

# RIEMANN-STIELTJES INTEGRATION ON TIME SCALES



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# Riemann\_Stieltjes Integration on Time Scales

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## 1: The Darboux-Stieltjes Delta Integral

Throughout this cycle,  $\mathbb{T}$  denotes a time scale with delta operator and forward jump operator  $\Delta$  and  $\sigma$ , respectively. Let  $a, b \in \mathbb{T}$ ,  $a < b$ . Define

$$I = [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}. \quad (1.1)$$

### Definition: Partition

A partition of the interval  $I$  is any finite ordered subset

$$P = \{t_0, t_1, \dots, t_n\} \subset [a, b]_{\mathbb{T}}, \quad (1.2)$$

where

$$a = t_0 < t_1 < \dots < t_n = b. \quad (1.3)$$

### ✓ Example 1.1

Let  $\mathbb{T} = 2^{\mathbb{N}_0}$  and  $a = 1$ ,  $b = 64$ . Then

$$P = \{1, 4, 16, 64\} \quad (1.4)$$

is a partition of the interval  $[1, 64]_{\mathbb{T}}$ .

### ✓ Example 1.2

Let  $U = \{\frac{1}{2^n}\}_{n \in \mathbb{N}_0}$ ,  $\mathbb{T} = (1 - U) \cup \{1\}$ ,  $a = 0$ ,  $b = 1$ . Then

$$P = \left\{0, \frac{1}{1024}, \frac{1}{256}, \frac{1}{32}, \frac{1}{2}, 1\right\} \quad (1.5)$$

is a partition of the interval  $[0, 1]_{\mathbb{T}}$ .

Each partition  $P = \{t_0, t_1, \dots, t_n\}$  of the interval  $I$  decomposes  $I$  into subintervals

$$I_j = I_j, \quad j \in \{1, \dots, n\}, \quad (1.6)$$

such that  $I_j \cap I_k = \emptyset$ ,  $j \neq k$ ,  $j, k \in \{1, \dots, n\}$ . By  $\Delta t_j = t_j - t_{j-1}$ ,  $j \in \{1, \dots, n\}$ , we denote the length of  $j$ 'th subinterval of the partition  $P$ . With  $\mathcal{P}(I)$  we denote the set of all partitions of  $I$ .

### Definition:

 Let  $P_m, P_n \in \mathcal{P}(I)$ . If  $P_m \subset P_n$ , then we say that  $P_n$  is a refinement of  $P_m$ .

If  $P_m$  and  $P_n$  are arbitrarily chosen partitions of  $I$ , then  $P_m \cup P_n$  is a common refinement of  $P_m$  and  $P_n$ .

 Definition:

 For a partition  $P \in \mathcal{P}(I)$  and a strictly increasing function  $g$  on  $I$ , we define

$$g(P) = \{g(a) = g(t_0), g(t_1), \dots, g(t_n) = g(b)\} \subset g(I) \quad (1.7)$$

and

$$\Delta g_j = g(t_j) - g(t_{j-1}), \quad j \in \{1, \dots, n\}. \quad (1.8)$$

Note that  $\Delta g_j > 0$ ,  $j \in \{1, \dots, n\}$ , and

$$\begin{aligned} \sum_{j=1}^n \Delta g_j &= \sum_{j=1}^n (g(t_j) - g(t_{j-1})) \\ &= g(t_1) - g(a) + g(t_2) - g(t_1) + \dots + g(b) - g(t_{n-1}) \\ &= g(b) - g(a). \end{aligned}$$

Moreover, we have that  $g(P)$  is a partition of the interval  $[g(a), g(b)]$ .

 Theorem 1.1

 For any strictly increasing function  $g$  on  $I$  and  $\delta \in (0, g(b) - g(a))$ , there is a partition

$$P_\delta = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(I) \quad (1.9)$$

such that for each  $j \in \{1, \dots, n\}$  one has

$$\Delta g_j \leq \delta \quad \text{or} \quad \Delta g_j \geq \delta, \quad \rho(t_j) \geq t_{j-1}. \quad (1.10)$$

 Proof

 Since  $g$  is strictly increasing, we have that from  $t \in \mathbb{T}$  and  $g(t) \in [g(a), g(b)]$ , it follows that  $t \in [a, b]_{\mathbb{T}}$ . For  $j \in \mathbb{N}$ , define the following families of sets

$$B_j = (g(t_{j-1}), g(t_{j-1}) + \delta],$$

$$A_j = \{t \in I : g(t) \in B_j\}.$$

Inductively, we construct the following partition:  $t_0 = a$  and

$$t_j = \begin{cases} \sup A_j & \text{if } A_j \neq \emptyset \\ \sigma(t_{j-1}) & \text{if } A_j = \emptyset, \quad t_{j-1} \text{ is right-scattered} \\ g(t_j) = \inf(g(t_{j-1}) + \delta, g(b)) & \text{if } A_j = \emptyset, \quad t_{j-1} \text{ is right-dense.} \end{cases} \quad (1.11)$$

Assume that the constructed partition has  $n \in \mathbb{N}$ ,  $n \geq 2$ , elements. If  $A_1 \neq \emptyset$ , then  $g(t_1) \in B_1 = (g(a), g(a) + \delta]$ . Hence,  $g(t_1) > g(a)$  and  $t_1 > t_0$ . Let  $A_1 = \emptyset$ . Then, we have the following cases.

1.  $t_0$  is right-scattered. Then  $t_1 > t_0$ .

2.  $t_0$  is right-dense. Then  $g(t_1) \in [g(a) + \delta, g(b)]$ . Hence,  $g(t_1) > g(a)$  and  $t_1 > t_0$ .

Assume that

$$a = t_0 < t_1 < \dots < t_j \quad (1.12)$$

for some  $j \in \{0, 1, \dots, n-1\}$ . If  $A_{j+1} \neq \emptyset$ , then  $g(t_{j+1}) \in (g(t_j), g(t_j) + \delta)$  and  $g(t_{j+1}) > g(t_j)$ , and then  $t_{j+1} > t_j$ . Let  $A_{j+1} = \emptyset$ . Then, we have the following cases.

1.  $t_j$  is right-scattered. Then  $t_{j+1} > t_j$ .

2.  $t_j$  is right-dense. Then  $g(t_{j+1}) \in [g(t_j) + \delta, g(b)]$  and  $g(t_{j+1}) > g(t_j)$ , and then  $t_{j+1} > t_j$ .

Therefore

$$a = t_0 < t_1 < \dots < t_n = b. \quad (1.13)$$

Now, take  $j \in \{1, \dots, n\}$  arbitrarily. Consider the following cases.

1.  $t_j = \sup A_j$ . Then  $g(t_j) \in (g(t_{j-1}), g(t_{j-1}) + \delta]$ . Hence,

$$\begin{aligned} \Delta g_j &= g(t_j) - g(t_{j-1}) \\ &\leq g(t_{j-1}) + \delta - g(t_{j-1}) \\ &= \delta. \end{aligned}$$

2.  $t_j = \sigma(t_{j-1})$  and  $t_{j-1}$  is right-scattered. Then  $A_j = \emptyset$  and

$$g(t_j) \geq g(t_{j-1}) + \delta, \quad (1.14)$$

and

$$\begin{aligned} \Delta g_j &= g(t_j) - g(t_{j-1}) \\ &\geq g(t_{j-1}) + \delta - g(t_{j-1}) \\ &= \delta. \end{aligned}$$

By the inequality (1.14), using that

$$g \quad (1.15)$$

is strictly increasing, we obtain that

$$\rho(t_j) \geq t_{j-1} \quad (1.16)$$

3.  $t_j$  is such that

$$g(t_j) = \inf(g(t_{j-1}) + \delta, g(b)). \quad (1.17)$$

Hence, we get (1.14) and then  $\Delta g_j \geq \delta$ , and  $\rho(t_j) \geq t_{j-1}$ . This completes the proof.

 Definition:

 We denote by  $\mathcal{P}_\delta(I)$  the set of all partitions  $P_\delta$  of the interval  $I$  that posses the property indicated in Theorem 1.1.

For a real-valued function  $f$  on  $I$ , denote

$$M_j = \inf_{t \in I_j} f(t) \quad \text{and} \quad M_j = \sup_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}. \quad (1.18)$$

 Definition:

 For any strictly increasing function  $g$  on  $I$  and function  $f$  on  $I$ , the upper Darboux-Stieltjes sum of  $f$  with respect to the partition  $P$ , denoted by  $U(P, f, g)$ , is defined by

$$U(P, f, g) = \sum_{j=1}^n M_j \Delta g_j. \quad (1.19)$$

The lower Darboux-Stieltjes sum of  $f$  with respect to the partition  $P$ , denoted by  $L(P, f, g)$ , is defined by

$$L(P, f, g) = \sum_{j=1}^n m_j \Delta g_j. \quad (1.20)$$

Evidently, we have

$$L(P, f, g) \leq U(P, f, g) \quad (1.21)$$

for any  $P \in \mathcal{P}(I)$ .

✓ Example 1.3

✓ Let  $\mathbb{T} = \overline{2^{\mathbb{Z}}}$ ,  $f(t) = t$ ,  $g(t) = t^2$  and  $I = [0, 1]_{\mathbb{T}}$ . For a partition, we take

$$P = \left\{ 0, \frac{1}{2^{n-1}}, \frac{1}{2^{n-2}}, \dots, \frac{1}{2}, 1 \right\}. \quad (1.22)$$

Here

$$t_j = 2^{j-n}, \quad j \in \{1, \dots, n\}. \quad (1.23)$$

We have

$$\begin{aligned}
 M_j &= \sup_{t \in I_j} f(t) \\
 &= t_{j-1} \\
 &= 2^{j-n-1}, \\
 m_j &= \inf_{t \in I_j} f(t) \\
 &= t_{j-1} \\
 &= 2^{j-1-n}, \quad j \in \{1, \dots, n\},
 \end{aligned}$$

and

$$\begin{aligned}
 \Delta g_j &= g(t_j) - g(t_{j-1}) \\
 &= t_j^2 - t_{j-1}^2 \\
 &= 2^{2j-2n} - 2^{2(j-1)-2n} \\
 &= 2^{2j-2n} - 2^{2j-2n-2} \\
 &= 2^{2j-2n} \left(1 - \frac{1}{4}\right) \\
 &= \frac{3}{4} 2^{2j-2n} \\
 &= 3 \cdot 2^{2j-2n-2}, \quad j \in \{1, \dots, n\}.
 \end{aligned}$$

Consequently, for the upper and lower Darboux-Stieltjes sums, we get

$$\begin{aligned}
 U(P, f, g) &= \sum_{j=1}^n M_j \Delta g_j \\
 &= \sum_{j=1}^n 2^{j-n-1} (3 \cdot 2^{2j-2n-2}) \\
 &= \frac{3}{8} \sum_{j=1}^n 2^{3j-3n}
 \end{aligned}$$

and

$$\begin{aligned}
 L(P, f, g) &= \sum_{j=1}^n m_j \Delta g_j \\
 &= \sum_{j=1}^n 2^{j-n-1} (3 \cdot 2^{2j-2n-2}) \\
 &= \frac{3}{8} \sum_{j=1}^n 2^{3j-3n},
 \end{aligned}$$

respectively.

### Theorem 1.2

 Let  $f$  be a bounded function on  $I$  and  $g$  be a strictly increasing function on  $I$ ,  $P \in \mathcal{P}(I)$  and

$$m \leq f(t) \leq M, \quad t \in I. \quad (1.24)$$

Then

$$m(g(b) - g(a)) \leq L(P, f, g) \leq U(P, f, g) \leq M(g(b) - g(a)) \quad \text{for any } P \in \mathcal{P}(I). \quad (1.25)$$

### Proof

 Take  $P \in \mathcal{P}(I)$  arbitrarily. Denote

$$M_j = \sup_{t \in I_j} f(t),$$

$$m_j = \inf_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}.$$

We have

$$m_j \geq m \quad \text{and} \quad M_j \leq M, \quad j \in \{1, \dots, n\}. \quad (1.26)$$

Then

$$\begin{aligned} L(P, f, g) &= \sum_{j=1}^n m_j \Delta g_j \\ &\geq m \sum_{j=1}^n \Delta g_j \\ &= m(g(b) - g(a)) \end{aligned}$$

and

$$\begin{aligned} U(P, f, g) &= \sum_{j=1}^n M_j \Delta g_j \\ &\leq M \sum_{j=1}^n \Delta g_j \\ &= M(g(b) - g(a)). \end{aligned}$$

This completes the proof.

### Theorem 1.3

 Let  $f$  be a bounded function on  $I$ ,  $g$  be a strictly increasing function on  $I$ ,  $P, Q \in \mathcal{P}(I)$  and  $Q$  be a refinement of  $P$ . Then

$$L(P, f, g) \leq L(Q, f, g) \leq U(Q, f, g) \leq U(P, f, g). \quad (1.27)$$

 **Proof**

 Let

$$P = \{t_0, t_1, \dots, t_n\}. \quad (1.28)$$

Without loss of generality, assume that

$$Q = P \cup \{t_m\}, \quad t_k < t_m < t_{k+1}, \quad (1.29)$$

for some  $k \in \{0, 1, \dots, n-1\}$ . Let

$$M_j = \sup_{t \in I_j} f(t),$$

$$m_j = \inf_{t \in I_j} f(t),$$

$$m_{k1} = \inf_{t \in [t_k, t_m)} f(t),$$

$$m_{k2} = \inf_{t \in [t_m, t_{k+1})} f(t),$$

$$M_{k1} = \sup_{t \in [t_k, t_m)} f(t),$$

$$M_{k2} = \sup_{t \in [t_m, t_{k+1})} f(t),$$

$$\Delta g_{k1} = g(t_m) - g(t_k),$$

$$\Delta g_{k2} = g(t_{k+1}) - g(t_m).$$

We have

$$L(P, f, g) = \sum_{j=1}^n m_j \Delta g_j \quad \text{and} \quad U(P, f, g) = \sum_{j=1}^n M_j \Delta g_j,$$

and

$$m_{k1} \geq m_{k+1}, \quad m_{k2} \geq m_{k+1}, \quad M_{k1} \leq M_{k+1}, \quad M_{k2} \leq M_{k+1}. \quad (1.30)$$

Hence,

$$\begin{aligned}
 L(Q, f, g) &= \sum_{j=1}^k m_j \Delta g_j + m_{k1} \Delta g_{k1} + m_{k2} \Delta g_{k2} + \sum_{j=k+2}^n m_j \Delta g_j \\
 &\geq \sum_{j=1}^k m_j \Delta g_j + m_{k+1} \Delta g_{k1} + m_{k+1} \Delta g_{k2} + \sum_{j=k+2}^n m_j \Delta g_j \\
 &= \sum_{j=1}^k m_j \Delta g_j + m_{k+1} \Delta g_{k+1} + \sum_{j=k+2}^n m_j \Delta g_j \\
 &= \sum_{j=1}^n m_j \Delta g_j \\
 &= L(P, f, g)
 \end{aligned}$$

and

$$\begin{aligned}
 U(Q, f, g) &= \sum_{j=1}^k M_j \Delta g_j + M_{k1} \Delta g_{k1} + M_{k2} \Delta g_{k2} + \sum_{j=k+2}^n M_j \Delta g_j \\
 &\leq \sum_{j=1}^k M_j \Delta g_j + M_{k+1} \Delta g_{k1} + M_{k+1} \Delta g_{k2} + \sum_{j=k+2}^n M_j \Delta g_j \\
 &= \sum_{j=1}^k M_j \Delta g_j + M_{k+1} \Delta g_{k+1} + \sum_{j=k+2}^n M_j \Delta g_j \\
 &= \sum_{j=1}^n M_j \Delta g_j \\
 &= U(P, f, g).
 \end{aligned}$$

This completes the proof.

#### Definition:

 Let  $g$  be a strictly increasing function on  $I$  and  $f$  be a function on  $I$ .

The upper Darboux-Stieltjes delta integral of the function  $f$  from  $a$  to  $b$  with respect to the function  $g$  is defined by

$$\overline{\int_a^b f(t) \Delta g(t)} = \inf_{P \in \mathcal{P}(I)} U(P, f, g). \quad (1.31)$$

The lower Darboux-Stieltjes delta integral of the function  $f$  from  $a$  to  $b$  with respect to the function  $g$  is defined as follows

$$\underline{\int_a^b f(t) \Delta g(t)} = \sup_{P \in \mathcal{P}(I)} L(P, f, g). \quad (1.32)$$

### Theorem 1.4

 Let  $f$  be a bounded function on  $I$  and  $g$  be a strictly increasing function on  $I$ . Then

$$\int_a^b f(t) \Delta g(t) \leq \overline{\int_a^b f(t) \Delta g(t)}. \quad (1.33)$$

### Proof

 Let  $P_1$  and  $P_2$  be two different partitions of  $I$ . Let also,  $P_3$  be their common refinement. Then, applying Theorem 1.4, we have

$$L(P_1, f, g) \leq L(P_3, f, g) \leq U(P_3, f, g) \leq U(P_2, f, g). \quad (1.34)$$

This implies that the lower Darboux-Stieltjes sum for any partition  $P_1$  of  $I$  is always bounded from above by the upper Darboux-Stieltjes sum of any other partition  $P_2$  of  $I$ .

 Fix an arbitrary partition  $P$  of  $I$ . By the inequality (1.34), we have

$$L(P', f, g) \leq U(P, f, g) \quad (1.35)$$

for any partition  $P'$  of  $I$ . Hence,

$$\sup_{P' \in \mathcal{P}(I)} L(P', f, g) \leq U(P, f, g), \quad (1.36)$$

whereupon

$$\int_a^b f(t) \Delta g(t) \leq U(P, f, g). \quad (1.37)$$

From the last inequality, we arrive at

$$\begin{aligned} \int_a^b f(t) \Delta g(t) &\leq \inf_{P \in \mathcal{P}(I)} U(P, f, g) \\ &= \overline{\int_a^b f(t) \Delta g(t)}. \end{aligned}$$

This completes the proof.

### Definition:

 Let  $f$  be a function on  $I$  and  $g$  be a strictly increasing function on  $I$ .  
If

$$\overline{\int_a^b f(t) \Delta g(t)} = \underline{\int_a^b f(t) \Delta g(t)}, \quad (1.38)$$

then we say that  $f$  is Darboux-Stieltjes delta integrable from  $a$  to  $b$  with respect to the function  $g$ . The common value of both integrals is denoted by

$$\text{DS} \int_a^b f(t) \Delta g(t) \quad (1.39)$$

and it is said to be the Darboux-Stieltjes delta integral of  $f$  from  $a$  to  $b$  with respect to  $g$ . The set of all functions that are Darboux-Stieltjes delta integrable from  $a$  to  $b$  with respect to  $g$  is denoted by  $\mathcal{D}(g, I)$ .

### Theorem 1.5 ( $\epsilon$ -Criterion for Darboux-Stieltjes Delta Integrability)

 Let  $f$  be a bounded function on  $I$  and  $g$  be a strictly increasing function on  $I$ . Then  $f \in \mathcal{D}(g, I)$  if and only if for any  $\epsilon > 0$  there exists a  $\delta = \delta(\epsilon) > 0$  such that  $P \in \mathcal{P}_\delta(I)$  implies

$$U(P, f, g) - L(P, f, g) < \epsilon. \quad (1.40)$$

#### Proof

 We prove the implications separately.

 1. Suppose that for any  $\epsilon > 0$  there exists a  $\delta = \delta(\epsilon) > 0$  such that  $P \in \mathcal{P}_\delta(I)$  implies

$$U(P, f, g) - L(P, f, g) < \epsilon. \quad (1.41)$$

By Theorem [PageIndex4](#), it follows that

$$\begin{aligned} 0 &\leq \overline{\int_a^b f(t) \Delta g(t)} - \underline{\int_a^b f(t) \Delta g(t)} \\ &\leq U(P, f, g) - L(P, f, g) \\ &< \epsilon. \end{aligned}$$

This implies that

$$\overline{\int_a^b f(t) \Delta g(t)} = \underline{\int_a^b f(t) \Delta g(t)} \quad (1.42)$$

and  $f \in \mathcal{D}(g, I)$ .

 2. Suppose that  $f \in \mathcal{D}(g, I)$ . Then

$$\sup_{P \in \mathcal{P}(I)} L(P, f, g) = \inf_{P \in \mathcal{P}(I)} U(P, f, g). \quad (1.43)$$

Fix  $\epsilon > 0$  arbitrarily. By the definition for infimum, we find that there is a partition  $P_1$  of  $I$  such that

$$\begin{aligned} \sup_{P \in \mathcal{P}(I)} L(P, f, g) &\leq U(P_1, f, g) \\ &< \sup_{P \in \mathcal{P}(I)} L(P, f, g) + \frac{\epsilon}{2} \end{aligned}$$

or

$$U(P_1, f, g) - \sup_{P \in \mathcal{P}(I)} L(P, f, g) < \frac{\epsilon}{2}. \quad (1.44)$$

Now, using the definition for supremum, there is a partition  $P_2$  of  $I$  such that

$$L(P_2, f, g) > \sup_{P \in \mathcal{P}(I)} L(P, f, g) - \frac{\epsilon}{2}. \quad (1.45)$$

Hence,

$$\sup_{P \in \mathcal{P}(I)} L(P, f, g) < L(P_2, f, g) + \frac{\epsilon}{2} \quad (1.46)$$

or

$$- \sup_{P \in \mathcal{P}(I)} L(P, f, g) > -L(P_2, f, g) - \frac{\epsilon}{2}. \quad (1.47)$$

Now, applying the inequality (1.44), we find

$$\begin{aligned} U(P_1, f, g) - L(P_2, f, g) - \frac{\epsilon}{2} &< U(P_1, f, g) - \sup_{P \in \mathcal{P}(I)} L(P, f, g) \\ &< \frac{\epsilon}{2}, \end{aligned}$$

whereupon

$$U(P_1, f, g) - L(P_2, f, g) < \epsilon. \quad (1.48)$$

Let  $P_3 = P_1 \cup P_2$ . Then  $P_3$  is a refinement of  $P_1$  and  $P_2$ . By Theorem 1.3, we get

$$L(P_2, f, g) \leq L(P_3, f, g) \quad (1.49)$$

and

$$U(P_1, f, g) \geq U(P_3, f, g). \quad (1.50)$$

Now, using (1.48), we arrive at

$$\begin{aligned} U(P_3, f, g) - L(P_3, f, g) &\leq U(P_1, f, g) - L(P_2, f, g) \\ &< \epsilon. \end{aligned}$$

This completes the proof.

### Corollary: 1.6

 Let  $I$  contains infinitely many points of  $\mathbb{T}$  and  $f$  be a bounded function on  $I$ , and  $g$  be a strictly increasing function on  $I$ . Then  $f \in \mathcal{D}(g, I)$  if and only if there is a sequence  $\{P_n\}_{n \in \mathbb{N}}$  of partitions of  $I$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} U(P_n, f, g) &= \lim_{n \rightarrow \infty} L(P_n, f, g) \\ &= \text{DS} \int_a^b f(t) \Delta g(t). \end{aligned}$$

### Prroof

 We prove the implications separately.

1. Let  $f \in \mathcal{D}(g, I)$ . By the  $\epsilon$ -criterion for Darboux-Stieltjes delta integration, we get that for any  $n \in \mathbb{N}$  there is a partition  $P_n$  of  $I$  such that

$$U(P_n, f, g) - L(P_n, f, g) < \frac{1}{n}. \quad (1.51)$$

Define the sequence  $\{Q_n\}_{n \in \mathbb{N}}$  of partitions of  $I$  in the following manner

$$\begin{aligned} Q_1 &= P_1, \\ Q_n &= Q_{n-1} \cup P_n, \quad n \in \mathbb{N}, \quad n \geq 2. \end{aligned}$$

Then

$$\begin{aligned} Q_1 &= P_1 \\ &\subset P_1 \cup P_2 \\ &= Q_1 \cup P_2 \\ &= Q_2. \end{aligned}$$

Assume that

$$Q_n \subset Q_{n+1} \quad (1.52)$$

for some  $n \in \mathbb{N}$ . We will prove that

$$Q_{n+1} \subset Q_{n+2}. \quad (1.53)$$

Really, we have

$$\begin{aligned}
 Q_{n+1} &= Q_n \cup P_{n+1} \\
 &\subset Q_n \cup P_{n+1} \cup P_{n+2} \\
 &= Q_{n+1} \cup P_{n+2} \\
 &= Q_{n+2}.
 \end{aligned}$$

Therefore the sequence  $\{Q_n\}_{n \in \mathbb{N}}$  is a sequence of refinements of the sequence  $\{P_n\}_{n \in \mathbb{N}}$ . By Theorem 1.3, it follows that the sequences  $\{U(Q_n, f, g)\}_{n \in \mathbb{N}}$  and  $\{L(Q_n, f, g)\}_{n \in \mathbb{N}}$  are bounded and monotone. Thus, both of them must converge. Since

$$\begin{aligned}
 U(Q_n, f, g) &\leq U(P_n, f, g), \\
 L(Q_n, f, g) &\geq L(P_n, f, g), \quad n \in \mathbb{N},
 \end{aligned}$$

we have

$$\begin{aligned}
 U(Q_n, f, g) - L(Q_n, f, g) &\leq U(P_n, f, g) - L(P_n, f, g) \\
 &< \frac{1}{n}, \quad n \in \mathbb{N}.
 \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  of both sides, we can deduce that

$$\lim_{n \rightarrow \infty} (U(Q_n, f, g) - L(Q_n, f, g)) = 0. \quad (1.54)$$

The algebra of limits then implies

$$\lim_{n \rightarrow \infty} U(Q_n, f, g) = \lim_{n \rightarrow \infty} L(Q_n, f, g). \quad (1.55)$$

2. Suppose that there exists a sequence  $\{P_n\}_{n \in \mathbb{N}}$  of partitions of  $I$  such that

$$\lim_{n \rightarrow \infty} U(P_n, f, g) = \lim_{n \rightarrow \infty} L(P_n, f, g). \quad (1.56)$$

By the algebra of limits, we have

$$\lim_{n \rightarrow \infty} (U(P_n, f, g) - L(P_n, f, g)) = 0. \quad (1.57)$$

This means for any  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that

$$\begin{aligned}
 |U(P_n, f, g) - L(P_n, f, g) - 0| &= |U(P_n, f, g) - L(P_n, f, g)| \\
 &< \epsilon \quad \text{for all } n \in \mathbb{N}.
 \end{aligned}$$

This gives us the  $\epsilon$ -criterion for Darboux-Stieltjes delta integral and so  $f \in \mathcal{D}(g, I)$ . Finally, for any  $n \in \mathbb{N}$ , we have

$$\begin{aligned} L(P_n, f, g) &\leq \int_a^b f(t) \Delta g(t) \\ &\leq \overline{\int_a^b f(t) \Delta g(t)} \\ &\leq U(P_n, f, g). \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ , and using the assumptions, we then have

$$\begin{aligned} \lim_{n \rightarrow \infty} L(P_n, f, g) &\leq \int_a^b f(t) \Delta g(t) \\ &\leq \overline{\int_a^b f(t) \Delta g(t)} \\ &\leq \lim_{n \rightarrow \infty} U(P_n, f, g). \end{aligned}$$

So,

$$\begin{aligned} \int_a^b f(t) \Delta g(t) &= \overline{\int_a^b f(t) \Delta g(t)} \\ &= \text{DS} \int_a^b f(t) \Delta g(t), \end{aligned}$$

which is what we claimed.

#### ✓ Example 1.4

✓ Let  $\mathbf{T}$ ,  $f$  and  $g$  be as in Example 1.3. By Example 1.3, we have

$$\begin{aligned} U(P, f, g) &= L(P, f, g) \\ &= \frac{3}{8} 2^{-3n} \sum_{j=1}^n 8^j \\ &= \frac{3}{8} 8^{-n} 8 \frac{8^n - 1}{8 - 1} \\ &= \frac{3}{7} (1 - 8^{-n}). \end{aligned}$$

Hence,

$$\begin{aligned}\inf_{P \in \mathcal{P}(I)} U(P, f, g) &= \sup_{P \in \mathcal{P}(I)} L(P, f, g) \\ &= \lim_{n \rightarrow \infty} \frac{3}{7} (1 - 8^{-n}) \\ &= \frac{3}{7}.\end{aligned}$$

Therefore

$$\text{DS} \int_0^1 t \Delta t^2 = \frac{3}{7}. \quad (1.58)$$

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## 2: The Riemann-Stieltjes Delta Integral

Consider a partition  $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(I)$ . For any  $j \in \{1, \dots, n\}$ , take  $\xi_j \in I_j$ .

### Definition:

Let  $f$  be a function on  $I$  and  $g$  be a strictly increasing function on  $I$ .

The Riemann-Stieltjes sum  $R(P, f, g)$  for the function  $f$  on the interval  $I$  with respect to the function  $g$  is defined as follows

$$R(P, f, g) = \sum_{j=1}^n f(\xi_j) \Delta g_j. \quad (2.1)$$

### Definition:

Let  $g$  be a strictly increasing function on  $I$ .

A function  $f$  on  $I$  is said to be Riemann-Stieltjes delta integrable on the interval  $I$  with respect to the function  $g$  if there exists a real number  $J$  such that for any  $\epsilon > 0$  there exists a  $\delta = \delta(\epsilon) > 0$  such that for any  $P \in \mathcal{P}_\delta(I)$  one has

$$|R(P, f, g) - J| < \epsilon \quad (2.2)$$

independent of the choice of  $\xi_j \in I_j$ ,  $j \in \{1, \dots, n\}$ .

The number  $J$  is called the Riemann-Stieltjes delta integral of the function  $f$  on the interval  $I$  with respect to the function  $g$ , also denoted as

$$\text{RS} \int_a^b f(t) \Delta g(t). \quad (2.3)$$

The set of all Riemann-Stieltjes delta integrable functions on the interval  $I$  with respect to the function  $g$  is denoted by  $\mathcal{R}(g, I)$ .

### Theorem 2.7

\begin{theorem}

Let  $f$  be a function on  $I$  and  $g$  be a strictly increasing function on  $I$ . If  $f \in \mathcal{R}(g, I)$ , then its Riemann-Stieltjes delta integral on the interval  $I$  with respect to the function  $g$  is unique.

#### Proof

Suppose the contrary, i.e., assume that we have two values for  $\text{RS} \int_a^b f(t) \Delta g(t)$ , namely  $J_1$  and  $J_2$ . Fix  $\epsilon > 0$  arbitrarily. By the definition for the Riemann-Stieltjes delta integral, it follows that there are  $\delta_1 = \delta_1(\epsilon) > 0$  and  $\delta_2 = \delta_2(\epsilon) > 0$  such that if  $P_1 \in \mathcal{P}_{\delta_1}(I)$ ,  $P_2 \in \mathcal{P}_{\delta_2}(I)$ , then one has

$$|R(P_1, f, g) - J_1| < \frac{\epsilon}{2} \quad (2.4)$$

and

$$|R(P_2, f, g) - J_2| < \frac{\epsilon}{2}. \quad (2.5)$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then, if  $P \in \mathcal{P}_\delta(I)$ , one has

$$|R(P, f, g) - J_1| < \frac{\epsilon}{2} \quad (2.6)$$

and

$$|R(P, f, g) - J_2| < \frac{\epsilon}{2}. \quad (2.7)$$

Hence,

$$\begin{aligned} |J_1 - J_2| &= |J_1 - R(P, f, g) + R(P, f, g) - J_2| \\ &\leq |R(P, f, g) - J_1| + |R(P, f, g) - J_2| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus,

$$|J_1 - J_2| < \epsilon \quad \text{for any } \epsilon > 0. \quad (2.8)$$

This means

$$|J_1 - J_2| = 0, \quad (2.9)$$

which says that the two values of the Riemann-Stieltjes delta integral must be equal to each other. This completes the proof.

### ✓ Example 2.5

Let  $\mathbb{T}$ ,  $f$  and  $g$  be as in the third example of "The Darboux-Stieltjes Integral". Let also,

$$\begin{aligned} \xi_j &\in I_j \\ &= [2^{j-1-n}, 2^{j-n})_{\mathbb{T}} \\ &= 2^{j-1-n}. \end{aligned}$$

Then, using the computations in the third and fourth examples of "The Darboux-Stieltjes Integral", we get

$$\begin{aligned}
 R(P, f, g) &= \sum_{j=1}^n f(\xi_j) \Delta g_j \\
 &= \sum_{j=1}^n 2^{j-1-n} (3 \cdot 2^{2j-2n-2}) \\
 &= \frac{3}{8} \sum_{j=1}^n 2^{3j-3n} \\
 &= \frac{3}{8} 2^{-3n} \sum_{j=1}^n 2^{3j} \\
 &= \frac{3}{8} 8^{-n} \frac{8^n - 1}{8 - 1} \\
 &= \frac{3}{7} (1 - 8^{-n}).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} R(P, f, g) &= \lim_{n \rightarrow \infty} \frac{3}{7} (1 - 8^{-n}) \\
 &= \frac{3}{7}.
 \end{aligned}$$

Thus,

$$\text{RS} \int_0^1 t \Delta t^2 = \frac{3}{7}. \quad (2.10)$$

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### 3: Correspondence between the Darboux-Stieltjes Delta Integral and the Riemann-Stieltjes Delta Integral

From the examples of the previous two sections, we saw that the values of the Darboux-Stieltjes and Riemann-Stieltjes delta integrals are exactly the same. This is because the construction by the Riemann-Stieltjes delta integral can be shown to be equivalent to the construction by the Darboux-Stieltjes delta integral. Furthermore, if the limits of the Riemann-Stieltjes sum and the Darboux-Stieltjes sums exist, we can show that they would have the same value.

#### Theorem 3.8

Let  $f$  be a bounded function on  $I$  and  $g$  be a strictly increasing function on  $I$ . Then  $f \in \mathcal{R}(g, I)$  if and only if  $f \in \mathcal{D}(g, I)$ . Moreover, the value of Riemann-Stieltjes and Darboux-Stieltjes delta integrals coincide and this is denoted by

$$\int_a^b f(t) \Delta g(t). \quad (3.1)$$

#### Proof

We will prove the implications separately.

1. Suppose that  $f \in \mathcal{D}(g, I)$ . Fix  $\epsilon > 0$  arbitrarily. By the  $\epsilon$ -criterion for Darboux-Stieltjes delta integrability, it follows that there is a  $\delta = \delta(\epsilon) > 0$  such that if  $P \in \mathcal{P}_\delta(I)$ , then

$$U(P, f, g) - L(P, f, g) < \epsilon. \quad (3.2)$$

Clearly, we have

$$L(P, f, g) \leq R(P, f, g) \leq U(P, f, g).$$

Then

$$\begin{aligned} R(P, f, g) &\leq U(P, f, g) \\ &< \epsilon + L(P, f, g) \\ &\leq \epsilon + \text{DS} \int_a^b f(t) \Delta g(t) \end{aligned}$$

and

$$\begin{aligned} R(P, f, g) &\geq L(P, f, g) \\ &> U(P, f, g) - \epsilon \\ &\geq \text{DS} \int_a^b f(t) \Delta g(t) - \epsilon. \end{aligned}$$

Consequently

$$\left| R(P, f, g) - \text{DS} \int_a^b f(t) \Delta g(t) \right| < \epsilon \quad (3.3)$$

for any  $\epsilon > 0$ . This means that

$$\text{RS} \int_a^b f(t) \Delta g(t) = \text{DS} \int_a^b f(t) \Delta g(t). \quad (3.4)$$

2. Now, suppose that  $f \in \mathcal{R}(g, I)$ . Fix  $\epsilon > 0$  arbitrarily. Then there is a  $\delta = \delta(\epsilon) > 0$  such that

$$\left| \text{RS} \int_a^b f(t) \Delta g(t) - R(P, f, g) \right| < \epsilon. \quad (3.5)$$

Let

$$P = \{t_0, t_1, \dots, t_n\}. \quad (3.6)$$

Let also,

$$m_j = \inf_{t \in I_j} f(t) \quad \text{and} \quad M_j = \sup_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}. \quad (3.7)$$

By the definitions of  $m_j$ ,  $j \in \{1, \dots, n\}$ , it follows that there exists a  $\xi_j \in I_j$  so that

$$f(\xi_j) < m_j + \epsilon, \quad j \in \{1, \dots, n\}. \quad (3.8)$$

Then

$$\begin{aligned} R(P, f, g) &= \sum_{j=1}^n f(\xi_j) \Delta g_j \\ &< \sum_{j=1}^n (m_j + \epsilon) \Delta g_j \\ &= \sum_{j=1}^n m_j \Delta g_j + \epsilon \sum_{j=1}^n \Delta g_j \\ &= L(P, f, g) + \epsilon(g(b) - g(a)). \end{aligned}$$

Hence,

$$\begin{aligned} \int_a^b f(t) \Delta g(t) &\geq L(P, f, g) \\ &> R(P, f, g) - \epsilon(g(b) - g(a)). \end{aligned}$$

By the definitions of  $M_j$ ,  $j \in \{1, \dots, n\}$ , it follows that there exists a  $\eta_j \in I_j$  so that

$$f(\eta_j) > M_j - \epsilon, \quad j \in \{1, \dots, n\}. \quad (3.9)$$

Then

$$\begin{aligned}
 R(P, f, g) &= \sum_{j=1}^n f(\eta_j) \Delta g_j \\
 &> \sum_{j=1}^n (M_j - \epsilon) \Delta g_j \\
 &= \sum_{j=1}^n M_j \Delta g_j - \epsilon \sum_{j=1}^n \Delta g_j \\
 &= U(P, f, g) - \epsilon(g(b) - g(a)).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \overline{\int_a^b f(t) \Delta g(t)} &\leq U(P, f, g) \\
 &< R(P, f, g) + \epsilon(g(b) - g(a)).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 R(P, f, g) - \epsilon(g(b) - g(a)) &< \int_a^b f(t) \Delta g(t) \\
 &\leq \overline{\int_a^b f(t) \Delta g(t)} \\
 &< R(P, f, g) + \epsilon(g(b) - g(a))
 \end{aligned}$$

for any  $\epsilon > 0$ . This means that

$$\begin{aligned}
 R(P, f, g) &\leq \int_a^b f(t) \Delta g(t) \\
 &\leq \overline{\int_a^b f(t) \Delta g(t)} \\
 &\leq R(P, f, g)
 \end{aligned}$$

and  $f \in \mathcal{D}(g, I)$ . This completes the proof.

### Theorem 3.9

Let  $g$  be a strictly increasing function on  $I$ . We have

$$\int_a^b c \Delta g(t) = c(g(b) - g(a)), \quad (3.10)$$

where  $c$  is a constant.

**Proof**

Let  $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(I)$ . Then

$$\begin{aligned} M_j &= \sup_{t \in I_j} f(t) \\ &= c, \\ m_j &= \inf_{t \in I_j} f(t) \\ &= c, \quad j \in \{1, \dots, n\}. \end{aligned}$$

Hence,

$$\begin{aligned} U(P, f, g) &= \sum_{j=1}^n M_j \Delta g_j \\ &= c \sum_{j=1}^n \Delta g_j \\ &= c(g(b) - g(a)) \end{aligned}$$

and

$$\begin{aligned} L(P, f, g) &= \sum_{j=1}^n m_j \Delta g_j \\ &= c \sum_{j=1}^n \Delta g_j \\ &= c(g(b) - g(a)). \end{aligned}$$

Note that there exists  $\int_a^b f(t) \Delta g(t)$  and (3.10) holds. This completes the proof.

 **Theorem 3.10**

Let  $f$  be a function on  $I$  and  $g$  be a strictly increasing function on  $I$ . Let also,  $t \in \mathbb{T}$  and  $f: \mathbb{T} \rightarrow \mathbb{R}$ . Then

$$\int_t^{\sigma(t)} f(t) \Delta g(t) = f(t)(g(\sigma(t)) - g(t)). \quad (3.11)$$

**Proof**

If  $\sigma(t) = t$ , then the assertion is evident. Let  $\sigma(t) > t$ . Then

$$\mathcal{P}([t, \sigma(t)]) = \{t, \sigma(t)\}. \quad (3.12)$$

We have

$$\begin{aligned} M &= \sup_{s \in [t, \sigma(t))} f(s) \\ &= f(t) \end{aligned}$$

and

$$\begin{aligned} m &= \inf_{s \in [t, \sigma(t))} f(s) \\ &= f(t). \end{aligned}$$

Hence,

$$\begin{aligned} U(P, f, g) &= f(t)(g(\sigma(t)) - g(t)) \\ &= L(P, f, g). \end{aligned}$$

Note that  $\int_t^{\sigma(t)} f(t) \Delta g(t)$  exists and (3.11) holds. This completes the proof.

### Theorem 3.11

Let  $g$  be a strictly increasing function on  $I$ . If  $f : I \rightarrow \mathbb{R}$  is a Riemann-Stieltjes delta integrable function, then  $f$  is bounded.

#### Proof

Suppose the contrary, i.e., let  $f : I \rightarrow \mathbb{R}$  be a Riemann-Stieltjes delta integrable on  $I$  and unbounded on  $I$ . Take  $\epsilon = 1$ . Then, using that  $f$  is Riemann-Stieltjes delta integrable on  $I$ , there exist a  $\delta > 0$  and a partition  $P \in \mathcal{P}_\delta(I)$  such that

$$\left| R(P, f, g) - \int_a^b f(t) \Delta g(t) \right| < 1. \quad (3.13)$$

From here, we find

$$\begin{aligned} |R(P, f, g)| &= \left| R(P, f, g) - \int_a^b f(t) \Delta g(t) + \int_a^b f(t) \Delta g(t) \right| \\ &\leq \left| R(P, f, g) - \int_a^b f(t) \Delta g(t) \right| + \left| \int_a^b f(t) \Delta g(t) \right| \\ &< 1 + \left| \int_a^b f(t) \Delta g(t) \right| \end{aligned} \quad (3.14)$$

Let  $P = \{t_0, t_1, \dots, t_n\}$ . Since  $f$  is unbounded on  $I$ , then there exist at least one subinterval of  $P$  where the function  $f$  is unbounded. Without loss of generality, assume that it is  $I_1 = [t_0, t_1]$ . Take  $p_j \in I_j$ ,  $j \in \{2, 3, \dots, n\}$ . Since  $f$  is unbounded on  $I_1$ , there is a  $p_1 \in I_1$  such that

$$|f(p_1)| > \frac{\left| \int_a^b f(t) \Delta g(t) \right| + 1 + \sum_{j=2}^n |f(p_j)| \Delta g_j}{\Delta g_1}. \quad (3.15)$$

Hence,

$$R(P, f, g) = f(p_1) \Delta g_1 + \sum_{j=2}^n f(p_j) \Delta g_j \quad (3.16)$$

and applying the reverse triangle inequality, we get

$$\begin{aligned} |R(P, f, g)| &= \left| f(p_1) \Delta g_1 + \sum_{j=2}^n f(p_j) \Delta g_j \right| \\ &\geq |f(p_1)| \Delta g_1 - \sum_{j=2}^n |f(p_j)| \Delta g_j \\ &> \left| \int_a^b f(t) \Delta g(t) \right| + 1 + \sum_{j=2}^n |f(p_j)| \Delta g_j - \sum_{j=2}^n |f(p_j)| \Delta g_j \\ &= 1 + \left| \int_a^b f(t) \Delta g(t) \right|. \end{aligned}$$

This contradicts with (3.14). Therefore  $f$  is bounded on  $I$ . This completes the proof.

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## 4: Properties of the Riemann-Stieltjes Delta Integral

Now, that we have clarified the definitions of Riemann-Stieltjes and Darboux-Stieltjes delta integrals, we move on to study their properties. Since the construction for Darboux-Stieltjes delta integral is more instructive, we shall prove these results using the Darboux-Stieltjes definition.

We start with the following preliminary result.

### Lemma:4.12

Let  $g$  be a strictly increasing function on  $I$ . For any two bounded functions  $f_1, f_2 : I \rightarrow \mathbb{R}$ , we have

$$\int_a^b (f_1 + f_2)(t) \Delta g(t) \leq \int_a^b f_1(t) \Delta g(t) + \int_a^b f_2(t) \Delta g(t) \quad (4.1)$$

and

$$\int_a^b (f_1 + f_2)(t) \Delta g(t) \geq \int_a^b f_1(t) \Delta g(t) + \int_a^b f_2(t) \Delta g(t). \quad (4.2)$$

### Proof

Take  $P = \{t_0, t_1, \dots, t_n\}$  and

$$M_j^{f_1} = \sup_{t \in I_j} f_1(t),$$

$$M_j^{f_2} = \sup_{t \in I_j} f_2(t),$$

$$M_j^{f_1+f_2} = \sup_{t \in I_j} (f_1 + f_2)(t),$$

$$m_j^{f_1} = \inf_{t \in I_j} f_1(t),$$

$$m_j^{f_2} = \inf_{t \in I_j} f_2(t),$$

$$m_j^{f_1+f_2} = \inf_{t \in I_j} (f_1 + f_2)(t), \quad j \in \{1, \dots, n\}.$$

Note that for any  $t \in I_j$ ,  $j \in \{1, \dots, n\}$ , one has

$$f_1(t) + f_2(t) \leq M_j^{f_1} + M_j^{f_2}, \quad j \in \{1, \dots, n\}, \quad (4.3)$$

and

$$f_1(t) + f_2(t) \geq m_j^{f_1} + m_j^{f_2}, \quad j \in \{1, \dots, n\}. \quad (4.4)$$

By taking supremum in the first inequality over  $t \in I_j$ ,  $j \in \{1, \dots, n\}$ , and infimum in the second inequality over  $t \in I_j$ ,  $j \in \{1, \dots, n\}$ , we get

$$M_j^{f_1+f_2} \leq M_j^{f_1} + M_j^{f_2}, \quad j \in \{1, \dots, n\}, \quad (4.5)$$

and

$$m_j^{f_1+f_2} \geq m_j^{f_1} + m_j^{f_2}, \quad j \in \{1, \dots, n\}, \quad (4.6)$$

respectively. Hence,

$$U(P, f_1 + f_2, g) \leq U(P, f_1, g) + U(P, f_2, g) \quad (4.7)$$

and

$$L(P, f_1 + f_2, g) \geq L(P, f_1, g) + L(P, f_2, g) \quad (4.8)$$

for any partition  $P \in \mathcal{P}(I)$ . By taking infimum in the first inequality over  $P \in \mathcal{P}(I)$  and supremum in the second inequality over  $P \in \mathcal{P}(I)$ , we obtain (4.1) and (4.2), respectively. This completes the proof.

### Theorem 4.13

Let  $g$  be a strictly increasing function on  $I$ . Let also,  $f_1, f_2 \in \mathcal{R}(g, I)$ . Then, for any  $\alpha, \beta \in \mathbb{R}$ , one has  $\alpha f_1 + \beta f_2 \in \mathcal{R}(g, I)$ .

**Proof**

Firstly, we will prove the assertion for  $\alpha = \beta = 1$ . By Lemma 4.12, we get

$$\begin{aligned} \int_a^b (f_1 + f_2)(t) \Delta g(t) &\leq \int_a^b f_1(t) \Delta g(t) + \int_a^b f_2(t) \Delta g(t) \\ &= \int_a^b f_1(t) \Delta g(t) + \int_a^b f_2(t) \Delta g(t) \end{aligned}$$

and

$$\begin{aligned} \int_a^b (f_1 + f_2)(t) \Delta g(t) &\geq \int_a^b f_1(t) \Delta g(t) + \int_a^b f_2(t) \Delta g(t) \\ &= \int_a^b f_1(t) \Delta g(t) + \int_a^b f_2(t) \Delta g(t). \end{aligned}$$

Therefore

$$\begin{aligned} \int_a^b (f_1 + f_2)(t) \Delta g(t) &= \int_a^b (f_1 + f_2)(t) \Delta g(t) \\ &= \int_a^b f_1(t) \Delta g(t) + \int_a^b f_2(t) \Delta g(t) \end{aligned}$$

and  $f_1 + f_2 \in \mathcal{R}(g, I)$ .

Now, we will prove that  $\alpha f_1 \in \mathcal{R}(g, I)$  for any  $\alpha \in \mathbb{R}$ . If  $\alpha = 0$ , then the assertion is evident. Let  $\alpha > 0$  and  $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(I)$ . Let also,

$$M_j = \sup_{t \in I_j} f(t),$$

$$m_j = \inf_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}.$$

Then

$$\begin{aligned} \alpha M_j &= \alpha \sup_{t \in I_j} f_1(t) \\ &= \sup_{t \in I_j} (\alpha f_1)(t), \quad j \in \{1, \dots, n\}, \end{aligned}$$

and

$$\begin{aligned} \alpha m_j &= \alpha \inf_{t \in I_j} f_1(t) \\ &= \inf_{t \in I_j} (\alpha f_1)(t), \quad j \in \{1, \dots, n\}. \end{aligned}$$

Hence, we conclude that

$$\alpha L(P, f_1, g) = L(P, \alpha f_1, g) \quad (4.9)$$

and

$$\alpha U(P, f_1, g) = U(P, \alpha f_1, g). \quad (4.10)$$

Since  $f_1 \in \mathcal{R}(g, I)$ , we have that

$$L(P, f_1, g) = U(P, f_1, g) \quad (4.11)$$

and

$$\alpha L(P, f_1, g) = \alpha U(P, f_1, g). \quad (4.12)$$

Therefore

$$U(P, \alpha f_1, g) = L(P, \alpha f_1, g) \quad (4.13)$$

for any partition  $P \in \mathcal{P}(I)$ . Therefore  $\alpha f_1 \in \mathcal{R}_1(g, I)$  and

$$\int_a^b (\alpha f_1)(t) \Delta g(t) = \alpha \int_a^b f_1(t) \Delta g(t). \quad (4.14)$$

Let now,  $\alpha < 0$ . Then

$$\alpha M_j = \inf_{t \in I_j} (\alpha f_1)(t), \quad j \in \{1, \dots, n\}, \quad (4.15)$$

and

$$\alpha m_j = \sup_{t \in I_j} (\alpha f_1)(t), \quad j \in \{1, \dots, n\}. \quad (4.16)$$

From here, we conclude that

$$\begin{aligned} \alpha U(P, f_1, g) &= L(P, \alpha f_1, g), \\ \alpha L(P, f_1, g) &= U(P, \alpha f_1, g). \end{aligned}$$

Consequently

$$U(P, \alpha f_1, g) = L(P, \alpha f_1, g) \quad (4.17)$$

for any  $P \in \mathcal{P}(I)$  and  $\alpha f_1 \in \mathcal{R}(g, I)$ , and

$$\int_a^b (\alpha f_1)(t) \Delta g(t) = \alpha \int_a^b f_1(t) \Delta g(t). \quad (4.18)$$

As above, one has  $\beta f_2 \in \mathcal{R}(g, I)$  for any  $\beta \in \mathbb{R}$ . Then  $\alpha f_1 + \beta f_2 \in \mathcal{R}(g, I)$  and

$$\int_a^b (\alpha f_1 + \beta f_2)(t) \Delta g(t) = \alpha \int_a^b f_1(t) \Delta g(t) + \beta \int_a^b f_2(t) \Delta g(t). \quad (4.19)$$

This completes the proof.

#### Theorem 4.14

Let  $g$  be a strictly increasing function on  $I$ . Let also,  $f \in \mathcal{R}(g, I)$ . Then  $f^2 \in \mathcal{R}(g, I)$ .

**Proof**

Since  $f \in \mathcal{R}(g, I)$ , then  $f$  is a bounded function on  $I$ . Thus, there exists a constant  $K > 0$  such that

$$|f(t)| \leq K, \quad t \in I. \quad (4.20)$$

Take  $\epsilon > 0$  arbitrarily. Then, applying the  $\epsilon$ -criterion for Darboux-Stieltjes delta integrability, we get that there is a  $P \in \mathcal{P}(I)$  such that

$$U(P, f, g) - L(P, f, g) < \frac{\epsilon}{2K}. \quad (4.21)$$

Let

$$\begin{aligned} M_j &= \sup_{t \in I_j} f(t), \\ m_j &= \inf_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}. \end{aligned}$$

Note that for any  $t, y \in I_j$ ,  $j \in \{1, \dots, n\}$ , one has

$$\begin{aligned}
 (f(t))^2 - (f(y))^2 &= (f(t) - f(y))(f(t) + f(y)) \\
 &\leq (f(t) + f(y))|f(t) - f(y)| \\
 &\leq 2K|f(t) - f(y)| \\
 &\leq 2K(M_j - m_j), \quad j \in \{1, \dots, n\}.
 \end{aligned}$$

Since  $t$  and  $y$  are arbitrary in  $I$ , we conclude that  $2K(M_j - m_j)$  is an upper bound for the difference  $(f(t))^2 - (f(y))^2$  for any  $t, y \in I_j$ ,  $j \in \{1, \dots, n\}$ . On the other hand, the supremum of this quantity over all pairs  $t, y \in I_j$ ,  $j \in \{1, \dots, n\}$ , is

$$\begin{aligned}
 \sup_{t, y \in I_j} ((f(t))^2 - (f(y))^2) &= \sup_{t \in I_j} (f(t))^2 + \sup_{y \in I_j} (-(f(y))^2) \\
 &= \sup_{t \in I_j} (f(t))^2 - \inf_{t \in I_j} (f(t))^2 \\
 &\leq M_j^{f^2} - m_j^{f^2}, \quad j \in \{1, \dots, n\},
 \end{aligned}$$

where

$$\begin{aligned}
 M_j^{f^2} &= \sup_{t \in I_j} (f(t))^2, \\
 m_j^{f^2} &= \inf_{t \in I_j} (f(t))^2, \quad j \in \{1, \dots, n\}.
 \end{aligned}$$

Therefore

$$M_j^{f^2} - m_j^{f^2} \leq 2K(M_j - m_j), \quad j \in \{1, \dots, n\}. \quad (4.22)$$

Thus, by constructing the Darboux-Stieltjes sum, for the function  $f^2$  with respect to the partition  $P$ , we get

$$\begin{aligned}
 U(P, f^2, g) - L(P, f^2, g) &= \sum_{j=1}^n (M_j^{f^2} - m_j^{f^2}) \Delta g_j \\
 &\leq 2K \sum_{j=1}^n (M_j - m_j) \Delta g_j \\
 &= 2K(U(P, f, g) - L(P, f, g)) \\
 &< 2K \frac{\epsilon}{2K} \\
 &= \epsilon,
 \end{aligned}$$

which is the  $\epsilon$ -criterion for Darboux-Stieltjes delta integrability. This proves  $f^2 \in \mathcal{R}(g, I)$ .

#### Theorem 4.15

Let  $f_1, f_2 \in \mathcal{R}(g, I)$  and  $g$  be a strictly increasing function on  $I$ . Then  $f_1 f_2 \in \mathcal{R}(g, I)$ .

**Proof**

Observe that

$$f_1 f_2 = \frac{1}{4}(f_1 + f_2)^2 - \frac{1}{4}(f_1 - f_2)^2 \quad \text{on } I. \quad (4.23)$$

By Theorem \(\PageIndex{13}\), we have that

$$f_1 + f_2, f_1 - f_2 \in \mathcal{R}(g, I). \quad (4.24)$$

Hence, applying Theorem \(\PageIndex{14}\), we find that

$$(f_1 + f_2)^2, (f_1 - f_2)^2 \in \mathcal{R}(g, I). \quad (4.25)$$

Again applying Theorem 4.13, we obtain that

$$\frac{1}{4}(f_1 + f_2)^2 - \frac{1}{4}(f_1 - f_2)^2 \in \mathcal{R}(g, I). \quad (4.26)$$

i.e.,  $f_1 f_2 \in \mathcal{R}(g, I)$ . This completes the proof.

#### Theorem 4.16

Let  $f \in \mathcal{R}(g, I)$ ,  $g$  be a strictly increasing function on  $I$  and  $c \in I$ . Then  $f \in \mathcal{R}(g, [a, c])$  and  $f \in \mathcal{R}(g, [c, b])$ , and

$$\int_a^b f(t) \Delta g(t) = \int_a^c f(t) \Delta g(t) + \int_c^b f(t) \Delta g(t). \quad (4.27)$$

#### Proof

Take  $\epsilon > 0$  arbitrarily. Since  $f \in \mathcal{R}(g, I)$ , by the  $\epsilon$ -criterion for Darboux-Stieltjes delta integrability, it follows that there is a partition  $P \in \mathcal{P}(I)$  such that

$$U(P, f, g) - L(P, f, g) < \epsilon. \quad (4.28)$$

Define  $P' = P \cup \{c\}$  and

$$P_1 = P' \cap [a, c],$$

$$P_2 = P' \cap [c, b].$$

Thus, by definition, we have

$$U(P_1, f, g) + U(P_2, f, g) = U(P', f, g)$$

$$\leq \overline{\int_a^b f(t) \Delta g(t)}$$

and

$$L(P_1, f, g) + L(P_2, f, g) = L(P', f, g)$$

$$\geq \underline{\int_a^b f(t) \Delta g(t)}.$$

Hence,

$$\begin{aligned}
 \epsilon &> U(P, f, g) - L(P, f, g) \\
 &\geq \overline{\int_a^b f(t) \Delta g(t)} - \underline{\int_a^b f(t) \Delta g(t)} \\
 &\geq U(P_1, f, g) + U(P_2, f, g) - L(P_1, f, g) - L(P_2, f, g) \\
 &= (U(P_1, f, g) - L(P_1, f, g)) + (U(P_2, f, g) - L(P_2, f, g)).
 \end{aligned}$$

Since

$$U(P_1, f, g) - L(P_1, f, g) \geq 0 \quad (4.29)$$

and

$$U(P_2, f, g) - L(P_2, f, g) \geq 0, \quad (4.30)$$

we get that

$$U(P_1, f, g) - L(P_1, f, g) < \epsilon \quad (4.31)$$

and

$$U(P_2, f, g) - L(P_2, f, g) < \epsilon, \quad (4.32)$$

which, by the  $\epsilon$ -criterion, imply that  $f \in \mathcal{R}(g, [a, c])$  and  $f \in \mathcal{R}(g, [c, b])$ . Now, we will show (4.27). Let  $Q = \{y_0, y_1, \dots, y_n\}$  be an arbitrary partition of  $I$ . Define  $Q' = Q \cup \{c\}$  and

$$Q_1 = Q' \cap [a, c],$$

$$Q_2 = Q' \cap [c, b].$$

By the definition of the Darboux-Stieltjes delta integral, we have

$$\begin{aligned}
 \int_a^c f(t) \Delta g(t) + \int_c^b f(t) \Delta g(t) &\leq U(Q_1, f, g) + U(Q_2, f, g) \\
 &= U(Q', f, g) \\
 &\leq U(Q, f, g)
 \end{aligned}$$

and

$$\begin{aligned}
 \int_a^c f(t) \Delta g(t) + \int_c^b f(t) \Delta g(t) &\geq L(Q_1, f, g) + L(Q_2, f, g) \\
 &= L(Q', f, g) \\
 &\geq L(Q, f, g),
 \end{aligned}$$

where we have used that  $Q'$  is a refinement of  $Q$ . Because  $Q \in \mathcal{P}(I)$  was arbitrarily chosen, we find

$$\begin{aligned}\int_a^c f(t)\Delta g(t) + \int_c^b f(t)\Delta g(t) &\leq \inf_{Q \in \mathcal{P}(I)} U(Q, f, g) \\ &= \int_a^b f(t)\Delta g(t)\end{aligned}$$

and

$$\begin{aligned}\int_a^c f(t)\Delta g(t) + \int_c^b f(t)\Delta g(t) &\geq \sup_{Q \in \mathcal{P}(I)} L(Q, f, g) \\ &= \int_a^b f(t)\Delta g(t)\end{aligned}$$

and hence, by sandwiching, we get the desired equality. This completes the proof.

#### Definition:

Let  $g$  be a strictly increasing function on  $I$ . For  $f \in \mathcal{R}(g, I)$ , define

$$\int_b^a f(t)\Delta g(t) = - \int_a^b f(t)\Delta g(t). \quad (4.33)$$

#### Note

This is motivated by Theorem 4.16. Namely, if we want to extend Theorem 4.16 to work for any  $c, d, e \in I$  in any order as well, we would have

$$\int_c^e f(t)\Delta g(t) = \int_c^d f(t)\Delta g(t) + \int_d^e f(t)\Delta g(t). \quad (4.34)$$

Since this is extended to be defined for  $c, d, e$  in any order, we can also set  $e = c$  to get

$$\begin{aligned}0 &= \int_c^c f(t)\Delta g(t) \\ &= \int_c^d f(t)\Delta g(t) + \int_d^c f(t)\Delta g(t),\end{aligned}$$

whereupon

$$\int_c^d f(t)\Delta g(t) = - \int_d^c f(t)\Delta g(t), \quad (4.35)$$

agreeing with the above definition.

### Theorem 4.17

Let  $g_1$  and  $g_2$  be strictly increasing functions on  $I$  and  $f \in \mathcal{R}(g_1, I)$ ,  $f \in \mathcal{R}(g_2, I)$ . Then  $f \in \mathcal{R}(g_1 + g_2, I)$  and

$$\int_a^b f(t) \Delta(g_1 + g_2)(t) = \int_a^b f(t) \Delta g_1(t) + \int_a^b f(t) \Delta g_2(t). \quad (4.36)$$

**Proof**

Let  $\epsilon > 0$  be arbitrarily chose. By the  $\epsilon$ -criterion, it follows that there exists a partition  $P = \{t_0, t_1, \dots, t_n\}$  of  $I$  such that

$$U(P, f, g_1) - L(P, f, g_1) < \frac{\epsilon}{2} \quad \text{and} \quad U(P, f, g_2) - L(P, f, g_2) < \frac{\epsilon}{2}. \quad (4.37)$$

Note that

$$(\Delta(g_1 + g_2))_j = (\Delta g_1)_j + (\Delta g_2)_j, \quad j \in \{1, \dots, n\}. \quad (4.38)$$

Let

$$M_j = \sup_{t \in I_j} f(t),$$

$$m_j = \inf_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}.$$

Then

$$\begin{aligned} U(P, f, g_1 + g_2) &= \sum_{j=1}^n M_j (\Delta(g_1 + g_2))_j \\ &= \sum_{j=1}^n M_j ((\Delta g_1)_j + (\Delta g_2)_j) \\ &= \sum_{j=1}^n M_j (\Delta g_1)_j + \sum_{j=1}^n M_j (\Delta g_2)_j \\ &= U(P, f, g_1) + U(P, f, g_2) \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} L(P, f, g_1 + g_2) &= \sum_{j=1}^n m_j (\Delta(g_1 + g_2))_j \\ &= \sum_{j=1}^n m_j ((\Delta g_1)_j + (\Delta g_2)_j) \\ &= \sum_{j=1}^n m_j (\Delta g_1)_j + \sum_{j=1}^n m_j (\Delta g_2)_j \\ &= L(P, f, g_1) + L(P, f, g_2). \end{aligned} \quad (4.40)$$

Therefore

$$\begin{aligned}
 & U(p, f, g_1 + g_2) + L(P, f, g_1 + g_2) \\
 &= U(P, f, g_1) + U(P, f, g_2) - L(P, f, g_1) - L(P, f, g_2) \\
 &= (U(P, f, g_1) - L(P, f, g_1)) + (U(P, f, g_2) - L(P, f, g_2)) \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon,
 \end{aligned} \tag{4.41}$$

which is the  $\epsilon$ -criterion for Darboux-Stieltjes delta integrability. Thus,  $f \in \mathcal{R}(g_1 + g_2, I)$ . Next, taking infimum in (4.39) over  $P \in \mathcal{P}(I)$ , we arrive at

$$\begin{aligned}
 \int_a^b f(t) \Delta(g_1 + g_2)(t) &\leq \int_a^b f(t) \Delta g_1(t) + \int_a^b f(t) \Delta g_2(t) \\
 &= \int_a^b f(t) \Delta g_1(t) + \int_a^b f(t) \Delta g_2(t)
 \end{aligned}$$

and taking supremum in (4.40) over  $P \in \mathcal{P}(I)$ , we get

$$\begin{aligned}
 \int_a^b f(t) \Delta(g_1 + g_2)(t) &\leq \int_a^b f(t) \Delta g_1(t) + \int_a^b f(t) \Delta g_2(t) \\
 &= \int_a^b f(t) \Delta g_1(t) + \int_a^b f(t) \Delta g_2(t).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \int_a^b f(t) \Delta g_1(t) + \int_a^b f(t) \Delta g_2(t) &\leq \int_a^b f(t) \Delta(g_1 + g_2)(t) \\
 &\leq \int_a^b f(t) \Delta(g_1 + g_2)(t) \\
 &\leq \int_a^b f(t) \Delta g_1(t) + \int_a^b f(t) \Delta g_2(t),
 \end{aligned}$$

whereupon we get (4.36). This completes the proof.

### ? Exercise 4.1

Let  $g_1$  and  $g_2$  be strictly increasing functions on  $I$  and  $f \in \mathcal{R}(g_1, I)$ ,  $f \in \mathcal{R}(g_2, I)$ . Prove that for any  $\alpha, \beta \in \mathbb{R}$ ,  $\alpha, \beta \geq 0$ , one has  $f \in \mathcal{R}(\alpha g_1 + \beta g_2, I)$ .

### Theorem 4.18

Let  $g$  be a strictly increasing function on  $I$ ,  $\tilde{T}$  be a time scale and  $\phi : \tilde{T} \rightarrow \mathbb{R}$  be a strictly increasing continuous function, and  $\sqrt{T} = \phi(\tilde{T})$ , and  $\phi : [A, B] \rightarrow I$  is onto, where  $A, B \in \tilde{T}$ ,  $A \leq B$ . Let also,  $f \in \mathcal{R}(g, I)$ . Then  $f \circ \phi \in \mathcal{R}(g \circ \phi, [A, B]_{\tilde{T}})$  and

$$\int_a^b f(t) \Delta g(t) = \int_A^B f(\phi(s)) \tilde{\Delta} g(\phi(s)). \quad (4.42)$$

#### Proof

Take  $\epsilon > 0$  arbitrarily. Since  $f \in \mathcal{R}(g, I)$ , by the  $\epsilon$ -criterion, it follows that there is a partition  $P = \{t_0, t_1, \dots, t_n\}$  of  $I$  such that

$$U(P, f, g) - L(P, f, g) < \epsilon. \quad (4.43)$$

Since  $\phi : [A, B]_{\tilde{T}} \rightarrow I$  is strictly increasing and onto, there is a partition  $Q = \{s_0, s_1, \dots, s_n\}$  of  $[A, B]_{\tilde{T}}$  such that

$$\phi(s_j) = t_j, \quad j \in \{0, 1, \dots, n\}. \quad (4.44)$$

Thus,

$$\phi([s_{j-1}, s_j]_{\tilde{T}}) = [t_{j-1}, t_j], \quad j \in \{1, \dots, n\}. \quad (4.45)$$

Therefore

$$\begin{aligned} L(P, f, g) &= L(Q, f \circ \phi, g \circ \phi), \\ U(P, f, g) &= U(Q, f \circ \phi, g \circ \phi). \end{aligned} \quad (4.46)$$

Hence, applying (4.43), we arrive at

$$\begin{aligned} U(Q, f \circ \phi, g \circ \phi) - L(Q, f \circ \phi, g \circ \phi) &= U(P, f, g) - L(P, f, g) \\ &< \epsilon, \end{aligned}$$

which is the  $\epsilon$ -criterion. This proves  $f \circ \phi \in \mathcal{R}(g \circ \phi, [A, B]_{\tilde{T}})$ . By (4.46), taking supremum in the first equation over  $P \in \mathcal{P}(I)$  and  $Q \in \mathcal{P}([A, B]_{\tilde{T}})$ , and taking infimum in the second equation over  $P \in \mathcal{P}(I)$  and  $Q \in \mathcal{P}([A, B]_{\tilde{T}})$ , we obtain (4.42). This completes the proof.

## 5: Some Inequalities for the Riemann-Stieltjes Delta Integral

We will deduce some inequalities for Riemann-Stieltjes delta integrals. We will start with following one.

### Theorem 5.19

Let  $g$  be a strictly increasing function on  $I$ ,  $f_1, f_2 \in \mathcal{R}(g, I)$  and  $f_1 \leq f_2$  on  $I$ . Then

$$\int_a^b f_1(t) \Delta g(t) \leq \int_a^b f_2(t) \Delta g(t). \quad (5.1)$$

#### Proof

Since  $f_1, f_2 \in \mathcal{R}(g, I)$ , it follows that  $f_2 - f_1 \in \mathcal{R}(g, I)$ . Moreover, we have

$$f_2(t) - f_1(t) \geq 0, \quad t \in I. \quad (5.2)$$

Then, we obtain

$$\int_a^b (f_2(t) - f_1(t)) \Delta g(t) \geq 0. \quad (5.3)$$

Now, we find

$$\begin{aligned} 0 &\leq \int_a^b (f_2(t) - f_1(t)) \Delta g(t) \\ &= \int_a^b f_2(t) \Delta g(t) - \int_a^b f_1(t) \Delta g(t), \end{aligned}$$

whereupon we get the inequality (5.1). This completes the proof.

### Theorem 5.20

Let  $g$  be a strictly increasing function on  $I$  and  $f \in \mathcal{R}(g, I)$ . Then  $|f| \in \mathcal{R}(g, I)$  and we have the "triangle inequality" for Riemann-Stieltjes delta integrals

$$\left| \int_a^b f(t) \Delta g(t) \right| \leq \int_a^b |f(t)| \Delta g(t). \quad (5.4)$$

#### Proof

Fix  $\epsilon > 0$  arbitrarily. Since  $f \in \mathcal{R}(g, I)$ , there is a partition  $P = \{t_0, t_1, \dots, t_n\}$  of  $I$  such that

$$U(P, f, g) - L(P, f, g) < \epsilon. \quad (5.5)$$

Define

$$m_j = \inf_{t \in I_j} f(t),$$

$$M_j = \sup_{t \in I_j} f(t),$$

$$m'_j = \inf_{t \in I_j} |f(t)|,$$

$$M'_j = \sup_{t \in I_j} |f(t)|, \quad j \in \{1, \dots, n\}.$$

Then

$$m_j \leq f(t) \leq M_j,$$

$$-m_j \geq f(t) \geq -M_j, \quad t \in I_j, \quad j \in \{1, \dots, n\}.$$

Combining these two inequalities, we find

$$m_j - M_j \leq f(t) - f(y) \leq M_j - m_j, \quad t, y \in I_j, \quad j \in \{1, \dots, n\}. \quad (5.6)$$

Hence,

$$|f(t) - f(y)| \leq M_j - m_j, \quad t, y \in I_j, \quad j \in \{1, \dots, n\}. \quad (5.7)$$

Applying the reverse triangle inequality

$$\begin{aligned} |f(t)| - |f(y)| &\leq |f(t) - f(y)| \\ &\leq M_j - m_j, \quad t, y \in I_j, \quad j \in \{1, \dots, n\}. \end{aligned}$$

Thus,  $M_j - m_j$  is an upper bound of  $|f(t)| - |f(y)|$ ,  $t, y \in I_j$ ,  $j \in \{1, \dots, n\}$ . On the other hand, we have

$$\begin{aligned} \sup_{t, y \in I_j} (|f(t)| - |f(y)|) &= \sup_{t \in I_j} |f(t)| - \inf_{y \in I_j} |f(y)| \\ &= M'_j - m'_j, \quad j \in \{1, \dots, n\}. \end{aligned}$$

Hence, we have the inequality

$$M'_j - m'_j \leq M_j - m_j, \quad j \in \{1, \dots, n\}. \quad (5.8)$$

For the difference of the upper and lower Darboux-Stieltjes sums of

$$|f| \quad (5.9)$$

with respect to the partition

$$P \quad (5.10)$$

, we get

$$\begin{aligned}
 U(P, |f|, g) - L(P, |f|, g) &= \sum_{j=1}^n (M'_j - m'_j) \Delta g_j \\
 &\leq \sum_{j=1}^n (M_j - m_j) \Delta g_j \\
 &= U(P, f, g) - L(P, f, g) \\
 &< \epsilon,
 \end{aligned}$$

which is the  $\epsilon$ -criterion. Thus,  $|f| \in \mathcal{R}(g, I)$ . Now, we will prove the "triangle inequality" for Riemann-Stieltjes delta integrals. Note that

$$-|f| \leq f \leq |f| \quad \text{on } I. \quad (5.11)$$

Now, we find

$$-\int_a^b |f(t)| \Delta g(t) \leq \int_a^b f(t) \Delta g(t) \leq \int_a^b |f(t)| \Delta g(t), \quad (5.12)$$

whereupon

$$\left| \int_a^b f(t) \Delta g(t) \right| \leq \int_a^b |f(t)| \Delta g(t). \quad (5.13)$$

This completes the proof.

### Theorem 5.21

Let  $g$  be a strictly increasing function on  $I$ ,  $f \in \mathcal{R}(g, I)$  and

$$m \leq f(t) \leq M, \quad t \in I. \quad (5.14)$$

Then

$$m(g(b) - g(a)) \leq \int_a^b f(t) \Delta g(t) \leq M(g(b) - g(a)). \quad (5.15)$$

**Proof**

We have

$$\begin{aligned}
 m(g(b) - g(a)) &\leq m \sum_{j=1}^n \Delta g_j \\
 &\leq L(P, f, g) \\
 &\leq U(P, f, g) \\
 &\leq M \sum_{j=1}^n \Delta g_j \\
 &= M(g(b) - g(a)).
 \end{aligned}$$

Using that  $f \in \mathcal{R}(g, I)$ , we get

$$\begin{aligned} m(g(b) - g(a)) &\leq \sup_{P \in \mathcal{P}(I)} L(P, f, g) \\ &= \int_a^b f(t) \Delta g(t) \\ &\leq \inf_{P \in \mathcal{P}(I)} U(P, f, g) \\ &\leq M(g(b) - g(a)). \end{aligned}$$

This completes the proof.

### Theorem 5.22

Let  $g$  be a strictly increasing function on  $I$ ,  $g^\Delta$  exists and is continuous on  $I$ , and  $f$  is a bounded function on  $I$ . Then  $f \in \mathcal{R}(g, I)$  if and only if  $fg^\Delta \in \mathcal{R}(E, I)$ . Moreover,

$$\int_a^b f(t) \Delta g(t) = \int_a^b f(t) g^\Delta(t) \Delta t. \quad (5.16)$$

Here  $E$  is the identity function, i.e.,  $E(t) = t, t \in I$ .

#### Proof

We will prove the implications separately.

1. Let  $f \in \mathcal{R}(g, I)$ . Because  $g^\Delta$  is continuous on  $I$ , it is bounded on  $I$  and Riemann delta integrable on  $I$ . Since  $f$  is a bounded function on  $I$ , there is a positive constant  $K$  such that

$$|f(t)| \leq K, \quad t \in I. \quad (5.17)$$

Fix  $\epsilon > 0$ . Since  $f \in \mathcal{R}(g, I)$ , there is a partition  $P_1 \in \mathcal{P}(I)$  such that

$$U(P_1, f, g) - L(P_1, f, g) < \frac{\epsilon}{2}. \quad (5.18)$$

Because  $g^\Delta$  is Riemann delta integrable on  $I$ , there is a partition  $P_2 \in \mathcal{P}(I)$  so that

$$U(P_2, g^\Delta, E) - L(P_2, g^\Delta, E) < \frac{\epsilon}{2K}. \quad (5.19)$$

Let  $P = P_1 \cup P_2 = \{t_0, t_1, \dots, t_n\}$ . Then  $P$  is a refinement of  $P_1$  and  $P_2$ . Then, we have the following inequalities

$$U(P, f, g) \leq U(P_1, f, g),$$

$$L(P, f, g) \geq L(P_1, f, g),$$

$$U(P, g^\Delta, E) \leq U(P_2, g^\Delta, E),$$

$$L(P, g^\Delta, E) \geq L(P_2, g^\Delta, E).$$

Hence, using (5.18) and (5.19), we find

$$\begin{aligned} U(P, f, g) - L(P, f, g) &\leq U(P_1, f, g) - L(P_1, f, g) \\ &< \frac{\epsilon}{2} \end{aligned} \quad (5.20)$$

and

$$\begin{aligned} U(P, g^\Delta, E) - L(P, g^\Delta, E) &\leq U(P_2, g^\Delta, E) - L(P_2, g^\Delta, E) \\ &< \frac{\epsilon}{2K}, \end{aligned} \quad (5.21)$$

respectively. Introduce the quantities

$$\begin{aligned} M_j^f &= \sup_{t \in I_j} f(t), \\ M_j^{g^\Delta} &= \sup_{t \in I_j} g^\Delta(t), \\ MP_j^{fg^\Delta} &= \sup_{t \in I_j} (fg^\Delta)(t), \\ m_j^f &= \inf_{t \in I_j} f(t), \\ m_j^{g^\Delta} &= \inf_{t \in I_j} g^\Delta(t), \\ m_j^{fg^\Delta} &= \inf_{t \in I_j} (fg^\Delta)(t), \quad j \in \{1, \dots, n\}. \end{aligned}$$

By the delta mean value theorem, it follows that for each  $j \in \{1, \dots, n\}$  there are  $\xi_j, \tau_j \in [t_{j-1}, t_j]$  such that

$$g^\Delta(\tau_j) \Delta t_j \leq \Delta g_j \leq g^\Delta(\xi_j) \Delta t_j. \quad (5.22)$$

Then, for any  $j \in \{1, \dots, n\}$  and any  $y \in I_j$ , one has

$$\begin{aligned} (fg^\Delta)(\tau_j) - (fg^\Delta)(y) &= f(\tau_j)g^\Delta(\tau_j) - f(y)g^\Delta(y) \\ &= f(\tau_j)g^\Delta(\tau_j) - f(y)g^\Delta(\tau_j) + f(y)g^\Delta(\tau_j) - f(y)g^\Delta(y) \\ &= (f(\tau_j) - f(y))g^\Delta(\tau_j) + f(y)(g^\Delta(\tau_j) - g^\Delta(y)) \\ &\leq |f(\tau_j) - f(y)|g^\Delta(\tau_j) + |f(y)||g^\Delta(\tau_j) - g^\Delta(y)| \\ &\leq (M_j^f - m_j^f) \frac{\Delta g_j}{\Delta t_j} + K (M_j^{g^\Delta} - m_j^{g^\Delta}), \end{aligned}$$

which is an upper bound of  $(fg^\Delta)(t) - (fg^\Delta)(z)$  for  $t, z \in I_j, j \in \{1, \dots, n\}$ . On the other hand, we have

$$\begin{aligned}
 \sup_{t,z \in I_j} ((fg^\Delta)(t) - (fg^\Delta)(z)) &= \sup_{t \in I_j} (fg^\Delta)(t) + \sup_{z \in I_j} (-(fg^\Delta)(z)) \\
 &= M_j^{fg^\Delta} - \inf_{z \in I_j} (fg^\Delta)(z) \\
 &= M_j^{fg^\Delta} - m_j^{fg^\Delta}, \quad j \in \{1, \dots, n\}.
 \end{aligned}$$

Consequently

$$M_j^{fg^\Delta} - m_j^{fg^\Delta} \leq (M_j^f - m_j^f) \frac{\Delta g_j}{\Delta t_j} + K (M_j^{g^\Delta} - m_j^{g^\Delta}), \quad j \in \{1, \dots, n\}. \quad (5.23)$$

Hence, applying (5.20) and (5.21), we arrive at

$$\begin{aligned}
 U(P, fg^\Delta, E) - L(P, fg^\Delta, E) &= \sum_{j=1}^n (M_j^{fg^\Delta} - m_j^{fg^\Delta}) \Delta t_j \\
 &\leq \sum_{j=1}^n \left( (M_j^f - m_j^f) \frac{\Delta g_j}{\Delta t_j} + K (M_j^{g^\Delta} - m_j^{g^\Delta}) \right) \Delta t_j \\
 &= \sum_{j=1}^n (M_j^f - m_j^f) \Delta g_j + K \sum_{j=1}^n (M_j^{g^\Delta} - m_j^{g^\Delta}) \Delta t_j \\
 &= U(P, f, g) - L(P, f, g) + K (U(P, g^\Delta, E) - L(P, g^\Delta, E)) \\
 &< \frac{\epsilon}{2} + K \frac{\epsilon}{2K} \\
 &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon,
 \end{aligned}$$

which is the  $\epsilon$ -criterion. This implies that  $fg^\Delta \in \mathcal{R}(E, I)$ . Now, we will prove that (5.16) holds. We have

$$\begin{aligned}
 f(\tau_j)g^\Delta(\tau_j) &\leq M_j^f g^\Delta(\tau_j) \\
 &\leq M_j^f \frac{\Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}.
 \end{aligned}$$

Thus,  $M_j^f \frac{\Delta g_j}{\Delta t_j}$  is an upper bound of  $f(t)g^\Delta(t)$ ,  $t \in I_j$ ,  $j \in \{1, \dots, n\}$ . Therefore

$$M_j^{fg^\Delta} \leq M_j^f \frac{\Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}. \quad (5.24)$$

Hence,

$$U(P, fg^\Delta, E) \leq U(P, f, g). \quad (5.25)$$

Taking infimum of both sides of the last inequality over  $P \in \mathcal{P}(I)$ , we get

$$\int_a^b f(t)g^\Delta(t)\Delta t \leq \int_a^b f(t)\Delta g(t). \quad (5.26)$$

On the other hand, we have

$$f(\xi_j)g^\Delta(\xi_j) \geq m_j^f \frac{\Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}. \quad (5.27)$$

Thus,  $m_j^f \frac{\Delta g_j}{\Delta t_j}$ ,  $j \in \{1, \dots, n\}$ , is a lower bound of  $f(t)g^\Delta(t)$ ,  $t \in I_j$ ,  $j \in \{1, \dots, n\}$ . Therefore

$$m_j^{fg^\Delta} \geq m_j^f \frac{\Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}, \quad (5.28)$$

and

$$L(P, fg^\Delta, E) \geq L(P, f, g). \quad (5.29)$$

Taking supremum of both sides of the last inequality over  $P \in \mathcal{P}(I)$ , we get

$$\int_a^b f(t)g^\Delta(t)\Delta t \geq \int_a^b f(t)\Delta g(t). \quad (5.30)$$

By the last inequality and the inequality (5.26), we obtain the equation (5.16).

2. Let  $fg^\Delta \in \mathcal{R}(E, I)$ . We will use the notations in the previous point. Fix  $\epsilon > 0$  arbitrarily. Then, there exists a partition  $P = \{t_0, t_1, \dots, t_n\}$  of the interval  $I$  so that

$$U(P, fg^\Delta, E) - L(P, fg^\Delta, E) < \frac{\epsilon}{2}. \quad (5.31)$$

Note that

$$\begin{aligned} M_j^{fg^\Delta} &\geq f(\xi_j)g^\Delta(\xi_j) \\ &\geq f(\xi_j) \frac{\Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}, \end{aligned}$$

whereupon  $M_j^{fg^\Delta}$  is an upper bound of  $f(t) \frac{\Delta g_j}{\Delta t_j}$ ,  $t \in I_j$ ,  $j \in \{1, \dots, n\}$ . On the other hand, we have

$$\sup_{t \in I_j} \left( f(t) \frac{\Delta g_j}{\Delta t_j} \right) = \frac{M_j^f \Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}. \quad (5.32)$$

Consequently

$$M_j^{fg^\Delta} \geq \frac{M_j^f \Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}, \quad (5.33)$$

and

$$M_j^f \Delta g_j \leq M_j^{fg^\Delta} \Delta t_j, \quad j \in \{1, \dots, n\}. \quad (5.34)$$

Next, we have

$$\begin{aligned} m_j^{fg^\Delta} &\leq f(\tau_j)g^\Delta(\tau_j) \\ &\leq \frac{f(\tau_j)\Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}. \end{aligned}$$

So,  $m_j^{fg^\Delta}$  is a lower bound of  $\frac{f(t)\Delta g_j}{\Delta t_j}$ ,  $t \in I_j$ ,  $j \in \{1, \dots, n\}$ . On the other hand,

$$\inf_{t \in I_j} \frac{f(t)\Delta g_j}{\Delta t_j} = \frac{m_j^f \Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}. \quad (5.35)$$

Therefore

$$m_j^{fg^\Delta} \leq \frac{m_j^f \Delta g_j}{\Delta t_j}, \quad j \in \{1, \dots, n\}, \quad (5.36)$$

and

$$m_j^{fg^\Delta} \Delta t_j \leq m_j^f \Delta g_j, \quad j \in \{1, \dots, n\}. \quad (5.37)$$

Now, applying (5.31), (5.34) and (5.37), we find

$$\begin{aligned} U(P, f, g) - L(P, f, g) &= \sum_{j=1}^n M_j^f \Delta g_j - \sum_{j=1}^n m_j^f \Delta g_j \\ &\leq \sum_{j=1}^n M_j^{fg^\Delta} \Delta t_j - \sum_{j=1}^n m_j^{fg^\Delta} \Delta t_j \\ &= U(P, fg^\Delta, E) - L(P, fg^\Delta, E) \\ &< \epsilon, \end{aligned}$$

which is the  $\epsilon$ -criterion. Therefore  $f \in \mathcal{R}(g, I)$ . By (5.34), we get

$$U(P, f, g) \leq U(P, fg^\Delta, E). \quad (5.38)$$

We take infimum of both sides of the last inequality over  $P \in \mathcal{P}(I)$  and we find

$$\int_a^b f(t) \Delta g(t) \leq \int_a^b f(t) g^\Delta(t) \Delta t. \quad (5.39)$$

By (5.37), we find

$$L(P, f, g) \geq L(P, fg^\Delta, E). \quad (5.40)$$

We take supremum of both sides of the last inequality over  $P \in \mathcal{P}(I)$  and we obtain

$$\int_a^b f(t) \Delta g(t) \geq \int_a^g f(t) g^\Delta(t) \Delta t. \quad (5.41)$$

By the last inequality and the inequality (5.39), we get (5.16). This completes the proof.

### ✓ Example 5.1

Let  $\mathbb{T} = 2^{\mathbb{N}_0}$ . We will evaluate the integral

$$I = \int_1^8 (t^2 - t) \Delta t^2. \quad (5.42)$$

Here

$$\sigma(t) = 2t,$$

$$f(t) = t^2 - t,$$

$$g(t) = t^2, \quad t \in \mathbb{T},$$

and  $a = 1, b = 8, I = [1, 8]$ . We have

$$\begin{aligned} g^\Delta(t) &= \sigma(t) + t \\ &= 2t + t \\ &= 3t \\ &> 0, \quad t \in \mathbb{T}. \end{aligned}$$

Thus,  $g$  is a strictly increasing function on  $I$ . Therefore, we can apply Theorem 5.22 and we get

$$\begin{aligned} I &= \int_1^8 (t^2 - t)(3t) \Delta t \\ &= 3 \int_1^8 (t^3 - t^2) \Delta t. \end{aligned}$$

Let

$$h(t) = \frac{t^4}{15} - \frac{t^3}{7}, \quad t \in I. \quad (5.43)$$

Then

$$\begin{aligned}
 h^{\Delta}(t) &= \frac{(\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3}{15} - \frac{(\sigma(t))^2 + t\sigma(t) + t^2}{7} \\
 &= \frac{(2t)^3 + t(2t)^2 + t^2(2t) + t^3}{15} - \frac{(2t)^2 + t(2t) + t^2}{7} \\
 &= \frac{8t^3 + 4t^3 + 2t^3 + t^3}{15} - \frac{4t^2 + 2t^2 + t^2}{7} \\
 &= t^3 - t^2, \quad t \in I.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 I &= 3 \int_1^8 h^{\Delta}(t) \Delta t \\
 &= 3h(t) \Big|_{t=1}^{t=8} \\
 &= 3 \left( \frac{8^4}{15} - \frac{8^3}{7} - \frac{1}{15} + \frac{1}{7} \right) \\
 &= 3 \left( \frac{4096}{15} - \frac{1}{15} - \frac{512}{7} + \frac{1}{7} \right) \\
 &= 3 \left( \frac{4095}{15} - \frac{511}{7} \right) \\
 &= 3(273 - 73) \\
 &= 3 \cdot 200 \\
 &= 600.
 \end{aligned}$$

### Theorem 5.23 (Delta Integration by Parts)

Let  $g$  be a strictly increasing function on  $I$  such that  $g^{\Delta}$  exists and is continuous on  $I$ . Let also,  $f$  be a strictly increasing bounded function on  $I$  such that  $f^{\Delta}$  exists on  $I$  and is continuous on  $I$ . Then

$$\int_a^b f(t) \Delta g(t) = f(t)g(t) \Big|_{t=a}^{t=b} - \int_a^b g^{\sigma}(t) \Delta f(t). \quad (5.44)$$

#### Proof

Since  $f^{\Delta}$  exists and is continuous on  $I$ , we have that  $f$  is Riemann delta integrable on  $I$ . Because  $g^{\Delta}$  is continuous on  $I$ , it is Riemann delta integrable on  $I$ . From here, we get that  $fg^{\Delta}$  is Riemann delta integrable on  $I$ , i.e.,  $fg^{\Delta} \in \mathcal{R}(E, I)$ . Now, applying Theorem 5.22, we find that  $f$  is Riemann-Stieltjes delta integrable on  $I$  and

$$\int_a^b f(t) \Delta g(t) = \int_a^b f(t) g^{\Delta}(t) \Delta t. \quad (5.45)$$

Now, delta integration by parts gives

$$\int_a^b f(t)g^\Delta(t)\Delta t = f(t)g(t)\Big|_{t=a}^{t=b} - \int_a^b g^\sigma(t)f^\Delta(t)\Delta t. \quad (5.46)$$

Since  $g^\sigma f^\Delta \in \mathcal{R}(E, I)$  and  $f$  is a strictly increasing function on  $I$ , applying Theorem 5.22, we get  $g^\sigma \in \mathcal{R}(f, I)$  and

$$\int_a^b g^\sigma(t)f^\Delta(t)\Delta t = \int_a^b g^\sigma(t)\Delta f(t). \quad (5.47)$$

By the last relation and the relations (5.45), (5.46), we arrive at (5.44). This completes the proof.

For the next result we have a need of the following auxiliary result.

### Theorem 5.24

Let  $f \in \mathcal{C}(I)$ ,  $g$  be a strictly increasing function on  $I$  and  $f \in \mathcal{R}(g, I)$ . Then, we have the following estimate

$$\left| \int_a^b f(t)\Delta g(t) \right| \leq \max_{t \in I} |f(t)|(g(b) - g(a)). \quad (5.48)$$

#### Proof

Since  $f \in \mathcal{C}(I)$  and  $I$  is a compact set, there exists  $\max_{t \in I} |f(t)|$ . Applying the "triangle inequality" for Riemann-Stieltjes delta integrals, Theorem 5.20, we obtain

$$\left| \int_a^b f(t)\Delta g(t) \right| \leq \int_a^b |f(t)|\Delta g(t). \quad (5.49)$$

Now, we find

$$\int_a^b |f(t)|\Delta g(t) \leq \max_{t \in I} |f(t)| \int_a^b 1\Delta g(t). \quad (5.50)$$

Next, we find

$$\int_a^b 1\Delta g(t) = g(b) - g(a). \quad (5.51)$$

By the last equation and the equations (5.49), (5.50), we arrive at

$$\begin{aligned} \left| \int_a^b f(t)\Delta g(t) \right| &\leq \int_a^b |f(t)|\Delta g(t) \\ &\leq \max_{t \in I} |f(t)| \int_a^b 1\Delta g(t) \\ &= \max_{t \in [a, b]} |f(t)|(g(b) - g(a)). \end{aligned}$$

This completes the proof.

### Theorem 5.1

Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of continuous functions on  $I$  that converges uniformly to the continuous function  $f$  on  $I$ . Let also,  $g$  be a strictly increasing function on  $I$  and  $f_n, f \in \mathcal{R}(g, I)$ ,  $n \in \mathbb{N}$ . Then

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) \Delta g(t) = \int_a^b f(t) \Delta g(t). \quad (5.52)$$

#### Proof

Applying Theorem 5.24, we get

$$\begin{aligned} \left| \int_a^b f_n(t) \Delta g(t) - \int_a^b f(t) \Delta g(t) \right| &= \left| \int_a^b (f_n(t) - f(t)) \Delta g(t) \right| \\ &\leq \max_{t \in [a, b]} |f_n(t) - f(t)| (g(b) - g(a)) \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This completes the proof.

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## 6: Mean Value Theorems for the Riemann-Stieltjes Delta Integral

### Theorem 6.25

Let  $g$  be a strictly increasing function on  $I$ ,  $f \in \mathcal{R}(g, I)$  be a bounded function on  $I$  and

$$m = \inf_{t \in [a, b]} f(t) \quad \text{and} \quad M = \sup_{t \in [a, b]} f(t). \quad (6.1)$$

Then there is a real number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t) \Delta g(t) = \Lambda (g(b) - g(a)). \quad (6.2)$$

#### Proof

We have

$$m \leq f(t) \leq M, \quad t \in I. \quad (6.3)$$

Then, we find

$$\begin{aligned} m(g(b) - g(a)) &= \int_a^b m \Delta g(t) \\ &\leq \int_a^b f(t) \Delta g(t) \\ &\leq M \int_a^b 1 \Delta g(t) \\ &= M(g(b) - g(a)). \end{aligned}$$

Hence, we get

$$m \leq \frac{\int_a^b f(t) \Delta g(t)}{g(b) - g(a)} \leq M. \quad (6.4)$$

Set

$$\Lambda = \frac{\int_a^b f(t) \Delta g(t)}{g(b) - g(a)}. \quad (6.5)$$

Then  $\Lambda \in [m, M]$  and (6.2) holds. This completes the proof.

For the next mean value theorem we have a need of the following auxiliary result.

### Lemma 6.26(The Abel Lemma)

Let  $p_j, j \in \{1, \dots, n\}$ , be nonnegative real numbers such that

$$p_1 \geq p_2 \geq \dots \geq p_n \geq 0. \quad (6.6)$$

Let also,  $q_j, j \in \{1, \dots, n\}$ , be nonnegative real numbers and the real numbers

$$S_k = \sum_{j=1}^k q_j, \quad k \in \{1, \dots, n\}, \quad (6.7)$$

satisfy the inequalities

$$m \leq S_k \leq M, \quad k \in \{1, \dots, n\}, \quad (6.8)$$

for some nonnegative real numbers  $m$  and  $M$ . Then one has

$$mp_1 \leq \sum_{j=1}^n p_j q_j \leq Mp_1. \quad (6.9)$$

#### **Proof**

We have

$$\begin{aligned} \sum_{j=1}^n p_j q_j &\geq p_1 q_1 \\ &= p_1 S_1 \\ &\geq mp_1 \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^n p_j q_j &\leq p_1 \sum_{j=1}^n q_j \\ &= p_1 S_n \\ &\leq Mp_1. \end{aligned}$$

This completes the proof.

### Theorem 6.27

Let  $g$  be a strictly increasing function on  $I$ ,  $f \in \mathcal{R}(g, I)$  be a nonnegative bounded function and the function

$$F(t) = \int_a^t f(s) \Delta s, \quad t \in I, \quad (6.10)$$

satisfies the inequalities

$$m(g(b) - g(a)) \leq F(t) \leq M(g(b) - g(a)), \quad t \in I, \quad (6.11)$$

for some nonnegative constants  $m$  and  $M$ . If  $h \in \mathcal{R}(g, I)$  is a nonnegative nonincreasing function on  $I$ , then there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda h(a)(g(b) - g(a)). \quad (6.12)$$

**Proof**

Fix  $\epsilon > 0$  arbitrarily. Since  $f, h \in \mathcal{R}(g, I)$ , we get  $fh \in \mathcal{R}(g, I)$ . By the  $\epsilon$ -criterion and the definition for Riemann-Stieltjes delta integral, it follows that there is a partition  $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(I)$  such that

$$U(P, f, g) - L(P, f, g) < \epsilon \quad (6.13)$$

and

$$\left| \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j - \int_a^b f(t)h(t)\Delta g(t) \right| < \epsilon. \quad (6.14)$$

By the first inequality, we find

$$\sum_{j=1}^n (M_j - m_j)\Delta g_j < \epsilon, \quad (6.15)$$

where

$$M_j = \sup_{t \in I_j} f(t),$$

$$m_j = \inf_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}.$$

Now, using that

$$m_j \leq f(t) \leq M_j, \quad t \in I_j, \quad j \in \{1, \dots, n\}, \quad (6.16)$$

and  $h \geq 0$  on  $I$ , we find

$$\begin{aligned} \sum_{j=1}^n m_j h(t_{j-1})\Delta g_j &\leq \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j \\ &\leq \sum_{j=1}^n M_j h(t_{j-1})\Delta g_j. \end{aligned} \quad (6.17)$$

By Theorem 6.26, it follows that for any  $j \in \{1, \dots, n\}$  there is  $\Lambda_j \in [m_j, M_j]$  such that

$$\int_{t_{j-1}}^{t_j} f(t)\Delta g(t) = \Lambda_j \Delta g_j. \quad (6.18)$$

Consider the numbers

$$S_k = \sum_{j=1}^k \Lambda_j \Delta g_j, \quad k \in \{1, \dots, n\}. \quad (6.19)$$

We have

$$\begin{aligned} S_k &= \sum_{j=1}^k \left( \int_{t_{j-1}}^{t_j} f(t) \Delta g(t) \right) \\ &= \int_a^{t_k} f(t) \Delta g(t), \quad k \in \{1, \dots, n\}. \end{aligned}$$

Hence,

$$\begin{aligned} m(g(b) - g(a)) &\leq S_k \\ &\leq M(g(b) - g(a)), \quad k \in \{1, \dots, n\}. \end{aligned}$$

Set

$$\begin{aligned} p_j &= h(t_{j-1}), \\ q_j &= \Lambda_j \Delta g_j, \quad j \in \{1, \dots, n\}. \end{aligned}$$

Since  $h$  is nonincreasing on  $I$ , we have

$$p_1 \geq p_2 \geq \dots \geq p_n. \quad (6.20)$$

Thus, the numbers  $p_j, S_j, q_j, j \in \{1, \dots, n\}$ , satisfy all conditions of the lemma of Abel. Therefore

$$\begin{aligned} m(g(b) - g(a))h(a) &\leq \sum_{j=1}^n h(t_{j-1}) \Lambda_j \Delta g_j \\ &\leq M(g(b) - g(a))h(a). \end{aligned} \quad (6.21)$$

On the other hand, we have the inequalities

$$\begin{aligned} \sum_{j=1}^n m_j h(t_{j-1}) \Delta g_j &\leq \sum_{j=1}^n \Lambda_j h(t_{j-1}) \Delta g_j \\ &\leq \sum_{j=1}^n M_j h(t_{j-1}) \Delta g_j. \end{aligned}$$

From the last inequalities and (6.17), we find

$$\begin{aligned} \sum_{j=1}^n h(t_{j-1}) (f(t_{j-1}) - \Lambda_j) \Delta g_j &\leq \sum_{j=1}^n h(t_{j-1}) (M_j - \Lambda_j) \Delta g_j \\ &\leq \sum_{j=1}^n h(t_{j-1}) (M_j - m_j) \Delta g_j \\ &\leq h(a) \sum_{j=1}^n (M_j - m_j) \Delta g_j \end{aligned}$$

and

$$\begin{aligned}
 \sum_{j=1}^n h(t_{j-1})(f(t_{j-1}) - \Lambda_j)\Delta g_j &\geq \sum_{j=1}^n h(t_{j-1})(m_j - \Lambda_j)\Delta g_j \\
 &\geq \sum_{j=1}^n h(t_{j-1})(m_j - M_j)\Delta g_j \\
 &\geq h(a) \sum_{j=1}^n (m_j - M_j)\Delta g_j.
 \end{aligned}$$

Therefore

$$\left| \sum_{j=1}^n h(t_{j-1})(f(t_{j-1}) - \Lambda_j)\Delta g_j \right| \leq h(a) \sum_{j=1}^n (M_j - m_j)\Delta g_j. \quad (6.22)$$

Now, applying the inequality (6.15), we arrive at the inequality

$$\left| \sum_{j=1}^n h(t_{j-1})(f(t_{j-1}) - \Lambda_j)\Delta g_j \right| < \epsilon h(a). \quad (6.23)$$

From here and (6.14), we get

$$\begin{aligned}
 &\left| \int_a^b f(t)h(t)\Delta g(t) - \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j \right| \\
 &= \left| \int_a^b f(t)h(t)\Delta g(t) - \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j \right. \\
 &\quad \left. + \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j - \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j \right| \\
 &\leq \left| \int_a^b f(t)h(t)\Delta g(t) - \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j \right| \\
 &\quad + \left| \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j - \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j \right| \\
 &< \epsilon + \epsilon h(a).
 \end{aligned} \quad (6.24)$$

Now, applying (6.21), we obtain

$$\begin{aligned}
 \int_a^b f(t)h(t)\Delta g(t) &\leq \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j + \epsilon + \epsilon h(a) \\
 &< M(g(b) - g(a))h(a) + \epsilon + \epsilon h(a)
 \end{aligned}$$

and

$$\begin{aligned}\int_a^b f(t)h(t)\Delta g(t) &\geq \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j - \epsilon - \epsilon h(a) \\ &> m(g(b) - g(a))h(a) - \epsilon - \epsilon h(a),\end{aligned}$$

i.e.,

$$\begin{aligned}mh(a)(g(b) - g(a)) - \epsilon - \epsilon h(a) &< \int_a^b f(t)h(t)\Delta g(t) \\ &< Mh(a)(g(b) - g(a)) + \epsilon + \epsilon h(a).\end{aligned}\tag{6.25}$$

Taking the limit as  $\epsilon \rightarrow 0$  in the last inequalities, we find

$$mh(a)(g(b) - g(a)) \leq \int_a^b f(t)h(t)\Delta g(t) \leq Mh(a)(g(b) - g(a)).\tag{6.26}$$

If  $h(a) = 0$ , then the assertion is evident. Let  $h(a) \neq 0$ . Then

$$m \leq \frac{\int_a^b f(t)h(t)\Delta g(t)}{h(a)(g(b) - g(a))} \leq M.\tag{6.27}$$

Let

$$\Lambda = \frac{\int_a^b f(t)h(t)\Delta g(t)}{h(a)(g(b) - g(a))}.\tag{6.28}$$

Then  $\Lambda \in [m, M]$  and

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda h(a)(g(b) - g(a)).\tag{6.29}$$

This completes the proof.

### Theorem 6.28

Let  $g$  be a strictly increasing function on  $I$ ,  $f \in \mathcal{R}(g, I)$  be a bounded nonnegative function and the function

$$F(t) = \int_a^t f(s)\Delta s, \quad t \in I,\tag{6.30}$$

satisfies the inequalities

$$m(g(b) - g(a)) \leq F(t) \leq M(g(b) - g(a)), \quad t \in I,\tag{6.31}$$

for some nonnegative constants  $m$  and  $M$ . If  $h \in \mathcal{R}(g, I)$  is any monotone function on  $I$ , then there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda(h(a) - h(b))(g(b) - g(a)) + h(b) \int_a^b f(t)\Delta g(t). \quad (6.32)$$

**Proof**

Suppose that  $h \in \mathcal{R}(g, I)$  is any nonincreasing nonnegative function on  $I$ . Define the function

$$l(t) = h(t) - h(b), \quad t \in I. \quad (6.33)$$

Then  $l$  is a nonincreasing function on  $I$  and

$$\begin{aligned} l(t) &\geq l(b) \\ &= h(b) - h(b) \\ &= 0, \quad t \in I. \end{aligned}$$

Then, we can apply Theorem 6.27. Hence, there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)l(t)\Delta g(t) = \Lambda l(a)(g(b) - g(a)), \quad (6.34)$$

or

$$\int_a^b f(t)(h(t) - h(b))\Delta g(t) = \Lambda(h(a) - h(b))(g(b) - g(a)), \quad (6.35)$$

or

$$\int_a^b f(t)h(t)\Delta g(t) - h(b) \int_a^b f(t)\Delta g(t) = \Lambda(h(a) - h(b))(g(b) - g(a)), \quad (6.36)$$

whereupon

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda(h(a) - h(b))(g(b) - g(a)) + h(b) \int_a^b f(t)\Delta g(t). \quad (6.37)$$

Now, assume that  $h \in \mathcal{R}(g, I)$  is any nondecreasing function. Define the function

$$h_1(t) = -h(t), \quad t \in I. \quad (6.38)$$

Then  $h_1$  is nonincreasing on  $I$ . Set

$$m(t) = h_1(t) - h_1(b), \quad t \in I. \quad (6.39)$$

We have that  $m$  is nondecreasing on  $I$  and

$$\begin{aligned} m(t) &\geq m(b) \\ &= h_1(b) - h_1(b) \\ &= 0, \quad t \in I. \end{aligned}$$

Thus, we can apply Theorem 6.27. Then there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)m(t)\Delta g(t) = \Lambda m(a)(g(b) - g(a)), \quad (6.40)$$

or

$$\begin{aligned} \int_a^b f(t)(h_1(t) - h_1(b))\Delta g(t) &= \Lambda(h_1(a) - h_1(b))(g(b) - g(a)) \\ &= \Lambda(-h(a) + h(b))(g(b) - g(a)), \end{aligned}$$

or

$$\int_a^b f(t)(h(t) - h(b))\Delta g(t) = \Lambda(h(a) - h(b))(g(b) - g(a)), \quad (6.41)$$

or

$$\int_a^b f(t)h(t)\Delta g(t) - h(b) \int_a^b f(t)\Delta g(t) = \Lambda(h(a) - h(b))(g(b) - g(a)), \quad (6.42)$$

whereupon

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda(h(a) - h(b))(g(b) - g(a)) + h(b) \int_a^b f(t)\Delta g(t). \quad (6.43)$$

This completes the proof.

### Theorem 6.29

Let  $g$  be a strictly increasing function on  $I$ ,  $f \in \mathcal{R}(g, I)$  be a nonnegative bounded function and the function

$$\Phi(t) = \int_t^b f(s)\Delta s, \quad t \in I, \quad (6.44)$$

satisfies the inequalities

$$m(g(b) - g(a)) \leq \Phi(t) \leq M(g(b) - g(a)), \quad t \in I, \quad (6.45)$$

for some nonnegative constants  $m$  and  $M$ . If  $h \in \mathcal{R}(g, I)$  is a nonnegative nondecreasing function on  $I$ , then there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda h(b)(g(b) - g(a)). \quad (6.46)$$

### Proof

Fix  $\epsilon > 0$  arbitrarily. Because  $f, h \in \mathcal{R}(g, I)$ , applying we get  $fh \in \mathcal{R}(g, I)$ . From the  $\epsilon$ -criterion and the definition for Riemann-Stieltjes delta integral, it follows that there is a partition  $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(I)$  such that

$$U(P, f, g) - L(P, f, g) < \epsilon \quad (6.47)$$

and

$$\left| \sum_{j=1}^n h(t_{j-1})\Delta g_j - \int_a^b f(t)h(t)\Delta g(t) \right| < \epsilon. \quad (6.48)$$

From the first inequality, we find

$$\sum_{j=1}^n (M_j - m_j)\Delta g_j < \epsilon, \quad (6.49)$$

where

$$M_j = \sup_{t \in I_j} f(t),$$

$$m_j = \inf_{t \in I_j} f(t), \quad j \in \{1, \dots, n\}.$$

Hence, using that

$$m_j \leq f(t) \leq M_j, \quad t \in I_j, \quad j \in \{1, \dots, n\}, \quad (6.50)$$

and  $h \geq 0$  on  $I$ , we find

$$\begin{aligned} \sum_{j=1}^n m_j h(t_{j-1})\Delta g_j &\leq \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j \\ &\leq \sum_{j=1}^n M_j h(t_{j-1})\Delta g_j. \end{aligned} \quad (6.51)$$

By Theorem 6.26, it follows that for any  $j \in \{1, \dots, n\}$  there is  $\Lambda_j \in [m_j, M_j]$  such that

$$\int_{t_{j-1}}^{t_j} f(t)\Delta g(t) = \Lambda_j \Delta g_j. \quad (6.52)$$

Consider the numbers

$$S_k = \sum_{j=k+1}^n \Lambda_j \Delta g_j, \quad k \in \{1, \dots, n\}. \quad (6.53)$$

We have

$$S_k = \sum_{j=k+1}^n \left( \int_{t_{j-1}}^{t_j} f(t) \Delta g(t) \right) \\ = \int_{t_k}^b f(t) \Delta g(t), \quad k \in \{1, \dots, n\}.$$

Hence,

$$m(g(b) - g(a)) \leq S_k \\ \leq M(g(b) - g(a)), \quad k \in \{1, \dots, n\}.$$

Set

$$p_j = h(t_{n-j+1}), \\ q_j = \Lambda_j \Delta g_j, \quad j \in \{1, \dots, n\}.$$

Since  $h$  is nondecreasing on  $I$ , we have

$$p_1 \geq p_2 \geq \dots \geq p_n. \quad (6.54)$$

Thus, the numbers  $p_j, S_j, q_j, j \in \{1, \dots, n\}$ , satisfy all conditions of the lemma of Abel. Therefore

$$m(g(b) - g(a))h(b) \leq \sum_{j=1}^n h(t_{j-1})\Lambda_j \Delta g_j \\ \leq M(g(b) - g(a))h(b). \quad (6.55)$$

On the other hand, we have the inequalities

$$\sum_{j=1}^n m_j h(t_{j-1}) \Delta g_j \leq \sum_{j=1}^n \Lambda_j h(t_{j-1}) \Delta g_j \\ \leq \sum_{j=1}^n M_j h(t_{j-1}) \Delta g_j.$$

From the last inequalities and (6.51), we find

$$\sum_{j=1}^n h(t_{j-1})(f(t_{j-1}) - \Lambda_j) \Delta g_j \leq \sum_{j=1}^n h(t_{j-1})(M_j - \Lambda_j) \Delta g_j \\ \leq \sum_{j=1}^n h(t_{j-1})(M_j - m_j) \Delta g_j \\ \leq h(b) \sum_{j=1}^n (M_j - m_j) \Delta g_j$$

and

$$\begin{aligned}
 \sum_{j=1}^n h(t_{j-1})(f(t_{j-1}) - \Lambda_j)\Delta g_j &\geq \sum_{j=1}^n h(t_{j-1})(m_j - \Lambda_j)\Delta g_j \\
 &\geq \sum_{j=1}^n h(t_{j-1})(m_j - M_j)\Delta g_j \\
 &\geq h(b) \sum_{j=1}^n (m_j - M_j)\Delta g_j.
 \end{aligned}$$

Therefore

$$\left| \sum_{j=1}^n h(t_{j-1})(f(t_{j-1}) - \Lambda_j)\Delta g_j \right| \leq h(b) \sum_{j=1}^n (M_j - m_j)\Delta g_j. \quad (6.56)$$

Now, applying the inequality (6.49), we arrive at the inequality

$$\left| \sum_{j=1}^n h(t_{j-1})(f(t_{j-1}) - \Lambda_j)\Delta g_j \right| < \epsilon h(b). \quad (6.57)$$

From here and (6.48), we get

$$\begin{aligned}
 &\left| \int_a^b f(t)h(t)\Delta g(t) - \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j \right| \quad (6.58) \\
 &= \left| \int_a^b f(t)h(t)\Delta g(t) - \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j \right. \\
 &\quad \left. + \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j - \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j \right| \\
 &\leq \left| \int_a^b f(t)h(t)\Delta g(t) - \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j \right| \\
 &\quad + \left| \sum_{j=1}^n f(t_{j-1})h(t_{j-1})\Delta g_j - \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j \right| \\
 &< \epsilon + \epsilon h(b).
 \end{aligned}$$

Now, applying (6.55), we obtain

$$\begin{aligned}
 \int_a^b f(t)h(t)\Delta g(t) &\leq \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j + \epsilon + \epsilon h(b) \\
 &< M(g(b) - g(a))h(b) + \epsilon + \epsilon h(b)
 \end{aligned}$$

and

$$\begin{aligned}\int_a^b f(t)h(t)\Delta g(t) &\geq \sum_{j=1}^n h(t_{j-1})\Lambda_j\Delta g_j - \epsilon - \epsilon h(b) \\ &> m(g(b) - g(a))h(b) - \epsilon - \epsilon h(b),\end{aligned}$$

i.e.,

$$\begin{aligned}mh(b)(g(b) - g(a)) - \epsilon - \epsilon h(b) &< \int_a^b f(t)h(t)\Delta g(t) \\ &< Mh(b)(g(b) - g(a)) + \epsilon + \epsilon h(b).\end{aligned}\tag{6.59}$$

Taking the limit as  $\epsilon \rightarrow 0$  in the last inequalities, we find

$$mh(b)(g(b) - g(a)) \leq \int_a^b f(t)h(t)\Delta g(t) \leq Mh(b)(g(b) - g(a)).\tag{6.60}$$

If  $h(b) = 0$ , then the assertion is evident. Let  $h(b) \neq 0$ . Then

$$m \leq \frac{\int_a^b f(t)h(t)\Delta g(t)}{h(b)(g(b) - g(a))} \leq M.\tag{6.61}$$

Let

$$\Lambda = \frac{\int_a^b f(t)h(t)\Delta g(t)}{h(b)(g(b) - g(a))}.\tag{6.62}$$

Then  $\Lambda \in [m, M]$  and

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda h(b)(g(b) - g(a)).\tag{6.63}$$

This completes the proof.

### Theorem 6.1

Let  $g$  be a strictly increasing function on  $I$ ,  $f \in \mathcal{R}(g, I)$  be a nonnegative bounded function and the function

$$\Phi(t) = \int_t^b f(s)\Delta s, \quad t \in I,\tag{6.64}$$

satisfies the inequalities

$$m(g(b) - g(a)) \leq \Phi(t) \leq M(g(b) - g(a)), \quad t \in I,\tag{6.65}$$

for some nonnegative constants  $m$  and  $M$ . If  $h \in \mathcal{R}(g, I)$  is any monotone function on  $I$ , then there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda(h(b) - h(a))(g(b) - g(a)) + h(a) \int_a^b f(t)\Delta g(t). \quad (6.66)$$

**Proof**

Suppose that  $h \in \mathcal{R}(g, I)$  is any nondecreasing function on  $I$ . Define the function

$$l(t) = h(t) - h(a), \quad t \in I. \quad (6.67)$$

Then  $l$  is a nondecreasing function on  $I$  and

$$\begin{aligned} l(t) &\geq l(a) \\ &= h(a) - h(a) \\ &= 0, \quad t \in I. \end{aligned}$$

Then, we can apply Theorem 6.29. Hence, there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)l(t)\Delta g(t) = \Lambda l(b)(g(b) - g(a)), \quad (6.68)$$

or

$$\int_a^b f(t)(h(t) - h(a))\Delta g(t) = \Lambda(h(b) - h(a))(g(b) - g(a)), \quad (6.69)$$

or

$$\int_a^b f(t)h(t)\Delta g(t) - h(a) \int_a^b f(t)\Delta g(t) = \Lambda(h(b) - h(a))(g(b) - g(a)), \quad (6.70)$$

whereupon

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda(h(b) - h(a))(g(b) - g(a)) + h(a) \int_a^b f(t)\Delta g(t). \quad (6.71)$$

Now, assume that  $h \in \mathcal{R}(g, I)$  is any nonincreasing function. Define the function

$$h_1(t) = -h(t), \quad t \in I. \quad (6.72)$$

Then  $h_1$  is nondecreasing on  $I$ . Set

$$m(t) = h_1(t) - h_1(a), \quad t \in I. \quad (6.73)$$

We have that  $m$  is nondecreasing on  $I$  and

$$\begin{aligned}
 m(t) &\geq m(a) \\
 &= h_1(a) - h_1(a) \\
 &= 0, \quad t \in I.
 \end{aligned}$$

Thus, we can apply Theorem 6.29. Then there is a number  $\Lambda \in [m, M]$  such that

$$\int_a^b f(t)m(t)\Delta g(t) = \Lambda m(b)(g(b) - g(a)), \quad (6.74)$$

or

$$\begin{aligned}
 \int_a^b f(t)(h_1(t) - h_1(a))\Delta g(t) &= \Lambda(h_1(b) - h_1(a))(g(b) - g(a)) \\
 &= \Lambda(-h(b) + h(a))(g(b) - g(a)),
 \end{aligned}$$

or

$$\int_a^b f(t)(h(t) - h(a))\Delta g(t) = \Lambda(h(b) - h(a))(g(b) - g(a)), \quad (6.75)$$

or

$$\int_a^b f(t)h(t)\Delta g(t) - h(a) \int_a^b f(t)\Delta g(t) = \Lambda(h(b) - h(a))(g(b) - g(a)), \quad (6.76)$$

whereupon

$$\int_a^b f(t)h(t)\Delta g(t) = \Lambda(h(b) - h(a))(g(b) - g(a)) + h(a) \int_a^b f(t)\Delta g(t). \quad (6.77)$$

This completes the proof.

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