TIME SCALES ANALYSIS



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Analysis

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1: Definition. Examples

Definition: 1.1

A time scale is an arbitrary nonempty closed subset of the real numbers.

We will denote a time scale by the symbol $\mathbb{T}$. We suppose that a time scale $\mathbb{T}$ has the topology that inherits from the real numbers with the standard topology.

Example 1.1

The sets $[-1,4], \mathbb{R}, \mathbb{Z}, \mathbb{N}$,

$$\left\{ -2, \quad -1, \quad -\frac{1}{2}, \quad 0, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad 2, \quad 3, \quad 6 \right\},$$

and

$$\{1\} \bigcup \left\{ \frac{1}{n} + 1 \right\}_{n \in \mathbb{N}} \bigcup \{3\} \bigcup \left\{ \frac{4}{n^2} + 3 \right\} \bigcup \{9\} \bigcup \left\{ \frac{7}{n^4} + 9 \right\}_{n \in \mathbb{N}}$$

are time scales.

Example 1.2

The sets $(-3,7), [0,5), (1,7]$ and $\left\{ \frac{5}{n} + 3 \right\}_{n \in \mathbb{N}}$ are not time scales.

Example 1.3

Let $a, b > 0$. The sets

$$P_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b) + a]$$

are time scales.

Example 1.4

The set of harmonic numbers

$$H_0 = 0$$
$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}$$

is a time scale.

Example 1.5

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers with $\alpha_n > 0, n \in \mathbb{N}_0$. Define

$$t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbb{N}.$$

Then the set

$$\mathbb{T} = \{t_n : n \in \mathbb{N}\}$$

is a time scale.

Example 1.6

(Cantor Set) Consider the interval $K_0 = [0, 1]$. We obtain a subset K_1 of K_0 by removing the open "middle third" of K_0, i.e., the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from K_0. The set K_2 is obtained by removing the two open middle thirds of K_1, i.e., the two open inetrvals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from K_1. Proceeding in this manner, we obtain a sequence $\{K_n\}_{n\in\mathbb{N}_0}$ of sub sets of the interval [0,1]. In Fig. 0.1 are shown the sets K_0, K_1, K_2, K_3 and so forth. The Cantor set C is now defined as follows

$$C = \bigcap_{n=0}^{\infty} K_n.$$

The Cantor set is a time scale.

Any its element x can be represented in its ternary expansion as follows

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \quad \text{where} \quad a_j \in \{1, 2, 3\}, \quad j \in \mathbb{N}.$$

This expansion is unique unless x is of the form $p3^{-k}$ for some integers p and k. In this case, x has two expansions

1. $a_j = 0$ for $j > k$.
2. $a_j = 2$ for $j > k$.

Assume that p is not divisible by 3 . One of these expansions will have $a_k = 1$ and the other wiill $a_k = 0$ or $a_k = 2$. We have that

$$a_1 = 1 \quad \text{if and only if} \quad \frac{1}{3} < x < \frac{2}{3}$$

and

$$a_1 \neq 1 \quad \text{and} \quad a_2 = 1 \quad \text{if and only if} \quad \frac{1}{9} < x < \frac{2}{9} \quad \text{or} \quad \frac{7}{9} < x < \frac{8}{9}$$

and so forth. If

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j} \quad \text{and} \quad y = \sum_{j=1}^{\infty} \frac{b_j}{3^j},$$

then $x < y$ if and only if there exists an $n \in \mathbb{N}$ such that $a_n < b_n$ and $a_j = b_j$ for $j < n$. Thus, the Cantor set C is the set of all $0 \leq x \leq 1$ that have a base- 3 expansion

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j} \quad \text{with} \quad a_j \neq 1 \quad \text{for any} \quad j.$$

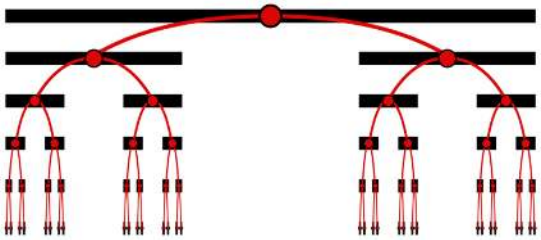


Figure 1.1: 1 Expansion of a Cantor Set.

Example 1.7

The set

$$\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0$$

is a time scale.

Example 1.8

The set

$$[0,1] \cup \left\{ 1 + \frac{1}{n} \right\}_{n \in \mathbb{N}} \cup (2,3] \cup \left\{ 3 + \frac{1}{n} \right\}_{n \in \mathbb{N}}$$

is a time scale.

Example 1.9

Let

$$U = \left\{ \frac{1}{2^n} : n \in \mathbb{N}_0 \right\}.$$

Then the set

$$\{0\} \cup U \cup (1-u) \cup (1+U) \cup (2-U) \cup (2+U) \cup (3-U) \cup (3+U) \cup \{1,2,3,4\}$$

is a time scale.

Exercise 1.1

Check if the following sets are time scales

1. $2^{\mathbb{N}_0}$.
2. $(-1,1] \cup [2,3] \cup [4,8]$.
3. $\left\{ -\frac{1}{2n} : n \in \mathbb{N} \right\} \cup 2\mathbb{N}_0$.
4. $U \cup (2-U) \cup (2+U), U = \left\{ \frac{1}{4^n} : n \in \mathbb{N}_0 \right\}$.
5. $[0,2] \cup \left\{ 2 + \frac{1}{n} \right\}_{n \in \mathbb{N}} \cup (3,5] \cup 7^{\mathbb{N}_0}$.

Answer

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2: Forward Jump Operators, Backward Jump Operators and Graininess Functions

We start by defining the forward jump operator.

Definition: Forward Jump Operator

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ in the following manner

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$. Then, $t = \sigma(t)$ if t is a maximum of $\mathbb{T}$.

Note

Note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

Example 2.1

Let $\mathbb{T} = h\mathbb{Z}, h > 0$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{Z}$ such that $t = hn$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned}\sigma(t) &= \inf\{s = hp, p \in \mathbb{Z} : hp > hn\} \\ &= h(n+1) \\ &= hn + h \\ &= t + h\end{aligned}$$

Example 2.2

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = 3^n$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned}\sigma(t) &= \inf\{3^s, s \in \mathbb{N}_0 : 3^s > 3^n\} \\ &= 3^{n+1} \\ &= 3 \cdot 3^n \\ &= 3t\end{aligned}$$

Example 2.3

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$ is fixed. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = n^k$. Hence, $n = \sqrt[k]{t}$. Now, applying the definition for forward jump operators, we arrive at

$$\begin{aligned}\sigma(t) &= \inf\{s^k, s \in \mathbb{N}_0 : s^k > n^k\} \\ &= (n+1)^k \\ &= (\sqrt[k]{t} + 1)^k.\end{aligned}$$

Example 2.4

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_n, n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$ arbitrarily. Then, applying the definition for forward jump operators, we find

$$\begin{aligned}\sigma(H_n) &= \inf\{H_s, s \in \mathbb{N}_0 : H_s > H_n\} \\ &= \inf\left\{H_s, s \in \mathbb{N}_0 : \sum_{k=1}^{s} \frac{1}{k} > \sum_{k=1}^{n} \frac{1}{k}\right\} \\ &= \sum_{k=1}^{n+1} \frac{1}{k} \\ &= H_{n+1}.\end{aligned}$$

Example 2.5

Let $\mathbb{T} = P_{1,3}$. Then

$$\begin{aligned}\mathbb{T} &= \bigcup_{k=0}^{\infty} [4k, 4k+1] \\ &= [0,1] \cup [4,5] \cup [8,9] \cup [12,13] \cup \ldots.\end{aligned}$$

If $t \in [0,1)$, then, applying the definition for forward jump operators, we find

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

If $t = 1$, then

$$\begin{aligned}\sigma(1) &= \inf\{s \in \mathbb{T} : s > 1\} \\ &= 4\end{aligned}$$

Let now, $k \in \mathbb{N}$ be arbitrarily chosen. If $t \in [4k, 4k+1)$, then we have

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t\end{aligned}$$

If $t = 4k+1$, then

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbf{T} : s > 4k+1\} \\ &= 4(k+1) \\ &= 4k+4 \\ &= 4k+1+3 \\ &= t+3\end{aligned}$$

Therefore

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[4k, 4k+1) \\ t+3 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}\{4k+1\} \end{cases}$$

? Exercise 2.1

Let $\mathbf{T} = P_{a,b}$, where $a, b > 0$. Find $\sigma(t)$.

Answer

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✓ Example 2.6

Let $\mathbf{T} = C$, where C is the Cantor set. We will find $\sigma(t)$ for $t \in \mathbf{T}$. For this aim, let C_1 denote the set of all left-hand end points of the open intervals that are removed. Then

$$C_1 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in \mathbf{N}, \quad a_k \in \{0,2\} \quad \text{for any} \quad 1 \le k \le m \right\}.$$

With C_2 we will denote the set of all right-hand end points of the open intervals that are removed. We have

$$C_2 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : m \in \mathbf{N}, \quad a_k \in \{0,2\} \quad \text{for any} \quad 1 \le k \le m \right\}$$

Take $t \in C$ arbitrarily. We have the following cases.

1. $t \in C_1$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}$$

Hence, we obtain

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbf{T} : s > t\} \\ &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\ &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} + \frac{1}{3^{m+1}} \\ &= t + \frac{1}{3^{m+1}}\end{aligned}$$

2. $t \in C_2$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}$$

Hence,

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbf{T} : s > t\} \\ &= t\end{aligned}$$

3. $t \in \mathbf{T} \setminus (C_1 \cup C_2)$. Then

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbf{T} : s > t\} \\ &= t\end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}} & \text{if} \quad t \in C_1, t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ t \text{ if } t \in \mathbf{T} \setminus C_1 \end{cases}$$

✓ Example 2.7

Let $\{\alpha_n\}_{n \in \mathbf{N}_0}$ be a sequence of real numbers with $\alpha_n > 0$, and

$$t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbf{N},$$

and

$$\mathbf{T} = \{t_n : n \in \mathbf{N}\}.$$

We will find $\sigma(t), t \in \mathbf{T}$. Take $n \in \mathbf{N}$ arbitrarily. Then

$$\begin{aligned}\sigma(t_n) &= \inf\left\{x \in \mathbf{T} : s = \sum_{k=0}^{n-1} \alpha_k, \quad s > t_n\right\} \\ &= \sum_{k=0}^{n} \alpha_k \\ &= \sum_{k=0}^{n-1} + \alpha_n \\ &= t_n + \alpha_n\end{aligned}$$

Example 2.8

Let

$$\mathbf{T} = \left\{t_n = -\frac{1}{n} : n \in \mathbf{N}\right\} \cup \mathbf{N}_0.$$

We will find $\sigma(t), t \in \mathbf{T}$. Take $n \in \mathbf{N}$ arbitrarily. Then

$$n = -\frac{1}{t_n}$$

and

$$\begin{aligned}\sigma(t_n) &= \inf\left\{s \in \mathbf{T} : s = -\frac{1}{m}, m \in \mathbf{N}, s > t_0\right\} \\ &= -\frac{1}{n+1} \\ &= -\frac{1}{-\frac{1}{t_n}+1} \\ &= -\frac{t_n}{t_n - 1}\end{aligned}$$

Next, if $t \in \mathbf{N}_0$, then

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbf{T} : s > t\} \\ &= t + 1.\end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} -\frac{t}{t-1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbf{N}\right\}, \quad t = t_n \\ t+1 \text{ if } t \in \mathbf{N}_0. \end{cases}$$

Example 2.9

Let

$$\mathbf{T} = \left\{t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbf{N}_0\right\} \cup \{0, 1\}$$

We will find $\sigma(t), t \in \mathbf{T}$. Take $n \in \mathbf{N}$ arbitrarily. Then

$$\begin{aligned}\sigma(t_n) &= \inf\{s \in \mathbf{T} : s > t_n\} \\ &= \left(\frac{1}{2}\right)^{2^{n-1}} \\ &= \left(\frac{1}{2}\right)^{2^n \cdot \frac{1}{2}} \\ &= \left(\left(\frac{1}{2}\right)^{2^n}\right)^{\frac{1}{2}} \\ &= \sqrt{t_n}\end{aligned}$$

Next,

$$t_0 = \frac{1}{2} \quad \text{and} \quad \sigma(t_0) = 1$$

and

$$\begin{aligned}\sigma(0) &= 0, \\ \sigma(1) &= 1.\end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} \sqrt{t} & \text{if} \quad t \in \left\{t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbf{N}\right\} \\ 1 & \text{if} \quad t = \frac{1}{2} \\ 0 & \text{if} \quad t = 0 \\ 1 & \text{if} \quad t = 1 \end{cases}$$

Example 2.10

Let $U = \left\{\frac{1}{2^n} : n \in \mathbf{N}\right\}$ and

$$\mathbf{T} = U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{0,1,2\}$$

We will find $\sigma(t), t \in \mathbf{T}$. We have the following cases.

1. Let $t = 0$. Then

$$\sigma(0) = 0$$

2. Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$$

3. Let $t = 1$. Then

$$\sigma(1) = 1$$

4. Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$$

5. Let $t = 2$. Then

$$\sigma(2) = 2.$$

6. Let $t = \frac{5}{2}$. Then

$$\sigma\left(\frac{5}{2}\right) = \frac{5}{2}$$

7. Let $t \in U \backslash \left\{\frac{1}{2}\right\}$. Then

$$t = \frac{1}{2^n}$$

and

$$\begin{aligned}\sigma(t) &= \frac{1}{2^{n-1}} \\ &= \frac{2}{2^n} \\ &= 2t\end{aligned}$$

8. Let $t \in (1-U) \backslash \left\{\frac{1}{2}\right\}$. Then

$$t = 1 - \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = 1 - t$$

Hence,

$$\begin{aligned}\sigma(t) &= 1 - \frac{1}{2^{n+1}} \\ &= 1 - \frac{1}{2} \cdot \frac{1}{2^n} \\ &= 1 - \frac{1-t}{2} \\ &= \frac{1+t}{2}\end{aligned}$$

9. Let $t \in (1+U) \backslash \left\{\frac{3}{2}\right\}$. Then

$$t = 1 + \frac{1}{2^n}$$

Hence,

$$\frac{1}{2^n} = t - 1$$

and

$$\begin{aligned}\sigma(t) &= 1 + \frac{1}{2^{n-1}} \\ &= 1 + \frac{2}{2^n} \\ &= 1 + 2(t-1) \\ &= 2t - 1\end{aligned}$$

10. Let $t \in (2-U) \backslash \left\{\frac{3}{2}\right\}$. Then

$$t = 2 - \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = 2 - t$$

Hence,

$$\begin{aligned}\sigma(t) &= 2 - \frac{1}{2^{n+1}} \\ &= 2 - \frac{1}{2} \cdot \frac{1}{2^n} \\ &= 2 - \frac{2-t}{2} \\ &= \frac{t+2}{2}.\end{aligned}$$

11. Let $t \in (2+U)\backslash\left\{\frac{5}{2}\right\}$. Then

$$t = 2 + \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = t - 2.$$

Hence,

$$\begin{aligned}\sigma(t) &= 2 + \frac{1}{2^{n-1}} \\ &= 2 + \frac{2}{2^n} \\ &= 2 + 2(t-2) \\ &= 2(t-1).\end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \frac{3}{4} & \text{if} \quad t = \frac{1}{2} \\ 1 & \text{if} \quad t = 1 \\ \frac{7}{4} & \text{if} \quad t = \frac{3}{2} \\ 2 & \text{if} \quad t = 2 \\ \frac{5}{2} & \text{if} \quad t = \frac{5}{2} \\ 2t \quad \text{if} \quad t \in U\backslash\left\{\frac{1}{2}\right\} & \\ \frac{1+t}{2} & \text{if} \quad t \in (1-U)\backslash\left\{\frac{1}{2}\right\} \\ 2t-1 & \text{if} \quad t \in (1+U)\backslash\left\{\frac{3}{2}\right\} \\ \frac{t+2}{2} & \text{if} \quad t \in (2-U)\backslash\left\{\frac{3}{2}\right\} \\ 2(t-1) & \text{if} \quad t \in (2+U)\backslash\left\{\frac{5}{2}\right\} \end{cases}$$

? Exercise 2.2

Find $\sigma(t), t \in \mathbb{T}$, where

1. $\mathbb{T} = h\mathbb{Z} + k, h > 0, k \in \mathbb{R}$.
2. $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}$.
3. $\mathbb{T} = P_{3,7} \cup [4,6]$.
4. $\mathbb{T} = 11^{\mathbb{N}_0} \cup \{0\}$.
5. $\mathbb{T} = [1,2] \cup [3,4] \cup [7,8] \cup 9^{\mathbb{N}}$.

Answer

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Definition: Forward Graininess Function

Define the forward graininess function $\mu : \mathbb{T} \to \mathbb{R}$ as follows

$$\mu(t) = \sigma(t) - t, \quad t \in \mathbb{T}.$$

Example 2.11

Example 0.20. Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then, using the computations in Example 0.10, we find

$$\begin{aligned}\mu(t) &= \sigma(t) - t \\ &= t + h - t \\ &= h, \quad t \in \mathbb{T}.\end{aligned}$$

Example 2.12

Example 0.21. Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then, using the computations in Example 0.11, we get

$$\begin{aligned}\mu(t) &= \sigma(t) - t \\ &= 3t - t \\ &= 2t, \quad t \in \mathbb{T}.\end{aligned}$$

Example 2.13

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$. Then, using the computations in Example 0.12, we find

$$\begin{aligned}\mu(t) &= \sigma(t) - t \\ &= (\sqrt[k]{t}+1)^k - t \\ &= t + \binom{k}{1}\sqrt[k]{t^{k-1}} + \binom{k}{2}\sqrt[k]{t^{k-2}} + \cdots + \binom{k}{k-1}\sqrt[k]{t} - t \\ &= \binom{k}{1}\sqrt[k]{t^{k-1}} + \binom{k}{2}\sqrt[k]{t^{k-2}} + \cdots + \binom{k}{k-1}\sqrt[k]{t}, \quad t \in \mathbb{T}\end{aligned}$$

Example 2.14

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_n, n \in \mathbb{N}_0$ are the harmonic numbers. Then, using the computations in Example 0.13, we find

$$\begin{aligned}\mu(H_n) &= \sigma(H_n) - H_n \\ &= H_{n+1} - H_n \\ &= \frac{1}{n+1}, \quad n \in \mathbb{N}.\end{aligned}$$

Example 2.15

Let $\mathbb{T} = P_{1,3}$. Then, using the computations in Example 0.14, we find

$$\begin{aligned}\mu(t) &= \sigma(t) - t \\ &= \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[4k, 4k+1) \\ 3 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}\{4k+1\} \end{cases}\end{aligned}$$

Example 2.16

Let $\mathbb{T} = C$, where C is the Cantor set. Then, using the computations in Example 0.15, we find

$$\begin{aligned}\mu(t) &= \sigma(t) - t \\ &= \begin{cases} \frac{1}{3^{m+1}} & \text{if} \quad t \in C_1, \quad t = \sum_{k=1}^{m}\frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ 0 & \text{if} \quad t \in \mathbb{T}\backslash C_1 \end{cases}\end{aligned}$$

Example 2.17

Let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n > 0, nn \in \mathbb{N}$,

$$t_n = \sum_{k=0}^{n-1}\alpha_k, \quad n \in \mathbb{N}, \quad \text{and} \quad \mathbb{T} = \{t_n : n \in \mathbb{N}\}$$

Then, using the computations in Example 0.16, we find

$$\begin{aligned}\mu(t_n) &= \sigma(t_n) - t_n \\ &= t_{n+1} - t_n \\ &= \alpha_n, \quad n \in \mathbb{N}.\end{aligned}$$

Example 0.26.

Example 2.18

Let $\mathbb{T} = \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$. Then, using the computations in Example 0.17 , we find

$$\begin{aligned}\mu(t) &= \sigma(t) - t \\ &= \begin{cases} -\frac{t^2}{t-1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\}, \quad t = t_n, \\ 1 & \text{if} \quad t \in \mathbb{N}_0. \end{cases}\end{aligned}$$

Example 2.19

Let $\mathbb{T}=\left\{t_n=\left(\frac{1}{2}\right)^{2^n}:n\in\mathbb{N}_0\right\}\cup\{0,1\}$. Then, using the computations in Example 0.18 , we find

$$\begin{aligned}\mu(t)&=\sigma(t)-t\\&=\begin{cases}\sqrt{t}-t & \text{if} \quad t\in\left\{t_n=\left(\frac{1}{2}\right)^{2^n}:n\in\mathbb{N}\right\}\\ \frac{1}{2} & \text{if} \quad t=\frac{1}{2}\\ 0 & \text{if} \quad t\in\{0,1\}.\end{cases}\end{aligned}$$

Example 2.20

Let $U=\left\{\frac{1}{2^n}:n\in\mathbb{N}\right\}$ and

$$\mathbb{T}=\{0\}\cup U\cup(1-U)\cup(1+U)\cup(2-U)\cup(2+U)\cup\{1,2\}.$$

Then, using the computations in Example 0.19, we find

$$\begin{aligned}\mu(t)&=\sigma(t)-t\\&=\begin{cases}0 & \text{if} \quad t=0\\ \frac{1}{4} & \text{if} \quad t=\frac{1}{2}\\ 0 & \text{if} \quad t=1\\ \frac{1}{4} & \text{if} \quad t=\frac{3}{2}\\ 0 & \text{if} \quad t=2\\ 0 & \text{if} \quad t=\frac{5}{2}\\ t & \text{if} \quad t\in U\backslash\left\{\frac{1}{2}\right\}\\ \frac{t-1}{2} & \text{if} \quad t\in(1-U)\backslash\left\{\frac{1}{2}\right\}\\ t-1 & \text{if} \quad t\in(1+U)\backslash\left\{\frac{3}{2}\right\}\\ \frac{t-2}{2} & \text{if} \quad t\in(2-U)\backslash\left\{\frac{3}{2}\right\}\\ t-2 & \text{if} \quad t\in(2+U)\backslash\left\{\frac{5}{2}\right\}\end{cases}\end{aligned}$$

? Exercise 2.3

Find $\mu(t), t\in\mathbb{T}$, where $\mathbb{T}$ are the time scales in Exercise ? 2.2.

Answer

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Definition: Backward Jump Operator

The backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined as follows

$$\rho(t)=\sup\{s\in\mathbb{T}:s<t\}.$$

In this definition we put $\sup\emptyset=\inf\mathbb{T}$. Then, $\rho(t)=t$ if t is a minimum of $\mathbb{T}$.

Note

Note that $\rho(t)\leq t$ for any $t\in\mathbb{T}$.

Example 2.21

Let $\mathbb{T}=h\mathbb{Z}, h>0$. Take $t\in\mathbb{T}$ arbitrarily. Then, there is a $n\in\mathbb{Z}$ such that $t=hn$. Hence, applying the definition for backward jump operator, we find

$$\begin{aligned}\rho(t)&=\sup\{s\in\mathbb{T}:s=hm,\quad m\in\mathbb{Z},\quad s<hn\}\\&=h(n-1)\\&=hn-h\\&=t-h\end{aligned}$$

Example 2.22

Let $\mathbb{T}=3^{\mathbb{N}_0}$. Take $t\in\mathbb{T}$ arbitrarily. We have the following cases.
1. Assume that $t=1$. Then

$$\rho(1) = 1$$

2. Assume that $t > 1$. Then $t = 3^l$ for some $l \in \mathbf{N}$. Hence, applying the definition for backward jump operator, we find

$$\begin{aligned}\rho(t) &= \sup\{s \in \mathbf{T} : s = 3^k, \quad k \in \mathbf{N}_0, \quad s < 3^l\} \\ &= 3^{l-1} \\ &= \frac{3^l}{3} \\ &= \frac{t}{3}\end{aligned}$$

Consequently

$$\rho(t) = \begin{cases} 1 & \text{if} \quad t = 1 \\ \frac{t}{3} & \text{if} \quad t > 1 \end{cases}$$

Example 2.23

Let $\mathbf{T} = \mathbf{N}_0^k, k \in \mathbf{N}$. Take $t \in \mathbf{T}$ arbitrarily. We have the following cases.

1. Assume that $t = 0$. Then

$$\rho(0) = 0$$

2. Let $t > 0$. Then there is an $n \in \mathbf{N}$ such that $t = n^k$. Hence, $n = \sqrt[k]{t}$. Now, applying the definition for backward jump operator, we find

$$\begin{aligned}\rho(t) &= \sup\{s \in \mathbf{T} : s = l^k, \quad l \in \mathbf{N}_0, \quad s < n^k\} \\ &= (n-1)^k \\ &= (\sqrt[k]{t} - 1)^k\end{aligned}$$

Consequently

$$\rho(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ (\sqrt[k]{t} - 1)^k & \text{if} \quad t > 0 \end{cases}$$

Example 2.24

Let $\mathbf{T} = \{H_n : n \in \mathbf{N}_0\}$, where $H_n, n \in \mathbf{N}_0$, are the harmonic numbers. Take $n \in \mathbf{N}_0$ arbitrarily. We have the following cases.

1. Assume that $n = 0$. Then

$$\rho(H_0) = H_0$$

2. Assume that $n \geq$. Then, applying the definition for backward jump operator, we find

$$\begin{aligned}\rho(H_n) &= \sup\{s \in \mathbf{T} : s = H_l, \quad l \in \mathbf{N}_0, \quad s < H_n\} \\ &= H_{n-1}\end{aligned}$$

Consequently

$$\rho(t) = \begin{cases} H_0 & \text{if} \quad n = 0 \\ H_{n-1} & \text{if} \quad n \geq 1 \end{cases}$$

Example 2.25

Let $\mathbf{T} = P_{1,3}$. Take $t \in \mathbf{T}$ arbitrarily. We have the following cases.

1. Assume that $t = 0$. Then

$$\rho(0) = 0$$

2. Assume that $t > 0$. Then we have the following subcases.

a. $t \in (0, 1] \bigcup \bigcup_{k=1}^{\infty} (4k, 4k+1]$. Then

$$\rho(t) = t$$

b. $t = 4k, k > 0$. Then $k = \frac{t}{4}$ and

$$\begin{aligned}\rho(t) &= 4(k-1) + 1 \\ &= 4k - 4 + 1 \\ &= 4k - 3 \\ &= t - 3\end{aligned}$$

Consequently

$$\rho(t) = \begin{cases} t & \text{if} \quad t \in [0, 1] \bigcup_{k=1}^{\infty} (4k, 4k+1] \\ t-3 & \text{if} \quad t \in \bigcup_{k=1}^{\infty} \{4k\} \end{cases}$$

Example 2.26

Let $\mathbf{T} = C$, where C is the Cantor set. Let also, C_2 be the set of all right-hand end points of the intervals that are removed from the interval $[0,1]$, i.e.,

$$C_2 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : m \in \mathbf{N}, \quad a_k \in \{0,2\}, \quad 1 \le k \le m \right\}$$

Take $t \in \mathbf{T}$ arbitrarily. Then we have the following cases.

1. Assume that $t \in C_2$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}$$

and

$$\begin{aligned} \rho(t) &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} - \frac{1}{3^{m+1}} \\ &= t - \frac{1}{3^{m+1}}, \quad \text{where} \quad a_k \in \{0,2\}, \quad 1 \le k \le m \end{aligned}$$

2. If $t \in \mathbf{T} \backslash C_2$, then $\rho(t) = t$.

Consequently

$$\rho(t) = \begin{cases} t - \frac{1}{3^{m+1}} & \text{if} \quad t \in C_2, \quad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\ t \text{ if} \quad t \in \mathbf{T} \backslash C_2 \end{cases}$$

Example 2.27

Let $\mathbf{T} = \left\{ t_n = \sum_{k=0}^{n-1} \alpha_k : n \in \mathbf{N} \right\}$, where $\alpha_n > 0, n \in \mathbf{N}_0$. Take $t_n \in \mathbf{T}$ arbitrarily for some $n \in \mathbf{N}$. We have the following cases.

1. Assume that $n = 1$. Then

$$\rho(t_1) = t_1.$$

2. Assume that $n > 1$. Then

$$\begin{aligned} \rho(t_n) &= \sup \left\{ t_m = \sum_{k=0}^{m-1}, \quad m \in \mathbf{N}, : t_m < t_n \right\} \\ &= \sum_{k=0}^{n-2} \alpha_k \\ &= \sum_{k=0}^{n-1} \alpha_k - \alpha_{n-1} \\ &= t_n - \alpha_{n-1}. \end{aligned}$$

Consequently

$$\rho(t_n) = \begin{cases} t_1 & \text{if} \quad n = 1 \\ t_n - \alpha_{n-1} & \text{if} \quad n \ge 2. \end{cases}$$

Example 2.28

Let $\mathbf{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbf{N} \right\} \cup \mathbf{N}_0$. Take $t \in \mathbf{T}$ arbitrarily. We have the following cases.

1. $t = t_n$ for some $n \in \mathbf{N}$.

a. If $n = 1$, then

$$\rho(t_1) = t_1.$$

b. If $n \ge 2$, then $n = -\frac{1}{t_n}$ and

$$\begin{aligned} \rho(t_n) &= -\frac{1}{n-1} \\ &= -\frac{1}{-\frac{1}{t_n} - 1} \\ &= \frac{t_n}{t_n + 1} \end{aligned}$$

2. Let $t \in \mathbf{N}_0$.

a. If $t = 0$, then

$$\rho(0) = 0.$$

b. If $t \ge 1$, then

$$\rho(t) = t - 1.$$

Consequently

$$\rho(t)=\begin{cases} t & \text{if} \quad t\in\{-1,0\} \\ \dfrac{t}{1+t} & \text{if} \quad t\in\left\{t_n=-\dfrac{1}{n}:n\in\mathbb{N},\quad n\geq 2\right\} \\ t-1 & \text{if} \quad t\in\mathbb{N}. \end{cases}$$

Example 2.29

Let $\mathbf{T}=\left\{t_n=\left(\frac{1}{2}\right)^{2}:n\in\mathbb{N}_0\right\}\cup\{0,1\}$. Take $t\in\mathbf{T}$ arbitrarily. We have the following cases.

1. If $t=\left(\frac{1}{2}\right)^{2^n}$ for some $n\in\mathbb{N}_0$, then

$$\begin{aligned}\rho(t)&=\left(\frac{1}{2}\right)^{2^{n+1}}\\&=\left(\frac{1}{2}\right)^{2^n\cdot 2}\\&=\left(\left(\frac{1}{2}\right)^{2^n}\right)^2\\&=t^2\end{aligned}$$

2. Let $t=0$. Then

$$\rho(0)=0.$$

3. Let $t=1$. Then

$$\rho(1)=\frac{1}{2}.$$

Consequently

$$\rho(t)=\begin{cases} t^2 & \text{if} \quad t\in\left\{t_n=\left(\frac{1}{2}\right)^{2^n}:n\in\mathbb{N}_0\right\} \\ 0 & \text{if} \quad t=0 \\ \frac{1}{2} & \text{if} \quad t=1. \end{cases}$$

Example 2.30

Let $U=\left\{\frac{1}{2^n}:n\in\mathbb{N}\right\}$ and

$$\mathbf{T}=\{0\}\cup U\cup(1-U)\cup(1+U)\cup(2-U)\cup(2+U)\cup\{1,2\}.$$

We will find $\rho(t),t\in\mathbf{T}$. Taket $t\in\mathbf{T}$ arbitrarily. We have the following cases.

1. If $t=\frac{1}{2^n}$ for some $n\in\mathbb{N}$, then

$$\begin{aligned}\rho(t)&=\frac{1}{2^{n+1}}\\&=\frac{1}{2}\cdot\frac{1}{2^n}\\&=\frac{t}{2}.\end{aligned}$$

2. Let $t\in(1-U)$.

a. If $t=\frac{1}{2}$, then

$$\rho\left(\frac{1}{2}\right)=\frac{1}{4}$$

b. If

$$t=1-\frac{1}{2^n},\quad n\geq 2$$

then

$$\frac{1}{2^n}=1-t,\quad n\geq 2$$

and

$$\begin{aligned}\rho(t)&=1-\frac{1}{2^{n-1}}\\&=1-\frac{2}{2^n}\\&=1-2(1-t)\\&=2t-1\end{aligned}$$

3. Let $t\in(1+U)$ and $t=1+\frac{1}{2^n}$ for some $n\in\mathbb{N}$. Then

$$\frac{1}{2^n}=t-1$$

and

$$\begin{aligned}\rho(t) &= 1+\frac{1}{2^{n+1}}\\ &= 1+\frac{1}{2}\cdot\frac{1}{2^n}\\ &= 1+\frac{t-1}{2}\\ &= \frac{t+1}{2}\end{aligned}$$

4. Let $t \in (2-U)$.

a. If $t=\frac{3}{2}$, then

$$\rho\left(\frac{3}{2}\right)=\frac{5}{4}$$

b. If $t=2-\frac{1}{2^n}$ for some $n \in \mathbf{N}, n \geq 2$, then

$$\frac{1}{2^n}=2-t$$

and

$$\begin{aligned}\rho(t) &= 2-\frac{1}{2^{n-1}}\\ &= 2-\frac{2}{2^n}\\ &= 2-2(2-t)\\ &= 2t-2\end{aligned}$$

5. Let $t \in (2+U)$ and $t=2+\frac{1}{2^n}, n \in \mathbf{N}$. Then

$$\frac{1}{2^n}=t-2$$

and

$$\begin{aligned}\rho(t) &= 2+\frac{1}{2^{n+1}}\\ &= 2+\frac{1}{2}\cdot\frac{1}{2^n}\\ &= 2+\frac{t-2}{2}\\ &= \frac{t+2}{2}\end{aligned}$$

6. If $t=0$, then

$$\rho(0)=0$$

7. If $t=1$, then

$$\rho(1)=1.$$

8. If $t=2$, then

$$\rho(2)=2$$

Consequently

$$\rho(t)=\begin{cases}\frac{1}{2}t & \text{if} \quad t \in U\\ \frac{1}{4} & \text{if} \quad t=\frac{1}{2}\\ 2t-1 & \text{if} \quad t \in (1-U)\backslash\left\{\frac{1}{2}\right\}\\ \frac{t+1}{2} & \text{if} \quad t \in (1+U)\\ \frac{5}{4} & \text{if} \quad t=\frac{3}{2}\\ 2(t-1) & \text{if} \quad t \in (2-U)\backslash\left\{\frac{3}{2}\right\}\\ \frac{t+2}{2} & \text{if} \quad t \in (2+U)\\ 0 & \text{if} \quad t=0\\ 1 & \text{if} \quad t=1\\ 2 & \text{if} \quad t=2.\end{cases}$$

? Exercise 2.4

Find $\rho(t), t \in \mathbf{T}$, where $\mathbf{T}$ are the time scales in Exercise ? 2.3.

Answer

Add texts here. Do not delete this text first.

Definition: Backward Graininess Function

The backward graininess function $v : \mathbb{T} \to \mathbb{R}$ is defined as follows

$$v(t) = t - \rho(t), \quad t \in \mathbb{T}.$$

Note

Note that $v(t) \geq 0, t \in \mathbb{T}$.

Example 2.31

Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then, using Example 2.21, we find

$$\begin{aligned} v(t) &= t - \rho(t) \\ &= t - (t - h) \\ &= h, \quad t \in \mathbb{T} \end{aligned}$$

Example 2.32

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then, using Example 2.22, we find

$$\begin{aligned} v(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t = 1 \\ \frac{2t}{3} & \text{if} \quad t > 1 \end{cases} \end{aligned}$$

Example 2.33

Let $\mathbb{T} = \mathbb{N}_0^k, k \in \mathbb{N}$. Then, using Example 2.23, we find

$$\begin{aligned} v(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t = 0 \\ t - (\sqrt[k]{t} - 1)^k & \text{if} \quad t > 0 \end{cases} \end{aligned}$$

Observe that

$$\begin{aligned} t - (\sqrt[k]{t} - 1)^k &= t - \left(t - \binom{k}{1} \sqrt[k]{t^{k-1}} + \binom{k}{2} \sqrt[k]{t^{k-2}} - \cdots - 1 \right) \\ &= \binom{k}{1} \sqrt[k]{t^{k-1}} - \binom{k}{2} \sqrt[k]{t^{k-2}} + \cdots + 1 \end{aligned}$$

Consequently

$$v(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \binom{k}{1} \sqrt[k]{t^{k-1}} - \binom{k}{2} \sqrt[k]{t^{k-2}} + \cdots + 1 & \text{if} \quad t > 0 \end{cases}$$

Example 2.34

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_n, n \in \mathbb{N}_0$, are the harmonic numbers. Using Example 2.24, we find

$$\begin{aligned} v(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad n = 0 \\ H_n - H_{n-1} & \text{if} \quad n \geq 1 \end{cases} \end{aligned}$$

Example 2.35

Let $\mathbb{T} = P_{1,3}$. Using Example 2.25, we find

$$\begin{aligned} v(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t \in [0,1] \bigcup \bigcup_{k=1}^{\infty} (4k, 4k+1] \\ 3 & \text{if} \quad t \in \bigcup_{k=1}^{\infty} \{4k\} \end{cases} \end{aligned}$$

Example 2.36

Let $\mathbb{T} = C$, where C is the Cantor set. Using Example 2.26, we find

$$v(t) = t - \rho(t)$$
$$= \begin{cases} \frac{1}{3^{m+1}} & \text{if} \quad t \in C_2, \quad t = \sum_{k=1}^{k} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\ 0 & \text{if} \quad t \in \mathbb{T} \backslash C_2 \end{cases}$$

Example 2.37

Let

$$\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N} \right\}$$

Using Example 2.27 find

$$v(t) = t - \rho(t)$$
$$= \begin{cases} 0 & \text{if} \quad n = 1 \\ \alpha_{n-1} & \text{if} \quad n \geq 2 \end{cases}$$

Example 2.38

Let $\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0$. Using Example 0.37, we find

\begin{aligned} v(t) & =t-\rho(t) \\ & = \begin{cases}0 & \text { if } \quad t \in\{-1,0\} \\ \dfrac{t^2}{t+1} & \text { if } \quad t \in\left\{t_n=-\dfrac{1}{n}: n \in \mathbb{N}, \quad n \geq 2\right\} \\ 1 & \text { if } \quad t \in \mathbb{N}\end{cases} \notag

Example 2.39

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$. Using Example 0.38, we find

$$v(t) = t = \rho(t)$$
$$= \begin{cases} t - gt^2 & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \\ 0 & \text{if} \quad t \in \{0, 1\} \end{cases}$$

Example 2.40

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

\[\mathbb{T}=\{0\} \cup U \cup(1-U) \cup(1+U) \cup(2-U) \cupThen, using Example 0.39, we find

$$v(t) = t - \rho(t)$$
$$= \begin{cases} \frac{t}{2} & \text{if} \quad t \in U \\ 0 & \text{if} \quad t = 0 \\ 1 - t & \text{if} \quad t \in (1-U) \backslash \left\{ \frac{1}{2} \right\} \\ \frac{t-1}{2} & \text{if} \quad t \in (1+U) \\ \frac{1}{4} & \text{if} \quad t = \frac{3}{2} \\ 2 - t & \text{if} \quad t \in (2-U) \backslash \left\{ \frac{3}{2} \right\} \\ \frac{t-2}{2} & \text{if} \quad t \in (2+U) \\ 0 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t = 2 \end{cases}$$

Exercise 2.5

Find $v(t), t \in \mathbb{T}$, where $\mathbb{T}$ are the time scales in Exercise 2.2.

Answer

Add texts here.

Example 2.41

We will show that σ is in general no continuous. Consider

$$\mathbb{T} = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0.$$

We have

$$\sigma(0) = 1$$
$$\sigma\left(-\frac{1}{n}\right) = -\frac{1}{n+1}, \quad n \in \mathbb{N}$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \sigma\left(-\frac{1}{n}\right) &= -\lim_{n\to\infty} \frac{1}{n+1} \\ &= 0 \\ &\neq 1 \\ &= \sigma(0) \\ &= \sigma\left(\lim_{n\to\infty}\left(-\frac{1}{n}\right)\right)\end{aligned}$$

Example \(\PageIndex{42}\)

We will show that ρ is in general not continuous. Consider

$$\mathbb{T} == -2, -1] \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \mathbb{N}_0$$

Then

$$\rho(0) = -1$$
$$\rho\left(\frac{1}{n}\right) = \frac{1}{n+1}, \quad n \in \mathbb{N}$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \rho\left(\frac{1}{n}\right) &= \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0 \\ &\neq -1 \\ &= \rho(0) \\ &= \rho\left(\lim_{n\to\infty} \frac{1}{n}\right)\end{aligned}$$

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3: A Classification of Points

For any element of any time scale the following classification holds.

Definitions:

For $t \in \mathbb{T}$ we have the following cases.

1. If $\sigma(t) > t$, then we say that t is right-scattered.
2. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then we say that t is right-dense.
3. If $\rho(t) < t$, then we say that t is left-scattered.
4. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that t is left-dense.
5. If t is left-scattered and right-scattered at the same time, then we say that t is isolated.
6. If t is left-dense and right-dense at the same time, then we say that t is dense.

Example 3.1

Let $\mathbb{T} = h\mathbb{Z}, h > 0$. By Example 0.10, we have that

$$\begin{aligned} \sigma(t) &= t + h \\ &> t, \quad t \in \mathbb{T}. \end{aligned}$$

Thus, any point of $\mathbb{T}$ is right-scattered. Now, using Example 0.30, we get

$$\begin{aligned} \rho(t) &= t - h \\ &< t, \quad t \in \mathbb{T}. \end{aligned}$$

Therefore any point of $\mathbb{T}$ is left-scattered. Hence, we conclude that any point of $\mathbb{T}$ is isolated.

Example 3.2

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Assume that $t = 1$. By Example 0.11 , we have

$$\begin{aligned} \sigma(1) &= 3 \\ &> 1, \end{aligned}$$

i.e., $t = 1$ is right-scattered. By Example 0.31, we have

$$\sigma(1) = 1.$$

Since $1 = \inf \mathbb{T}$, we conclude that $t = 1$ is not left-dense.

2. Let $t > 1$. By Example 0.11, we have

$$\begin{aligned} \sigma(t) &= 3t \\ &> t. \end{aligned}$$

Thus, t is right-scattered. By Example 0.31, we get

$$\begin{aligned} \rho(t) &= \frac{t}{3} \\ &< t \end{aligned}$$

i.e., t is left scattered. Hence, we conclude that t is isolated.

Example 3.3

Let $\mathbb{T} = \mathbb{N}_0^k, k \in \mathbb{N}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Let $t = 0$. By Example 0.12, we have

$$\begin{aligned} \sigma(0) &= 1 \\ &> 1, \end{aligned}$$

i.e., $t = 0$ is right-scattered. By Example 0.32, we obtain

$$\rho(0) = 0.$$

Since $0 = \inf \mathbb{T}$, we conclude that $t = 0$ is not left-dense.

2. Let $t > 0$. By Example 0.12, we get

$$\begin{aligned} \sigma(t) &= (\sqrt[k]{t} + 1)^k \\ &> t \end{aligned}$$

i.e., t is right-scattered. B y Example 0.32, we find

$$\begin{aligned} \rho(t) &= (\sqrt[k]{t} - 1)^k \\ &< t \end{aligned}$$

i.e., t is left-scattered. Therefore t is isolated.

Example 3.4

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_n, n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$. We have the following cases.

1. Let $n = 0$. Then, by Example 0.13 , we get

$$\sigma(H_0) = H_1,$$

i.e., H_0 is right-scattered. By Example 0.33, we have

$$\rho(H_0) = H_0.$$

Since $H_0 = \inf \mathbb{T}$, we conclude that H_0 is not left-dense.

2. Let $n > 0$. By Example 0.13, we get

$$\begin{aligned} \sigma(H_n) &= H_{n+1} \\ &> H_n. \end{aligned}$$

Then H_n is right-scattered. By Example 0.33, we get

$$\begin{aligned} \rho(H_n) &= H_{n-1} \\ &< H_n \end{aligned}$$

i.e., H_n is left-scattered. Then H_n is isolated.

Example 3.5

Let $\mathbb{T} = P_{1,3}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Let $t \in \bigcup_{k=0}^{\infty}$. By Example 0.14 , we get

$$\sigma(t) = t,$$

i.e., t is right-dense. By Example 0.34, we find

$$\rho(t) = t,$$

i.e., t is left-dense. Thus, t is dense.

2. Let $t = 0$. By Example 0.14 , we obtain

$$\sigma(0) = 0,$$

i.e, $t = 0$ is right-dense. By Example 0.34, we find

$$\rho(0) = 0.$$

Since $0 = \inf \mathbb{T}$, we conclude that 0 is not left-dense.

3. Let $t \in \bigcup_{k=1}\{4k\}$. By Example 0.14, we get

$$\sigma(t) = t,$$

i.e., t is right-dense. By Example 0.34, we find

$$\begin{aligned}\rho(t) &= t - 3 \\ &< t,\end{aligned}$$

i.e., t is left-scattered.

4. Let $t \in \bigcup_{k=0}^{\infty}\{4k+1\}$. By Example 0.14, we find

$$\begin{aligned}\sigma(t) &= t + 3 \\ &> t,\end{aligned}$$

i.e., t is right-scattered. By Example 0.34, we find

$$\rho(t) = t$$

i.e., t is left-dense.

Example 3.6

Let $\mathbb{T} = C$, where C is the Cantor set. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Let $t \in C_1$. By Example 0.15, we have

$$\begin{aligned}\sigma(t) &= t + \frac{1}{3^{m+1}} \\ &> t\end{aligned}$$

i.e., t is right-scattered. By Example 0.35, we find

$$\rho(t) = t.$$

If $t \neq 0$, then it is left-dense. If $t = 0$, then it is not left-dense because $0 = \inf \mathbb{T}$.

v2 . Let $t \in C_2$. By Example 0.15, we get

$$\sigma(t) = t,$$

i.e., t is right-dense. By Example 0.35, we find

$$\begin{aligned}\rho(t) &= t - \frac{1}{3^{m+1}} \\ &< t\end{aligned}$$

i.e., t is left-scattered.

3. Let $t \in T \backslash C_1$. We have the following subcases.

a. Let $t \in C_2$. By Example 0.15, we find

$$\sigma(t) = t$$

i.e., t is right-dense. By Example 0.35, we obtain

$$\begin{aligned} \rho(t) &= t - \frac{1}{3^{m+1}} \\ &< t, \end{aligned}$$

i.e., t is left-scattered.

b. Let $t \in T \backslash C_2$. By Example 0.15, we arrive at

$$\sigma(t) = t,$$

i.e., t is right-dense. By Example 0.35, we have

$$\rho(t) = t.$$

If $t \neq 0$, then it is left-dense. If $t = 0$, then it is not left-dense.

4. Let $t \in \mathbb{T} \backslash C_2$. We have the following subcases.

a. Let $t \in C_1$. By Example 0.15 , we find

$$\begin{aligned} \sigma(t) &= t + \frac{1}{3^{m+1}} \\ &> t \end{aligned}$$

i.e., t is right-scattered. By Example 0.35, we find

$$\rho(t) = t.$$

If $t \neq 0$, then it is left-dense. If $t = 0$, then is left-dense.

b. Let $t \in T \backslash C_1$. By Example 0.15, we have

$$\sigma(t) = t,$$

i.e., t is right-dense. By Example 0.35, we have

$$\rho(t) = t.$$

If $t \neq 0$, then it is left-dense and hence dense. If $t = 0$, it is not left-dense

Example 3.7

Let

$$\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : \alpha_k > 0, \quad k \in \mathbb{N}_0, \quad n \in \mathbb{N} \right\}$$

Take $t \in \mathbb{T}$ arbitrarily. Then there is a $n \in \mathbb{N}$ such that

$$t = \sum_{k=0}^{n-1} \alpha_k$$

We have the following cases.

1. Let $n = 1$. By Example 0.16, we have

$$\begin{aligned}\sigma(t) &= \sum_{k=0}^{n} \alpha_k \\ &> t\end{aligned}$$

i.e., t is right-scattered. By Example 0.36, we arrive at

$$\rho(t) = t.$$

Since $t = \inf^{\mathsf{T}} \mathbb{T}$, we conclude that t is not left-dense.

2 . Let $n > 1$. By Example 0.16, we find

$$\begin{aligned}\sigma(t) &= \sum_{k=0}^{n} \alpha_k \\ &> t\end{aligned}$$

i.e., t is right-scattered. By Example 0.36, we find

$$\begin{aligned}\rho(t) &= t - \alpha_{n-1} \\ &< t\end{aligned}$$

i.e., t is left-scattered. Therefore t is isolated.

Example 3.8

Let $\mathbb{T} = \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Let $t = -1$. By Example 0.17, we have

$$\begin{aligned}\sigma(-1) &= -\frac{1}{2} \\ &> -1\end{aligned}$$

i.e., $t = -1$ is right-scattered. By Example 0.37, we find

$$\rho(-1) = -1.$$

Since $-1 = \inf^{T} \mathbb{T}$, we conclude that $t = -1$ is not left-dense.

2 . Let $t = 0$. By Example 0.17, we have

$$\begin{aligned}\sigma(0) &= 1 \\ &> 0\end{aligned}$$

i.e., $t = 0$ is right-scattered. By Example 0.37, we obtain

$$\rho(0) = 0.$$

Since $0 > \inf \mathbb{T}$, we conclude that $t = 0$ is left-dense.

3. Let $t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\} \setminus \{-1\}$. By Example 0.17 , we have

$$\begin{aligned}\sigma(t) &= -\frac{t}{t-1} \\ &> t\end{aligned}$$

i.e., t is right-scattered. By Example 0.37, we get

$$\begin{aligned}\rho(t) &= \frac{t}{t+1} \\ &< t\end{aligned}$$

i.e., t is left-scattered. Thus, t is isolated.

4. Let $t \in \mathbb{N}$. By Example 0.17, we have

$$\begin{aligned}\sigma(t) &= t+1 \\ &> t\end{aligned}$$

i.e., t is right-scattered. By Example 0.37, we obtain

$$\begin{aligned}\rho(t) &= t-1 \\ &< t\end{aligned}$$

i.e., t is left-scattered. Thus, t is isolated.

Example 3.9

Let $\mathbb{T} = \left\{t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \bigcup \{0,1\}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Let $t = 0$. By Example 0.18, we have

$$\sigma(0) = 0$$

i.e., $t = 0$ is right-dense. By Example 0.38, we get

$$\rho(0) = 0$$

Since $0 = \inf \mathbb{T}$, we conclude that $t = 0$ is not left-dense.

2 . Let $t = \frac{1}{2}$. By Example 0.18 , we have

$$\sigma\left(\frac{1}{2}\right) = 1$$

i.e., $t = \frac{1}{2}$ is right-scattered. By Example 0.38, we get

$$\begin{aligned}\rho\left(\frac{1}{2}\right) &= \frac{1}{4} \\ &< \frac{1}{2}\end{aligned}$$

i.e., $t = \frac{1}{2}$ is left-scattered. Thus, $t = \frac{1}{2}$ is isolated.

3. Let $t = 1$. By Example 0.18, we have

$$\sigma(1) = 1.$$

Since $1 = \sup \mathbb{T}$, we conclude that $t = 1$ is not right-dense. By Example 0.38, we obtain

$$\begin{aligned}\rho(1) &= \frac{1}{2} \\ &< 1\end{aligned}$$

i.e., $t = 1$ is left-scattered.

4. Let $t \in \left\{t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}\right\}$. Then there is a $n \in \mathbb{N}$ such that

$$t = \left(\frac{1}{2}\right)^{2^n}$$

By Example 0.18, we have

$$\begin{aligned}\sigma(t) &= \sqrt{t} \\ &> t\end{aligned}$$

i.e., t is right-scattered. By Example 0.38, we obtain

$$\rho(t) = t^2 < t$$

i.e., t is left-scattered. Thus, t is isolated.

Example 3.10

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{1,2\}.$$

Take $t \in \mathbb{T}$ arbitrarily. Then, we have the following cases.

1. Let $t = 0$. By Example 0.19 , we have

$$\sigma(0) = 0$$

i.e., $t = 0$ is right-dense, By Example 0.39, we get

$$\rho(0) = 0$$

Since $0 = \inf \mathbb{T}$, we conclude that $t = 0$ is not left-dense.

2 . Let $t = \frac{1}{2}$. By Example 0.19 , we have

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4},$$

i.e., $t = \frac{1}{2}$ is right-scattered. By Example 0.39, we get

$$\rho\left(\frac{1}{2}\right) = \frac{1}{4}$$

i.e., $t = \frac{1}{2}$ is right-scattered. Thus, $t = \frac{1}{2}$ is isolated.

3. Let $t = 1$. By Example 0.19, we have

$$\sigma(1) = 1,$$

i.e., $t = 1$ is right-dense. By Example 0.39, we find

$$\rho(1) = 1,$$

i.e., $t = 1$ is left-dense. Thus, $t = 1$ is dense.

4 . Let $t = \frac{3}{2}$. By Example 0.19 , we have

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$$

i.e., $t = \frac{3}{2}$ is right-scattered. By Example 0.39, we find

$$\rho\left(\frac{3}{2}\right) = \frac{5}{4}$$

i.e., $t = \frac{3}{2}$ is left-scattered. Thus, $t = \frac{3}{2}$ is isolated.

5 . Let $t = 2$. By Example 0.19, we have

$$\sigma(2) = 2,$$

i.e., $t = 2$ is right-dense. By Example 0.39, we get

$$\rho(2) = 2$$

i.e., $t = 2$ is left-dense. Thus, $t = 2$ is dense.

6. Let $t = \frac{5}{2}$. Then, by Example 0.19 , we get

$$\sigma\left(\frac{5}{2}\right) = \frac{5}{2}$$

Since $\frac{5}{2} = \sup \mathbb{T}$, we conclude that $t = \frac{5}{2}$ is not right-dense. By Example 0.39 , we find

$$\rho\left(\frac{5}{2}\right) = \frac{9}{8}$$

Thus, $t = \frac{5}{2}$ is left-scattered.

7. Let $t \in U \backslash \left\{\frac{1}{2}\right\}$. By Example 0.19 , we have

$$\begin{aligned} \sigma(t) &= 2t \\ &> t \end{aligned}$$

i.e., t is right-scattered. By Example 0.39, we get

$$\begin{aligned} \rho(t) &= \frac{t}{2} \\ &< t, \end{aligned}$$

i.e., t is left-scattered. Thus, t is isolated.

8. Let $t \in (1 - U) \backslash \left\{\frac{1}{2}\right\}$. By Example 0.19, we have

$$\begin{aligned} \sigma(t) &= \frac{1+t}{2} \\ &> t \end{aligned}$$

i.e., t is right-scattered. By Example 0.39, we find

$$\begin{aligned} \rho(t) &= 2t - 1 \\ &< t \end{aligned}$$

i.e., t is left-scattered. Thus, t is isolated.

9. Let $t \in (1 + U) \backslash \left\{\frac{3}{2}\right\}$. By Example 0.19 , we have

$$\begin{aligned} \sigma(t) &= 2t - 1 \\ &> t \end{aligned}$$

i.e., t is right-scattered. By Example 0.39, we find

$$\begin{aligned} \rho(t) &= \frac{t+1}{2} \\ &< t \end{aligned}$$

i.e., t is left-scattered. Thus, t is isolated.

10. Let $t \in (2 - U) \backslash \left\{\frac{3}{2}\right\}$. By Example 0.19, we have

$$\begin{aligned} \sigma(t) &= \frac{2+t}{2} \\ &> t \end{aligned}$$

i.e., t is right-scattered. By Example 0.39, we find

$$\begin{aligned} \rho(t) &= 2t - 2 \\ &< t \end{aligned}$$

i.e., t is left-scattered. Thus, t is isolated.

11. Let $t \in (2+U)\backslash\left\{\frac{5}{2}\right\}$. By Example 0.19 , we have

$$\begin{aligned}\sigma(t) &= 2(t-1)\\ &> t\end{aligned}$$

i.e., t is right-scattered. By Example 0.39, we find

$$\begin{aligned}\rho(t) &= \frac{t+2}{2}\\ &< t\end{aligned}$$

i.e., t is left-scattered. Thus, t is isolated.

? Exercise 3.1

Classify each point of $\mathbb{T}$, where $\mathbb{T}$ are the time scales in Exercise 0.3.

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4: The Topology of Time Scales

Throughout this book, we assume that any time scale has the topology that it inherits from gthe real numbers with the standard topology.

Definition: Term

Let $a, b \in \mathbb{T}, a \leq b$. Define closed, half open and open time scales intervals as follows

$$[a,b]_{\mathbb{T}} = \{x \in \mathbb{T} : a \leq x \leq b\}$$
$$[a,b)_{\mathbb{T}} = \{x \in \mathbb{T} : a \leq x < b\}$$
$$(a,b]_{\mathbb{T}} = \{x \in \mathbb{T} : a < x \leq b\}$$
$$(a,b)_{\mathbb{T}} = \{x \in \mathbb{T} : a < x < b\}$$

respectively.

Example 4.1

Let $\mathbb{T} = 3\mathbb{Z}$.

Then

$$[-3,12]_{\mathbb{T}} = \{-3,0,3,6,9,12\}$$
$$[-3,12)_{\mathbb{T}} = \{-3,0,3,6,9\}$$
$$(-3,12]_{\mathbb{T}} = \{0,3,6,9,12\}$$
$$(-3,12)_{\mathbb{T}} = \{0,3,6,9\}$$

Example 4.2

Let $\mathbb{T} = 3^{\mathbb{N}_0}$.

Then

$$[3,243]_{\mathbb{T}} = \{3,9,27,81,243\}$$
$$[3,243)_{\mathbb{T}} = \{3,9,27,81\}$$
$$(3,243]_{\mathbb{T}} = \{9,27,81,243\}$$
$$(3,243)_{\mathbb{T}} = \{9,27,81\}$$

Example 4.3

Let $\mathbb{T} = \mathbb{N}_0^3$.

Then

$$[1,27]_{\mathbb{T}} = \left\{1,2,9,(\sqrt[3]{2}+1)^3,(\sqrt[3]{3}+1)^3,(\sqrt[3]{4}+1)^3,(\sqrt[3]{5}+1)^3,(\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3,27\right\}$$
$$[1,27)_{\mathbb{T}} = \left\{1,2,9,(\sqrt[3]{2}+1)^3,(\sqrt[3]{3}+1)^3,(\sqrt[3]{4}+1)^3,(\sqrt[3]{5}+1)^3,(\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3\right\}$$

$$(1,27]_{\mathbb{T}} = \left\{2,9,(\sqrt[3]{2}+1)^3,(\sqrt[3]{3}+1)^3,(\sqrt[3]{4}+1)^3,(\sqrt[3]{5}+1)^3,(\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3,27\right\}$$
$$(1,27)_{\mathbb{T}} = \left\{2,9,(\sqrt[3]{2}+1)^3,(\sqrt[3]{3}+1)^3,(\sqrt[3]{4}+1)^3,(\sqrt[3]{5}+1)^3,(\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3\right\}$$

Example 4.4

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_n, n \in \mathbb{N}_0$, are the harmonic numbers.

Then

$$\left[1, \frac{147}{60}\right]_{\mathbb{T}} = \left\{1, \frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}\right\},$$
$$\left[1, \frac{147}{60}\right)_{\mathbb{T}} = \left\{1, \frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}\right\},$$
$$\left(1, \frac{147}{60}\right]_{\mathbb{T}} = \left\{\frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}\right\},$$
$$\left(1, \frac{147}{60}\right)_{\mathbb{T}} = \left\{\frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}\right\}.$$

Example 4.5

Let $\mathbb{T} = P_{1,3}$.

Then

$$[0, 12]_{\mathbb{T}} = [0, 1] \cup [4, 5] \cup [8, 9] \cup \{12\}$$
$$[0, 12)_{\mathbb{T}} = [0, 1] \cup [4, 5] \cup [8, 9]$$
$$(0, 12]_{\mathbb{T}} = (0, 1] \cup [4, 5] \cup [8, 9] \cup \{12\}$$
$$(0, 12)_{\mathbb{T}} = (0, 1] \cup [4, 5] \cup [8, 9]$$

Example 4.6

Let $\mathbb{T} = \{\sum_{k=0}^{n} k : n \in \mathbb{N}\}$

Then

$$[0, 28]_{\mathbb{T}} = \{0, 1, 3, 6, 10, 15, 21, 28\}$$
$$[0, 28)_{\mathbb{T}} = \{0, 1, 3, 6, 10, 15, 21\}$$
$$(0, 28]_{\mathbb{T}} = \{1, 3, 6, 10, 15, 21, 28\}$$
$$(0, 28)_{\mathbb{T}} = \{1, 3, 6, 10, 15, 21\}$$

Example 4.7

Let $\mathbb{T} = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$.

Then

$$\left[-\frac{1}{3}, 3\right]_{\mathbb{T}} = \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 3\right\} \cup \{0, 1, 2, 3\},$$
$$\left[-\frac{1}{3}, 3\right)_{\mathbb{T}} = \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 3\right\} \cup \{0, 1, 2\},$$
$$\left(-\frac{1}{3}, 3\right]_{\mathbb{T}} = \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 4\right\} \cup \{0, 1, 2, 3\},$$
$$\left(-\frac{1}{3}, 3\right)_{\mathbb{T}} = \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 4\right\} \cup \{0, 1, 2\}.$$

Example 4.8

Let $\mathbb{T} = \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0,1\}$.

Then

$$\left[0,\frac{1}{2}\right]_{\mathbb{T}} = \left\{0,\frac{1}{16},\frac{1}{4},\frac{1}{2}\right\},$$
$$\left[0,\frac{1}{2}\right)_{\mathbb{T}} = \left\{0,\frac{1}{16},\frac{1}{4}\right\},$$
$$\left(0,\frac{1}{2}\right]_{\mathbb{T}} = \left\{\frac{1}{16},\frac{1}{4},\frac{1}{2}\right\},$$
$$\left(0,\frac{1}{2}\right)_{\mathbb{T}} = \left\{\frac{1}{16},\frac{1}{4}\right\}.$$

Example 4.9

Let $U = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{1,2\}.$$

Then

$$\left[0,\frac{7}{8}\right]_{\mathbb{T}} = \{0\} \cup U \cup \left\{\frac{3}{4},\frac{7}{8}\right\},$$
$$\left[0,\frac{7}{8}\right)_{\mathbb{T}} = \{0\} \cup U \cup \left\{\frac{3}{4}\right\},$$
$$\left(0,\frac{7}{8}\right]_{\mathbb{T}} = U \cup \left\{\frac{3}{4},\frac{7}{8}\right\},$$
$$\left(0,\frac{7}{8}\right)_{\mathbb{T}} = U \cup \left\{\frac{3}{4}\right\}.$$

Exercise 4.1

Find

$$[a,b]_{\mathbb{T}}, \quad [a,b)_{\mathbb{T}}, \quad (a,b]_{\mathbb{T}}, \quad (a,b)_{\mathbb{T}}$$

where

1. $\mathbb{T} = 4\mathbb{Z} + 1, a = -3, b = 9$.
2. $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}, a = -4, b = 9$.
3. $\mathbb{T} = P_{3,7} \cup [4,6], a = 0, b = 12$.
4. $\mathbb{T} = \{0\} \cup 11^{\mathbb{N}_0}, a = 0, b = 121$.
5. $\mathbb{T} = [1,2] \cup [3,4] \cup [7,8] \cup 9^{\mathbb{N}}, a = 2, b = 9$.

Answer

Add texts here. Do

Definition: Sets

Define the sets

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}]_{\mathbb{T}} & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} \quad \text{if} \quad \sup \mathbb{T} = \infty \end{cases}$$

and

$$\mathbb{T}_{\kappa} = \begin{cases} \mathbb{T}\backslash[\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} & \text{if} \quad \inf \mathbb{T} > -\infty \\ \mathbb{T} \quad \text{if} \quad \inf \mathbb{T} > -\infty. \end{cases}$$

Example 4.10

Let $\mathbb{T} = h\mathbb{Z}, h > 0$

Then

$$\begin{aligned} \inf \mathbb{T} &= -\infty, \\ \sup \mathbb{T} &= n\infty \end{aligned}$$

and

$$\begin{aligned} \mathbb{T}^{\kappa} &= \mathbb{T}, \\ \mathbb{T}_{\kappa} &= \mathbb{T}. \end{aligned}$$

Example 4.11

Let $\mathbb{T} = 3^{\mathbb{N}_0}$.

Then

$$\begin{aligned} \sup \mathbb{T} &= \infty, \\ \inf \mathbb{T} &= 1, \\ \sigma(\inf \mathbb{T}) &= \sigma(1) \\ &= 3. \end{aligned}$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\begin{aligned} \mathbb{T}_{\kappa} &= \mathbb{T}\backslash[\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} \\ &= \mathbb{T}\backslash[1, 3)_{\mathbb{T}} \\ &= \mathbb{T}\backslash\{1\} \\ &= 3^{\mathbb{N}} \end{aligned}$$

Example 4.12

Let $\mathbb{T} = \mathbb{N}_0^k, k \in \mathbb{N}$.

Then

$$\begin{aligned}\sup \mathbb{T} &= \infty,\\ \inf \mathbb{T} &= 0\\ &= H_0,\\ \sigma(\inf \mathbb{T}) &= \sigma\left(H_0\right)\\ &= H_1.\end{aligned}$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\begin{aligned}\mathbb{T}_{\kappa} &= \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}}\\ &= \mathbb{T} \backslash [H_0, H_1)_{\mathbb{T}}\\ &= \mathbb{T} \backslash \{H_0\}\\ &= \mathbb{N}^k\end{aligned}$$

Example 4.13

Let $\mathbb{T} = P_{1,3}$.

Then

$$\begin{aligned}\sup \mathbb{T} &= \infty\\ \inf \mathbb{T} &= 0\\ \sigma(\inf \mathbb{T}) &= \sigma(0)\\ &= 0.\end{aligned}$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\begin{aligned}\mathbb{T}_{\kappa} &= \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}}\\ &= \mathbb{T} \backslash [0, 0)_{\mathbb{T}}\\ &= \mathbb{T} \backslash \emptyset\\ &= \mathbb{T}\end{aligned}$$

Example 4.14

Let $\mathbb{T} = C$, where C is the Cantor set.

Then

$$\begin{aligned}\sup \mathbb{T} &= 1,\\ \rho(\sup \mathbb{T}) &= \rho(1)\\ &= 1,\\ \inf \mathbb{T} &= 0,\\ \sigma(\inf \mathbb{T}) &= \sigma(0)\\ &= 0.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{T}^{\kappa} &= \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] \\ &= \mathbb{T}\backslash(1,1] \\ &= \mathbb{T}\backslash\emptyset \\ &= \mathbb{T}\end{aligned}$$

and

$$\begin{aligned}\mathbb{T}_{\kappa} &= \mathbb{T}\backslash[\inf\mathbb{T}, \sigma(\inf\mathbb{T})_{\mathbb{T}} \\ &= \mathbb{T}\backslash[0,0)_{\mathbb{T}} \\ &= \mathbb{T}\backslash\emptyset \\ &= \mathbb{T}\end{aligned}$$

Example 4.15

Let $\mathbb{T} = \left\{\sum_{k=1}^{n-1}\alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N}_0\right\}$.

Then

$$\begin{aligned}\sup\mathbb{T} &= \infty, \\ \inf\mathbb{T} &= \alpha_0, \\ \sigma(\inf\mathbb{T}) &= \sigma(\alpha_0) \\ &= \alpha_1.\end{aligned}$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\begin{aligned}\mathbb{T}_{\kappa} &= \mathbb{T}\backslash[\inf\mathbb{T}, \sigma(\inf\mathbb{T}))_{\mathbb{T}} \\ &= \mathbb{T}\backslash[\alpha_0, \alpha_1)_{\mathbb{T}} \\ &= \mathbb{T}\backslash\{\alpha_0\} \\ &= \left\{\sum_{k=0}^{n-1}\alpha_k : n \in \mathbb{N}, \quad n \geq 2, \quad \alpha_k >, \quad k \in \mathbb{N}_0\right\}.\end{aligned}$$

Example 4.16

Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$.

Then

$$\begin{aligned}\sup\mathbb{T} &= \infty, \\ \inf\mathbb{T} &= 0,\end{aligned}$$

$$\begin{aligned}\sigma(\inf\mathbb{T}) &= \sigma(0) \\ &= 0\end{aligned}$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\begin{aligned}\mathbb{T}_{\kappa} &= \mathbb{T}\backslash[\inf\mathbb{T}, \sigma(\inf\mathbb{T}))_{\mathbb{T}} \\ &= \mathbb{T}\backslash[0,0)_{\mathbb{T}} \\ &= \mathbb{T}\backslash\emptyset \\ &= \mathbb{T}\end{aligned}$$

Example 4.17

Let $\mathbb{T} = \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0, 1\}$.

Then

$$\begin{aligned} \sup \mathbb{T} &= 1, \\ \rho(\sup \mathbb{T}) &= \rho(1) \\ &= \frac{1}{2}, \\ \inf \mathbb{T} &= 0, \\ \sigma(\inf \mathbb{T}) &= \sigma(0) \\ &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{T}^{\kappa} &= \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] \\ &= \mathbb{T} \backslash \left(\frac{1}{2}, 1\right] \end{aligned}$$

$$\begin{aligned} &= \mathbb{T} \backslash \{1\} \\ &= \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0\}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{T}_{\kappa} &= \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T})_{\mathbb{T}} \\ &= \mathbb{T} \backslash [0, 0)_{\mathbb{T}} \\ &= \mathbb{T} \backslash \emptyset \\ &= \mathbb{T} \end{aligned}$$

Example 4.18

Let $U = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

Then

$$\begin{aligned} \sup \mathbb{T} &= \frac{5}{2}, \\ \rho(\sup \mathbb{T}) &= \rho\left(\frac{5}{2}\right) \\ &= \frac{9}{4}, \\ \inf \mathbb{T} &= 0, \\ \sigma(\inf \mathbb{T}) &= \sigma(0) \\ &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{T}^{\kappa} &= \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] \\ &= \mathbb{T} \backslash \left(\frac{9}{4}, \frac{5}{2}\right] \end{aligned}$$

$$= \mathbb{T} \backslash \left\{\frac{5}{2}\right\}$$

and

$$\begin{aligned}\mathbb{T}_\kappa &= \mathbb{T}\backslash[\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} \\ &= \mathbb{T}\backslash[0,0)_{\mathbb{T}} \\ &= \mathbb{T}\backslash\emptyset \\ &= \mathbb{T}\end{aligned}$$

? Exercise 4.1

Add exercises text here.

Answer

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Exercise 0.9. Find $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$, where $\mathbb{T}$ are the time scales in Exercise 0.3.

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5: Functions and Jump Operators

In this section, we will consider compositions of real-valued functions with jump operators.

Definition: Term

For a function $f:\mathbb{T}\to\mathbb{R}$ and a $k\in\mathbb{N}_0$, define

$$f^{\sigma^k}(t)=f\left(\sigma^k(t)\right),\quad t\in\mathbb{T}$$

Example 5.1

Let $\mathbb{T}=h\mathbb{Z}, h>0$ and

$$f(t)=1=2t+3t^2,\quad t\in\mathbb{T}$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $ty\in\mathbb{T}$.

Solution

By Example 2.1 , we have

$$\sigma(t)=t+h,\quad t\in\mathbb{T}$$

and

$$\begin{aligned}f^{\sigma}(t)&=f(\sigma(t))\\&=1+2\sigma(t)+3(\sigma(t))^2\\&=1+2(t+h)+3(t+h)^2\\&=1+2t+2h+3t^2+6ht+3h^2\end{aligned}$$

$$=1+2h+3h^2+2(1+3h)t+3t^2,\quad t\in\mathbb{T}.$$

Hence,

$$\begin{aligned}f^{\sigma^2}(t)&=f\left(\sigma^2(t)\right)\\&=f(\sigma(\sigma(t)))\\&=f^{\sigma}(\sigma(t))\\&=1+2h+3h^2+2(1+3h)\sigma(t)+3(\sigma(t))^2\\&=1+2h+3h^2+2(1+3h)(t+h)+3(t+h)^2\\&=1+2h+3h^2+2h(1+3h)+2(1+3h)t+3t^2+6ht+3h^2\\&=1+2h+6h^2+2h+6h^2+2(1+3h+3h)t+3t^2\\&=1+4h+12h^2+2(1+6h)t+3t^2,\quad t\in\mathbb{T}\end{aligned}$$

Example 5.2

Let $\mathbb{T}=3^{\mathbb{N}_0}$ and

$$f(t)=\frac{1+t}{2+t},\quad t\in\mathbb{T}$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t\in\mathbb{T}$.

Solution

By Example 2.2 , we have

$$\sigma(t)=3t,\quad t\in\mathbb{T},$$

and

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= \frac{1+\sigma(t)}{2+\sigma(t)} \\ &= \frac{1+3t}{2+3t}, \quad t \in \mathbb{T} \end{aligned}$$

Hence,

$$\begin{aligned} f^{\sigma^2}(t) &= f\left(\sigma^2(t)\right) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= \frac{1+3\sigma(t)}{2+3\sigma(t)} \\ &= \frac{1+9t}{2+9t}, \quad t \in \mathbb{T}. \end{aligned}$$

Example 5.3

Let $\mathbb{T} = \mathbb{N}_0^2$ and

$$f(t) = \frac{1+2t}{1+t^2}, \quad t \in \mathbb{T}$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$.

Solution

By Example 2.3 , we have

$$\sigma(t) = (\sqrt{t}+1)^2, \quad t \in \mathbb{T},$$

Then

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= \frac{1+2\sigma(t)}{1+(\sigma(t))^2} \\ &= \frac{1+2(\sqrt{t}+1)^2}{1+(\sqrt{t}+1)^4}, \quad t \in \mathbb{T} \end{aligned}$$

Hence,

$$\begin{aligned} f^{\sigma^2}(t) &= f\left(\sigma^2(t)\right) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= \frac{1+2(\sqrt{\sigma(t)}+1)^2}{1+(\sqrt{\sigma(t)}+1)^4} \\ &= \frac{1+\left(\sqrt{(\sqrt{t}+1)^2}+1\right)^2}{1+\left(\sqrt{(\sqrt{t}+1)^4}+1\right)^4} \\ &= \frac{1+2(\sqrt{t}+1+1)^2}{1+(\sqrt{t}+1+1)^4} \\ &= \frac{1+2(\sqrt{t}+2)^2}{1+(\sqrt{t}+2)^4}, \quad t \in \mathbb{T} \end{aligned}$$

Example 5.4

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_n, n \in \mathbb{N}_0$, are the harmonic numbers, and

$$f(t) = \frac{1-t}{1+4t}, \quad t \in \mathbb{T}.$$

\

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$.

Solution

By Example 2.4 , we have

$$\sigma(H_n) = H_{n+1}, \quad n \in \mathbb{N}_0.$$

Then

$$\begin{aligned} f^{\sigma}(H_n) &= f(\sigma(H_n)) \\ &= \frac{1-\sigma(H_n)}{1+4\sigma(H_n)} \\ &= \frac{1-H_{n+1}}{1+4H_{n+1}}, \quad n \in \mathbb{N}_0 \end{aligned}$$

Hence,

$$\begin{aligned} f^{\sigma^2(H_n)} &= f(\sigma^2(H_n)) \\ &= f(\sigma(\sigma(H_n))) \\ &= f^{\sigma}(\sigma(H_n)) \\ &= \frac{1-\sigma(H_{n+1})}{1+4\sigma(H_{n+1})} \\ &= \frac{1-H_{n+2}}{1+4H_{n+2}}, \quad n \in \mathbb{N}_0 \end{aligned}$$

Example 5.5

Let $\mathbb{T} = P_{1,3}$ and

$$f(t) = 1 - 2t - t^2, \quad t \in \mathbb{T}.$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$.

Solution

By Example 2.5, we have

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[4k, 4k+1) \\ t+3 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}\{4k+1\}. \end{cases}$$

Then, we will consider the following cases.

1. Let $t \in \bigcup_{k=0}^{\infty}[4k, 4k+1)$. Then $\sigma = I$, where I is the identity operator, and then

$$\begin{aligned} f^{\sigma}(t) &= f(t) \\ f^{\sigma^2}(t) &= f(t), \quad t \in \mathbb{T} \end{aligned}$$

2. Let $t \bigcup_{k=0}^{\infty}\{4k+1\}$. Then

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= 1 - 2\sigma(t) - (\sigma(t))^2 \\ &= 1 - 2(t+2) - (y+3)^2 \\ &= 1 - 2t - 6 - t^2 - 6t - 9 \\ &= -14 - 8t - t^2, \quad t \in \mathbb{T} \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= f\left(\sigma^2(t)\right) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= -14 - 8\sigma(t) - (\sigma(t))^2 \\ &= -14 - 8(t+3) - (t+3)^2 \\ &= -14 - 8t - 24 - t^2 - 6t - 9 \\ &= -47 - 14t - t^2, \quad t \in \mathbb{T}. \end{aligned}$$

Example 5.6

Let $\mathbb{T} = C$, where C is the Cantor set, and

$$f(t) = t^3, \quad t \in \mathbb{T}.$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$.

Solution

By Example 2.6 , we have

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}}, & t \in C_1, \quad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}, \\ t, & t \in \mathbb{T} \backslash C_1. \end{cases}$$

Then, we have the following cases.

1. Let $t \in C_1$ and

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}$$

Then

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= (\sigma(t))^3 \\ &= \left(t + \frac{1}{3^{m+1}}\right)^3 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= f\left(\sigma^2(t)\right) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= \left(\sigma(t) + \frac{1}{3^{m+1}}\right)^3 \\ &= \left(t + \frac{1}{3^{m+1}} + \frac{1}{3^{m+1}}\right)^3 \\ &= \left(t + \frac{2}{3^{m+1}}\right)^3. \end{aligned}$$

Example 5.7

Let

$$\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N}_0 \right\},$$

and

$$f(t) = \frac{1+t^3}{3+t^4}, \quad t \in \mathbb{T}$$

Solution

By Example 2.7, we have

$$\sigma(t_n) = t_{n+1}, \quad n \in \mathbb{N}$$

Then

$$\begin{aligned} f^{\sigma}(t_n) &= f(\sigma(t_n)) \\ &= \frac{1+(\sigma(t_n))^3}{3+(\sigma(t_n))^4} \\ &= \frac{1+t_{n+1}^3}{3+t_{n+1}^4}, \quad n \in \mathbb{N} \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t_n) &= f(\sigma^2(t_n)) \\ &= f(\sigma(\sigma(t_n))) \\ &= f^{\sigma}(\sigma(t_n)) \\ &= \frac{1+(\sigma(t_{n+1}))^3}{3+(\sigma(t_{n+1}))^4} \\ &= \frac{1+t_{n+2}^3}{3+t_{n+2}^4}, \quad n \in \mathbb{N} \end{aligned}$$

Example 5.8

Let $\mathbb{T} = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0$ and

$$f(t) = 1 + t - t^2, \quad t \in \mathbb{T}$$

Solution

By Example 2.8, we have

$$\sigma(t) = \begin{cases} -\dfrac{t}{t-1} \text{ if } t \in \left\{ -\dfrac{1}{n} : n \in \mathbb{N} \right\} \\ t+1 \text{ if } t \in \mathbb{N}_0. \end{cases}$$

Then, we will consider the following cases.

1. Let $t \in \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$. Then

$$\begin{aligned} f\sigma(t) &= f(\sigma(t)) \\ &= 1+\sigma(t)-(\sigma(t))^2 \\ &= 1+\frac{t}{t-1}-\frac{t^2}{(t-1)^2} \\ &= \frac{(t-1)^2+t(t-1)-t^2}{(t-1)^2} \\ &= \frac{t^2-2t+1+t^2-t-t^2}{(t-1)^2} \\ &= \frac{t^2-3t+1}{(t-1)^2} \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= f\left(\sigma^2(t)\right) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= \frac{(\sigma(t))^2-3\sigma(t)+1}{(\sigma(t)-1)^2} \\ &= \frac{\frac{t^2}{(t-1)^2}+\frac{3t}{t-1}+1}{\left(-\frac{t}{t-1}-1\right)^2} \\ &= \frac{\frac{t^2+3t(t-1)+(t-1)^2}{(t-1)^2}}{\left(\frac{t+1-1}{t-1}\right)^2} \\ &= \frac{t^2+3gt^2-3t+t^2-2t+1}{(2t-1)^2} \\ &= \frac{5t^2-5t+1}{(2t-1)^2}. \end{aligned}$$

2. Let $t \in \mathbb{N}_0$. Then

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= 1+\sigma(t)-(\sigma(t))^2 \\ &= 1+t+1-(t+1)^2 \\ &= 2+t-t^2-2t-1 \\ &= -t^2-t-1 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2(t)} &= f\left(\sigma^2(t)\right) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= -(\sigma(t))^2-\sigma(t)+1 \\ &= -(t+1)^2-(t+1)+1 \\ &= -t^2-2t-1-t-1+1 \\ &= -t^2-3t-1 \end{aligned}$$

Example 5.9

Let $\mathbb{T}\left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0,1\}$ and

$$f(t) = t - t^3, \quad t \in \mathbb{T}.$$

Then

$$f^{\sigma}(t) = f(\sigma(t))$$
$$= \sigma(t) - (\sigma(t))^3, \quad t \in \mathbb{T},$$

and

$$f^{\sigma^2}(t) = f\left(\sigma^2(t)\right)$$
$$= \sigma^2(t) - \left(\sigma^2(t)\right)^3, \quad t \in \mathbb{T}$$

By Example 0.18, we get the following cases.

1. Let $t \in \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\}$. Then

$$\sigma(t) = \sqrt{t}$$
$$\sigma^2(t) = \sqrt{\sigma(t)}$$
$$= \sqrt{\sqrt{t}}$$
$$= \sqrt[4]{t}, \quad t \in \mathbb{T}$$

Hence,

$$f\sigma(t) = \sqrt{t} - (\sqrt{t})^3$$

and

$$f^{\sigma^2}(t) = \sqrt[4]{t} - (\sqrt[4]{t})^3$$

2. Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = 1$$

and

$$\sigma^2\left(\frac{1}{2}\right) = \sigma\left(\sigma\left(\frac{1}{2}\right)\right)$$
$$= \sigma(1)$$
$$= 1$$

Hence,

$$f^{\sigma}\left(\frac{1}{2}\right) = \sigma\left(\frac{1}{2}\right) - \left(\sigma\left(\frac{1}{2}\right)\right)^3$$
$$= 1 - 1^3$$
$$= 0$$

and

$$f^{\sigma^2}\left(\frac{1}{2}\right) = \sigma^2\left(\frac{1}{2}\right) - \left(\sigma^2\left(\frac{1}{2}\right)\right)^3$$
$$= 1 - 1^3$$
$$= 0$$

3. Let $t = 0$. Then

$$\sigma(0) = 0$$

and

$$\begin{aligned}\sigma^2(0) &= \sigma(\sigma(0)) \\ &= \sigma(0) \\ &= 0\end{aligned}$$

Hence,

$$\begin{aligned}f^\sigma(0) &= \sigma(0) - (\sigma(0))^3 \\ &= 0 - 0^3 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2}(0) &= \sigma^2(0) - \left(\sigma^2(0)\right)^3 \\ &= 0 - 0^3 \\ &= 0\end{aligned}$$

4. Let $t = 1$. Then

$$\sigma(1) = 1$$

and

$$\begin{aligned}\sigma^2(1) &= \sigma(\sigma(1)) \\ &= \sigma(1) \\ &= 1\end{aligned}$$

Hence,

$$\begin{aligned}f^\sigma(1) &= \sigma(1) - (\sigma(1))^3 \\ &= 1 - 1^3 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2(1)} &= \sigma^2(1) - \left(\sigma^2(1)\right)^3 \\ &= 1 - 1^3 \\ &= 0\end{aligned}$$

Example 5.10

Let $U = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{1,2\},$$

and

$$f(t) = \frac{1+t}{1+4t}, \quad t \in \mathbb{T}$$

We have

$$\begin{aligned}f^\sigma(t) &= f(\sigma(t)) \\ &= \frac{1+\sigma(t)}{1+4\sigma(t)}, \quad t \in \mathbb{T}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2(t)} &= f\left(\sigma^2(t)\right) \\ &= \frac{1+\sigma^2(t)}{1+4\sigma^2(t)}, \quad t \in \mathbb{T}\end{aligned}$$

By Example 2.9, we have the following cases.

1. Let $t \in U \setminus \left\{\frac{1}{2}\right\}$. Then

$$\sigma(t) = 2t$$

and

$$\begin{aligned}\sigma^2(t) &= \sigma(\sigma(t)) \\ &= 2\sigma(t) \\ &= 2(2t) \\ &= 4t\end{aligned}$$

Hence,

$$\begin{aligned}f^{\sigma}(t) &= \frac{1+2t}{1+4(2t)} \\ &= \frac{1+2t}{1+8t}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2}(t) &= \frac{1+4t}{1+4(4t)} \\ &= \frac{1+4t}{1+16t}\end{aligned}$$

2. Let $t \in (1-U) \setminus \left\{\frac{1}{2}\right\}$. Then

$$\sigma(t) = \frac{1+t}{2}$$

and

$$\begin{aligned}\sigma^2(t) &= \sigma(\sigma(t)) \\ &= \frac{1\sigma(t)}{2} \\ &= \frac{1+\frac{1+t}{2}}{2} \\ &= \frac{2+1+t}{4} \\ &= \frac{3+t}{4}.\end{aligned}$$

Hence,

$$\begin{aligned}f^{\sigma}(t) &= \frac{1+\frac{1+t}{2}}{1+4\frac{1+t}{2}} \\ &= \frac{2+1+t}{2+4+4t} \\ &= \frac{3+t}{6+4t}\end{aligned}$$

and

$$f^{\sigma^2}(t) = \frac{1+\frac{3+t}{4}}{2+4\frac{3+t}{4}} = \frac{4+3+t}{8+12+4t} = \frac{7+t}{20+4t}$$

3. Let $t \in (1+U)\backslash\left\{\frac{3}{2}\right\}$. Then

$$\sigma(t) = 2t-1$$

and

$$\begin{aligned}\sigma^2(t) &= \sigma(\sigma(t)) \\ &= 2\sigma(t)-1 \\ &= 2(2t-1)-1 \\ &= 4t-2-1 \\ &= 4t-3\end{aligned}$$

Hence,

$$f^{\sigma}(t) = \frac{1+2t-1}{1+4(2t-1)} = \frac{2t}{1+8t-4} = \frac{2t}{8t-3}$$

and

$$f^{\sigma^2}(t) = \frac{1+4t-3}{1+4(4t-3)} = \frac{4t-2}{1+16t-12} = \frac{4t-2}{16t-11}$$

4. Let $t \in (2-U)\backslash\left\{\frac{3}{2}\right\}$. Then

$$\sigma(t) = \frac{t+2}{2}$$

and

$$\begin{aligned}\sigma^2(t) &= \sigma(\sigma(t)) \\ &= \frac{\sigma(t)+2}{2} \\ &= \frac{\frac{t+2}{2}+2}{2} \\ &= \frac{t+2+4}{4} \\ &= \frac{t+6}{4}\end{aligned}$$

Hence,

$$f^{\sigma}(t) = \frac{1+\frac{t+2}{2}}{1+4\frac{t+2}{2}} = \frac{2+t+2}{2(1+2t+4)}$$

$$= \frac{t+4}{2(2t+5)}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= \frac{1+\frac{t+6}{4}}{1+4\frac{t+6}{4}} \\ &= \frac{4+t+6}{4(t+6+1)} \\ &= \frac{t+10}{4(t+7)} \end{aligned}$$

5. Let $t \in (2+U)\backslash\left\{\frac{5}{2}\right\}$. Then

$$\sigma(t) = 2(t-1)$$

and

$$\begin{aligned} \sigma^2(t) &= \sigma(\sigma(t)) \\ &= 2(\sigma(t)-1) \\ &= 2(2(t-1)-1) \\ &= 2(2t-2-1) \\ &= 2(2t-3) \end{aligned}$$

Hence,

$$\begin{aligned} f^{\sigma}(t) &= \frac{1+2(t-1)}{1+4(t-1)} \\ &= \frac{1+2t-2}{1+8t-8} \\ &= \frac{2t-1}{8t-7} \end{aligned}$$

and

$$f^{\sigma^2}(t) = \frac{1+2(2t-3)}{1+4(2(2t-3))}$$

$$\begin{aligned} &= \frac{1+4t-6}{1+16t-24} \\ &= \frac{P4t-5}{16t-23}. \end{aligned}$$

6. Let $t = 0$. Then

$$\sigma(0) = 0$$

and

$$\begin{aligned} \sigma^2(0) &= \sigma(\sigma(0)) \\ &= \sigma(0) \\ &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} f^{\sigma}(0) &= \frac{1+0}{1+4\cdot 0} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(0) &= \frac{1+0}{1+4\cdot 0} \\ &= 1 \end{aligned}$$

7. Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$$

and

$$\begin{aligned}\sigma^2\left(\frac{1}{2}\right) &= \sigma\left(\sigma\left(\frac{1}{2}\right)\right)\\ &= \sigma\left(\frac{3}{4}\right)\\ &= \frac{7}{8}\end{aligned}$$

Hence,

$$\begin{aligned}f^{\sigma}\left(\frac{1}{2}\right) &= \frac{1+\frac{3}{4}}{1+4\cdot\frac{3}{4}}\\ &= \frac{4+3}{4(1+3)}\\ &= \frac{7}{16}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2}\left(\frac{1}{2}\right) &= \frac{1+\frac{7}{8}}{1+4\cdot\frac{7}{8}}\\ &= \frac{8+7}{8+28}\\ &= \frac{15}{36}\\ &= \frac{5}{12}\end{aligned}$$

8. Let $t = 1$. Then

$$\sigma(1) = 1 \tag{5.1}$$

and

$$\begin{aligned}\sigma^2(1) &= \sigma(\sigma(1))\\ &= \sigma(1)\\ &= 1\end{aligned} \tag{5.2}$$

Hence,

$$\begin{aligned}f^{\sigma}(1) &= \frac{1+1}{1+4\cdot 1}\\ &= \frac{2}{5}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2}(1) &= \frac{1+1}{1+4\cdot 1}\\ &= \frac{2}{5}\end{aligned}$$

9. Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$$

and

$$\begin{aligned}\sigma^2\left(\frac{3}{2}\right) &= \sigma\left(\sigma\left(\frac{3}{2}\right)\right)\\ &= \sigma\left(\frac{7}{4}\right)\\ &= \frac{15}{8}\end{aligned}$$

Hence,

$$\begin{aligned}f^{\sigma}\left(\frac{3}{2}\right) &= \frac{1+\frac{7}{4}}{1+4\cdot\frac{7}{4}}\\ &= \frac{4+7}{4(1+7)}\\ &= \frac{11}{32}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2}\left(\frac{3}{2}\right) &= \frac{1+\frac{15}{8}}{1+4\cdot\frac{15}{8}}\\ &= \frac{8+15}{8+60}\\ &= \frac{23}{68}\end{aligned}$$

10. Let $t = 2$. Then

$$\sigma(2) = 2$$

and

$$\begin{aligned}\sigma^2(2) &= \sigma(\sigma(2))\\ &= \sigma(2)\\ &= 2\end{aligned}$$

Hence,

$$\begin{aligned}f^{\sigma}(2) &= \frac{1+2}{1+4\cdot 2}\\ &= \frac{3}{9}\\ &= \frac{1}{3}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^2}(2) &= \frac{1+2}{1+4\cdot 2}\\ &= \frac{1}{3}\end{aligned}$$

11. Let $t = \frac{5}{2}$. Then

$$\sigma\left(\frac{5}{2}\right) = \frac{5}{2}$$

and

$$\sigma^2\left(\frac{5}{2}\right) = \sigma\left(\sigma\left(\frac{5}{2}\right)\right)$$
$$= \sigma\left(\frac{5}{2}\right)$$
$$= \frac{5}{2}$$

Hence,

$$f^{\sigma}\left(\frac{5}{2}\right) = \frac{1+\frac{5}{2}}{1+4\cdot\frac{5}{2}}$$
$$= \frac{2+5}{2(1+10)}$$
$$= \frac{7}{22}$$

and

$$f^{\sigma^2}\left(\frac{5}{2}\right) = \frac{1+\frac{5}{2}}{1+4\cdot\frac{5}{2}}$$
$$= \frac{7}{22}.$$

? Exercise 5.1

Let

$$f(t) = 2 - 4t + t^2, \quad t \text{in} \mathbb{T}$$

where $\mathbb{T}$ are the time scales from Exercise 0.3. Find

$$f^{\sigma}(t) \text{ and } f^{\sigma^2}(t) \text{ for } t \in \mathbb{T}.$$

Answer

Add texts here. Do not delete this text first.

Definition:

For a function $f : \mathbb{T} \to \mathbb{R}$ and $k, l \in \mathbb{N}_0$, define

$$f^{\rho^k}(t) = f\left(\rho^k(t)\right), \quad t \in \mathbb{T},$$

and

$$f^{\rho^k\sigma^l}(t) = f\left(\sigma^l\left(\rho^k(t)\right)\right), \quad t \in \mathbb{T},$$

and

$$f^{\sigma^l\rho^k}(t) = f\left(\rho^k\left(\sigma^l(t)\right)\right), \quad t \in \mathbb{T}.$$

Definition 0.10.

Remark

Note that in the general case, we have

$$f^{p^k\sigma^l}(t) \neq f^{\sigma^l\rho^k}(t), \quad t \in \mathbb{T}$$

Really, let $\mathbb{T} = \{-1, 0\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$, and

$$f(t) = t, \quad t \in \mathbb{T}$$

Then

$$\begin{aligned}\rho(\sigma(0)) &= \rho(0)\\ &= -1\end{aligned}$$

and

$$\begin{aligned}\sigma(\rho(0)) &= \sigma(-1)\\ &= 0\end{aligned}$$

Thus,

$$\rho(\sigma(0)) \neq \sigma(\rho(0)).$$

Example 5.12

Let $\mathbb{T} = \{-1, 0\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ and

$$f(t) = 1 + 4t^2, \quad t \in \mathbb{T}.$$

We will find

$$f^{\rho^2}(t), \quad f^{\sigma\rho\sigma}(t) \quad \text{and} \quad f^{\sigma^2\rho^3}(t) \quad \text{for} \quad t \in \mathbb{T}.$$

Firstly, we will determine the forward and backward jump operators for the time scale T . We ha cases.

1. Let $t = \frac{1}{n}, n \in \mathbb{N}, n \geq 2$. Then

$$\begin{aligned}\sigma(t) &= \frac{1}{n-1}\\ &= \frac{1}{\frac{1}{t} - 1}\\ &= \frac{t}{1-t}\end{aligned}$$

and

$$\begin{aligned}\rho(t) &= \frac{1}{n+1}\\ &= \frac{1}{\frac{1}{t} + 1}\\ &= \frac{t}{1+t}\end{aligned}$$

2. Let $t = 1$. Then

$$\begin{aligned}\sigma(1) &= 1\\ \rho(1) &= \frac{1}{2}\end{aligned}$$

3. Let $t = 0$. Then]

$$\sigma(0) = 0$$
$$\rho(0) = -1$$

4. Let $t = -1$. Then

$$\sigma(-1) = 0$$
$$\rho(-1) = -1$$

Therefore

$$\sigma(t) = \begin{cases} \frac{t}{1-t} & \text{if} \quad t \in \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}, n\geq 2} \\ 1 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t = -1 \end{cases} \tag{5.3}$$

and

$$\rho(t) = \begin{cases} \frac{t}{1+t} & \text{if} \quad t \in \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}, n\geq 2} \\ \frac{1}{-2} & \text{if} \quad t = 1 \\ -1 & \text{if} \quad t = 0 \\ -1 & \text{if} \quad t = -1 \end{cases} \tag{5.4}$$

Hence, we have the following cases.

1. Let $t \in \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}, n\geq 2}$. Then

$$\begin{aligned} \rho^2(t) &= \rho(\rho(t)) \\ &= \rho\left(\frac{t}{1+t}\right) \\ &= \frac{\rho(t)}{1+\rho(t)} \\ &= \frac{\frac{t}{1+t}}{1+\frac{t}{1+t}} \\ &= \frac{t}{1+2t} \end{aligned}$$

and

$$\begin{aligned} f^{p^2}(t) &= f\left(\rho^2(t)\right) \\ &= f\left(\frac{t}{1+2t}\right) \\ &= 1 + 4\frac{t^2}{(1+2t)^2} \end{aligned}$$

$$\begin{aligned} &= \frac{(1+2t)^2 + 4t^2}{(1+2t)^2} \\ &= \frac{1+4t+4t^2+4t^2}{(1+2t)^2} \\ &= \frac{1+4t+8t^2}{(1+2t)^2} \end{aligned}$$

Next,

$$\begin{aligned}\sigma(\rho(\sigma(t))) &= \sigma\left(\rho\left(\frac{t}{1-t}\right)\right)\\ &= \sigma\left(\frac{\rho(t)}{1-\rho(t)}\right)\\ &= \sigma\left(\frac{\frac{t}{1+t}}{1-\frac{t}{1+t}}\right)\\ &= \sigma(t)\\ &= \frac{t}{1-t}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma\rho\sigma}(t) &= f(\sigma(\rho(\sigma(t))))\\ &= f\left(\frac{t}{1-t}\right)\\ &= 1+\frac{4t^2}{(1-t)^2}\\ &= \frac{(1-t)^2+4t^2}{(1-t)^2}\\ &= \frac{1-2t+t^2+4t^2}{(1-t)^2}\\ &= \frac{1-2t+5t^2}{(1-t)^2}\end{aligned}$$

Moreover,

$$\begin{aligned}\rho^2\left(\sigma^2(t)\right) &= \rho^2(\sigma(\sigma(t)))\\ &= \rho^3\left(\sigma\left(\frac{t}{1-t}\right)\right)\\ &= \rho^3\left(\frac{\sigma(t)}{1-\sigma(t)}\right)\\ &= \rho^3\left(\frac{\frac{t}{1-t}}{1-\frac{t}{1-t}}\right)\\ &= \rho^3\left(\frac{t}{1-2t}\right)\\ &= \rho^2\left(\rho\left(\frac{t}{1-2t}\right)\right)\\ &= \rho^2\left(\frac{\rho(t)}{1-2\rho(t)}\right)\\ &= \rho^2\left(\frac{t}{1+t}\right)\\ &= \rho^2\left(\frac{t}{1-t}\right)\\ &= \rho\left(\rho\left(\frac{t}{1-t}\right)\right)\\ &= \rho\left(\frac{\rho(t)}{1-\rho(t)}\right)\\ &= \rho\left(\frac{\frac{t}{1+t}}{1-\frac{t}{1+t}}\right)\\ &= \rho(t)\\ &= \frac{t}{t+1}\end{aligned}$$

and

$$f^{\sigma^2\rho^3}(t) = f\left(\rho^3\left(\sigma^2(t)\right)\right)$$
$$= f\left(\frac{t}{1+t}\right)$$

$$= 1 + \frac{4t^2}{(1+t)^2}$$
$$= \frac{(1+t)^2 + 4t^2}{(1+t)^2}$$
$$= \frac{1+2t+t^2+4t^2}{(1+t)^2}$$
$$= \frac{1+2t+5t^2}{(1+t)^2}.$$

2. Let $t = 1$. Then

$$\rho^2(1) = \rho(\rho(1))$$
$$= \rho\left(\frac{1}{2}\right)$$
$$= \frac{1}{3}$$

and

$$f^{\rho^2}(1) = f\left(\rho^2(1)\right)$$
$$= f\left(\frac{1}{3}\right)$$
$$= 1 + \frac{4}{9}$$
$$= \frac{13}{9}$$

Next,

$$\sigma\rho\sigma(1) = \sigma(\rho(\sigma(1)))$$
$$= \sigma(\rho(1))$$
$$= \sigma\left(\frac{1}{2}\right)$$
$$= 1$$

and

$$f^{\sigma\rho\sigma}(1) = f(\sigma(\rho(\sigma(1))))$$
$$= f(1)$$
$$= 1 + 4$$
$$= 5$$

Moreover,

$$\rho^3\sigma^2(1) = \rho^3(\sigma(\sigma(1)))$$
$$= \rho^3(\sigma(1))$$
$$= \rho^3(1)$$
$$= \rho(\rho(\rho(1)))$$
$$= \rho\left(\rho\left(\frac{1}{2}\right)\right)$$
$$= \rho\left(\frac{1}{3}\right)$$
$$= \frac{1}{4}$$

and

$$\begin{aligned} f^{\sigma^2\rho^3}(1) &= f\left(\rho^3\sigma^2(1)\right) \\ &= f\left(\frac{1}{4}\right) \\ &= 1 + 4 \cdot \frac{1}{16} \\ &= 1 + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

3. Let $t = 0$. Then

$$\begin{aligned} \sigma\rho\sigma(0) &= \sigma\rho(\sigma(0)) \\ &= \sigma(\rho(0)) \\ &= \sigma(-1) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma\rho\sigma}(0) &= f(\sigma\rho\sigma(0)) \\ &= f(0) \\ &= 1 \end{aligned}$$

Next,

$$\begin{aligned} \rho^2(0) &= \rho(\rho(0)) \\ &= \rho(-1) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} f^{+}\rho^2(0) &= f\left(\rho^2(0)\right) \\ &= f(-1) \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

Moreover,

$$\begin{aligned} \rho^3\sigma^2(0) &= \rho^3\sigma(\sigma(0)) \\ &= \rho^3(\sigma(0)) \end{aligned}$$

$$\begin{aligned} &= \rho^3(0) \\ &= \rho(\rho(\rho(0)) \\ &= \rho(\rho(-1)) \\ &= \rho(-1) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2\rho^3}(0) &= f\left(\rho^3\sigma^2(0)\right) \\ &= f(-1) \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

4. Let $t = -1$. Then

$$\begin{aligned} \rho^2(-1) &= \rho(\rho(-1)) \\ &= \rho(-1) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} f^{\rho^3}(-1) &= f\left(\rho^2(-1)\right) \\ &= f(-1) \\ &= 1+4 \\ &= 5 \end{aligned}$$

Next,

$$\begin{aligned} \sigma\rho\sigma(-1) &= \sigma(\rho(\sigma(-1))) \\ &= \sigma(\rho(0)) \\ &= \sigma(-1) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma\rho\sigma}(-1) &= f(\sigma\rho\sigma(-1)) \\ &= f(0) \\ &= 1 \end{aligned}$$

Moreover,

$$\begin{aligned} \rho^3\sigma^2(-1) &= \rho^3\sigma(\sigma(-1)) \\ &= \rho^3(\sigma(0)) \\ &= \rho^3(0) \\ &= \rho^2(\rho(0)) \\ &= \rho^2(-1) \\ &= \rho(\rho(-1)) \\ &= \rho(-1) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2\rho^3}(-1) &= f\left(\rho^3\sigma^2(-1)\right) \\ &= f(-1) \\ &= 1+4 \\ &= 5 \end{aligned}$$

Example 5.13

Let $\mathbb{T} = 3^{\mathbb{N}_0}$ and

$$f(t) = \frac{2-t}{4+3t}, \quad t \in \mathbb{T}$$

We will find

$f^{\sigma\rho^2\sigma^4}(1)$ and $f^{\rho\sigma\rho}(27)$. By Example 2.8, we have

$$\sigma(t) = 3t, \quad t \in \mathbb{T},$$

and by Example 0.31, we find

$$\rho(t) = \begin{cases} 1 & \text{if} \quad t = 1 \\ \dfrac{t}{3} & \text{if} \quad t \in 3^{\mathbb{N}}. \end{cases}$$

Then

$$\begin{aligned}\sigma^4\rho^2\sigma(1) &= \sigma^4\rho^2(\sigma(1))\\ &= \sigma^4\rho^2(3)\\ &= \sigma^4\rho(\rho(3))\\ &= \sigma^4(\rho(1))\\ &= \sigma^4(1)\\ &= \sigma^3(\sigma(1))\\ &= \sigma^3(3)\\ &= \sigma^2(\sigma(3))\\ &= \sigma^2(9)\\ &= \sigma(\sigma(9))\\ &= \sigma(27)\\ &= 81\end{aligned}$$

and

$$\begin{aligned}\rho\sigma\rho(27) &= \rho(\sigma(\rho(27)))\\ &= \rho(\sigma(9))\\ &= \rho(27)\\ &= 9\end{aligned}$$

Hence,

$$\begin{aligned}f^{\sigma\rho^2\sigma^4}(1) &= f\left(\sigma^4\rho^2\sigma(1)\right)\\ &= f(81)\\ &= \frac{2-81}{4+3\cdot 81}\\ &= \frac{-79}{4+243}\\ &= -\frac{79}{247}\end{aligned}$$

and

$$\begin{aligned}f^{\rho\sigma\rho}(27) &= f(\rho\sigma\rho(27))\\ &= f(9)\\ &= \frac{2-9}{4+3\cdot 7}\\ &= \frac{-7}{4+27}\\ &= -\frac{7}{31}\end{aligned}$$

Example 5.14

Let $\mathbb{T} = \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0,1\}$ and

$$f(t) = \sqrt{1+t+t^2}, \quad t \in \mathbb{T}$$

We will find

$$f^{\rho^2}(t), \quad f^{\sigma^3\rho^2}(t), \quad f^{\rho^2\sigma^3}(t), \quad t \in \mathbb{T}.$$

Solution

Add example text here.

Exercise 5.2

Let

$$f(t) = \sqrt[3]{\frac{1+7}{2+7t}}, \quad t \in \mathbb{T},$$

where $\mathbb{T}$ are the time scales in Exercise 0.3.

Find

$$f^{\rho^2}(t), \quad f^{\sigma^3\rho^2}(t), \quad f^{\rho^2\sigma^3}(t), \quad t \in \mathbb{T}.$$

Answer

$$f^{\rho^2}(t), \quad f^{\sigma^3\rho^2}(t), \quad f^{\rho^2\sigma^3}(t), \quad t \in \mathbb{T}.$$

We have

$$\sigma(t) = \begin{cases} \sqrt{t} & \text{if} \quad t \in \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}\right\} \\ 1 & \text{if} \quad t = \frac{1}{2} \\ 0 & \text{if} \quad t = 0 \\ 1 & \text{if} \quad t = 1 \end{cases}$$

and by Example 0.38, we have

$$\rho(t) = \begin{cases} t^2 & \text{if} \quad t \in \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \\ 0 & \text{if} \quad t = 0 \\ \frac{1}{2} & \text{if} \quad t = 1 \end{cases}$$

Thus, we have the following cases.

1. Let $t \in \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}\right\}$. Then

$$\begin{aligned} \rho^2(t) &= \rho(\rho(t)) \\ &= \rho\left(t^2\right) \\ &= (\rho(t))^2 \\ &= t^4 \end{aligned}$$

and

$$\begin{aligned} \sigma^3\rho^2(t) &= \sigma^3\left(\rho^2(t)\right) \\ &= \sigma^3\left(t^4\right) \\ &= n\sigma^2\left(\sigma\left(t^4\right)\right) \\ &= \sigma^2\left(t^2\right) \\ &= \sigma\left(\sigma\left(t^2\right)\right) \\ &= \sigma(t) \\ &= \sqrt{t}, \end{aligned}$$

and

$$\begin{aligned}\rho^2\sigma^3(t) &= \rho^2\sigma^2(\sigma(t)) \\ &= \rho^2\sigma^2(\sqrt{t}) \\ &= \rho^2\sigma(\sigma(\sqrt{t})) \\ &= \rho^2(\sigma(\sqrt[4]{t})) \\ &= \rho^2(\sqrt[8]{t}) \\ &= \rho(\rho(\sqrt[8]{t})) \\ &= \rho(\sqrt[4]{t}) \\ &= \sqrt{t}\end{aligned}$$

Hence,

$$\begin{aligned}f^{p^2}(t) &= f\left(\rho^2(t)\right) \\ &= f\left(t^2\right) \\ &= \sqrt{1+t^4+t^8}\end{aligned}$$

and

$$\begin{aligned}f^{\sigma^3\rho^2(t)} &= f\left(\rho^2\left(\sigma^3(t)\right)\right) \\ &= f(\sqrt{t}) \\ &= \sqrt{1+\sqrt{t}+t}\end{aligned}$$

and

$$\begin{aligned}f^{\rho^2\sigma^3}(t) &= f\left(\sigma^3\rho^2(t)\right) \\ &= f(\sqrt{t}) \\ &= \sqrt{1+\sqrt{t}+t}\end{aligned}$$

2. Let $t = 0$. Then

$$\begin{aligned}\rho^2(0) &= \rho(\rho(0)) \\ &= \rho(0) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\sigma^3\rho^2(0) &= \sigma^3\left(\rho^2(0)\right) \\ &= \sigma^3(0) \\ &= \sigma^2(\sigma(0)) \\ &= \sigma^2(0) \\ &= \sigma(\sigma(0)) \\ &= \sigma(0) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\rho^2\sigma^3(0) &= \rho^2\sigma^2(\sigma(0))\| \\ &= \rho^2\sigma(\sigma(0)) \\ &= \rho^2(\sigma(0)) \\ &= \rho^2(0) \\ &= \rho(\rho(0)) \\ &= \rho(0) \\ &= 0\end{aligned}$$

Hence,

$$\begin{aligned}f^{p^2}(0) &= f\left(\rho^2(0)\right) \\ &= f(0) \\ &= 1\end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2 3\rho^2}(0) &= f\left(\rho^2\sigma^3(0)\right) \\ &= f(0) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f^{\rho^2\sigma^3}(0) &= f\left(\sigma^3\rho^2(0)\right) \\ &= f(0) \\ &= 1 \end{aligned}$$

3. Let $t = \frac{1}{2}$. Then

$$\begin{aligned} \rho^2\left(\frac{1}{2}\right) &= \rho\left(\rho\left(\frac{1}{2}\right)\right) \\ &= \rho\left(\frac{1}{4}\right) \\ &= \frac{1}{16} \end{aligned}$$

and

$$\begin{aligned} \sigma^3\rho^2\left(\frac{1}{2}\right) &= \sigma^3\left(\rho^2\left(\frac{1}{2}\right)\right) \\ &= \sigma^3\left(\frac{1}{16}\right) \\ &= \sigma^2\left(\sigma\left(\frac{1}{16}\right)\right) \\ &= \sigma^2\left(\frac{1}{4}\right) \\ &= \sigma\left(\sigma\left(\frac{1}{4}\right)\right) \\ &= \sigma\left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \rho^2\sigma^3\left(\frac{1}{2}\right) &= \rho^2\sigma^2\left(\sigma\left(\frac{1}{2}\right)\right) \\ &= \rho^2\sigma(\sigma(1)) \\ &= \rho^2\sigma(1) \\ &= \rho^2(\sigma(1)) \\ &= \rho^2(1) \\ &= \rho(\rho(1)) \\ &= \rho\left(\frac{1}{2}\right) \\ &= \frac{1}{4} \end{aligned}$$

Hence,

$$\begin{aligned} f^{\rho^2}\left(\frac{1}{2}\right) &= f\left(\rho^2\left(\frac{1}{2}\right)\right) \\ &= f\left(\frac{1}{16}\right) \\ &= \sqrt{1+\frac{1}{16}+\frac{1}{256}} \\ &= \sqrt{\frac{256+16+1}{256}} \\ &= \frac{\sqrt{273}}{16} \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^3\rho^2}\left(\frac{1}{2}\right) &= f\left(\rho^2\sigma^3\left(\frac{1}{2}\right)\right) \\ &= f\left(\frac{1}{4}\right) \\ &= \sqrt{1+\frac{1}{4}+\frac{1}{16}} \\ &= \sqrt{\frac{16+4+1}{16}} \\ &= \frac{\sqrt{21}}{4} \end{aligned}$$

and

$$\begin{aligned} f^{\rho^2\sigma^3}\left(\frac{1}{2}\right) &= f\left(\sigma^3\rho^2\left(\frac{1}{2}\right)\right) \\ &= f(1) \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \end{aligned}$$

4. Let $t = 1$. Then

$$\begin{aligned} \rho^2(1) &= \rho(\rho(1)) \\ &= \rho\left(\frac{1}{2}\right) \\ &= \frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} \sigma^3\rho^2(1) &= \sigma^3\left(\rho^2(1)\right) \\ &= \sigma^3\left(\frac{1}{4}\right) \\ &= \sigma^2\left(\sigma\left(\frac{1}{4}\right)\right) \\ &= \sigma^2\left(\frac{1}{2}\right) \\ &= \sigma\left(\sigma\left(\frac{1}{2}\right)\right) \\ &= \sigma(1) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \rho^2\sigma^3(1) &= \rho^2\sigma^2(\sigma(1)) \\ &= \rho^2\sigma(\sigma(1)) \\ &= \rho^2(\sigma(1)) \\ &= \rho^2(1) \\ &= \frac{1}{4}. \end{aligned}$$

Hence,

$$\begin{aligned} f^{p^2}(1) &= f\left(\rho^2(1)\right) \\ &= f\left(\frac{1}{4}\right) \\ &= \sqrt{1+\frac{1}{4}+\frac{1}{16}} \\ &= \frac{\sqrt{21}}{4} \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^3\rho^2}(1) &= f\left(\rho^2\sigma^3(1)\right) \\ &= f\left(\frac{1}{4}\right) \\ &= \frac{\sqrt{21}}{4} \end{aligned}$$

and

$$\begin{aligned} f^{\rho^2\sigma^3(1)} &= f\left(\sigma^3\rho^2(1)\right) \\ &= f(1) \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \end{aligned}$$

Example 5.15

We will simplify

$$A(t) = \left(\frac{5\sigma(t) - 2(\sigma(t))^2}{(\sigma(t)-2)^2} + \sigma(t)\right) : \left(\sigma(t) - \frac{3\sigma(t)-8}{(\sigma(t)-2)^2} - 2\right), \quad t \in \mathbb{T}, \quad \sigma(t) \neq 2.$$

We have

$$\begin{aligned} A(t) =& \left(\frac{5\sigma(t) - 2(\sigma(t))^2 + \sigma(t)(\sigma(t)-2)^2}{(\sigma(t)-2)^2}\right) : \left(\frac{(\sigma(t)-2)^3 - 3\sigma(t) + 8}{(\sigma(t)-2)^2}\right) \\ =& \left(\frac{5\sigma(t) - 2(\sigma(t))^2 + (\sigma(t))^3 - 4(\sigma(t))^2 + 4\sigma(t)}{(\sigma(t)-2)^2}\right) \\ & \times \left(\frac{(\sigma(t)-2)^2}{(\sigma(t))^3 - 6(\sigma(t))^2 + 12\sigma(t) - 8 - 3\sigma(t) + 8}\right) \\ =& \frac{9\sigma(t) - 6(\sigma(t))^2 + (\sigma(t))^3}{(\sigma(t))^3 - 6(\sigma(t))^2 + 9\sigma(t)} \\ =& \frac{\sigma(t)\left(9 - 6\sigma(t) + (\sigma(t))^2\right)}{\sigma(t)\left(9 - 6\sigma(t) + (\sigma(t))^2\right)} \\ =& 1, \quad t \in \mathbb{T}, \quad \sigma(t) \neq 2 \end{aligned}$$

Example 5.16

We will simplify

$$\begin{aligned} A(t) =& \left(\frac{2\rho(t)}{\sigma(t) - 4\rho(t)} - \frac{10\sigma(t)(\rho(t))^2 - 32(\rho(t))^3}{(4\rho(t) - \sigma(t))^3}\right) \\ & \times \left(\frac{2\sigma(t)(\rho(t))^2 - 64(\rho(t))^3}{(\sigma(t))^2 - 3\sigma(t)\rho(t)} + \sigma(t) - 9\rho(t)\right), \quad t \in \mathbb{T}, \quad \sigma(t) \neq 0, 3\rho(t), 4\rho(t) \end{aligned}$$

We have

$$\begin{aligned}
A(t) =& \left(\frac{2\rho(t)}{\sigma(t)-4\rho(t)}+\frac{10\sigma(t)(\rho(t))^2-32(\rho(t))^3}{(\sigma(t)-4\rho(t))^3}\right)\\
&\times\left(\frac{2\sigma(t)(\rho(t))^2-64(\rho(t))^3}{(\sigma(t))^2-3\sigma(t)\rho(t)}+\sigma(t)-9\rho(t)\right)\\
=&\left(\frac{2\rho(t)(\sigma(t)-4\rho(t))^2+10\sigma(t)(\rho(t))^2-32(\rho(t))^3}{(\sigma(t)-4\rho(t))^3}\right)\\
&\times\left(\frac{21\sigma(t)(\rho(t))^2-64(\rho(t))^3+(\sigma(t)-9\rho(t))\left((\sigma(t))^2-3\sigma(t)\rho(t)\right)}{\sigma(t)(\sigma(t)-3\rho(t))}\right)\\
=&\left(\frac{2\rho(t)\left((\sigma(t))^2-8\sigma(t)\rho(t)+16(\rho(t))^2\right)+10\sigma(t)(\rho(t))^2-32(\rho(t))^3}{(\sigma(t)-4\rho(t))^3}\right)
\end{aligned}$$

$$\begin{aligned}
&\times\\
&\left(\frac{21\sigma(t)(\rho(t))^2-64(\rho(t))^3+(\sigma(t))^3-3(\sigma(t))^2\rho(t)-9(\sigma(t))^2\rho(t)+27\sigma(t)(\rho(t))^2}{\sigma(t)(\sigma(t)-3\rho(t))}\right)\\
=&\left(\frac{2\rho(t)(\sigma(t))^2-16\sigma(t)(\rho(t))^2+32(\rho(t))^3+10\sigma(t)(\rho(t))^2-32(\rho(t))^3}{(\sigma(t)-4\rho(t))^3}\right)\\
&\times\left(\frac{48\sigma(t)(\rho(t))^2-12(\sigma(t))^2\rho(t)-64(\rho(t))^3+(\sigma(t))^3}{\sigma(t)(\sigma(t)-3\rho(t))}\right)\\
=&\left(\frac{2\rho(t)(\sigma(t))^2+6\sigma(t)(\rho(t))^2}{(\sigma(t)-4\rho(t))^3}\right)\left(\frac{(\sigma(t)-4\rho(t))^3}{\sigma(t)(\sigma(t)-3\rho(t))}\right)\\
=&\frac{2\rho(t)\sigma(t)(\sigma(t)-3\rho(t))}{\sigma(t)(\sigma(t)-3\rho(t))}\\
=&2\rho(t),\quad t\in\mathbb{T},\quad \sigma(t)\neq 0,3\rho(t),4\rho(t).
\end{aligned}$$

Example 5.17

We will simplify

$$A(t)=\frac{3}{\sigma(t)+\rho(t)}-\frac{3\sigma(t)-3\rho(t)}{2\sigma(t)-3\rho(t)}\left(\frac{2\sigma(t)-3\rho(t)}{(\sigma(t))^2-(\rho(t))^2}-2\sigma(t)+3\rho(t)\right)$$

$t\in\mathbb{T},\sigma(t)\neq\pm\rho(t),\frac{3}{2}\rho(t).$

We have

$$\begin{aligned}
A(t)&=\frac{3}{\sigma(t)+\rho(t)}-\frac{3(\sigma(t)-\rho(t))}{2\sigma(t)-3\rho(t)}(2\sigma(t)-3\rho(t))\left(\frac{1}{(\sigma(t))^2-(\rho(t))^2}-1\right)\\
&=\frac{3}{\sigma(t)+\rho(t)}-3(\sigma(t)-\rho(t))\left(\frac{1-(\sigma(t))^2+(\rho(t))^2}{(\sigma(t))^2-(\rho(t))^2}\right)\\
&=3\frac{(\sigma(t))^2-(\rho(t))^2-(\sigma(t)+\rho(t))(\sigma(t)-\rho(t))\left(1-(\sigma(t))^2+(\rho(t))^2\right)}{(\sigma(t)+\rho(t))\left((\sigma(t))^2-(\rho(t))^2\right)}\\
&=3\frac{(\sigma(t))^2-(\rho(t))^2-\left((\sigma(t))^2-(\rho(t))^2\right)\left(1-(\sigma(t))^2+(\rho(t))^2\right)}{(\sigma(t)+\rho(t))\left((\sigma(t))^2-(\rho(t))^2\right)}\\
&=3\frac{(\sigma(t))^2-(\rho(t))^2-(\sigma(t))^2+(\rho(t))^2+\left((\sigma(t))^2-(\rho(t))^2\right)^2}{(\sigma(t)+\rho(t))\left((\sigma(t))^2-(\rho(t))^2\right)}\\
&=3\frac{(\sigma(t)-\rho(t))^2(\sigma(t)+\rho(t))^2}{(\sigma(t)+\rho(t))^2(\sigma(t)-\rho(t))}\\
&=3(\sigma(t)-\rho(t)),\quad t\in\mathbb{T},\quad \sigma(t)\neq\pm\rho(t),\frac{3}{2}\rho(t)
\end{aligned}$$

Exercise 5.3

Simplify

1.

$$\left(\rho(t)+1-\frac{5(\rho(t))^2+2\rho(t)+1}{(\rho(t)+1)^2}\right)\left(\frac{1}{\rho(t)}+\frac{4}{(\rho(t)-1)^2}\right),$$

$t\in\mathbb{T},\rho(t)\neq 0,\pm 1.$

2.

$$\left(\frac{2\sigma(t)+\rho(t)}{(\sigma(t))^2-4(\rho(t))^2}-2\sigma(t)-\rho(t)\right)\frac{4\rho(t)+2\sigma(t)}{2\sigma(t)+\rho(t)}-\frac{2}{\sigma(t)-2\rho(t)}$$

$r\in\mathbb{T},\sigma(t)\neq\pm 2\rho(t),-\frac{\rho(t)}{2}.$

3.

$$\frac{\rho(t)}{v(t)}+\frac{\sigma(t)\rho(t)}{\mu(t)}\left(\frac{t+\sigma(t)}{t\sigma(t)-\sigma(t)\rho(t)}+\frac{t+\rho(t)}{(\rho(t))^2-t\rho(t)}+\frac{t}{\sigma(t)\rho(t)}\right),$$

$\sigma(t)\neq 0,t,\rho(t)\neq 0,t,t\in\mathbb{T}.$

4.

$$\left(\frac{\sigma(t)-\rho(t)}{\sigma(t)\rho(t)}-\frac{3\sigma(t)+\rho(t)}{\sigma(t)\rho(t)-(\sigma(t))^2}+\frac{3\rho(t)+\sigma(t)}{\sigma(t)\rho(t)-(\rho(t))^2}\right)$$
$$:\frac{2\sigma(t)+2\rho(t)}{\sigma(t)\rho(t)}+\frac{2\sigma(t)}{\rho(t)-\sigma(t)},$$

$t\in\mathbb{T},\sigma(t)\neq 0,\pm\rho(t),\rho(t)\neq 0.$

5.

$$\frac{\sigma(t)-2}{(2\sigma(t)+4)^2}:\left(\frac{\sigma(t)}{2\sigma(t)-4}-\frac{(\sigma(t))^2+4}{2(\sigma(t))^2-8}-\frac{2}{(\sigma(t))^2+2\sigma(t)}\right),$$

$t\in\mathbb{T},\sigma(t)\neq\pm 2,0.$

Answer

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6: The Induction Principle

The classical mathematical induction is a concept that helps to prove mathematical results and theorems for all natural numbers. The principle of the classical mathematical induction is a specific technique that is used to prove certain statements in algebra which are formulated in terms of n, where n is a natural number. Any mathematical statement, expression is proved based on the premise that it is true for $n = 1, n = k$, and then it is proved for $n = k + 1$.

Theorem 6.1: Induction Principle

Let $t_0 \in \mathbb{T}$ and assume that

$$\{S(t) : t \in [t_0, \infty)\}$$

is a family of statements satisfying
(i) $S(t_0)$ is true.
(ii) If $t \in [t_0, \infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is true.
(iii) If $t \in [t_0, \infty)$ is right-dense and $S(t)$ is true, then there exists a neighbourhood U of t such that $S(s)$ is true for all $s \in U \cap (t, \infty)$.
(iv) If $t \in (t_0, \infty)$ is left-dense and $S(s)$ is true for $s \in [t_0, t)$, then $S(t)$ is true.

Then $S(t)$ is true for all $t \in [t_0, \infty)$.

Proof.

Let

$$S^* = \{t \in [t_0, \infty) : S(t) \quad \text{is not true} \quad \}$$

We assume that $S^* \neq \emptyset$. Let $\inf S^* = t^*$. Because $\mathbb{T}$ is closed, we have $t^* \in \mathbb{T}$.

1. If $t^* = t_0$, then $S(t^*)$ is true.
2. If $t^* \neq t_0$ and $t^* = \rho(t^*)$, then, using (iv), we get that $S(t^*)$ is true.
3. If $t^* \neq t_0$ and $\rho(t^*) < t^*$, then $\rho(t^*)$ is right-scattered. Since $S(\rho(t^*))$ is true, we get that $S(t^*)$ is true.

Consequently, $t^* \notin S^*$. If we suppose that t^* is right-scattered, then, using that $S(t^*)$ is true and (ii), we conclude that $S(\sigma(t^*))$ is true, which is a contradiction. From the definition of t^*, it follows that $t^* \neq \sup \mathbb{T}$. Since t^* is not right-scattered and $t^* \neq \sup \mathbb{T}$, we obtain that t^* is right-dense. Because $S(t^*)$ is true, using (iii), there exists a neighbourhood U of t^* such that $S(s)$ is true for all $s \in U \cap (t^*, \infty)$, which is a contradiction. Consequently, $S^* = \emptyset$.

Theorem 6.2: Dual Version of the Induction Principle

Let $t_0 \in \mathbb{T}$ and assume that

$$\{S(t) : t \in (-\infty, t_0]\}$$

is a family of statements satisfying
(i) $S(t_0)$ is true.
(ii) If $t \in (-\infty, t_0]$ is left-scattered and $S(t)$ is true, then $S(\rho(t))$ is true.
(iii) If $t \in (-\infty, t_0]$ is left-dense and $S(t)$ is true, then there exists a neighbourhood U of t such that $S(s)$ is true for all $s \in U \cap (-\infty, t)$.
(iv) If $t \in (-\infty, t_0)$ is right-dense and $S(s)$ is true for $s \in (t, t_0)$, then $S(t)$ is true.

Then $S(t)$ is true for all $t \in (-\infty, t_0]$.

Proof.

Let

$$S^* = \{t \in (-\infty, t_0] : S(t) \quad \text{is not true} \quad \}$$

Assume that $S^* \neq \emptyset$. Note that $t_0 \notin S^*$. Let

$$t^* = \sup S^* \tag{6.1}$$

1. If $t^* = t_0$, then $S(t^*)$ is true.
2. If $t^* \neq t_0$ and $t^* = \sigma(t^*)$, then, using (iv), we obtain that $S(t^*)$ is true.
3. If $t^* \neq t_0$ and $t^* < \sigma(t^*)$, then $\sigma(t^*)$ is left-scattered, $\rho(\sigma(t^*)) = t^*$, and $S(\sigma(t^*))$ is true. Therefore, we conclude that $S(t^*)$ is true.

Therefore, $t^* \notin S^*$. If t^* is left-scattered, then, using (ii), we get that $S(\rho(t^*))$ is true, which is a contradiction. If t^* is left-dense, then, using (iii), there exists a neighbourhood U of t^* such that $S(s)$ is true for all $s \in U \cap (-\infty, t^*)$, which is a contradiction. Consequently, $S^* = \emptyset$.

✔ Example 6.1

Let $\mathbb{T} = 3^{\mathbb{N}}$. We will prove the inequality

$$\frac{1}{4}(s^2 - r^2) \leq \frac{1}{13}(s^3 - r^3) \quad \text{for any} \quad r \leq s, \quad r, s \in \mathbb{T}. \tag{6.2}$$

Solution

Fix $r, s \in \mathbb{T}, r \leq s$. Let

$$S(t) : \frac{1}{4}(t^2 - r^2) \leq \frac{1}{13}(t^3 - r^3), \quad t \in [r, s]_{\mathbb{T}}$$

1. The statement $S(r)$ is true.
2. Observe that any $t \in \mathbb{T}$ is right-scattered. Assume that $S(t)$ is true. Then

$$\sigma(t) = 3t$$

and

$$t^2 \leq t^3$$

Hence, using (✔ 6.2), we find

$$\frac{1}{4}(t^2 - r^2) + 2t^2 \leq \frac{1}{13}(t^3 - r^3) + 2t^3$$

or

$$\frac{1}{4}(9t^2 - r^2) \leq \frac{1}{13}(27t^3 - r^3)$$

Thus,

$$\frac{1}{4}((\sigma(t))^2 - r^2) \leq \frac{1}{3}((\sigma(t))^3 - r^3)$$

Consequently $S(\sigma(t))$ is true.

Applying the induction principle, we conclude that (0.2) is true for any $t \in [r, s]_{\mathbb{T}}$ and then (0.1) is true for any $r, s \in \mathbb{T}, r \leq s$.

Example 6.2

Example 0.97. Let $\mathbb{T} = [-2,-1) \cup \left\{-\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{0\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{4}, \frac{11}{6}\right]$. We will prove

$$S(t) : \sqrt{t+2} > t \quad \text{for any} \quad t \in \mathbb{T}.$$

Solution

We have the following cases.

1. For $t = -2$, we have

$$0 > -2$$

i.e., the given inequality is true.

2. Let $t \in \mathbb{T}$ be right0scattered. Then, we have the following cases.

a. Let $t = -\frac{1}{n}$ for some $n \in \mathbb{N}$. Then

$$\sqrt{-\frac{1}{n}+2} > 0$$
$$> -\frac{1}{n}$$

i.e., $S(t)$ is true. Note that

$$\sigma(t) = -\frac{1}{n+1}$$

and

$$\sqrt{-\frac{1}{n+1}+2} > 0$$
$$> -\frac{1}{n+1}$$

i.e., $S(\sigma(t))$ is true.

b. Let $t = \frac{1}{n}$ for some $n \in \mathbb{N}, n \geq 2$. Then

$$\frac{1}{n} > \frac{1}{n^2}$$

and

$$\frac{1}{n} + 2 > \frac{1}{n^2},$$

whereupon

$$\sqrt{\frac{1}{n}+2} > \frac{1}{n},$$

i.e., $S(t)$ is true. Note that

$$\sigma(t) = \frac{1}{n-1}.$$

Then

$$\frac{1}{n-1} > \frac{1}{(n-1)^2}$$

and

$$\frac{1}{n-1} + 2 > \frac{1}{(n-1)^2},$$

whereupon

$$\sqrt{\frac{1}{n-1}+2}>\frac{1}{n-1}.$$

Thus, $S(\sigma(t))$ is true.

c. Let $t=\frac{3}{2}$. Then

$$\begin{aligned}\sqrt{\frac{3}{2}+2}&=\sqrt{\frac{7}{2}}\\&>\sqrt{\frac{9}{4}}\\&=\frac{3}{2}\end{aligned}$$

i.e., $S(t)$ is true. Note that

$$\sigma(t)=\frac{7}{4}$$

and

$$\begin{aligned}\sqrt{\frac{7}{4}+2}&=\sqrt{\frac{15}{4}}\\&>\sqrt{\frac{49}{16}}\\&=\frac{7}{4}\end{aligned}$$

i.e., $S(\sigma(t))$ is true.

3. Let t is right-dense. Then $t\in[-2,-1)\cup\{0\}\cup\left(1,\frac{3}{2}\right]\cup\left[\frac{7}{4},\frac{11}{6}\right]$. We have the following cases.

a. Let $t\in[-2,-1)\cup\left(1,\frac{3}{2}\right]\cup\left[\frac{7}{4},\frac{11}{6}\right]$. Then

$$t^2-t-2<0,\quad t\in\left(1,\frac{3}{2}\right]\cup\left[\frac{7}{4},\frac{11}{6}\right]$$

and hence,

$$t^2<t+2,\quad t\in\left(1,\frac{3}{2}\right]\cup\left[\frac{7}{4},\frac{11}{6}\right],$$

whereupon

$$\sqrt{t+2}>t,\quad t\in\left(1,\frac{3}{2}\right]\cup\left[\frac{7}{4},\frac{11}{6}\right].$$

For $t\in[-2,-1)$ the inequality is true.

b. Let $t=0$. Then

$$\sqrt{2}>0$$

and by point 2 it follows that there is a neighbourhood U of 0 so that $S(s)$ is true for any $s\in U\cap\mathbb{T}$.

c. Let t is left0dense. Then $t\in[-2,-1)\cup\{0\}\cup\left(1,\frac{3}{2}\right]\cup\left(\frac{7}{4},\frac{11}{6}\right]$. By points 2 and 3 , we get that if $s\in[t_0,t)_{\mathbb{T}}$ for some $t_0\in\mathbb{T}$ and $S(s)$ is true, then $S(t)$ is true.

By the induction principle, it follows that $S(t)$ is true for any $t\in\mathbb{T}$.

Example 6.3

Let $\mathbb{T} = (-\infty, -8) \cup \left\{-7 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{-7\} \cup \left\{-7 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{8\} \cup \left\{8 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup 10^{\mathbb{N}}$. We will prove

$$S(t) : \frac{2t-15}{t+5} \geq 0 \quad \text{for any} \quad t \in \mathbb{T}.$$

1. We will prove that (0.3) holds for any $t \in (-\infty, -6]_{\mathbb{T}}$ using the dual induction principle.

a. We have

$$\begin{aligned} \frac{2\cdot(-6)-15}{-6+5} &= \frac{-12-15}{-1} \\ &= 27 \\ &> 0 \end{aligned}$$

i.e., $S(-6)$ is true.

b. Let $t \in (-\infty, -6]_{\mathbb{T}}$ is left-scattered. Then we have the following cases.

i. Let $t = -7 - \frac{1}{n}$ for some $n \in \mathbb{N}, n \geq 2$. Then

$$\rho(t) = -7 - \frac{1}{n-1}.$$

We have

$$\begin{aligned} \frac{2\left(-7-\frac{1}{n}\right)-15}{-7-\frac{1}{n}+5} &= \frac{-14-\frac{2}{n}-15}{-2-\frac{1}{n}} \\ &= \frac{29+\frac{2}{n}}{2+\frac{1}{n}} \\ &> 0 \end{aligned}$$

i.e., $S(t)$ is true. Moreover, we have

$$\begin{aligned} \frac{2\left(-7-\frac{1}{n-1}\right)-15}{-7-\frac{1}{n-1}+5} &= \frac{-14-\frac{2}{n-1}-15}{-2-\frac{1}{n-1}} \\ &= \frac{29+\frac{2}{n-1}}{2+\frac{1}{n-1}} \\ &> 0 \end{aligned}$$

Thus, $S(\rho(t))$ is true.

ii. Let $t = -8$. Then t is left-dense and (0.3) holds for any $t \in (-\infty, -8]_{\mathbb{T}}$. Thus, there is a neighbourhood U of -8 such that (0.3) holds for any $t \in U \cap (-\infty, -6]$ T.

iii. Let $t = -7 + \frac{1}{n}$ for some $n \in \mathbb{N}$. Then

$$\rho(t) = -7 + \frac{1}{n+1}.$$

$$\frac{2\left(-7+\frac{1}{n}\right)-15}{-7+\frac{1}{n}+5} = \frac{-14+\frac{2}{n}-15}{-2+\frac{1}{n}}$$
$$= \frac{29-\frac{2}{n}}{2-\frac{1}{n}}$$
$$> 0$$

because $29-\frac{2}{n}>0$ and $2-\frac{1}{n}>0$. Thus, $S(t)$ is true. Next,

$$\frac{2\left(-7+\frac{1}{n+1}\right)-15}{-7+\frac{1}{n+1}+5} = \frac{-14+\frac{2}{n+1}-15}{-2+\frac{1}{n+1}}$$
$$= \frac{29-\frac{2}{n+1}}{2-\frac{1}{n+1}}$$
$$> 0$$

$29-\frac{2}{n+1}>0$ and $2-\frac{1}{n+1}>0$. Therefore $S(\rho(t))$ is true.

c. Let $t \in (-\infty,-6]_{\mathbb{T}}$ is left-dense. Then we have the following cases.

i. Let $t \in (-\infty,-8)_{\mathbb{T}}$. Then

$$\frac{2t-15}{t+5}>0$$

and

$$\frac{2s-15}{s+5}>0 \quad \text{for any} \quad s \in (-\infty,t)_{\mathbb{T}}.$$

ii. Let $t=-7$. By 1.2.1, it follows that there is a neighbourhood U of -7 such gthat $S(s)$ holds for any $s \in U \cap (-\infty,-7)_{\mathbb{T}}$.

d. Let $t \in (-\infty,-6]_{\mathbb{T}}$ is right-dense. Then we have the following cases.

i. Let $t \in (-\infty,-8)_{\mathbb{T}}$. By the previous cases, we get that $S(s)$ is true for any $s \in (t,-6)_{\mathbb{T}}$ and $S(t)$ is true.

ii. Let $t=-7$. By 1.2 .1 , it follows that there is a neighbourhood U of -7 such gthat $S(s)$ holds for any $s \in U \cap (-\infty,-7)_{\mathbb{T}}$.

Now, applying the dual induction principle, we conclude that $S(t)$ is true for any $t \in (-\infty,-6]_{\mathbb{T}}$.

2. Now, we will prove that (0.3) holds for any $t \in [8,\infty)_{\mathbb{T}}$ using the induction principle.

a. We have

$$\frac{2\cdot 8-15}{8+5} = \frac{16-15}{13}$$
$$= \frac{1}{13}$$
$$> 0$$

Thus, $S(8)$ is true.

b. Let $t \in [8,\infty)_{\mathbb{T}}$ is right-scattered and $S(t)$ is true. Then we have the following cases.

i. Let $t=8+\frac{1}{n}$ for some $n \in \mathbb{N}, n \geq 2$. Then

$$\sigma(t) = 8+\frac{1}{n-1}$$

We have

$$\frac{2\left(8+\frac{1}{n}\right)-15}{8+\frac{1}{n}+5} = \frac{16+\frac{2}{n}-15}{13+\frac{1}{n}}$$
$$= \frac{1+\frac{2}{n}}{13+\frac{1}{n}}$$
$$> 0$$

and

$$\frac{2\left(8+\frac{1}{n-1}\right)-15}{8+\frac{1}{n-1}+5} = \frac{16+\frac{2}{n-1}-15}{13+\frac{1}{n-1}}$$
$$= \frac{1+\frac{2}{n-1}}{13+\frac{1}{n-1}}$$
$$> 0$$

i.e., $\mathcal{S}(t)$ and $\mathcal{S}(\sigma(t))$ is true.

ii. Let $t = 9$. Then

$$\sigma(9) = 10$$

We have

$$\frac{2\cdot 9-15}{9+5} = \frac{18-15}{14}$$
$$= \frac{3}{14}$$
$$> 0$$

and

$$frac2\cdot 10-1510+5 = \frac{20-15}{15}$$
$$= \frac{5}{15}$$
$$= \frac{1}{3}$$
$$> 0.$$

So, $\mathcal{S}(9)$ and $\mathcal{S}(10)$ are true.

iii. Let $t = 10^n$ for some $n \in \mathbb{N}$. Then

$$\sigma(t) = 10^{n+1}$$

We have

$$\frac{2\cdot 10^n-15}{10^n+5} > 0$$

and

$$\frac{2\cdot 10^{n+1}-15}{10^{n+1}+5} > 0.$$

Thus, $\mathcal{S}(t)$ and $\mathcal{S}(\sigma(t))$ are true.

c. Let $t = 8$. We have

$$\begin{aligned}\frac{2\cdot 8-15}{8+5} &= \frac{16-15}{13}\\ &= \frac{1}{13}\\ &> 0\end{aligned}$$

i.e., $S(8)$ is true. By 2.2.1, it follows that there is a neighbourhood U of 8 such that $S(t)$ is true for any $t \in U \cap [8,\infty)$ T.

d. Note that there is no left-dense points in $(8,\infty)_{\mathbb{T}}$.

Now, applying the induction principle, we conclude that $S(t)$ is true for any $t \in [8,\infty)_{\mathbb{T}}$.

Example 6.4

Let $\mathbb{T} = \{-1\} \cup \left\{-1-\frac{1}{n^2}\right\}_{n\in\mathbb{N}} \cup \left\{\frac{1}{n^3}\right\}_{n\in\mathbb{N}} \cup (1,2]$. We will prove that

$$S(t): t^2 - t - 2 \leq 2 \text{ for any } t \in \mathbb{T}.$$

1. We have

$$\begin{aligned}(-1)^2 - (1-1) - 2 &= 1+1-2\\ &= 0\end{aligned}$$

i.e., $S(-1)$ is true.

2. Let $t \in \mathbb{T}$ be right-scattered. We have the following cases.

a. Let $t = -1 + \frac{1}{n^2}$ for some $n \in \mathbb{N}, n \geq 2$. Then

$$\sigma(t) = -1 + \frac{1}{(n-1)^2}$$

We have

$$\begin{aligned}\left(-1+\frac{1}{n^2}\right)^2 - \left(-1+\frac{1}{n^2}\right) - 2 &= 1 - \frac{2}{n^2} + \frac{1}{n^4} + 1 - \frac{1}{n^2} - 2\\ &= \frac{1}{n^4} - \frac{3}{n^2}\\ &= \frac{1-3n^2}{n^4}\\ &\leq 0\end{aligned}$$

i.e., $S(t)$ is true. Next,

$$\begin{aligned}\left(-1+\frac{1}{(n-1)^2}\right)^2 - \left(-1+\frac{1}{(n-1)^2}\right) - 2 &= 1 - \frac{2}{(n-1)^2} + \frac{1}{(n-1)^4} + 1\\ &\quad - \frac{1}{(n-1)^2} - 2\\ &= \frac{1}{(n-1)^4} - \frac{3}{(n-1)^2}\\ &= \frac{1-3(n-1)^2}{(n-1)^4}\\ &\leq 0\end{aligned}$$

Thus, $S(\sigma(t))$ is true.

b. Let $t = \frac{1}{n^3}$ for some $n \in \mathbb{N}, \geq 2$. Then

$$\sigma(t) = \frac{1}{(n-1)^3}$$

We have

$$\left(-1+\frac{1}{n^3}\right)^2-\left(-1+\frac{1}{n^3}\right)-2=1-\frac{2}{n^3}+\frac{1}{n^6}+1-\frac{1}{n^3}-2$$
$$=\frac{1}{n^6}-\frac{3}{n^3}$$
$$=\frac{1-3n^3}{n^6}$$
$$\leq 0$$

i.e., $S(t)$ is true. Moreover,

$$\left(-1+\frac{1}{(n-1)^3}\right)^2-\left(-1+\frac{1}{(n-1)^3}\right)-2=1-\frac{2}{(n-1)^3}+\frac{1}{(n-1)^6}+1$$
$$-\frac{1}{(n-1)^3}-2$$
$$=\frac{1}{(n-1)^6}-\frac{3}{(n-1)^3}$$
$$=\frac{1-3(n-1)^3}{(n-1)^6}$$
$$\leq 0$$

and then $S(\sigma(t))$ is true.

3. Let $t\in\mathbb{T}$ be right-dense and $S(t)$ be true. We have the following cases.

a. Let $t=-1$. Then, by 2.1 , it follows that there is a neighbourhood U of -1 such that $S(s)$ is true for any $s\in U\cap\mathbb{T}$.

b. Let $t=0$. Then, by 2.2 , it follows that there is a neighbourhood U of 0 such that $S(s)$ is true for any $s\in U\cap\mathbb{T}$.

c. Let $t\in(1,2]$. Then

$$t^2-t-2\leq 0$$

and there is a neighbourhood U of t such that $S(s)$ is true for any $s\in U\cap\mathbb{T}$.

d. Let $t\in\mathbb{T}$ be left0dense and $S(s)$ is true for any $s\in[-1,t)$. We have that $t\in(1,2]$. By the previous cases it follows that $S(t)$ is true.

Now, applying the induction principle, we conclude that $S(t)$ is true for any $t\in\mathbb{T}$.

Example 6.5

Let $\mathbb{T}=\mathbb{N}$. We will prove that

$$S(n):1^4+2^4+\cdots+n^4=\frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)\quad\text{for any}\quad n\in\mathbb{T}.$$

Solution

1. We have

$$\frac{1}{30}\cdot 1\cdot(1+1)\cdot(2\cdot 1+1)\left(3\cdot 1^2+3\cdot 1-1\right)$$
$$=\frac{1}{30}\cdot 2\cdot 3\cdot 5$$
$$=1$$
$$=1^4$$

Thus, $S(1)$ is true.

2. Firstly, note that any point of $\mathbb{T}$ is right-scattered. Assume that $S(n)$ is true for some $n\in\mathbb{T}$. We have

$$\sigma(n)=n+1.$$

Then

$$
\begin{aligned}
&1^4+2^4+\cdots+(n+1)^4 = 1^4+2^4+\cdots+n^4+(n+1)^4\\
=&\frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)+(n+1)^4\\
=&(n+1)\left(\frac{1}{30}n(2n+1)\left(3n^2+3n-1\right)+(n+1)^3\right)\\
=&\frac{1}{30}(n+1)\left(\left(2n^2+n\right)\left(3n^2+3n-1\right)+30(n+1)^3\right)\\
=&\frac{1}{30}(n+1)\left(6n^4+6n^3-2n^2+3n^3+3n^2-n+30n^3+90n^2+90n+30\right)\\
=&\frac{1}{30}(n+1)\left(6n^4+39n^3+91n^2+89n+30\right)\\
=&\frac{1}{30}(n+1)(n+2)\left(6n^3+27n^2+37n+15\right)\\
=&\frac{1}{30}(n+1)(n+2)(2n+3)\left(3n^2+9n+5\right)\\
=&\frac{1}{30}(n+1)(n+2)(2n+3)\left(3n^2+6n+3+3n+3-1\right)\\
=&\frac{1}{30}(n+1)(n+2)(2n+3)\left(3(n+1)^2+3(n+1)-1\right)
\end{aligned}
$$

Thus, $S(\sigma(n))$ is true.

Applying the induction principle, we conclude that $S(n)$ is true for any $n \in \mathbb{N}$.

? Exercise 6.1

Let

$$\mathbb{T} = -15^{\mathbb{N}} \cup \left\{-14-\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{-14+\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup (-12,-10] \cup \{-9\} \cup \left\{-9+\frac{1}{n}\right\}_{n\in\mathbb{N}}$$

Prove that

$$\frac{2t-1}{3} < \frac{t-3}{2} \text{ for any } t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

? Exercise 6.2

Let

$$\mathbb{T} = \{2\} \cup \left\{2+\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{3-\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{3+\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{3\} \cup 4^{\mathbb{N}}$$

Prove that

$$\frac{2-t}{t+1}+\frac{t+2}{t-1} > \frac{t-3}{2} \text{ for any } t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 6.3

Let

$$\mathbb{T} = (-\infty, 0] \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{2\} \cup \left\{2 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{5\} \cup \left\{5 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{5 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup 7^{\mathbb{N}}$$

Prove that

$$5(t-3)(t-4) > 0 \quad \text{for any} \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 6.4

$$\begin{aligned}\mathbb{T} = &(-\infty, -10] \cup \{-9\} \cup \left\{-9 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{-9 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \\ &\cup (-8, -1] \cup \{4\} \cup \left\{4 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \\ &\cup \{6\} \cup \left\{6 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{6 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup 9^{\mathbb{N}}.\end{aligned}$$

Prove that

$$5t^2 - 2(t+3)(t-1) > 4t + 9 \quad \text{for any} \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 6.1

Let $\mathbb{T} = \mathbb{N}$. Prove that

$$1^5 + 2^5 + \cdots + n^5 = \frac{1}{12} n^2 (n+1)^2 \left(2n^2 + 2n - 1\right) \quad \text{for any} \quad n \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

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7: Differentiable Functions of One Variable

Definition: Delta derivative

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. We define $f^\Delta(t)$ to be the number, provided it exists, with the property that for any $\varepsilon > 0$, there exists a neighbourhood U of $t, U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U$$

We call $f^\Delta(t)$ the delta or Hilger derivative of f at t. We say that f is delta or Hilger differentiable, shortly differentiable, in $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T} \to \mathbb{R}$ is said to be the delta derivative or Hilger derivative, shortly derivative, of f in $\mathbb{T}^\kappa$.

Remark

If $\mathbb{T} = \mathbb{R}$, then the delta derivative coincides with the classical derivative.

Theorem 7.1

The delta derivative is well defined.

Proof. Let $t \in \mathbb{T}^\kappa$. Suppose $f_1^\Delta(t)$ and $f_2^\Delta(t)$ are such that

$$\left|f(\sigma(t)) - f(s) - f_1^\Delta(t)(\sigma(t) - s)\right| \leq \frac{\varepsilon}{2}|\sigma(t) - s|$$

and

$$\left|f(\sigma(t)) - f(s) - f_2^\Delta(t)(\sigma(t) - s)\right| \leq \frac{\varepsilon}{2}|\sigma(t) - s|$$

for any $\varepsilon > 0$ and any s belonging to a neighbourhood U of $t, U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$. Hence, if $s \neq \sigma(t)$, then

$$\begin{aligned}
|f_1^\Delta(t) - f_2^\Delta(t)| &= \left|f_1^\Delta(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} + \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^\Delta(t)\right| \\
&\leq \left|f_1^\Delta(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}\right| + \left|\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^\Delta(t)\right| \\
&= \frac{\left|f(\sigma(t)) - f(s) - f_1^\Delta(t)(\sigma(t) - s)\right|}{|\sigma(t) - s|} + \frac{\left|f(\sigma(t)) - f(s) - f_2^\Delta(t)(\sigma(t) - s)\right|}{|\sigma(t) - s|} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$f_1^\Delta(t) = f_2^\Delta(t)$$

This completes the proof.Remark 0.3.

Remark

Let us assume that $\sup \mathbb{T} < \infty$ and $f^\Delta(t)$ is defined at a point $t \in \mathbb{T}\backslash\mathbb{T}^\kappa$ with the same definition as given in Definition 0.11. Then the unique point $t \in \mathbb{T}\backslash\mathbb{T}^K$ is $\sup \mathbb{T}$. Hence, for any $\varepsilon > 0$, there is a neighbourhood $U = (t-\delta, t+\delta) \cap (\mathbb{T}\backslash\mathbb{T}^K)$, for some δ

> 0, such that

$$f(\sigma(t)) = f(s) = f(\sigma(\sup \mathbb{T})) = f(\sup \mathbb{T}), \quad s \in U.$$

Therefore, for any $\alpha \in \mathbb{R}$ and $s \in U$, we have

$$\begin{aligned} |f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| &= |f(\sup \mathbb{T}) - f(\sup \mathbb{T}) - \alpha(\sup \mathbb{T} - \sup \mathbb{T})| \\ &\leq \varepsilon |\sigma(t) - s| \end{aligned}$$

i.e., any $\alpha \in \mathbb{R}$ is the delta derivative of f at the point $t \in \mathbb{T} \backslash \mathbb{T}^K$.

Example 7.1

Let $f(t) = \alpha \in \mathbb{R}$. We will prove that $f^{\Delta}(t) = 0$ for any $t \in \mathbb{T}^{\kappa}$. Indeed, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0, s \in (t-1, t+1) \cap \mathbb{T}$

Implies

$$|f(\sigma(t)) - f(s) - 0(\sigma(t) - s)| = |\alpha - \alpha| = 0 \leq \varepsilon |\sigma(t) - s|$$

Example 7.2

Let $f(t) = t, t \in \mathbb{T}$. We will prove that $f^{\Delta}(t) = 1$ for any $t \in \mathbb{T}^{\kappa}$. Indeed, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0, s \in (t-1, t+1) \cap \mathbb{T}$

Implies

$$\begin{aligned} |f(\sigma(t)) - f(s) - 1(\sigma(t) - s)| &= |\sigma(t) - s - (\sigma(t) - s)| \\ &= 0 \leq \varepsilon |\sigma(t) - s|. \end{aligned}$$

Example 7.3

Let $f(t) = t^2, t \in \mathbb{T}$. We will prove that $f^{\Delta}(t) = \sigma(t) + t, t \in \mathbb{T}^{\kappa}$. Indeed, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0, s \in (t-\varepsilon, t+\varepsilon) \cap \mathbb{T}$ implies $|t - s| < \varepsilon$ and

$$\begin{aligned} &|f(\sigma(t)) - f(s) - (\sigma(t) + t)(\sigma(t) - s)| \\ &= \left|(\sigma(t))^2 - s^2 - (\sigma(t) + t)(\sigma(t) - s)\right| \\ &= |(\sigma(t) - s)(\sigma(t) + s) - (\sigma(t) + t)(\sigma(t) - s)| \\ &= |\sigma(t) - s||t - s| \\ &\leq \varepsilon |\sigma(t) - s| \end{aligned}$$

Exercise 7.1

Let $f(t) = \sqrt{t}, t \in \mathbb{T}, t > 0$. Prove that

$$f^{\Delta}(t) = \frac{1}{\sqrt{t} + \sqrt{\sigma(t)}} \quad \text{for all} \quad t \in \mathbb{T}^{\kappa} \cap (0, \infty)$$

Answer

Add texts here. Do not delete this text first.

? Exercise 7.2

Let $f(t) = t^3, t \in \mathbb{T}$. Prove that

$$f^{\Delta}(t) = (\sigma(t))^2 + t\sigma(t) + t^2 \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}$$

Answer

Add texts here. Do not delete this text first.

Theorem 7.2

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^K$. Then we have the following.

1. If f is differentiable at t, then f is continuous at t.
2. If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

3. If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

4. If f is differentiable at t, then the "simple useful formula"

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

holds.

Proof. 1. Assume that f is differentiable at $t \in \mathbb{T}^K$. Let $\varepsilon \in (0, 1)$ be arbitrarily chosen. Set

$$\varepsilon^* = \frac{\varepsilon}{1 + |f^{\Delta}(t)| + 2\mu(t)}.$$

Since f is differentiable at t, there exists a neighbourhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \leq \varepsilon^* |\sigma(t) - s|.$$

Hence, for all $s \in U \cap (t - \varepsilon^*, t + \varepsilon^*)$, we have

$$
\begin{aligned}
|f(t)-f(s)| &= |\, f(t)+f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s) \\
&\quad -f(\sigma(t))+f^{\Delta}(t)(\sigma(t)-s)\,| \\
&\le \left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right| \\
&\quad +\left|f(\sigma(t))-f(t)-f^{\Delta}(t)(\sigma(t)-s)\right| \\
&\le \varepsilon^{*}|\sigma(t)-s|+|\, f(\sigma(t))-f(t)-f^{\Delta}(t)(\sigma(t)-t) \\
&\quad +f^{\Delta}(t)(s-t)\,| \\
&\le \varepsilon^{*}|\sigma(t)-s|+\left|f(\sigma(t))-f(t)-f^{\Delta}(t)(\sigma(t)-t)\right| \\
&\quad +\left|f^{\Delta}(t)\right||s-t| \\
&\le \varepsilon^{*}|\sigma(t)-s|+\varepsilon^{*}\mu(t)+\varepsilon^{*}\left|f^{\Delta}(t)\right| \\
&= \varepsilon^{*}\left(|\sigma(t)-s|+\mu(t)+\left|f^{\Delta}(t)\right|\right) \\
&= \varepsilon^{*}\left(|\sigma(t)-t+t-s|+\mu(t)+\left|f^{\Delta}(t)\right|\right) \\
&\le \varepsilon^{*}\left(\sigma(t)-t+|t-s|+\mu(t)+\left|f^{\Delta}(t)\right|\right) \\
&= \varepsilon^{*}\left(2\mu(t)+|t-s|+\left|f^{\Delta}(t)\right|\right) \\
&\le \varepsilon^{*}\left(2\mu(t)+\varepsilon^{*}+\left|f^{\Delta}(t)\right|\right) \\
&\le \varepsilon^{*}\left(1+2\mu(t)+\left|f^{\Delta}(t)\right|\right) \\
&= \varepsilon
\end{aligned}
$$

which completes the proof.

2. Assume that f is continuous at t and t is right-scattered. By continuity, we have

$$
\begin{aligned}
\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s} &= \frac{f(\sigma(t))-f(t)}{\sigma(t)-t} \\
&= \frac{f(\sigma(t))-f(t)}{\mu(t)}.
\end{aligned}
$$

Therefore, for any $\varepsilon>0$, there exists a neighbourhood U of t such that

$$
\left|\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}-\frac{f(\sigma(t))-f(t)}{\mu(t)}\right|\le\varepsilon
$$

for all $s\in U$, i.e.,

$$
\left|f(\sigma(t))-f(s)-\frac{f(\sigma(t))-f(t)}{\mu(t)}(\sigma(t)-s)\right|\le\varepsilon|\sigma(t)-s|
$$

for all $s\in U$. Hence,

$$
f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}.
$$

3. Assume that t is right-dense. Let $\varepsilon>0$ be arbitrarily chosen. Then f is differentiable at t iff there is a neighbourhood U of t such that

$$
\left|f(t)-f(s)-f^{\Delta}(t)(t-s)\right|\le\varepsilon|t-s| \quad \text{for all} \quad s\in U
$$

i.e., iff

$$
\left|\frac{f(t)-f(s)}{t-s}-f^{\Delta}(t)\right|\le\varepsilon \quad \text{for all} \quad s\in U
$$

i.e., iff

$$
\lim_{s\to t}\frac{f(t)-f(s)}{t-s}=f^{\Delta}(t)
$$

4. Assume that f is differentiable at t.

a. If t is right-dense, then $\sigma(t)=t, \mu(t)=0$ and

$$f(\sigma(t)) = f(t) = f(t) + \mu(t) f^{\Delta}(t).$$

b. If t is right-scattered, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

whereupon

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

This completes the proof.

Example 7.4

Let

$$\mathbb{T} = \left\{ \frac{1}{2n+1} : n \in \mathbb{N}_0 \right\} \cup \{0\}$$

and $f(t) = \sigma(t)$ for $t \in \mathbb{T}$. We will find $f^{\Delta}(t)$ for $t \in \mathbb{T}^{\kappa} = \mathbb{T} \backslash \{1\}$. For

$$t \in \mathbb{T}^{\kappa} \backslash \{0\}, \quad t = \frac{1}{2n+1}, \quad n = \frac{1-t}{2t}, \quad n \in \mathbb{N},$$

we have

$$\begin{aligned} \sigma(t) &= \inf \left\{ s \in \mathbb{T} : s > \frac{1}{2n+1} \right\} \\ &= \frac{1}{2n-1} \\ &= \frac{1}{2\dfrac{1-t}{2t} - 1} \\ &= \frac{t}{1-2t} > t \end{aligned}$$

i.e., any point $t = \dfrac{1}{2n+1}, n \in \mathbb{N}$, is right-scattered. At these points,

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\ &= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\ &= 2\frac{(\sigma(t))^2}{(1 - 2\sigma(t))(\sigma(t) - t)} \\ &= 2\frac{\left(\dfrac{t}{1-2t}\right)^2}{\left(1 - \dfrac{2t}{1-2t}\right)\left(\dfrac{t}{1-2t} - t\right)} \\ &= 2\frac{\dfrac{t^2}{(1-2t)^2}}{\dfrac{1-4t}{1-2t}\dfrac{2t^2}{1-2t}} \\ &= 2\frac{t^2}{2t^2(1-4t)} \\ &= \frac{1}{1-4t}. \end{aligned}$$

Let now $t = 0$. Then

$$\sigma(0) = \inf\{s \in \mathbb{T} : s > 0\} = 0.$$

Consequently, $t = 0$ is right-dense. Also,

$$\lim_{h \to 0} \frac{\sigma(h) - \sigma(0)}{h} = \lim_{h \to 0} \frac{\frac{h}{1-2h} - 0}{h} = \lim_{h \to 0} \frac{1}{1-2h} = 1.$$

Therefore, $f^{\Delta}(0) = 1$. Altogether, $f^{\Delta}(t) = \dfrac{1}{1-4t}$ for all $t \in \mathbb{T}^{\kappa}$.

Example 7.5

Let $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$ and $f(t) = t^2, g(t) = \sigma(t)$ for $t \in \mathbb{T}$. We will find $f^{\Delta}(t)$ and $g^{\Delta}(t)$ for $t \in \mathbb{T}^{\kappa} = \mathbb{T}$. For $t \in \mathbb{T}, t = n^2, n = \sqrt{t}, n \in \mathbb{N}_0$, we have

$$\sigma(t) = \inf\left\{l^2 : l^2 > n^2, l \in \mathbb{N}_0\right\} = (n+1)^2 = (1+\sqrt{t})^2 > t.$$

Therefore, all points of $\mathbb{T}$ are right-scattered. We note that f and g are continuous functions in $\mathbb{T}$. Hence,

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\ &= \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\ &= \sigma(t) + t \\ &= (1+\sqrt{t})^2 + t \\ &= t + 2\sqrt{t} + 1 + t \\ &= 1 + 2\sqrt{t} + 2t \end{aligned}$$

and

$$\begin{aligned} g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\ &= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\ &= \frac{(1+\sqrt{\sigma(t)})^2 - \sigma(t)}{\sigma(t) - t} \\ &= \frac{\sigma(t) + 2\sqrt{\sigma(t)} + 1 - \sigma(t)}{\sigma(t) - t} \\ &= \frac{1 + 2\sqrt{\sigma(t)}}{\sigma(t) - t} \\ &= \frac{1 + 2(1+\sqrt{t})}{(1+\sqrt{t})^2 - t} \\ &= \frac{3 + 2\sqrt{t}}{1 + 2\sqrt{t}} \end{aligned}$$

Example 7.6

Let $\mathbb{T} = \left\{\sqrt[4]{2n+1} : n \in \mathbb{N}_0\right\}$ and $f(t) = t^4$ for $t \in \mathbb{T}$. We will find $f^{\Delta}(t)$ for $t \in \mathbb{T}$. For $t \in \mathbb{T}$, $t = \sqrt[4]{2n+1}, n = \dfrac{t^4-1}{2}, n \in \mathbb{N}_0$, we have

Solution

$$\begin{aligned}\sigma(t) &= \inf\left\{\sqrt[4]{2l+1} : \sqrt[4]{2l+1} > \sqrt[4]{2n+1}, l \in \mathbb{N}_0\right\}\\ &= \sqrt[4]{2n+3}\\ &= \sqrt[4]{t^4+2} > t\end{aligned}$$

Therefore, every point of $\mathbb{T}$ is right-scattered. We note that the function f is continuous in $\mathbb{T}$. Hence,

$$\begin{aligned}f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}\\ &= \frac{(\sigma(t))^4 - t^4}{\sigma(t) - t}\\ &= (\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3\\ &= \sqrt[4]{(t^4+2)^3} + t^2\sqrt[4]{t^4+2} + t\sqrt{t^4+2} + t^3\end{aligned}$$

Exercise 7.3

Let $\mathbb{T} = \left\{\sqrt[5]{n+1} : n \in \mathbb{N}_0\right\}$ and $f(t) = t + t^3$ for $t \in \mathbb{T}$. Find $f^{\Delta}(t)$ for $t \in \mathbb{T}^{\kappa}$.

Solution. $1 + \sqrt[5]{(t^5+1)^2} + t\sqrt[5]{t^5+1} + t^2$.

Answer

Add texts here. Do not delete this text first.

Example 7.7

Let $\mathbb{T} = \mathbb{Z}$ and f be differentiable at t. Note that all points of t are right-scattered and $\sigma(t) = t + 1$.

Therefore,

$$\begin{aligned}f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}\\ &= \frac{f(t+1) - f(t)}{t+1-t}\\ &= f(t+1) - f(t)\\ &= \Delta f(t)\end{aligned}$$

where Δ is the usual forward difference operator.

Exercise 7.4

Let $\mathbb{T} = h\mathbb{Z}$ for some $h > 0$. Prove that

$$\begin{aligned}t &= \frac{1}{2}f_2^{\Delta}(t) - \frac{h}{2}\\ t^2 &= \frac{1}{3}f_3^{\Delta}(t) - \frac{h}{2}f_2^{\Delta}(t) + \frac{1}{6}h^2\\ t^3 &= \frac{1}{4}f_4^{\Delta}(t) - \frac{1}{2}hf_3^{\Delta}(t) + \frac{1}{4}h^2f_2^{\Delta}(t)\\ t^4 &= \frac{1}{5}f_5^{\Delta}(t) - \frac{1}{2}hf_4^{\Delta}(t) + \frac{1}{3}h^2f_3^{\Delta}(t) - \frac{1}{30}h^4\\ t^5 &= \frac{1}{6}f_6^{\Delta}(t) - \frac{1}{2}hf_5^{\Delta}(t) + \frac{5}{12}h^2f_4^{\Delta}(t) - \frac{1}{12}h^4f_2^{\Delta}(t)\end{aligned}$$

where $f_i(t) = t^i, i = 1, \ldots, 6, t \in \mathbb{T}$.

Answer

We note that all points of $\mathbb{T}$ are right-scattered and $\sigma(t) = t + h, \mu(t) = h, t \in \mathbb{T}$. Then

$$\begin{aligned}
\frac{1}{2} f_2^{\Delta}(t) - \frac{h}{2} &= \frac{1}{2} \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} - \frac{h}{2} \\
&= \frac{1}{2} \frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} - \frac{h}{2} \\
&= \frac{1}{2}(\sigma(t) + t) - \frac{h}{2} \\
&= \frac{1}{2}(2t + h) - \frac{h}{2} \\
&= t + \frac{h}{2} - \frac{h}{2} \\
&= t
\end{aligned}$$

$$\frac{1}{3} f_3^{\Delta}(t) - \frac{h}{2} f_2^{\Delta}(t) + \frac{1}{6} h^2 = \frac{1}{3} \frac{(\sigma(t))^3 - t^3}{\sigma(t) - t} - \frac{h}{2} \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} + \frac{1}{6} h^2$$

$$\begin{aligned}
&= \frac{1}{3} \frac{(\sigma(t) - t)\left((\sigma(t))^2 + t\sigma(t) + t^2\right)}{\sigma(t) - t} \\
&\quad - \frac{h}{2} \frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} + \frac{1}{6} h^2 \\
&= \frac{1}{3}\left((\sigma(t))^2 + t\sigma(t) + t^2\right) - \frac{h}{2}(\sigma(t) + t) + \frac{1}{6} h^2 \\
&= \frac{1}{3}\left((t + h)^2 + t(t + h) + t^2\right) - \frac{h}{2}(t + h + t) + \frac{1}{6} h^2 \\
&= \frac{1}{3}\left(t^2 + 2ht + h^2 + t^2 + ht + t^2\right) - \frac{h}{2}(2t + h) + \frac{1}{6} h^2 \\
&= \frac{1}{3}\left(3t^2 + 3ht + h^2\right) - ht - \frac{h^2}{2} + \frac{1}{6} h^2 \\
&= t^2 + ht + \frac{h^2}{3} - ht - \frac{h^2}{3} \\
&= t^2,
\end{aligned}$$

$$\frac{1}{4} f_4^{\Delta}(t) - \frac{1}{2} h f_3^{\Delta}(t) + \frac{1}{4} h^2 f_2^{\Delta}(t) = \frac{1}{4} \frac{(\sigma(t))^4 - t^4}{\sigma(t) - t} - \frac{1}{2} h \frac{(\sigma(t))^3 - t^3}{\sigma(t) - t} \tag{7.1}$$

$$\begin{aligned}
&\quad + \frac{1}{4} h^2 \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\
&= \frac{1}{4} \frac{(\sigma(t) - t)\left((\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3\right)}{\sigma(t) - t} \\
&\quad - \frac{1}{2} h \frac{(\sigma(t) - t)\left((\sigma(t))^2 + t\sigma(t) + t^2\right)}{\sigma(t) - t} \\
&\quad + \frac{1}{4} h^2 \frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} \\
&= \frac{1}{4}\left((\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3\right) - \frac{1}{2} h\left((\sigma(t))^2 + t\sigma(t) + t^2\right) \\
&\quad + \frac{1}{4} h^2(\sigma(t) + t) \\
&= \frac{1}{4}\left((t + h)^3 + t(t + h)^2 + t^2(t + h) + t^3\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}h\left((t+h)^2+t(t+h)+t^2\right)+\frac{1}{4}h^2(t+h+t)\\
=&\frac{1}{4}\left(t^3+3t^2h+3th^2+h^3+t\left(t^2+2th+h^2\right)+t^3+ht^2+t^3\right)\\
&-\frac{1}{2}h\left(t^2+2ht+h^2+t^2+ht+t^2\right)+\frac{1}{4}h^2(2t+h)\\
=&\frac{1}{4}\left(3t^3+4t^2h+3th^2+h^3+t^3+2t^2h+th^2\right)\\
&-\frac{1}{2}h\left(3t^2+3ht+h^2\right)+\frac{1}{2}th^2+\frac{1}{4}h^3\\
=&\frac{1}{4}\left(4t^3+6t^2h+4th^2+h^3\right)-\frac{3}{2}ht^2-\frac{3}{2}h^2t-\frac{1}{2}h^3+\frac{1}{2}h^2t+\frac{1}{4}h^3\\
=&t^3+\frac{3}{2}ht^2+h^2t+\frac{1}{4}h^3-\frac{3}{2}ht^2-\frac{3}{2}h^2t-\frac{1}{2}h^3+\frac{1}{2}h^2t+\frac{1}{4}h^3\\
=&t^3,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{5}f_5^{\Delta}-\frac{h}{2}f_4^{\Delta}+\frac{1}{3}h^2f_3^{\Delta}-\frac{1}{30}h^4=\frac{1}{5}\frac{(\sigma(t))^5-t^5}{\sigma(t)-t}\\
&-\frac{h}{2}\frac{(\sigma(t))^4-t^4}{\sigma(t)-t}+\frac{1}{3}h^2\frac{(\sigma(t))^3-t^3}{\sigma(t)-t}-\frac{1}{30}h^4\\
=&\frac{1}{5}\frac{(\sigma(t)-t)\left((\sigma(t))^4+t(\sigma(t))^3+t^2(\sigma(t))^2+t^3\sigma(t)+t^4\right)}{\sigma(t)-t}\\
&-\frac{h}{2}\frac{(\sigma(t)-t)\left((\sigma(t))^3+t(\sigma(t))^2+t^2\sigma(t)+t^3\right)}{\sigma(t)-t}\\
&+\frac{1}{3}h^2\frac{(\sigma(t)-t)\left((\sigma(t))^2+t\sigma(t)+t^2\right)}{\sigma(t)-t}-\frac{1}{30}h^4\\
=&\frac{1}{5}\left((t+h)^4+t(t+h)^3+t^2(t+h)^2+t^3(t+h)+t^4\right)\\
&-\frac{h}{2}\left((t+h)^3+t(t+h)^2+t^2(t+h)+t^3\right)\\
&+\frac{1}{3}h^2\left((t+h)^2+t(t+h)+t^2\right)-\frac{1}{30}h^4\\
=&\frac{1}{5}\left(t^4+4t^2h^2+h^4+4t^3h+2t^2h^2+4th^3+t\left(t^3+3t^2h+3th^2+h^3\right)\right.\\
&\left.+t^2\left(t^2+2th+h^2\right)+t^4+ht^3+t^4\right)\\
&-\frac{h}{2}\left(t^3+3t^2h+3th^2+h^3+t\left(t^2+2th+h^2\right)+t^3+ht^2+t^3\right)\\
&+\frac{1}{3}h^2\left(t^2+2th+h^2+t^2+ht+t^2\right)-\frac{1}{30}h^4\\
=&\frac{1}{5}\left(t^4+6t^2h^2+h^4+4t^3h+4th^3+t^4+3t^3h+3t^2h^2+th^3\right.\\
&\left.+t^4+2t^3h+t^2h^2+2t^4+ht^3\right)\\
&-\frac{h}{2}\left(3t^3+4t^2h+3th^2+h^3+t^3+2t^2h+th^2\right)\\
&+\frac{1}{3}h^2\left(3t^2+3ht+h^2\right)-\frac{1}{30}h^4\\
=&\frac{1}{5}\left(5t^4+10t^3h+10t^2h^2+5th^3+h^4\right)-\frac{h}{2}\left(4t^3+6t^2h+4th^2+h^3\right)\\
&+h^2t^2+h^3t+\frac{1}{3}h^4-\frac{1}{30}h^4\\
=&t^4+2t^3h+2t^2h^2+th^3+\frac{1}{5}h^4-2ht^3-3t^2h^2-2th^3-\frac{h^4}{2}\\
&+h^2t^2+h^3t+\frac{1}{3}h^4-\frac{1}{30}h^4\\
=&t^4
\end{aligned}
\tag{7.2}
$$

The last equality is left to the reader for exercise.

? Exercise 7.5

Let $\mathbb{T} = q^{\mathbb{N}}$ for some $q > 1$. Prove

$$t = \frac{1}{1+q} f_2^{\Delta}(t)$$
$$t^2 = \frac{1}{1+q+q^2} f_3^{\Delta}(t)$$
$$t^3 = \frac{1}{1+q+q^2+q^3} f_4^{\Delta}(t)$$
$$t^4 = \frac{1}{1+q+q^2+q^3+q^4} f_5^{\Delta}(t)$$
$$t^5 = \frac{1}{1+q+q^2+q^3+q^4+q^5} f_6^{\Delta}(t)$$

where $f_i(t) = t^i, i = 1, \dots, 6, t \in \mathbb{T}$.

Answer

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Theorem 7.3

Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then we have the following.

1. The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$$

2. For any constant $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$$

3. The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t, and the "product rule"

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$$

holds.

4. If $g(t)g(\sigma(t)) \neq 0$, then the quotient $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is differentiable at t, and the "quotient rule"

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}$$

holds.

Proof. 1. Let $\varepsilon > 0$ be arbitrarily chosen. Since f and g are differentiable at t, there exist neighbourhoods U_1 and U_2 of t so that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq \frac{\varepsilon}{2}|\sigma(t) - s| \quad \text{for all} \quad s \in U_1$$

and

$$\left|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)\right| \leq \frac{\varepsilon}{2}|\sigma(t) - s| \quad \text{for all} \quad s \in U_2$$

Hence, for $s \in U_1 \cap U_2$, we have

$$
\begin{aligned}
&\left|f(\sigma(t))+g(\sigma(t))-f(s)-g(s)-\left(f^{\Delta}(t)+g^{\Delta}(t)\right)(\sigma(t)-s)\right| \\
&\leq\left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right|+\left|g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s)\right| \\
&\leq \frac{\varepsilon}{2}|\sigma(t)-s|+\frac{\varepsilon}{2}|\sigma(t)-s| \\
&=\varepsilon|\sigma(t)-s|
\end{aligned}
$$

which completes the proof.

2. Let $\alpha \neq 0$. Assume that $\varepsilon>0$ is arbitrarily chosen. Since f is differentiable at t, there exists a neighbourhood U of t such that

$$
\left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right| \leq \frac{\varepsilon}{|\alpha|}|\sigma(t)-s| \quad \text{for all} \quad s \in U
$$

Hence, for $s \in U$, we have

$$
\begin{aligned}
&\left|\alpha f(\sigma(t))-\alpha f(s)-\alpha f^{\Delta}(t)(\sigma(t)-s)\right| \\
&\quad=|\alpha|\left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right| \\
&\quad \leq|\alpha| \frac{\varepsilon}{|\alpha|} \\
&\quad=\varepsilon
\end{aligned}
$$

which completes the proof.

3. Let $\varepsilon>0$ be arbitrarily chosen. Let also $[a, b] \subset \mathbb{T}$ be such that $t, \sigma(t) \in[a, b]$. Set

$$
M=\max_{t \in[a, b]}|f(t)|
$$

and

$$
\varepsilon^{*}=\frac{\varepsilon}{1+M+|g(\sigma(t))|+|g^{\Delta}(t)|}
$$

Since f is differentiable at t, there exists a neighbourhood U_1 of t such that

$$
\left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right| \leq \varepsilon^{*}|\sigma(t)-s| \quad \text{for any} \quad s \in U_1.
$$

Since g is differentiable at t, there exists a neighbourhood U_2 of t such that

$$
\left|g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s)\right| \leq \varepsilon^{*}|\sigma(t)-s| \quad \text{for any} \quad s \in U_2.
$$

Since f is differentiable at t, f is continuous at t. Therefore, there exists a neighbourhood U_3 of t such that

$$
|f(t)-f(s)| \leq \varepsilon^{*} \quad \text{for any} \quad s \in U_3.
$$

Then, for any $s \in U_1 \cap U_2 \cap U_3 \cap[a, b]$, we get

$$
\begin{aligned}
&\left|f(\sigma(t))g(\sigma(t))-f(s)g(s)-\left(f^{\Delta}(t)g(\sigma(t))+f(t)g^{\Delta}(t)\right)(\sigma(t)-s)\right| \\
&=\mid f(\sigma(t))g(\sigma(t))-f(s)g(\sigma(t))-f^{\Delta}(t)g(\sigma(t))(\sigma(t)-s) \\
&\quad+f(s)g(\sigma(t))+f^{\Delta}(t)g(\sigma(t))(\sigma(t)-s)+g^{\Delta}(t)f(s)(\sigma(t)-s) \\
&\quad-\left(f^{\Delta}(t)g(\sigma(t))+f(t)g^{\Delta}(t)\right)(\sigma(t)-s)-f(s)g(s)-g^{\Delta}(t)f(s)(\sigma(t)-s) \mid \\
&=\mid\left(f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right)g(\sigma(t)) \\
&\quad+f(s)\left(g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s)\right)-(f(t)-f(s))g^{\Delta}(t)(\sigma(t)-s) \mid \\
&\leq|g(\sigma(t))|\left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right| \\
&\quad+|f(s)|\left|g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s)\right|+|f(t)-f(s)|\left|g^{\Delta}(t)\right||\sigma(t)-s| \\
&\leq \varepsilon^{*}|g(\sigma(t))||\sigma(t)-s| \\
&\quad+\varepsilon^{*}|f(s)||\sigma(t)-s|+\varepsilon^{*}\left|g^{\Delta}(t)\right||\sigma(t)-s| \\
&\leq \varepsilon^{*}\left(|g(\sigma(t))|+M+\left|g^{\Delta}(t)\right|\right)|\sigma(t)-s| \\
&\leq \varepsilon|\sigma(t)-s|
\end{aligned}
$$

which completes the proof.

4. Let $\varepsilon > 0$ be arbitrarily chosen. Since f and g are differentiable at t, they are continuous at t. Because $g(t) \neq 0$, there exists a neighbourhood U of t such that

$$|g(s)| \geq m_1 > 0 \quad \text{for all} \quad s \in U$$

for some constant m_1. Let $[a, b] \subset \mathbb{T}$ be such that $\sigma(t), t \in [a, b]$. Set

$$M_1 = \sup_{t\in[a,b]} |f(t)|, \quad M_2 = \sup_{t\in[a,b]} |g(t)|$$

and

$$\varepsilon^* = \varepsilon \frac{m_1 |g(\sigma(t))g(t)|}{1 + 2M_1M_2 + |g^\Delta(t)|}$$

Since f and g are continuous at t, there exists a neighbourhood U_1 of t such that

$$|f(t)g(s) - f(s)g(t)| \leq \varepsilon^* \quad \text{and} \quad |g(s) - g(t)| \leq \varepsilon^*$$

for any $s \in U_1$. Since f is differentiable at t, there exists a neighbourhood U_2 of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon^* |\sigma(t) - s| \quad \text{for any} \quad s \in U_2.$$

Since g is differentiable at t, there exists a neighbourhood U_3 of t such that

$$\left|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon^* |\sigma(t) - s| \quad \text{for any} \quad s \in U_3.$$

Hence, for $s \in U_1 \cap U_2 \cap U_3 \cap U \cap [a, b]$, we get

$$\begin{aligned}
&\left|\tfrac{f(\sigma(t))}{g(\sigma(t))} - \tfrac{f(s)}{g(s)} - \tfrac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}(\sigma(t) - s)\right| \\
&= \frac{1}{|g(\sigma(t))g(s)g(t)|} \Big| f(\sigma(t))g(t)g(s) \\
&\quad - f(s)g(t)g(\sigma(t)) \\
&\quad - \left(f^\Delta(t)g(t) - f(t)g^\Delta(t)\right) g(s)(\sigma(t) - s) \Big| \\
&= \frac{1}{|g(\sigma(t))g(s)g(t)|} \Big| \\
&\quad \left(f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right) g(t)g(s) \\
&\quad + f(s)g(t)g(s) + f^\Delta(t)g(t)g(s)(\sigma(t) - s) \\
&\quad - \left(g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)\right) f(s)g(t) \\
&\quad - g(s)f(s)g(t) - f(s)g^\Delta(t)g(t)(\sigma(t) - s) \\
&\quad - f^\Delta(t)g(t)g(s)(\sigma(t) - s) \\
&\quad + f(t)g^\Delta(t)g(s)(\sigma(t) - s) \Big| \\
&= \frac{1}{|g(\sigma(t))g(s)g(t)|} \Big| \\
&\quad \left(f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right) g(t)g(s) \\
&\quad - \left(g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)\right) f(s)g(t) \\
&\quad + g^\Delta(t)(f(t)g(s) - f(s)g(t))(\sigma(t) - s) \Big| \\
&\leq \frac{1}{|g(\sigma(t))g(s)g(t)|} \Big(\Big|f(\sigma(t)) - f(s) \\
&\quad - f^\Delta(t)(\sigma(t) - s)\Big| |g(t)||g(s)| \\
&\quad + \left|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)\right| |f(s)||g(t)| \\
&\quad + \left|g^\Delta(t)\right| |f(t)g(s) - f(s)g(t)||\sigma(t) - s| \Big) \\
&\leq \frac{1}{|g(\sigma(t))g(s)g(t)|} \\
&\quad (\varepsilon^* M_1M_2|\sigma(t) - s| + \varepsilon^*|\sigma(t) - s|M_1M_2 \\
&\quad + \left|g^\Delta(t)\right| \varepsilon^*|\sigma(t) - s|\Big) \\
&\leq \varepsilon^* \frac{1}{|g(\sigma(t))g(s)g(t)|} \\
&\quad \left(1 + 2M_1M_2 + \left|g^\Delta(t)\right|\right) |\sigma(t) - s|
\end{aligned}$$

$$\leq \varepsilon^* \frac{1}{m_1|g(\sigma(t))g(t)|}\left(1+2M_1M_2+\left|g^{\Delta}(t)\right|\right)|\sigma(t)-s|$$
$$=\varepsilon|\sigma(t)-s|$$

which completes the proof.

Example 7.8

Let $\mathbb{T}=\left\{t_n=-\frac{1}{n}:n\in\mathbb{N}\right\}\cup\mathbb{N}_0$ and

$$f(t)=\frac{1+t^2}{1+7t},\quad t\in\mathbb{T}$$

We will find $f^{\Delta}(t), t\in\mathbb{T}$. Note that any points $t\in\mathbb{T}$ are right-scattered.

Solution

Let

$$g(t)=1+t^2$$
$$h(t)=1+7t,\quad t\in\mathbb{T}$$

Then

$$\begin{aligned}g^{\Delta}(t)&=\frac{g(\sigma(t))-g(t)}{\sigma(t)-t}\\&=\frac{1+(\sigma(t))^2-1-t^2}{\sigma(t)-t}\\&=\frac{(\sigma(t))^2-t^2}{\sigma(t)-t}\\&=\frac{(\sigma(t)-t)(\sigma(t)+t)}{\sigma(t)-t}\\&=\sigma(t)+t,\quad t\in\mathbb{T}\end{aligned}$$

and

$$\begin{aligned}h^{\Delta}(t)&=\frac{h(\sigma(t))-h(t)}{\sigma(t)-t}\\&=\frac{1+7\sigma(t)-1-7t}{\sigma(t)-t}\\&=\frac{7(\sigma(t)-t)}{\sigma(t)-t}\\&=7,\quad t\in\mathbb{T}\end{aligned}$$

and

$$h(\sigma(t))=1+7\sigma(t),\quad t\in\mathbb{T}$$

Now, applying the rule for delta derivative of a quotient of two functions, we arrive at

$$\begin{aligned}f^{\Delta}(t)&=\left(\frac{g}{h}\right)^{\Delta}(t)\\&=\frac{g^{\Delta}(t)h(t)-g(t)h^{\Delta}(t)}{h(t)h(\sigma(t))}\\&=\frac{(\sigma(t)+t)(1+7t)-(1+t^2)7}{(1+7t)(1+7\sigma(t))}\\&=\frac{\sigma(t)+7t\sigma(t)+t+7t^2-7-7t^2}{(1+7t)(1+7\sigma(t))}\\&=\frac{-7+t+\sigma(t)+7t\sigma(t)}{(1+7t)(1+7\sigma(t))},\quad t\in\mathbb{T}\end{aligned}$$

Then, we have the following cases.

1. Let $t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$. Then

$$\sigma(t) = -\frac{t}{t-1}$$

and

$$\begin{aligned} f^{\Delta}(t) &= \frac{-7+t-\frac{t}{t-1}-\frac{7t^2}{t-1}}{(1+7t)\left(1-\frac{7t}{t-1}\right)} \\ &= \frac{(-7+t)(t-1)-t-7t^2}{(1+7t)(t-1-7t)} \\ &= \frac{-7t+7+t^2-t-t-7t^2}{-(1+7t)(1+6t)} \\ &= \frac{-6t^2-9t+7}{-(1+7t)(1+6t)} \\ &= \frac{6t^2+9t-7}{(1+7t)(1+6t)} \end{aligned}$$

2. Let $t \in \mathbb{N}_0$. Then

$$\sigma(t) = t+1$$

and

$$\begin{aligned} f^{\Delta}(t) &= \frac{-7+t+t+1+7(t+1)t}{(1+7t)(1+7t+7)} \\ &= \frac{-6+2t+7t^2+7t}{(1+7t)(8+7t)} \\ &= \frac{-6+9t+7t^2}{(1+7t)(8+7t)} \end{aligned}$$

Consequently

$$f^{\Delta}(t) = \begin{cases} \frac{6t^2+9t-7}{(1+7t)(1+6t)} & \text{if} \quad t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \\ \frac{7t^2+9t-6}{(1+7t)(8+7t)} & \text{if} \quad t \in \mathbb{N}_0 \end{cases}$$

Example 7.9

Let $\mathbb{T} = \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0,1\}$ and

$$f(t) = \frac{1+7t}{4+5t}, \quad t \in \mathbb{T}$$

We will find $f^{\Delta}(t), t \in \mathbb{T}^{\kappa}$.

Solution

Here

$$\mathbb{T}^{\kappa} = \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0\}.$$

We have the following cases.

1. Let $t \in \mathbb{T}$ be right-scattered. Set

$$g(t) = 1 + 7t$$
$$h(t) = 4 + 5t, \quad t \in \mathbb{T}^{\kappa}$$

Then

$$\begin{aligned} g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\ &= \frac{1 + 7\sigma(t) - 1 - 7t}{\sigma(t) - t} \\ &= \frac{7(\sigma(t) - t)}{\sigma(t) - t} \\ &= 7, \quad t \in \mathbb{T}^{\kappa} \end{aligned}$$

and

$$\begin{aligned} h^{\Delta}(t) &= \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} \\ &= \frac{4 + 5\sigma(t) - 4 - 5t}{\sigma(t) - t} \\ &= \frac{5(\sigma(t) - t)}{\sigma(t) - t} \\ &= 5, \quad t \in \mathbb{T}^{\kappa}, \\ h(\sigma(t)) &= 4 + 5\sigma(t), \quad t \in \mathbb{T}^{\kappa}. \end{aligned}$$

Now, applying the rule for delta derivative of quotient of two functions, we arrive at

$$\begin{aligned} f^{\Delta}(t) &= \left(\frac{g}{h}\right)^{\Delta}(t) \\ &= \frac{g^{\Delta}(t)h(t) - g(t)h^{\Delta}(t)}{h(t)h(\sigma(t))} \\ &= \frac{7(4 + 5t) - 5(1 + 7t)}{(4 + 5t)(4 + 5\sigma(t))} \\ &= \frac{28 + 35t - 5 - 35t}{(4 + 5t)(4 + 5\sigma(t))} \\ &= \frac{23}{(4 + 5t)(4 + 5\sigma(t))} \end{aligned}$$

We have the following subcases.

a. Let $t \in \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}\right\}$. Then

$$\sigma(t) = \sqrt{t}$$

and

$$f^{\Delta}(t) = \frac{23}{(4 + 5t)(4 + 5\sqrt{t})}$$

b. Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = 1$$

and

$$\begin{aligned} f^{\Delta}\left(\frac{1}{2}\right) &= \frac{23}{\left(4 + \frac{5}{2}\right)(4 + 5)} \\ &= \frac{23}{\frac{13}{2} \cdot 9} \\ &= b\frac{46}{117} \end{aligned}$$

2. Let $t \in \mathbb{T}$ be right-dense. Then $t = 0$. Hence,

$$\begin{aligned} f^{\Delta}(0) &= \lim_{s \to 0} \frac{f(s) - f(0)}{s} \\ &= \lim_{s \to 0} \frac{\frac{1+7s}{4+5s} - \frac{1}{4}}{s} \\ &= \lim_{s \to 0} \frac{4 + 28s - 4 - 5s}{4s(4+5s)} \\ &= \lim_{s \to 0} \frac{23s}{4s(4+5s)} \\ &= \lim_{s \to 0} \frac{23}{4(4+5s)} \\ &= \frac{23}{16} \end{aligned}$$

Consequently

$$f^{\Delta}(t) = \begin{cases} \dfrac{23}{(4+5t)(4+5\sqrt{t})} & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ \dfrac{46}{117} & \text{if} \quad t = \dfrac{1}{2} \\ \dfrac{23}{16} & \text{if} \quad t = 0. \end{cases}$$

Example 7.10

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{1, 2\}$$

and

$$f(t) = \left(2 + t^2\right)\left(1 - t^2\right), \quad t \in \mathbb{T}.$$

We will find $f^{\Delta}(t), t \in \mathbb{T}^{\kappa}$.

Solution

Here

$$\mathbb{T}^{\kappa} = \mathbb{T} \backslash \left\{ \frac{5}{2} \right\}$$

We have the following cases.

1. Let $t \in \mathbb{T}^{\kappa}$ be right-scattered. Set

$$\begin{aligned} g(t) &= 2 - t^2 \\ h(t) &= 1 + t^2, \quad t \in \mathbb{T}^{\kappa} \end{aligned}$$

Then

$$\begin{aligned} g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\ &= \frac{2 + (\sigma(t))^2 - 2 - t^2}{\sigma(t) - t} \\ &= \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\ &= \frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} \\ &= \sigma(t) + t, \quad t \in \mathbb{T}^{\kappa} \end{aligned}$$

and

$$\begin{aligned} h^{\Delta}(t) &= \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} \\ &= \frac{1 - (\sigma(t))^2 - 1 + t^2}{\sigma(t) - t} \\ &= -\frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\ &= -\frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} \\ &= -\sigma(t) - t, \quad t \in \mathbb{T}^{\kappa} \end{aligned}$$

and

$$h(\sigma(t)) = 1 - (\sigma(t))^2, \quad t \in \mathbb{T}^{\kappa}$$

Now, applying the rule for delta derivative of a multiplication of two functions, we find

$$\begin{aligned} f^{\Delta}(t) &= (gh)^{\Delta}(t) \\ &= g^{\Delta}(t)h(\sigma(t)) + g(t)h^{\Delta}(t) \end{aligned}$$

$$\begin{aligned} &= (\sigma(t+t)\,(1 - (\sigma(t))^2) + (2 + t^2)\,(-\sigma(t) - t) \\ &= \sigma(t) - (\sigma(t))^3 + t - t(\sigma(t))^2 - 2\sigma(t) - 2t - t^2\sigma(t) \\ &= -(\sigma(t))^3 - \sigma(t) - t - t^3 - t(\sigma(t))^2 - t^2\sigma(t), \quad t \in \mathbb{T}^{\kappa} \end{aligned}$$

Now, we will consider the following subcases.

a. Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$$

and

$$\begin{aligned} f^{\Delta}\left(\frac{1}{2}\right) &= -\left(\sigma\left(\frac{1}{2}\right)\right)^3 - \sigma\left(\frac{1}{2}\right) - \frac{1}{2} - \left(\frac{1}{2}\right)^3 - \frac{1}{2}\left(\sigma\left(\frac{1}{2}\right)\right)^2 - \left(\frac{1}{2}\right)\sigma\left(\frac{1}{2}\right) \\ &= -\left(\frac{3}{4}\right)^3 - \frac{3}{4} - \frac{1}{2} - \frac{1}{8} - \frac{1}{2} \cdot \frac{3}{4} - \frac{1}{4} \cdot \frac{3}{4} \\ &= -\frac{27}{64} - \frac{3}{4} - \frac{1}{2} - \frac{1}{8} - \frac{3}{8} - \frac{3}{16} \\ &= -\frac{27}{64} - \frac{3}{4} - \frac{1}{2} - \frac{1}{2} - \frac{3}{16} \\ &= -\frac{27}{64} - \frac{3}{4} - 1 - \frac{3}{16} \\ &= \frac{-27 - 48 - 64 - 12}{64} \\ &= -\frac{151}{64} \end{aligned}$$

b. Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$$

and

$$f^{\Delta}\left(\frac{3}{2}\right) = \left(\sigma\left(\frac{3}{2}\right)\right)^3 - \sigma\left(\frac{3}{2}\right) - \frac{3}{2} - \left(\frac{3}{2}\right)^3 - \frac{3}{2}\left(\sigma\left(\frac{3}{2}\right)\right)^2 - \left(\frac{3}{2}\right)^2\sigma\left(\frac{3}{2}\right)$$

$$
\begin{aligned}
&= -\left(\frac{7}{4}\right)^3 - \frac{7}{4} - \frac{3}{2} - \frac{27}{8} - \frac{3}{2}\cdot\frac{7}{4} - \frac{9}{4}\cdot\frac{7}{4} \\
&= -\frac{343}{64} - \frac{7}{4} - \frac{3}{2} - \frac{27}{8} - \frac{21}{8} - \frac{63}{16} \\
&= -\frac{343}{64} - \frac{7}{4} - \frac{3}{2} - \frac{49}{8} - \frac{63}{16} \\
&= \frac{-343 - 112 - 96 - 392 - 252}{64} \\
&= -\frac{1195}{64}
\end{aligned}
$$

c. Let $t \in U \backslash \left\{\frac{1}{2}\right\}$. Then

$$\sigma(t) = 2t$$

and

$$
\begin{aligned}
f^{\Delta}(t) &= -(2t)^3 - 2t - t - t^3 - t(2t)^2 - t^2(2t) \\
&= -8t^3 - 3t - t^3 - 4t^3 - 2t^3 \\
&= -15t^3 - 3t
\end{aligned}
$$

d. Let $t \in (1-U) \backslash \left\{\frac{1}{2}\right\}$. Then

$$\sigma(t) = \frac{1+t}{2}$$

and

$$
\begin{aligned}
f^{\Delta}(t) &= -\left(\frac{1+t}{2}\right)^3 - \frac{1+t}{2} - t - t^3 - t\left(\frac{1+t}{2}\right)^2 - t^2\frac{1+t}{2} \\
&= -\frac{(1+t)^3}{8} - \frac{1+t}{2} - t - t^3 - t\frac{(1+t)^2}{4} - t^2\frac{1+t}{2} \\
&= -\frac{(1+t)^3 + 4(1+t) + 8t + 8t^3 + 2t(1+t)^2 + 4t^2(1+t)}{8} \\
&= -\frac{t^3 + 3t^2 + 3t + 1 + 4t + 4 + 8t + 8t^3 + 2t\left(t^2 + 2t + 1\right) + 4t^2 + 4t^3}{8} \\
&= -\frac{13t^3 + 7t^2 + 15t + 5 + 2t^3 + 4t^2 + 2t}{8} \\
&= -\frac{15t^3 + 11t^2 + 17t + 5}{8}.
\end{aligned}
$$

e. Let $t \in (1+U) \backslash \left\{\frac{3}{2}\right\}$. Then

$$\sigma(t) = 2t - 1$$

and

$$
\begin{aligned}
f^{\Delta}(t) &= -(2t-1)^3 - (2t-1) - t - t^3 - t(2t-1)^2 - t^2(2t-1) \\
&= -8t^3 + 12t^2 - 6t + 1 - 2t + 1 - t - t^3 - t\left(4t^2 - 4t + 1\right) - 2t^3 + t^2 \\
&= -11t^3 + 13t^2 - 9t + 2 - 4t^3 + 4t^2 - t \\
&= -15t^3 + 17t^2 - 10t + 2
\end{aligned}
$$

f. Let $t \in (2-U) \backslash \left\{\frac{3}{2}\right\}$. Then

$$\sigma(t) = \frac{t+2}{2}$$

and

$$
\begin{aligned}
f^{\Delta}(t) &= -\left(\frac{t+2}{2}\right)^3 - \frac{t+2}{2} - t - t^3 - t\left(\frac{t+2}{2}\right)^2 - t^2\left(\frac{t+2}{2}\right) \\
&= -\frac{(t+2)^3}{8} - \frac{t+2}{2} - t - t^3 - t\frac{(t+2)^2}{4} - t^2\frac{t+2}{2} \\
&= -\frac{(t+2)^3 + 4(t+2) + 8t + 8t^3 + 2t(t+2)^2 + 4t^2(t+2)}{8} \\
&= -\frac{t^3 + 6t^2 + 12t + 8 + 4t + 8 + 8t + 8t^3 + 2t\left(t^2 + 4t + 4\right) + 4t^3 + 8t^2}{8} \\
&= -\frac{13t^3 + 14t^2 + 24t + 8 + 2t^3 + 8t^2 + 8t}{8} \\
&= -\frac{15t^3 + 22t^2 + 32t + 8}{8}
\end{aligned}
$$

g. Let $t \in (2+U)\backslash\left\{\frac{5}{2}\right\}$. Then

$$\sigma(t) = 2(t-1)$$

and

$$
\begin{aligned}
f^{\Delta}(t) &= -(2(t-1))^3 - 2(t-1) - t - t^3 - t(2(t-1))^2 - t^2(2(t-1)) \\
&= -8(t-1)^3 - 2(t-1) - t - t^3 - 4t(t-1)^2 - 2t^2(t-1) \\
&= -8t^3 + 24t^2 - 24t + 8 - 2t + 2 - t - t^3 - 4t\left(t^2 - 2t + 1\right) - 2t^3 + 2t^2 \\
&= -11t^3 + 26t^2 - 27t + 10 - 4t^3 + 8t^2 - 4t \\
&= -15t^3 + 34t^2 - 31t + 10
\end{aligned}
$$

2. Let $t \in \mathbb{T}^{\kappa}$ be right-dense. Then $t = 0$ or $t = 1$, or $t = 1$.

a. Let $t = 0$. Then

$$
\begin{aligned}
f^{\Delta}(0) &= \lim_{s \to 0} \frac{f(s) - f(0)}{s} \\
&= \lim_{s \to 0} \frac{\left(2 + s^2\right)\left(1 - s^2\right) - 2}{s} \\
&= \lim_{s \to 0} \frac{2 - 2s^2 + s^2 - s^4 - 2}{s} \\
&= -\lim_{s \to 0} \frac{2\left(s + s^3\right)}{s} \\
&= -\lim_{s \to 0} \left(s + s^3\right) \\
&= 0
\end{aligned}
$$

b. Let $t = 1$. Then

$$
\begin{aligned}
f^{\Delta}(1) &= \lim_{s \to 1} \frac{f(s) - f(1)}{s - 1} \\
&= \lim_{s \to 1} \frac{\left(2 + s^2\right)\left(1 - s^2\right) - (2+1)(1-1)}{s - 1} \\
&= -\lim_{s \to 1} \frac{\left(2 + s^2\right)(s-1)(s+1)}{s - 1} \\
&= -\lim_{s \to 1} \left(2 + s^2\right)(s+1) \\
&= -(2+1)(1+1) \\
&= -6
\end{aligned}
$$

c. Let $t = 2$. Then

$$
\begin{aligned}
f^{\Delta}(2) &= \lim_{s\to 2}\frac{f(s)-f(2)}{s-2}\\
&= \lim_{s\to 2}\frac{\left(2+s^2\right)\left(1-s^2\right)-(2+4)(1-4)}{s-2}\\
&= \lim_{s\to 2}\frac{2-2s^2+-s^2-s^4+18}{s-2}\\
&= \lim_{s\to 2}\frac{s^4+s^2-20}{s-2}\\
&= -\lim_{s\to 2}\frac{(s-2)\left(s^3+2s^2+5s+10\right)}{s-2}\\
&= -\lim_{s\to 2}\left(s^3+2s^2+5s+10\right)\\
&= -\left(2^3+2\cdot 2^2+5\cdot 2+10\right)\\
&= -(8+8+10+10)\\
&= -36
\end{aligned}
$$

Consequently

$$
f^{\Delta}(t)=\begin{cases}
-\frac{151}{64} \text{ if } t=\frac{1}{2}\\
-\frac{1195}{64} \text{ if } t=\frac{3}{2}\\
-15t^3-3t \quad \text{if} \quad t\in U\backslash\left\{\frac{1}{2}\right\}\\
-\frac{15t^3+11t^2+17t+5}{8} \quad \text{if} \quad t\in(1-U)\backslash\left\{\frac{1}{2}\right\}\\
-15t^3+17t^2-10t+2 \quad \text{if} \quad t\in(1+U)\backslash\left\{\frac{3}{2}\right\}\\
-\frac{15t^3+22t^2+32t+8}{8} \quad \text{if} \quad t\in(2-U)\backslash\left\{\frac{3}{2}\right\}\\
-15t^3+34t^2-31t+10 \quad \text{if} \quad t\in(2+U)\backslash\left\{\frac{5}{2}\right\}\\
0 \quad \text{if} \quad t=0\\
-6 \text{ if} \quad t=1\\
-36 \text{ if} \quad t=2.
\end{cases}
$$

Example 0.110.

Example 7.11

Let

$$f(t)=t^n, \quad t\in\mathbb{T}, \quad n\in\mathbb{N}.$$

We will prove that

$$f^{\Delta}(t)=(\sigma(t))^{n-1}+t(\sigma(t))^{n-2}+\cdots+t^{n-2}\sigma(t)+t^{n-1}$$

if $t\in\mathbb{T}$ is right-scattered.

Solution

Really, let $t\in\mathbb{T}$ be right-scattered. Then

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t))-f(t)}{\sigma(t)-t}\\
&= \frac{(\sigma(t))^n-t^n}{\sigma(t)-t}\\
&= \frac{(\sigma(t)-t)\left((\sigma(t))^{n-1}+t(\sigma(t))^{n-2}+\cdots+t^{n-2}\sigma(t)+t^{n-1}\right)}{\sigma(t)-t}\\
&= (\sigma(t))^{n-1}+t(\sigma(t))^{n-2}+\cdots+t^{n-2}\sigma(t)+t^{n-1}
\end{aligned}
$$

Example 7.12

Let

$$\mathbb{T} = \left\{ \frac{1}{2n+1} : n \in \mathbb{N}_0 \right\} \cup \{0\}$$

$f(t) = \sigma(t), t \in \mathbb{T}$. We will find $f^{\Delta}(t), t \in \mathbb{T}$. For

$$t \in \mathbb{T}, \quad t = \frac{1}{2n+1}, \quad n = \frac{1-t}{2t}, \quad n \geq 1$$

Solution

We have

$$\begin{aligned}
\sigma(t) &= \inf \left\{ \frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > \frac{1}{2n+1}, l \in \mathbb{N}_0 \right\} \\
&= \frac{1}{2n-1} \\
&= \frac{1}{2\dfrac{1-t}{2t} - 1} \\
&= \frac{t}{1-2t} > t
\end{aligned}$$

i.e., any point $t = \dfrac{1}{2n+1}, n \geq 1$, is right-scattered. At these points

$$\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\
&= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\
&= 2\frac{(\sigma(t))^2}{(1-2\sigma(t))(\sigma(t)-t)} \\
&= 2\frac{\left(\dfrac{t}{1-2t}\right)^2}{\left(1 - \dfrac{2t}{1-2t}\right)\left(\dfrac{t}{1-2t} - t\right)} \\
&= 2\frac{\dfrac{t^2}{(1-2t)^2}}{\dfrac{1-4t}{1-2t}\dfrac{2t^2}{1-2t}} \\
&= 2\frac{t^2}{2t^2(1-4t)} \\
&= \frac{1}{1-4t}
\end{aligned}$$

Let $n = 0$, i.e., $t = 1$. Then

$$\begin{aligned}
\sigma(1) &= \inf \left\{ \frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > 1, l \in \mathbb{N}_0 \right\} \\
&= \inf \emptyset \\
&= \sup \mathbb{T} \\
&= 1
\end{aligned}$$

i.e., $t = 1$ is a right-dense point. Also,

$$\begin{aligned}\lim_{h\to 0}\frac{f(1+h)-f(h)}{h} &= \lim_{h\to 0}\frac{\sigma(1+h)-\sigma(h)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1+h}{1-2(1+h)}-\frac{h}{1-2h}}{h}\\ &= -\lim_{h\to 0}\frac{\frac{1+h}{1+2h}+\frac{h}{1-2h}}{h}\\ &= -\lim_{h\to 0}\frac{(1+h)(1-2h)+h(1+2h)}{(1+2h)(1-2h)h}\\ &= -\lim_{h\to 0}\frac{1}{(1+2h)(1-2h)h}\\ &= -\infty\end{aligned}$$

Therefore $\sigma^{\Delta}(1)$ does not exist.

Let now $t=0$. Then

$$\begin{aligned}\sigma(0) &= \inf\left\{\frac{1}{2l+1},0:\frac{1}{2l+1},0>0,l\in\mathbb{N}_0\right\}\\ &= 0\end{aligned}$$

Consequently, $t=0$ is right-dense. Also,

$$\begin{aligned}\lim_{h\to 0}\frac{\sigma(h)-\sigma(0)}{h} &= \lim_{h\to 0}\frac{\frac{h}{1-2h}-0}{h}\\ &= \lim_{h\to 0}\frac{1}{1-2h}\\ &= 1\end{aligned}$$

Therefore $\sigma^{\Delta}(0)=1$.

0.112.

Example 7.13

Let $f,g,h:\mathbb{T}\to\mathbb{R}$ be differentiable at $t\in\mathbb{T}^{\kappa}$.

Solution

Then

$$\begin{aligned}(fgh)^{\Delta}(t) &= ((fg)h)^{\Delta}(t)\\ &= (fg)^{\Delta}(t)h(t)+(fg)(\sigma(t))h^{\Delta}(t)\\ &= \left(f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)\right)h(t)+f^{\sigma}(t)g^{\sigma}(t)h^{\Delta}(t)\\ &= f^{\Delta}(t)g(t)h(t)+f^{\sigma}(t)g^{\Delta}(t)h(t)+f^{\sigma}(t)g^{\sigma}(t)h^{\Delta}(t)\end{aligned}$$

Example 7.14

Let $f:\mathbb{T}\to\mathbb{R}$ be differentiable at $t\in\mathbb{T}^{\kappa}$. Then

$$\begin{aligned}\left(f^2\right)^{\Delta}(t) &= (ff)^{\Delta}(t)\\ &= f^{\Delta}(t)f(t)+f(\sigma(t))f^{\Delta}(t)\\ &= f^{\Delta}(t)\left(f^{\sigma}(t)+f(t)\right)\end{aligned}$$

Also,

$$
\begin{aligned}
\left(f^3\right)^{\Delta}(t) &= \left(ff^2\right)^{\Delta}(t) \\
&= f^{\Delta}(t)f^2(t) + f(\sigma(t))\left(f^2\right)^{\Delta}(t) \\
&= f^{\Delta}(t)f^2(t) + f^{\sigma}(t)f^{\Delta}(t)\left(f^{\sigma}(t) + f(t)\right) \\
&= f^{\Delta}(t)\left(f^2(t) + f(t)f^{\sigma}(t) + (f^{\sigma})^2(t)\right)
\end{aligned}
$$

We assume that

$$
\left(f^n\right)^{\Delta}(t) = f^{\Delta}(t)\sum_{k=0}^{n-1} f^k(t)(f^{\sigma})^{n-1-k}(t)
$$

for some $n \in \mathbb{N}$. We will prove that

$$
\left(f^{n+1}\right)^{\Delta}(t) = f^{\Delta}(t)\sum_{k=0}^{n} f^k(t)(f^{\sigma})^{n-k}(t)
$$

Solution

Indeed,

$$
\left(f^{n+1}\right)^{\Delta}(t) = \left(ff^n\right)^{\Delta}(t)
$$

$$
\begin{aligned}
&= f^{\Delta}(t)f^n(t) + f^{\sigma}(t)\left(f^n\right)^{\Delta}(t) \\
&= f^{\Delta}(t)f^n(t) + f^{\Delta}(t)f^{\sigma}(t)\left(f^{n-1}(t) + f^{n-2}(t)f^{\sigma}(t)\right. \\
&\quad \left. + \cdots + f(t)(f^{\sigma})^{n-2}(t) + (f^{\sigma})^{n-1}(t)\right) f^{\sigma}(t) \\
&= f^{\Delta}(t)\left(f^n(t) + f^{n-1}(t)f^{\sigma}(t) + f^{n-2}(t)(f^{\sigma})^2(t) + \cdots + (f^{\sigma})^n(t)\right) \\
&= f^{\Delta}(t)\sum_{k=0}^{n} f^k(t)(f^{\sigma})^{n-k}(t)
\end{aligned}
$$

Example 0.114.

Example 7.15

Now we consider $f(t) = (t-a)^m$ for $a \in \mathbb{R}$ and $m \in \mathbb{N}$. We set

$$
h(t) = (t-a).
$$

Then

$$
h^{\Delta}(t) = 1
$$

Solution

We get

$$
\begin{aligned}
\left(h^m\right)^{\Delta}(t) &= h^{\Delta}(t)\sum_{k=0}^{m-1} h^k(t)(h^{\sigma})^{m-1-k}(t) \\
&= \sum_{k=0}^{m-1} (t-a)^k(\sigma(t)-a)^{m-1-k}
\end{aligned}
$$

Let now $g(t) = \dfrac{1}{f(t)}$. Then

$$
g^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(\sigma(t))f(t)},
$$

whereupon

$$\left(\frac{1}{h^m}\right)^{\Delta}(t) = -\frac{1}{(\sigma(t)-a)^m(t-a)^m}\sum_{k=0}^{m-1}(t-a)^k(\sigma(t)-a)^{m-1-k}$$
$$= -\sum_{k=0}^{m-1}\frac{1}{(t-a)^{m-k}}\frac{1}{(\sigma(t)-a)^{k+1}}$$

Exercise 7.6

Let $\mathbb{T} = k\mathbb{Z}, \quad k > 0$. Prove that

$$t = \frac{1}{2}f_2^{\Delta}(t) - \frac{k}{2},$$

Answer

$$t^2 = \frac{1}{3}f_3^{\Delta}(t) - \frac{k}{2}f_2^{\Delta}(t) + \frac{1}{6}k^2$$
$$t^3 = \frac{1}{4}f_4^{\Delta}(t) - \frac{1}{2}kf_3^{\Delta}(t) + \frac{1}{4}k^2f_2^{\Delta}(t)$$
$$t^4 = \frac{1}{5}f_5^{\Delta}(t) - \frac{1}{2}kf_4^{\Delta}(t) + \frac{1}{3}k^2f_3^{\Delta}(t) - \frac{1}{30}k^4$$
$$t^5 = \frac{1}{6}f_6^{\Delta}(t) - \frac{1}{2}kf_5^{\Delta}(t) + \frac{5}{12}k^2f_4^{\Delta}(t) - \frac{1}{12}k^4f_2^{\Delta}(t)$$

where $f_i(t) = t^i, i = 1, \ldots, 6, t \in \mathbb{T}$.

Exercise 7.7

Let $t \in \mathbb{T}$ be right-scattered. Find $f^{\Delta}(t)$, where

$$f(t) = 2\sin t + t^2 - 3t^3, \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 7.8

Let $t \in 2^{\mathbb{N}_0}$. Find $f^{\Delta}(t), t \in \mathbb{T}$, where

$$f(t) = \frac{t^3 + t^2 - 2t}{2t^2 + 3t + 1}, \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 7.9

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Find $f^{\Delta}(t)$, where

$$f(t) = \frac{1}{4}t^4 + \frac{1}{3}t^3 - t^2 + 1, \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 7.10

Let $\mathbb{T} = \mathbb{Z}$. Find $f^{\Delta}(t)$, where

$$f(t) = \frac{1}{3}t^3 - \frac{5}{2}t^2 - 4t, \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 7.11

Let $\mathbb{T} = \{\sqrt[4]{n} : n \in \mathbb{N}\}$. Find $f^{\Delta}(t)$, where

$$f(t) = \frac{1}{3}t^3 - 2t^2 - 3t + 2, \quad t \in \mathbb{T}$$

Answer

Exercise 7.12

Let $\mathbb{T} = [1,2] \cup [3,4] \cup [7,8] \cup 9^{\mathbb{N}}$. Find $f^{\Delta}(t)$, where

$$f(t) = \frac{2-t}{3+7t}, \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

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8: Higher Order Delta Differentiation

Definition: Second derivative and Higher order derivatives

Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in(\mathbb{T}^{\kappa})^{\kappa}=\mathbb{T}^{\kappa^2}$. We define the second derivative of f at t, provided it exists, by

$$f^{\Delta^2}=\left(f^{\Delta}\right)^{\Delta}:\mathbb{T}^{\kappa^2}\to\mathbb{R}$$

Similarly, we define higher order derivatives $f^{\Delta^n}:\mathbb{T}^{\kappa^n}\to\mathbb{R}$.

Example 8.1

Let $\mathbb{T}=\left\{-\frac{1}{n}:n\in\mathbb{N}\right\}\cup\mathbb{N}_0$ and

$$f(t)=\frac{1+t^2}{1+7t},\quad t\in\mathbb{T}$$

We will find $f^{\Delta^2}(t), t\in\mathbb{T}$.

Solution

We have the following cases.

1. Let $t\in\left\{-\frac{1}{n}:n\in\mathbb{N}\right\}$. Then

$$\sigma(t)=-\frac{t}{t-1}$$

By Example 7.8, we have

$$f^{\Delta}(t)=\frac{6t^2+9t-7}{(1+7t)(1+6t)}$$

Then

$$\begin{aligned}
f^{\Delta}(\sigma(t)) &= \frac{6(\sigma(t))^2+9\sigma(t)-7}{(1+7\sigma(t))(1+6\sigma(t))}\\
&= \frac{6\frac{t^2}{(t-1)^2}-\frac{9t}{t-1}-7}{\left(1-\frac{7t}{t-1}\right)\left(1-\frac{6t}{t-1}\right)}\\
&= \frac{6t^2-9t(t-1)-7(t-1)^2}{(y-1-7t)(t-1-6t)}\\
&= \frac{6t^2-9t^2+9t-7t^2+14t-7}{(1+6t)(1+5t)}\\
&= \frac{-10t^2+23t-7}{(1+6t)(1+5t)}
\end{aligned}$$

and

$$\begin{aligned}
f^{\Delta^2}(t) &= \frac{gf^{\Delta}(\sigma(t))-f^{\Delta}(t)}{\sigma(t)-t}\\
&= \frac{1}{-\frac{t}{t-1}-t}\left(\frac{-10t^2+23t-7}{(1+6t)(1+5t)}-\frac{6t^2+9t-7}{(1+6t)(1+7t)}\right)\\
&= -\frac{(t-1)\left(\left(-10t^2+23t-7\right)(1+7t)-\left(6t^2+9t-7\right)(1+5t)\right)}{t^2(1+5t)(1+6t)(1+7t)}
\end{aligned}$$

$$
\begin{aligned}
&= \\
&-\frac{(t-1)\left(-10t^2-70t^3+23t+161t^2-7-49t-6t^2-30t^3-9t-45t^2+7+35t\right)}{t^2(1+5t)(1+6t)(1+7t)} \\
&= -\frac{(t-1)\left(-100t^3+100t^2\right)}{t^2(1+5t)(1+6t)(1+7t)} \\
&= \frac{100(t-1)^2}{(1+5t)(1+6t)(1+7t)}
\end{aligned}
$$

2. Let $t \in \mathbb{N}_0$. Then

$$\sigma(t) = t+1$$

By Example 7.8, we have

$$f^{\Delta}(t) = \frac{7t^2+9t-6}{(1+7t)(8+7t)}$$

Hence,

$$
\begin{aligned}
f^{\Delta}(\sigma(t) &= \frac{7(\sigma(t))^2+9\sigma(t)-6}{(1+7\sigma(t))(8+7\sigma(t))} \\
&= \frac{7(t+1)^2+9(t+1)-6}{(1+7t+7)(8+7t+t)} \\
&= \frac{7t^2+14t+7+9t+9-6}{(8+7t)(15+7t)} \\
&= \frac{7t^2+23t+10}{(8+7t)(15+7t)}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t))-f^{\Delta}(t)}{\sigma(t)-t} \\
&= \frac{7t^2+23t+10}{(8+7t)(15+7t)} - \frac{7t^2+9t-6}{(1+7t)(8+7t)} \\
&= \frac{\left(7t^2+23t+10\right)(1+7t)-\left(7t^2+9t-6\right)(15+7t)}{(1+7t)(8+7t)(15+7t)} \\
&= \frac{7t^2+23t+10+49t^3+161t^2+70t-105t^2-49t^3-135t-63t^2+90+42t}{(1+7t)(8+7t)(15+7t)} \\
&= \frac{10(t+10)}{(1+7t)(8+7t)(15+7t)}
\end{aligned}
$$

Consequently

$$
f^{\Delta^2}(t) = \begin{cases} \dfrac{100(t-1)^2}{(1+5t)(1+6t)(1+7t)} & \text{if} \quad t \in \left\{-\dfrac{1}{n} : n \in \mathbb{N}\right\} \\ \dfrac{10(t+10)}{(1+7t)(8+7t)(15+7t)} & \text{if} \quad t \in \mathbb{N}_0 \end{cases}
$$

Example 8.2

Let $\mathbb{T} = \left\{\left(\frac{1}{2}\right)^{2^n}; n \in \mathbb{N}_0\right\} \cup \{0,1\}$ and

$$f(t) = \frac{1+7t}{4+5t}, \quad t \in \mathbb{T}$$

We will find $f^{\Delta^2}(t), t \in \mathbb{T}^{\kappa^2}$.

Solution

Here

$$\mathbb{T}^{\kappa^2} = \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}\right\} \cup \{0\}$$

We have the following cases.

1. Let $t \in \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}\right\}$. Then

$$\sigma(t) = \sqrt{t}.$$

By Example 0.109, we have

$$f^{\Delta}(t) = \frac{23}{(4+5t)(4+5\sqrt{t})}$$

Then

$$\begin{aligned} f(\sigma(t)) &= \frac{23}{(4+5\sigma(t))(4+5\sqrt{\sigma(t)})} \\ &= \frac{23}{(4+5\sqrt{t})(4+5\sqrt[4]{t})} \end{aligned}$$

and

$$\begin{aligned} f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\ &= \frac{1}{\sqrt{t} - t}\left(\frac{23}{(4+5\sqrt{t})(4+5\sqrt[4]{t})} - \frac{23}{(4+5t)(4+5\sqrt{t})}\right) \\ &= \frac{23(4+5t-4-5\sqrt[4]{t})}{\sqrt{t}(1-\sqrt{t})(4+5t)(4+5\sqrt{t})(4+5\sqrt[4]{t})} \\ &= \frac{115(t-\sqrt[4]{t})}{\sqrt{t}(1-\sqrt{t})(4+5t)(4+5\sqrt{t})(4+5\sqrt[4]{t})} \end{aligned}$$

2. Let $t = 0$. Then

$$\sigma(0) = 0.$$

By Example 0.109, we have

$$f^{\Delta}(0) = \frac{23}{16}.$$

Hence,

$$
\begin{aligned}
f^{\Delta^2}(0) &= \lim_{s\to 0}\frac{f^{\Delta}(s)-f^{\Delta}(0)}{s}\\
&= \lim_{s\to 0}\frac{\dfrac{23}{(4+5s)(4+5\sqrt{s})}-\dfrac{23}{16}}{s}\\
&= \lim_{s\to 0}\frac{23(16-(4+5s)(4+5\sqrt{s}))}{s(4+5s)(4+5\sqrt{s})}\\
&= \lim_{s\to 0}\frac{23\left(16-16-20\sqrt{s}-20s-25s^{\frac{3}{2}}\right)}{s(4+5s)(4+5\sqrt{s})}\\
&= -\lim_{s\to 0}\frac{115\left(5\sqrt{s}+5s+5s^{\frac{3}{2}}\right)}{s(4+5s)(4+5\sqrt{s})}\\
&= -\lim_{s\to 0}\frac{115(5+5\sqrt{s}+5s)}{\sqrt{s}(4+5s)(4+5\sqrt{s})}\\
&= -\infty
\end{aligned}
$$

Thus, $f^{\Delta^2}(0)$ does not exist.

Consequently

$$
f^{\Delta^2}(t)=\begin{cases}\dfrac{115(t-\sqrt[4]{t})}{\sqrt{t}(1-\sqrt{t})(4+5t)(4+5\sqrt{t})(4+5\sqrt[4]{t})} & \text{if} \quad t\in\left\{\left(\frac{1}{2}\right)^{2^n}:n\in\mathbb{N}\right\}\\ \text{does not exist if} \quad t=0\end{cases}
$$

Example 8.3

Let $U=\left\{\frac{1}{2^n}:n\in\mathbb{N}\right\}$ and

$$
\mathbb{T}=U\cup(1-U)\cup(1+U)\cup(2-U)\cup(2+U)\cup\{0,1,2\},
$$

and

$$
f(t)=\left(2+t^2\right)\left(1-t^2\right), \quad t\in\mathbb{T}.
$$

We will find $f^{\Delta^2}(t), t\in\mathbb{T}^{\kappa^2}$.

Solution

Here $\mathbb{T}^{\kappa^2}=\mathbb{T}\backslash\left\{\frac{5}{2},\frac{9}{4}\right\}$. We have the following cases.

1. Let $t\in U\backslash\left\{\frac{1}{2}\right\}$. Then

$$
\sigma(t)=2t
$$

By Example 7.10, we have

$$
f^{\Delta}(t)=-15t^3-3t
$$

Then

$$
\begin{aligned}
f^{\Delta}(\sigma(t)) &= -15(\sigma(t))^3-3\sigma(t)\\
&= -15\left(8t^3\right)-3(2t)\\
&= -120t^3-6t
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{-120t^3 - 6t + 15t^3 + 2t}{2t - t} \\
&= \frac{-105t^3 - 3t}{t} \\
&= -105t^2 - 3
\end{aligned}
$$

2. Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$$

By Example 7.10, we have

$$f^{\Delta}\left(\frac{1}{2}\right) = -\frac{151}{64}$$

and

$$
\begin{aligned}
f^{\Delta}\left(\sigma\left(\frac{1}{2}\right)\right) &= f^{\Delta}\left(\frac{3}{4}\right) \\
&= -\frac{15\left(\frac{3}{4}\right)^3 + 11\left(\frac{3}{4}\right)^2 + 17\left(\frac{3}{4}\right) + 5}{8} \\
&= -\frac{15\left(\frac{27}{64}\right) + 11\left(\frac{9}{16}\right) + \frac{51}{4} + 5}{8} \\
&= -\frac{15 \cdot 27 + 11 \cdot 36 + 51 \cdot 16 + 5 \cdot 64}{512} \\
&= -\frac{405 + 396 + 816 + 320}{512} \\
&= -\frac{1937}{512}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f^{\Delta^2}\left(\frac{1}{2}\right) &= \frac{f^{\Delta}\left(\sigma\left(\frac{1}{2}\right)\right) - f^{\Delta}\left(\frac{1}{2}\right)}{\sigma\left(\frac{3}{4}\right) - \frac{1}{2}} \\
&= \frac{f^{\Delta}\left(\frac{3}{4}\right) - f^{\Delta}\left(\frac{1}{2}\right)}{\frac{3}{4} - \frac{1}{2}} \\
&= \frac{-\frac{1937}{512} + \frac{151}{64}}{\frac{1}{4}} \\
&= \frac{-1937 + 1208}{128} \\
&= -\frac{729}{128}
\end{aligned}
$$

3. Let $t \in (1 - U) \backslash \left\{\frac{1}{2}\right\}$. Then

$$\sigma(t) = \frac{1 + t}{2}.$$

By Example 7.10, we have

$$f^{\Delta}(t) = -\frac{15t^3 + 11t^2 + 17t + 5}{8}$$

Then

$$
\begin{aligned}
f^{\Delta}(\sigma(t)) &= -\frac{15\left(\frac{1+t}{2}\right)^3 + 11\left(\frac{1+t}{2}\right)^2 + 17\left(\frac{1+t}{2}\right) + 5}{8} \\
&= -\frac{15(1+t)^3 + 22(1+t)^2 + 68(1+t) + 40}{64} \\
&= -\frac{15 + 45t^2 + 45t + 15t^3 + 22 + 44t + 22t^2 + 68 + 68t + 40}{64} \\
&= -\frac{15t^3 + 67t^2 + 157t + 77}{64}
\end{aligned}
$$

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{1}{\frac{1+t}{2} - t}\left(-\frac{15t^3 + 67t^2 + 157t + 77}{64} + \frac{15t^3 + 11t^2 + 17t + 5}{8}\right) \\
&= \frac{2}{1-t}\left(\frac{-15t^3 - 67t^2 - 157t - 77 + 120t^3 + 88t^2 + 136t + 40}{64}\right) \\
&= \frac{105t^3 + 21t^2 - 21t - 37}{32(1-t)}
\end{aligned}
$$

4. Let $t \in (1+U)\backslash\left\{\frac{3}{2}\right\}$. Then

$$\sigma(t) = 2t - 1$$

By Example 7.10, we have

$$f^{\Delta}(t) = -15t^3 + 17t^2 - 10y + 2$$

Then

$$
\begin{aligned}
f^{\Delta}(\sigma(t)) &= -15(\sigma(t))^3 + 17\left(\sigma(t)0^2 - 10\sigma(t) + 2\right. \\
&= -15(2t-1)^3 + 17(2t-1)^2 - 10(2t-1) + 2 \\
&= -15\left(8t^3 - 12t^2 + 6t - 1\right) + 17\left(4t^2 - 4t + 1\right) - 20t + 10 + 2 \\
&= -120t^3 + 180t^2 - 90t + 15 + 68t^2 - 68t + 17 - 20t + 12 \\
&= -120t^3 + 248t^2 - 178t + 44
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{-120t^3 + 248t^2 - 178t + 44 + 15t^3 - 17t^2 + 10t - 2}{2t - 1 - t} \\
&= \frac{-105t^3 + 231t^2 - 168t + 42}{t - 1}
\end{aligned}
$$

5. Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$$

$$f^{\Delta}\left(\frac{3}{2}\right) = -\frac{1195}{64}$$

and

$$\begin{aligned} f^{\Delta}\left(\sigma\left(\frac{3}{2}\right)\right) &= f^{\Delta}\left(\frac{7}{4}\right) \\ &= -\frac{15\left(\frac{7}{4}\right)^3 + 22\left(\frac{7}{4}\right)^2 + 32\left(\frac{7}{4}\right) + 8}{8} \\ &= -\frac{15\left(\frac{343}{64}\right) + 22\left(\frac{49}{16}\right) + 32\left(\frac{7}{4}\right) + 8}{8} \\ &= -\frac{15 \cdot 343 + 176 \cdot 49 + 32 \cdot 112 + 8 \cdot 64}{512} \\ &= -\frac{5145 + 8624 + 3584 + 512}{512} \\ &= -\frac{17865}{512} \end{aligned}$$

Hence,

$$\begin{aligned} f^{\Delta^2}\left(\frac{3}{2}\right) &= \frac{f^{\Delta}\left(\sigma\left(\frac{3}{2}\right)\right) - f^{\Delta}\left(\frac{3}{2}\right)}{\sigma\left(\frac{3}{2}\right) - \frac{3}{2}} \\ &= \frac{-\frac{17865}{512} + \frac{1195}{64}}{\frac{7}{4} - \frac{3}{2}} \\ &= \frac{-17865 + 9560}{128} \\ &= -\frac{8305}{128}. \end{aligned}$$

6. Let $t \in (2-U)\backslash\left\{\frac{3}{2}\right\}$. Then

$$\sigma(t) = \frac{t+2}{2}.$$

By Example 7.10, we have

$$f^{\Delta}(t) = -\frac{15t^3 + 22t^2 + 32t + 8}{8}$$

Then

$$\begin{aligned} f^{\Delta^2}(\sigma(t)) &= -\frac{15(\sigma(t))^3 + 22(\sigma(t))^2 + 32\sigma(t) + 8}{8} \\ &= -\frac{15\left(\frac{t+2}{2}\right)^3 + 22\left(\frac{t+2}{2}\right)^2 + 32\left(\frac{t+2}{2}\right) + 8}{8} \\ &= -\frac{15(t+2)^3 + 88(t+2)^2 + 128(t+2) + 64}{64} \\ &= -\frac{15t^3 + 90t^2 + 180t + 120 + 88t^2 + 352t + 352 + 128t + 256 + 64}{64} \\ &= -\frac{15t^3 + 178t^2 + 660t + 792}{64} \end{aligned}$$

and

$$\begin{aligned} f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\ &= \frac{1}{\frac{t+2}{2} - t}\left(-\frac{15t^3 + 178t^2 + 660t + 792}{64} + \frac{15t^3 + 22t^2 + 32t + 8}{8}\right) \\ &= \frac{-15t^3 - 178t^2 - 660t - 792 + 120t^3 + 176t^2 + 256t + 64}{32(2-t)} \\ &= \frac{105t^3 - 2t^2 - 404t - 728}{32(2-t)} \end{aligned}$$

7. Let $t \in (2+U)\backslash\left\{\frac{5}{2}\right\}$. Then

$$\sigma(t) = 2(t-1)$$

By Example 7.10 , we have

$$f^{\Delta}(t) = -15t^3 + 34t^2 - 31t + 10$$

Then

$$\begin{aligned} f^{\Delta}(\sigma(t)) &= -15\left(\sigma(t)0^3 + 34(\sigma(t))^2 - 31\sigma(t) + 10\right. \\ &= -120(t-1)^3 + 136(t-1)^2 - 62(t-1) + 10 \\ &= -120t^3 + 360t^2 - 360t - 120 + 136t^2 - 272t + 136 - 62t + 62 + 10 \\ &= -120t^3 + 496t^2 - 694t + 88 \end{aligned}$$

and

$$\begin{aligned} f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\ &= \frac{-120t^3 + 496t^2 - 694t + 88 + 15t^3 - 34t^2 + 31t - 10}{t-2} \\ &= \frac{-105t^3 + 461t^2 - 663t + 78}{t-2} \end{aligned}$$

8. Let $t = 0$. Then

$$\sigma(0) = 0.$$

By Example 7.10, we have

$$f^{\Delta}(0) = 0$$

Hence,

$$\begin{aligned} f^{\Delta^2}(0) &= \lim_{s\to 0} \frac{f^{\Delta}(s) - f^{\Delta}(0)}{s} \\ &= \lim_{s\to 0} \frac{-15s^3 - 3s}{s} \\ &= \lim_{s\to 0} \left(-15s^2 - 3\right) \\ &= -3 \end{aligned}$$

9. Let $t = 1$. Then

$$\sigma(1) = 1.$$

By Example 7.10, we have

$$f^{\Delta}(1) = -6.$$

Hence,

$$\begin{aligned} \lim_{s\to 1, s>1} \frac{f^{\Delta}(s) - f^{\Delta}(1)}{s-1} &= \lim_{s\to 1, s>1} \frac{-15s^3 + 17s^2 - 10s + 2 + 6}{s-1} \\ &= \lim_{s\to 1, s>1} \frac{15s^3 + 17s^2 - 10s + 8}{s-1} \\ &= \lim_{s\to 1, s>1} \frac{(s-1)\left(-15s^2 + 2s - 8\right)}{s-1} \\ &= \lim_{s\to 1, s>1} \left(-15s^2 + 2s - 8\right) \\ &= -21 \end{aligned}$$

and

$$\begin{aligned}\lim_{s\to 1,s<1}\frac{f^{\Delta}(s)-f^{\Delta}(1)}{s-1} &= \lim_{s\to 1,s<1}\frac{-\dfrac{15s^3+11s^2+17s+5}{8}+6}{s-1}\\ &= \lim_{s\to 1,s<1}\frac{-15s^3-11s^2-17s-5+48}{8(s-1)}\\ &= \lim_{s\to 1,s<1}\frac{-15s^3-11s^2-17s+43}{8(s-1)}\\ &= \lim_{s\to 1,s<1}\frac{(s-1)\left(-15s^2-26s-43\right)}{8(s-1)}\\ &= \lim_{s\to 1,s<1}\frac{-15s^2-26s-43}{8}\\ &= -\frac{21}{2}\end{aligned}$$

Since

$$\lim_{s\to 1,s<1}\frac{f^{\Delta}(s)-f^{\Delta}(1)}{s-1}\neq\lim_{s\to 1,s>1}\frac{f^{\Delta}(s)-f^{\Delta}(1)}{s-1}$$

we conclude that $f^{\Delta^2}(1)$ does not exist.

10 . Let $t=2$. Then

$$\sigma(2)=2.$$

By Example 7.10, we have

$$f^{\Delta}(2)=-36$$

Then

$$\begin{aligned}\lim_{s\to 2,s<2}\frac{f^{\Delta}(s)-f^{\Delta}(1)}{s-2} &= \lim_{s\to 2,s<2}\frac{1}{s-2}\left(-\frac{15s^3+22s^2+32s+8}{8}+36\right)\\ &= \lim_{s\to 2,s<2}\frac{1}{s-2}\left(\frac{-15s^3-22s^2-32s-8+288}{8}\right)\\ &= \lim_{s\to 2,s<2}\frac{-15s^3-22s^2-32s+280}{8(s-2)}\\ &= -\infty\end{aligned}$$

Therefore $f^{\Delta^2}(2)$ does not exist.

Consequently

$$f^{\Delta^2}(t) = \begin{cases} -105t^2 - 3 \text{ if } t \in U \setminus \left\{\frac{1}{2}\right\} \\ -\frac{729}{128} \text{ if } t = \frac{1}{2} \\ \frac{105t^3 + 21t^2 - 21t - 37}{32(1-t)} \text{ if } \quad t \in (1-U) \setminus \left\{\frac{1}{2}\right\} \\ \frac{-105t^3 + 231t^2 - 168t + 42}{t-1} \quad \text{if} \\ \qquad\qquad\qquad\qquad\quad t \in (1+U) \setminus \left\{\frac{3}{2}\right\} \\ -\frac{8305}{128} \text{ if } t = \frac{3}{2} \\ \frac{105t^3 - 2t^2 - 404t - 728}{32(2-t)} \quad \text{if} \\ \qquad\qquad\qquad\qquad\quad t \in (2-U) \setminus \left\{\frac{3}{2}\right\} \\ \frac{-105t^3 + 461t^2 - 663t + 78}{t-2} \quad \text{if} \\ \qquad\qquad\qquad\qquad\quad t \in (2+U) \setminus \left\{\frac{5}{2}\right\} \\ -3 \text{ if } t = 0 \\ \text{does not exist if } t = 1 \\ \text{does not exist if } t = 2. \end{cases}$$

Example 0.118.

Example 8.4

(Leibniz Formula). Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n-k$ times Δ. Let also,

$$f^{\Lambda} \text{ exists for all } \Lambda \in S_k^{(n)}.$$

We will prove that

$$(fg)^{\Delta^n} = \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k} \tag{8.1}$$

Solution

For this aim, we will use the classical induction principle.

1. $n = 1$. Then

$$S_0^{(1)} = 0, \quad S_1^{(1)} = \sigma.$$

Hence,

$$\begin{aligned} \sum_{k=0}^{1} \left(\sum_{\Lambda \in S_k^{(1)}} f^{\Lambda} \right) g^{\Delta^2} &= \sum_{\Lambda \in S_0^{(1)}} f^{\Lambda} g + \sum_{\Lambda \in S_1^{(1)}} f^{\Lambda} g^{\Delta} \\ &= f^{\Delta} g + f^{\sigma} g^{\Delta} \\ &= (fg)^{\Delta}, \end{aligned}$$

i.e., the assertion holds for $n = 1$.

2. Assume that the assertion is valid for some $n \in \mathbb{N}$.

We will prove that

$$(fg)^{\Delta^{n+1}} = \left(\sum_{\Lambda \in S_k^{(n+1)}} f^{\Lambda} \right) g^{\Delta^k}.$$

Using (0.4), we have

$$\begin{aligned}
(fg)^{\Delta^{n+1}} &= \left((fg)^{\Delta^n}\right)^{\Delta} \\
&= \left(\sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k} \right)^{\Delta} \\
&= \sum_{k=0}^{n} \left(\left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k} \right)^{\Delta} \\
&= \sum_{k=0}^{n} \left(\left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\sigma} \right) g^{\Delta^{k+1}} + \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\Delta} \right) g^{\Delta^k} \right) \\
&= \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\sigma} \right) g^{\Delta^{k+1}} + \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\Delta} \right) g^{\Delta^k} \\
&= \sum_{k=1}^{n+1} \left(\sum_{\Lambda \in S_{k-1}^{(n)}} f^{\Lambda\sigma} \right) g^{\Delta^k} + \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\Delta} \right) g^{\Delta^k} \\
&= \sum_{k=1}^{n} \left(\sum_{\Lambda \in S_{k-1}^{(n)}} f^{\Lambda\sigma} \right) g^{\Delta^k} + \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\sigma} \right) g^{\Delta^{n+1}} \\
&\quad + \sum_{k=1}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\Delta} \right) g^{\Delta^k} + \left(\sum_{\Lambda \in S_0^{(n)}} f^{\Lambda\Delta} \right) g \\
&= \sum_{k=1}^{n} \left(\sum_{\Lambda \in S_{k-1}^{(n)}} f^{\Lambda\sigma} + \sum_{\Lambda \in S_k^{(n)}} f^{\Lambda\Delta} \right) g^{\Delta^k} \\
&\quad + \left(\sum_{\Lambda \in S_n^{(n)}} f^{\Lambda\sigma} + \sum_{\Lambda \in S_0^{(n)}} f^{\Lambda\Delta} \right) g \\
&= \sum_{k=1}^{n} \left(\sum_{\Lambda \in S_k^{(n+1)}} f^{\Lambda} \right) g^{\Delta^k} + \left(\sum_{\Lambda \in S_{n+1}^{(n+1)}} f^{\Lambda} \right) g^{\Delta^{n+1}} + \left(\sum_{\Lambda \in S_0^{(n+1)}} f^{\Lambda} \right) g \\
&= \sum_{k=0}^{n+1} \left(\sum_{\Lambda \in S_k^{(n+1)}} f^{\Lambda} \right) g^{\Delta^k}.
\end{aligned}$$

Example 8.5

Let f and μ be differentiable in $\mathbb{T}^{\kappa}$. Then

$$f^{\Delta\sigma} = \frac{f^{\sigma^2} - f^{\sigma}}{\mu^{\sigma}} \quad \text{and} \quad f^{\sigma\Delta} = \frac{f^{\sigma^2} - f^{\sigma}}{\mu}$$

Solution

Therefore

$$f^{\sigma\Delta} = \frac{f^{\Delta\sigma}\mu^{\sigma}}{\mu}$$
$$= \frac{f^{\Delta\sigma}\mu\left(1+\mu^{\Delta}\right)}{\mu}$$
$$= \left(1+\mu^{\Delta}\right) f^{\Delta\sigma}$$

Also,

$$f^{\sigma^2\Delta} = \frac{f^{\sigma^3} - f^{\sigma^2}}{\mu} \quad \text{and} \quad f^{\sigma\Delta\sigma} = \frac{f^{\sigma^3} - f^{\sigma^2}}{\mu^{\sigma}}$$

Therefore

$$f^{\sigma\Delta\sigma} = \frac{f^{\sigma^2\Delta}\mu}{\mu^{\sigma}}$$
$$= \frac{f^{\sigma^2\Delta}\mu}{\mu\left(1+\mu^{\Delta}\right)}$$
$$= \frac{f^{\sigma^2\Delta}}{1+\mu^{\Delta}}$$

whereupon

$$f^{\sigma^2\Delta} = \left(1+\mu^{\Delta}\right) f^{\sigma\Delta\sigma}$$

We have

$$f^{\Delta\sigma^2} = \frac{f^{\sigma^3} - f^{\sigma^2}}{\mu^{\sigma^2}}$$
$$= \frac{f^{\sigma^2\Delta}\mu}{\mu^{\sigma^2}}$$
$$= \frac{\mu f^{\sigma^2\Delta}}{\mu\left(1+\mu^{\Delta}\right)\left(1-\mu^{\Delta\sigma}\right)}$$
$$= \frac{f^{\sigma^2\Delta}}{\left(1+\mu^{\Delta}\right)\left(1+\mu^{\Delta\sigma}\right)}$$

from where

$$f^{\sigma^2\Delta} = \left(1+\mu^{\Delta}\right)\left(1+\mu^{\Delta\sigma}\right) f^{\Delta\sigma^2}.$$

? Exercise 8.1

Let $\mathbb{T} = \mathbb{Z}$. Find $f^{\Delta^2}(t), t \in \mathbb{T}$, where

$$f(t) = t^2 - 3t + 1, \quad t \in \mathbb{T}.$$

Answer

Add texts here. Do not delete this text first.

Exercise 8.2

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Find $f^{\Delta^2}(t), t \in \mathbb{T}$, where

$$f(t) = t^3 + 3t^2 + t + 1, \quad t \in \mathbb{T}.$$

Answer

Add texts here. Do not delete this text first.

Exercise 8.3

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Find $f^{\Delta^2}(t), t \in \mathbb{T}$, where

$$f(t) = \frac{t^2+1}{t+1}, \quad t \in \mathbb{T}.$$

Answer

Add texts here. Do not delete this text first.

Exercise 8.4

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Find $f^{\Delta^2}(t), t \in \mathbb{T}$, where

$$f(t) = \frac{t+2}{t+3}, \quad t \in \mathbb{T}.$$

Answer

Add texts here. Do not delete this text first.

Exercise 8.5

Let $\mathbb{T} = \mathbb{N}_0$. Find $f^{\Delta^2}(t), t \in \mathbb{T}$, where

$$f(t) = \frac{1}{t+1}, \quad t \in \mathbb{T}.$$

Answer

Add texts here. Do not delete this text first.

Exercise 8.6

Let $\mathbb{T} = P_{3,4} \cup [4, 6]$. Find $f^{\Delta^2}(t), t \in \mathbb{T}^{\kappa^2}$, where

$$f(t) = \frac{t-2}{t-4}, \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

Exercise 8.7

Let $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}$. Find $f^{\Delta^2}(t), t \in \mathbb{T}$, where

$$f(t) = t^3 - t^2 + t - 1, \quad t \in \mathbb{T}$$

Answer

Add texts here. Do not delete this text first.

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9: Nabla Derivatives

Definition: The Nabla Derivative

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be nabla differentiable at $t\in\mathbb{T}_\kappa$ if
1. f is defined in a neighbourhood U of t,
2. f is defined at $\rho(t)$,
3. there exists a unique real number $f^\nabla(t)$, called the nabla derivative of f at t, such that for each $\varepsilon>0$, there exists a neighbourhood N of t with $N\subseteq U$ and

$$\left|f(\rho(t))-f(s)-(\rho(t)-s)f^\nabla(t)\right|\leq\varepsilon|\rho(t)-s| \quad \text{for all} \quad s\in N.$$

The basic rules for nabla differentiation read as follows.

Let $f,g:\mathbb{T}\to\mathbb{R}$ be functions and let $t\in\mathbb{T}_\kappa$. Then we have the following.
1. The nabla derivative is well defined.
2. If f is nabla differentiable at t, then f is continuous at t.
3. If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with

$$f^\nabla(t)=\frac{f(t)-f(\rho(t))}{v(t)}$$

4. If t is left-dense, then f is nabla differentiable at t iff the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case,

$$f^\nabla(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

5. If f is differentiable at t, then

$$f(\rho(t))=f(t)+v(t)f^\nabla(t)$$

6. If f and g are nabla differentiable at t, then

a. the sum $f+g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with

$$(f+g)^\nabla(t)=f^\nabla(t)+g^\nabla(t)$$

b. For any constant $\alpha,\alpha f:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with

$$(\alpha f)^\nabla(t)=\alpha f^\nabla(t).$$

c. The product $fg:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with

$$(fg)^\nabla(t)=f^\nabla(t)g(t)+f(\rho(t))g^\nabla(t)=f(t)g^\nabla(t)+f^\nabla(t)g(\rho(t)).$$

d. If $g(t)g(\rho(t))\neq 0$, then $\frac{f}{g}:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with

$$\left(\frac{f}{g}\right)^\nabla(t)=\frac{f^\nabla(t)g(t)-f(t)g^\nabla(t)}{g(t)g(\rho(t))}.$$

Definition: Term

Let $f:\mathbb{T}\to\mathbb{R}$ and let $t\in(\mathbb{T}_\kappa)_\kappa=\mathbb{T}_{\kappa^2}$. We define the second nabla derivative of f at t, provided it exists, by

$$f^{\nabla\nabla}=\left(f^\nabla\right)^\nabla:\mathbb{T}_{\kappa^2}\to\mathbb{R}$$

Similarly, we define higher-order nabla derivatives $f^{\nabla^n}:\mathbb{T}_{\kappa^n}\to\mathbb{R}$. Let $t\in\mathbb{T}_\kappa^\kappa$. We define the second mixed derivative of f at t, provided it exists, by

$$f^{\nabla\Delta}=\left(f^{\nabla}\right)^{\Delta}:\mathbb{T}_\kappa^\kappa\to\mathbb{R}$$

and

$$f^{\Delta\nabla}=\left(f^{\Delta}\right)^{\nabla}:\mathbb{T}_\kappa^\kappa\to\mathbb{R}$$

Example 9.1

Let $\mathbb{T}=\mathbb{Z}$ and $f:\mathbb{T}\to\mathbb{R}$ be a function. Then

$$f^{\nabla}(t)=f(t)-f(t-1)\quad\text{for any}\quad t\in\mathbb{T}.$$

Example 9.2

Let $\mathbb{T}=2^{\mathbb{N}_0}$ and

$$f(t)=t^3+t^2+t+1,\quad t\in\mathbb{T}_\kappa.$$

We will find $f^{\nabla}(t)$ for $t\in\mathbb{T}_\kappa$. We have that $\mathbb{T}_\kappa=\mathbb{T}\backslash\{1\}$ and $\rho(t)=\frac{t}{2}, t\in\mathbb{T}_\kappa$. Hence,

$$\begin{aligned}f^{\nabla}(t)&=(\rho(t))^2+t\rho(t)+t^2+\rho(t)+t+1\\&=\frac{t^2}{4}+\frac{t^2}{2}+t^2+\frac{t}{2}+t+1\\&=\frac{7}{4}t^2+\frac{3}{2}t+1,\quad t\in\mathbb{T}_\kappa.\end{aligned}$$

Example 9.3

Let $\mathbb{T}=P_{1,3}$ and

$$f(t)=\frac{1-t}{1+t},\quad t\in\mathbb{T}.$$

We will find $f^{\nabla}(t)$ and $t\in\mathbb{T}_\kappa$ and $f^{\Delta\nabla}(t)$ for $t\in\mathbb{T}_\kappa^\kappa$. Firstly, we will find $f^{\nabla}(t)$ for $t\in\mathbb{T}_\kappa$. We have the following cases.

1. Let $t\in\mathbb{T}_\kappa$ be left-scattered. Then $\rho(t)=t-3$ and

$$\begin{aligned}f^{\nabla}(t)&=\frac{f(\rho(t))-f(t)}{\rho(t)-t}\\&=\frac{f(t-3)-f(t)}{t-3-t}\\&=\frac{\frac{1-t+3}{1+t-3}-\frac{1-t}{1+t}}{-3}\\&=\frac{1}{3}\left(\frac{1-t}{1+t}-\frac{4-t}{t-2}\right)\\&=\frac{1}{3}\left(\frac{(1-t)(t-2)-(1+t)(4-t)}{(t+1)(t-2)}\right)\\&=\frac{1}{3}\left(\frac{t-2-t^2+2t-4+t-4t+t^2}{(t+1)(t-2)}\right)\\&=\frac{1}{3}\left(-\frac{6}{(t+1)(t-2)}\right)\\&=-\frac{2}{(t+1)(t-2)}\end{aligned}$$

2. Let $t \in \mathbb{T}_\kappa$ be left-dense. Then

$$\begin{aligned} f^{\nabla}(t) &= \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \\ &= \lim_{s \to t} \frac{\frac{1-t}{1+t} - \frac{1-s}{1+s}}{t - s} \\ &= \lim_{s \to t} \frac{(1-t)(1+s) - (1-s)(1+t)}{(t-s)(1+t)(1+s)} \\ &= \lim_{s \to t} \frac{1+s-t-st-1-t+s+st}{(t-s)(1+t)(1+s)} \\ &= \lim_{s \to t} \frac{2s-2t}{(t-s)(1+t)(1+s)} \\ &= -2 \lim_{s \to t} \frac{1}{(1+t)(1+s)} \\ &= -\frac{2}{(1+t)^2} \end{aligned}$$

Now, we will find $f^{\Delta\nabla}(t)$ for $t \in \mathbb{T}_\kappa^\kappa$. We have the following cases.

1. Let $t \in \mathbb{T}_\kappa^\kappa$ be right-scattered. We have

$$f^{\Delta}(t) = -\frac{2}{(1+t)(4+t)}$$

a. Let t be left-scattered. Then

$$\begin{aligned} f^{\Delta\nabla}(t) &= \frac{f^{\Delta}(\rho(t)) - f^{\Delta}(t)}{\rho(t) - t} \\ &= \frac{f^{\Delta}(t-3) - f^{\Delta}(t)}{t-3-t} \\ &= \frac{-\frac{2}{(1+t-3)(4+t-3)} + \frac{2}{(1+t)(4+t)}}{-3} \\ &= \frac{-\frac{2}{(t-2)(t+1)} + \frac{2}{(t+1)(t+4)}}{-3} \\ &= \frac{2}{3}\left(\frac{t+4-t+2}{(t-2)(t+1)(t+4)}\right) \\ &= \frac{2}{3}\left(\frac{6}{(t-2)(t+1)(t+4)}\right) \\ &= \frac{4}{(t-2)(t+1)(t+4)} \end{aligned}$$

b. Let t be left-dense. Then

$$\begin{aligned} f^{\Delta\nabla}(t) &= \lim_{s \to t} \frac{f^{\Delta}(t) - f^{\Delta}(s)}{t - s} \\ &= \lim_{s \to t} \frac{-\frac{2}{(1+t)(4+t)} + \frac{2}{(1+s)(4+s)}}{t - s} \\ &= -2 \lim_{s \to t} \frac{\frac{1}{(1+t)(4+t)} - \frac{1}{(1+s)(4+s)}}{t - s} \\ &= -2 \lim_{s \to t} \left(\frac{(1+s)(4+s) - (1+t)(4+t)}{(t-s)(1+t)(4+t)(1+s)(4+s)}\right) \end{aligned}$$

$$
\begin{aligned}
&= -2\lim_{s\to t}\frac{4+s+4s+s^2-4-t-4t-t^2}{(t-s)(1+t)(4+t)(1+s)(4+s)}\\
&= -2\lim_{s\to t}\frac{5(s-t)+(s-t)(s+t)}{(t-s)(1+t)(4+t)(1+s)(4+s)}\\
&= 2\lim_{s\to t}\frac{5+s+t}{(1+t)(4+t)(1+s)(4+s)}\\
&= \frac{2(5+2t)}{(1+t)^2(4+t)^2}
\end{aligned}
$$

2. Let $t \in \mathbb{T}_\kappa^\kappa$ be right-dense. We have

$$
f^\Delta(t) = -\frac{2}{(1+t)^2}
$$

a. Let t be left-scattered. Then

$$
\begin{aligned}
f^{\Delta\nabla}(t) &= \frac{f^\Delta(\rho(t)) - f^\Delta(t)}{\rho(t)-t}\\
&= \frac{f^\Delta(t-3) - f^\Delta(t)}{t-3-t}\\
&= \frac{-\frac{2}{(1+t-3)^2} + \frac{2}{(1+t)^2}}{-3}\\
&= \frac{2}{3}\left(\frac{1}{(t-2)^2} - \frac{1}{(t+1)^2}\right)\\
&= \frac{2}{3}\left(\frac{(t+1)^2-(t-2)^2}{(t+1)^2(t-2)^2}\right)\\
&= \frac{2}{3}\left(\frac{t^2+2t+1-t^2+4t-4}{(t+1)^2(t-2)^2}\right)\\
&= \frac{2}{3}\left(\frac{6t-3}{(t+1)^2(t-2)^2}\right)\\
&= \frac{2(2t-1)}{(t+1)^2(t-2)^2}
\end{aligned}
$$

b. Let t be left-dense. Then

$$
\begin{aligned}
f^{\Delta\nabla}(t) &= \lim_{s\to t}\frac{f^\Delta(t) - f^\Delta(s)}{t-s}\\
&= \lim_{s\to t}\frac{-\frac{2}{(1+t)^2} + \frac{2}{(1+s)^2}}{t-s}\\
&= \lim_{s\to t}\frac{(1+t)^2-(1+s)^2}{(t-s)(1+t)^2(1+s)^2}\\
&= 2\lim_{s\to t}\frac{1+2t+t^2-1-2s-s^2}{(t-s)(1+t)^2(1+s)^2}\\
&= 2\lim_{s\to t}\frac{2(t-s)+(t-s)(t+s)}{(t-s)(1+t)^2(1+s)^2}\\
&= 2\lim_{s\to t}\frac{2+t+s}{(1+t)^2(1+s)^2}\\
&= \frac{4(1+t)}{(1+t)^4}\\
&= \frac{4}{(1+t)^3}
\end{aligned}
$$

Example 9.4

Let $\mathbb{T} = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$ and

$$f(t) = \frac{1+t^2}{1+7t}, \quad t \in \mathbb{T}.$$

We will find $f^{\nabla}(t), t \in \mathbb{T}_{\kappa}$. We have the following cases.

1. Let $t \in \left\{-\frac{1}{n} : n \in \mathbb{N}, n \geq 2\right\}$. Then

$$\rho(t) = \frac{t}{t+1}$$

Hence,

$$\begin{aligned}
f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\
&= \frac{\frac{1+(\rho(t))^2}{1+7\rho(t)} - \frac{1+t^2}{1+7t}}{\frac{t}{t+1} - t} \\
&= \frac{1 + \frac{t^2}{(t+1)^2} - \frac{1+t^2}{1+7t}}{1+7\frac{t}{t+1}} \\
&= \frac{1 + \frac{t^2}{(t+1)^2} - \frac{1+t^2}{1+7t}}{t\left(\frac{1}{t+1} - 1\right)} \\
&= \frac{\frac{(t+1)^2+t^2}{(t+1)(t+1+7t)} - \frac{1+t^2}{1+7t}}{t\left(\frac{1-t-1}{t+1}\right)} \\
&= \frac{\frac{2t^2+2t+1}{(t+1)(8t+1)} - \frac{1+t^2}{1+7t}}{-\frac{t^2}{t+1}} \\
&= -\frac{(2t^2+2t+1)(1+7t) - (1+t^2)(t+1)(8t+1)}{t^2(8t+1)(7t+1)} \\
&= -\frac{2t^2+14t^3+2t+14t^2+1+7t-(1+t^2)(8t^2+9t+1)}{t^2(8t+1)(7t+1)} \\
&= -\frac{14t^3+16t^2+9t+1-8t^2-9t-1-8t^4-9t^3-t^2}{t^2(8t+1)(7t+1)} \\
&= -\frac{-8t^4+5t^3+7t^2}{t^2(8t+1)(7t+1)} \\
&= \frac{8t^2-5t-7}{(8t+1)(7t+1)}
\end{aligned}$$

2. Let $t = 0$. Then $\rho(0) = 0$ and

$$
\begin{aligned}
f^{\nabla}(t) &= \lim_{s\to 0}\frac{f(s)-f(0)}{s}\\
&= \lim_{s\to 0}\frac{\frac{1+s^2}{1+7s}-1}{s}\\
&= \lim_{s\to 0}\frac{1+s^2-1-7s}{s(1+7s)}\\
&= \lim_{s\to 0}\frac{s(s-7)}{s(1+7s)}\\
&= \lim_{s\to 0}\frac{s-7}{1+7s}\\
&= -7
\end{aligned}
$$

3. Let $t\in\mathbb{N}$. Then $\rho(t)=t-1$ and

$$
f^{\nabla}(t)=\frac{f(\rho(t))-f(t)}{\rho(t)-t}
$$

$$
\begin{aligned}
&= \frac{\frac{1+(t-1)^2}{1+7(t-1)}-\frac{1+t^2}{1+7t}}{t-1-t}\\
&= \frac{1+t^2}{1+7t}-\frac{1+(t-1)^2}{1+7(t-1)}\\
&= \frac{1+t^2}{1+7t}-\frac{t^2-2t+2}{7t-6}\\
&= \frac{\left(1+t^2\right)(7t-6)-(1+7t)\left(t^2-2t+2\right)}{(7t+1)(7t-6)}\\
&= \frac{7t-6+7t^3-6t^2-t^2+2t-2-7t^3+14t^2-14t}{(7t+1)(7t-6)}\\
&= \frac{7t^2-5t-8}{(7t-1)(7t+6)}
\end{aligned}
$$

Example 9.5

Let $U=\left\{\frac{1}{2^n}:n\in\mathbb{N}\right\}$, $\mathbb{T}=\{0\}\cup U\cup(1-U)\cup\{1\}$ and

$$
f(t)=\left(2+t^2\right)\left(1-t^2\right),\quad t\in\mathbb{T}.
$$

We will find $f^{\nabla}(t), t\in\mathbb{T}_{\kappa}$ and $f^{\nabla^2}(t), t\in\mathbb{T}_{\kappa^2}$. We have the following cases.

1. Let $t\in U$. By Example 0.39, we have

$$
\rho(t)=\frac{1}{2}t.
$$

Hence,

$$
\begin{aligned}
f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\
&= \frac{\left(2 + \frac{t^2}{4}\right)\left(1 - \frac{1}{t}^2\right]4\right) - (2 + t^2)(1 - t^2)}{\frac{t}{2} - t} \\
&= \frac{(2 + t^2)(1 - t^2) - \frac{1}{16}(8 + t^2)(4 - t^2)}{\frac{t}{2}} \\
&= \frac{16(2 + gt^2)(1 - t^2) - (8 + t^2)(4 - t^2)}{8t} \\
&= \frac{32 - 32t^2 + 16t^2 - 16t^4 - 32 + 8t^2 - 4t^2 + t^4}{8t} \\
&= \frac{-15t^4 - 12t^2}{8t} \\
&= -\frac{15}{8}t^3 - \frac{3}{2}t
\end{aligned}
$$

Then

$$
\begin{aligned}
f^{\nabla^2}(t) &= \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t} \\
&= \frac{-\frac{15}{64}t^3 - \frac{3}{4}t}{\frac{t}{2} - t} \\
&= \frac{\frac{15}{64}t^3 + \frac{3}{4}t}{\frac{t}{2}} \\
&= \frac{15}{32}t^2 + \frac{3}{2}
\end{aligned}
$$

2. Let $t \in (1 - U) \backslash \left\{\frac{1}{2}\right\}$. By Example 0.39 , we get

$$
\rho(t) = 2t - 1
$$

and

$$
\begin{aligned}
f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\
&= \frac{(2 + (2t - 1)^2)(1 - (2t - 1)^2) - (2 + t^2)(1 - t^2)}{2t - 1 - t} \\
&= \frac{(4t^2 - 4t + 3)(1 - 2t + 1)(1 + 2t - 1) - (2 + t^2)(1 - t)(1 + t)}{t - 1} \\
&= \frac{-4(4t^2 - 4t + 3)t(t - 1) + (2 + t^2)(t - 1)(t + 1)}{t - 1} \\
&= -4t(4t^2 - 4t + 3) + (2 + t^2)(t + 1) \\
&= -16tr^3 + 16t^2 - 12t + 2t + 2 + t^3 + t^2 \\
&= -15t^3 + 17t^2 - 10t + 2
\end{aligned}
$$

Then

$$
\begin{aligned}
f^{\nabla^2}(t) &= \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t} \\
&= \frac{-15(2t-1)^3 + 17(2t-1)^2 - 10(2t-1) + 2 + 15t^3 - 17t^2 + 10t - 2}{2t - 1 - t} \\
&= \frac{-15(t-1)\left(4t^2 - 4t + 1 + 2t^2 - t + t^2\right) + 17(t-1)(2t-1+t) - 10(t-1)}{t-1} \\
&= -15\left(7t^2 - 5t + 1\right) + 17(3t-1) - 10 \\
&= -105t^2 + 75t - 15 + 51t - 17 - 10 \\
&= -105t^2 + 126t - 42
\end{aligned}
$$

3. Let $t = \frac{1}{2}$. By Example 0.39 , we have

$$
\rho\left(\frac{1}{2}\right) = \frac{1}{4}
$$

and then

$$
\begin{aligned}
f^{\nabla}\left(\frac{1}{2}\right) &= \frac{f\left(\rho\left(\frac{1}{2}\right)\right) - f\left(\frac{1}{2}\right)}{\rho\left(\frac{1}{2}\right) - \frac{1}{2}} \\
&= \frac{\left(2 + \frac{1}{16}\right)\left(1 - \frac{1}{16}\right) - \left(2 + \frac{1}{4}\right)\left(1 - \frac{1}{4}\right)}{\frac{1}{4} - \frac{1}{2}} \\
&= \frac{\frac{33}{16} \cdot \frac{15}{16} - \frac{9}{4} \cdot \frac{3}{4}}{-\frac{1}{4}} \\
&= \frac{27 \cdot 16 - 33 \cdot 15}{4} \\
&= \frac{432 - 495}{4} \\
&= -\frac{63}{4}
\end{aligned}
$$

Note that

$$
\begin{aligned}
f^{\nabla^2}\left(\frac{1}{4}\right) &= \frac{15}{32} \cdot \frac{1}{4} + \frac{3}{2} \\
&= \frac{15 + 3 \cdot 64}{128} \\
&= \frac{15 + 192}{128} \\
&= \frac{207}{128}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f^{\nabla^2}\left(\frac{1}{2}\right) &= \frac{f^{\nabla}\left(\rho\left(\frac{1}{2}\right)\right) - f^{\nabla}\left(\frac{1}{2}\right)}{\rho\left(\frac{1}{2}\right) - \frac{1}{2}} \\
&= \frac{\frac{207}{128} + \frac{63}{4}}{\frac{1}{4} - \frac{1}{2}} \\
&= -\frac{207 + 63 \cdot 32}{32} \\
&= -\frac{207 + 2016}{32} \\
&= -\frac{2223}{32}
\end{aligned}
$$

4. Let $t = 1$.By Example 0.39 , we have $\rho(1) = 1$ and then

$$\begin{aligned} f^{\nabla}(1) &= \lim_{s\to 1} \frac{f(s) - f(1)}{s-1} \\ &= \lim_{s\to 1} \frac{(2+s^2)(1-s^2)}{s-1} \\ &= -\lim_{s\to 1} (2+s^2)(1+s) \\ &= -6 \end{aligned}$$

Hence,

$$\begin{aligned} f^{\nabla^2}(1) &= \lim_{s\to 1} \frac{f^{\nabla}(s) - f^{\nabla}(1)}{s-1} \\ &= \lim_{s\to 1} \frac{-15s^3 + 17s^2 - 10s + 2 + 6}{s-1} \\ &= -\lim_{s\to 1} \frac{15s^3 - 17s^2 + 10s - 8}{s-1} \\ &= -\lim_{s\to 1} \frac{(15s^2 - 2s + 8)(s-1)}{s-1} \\ &= -\lim_{s\to 1} (15s^2 - 2s + 8) \\ &= -21 \end{aligned}$$

Example 9.6

Let $\mathbb{T} = (-\mathbb{N}_0) \cup [1,2] \cup 4^{\mathbb{N}}$ and

$$f(t) = t^3 + t, \quad t \in \mathbb{T}$$

We will find $f^{\nabla}(t), f^{\nabla^2}(t), f^{\nabla^3}(t), t \in \mathbb{T}$. We have the following cases.

1. Let $t \in (-\mathbb{N}_0)$. Then

$$\rho(t) = t - 1$$

and

$$\begin{aligned} f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\ &= \frac{f(t-1) - f(t)}{t - 1 - t} \\ &= t^3 + t - (t-1)^3 - (t-1) \\ &= t^3 - t^3 + 3t^2 - 3t + 1 + t - t + 1 \\ &= 3t^2 - 3t + 2 \end{aligned}$$

and

$$\begin{aligned} f^{\nabla^2}(t) &= \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t} \\ &= \frac{f^{\nabla}(t-1) - f^{\nabla}(t)}{t - 1 - t} \\ &= 3t^2 - 3t + 2 - 3(t-1)^2 + 3(t-1) - 2 \\ &= 3t^2 - 3t^2 + 6t - 3 - 3t + 3t - 3 \\ &= 6t - 6 \end{aligned}$$

and

$$
\begin{aligned}
f^{\nabla^3}(t) &= \frac{f^{\nabla^2}(\rho(t)) - f^{\nabla^2}(t)}{\rho(t) - t} \\
&= \frac{f^{\nabla^2}(t-1) - f^{\nabla^2}(t)}{t-1-t} \\
&= 6t - 6 - 6(t-1) + 6 \\
&= 6t - 6t + 6 \\
&= 6
\end{aligned}
$$

2. Let $t = 1$. Then $\rho(1) = 0$ and

$$
\begin{aligned}
f^{\nabla}(1) &= \frac{f(\rho(1)) - f(1)}{\rho(1) - 1} \\
&= \frac{f(0) - f(1)}{0 - 1} \\
&= f(1) - f(0) \\
&= 2
\end{aligned}
$$

Note that

$$
f^{\nabla}(0) = 1.
$$

Then

$$
\begin{aligned}
f^{\nabla^2}(1) &= \frac{f^{\nabla}(\rho(1)) - f^{\nabla}(1)}{\rho(1) - 1} \\
&= \frac{f^{\nabla}(0) - f^{\nabla}(1)}{0 - 1} \\
&= 2 - 1 \\
&= 1
\end{aligned}
$$

Next,

$$
f^{\nabla^2}(0) = -6
$$

and

$$
\begin{aligned}
f^{\nabla^3}(1) &= \frac{f^{\nabla^2}(\rho(1)) - f^{\nabla^2}(1)}{\rho(1) - 1} \\
&\quad \frac{f^{\nabla^2}(0) - f^{\nabla^2}(1)}{0 - 1} \\
&= 1 - (-6) \\
&= 7.
\end{aligned}
$$

3. Let $t \in (1, 2]$. Then

$$
\begin{aligned}
f^{\nabla}(t) &= 3t^2 + 1, \\
f^{\nabla^2}(t) &= 6t, \\
f^{\nabla^3}(t) &= 6.
\end{aligned}
$$

4. Let $t = 4$. Then $\rho(4) = 2$ and

$$
\begin{aligned}
f^{\nabla}(4) &= \frac{f(\rho(4)) - f(4)}{\rho(4) - 4} \\
&= \frac{f(2) - f(4)}{2 - 4} \\
&= \frac{68 - 10}{2} \\
&= 29.
\end{aligned}
$$

Note that

$$
\begin{aligned}
f^{\nabla}(2) &= 3 \cot 4 + 1 \\
&= 13
\end{aligned}
$$

and then

$$\begin{aligned} f^{\nabla^2}(4) &= \frac{f^{\nabla}(\rho(4)) - f^{\nabla}(4)}{\rho(4) - 4} \\ &= \frac{f^{\nabla}(2) - f^{\nabla}(4)}{2 - 4} \\ &= \frac{29 - 13}{2} \\ &= 8. \end{aligned}$$

Next,

$$f^{\nabla^2}(2) = 12$$

and

$$\begin{aligned} f^{\nabla^3}(4) &= \frac{f^{\nabla^2}(\rho(4)) - f^{\nabla^2}(4)}{\rho(4) - 4} \\ &= \frac{f^{\nabla^2}(2) - f^{\nabla^2}(4)}{2 - 4} \\ &= \frac{12 - 8}{2} \\ &= 2 \end{aligned}$$

5. Let $t \in 4^{\mathbb{N}}, t \geq 16$. Then

$$\rho(t) = \frac{t}{4}$$

and

$$\begin{aligned} f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\ &= \frac{t^3 + t - \frac{t^3}{64} - \frac{t}{4}}{t - \frac{t}{4}} \\ &= \frac{\frac{63}{64}t^3 + \frac{3}{4}t}{\frac{3}{4}t} \\ &= \frac{21}{16}t^2 + 1 \end{aligned}$$

and

$$\begin{aligned} f^{\nabla^2}(t) &= \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t} \\ &= \frac{\frac{21}{16}t^2 + 1 - \frac{21}{256}t^2 - 1}{t - \frac{t}{4}} \\ &= \frac{\frac{315}{256}t^2}{\frac{3}{4}t} \\ &= \frac{105}{64}t, \end{aligned}$$

and

$$
\begin{aligned}
f^{\nabla^3}(t) &= \frac{f^{\nabla^2}(\rho(t)) - f^{\nabla^2}(t)}{\rho(t) - t} \\
&= \frac{\frac{105}{64}t - \frac{105}{256}t}{t - \frac{t}{4}} \\
&= \frac{\frac{315}{256}t}{\frac{3}{4}t} \\
&= \frac{105}{64}
\end{aligned}
$$

? Exercise 9.1

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Find $f^{\nabla}(t)$ for $t \in \mathbb{T}_{\kappa}$, where

$$f(t) = \frac{t^2 + 2t - 3}{t - 7}$$

Answer

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✓ Example 9.2

Let $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}$ and

$$f(t) = t^3 - 3t^2 + 2t, \quad t \in \mathbb{T}$$

Find $f^{\nabla}(t)$ and $f^{\nabla^2}(t)$ for $t \in \mathbb{T}$.

Solution

Add example text here.

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10: Regulated, rd-Continuous, and Pre-Differentiable Functions

Definition: Regulated functions

A function $f:\mathbb{T}\to\mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all rightdense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Example 10.1

Let $\mathbb{T}=\mathbb{N}$ and

$$f(t)=\frac{t^2}{t-1},\quad g(t)=\frac{t}{t+1},\quad t\in\mathbb{T}$$

We note that all points of $\mathbb{T}$ are right-scattered. The points $t\in\mathbb{T}, t\neq 1$, are left-scattered. Also, $\lim_{t\to 1-} f(t)$ is not finite and $\lim_{t\to 1-} g(t)$ exists and it is finite. Therefore, the function f is not regulated and the function g is regulated.

Example 10.2

Let $\mathbb{T}=\mathbb{R}$ and

$$f(t)=\begin{cases}\frac{1}{t} & \text{for} \quad t\in\mathbb{R}\backslash\{0\}\\ 0 & \text{for} \quad t=0\end{cases}$$

We have that all points of $\mathbb{T}$ are dense and $\lim_{t\to 0-} f(t), \lim_{t\to 0+} f(t)$ are not finite. Therefore, the function f is not regulated.

Exercise 10.1

Let $\mathbb{T}=\mathbb{R}$ and

$$f(t)=\begin{cases}\frac{1}{t-1} & \text{for} \quad t\in\mathbb{R}\backslash\{1\}\\ 11 & \text{for} \quad t=1\end{cases}$$

Determine if f is regulated.

Answer

The function f is not regulated.

Definition: Pre-differentiable function

A continuous function $f:\mathbb{T}\to\mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided
1. $D\subset\mathbb{T}^{\kappa}$,
2. $\mathbb{T}^{\kappa}\backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$,
3. f is differentiable at each $t\in D$.

Example 10.3

Let $\mathbb{T} = P_{a,b} = \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+a]$ for $a > b > 0$. Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+b] \\ t-(a+b)k-b & \text{if} \quad t \in [(a+b)k+b, (a+b)k+a] \end{cases}$$

Then f is pre-differentiable with $D = \mathbb{T}\backslash\bigcup_{k=0}^{\infty}\{(a+b)k+b\}$.

Example 10.4

Let $\mathbb{T} = \mathbb{R}$ and

$$f(t) = \begin{cases} \dfrac{1}{t-3} & \text{if} \quad \mathbb{R}\backslash\{3\} \\ 0 & \text{if} \quad t = 3 \end{cases}$$

Since $f : \mathbb{T} \to \mathbb{R}$ is not continuous at $t = 3$, the function f is not pre-differentiable.

Example 10.5

Let $\mathbb{T} = \mathbb{N}_0 \bigcup \left\{1 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \mathbb{N} \\ t & \text{otherwise} \end{cases}$$

Since f is not continuous at $t = 1$, it is not pre-differentiable.

Exercise 10.2

Let $\mathbb{T} = \mathbb{R}$ and

$$f(t) = \begin{cases} \dfrac{1}{t+3} & \text{if} \quad t \in \mathbb{R}\backslash\{-3\} \\ 0 & \text{if} \quad t = -3 \end{cases}$$

Check if $f : \mathbb{T} \to \mathbb{R}$ is pre-differentiable, and if it is, then find the region of differentiation.
Definition 0.17. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathbf{C}_{\mathbf{rd}}$ or $\mathbf{C}_{\mathbf{rd}}(\mathbb{T})$ or $\mathbf{C}_{\mathbf{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathbf{C}^{\mathbf{1}}_{\mathbf{rd}}(\mathbb{T})$.

Some results concerning rd-continuous and regulated functions are contained in the following theorem. Since its statements follow directly from the definitions, we leave the proofs to the reader.

Theorem 10.1

Assume $f : \mathbb{T} \to \mathbb{R}$.
1. If f is continuous, then f is rd-continuous.
2. If f is rd-continuous, then f is regulated.
3. The jump operator σ is rd-continuous.

4. If f is regulated or rd-continuous, then so is f^{σ}.
5. Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property.

Theorem 10.2

Every regulated function on a compact interval is bounded.

Proof. Assume that $f : [a, b] \to \mathbb{R}, [a, b] \subset \mathbb{T}$, is unbounded. Then, for each $n \in \mathbb{N}$, there exists $t_n \in [a, b]$ such that $|f(t_n)| > n$. Because $\{t_n\}_{n\in\mathbb{N}} \subset [a, b]$, there exists a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}} \subset \{t_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} t_{n_k} = t_0$$

Since $\mathbb{T}$ is closed, we have that $t_0 \in \mathbb{T}$. Also, t_0 is a right-dense point or a left-dense point. Using that f is regulated, we get

$$\left|\lim_{k\to\infty} f(t_{n_k})\right| \neq \infty,$$

which is a contradiction.

Theorem 10.3; Mean Value Theorem

If $f, g : \mathbb{T} \to \mathbb{R}$ are pre-differentiable with D, then

$$\left|f^{\Delta}(t)\right| \leq \left|g^{\Delta}(t)\right| \quad \text{for all} \quad t \in D$$

implies

$$|f(s) - f(r)| \leq g(s) - g(r) \quad \text{for all} \quad r, s \in \mathbb{T}, \quad r \leq s. \tag{10.1}$$

Proof. Let $r, s \in \mathbb{T}$ with $r \leq s$. Assume also

$$[r, s)\backslash D = \{t_n : n \in \mathbb{N}\}$$

We take $\varepsilon > 0$. We consider the statements

$$S(t) : |f(t) - f(r)| \leq g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t} 2^{-n}\right)$$

for $t \in [r, s]$. We will prove, using the induction principle, that $S(t)$ is true for all $t \in [r, s]$.

1. Since $0 \leq \varepsilon \sum_{t_n<t} 2^{-n}$, the statement $S(r)$ is true.
2. Let $t \in [r, s]$ be right-scattered and assume that $S(t)$ is true. Then we have

$$\begin{aligned}
|f(\sigma(t)) - f(r)| &= \left|f(t) + \mu(t)f^{\Delta}(t) - f(r)\right| \\
&\leq \mu(t)\left|f^{\Delta}(t)\right| + |f(t) - f(r)| \\
&\leq \mu(t)\left|f^{\Delta}(t)\right| + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t} 2^{-n}\right) \\
&\leq \mu(t)g^{\Delta}(t) + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t} 2^{-n}\right) \\
&\leq g(\sigma(t)) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t} 2^{-n}\right) \\
&< g(\sigma(t)) - g(r) + \varepsilon\left(\sigma(t) - r + \sum_{t_n<\sigma(t)} 2^{-n}\right)
\end{aligned}$$

since $t < \sigma(t)$. Hence, $S(\sigma(t))$ is true.

3. Let $t \in [r, s)$ be right-dense.

a. Let $t \in D$. Then f and g are differentiable at t. Hence, there exists a neighbourhood U of t such that

$$\left|f(t) - f(\tau) - f^{\Delta}(t)(t-\tau)\right| \leq \frac{\varepsilon}{2}|t-\tau|$$

and

$$\left|g(t) - g(\tau) - g^{\Delta}(t)(t-\tau)\right| \leq \frac{\varepsilon}{2}|t-\tau|$$

for all $\tau \in U$. Thus,

$$|f(t) - f(\tau)| \leq \left(\left|f^{\Delta}(t)\right| + \frac{\varepsilon}{2}\right)|t-\tau|$$

for all $\tau \in U$ and

$$g(\tau) - g(t) + g^{\Delta}(t)(t-\tau) \geq -\frac{\varepsilon}{2}|t-\tau|$$

for all $\tau \in U$, i.e.,

$$g(\tau) - g(t) - g^{\Delta}(t)(\tau-t) \geq -\frac{\varepsilon}{2}|t-\tau|$$

for all $\tau \in U$. Hence, for all $\tau \in U \cap (t, \infty)$, we have

$$\begin{aligned}
&|f(\tau) - f(r)| = |f(\tau) - f(t) + f(t) - f(r)| \\
&\leq |f(\tau) - f(t)| + |f(t) - f(r)| \\
&\leq \left(\left|f^{\Delta}(t)\right| + \frac{\varepsilon}{2}\right)|t-\tau| + g(t) - g(r) + \varepsilon\left(t - r + \textstyle\sum_{t_n<t} 2^{-n}\right) \\
&\leq \left(g^{\Delta}(t) + \frac{\varepsilon}{2}\right)|t-\tau| + g(t) - g(r) + \varepsilon\left(t - r + \textstyle\sum_{t_n<t} 2^{-n}\right) \\
&= g^{\Delta}(t)(\tau-t) + \frac{\varepsilon}{2}(\tau-t) + g(t) - g(r) + \varepsilon\left(t - r + \textstyle\sum_{t_n<t} 2^{-n}\right) \\
&\leq g(\tau) - G(T) + \frac{\varepsilon}{2}|t-\tau| + \frac{\varepsilon}{2}(\tau-t) + g(t) - g(r) + \varepsilon\left(t - r + \textstyle\sum_{t_n<t} 2^{-n}\right) \\
&= g(\tau) - g(r) + \varepsilon(\tau-t) + \varepsilon\left(t - r + \textstyle\sum_{t_n<t} 2^{-n}\right) \\
&= g(\tau) - g(r) + \varepsilon\left(\tau - r + \textstyle\sum_{t_n<t} 2^{-n}\right)
\end{aligned}$$

Therefore, $S(\tau)$ is true for all $\tau \in U \cap (t, \infty)$.

b. Let $t \notin D$. Then $t = t_m$ for some $m \in \mathbb{N}$. Since f and g are pre-differentiable, they both are continuous. Therefore, there exists a neighbourhood U of t such that

$$|f(\tau) - f(t)| \leq \frac{\varepsilon}{2}2^{-m} \quad \text{for all} \quad \tau \in U$$

and

$$|g(\tau) - g(t)| \leq \frac{\varepsilon}{2}2^{-m} \quad \text{for all} \quad \tau \in U$$

Therefore,

$$g(\tau) - g(t) \geq -\frac{\varepsilon}{2}2^{-m} \quad \text{for all} \quad \tau \in U$$

Consequently,

$$\begin{aligned}
|f(\tau) - f(r)| &= |f(\tau) - f(t) + f(t) - f(r)| \\
&\leq |f(\tau) - f(t)| + |f(t) - f(r)| \\
&\leq \frac{\varepsilon}{2}2^{-m} + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t} 2^{-n}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\varepsilon}{2}2^{-m} + g(\tau) + \frac{\varepsilon}{2}2^{-m} - g(r) + \varepsilon\left(t - r + \textstyle\sum_{t_n<t} 2^{-n}\right)\\
&= \varepsilon 2^{-m} + g(\tau) - g(r) + \varepsilon\left(t - r + \textstyle\sum_{t_n<t} 2^{-n}\right)\\
&\leq \varepsilon 2^{-m} + g(\tau) - g(r) + \varepsilon\left(\tau - r + \textstyle\sum_{t_n<\tau} 2^{-n}\right),
\end{aligned}$$

so $S(\tau)$ is true for all $\tau \in U \cap (t, \infty)$.

4. Let t be left-dense and assume that $S(\tau)$ is true for $\tau \in [r, t)$. Then

$$\begin{aligned}
\lim_{\tau\to t-} |f(\tau) - f(r)| &\leq \lim_{\tau\to t-}\left\{g(\tau) - g(r) + \varepsilon\left(\tau - r + \sum_{t_n<\tau} 2^{-n}\right)\right\}\\
&\leq \lim_{\tau\to t-}\left\{g(\tau) - g(r) + \varepsilon\left(\tau - r + \sum_{t_n<t} 2^{-n}\right)\right\}
\end{aligned}$$

implies that $S(t)$ is true since f and g are continuous at t.

Thus, using the induction principle, if follows that $S(t)$ is true for all $t \in [r, s]$. Consequently, (10.1) holds for all $r \leq s, r, s \in \mathbb{T}$.

Theorem 10.4

Suppose $f : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D. If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then

$$|f(s) - f(r)| \leq \left\{\sup_{t\in U^\kappa\cap D} \left|f^\Delta(t)\right|\right\} |s - r|.$$

Proof. Without loss of generality, we suppose that $r \leq s$. We set

$$g(t) = \left\{\sup_{\tau\in U^\kappa\cap D} \left|f^\Delta(\tau)\right|\right\} (t - r), \quad t \in \mathbb{T}$$

Then

$$g^\Delta(t) = \left\{\sup_{\tau\in U^\kappa\cap D} \left|f^\Delta(\tau)\right|\right\} \geq \left|f^\Delta(t)\right|$$

for all $t \in D \cap [r, s]^\kappa$. Thus, using Theorem 0.8, we obtain

$$|f(t) - f(r)| \leq g(t) - g(r) \quad \text{for all} \quad t \in [r, s]$$

whereupon

$$|f(s) - f(r)| \leq g(s) - g(r) = g(s) = \left\{\sup_{t\in U^\kappa\cap D} \left|f^\Delta(t)\right|\right\} (s - r)$$

Theorem 10.5

Let f be pre-differentiable with D. If $f^\Delta(t) = 0$ for all $t \in D$, then f is a constant function.

Proof. Let U be a compact interval with endpoints $r, s \in \mathbb{T}$. From Theorem 0.9, it follows that for all $r, s \in \mathbb{T}$,

$$|f(s) - f(r)| \leq \left\{\sup_{t\in U^\kappa\cap D} \left|f^\Delta(t)\right|\right\} |s - r| = 0$$

i.e., $f(s) = f(r)$. Therefore, f is a constant function.

Theorem 10.6

Let f and g be pre-differentiable with D. If $f^{\Delta}(t) = g^{\Delta}(t)$ for all $t \in D$, then

$$g(t) = f(t) + C \quad \text{for all} \quad t \in \mathbb{T}$$

where C is a constant.

Proof. Let $h(t) = f(t) - g(t), t \in \mathbb{T}$. Then

$$h^{\Delta}(t) = f^{\Delta}(t) - g^{\Delta}(t) = 0 \quad \text{for all} \quad t \in D$$

Thus, using Theorem **10.5**, it follows that h is a constant function.

Theorem 10.7

Suppose $f_n : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D for each $n \in \mathbb{N}$. Assume that for each $t \in \mathbb{T}^{\kappa}$, there exists a compact interval neighbourhood $U(t)$ such that the sequence $\{f_n^{\Delta}\}_{n\in\mathbb{N}}$ converges uniformly on $U(t) \cap D$.

(i) If $\{f_n\}_{n\in\mathbb{N}}$ converges at some $t_0 \in U(t)$ for some $t \in \mathbb{T}^{\kappa}$, then it converges uniformly on $U(t)$.
(ii) If $\{f_n\}_{n\in\mathbb{N}}$ converges at some $t_0 \in \mathbb{T}$, then it converges uniformly on $U(t)$ for all $t \in \mathbb{T}^{\kappa}$.
(iii) The limit mapping $f = \lim_{n\to\infty} f_n$ is pre-differentiable with D and

$$f^{\Delta}(t) = \lim_{n\to\infty} f_n^{\Delta}(t) \quad \text{for all} \quad t \in D$$

Proof.

1. (i)] Since $\{f_n^{\Delta}\}_{n\in\mathbb{N}}$ converges uniformly on $U(t) \cap D$, there exists $N \in \mathbb{N}$ such that

$$\sup_{s\in U(t)\cap D} \left|(f_m - f_n)^{\Delta}(s)\right|$$

is finite for all $m, n \in \mathbb{N}$. Let $m, n \geq N$ and $r \in U(t)$. Then

$$\begin{aligned} |f_n(r) - f_m(r)| &= |f_n(r) - f_m(r) - (f_n(t_0) - f_m(t_0)) + (f_n(t_0) - f_m(t_0))| \\ &\leq |f_n(t_0) - f_m(t_0)| + \left\{ \sup_{s\in U(t)\cap D} \left|(f_n - f_m)^{\Delta}(s)\right| \right\} |r - t_0| \end{aligned}$$

Hence, $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $U(t)$, i.e., $\{f_n\}_{n\in\mathbb{N}}$ is a locally uniformly convergent sequence.

(ii) Assume $\{f_n(t_0)\}_{n\in\mathbb{N}}$ converges for some $t_0 \in \mathbb{T}$. Consider the statement

$$S(t) : \{f_n(t)\}_{n\in\mathbb{N}} \quad \text{converges.}$$

a. Since $\{f_n(t_0)\}$ converges, the statement $S(t_0)$ is true.

b. Let t be right-scattered and assume that $S(t)$ is true. Then

$$f_n(\sigma(t)) = f_n(t) + \mu(t) f_n^{\Delta}(t)$$

converges by the assumption, i.e., $S(\sigma(t))$ is true.

c. Let t be right-dense and assume that $S(t)$ is true. Then, by (i), $\{f_n\}_{n\in\mathbb{N}}$ converges on $U(t)$, and so $S(r)$ is true for all $r \in U(t) \cap (t, \infty)$.

d. Let t be left-dense and assume that $S(r)$ is true for all $t_0 \leq r < t$. Since $U(t) \cap [t_0, t) \neq \emptyset$, using again part (i), we have that $\{f_n\}_{n\in\mathbb{N}}$ converges on $U(t)$; in particular, $S(t)$ is true.

Consequently, $S(t)$ is true for all $t \in [t_0, \infty)$. Using the dual version of the induction principle for the negative direction, we have that $S(t)$ is also true for all $t \in (-\infty, t_0]$ (we note that the first part of this has already been shown, the second part follows by $f_n(\rho(t)) = f_n(t) - \mu(\rho(t)) f_n^{\Delta}(\rho(t))$, the third part and the fourth part follow again by (i)).

(iii) Let $t \in D$. Without loss of generality, we can assume that $\sigma(t) \in U(t)$. We take $\varepsilon > 0$ arbitrarily. Using (i), there exists $N \in \mathbb{N}$ such that

$$|(f_n - f_m)(r) - (f_n - f_m)(\sigma(t))| \leq \left\{ \sup_{s \in U(t) \cap D} \left| (f_n - f_m)^{\Delta}(s) \right| \right\} |r - \sigma(t)|$$

for all $r \in U(t)$ and all $m, n \geq N$. Since

$$\left\{ f_n^{\Delta} \right\}_{n \in \mathbb{N}}$$

converges uniformly on $U(t) \cap D$, there exists $N_1 \geq N$ such that

$$\sup_{s \in U(t) \cap D} \left| (f_n - f_m)^{\Delta}(s) \right| \leq \frac{\varepsilon}{3} \quad \text{for all} \quad m, n \geq N_1$$

Hence,

$$|(f_n - f_m)(r) - (f_n - f_m)(\sigma(t))| \leq \frac{\varepsilon}{3} |r - \sigma(t)|$$

for all $r \in U(t)$ and for all $m, n \geq N_1$. Now, letting $m \to \infty$, we obtain

$$|(f_n - f)(r) - (f_n - f)(\sigma(t))| \leq \frac{\varepsilon}{3} |r - \sigma(t)|$$

for all $r \in U(t)$ and all $n \geq N_1$. Let

$$g = \lim_{n \to \infty} f_n^{\Delta}$$

Then there exists $M \geq N_1$ such that

$$\left| f_M^{\Delta}(t) - g(t) \right| \leq \frac{\varepsilon}{3}$$

and since f_M is differentiable at t, there also exists a neighbourhood W of t with

$$\left| f_M(\sigma(t)) - f_M(r) - f_M^{\Delta}(t)(\sigma(t) - r) \right| \leq \frac{\varepsilon}{3} |\sigma(t) - r|$$

for all $r \in W$. From here, for all $r \in U(t) \cap W$, we get

$$\begin{aligned} |f(\sigma(t)) - f(r) - g(t)|\sigma(t) - r|| &\leq |(f_M - f)(\sigma(t)) - (f_M - f)(r)| \\ &+ \left| f_M^{\Delta}(t) - g(t) \right| |\sigma(t) - r| + \left| f_M(\sigma(t)) - f_M(r) - f_M^{\Delta}(t) \right| \sigma(t) - r|| \\ &\leq \frac{\varepsilon}{3} |\sigma(t) - r| + \frac{\varepsilon}{3} |\sigma(t) - r| + \frac{\varepsilon}{3} |\sigma(t) - r| \\ &= \varepsilon |\sigma(t) - r| \end{aligned}$$

Consequently, f is differentiable at t with $f^{\Delta}(t) = g(t)$.

Theorem 10.8

Let $t_0 \in \mathbb{T}, x_0 \in \mathbb{R}$. Assume $f : \mathbb{T}^{\kappa} \to \mathbb{R}$ is regulated. Then there exists exactly one pre-differentiable function F with D satisfying

$$F^{\Delta}(t) = f(t) \quad \text{for all} \quad t \in D, \quad F(t_0) = x_0.$$

Proof. Let $n \in \mathbb{N}$ and consider the statement

$$S(t) : \begin{cases} \text{there exists a pre-differentiable} & (F_{nt}, D_{nt}), \\ F_{nt} : [t_0, t] \to \mathbb{R} \quad \text{with} & F_{nt}(t_0) = x_0 \quad \text{and} \\ \left| F_{nt}^{\Delta}(s) - f(s) \right| \leq \dfrac{1}{n} & \text{for} \quad s \in D_{nt}. \end{cases}$$

1. Let $t = t_0, D_{nt_0} = \emptyset$ and $F_{nt_0}(t_0) = x_0$. Then the statement $S(t_0)$ is true.

2. Let t be right-scattered and assume that $S(t)$ is true. Define

$$D_{n\sigma(t)} = D_{nt} \cup \{t\}$$

and $F_{n\sigma(t)}$ on $[t_0, \sigma(t)]$ by

$$F_{n\sigma(t)}(s) = \begin{cases} F_{nt}(s) & \text{if} \quad s \in [t_0, t] \\ F_{nt}(t) + \mu(t) f(t) & \text{if} \quad s = \sigma(t) \end{cases}$$

Then

$$F_{n\sigma(t)}(t_0) = F_{nt}(t_0) = x_0$$
$$\left|F^{\Delta}_{n\sigma(t)}(s) - f(s)\right| = \left|F^{\Delta}_{nt}(s) - f(s)\right| \leq \frac{1}{n} \quad \text{for} \quad s \in D_{nt}$$

and

$$\begin{aligned} \left|F^{\Delta}_{n\sigma(t)}(t) - f(t)\right| &= \left|\frac{F_{n\sigma(t)}(\sigma(t)) - F_{n\sigma(t)}(t)}{\mu(t)} - f(t)\right| \\ &= \left|\frac{F_{nt}(t) + \mu(t) f(t) - F_{n\sigma(t)}(t)}{\mu(t)} - f(t)\right| \\ &= \left|\frac{F_{nt}(t) + \mu(t) f(t) - F_{nt}(t)}{\mu(t)} - f(t)\right| \\ &= \left|\frac{\mu(t) f(t)}{\mu(t)} - f(t)\right| \\ &= 0 \\ &\leq \frac{1}{n} \end{aligned}$$

and therefore the statement $S(\sigma(t))$ is true.

3. Let t be right-dense and assume that $S(t)$ is true. Since t is right-dense and f is regulated,

$$f(t^+) = \lim_{s \to t, s > t} f(s) \quad \text{exists.}$$

Hence, there exists a neighbourhood U of t with

$$\left|f(s) - f(t^+)\right| \leq \frac{1}{n} \quad \text{for all} \quad s \in U \cap (t, \infty). \tag{10.2}$$

Let $r \in U \cap (t, \infty)$. Define

$$D_{nr} = (D_{nt} \backslash \{t\}) \cup [t, r]^{\kappa}$$

and F_{nr} on $[t_0, r]$ by

$$F_{nr}(s) = \begin{cases} F_{nt}(s) & \text{if} \quad s \in [t_0, t] \\ F_{nt}(t) + f(t^+)(s - t) & \text{if} \quad s \in (t, r]. \end{cases}$$

Then F_{nr} is continuous at t and hence on $[t_0, r]$. Also, F_{nr} is differentiable on $(t, r)^{\kappa}$ with

$$F^{\Delta}_{nr}(s) = f(t^+) \quad \text{for all} \quad s \in (t, r]^{\kappa}.$$

Hence, F_{nr} is pre-differentiable on $[t_0, t)$. Since t is right-dense, we have that F_{nr} is pre-differentiable with D_{nr}.

From here and from (10.2), we also have

$$\left|F^{\Delta}_{nr}(s) - f(s)\right| \leq \frac{1}{n} \quad \text{for all} \quad s \in D_{nr}$$

Therefore, the statement $S(r)$ is true for all $r \in U \cap (t, \infty)$.

4. Now, let t be left-dense and suppose that $S(r)$ is true for $r < t$. Since f is regulated,

$$f(t^-) = \lim_{s \to t, s<t} f(s) \quad \text{exists.} \tag{10.3}$$

Hence, there exists a neighbourhood U of t with

$$|f(s) - f(t^-)| \leq \frac{1}{n} \quad \text{for all} \quad s \in U \cap (-\infty, t).$$

Fix some $r \in U \cap (-\infty, t)$ and define

$$D_{nt} = \begin{cases} D_{nr} \cup (r, t) & \text{if } r \text{ is right-dense} \\ D_{nr} \cup [r, t) & \text{if } r \text{ is right-scattered} \end{cases}$$

and F_{nt} on $[t_0, t]$ by

$$F_{nt}(s) = \begin{cases} F_{nr}(s) & \text{if} \quad s \in (t_0, r] \\ F_{nr}(t) + f(t^-)(s - r) & \text{if} \quad s \in (r, t] \end{cases}$$

We note that F_{nt} is continuous at r and hence in $[t_0, t]$. Since

$$F_{nt}^{\Delta}(s) = f(t^-) \quad \text{for all} \quad s \in (r, t]$$

F_{nt} is pre-differentiable with D_{nt} and

$$\left|F_{nt}^{\Delta}(s) - f(s)\right| \leq \frac{1}{n} \quad \text{for all} \quad s \in D_{nt}$$

Hence, the statement $S(t)$ is true.

By the induction principle, it follows that $S(t)$ is true for all $t \geq t_0, t \in \mathbb{T}$. Similarly, we can show that $S(t)$ is valid for $t \leq t_0$. Hence, F_n is pre-differentiable with $D_n, F_n(t_0) = x_0$, and

$$\left|F_n^{\Delta}(t) - f(t)\right| \leq \frac{1}{n} \quad \text{for all} \quad t \in D_n.$$

Now, let

$$F = \lim_{n \to \infty} F_n \quad \text{and} \quad D = \bigcap_{n \in \mathbb{N}} D_n.$$

Then $F(t_0) = x_0, f$ is pre-differentiable on D, and, using Theorem 10.7,

$$F^{\Delta}(t) = \lim_{n \to \infty} F_n^{\Delta}(t) = f(t) \quad \text{for all} \quad t \in D$$

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11: The Delta Integral

Definition: Cauchy Integral

Assume $f:\mathbb{T}\to\mathbb{R}$ is a regulated function. Any function F as in Theorem 10.8 is called a preantiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t)+c$$

where c is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_s^t f(\tau)\Delta\tau = F(t)-F(s) \quad \text{for all} \quad s,t\in\mathbb{T}$$

Definition: Antiderivative of a function

A function $F:\mathbb{T}\to\mathbb{R}$ is called an antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided

$$F^{\Delta}(t)=f(t) \quad \text{holds for all} \quad t\in\mathbb{T}^{\kappa}.$$

Example 11.1

Let $\mathbb{T}=\mathbb{Z}$. Then $\sigma(t)=t+1, t\in\mathbb{T}$. Assume $f(t)=3t^2+5t+2, t\in\mathbb{T}$. Since $g(t)=t^3+t^2$ satisfies

$$\begin{aligned} g^{\Delta}(t) &= (\sigma(t))^2+t\sigma(t)+t^2+\sigma(t)+t \\ &= (t+1)^2+t(t+1)+t^2+t+1+t \\ &= t^2+2t+1+t^2+t+t^2+2t+1 \\ &= 3t^2+5t+2 \end{aligned}$$

we have

$$\int \left(3t^2+5t+2\right)\Delta t = t^3+t^2+c$$

Example 11.2

Let $\mathbb{T}=2^{\mathbb{N}}$ and define $f:\mathbb{T}\to\mathbb{R}$ by $f(t)=\frac{2}{t}\sin\frac{t}{2}\cos\frac{3t}{2}, t\in\mathbb{T}$. In this case, we have that $\sigma(t)=2t$. Since $g(t)=\sin t$ satisfies

$$\begin{aligned} g^{\Delta}(t) &= \frac{\sin\sigma(t)-\sin t}{\sigma(t)-t} \\ &= \frac{\sin(2t)-\sin t}{t} \\ &= \frac{2}{t}\sin\frac{t}{2}\cos\frac{3t}{2} \end{aligned}$$

we get

$$\int \frac{2}{t}\sin\frac{t}{2}\cos\frac{3t}{2}\Delta t = \sin t+c$$

Example 11.3

Let $\mathbb{T}=\mathbb{N}_0^2$ and define $f:\mathbb{T}\to\mathbb{R}$ by $f(t)=\dfrac{1}{1+2\sqrt{t}}\log\dfrac{(1+\sqrt{t})^2}{t}, t\in\mathbb{T}$. Here, $\sigma(t)=(1+\sqrt{t})^2$.

Since $g(t)=\log t$ satisfies

$$\begin{aligned} g^{\Delta}(t) &= \frac{\log\sigma(t)-\log t}{\sigma(t)-t} \\ &= \frac{\log(1+\sqrt{t})^2-\log t}{(1+\sqrt{t})^2-t} \\ &= \frac{1}{1+2\sqrt{t}}\log\frac{(1+\sqrt{t})^2}{t} \end{aligned}$$

we get

$$\int \frac{1}{1+2\sqrt{t}}\log\frac{(1+\sqrt{t})^2}{t}\Delta t=\log t+c$$

Example 11.1

Let $\mathbb{T}=\left\{-\dfrac{1}{n}:n\in\mathbb{N}\right\}\cup\mathbb{N}_0$ and

$$f(t)=\begin{cases} \dfrac{6t^2+9t-7}{(1+7t)(1+6t)} & \text{if} \quad t\in\left\{-\dfrac{1}{n}:n\in\mathbb{N}\right\} \\ \dfrac{7t^2+9t-6}{(1+7t)(8+7t)} & \text{if} \quad t\in\mathbb{N}_0 \end{cases}$$

Set

$$\frac{1+t^2}{1+7t},\quad t\in\mathbb{T}$$

By Example 7.8, we get

$$F^{\Delta}(t)=f(t),\quad t\in\mathbb{T}$$

Therefore

$$\int f(t)\Delta t=\frac{1+t^2}{1+7t}+C,\quad t\in\mathbb{T}$$

where C is a constant.

Example 11.1

Let $\mathbb{T}=\left\{\left(\dfrac{1}{2}\right)^{2^n}:n\in\mathbb{N}_0\right\}\cup\{0,1\}$ and

$$f(t)=\begin{cases} \dfrac{23}{(4+5t)(4+5\sqrt{t})} & \text{if} \quad t\in\left\{\left(\dfrac{1}{2}\right)^{2^n}:n\in\mathbb{N}\right\} \\ \dfrac{46}{117} \quad \text{if} \quad t=\dfrac{1}{2} \\ \dfrac{23}{16} \quad \text{if} \quad t=0. \end{cases}$$

Set

$$F(t)=\frac{1+7t}{4+5t},\quad t\in\mathbb{T}$$

By Example 7.9, we get

$$F^{\Delta}(t)=f(t),\quad t\in\mathbb{T}$$

Therefore

$$\int f(t)\Delta t=\frac{1+7t}{4+5t}+C,\quad t\in\mathbb{T}$$

where C is a constant.

Example 11.1

Let $U=\left\{\frac{1}{2^n}:n\in\mathbb{N}\right\}$ and

$$\mathbb{T}=U\cup(1-U)\cup(1+U)\cup(2-U)\cup(2+U)\cup\{0,1,2\},$$

and

$$f(t)=\begin{cases}-\frac{151}{64} & \text{if} \quad t=\frac{1}{2}\\ -\frac{1195}{64} & \text{if} \quad t=\frac{3}{2}\\ -15t^3-3t & \text{if} \quad t\in U\setminus\left\{\frac{1}{2}\right\}\\ -\frac{15t^3+11t^2+17t+5}{8} & \text{if} \quad t\in(1-U)\setminus\left\{\frac{1}{2}\right\}\\ -15t^3+17t^2-10t+2 & \text{if} \quad t\in(1+U)\setminus\left\{\frac{3}{2}\right\}\\ -\frac{15t^3+22t^2+32t+8}{8} & \text{if} \quad t\in(2-U)\setminus\left\{\frac{3}{2}\right\}\\ -15t^3+34t^2-31t+10 & \text{if} \quad t\in(2+U)\setminus\left\{\frac{5}{2}\right\}\\ 0 \quad \text{if} \quad t=0\\ -6 \text{ if} \quad t=1\\ -36 \text{ if} \quad t=2.\end{cases}$$

Set

$$F(t)=\left(2+t^2\right)\left(1-t^2\right),\quad t\in\mathbb{T}.$$

By Example 7.10, we get

$$F^{\Delta}(t)=f(t),\quad t\in\mathbb{T}^{\kappa}$$

Therefore

$$\int f(t)\Delta t=\left(2+t^2\right)\left(1-t^2\right)+C,\quad t\in\mathbb{T}^{\kappa}$$

where C is a constant.

? Exercise 11.1

Let $\mathbb{T} = \mathbb{N}_0^3$. Prove that

$$\int \left(2t + 3\sqrt[3]{t^2} + 3\sqrt[3]{t} + 2\right) \Delta t = t^2 + t + c$$

Answer

Add texts here. Do not delete this text first.

Theorem 11.1

Every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then f defined by

$$F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau \quad \text{for} \quad t \in \mathbb{T},$$

is an antiderivative of f.

Proof. Since f is rd-continuous, it is regulated. Let F be a function guaranteed to exist by Theorem 0.13, together with D, satisfying

$$F^{\Delta}(t) = f(t) \quad \text{for} \quad t \in D$$

We have that F is pre-differentiable with D. Let $t \in \mathbb{T}^{\kappa} \backslash D$. Then t is right-dense. Since f is rd-continuous, f is continuous at t. Let $\varepsilon > 0$ be arbitrarily chosen. Then there exists a neighbourhood U of t such that

$$|f(s) - f(t)| \leq \varepsilon \quad \text{for all} \quad s \in$$

We define

$$h(\tau) = F(\tau) - f(t)\,(\tau - t_0) \quad \text{for} \quad \tau \in \mathbb{T}.$$

Then h is pre-differentiable with D and

$$\begin{aligned} h^{\Delta}(\tau) &= F^{\Delta}(\tau) - f(t) \\ &= f(\tau) - f(t) \quad \text{for all} \quad \tau \in D \end{aligned}$$

Hence,

$$\begin{aligned} \left|h^{\Delta}(s)\right| &= |f(s) - f(t)| \\ &\leq \varepsilon \quad \text{for all} \quad s \in D \cap U \end{aligned}$$

Therefore,

$$\sup_{s \in D \cap U} \left|h^{\Delta}(s)\right| \leq \varepsilon,$$

whereupon

$$\begin{aligned} |F(t) - F(r) - f(t)(t - r)| = |\, & h(t) + f(t)\,(t - t_0) - (h(r) \\ & + f(t)\,(r - t_0)) - f(t)(t - r)\,| \\ = & |h(t) - h(r)| \\ \leq & \left\{ \sup_{s \in D \cap U} \left|h^{\Delta}(s)\right| \right\} |t - r| \\ \leq & \varepsilon |t - r| \end{aligned}$$

which shows that f is differentiable at t and $F^{\Delta}(t) = f(t)$.

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