a, b]_{\mathbb{T}}) &= \mu_{\Delta}([a, b)_{\mathbb{T}} \cup \{b\}) \\ &= \mu_{\Delta}([a, b)_{\mathbb{T}}) + \mu_{\Delta}(\{b\}) \\ &= b - a + \sigma(b) - b \\ &= \sigma(b) - a.\end{aligned}$$

This completes the proof.

✓ Example

Let $\mathbf{T} = \mathbb{Z}$ and $a = -5$, $b = 7$. Here

$$\sigma(t) = t + 1, \quad t \in \mathbf{T}.$$

Then

$$\begin{aligned}\sigma(-5) &= -5 + 1 \\ &= -4, \\ \sigma(7) &= 7 + 1 \\ &= 8.\end{aligned}$$

Hence,

$$\begin{aligned}\mu_{\Delta}([-5, 7)_{\mathbf{T}}) &= 7 - (-5) \\ &= 7 + 5 \\ &= 12, \\ \mu_{\Delta}([-5, 7]_{\mathbf{T}}) &= \sigma(7) - (-5) \\ &= 8 + 5 \\ &= 13, \\ \mu_{\Delta}((-5, 7)_{\mathbf{T}}) &= 7 - \sigma(-5) \\ &= 7 - (-4) \\ &= 7 + 4 \\ &= 11\end{aligned}$$

and

$$\begin{aligned}\mu_{\Delta}((-5, 7]_{\mathbf{T}}) &= \sigma(7) - \sigma(-5) \\ &= 8 - (-4) \\ &= 8 + 4 \\ &= 12.\end{aligned}$$

✓ Example

Let $\mathbf{T} = 4^{\mathbb{N}_0}$ and $a = 1$, $b = 256$. Here

$$\sigma(t) = 4t, \quad t \in \mathbf{T}.$$

Then

$$\begin{aligned}\sigma(1) &= 4 \cdot 1 \\ &= 4, \\ \sigma(256) &= 4 \cdot 256 \\ &= 1024.\end{aligned}$$

Hence,

$$\begin{aligned}\mu_{\Delta}([1, 256)_{\mathbf{T}}) &= 256 - 1 \\ &= 255, \\ \mu_{\Delta}([1, 256]_{\mathbf{T}}) &= \sigma(256) - 1 \\ &= 1024 - 1 \\ &= 1023, \\ \mu_{\Delta}((1, 256)_{\mathbf{T}}) &= 256 - \sigma(1) \\ &= 255\end{aligned}$$

and

$$\begin{aligned}\mu_{\Delta}((1, 256]_{\mathbf{T}}) &= \sigma(256) - \sigma(1) \\ &= 1024 - 4 \\ &= 1020.\end{aligned}$$

✓ Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2} \right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$ and $a = \frac{1}{256}$, $b = \frac{1}{2}$. Here

$$\sigma(t) = \begin{cases} \sqrt{t} & \text{if } t \in \left\{ \left(\frac{1}{2} \right)^{2^n} : n \in \mathbb{N} \right\} \\ 1 & \text{if } t = \frac{1}{2} \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t = 1. \end{cases}$$

Then

$$\begin{aligned} \sigma\left(\frac{1}{256}\right) &= \sqrt{\frac{1}{256}} \\ &= \frac{1}{16}, \\ \sigma\left(\frac{1}{2}\right) &= 1. \end{aligned}$$

Hence,

$$\begin{aligned} \mu_{\Delta}\left(\left[\frac{1}{256}, \frac{1}{2}\right]_{\mathbb{T}}\right) &= \frac{1}{2} - \frac{1}{256} \\ &= \frac{128-1}{256} \\ &= \frac{127}{256}, \\ \mu_{\Delta}\left(\left[\frac{1}{256}, \frac{1}{2}\right]_{\mathbb{T}}\right) &= \sigma\left(\frac{1}{2}\right) - \frac{1}{256} \\ &= 1 - \frac{1}{256} \\ &= \frac{256-1}{256} \\ &= \frac{255}{256}, \\ \mu_{\Delta}\left(\left(\frac{1}{256}, \frac{1}{2}\right)_{\mathbb{T}}\right) &= \frac{1}{2} - \sigma\left(\frac{1}{256}\right) \\ &= \frac{1}{2} - \frac{1}{16} \\ &= \frac{8-1}{16} \\ &= \frac{7}{16} \end{aligned}$$

and

$$\begin{aligned} \mu_{\Delta}\left(\left(\frac{1}{256}, \frac{1}{2}\right]_{\mathbb{T}}\right) &= \sigma\left(\frac{1}{2}\right) - \sigma\left(\frac{1}{256}\right) \\ &= 1 - \frac{1}{16} \\ &= \frac{16-1}{16} \\ &= \frac{15}{16}. \end{aligned}$$

✓ Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N}_0 \right\}$ and $\mathbb{T} = U \cup (1 - U) \cup \{1\}$, and $a = \frac{1}{32}$, $b = \frac{7}{8}$. Here

$$\sigma(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{3}{4} & \text{if } t = \frac{1}{2} \\ 2t & \text{if } t \in U \setminus \left\{ \frac{1}{2} \right\} \\ \frac{1+t}{2} & \text{if } t \in (1 - U) \setminus \left\{ \frac{1}{2} \right\} \\ 1 & \text{if } t = 1. \end{cases}$$

Then

$$\begin{aligned}\sigma\left(\frac{1}{32}\right) &= 2 \cdot \frac{1}{32} \\ &= \frac{1}{16}, \\ \sigma\left(\frac{7}{8}\right) &= \frac{1 + \frac{7}{8}}{2} \\ &= \frac{7+8}{2 \cdot 8} \\ &= \frac{15}{16}.\end{aligned}$$

Hence,

$$\begin{aligned}\mu_{\delta}\left(\left[\frac{1}{32}, \frac{7}{8}\right]_{\mathbf{T}}\right) &= \frac{7}{8} - \frac{1}{32} \\ &= \frac{28-1}{32} \\ &= \frac{27}{32}, \\ \mu_{\Delta}\left(\left[\frac{1}{32}, \frac{7}{8}\right]_{\mathbf{T}}\right) &= \sigma\left(\frac{7}{8}\right) - \frac{1}{32} \\ &= \frac{15}{16} - \frac{1}{32} \\ &= \frac{30-1}{32} \\ &= \frac{29}{32}, \\ \mu_{\Delta}\left(\left(\frac{1}{32}, \frac{7}{8}\right)_{\mathbf{T}}\right) &= \frac{7}{8} - \sigma\left(\frac{1}{32}\right) \\ &= \frac{7}{8} - \frac{1}{16} \\ &= \frac{14-1}{16} \\ &= \frac{13}{16}\end{aligned}$$

and

$$\begin{aligned}\mu_{\Delta}\left(\left(\frac{1}{32}, \frac{7}{8}\right]_{\mathbf{T}}\right) &= \sigma\left(\frac{7}{8}\right) - \sigma\left(\frac{1}{32}\right) \\ &= \frac{15}{16} - \frac{1}{16} \\ &= \frac{14}{16} \\ &= \frac{7}{8}.\end{aligned}$$

? Exercise

Let $\mathbf{T} = h\mathbb{Z} + k$, where $h > 0$ and $k \in \mathbb{R}$. Find

$$\mu_{\Delta}([-4h + k, 10h + k]_{\mathbf{T}}).$$

? Exercise

Let $\mathbf{T} = (-2\mathbb{N}_0) \cup 3\mathbb{N}_0$. Find

$$\mu_{\Delta}((-10, 27)_{\mathbf{T}}).$$

? Exercise

Let $\mathbb{T} = \{0\} \cup 11^{\mathbb{N}_0}$. Find

$$\mu_{\Delta}([0, 121]_{\mathbb{T}}).$$

? Exercise

Let $\mathbb{T} = [1, 2] \cup [3, 4] \cup [7, 8] \cup 9^{\mathbb{N}}$. Find

$$\mu_{\Delta}([1, 8]_{\mathbb{T}}).$$

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Chapter 3: Lebesgue Delta Measurable Functions

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Now, we proceed to define functions which we want to integrate. These functions are called Lebesgue delta measurable functions, which we define as follows.

Definition

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be Lebesgue delta measurable if the preimage of any $E \in \mathcal{B}$ under f is in $\mathbb{M}(m^*)$, namely

$$f^{-1}(E) \in \mathbb{M}(m^*)$$

Note

The Lebesgue delta measurable functions do not require measures but just $\mathbb{M}(m^*)$ as the domain.

Theorem

Let $A \subset \mathcal{F}$. Then the indication function $\mathbf{1}_A : \mathbb{T} \rightarrow \mathbb{R}$ is Lebesgue delta measurable if and only if $A \in \mathbb{M}(m^*)$. Here $\mathcal{B}$ is the Borel σ -algebra generated by the open sets in $\mathbb{R}$.

Proof

Note that for any Borel set $E \in \mathcal{B}$ we have the following cases 1. $0, 1 \in E$, 2. $0, 1 \notin E$, 3. $0 \in E, 1 \notin E$, 4. $0 \notin E, 1 \in E$. Therefore $f^{-1}(E)$ is either $\mathbb{T}$, $\emptyset$, $\mathbb{T} \setminus A$, or A . Thus, the assertion follows.

Theorem

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be Lebesgue delta measurable and $g : \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable. Then the composition $g \circ f : \mathbb{T} \rightarrow \mathbb{R}$ is Lebesgue delta measurable.

Proof

For any Borel measurable set $B \in \mathcal{B}$, we have $g^{-1}(B) \in \mathcal{B}$. Hence,

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)) \in \mathbb{M}(m^*).$$

This completes the proof.

Theorem

Suppose that $f : \mathbb{T} \rightarrow \mathbb{R}$. The following are equivalent. 1. The function f is Lebesgue delta integrable. 2. $f^{-1}(I) \in \mathbb{M}(m^*)$ for any open interval $I \subset \mathbb{R}$. 3. $f^{-1}(I_a) \in \mathbb{M}(m^*)$ for any set I_a of the form $I_a = (-\infty, a]$, where $a \in \mathbb{R}$. 4. $f^{-1}(J_a) \in \mathbb{M}(m^*)$ for any set J_a of the form $J_a = (-\infty, a)$, where $a \in \mathbb{R}$. 5. $f^{-1}(K_a) \in \mathbb{M}(m^*)$ for any set K_a of the form $K_a = [a, \infty)$, where $a \in \mathbb{R}$. 6. $f^{-1}(L_a) \in \mathbb{M}(m^*)$ for any set L_a of the form $L_a = (a, \infty)$, where $a \in \mathbb{R}$.

Proof

We prove only the equivalence of some of the statements above which can then be combined together to give the full equivalence. **(1) $\implies$ (2)** By the definition for Lebesgue delta measurable functions, we get that $f^{-1}(I) \in \mathbb{M}(m^*)$ for any open interval $I \subset \mathbb{R}$. **(2) $\implies$ (1)** By the fact that the Borel σ -algebra is generated by the open sets of $\mathbb{R}$ and the preimage operation preserves countable unions, intersections, and complements, we get that $f^{-1}(A) \in \mathbb{M}(m^*)$ for any $A \in \mathcal{B}$. Now, applying the

definition for Lebesgue delta measurable functions, we conclude that the function f is Lebesgue delta measurable. **(2) $\implies$ (3)**
 For a fixed $a \in \mathbb{R}$, define the open intervals

$$I_n = \left(-\infty, a + \frac{1}{n}\right).$$

Then, the set $(-\infty, a]$ can be written as the countable intersection

$$(-\infty, a] = \bigcap_{n=1}^{\infty} I_n.$$

Hence, we have

$$\begin{aligned} f^{-1}((-\infty, a]) &= f^{-1}\left(\bigcap_{n=1}^{\infty} I_n\right) \\ &= \bigcap_{n=1}^{\infty} f^{-1}(I_n) \\ &\subset \mathbb{M}(m^*). \end{aligned}$$

(3) $\implies$ (2) We first show that we can produce any open interval (a, b) for $a, b \in \mathbb{R}$ with $a < b$ from these half-infinity intervals by countable unions, intersection, and complements. For finite $a, b \in \mathbb{R}$, we have that

$$\begin{aligned} (a, b] &= (-\infty, b] \setminus (-\infty, a], \\ \{b\} &= \bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, b\right] \\ &= \bigcap \left((-\infty, b] \setminus \left(-\infty, b - \frac{1}{n}\right]\right). \end{aligned}$$

Therefore, the set $(a, b) = (a, b] \setminus \{b\}$ can be made up of countable unions, intersections, and complements of intervals of the form $(-\infty, a)$. Next, the intervals $(-\infty, a)$ and (a, ∞) can be represented as follows

$$(-\infty, a) = (\dots \cup (a-3, a-2] \cup (a-2, a-1]) \cup (a-1, a)$$

and

$$(a, \infty) = (a, a+1) \cup (a+1, a+2) \cup (a+2, a+3) \cup \dots.$$

Thus, by using the fact that the inverse functions preserve unions, intersections, and complements, we conclude the result.

(2) $\implies$ (4) This is trivially true since $(-\infty, a)$ are open intervals in $\mathbb{R}$. **(4) $\implies$ (2)** We need to show that for any $a, b \in \mathbb{R}$ with $a < b$, the interval (a, b) can be constructed by countable intersections. For finite $a, b \in \mathbb{R}$ with $a < b$, we note that

$$[a, b) = (-\infty, b) \setminus (-\infty, a)$$

and

$$\begin{aligned} \{a\} &= \bigcap_{n=1}^{\infty} \left[a, a + \frac{1}{n}\right) \\ &= \bigcap_{n=1}^{\infty} \left(\left(-\infty, a + \frac{1}{n}\right) \setminus (-\infty, a)\right). \end{aligned}$$

Therefore, the set

$$(a, b) = (-\infty, b) \setminus (-\infty, a) \cap \{a\}^c$$

is made up of countable unions, intersections, and complements of intervals of the forms $(-\infty, a)$ and (a, ∞) . Thus, by using the fact that the inverse functions preserve intersections, we conclude the result. **(3) $\iff$ (6)**

Note that for any $a \in \mathbb{R}$, we have $(a, \infty) = (-\infty, a]^c$. Thus,

$$\begin{aligned} f^{-1}((a, \infty)) &= f^{-1}((-\infty, a]^c) \\ &= \mathbb{T} \setminus f^{-1}((-\infty, a]). \end{aligned}$$

This implies that $f^{-1}((-\infty, a)) \in \mathbb{M}(m^*)$ for any $a \in \mathbb{R}$ if and only if $f^{-1}((-\infty, a]) \in \mathbb{M}(m^*)$ for any $a \in \mathbb{R}$. (4) $\iff$ (5) Note that for any $a \in \mathbb{R}$, we have

$$[a, \infty) = (-\infty, a)^c.$$

Hence,

$$\begin{aligned} f^{-1}([a, \infty)) &= f^{-1}((-\infty, a)^c) \\ &= \mathbb{T} \setminus f^{-1}((-\infty, a)). \end{aligned}$$

From here, we get that $f^{-1}((-\infty, a)) \in \mathbb{M}(m^*)$ for any $a \in \mathbb{R}$ if and only if $f^{-1}([a, \infty)) \in \mathbb{M}(m^*)$. This completes the proof. $\square$

Using the above characterization, we show that the Lebesgue delta measurable functions are preserved under many algebraic operations.

Theorem

Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be Lebesgue delta measurable functions. Then so are the functions 1. λf for some constant $\lambda \in \mathbb{R}$. 2. $f \pm g$. 3. fg . 4. $\frac{f}{g}$ defined on $\{t \in \mathbb{T} : \frac{1}{g(t)} \neq 0\}$. 5. $\max\{f, g\}$ and $\min\{f, g\}$. 6. f^+ , f^- and $|f|$, where

$$\begin{aligned} f^+ &= \max\{f, 0\}, \\ f^- &= \max\{-f, 0\}. \end{aligned}$$

Proof

1. If $\lambda = 0$, then there is nothing to check. Suppose that $\lambda > 0$. Since $f : \mathbb{T} \rightarrow \mathbb{R}$ is Lebesgue delta measurable, then all preimages of the intervals $(-\infty, b)$ for any $b \in \mathbb{R}$ are contained in $\mathbb{M}(m^*)$. In other words, we know that the sets

$$\{t \in \mathbb{T} : f(t) < b\}$$

lie in $\mathbb{M}(m^*)$ for any $b \in \mathbb{R}$. To show that $\lambda f : \mathbb{T} \rightarrow \mathbb{R}$ is Lebesgue delta measurable, we need to show that

$$\{t \in \mathbb{T} : \lambda f(t) < c\} \subset \mathbb{M}(m^*)$$

for any $c \in \mathbb{R}$. However, for each $c \in \mathbb{R}$, this is simply

$$\begin{aligned} \{t \in \mathbb{T} : \lambda f(t) < c\} &= \left\{t \in \mathbb{T} : f(t) < \frac{c}{\lambda}\right\} \\ &= \{t \in \mathbb{T} : f(t) < b\} \quad \text{for } b = \frac{c}{\lambda} \in \mathbb{R}, \end{aligned}$$

which we know lies in $\mathbb{M}(m^*)$. Finally, for the case $\lambda < 0$, we have

$$\{t \in \mathbb{T} : \lambda f(t) < c\} = \left\{t \in \mathbb{T} : f(t) > \frac{c}{\lambda}\right\},$$

which is also contained in $\mathbb{M}(m^*)$. 2. We need to check that for all $c \in \mathbb{R}$, the sets

$$\{t \in \mathbb{T} : f(t) + g(t) > c\}$$

lie in $\mathbb{M}(m^*)$. Observe that

$$\{t \in \mathbb{T} : f(t) + g(t) > c\} = \bigcup_{q \in \mathbb{Q}} (\{t \in \mathbb{T} : f(t) > q\} \cap \{t \in \mathbb{T} : g(t) > c - q\}).$$

Since $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are Lebesgue delta measurable we know that the sets

$$\{t \in \mathbb{T} : f(t) > q\} \quad \text{and} \quad \{t \in \mathbb{T} : g(t) > c - q\}$$

lie in $\mathbb{M}(m^*)$ for each $c \in \mathbb{R}$ and $q \in \mathbb{Q}$. Since $\mathbb{M}(m^*)$ is a σ -algebra, this countable union is also in $\mathbb{M}(m^*)$. Similarly, we have

$$\begin{aligned} \{t \in \mathbb{T} : f(t) - g(t) > c\} &= \bigcup_{q \in \mathbb{Q}} (\{t \in \mathbb{T} : f(t) > q\} \cap \{t \in \mathbb{T} : g(t) < q - c\}) \\ &\subset \mathbb{M}(m^*). \end{aligned}$$

So, the function $f - g$ is also Lebesgue delta measurable. 3. To show that $f g : \mathbb{T} \rightarrow \mathbb{R}$ is Lebesgue delta measurable, firstly we will show that the function $f^2 : \mathbb{T} \rightarrow \mathbb{R}$ is Lebesgue delta measurable. This is equivalent to showing that the set

$$\{t \in \mathbb{T} : f(t)^2 < c\}$$

is in $\mathbb{M}(m^*)$ for any $c \in \mathbb{R}$. We split this into two cases. (a) For $c < 0$, we get

$$\begin{aligned} \{t \in \mathbb{T} : f(t)^2 < c\} &= \emptyset \\ &\subset \mathbb{M}(m^*). \end{aligned}$$

(b) For $c \geq 0$, we find

$$\begin{aligned} \{t \in \mathbb{T} : f(t)^2 < c\} &= \{t \in \mathbb{T} : (f(t) - \sqrt{c})(f(t) + \sqrt{c}) < 0\} \\ &= (\{t \in \mathbb{T} : f(t) > -\sqrt{c}\} \cap \{t \in \mathbb{T} : f(t) < \sqrt{c}\}) \\ &\quad \cup (\{t \in \mathbb{T} : f(t) < -\sqrt{c}\} \cap \{t \in \mathbb{T} : f(t) > \sqrt{c}\}), \end{aligned}$$

which lies in $\mathbb{M}(m^*)$ since f is Lebesgue delta measurable.

To show that $f g : \mathbb{T} \rightarrow \mathbb{R}$ is Lebesgue delta measurable, we note that

$$f g = \frac{1}{4} ((f + g)^2 - (f - g)^2).$$

Since f and g are Lebesgue delta measurable, necessarily $f \pm g$ are Lebesgue delta measurable by the second assertion. Hence, $\frac{1}{4}(f \pm g)^2$ are Lebesgue delta measurable by the above and the first assertion. Finally, we conclude that their difference, which is $f g$, is also Lebesgue delta measurable by the second assertion.

4. To show that $\frac{f}{g} : \{t \in \mathbb{T} : g(t) \neq 0\}$ is Lebesgue delta measurable, it suffices to show that the function $\frac{1}{g} : \{t \in \mathbb{T} : g(t) \neq 0\} \rightarrow \mathbb{R}$ is Lebesgue delta measurable. We check for different cases of the value c .

(a) For $c > 0$, we have

$$\left\{t \in \mathbb{T} : \frac{1}{g(t)} < c\right\} = \left\{t \in \mathbb{T} : \frac{1}{c} < g(t)\right\} \cap \{t \in \mathbb{T} : g(t) < 0\} \subset \mathbb{M}(m^*).$$

(b) For $c = 0$, we have

$$\left\{t \in \mathbb{T} : \frac{1}{g(t)} < 0\right\} = \{t \in \mathbb{T} : g(t) < 0\} \subset \mathbb{M}(m^*).$$

(c) For $c > 0$, we get

$$\left\{t \in \mathbb{T} : \frac{1}{g(t)} < c\right\} = \left\{t \in \mathbb{T} : \frac{1}{c} < g(t)\right\} \cup \{t \in \mathbb{T} : g(t) < 0\} \subset \mathbb{M}(m^*).$$

Thus, the function $\frac{1}{g}$ is also Lebesgue delta measurable. Since the product of Lebesgue delta measurable functions is also Lebesgue delta measurable, by the third assertion, we conclude that $\frac{f}{g}$ is also Lebesgue delta measurable.

5. Note that

$$\{t \in \mathbb{T} : \max\{f, g\}(t) < c\} = \{t \in \mathbb{T} : f(t) < c\} \cap \{t \in \mathbb{T} : g(t) < c\}$$

and

$$\{t \in \mathbb{T} : \min\{f, g\}(t) > c\} = \{t \in \mathbb{T} : f(t) > c\} \cap \{t \in \mathbb{T} : g(t) > c\}$$

for any $c \in \mathbb{R}$. Thus, the functions $\max\{f, g\}$ and $\min\{f, g\}$ are Lebesgue delta measurable.

6. Since 0 is Lebesgue delta measurable, by the previous assertion, we conclude that f^+ and f^- are Lebesgue delta measurable. Next,

$$\{t \in \mathbb{T} : |f(t)| < c\} = \{t \in \mathbb{T} : f(t) < c\} \cap \{t \in \mathbb{T} : f(t) > -c\},$$

whereupon we conclude that $|f|$ is Lebesgue delta measurable. This completes the proof.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We can extend the definition for Lebesgue delta measurable functions to functions with images including $\pm\infty$.

Definition

A function $f : \mathbb{T} \rightarrow \overline{\mathbb{R}}$ is said to be Lebesgue delta measurable if

1. for any $E \subset \mathcal{B}$, its preimage under f is in $\mathbb{M}(m^*)$, that is $f^{-1}(E) \subset \mathbb{M}(m^*)$, and
2. the preimage sets

$$f^{-1}(\{\infty\}) = \{t \in \mathbb{T} : f(t) = \infty\}$$

and

$$f^{-1}(\{-\infty\}) = \{t \in \mathbb{T} : f(t) = -\infty\}$$

lie in $\mathbb{M}(m^*)$.

Using the above definition, we have a way to check this.

Theorem

Suppose that $f : \mathbb{T} \rightarrow \overline{\mathbb{R}}$. Then the following are equivalent.

1. The function f is Lebesgue delta measurable.
2. $f^{-1}(I_a) \subset \mathbb{M}(m^*)$ for any set I_a of the form

$$I_a = [-\infty, a], \quad \text{where } a \in \mathbb{R}.$$

Proof

(1) $\Rightarrow$ (2)

By assumption, we have that $f : \mathbb{T} \rightarrow \overline{\mathbb{R}}$ is Lebesgue delta measurable. Then, we have $f^{-1}(\{-\infty\}) \subset \mathbb{M}(m^*)$. Since $(-\infty, a] \subset \mathcal{B}$, we get that $f^{-1}((-\infty, a]) \subset \mathbb{M}(m^*)$. Hence,

$$\begin{aligned} f^{-1}([-\infty, a]) &= f^{-1}(\{-\infty\} \cup (-\infty, a]) \\ &= f^{-1}(\{-\infty\}) \cup f^{-1}((-\infty, a]) \\ &\subset \mathbb{M}(m^*). \end{aligned}$$

(2) $\Rightarrow$ (1)

By assumption, we have $f^{-1}([-\infty, a]) \subset \mathbb{M}(m^*)$. We first show that $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \subset \mathbb{M}(m^*)$.

(a) Note that

$$\{-\infty\} = \bigcap_{n \in \mathbb{Z}} I_n.$$

Then

$$\begin{aligned} f^{-1}(\{-\infty\}) &= f^{-1}\left(\bigcap_{n \in \mathbb{Z}} I_n\right) \\ &= \bigcap_{n \in \mathbb{Z}} f^{-1}(I_n) \\ &\subset \mathbb{M}(m^*). \end{aligned}$$

Now, using De Morgans law, we have

$$\begin{aligned}\{\infty\} &= \overline{\mathbb{R}} \setminus \bigcup_{n \in \mathbb{Z}} I_n \\ &= \bigcap_{n \in \mathbb{Z}} I_n^c.\end{aligned}$$

Hence,

$$\begin{aligned}f^{-1}(\{\infty\}) &= f^{-1}\left(\bigcap_{n \in \mathbb{Z}} I_n^c\right) \\ &= \bigcap_{n \in \mathbb{Z}} f^{-1}(I_n^c) \\ &= \bigcap_{n \in \mathbb{Z}} (\mathbb{T} \setminus f^{-1}(I_n)) \\ &= \mathbb{T} \setminus \bigcup_{n \in \mathbb{Z}} f^{-1}(I_n) \\ &\subset \mathbb{M}(m^*).\end{aligned}$$

Note that

$$(-\infty, a] = [-\infty, a] \setminus \{-\infty\}.$$

Then

$$\begin{aligned}f^{-1}((-\infty, a]) &= f^{-1}([- \infty, a] \setminus \{-\infty\}) \\ &= f^{-1}([- \infty, a]) \setminus f^{-1}(\{-\infty\}) \\ &\subset \mathbb{M}(m^*).\end{aligned}$$

From here, we get that for any $A \subset \mathcal{B}$ one has $f^{-1}(A) \subset \mathbb{M}(m^*)$. Therefore f is Lebesgue delta measurable. This completes the proof. \end{proof}

Theorem

Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of Lebesgue delta measurable functions $f_n : \mathbb{T} \rightarrow \overline{\mathbb{R}}$. Then the following functions on $\mathbb{T}$ are Lebesgue delta measurable.

1. $\sup_{n \in \mathbb{N}} f_n$ and $\inf_{n \in \mathbb{N}} f_n$.
2. $\limsup_{n \rightarrow \infty} f_n$ and $\lim_{n \rightarrow \infty} \inf_{n \rightarrow \infty} f_n$.

Proof

1. Note that

$$\begin{aligned}\{t \in \mathbb{T} : \sup_{n \in \mathbb{N}} f_n(t) \leq c\} &= \bigcap_{n \in \mathbb{N}} \{t \in \mathbb{T} : f_n(t) \leq c\} \\ &\subset \mathbb{M}(m^*) \quad \text{for any } c \in \mathbb{R}\end{aligned}$$

and

$$\begin{aligned}\{t \in \mathbb{T} : \inf_{n \in \mathbb{N}} f_n(t) \leq c\} &= \bigcup_{n \in \mathbb{N}} \{t \in \mathbb{T} : f_n(t) \leq c\} \\ &\subset \mathbb{M}(m^*) \quad \text{for any } c \in \mathbb{R}.\end{aligned}$$

Therefore $\sup_{n \in \mathbb{N}} f_n$ and $\inf_{n \in \mathbb{N}} f_n$ are Lebesgue delta measurable.

2. Define the sequences $\{\sup_{m \geq n} f_m(t)\}_{n \in \mathbb{N}}$ and $\{\inf_{m \geq n} f_m(t)\}_{n \in \mathbb{N}}$ for $t \in \mathbb{T}$. By the first assertion, we have that the functions $\sup_{m \geq n} f_m(t)$ and $\inf_{m \geq n} f_m(t)$, $n \in \mathbb{N}$, are Lebesgue delta measurable for any $n \in \mathbb{N}$. Furthermore, since

$$\begin{aligned}\limsup_{n \rightarrow \infty} f_n(t) &= \inf_{n \in \mathbb{N}} \left(\sup_{m \geq n} f_m(t) \right), \\ \liminf_{n \rightarrow \infty} f_n(t) &= \sup_{n \in \mathbb{N}} \left(\inf_{m \geq n} f_m(t) \right), \quad t \in \mathbb{T},\end{aligned}$$

applying the first assertion, we get that $\limsup_{n \rightarrow \infty} f_n$ and $\liminf_{n \rightarrow \infty} f_n$ are Lebesgue delta measurable. This completes the proof.

\end{proof}

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Chapter 4: Lebesgue Delta Integration of Simple Functions

Theorem *Chapter4.1*

Theorem text

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After seeing how we can define a valid Lebesgue delta measure, we are now ready to approximate the Lebesgue delta measurable functions by an analogue of step functions. These functions are called simple functions.

Definition

A function $\phi : \mathbb{T} \rightarrow \mathbb{R}$ is called a simple function if there exists an $n \in \mathbb{N}$ such that for any $j \in \{1, 2, \dots, n\}$ there are constants $c_j \in \mathbb{R}$ and Lebesgue delta measurable sets $E_j \subset \mathbb{M}(m^*)$ such that

$$\phi(t) = \sum_{j=1}^n c_j \mathbf{1}_{E_j}(t), \quad t \in \mathbb{T},$$

where $\mathbf{1}_{E_j} : \mathbb{T} \rightarrow \mathbb{R}$ are the indicator functions on the sets $E_j, j \in \{1, 2, \dots, n\}$.

In other words, a simple function is a Lebesgue delta measurable function that attains only finitely many values of $\mathbb{R}$. Clearly, finite sums and scalar multiples of a simple function is also a simple function. Without loss of generality, assume that the sets $E_j, j \in \{1, 2, \dots, n\}$, are pairwise disjoint.

Theorem

Let $E, F \subset \mathbb{M}(m^*)$ and $\mathbf{1}_E, \mathbf{1}_F : \mathbb{T} \rightarrow \mathbb{R}$ be indicator functions on these sets. Then we have the following.

1. $\mathbf{1}_E \cdot \mathbf{1}_F = \mathbf{1}_{E \cap F}$.
2. $\mathbf{1}_E + \mathbf{1}_F = \mathbf{1}_{E \cup F} + \mathbf{1}_{E \cap F}$.
3. $|\mathbf{1}_E - \mathbf{1}_F| = \mathbf{1}_{E \Delta F}$.

Proof

1. Let $t \in \mathbb{T}$ be arbitrarily chosen. We have the following cases.

(a) $t \in E, t \in F$. Then $t \in E \cap F$ and

$$\begin{aligned} \mathbf{1}_E(t) \cdot \mathbf{1}_F(t) &= 1 \cdot 1 \\ &= 1, \\ \mathbf{1}_{E \cap F}(t) &= 1. \end{aligned}$$

(b) $t \in E, t \notin F$. Then $t \notin E \cap F$ and

$$\begin{aligned} \mathbf{1}_E(t) \cdot \mathbf{1}_F(t) &= 1 \cdot 0 \\ &= 0, \\ \mathbf{1}_{E \cap F}(t) &= 0. \end{aligned}$$

(c) $t \notin E, t \in F$. Then $t \notin E \cap F$ and

$$\begin{aligned} \mathbf{1}_E(t) \cdot \mathbf{1}_F(t) &= 0 \cdot 1 \\ &= 0, \\ \mathbf{1}_{E \cap F}(t) &= 0. \end{aligned}$$

(d) $t \notin E, t \notin F$. Then $t \notin E \cap F$ and

$$\mathbf{1}_{E \cap F}(t) = 0.$$

$$\begin{aligned} 1_E(t) \cdot 1_F(t) &= 0 \cdot 0 \\ &= 0, \\ 1_{E \cap F}(t) &= 0. \end{aligned}$$

2. Let $t \in \mathbb{T}$ be arbitrarily chosen. We have the following cases.

(a) $t \in E, t \in F$. Then $t \in E \cup F$ and $t \in E \cap F$. Hence,

$$\begin{aligned} 1(t) + 1_F(t) &= 1 + 1 \\ &= 2, \\ 1_{E \cup F}(t) + 1_{E \cap F}(t) &= 1 + 1 \\ &= 2. \end{aligned}$$

(b) $t \in E, t \notin F$. Then $t \in E \cup F$ and $t \notin E \cap F$. Hence,

$$\begin{aligned} 1(t) + 1_F(t) &= 1 + 0 \\ &= 1, \\ 1_{E \cup F}(t) + 1_{E \cap F}(t) &= 1 + 0 \\ &= 1. \end{aligned}$$

(c) $t \notin E, t \in F$. Then $t \in E \cup F$ and $t \notin E \cap F$. Hence,

$$\begin{aligned} 1_E(t) + 1_F(t) &= 0 + 1 \\ &= 1, \\ 1_{E \cup F}(t) + 1_{E \cap F}(t) &= 1 + 0 \\ &= 1. \end{aligned}$$

(d) $t \notin E, t \notin F$. Then $t \notin E \cup F, t \notin E \cap F$. Hence,

$$\begin{aligned} 1_E(t) + 1_F(t) &= 0 + 0 \\ &= 0, \\ 1_{E \cup F}(t) + 1_{E \cap F}(t) &= 0 + 0 \\ &= 0. \end{aligned}$$

3. Let $t \in \mathbb{T}$ be arbitrarily chosen. We have the following cases.

(a) $t \in E, t \in F$. Then $t \in E \cup F$ and $t \in E \cap F$. Hence, $t \notin E \Delta F$ and

$$\begin{aligned} |1_E(t) - 1_F(t)| &= |1 - 1| \\ &= 0, \\ 1_{E \Delta F}(t) &= 0. \end{aligned}$$

(b) $t \in E, t \notin F$. Then $t \in E \cup F$ and $t \notin E \cap F$. Hence, $t \in E \Delta F$ and

$$\begin{aligned} |1_E(t) - 1_F(t)| &= |1 - 0| \\ &= 1, \\ 1_{E \Delta F} &= 1. \end{aligned}$$

(c) $t \notin E, t \in F$. Then $t \in E \cup F$ and $t \notin E \cap F$. Hence,

$$\begin{aligned} |1_E(t) - 1_F(t)| &= |0 - 1| \\ &= 1, \\ 1_{E \Delta F} &= 1. \end{aligned}$$

(d) $t \notin E, t \notin F$. Then $t \notin E \cup F$ and $t \notin E \cap F$. Hence, $t \notin E \Delta F$ and

$$\begin{aligned} |1_E(t) - 1_F(t)| &= |1 - 1| \\ &= 0, \\ 1_{E \Delta F} &= 0. \end{aligned}$$

This completes the proof.

The next question is: how do we approximate a nonnegative Lebesgue delta measurable function using simple functions? In this regard, we have the following result.

Theorem

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a nonnegative Lebesgue delta measurable function. Then, there is a pointwise increasing sequence $\{f_n\}_{n \in \mathbb{N}}$ of simple functions $f_n : \mathbb{T} \rightarrow \mathbb{R}$ such that $f_n \rightarrow f$ as $n \rightarrow \infty$ pointwise.

Proof

For each $n \in \mathbb{N}$, define

$$f_n(t) = \sum_{j=0}^{2^n-1} \frac{j}{2^n} \mathbf{1}_{E_{n,j}}(t) + 2^n \mathbf{1}_{A_n}(t), \quad t \in \mathbb{T},$$

where

$$A_n = \{t \in \mathbb{T} : f(t) \geq 2^n\}$$

and

$$E_{n,j} = \left\{ t \in \mathbb{T} : \frac{j}{2^n} \leq f(t) < \frac{j+1}{2^n} \right\}, \quad j \in \{0, 1, \dots, 2^n - 1\}.$$

First, we will show that this sequence of functions is pointwise increasing. Fix $t \in \mathbb{T}$ and $n \in \mathbb{N}$. We want to show that

$$f_{n+1}(t) \geq f_n(t).$$

We have several cases for t .

1. If $t \in E_{n,j}$ for some $j \in \{0, 1, \dots, 2^n - 1\}$, then

$$f_n(t) = \frac{j}{2^n}$$

and

$$\frac{j}{2^n} \leq f(t) < \frac{j+1}{2^n}.$$

This implies

$$\frac{2j}{2^{n+1}} \leq f(t) < \frac{2j+2}{2^{n+1}}.$$

Thus, $t \in E_{n+1,k}$ for $k = 2j$ or $k = 2j + 1$. The former means

$$\begin{aligned} f_{n+1}(t) &= \frac{2j}{2^{n+1}} \\ &= \frac{j}{2^n} \\ &= f_n(t), \end{aligned}$$

whilst the latter means

$$\begin{aligned} f_{n+1}(t) &= \frac{2j+1}{2^{n+1}} \\ &> \frac{j}{2^n} \\ &= f_n(t). \end{aligned}$$

In either case, we have

$$f_{n+1}(t) \geq f_n(t).$$

2. If $t \in A_n$, then $f(t) \geq 2^n$ and $f_n(t) = 2^n$. Thus, we either have subcases

$$f(t) \geq 2^{n+1} \quad \text{or} \quad 2^n \leq f(t) < 2^{n+1}.$$

(a) For the first subcase, we get

$$\begin{aligned} f_{n+1}(t) &\geq 2^{n+1} \\ &> 2^n \\ &= f_n(t). \end{aligned}$$

(b) On the other hand,

$$\frac{2^n 2^{n+1}}{2^{n+1}} \leq f(t) < \frac{2^{n+1} 2^{n+1}}{2^{n+1}},$$

whereupon

$$\frac{2^{2n+1}}{2^{n+1}} \leq f(t) < \frac{2^{2n+2}}{2^{n+1}},$$

and $t \in E_{n+1,j}$ for some $j \in \{2^{2n+1}, \dots, 2^{2n+2} - 1\}$. Therefore

$$\begin{aligned} f_{n+1}(t) &\geq \frac{2^{2n+1}}{2^{n+1}} \\ &= 2^n \\ &= f_n(t). \end{aligned}$$

Hence, in all subcases we have

$$f_{n+1}(t) \geq f_n(t).$$

Thus, the sequence $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of pointwise increasing simple functions. Moreover, by construction, we have

$$0 \leq f - f_n \leq \frac{1}{2^n} \quad \text{on the set} \quad \{t \in \mathbb{T} : f(t) < 2^n\}$$

and

$$f_n = 2^n \quad \text{on the set} \quad \{t \in \mathbb{T} : f(t) \geq 2^n\}.$$

So, for any $t \in \mathbb{T}$, we have two possible cases.

1. If $f(t)$ is finite, then there exists an $N \in \mathbb{N}$ such that $f(t) \leq 2^N$. Thus, for any $n \geq N$, at this point we get

$$\begin{aligned} 0 &\leq f(t) - f_n(t) \\ &\leq \frac{1}{2^n} \\ &\leq \frac{1}{2^N}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we then have $f_n(t) \rightarrow f(t)$.

2. If $f(t) = \infty$, then $\{f_n(t)\}_{n \in \mathbb{N}} = \{2^n\}_{n \in \mathbb{N}}$ which diverges to ∞ .

In either case, we have $f_n \rightarrow f$ as $n \rightarrow \infty$ pointwise. This completes the proof.

Now, we consider a simple function $\phi : \mathbb{T} \rightarrow \mathbb{R}$ of the form

$$\phi(t) = \sum_{j=1}^n c_j 1_{E_j}(t), \quad t \in \mathbb{T},$$

where $E_j \subset \mathbb{M}(m^*)$, $j \in \{1, 2, \dots, n\}$, are pairwise disjoint and $c_j \in \mathbb{R}$, $j \in \{1, 2, \dots, n\}$. Define

$$I(\phi) = \sum_{j=1}^n c_j \mu_{\Delta}(E_j). \quad (\text{Chapter 4.1})$$

We note that the function ϕ may be expressed in a different way using different collections of Lebesgue delta measurable sets E_j and constants c_j . But the value of $I(\phi)$ in (Chapter 4.1) is defined to be explicitly dependent on the way it is represented.

Theorem

The value of $I(\phi)$ in (Chapter 4.1) is well-defined regardless of the representation of ϕ .

Proof

Indeed, suppose that we write ϕ in two different representations

$$\phi = \sum_{j=1}^n c_j \mathbf{1}_{E_j}$$

and

$$\phi = \sum_{i=1}^m d_i \mathbf{1}_{F_i},$$

where $\{E_j\}_{j=1}^n$ and $\{F_i\}_{i=1}^m$ are pairwise disjoint collections of sets of $\mathbb{M}(m^*)$ and $c_j, d_i \in \mathbb{R}$. Define

$$E_0 = \mathbb{T} \setminus \bigcup_{j=1}^n E_j,$$

$$F_0 = \mathbb{T} \setminus \bigcup_{i=1}^m F_i.$$

Furthermore, let

$$G_{ij} = F_i \cap E_j \subset \mathbb{M}(m^*)$$

for any $j \in \{0, 1, 2, \dots, n\}$ and $i \in \{0, 1, 2, \dots, m\}$. For each $j \in \{0, 1, 2, \dots, n\}$, we have

$$E_j = \bigcup_{i=0}^m (F_i \cap E_j)$$

$$= \bigcup_{i=0}^m G_{ij}$$

and for each $i \in \{0, 1, 2, \dots, m\}$, we have

$$F_i = \bigcup_{j=0}^n G_{ij}.$$

Moreover, G_{ij} are pairwise disjoint for $i \in \{0, 1, 2, \dots, m\}$ and $j \in \{0, 1, 2, \dots, n\}$. If we set $c_0 = d_0 = 0$, we can write ϕ as follows.

$$\begin{aligned} \phi &= \sum_{j=0}^n c_j \mathbf{1}_{E_j} \\ &= \sum_{j=0}^n c_j \sum_{i=0}^m \mathbf{1}_{F_i \cap E_j} \\ &= \sum_{j=0, i=0}^{n, m} c_j \mathbf{1}_{G_{ij}} \end{aligned}$$

and

$$\begin{aligned} \phi &= \sum_{i=0}^m d_i \mathbf{1}_{F_i} \\ &= \sum_{i=0}^m d_i \sum_{j=0}^n \mathbf{1}_{F_i \cap E_j} \\ &= \sum_{j=0, i=0}^{n, m} d_i \mathbf{1}_{G_{ij}}. \end{aligned}$$

Whenever $F_i \cap E_j = G_{ij} \neq \emptyset$, evaluation the function ϕ at any $t \in G_{ij}$ we must have $c_j = d_i$. Via contra opposite, if $c_j \neq d_i$, then $G_{ij} = \emptyset$ and hence, $\mu_\Delta(G_{ij}) = 0$. Using this observation, we arrive at

$$\begin{aligned}
 I\left(\sum_{j=0}^n c_j 1_{E_j}\right) &= \sum_{j=0}^n c_j \mu_{\Delta}(E_j) \\
 &= \sum_{j=0, i=0}^{n, m} c_j \mu_{\Delta}(G_{ij}) \\
 &= \sum_{j=0, i=0}^{n, m} d_i \mu_{\Delta}(G_{ij}) \\
 &= \sum_{i=0}^m d_i \mu_{\Delta}(F_i) \\
 &= I\left(\sum_{i=0}^m d_i 1_{F_i}\right).
 \end{aligned}$$

This completes the proof.

✓ Example

Let $\mathbb{T} = \mathbb{Z}$ and $\phi : [0, 5]_{\mathbb{T}} \rightarrow [0, \infty]$ be defined as follows

$$\phi(t) = \begin{cases} 1 & \text{if } [0, 2]_{\mathbb{T}} \cup [4, 5]_{\mathbb{T}} \\ 2 & \text{if } t \in (2, 3]_{\mathbb{T}} \\ 0 & \text{if } t \in (3, 4)_{\mathbb{T}}. \end{cases}$$

Let

$$\begin{aligned}
 E_1 &= [0, 2]_{\mathbb{T}}, \\
 E_2 &= (2, 3]_{\mathbb{T}}, \\
 E_3 &= (3, 4)_{\mathbb{T}}, \\
 E_4 &= (4, 5]_{\mathbb{T}}.
 \end{aligned}$$

Here

$$\sigma(t) = t + 1, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned}
 \mu_{\Delta}(E_1) &= \sigma(2) - 0 \\
 &= 3 - 0 \\
 &= 3, \\
 \mu_{\Delta}(E_2) &= \sigma(3) - \sigma(2) \\
 &= 4 - 3 \\
 &= 1, \\
 \mu_{\Delta}(E_3) &= 4 - \sigma(3) \\
 &= 4 - 4 \\
 &= 0, \\
 \mu_{\Delta}(E_4) &= \sigma(5) - \sigma(4) \\
 &= 6 - 5 \\
 &= 1.
 \end{aligned}$$

The simple function ϕ can be represented in the form

$$\phi(t) = 1 \cdot 1_{E_1} + 2 \cdot 1_{E_2} + 0 \cdot 1_{E_3} + 1 \cdot 1_{E_4}, \quad t \in [0, 5]_{\mathbb{T}}.$$

Hence,

$$\begin{aligned}
 I(\phi) &= 1 \cdot \mu_{\Delta}(E_1) + 2 \cdot \mu_{\Delta}(E_2) + 1 \cdot \mu_{\Delta}(E_4) \\
 &= 1 \cdot 3 + 2 \cdot 1 + 1 \cdot 1 \\
 &= 3 + 2 + 1 \\
 &= 6.
 \end{aligned}$$

From definitions, we can prove the following properties.

Theorem

Let $\phi, \psi : \mathbb{T} \rightarrow \mathbb{R}$ be simple functions.

1. For any constant $\lambda \in \mathbb{R}$ one has

$$I(\lambda\phi) = \lambda I(\phi).$$

2. $I(\phi + \psi) = I(\phi) + I(\psi)$.

3. If $0 \leq \phi \leq \psi$ on $\mathbb{T}$, then

$$0 \leq I(\phi) \leq I(\psi).$$

4. $|I(\phi)| \leq I(|\phi|)$.

5. If $0 \leq \phi$ on $\mathbb{T}$ and $E \subset \mathbb{M}(m^*)$, then

$$I_E(\phi) = I(\mathbf{1}_E \cdot \phi) \leq I(\phi).$$

6. If $E, F \subset \mathbb{M}(m^*)$ are disjoint, then

$$I_{E \cup F}(\phi) = I_E(\phi) + I_F(\phi).$$

Proof

Let

$$\begin{aligned}\phi(t) &= \sum_{j=1}^n c_j \mathbf{1}_{E_j}, \\ \psi(t) &= \sum_{i=1}^m d_i \mathbf{1}_{F_i}, \quad t \in \mathbb{T},\end{aligned}$$

where $\{E_j\}_{j=1}^n, \{F_i\}_{i=1}^m \subset \mathbb{M}(m^*)$ are each pairwise disjoint collections of sets in $\mathbb{T}$ and $c_j, d_i \in \mathbb{R}$.

1. We have

$$\begin{aligned}\lambda\phi(t) &= \lambda \left(\sum_{j=1}^n c_j \mathbf{1}_{E_j}(t) \right) \\ &= \sum_{j=1}^n (\lambda c_j) \mathbf{1}_{E_j}(t), \quad t \in \mathbb{T}, \\ I(\phi) &= \sum_{j=1}^n c_j \mu_{\Delta}(E_j), \\ I(\lambda\phi) &= \sum_{j=1}^n (\lambda c_j) \mu_{\Delta}(E_j).\end{aligned}$$

Hence,

$$\begin{aligned}\lambda I(\phi) &= \lambda \left(\sum_{j=1}^n c_j \mu_{\Delta}(E_j) \right) \\ &= \sum_{j=1}^n (\lambda c_j) \mu_{\Delta}(E_j) \\ &= I(\lambda\phi).\end{aligned}$$

2. Define

$$\begin{aligned}E_0 &= \mathbb{T} \setminus \bigcup_{j=1}^n E_j, \\ F_0 &= \mathbb{T} \setminus \bigcup_{i=1}^m F_i, \\ G_{ij} &= F_i \cap E_j, \quad i \in \{0, 1, 2, \dots, m\}, \quad j \in \{0, 1, 2, \dots, n\}.\end{aligned}$$

Then

$$E_j = \bigcup_{i=0}^m G_{ij},$$

$$F_i = \bigcup_{j=0}^n G_{ij}, \quad i \in \{0, 1, 2, \dots, m\}, \quad j \in \{0, 1, 2, \dots, n\},$$

and

$$\phi(t) = \sum_{j=0, i=0}^{n, m} c_j \mathbf{1}_{G_{ij}}(t),$$

$$\psi(t) = \sum_{j=0, i=0}^{n, m} d_i \mathbf{1}_{G_{ij}}(t), \quad t \in \mathbb{T}.$$

Hence,

$$I(\phi) = \sum_{j=0, i=0}^{n, m} c_j \mu_{\Delta}(G_{ij}),$$

$$I(\psi) = \sum_{j=0, i=0}^{n, m} d_i \mu_{\Delta}(G_{ij}).$$

Using the above, we get

$$\begin{aligned} I(\phi + \psi) &= I\left(\sum_{j=0, i=0}^{n, m} c_j \mathbf{1}_{G_{ij}} + \sum_{j=0, i=0}^{n, m} d_i \mathbf{1}_{G_{ij}}\right) \\ &= I\left(\sum_{j=0, i=0}^{n, m} (c_j + d_i) \mathbf{1}_{G_{ij}}\right) \\ &= \sum_{j=0, i=0}^{n, m} (c_j + d_i) \mu_{\Delta}(G_{ij}) \\ &= \sum_{j=0, i=0}^{n, m} c_j \mu_{\Delta}(G_{ij}) + \sum_{j=0, i=0}^{n, m} d_i \mu_{\Delta}(G_{ij}) \\ &= I(\phi) + I(\psi). \end{aligned}$$

3. Since $\phi \geq 0$ on $\mathbb{T}$, we have that $c_j \geq 0, j \in \{0, 1, 2, \dots, n\}$. Hence,

$$I(\phi) = \sum_{j=1}^n c_j \mu_{\Delta}(E_j) \geq 0.$$

If $\phi \leq \psi$ pointwise, the simple function $\psi - \phi$ is nonnegative. We have the following cases.

- (a) If $I(\phi) = I(\psi) = \infty$ or $I(\phi) < \infty$ and $I(\psi) = \infty$, then the inequality is clearly true.
- (b) The case $I(\phi) = \infty$ and $I(\psi) < \infty$ is impossible.
- (c) Assume that $I(\phi)$ and $I(\psi)$ are finite. Note that

$$I(\psi - \phi) \geq 0.$$

By using the first two assertions, we get

$$\begin{aligned} I(\psi - \phi) &= I(\psi + (-1) \cdot \phi) \\ &= I(\psi) + I((-1) \cdot \phi) \\ &= I(\psi) + (-1) \cdot I(\phi) \\ &= I(\psi) - I(\phi) \\ &\geq 0, \end{aligned}$$

which implies the desired inequality.

(4) We have

$$\begin{aligned}
 |I(\phi)| &= \left| \sum_{j=1}^n c_j \mu_{\Delta}(E_j) \right| \\
 &\leq \sum_{j=1}^n |c_j \mu_{\Delta}(E_j)| \\
 &= \sum_{j=1}^n |c_j| \mu_{\Delta}(E_j) \\
 &= I(|\phi|).
 \end{aligned}$$

(5) Since $\phi \geq 0$ on $\mathbf{T}$, we have that $c_j \geq 0$, $j \in \{0, 1, 2, \dots, n\}$. Note that

$$\begin{aligned}
 \mathbf{1}_E \cdot \phi &= \mathbf{1}_E \cdot \left(\sum_{j=1}^n c_j \mathbf{1}_{E_j} \right) \\
 &= \sum_{j=1}^n c_j \mathbf{1}_E \cdot \mathbf{1}_{E_j} \\
 &= \sum_{j=1}^n c_j \mathbf{1}_{E \cap E_j}
 \end{aligned}$$

and because $E \cap E_j \subset E_j$, $j \in \{1, 2, \dots, n\}$, we get

$$\mu_{\Delta}(E \cap E_j) \leq \mu_{\Delta}(E_j), \quad j \in \{1, 2, \dots, n\},$$

and

$$\begin{aligned}
 I(\mathbf{1}_E \cdot \phi) &= \sum_{j=1}^n c_j \mu_{\Delta}(E \cap E_j) \\
 &\leq \sum_{j=1}^n c_j \mu_{\Delta}(E_j) \\
 &= I(\phi).
 \end{aligned}$$

6. Since E and F are disjoint, we get

$$\mathbf{1}_{E \cup F} = \mathbf{1}_E + \mathbf{1}_F.$$

Then

$$\begin{aligned}
 I_{E \cup F}(\phi) &= I(\mathbf{1}_{E \cup F} \cdot \phi) \\
 &= I(\mathbf{1}_E + \mathbf{1}_F) \cdot \phi \\
 &= I(\mathbf{1}_E \cdot \phi + \mathbf{1}_F \cdot \phi) \\
 &= I(\mathbf{1}_E \cdot \phi) + I(\mathbf{1}_F \cdot \phi) \\
 &= I_E(\phi) + I_F(\phi).
 \end{aligned}$$

This completes the proof.

Chapter 5: Lebesgue Delta Integration of Nonnegative Functions

For a Lebesgue delta measurable function $f : \mathbb{T} \rightarrow \mathbb{R}$, denote

$$U_f = \{\phi \text{ is a nonnegative simple function with } \phi \leq f\}.$$

Definition

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a nonnegative Lebesgue delta measurable function. We define the Lebesgue delta integral of f over $\mathbb{T}$ as follows

$$\int_{\mathbb{T}} f \Delta \mu_{\Delta} = \sup\{I(\phi) : \phi \in U_f\}.$$

Definition

A nonnegative Lebesgue measurable function $f : \mathbb{T} \rightarrow [0, \infty]$ is called Lebesgue delta integrable if

$$\int_{\mathbb{T}} f \Delta \mu_{\Delta} < \infty.$$

The space of all Lebesgue delta integrable functions over $\mathbb{T}$ is denoted by $\mathcal{L}_{\Delta}^1(\mathbb{T})$.

Definition

We can also define a Lebesgue delta integral of a nonnegative Lebesgue delta measurable function f over some subset $E \subset \mathbb{T}$ in the σ -algebra $\mathbf{M}(m^*)$. This is defined using the integral above as follows

$$\begin{aligned} \int_E f \Delta \mu_{\Delta} &= \int_{\mathbb{T}} \mathbf{1}_E \cdot f \Delta \mu_{\Delta} \\ &= \sup\{I(\phi) : \phi \in U_{\mathbf{1}_E f}\} \\ &= \sup\{I(\mathbf{1}_E \phi) : \phi \in U_{\mathbf{1}_E f}\} \\ &= \sup\{I_E(\phi) : \phi \in U_f\}. \end{aligned}$$

Theorem

Let $f, g : \mathbb{T} \rightarrow [0, \infty]$ be nonnegative Lebesgue delta integrable functions.

1. If $\phi : \mathbb{T} \rightarrow \mathbb{R}$ is a simple function, then

$$\int_{\mathbb{T}} \phi \Delta \mu_{\Delta} = I(\phi).$$

2. If $\lambda > 0$ is a real constant, then

$$\int_{\mathbb{T}} \lambda f \Delta \mu_{\Delta} = \lambda \int_{\mathbb{T}} f \Delta \mu_{\Delta}.$$

3. If $0 \leq f \leq g$, then

$$\int_{\mathbb{T}} f \Delta \mu_{\Delta} \leq \int_{\mathbb{T}} g \Delta \mu_{\Delta}.$$

4. If $E, F \subset \mathbf{M}(m^*)$ are disjoint and $f \in \mathcal{L}_{\Delta}^1(\mathbb{T})$, then

$$\int_{E \cup F} f \Delta \mu_{\Delta} = \int_E f \Delta \mu_{\Delta} + \int_F f \Delta \mu_{\Delta}.$$

5. If $E \subset \mathbb{M}(m^*)$ and $\mu_\Delta(E) = 0$, then

$$\int_E f \Delta \mu_\Delta = 0.$$

6. If $f = 0$ μ_Δ -a.e., then

$$\int_T f \Delta \mu_\Delta = 0.$$

\end{theorem} \begin{proof}

1. We have

$$\begin{aligned} \int_T \phi \Delta \mu_\Delta &= \sup\{I(\phi) : \phi \in U_\phi\} \\ &= I(\phi). \end{aligned}$$

2. Since $I(\lambda\phi) = \lambda I(\phi)$ for any $\phi \in U_f$, we get

$$\begin{aligned} \int_T \lambda f \Delta \mu_\Delta &= \sup\{I(\phi) : \phi \in U_{\lambda f}\} \\ &= \sup\{I(\lambda\phi) : \lambda\phi \in U_{\lambda f}\} \\ &= \sup\{\lambda I(\phi) : \phi \in U_f\} \\ &= \lambda \sup\{I(\phi) : \phi \in U_f\} \\ &= \lambda \int_T f \Delta \mu_\Delta. \end{aligned}$$

3. Since $f \leq g$, we get $U_f \subset U_g$. Hence,

$$\begin{aligned} \int_T f \Delta \mu_\Delta &= \sup\{I(\phi) : \phi \in U_f\} \\ &\leq \sup\{I(\phi) : \phi \in U_g\} \\ &= \int_T g \Delta \mu_\Delta. \end{aligned}$$

4. First, by subadditivity of supremum, we have

$$\begin{aligned} \int_{E \cup F} f \Delta \mu_\Delta &= \sup\{I_{E \cup F}(\phi) : \phi \in U_f\} \\ &= \sup\{I_E(\phi) + I_F(\phi) : \phi \in U_f\} \\ &\leq \sup\{I_E(\phi) : \phi \in U_f\} + \sup\{I_F(\phi) : \phi \in U_f\} \\ &= \int_E f \Delta \mu_\Delta + \int_F f \Delta \mu_\Delta. \end{aligned}$$

For any simple functions $\phi, \psi \in U_f$, consider the function

$$\kappa = \mathbf{1}_E \phi + \mathbf{1}_F \psi,$$

which is also a simple function. If $t \in F$, then

$$\begin{aligned} \kappa(t) &= \mathbf{1}_E(t)\phi(t) + \mathbf{1}_F(t)\psi(t) \\ &= \psi(t) \\ &\leq f(t). \end{aligned}$$

Likewise, for $t \in E$, we have

$$\begin{aligned} \kappa(t) &= \phi(t) \\ &\leq f(t). \end{aligned}$$

For $t \in (E \cup F)^c$, we have

$$\begin{aligned}\kappa(t) &= 0 \\ &\leq f(t).\end{aligned}$$

Thus, κ is a simple function for which $\kappa \in U_f$. Therefore

$$\begin{aligned}I_E(\phi) + I_F(\psi) &= I(\mathbf{1}_E\phi + \mathbf{1}_F\psi) \\ &= I(\kappa) \\ &\leq \int_{E \cup F} f \Delta \mu_\Delta.\end{aligned}$$

Taking supremum of $I_E(\phi)$ and $I_F(\psi)$, we arrive at

$$\int_E f \Delta \mu_\Delta + \int_F f \Delta \mu_\Delta \leq \int_{E \cup F} f \Delta \mu_\Delta.$$

Therefore

$$\begin{aligned}\int_{E \cup F} f \Delta \mu_\Delta &\leq \int_E f \Delta \mu_\Delta + \int_F f \Delta \mu_\Delta \\ &\leq \int_{E \cup F} f \Delta \mu_\Delta,\end{aligned}$$

whereupon we get the desired result.

5. For any $\phi \in U_f$ for which

$$\phi(t) = \sum_{j=1}^n c_j \mathbf{1}_{E_j}(t), \quad t \in \mathbb{T},$$

where the sets $E_j \subset \mathbb{M}(m^*)$ are disjoint and c_j are real constants, $j \in \{1, 2, \dots, n\}$. Since $E_j \cap E \subset E$ and $\mu_\Delta(E) = 0$, we obtain $\mu_\Delta(E_j \cap E) = 0$, $j \in \{1, 2, \dots, n\}$. Hence,

$$\begin{aligned}I_E(\phi) &= \sum_{j=1}^n c_j \mu_\Delta(E_j \cap E) \\ &= 0.\end{aligned}$$

Therefore

$$\int_E f \Delta \mu_\Delta = 0.$$

6. Let

$$E = \{t \in \mathbb{T} : f(t) = 0\}.$$

Since $f = 0$ μ_Δ -a.e., we have

$$\mu_\Delta(\mathbb{T} \setminus E) = 0.$$

From the fourth and fifth assertions, we get

$$\begin{aligned}\int_{\mathbb{T}} f \Delta \mu_\Delta &= \int_{E \cup (\mathbb{T} \setminus E)} f \Delta \mu_\Delta \\ &= \int_E f \Delta \mu_\Delta + \int_{\mathbb{T} \setminus E} f \Delta \mu_\Delta \\ &= 0.\end{aligned}$$

This completes the proof.

Theorem (The Markov Inequality)

If $f : \mathbb{T} \rightarrow [0, \infty]$ is Lebesgue delta integrable, then for any constant $c > 0$ we have

$$\mu_{\Delta}(\{t \in \mathbb{T} : f(t) \geq c\}) \leq \frac{1}{c} \int_{\mathbb{T}} f \Delta \mu_{\Delta}.$$

Proof

Define

$$\phi = c \mathbf{1}_{\{t \in \mathbb{T} : f(t) \geq c\}}.$$

Then

$$\phi \leq f \quad \text{on} \quad \{t \in \mathbb{T} : f(t) \geq c\}$$

and

$$\begin{aligned} c \mu_{\Delta}(\{t \in \mathbb{T} : f(t) \geq c\}) &= \int_{\{t \in \mathbb{T} : f(t) \geq c\}} c \Delta \mu_{\Delta} \\ &= \int_{\mathbb{T}} c \mathbf{1}_{\{t \in \mathbb{T} : f(t) \geq c\}} \Delta \mu_{\Delta} \\ &= \int_{\mathbb{T}} \phi \Delta \mu_{\Delta} \\ &\leq \int_{\mathbb{T}} f \Delta \mu_{\Delta}. \end{aligned}$$

This completes the proof.

Theorem

Let $f : \mathbb{T} \rightarrow [0, \infty]$ be Lebesgue delta integrable.

1. If $f \in \mathcal{L}_{\Delta}^1(\mathbb{T})$, then

$$\mu_{\Delta}(\{t \in \mathbb{T} : f(t) = \infty\}) = 0.$$

2. If

$$\int_{\mathbb{T}} f \Delta \mu_{\Delta} = 0,$$

then $f = 0$ μ_{Δ} -a.e. on $\mathbb{T}$.

Proof

1. Since $f \in \mathcal{L}_{\Delta}^1(\mathbb{T})$, we have that

$$\int_{\mathbb{T}} f \Delta \mu_{\Delta} \quad \text{is a finite constant.}$$

Define

$$E_n = \{t \in \mathbb{T} : f(t) \geq n\}, \quad n \in \mathbb{N}.$$

Then

$$E_{n+1} \subset E_n \quad n \in \mathbb{N}.$$

Applying the Markov inequality, we find

$$\mu_{\Delta}(E_n) \leq \frac{1}{n} \int_{\mathbb{T}} f \Delta \mu_{\Delta}.$$

From here, we arrive at

$$\begin{aligned} 0 &\leq \mu_{\Delta}(\{t \in \mathbb{T} : f(t) = \infty\}) \\ &= \mu_{\Delta}\left(\bigcap_{j=1}^{\infty} E_j\right) \\ &= \lim_{n \rightarrow \infty} \mu_{\Delta}(E_n) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathbb{T}} f \Delta \mu_{\Delta} \\ &= 0. \end{aligned}$$

2. Define

$$F_n = \{t \in \mathbb{T} : f(t) > \frac{1}{n}\}, \quad n \in \mathbb{N}.$$

We have

$$F_n \subset F_{n+1}, \quad n \in \mathbb{N}.$$

Applying the Markov inequality, we find

$$\begin{aligned} \mu_{\Delta}(F_n) &\leq n \int_{\mathbb{T}} f \Delta \mu_{\Delta} \\ &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} \mu_{\Delta}(\{t \in \mathbb{T} : f(t) > 0\}) &= \mu_{\Delta}\left(\bigcup_{j=1}^{\infty} E_j\right) \\ &= \lim_{n \rightarrow \infty} \mu_{\Delta}(F_n) \\ &= 0, \end{aligned}$$

giving us the result.

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Chapter 6: The Monotone Convergence Theorem

A very useful result for nonnegative delta measurable functions is the monotone convergence theorem. This is a precursor to many convergence theorems which we shall prove later.

Theorem

Let $\{f_n\}_{n \in \mathbb{N}}$ be an increasing sequence of nonnegative Lebesgue delta measurable functions $f_n : \mathbb{T} \rightarrow [0, \infty]$ that converges to some nonnegative Lebesgue delta measurable function $f : \mathbb{T} \rightarrow [0, \infty]$. Then

$$\begin{aligned} \int_{\mathbb{T}} f \Delta \mu_{\Delta} &= \int_{\mathbb{T}} \lim_{n \rightarrow \infty} f_n \Delta \mu_{\Delta} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}. \end{aligned}$$

Proof

We have

$$f_n \leq f_{n+1} \leq f, \quad n \in \mathbb{N}.$$

Then

$$\begin{aligned} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} &\leq \int_{\mathbb{T}} f_{n+1} \Delta \mu_{\Delta} \\ &\leq \int_{\mathbb{T}} f \Delta \mu_{\Delta}, \quad n \in \mathbb{N}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} &= \sup_{n \in \mathbb{N}} \left(\int_{\mathbb{T}} f_n \Delta \mu_{\Delta} \right) \\ &\leq \int_{\mathbb{T}} f \Delta \mu_{\Delta}. \end{aligned}$$

Now, we need to prove the inverse inequality. Let

$$\phi(t) = \sum_{j=1}^m c_j 1_{E_j}(t), \quad t \in \mathbb{T},$$

where $\{E_j\}_{j=1}^m$ are pairwise disjoint and $c_j, j \in \{1, \dots, m\}$, are real constants, and

$$0 \leq \phi \leq f.$$

Fix $\alpha \in (0, 1)$ arbitrarily and define

$$B_n = \{t \in \mathbb{T} : f_n(t) > \alpha \phi(t)\}, \quad n \in \mathbb{N}.$$

The sets $B_n, n \in \mathbb{N}$, are Lebesgue delta measurable and since the sequence $\{f_n\}_{n \in \mathbb{N}}$ is increasing, we get

$$B_n \subset B_{n+1}, \quad n \in \mathbb{N}.$$

Moreover, we have

$$\mathbb{T} = \bigcup_{n=1}^{\infty} B_n$$

because

$$\begin{aligned} f_n(t) &\rightarrow f(t) \\ &> \alpha \phi(t) \quad \text{as } n \rightarrow \infty, \quad t \in \mathbb{T}. \end{aligned}$$

Since

$$\begin{aligned}\alpha\phi 1_{B_n} &\leq f_n 1_{B_n} \\ &\leq f_n, \quad n \in \mathbb{N},\end{aligned}$$

by integrating over $\mathbb{T}$, we get

$$\alpha I_{B_n}(\phi) \leq \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}, \quad n \in \mathbb{N}. \quad (\text{Chapter 6.1})$$

Since

$$E_j \cap B_n \subset E_j \cap B_{n+1}, \quad j, n \in \mathbb{N},$$

and

$$\bigcup_{n=1}^{\infty} (E_j \cap B_n) = E_j,$$

we have the following limit

$$\begin{aligned}\lim_{n \rightarrow \infty} I_{B_n}(\phi) &= \lim_{n \rightarrow \infty} \sum_{j=1}^m c_j \mu_{\Delta}(E_j \cap B_n) \\ &= \sum_{j=1}^m c_j \lim_{n \rightarrow \infty} \mu_{\Delta}(E_j \cap B_n) \\ &= \sum_{j=1}^m c_j \mu_{\Delta}(E_j) \\ &= I(\phi).\end{aligned}$$

Hence, taking the limit as $n \rightarrow \infty$ in (Chapter 6.1), we arrive at

$$\alpha I(\phi) \leq \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}$$

for any arbitrary $\alpha \in (0, 1)$. Taking the limit as $\alpha \rightarrow 1$ then yields

$$I(\phi) \leq \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}.$$

Now, taking the supremum over all simple functions ϕ with $0 \leq \phi \leq f$, we find

$$\int_{\mathbb{T}} f \Delta \mu_{\Delta} \leq \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}.$$

Putting the two inequalities together, we obtain the desired result.

Chapter 7: The Fatou Lemmas

Since integrals can be thought of as limits of sums, the finite subadditivity of limit superior and finite superadditivity of limit inferior carry forward to Lebesgue delta integrals. These generalisations are called the Fatou lemmas.

Theorem (The Fatou Lemma)

Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of Lebesgue delta measurable functions $f_n : \mathbb{T} \rightarrow [0, \infty]$. Then

$$\int_{\mathbb{T}} \liminf_{n \rightarrow \infty} f_n \Delta \mu_{\Delta} \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}.$$

In particular, if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of nonnegative Lebesgue delta measurable functions and $f_n \rightarrow f$ as $n \rightarrow \infty$ μ_{Δ} -a.e., then

$$\int_{\mathbb{T}} f \Delta \mu_{\Delta} \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}.$$

Proof

Define

$$g_n(t) = \inf_{m \geq n} f_m(t), \quad t \in \mathbb{T}, \quad n \in \mathbb{N}.$$

Then $\{g_n\}_{n \in \mathbb{N}}$ is a pointwise increasing sequence and

$$\lim_{n \rightarrow \infty} g_n = \liminf_{n \rightarrow \infty} f_n.$$

We apply the Monotone Convergence Theorem to the sequence $\{g_n\}_{n \in \mathbb{N}}$ to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{T}} g_n \Delta \mu_{\Delta} &= \int_{\mathbb{T}} \lim_{n \rightarrow \infty} g_n \Delta \mu_{\Delta} \\ &= \int_{\mathbb{T}} \liminf_{n \rightarrow \infty} f_n \Delta \mu_{\Delta}. \end{aligned}$$

On the other hand, we have

$$0 \leq g_n \leq f_m, \quad m \geq n.$$

Then

$$\int_{\mathbb{T}} g_n \Delta \mu_{\Delta} \leq \int_{\mathbb{T}} f_m \Delta \mu_{\Delta}, \quad m \geq n.$$

Thus, we have

$$\int_{\mathbb{T}} g_n \Delta \mu_{\Delta} \leq \inf_{m \geq n} \int_{\mathbb{T}} f_m \Delta \mu_{\Delta},$$

whereupon

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} g_n \Delta \mu_{\Delta} \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta}$$

and

$$\int_{\mathbb{T}} \liminf_{n \rightarrow \infty} f_n \Delta \mu_{\Delta} \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta},$$

which completes the proof.

Corollary (The Reverse Fatou Lemma)

Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of Lebesgue delta measurable functions $f_n : \mathbb{T} \rightarrow [0, \infty]$ such that there exists a nonnegative Lebesgue delta measurable function g for which

$$f_n \leq g, \quad n \in \mathbb{N}.$$

Then

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} \leq \int_{\mathbb{T}} \limsup_{n \rightarrow \infty} f_n \Delta \mu_{\Delta}.$$

Proof

Define

$$h_n = g - f_n, \quad n \in \mathbb{N}.$$

Applying the Fatou Lemma to the sequence $\{h_n\}_{n \in \mathbb{N}}$, we find

$$\int_{\mathbb{T}} \liminf_{n \rightarrow \infty} (g - f_n) \Delta \mu_{\Delta} \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{T}} (g - f_n) \Delta \mu_{\Delta},$$

or

$$\int_{\mathbb{T}} g \Delta \mu_{\Delta} + \int_{\mathbb{T}} \liminf_{n \rightarrow \infty} (-f_n) \Delta \mu_{\Delta} \leq \int_{\mathbb{T}} g \Delta \mu_{\Delta} + \liminf_{n \rightarrow \infty} \left(- \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} \right),$$

or

$$- \int_{\mathbb{T}} \limsup_{n \rightarrow \infty} f_n \Delta \mu_{\Delta} \leq - \limsup_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta},$$

whereupon we get the desired result.

Corollary

Let $f, g : \mathbb{T} \rightarrow [0, \infty]$ be nonnegative Lebesgue delta measurable functions and $\alpha, \beta \geq 0$. Then

$$\int_{\mathbb{T}} (\alpha f + \beta g) \Delta \mu_{\Delta} = \alpha \int_{\mathbb{T}} f \Delta \mu_{\Delta} + \beta \int_{\mathbb{T}} g \Delta \mu_{\Delta}.$$

Proof

Firstly, we will prove that

$$\int_{\mathbb{T}} (f + g) \Delta \mu_{\Delta} = \int_{\mathbb{T}} f \Delta \mu_{\Delta} + \int_{\mathbb{T}} g \Delta \mu_{\Delta}. \quad (\text{Chapter 7.1})$$

Consider the pointwise increasing sequences of simple functions $\{\phi_n\}_{n \in \mathbb{N}}$ and $\{\psi_n\}_{n \in \mathbb{N}}$ such that $\phi_n \rightarrow f$ and $\psi_n \rightarrow g$ as $n \rightarrow \infty$. Then

$$\phi_n + \psi_n \rightarrow f + g \quad \text{as } n \rightarrow \infty.$$

By the Monotone Convergence Theorem, we get

$$\begin{aligned}
 \int_{\mathbf{T}} (f + g) \Delta\mu_{\Delta} &= \lim_{n \rightarrow \infty} I(\phi_n + \psi_n) \\
 &= \lim_{n \rightarrow \infty} (I(\phi_n) + I(\psi_n)) \\
 &= \lim_{n \rightarrow \infty} I(\phi_n) + \lim_{n \rightarrow \infty} I(\psi_n) \\
 &= \int_{\mathbf{T}} f \Delta\mu_{\Delta} + \int_{\mathbf{T}} g \Delta\mu_{\Delta}.
 \end{aligned}$$

Now, we apply [\(Chapter 7.1\)](#) for the functions αf and βg and using the homogeneity of the Lebesgue delta integral, we find

$$\begin{aligned}
 \int_{\mathbf{T}} ((\alpha f) + (\beta g)) \Delta\mu_{\Delta} &= \int_{\mathbf{T}} \alpha f \Delta\mu_{\Delta} + \int_{\mathbf{T}} \beta g \Delta\mu_{\Delta} \\
 &= \alpha \int_{\mathbf{T}} f \Delta\mu_{\Delta} + \beta \int_{\mathbf{T}} g \Delta\mu_{\Delta}.
 \end{aligned}$$

This completes the proof.

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Chapter 8: Lebesgue Delta Integration of Nonnegative Functions Series

Since the Lebesgue delta integral behaves well under limits of nonnegative functions, we can do much more with it than we could with the Riemann delta integral.

Recall that if $\{f_j\}_{j \in \mathbb{N}}$ is a sequence of functions $f_j : \mathbb{T} \rightarrow \mathbb{R}$, we define the infinite sum/series pointwise via the limit of partial sums

$$s_n = \sum_{j=1}^n f_j,$$

that is

$$\begin{aligned} \sum_{j=1}^{\infty} f_j(t) &= \lim_{n \rightarrow \infty} \sum_{j=1}^n f_j(t) \\ &= \lim_{n \rightarrow \infty} s_n(t), \quad t \in \mathbb{T}. \end{aligned}$$

Thus, we can prove the following result.

Theorem

Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative Lebesgue delta measurable functions $f_n : \mathbb{T} \rightarrow [0, \infty]$. Then

$$\sum_{n=1}^{\infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} = \int_{\mathbb{T}} \sum_{n=1}^{\infty} f_n \Delta \mu_{\Delta}.$$

In particular,

$$\sum_{n=1}^{\infty} f_n \in \mathcal{L}_{\Delta}^1(\mathbb{T}) \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} < \infty.$$

Proof

Since the functions f_j , $j \in \mathbb{N}$, in the series are nonnegative, the partial sums s_n , $n \in \mathbb{N}$, are nonnegative and the sequence $\{s_n\}_{n \in \mathbb{N}}$ is increasing pointwise. By using the Monotone Convergence Theorem, we get

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} &= \lim_{m \rightarrow \infty} \sum_{n=1}^m \int_{\mathbb{T}} f_n \Delta \mu_{\Delta} \\ &= \lim_{m \rightarrow \infty} \int_{\mathbb{T}} \sum_{n=1}^m f_n \Delta \mu_{\Delta} \\ &= \int_{\mathbb{T}} \lim_{m \rightarrow \infty} \sum_{n=1}^m f_n \Delta \mu_{\Delta} \\ &= \int_{\mathbb{T}} \sum_{n=1}^{\infty} f_n \Delta \mu_{\Delta}, \end{aligned}$$

and we are done.

A variant of this is to split the domain of integration E into countably many disjoint subsets. We have the following domain-additive property of the Lebesgue delta integral of nonnegative functions.

Theorem

Suppose that $\{E_n\}_{n \in \mathbb{N}}$ is a sequence of Lebesgue delta measurable sets with E_n all pairwise disjoint, and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Let $f : \mathbb{T} \rightarrow [0, \infty]$ be a nonnegative Lebesgue delta measurable function. Then

$$\int_E f \Delta \mu_\Delta = \sum_{n=1}^{\infty} \int_{E_n} f \Delta \mu_\Delta.$$

Proof

For any $n \in \mathbb{N}$, define

$$f_n = f \mathbf{1}_{\cup_{j=1}^n E_j}.$$

The sequence $\{f_n\}_{n \in \mathbb{N}}$ is pointwise increasing to the limit f . So, we can apply the Monotone Convergence Theorem. to obtain

$$\begin{aligned} \int_T f \Delta \mu_\Delta &= \int_T \lim_{n \rightarrow \infty} f_n \Delta \mu_\Delta \\ &= \int_T \lim_{n \rightarrow \infty} f \mathbf{1}_{\cup_{j=1}^n E_j} \Delta \mu_\Delta \\ &= \lim_{n \rightarrow \infty} \int_T f \mathbf{1}_{\cup_{j=1}^n E_j} \Delta \mu_\Delta \\ &= \lim_{n \rightarrow \infty} \int_T \sum_{j=1}^n f \mathbf{1}_{E_j} \Delta \mu_\Delta \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{E_j} f \Delta \mu_\Delta \\ &= \sum_{j=1}^{\infty} \int_{E_j} f \Delta \mu_\Delta, \end{aligned}$$

and we are done.

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Chapter 9: The Lebesgue Delta Integral

The construction of the Lebesgue delta integral can be generalised from nonnegative functions to any Lebesgue delta measurable functions with image in $\widetilde{\mathbb{R}} = [-\infty, \infty]$. This is done by splitting an arbitrary Lebesgue delta measurable function $f : \mathbb{T} \rightarrow \widetilde{\mathbb{R}}$ into its positive and negative parts, namely

$$f = f^+ - f^-,$$

which are two nonnegative functions.

Definition

If at least one of the two integrals

$$\int_{\mathbb{T}} f^+ \Delta\mu_{\Delta} \quad \text{and} \quad \int_{\mathbb{T}} f^- \Delta\mu_{\Delta}$$

is finite, we can define the Lebesgue delta integral as follows

$$\int_{\mathbb{T}} f \Delta\mu_{\Delta} = \int_{\mathbb{T}} f^+ \Delta\mu_{\Delta} - \int_{\mathbb{T}} f^- \Delta\mu_{\Delta}.$$

If both integrals are finite, then we say that the function f is Lebesgue delta integrable over $\mathbb{T}$. The space of all Lebesgue delta integrable functions over $\mathbb{T}$ is denoted by $\mathcal{L}_{\Delta}^1(\mathbb{T})$.

Theorem

Suppose that $f, g : \mathbb{T} \rightarrow \widetilde{\mathbb{R}}$ are Lebesgue delta measurable functions such that $f, g \in \mathcal{L}_{\Delta}^1(\mathbb{T})$ and $\alpha, \beta \in \mathbb{R}$ are constants. Then, we have the following.

1. $\alpha f \in \mathcal{L}_{\Delta}^1(\mathbb{T})$ and

$$\int_{\mathbb{T}} \alpha f \Delta\mu_{\Delta} = \alpha \int_{\mathbb{T}} f \Delta\mu_{\Delta}.$$

2. $\alpha f + \beta g \in \mathcal{L}_{\Delta}^1(\mathbb{T})$ and

$$\int_{\mathbb{T}} (\alpha f + \beta g) \Delta\mu_{\Delta} = \alpha \int_{\mathbb{T}} f \Delta\mu_{\Delta} + \beta \int_{\mathbb{T}} g \Delta\mu_{\Delta}.$$

Proof

1. If $\alpha = 0$, then there is nothing to prove. Otherwise, we have two cases.

- (a) $\alpha > 0$. Then \

$$\begin{aligned} (\alpha f)^+ &= \alpha f^+, \\ (\alpha f)^- &= \alpha f^-. \end{aligned}$$

Hence, using the homogeneity of the Lebesgue delta integral of nonnegative Lebesgue delta measurable functions, we arrive at

$$\begin{aligned}
 \int_T \alpha f \Delta \mu_\Delta &= \int_T (\alpha f)^+ \Delta \mu_\Delta - \int_T (\alpha f)^- \Delta \mu_\Delta \\
 &= \int_T \alpha f^+ \Delta \mu_\Delta - \int_T \alpha f^- \Delta \mu_\Delta \\
 &= \alpha \int_T f^+ \Delta \mu_\Delta - \alpha \int_T f^- \Delta \mu_\Delta \\
 &= \alpha \left(\int_T f^+ \Delta \mu_\Delta - \int_T f^- \Delta \mu_\Delta \right) \\
 &= \alpha \int_T f \Delta \mu_\Delta.
 \end{aligned}$$

(b) $\alpha < 0$. Then

$$\begin{aligned}
 (\alpha f)^+ &= \max(\alpha f, 0) \\
 &= \max((- \alpha)(-f), 0) \\
 &= (-\alpha) \max(-f, 0) \\
 &= (-\alpha) f^-,
 \end{aligned}$$

and likewise,

$$(\alpha f)^- = (-\alpha) f^+.$$

Hence, using that $-\alpha > 0$, we get

$$\begin{aligned}
 \int_T \alpha f \Delta \mu_\Delta &= \int_T (\alpha f)^+ \Delta \mu_\Delta - \int_T (\alpha f)^- \Delta \mu_\Delta \\
 &= \int_T (-\alpha) f^- \Delta \mu_\Delta - \int_T (-\alpha) f^+ \Delta \mu_\Delta \\
 &= (-\alpha) \int_T f^- \Delta \mu_\Delta + \alpha \int_T f^+ \Delta \mu_\Delta \\
 &= \alpha \left(\int_T f^+ \Delta \mu_\Delta - \int_T f^- \Delta \mu_\Delta \right) \\
 &= \alpha \int_T f \Delta \mu_\Delta.
 \end{aligned}$$

2. First, assume that $\alpha = \beta = 1$. We have

$$f + g = (f + g)^+ - (f + g)^-$$

and

$$f + g = (f^+ - f^-) + (g^+ - g^-).$$

Combining this, we find

$$(f^+ - f^-) + (g^+ - g^-) = (f + g)^+ - (f + g)^-,$$

or

$$f^+ + g^+ + (f + g)^- = (f + g)^+ + f^- + g^-,$$

where all terms of both sides are nonnegative.

Thus,

$$\int_T ((f + g)^+ + f^- + g^-) \Delta \mu_\Delta = \int_T (f^+ + g^+ + (f + g)^-) \Delta \mu_\Delta,$$

or

$$\int_{\mathbf{T}} (f+g)^+ \Delta\mu_{\Delta} + \int_{\mathbf{T}} f^- \Delta\mu_{\Delta} + \int_{\mathbf{T}} g^- \Delta\mu_{\Delta} = \int_{\mathbf{T}} f^+ \Delta\mu_{\Delta} + \int_{\mathbf{T}} g^+ \Delta\mu_{\Delta} + \int_{\mathbf{T}} (f+g)^- \Delta\mu_{\Delta},$$

whereupon

$$\int_{\mathbf{T}} (f+g)^+ \Delta\mu_{\Delta} - \int_{\mathbf{T}} (f+g)^- \Delta\mu_{\Delta} = \left(\int_{\mathbf{T}} f^+ \Delta\mu_{\Delta} - \int_{\mathbf{T}} f^- \Delta\mu_{\Delta} \right) + \left(\int_{\mathbf{T}} g^+ \Delta\mu_{\Delta} - \int_{\mathbf{T}} g^- \Delta\mu_{\Delta} \right),$$

or

$$\int_{\mathbf{T}} (f+g) \Delta\mu_{\Delta} = \int_{\mathbf{T}} f \Delta\mu_{\Delta} + \int_{\mathbf{T}} g \Delta\mu_{\Delta}.$$

Now, we apply this for the functions (αf) and (βg) , and using the first assertion, we obtain

$$\begin{aligned} \int_{\mathbf{T}} ((\alpha f) + (\beta g)) \Delta\mu_{\Delta} &= \int_{\mathbf{T}} \alpha f \Delta\mu_{\Delta} + \int_{\mathbf{T}} \beta g \Delta\mu_{\Delta} \\ &= \alpha \int_{\mathbf{T}} f \Delta\mu_{\Delta} + \beta \int_{\mathbf{T}} g \Delta\mu_{\Delta}. \end{aligned}$$

This completes the proof.

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Chapter 10: Convergence Theorems

We can improve the Monotone Convergence Theorem to general Lebesgue delta measurable functions.

Theorem

Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of Lebesgue delta measurable functions $f_n : \mathbb{T} \rightarrow \widetilde{\mathbb{R}} = [-\infty, \infty]$ that is pointwise increasing to some function $f : \mathbb{T} \rightarrow \mathbb{R}$ and $f_n \rightarrow f$ as $n \rightarrow \infty$ μ_Δ -a.e. Suppose further that the set of Lebesgue delta integrals

$$\left\{ \int_{\mathbb{T}} f_n \Delta \mu_\Delta \right\}_{n \in \mathbb{N}}$$

is bounded. Then $f \in \mathcal{L}_\Delta^1(\mathbb{T})$ and

$$\begin{aligned} \int_{\mathbb{T}} f \Delta \mu_\Delta &= \int_{\mathbb{T}} \lim_{n \rightarrow \infty} f_n \Delta \mu_\Delta \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta. \end{aligned}$$

Proof

Define

$$g_n = f_n - f_1, \quad n \in \mathbb{N}.$$

We have that $g_n : \mathbb{T} \rightarrow [0, \infty]$ and the sequence $\{g_n\}_{n \in \mathbb{N}}$ is increasing. Then we can apply the Monotone Convergence Theorem to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{T}} g_n \Delta \mu_\Delta &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}} (f_n - f_1) \Delta \mu_\Delta \\ &= \int_{\mathbb{T}} \lim_{n \rightarrow \infty} (f_n - f_1) \Delta \mu_\Delta \\ &= \int_{\mathbb{T}} (f - f_1) \Delta \mu_\Delta. \end{aligned}$$

Since

$$\int_{\mathbb{T}} f_1 \Delta \mu_\Delta$$

is finite, we can use algebra to get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta - \int_{\mathbb{T}} f_1 \Delta \mu_\Delta = \int_{\mathbb{T}} f \Delta \mu_\Delta - \int_{\mathbb{T}} f_1 \Delta \mu_\Delta,$$

or

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta = \int_{\mathbb{T}} f \Delta \mu_\Delta.$$

Finally, since the sequence of Lebesgue delta integrals

$$\int_{\mathbb{T}} f_n \Delta \mu_\Delta$$

is bounded above and since limits preserve weak inequalities, the resulting limit

$$\int_{\mathbf{T}} f \Delta \mu_{\Delta}$$

must be finite and so $f \in \mathcal{L}_{\Delta}^1(\mathbf{T})$. This completes the proof.

Theorem

Let $f : \mathbf{T} \rightarrow \widetilde{\mathbb{R}}$ be a Lebesgue delta measurable function. For any $\epsilon > 0$ there exists a $\delta > 0$ such that for any set $E \subset \mathbf{M}(m^*)$ with $\mu_{\Delta}(E) < \delta$, we must have

$$\int_E |f| \Delta \mu_{\Delta} < \epsilon.$$

Proof

Note that $|f|$ is a Lebesgue delta measurable function. Define the sets

$$E_n = \{t \in \mathbf{T} : |f(t)| \leq n\}, \quad n \in \mathbb{N},$$

and the functions

$$g_n = |f| 1_{E_n}, \quad n \in \mathbb{N}.$$

Note that $\{g_n\}_{n \in \mathbb{N}}$ is pointwise increasing to $|f|$. By the Monotone Convergence Theorem, it follows that there exists an index $N \in \mathbb{N}$ such that

$$\left| \int_{\mathbf{T}} g_N \Delta \mu_{\Delta} - \int_{\mathbf{T}} |f| \Delta \mu_{\Delta} \right| = \int_{\mathbf{T}} (|f| - g_N) \Delta \mu_{\Delta} < \frac{\epsilon}{2}.$$

For any $E \subset \mathbf{M}(m^*)$ with $\mu_{\Delta}(E) < \infty$, since $|f| \geq g_N$ on $\mathbf{T}$ and $g_N \leq N$ on E , we get

$$\begin{aligned} \int_E |f| \Delta \mu_{\Delta} &= \int_E (|f| - g_N) \Delta \mu_{\Delta} + \int_E g_N \Delta \mu_{\Delta} \\ &\leq \int_{\mathbf{T}} (|f| - g_N) \Delta \mu_{\Delta} + \int_E N \Delta \mu_{\Delta} \\ &= \int_{\mathbf{T}} (|f| - g_N) \Delta \mu_{\Delta} + N \mu_{\Delta}(E) \\ &< \frac{\epsilon}{2} + N \mu_{\Delta}(E). \end{aligned}$$

Now, we take $\delta = \frac{\epsilon}{2N} > 0$. Then, for any $E \subset \mathbf{M}(m^*)$ with $\mu_{\Delta}(E) < \delta$, we find

$$\begin{aligned} \int_E |f| \Delta \mu_{\Delta} &< \frac{\epsilon}{2} + N \left(\frac{\epsilon}{2N} \right) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

This completes the proof.

Theorem (Dominated Convergence Theorem)

Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of Lebesgue delta measurable functions where $f_n : \mathbf{T} \rightarrow \widetilde{\mathbb{R}}$ and $f_n \rightarrow f$ on $\mathbf{T}$ pointwise to some function $f : \mathbf{T} \rightarrow \widetilde{\mathbb{R}}$. Suppose that there exists a function $g \in \mathcal{L}_{\Delta}^1(\mathbf{T})$ such that $|f_n| \leq g$ μ_{Δ} -a.e. on $\mathbf{T}$ for any $n \in \mathbb{N}$. Then $f_n, f \in \mathcal{L}_{\Delta}^1(\mathbf{T})$ and

$$\lim_{n \rightarrow \infty} \int_{\mathbf{T}} f_n \Delta \mu_{\Delta} = \int_{\mathbf{T}} f \Delta \mu_{\Delta}.$$

Зиддо

Since f_n , $n \in \mathbb{N}$, are Lebesgue delta measurable, we have that the limiting function f is Lebesgue delta measurable. By comparison, since $|f_n| \leq g \mu_\Delta$ -a.e., f_n , $n \in \mathbb{N}$, are Lebesgue delta integrable over $\mathbb{T}$ for any $n \in \mathbb{N}$. Next,

$$\begin{aligned} |f| &= \lim_{n \rightarrow \infty} |f_n| \\ &\leq g \quad \mu_\Delta\text{-a.e.} \end{aligned}$$

Hence, the function f must be Lebesgue delta integrable over $\mathbb{T}$, as well. Define

$$\begin{aligned} h_n &= g - f_n, \\ p_n &= g + f_n, \quad n \in \mathbb{N}. \end{aligned}$$

We apply the Fatou lemma to the sequences $\{h_n\}_{n \in \mathbb{N}}$ and $\{p_n\}_{n \in \mathbb{N}}$ and we find

$$\begin{aligned} \int_{\mathbb{T}} (g - f) \Delta \mu_\Delta &= \int_{\mathbb{T}} \liminf_{n \rightarrow \infty} (g - f_n) \Delta \mu_\Delta \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} (g - f_n) \Delta \mu_\Delta \\ &= \int_{\mathbb{T}} g \Delta \mu_\Delta - \limsup_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{T}} (g + f) \Delta \mu_\Delta &= \int_{\mathbb{T}} \liminf_{n \rightarrow \infty} (g + f_n) \Delta \mu_\Delta \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} (g + f_n) \Delta \mu_\Delta \\ &= \int_{\mathbb{T}} g \Delta \mu_\Delta + \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta. \end{aligned}$$

Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta &\leq \int_{\mathbb{T}} f \Delta \mu_\Delta \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta \\ &\leq \limsup_{n \rightarrow \infty} \int_{\mathbb{T}} f_n \Delta \mu_\Delta, \end{aligned}$$

whereupon we get the desired result. \end{proof}

Chapter 11: Absolute Continuous Functions

Let $a, b \in \mathbb{T}$, $a < b$. For a given function $f : \mathbb{T} \rightarrow \mathbb{R}$, consider an auxiliary function $\bar{f} : [a, b] \rightarrow \mathbb{R}$ defined by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in \mathbb{T} \\ f(t_i) + \frac{f(\sigma(t_i)) - f(t_i)}{\mu(t_i)}(t - t_i) & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I, \end{cases} \quad (\text{Chapter 11.1})$$

with $I \subset \mathbb{N}$ and $\{t_i\}_{i \in I}$ given by

$$R = \{t \in \mathbb{T} : t < \sigma(t)\} = \{t_i\}_{i \in I}.$$

Theorem

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $\bar{f} : [a, b] \rightarrow \mathbb{R}$ is the extension of f , defined by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in \mathbb{T} \\ f(t_i) + \frac{f(\sigma(t_i)) - f(t_i)}{\mu(t_i)}(t - t_i) & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I, \end{cases}$$

Then the following statements are equivalent

1. f maps every Δ -null subset of $\mathbb{T}$ into a null set.
2. $\bar{f}$ maps every null subset of $[a, b]$ into a null set.

Proof

1. Let $E \subset [a, b]$ be a null set. We will prove that $\bar{f}(E)$ is a null set.

(a) Let $b \notin E$. Then

$$E = \left(\bigcup_{i \in I} (E \cap [t_i, \sigma(t_i))) \right) \cup (E \cap (\mathbb{T} \setminus R))$$

and

$$\mu(\bar{f}(E)) \leq \sum_{i \in I} \mu(\bar{f}(E \cap [t_i, \sigma(t_i)))) + \mu(\bar{f}(E \cap (\mathbb{T} \setminus R))). \quad (\text{Chapter 11.2})$$

Fix $i \in I$. By the definition of the function $\bar{f}$, it follows that $\bar{f}|_{[t_i, \sigma(t_i)]}$ is absolutely continuous. Hence,

$$\mu(E \cap [t_i, \sigma(t_i))) = 0.$$

Now, applying the Banach-Zarecki Theorem, we get

$$\mu(\bar{f}(E \cap [t_i, \sigma(t_i)))) = 0. \quad (\text{Chapter 11.3})$$

Moreover, since the set $E \cap (\mathbb{T} \setminus R)$ does not have any right-scattered points and $b \notin E \cap (\mathbb{T} \setminus R)$, we get

$$\mu_\Delta(E \cap (\mathbb{T} \setminus R)) = \mu(E \cap (\mathbb{T} \setminus R)) = 0.$$

Hence and $\bar{f}|_{\mathbb{T}} = f$ and our assumptions, we obtain

$$\mu(\bar{f}(E \cap (\mathbb{T} \setminus R))) = \mu(f(E \cap (\mathbb{T} \setminus R))) = 0.$$

From the last relation and from (Chapter 11.2), (Chapter 11.3), we get

$$\mu(\bar{f}(E)) = 0.$$

(b) Let $b \in E$. Then we consider both sets $E \setminus \{b\}$ and $\{b\}$. Since

$$\mu(\bar{f}(\{b\})) = 0,$$

we obtain

$$\begin{aligned} \mu(\bar{f}(E)) &\leq \mu(\bar{f}(E \setminus \{b\})) + \mu(\bar{f}(\{b\})) \\ &= 0, \end{aligned}$$

i.e.,

$$\mu(\bar{f}(E)) = 0.$$

2. Let $E \subset \mathbb{T}$ be a Δ -null set.

(a) Let $b \notin E$. Then, we obtain

$$\begin{aligned} 0 &= \mu_{\Delta}(E) \\ &= \sum_{i \in I_E} (\sigma(t_i) - t_i) + \mu(E) \\ &= \sum_{i \in I_E} (\sigma(t_i) - t_i). \end{aligned}$$

Therefore, the set E does not have any right-scattered points. So,

$$f(E) = \bar{f}(E)$$

and the assertion follows.

(b) Let $b \in E$. Consider both sets $E \setminus \{b\}$ and $\{b\}$. By the previous case, it follows that $E \setminus \{b\}$ does not have any right-scattered points. Then $\bar{f}(E \setminus \{b\})$ is a null set. Since

$$\mu(\{b\}) = 0,$$

then, we have that $\bar{f}(\{b\})$ is a null set. Consequently $\bar{f}(E)$ is a null set. This completes the proof.

Definition

For a given function $f : \mathbb{T} \rightarrow \mathbb{R}$ and a partition $P = \{x_0, x_1, \dots, x_n\}$ of $\mathbb{T}$, define

$$V(P, f) = \sum_{k=1}^n |f(x_k) - f(x_{k-1})|.$$

We define the total variation of the function f as follows.

$$\bigvee_a^b(f) = \sup\{V(P, f) : P \text{ partition of } \mathbb{T}\}.$$

If $\bigvee_a^b(f) < \infty$, we say that the function f is of bounded variation on the time scale $\mathbb{T}$.

Note that if $f : \mathbb{T} \rightarrow \mathbb{R}$ is of bounded variation, then $\bar{f} : \mathbb{T} \rightarrow \mathbb{R}$, defined by (Chapter 11.1), is of bounded variation.

Definition

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be absolutely continuous on $\mathbb{T}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\{(a_k, b_k) \cap \mathbb{T}\}_{k=1}^n$, with $a_k, b_k \in \mathbb{T}$, is a finite partition disjoint family of subintervals of $\mathbb{T}$ satisfying

$$\sum_{k=1}^n (b_k - a_k) < \delta,$$

then

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

Theorem

A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is absolutely continuous on $\mathbb{T}$ if and only if the following conditions hold.

1. f is continuous and of bounded variation on $\mathbb{T}$.
2. f maps every Δ -null subset of $\mathbb{T}$ into a null set.

Proof

Let $\bar{f}$ be the extension of the function f , defined by (Chapter 11.1).

1. Let f is absolutely continuous on $\mathbb{T}$. Then f is continuous on $\mathbb{T}$. Now we will prove that it is of bounded variation on $\mathbb{T}$. Let $\delta > 0$ corresponds to $\epsilon = 1$ as in the definition of absolutely continuous function on $\mathbb{T}$. Then there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ such that for any $k \in \{1, \dots, n\}$ either

$$x_k - x_{k-1} \leq \frac{\delta}{2}$$

or

$$x_k - x_{k-1} > \frac{\delta}{2} \quad \text{and} \quad \sigma(x_{k-1}) = x_k.$$

If $x_k - x_{k-1} \leq \frac{\delta}{2}$, then

$$\bigvee_{x_{k-1}}^{x_k} (f) \leq 1.$$

Thus, $f|_{[x_{k-1}, x_k] \cap \mathbb{T}}$ is of bounded variation on $[x_{k-1}, x_k] \cap \mathbb{T}$. If $x_k - x_{k-1} > \frac{\delta}{2}$, then

$$\bigvee_{x_{k-1}}^{x_k} (f) = |f(x_k) - f(x_{k-1})|$$

and $f|_{[x_{k-1}, x_k] \cap \mathbb{T}}$ is of bounded variation on $[x_{k-1}, x_k] \cap \mathbb{T}$. Therefore $\bar{f}|_{[x_{k-1}, x_k]}$ is of bounded variation on $[x_{k-1}, x_k]$. Consequently $\bar{f}$ is of bounded variation on $\mathbb{T}$. Let now $E \subset \mathbb{T}$ is a Δ -null set. We will prove that $f(E)$ is a null set. Let $\epsilon > 0$ be arbitrarily chosen and $\delta > 0$ corresponds to ϵ as in the definition for absolutely continuous function. Because $\mu_\Delta(E) = 0$, we have that $b \notin E$ and it has not any right-scattered points. Then we may choose a family $\{[a_k, b_k] \cap \mathbb{T}\}_{k=1}^\infty$ with $a_k, b_k \in \mathbb{T}$, of pairwise disjoint subintervals of $\mathbb{T}$ such that

$$E \subset \bigcup_{k=1}^\infty ([a_k, b_k] \cap \mathbb{T})$$

and

$$\sum_{k=1}^\infty (b_k - a_k) < \delta.$$

Hence,

$$\begin{aligned} f(E) &\subset f\left(\bigcup_{k=1}^\infty ([a_k, b_k] \cap \mathbb{T})\right) \\ &= \bigcup_{k=1}^\infty f([a_k, b_k] \cap \mathbb{T}) \end{aligned}$$

and

$$\begin{aligned}\mu(f(E)) &\subset \mu\left(f\left(\bigcup_{k=1}^{\infty}([a_k, b_k] \cap \mathbb{T})\right)\right) \\ &= \bigcup_{k=1}^{\infty} \mu(f([a_k, b_k] \cap \mathbb{T})).\end{aligned}$$

Because f is continuous on $[a_k, b_k] \cap \mathbb{T}$, it follows that there exist $c_k, d_k \in [a_k, b_k] \cap \mathbb{T}$, $c_k < d_k$, such that

$$\begin{aligned}\mu(f([a_k, b_k] \cap \mathbb{T})) &= \mu(f([a_k, b_k] \cap \mathbb{T})) \\ &\leq |f(d_k) - f(c_k)|.\end{aligned}$$

Since $[c_k, d_k] \subset [a_k, b_k]$ and f is absolutely continuous on $\mathbb{T}$, we have

$$\sum_{k=1}^n |f(d_k) - f(c_k)| < \epsilon$$

and

$$\sum_{k=1}^{\infty} |f(d_k) - f(c_k)| \leq \epsilon.$$

From here,

$$\begin{aligned}\mu(f(E)) &\leq \sum_{k=1}^{\infty} \mu(f([a_k, b_k] \cap \mathbb{T})) \\ &\leq \sum_{k=1}^{\infty} |f(d_k) - f(c_k)| \\ &\leq \epsilon.\end{aligned}$$

Because $\epsilon > 0$ was arbitrarily chosen, we get that

$$\mu(f(E)) = 0.$$

2. It follows that $\bar{f}$ maps every null subset of $[a, b]$ into a null set. Hence and the Banach-Zarecki theorem, it follows that $\bar{f}$ is absolutely continuous on $[a, b]$. By the equality $f = \bar{f}|_{\mathbb{T}}$, we conclude that f is absolutely continuous on $\mathbb{T}$. This completes the proof.

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Chapter 12: Alternative Way for Defining of Lebesgue Type Measure and Integration on Time Scales

Let $a = \inf \mathbb{T}$ and $b = \sup \mathbb{T}$. We assume that $-\infty < a < \rho(b) \leq b < \infty$. Define the function $E : [a, b) \rightarrow \mathbb{R}$ as follows

$$E(t) = \sup \{s \in \mathbb{T} : s \leq t\}, \quad t \in [a, b].$$

Suppose that $u : \mathbb{T}^{\kappa} \rightarrow \mathbb{R}$ is an arbitrary function. Then the function $u \circ E : [a, b) \rightarrow \mathbb{R}$ is an extension of u to $[a, b)$ and it is constant on "gaps" in $\mathbb{T}$, with value equal to the value of u at left-hand end of the gap.

Definition

We say that u is Lebesgue delta measurable, respectively Lebesgue delta integrable, if $u \circ E$ is Lebesgue measurable, respectively Lebesgue integrable, on the real interval $[a, b)$ in the usual Lebesgue sense. Let $\mathcal{L}_{\Delta}^1(\mathbb{T})$ denotes the set of such integrable functions on $\mathbb{T}$.

Definition

For any $u \in \mathcal{L}_{\Delta}^1(\mathbb{T})$, we define the integral of u by

$$\int_s^t u \Delta \mu_{\Delta} = \int_s^t u \circ E dx, \quad s, t \in \mathbb{T},$$

where the integration on the right-hand side is the standard Lebesgue integral over the real interval $[s, t] \subset [a, b]$.

Using the above construction, we can also define a Lebesgue delta measure and a class of Lebesgue delta measurable subsets of $\mathbb{T}$, and Lebesgue delta integrals over such sets. For a set $A \subseteq \mathbb{T}$, with χ_A we will denote the characteristic function of A .

Definition

We say that A is Lebesgue delta measurable if χ_A is Lebesgue delta measurable in the sense of Definition ???.

Definition

If $A \subseteq \mathbb{T}$ is measurable, we define the integral of $u \in \mathcal{L}_{\Delta}^1(\mathbb{T})$ over A to be

$$\int_A u \Delta \mu_{\Delta} = \int_a^b u \chi_A \Delta \mu_{\Delta}.$$

We define the measure $\mu_{\Delta}(A)$ of A to be

$$\mu_{\Delta}(A) = \int_A 1 \Delta \mu_{\Delta}.$$

From the definition, it follows

$$\mu_{\Delta}([s, t)_{\mathbb{T}}) = t - s, \quad s, t \in \mathbb{T}, \quad s < t,$$

and

$$\mu_{\Delta}(\{b\}) = 0, \quad \mu_{\mathbb{T}}(\{t\}) = \sigma(t) - t, \quad t \in \mathbb{T}.$$

Also, if $t \in \mathbb{T}$ is right-scattered and $u \in \mathcal{L}_{\Delta}^1(\mathbb{T})$, then $u(t)$ is well-defined and

$$\int_t^{\sigma(t)} u \Delta = u(t) (\sigma(t) - t).$$

Note

From the definition ???, it follows that the most of the standard Lebesgue integration results on $[a, b]$ hold immediately for the above integral on $\mathbb{T}$. In particular, Minkowsky's type inequality and Hölder's type inequality, and the dominated convergence theorem hold.

For $u \in \mathcal{C}_{rd}(\mathbb{T})$, we define

$$|u|_{0,\mathbb{T}} = \sup \{|u(t)| : t \in \mathbb{T}\}.$$

For $u \in \mathcal{C}_{rd}^1(\mathbb{T})$, we define

$$|u|_{1,\mathbb{T}} = |u|_{0,\mathbb{T}} + |u^\Delta|_{0,\mathbb{T}^\kappa}.$$

Theorem

If $u \in \mathcal{L}^1(\mathbb{T})$ and $s \in \mathbb{T}$, then the function

$$U_s(t) = \int_s^t u \Delta \mu_\Delta, \quad t \in \mathbb{T},$$

is well-defined and $U_s \in \mathcal{C}(\mathbb{T})$ with

$$|U_s|_{0,\mathbb{T}} \leq \int_a^b |u| \Delta \mu_\Delta.$$

Proof

By the definition, we get

$$\begin{aligned} U_s(t) &= \int_s^t u \Delta \mu_\Delta \\ &= \int_s^t u \circ E dx, \quad t \in \mathbb{T}. \end{aligned}$$

Hence, using the standard Lebesgue integration theory, the integral $\int_s^t u \circ E dx$ defines an absolutely continuous function of $t \in [a, b]$. Therefore U_s is well-defined on $\mathbb{T}$. Also,

$$\begin{aligned} |U_s(t)| &= \left| \int_s^t u \circ E dx \right| \\ &\leq \int_s^t |u \circ E| dx \\ &\leq \int_a^b |u \circ E| dx \\ &\leq \int_a^b |u| \Delta, \quad t \in \mathbb{T}. \end{aligned}$$

Hence,

$$\begin{aligned} |U_s|_{0,\mathbb{T}} &= \sup \{|U_s(t)| : t \in \mathbb{T}\} \\ &\leq \int_a^b |u| \Delta \mu_\Delta. \end{aligned}$$

This completes the proof.

Theorem

If $u \in \mathcal{C}^1(\mathbb{T})$, then

$$u(t) - u(s) = \int_s^t u^\Delta \Delta \mu_\Delta, \quad s, t \in \mathbb{T}.$$

Proof

Let $s \in \mathbb{T}$ be fixed. We define the function

$$H(t) = u(t) - u(s) - \int_s^t u^\Delta \Delta \mu_\Delta, \quad t \in \mathbb{T}.$$

Since $u^\Delta \in \mathcal{C}(\mathbb{T})$, by Theorem ???, it follows that $\int_s^t u^\Delta \Delta \mu_\Delta$ is a continuous function of $t \in \mathbb{T}$. Therefore H is a continuous function on $\mathbb{T}$. Let $t \in \mathbb{T}$ be arbitrarily chosen and fixed. We take $\epsilon > 0$ arbitrarily. Then

$$\begin{aligned} |H(\sigma(t)) - H(t)| &= \left| u(\sigma(t)) - u(s) - \int_s^{\sigma(t)} u^\Delta \Delta \mu_\Delta - u(t) + u(s) + \int_s^t u^\Delta \Delta \mu_\Delta \right| \\ &= \left| u(\sigma(t)) - u(t) - \int_t^{\sigma(t)} u^\Delta \Delta \mu_\Delta \right| \\ &= \left| u(\sigma(t)) - u(t) - u^\Delta(t)(\sigma(t) - t) \right| \\ &\leq \epsilon(\sigma(t) - t). \end{aligned}$$

In the last inequality we have used that $u \in \mathcal{C}^1(\mathbb{T})$. Since H is a continuous function on $\mathbb{T}$, there is a $\delta > 0$ such that for any $z \in (t - \delta, t + \delta)$ we have

$$|H(t) - H(z)| < \epsilon.$$

Therefore

$$\begin{aligned} |H(\sigma(t)) - H(z)| &= |H(\sigma(t)) - H(t) + H(t) - H(z)| \\ &\leq |H(\sigma(t)) - H(t)| + |H(t) - H(z)| \\ &< \epsilon(\sigma(t) - t) + \epsilon \end{aligned}$$

for any $z \in (t - \delta, t + \delta)$. Since $t \in \mathbb{T}$ was arbitrarily chosen, we conclude that $H^\Delta(t)$ exists and $H^\Delta(t) = 0$ for any $t \in \mathbb{T}$. Hence, using the Mean Value Theorem, we conclude that H is a constant on $\mathbb{T}$. Because $H(s) = 0$, we obtain $H \equiv 0$ on $\mathbb{T}$. This completes the proof.

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Chapter 13: The Fundamental Theorem of Calculus

Firstly, we will show some properties about Δ -differentiability and differentiability.

Theorem

Let $I \subset \mathbb{N}$ and $R = \{t_i\}_{i \in I}$ be given as follows

$$R = \{t \in \mathbb{T} : t < \sigma(t)\} = \{t_i\}_{i \in I}. \quad (\text{Chapter 13.1})$$

Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ and $\bar{f} : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is the extension of f to $[a, b]_{\mathbb{T}}$, defined by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in \mathbb{T} \\ f(t_i) + \frac{f(\sigma(t_i)) - f(t_i)}{\mu(t_i)}(t - t_i) & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I, \end{cases} \quad (\text{Chapter 13.2})$$

Then we have the following statements.

1. $\bar{f}$ is differentiable on the set $\bigcup_{i \in I} (t_i, \sigma(t_i))$ and

$$\bar{f}'(t) = \frac{f(\sigma(t_i)) - f(t_i)}{\mu(t_i)},$$

$t \in (t_i, \sigma(t_i))$, for some $i \in I$.

2. If f is differentiable at $t \in [a, b) \cap \mathbb{T}$, then f is Δ -differentiable at t and its Δ -derivative at t satisfies

$$f^\Delta(t) = \bar{f}'(t).$$

3. If f is Δ -differentiable at $([a, b]_{\mathbb{T}} \setminus (R \cup \sigma(R)))$, then $\bar{f}$ is differentiable at t and its derivative at t satisfies the equality

$$\bar{f}'(t) = f^\Delta(t).$$

Proof

1. By the definition of $\bar{f}$, it follows that there exists $\bar{f}'(t)$ at any $t \in \bigcup_{i \in I} (t_i, \sigma(t_i))$ and

$$\bar{f}'(t) = \frac{f(\sigma(t_i)) - f(t_i)}{\mu(t_i)}.$$

2. Let $t \in [a, b)_{\mathbb{T}} \cap \mathbb{T}$ be such that $\bar{f}$ is differentiable at t .

(a) Let t be right-scattered. Then there is a $t_i \in R$ such that $t = t_i$. Since $\bar{f}$ is continuous at t_i , then f is continuous at t_i and so, f is Δ -differentiable at t_i , and

$$\begin{aligned} f^\Delta(t_i) &= \frac{f(\sigma(t_i)) - f(t_i)}{\mu(t_i)} \\ &= \bar{f}'(t_i). \end{aligned}$$

(b) Let t is right-dense. We have

$$\begin{aligned} \bar{f}'(t) &= \lim_{s \rightarrow t} \frac{\bar{f}(t) - \bar{f}(s)}{t - s} \\ &= \lim_{s \rightarrow t, s \in \mathbb{T}} \frac{\bar{f}(t) - \bar{f}(s)}{t - s} \\ &= \lim_{s \rightarrow t, s \in \mathbb{T}} \frac{f(t) - f(s)}{t - s} \\ &= f^\Delta(t). \end{aligned}$$

Therefore f is Δ -differentiable at t and

$$f^\Delta(t) = \bar{f}'(t).$$

3. Let $t \in ([a, b] \cap \mathbb{T}) \setminus (R \cup \sigma(R))$ and f is Δ -differentiable at t . Take $\epsilon > 0$ arbitrarily. Then there exists a $\delta > 0$ such that

$$\left| \frac{\bar{f}(t) - \bar{f}(s)}{t - s} - f^\Delta(t) \right| = \left| \frac{f(t) - f(s)}{t - s} - f^\Delta(t) \right| < \epsilon \quad (\text{Chapter 13.3})$$

for all $s \in (t - \delta, t + \delta)_{\mathbb{T}} \cap \mathbb{T}$, $s \neq t$. Let $s \in (t - \delta, t + \delta)_{\mathbb{T}} \setminus \mathbb{T}$. Then there is an $i \in I$ such that $s \in (t_i, \sigma(t_i))$ and $t_i, \sigma(t_i) \in (t - \delta, t + \delta)$. Define

$$m = \min \left\{ \frac{f(t) - f(t_i)}{t - t_i}, \frac{f(t) - f(\sigma(t_i))}{t - \sigma(t_i)} \right\},$$

$$M = \max \left\{ \frac{f(t) - f(t_i)}{t - t_i}, \frac{f(t) - f(\sigma(t_i))}{t - \sigma(t_i)} \right\}.$$

We have that

$$m \leq \frac{\bar{f}(t) - \bar{f}(s)}{t - s} \leq M, \quad s \in (t_i, \sigma(t_i)).$$

Hence and (Chapter 13.3), we get that $\bar{f}$ is differentiable at t and

$$\bar{f}'(t) = f^\Delta(t).$$

This completes the proof.

Theorem (The Fundamental Theorem of Calculus)

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is absolutely continuous on $\mathbb{T}$ if and only if the following conditions are satisfied.

1. f is Δ -differentiable μ_Δ -a.e. on $[a, b]_{\mathbb{T}}$ and $f^\Delta \in \mathcal{L}_\Delta^1(\mathbb{T})$.
2. The equality

$$f(t) = f(a) + \int_{[a, t]_{\mathbb{T}}} f^\Delta \Delta \mu_\Delta$$

holds for every $t \in \mathbb{T}$.

Proof

Let $\bar{f} : [a, b] \rightarrow \mathbb{R}$ be the extension of the function f , defined by (Chapter 13.2).

1. Let f be an absolutely continuous function on $\mathbb{T}$. Then $\bar{f}$ is an absolutely continuous function on $[a, b]$. Hence and the classical fundamental theorem of calculus, it follows that

(a) $\bar{f}$ is differentiable a.e. on $[a, b]$ and $\bar{f}' \in \mathcal{L}^1([a, b])$. (b) it is valid the following equality

$$\bar{f}(t) = \bar{f}(a) + \int_{[a, t]} \bar{f}'(s) ds$$

for any $t \in [a, b]$. By Theorem ???, (2), it follows that

$$\begin{aligned} E_1 &= \{t \in [a, b] \cap \mathbb{T} : \nexists f^\Delta(t)\} \\ &\subset \{t \in [a, b] \cap \mathbb{T} : \nexists \bar{f}'(t)\} \\ &= E_2. \end{aligned} \quad (\text{Chapter 13.4})$$

Since f is continuous on $\mathbb{T}$, we have that f is Δ -differentiable at any right-scattered points of $\mathbb{T}$. Thus, the set E_1 does not have any right-scattered points. Therefore

$$\mu_\Delta(E_1) = \mu(E_1). \quad (\text{Chapter 13.5})$$

Because $\bar{f}$ is differentiable a.e. on $[a, b]$, we have that

$$\mu(E_2) = 0.$$

Hence and the relations (Chapter 13.4), (Chapter 13.5), we get

$$\begin{aligned}\mu_\Delta(E_1) &= \mu(E_1) \\ &\leq \mu(E_2) \\ &= 0,\end{aligned}$$

i.e.,

$$\mu_\Delta(E_1) = 0$$

and f is Δ -differentiable a.e. on $[a, b]_{\mathbb{T}}$. Now, we redefine $f^\Delta : \mathbb{T} \rightarrow \mathbb{R}$ in the following manner.

$$f^\Delta(t) = \begin{cases} f^\Delta(t) & \text{if } t \in ([a, b] \cap \mathbb{T}) \setminus E_1 \\ 0 & \text{if } t \in E_1 \cup \{b\}. \end{cases}$$

Define

$$\widetilde{f^\Delta}(t) = \begin{cases} f^\Delta(t) & \text{if } t \in \mathbb{T} \\ f^\Delta(t_i) & \text{if } t \in (t_i, \sigma(t_i)) \text{ for some } i \in I. \end{cases} \quad (\text{Chapter 13.6})$$

Then, by Theorem ??? (2), it follows that

$$\widetilde{f^\Delta}(t) = f'(t), \quad t \in [a, b] \setminus E_2.$$

Thus, $\mu(E_2) = 0$ and $\bar{f}' \in \mathbb{L}^1([a, b]_{\mathbb{T}})$, whereupon $\widetilde{f^\Delta} \in \mathcal{L}_\Delta^1([a, b]_{\mathbb{T}})$ and

$$\int_{[a, t]_{\mathbb{T}}} \widetilde{f^\Delta}(s) ds = \int_{[a, t]} \bar{f}'(s) ds, \quad t \in [a, b]_{\mathbb{T}}.$$

Therefore $f^\Delta \in \mathcal{L}_\Delta^1(\mathbb{T})$ and

$$\begin{aligned}\int_{[a, t]_{\mathbb{T}}} f^\Delta \Delta \mu_\Delta &= \int_{[a, t]} \widetilde{f^\Delta}(s) ds \\ &= \int_{[a, t]} \bar{f}'(s) ds, \quad t \in \mathbb{T},\end{aligned}$$

and

$$\begin{aligned}f(t) - f(a) &= \bar{f}(t) - \bar{f}(a) \\ &= \int_{[a, t]} \bar{f}'(s) ds \\ &= \int_{[a, t] \cap \mathbb{T}} f^\Delta(s) \Delta s, \quad t \in \mathbb{T}.\end{aligned}$$

2. Suppose (1) and (2). Let $\widetilde{f^\Delta}$ be the extension of f^Δ , defined by (Chapter 13.6). Since $f^\Delta \in \mathcal{L}_\Delta^1(\mathbb{T})$, we get $\widetilde{f^\Delta} \in \mathcal{L}_\Delta^1([a, b]_{\mathbb{T}})$. From here,

$$\begin{aligned}f(t) - f(a) &= \int_{[a, t]_{\mathbb{T}}} f^\Delta(s) \Delta s \\ &= \int_{[a, t]} \widetilde{f^\Delta}(s) ds, \quad t \in \mathbb{T},\end{aligned}$$

and

$$\bar{f}(t) = \bar{f}(a) + \int_{[a, t]} \widetilde{f^\Delta}(s) ds, \quad t \in [a, b]_{\mathbb{T}}.$$

Hence and the classical fundamental theorem of calculus, it follows that $\bar{f}$ is an absolutely continuous function on $[a, b]$. Therefore f is absolutely continuous on $[a, b]_{\mathbf{T}}$. This completes the proof.

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