

MATH 384: LECTURE NOTES



Roy Simpson
Cosumnes River College

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (<https://LibreTexts.org>) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden borne from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by [NICE CXone Expert](#) and were supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptations contact info@LibreTexts.org or visit our main website at <https://LibreTexts.org>.

The LibreTexts name, logo, and book covers are trademarked by LibreTexts, Inc. (a 501(c)(3) not-for-profit organization) and protected against unauthorized use.

This text was compiled on 10/16/2026

TABLE OF CONTENTS

Licensing

1: Triangles and Circles

- 1.1: Angles and Basic Geometry
- 1.2: Triangles
- 1.3: Similar Triangles
- 1.4: Circles and Angles in the Rectangular Coordinate System

2: Functions

- 2.1: Functions and Function Notation
- 2.2: Domain and Range
- 2.3: The Arithmetic and Composition of Functions
- 2.4: Behavior of Graphs of Functions
- 2.5: Transformations of Functions
- 2.6: Absolute Value Functions
- 2.7: Functions and Function Notation (1)**
 - 2.7.1: Resources and Key Concepts
 - 2.7.2: Homework

3: Polynomial and Rational Functions

- 3.1: Quadratic Functions
- 3.2: Power Functions and Polynomial Functions
- 3.3: Graphs of Polynomial Functions
- 3.4: Dividing Polynomials
- 3.5: Rational Functions

4: Inverse and Radical Functions

- 4.1: Inverse Functions
- 4.2: Radical Functions
- 4.3: Radical Equations
- 4.4: Rational Exponent Expressions Encountered in Calculus

5: Coordinate Trigonometry

- 5.1: The Trigonometric Functions - Coordinate Definition
- 5.2: Algebra with Trigonometric Functions
- 5.3: Introduction to Trigonometric Identities
- 5.4: Introduction to Proofs in Trigonometry

6: Right Triangle Trigonometry

- 6.1: The Trigonometric Functions - Right Triangle Definition
- 6.2: Technology and Right Triangle Trigonometry
- 6.3: Solving Right Triangles

6.4: Applications Involving Right Triangles

7: Radian Measure and the Circular Functions

7.1: Reference Angles

7.2: Radian Measure, Arc Length, and Sector Area

7.3: The Trigonometric Functions - Unit Circle Definition

8: Graphs of the Trigonometric Functions

8.1: Base Graphs of the Fundamental Trigonometric Functions

8.2: Non-Rigid Transformations

8.3: Rigid Transformations

8.4: Graphs of the Remaining Trigonometric Functions

9: Analytic Trigonometry

9.1: Basic Trigonometric Identities and Proof Techniques

9.2: Sum and Difference Identities

9.3: Double-Angle Identities

10: Trigonometric Equations

10.1: The Fundamental Inverse Trigonometric Functions

10.2: The Remaining Inverse Trigonometric Functions

10.3: Solving Trigonometric Equations - Algebraic Techniques

10.4: Solving Trigonometric Equations Using Identities

11: Non-Right Triangle Trigonometry

11.1: The Law of Sines

11.2: The Law of Cosines

12: Exponential and Logarithmic Functions

12.1: Exponential Functions

12.2: Graphs of Exponential Functions

12.3: Logarithmic Functions

12.4: Graphs of Logarithmic Functions

12.5: Logarithmic Properties

12.6: Exponential and Logarithmic Equations

12.7: Exponential and Logarithmic Models

Index

Glossary

Detailed Licensing

Detailed Licensing

This page titled [Math 384: Lecture Notes](#) is shared under a [CC BY-SA 4.0](#) license and was authored, remixed, and/or curated by Roy Simpson.

Licensing

A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.

1: TRIANGLES AND CIRCLES

1.1: Angles and Basic Geometry

1.2: Triangles

1.3: Similar Triangles

1.4: Circles and Angles in the Rectangular Coordinate System

1.1: Angles and Basic Geometry

Note to the Instructor - Non-Triangle Geometry

The material in this section focuses on the basic Geometry needed for the rest of this course. Angles (in degrees), circles, and areas of circles are the main focus (anything to do with triangles is reserved for the next section).

Prerequisite Topics

- Sets and Numbers
 - The Language of Algebra
 - Sets of Numbers
- Arithmetic
 - Mathematical Operations - Addition, Subtraction, Multiplication, and Division
 - The Order of Operations
- Simplifying Expressions
 - Algebraic Expressions - An Introduction and Using in Applications
- Solving Equations
 - Solving Linear Equations
- Systems
 - Solving 2×2 Systems of Linear Equations

Angles

Definition: Ray

A **ray** is a line segment that has one endpoint and extends infinitely in one direction. The point from which the ray originates is called the **initial point** of the ray. A ray can also be described as a **half-line**.

Definition: Angle

When two rays share a common initial point, they form an **angle**, and the shared initial point is called the angle's **vertex**.

Definition: Straight Angle

A **straight angle** is defined as an angle that is formed when the two rays forming the angle extend in opposite directions.

Definition: Angle Measure

The **measure of an angle** is a number that indicates the amount of rotation that separates the rays of the angle.

Definition: Oriented Angle

An **oriented angle** refers to the measure of rotation between two rays with a common vertex, taking into account the direction of rotation (clockwise or counterclockwise). The angle is swept out starting from an **initial side** and ending at a **terminal side**. When the rotation is counterclockwise from the initial side to the terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

Alternative Angle Notations

Common Angle Notations

The common notations used for angles are as follows:

- Lowercase Greek letters (most commonly α , β , and θ)
- $\angle A$ or $m\angle A$, where A is the vertex of the rays where the angle is formed
- $\angle BAC$, where A is the vertex where the rays emanating from A and going through B and C meet.

Degree Measure

Definition: Degree Measure

We define **degree measure** by dividing the full rotation around the circle into 360 segments. The angular measure of one of these segments is defined to be one degree, denoted 1° . That is, 1° represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution.

Theorem: Degree Measure of a Straight Angle

The degree measure of a straight angle is 180° .

Definition: Right Angle

A **right angle** is defined to be the angle measure of a quarter-circle, which is 90° .

Definition: Acute Angle

An angle that measures strictly between 0° and 90° is called an **acute angle**.

Definition: Obtuse Angle

An angle that measures strictly between 90° and 180° is called an **obtuse angle**.

Definition: Complementary Angles

Two acute angles are called **complementary angles** if their measures add to 90° .

Definition: Supplementary Angles

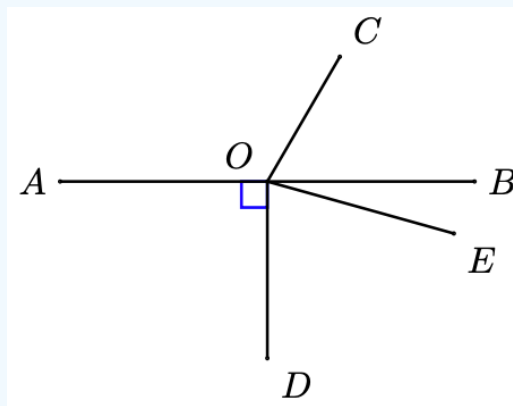
Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180° .

Caution

In this text, we will restrict the definition of complementary and supplementary angles to positive angles only. That is, despite their sum being 90° , we will *not* say that $\alpha = -20^\circ$ and $\beta = 110^\circ$ are complementary. Other textbooks allow for this silliness, but in truth there is no need to expand these definitions to include negative angles.

Example 1.1.1

Draw a figure similar to (but not exactly like) the figure below.



- Which angle pairs are supplementary?
- Which angle pairs are complementary?
- Which angles are obtuse?
- Which angles are acute?

Definition: Decimal Degree

An angle written as a decimal is said to belong to the **decimal degree (DD)** system of measuring angles.

Angles in Rotations

Example 1.1.2

Through how many degrees does the minute hand rotate in the time lapsed during one of our class sessions?

Geometry Necessary for Trigonometry

Definition: Vertical Angles

When two lines intersect, they form four angles. The two pairs of nonadjacent angles—those lying opposite each other at the point of intersection—are called **vertical angles** (or **vertically opposite angles**).

Definition: Equality of Angles

Two angles are defined to be **equal** if their measures are equal.

Theorem: Congruency of Vertical Angles

Vertical angles are equal (also known in Geometry as **congruent**).

Proof available at the end of this section.

Definition: Transversal

A **transversal** is a line that intersects two or more lines at distinct points.

Definition: Corresponding Angles

Corresponding angles are a pair of angles, one at each intersection, that lie on the same side of the transversal with one angle interior and the other exterior. Equivalently, they occupy the same relative position at the two intersections.

Theorem: Congruency of Corresponding Angles

[3.02: Vertical Alternate and Corresponding Angles](#)

Definition: Alternate Interior Angles

Alternate interior angles are a pair of interior angles that lie on opposite sides of the transversal, with one angle at each intersection.

Theorem: Alternate Interior Angles

[3.02: Vertical Alternate and Corresponding Angles](#)

Example 1.1.3

Take 21 from the MOM homework.

Circles

Definition: Circle

The shape created from all points in a plane that are equidistant from a given point is called a **circle**. The given point is called the **center** of the circle, and the distance from the center to any point along the circle is called the **radius** of the circle.

Definition: Radius (as a line segment)

A line segment with one endpoint at the center of a circle and the other on the circle is called a **radius** of the circle.

Definition: Diameter (as a distance)

The **diameter** of a circle is the length of any line segment going through the center of a circle and whose endpoints are on the circle.

Theorem: Diameter in Terms of Radius

Given a circle of radius r , the diameter, d , of the circle satisfies

$$d = 2r.$$

Definition: Diameter (as a line segment)

A line segment going through the center of a circle and whose endpoints are on the circle is called a **diameter** of the circle.

Circumference

Definition: Circumference

The **circumference** is the distance around a circle.

Theorem: Circumference

A circle's circumference, C , is proportional to its radius, r . That is,

$$C = kr.$$

The constant of proportionality is 2π . Hence,

$$C = 2\pi r.$$

Area of a Circle

Theorem: Area of a Circle

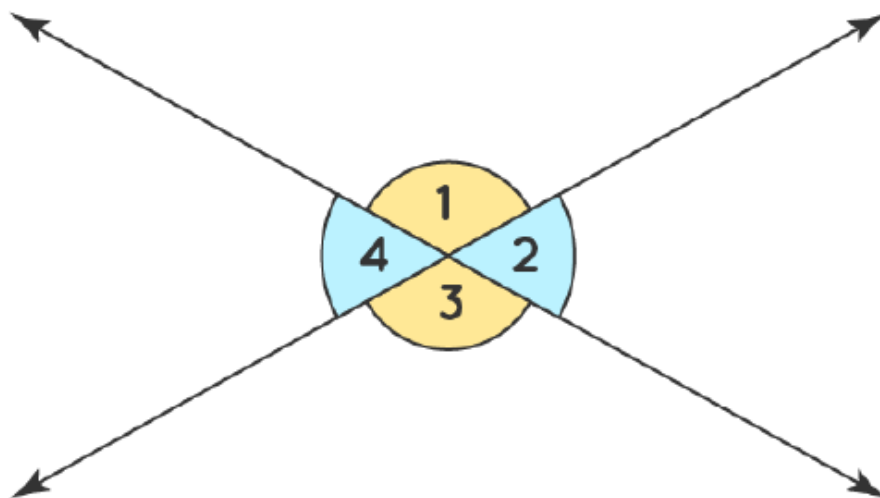
The area of a circle of radius r is given by

$$A = \pi r^2.$$

Answers

Theorem: Congruency of Vertical Angles

Suppose two lines intersect, as shown in the figure below.



Since $\angle 1 + \angle 2$ forms a straight angle,

$$\angle 1 + \angle 2 = 180^\circ.$$

By a similar argument,

$$\angle 1 + \angle 4 = 180^\circ.$$

Therefore,

$$\angle 1 + \angle 2 = 180^\circ = \angle 1 + \angle 4.$$

That is,

$$\angle 2 = \angle 4.$$

By a similar set of arguments, we can show that

$$\angle 1 = \angle 3.$$

Hence, it must always be the case that vertical angles are congruent.

1.2: Triangles

Note to the Instructor - Triangle Geometry

Triangles, being the focus of Trigonometry, deserve their own full section. Focus here is on types of triangles, the Pythagorean Theorem, the special triangles, and the area of a triangle (not requiring trigonometric functions). Many exercises involving the 30° - 60° - 90° and 45° - 45° - 90° that would normally be presented in Right Triangle Trigonometry are found in this section and its homework (but without reference to the trigonometric functions). Similar triangles are held off until next section.

Prerequisite Skills

- **Sets and Numbers**
 - Inequalities and Inequality Notation
- **Geometry**
 - Common Volumes
- **Simplifying Expressions**
 - Properties of Radicals and Simplifying Expressions Involving Roots
- **Solving Equations**
 - Solving Quadratic Equations (by Extraction of Roots)
 - Solving Radical Equations
- **Solving Inequalities**
 - Solving Linear Inequalities
 - Solving Compound Inequalities

Triangle Basics

Definition: Triangle

When three line segments bound a portion of the plane, the resulting shape is called a **triangle** (also known as a **three-sided polygon**). The line segments are called the **sides** of the triangle. A point where two sides meet is called a **vertex** of the triangle. Finally, the angle formed in the triangle's interior where two sides meet is called an **angle** of the triangle.

Theorem: Triangle Sum

[3.4: The Triangle Inequality Theorem](#)

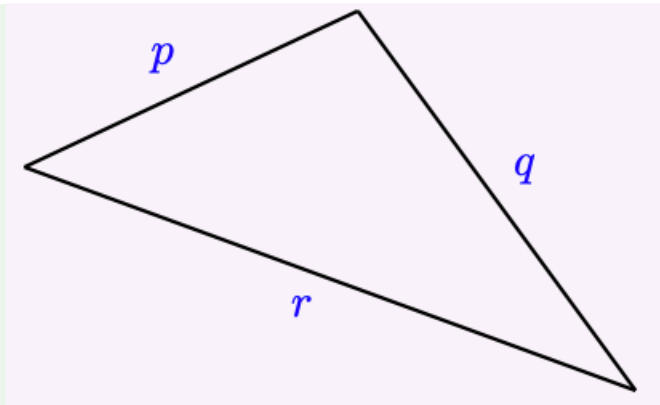
The Triangle Inequality

Theorem: Triangle Inequality

In any triangle, we must have that

$$p + q > r,$$

where p , q , and r are the lengths of the sides of the triangle (see the figure below).



That is, the sum of the lengths of any two sides of a triangle is greater than the length of the other side.

Types of Triangles

Equilateral Triangles

Definition: Equilateral Triangle

A triangle in which all three sides have the same length is called an **equilateral triangle**.

Theorem: Angles of an Equilateral Triangle

A triangle is equilateral if and only if it is equiangular, in which case each interior angle measures 60° .

Isosceles Triangles

Definition: Isosceles Triangle

A triangle in which at least two sides are of equal length is called an **isosceles triangle**. The angle between the equal sides is called the **vertex angle**, and the other two angles are called the **base angles**.

Theorem: Angles of an Isosceles Triangle

The base angles of an isosceles triangle are equal.

We introduce another common notation used throughout Geometry - the **tick mark**. We use tick marks on line segments to denote they have equal length. Similarly, tick marks on angles denote that the angles have the same measure.

Scalene Triangles

Definition: Scalene Triangle

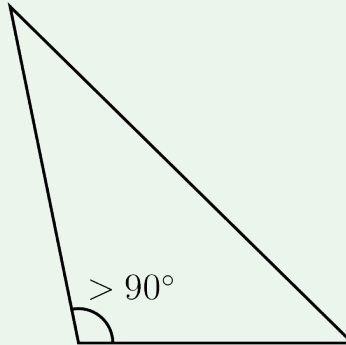
A triangle in which none of the sides have equal length (and, therefore, none of the angles are equal) is called a **scalene triangle**.

Classification of Triangles by Largest Angle

Obtuse Triangles

Definition: Obtuse Triangle

An **obtuse triangle** is a triangle that has one interior angle measuring more than 90° .



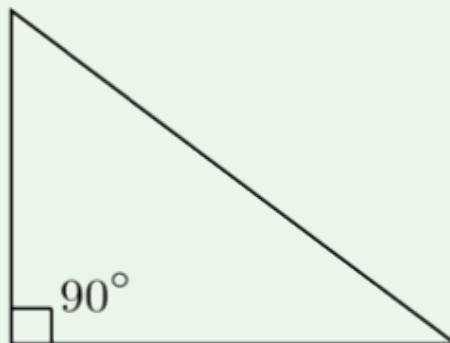
Theorem: Angles of an Obtuse Triangle

In an obtuse triangle, one angle must be greater than 90° , and the remaining two angles must each be less than 90° .

Right Triangles

Definition: Right Triangle

A **right triangle** is a triangle that has one interior angle measuring exactly 90° . The side opposite the right angle is the **hypotenuse**. The two remaining sides are the **legs**.

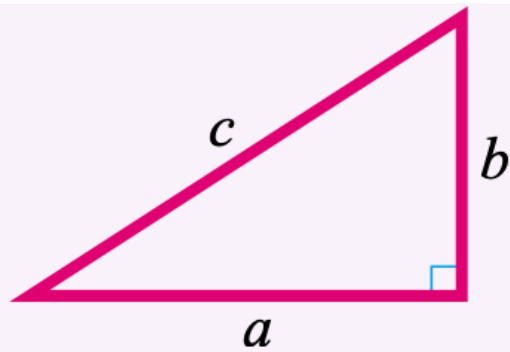


Right Triangles: The Pythagorean Theorem

Theorem: Pythagorean Theorem

In a right triangle, if c is the length of the hypotenuse, and the lengths of the two legs are denoted by a and b (as shown in the figure below), then

$$a^2 + b^2 = c^2.$$



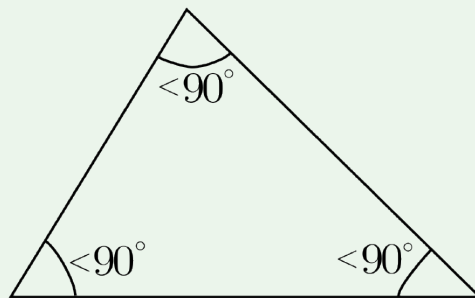
Caution

The Pythagorean Theorem applies only to right triangles, and c must always be the hypotenuse—the side opposite the right angle, which is also the longest side.

Acute Triangles

Definition: Acute Triangle

An **acute triangle** is a triangle whose three interior angles each measure less than 90° .



Special Triangles

Definition: Altitude of a Triangle

The **altitude** of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side (or the line containing the opposite side).

Theorem: Altitudes and Right Angles

The altitude of a triangle represents the shortest distance from the vertex to the opposite side and forms a 90° angle with that side.

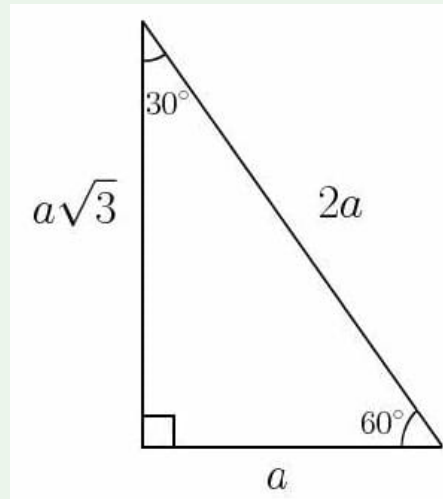
The 30°-60°-90° Triangle

Definition: 30°-60°-90° Triangle

A right triangle in which one angle is 30° and another angle is 60° is called a 30°-60°-90° **triangle**.

Theorem: Side Relationships for a 30°-60°-90° Triangle

In any right triangle in which the two acute angles are 30° and 60°, the hypotenuse is always twice the length of the shortest side (the side opposite the 30° angle), and the remaining side (opposite the 60° angle) is always $\sqrt{3}$ times the shortest side.



Proof available at the end of this section.

The 45°-45°-90° Triangle

Definition: Isosceles Right Triangle

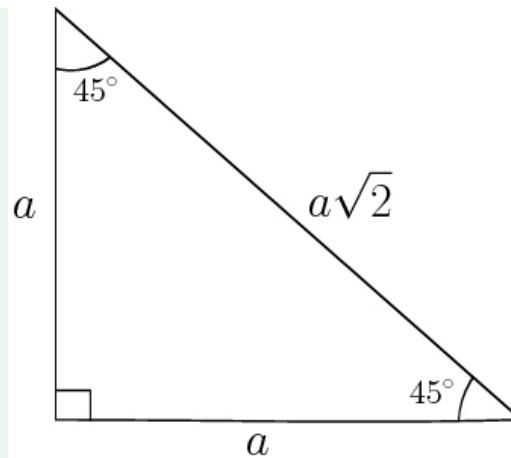
An isosceles triangle in which the vertex angle is 90° is called an **isosceles right triangle**.

Definition: 45°-45°-90° Triangle

Since an isosceles right triangle has two angles that are 45°, it is commonly called a 45°-45°-90° **triangle**.

Theorem: Side Relationship for a 45°-45°-90° Triangle

In any right triangle in which the two acute angles are 45°, the hypotenuse is always $\sqrt{2}$ times the side length.



Proof available at the end of this section.

Area of a Triangle

Theorem: Area of a Triangle

Let b be the length of one side of a triangle and h be the altitude height from that side to its opposite vertex. Then the area of the triangle is given by

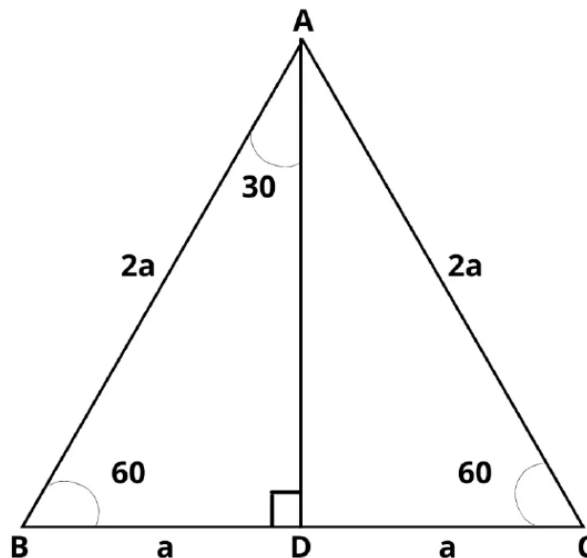
$$A = \frac{1}{2}bh.$$

Proof available at the end of this section.

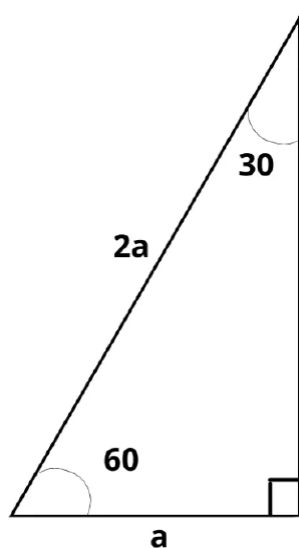
Answers

Theorem: Side Relationships for a 30° - 60° - 90° Triangle

Consider an equilateral triangle with side lengths $2a$, as shown in the figure below.



Since this triangle is equilateral, each angle is 60° . Drawing an altitude divides the triangle into two smaller triangles. Since the altitude is perpendicular to the base, each of these smaller triangles is a right triangle. Moreover, since one angle in each of these smaller right triangles is 60° , the remaining angle must be 30° . Furthermore, the side opposite the 30° angle must have length a (in the original equilateral triangle, the side opposite 60° had length $2a$, so it should make sense that the side opposite 30° is half of this). We now consider one of these smaller right triangles (see the figure below).

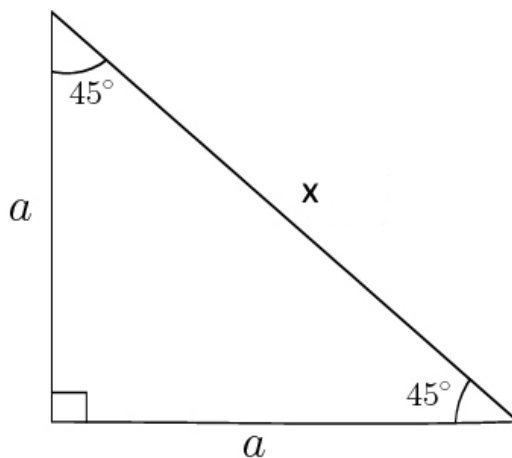


The missing side length, which we will call x for now, can be found using the Pythagorean Theorem.

$$\begin{aligned} x^2 + a^2 &= (2a)^2 \\ \Rightarrow x^2 + a^2 &= 4a^2 \\ \Rightarrow x^2 &= 3a^2 \\ \Rightarrow x &= a\sqrt{3} \end{aligned}$$

Theorem: Side Relationship for a 45° - 45° - 90° Triangle

The sides opposite the 45° angles in a 45° - 45° - 90° triangle must have the same length. Let a be the lengths of these sides and x be the length of the hypotenuse, as shown in the figure below.



Since this is a right triangle, we can use the Pythagorean Theorem to find the length of the hypotenuse in terms of a .

$$\begin{aligned} a^2 + a^2 &= x^2 \\ \Rightarrow 2a^2 &= x^2 \\ \Rightarrow a\sqrt{2} &= x \end{aligned}$$

Theorem: Area of a Triangle

To be done in homework.

1.3: Similar Triangles

Prerequisite Skills

- **Arithmetic**
 - Fractions
- **Solving Equations**
 - Solving Rational Equations

Congruent Triangles

Definition: Congruent Triangles

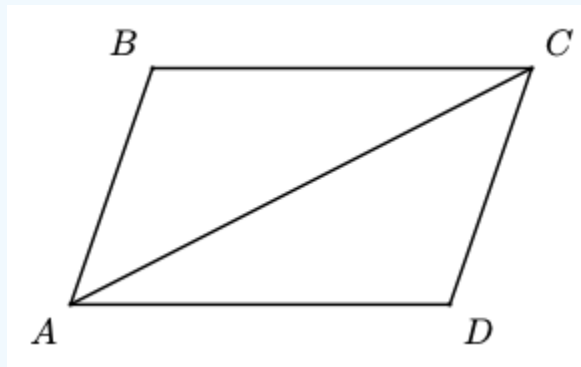
Two triangles are said to be **congruent** if their corresponding angles are equal, and their corresponding sides have the same lengths. Symbolically, we use the congruency symbol $\cong$ to state that two triangles are congruent.

Definition: Parallelogram

3.11: Other Common Geometric Shapes

Example 1.3.1

The diagonal of a parallelogram divides it into two congruent triangles. List the corresponding parts of the two triangles and explain why each pair is equal.



Similar Triangles

Definition: Similar Triangles

Two triangles are **similar** if their corresponding angles are congruent and the lengths of their corresponding sides are proportional. If $\triangle ABC$ is similar to $\triangle DEF$, you write $\triangle ABC \sim \triangle DEF$, where the order of the vertices specifies the correspondence: A with D , B with E , and C with F .

Fun Fact!

Congruent triangles are similar, but similar triangles need not be congruent.

Theorem: Similarity Conditions

Two triangles are similar whenever any one of the following conditions holds:

- **(AA)** Two angles of one triangle are congruent to two angles of the other. The third pair of angles is then automatically congruent.
- **(SSS)** All three pairs of corresponding sides are proportional.
- **(SAS)** Two pairs of corresponding sides are proportional, and the angles included between those sides are congruent.

MyOpenMath 1.3.5: Determining if triangles are similar

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Using Proportions with Similar Triangles

MyOpenMath 1.3.2: Taking advantage of similar triangles

MyOpenMath 1.3.15: Algebra in similar triangles

Similar Right Triangles

MyOpenMath 1.3.16: Separated similar right triangles in applications

Overlapping Triangles

MyOpenMath 1.3.17: Overlapping similar right triangles in applications

1.4: Circles and Angles in the Rectangular Coordinate System

Learning Objectives

- **Solving Equations**
 - Solving Quadratic Equations
- **Graphs**
 - The Cartesian Coordinate System and the Quadrants
 - Ordered Pairs and Graphing Relations by Point-Plotting

The Distance Formula

Theorem: Distance Formula

The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

MyOpenMath 1.4.6: Finding the straight-line distance between two points on a function

MyOpenMath 1.4.18: Solving for an unknown in a distance formula problem

Equation for a Circle

Theorem: Equation of a Circle Centered at the Origin

The equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Proof available at the end of this section.

MyOpenMath 1.4.26: Semi-circle equations

Definition: Unit Circle

The circle centered at the origin and having radius 1, is called the **unit circle**. The unit circle has equation

$$x^2 + y^2 = 1.$$

Theorem: Equation of a Circle Centered Off-Origin

The equation for a circle of radius r centered at the point (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is also known as the **standard form** of the equation of a circle.

Proof available at the end of this section.

MyOpenMath 1.4.9: Writing the equation of a circle and graphing

MyOpenMath 1.4.28: A challenging application

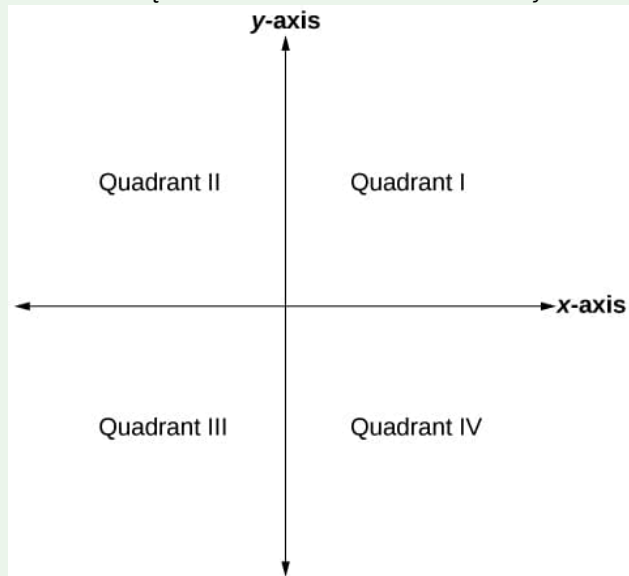
Angles in the Cartesian Coordinate System

Definition: Quadrants

In the Cartesian coordinate system, the section of the plane where

- x and y are both positive is called **Quadrant I (QI)**,
- $x < 0$ and $y > 0$ is called **Quadrant II (QII)**,
- x and y are both negative is called **Quadrant III (QIII)**, and
- $x > 0$ and $y < 0$ is called **Quadrant IV (QIV)**.

The four quadrants of the Cartesian coordinate system



Angles in Standard Position

Definition: Standard Position of an Angle

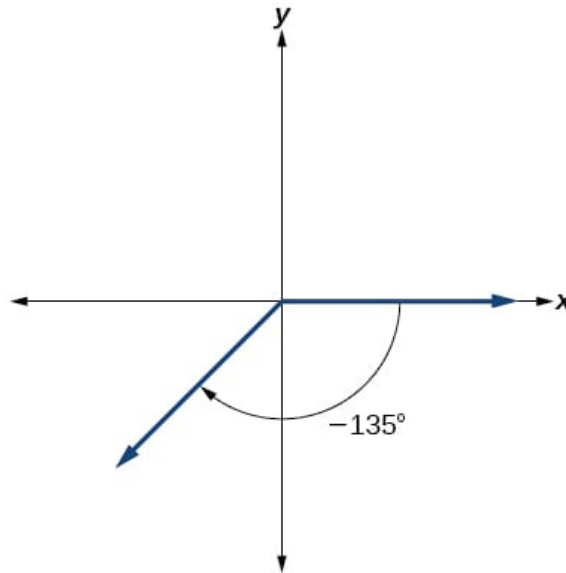
An angle is said to be in **standard position** if the vertex of the angle is at the origin and the initial side of the angle is on the positive x -axis.

Quadrantal Angles

Definition: Quadrantal Angle

An angle in standard position having terminal side on an axis is called a **quadrantal angle**.

Negative Angles



Coterminal Angles

Definition: Coterminal Angles

Angles in standard position that share a terminal side are called **coterminal angles**.

Theorem: Coterminal Angle Formula

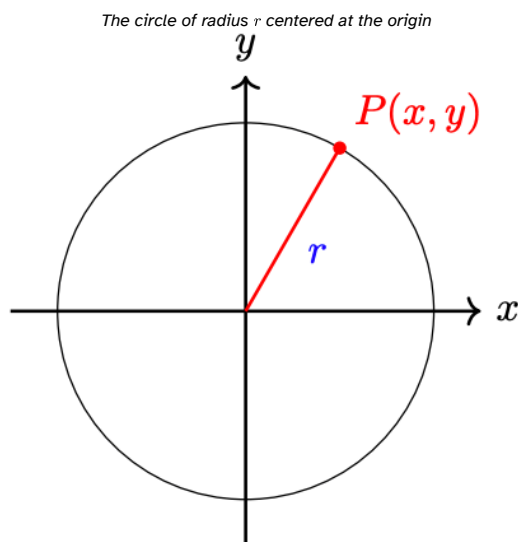
All angles coterminal to the angle α (in standard position) have the form

$$\alpha + 360^\circ k, \text{ where } k \in \mathbb{Z}.$$

Answers

Theorem: Equation of a Circle Centered at the Origin

Consider a circle of radius r centered at the origin, as shown in the following figure.



The distance from the origin to any point $P(x, y)$ on the circle is r . Therefore, by the Distance Formula,

$$\sqrt{(x-0)^2 + (y-0)^2} = r$$

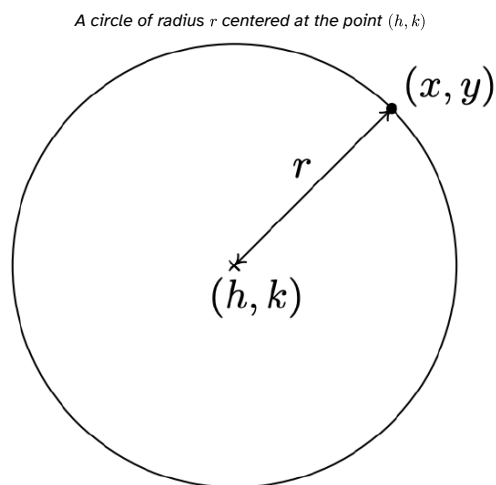
or, squaring both sides,

$$(x-0)^2 + (y-0)^2 = r^2.$$

Because every point on the circle must satisfy this equation, we have found an equation for the circle.

Theorem: Equation of a Circle Centered Off-Origin

Consider a circle of radius r centered at the point (h, k) , as shown in the following figure.



Since this is a circle of radius r , the point (x, y) is on the circle if and only if its distance to (h, k) is r . By the Distance Formula, this means

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

By squaring both sides of this equation, we get

$$r^2 = (x-h)^2 + (y-k)^2.$$

2: FUNCTIONS

2.1: Functions and Function Notation

2.2: Domain and Range

2.3: The Arithmetic and Composition of Functions

2.4: Behavior of Graphs of Functions

2.5: Transformations of Functions

2.6: Absolute Value Functions

2.7: Functions and Function Notation (1)

2.7.1: Resources and Key Concepts

2.7.2: Homework

2.1: Functions and Function Notation

Prerequisite Skills

- **Sets and Numbers**
 - The Language of Algebra
 - Sets and Set Notation
 - Sets of Numbers (natural numbers, integers, real numbers)
 - Properties of Real Numbers (distributive properties)
- **Arithmetic**
 - Mathematical Operations - Addition, Subtraction, Multiplication, and Division
 - Mathematical Operations - Powers and Square Roots
 - The Order of Operations
 - Fractions
- **Simplifying Expressions**
 - Algebraic Expressions - An Introduction and Using in Applications
 - Multiplying and Distributing Polynomial Expressions
 - Simplifying Ratios of Polynomials (simple cases)
- **Factoring Techniques**
 - Greatest Common Factor
 - Factoring Trinomials
- **Solving Equations**
 - Solving Literal Equations
 - Solving Linear Equations
 - Solving Quadratic Equations
 - Solving Radical Equations
- **Graphs**
 - The Cartesian Coordinate System and the Quadrants
 - Ordered Pairs and Graphing Relations by Point-Plotting
 - Reading Values from Graphs

Determining Whether a Relation Represents a Function

Definition: Relation

A **relation** from a set A to a set B is a subset of all possible ordered pairs (a, b) with $a \in A$ and $b \in B$. When $A = B$, we call it a relation **on** A .

Definition: Input and Output Values

Each value in the domain is also known as an **input value**, and each value in the range is also known as an **output value**.

Definition: Independent and Dependent Variables

If we are treating the input value as a variable, x , then we call x the **independent variable**. Likewise, the variable y is often used when referencing an arbitrary element from the set of output values, and y is called the **dependent variable**.

Definition: Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say the output is a function of the input. The set of input values makes up the **domain** of the function, and the corresponding set of output values makes up the **range** of the function.

MyOpenMath 2.1.2: Is a mapping a function?

MyOpenMath 2.1.5: Does an equation represent a function?

Using Function Notation

Definition: Function Notation

For the notation $y = f(x)$, f is the **name** of the function, x is the **domain value**, also known as the **argument** or the **independent variable**, and $f(x)$ is the **range value** of the function f at x . Since $y = f(x)$, we can also say that the range value of the function f at x is y . In this case, we say that y is the **dependent variable** (as its value *depends* on x). We read $f(x)$ as " f of x " or the value of f at x .

MyOpenMath 2.1.4: Using function notation with a table

Finding Input and Output Values of a Function

Evaluation of Functions in Algebraic Forms

MyOpenMath 2.1.10: Evaluating a function at given values

MyOpenMath 2.1.16: Solving equations involving functions

MyOpenMath 2.1.13: Evaluating a function at an expression

MyOpenMath 2.1.28: An application requiring function notation (and your brain)

Finding Function Values from a Graph

MyOpenMath 2.1.20: Reading function values from a graph

Determining Whether a Function is One-to-One

Definition: One-to-One Function

A function f is **one-to-one** (or **injective**) if distinct inputs always produce distinct outputs; that is, whenever $a \neq b$ in the domain of f , then $f(a) \neq f(b)$. Equivalently, $f(a) = f(b)$ forces $a = b$.

MyOpenMath 2.1.21: Determining whether a relationship is a one-to-one function

Using the Vertical Line Test

Theorem: Vertical Line Test

10.01: Graphs of Functions

MyOpenMath 2.1.18: Applying the Vertical Line Test

Using the Horizontal Line Test

Theorem: Horizontal Line Test

A set of points representing a function in a rectangular coordinate system is the graph of a one-to-one function if and only if every horizontal line intersects the graph in at most one point.

MyOpenMath 2.1.24: Forcing a relation to be a one-to-one function

Identifying Basic Toolkit Functions

Let's graph the **Toolkit** functions.

2.2: Domain and Range

Prerequisite Skills

- **Sets and Numbers**
 - Inequalities and Inequality Notation
 - Intervals and Interval Notation
- **Factoring Techniques**
 - Factoring Difference of Squares
- **Solving Inequalities**
 - Solving Linear Inequalities
 - Solving Compound Inequalities
 - Solving Quadratic Inequalities (sign chart method)
- **Graphs**
 - Closed- and Open-Circle Notations

Finding the Domain of a Function Defined by an Equation

MyOpenMath 2.2.3: Finding the domain and range of a relation given as a set of ordered pairs

MyOpenMath 2.2.9: Finding the domain of a quadratic function

MyOpenMath 2.2.11: Finding the domain of a rational function

MyOpenMath 2.2.15: Finding the domain of a function with an even root

Using Notations to Specify Domain and Range

Definition: Set-Builder Notation

Set-builder notation is a notation for specifying a set by a property that characterizes its members.

Example: Using different notations

State the solution to the last example using interval, inequality, and set-builder notations.

Finding Domain and Range from Graphs

MyOpenMath 2.2.6: Finding domain and range from a graph

Finding Domains and Ranges of the Toolkit Functions

Let's state the domains and ranges of the **Toolkit functions**.

Graphing Piecewise-Defined Functions

Definition: Piecewise Function

A **piecewise-defined function** (also called a **piecewise function** or a **function defined by cases**) is a function specified by two or more sub-functions, where each sub-function applies on its own portion of the domain. The portion of the domain on which a given sub-function applies is called its **subdomain**.

MyOpenMath 2.2.16: Evaluating piecewise functions

Example: Writing a piecewise function

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a function relating the number of people, n , to the cost, C .

MyOpenMath 2.2.18: Graphing a piecewise function

Definition: Absolute Value Function

19.1: The Absolute Value Function

2.3: The Arithmetic and Composition of Functions

Prerequisite Skills

- **Simplifying Expressions**
 - Simplifying Ratios of Polynomials
 - Adding and Subtracting Rational Expressions
 - Multiplying and Dividing Rational Expressions
 - Simplifying Compound Rational Expressions
- **Solving Equations**
 - Solving Rational Equations

Combining Functions Using Algebraic Operations

Definition: Function Arithmetic

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f + g$, $f - g$, fg , and $\frac{f}{g}$ by the relations

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ if } g(x) \neq 0$$

MyOpenMath 2.3.6: Performing algebraic operations on functions

MyOpenMath 2.3.4: Performing arithmetic on graphs of functions

Focus on Calculus - Difference Quotients

Definition: Difference Quotient

For a function f , the **difference quotient** of f at x with increment h , where $h \neq 0$, is

$$\frac{f(x+h) - f(x)}{h}.$$

It is the average rate of change of f over the interval $[x, x+h]$, whose length is h .

Create a Function by Composition of Functions

Definition: Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. Given functions f and g , the **composite function**, written as $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

The function g is the **inner function** (applied first), and f is the **outer function** (applied second).

Evaluating Composite Functions

Evaluating Composite Functions Using Tables

Evaluating Composite Functions Using Graphs

Evaluating Composite Functions Using Formulas

Finding the Domain of a Composite Function

Theorem: Domain of a Composite Function

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f :

$$\{x \mid x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f\}.$$

MyOpenMath 2.3.17: Finding the domain of a composed function

MyOpenMath 2.3.24: A difficult domain to find

Decomposing a Composite Function into its Component Functions

MyOpenMath 2.3.18: Decomposing a function

This page titled [2.3: The Arithmetic and Composition of Functions](#) is shared under a [CC BY-SA 4.0](#) license and was authored, remixed, and/or curated by [Roy Simpson](#), Cosumnes River College via [source content](#) that was edited to the style and standards of the LibreTexts platform.

2.4: Behavior of Graphs of Functions

Prerequisite Skills

- **Graphing Functions**
 - Base Graphs
- **Linear Functions**
 - Slope and Average Rate of Change

Average Rate of Change

Definition: Average Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

MyOpenMath 2.4.4: Computing an average rate of change given a table

MyOpenMath 2.4.3: Computing average rate of change from a graph

MyOpenMath 2.4.17: Computing average rate of change for a function expressed as a formula

Focus on Calculus - Local Extrema and Intervals of Increase and Decrease

Definition: Increasing

Let f be a function defined on an interval. The function f is **increasing** on that interval if, for every pair of numbers a and b in the interval with $a < b$, it follows that $f(a) < f(b)$.

Definition: Decreasing

Let f be a function defined on an interval. The function f is **decreasing** on that interval if, for every pair of numbers a and b in the interval with $a < b$, it follows that $f(a) > f(b)$.

Definition: Local Extrema

A function f has a **local maximum** at $x = b$ in an open interval (a, c) if $f(b) \geq f(x)$ for all $x \neq b$ in the interval. Likewise, f has a **local minimum** at $x = b$ in (a, c) if $f(b) \leq f(x)$ for all $x \neq b$ in the interval.

If f has more than one local maximum on a given interval, we call the collection of these maximums the **local maxima**. Likewise, if f has more than one local minimum on a given interval, we call the collection of these minimums the **local minima**.

Together, local maxima and minima are called **local extrema** (or **local extreme values**).

Some texts use the word "relative" instead of "local." We will use the latter throughout this text.

Example: Finding Increasing and Decreasing Intervals on a Graph

Given the graph of a function, state the intervals of increase and decrease, and any local extrema.

MyOpenMath 2.4.19: Finding local extrema using technology

Revisiting the Toolkit Functions

Let's look at the toolkit functions and discuss increasing/decreasing behavior and local extrema.

Focus on Calculus - Absolute Extrema

Definition: Absolute Extrema

A function f has an **absolute maximum** at $x = b$ if $f(b) \geq f(x)$ for all x in the domain of f . Likewise, a function f has an **absolute minimum** at $x = b$ if $f(b) \leq f(x)$ for all x in the domain of f .

If f has more than one local absolute maximum, we call the collection of these maximums the **absolute maxima**. Likewise, if f has more than one absolute minimum, we call the collection of these minimums the **absolute minima**.

Together, absolute maxima and minima are called **absolute extrema**.

Some texts use the word "global" instead of "absolute." We will use the latter throughout this text.

Example: Finding absolute extrema from a graph

For the function f shown below, find all absolute maxima and minima.

Symmetry - Even and Odd Functions

Definition: Even and Odd Functions

The function f is called an **even function** if for every input x in the domain of f

$$f(-x) = f(x).$$

The function f is called an **odd function** if for every input x in the domain of f

$$f(-x) = -f(x).$$

Theorem: Symmetry of Even and Odd Functions

Let f be a real function.

- f is even if and only if its graph is symmetric with respect to the y -axis; that is, the graph is unchanged under reflection across the y -axis.
- f is odd if and only if its graph is symmetric with respect to the origin; that is, the graph is unchanged under a 180° rotation about the origin.

Equivalently, f is even when the point $(-x, y)$ lies on the graph whenever (x, y) does, and f is odd when the point $(-x, -y)$ lies on the graph whenever (x, y) does.

MyOpenMath 2.4.15: Determining whether a function Is even, odd, or neither

2.5: Transformations of Functions

Prerequisite Skills

- **Arithmetic**
 - The Order of Operations

Graphing Functions Using Vertical and Horizontal Shifts

Identifying Vertical Shifts

Theorem: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of $f(x)$. All the output values change by k units. If k is positive, the graph will shift up by k units. If k is negative, the graph will shift down by $|k|$ units.

Example: A basic vertical shift of a Toolkit function

Let's do a vertical shift.

Identifying Horizontal Shifts

Theorem: Horizontal Shift

Given a function $f(x)$, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of $f(x)$. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example: A vertical and horizontal shift

Let's do one.

Graphing Functions Using Reflections about the Axes

Theorem: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a vertical reflection of the function $f(x)$, sometimes called a reflection about (or over, or through) the x -axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a horizontal reflection of the function $f(x)$, sometimes called a reflection about the y -axis.

MyOpenMath 2.5.8: Playing with graphs

Example: Graphing a Toolkit function

Let's do this with a Toolkit function.

Answer available at the end of this section.

Graphing Functions Using Stretches and Compressions

Vertical Stretches and Compressions

Theorem: Vertical Stretches and Compressions

10.5: Transformations - Rigid Transformations

Example

Let's do one.

Horizontal Stretches and Compressions

Theorem: Horizontal Stretches and Compressions

10.5: Transformations - Rigid Transformations

Example

Let's do one.

Performing a Sequence of Transformations

Combining Transformations

10.5: Transformations - Rigid Transformations

Example

Let's do one.

Answers

Example: Graphing a Toolkit function

2.6: Absolute Value Functions

Prerequisite Skills

- **Arithmetic**
 - The Absolute Value of a Number
- **Absolute Value Functions**
 - The Concept of Absolute Value

Focus on Calculus - Piecewise Definition of the Absolute Value

Definition: Absolute Value Function

19.1: The Absolute Value Function

MyOpenMath 2.6.15

Graphing an Absolute Value Function

MyOpenMath 2.6.9

Solving an Absolute Value Equation

Definition: Absolute Value Equation

An **absolute value equation** is an equation in which the unknown variable appears within the absolute value bars.

MyOpenMath 2.6.2

MyOpenMath 2.6.11

Focus on Calculus - Solving Absolute Value Inequalities

Definition: Absolute Value Inequality

An **absolute value inequality** is an inequality in which the unknown variable appears within the absolute value bars.

MyOpenMath 2.6.4

MyOpenMath 2.6.6

This page titled 2.6: Absolute Value Functions is shared under a [CC BY-SA 4.0](#) license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College via [source content](#) that was edited to the style and standards of the LibreTexts platform.

2.7: FUNCTIONS AND FUNCTION NOTATION (1)

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

We begin our journey with a definition (and from this, progress to more definitions, then theorems, then more theorems, and so on until we will have derived all of College Algebra) and an exploration.

Definition: Ordered Pair

An **ordered pair** (a, b) is a listing of two objects in which the order is significant; a is called the **first coordinate** and b the **second coordinate**. Two ordered pairs, (a, b) and (c, d) , are equal (written as $(a, b) = (c, d)$) if and only if $a = c$ and $b = d$.

Consider the following set of ordered pairs:

$$D = \{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}. \quad (2.7.1)$$

The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first. It is logical to want to describe this link between the first and second values of these ordered pairs. We do so by defining this link as a **relation**.

Definition: Relation

A **relation** from a set A to a set B is a subset of all possible ordered pairs (a, b) with $a \in A$ and $b \in B$. When $A = B$, we call it a relation **on** A .

We should spend a moment unpacking the language in that last definition.

- "A relation *from* a set A to a set B ," implies there is a direction—we are going *from* somewhere *to* somewhere else.
- "...is a *subset* of *all possible* ordered pairs (a, b) ," means that the relation might not contain every possible combination of elements from A and B —it only needs to contain a *subset* of these combinations. It is possible that it contains all

combinations of ordered pairs between these two sets, but it is not necessary.

For example, let A be the set of names of states in the United States, and let B be the set of flavors of Ben & Jerry's ice cream. We could create a (non-sensical) relation from A to B containing only the following ordered pairs: {(California, Chunky Monkey), (New York, Americone Dream), (Texas, Cherry Garcia)}. Notice that this relation definitely didn't use *all* of the states, nor did it use *all* of the Ben & Jerry's ice cream flavors—that's fine! By definition, it is still a relation between A and B .

Relations *might* be able to be described in terms of formulas (like, $y = 2x$ in the set 2.7.1), but this property need not exist. That is, we might be given a set of ordered pairs where there is no formula to the relationship between x and y .

Note

Before we go too far, it is important to mention that relations can involve more than two variables, but until you reach Calculus III, we will try our best to restrict our exploration to relations between two unknowns.

Now that we have a name for the "machine" linking the first and second values of these ordered pairs, let's introduce more language so that we can stop saying, "the first and second values of the ordered pairs."

Definition: Domain and Range (of a relation)

Let R be a relation. The **domain** of R is the set of all first coordinates that appear in R , and the **range** of R is the set of all second coordinates that appear in R :

$$\text{domain} = \{a \mid (a, b) \in R \text{ for some } b\}, \quad \text{range} = \{b \mid (a, b) \in R \text{ for some } a\}.$$

Therefore, the domain of the relation 2.7.1 is $\{1, 2, 3, 4, 5\}$ and the range is $\{2, 4, 6, 8, 10\}$.

Definition: Input and Output Values

Each value in the domain is also known as an **input value**, and each value in the range is also known as an **output value**.

Input and output values are traditionally considered constants. That is, in 2.7.1 the input value of 3 mapped to an output value of 6. However, when referring to an *arbitrary* element from the set of input or output values, we use a slightly modified language (because, in Mathematics, the word "arbitrary" means "variable"—or non-constant).

Definition: Independent and Dependent Variables

If we are treating the input value as a variable, x , then we call x the **independent variable**. Likewise, the variable y is often used when referencing an arbitrary element from the set of output values, and y is called the **dependent variable**.

A Variable By Any Other Name...

While x and y are traditionally used in lower-level math courses as the independent and dependent variables, respectively, this need not always be the case. We can use any letter (or picture, for that matter) to represent these variables. The choice is literally limitless.

Relations are incredibly flexible—as long as there is some link between two variables, we can form a relation between them. For example, there is a relation between social security number and name. That is, the social security number 123-45-6789 (which, in truth, is not a real social security number) is related to a person's name. A natural question that arises when we do some thinking is whether the relation is unique. That is, if we grab a random social security number, will it be related to one name? This type of

special relation is called a **function**.

Definition: Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say the output is a function of the input. The set of input values makes up the **domain** of the function, and the corresponding set of output values makes up the **range** of the function.

In a very real sense, functions are a special subclass of relations. While a relation can relate the input of 5 to the output 7 and then the input of 5 to the output of 10, functions are not allowed such silliness.

The relation from 2.7.1 is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$, is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$. For the social security numbers, the relation in one direction is a function—each social security number leads to a single name. However, if we reversed the direction of the relation so that the domain is the set of all names and the range is the set of all social security numbers, the relation would *not* be a function. This is because the name, Roy Simpson, leads to more than one social security number (there are more Roy Simpsons in the United States than just me!).

Let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd}, 1), (\text{even}, 2), (\text{odd}, 3), (\text{even}, 4), (\text{odd}, 5)\}.$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$, is not paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the range, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function. Figure 2.7.1 compares relations that are functions and not functions.

Three relations that demonstrate what constitute a function.

Figure 2.7.1

Figure 2.7.1(a) is a function because each input is associated with a single output. Note that inputs q and r both give output n . This is perfectly fine. The definition of a function only restricts a single input from leading to two distinctly different outputs. Figure 2.7.1(b) is also a function. In this case, each input is associated with a single output. Finally, Figure 2.7.1(c) is *not* a function because input q is associated with two different outputs (y and z).

Example 2.7.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure 2.7.2, consists of items and their prices.

- Is price a function of the item?
- Is the item a function of the price?

A menu of donut prices from a coffee shop where a plain donut is \$1.49 and a jelly donut and chocolate donut are \$1.99.

Figure 2.7.2

Solutions available at the end of this section.

Example 2.7.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 2.7.1 shows a possible rule for assigning grade points.

Table 2.7.1

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
---------------	------	-------	-------	-------	-------	-------	-------	--------

Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0
---------------------	-----	-----	-----	-----	-----	-----	-----	-----

Solution available at the end of this section.

Checkpoint 2.7.2

Table 2.7.2 lists the five greatest baseball players of all time in order of rank.

Table 2.7.2

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

- Is the rank a function of the player name?
- Is the player name a function of the rank?

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables: h for height and a for age. The letters f , g , and h are often used to represent functions just as we use x , y , and z to represent numbers and A , B , and C to represent sets.

We then build a structure for the function notation. Since height is a function of age, we would like our notation to match.

$$\begin{aligned} \text{height is a function of age} \\ \Rightarrow h = f(a) \end{aligned}$$

Remember, we can use any letter to name the function. The notation $f(a)$ shows us that f depends on a . The value a must be put into the function f to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example, $f(a + b)$ means, "first add a and b , and then

input the result into the function f ." The operations must be performed in this order to obtain the correct result.

Definition: Function Notation

For the notation $y = f(x)$, f is the **name** of the function, x is the **domain value**, also known as the **argument** or the **independent variable**, and $f(x)$ is the **range value** of the function f at x . Since $y = f(x)$, we can also say that the range value of the function f at x is y . In this case, we say that y is the **dependent variable** (as its value *depends* on x). We read $f(x)$ as " f of x " or the value of f at x .

Example 2.7.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution available at the end of this section.

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 2.7.4: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N , in a town in year y . What does $f(2005) = 300$ represent?

Solution available at the end of this section.

Checkpoint 2.7.4

Use function notation to express the weight of a pig in pounds as a function of its age in days d .

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning " y is a function of x ?"

Yes, this is often done, especially in applied subjects that use higher math, such as Physics and Engineering. However, in exploring Mathematics itself, we like to maintain a distinction between a function such as f , which is a rule or procedure, and the output y we get by applying f to a particular input x . This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 2.7.3 lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table 2.7.3

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 2.7.4 defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q .

Table 2.7.4

n	1	2	3	4	5
Q	8	6	7	6	8

Table 2.7.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 2.7.5

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Example 2.7.5: Identifying Tables that Represent Functions

Which table, Table 2.7.6A, Table 2.7.6B, or Table 2.7.6C, represents a function (if any)?

Table 2.7.6A

Input	Output
2	1
5	3
8	6

Table 2.7.6B

Input	Output
-3	5
0	1
4	5

Table 2.7.6C

Input	Output
1	0
5	2
5	4

Solution available at the end of this section.

Checkpoint 2.7.5

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Preview of Future Material

Each of the *types* of functions in the following examples will be studied in detail as we move through the textbook. The functions in Examples 2.7.6, 2.7.7, and 2.7.8 are called **quadratic functions**, and are the focus of an entire section later on. The functions in Checkpoints 2.7.7 and 2.7.8 are called **square root functions** (a subset of the **radical functions**). These also get their own sections later in the text.

Example 2.7.6: Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at:

- 2
- a
- $a + h$
- Now evaluate $\frac{f(a+h) - f(a)}{h}$

Solutions available at the end of this section.

Example 2.7.7: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution available at the end of this section.

Checkpoint 2.7.7

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Example 2.7.8: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve $h(p) = 3$.

Solution available at the end of this section.

Checkpoint 2.7.8

Given the function $g(m) = \sqrt{m - 4}$, solve $g(m) = 2$.

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between n and p . We can rewrite it to decide if p is a function of n .

Example 2.7.9: Finding the Algebraic Form of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Solution available at the end of this section.

Caution: Not all relations are functions

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 2.7.10: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution available at the end of this section.

Checkpoint 2.7.10

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Q&A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x , there is no simple algebraic formula involving only x that equals y . However, each x does determine a unique value for y , and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x , even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table.

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. For the pet-memory span information, let's call the function P . The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Example 2.7.11: Evaluating and Solving a Tabular Function

Use the provided table to answer the following questions.

- Evaluate $g(3)$.

b. Solve $g(n) = 6$.

n	1	2	3	4	5
g	8	6	7	6	8

Solutions available at the end of this section.

Checkpoint 2.7.11

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 2.7.12: Reading Function Values from a Graph

Use the graph below to answer the following questions.

- Evaluate $f(2)$.
- Solve $f(x) = 4$.

Graph of a positive parabola centered at $(1, 0)$.

Figure 2.7.6

Solutions available at the end of this section.

Checkpoint 2.7.12

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the figure at the beginning of this chapter, the stock price was \$1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of \$1000.

However, some functions have only one input value for each output value, as well as having only one output for each input.

Definition: One-to-One Function

A function f is **one-to-one** (or **injective**) if distinct inputs always produce distinct outputs; that is, whenever $a \neq b$ in the domain of f , then $f(a) \neq f(b)$. Equivalently, $f(a) = f(b)$ forces $a = b$.

As an example, consider a school that uses only letter grades and decimal equivalents, as listed in the following table.

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figure 2.7.1(a) and Figure 2.7.1(b). The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n . The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

Example 2.7.13: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution available at the end of this section.

Checkpoint 2.7.13A

- Is a balance a function of the bank account number?
- Is a bank account number a function of the balance?
- Is a balance a one-to-one function of the bank account number?

Checkpoint 2.7.13B

Evaluate the following:

- If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
- If so, is the function one-to-one?

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y , and we say y is a function of x , or $y = f(x)$ when the function is named f . The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the

function is defined for only a few input values, then the graph of the function is only a few points, where the x -coordinate of each point is an input value and the y -coordinate of each point is the corresponding output value. For example, the black dots on the graph in the following figure tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points.

Graph of a polynomial.

Figure 2.7.9

If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value (see the following figure).

Three graphs visually showing what is and is not a function.

Figure 2.7.10

This leads us to the **vertical line test**.

Theorem: Vertical Line Test

A set of points in a rectangular coordinate system is the graph of a function if and only if every vertical line intersects the graph in at most one point.

Interactive Element: The Vertical Line Test

The following Interactive Element might help you visualize why a circle is not a function.

Interact: Drag the red point along the horizontal axis to sweep the vertical line across the circle.

Observation: Whenever the line passes through the interior of the circle it meets the circle at two points — one input x with two outputs — so the circle fails the Vertical Line Test and is not a function $y = f(x)$. A single vertical line with two crossings is enough to disqualify the graph.

Example 2.7.14: Applying the Vertical Line Test

Which of the graphs in the following figure represent(s) a function $y = f(x)$?

Graph of a polynomial.

Figure 2.7.11

Solution available at the end of this section.

Checkpoint 2.7.14

Does the graph in the following figure represent a function?

Graph of absolute value function.

Figure 2.7.13

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the

Horizontal Line Test.

Theorem: Horizontal Line Test

A set of points representing a function in a rectangular coordinate system is the graph of a one-to-one function if and only if every horizontal line intersects the graph in at most one point.

To use the Horizontal Line Test, draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Example 2.7.15: Applying the Horizontal Line Test

Consider the functions shown in parts (a) and (b) of Example 2.7.14. Are either of the functions one-to-one?

Solution available at the end of this section.

Checkpoint 2.7.15

Is the graph shown in part (c) of Example 2.7.14 one-to-one?

Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do Arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "**toolkit functions**," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in the following table.

Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$, where c is a constant	Graph of a constant function.
Identity	$f(x) = x$	Graph of a straight line.
Absolute value	$f(x) = x $	Graph of absolute function.
Quadratic	$f(x) = x^2$	Graph of a parabola.
Cubic	$f(x) = x^3$	Graph of $f(x) = x^3$.

Name	Function	Graph
Reciprocal	$f(x) = \frac{1}{x}$	Graph of $f(x)=1/x$.
Reciprocal squared	$f(x) = \frac{1}{x^2}$	Graph of $f(x)=1/x^2$.
Square root	$f(x) = \sqrt{x}$	Graph of $f(x)=\text{sqrt}(x)$.
Cube root	$f(x) = \sqrt[3]{x}$	Graph of $f(x)=x^{(1/3)}$.

Hidden Resources: Borrowing Technology

Having a computer is a requirement for any college class these days; however, what do you do if you don't have a computer? Cosumnes River College (and the Los Rios Community College District) has pulled together a great list of resources for students needing a computer. Whether you want to borrow a Chromebook from the CRC library for the semester or you prefer to purchase a discounted computer, the [CRC Library](#) has all the information you need!

MyOpenMath Advice: "Message Instructor" Within Questions

What is it?

If you are stuck on a homework question, and if your professor has set up a discussion area on Canvas, post your question to that area first; however, not all professors have enabled a discussions area. In this case, you can ask your professor for help directly by clicking on the "Message instructor" link at the bottom of the question.

Screenshot 2022-08-23 045226.jpg

Why use it?

The nice thing about using the "Message instructor" link is that it sends *your version* of the question you are asking about directly to your professor. This helps them better understand your question. Be sure to include some information as to what might be causing your confusion.

Screenshot 2022-08-23 045353.jpg

How do you know when your professor has responded?

This is the only tricky part. When your professor responds, you will *not* get an email notification. This is a weakness of the MyOpenMath embed into Canvas (the messaging systems do not work well together); however, when your professor responds, you will see a message flag within *any* homework assignment you open.

Screenshot 2022-08-23 045524-1.jpg

Just click on the envelope and you will see your professor's response.

Answers

Section Preview

Treat this section as a review of functions (represented as tables, algebraic functions, word problems, and graphs). As far as computation is concerned, the algebraic functions used within this section are restricted to linear, quadratic, radical, and very simple rational functions. Finally, the nine Toolkit functions are introduced, but do not cover transformations (as this is held off for later).

Section Preview

- **Sets and Numbers**
 - **Sets of Numbers:** The opening relation $D = \{(1, 2), (2, 4), \dots\}$ is built from the natural numbers, and domains/ranges throughout the section are described as sets of numbers; students must be able to recognize and classify these number types to read the examples.
- **Graphs**
 - **The Cartesian Coordinate System and the Quadrants:** All graphical representations of functions, the Vertical Line Test, and the Horizontal Line Test presuppose a working understanding of the coordinate plane and how points live on it.
 - **Ordered Pairs and Graphing Equations by Point-Plotting:** The section introduces relations and functions as sets of ordered pairs (x, y) ; a student must already understand what an ordered pair represents and how it corresponds to a plotted point to follow the development.

Section Preview

- **Arithmetic**
 - **Mathematical Operations — Powers and Square Roots:** Evaluating the quadratic $f(x) = x^2$ and square-root $f(x) = \sqrt{x}$ toolkit functions, and computing values such as $h(4) = 4^2 + 2(4)$ in Example 7, requires fluency with powers and roots.
 - **The Order of Operations:** Correctly evaluating expressions like $f(2) = 2^2 + 3(2) - 4$ in Example 6a depends on applying PEMDAS in the right sequence.
- **Geometry**
 - **Circles — Properties, Area, and Circumference:** Example 13 relies on the area formula $A = \pi r^2$ to argue that area is a one-to-one function of radius; students must already know and apply this formula.
- **Factoring Techniques**
 - **Greatest Common Factor:** Example 6d factors the common h out of $2ah + h^2 + 3h$ in order to cancel and simplify the difference-quotient result.
- **Simplifying Expressions**
 - **Algebraic Vocabulary and Evaluating Expressions:** Nearly every example (6, 7, 8) hinges on substituting a value or expression for the input variable and simplifying the result, the central operational skill of the section.
 - **Multiplying Polynomials and the Distributive Property:** Evaluating $f(a + h)$ in Example 6c requires expanding $(a + h)^2$ and distributing $3(a + h)$.
- **Solving Equations**
 - **Solving Literal Equations:** Example 9 rewrites $2n + 6p = 12$ as $p = f(n)$, which requires isolating one variable while treating the other as a constant.
 - **Solving Quadratic Equations by Factoring:** Example 8 solves $h(p) = 3$ by rewriting it as $p^2 + 2p - 3 = 0$, factoring, and applying the Zero Product Property.
 - **Solving Quadratic Equations Using Extraction of Roots:** Example 10 isolates y^2 in $x^2 + y^2 = 1$ and extracts roots to obtain $y = \pm\sqrt{1 - x^2}$.
- **Graphs**
 - **Reading Values from Graphs:** Example 12 asks students to evaluate $f(2)$ and solve $f(x) = 4$ by reading input and output coordinates directly off a plotted curve.

Section Preview

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the Vertical Line Test to identify functions.
- Graph the functions listed in the library of functions.

Example 2.7.1: Determining If Menu Price Lists Are Functions

- Let's begin by considering the input as the items on the menu. The output values are then the prices (see Figure 2.7.2). Each item on the menu has only one price, so the price is a function of the item.
- Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it (see the figure below).

Association of the prices to the donuts.

Figure 2.7.3

Example 2.7.2: Determining If Class Grade Rules Are Functions

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Example 2.7.3: Using Function Notation for Days in a Month

The number of days in a month is a function of the name of the month, so if we name the function f , we write

$$\text{days} = f(\text{month})$$

or

$$d = f(m).$$

The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

The function $31 = f(\text{January})$ where 31 is the output, f is the rule, and January is the input.

Figure 2.7.4

For example, $f(\text{March}) = 31$ because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Example 2.7.4: Interpreting Function Notation

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The

statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Example 2.7.5: Identifying Tables that Represent Functions

Table 2.7.6A and Table 2.7.6B define functions. In both, each input value corresponds to exactly one output value. Table 2.7.6C does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 2.7.6A can be represented by writing

$$f(2) = 1, f(5) = 3, \text{ and } f(8) = 6.$$

Similarly, the statements

$$g(-3) = 5, g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table 2.7.6B.

Table 2.7.6C cannot be expressed in a similar way because it does not represent a function.

Example 2.7.6: Evaluating Functions at Specific Values

Replace the x in the function with each specified value.

- a. Because the input value is a number, 2, we can use simple Algebra to simplify.

$$\begin{aligned} f(2) &= (2)^2 + 3(2) - 4 && \text{(substituting)} \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

- b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

- c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a + h) &= (a + h)^2 + 3(a + h) - 4 && \text{(substituting)} \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 && \text{(distributing)} \end{aligned}$$

- d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a + h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4.$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} && \text{(substituting)} \\ &= \frac{a^2 + 2ah + h^2 + 3a + 3h - 4 - a^2 - 3a + 4}{h} && \text{(distributing)} \\ &= \frac{2ah + h^2 + 3h}{h} && \text{(combining like terms)} \\ &= \frac{h(2a + h + 3)}{h} && \text{(factoring out the GCF)} \\ &= \frac{\cancel{h}^1(2a + h + 3)}{\cancel{h}_1} && \text{(canceling like factors)} \\ &= 2a + h + 3 \end{aligned}$$

Example 2.7.7: Evaluating Functions

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(4) &= (4)^2 + 2(4) && \text{(substituting)} \\ &= 16 + 8 && \text{(simplifying)} \\ &= 24 && \text{(simplifying)} \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Example 2.7.8: Solving Functions

$$\begin{aligned} h(p) &= 3 \\ \Rightarrow p^2 + 2p &= 3 && \text{(substituting)} \\ \Rightarrow p^2 + 2p - 3 &= 0 && \text{(subtracting 3 from both sides)} \\ \Rightarrow (p + 3)(p - 1) &= 0 && \text{(factoring)} \end{aligned}$$

If $(p + 3)(p - 1) = 0$, either $(p + 3) = 0$ or $(p - 1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case (recall from Elementary Algebra, this is called the **Zero Factor Property**).

$$(p + 3) = 0 \Rightarrow p = -3$$

$$(p - 1) = 0 \Rightarrow p = 1$$

This gives us two solutions, so we know that $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing (see the figure below). The graph verifies that $h(1) = h(-3) = 3$ and (for Example 2.7.7) $h(4) = 24$.

Graph of a parabola with labeled points $(-3, 3)$, $(1, 3)$, and $(4, 24)$.

Figure 2.7.5

Example 2.7.9: Finding the Algebraic Form of a Function

To express the relationship in this form, we need to be able to write the relationship where p is a function of n , which means writing it as

$$p = [\text{expression involving } n].$$

$$\begin{aligned} 2n + 6p &= 12 \\ \Rightarrow 6p &= 12 - 2n && (\text{subtracting } 2n \text{ from both sides}) \\ \Rightarrow p &= \frac{12 - 2n}{6} && (\text{dividing both sides by } 6) \\ \Rightarrow p &= \frac{12}{6} - \frac{2n}{6} \\ \Rightarrow p &= 2 - \frac{n}{3} && (\text{canceling like factors}) \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n.$$

Example 2.7.10: Expressing the Equation of a Circle as a Function

First we subtract x^2 from both sides, to get

$$y^2 = 1 - x^2.$$

We now try to solve for y in this equation.

$$\begin{aligned} y^2 &= 1 - x^2 \\ \Rightarrow y &= \pm\sqrt{1 - x^2} && (\text{Extraction of Roots}) \end{aligned}$$

Thus, either $y = \sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$. That is, we get two outputs corresponding to the same input. Therefore, this relationship cannot be represented as a single function $y = f(x)$.

Example 2.7.11: Evaluating and Solving a Tabular Function

- Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
- Solving $g(n) = 6$ means identifying the input values, n , that produce an output value of 6. The table shows two solutions: 2 and 4. That is, when we input 2 into the function g , our output is 6. When we input 4 into the function g , our output is also 6.

Example 2.7.12: Reading Function Values from a Graph

- To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y -coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$ (see the following figure).

Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

Figure 2.7.7

- To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3 . This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3 , the output is 4 (see the figure below).

Graph of an upward-facing parabola with a vertex at $(0, 1)$ and labeled points at $(-1, 4)$ and $(3, 4)$. A line at $y = 4$ intersects the parabola at the labeled points.

Figure 2.7.8

Example 2.7.13: Determining Whether a Relationship Is a One-to-One Function

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r , there is only one output, A . The area is a function of radius r .

If the function is one-to-one, the output value (the area) must correspond to a unique input value (the radius). Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Example 2.7.14: Applying the Vertical Line Test

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of the figure. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x -values, a vertical line would intersect the graph at more than one point, as shown in the figure below.

Graph of a circle.

Figure 2.7.12

Example 2.7.15: Applying the Horizontal Line Test

The function in part (a) is not one-to-one. The horizontal line shown in the following figure intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

140fa74177d512d2a0de3741db0fb44d27334d60

Figure 2.7.14

The function in part (b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.

2.7.1: Resources and Key Concepts

2.7.2: Homework

This page titled [2.7: Functions and Function Notation \(1\)](#) is shared under a [CC BY-SA 4.0](#) license and was authored, remixed, and/or curated by [Roy Simpson, Cosumnes River College](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.

2.7.1: Resources and Key Concepts

Resources

Videos

- Relations
 - [Functions - Overview - A Review for College Algebra: Relations and Functions](#)
- Functions
 - [Algebra in Review: Functions - Overview](#)
 - [Intermediate Algebra - Functions: An Introduction](#)
- Function notation
 - [Algebra in Review: Functions - Notation](#)
- Evaluating functions
 - [Functions - Overview - A Review for College Algebra: Evaluating Functions](#)
 - [Functions - Overview - A Review for Precalculus: Evaluating Functions](#)
 - [Algebra in Review: Functions - Evaluation](#)
- Representation of functions
 - [Intermediate Algebra - Functions: The Four Representations](#)
- One-to-one function
 - [Functions - Overview - A Review for College Algebra: One-to-One Functions](#)
- Vertical Line Test
 - [Functions - Overview - A Review for College Algebra: The Vertical Line Test](#)
- Horizontal Line Test
 - [Functions - Overview - A Review for College Algebra: The Horizontal Line Test](#)
- Toolkit functions
 - [Intermediate Algebra - Functions: Base Graphs](#)

Key Concepts

Definitions

- **Relation:** Any set of ordered pairs, denoted as (x, y) .
- **Domain (of a relation/function):** The set of all first components (x -values) from the ordered pairs in a relation; it represents all possible input values.
- **Range (of a relation/function):** The set of all second components (y -values) from the ordered pairs in a relation; it represents all possible output values.
- **Input Value:** A value from the domain of a relation or function.
- **Output Value:** A value from the range of a relation or function, resulting from a given input value.
- **Independent Variable:** The variable (commonly x) that represents the input values of a function.
- **Dependent Variable:** The variable (commonly y) that represents the output values of a function, as its value depends on the input.
- **Function:** A specific type of relation where each possible input value is associated with exactly one output value.
- **Function Notation:** A convention, typically written as $y = f(x)$, used to denote a function, where f is the name of the function, x is the input (independent variable), and $f(x)$ is the output (dependent variable) corresponding to the input x .
- **One-to-One Function:** A function for which every distinct input value produces a distinct output value, meaning each output value corresponds to exactly one input value.

Theorems

- **Vertical Line Test:** A graph in a rectangular coordinate system represents a function if and only if no vertical line intersects the graph at more than one point. If any vertical line intersects the graph at more than one point, the graph does not represent

a function.

- **Horizontal Line Test:** The graph of a function represents a one-to-one function if and only if no horizontal line intersects the graph at more than one point. If any horizontal line intersects the graph at more than one point, the function is not one-to-one.

Common Mistakes

- **Misinterpreting Parentheses in Function Notation:** The parentheses in function notation, such as in $f(a)$, signify that a is the input to the function f . They do not indicate multiplication of f by a .
- **Assuming All Equations Define Functions:** Not every relationship expressed by an equation can be written as a single function where one variable is a function of the other. For example, the equation of a circle $x^2 + y^2 = 1$ does not define y as a single function of x because for a given x -value (other than 1 or -1), there are two corresponding y -values.

2.7.2: Homework

Reading Questions

1. What is a relation?
2. What are the domain and range of a relation?
3. How does a function differ from a general relation?
4. In the function notation $y = f(x)$, what do x , f , and $f(x)$ represent?
5. What does it mean to "evaluate" a function at a specific value, for example, $f(2)$?
6. Describe the Vertical Line Test. What is it used for?
7. What is a one-to-one function?
8. How can the Horizontal Line Test be used?
9. Does the equation $x^2 + y^2 = 1$ represent y as a function of x ? Why or why not?
10. Name three of the "toolkit functions" mentioned in the section.

Homework

1. What is the difference between a relation and a function?
2. What is the difference between the input and the output of a function?
3. Why does the Vertical Line Test tell us whether the graph of a relation represents a function?
4. How can you determine if a relation is a one-to-one function?
5. Why does the Horizontal Line Test tell us whether the graph of a function is one-to-one?

For the following exercises, determine whether the relation represents a function.

6. $\{(a, b), (c, d), (a, c)\}$
7. $\{(a, b), (b, c), (c, c)\}$

For the following exercises, determine whether the relation represents y as a function of x .

8. $5x + 2y = 10$
9. $y = x^2$
10. $x = y^2$
11. $3x^2 + y = 14$
12. $2x + y^2 = 6$
13. $y = -2x^2 + 40x$
14. $y = \frac{1}{x}$
15. $x = \frac{3y+5}{7y-1}$
16. $x = \sqrt{1 - y^2}$
17. $y = \frac{3x+5}{7x-1}$
18. $x^2 + y^2 = 9$
19. $2xy = 1$
20. $x = y^3$
21. $y = x^3$
22. $y = \sqrt{1 - x^2}$
23. $x = \pm\sqrt{1 - y}$
24. $y = \pm\sqrt{1 - x}$
25. $y^2 = x^2$
26. $y^3 = x^2$

For the following exercises, evaluate $f(-3)$, $f(2)$, $f(-a)$, $-f(a)$, and $f(a + h)$.

27. $f(x) = 2x - 5$
28. $f(x) = -5x^2 + 2x - 1$

29. $f(x) = \sqrt{2-x} + 5$
 30. $f(x) = \frac{6x-1}{5x+2}$
 31. $f(x) = |x-1| - |x+1|$
 32. Given the function $g(x) = 5 - x^2$, evaluate

$$\frac{g(x+h) - g(x)}{h}, \text{ where } h \neq 0.$$

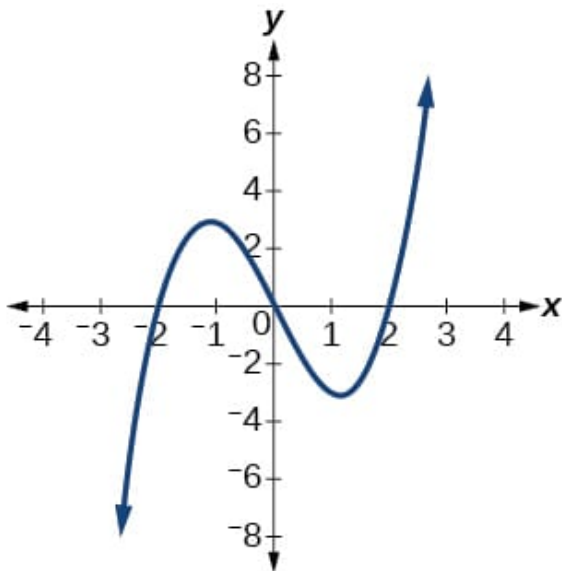
33. Given the function $g(x) = x^2 + 2x$, evaluate

$$\frac{g(x) - g(a)}{x - a}, \text{ where } x \neq a.$$

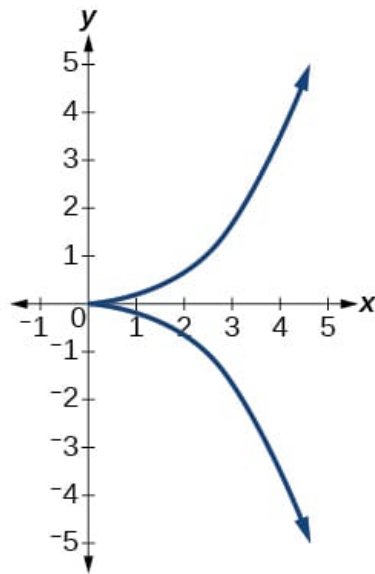
34. Given the function $k(t) = 2t - 1$:
 a. Evaluate $k(2)$.
 b. Solve $k(t) = 7$.
 35. Given the function $f(x) = 8 - 3x$:
 a. Evaluate $f(-2)$.
 b. Solve $f(x) = -1$.
 36. Given the function $p(c) = c^2 + c$:
 a. Evaluate $p(-3)$.
 b. Solve $p(c) = 2$.
 37. Given the function $f(x) = x^2 - 3x$:
 a. Evaluate $f(5)$.
 b. Solve $f(x) = 4$.
 38. Given the function $f(x) = \sqrt{x+2}$:
 a. Evaluate $f(7)$.
 b. Solve $f(x) = 4$.
 39. Consider the relationship $3r + 2t = 18$.
 a. Write the relationship as a function $r = f(t)$.
 b. Evaluate $f(-3)$.
 c. Solve $f(t) = 2$.

For the following exercises, use the Vertical Line Test to determine which graphs show relations that are functions.

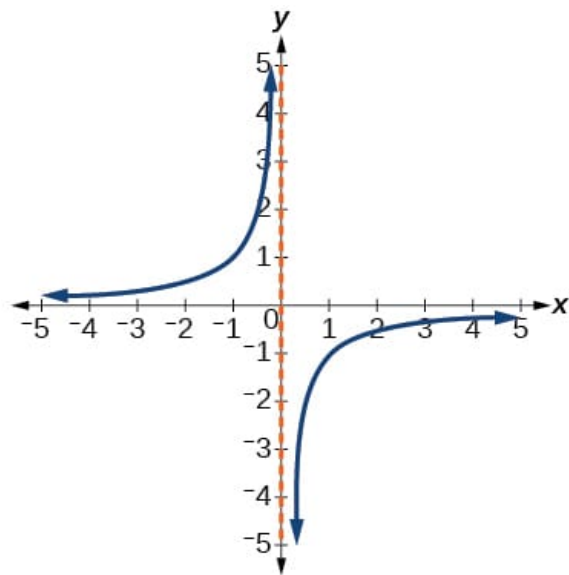
40.



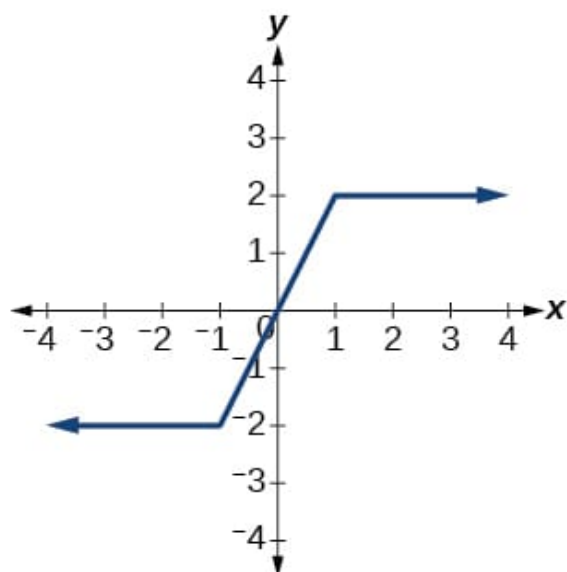
41.



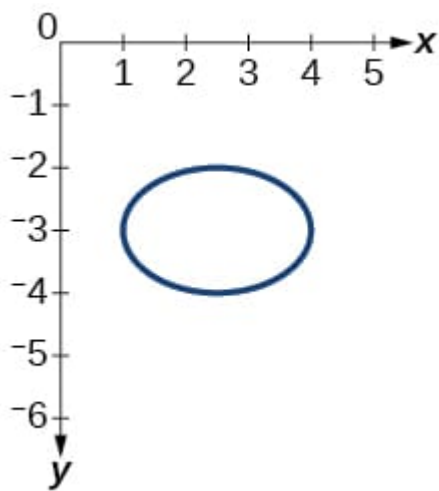
42.



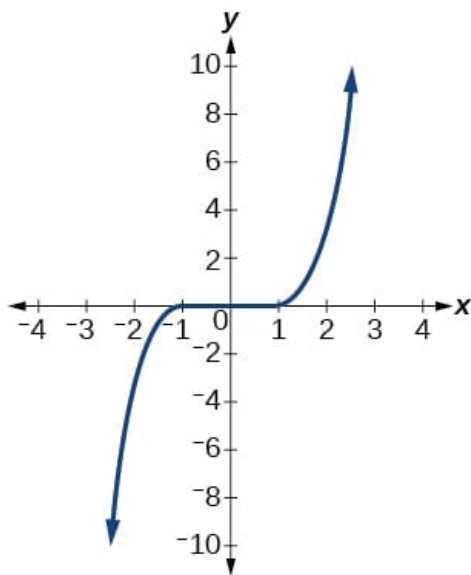
43.



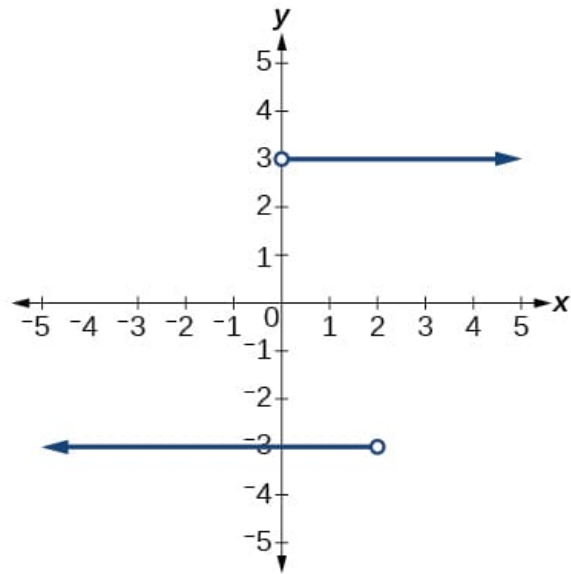
14.



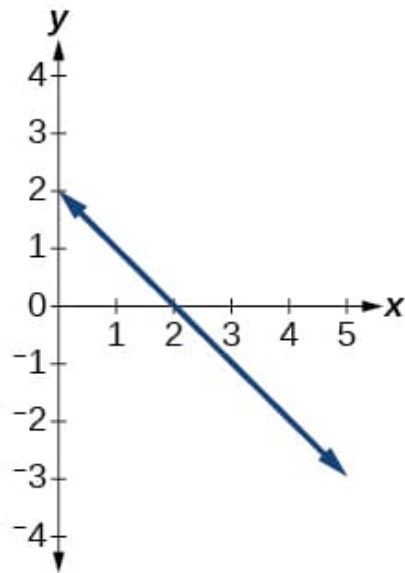
15.



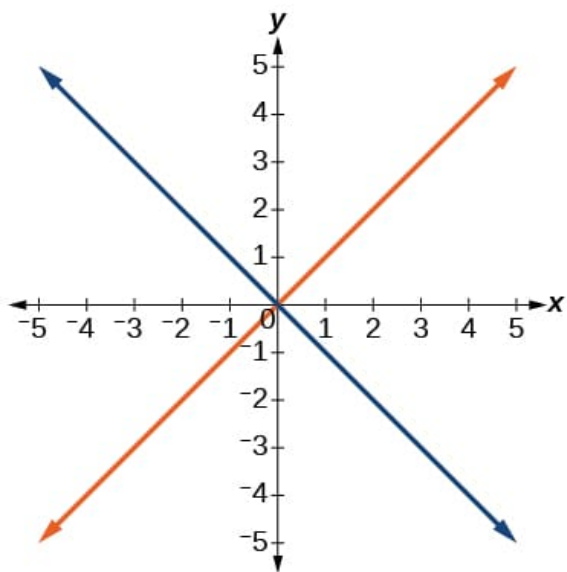
16.



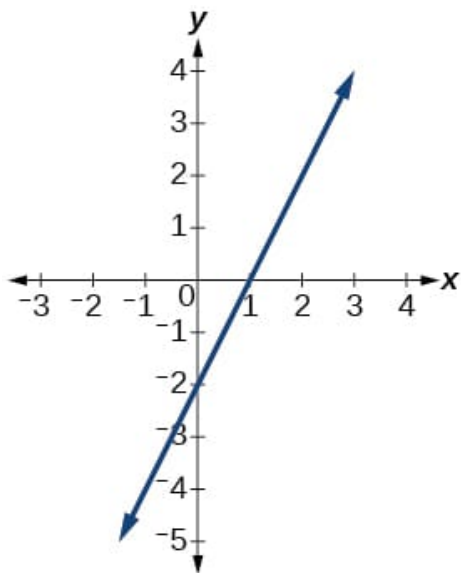
47.



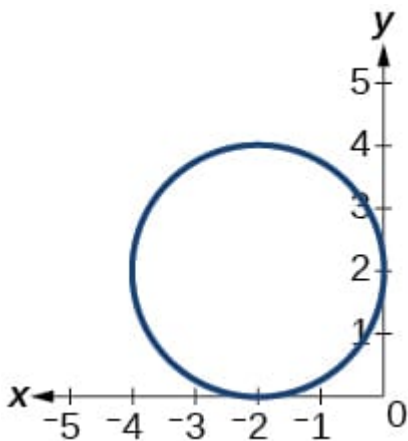
48.



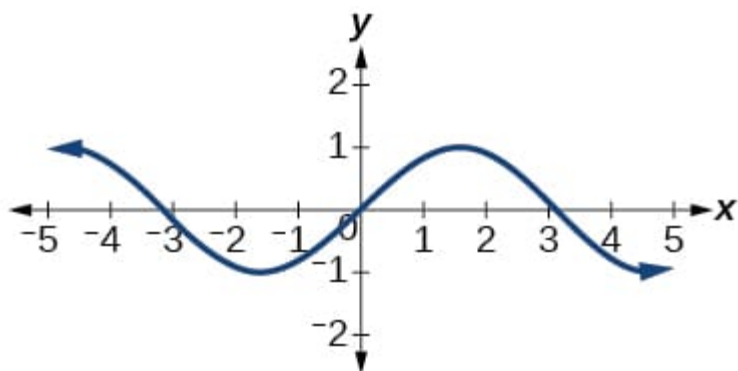
49.



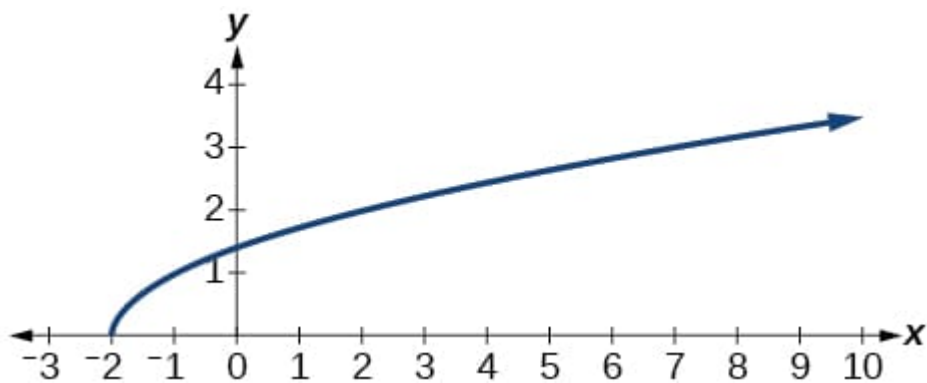
50.



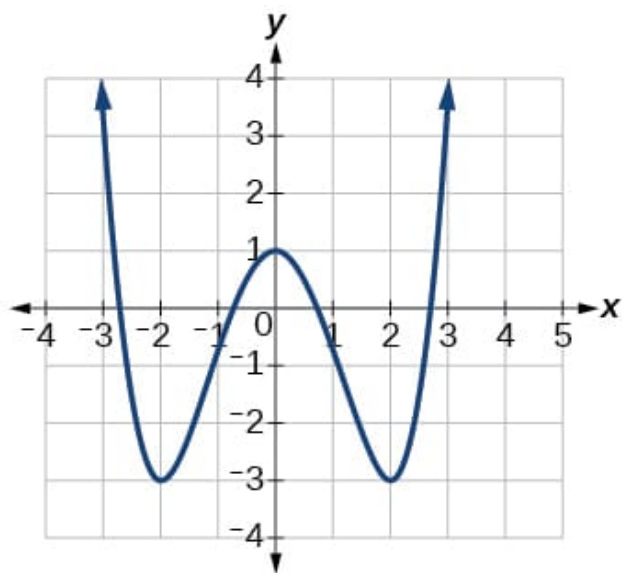
51.



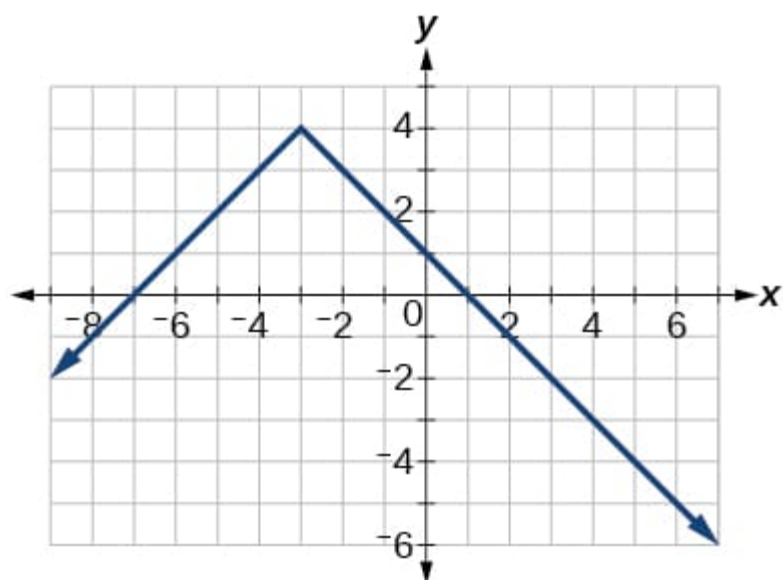
52. Given the following graph,
- Evaluate $f(-1)$.
 - Solve for $f(x) = 3$.



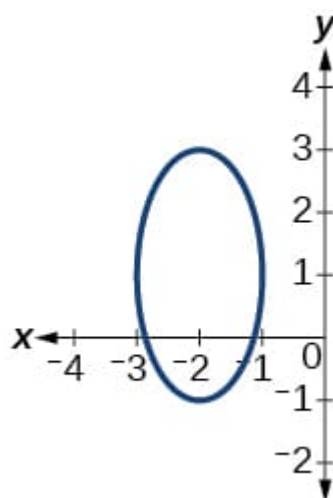
53. Given the following graph,
- Evaluate $f(0)$.
 - Solve for $f(x) = -3$.



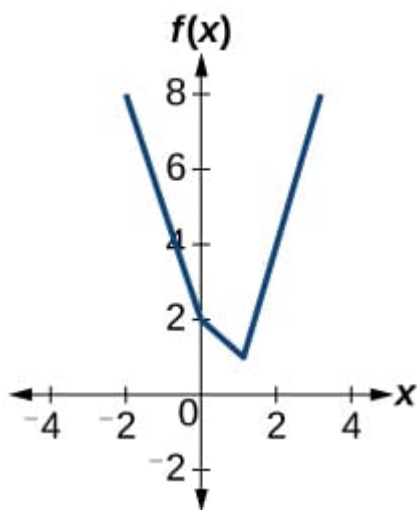
54. Given the following graph,
- Evaluate $f(4)$.
 - Solve for $f(x) = 1$.



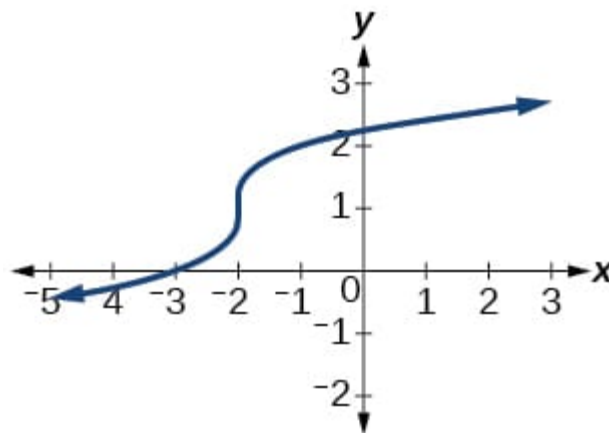
For the following exercises, determine if the given graph is a one-to-one function.



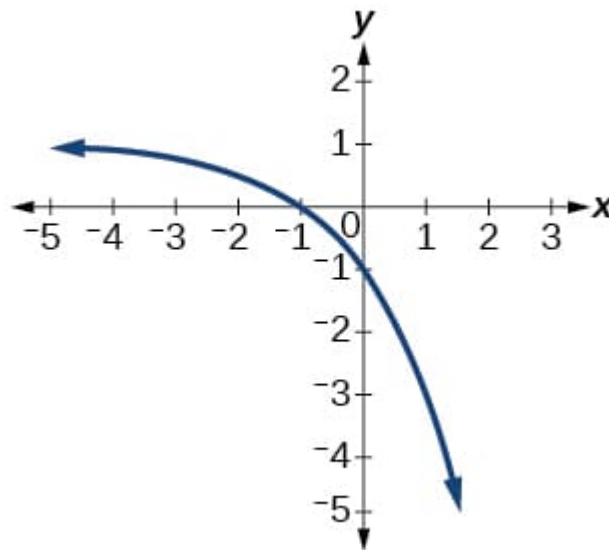
55.



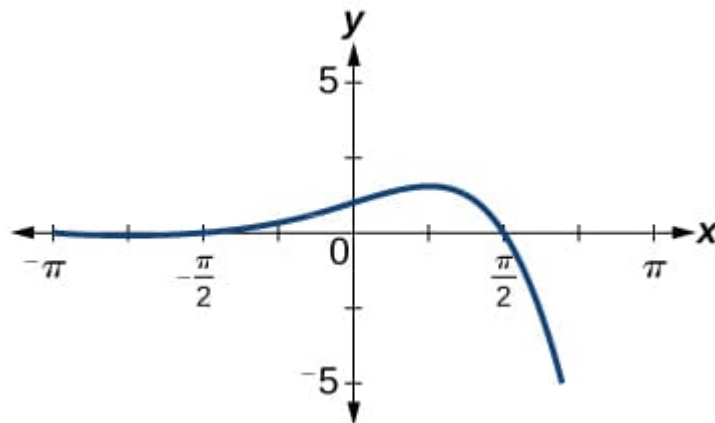
56.



57.



58.



59.

For the following exercises, determine whether the relation represents a function.

60. $\{(-1, -1), (-2, -2), (-3, -3)\}$

61. $\{(3, 4), (4, 5), (5, 6)\}$

62. $\{(2, 5), (7, 11), (15, 8), (7, 9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x .

63.	x	5	10	15
-----	-----	---	----	----

y	3	8	14
-----	---	---	----

34.

x	5	10	15
y	3	8	8

35.

x	5	10	10
y	3	8	14

For the following exercises, use the function f represented in the following table.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

36. Evaluate $f(3)$.
 37. Solve $f(x) = 1$.

For the following exercises, evaluate the function f at the values -2 , -1 , 0 , 1 , and 2 .

38. $f(x) = 4 - 2x$
 39. $f(x) = 8 - 3x$
 70. $f(x) = 8x^2 - 7x + 3$
 71. $f(x) = 3 + \sqrt{x+3}$
 72. $f(x) = \frac{x-2}{x+3}$
 73. $f(x) = 3^x$

For the following exercises, evaluate the expressions, given functions f , g , and h :

- $f(x) = 3x - 2$
 - $g(x) = 5 - x^2$
 - $h(x) = -2x^2 + 3x - 1$
74. $3f(1) - 4g(-2)$
 75. $f\left(\frac{1}{3}\right) - h(-2)$

For the following exercises, graph $y = x^2$ on the given domain. Determine the corresponding range. Show each graph.

76. $[-0.1, 0.1]$
 77. $[-10, 10]$
 78. $[-100, 100]$

For the following exercises, graph $y = x^3$ on the given domain. Determine the corresponding range. Show each graph.

79. $[-0.1, 0.1]$
 80. $[-10, 10]$
 81. $[-100, 100]$

For the following exercises, graph $y = \sqrt{x}$ on the given domain. Determine the corresponding range. Show each graph.

32. $[0, 0.01]$
 33. $[0, 100]$
 34. $[0, 10000]$

For the following exercises, graph $y = \sqrt[3]{x}$ on the given domain. Determine the corresponding range. Show each graph.

35. $[-0.001, 0.001]$
 36. $[-1000, 1000]$

37. $[-1000000, 1000000]$
38. The amount of garbage, G , produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.
- The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f .
 - Explain the meaning of the statement $f(5) = 2$.
39. The number of cubic yards of dirt, D , needed to cover a garden with area a square feet is given by $D = g(a)$.
- A garden with area 5000 ft² requires 50 yd³ of dirt. Express this information in terms of the function g .
 - Explain the meaning of the statement $g(100) = 1$.
40. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
- $f(5) = 30$
 - $f(10) = 40$
41. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
- $h(1) = 200$
 - $h(2) = 350$
42. Show that the function $f(x) = 3(x - 5)^2 + 7$ is not one-to-one.

3: POLYNOMIAL AND RATIONAL FUNCTIONS



35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography.
(credit “film”: modification of work by Horia Varlan; credit “memory cards”: modification of work by Paul Hudson)

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by Mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how Mathematics can be used in such applications.

3.1: Quadratic Functions

3.2: Power Functions and Polynomial Functions

3.3: Graphs of Polynomial Functions

3.4: Dividing Polynomials

3.5: Rational Functions

This page titled 3: Polynomial and Rational Functions is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by Roy Simpson via [source content](#) that was edited to the style and standards of the LibreTexts platform.

- 3: Polynomial and Rational Functions by OpenStax is licensed [CC BY 4.0](#). Original source: <https://openstax.org/details/books/precalculus>.

3.1: Quadratic Functions

Prerequisite Skills

- **Solving Equations**
 - Solving Absolute Value Equations
- **Quadratic Functions**
 - Vertex Form and Completing the Square

Recognizing Characteristics of Parabolas

Definition: Parabola

13.02: Graphing Quadratic Functions in General Form

Definition: Zeros of Functions and Roots of Equations

10.01: Graphs of Functions

MyOpenMath 3.1.9

Quadratic Functions in General Form

Definition: General Form of a Quadratic Function

13.02: Graphing Quadratic Functions in General Form

Theorem: Quadratic Formula

6.4: Solving Quadratic Equations by Factoring

Theorem: Axis of Symmetry for a Quadratic Function

13.02: Graphing Quadratic Functions in General Form

Theorem: Coordinates of the Vertex

13.02: Graphing Quadratic Functions in General Form

MyOpenMath 3.1.19

Quadratic Functions in Standard Form

Definition: Standard Form of a Quadratic Function

13.01: Vertex Form and Completing the Square

Theorem: Vertex of a Quadratic Function in Standard Form

13.01: Vertex Form and Completing the Square

Example

Write the equation of the graph from MyOpenMath 3.1.9.

Example

Rewrite the quadratic function $f(x) = -3x^2 + 6x - 2$ in standard (vertex) form by Completing the Square.

Finding the Domain and Range of a Quadratic Function

Theorem: Domain and Range of a Quadratic Function

13.02: Graphing Quadratic Functions in General Form

MyOpenMath 3.1.7

Determining the Maximum and Minimum Values of Quadratic Functions

MyOpenMath 3.1.18

MyOpenMath 3.1.18

Solving Quadratic Equations

MyOpenMath 3.1.2

Solving Quadratic Inequalities

Definition: Quadratic Inequality

A **quadratic inequality** in x is an inequality that can be written in one of the forms

$$ax^2 + bx + c < 0, \quad ax^2 + bx + c \leq 0, \quad ax^2 + bx + c > 0, \quad \text{or} \quad ax^2 + bx + c \geq 0,$$

where a , b , and c are real numbers with $a \neq 0$.

Graphing Absolute Values of Quadratics

Focus on Calculus - Solving Equations that are Quadratic-in-Form

Definition: Quadratic-in-Form

An equation in the variable x is **quadratic-in-form** if it can be rewritten as

$$au^2 + bu + c = 0,$$

where a , b , and c are constants with $a \neq 0$, and u represents an algebraic expression in x .

3.2: Power Functions and Polynomial Functions

Identifying Power Functions

Definition: Power Function

A **power function** is a function of the form

$$f(x) = ax^n,$$

where a and n are real numbers and $a \neq 0$. a is known as the **coefficient** of the power function, and n is called its **exponent**.

MyOpenMath 3.2.1

Identifying End Behavior of Power Functions

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x -value) approaches positive or negative infinity (denoted $x \rightarrow \infty$ or $x \rightarrow -\infty$).

MyopenMath 3.2.8

Identifying Polynomial Functions

Definition: Polynomial Function

A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0,$$

where $n \in \mathbb{N}$. This is called the **general form of a polynomial function**. Each a_i is a **coefficient** and can be any real number. Each product $a_i x_i$ is called a **term** of the function. The largest n for which $a_n \neq 0$ is called the **degree** of the polynomial function. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Example

Identify the degree, leading term, and leading coefficient of the function in the following example.

MyOpenMath 3.2.13

Identifying End Behavior of Polynomial Functions

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x -value) approaches positive or negative infinity (denoted $x \rightarrow \infty$ or $x \rightarrow -\infty$).

Theorem: End Behavior of a Polynomial Function

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$ and $n \geq 1$. The end behavior of the graph of $P(x)$ matches the end behavior of the graph of $y = a_n x^n$. Hence,

- If n is even and $a_n > 0$, then $P(x) \rightarrow \infty$ as $x \rightarrow -\infty$ and as $x \rightarrow \infty$.
- If n is even and $a_n < 0$, then $P(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and as $x \rightarrow \infty$.
- If n is odd and $a_n > 0$, then $P(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, and $P(x) \rightarrow \infty$ as $x \rightarrow \infty$.
- If n is odd and $a_n < 0$, then $P(x) \rightarrow \infty$ as $x \rightarrow -\infty$, and $P(x) \rightarrow -\infty$ as $x \rightarrow \infty$.

Proof available at the end of this section.

MyOpenMath 3.2.9

In addition to what is being asked, determine a possible degree (even or odd) of the polynomial function.

Identifying Local Behavior of Polynomial Functions

MyOpenMath 3.2.4

Comparing Smooth and Continuous Graphs

Definition: Turning Point

A **turning point** of a graph is a point at which the graph changes from increasing to decreasing, or from decreasing to increasing.

Theorem: Number of Turning Points for a Polynomial Function

The graph of a polynomial function of degree $n \geq 1$ will have, at most, $n - 1$ turning points and n horizontal intercepts.

Proof available at the end of this section.

MyOpenMath 3.2.5

Only do part c of the following problem:

Answers

Theorem: End Behavior of a Polynomial Function

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Then

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1} x^{n-1}}{a_n x^n} + \frac{a_{n-2} x^{n-2}}{a_n x^n} + \cdots + \frac{a_2 x^2}{a_n x^n} + \frac{a_1 x}{a_n x^n} + \frac{a_0}{a_n x^n} \right) && \text{(factoring out } a_n x^n) \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \frac{a_{n-2}}{a_n x^2} + \cdots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right) && \text{(Laws of Exponents)} \end{aligned}$$

Since $a_n \neq 0$, the fractions $\frac{a_k}{a_n}$ are finite constants. For notational efficiency, let's denote $\frac{a_k}{a_n}$ as b_k . Therefore,

$$P(x) = a_n x^n \left(1 + b_{n-1} \cdot \frac{1}{x} + b_{n-2} \cdot \frac{1}{x^2} + \cdots + b_2 \cdot \frac{1}{x^{n-2}} + b_1 \cdot \frac{1}{x^{n-1}} + b_0 \cdot \frac{1}{x^n} \right).$$

(The next step is proved in Calculus, but can be understood here as a strong possibility.) As long as the power on x is not negative, as $x \rightarrow \pm\infty$, $\frac{1}{x^p} \rightarrow 0$. Since all the powers on x in

$$P(x) = a_n x^n \left(1 + b_{n-1} \cdot \frac{1}{x} + b_{n-2} \cdot \frac{1}{x^2} + \cdots + b_2 \cdot \frac{1}{x^{n-2}} + b_1 \cdot \frac{1}{x^{n-1}} + b_0 \cdot \frac{1}{x^n} \right)$$

are not negative, all terms involving x in the denominator must tend to 0 as $x \rightarrow \pm\infty$. That is, as $x \rightarrow \pm\infty$, $P(x) \rightarrow a_n x^n (1 + 0 + 0 + \cdots + 0 + 0 + 0) = a_n x^n$. Hence, the end behavior of $P(x)$ is $y = a_n x^n$.

Theorem: Number of Turning Points for a Polynomial Function

This proof must wait until Calculus.

3.3: Graphs of Polynomial Functions

Prerequisite Skills

- **Solving Equations**
 - Solving Polynomial Equations

Recognizing Characteristics of Graphs of Polynomial Functions

MyOpenMath 3.3.11

Using Factoring to Find Zeros of Polynomial Functions

MyOpenMath 3.3.2

MyOpenMath 3.3.7 – 8

Identifying Zeros and Their Multiplicities

Definition: Multiplicity of a Zero

Let h be a zero of a nonzero polynomial function f . The **multiplicity** of h is the largest positive integer p for which $(x - h)^p$ is a factor of $f(x)$. We say that $x = h$ is a **zero of multiplicity** p . A zero of multiplicity 1 is called a **simple zero**; a zero of multiplicity greater than 1 is called a **repeated zero** (or **multiple zero**).

Theorem: Local Behavior Near a Zero

If $x = h$ is a zero of multiplicity p for a given polynomial, then the local behavior of the graph of the polynomial near $x = h$ matches the behavior of $y = \pm(x - h)^p$, where the sign of the coefficient is determined by other characteristics of the given polynomial.

Caution: Graph Accurately Near Zeros!

You will eventually be asked to make accurate sketches of polynomial functions. Part of this skill is to show that you know the impact of high multiplicity on a zero. The higher the multiplicity, the flatter the polynomial graph is near to the zero - be prepared to show this.

Theorem: Sum of Multiplicities and the Degree of a Polynomial

The degree of a polynomial factored into the product of linear and irreducible quadratic factors is sum of the multiplicities of the linear factors and the twice the sum of the multiplicities of the irreducible quadratic factors. That is, if

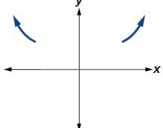
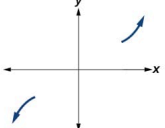
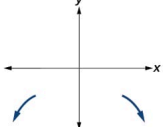
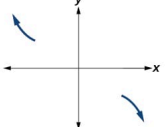
$$P(x) = (a_1x + b_1)^{m_1}(a_2x + b_2)^{m_2} \cdots (a_kx + b_k)^{m_k}(c_1x^2 + d_1)^{p_1}(c_2x^2 + d_2)^{p_2} \cdots (c_nx^2 + d_n)^{p_n},$$

then the degree of $P(x)$ is

$$m_1 + m_2 + \cdots + m_k + 2(p_1 + p_2 + \cdots p_n).$$

MyOpenMath 3.3.4

Determining End Behavior (a reminder)

Even Degree	Odd Degree
Positive Leading Coefficient, $a_n > 0$  End Behavior: $x \rightarrow \infty, f(x) \rightarrow \infty$ $x \rightarrow -\infty, f(x) \rightarrow \infty$	Positive Leading Coefficient, $a_n > 0$  End Behavior: $x \rightarrow \infty, f(x) \rightarrow \infty$ $x \rightarrow -\infty, f(x) \rightarrow -\infty$
Negative Leading Coefficient, $a_n < 0$  End Behavior: $x \rightarrow \infty, f(x) \rightarrow -\infty$ $x \rightarrow -\infty, f(x) \rightarrow -\infty$	Negative Leading Coefficient, $a_n < 0$  End Behavior: $x \rightarrow \infty, f(x) \rightarrow -\infty$ $x \rightarrow -\infty, f(x) \rightarrow \infty$

Graphing Polynomial Functions

MyOpenMath 3.3.5

MyOpenMath 3.3.10

Solving Polynomial Inequalities

MyOpenMath 3.3.11

3.4: Dividing Polynomials

Prerequisite Skills

- **Simplifying Expressions**
 - Dividing Polynomials - Long Division

Using Long Division to Divide Polynomials

Theorem: Division Algorithm

Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d . Then there exist two unique polynomials, $q(x)$ and $r(x)$, such that

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)},$$

where either $r(x) = 0$ or the degree of r is strictly less than the degree of d . By multiplying both sides of this equation by $d(x)$, we get the equivalent form

$$p(x) = q(x)d(x) + r(x).$$

Each of these polynomials is given a special name:

- $d(x)$ is called the **divisor**
- $p(x)$ is called the **dividend**
- $q(x)$ is called the **quotient**
- $r(x)$ is called the **remainder**

MyOpenMath 3.4.7

Focus on Calculus - Using Synthetic Division to Divide Polynomials

MyOpenMath 3.4.3

MyOpenMath 3.4.6

Example

Redo 3.4.7 using synthetic division.

3.5: Rational Functions

Prerequisite Skills

- **Solving Equations**
 - Solving Rational Equations

Rational Functions

Definition: Rational Function

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_px^p + a_{p-1}x^{p-1} + \dots + a_1x + a_0}{b_qx^q + b_{q-1}x^{q-1} + \dots + b_1x + b_0}, \text{ where } Q(x) \neq 0.$$

Identifying Vertical Asymptotes of Rational Functions

Removable Discontinuities

Definition: Removable Discontinuity

A single point where the graph of a function is not defined, indicated by an open circle on the graph, is called a **removable discontinuity**.

Definition: Removable Discontinuity of a Rational Function

A removable discontinuity for the graph of a rational function occurs at $x = a$ if

1. a is a zero for a factor in the denominator that is common with a factor in the numerator, and
2. after simplification, a is no longer a zero of the denominator.

Proof available at the end of this section.

MyOpenMath 3.5.3

Identifying End Behavior of Rational Functions

Definition: End Behavior

The **end behavior** of a graph is a description, usually as a function, of what the function values of the graph tend to approach as the inputs increase or decrease without bound.

Definition: Polynomial Degree Notation

The degree of a polynomial $f(x)$ is denoted $\deg(f(x))$.

Theorem: End Behavior of a Rational Function

Let $f(x) = \frac{N(x)}{D(x)}$ be a rational function. Then

- $f(x)$ has a horizontal asymptote of $y = 0$ if $\deg(D(x)) > \deg(N(x))$
- $f(x)$ has a horizontal asymptote of $y = \frac{a}{b}$ if $\deg(D(x)) = \deg(N(x))$, where a is the lead coefficient of $N(x)$ and b is the lead coefficient of $D(x)$
- $f(x)$ has a slant or oblique asymptote if $\deg(D(x)) < \deg(N(x))$. The asymptote is a slant asymptote if $\deg(N(x)) = \deg(D(x)) + 1$; otherwise, it is an oblique asymptote. In either case, the equation of the asymptote is the quotient after dividing $N(x)$ by $D(x)$.

MyOpenMath 3.5.4

MyOpenMath 3.5.7

Graphing Rational Functions

How To: Graph a rational function

1. Evaluate the function at 0 to find the y -intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x -intercepts.
4. Find the multiplicities of the x -intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

MyOpenMath 3.5.14

Writing Rational Functions

MyOpenMath 3.5.10

Revisiting Solving Inequalities

MyOpenMath 3.5.12

Answers

Definition: Removable Discontinuity of a Rational Function

The most elegant proof of this theorem occurs in Calculus.

4: INVERSE AND RADICAL FUNCTIONS

4.1: Inverse Functions

4.2: Radical Functions

4.3: Radical Equations

4.4: Rational Exponent Expressions Encountered in Calculus

4.1: Inverse Functions

Prerequisite Skills

- **Arithmetic**
 - Mathematical Operations - Higher-Index Roots
- **Geometry**
 - Common Volumes

Restricting the Domain to Find the Inverse of a Polynomial Function

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

How To: Restrict Domain and Force Invertibility

Given a function that is not one-to-one,

1. Restrict the domain by determining a domain on which the original function is one-to-one (i.e., a domain where the function passes the Horizontal Line Test).
2. Replace $f(x)$ with y .
3. Solve for x interchange x and y .
4. Rename the function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

MyOpenMath 4.1.4

MyOpenMath 4.1.5

4.2: Radical Functions

Prerequisite Skills

- **Solving Inequalities**
 - Solving Quadratic Inequalities

Radical Functions

Definition: Radical Function

A **radical function** is a function that is defined by a radical expression.

The Domain of a Radical Function

Theorem: Domain of a Radical Function

5.13: Simplifying Products of Radical Expressions

MyOpenMath 4.2.1

MyOpenMath 4.2.8

MyOpenMath 4.2.13

Answer available at the end of this section.

Answers

This page titled [4.2: Radical Functions](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by [Roy Simpson, Cosumnes River College](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.

4.3: Radical Equations

Solving Radical Equations Containing One Radical

Definition: Radical Equation

An equation in which variables are part of a radicand is called a **radical equation**.

MyOpenMath 4.3.2

Caution: Solving Equations Containing Even-Indexed Radicals

Care must be taken when working with expressions and equations involving even-indexed radicals. This is due to the nature of such beasts. Remember, the argument of an even-indexed radical cannot be negative, nor can an even-indexed radical return a negative value.

Definition: Extraneous Solution

An **extraneous solution**, also known as a **spurious solution**, is the solution of a transformed version of the original equation that is not an actual solution of the original equation because it was excluded from the domain (of the original equation).

Definition: Contradiction

25.09: Indirect Proofs - Contradiction

Solving Equations Containing Rational Exponents

MyOpenMath 4.3.5

Solve this if the constant on the right side is very negative.

MyOpenMath 4.3.4

Solving Radical Equations with Two Radicals

MyOpenMath 4.3.6

How To: Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
If yes, repeat Step 1 and Step 2 again.
If no, solve the new equation.

4. Check the answer in the original equation.

MyOpenMath 4.3.9

MyOpenMath 4.3.12

MyOpenMath 4.3.13

4.4: Rational Exponent Expressions Encountered in Calculus

Prerequisite Skills

- **Simplifying Expressions**
 - Simplifying Expressions Involving Positive Rational Exponents

MyOpenMath 4.4.3

MyOpenMath 4.4.5

Query 4.4.1

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Query 4.4.1

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



Query 4.4.1

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Interactive Content online](#)



5: COORDINATE TRIGONOMETRY

5.1: The Trigonometric Functions - Coordinate Definition

5.2: Algebra with Trigonometric Functions

5.3: Introduction to Trigonometric Identities

5.4: Introduction to Proofs in Trigonometry

This page titled [5: Coordinate Trigonometry](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

- [Chapter 4: Trigonometric Functions](#) by [Katherine Yoshiwara](#) is licensed [GNU FDL](#). Original source: <https://yoshiwarabooks.org/trig/trig.html>.

5.1: The Trigonometric Functions - Coordinate Definition

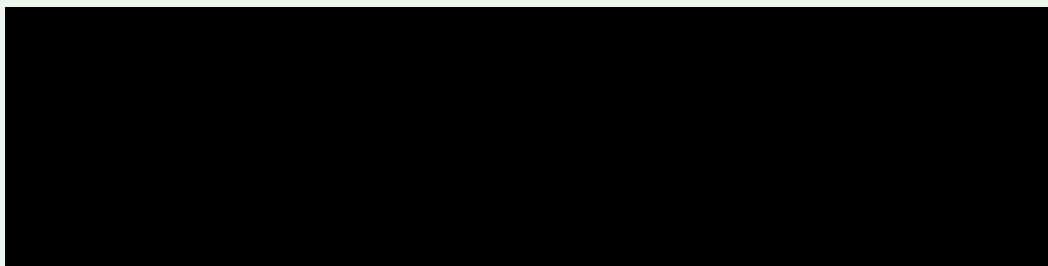
Prerequisite Skills

- **Simplifying Expressions**
 - Rationalizing Denominators and Numerators
- **An Overview of Functions**
 - The Definition of a Function
 - Evaluating Functions
 - The Domain and Range of a Function

Defining Trigonometric Functions in Terms of Coordinates

Definition: Trigonometric Functions (Coordinate Definition)

Let θ be an angle in standard position with the point $P(x, y)$ on the terminal side of θ , where P is not the origin. Define $r = \sqrt{x^2 + y^2}$. The trigonometric functions of the angle θ are defined as follows:



Note: Angles are in Standard Position

Unless otherwise stated, all angles from this point forward will be considered angles in standard position.

Example 5.1.1: Finding Trig Function Values at a Point

Find the values of the six trigonometric functions if the given point is on the terminal side of θ .

Note: Rationalizing Denominators

In Algebra, we learned to rationalize denominators so we could "simplify" expressions like $-\frac{8}{\sqrt{73}}$; however, in Trigonometry, such manipulations often are pointless. The rule of thumb I will adhere to in this textbook is to rationalize denominators *if* the denominator involves a variable (e.g., $\frac{3}{\sqrt{x-2}}$). In all other cases (specifically, when the numerator and denominator do *not* contain variables), I will only rationalize denominators if there is an "obvious" benefit.

In the case of $-\frac{8}{\sqrt{73}}$, if we tried to rationalize the denominator, we would get

$$-\frac{8}{\sqrt{73}} \cdot \frac{\sqrt{73}}{\sqrt{73}} = -\frac{8\sqrt{73}}{73},$$

which is no better than the original fraction.

Example 5.1.2: Finding Trig Function Values Given Another Trig Function Value

Find the values of two other trig functions if θ is an obtuse angle with...

Trigonometric Values at Special Angles

Example 5.1.3: Evaluating Trig Functions at Special Acute Angles

Find the values of all six trigonometric functions of...

Example 5.1.4: Evaluating Trig Functions at Obtuse Angles Related to Special Angles

Find exact values for all six trigonometric functions of...

Definition: Special Angles

The **special angles** are defined to be any angle that can be written as either $\pm 30^\circ k$ or $\pm 45^\circ k$, where $k \in \mathbb{Z}$.

Trigonometric Values at Quadrantal Angles

Example 5.1.5: Evaluating Trig Functions at Quadrantal Angles

Find the values of the six trigonometric functions at...

Quadrants and the Signs of the Trigonometric Functions

Example 5.1.6: Using the Quadrants to Find the Sign

Give the sign of the sine, cosine, and tangent of the angle...

Theorem: Signs of the Trigonometric Functions

The signs of the trigonometric functions depend on the trigonometric function in question and the quadrant where the terminal side of θ lies.

All trigonometric functions are positive when the terminal side of θ is in the first quadrant.

Sine and the cosecant functions are positive when the terminal side of θ is in the second quadrant.

Tangent and the cotangent functions are positive when the terminal side of θ is in the third quadrant.

Cosine and the secant functions are positive when the terminal side of θ is in the fourth quadrant.

Proof available at the end of this section.

Example 5.1.7: Finding the Remaining Trig Values Given One and a Quadrant

Find the values of the remaining trigonometric functions of θ given that...

Answers

Theorem: Signs of the Trigonometric Functions

Since the value of r is always positive (unless the point is the origin), the signs of the trigonometric functions are completely dependent on the signs of the x - and y -coordinates of the point on the terminal side of θ .



Therefore, the signs of the sine and cosecant functions completely depend on the signs of the y -value of the point on the terminal side of θ . Hence, the sine and cosecant are positive in QI and QII. On the other hand, these functions are negative when the angle terminates in either QIII or QIV.

Likewise, the signs of the cosine and secant functions completely depend on the signs of the x -value of the point on the terminal side of θ . Therefore, the cosine and secant functions are positive in QI and QIV - they are negative in QII and QIII.

Finally, since the tangent and cotangent functions rely on the values of both x and y , we must consider what would make these ratios positive and negative. If the values of x and y are both positive or both negative, their ratios will be positive. This occurs in the first and third quadrants. Thus, the tangent and cotangent functions are positive in QI and QIII - they are negative in QII and QIV.

5.2: Algebra with Trigonometric Functions

Prerequisite Skills

- **Simplifying Expressions**
 - Laws of Exponents
 - Multiplying and Distributing Polynomial Expressions
- **Factoring Techniques**
 - Greatest Common Factor
 - Factoring by Grouping
 - Factoring Trinomials
 - Factoring Difference of Squares
 - Factoring Sums and Differences of Cubes

Evaluating Trigonometric Expressions

Example 5.2.1

Evaluate each expression for $X = 30^\circ$ and $Y = 240^\circ$.

- a. $4 \sin(3X + 45^\circ)$
- b. $1 - \cos(4Y)$

Answer available at the end of this section.

Powers of Trigonometric Functions

Notation: Powers of Trigonometric Functions

$\sin^2(\theta)$	means	$(\sin(\theta))^2$	$\csc^2(\theta)$	means	$(\csc(\theta))^2$
$\cos^2(\theta)$	means	$(\cos(\theta))^2$	$\sec^2(\theta)$	means	$(\sec(\theta))^2$
$\tan^2(\theta)$	means	$(\tan(\theta))^2$	$\cot^2(\theta)$	means	$(\cot(\theta))^2$

Example 5.2.2

Evaluate $\tan^4(60^\circ)$.

Caution: Negative Exponents

We have already stated that

$$[\cos(\theta)]^2 = \cos^2(\theta)$$

and

$$[\sin(\theta)]^5 = \sin^5(\theta).$$

However, we will **not** adopt such notation for negative exponents. That is,

$$[\cos(\theta)]^{-1} \neq \cos^{-1}(\theta).$$

The reason for excluding the negative exponents from this "shorthand" notation will become apparent when we discuss inverse trigonometric functions.

Products of Trigonometric Functions

Example 5.2.3

Multiply $2 \tan(\beta) (4 \tan^2(\beta) + \tan(\alpha))$

Example 5.2.4

Expand $(4 \cos(\alpha) + 3)^2$

Factoring Expressions Involving Trigonometric Functions

Factoring Out a Common Expression

Example 5.2.5

Factor $2 \cos^2(\phi) - \cos(\phi) \sin(\phi)$.

Factoring by Grouping

Example 5.2.6

Factor $12 \sec(\alpha) \tan(\beta) + 9 \sec(\alpha) - 20 \tan(\beta) - 15$.

Factoring Trinomials

Example 5.2.7

Factor $3 \sin^2(\beta) - 2 \sin(\beta) - 1$.

Factoring Differences of Squares

Example 5.2.8

Factor completely.

$$\sin^4(\alpha) - 81 \cos^4(\beta)$$

Factoring Sums and Differences of Cubes

Example 5.2.9

Factor.

$$125 \csc^3(\theta) + 64 \cot^3(\phi)$$

Answers

Example 5.2.1

5.3: Introduction to Trigonometric Identities

Prerequisite Skills

- **Simplifying Expressions**
 - Multiplying and Dividing Rational Expressions
 - Adding and Subtracting Rational Expressions
 - Simplifying Compound Rational Expressions

What is an Identity?

Definition: Identity

1.1: The Language of Algebra

Ratio Identities

Theorem: Ratio Identities

For any angle not coterminal with 90° or 270° ,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

For any angle not coterminal with 0° or 180° ,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Example 5.3.1

If $\sin(\theta) = -\frac{1}{5}$ and $\cos(\theta) = -\frac{2\sqrt{6}}{5}$, find the values of $\tan(\theta)$ and $\cot(\theta)$.

Reciprocal Identities

Theorem: Reciprocal Identities

$\sin(\theta) = \frac{1}{\csc(\theta)}$	(provided $\csc(\theta) \neq 0$)	$\csc(\theta) = \frac{1}{\sin(\theta)}$	(provided $\sin(\theta) \neq 0$)
$\cos(\theta) = \frac{1}{\sec(\theta)}$	(provided $\sec(\theta) \neq 0$)	$\sec(\theta) = \frac{1}{\cos(\theta)}$	(provided $\cos(\theta) \neq 0$)
$\tan(\theta) = \frac{1}{\cot(\theta)}$	(provided $\cot(\theta) \neq 0$)	$\cot(\theta) = \frac{1}{\tan(\theta)}$	(provided $\tan(\theta) \neq 0$)

Example 5.3.2

If $\cot(\theta) = \frac{8}{5}$, find $\tan(\theta)$.

Pythagorean Identities

Theorem: The Pythagorean Identity

For any angle θ ,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Proof available at the end of this section.

Example 5.3.3

Evaluate $5 [\cos^2(12^\circ) + \sin^2(12^\circ)]$ without using a calculator.

Corollary: Alternate Forms of the Pythagorean Identity

$$\sin^2(\theta) = 1 - \cos^2(\theta) \implies \sin(\theta) = \pm \sqrt{1 - \cos^2(\theta)}$$

$$\cos^2(\theta) = 1 - \sin^2(\theta) \implies \cos(\theta) = \pm \sqrt{1 - \sin^2(\theta)}$$

Example 5.3.4

- If $\cos(\theta) = \frac{5}{13}$ and $-90^\circ < \theta < 0^\circ$, find $\sin(\theta)$ and $\tan(\theta)$.
- Find $\sin(\theta)$ and $\tan(\theta)$ by using an appropriate sketch.

Answer available at the end of this section.

Theorem: Pythagorean Identities

For any angle θ where the functions are defined,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

Answers

Theorem: The Pythagorean Identity

From the coordinate definition of the trigonometric functions, we know that $r = \sqrt{x^2 + y^2}$. Thus,

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \implies r^2 &= x^2 + y^2 && \text{(squaring both sides)} \\ \implies 1 &= \frac{x^2}{r^2} + \frac{y^2}{r^2} && \text{(dividing both sides by } r^2) \\ \implies 1 &= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 && \text{(Laws of Exponents)} \\ \implies 1 &= (\cos(\theta))^2 + (\sin(\theta))^2 && \text{(definitions of the cosine and sine functions)} \\ \implies 1 &= \cos^2(\theta) + \sin^2(\theta) && \text{(using a better notation)} \end{aligned}$$

Example 5.3.4

This page titled [5.3: Introduction to Trigonometric Identities](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

5.4: Introduction to Proofs in Trigonometry

Prerequisite Skills

- **Graphing Functions**
 - Graphs of Functions

Simplifying Expressions Involving Trigonometric Functions

Example 5.4.1

Simplify each expression.

- $(1 - \cos(\theta))(1 + \cos(\theta))$
- $2 \cos(t) - 4 \cos(w) \sin(w) + 3 \cos(t) - 2 \cos(w)$

Answer available at the end of this section.

Rewriting Trigonometric Expressions

Example 5.4.2

Rewrite $\sin^2(\alpha) \cos^2(\alpha)$ as an expression involving only sums or differences of powers of $\cos(\alpha)$.

Example 5.4.3

Simplify the expression

$$\frac{3}{\sqrt{x^2 + 4}}$$

as much as possible by substituting $2 \tan(\theta)$ for x .

Proving Identities

Example 5.4.4

Prove the identity.

$$(\sin(\theta) - \cos(\theta))^2 - 1 = -2 \sin(\theta) \cos(\theta)$$

Example 5.4.5

Prove the identity $2 \cos^2(x) - 1 = 1 - 2 \sin^2(x)$

Answers

Example 5.4.1

This page titled [5.4: Introduction to Proofs in Trigonometry](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

6: RIGHT TRIANGLE TRIGONOMETRY

6.1: The Trigonometric Functions - Right Triangle Definition

6.2: Technology and Right Triangle Trigonometry

6.3: Solving Right Triangles

6.4: Applications Involving Right Triangles

This page titled [6: Right Triangle Trigonometry](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Katherine Yoshiwara](#).

6.1: The Trigonometric Functions - Right Triangle Definition

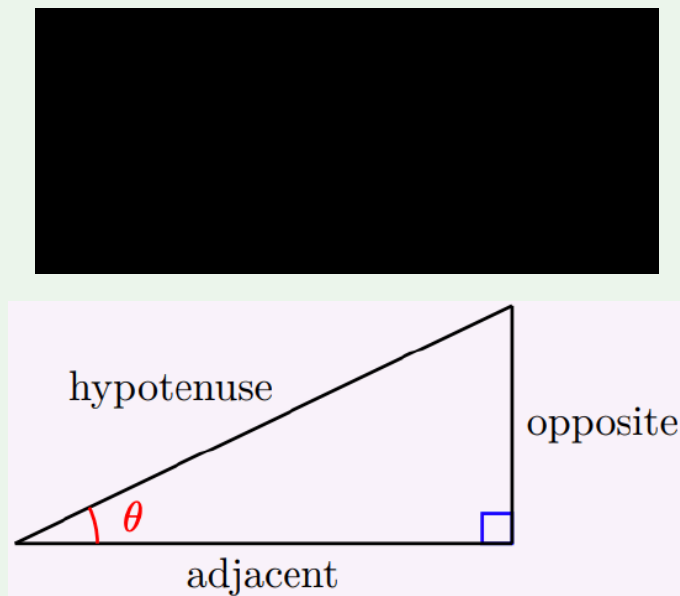
Prerequisite Skills

- **Geometry**
 - The Pythagorean Theorem, Distance Formula, and Midpoint Formula
- **Solving Equations**
 - Solving Literal Equations

Introduction to Right Triangle Trigonometry

Definition: Trigonometric Functions (Right Triangle Definition)

If θ is one of the angles in a right triangle, where the side opposite θ is called "opposite," the side adjacent to θ is called "adjacent," and the hypotenuse is named as such, we define the trigonometric functions of θ as follows:



Example 6.1.1: Finding the Values of Trig Functions Given Two Sides of a Right Triangle

Find all trigonometric ratios of the labeled angle in the triangle.

Definition: Pythagorean Triple

A **Pythagorean triple** consists of three positive integers a , b , and c , such that $a^2 + b^2 = c^2$.

Definition: Pythagorean Triangle

A right triangle whose sides form a Pythagorean triple is called a **Pythagorean triangle**.

Example 6.1.2

If the hypotenuse of a right triangle is h and the side opposite θ is L , write expressions in h and L for all six trigonometric functions of θ .

Revisiting the Fundamental Identities

Example 6.1.3

If β is an acute angle in a right triangle and $\csc(\beta) = \frac{2}{\sqrt{3}}$, compute the remaining trigonometric ratios of β .

Example 6.1.4

If λ is an acute angle in a right triangle, $\tan(\lambda) = \frac{2}{3}$, and $\cos(\lambda) = \frac{3}{\sqrt{13}}$, find $\sin(\lambda)$ using only the Ratio Identities.

Right Triangle Trigonometry and the Special Angles

Example 6.1.5: Stolen from the text

Let $x = 60^\circ$ and $y = 45^\circ$. Evaluate.

$$\sqrt{3} \csc(x) - \frac{5}{\sqrt{2}} \sec(y) + \sqrt{6} \tan(90^\circ - x) \sin(y)$$

Cofunction Identities

Definition: Cofunction

26.5: Trigonometric Identities - The Reciprocal and Ratio Identities

Theorem: Cofunction Identities

26.5: Trigonometric Identities - The Reciprocal and Ratio Identities

Example 6.1.6

Evaluate (by hand).

$$\cos^2(51^\circ) + \cos^2(39^\circ)$$

Observations of Values for Trigonometric Functions Using Right Triangle Trigonometry

Some Trigonometric Functions Have Limited Ranges

Example 6.1.7

If θ is an angle within a right triangle, is it possible for $\csc(\theta) = \frac{1}{2}$?

6.2: Technology and Right Triangle Trigonometry

Using a Calculator to Compute the Values of Trigonometric Functions

Note: Change to Degree Mode

Before evaluating any trigonometric function, ensure your technology is set to the correct "mode."

Example 6.2.1

Approximate $\cos(43^\circ)$ and a lot more...

Caution: Save Rounding for the Final Answer

Example 6.2.2c demonstrates a significant point that bears repeating - **rounding is for display purposes only!** Never round a result and then use that rounded result for other computations.

For example, had we rounded our approximation of $\sin(10^\circ)$ to be 0.1736 and then squared, we would have incorrectly stated $\sin^2(10^\circ) \approx 0.0301$. While the difference between this result and the more accurate result of 0.0304 may seem insignificant, these rounding errors can propagate in such a way as to make massive differences between answers.

Example 6.2.2

Simplify, and evaluate for $x = 25^\circ$, $y = 70^\circ$.

$$3 \cos(x) + \cos(y) - 2 \cos(y) + \cos(x)$$

Example 6.2.3

Use technology to approximate $\csc(82^\circ)$ to three decimal places.

Finding an Acute Angle Using a Calculator

Note: Use Degree Mode

Until later in this course, you should make sure your calculator is set to "degree" mode. If you do not do this, your answers, while correct in a different angular measure system, will be incorrect.

Caution: Do Not Use Rounded Values for Calculations

Example 6.2.4

Find the angle whose cosine is... (some decimal value).

Caution: Inverse Trigonometric Functions are NOT Reciprocals

The notation $\sin^{-1}(x)$ does *not* mean $\frac{1}{\sin(x)}$!

We indeed use negative exponents to indicate reciprocals of numbers, for example, $a^{-1} = \frac{1}{a}$ and $3^{-1} = \frac{1}{3}$. However, "sin" by itself is not a variable.

- $\sin^{-1}(x)$ means "the angle whose sine is x "
- $\frac{1}{\sin(x)}$ means "the reciprocal of the sine of angle x "

(You may recall that $f^{-1}(x)$ denotes the inverse function for $f(x)$. We will study inverse trigonometric functions in a later chapter.)

Example 6.2.5

Find the acute angle for which the equation is true. Round answers to the nearest tenth.

- $\sin(\beta) = 0.123$
- $\tan(\kappa) = 1.234$

Example 6.2.6

Suppose x is an acute angle within a right triangle. Solve the equation. Round answers to two decimal places.

$$5.156 - \sec(x) = 3.104$$

6.3: Solving Right Triangles

Prerequisite Skills

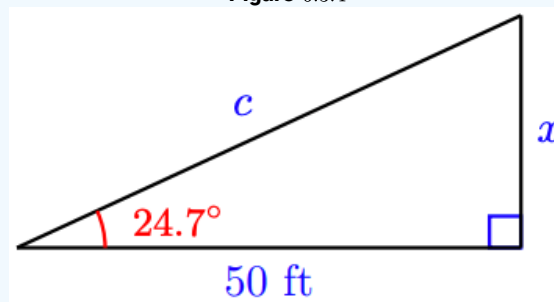
- **Simplifying Expressions**
 - Simplifying Radical Expressions

Using a Trigonometric Function to Find an Unknown Side

Example 6.3.1

- a. Use the tangent ratio to find x in Figure 6.3.4.

Figure 6.3.4



- b. Use the sine ratio to find the hypotenuse, c , of the triangle.
 c. Use the Pythagorean Theorem to find the hypotenuse of the triangle. Is the answer the same with both methods? Explain why the calculations might give (slightly) different answers.

Answer available at the end of this section.

Solving Triangles

Definition: Solving a Triangle

Solving a triangle means finding all the missing side lengths and angle measures, given some initial information about the triangle.

Example 6.3.2

In the right triangle $\triangle ABC$, $\angle A = 23^\circ$ and (non-hypotenuse) side $a = 11$. Solve the triangle.

Traditional Notation: The Right Angle of $\triangle ABC$

Given the right triangle $\triangle ABC$, we adopt the convention used throughout most math textbooks that $\angle C$ is the 90° angle and side c is the hypotenuse.

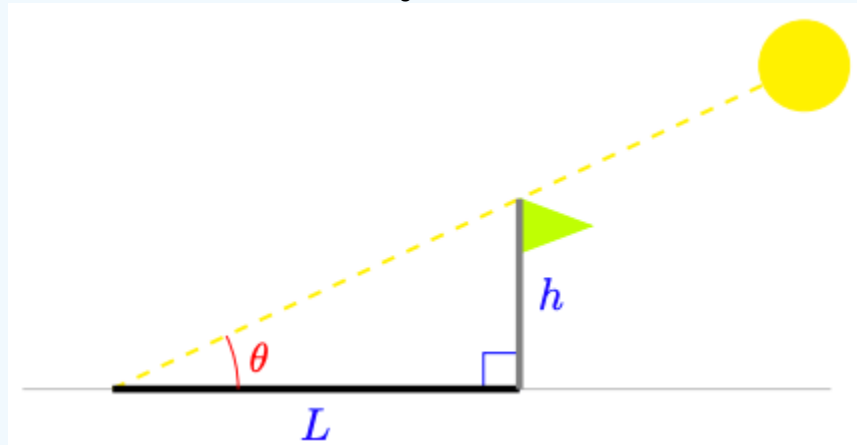
Example 6.3.3

Solve $\triangle ABC$ given that $b = 4$ and $c = 5$.

Example 6.3.4

The length, L , of the shadow cast by a flagpole on a sunny day depends on the height, h , of the flagpole and the angle, θ , that the sun's rays make with the ground (see Figure 6.3.8).

Figure 6.3.8



- Write an expression for the length, L , of the shadow cast by a flagpole of height h when the sun makes an angle of θ from the ground.
- Find the length (to the nearest 0.01 meter) of the shadow cast by a 3-meter flagpole when the sun makes an angle of 20° from the ground.

Answer available at the end of this section.

Introducing Right Triangles to Non-Right Triangles

Example 6.3.5

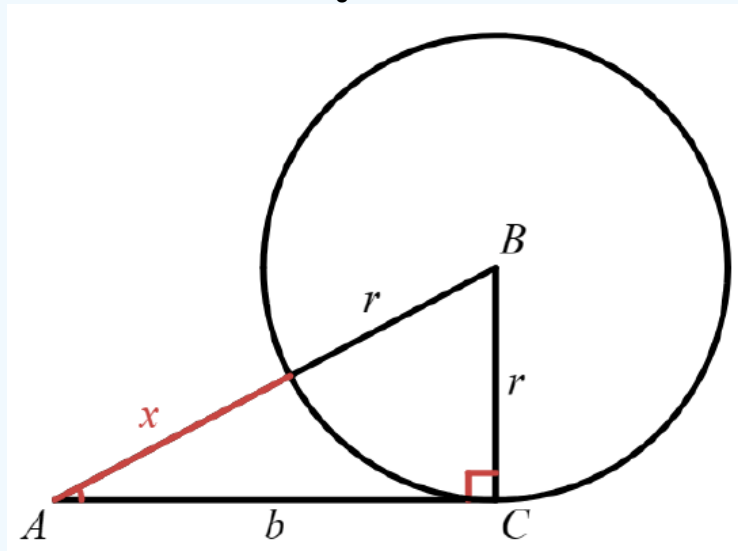
An isosceles triangle has base angles of 72° and equal sides of length 6.8 centimeters. Find the size of the base.

Thought-Provoking Right Triangle Geometry

Example 6.3.6

Figure 6.3.10 shows a circle of radius $r = 5$ and a right triangle with side a along the radius of the circle and side b tangent to the circle.

Figure 6.3.10

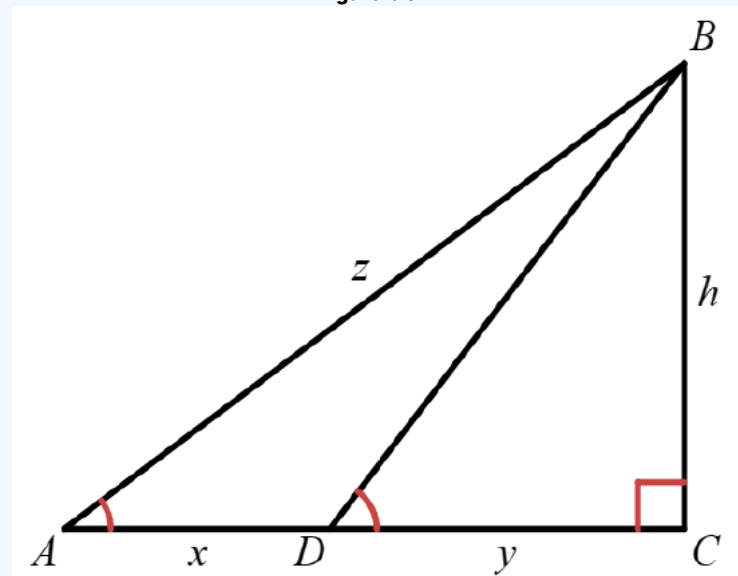


If $\angle A = 24^\circ$ and $b = 10$ inches, find the radius of the circle and the partial length of the hypotenuse, x , that sits outside the circle. Round your answers to the nearest tenth.

Example 6.3.7

In Figure 6.3.11, $y = 12$, $\angle A = 24^\circ$, and $\angle BDC = 62^\circ$. Find x .

Figure 6.3.11



Answers

Example 6.3.1

Example 6.3.4

6.4: Applications Involving Right Triangles

Prerequisite Skills

- **Linear Functions**
 - Equations of Lines

Angles of Elevation or Depression

Definition: Angle of Elevation

The **angle of elevation** (or **angle of inclination**) is the angle formed between a horizontal line and the line of sight when an observer looks upwards at an object. It represents the amount an observer would need to elevate their gaze (or line of sight) from a horizontal plane to see an object above them.

Example 6.4.1

The tallest living tree is a coastal redwood named Hyperion, measuring 378.1 feet tall. From a distance of 100 feet from the base of the tree, what is the angle of elevation to the top of the tree? Round the answer to the nearest degree.

Definition: Angle of Depression

The **angle of depression** (or **angle of declination**) is the angle formed between a horizontal line and the line of sight when an observer looks downwards at an object. It represents the amount an observer would need to lower their gaze (or line of sight) from a horizontal plane to see an object below them.

Example 6.4.2

On August 7th, 1974, French street performer Philippe Petit began the most famous high-wire walk in history, calmly traversing the space between the former Twin Towers (in New York City) at an elevation of 1,350 feet without any safety gear. Not long after stepping out onto the thin wire, the angle of depression to the base of one building was 88° , and the angle of depression to the base of the other building was 86° . How many feet was his tightrope walk?

Angles of Lines

Definition: Angle of Inclination of a Line

The **angle of inclination of a line** is the measure of the angle in the positive direction from the positive x -axis to the line.

Theorem: The Tangent and the Angle of Inclination of a Line

If the slope of the line is m , then the angle of inclination of the line, α , is related to the slope by the formula

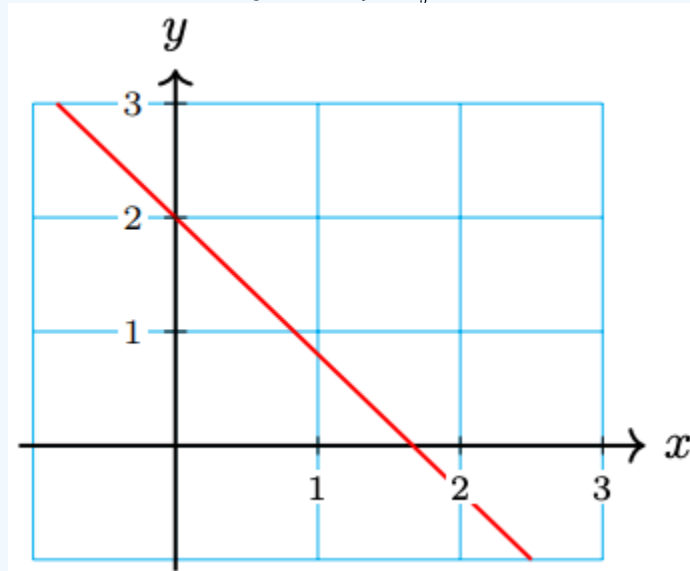
$$\tan(\alpha) = m$$

where $0^\circ \leq \alpha \leq 180^\circ$.

Example 6.4.3

Find the angle of inclination of the line in Figure 6.4.8.

Figure 6.4.8: $y = -\frac{6}{5}x + 2$



Navigation Problems

Heading

Definition: Heading (navigation)

In aviation and nautical navigation, the **heading** of a vessel or aircraft is the compass direction measured clockwise from due north in which the craft's bow or nose is pointed.

Example 6.4.4

Lap leaves the airport and flies 150 miles at a heading of 132° . How far east of the airport is he at that time?

Bearing

Definition: Bearing (navigational)

In land-based navigation, the **bearing** (also called the **quadrant bearing**) of an object (person, vehicle, etc.) represents the direction that object is facing or moving. The notation for bearing is a letter followed by an acute angle followed by a letter. These are known as **Meridian - Angle - Direction**.

- **Meridian:** The first letter indicates whether the angle is measured from North (N) or South (S), whichever is closer to the direction.
- **Angle:** This is the acute angle measured from the chosen meridian to the direction of interest.

- **Direction:** The second letter indicates whether the angle is measured towards the East (E) or West (W).

Example 6.4.5

Ivan leaves the airport and flies 150 miles at a bearing of S 48° E. How far east of the airport is he at that time?

Example 6.4.6

Sutter County Airport is 28.4 miles due north of Sacramento International Airport. If Placerville Airport is 45.3 miles due east of Sacramento International Airport, what is the bearing from Sutter County Airport to Placerville Airport, and what is the distance between these two airports?

Example 6.4.7

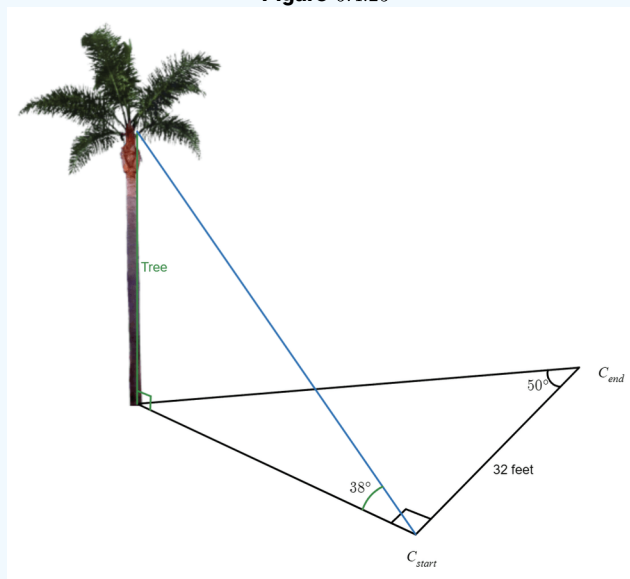
A ship leaves port bearing S 32° E. It travels in this direction for 1 mile and changes its bearing to S 58° W. It then travels an additional 10 miles. At this point, the captain remembers he left his phone back at port (and he wanted to post a ton of videos of this trip on his social media platforms). Find the direct distance and the bearing from this point back to port so that the "social-influencer" captain can get his phone. (Round answers to the nearest tenth.)

Compound Right Triangle Applications

Example 6.4.8

Camille is worried about her neighbor's tree eventually falling and hitting her house. Unfortunately, her neighbor has refused to allow her to pay someone to measure the exact height of the tree, so Camille resorts to her top-notch math skills. Since she cannot step onto her neighbor's property, Camille stands on her property, facing the tree, and finds the angle of elevation to the top of the tree to be 38° . She then turns 90° and walks for 32 feet (staying on her property the entire time). At this point, she turns to face her neighbor's tree and measures the angle between the path she just walked and her neighbor's tree to be 50° . Use this information and Figure 6.4.25 to find the tree's height.

Figure 6.4.25



This page titled [6.4: Applications Involving Right Triangles](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

7: RADIAN MEASURE AND THE CIRCULAR FUNCTIONS

7.1: Reference Angles

7.2: Radian Measure, Arc Length, and Sector Area

7.3: The Trigonometric Functions - Unit Circle Definition

7.1: Reference Angles

Reference Angles

Definition: Reference Angle

The **reference angle** for an angle θ given in standard form is the acute angle formed between the terminal side of θ and the x -axis. By convention, the reference angle is always considered positive (even if it opens in a clockwise direction). It is denoted with the "hat" notation, $\hat{\theta}$.

Example 7.1.1

Find and label the reference angle for the given angles...

Caution: Reference Angles are Acute

By definition, reference angles are acute angles. This means they are *not* 0° or 90° . In fact, quadrantal angles do not have reference angles.

Evaluating Trigonometric Functions Using Reference Angles

Definition: Reference Triangle

A **reference triangle** is a right triangle formed by drawing a perpendicular from a point on the terminal side of an angle (in standard position) to the x -axis.

Reference Angle Theorem

Theorem: Reference Angle Theorem

The value of a trigonometric function of any angle is equal to that of the function at its reference angle, except for sign. The quadrant determines the sign of the function.

Example 7.1.2

- Find the reference angle for 187° .
- Sketch 187° and its reference angle in standard position, along with the reference triangle. Verify that both angles have the same trigonometric values, up to sign.

Corollary: Rotational Symmetry

For any integer n ,

$$\begin{array}{lll} \sin(\theta + 360^\circ n) & = & \sin(\theta) & \csc(\theta + 360^\circ n) & = & \csc(\theta) \\ \cos(\theta + 360^\circ n) & = & \cos(\theta) & \sec(\theta + 360^\circ n) & = & \sec(\theta) \\ \tan(\theta + 360^\circ n) & = & \tan(\theta) & \cot(\theta + 360^\circ n) & = & \cot(\theta) \end{array}$$

An Additional Note About the Tangent

As we will investigate later, it turns out that

$$\tan(\theta + 180^\circ n) = \tan(\theta) \quad \text{and} \quad \cot(\theta + 180^\circ n) = \cot(\theta)$$

for any integer n ; however, this is not needed for our current discussion.

Example 7.1.3

Find the exact value of each trigonometric function.

- $\sin(210^\circ)$
- $\sec(-780^\circ)$
- $\sin(405^\circ)$

Answer available at the end of this section.

Finding Non-Acute Angles

Example 7.1.4

- Solve $\sec(\theta) = -1.8$, where $90^\circ < \theta < 180^\circ$. Round your answer to the nearest tenth of a degree.
- Find the sine of θ given that $\tan(\theta) = -2$, $\cos(\theta) > 0$, and $0^\circ < \theta < 360^\circ$. Round your answer to four decimal places.

Answer available at the end of this section.

Example 7.1.5

Solve $\tan(\beta) = -\sqrt{3}$ given $0^\circ < \beta < 360^\circ$ and $\cos(\beta) > 0$.

Answers

Example 7.1.3

Example 7.1.4

7.2: Radian Measure, Arc Length, and Sector Area

Definition: Arc and Arc Length

An **arc** along a circle is a portion of the circumference of a circle. It's the curved line connecting two points on the circle's edge. The length of this curve is called the **arc length**.

Definition: Central Angle

An angle with vertex at the center of a circle is called a **central angle** of the circle.

Definition: Subtend

A central angle whose initial and terminal sides meet the endpoints of an arc is said to **subtend** (or **span**) the arc.

Measuring Angles in Radians

Definition: Radian Measure

The **radian measure** of an angle is given by

$$\text{fraction of one revolution} \times 2\pi.$$

By definition, then, the radian measure of an angle is equal to the arc length subtended by the angle on a circle of radius 1.

Example 7.2.1

What is the radian measure of an angle of 150° ? What about some other special angles?

Example 7.2.2

In which quadrant would you find an angle of 2 radians? An angle of 5 radians?

Converting Between Degrees and Radians

Theorem: Angle Measure Conversion Theorem

$$\begin{aligned} 2\pi \text{ radians} &= 360^\circ \\ \Rightarrow 1 &= \frac{180^\circ}{\pi \text{ radians}} && \text{(dividing both sides by } 2\pi \text{ radians)} \\ &\text{and} \\ \Rightarrow \frac{\pi \text{ radians}}{180^\circ} &= 1 && \text{(dividing both sides instead by } 360^\circ) \end{aligned}$$

Example 7.2.3

- Convert 4 radians to degrees.
- Convert 4 degrees to radians.
- What about special angles?

Answer available at the end of this section.

Quadrants and Reference Angles in Radians

Example 7.2.4

State the quadrant in which $\beta = \frac{7\pi}{6}$ terminates, state its reference angle and sketch. What about $\alpha = -\frac{17\pi}{16}$?

Arc Length Formula

Theorem: Arc Length Formula

On a circle of radius r , the length s of an arc spanned by an angle θ in radians is

$$s = r\theta.$$

Caution: The Radian Requirement

It is crucial to remember that the arc length formula, $s = r\theta$, is only valid if θ is in radians.

Example 7.2.5

The Earth's radius is roughly 3,960 miles. If you travel 300 miles due south, how many degrees of latitude will you traverse? (Latitude is measured in degrees north or south of the equator.)

Example 7.2.6

A central angle of 135° in a circle of radius r cuts off an arc of length 24 cm. What is the radius of the circle?

Areas of Sectors

Definition: Sector (of a circle)

A **circular sector** (or simply a **sector**) is the region of a disk enclosed by two radii and the arc between their endpoints. The angle θ formed by the two radii at the center of the circle is the **central angle** of the sector. Two radii divide a disk into two sectors: the one with the smaller area is the **minor sector**, and the one with the larger area is the **major sector**.

Theorem: Area of a Sector

If a sector of a circle with radius r has central angle θ , measured in radians, where $0 < \theta \leq 2\pi$, then the area A of the sector is the fraction $\frac{\theta}{2\pi}$ of the area of the entire circle:

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta.$$

Example 7.2.7

Find the area of the sector subtended by a central angle of 240° in a circle of radius 10.

Let's also make another one up!

Trigonometric Functions of Angles in Radians

Table 7.2.2

θ in Degrees	θ in Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undefined
180°	π	0	-1	0
270°	$\frac{3\pi}{2}$	-1	0	undefined
360°	2π	0	1	0

Example 7.2.8

Compute the values of some trig functions at radian angles.

Answers

Example 7.2.3

7.3: The Trigonometric Functions - Unit Circle Definition

Connecting the Sine and Cosine Functions to the Unit Circle

Lemma

Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then

$$\cos(\theta) = x \quad \text{and} \quad \sin(\theta) = y.$$

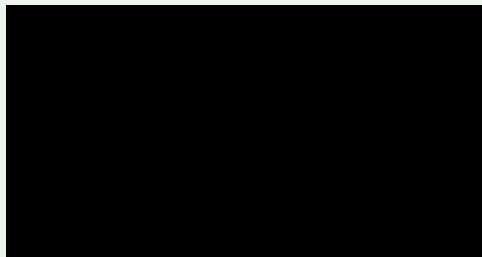
Example 7.3.1

Find the coordinates of point P on the unit circle below.

Trigonometric Functions of Real Numbers and the Unit Circle Definition

Definition: Trigonometric Functions (Unit Circle Definition)

Let $P(x, y)$ be a point on the unit circle, and let t be the arc length from the point $(1, 0)$ to P along the circumference of the unit circle. The trigonometric functions of the real number t (also called the **circular functions** in this context) are defined as follows:



Example 7.3.2

Let $t = \frac{5\pi}{6}$ be the length of the arc along the unit circle from the point $(1, 0)$ to $P(x, y)$. Evaluate each of the following functions.

- $\sin(t)$
- $\cot(t)$

Answer available at the end of this section.

Example 7.3.3

Find the coordinates of the terminal point, P , of an arc of length t starting at $(1, 0)$ on a unit circle.

- $t = \frac{3\pi}{2}$
- $t = \frac{4\pi}{3}$

Answer available at the end of this section.

Using a Calculator to Compute the Values of Trigonometric Functions

Note: Change to Radian Mode

If you use a scientific or graphing calculator, refer to your instruction manual on setting it to radian mode. The scientific calculator in Desmos is set to degree mode by default, so if you are using that technology, you need to click the "DEG" toggle so that "RAD" is showing; however, the *graphing* calculator in Desmos defaults to radian mode.

Example 7.3.4

Use your calculator to find the sine and cosine of the following angles in radians. Round your answers to four decimal places...

Domain and Range of the Circular Functions

Interactive Element: Geometric Interpretations of the Circular Functions

Interact: Move the point D to help you investigate the circular functions geometrically.

Theorem: Domains of the Circular Functions

Let t be a real number and k be an integer. The domains of the circular functions are

$\sin(t)$ and $\cos(t)$	All real numbers, or $(-\infty, \infty)$
$\tan(t)$ and $\sec(t)$	All real numbers except $t = \frac{\pi}{2} + k\pi$
$\cot(t)$ and $\csc(t)$	All real numbers except $t = k\pi$

Theorem: Ranges of the Circular Functions

Let t be a real number. The ranges of the circular functions are

$\sin(t)$ and $\cos(t)$	$[-1, 1]$
$\tan(t)$ and $\cot(t)$	$(-\infty, \infty)$
$\sec(t)$ and $\csc(t)$	$(-\infty, -1] \cup [1, \infty)$

Example 7.3.8

Is it possible for $\csc(t) = -\frac{1}{2}$?

Answers

Example 7.3.2

Example 7.3.3

This page titled [7.3: The Trigonometric Functions - Unit Circle Definition](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

8: GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

8.1: Base Graphs of the Fundamental Trigonometric Functions

8.2: Non-Rigid Transformations

8.3: Rigid Transformations

8.4: Graphs of the Remaining Trigonometric Functions

8.1: Base Graphs of the Fundamental Trigonometric Functions

Prerequisite Skills

- **Sets and Numbers**
 - Intervals and Interval Notation
 - Sets and Set Notation
- **Graphing Functions**
 - Symmetry
- **Rational Functions**
 - Asymptotes

Definition: Fundamental Trigonometric Functions

The sine, cosine, and tangent are called the **fundamental trigonometric functions**.

Motivating Periodic Functions of Angles

Definition: Periodic Motion

When a body repeats its movement along a path in a regular and predictable manner over a specific time interval, its movement is called **periodic motion**.

Definition: Periodic Function

A mathematical function modeling periodic motion is called a **periodic function**. A periodic function repeats its values at regular intervals, meaning its graph has a pattern that repeats itself over a specific horizontal distance or time interval.

Definition: Sine Curve

A smooth, wave-like curve that oscillates (moves up and down) periodically and which can be derived from either the sine or cosine function is called a **sine curve**. A curve of this nature is often called **sinusoidal**.

Graphing the Sine Function

Point-Plotting Approach

Example 8.1.1

Sketch a portion of the graph of $y = \sin(x)$ on the interval $[0, 2\pi]$ by making a table of values.

Unit Circle Approach

Interactive Element: Developing the Graph of the Sine Function via the Unit Circle

Observation: As the angle (or arc length) t increases, the y -coordinate increases to 1, then decreases to -1 , then increases back to $y = 0$.

Interactive Element: Extending the Graph of the Sine Function

Basic Features of the Sine Curve

Domain and Range

Theorem: Domain and Range of the Sine Function

The domain of $f(x) = \sin(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Period

Definition: Period

For any function $f(x)$, the smallest positive number p for which

$$f(x + p) = f(x)$$

for every x in the domain of f is called the **period** (or the **fundamental period**) of $f(x)$. If no such p exists, the function is not periodic.

Theorem: Period of the Sine Function

The period of $f(x) = \sin(x)$ is 2π . This is also known as the **natural period** of the sine function.

Definition: Principal Cycle (of a trigonometric function)

The **principal cycle** of a trigonometric function is the interval starting at $x = 0$ and terminating at $x = p$, where p is the period of the trigonometric function.

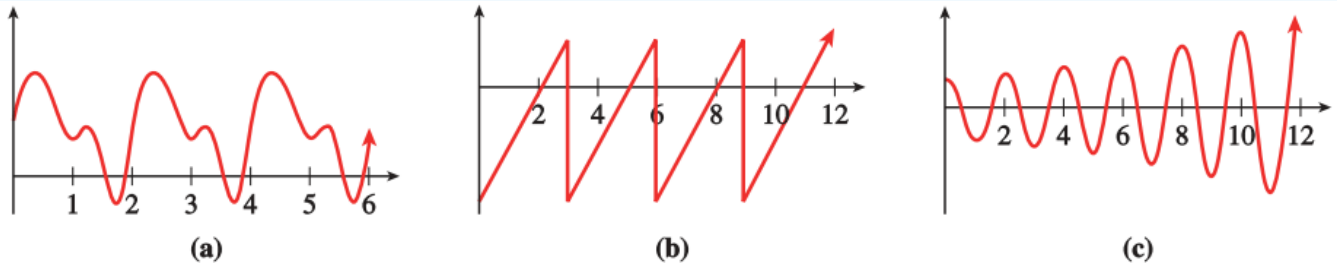
Theorem: Principal Cycle of the Sine Function

The principal cycle of the sine function is $[0, 2\pi]$.

Example 8.1.2

Which of the functions shown below are periodic? If the function is periodic, give its period.

Figure 8.1.12



Amplitude

Definition: Amplitude (of a function)

Let M and m be a function's greatest and least values, respectively. The **amplitude** of the graph of the function is defined to be

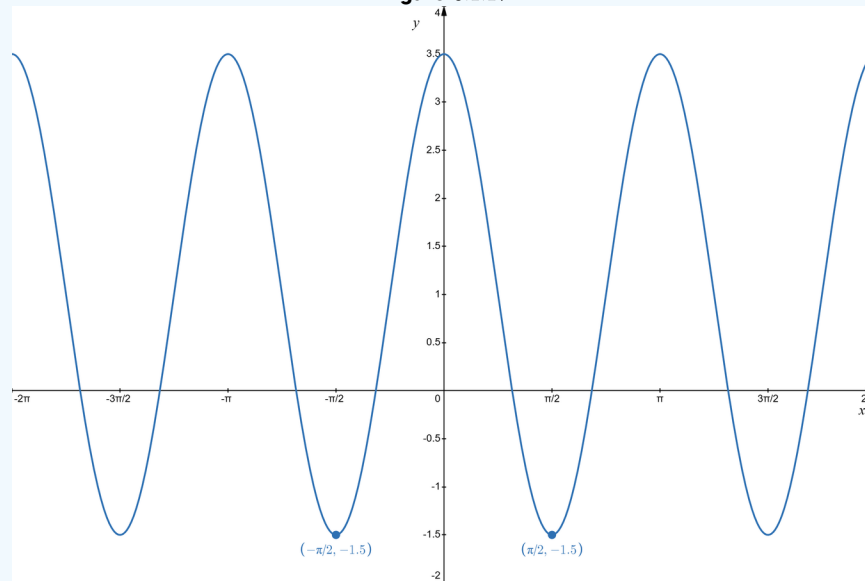
$$\text{Amplitude} = \frac{1}{2} |M - m|.$$

A function that is unbounded has no amplitude.

Example 8.1.3

Determine the amplitude and period of the function.

Figure 8.1.17



Theorem: Amplitude of the Sine Function

The amplitude of $f(x) = \sin(x)$ is 1.

Proof available at the end of this section.

Zeros

Theorem: Zeros of the Sine Function

The zeros of $f(x) = \sin(x)$ occur at $x = \pi k$, where $k \in \mathbb{Z}$.

Graphing the Cosine Function

Point-Plotting Approach

Example 8.1.4

Sketch a portion of the graph of $y = \cos(x)$ on the interval $[0, 2\pi]$ by making a table of values.

Unit Circle Approach

Interactive Element: Developing the Graph of the Cosine Function via the Unit Circle

Observation: As the angle (or arc length) t increases, the x -coordinate decreases from 1 to -1 , then increases from -1 back to 1.

Domain, Range, Period, Amplitude, and Zeros

Theorem: Domain and Range of the Cosine Function

The domain of $f(x) = \cos(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Theorem: Period of the Cosine Function

The period of $f(x) = \cos(x)$ is 2π . Thus, the natural period of the cosine function is 2π .

Theorem: Principal Cycle of the Cosine Function

The principal cycle of the cosine function is $[0, 2\pi]$.

Theorem: Amplitude of the Cosine Function

The amplitude of $f(x) = \cos(x)$ is 1.

Theorem: Zeros of the Cosine Function

The zeros of $f(x) = \cos(x)$ occur at $x = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$.

Graphing the Tangent Function

Point-Plotting Approach

Unit Circle Approach

Interactive Element

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Desmos Interactive online](#)



Conceptual Approach

Definition

The notation $x \rightarrow a^+$ is read as " x is approaching a from the right side of a ," and it means that x is strictly greater than a and is getting closer and closer to a .

The notation $x \rightarrow a^-$ is read as " x is approaching a from the left side of a ," and it means that x is strictly less than a and is getting closer and closer to a .

In both cases, x will never take on the value of a .

Domain, Range, Period, and Zeros

Theorem: Domain and Range of the Tangent Function

The domain of the tangent function is $\{x \mid x \neq \frac{\pi}{2} + \pi k\}$, where $k \in \mathbb{Z}$. The range is $(-\infty, \infty)$.

Theorem: Period of the Tangent Function

The period of $f(x) = \tan(x)$ is π . Thus, the natural period of the tangent function is π .

Theorem: Principal Cycle of the Tangent Function

The principal cycle of the tangent function is $[0, \pi]$.

Theorem: Zeros of the Tangent Function

The zeros of the tangent function occur at $x = \pi k$, where $k \in \mathbb{Z}$.

Symmetry Identities

Theorem: Symmetry Identities

26.5: Trigonometric Identities - The Reciprocal and Ratio Identities

Example 8.1.5

Determine if the function is even, odd, or neither.

$$g(x) = x^3 \cos(x) - \tan(x)$$

Example 8.1.6

Evaluate $\cot\left(-\frac{2\pi}{3}\right)$ using symmetry.

Solving Trigonometric Equations Using Graphs

Example 8.1.8

Use a graph to find the two angles between 0 and 2π that satisfy $\sin(t) = -0.85$.

Answers

Theorem: Amplitude of the Sine Function

The maximum value of $f(x) = \sin(x)$ is $M = 1$ and the minimum value is $m = -1$. Hence,

$$\text{Amplitude} = \frac{1}{2} |1 - (-1)| = 1.$$

8.2: Non-Rigid Transformations

Definition: Rigid Transformation

A **rigid transformation** of a graph is a transformation that changes the position of the graph without changing its shape or size.

Definition: Non-rigid Transformation

A **non-rigid transformation** of a graph is a transformation that retains the graph's original center position but scales the graph vertically or horizontally, or changes the graph's orientation by reflecting the graph about a line.

Amplitude and the Graph of $y = A \operatorname{trig}(x)$

Theorem: Vertical Scaling

The graph of

$$f(x) = A \operatorname{trig}(x)$$

vertically scales the graph of $y = \operatorname{trig}(x)$ by a factor of $|A|$.

- If $|A| > 1$, it is a vertical stretch by a factor of $|A|$.
- If $0 < |A| < 1$, it is a vertical compression by a factor of $|A|$.

Additionally, if $A < 0$, the graph is reflected about the horizontal axis.

In either case, the y -coordinates of points on the graph of $y = \operatorname{trig}(x)$ are multiplied by A to become the new points on the graph of $f(x) = A \operatorname{trig}(x)$.

Note: Vertical Scaling Changes the Range

Except for the tangent function (whose range is all real numbers), the range of a trigonometric function will change under vertical scaling. For example, in Example 8.2.1b the range of $g(x) = \frac{1}{5} \cos(x)$ is $[-\frac{1}{5}, \frac{1}{5}]$.

Theorem: Amplitude of a Sinusoidal Function

The amplitude of $f(x) = A \cos(x)$ or $g(x) = A \sin(x)$ is $|A|$.

Example 8.2.1

State the amplitude of each function.

- $f(x) = -2 \cos(x)$
- $g(x) = -\frac{1}{2} \sin(x)$
- $h(x) = 12 \tan(x)$

Answer available at the end of this section.

Interactive Element: Investigating the Effects of A on the Graph of $y = A \sin(x)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Desmos Interactive online](#)



Interaction: Adjust the slider for A to see the effect on the graph of $y = A \sin(x)$.

Period

Definition: Key Numbers

The **key numbers** of a cycle for a trigonometric function are the x -values of the initial, final, middle, and quarter-points in the principal cycle. These can be found by partitioning the cycle into four equally-spaced subintervals.

Definition: Step Size

The distance between any two consecutive key numbers is called the **step size**.

Theorem: Period of a Trigonometric Function

Let p be the natural period of a trigonometric function. Then the period of $y = A \operatorname{trig}(Bx)$ is $\frac{p}{B}$.

Theorem: Step Size Formula

The step size for a trigonometric function is $\frac{\text{Period}}{4}$.

Example 8.2.2

Sketch a single cycle of each function, stating its amplitude (if applicable), period, and step size.

- $f(x) = \cos(4x)$
- $g(x) = -2 \sin\left(\frac{x}{5}\right)$
- $y = \frac{1}{2} \tan(-3x)$

Answer available at the end of this section.

Note: Changes in the Period Affect the Domain of the Tangent

While the domain of the sine and cosine functions remain all real numbers no matter the period, the tangent's domain changes when we change the period. For example, in Example 8.2.3c the domain of the tangent does not include $\frac{\pi}{8}$ (whereas, in the base tangent function, $\frac{\pi}{8}$ is in the domain). This is because a change in the period of a function causes **horizontal scaling** (stretching or compressing along the horizontal axis). When this scaling occurs, any vertical asymptotes the function might have also move.

Interactive Element: Investigating the Effects of B on the Graph of $y = A \sin(Bx)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Desmos Interactive online](#)

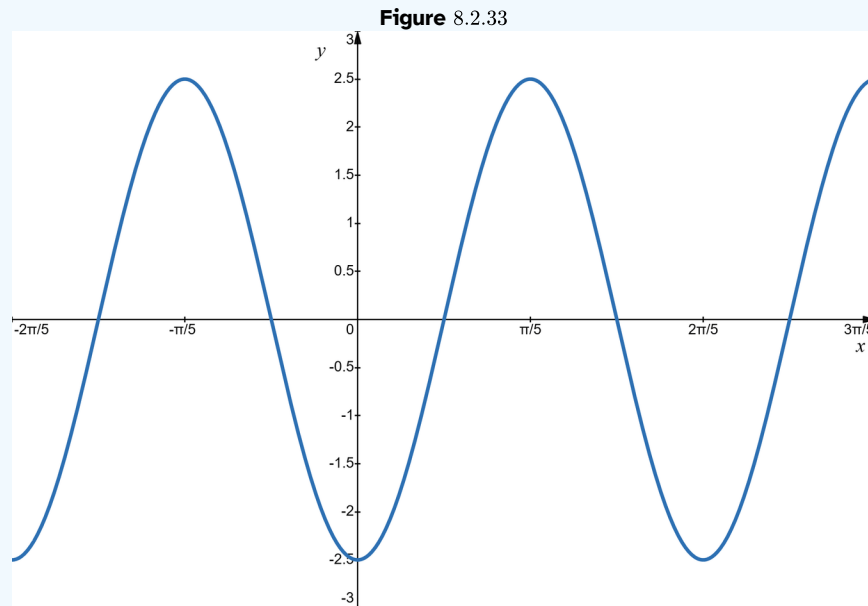


Interaction: Adjust the slider for A and B to see the effect on the graph of $y = A \sin(Bx)$.

Finding Equations Given Graphs

Example 8.2.3

Find the equation of the function graphed below.



Answers

Example 8.2.1

Example 8.2.2

8.3: Rigid Transformations

Phase Shifts (also known as Horizontal Shifts)

Definition: Phase Shift

The **phase shift** of a trigonometric function is where the original x -value from the principal cycle (which is $x = 0$) has been shifted.

Lemma: Phase Shift Formula for $y = \text{trig}(x + C)$

The phase shift for a trigonometric function of the form $y = \text{trig}(x + C)$ is $-C$.

Theorem: Phase Shift Formula

The phase shift of the function $y = A \text{ trig}(Bx + C)$ is $-\frac{C}{B}$.

Caution: C is Not Always the Phase Shift

When considering the function

$$f(x) = A \text{ trig}(Bx + C),$$

it is crucial to remember that B is **not** the period and C is **not necessarily** the phase shift.³

- Period = $\frac{\text{natural period}}{B}$
- Phase Shift = $-\frac{C}{B}$

Example 8.3.1

State the phase shift of $y = -2 \cos(2x - \frac{\pi}{3})$

Example 8.3.2

Graph a single cycle of the given trigonometric function, labeling key numbers, the y -axis, stating the period, step size, amplitude, and phase shift.

- $y = 3 \tan(3x - \pi)$
- $y = \sqrt{2} \sin(-\frac{\pi}{3}x + \frac{\pi}{6})$

Answer available at the end of this section.

Interactive Element: Investigating the Effects of C on the Graph of $y = A \sin(Bx + C)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Desmos Interactive online](#)



Interaction: Adjust the slider for A , B , and C to see the effect on the graph of $y = A \sin(Bx + C)$.

Vertical Shifts and the Midline

Definition: Midline

The **midline** of a trigonometric function is a horizontal line that represents the average value of the function. For sinusoidal functions, it is situated halfway between the maximum and minimum points. The midline for all base graphs of the trigonometric functions is $y = 0$.

Example 8.3.3

Graph each function. State the amplitude (if applicable), period, step size, phase shift, and vertical shift. Your graph should have five key points (or four key points and an asymptote) labeled, as well as the midline.

- $y = -2 + 2 \sin\left(5\pi x + \frac{\pi}{2}\right)$
- $y = \frac{1}{5} \tan(3x - 2) + 1$

Answer available at the end of this section.

Interactive Element: Investigating the Effects of D on the Graph of $y = A \sin(Bx + C) + D$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

[View this Desmos Interactive online](#)

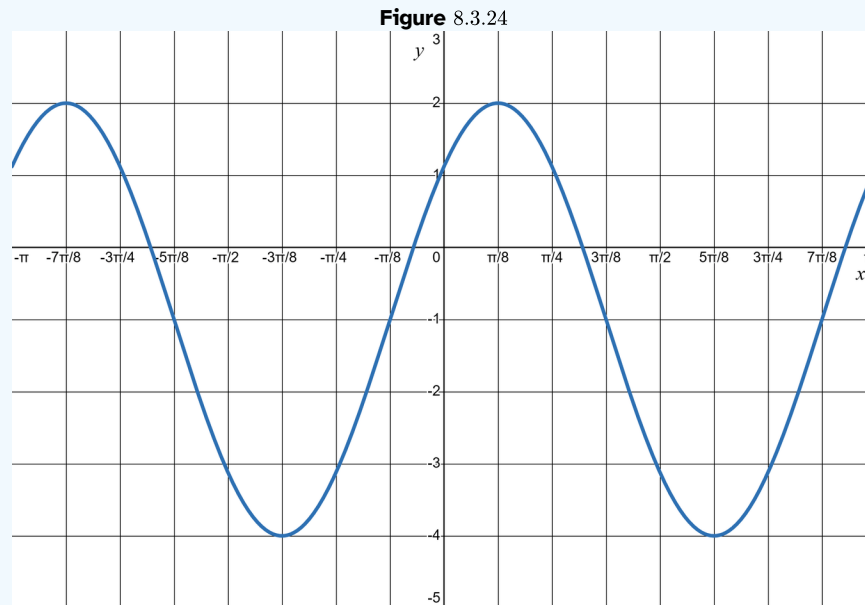


Interaction: Adjust the sliders for each parameter to see the effects on the graph of $y = A \sin(Bx + C) + D$.

Finding Equations Given Graphs

Example 8.3.4

Find the equation of the graphed function.



Revisiting the Cofunction Identities

Theorem: Cofunction Identities (radians)

26.5: Trigonometric Identities - The Reciprocal and Ratio Identities

Introduction to Sinusoidal Models

Example 8.3.5

The pistons in an automobile engine move up and down in the cylinders. If t is in milliseconds, the distance from the top of the piston to the top of the cylinder is given in centimeters by

$$D = 7 + 6 \sin(4\pi t)$$

- Graph the function by hand.
- State the midline, amplitude, and period of the graph.
- Find the largest and the smallest clearance between the piston and the top of the cylinder.

Answers

Example 8.3.2

Example 8.3.3

8.4: Graphs of the Remaining Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

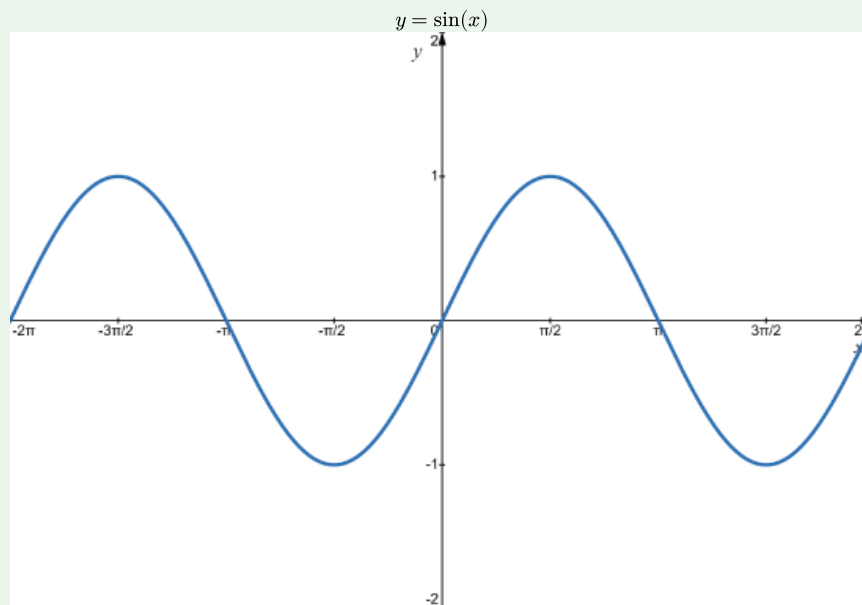
The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A Summary of the Graphs of the Fundamental Trigonometric Functions

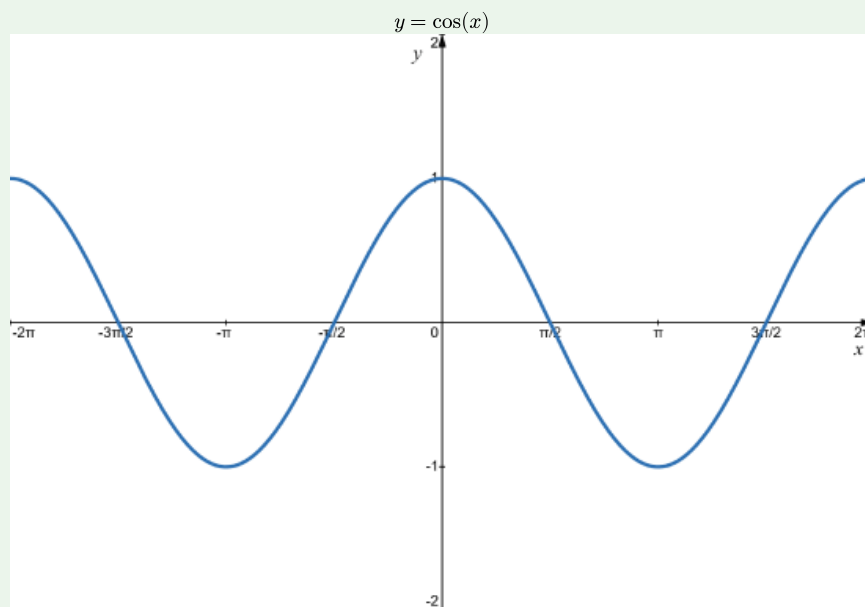
Before we go too far, let's summarize what we know about the fundamental trigonometric functions.

Theorem: Properties of the Graph of the Sine Function



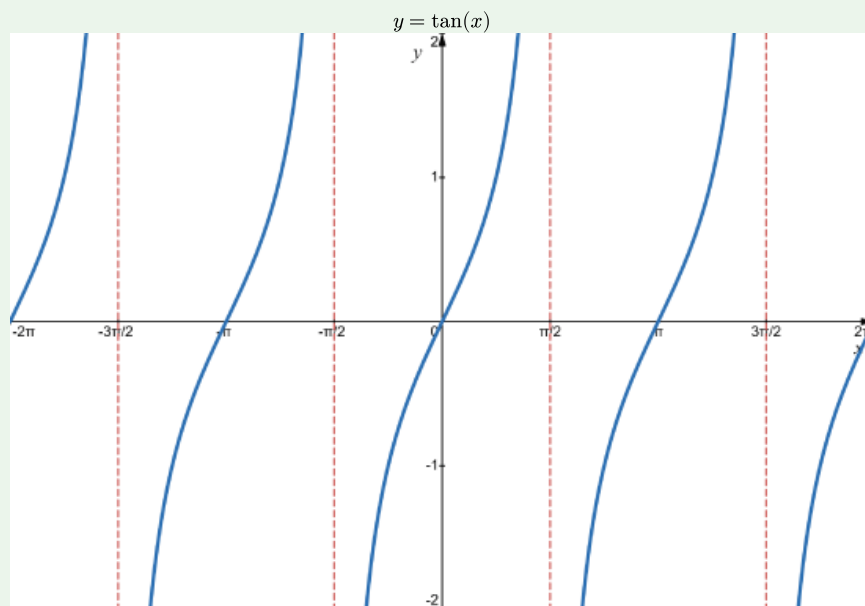
- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = k\pi$, where $k \in \mathbb{Z}$

Theorem: Properties of the Graph of the Cosine Function



- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Even (symmetric about the y -axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

Theorem: Properties of the Graph of the Tangent Function



- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)

- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A
- **Zeros:** $x = k\pi$, where $k \in \mathbb{Z}$

The principal cycle of the tangent is $[0, \pi]$, even though the tangent is undefined halfway through the cycle. It's just the cycle we use as the "standard" for our base graph, and it is okay that the tangent is undefined at some point within the cycle. The principal cycle is *not* the domain.

An Interlude—Vertical Asymptotes

We now turn our attention to the graphs of the remaining trigonometric functions— $\csc(x)$, $\sec(x)$, and $\cot(x)$. In each of these cases, we must understand the conceptual approach to graphing we developed in when we introduced the graphs of the fundamental trigonometric functions.

Recall, if given a fraction where the numerator is held constant and the denominator is *approaching* zero (but not equal to zero), then the value of the fraction is approaching either $+\infty$ or $-\infty$. To drive the point home, let's demonstrate this behavior both numerically and graphically.

For the numerical demonstration, you can create any rational function you want where the numerator is a constant and the denominator is an algebraic expression. All we require is that the denominator approaches zero as the values of x approach some value. For example, suppose we choose the numerator to be 7 and the denominator to be $6x^2 + 11x + 3$. Then the function we are working with is

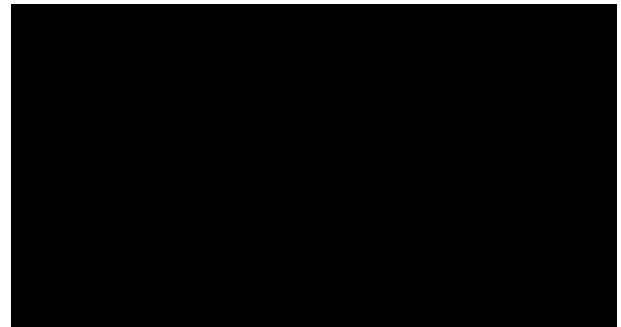
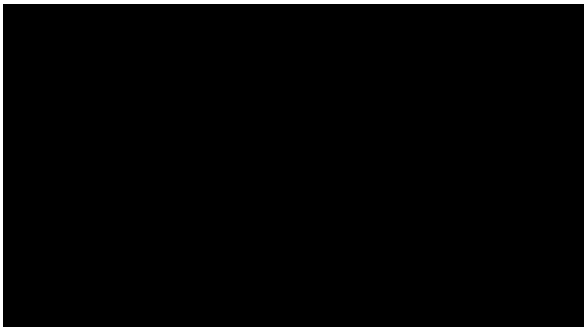
$$R(x) = \frac{7}{6x^2 + 11x + 3}.$$

We now want the values of x to approach some number that causes division by zero. To reveal what causes division by zero for this function, we factor the denominator as follows:

$$R(x) = \frac{7}{(3x + 1)(2x + 3)}.$$

We can see this function would have division by zero at $x = -\frac{1}{3}$ and $x = -\frac{3}{2}$.

Let's build two tables of values—the first to see what happens to the function as $x \rightarrow -\frac{3}{2}^-$, and the second to see what happens as $x \rightarrow -\frac{3}{2}^+$. For convenience, we will rewrite $-\frac{3}{2}$ in its decimal form, -1.5 .



In the first table, as x is approaching -1.5 from the left, the denominator of $R(x)$ is approaching zero from the right. Simultaneously, the function values are becoming larger and larger positive numbers. That is,

$$\text{As } x \rightarrow -\frac{3}{2}^-, R(x) \rightarrow \infty.$$

Similarly, the second table demonstrates that, as x is approaching -1.5 from the right, the denominator of $R(x)$ approaches zero from the left. This causes the function values to become larger and larger negative numbers. We would say,

$$\text{As } x \rightarrow -\frac{3}{2}^+, R(x) \rightarrow -\infty.$$

For the graphical demonstration, we quickly sketch the graph of $R(x)$ around $x = -\frac{3}{2}$.

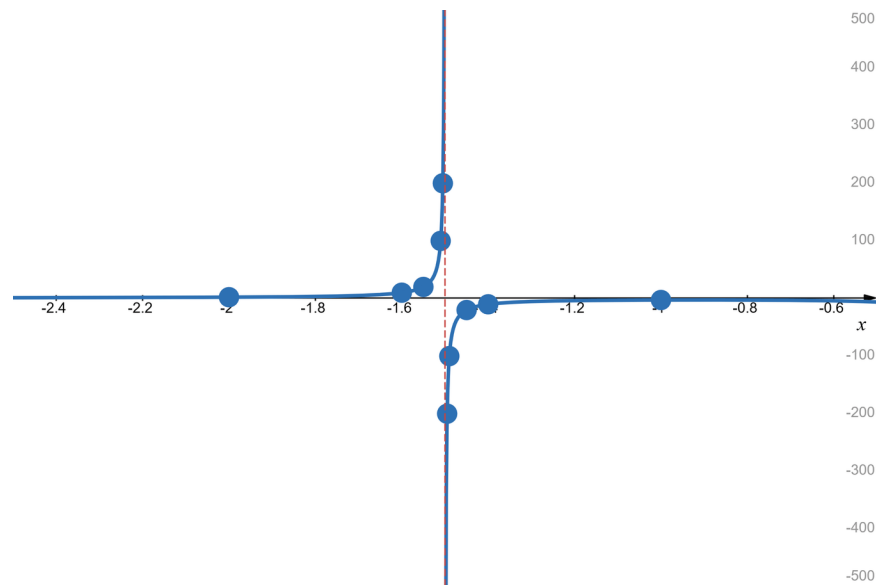


Figure 8.4.1

As you can see from the graph, the y -values of the function (also called the function values) become more and more positive as the x -values get closer and closer to -1.5 from the left. Likewise, the function values become increasingly more negative as the x -values approach -1.5 from the right.

The moral of the story can be stated as follows:

If the numerator is a nonzero constant and the denominator is tending to (but not becoming) zero, the ratio

$$\frac{\text{numerator}}{\text{denominator}}$$

approaches either $+\infty$ or $-\infty$. The sign can be determined by comparing the signs of the numerator and denominator as the denominator approaches zero. Graphically, this means there is a vertical asymptote at this value of x .

Graphing the Cosecant Function

We know that $\csc(x) = \frac{1}{\sin(x)}$. Therefore, by the preceding discussion, as the denominator tends to zero, the overall ratio (which is the cosecant) will tend to $+\infty$ or $-\infty$. The zeros of the sine function occur at $x = k\pi$, where $k \in \mathbb{Z}$ (see Figure 8.4.2). Let's start by focusing on the zero at $x = 0$.

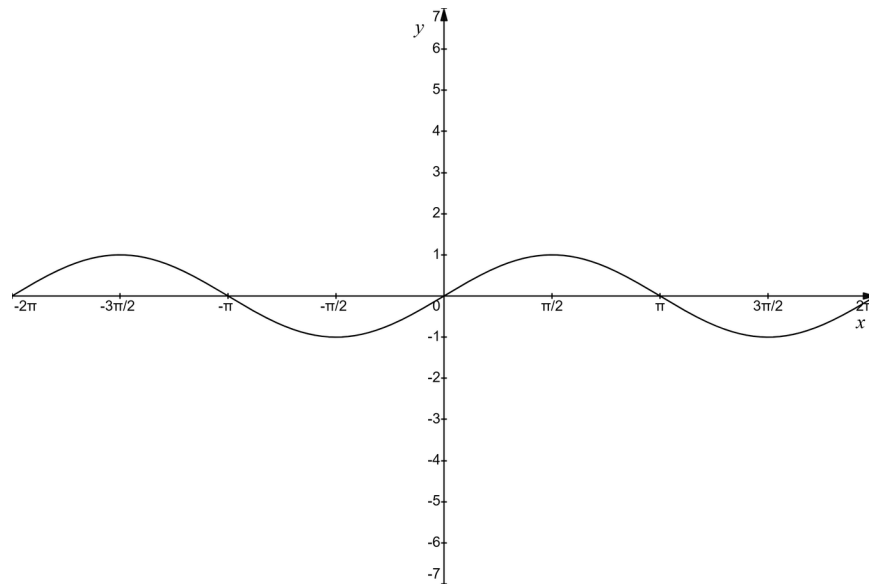


Figure 8.4.2: $y = \sin(x)$

From Figure 8.4.2, we can see as $x \rightarrow 0^-$, the y -values of $\sin(x)$ are approaching 0^- . Therefore,

$$\text{as } x \rightarrow 0^-, \frac{1}{\sin(x)} \rightarrow \frac{1}{0^-} \rightarrow -\infty.$$

It is *negative* infinity because the numerator is *positive*, but the denominator is negative (albeit, very close to zero). Similarly, as $x \rightarrow 0^+$, the y -values of $\sin(x)$ are approaching 0^+ . Therefore,

$$\text{as } x \rightarrow 0^+, \frac{1}{\sin(x)} \rightarrow \frac{1}{0^+} \rightarrow +\infty = \infty.$$

Again, we know the result is *positive* infinity because the numerator is positive and the denominator is positive. Sketching this behavior, we get the following:

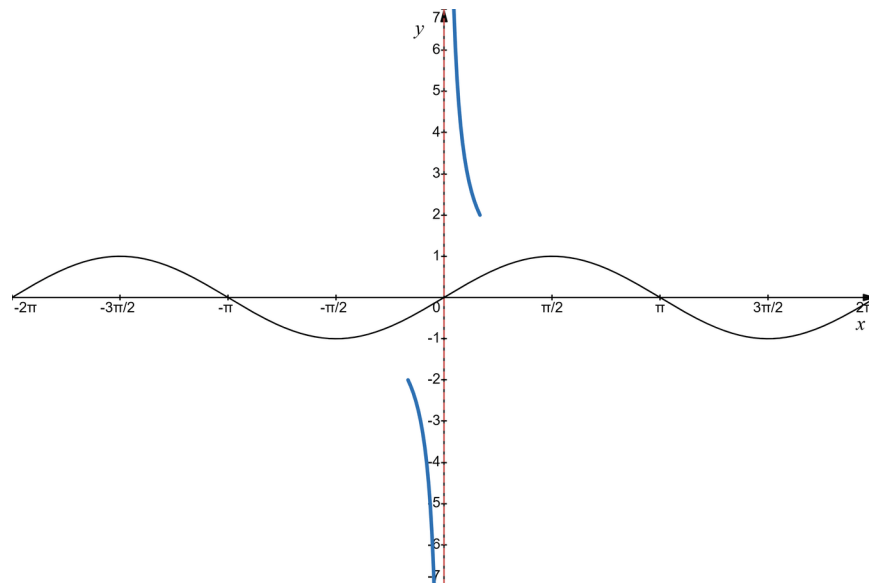


Figure 8.4.3: Beginning the sketch of $y = \csc(x)$

It's hard to see in Figure 8.4.3, but we drew a vertical asymptote at $x = 0$ (the y -axis). A similar investigation shows that

$$\text{as } x \rightarrow \pi^-, \frac{1}{\sin(x)} \rightarrow \frac{1}{0^+} \rightarrow \infty$$

and

$$\text{as } x \rightarrow \pi^+, \frac{1}{\sin(x)} \rightarrow \frac{1}{0^-} \rightarrow -\infty.$$

Using this to fill in a little more of the graph, we get the following:

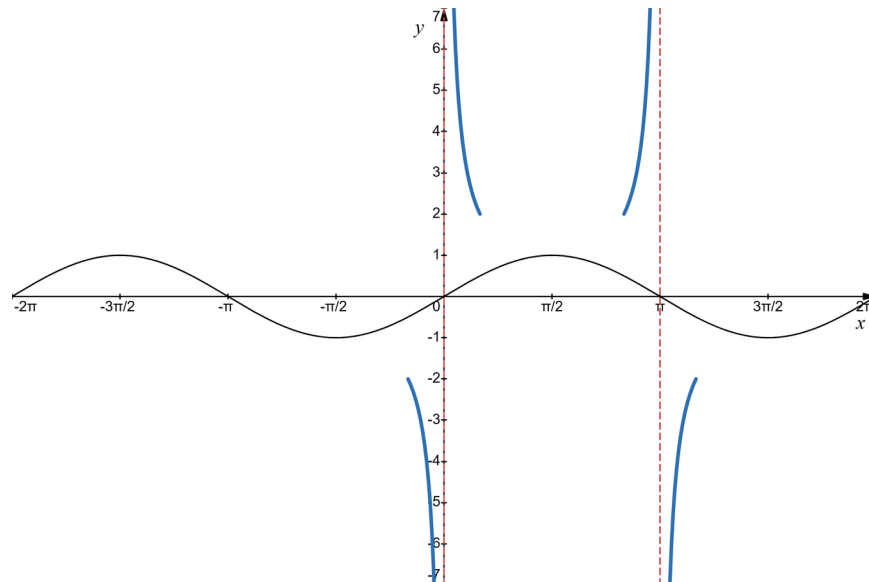


Figure 8.4.4: Getting closer to the graph of $y = \csc(x)$

From this, we expect the cosecant function to have vertical asymptotes at these zeros of $\sin(x)$ (due to the division by zero). Furthermore, we expect the cosecant to be positive when its reciprocal (the sine) is positive, and negative when the reciprocal is negative. This gets us very close to the base graph of the cosecant.

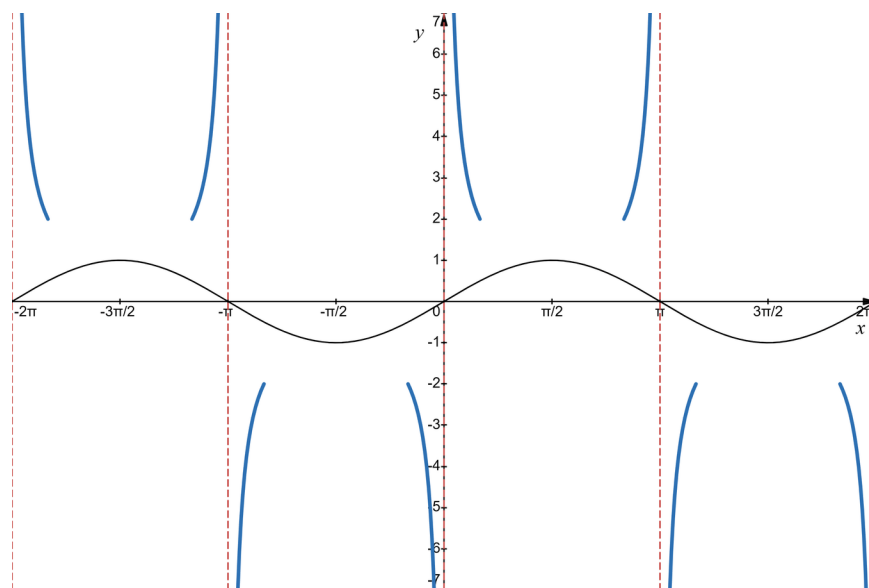


Figure 8.4.5: Almost there

To finish out the graph of the base cosecant function, we will build a table of values for the interval $[\frac{\pi}{6}, \frac{5\pi}{6}]$ and use this to extrapolate the features for the rest of the graph.



Plotting these values, we get the following:

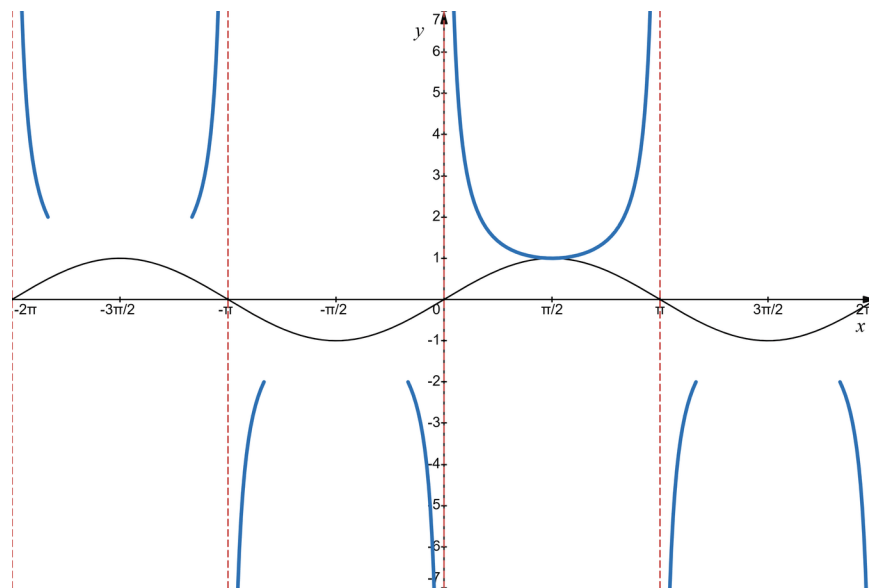


Figure 8.4.6: *So close!*

We can now see the pattern for the entire graph of the cosecant. Filling in the blanks, we get our final graph.

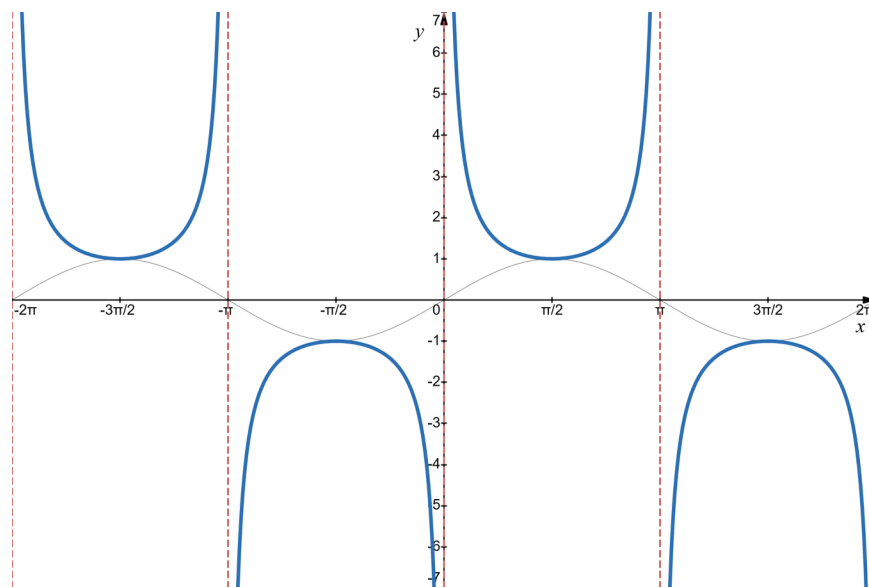
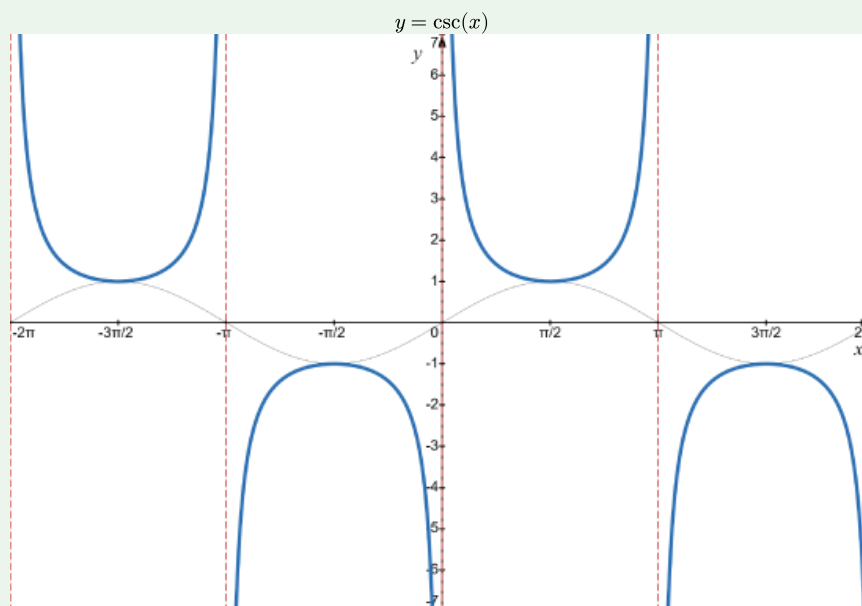


Figure 8.4.7: The base graph of the cosecant, $y = \csc(x)$

I have left a very lightly sketched sine function in this graph. This is because when you are graphing the cosecant, you will inevitably need to start by graphing its reciprocal.

Just as we did with the fundamental trigonometric functions, we now summarize the properties of the cosecant function (with the help of its graph as a visual aid).

Theorem: Properties of the Graph of the Cosecant Function

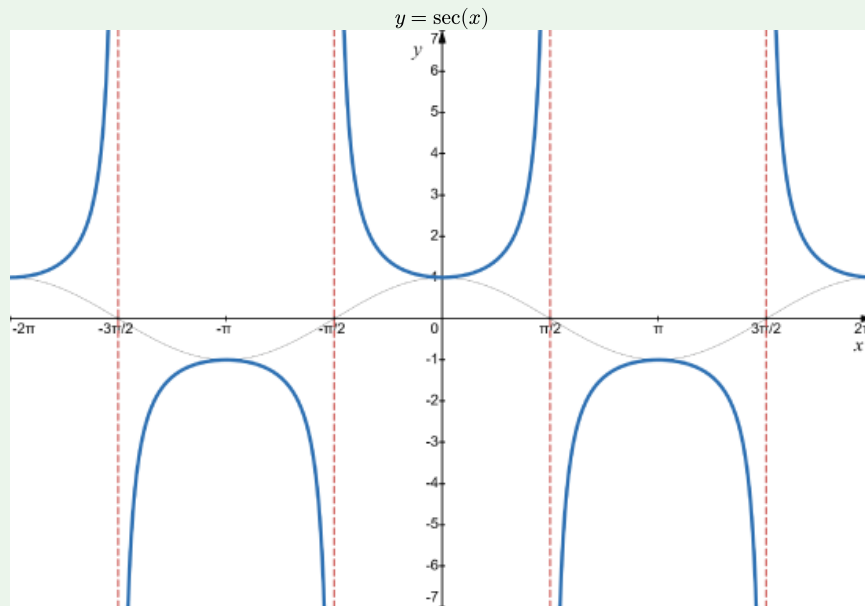


- **Domain:** All $x \neq k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

Graphing the Secant Function

The derivation of the graph of the secant function is similar to that of the cosecant function. It is left for you as an exercise.

Theorem: Properties of the Graph of the Secant Function



- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Even (symmetric about the y -axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

As with the graph of the cosecant, we usually lightly graph the reciprocal of the secant (which is the cosine) to help us in graphing the secant.

Graphing the Cotangent Function

As with the cosecant and the secant, to graph the cotangent function, we will use its reciprocal. That is, because $\cot(x) = \frac{1}{\tan(x)}$, we will first graph the tangent and discuss what happens as the tangent function tends to zero.

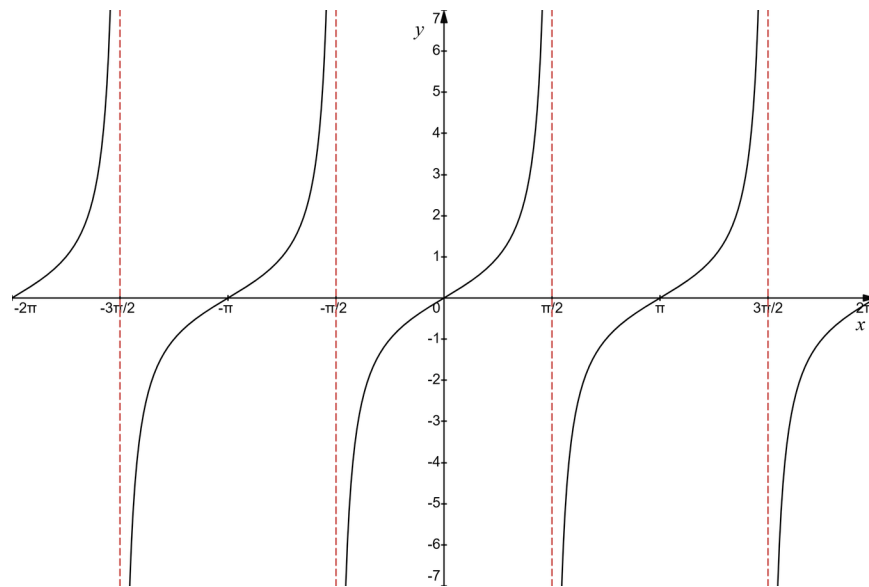


Figure 8.4.8: $y = \tan(x)$

Figure 8.4.8 shows the base graph of the tangent function. Using the same arguments as before, we can get the behavior of the reciprocal function, $\frac{1}{\tan(x)} = \cot(x)$, near the zeros of the tangent.

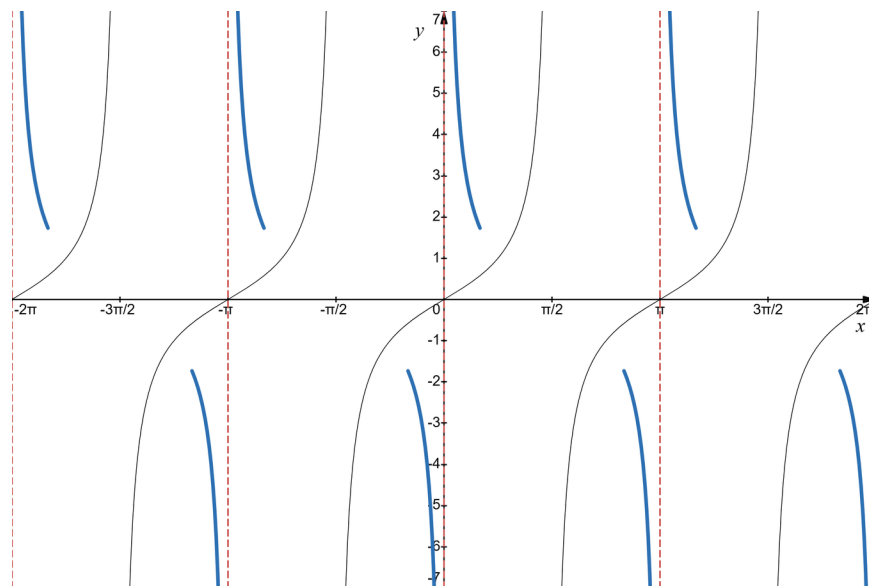
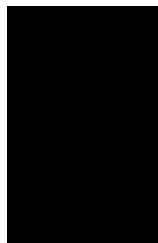


Figure 8.4.9: Almost done graphing the base graph of the cotangent

Notice in Figure 8.4.9 that the vertical asymptotes are not in the same places as the vertical asymptotes for the tangent. This is because the tangent is undefined when the cosine function is zero (at $x = \frac{\pi}{2} + k\pi$) and the cotangent is undefined when the sine is zero (at $x = k\pi$). Evaluating the cotangent function for a few special values between $x = 0$ and $x = \pi$ should give us the pattern of those missing middle regions.



Plotting these points, filling out the rest of the curve with the same pattern, and erasing the underlying tangent graph, we get the following:

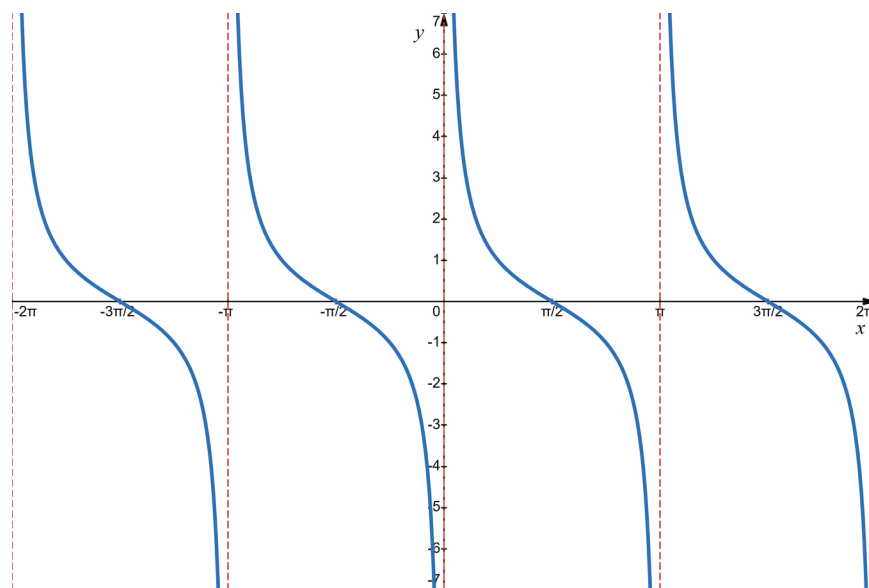
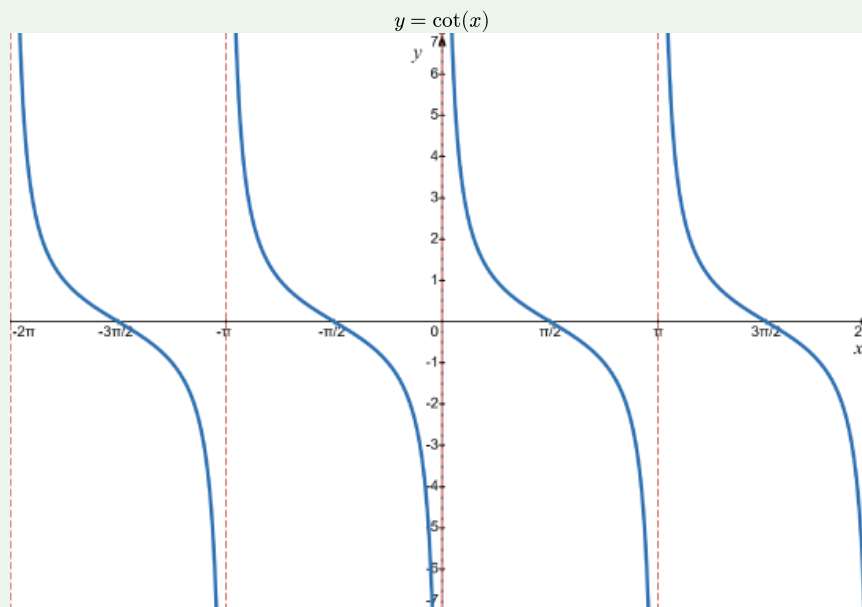


Figure 8.4.10: The base graph of the cotangent, $y = \cot(x)$

As before, we list all of the properties in a convenient theorem.

Theorem: Properties of the Graph of the Cotangent Function



- **Domain:** All $x \neq k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A
- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

Transformations of the Reciprocal Functions

Graphing transformations of the reciprocal functions is the same as graphing the transformations of the fundamental trigonometric functions. In fact, to graph transformations of the cosecant and secant functions, we first fully graph the transformations of their related fundamental functions. After that, graphing the transformed cosecant and secant functions is a somewhat trivial move. Let's see this in action.

Example 8.4.1

Graph a single cycle of each function. State all key characteristics of the graph (including domain, range, period, phase shift, etc.).

- a. $f(x) = -1.2 + 2.8 \sec\left(-3x - \frac{\pi}{2}\right)$
- b. $g(x) = 2 \cot(2x - 2) + 1$

Solutions available at the end of this section.

Checkpoint 8.4.1

Graph the function. Clearly state all key information (including domain and range).

$$y = 3 - 4 \csc\left(-\pi x + \frac{2}{3}\pi\right)$$

Answer available at the end of this section.

Finding Equations Given Graphs

Finding the equation of a reciprocal function given its graph requires a little bit more work than with the fundamental trigonometric functions.

Example 8.4.2

Find the equation of the graphed function.

- a.

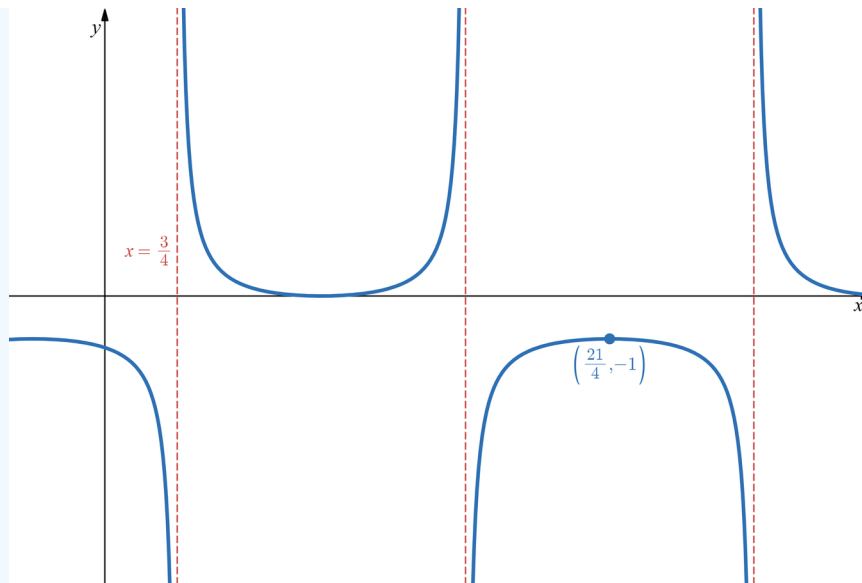


Figure 8.4.19

b.

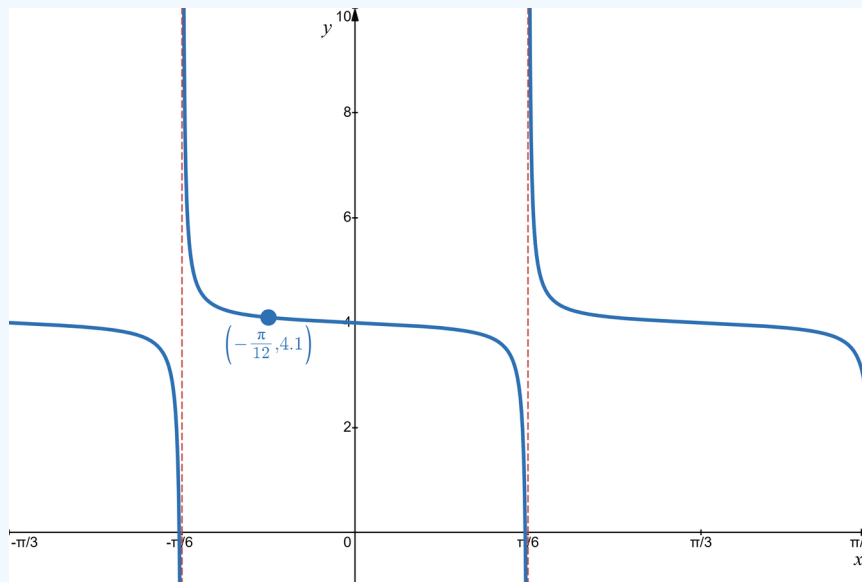


Figure 8.4.20

Solutions available at the end of this section.

Checkpoint 8.4.2

Find the equation of the graphed function.

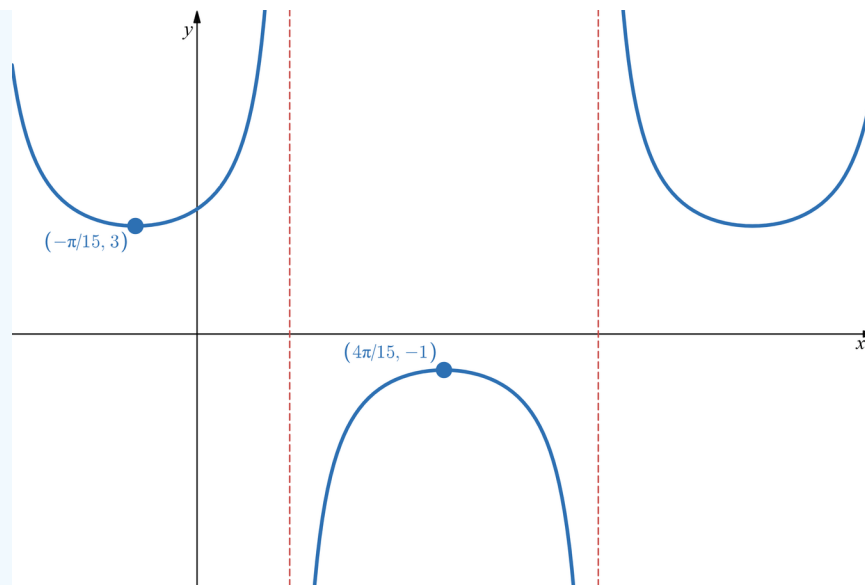


Figure 8.4.22

Answer available at the end of this section.

Success in Mathematics: Math Anxiety (Part 3)

For some math students, anxiety changes their ability to show what they know. They have physical and emotional symptoms. You are responsible for recognizing if math anxiety is interfering with your success. Do you need to make some changes to reduce your math anxiety? Do you need help to do so?

Researchers have identified various events that seem to result in math anxiety. Perhaps you had negative experiences learning math. For you, learning math meant feeling dumb or embarrassed. Perhaps a teacher or parent who struggled with math has passed on their fears to you. Or you believe that your gender or ethnic heritage means you cannot do the math.

Some students learn to cope on their own with their math anxiety. They use relaxation techniques. They practice problems over and over. At test time, their anxiety does not block them from doing the work. Others find help from counselors at their college.

Math anxiety is common among college students. It is your responsibility to think about it and, if needed, get help so that you can cope with your fears.

Math anxiety affects students in physical ways such as increasing blood pressure and heart rate, breathing faster, excess sweating, dry mouth, restlessness, or nausea. Emotionally, it can cause fear, anger, nervousness, worry, and feelings of hopelessness. Students may have trouble concentrating, sitting still, reading, or keeping their legs from bouncing. They may feel desperate - like they need to run away or cry.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Using a scale of 1 to 10, with 10 describing overwhelming anxiety that makes it extremely difficult to do math, rate your math anxiety. How does it affect you?
2. Most students have some anxiety about their classes. They worry about giving speeches, taking tests, writing a good paper, or completing all their work. Worrying can be motivating. It can help us get going and to get things done. Or, worrying can be a bad thing. It paralyzes us. In general, how does worrying affect you?

Applications of Mathematics: Poker

So, where does mathematics apply in poker? Poker players can use mathematics in many different situations; however, it is most commonly used when a player is on a draw, such as a flush or straight draw. If an opponent makes a bet and you are on a draw, you are faced with the decision of whether or not you should call to try and complete the draw by the next card or fold and let your opponent take the pot. In this situation, a player with good knowledge of poker mathematics will always know whether or not to call, while a player who does not know mathematics will be unsure.

Players unfamiliar with poker odds will make an educated guess on whether or not they should call. If the bet is large, they may feel it is too expensive to try and catch the right card, but if the bet is small, they will be more inclined to call. On the other hand, a player that can use mathematics correctly will be able to work out the pot odds they are getting on the hand and act accordingly. Pot odds consider the amount your opponent has bet in relation to the pot size and the likelihood of completing your draw to inform you about whether or not you should call or fold.

1. Can you think of three other games that rely explicitly or implicitly on mathematics? Explain.
2. Oddly enough, most of us are superstitious about numbers. Some people even play their "favorite" numbers when they play the lottery. Suppose you pick your favorite numbers, and I pick the numbers 1, 2, 3, 4, 5, and 6. Who has better odds of winning? Why are you still likely to win more if the odds are the same?

Answers

Section Preview

The graphs of the rest of the trigonometric functions are covered in this section. Try your best to avoid treating this material like an afterthought. The graphs of the secant and cosecant spring up in odd places in Calculus, so the students need to be aware that this material, while held off for the end of the chapter, is critical to understand.

Section Preview

- Graph the cosecant, secant, and cotangent functions, along with transformations of these functions.
- Determine the equation of a trigonometric function given its graph.

Example 8.4.1

- a. Just as we have done previously, we rewrite it in the better form

$$f(x) = 2.8 \sec\left(-3x - \frac{\pi}{2}\right) - 1.2$$

Using the fact that secant is an even function, we clean this up a little more to the following:

$$f(x) = 2.8 \sec\left(-\left(3x + \frac{\pi}{2}\right)\right) - 1.2 = 2.8 \sec\left(3x + \frac{\pi}{2}\right) - 1.2$$

Now that we are finished "preconditioning" the function, it's time we discussed how we are going to graph it. With the secant and cosecant, we perform the intermediate step of graphing their reciprocal functions, the cosine and sine, respectively. So, let's graph

$$y = 2.8 \cos\left(3x + \frac{\pi}{2}\right) - 1.2.$$

We list out all the essential information.

$A = 2.8$	$\Rightarrow$	Amplitude	$= 2.8$
$B = 3$	$\Rightarrow$	Period	$= \frac{\text{Natural Period}}{B} = \frac{2\pi}{3}$
	$\Rightarrow$	Step Size	$= \frac{\text{Period}}{4} = \frac{2\pi/3}{4} = \frac{\pi}{6}$
$C = \frac{\pi}{2}$	$\Rightarrow$	Phase Shift	$= -\frac{C}{B} = -\frac{\pi/2}{3} = -\frac{\pi}{6}$
$D = -1.2$	$\Rightarrow$	Vertical Shift	$= -1.2$

As usual, we graph and label the midline, label the initial number as the phase shift, and step to the other key numbers using the step size.

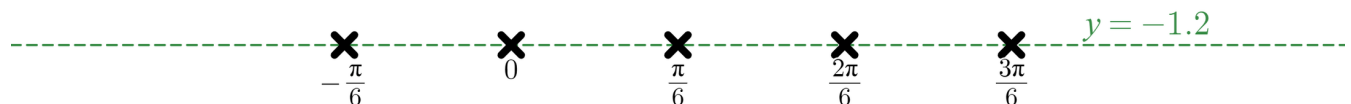


Figure 8.4.11

Note that the distance between the first and last key number is

$$\left| \frac{3\pi}{6} - \left(-\frac{\pi}{6}\right) \right| = \frac{2\pi}{3} = \text{Period},$$

so we have done something right! We then graph the non-reflected cosine, draw in and label the y -axis, and use that to place the x -axis.

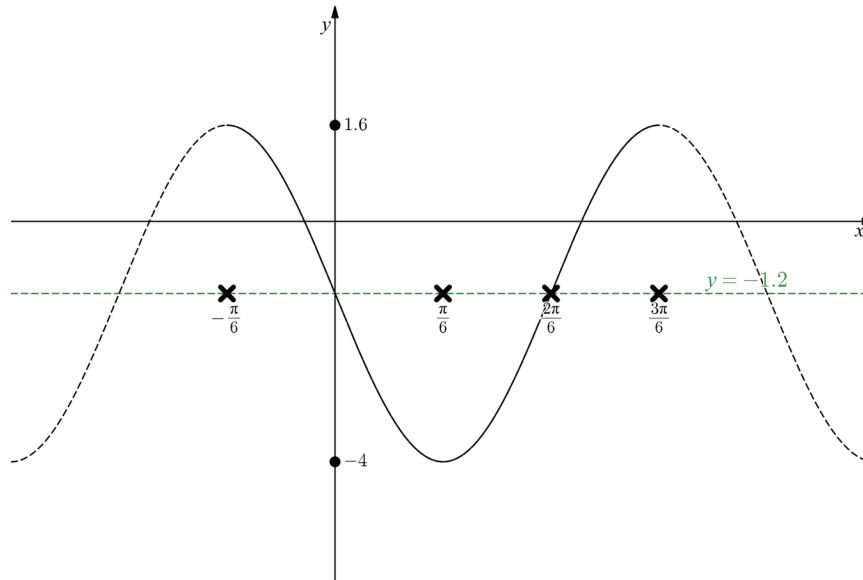


Figure 8.4.12

Now comes the new part. Wherever the cosine function crosses the midline, the original function is undefined. This gives us our vertical asymptotes at $x = 0$, $x = \pm \frac{2\pi}{6} = \pm \frac{\pi}{3}$, $x = \pm \frac{4\pi}{6} = \pm \frac{2\pi}{3}$, and so on. Graphing these vertical asymptotes allows us to graph the secant.

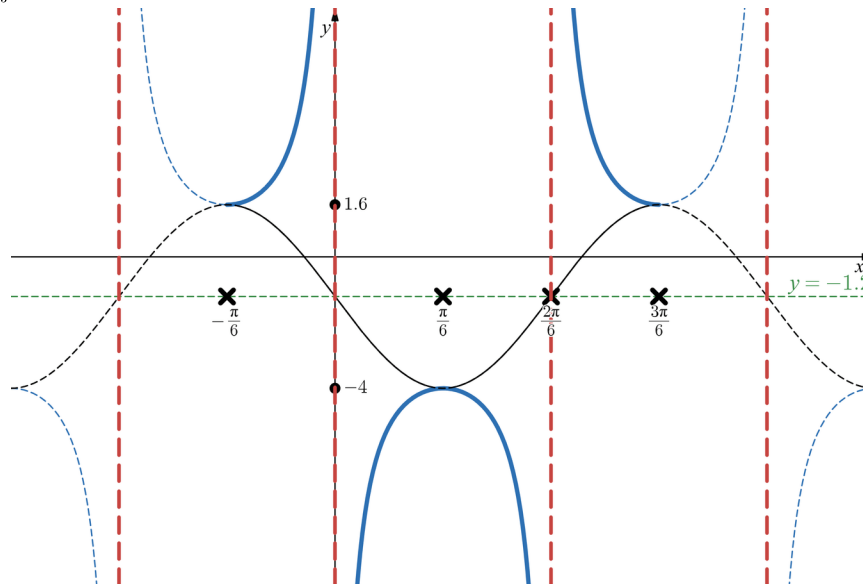


Figure 8.4.13

Notice that I didn't remove the sketch of the cosine. This is perfectly fine (and some instructors expect you to leave it). Finally, we should state as much about this function as we can. The domain (by looking at our spectacular graph) is all $x \neq \frac{\pi}{3}k$, where $k \in \mathbb{Z}$. The range (again, from the graph) is $(-\infty, -4] \cup [1.6, \infty)$. The only thing that we cannot state at this point are the zeros of this function (luckily, I didn't ask for them).

- b. Before we start, I need to be clear and state that the secant and cosecant functions are the only ones where we must first graph their reciprocal functions. The base graph for the cotangent is simple enough to handle without bothering with the tangent.

Now, there isn't any preconditioning necessary for this function, so we go ahead and jump right into the other stages. Just like the tangent, the cotangent does not have an amplitude and its natural period is π .

$$\begin{aligned} B &= 2 \Rightarrow \text{Period} = \frac{\text{Natural Period}}{B} = \frac{\pi}{2} \\ &\Rightarrow \text{Step Size} = \frac{\text{Period}}{4} = \frac{\pi/2}{4} = \frac{\pi}{8} \\ C &= -2 \Rightarrow \text{Phase Shift} = -\frac{C}{B} = -\frac{-2}{2} = 1 \\ D &= 1 \Rightarrow \text{Vertical Shift} = 1 \end{aligned}$$

Graphing and labeling the midline, plotting the phase shift as the first key number, and stepping to each subsequent key number using the step size, we get the following:

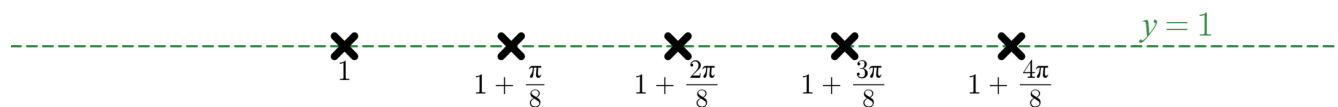


Figure 8.4.14

Because the phase shift was a rational number and the step size was an irrational number, we could not easily add them. Leaving them in the form above is preferred. If you look at the base graph for the cotangent, its principal cycle occurs from $x = 0$ to $x = \pi$. This cycle starts and ends with a vertical asymptote, decreases from left to right, and crosses the midline halfway through. That's precisely what we are going to do here.

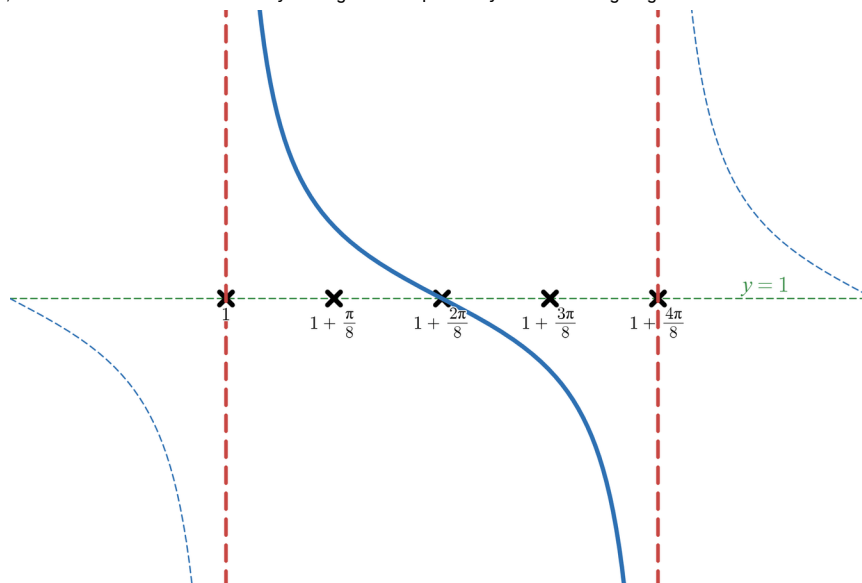


Figure 8.4.15

Now we place the y -axis one unit to the left of the phase shift and the x -axis one unit down from the midline. How do I know how far one unit is? It's more of an art of approximation.

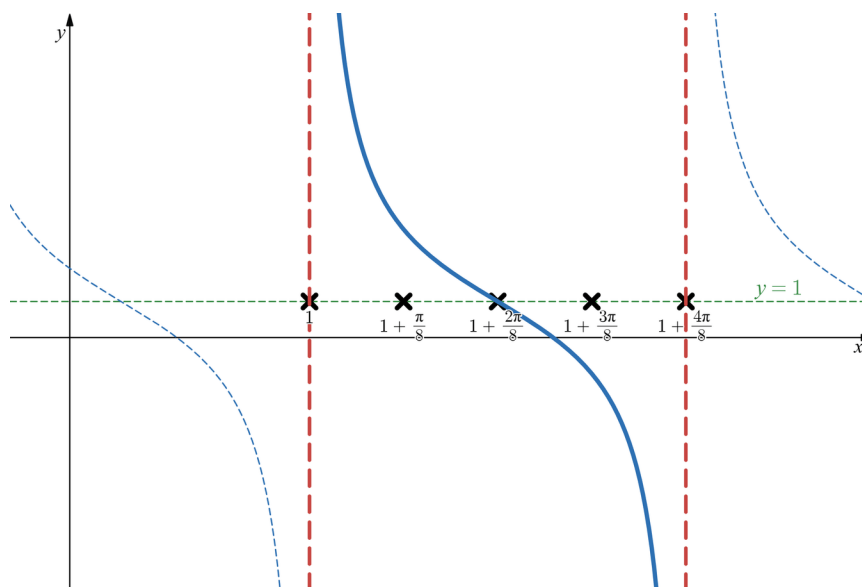


Figure 8.4.16

Finally, we *should* label the two points on the curve at $x = 1 + \frac{\pi}{8}$ and $x = 1 + \frac{3\pi}{8}$. The good news is that the y -values of these points vary from the midline by $|A|$. Thus, the corresponding y -values are $y = 1 + 2 = 3$ and $y = 1 - 2 = -1$.

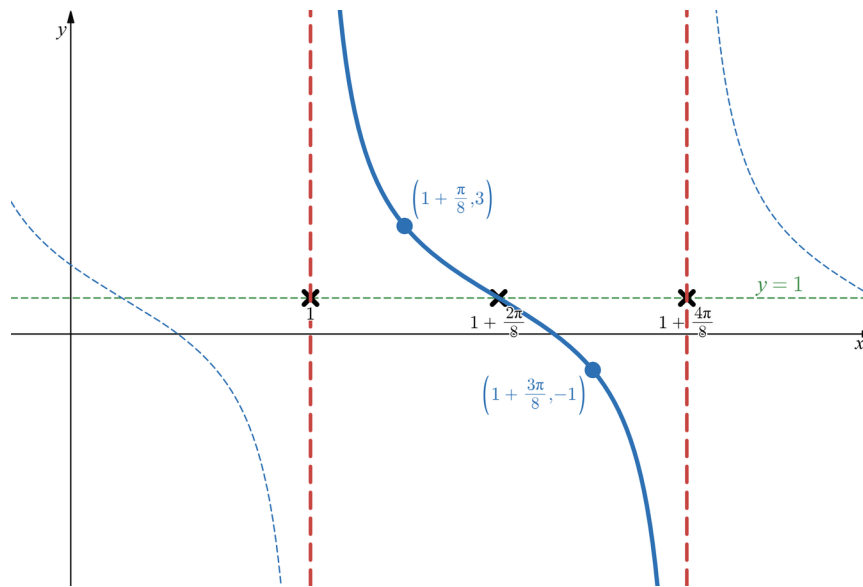


Figure 8.4.17

The domain of the function is all $x \neq 1 + \frac{\pi}{2}k$, where $k \in \mathbb{Z}$. The range is all real numbers, $\mathbb{R}$.

Checkpoint 8.4.1

Amplitude of the related sine function: 4
 Period: 2
 Step Size: $\frac{1}{2}$
 Phase Shift: $\frac{2}{3}$
 Vertical Shift: 3
 Domain: All $x \neq \frac{2}{3} + k$, where $k \in \mathbb{Z}$
 Range: $(-\infty, 1] \cup [7, \infty)$

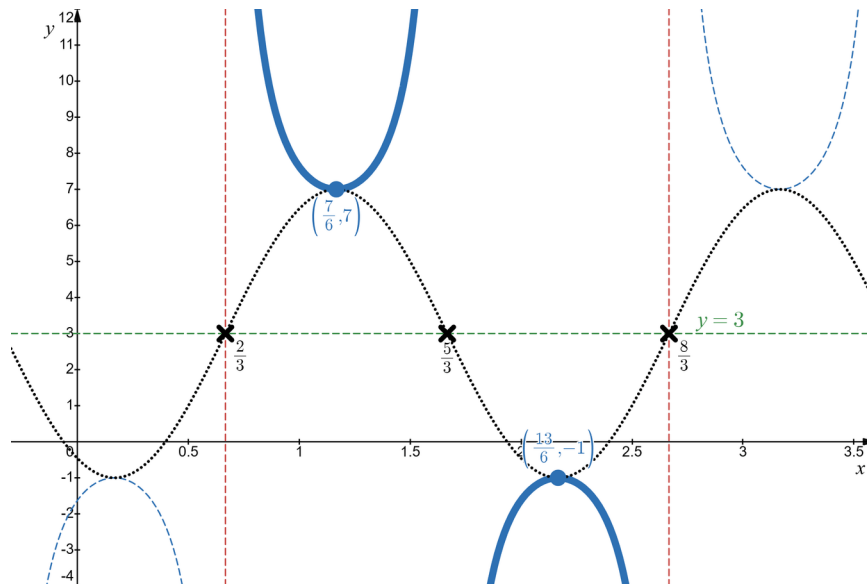


Figure 8.4.18

Example 8.4.2

- As with the sine and cosine functions, graphs of the cosecant and secant are so similar that you could use either as an equation for the given graph (they would just be shifts of one another). A good rule of thumb when building trigonometric equations from graphs is to read the graph from left to right. The piece of information (whether a point or an equation of an asymptote) you have been given will set the beginning of the cycle for the trigonometric function. Therefore, in this case, we will assume the cycle of the trigonometric function starts at $x = \frac{3}{4}$. With the cosecant and secant, it's beneficial to draw a sinusoid "helper" function. Looking at the given graph, we can see that the midline should be at $y = -\frac{1}{2}$. Drawing this in allows us to sketch our "helper" function.

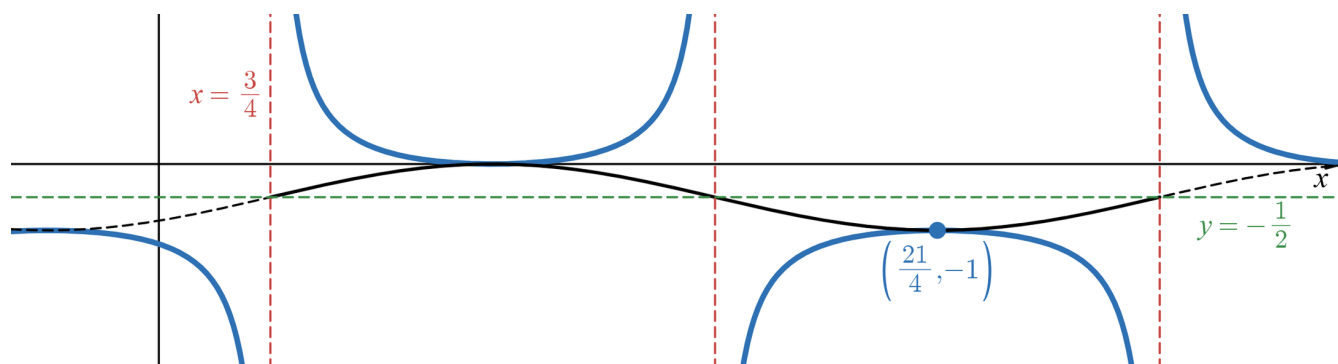


Figure 8.4.21

We now find the equation of the "helper" function. It is definitely a sine function starting at $x = \frac{3}{4}$, so the phase shift is

$$-\frac{C}{B} = \frac{3}{4}.$$

The midline is $y = -\frac{1}{2}$, so the vertical shift is

$$D = -\frac{1}{2}.$$

Additionally, the amplitude is the distance between the midline and the maximum or minimum values of the sinusoid. Since the maximum is $y = 0$ and the midline is $y = -\frac{1}{2}$, the amplitude is $\frac{1}{2}$. Moreover, since this sine function is not reflected,

$$A = \frac{1}{2}.$$

The only thing left to find is the period. This would give us an equation to find B , which in turn would allow us to find C . The cycle begins at $x = \frac{3}{4}$ and three-quarters of the way through the cycle, we are given the point $(\frac{21}{4}, -1)$. That is, the distance between $\frac{3}{4}$ and $\frac{21}{4}$ is three-quarters of the period.

$$\begin{aligned} \frac{3}{4} \text{ of the Period} &= \left| \frac{3}{4} - \frac{21}{4} \right| \\ &= \left| -\frac{18}{4} \right| \\ &= \frac{9}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Period} &= \left(\frac{9}{2} \right) \left(\frac{4}{3} \right) \\ &= 6 \end{aligned}$$

Since the period is $\frac{2\pi}{B}$,

$$\frac{2\pi}{B} = 6 \Rightarrow \frac{2\pi}{6} = B \Rightarrow \frac{\pi}{3} = B.$$

Earlier we stated that $-\frac{C}{B} = \frac{3}{4}$, so

$$-\frac{C}{\pi/3} = \frac{3}{4} \Rightarrow C = -\frac{\pi}{4}.$$

Thus, the "helper" function is

$$y = A \sin(Bx + C) + D = \frac{1}{2} \sin\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

Finally, the *actual* function is the reciprocal of the sine. That is, the function in the initial graph is

$$y = \frac{1}{2} \csc\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

- b. Again, you *could* write the equation for this in terms of the (reflected) tangent function; however, we will write our equation in terms of the cotangent. Sometimes, you have to "eyeball" the features of a graph. In this example, the midline *looks* to be $y = 4$, so we will say $D = 4$. The phase shift could be $-\frac{\pi}{6}$ or $+\frac{\pi}{6}$, but the extra point they gave us is in the interval $(-\frac{\pi}{6}, \frac{\pi}{6})$, so we will choose our phase shift to be $-\frac{\pi}{6}$. This means

$$-\frac{C}{B} = -\frac{\pi}{6}.$$

Furthermore, the period is easy to spot on this graph. From vertical asymptote to vertical asymptote, the distance is $\frac{2\pi}{6} = \frac{\pi}{3}$. Therefore, the period is $\frac{\pi}{3}$ and, as such,

$$\frac{\pi}{B} = \frac{\pi}{3} \Rightarrow B = 3.$$

With this, we can find C .

$$-\frac{C}{B} = -\frac{\pi}{6} \Rightarrow \frac{C}{3} = \frac{\pi}{6} \Rightarrow C = \frac{\pi}{2}.$$

We are close to the finish line. Our function is

$$y = A \cot\left(3x + \frac{\pi}{2}\right) + 4.$$

Now we use the given data point to help us find the value of A .

$$\begin{aligned}4.1 &= A \cot \left(3 \left(-\frac{\pi}{12} \right) + \frac{\pi}{2} \right) + 4 \\ \Rightarrow 0.1 &= A \cot \left(-\frac{\pi}{4} + \frac{\pi}{2} \right) \\ \Rightarrow 0.1 &= A \cot \left(\frac{\pi}{4} \right) \\ \Rightarrow 0.1 &= A (1) \\ \Rightarrow 0.1 &= A\end{aligned}$$

Thus, the function is

$$y = 0.1 \cot \left(3x + \frac{\pi}{2} \right) + 4.$$

Checkpoint 8.4.2

$$y = 2 \sec \left(3x + \frac{\pi}{5} \right) + 1$$

9: ANALYTIC TRIGONOMETRY

9.1: Basic Trigonometric Identities and Proof Techniques

9.2: Sum and Difference Identities

9.3: Double-Angle Identities

This page titled 9: Analytic Trigonometry is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Katherine Yoshiwara & Roy Simpson](#).

9.1: Basic Trigonometric Identities and Proof Techniques

A Review of Current Identities

- Reciprocal
- Ratio
- Pythagorean
- Symmetry
- Cofunction

Proving Identities

How To... Prove a Trigonometric Identity

26.21: Proving Trigonometric Identities

Example 9.1.1

Prove the identity.

- a. $\frac{\cot(\theta)}{\csc(\theta)} = \cos(\theta)$
- b. $\frac{\sin^2(\theta)-1}{\tan(\theta)\sin(\theta)-\tan(\theta)} = \frac{\sin(\theta)+1}{\tan(\theta)}$

This page titled [9.1: Basic Trigonometric Identities and Proof Techniques](#) is shared under a [CC BY-SA 3.0](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

9.2: Sum and Difference Identities

The Sum and Difference Identities for the Cosine

Example 9.2.1: Is $\text{trig}(A + B) = \text{trig}(A) + \text{trig}(B)$?

Let's find out (using special angles for practice)!

Caution: You Cannot "Split" a Trigonometric Function Across its Arguments

Because the values of the expressions in the Example 9.2.1 and Checkpoint 9.2.1 are different, it is *not* true that $\cos(\alpha + \beta)$ is equal to $\cos(\alpha) + \cos(\beta)$ for all angles α and β , or that $\sin(\alpha + \beta)$ is equal to $\sin(\alpha) + \sin(\beta)$.

This is a general property of almost all functions. That is, if f is a function, it is generally the case that

$$f(a \pm b) \neq f(a) \pm f(b).$$

Theorem: Sum and Difference Identities for the Cosine

For angles α and β ,

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta).$$

Proof available at the end of this section.

Example 9.2.2: We can now evaluate the trigonometric functions at new angles!

Find an exact value for $\cos(255^\circ)$.

Example 9.2.3: We are finally able to *prove* a Cofunction Identity!

Prove the cofunction identity for the cosine.

The Sum and Difference Identities for the Sine

Theorem: Sum and Difference Identities for the Sine

For angles α and β ,

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta).$$

Proof available at the end of this section.

Example 9.2.4: We can also evaluate trigonometric functions at new radian angles!

Evaluate $\sin\left(\frac{7\pi}{12}\right)$ exactly.

Example 9.2.5: A popular style of problem.

If α is a Quadrant III angle with $\sin(\alpha) = -\frac{12}{13}$, and β is a Quadrant IV angle with $\tan(\beta) = -5$, find the following

- $\sin(\alpha + \beta)$
- $\cos(\alpha - \beta)$

The Sum and Difference Identities for the Tangent

Theorem: Sum and Difference Identities for the Tangent

For angles α and β where the following formula is defined,

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha) \tan(\beta)}$$

Example 9.2.6: A required example.

Evaluate $\tan\left(\frac{\pi}{12}\right)$ exactly.

Revisiting Proofs of Identities

Example 9.2.7: We can do new proofs!

Prove.

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin(\alpha) \cos(\beta)$$

Revisiting Graphs of Trigonometric Functions

Example 9.2.8: We can graph combinations of trigonometric functions!

Graph the function.

$$g(x) = -2 \sin\left(\frac{\pi}{4}x + 1\right) \cos\left(\frac{\pi}{3}x - 1\right) + 2 \sin\left(\frac{\pi}{3}x - 1\right) \cos\left(\frac{\pi}{4}x + 1\right) - 3$$

Answers

Theorem: Sum and Difference Identities for the Cosine

Let α and β be angles in standard position and let α_0 and β_0 be the angles coterminal to α and β , respectively, each of which measure between 0 and 2π radians. Moreover, without loss of generality, assume that $\alpha_0 > \beta_0$ (if this assumption were not true, we would just relabel the variables accordingly).

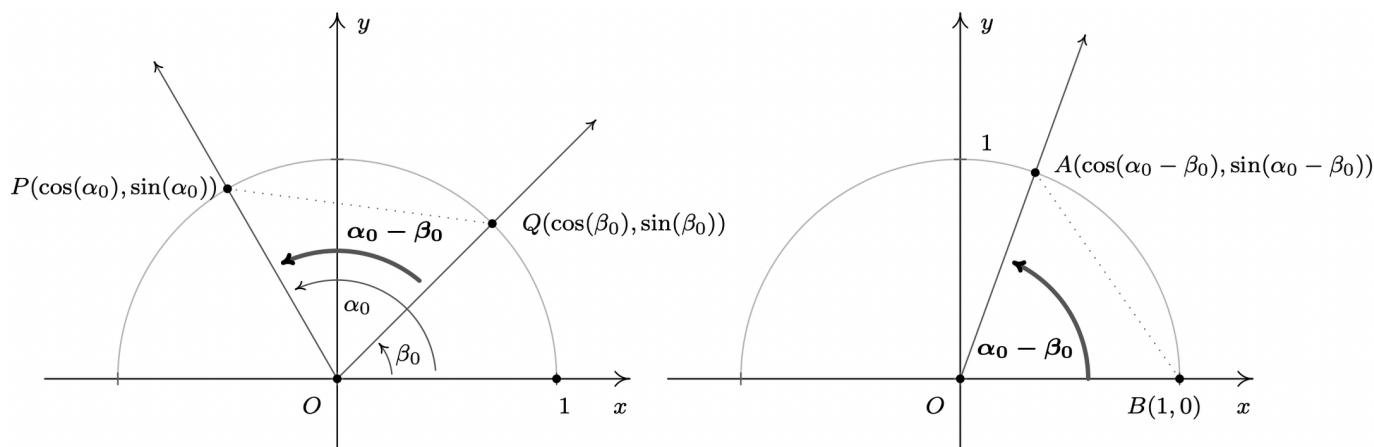
Since α_0 is coterminal with α , $\alpha = \alpha_0 + 2\pi m$ for some $m \in \mathbb{Z}$. Likewise, $\beta = \beta_0 + 2\pi n$ for some $n \in \mathbb{Z}$. Therefore,

$$\cos(\alpha - \beta) = \cos(\alpha_0 + 2\pi m - (\beta_0 + 2\pi n)) = \cos(\alpha_0 - \beta_0 + 2\pi(m - n)).$$

Since m and n are both integers, $m - n$ is also an integer. Moreover, cosine has a period of 2π means that $\cos(\alpha_0 - \beta_0 + 2\pi(m - n)) = \cos(\alpha_0 - \beta_0)$. Thus,

$$\cos(\alpha - \beta) = \cos(\alpha_0 - \beta_0).$$

All of this is to reduce our work to the angles α_0 and β_0 . We now draw and label the points P and Q on the unit circle which are on the terminal sides of α_0 and β_0 , respectively.



The image on the right side of this figure is a rotation of the image on the left clockwise by β_0 radians. Since the angles $\angle POQ$ and $\angle AOB$ are congruent, the distance between P and Q is equal to the distance between A and B . The Distance Formula yields

$$\text{Distance between } P \text{ and } Q = \text{Distance between } A \text{ and } B$$

$$\Rightarrow \sqrt{(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2} = \sqrt{(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2} \quad (\text{Distance Formula})$$

$$\Rightarrow (\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2 = (\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 \quad (\text{squaring both sides})$$

$$\Rightarrow \begin{aligned} \cos^2(\alpha_0) - 2\cos(\alpha_0)\cos(\beta_0) + \cos^2(\beta_0) &= \cos^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0) + 1 \\ + \sin^2(\alpha_0) - 2\sin(\alpha_0)\sin(\beta_0) + \sin^2(\beta_0) &= + \sin^2(\alpha_0 - \beta_0) \end{aligned} \quad (\text{distributing})$$

$$\Rightarrow \begin{aligned} \cos^2(\alpha_0) + \sin^2(\alpha_0) & \quad \cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) \\ -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) &= -2\cos(\alpha_0 - \beta_0) \\ + \cos^2(\beta_0) + \sin^2(\beta_0) & \quad +1 \end{aligned} \quad (\text{Commutative Property of Addition})$$

$$\Rightarrow \begin{aligned} 1 & \quad 1 \\ -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) &= -2\cos(\alpha_0 - \beta_0) \\ +1 & \quad +1 \end{aligned} \quad (\text{Pythagorean Identity})$$

$$\Rightarrow 2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) = 2 - 2\cos(\alpha_0 - \beta_0) \quad (\text{simplifying})$$

$$\Rightarrow -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) = -2\cos(\alpha_0 - \beta_0) \quad (\text{subtracting 2 from both sides})$$

$$\Rightarrow \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0) = \cos(\alpha_0 - \beta_0) \quad (\text{dividing both sides by } -2)$$

Combining this result with the fact that $\cos(\alpha - \beta) = \cos(\alpha_0 - \beta_0)$, we have the identity

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

To get the sum identity for cosine, we use this result along with the Symmetry Identities.

$$\cos(\alpha + \beta) = \cos(\alpha - (-\beta)) = \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Hence,

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Theorem: Sum and Difference Identities for the Sine

To prove the Sum and Difference Identities for the sine, we make use of the Cofunction Identities.

$$\begin{aligned} \sin(\alpha + \beta) &= \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right) && (\text{Cofunction Identities}) \\ &= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right) && (\text{Associative Property of Addition}) \\ &= \cos\left(\frac{\pi}{2} - \alpha\right)\cos(\beta) + \sin\left(\frac{\pi}{2} - \alpha\right)\sin(\beta) && (\text{Difference Identity for Cosine}) \\ &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) && (\text{Cofunction Identities}) \\ &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) && (\text{Cofunction Identities}) \end{aligned}$$

Thus, we get the identity.

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

A similar proof shows that

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta).$$

This page titled [9.2: Sum and Difference Identities](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

9.3: Double-Angle Identities

The Double Angle Identities

Theorem: Double-Angle Identities

$$\begin{aligned}\sin(2\theta) &= 2 \sin(\theta) \cos(\theta) \\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2 \cos^2(\theta) - 1 \\ &= 1 - 2 \sin^2(\theta) \\ \tan(2\theta) &= \frac{2 \tan(\theta)}{1 - \tan^2(\theta)}\end{aligned}$$

Proof available at the end of this section.

Caution: Don't Factor Out of Functions!

$$\text{trig}(2\theta) \neq 2\text{trig}(\theta)$$

Finding Exact Values of Trigonometric Functions Involving Double Angles

Example 9.3.1: Using double angles with triangles

Let's consider a right triangle with an interior unknown angle of θ , where we are given two sides. Find the values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

Example 9.3.2: A popular style of problem revisited.

If α is a Quadrant III angle with $\sin(\alpha) = -\frac{12}{13}$, and β is a Quadrant IV angle with $\tan(\beta) = -5$, find the following

- $\sin(2\alpha)$
- $\cos(2\alpha)$
- $\cos(2\beta)$

Revisiting Proofs of Identities

Example 9.3.3: We can now do even *more* proofs!

Prove each identity.

- $\cos^4(\theta) - \sin^4(\theta) = \cos(2\theta)$
- $\cos(2\theta) \cos \theta = \cos^3(\theta) - \cos(\theta) \sin^2(\theta)$

Revisiting Graphs of Trigonometric Functions

Example 9.3.4: We can graph more interesting functions!

Graph the function.

$$g(x) = \frac{6 \tan\left(\frac{\pi}{4}x\right)}{1 - \tan^2\left(\frac{\pi}{4}x\right)}$$

Answers

Theorem: Double-Angle Identities

Deriving the Double-Angle Identity for sine begins with the Sum Identity,

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta).$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned} \sin(\theta + \theta) &= \sin(\theta) \cos(\theta) + \cos(\theta) \sin(\theta) \\ \Rightarrow \sin(2\theta) &= 2 \sin(\theta) \cos(\theta) \end{aligned}$$

Deriving the Double-Angle Identity for cosine gives us three options. First, starting from the Sum Identity,

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta),$$

and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned} \cos(\theta + \theta) &= \cos(\theta) \cos(\theta) - \sin(\theta) \sin(\theta) \\ \Rightarrow \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \end{aligned}$$

Using the Pythagorean Identities, we can expand this Double-Angle Identity for cosine and get two more variations. The first variation is

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= (1 - \sin^2(\theta)) - \sin^2(\theta) \\ &= 1 - 2\sin^2(\theta) \end{aligned}$$

The second variation is

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - (1 - \cos^2(\theta)) \\ &= 2\cos^2(\theta) - 1 \end{aligned}$$

Similarly, to derive the Double-Angle Identity for the tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha) \tan(\beta)} \\ \Rightarrow \tan(\theta + \theta) &= \frac{\tan(\theta) + \tan(\theta)}{1 - \tan(\theta) \tan(\theta)} \\ \Rightarrow \tan(2\theta) &= \frac{2 \tan(\theta)}{1 - \tan^2(\theta)} \end{aligned}$$

10: TRIGONOMETRIC EQUATIONS

- 10.1: The Fundamental Inverse Trigonometric Functions
- 10.2: The Remaining Inverse Trigonometric Functions
- 10.3: Solving Trigonometric Equations - Algebraic Techniques
- 10.4: Solving Trigonometric Equations Using Identities

This page titled [10: Trigonometric Equations](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

- [Chapter 5: Equations and Identities](#) by Katherine Yoshiwara is licensed [GNU FDL](#). Original source: <https://yoshiwarabooks.org/trig/trig.html>.

10.1: The Fundamental Inverse Trigonometric Functions

Restricting the Domains to Force Invertibility

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

Definition: The Inverse Sine Function

The **inverse sine function**, written $\sin^{-1}(x)$ or $\arcsin(x)$, is defined for $-1 \leq x \leq 1$ by

$$\sin^{-1}(x) = \theta \quad \text{means} \quad \sin(\theta) = x \quad \text{and} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

Example 10.1.1

Simplify each expression without using a calculator.

- $\sin^{-1}\left(-\frac{\sqrt{1}}{2}\right)$
- $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$
- $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$
- $\sin^{-1}(\sin(3\pi))$
- $\sin(\sin^{-1}(0.8))$

Answer available at the end of this section.

Caution: $\sin^{-1}(\sin(\theta))$

$\sin^{-1}(\sin(\theta))$ is not necessarily θ .

Theorem: Properties of the Inverse Sine Function

Properties of $f(x) = \sin^{-1}(x)$:

- $\theta = \sin^{-1}(x)$ if and only if $\sin(\theta) = x$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- Domain: $[-1, 1]$
- Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\sin(\sin^{-1}(x)) = x$ for all $x \in [-1, 1]$
- $\sin^{-1}(\sin(x)) = x$ provided $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Definition: Inverse Cosine Function

The **inverse cosine function**, written $\cos^{-1}(x)$ or $\arccos(x)$, is defined for $-1 \leq x \leq 1$ by

$$\cos^{-1}(x) = \theta \quad \text{means} \quad \cos(\theta) = x \quad \text{and} \quad 0 \leq \theta \leq \pi.$$

Theorem: Properties of the Inverse Cosine Function

Properties of $f(x) = \cos^{-1}(x)$:

- $\theta = \cos^{-1}(x)$ if and only if $\cos(\theta) = x$ where $0 \leq \theta \leq \pi$
- Domain: $[-1, 1]$
- Range: $[0, \pi]$
- $\cos(\cos^{-1}(x)) = x$ for all $x \in [-1, 1]$
- $\cos^{-1}(\cos(x)) = x$ provided $x \in [0, \pi]$

Example 10.1.2

Simplify each expression without using a calculator.

- $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
- $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

Answer available at the end of this section.

Definition: The Inverse Tangent Function

The **inverse tangent function**, written $\tan^{-1}(x)$ or $\arctan(x)$, is defined for all real x by

$$\tan^{-1}(x) = \theta \quad \text{means} \quad \tan(\theta) = x \quad \text{and} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Theorem: Properties of the Inverse Tangent Function

Properties of $f(x) = \tan^{-1}(x)$:

- $\theta = \tan^{-1}(x)$ if and only if $\tan(\theta) = x$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- Domain: $(-\infty, \infty)$
- Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$
- As $x \rightarrow -\infty$, $\tan^{-1}(x) \rightarrow -\frac{\pi}{2}$, and as $x \rightarrow \infty$, $\tan^{-1}(x) \rightarrow \frac{\pi}{2}$
- $\tan(\tan^{-1}(x)) = x$ for all $x \in \mathbb{R}$
- $\tan^{-1}(\tan(x)) = x$ provided $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Example 10.1.3

Simplify each expression without using a calculator.

- $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$
- $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

Answer available at the end of this section.

Using a Calculator to Evaluate Inverse Trigonometric Functions

Example 10.1.4

Evaluate each of the following using a calculator. Your answers should be in radians.

- a. $\sin^{-1}(0.13)$
- b. $\cos^{-1}(-0.321)$
- c. $\tan^{-1}(-4)$

Answer available at the end of this section.

Alternate Notations

Example 10.1.5

Simplify each expression.

- a. $\tan\left(\arctan\left(-\frac{5}{2}\right)\right)$
- b. $\arctan\left(\tan\left(-\frac{5\pi}{6}\right)\right)$

Answer available at the end of this section.

Simplifying Expressions Involving Different Trigonometric Functions

Example 10.1.6

Evaluate $\cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$.

Example 10.1.7

Simplify $\sin\left(\tan^{-1}(z)\right)$, assuming that $z \geq 0$.

Simplifying Expressions Involving Identities

Example 10.1.8

Find the exact value of the expression.

$$\sin\left(2\cos^{-1}\left(-\frac{1}{3}\right)\right)$$

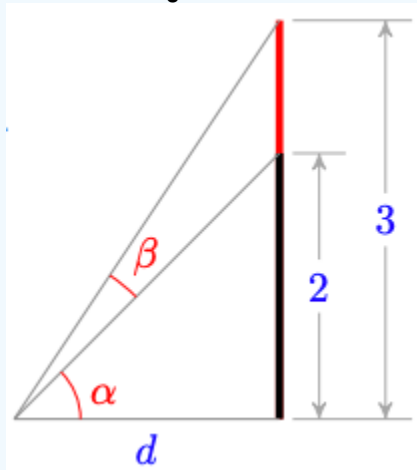
Modeling with Inverse Functions

Example 10.1.9

The tapestry from the previous example includes a 2-meter-tall unicorn with its feet at the bottom of the tapestry.

- Express α , the angle subtended vertically by the unicorn, as a function of d , your distance to the tapestry.
- Express β , the angle subtended by the portion of the tapestry above the unicorn, as a function of d . (See the figure below).

Figure 10.1.16



Answers

Example 10.1.1

Example 10.1.2

Example 10.1.3

Example 10.1.4

Example 10.1.5

10.2: The Remaining Inverse Trigonometric Functions

The Other Inverse Trigonometric Functions

Definition: Inverse Cotangent Function

The function $f(x) = \cot^{-1}(x)$ is defined as follows:

$$\cot^{-1}(x) = \theta \quad \text{if and only if} \quad \cot(\theta) = x \quad \text{and} \quad 0 < \theta < \pi.$$

Theorem: Properties of the Inverse Cotangent Function

Properties of $f(x) = \cot^{-1}(x)$

- $\theta = \cot^{-1}(x)$ if and only if $\cot(\theta) = x$ where $0 < \theta < \pi$
- Domain: $(-\infty, \infty)$
- Range: $(0, \pi)$
- As $x \rightarrow -\infty$, $\cot^{-1}(x) \rightarrow \pi^-$, and as $x \rightarrow \infty$, $\cot^{-1}(x) \rightarrow 0^+$
- $\cot(\cot^{-1}(x)) = x$ for all $x \in \mathbb{R}$
- $\cot^{-1}(\cot(x)) = x$ provided $0 < x < \pi$

Example 10.2.1

- Find the exact value of $\operatorname{arccot}(-\sqrt{3})$
- Find the exact value of the expression.

$$\sin\left(\cot^{-1}\left(-\frac{5}{2}\right)\right)$$

- Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

$$\cos(\operatorname{arccot}(2x))$$

Definition: The Inverse Secant Function

The function $f(x) = \sec^{-1}(x)$ is defined as follows:

$$\sec^{-1}(x) = \theta \quad \text{if and only if} \quad \sec(\theta) = x \quad \text{and} \quad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

Definition: The Inverse Cosecant Function

The function $f(x) = \csc^{-1}(x)$ is defined as follows:

$$\csc^{-1}(x) = \theta \quad \text{if and only if} \quad \csc(\theta) = x \quad \text{and} \quad \theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right].$$

Theorem: Properties of the Inverse Secant and Inverse Cosecant Functions

Properties of $f(x) = \operatorname{arcsec}(x)$

- $\sec^{-1}(x) = \theta$ if and only if $\sec(\theta) = x$ where $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

- Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
- Range: $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
- As $x \rightarrow -\infty$, $\operatorname{arcsec}(x) \rightarrow \frac{\pi}{2}^+$, and as $x \rightarrow \infty$, $\operatorname{arcsec}(x) \rightarrow \frac{\pi}{2}^-$
- $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
- $\operatorname{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \leq \pi$

Properties of $f(x) = \operatorname{arccsc}(x)$

- $\csc^{-1}(x) = \theta$ if and only if $\csc(\theta) = x$ where $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$
- Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
- Range: $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$
- As $x \rightarrow -\infty$, $\operatorname{arccsc}(x) \rightarrow 0^-$, and as $x \rightarrow \infty$, $\operatorname{arccsc}(x) \rightarrow 0^+$
- $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
- $\operatorname{arccsc}(\csc(x)) = x$ provided $-\frac{\pi}{2} \leq x < 0$ or $0 < x \leq \frac{\pi}{2}$
- Additionally, arccosecant is an odd function.

Example 10.2.2

- Find the exact values of the following.
 - $\csc^{-1}(2)$
 - $\operatorname{arcsec}(-2)$
 - $\operatorname{arccsc}(\csc(\frac{18\pi}{17}))$
 - $\cot(\operatorname{arcsec}(-3))$
- Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.
 - $\tan(\operatorname{arccsc}(x))$
 - $\cos(\operatorname{arcsec}(3x))$

Answer available at the end of this section.

Calculators and the Non-Fundamental Inverse Trigonometric Functions

Example 10.2.3

- Use a calculator to approximate the following values to four decimal places.
 - $\cot^{-1}(5)$
 - $\operatorname{arcsec}(15)$
 - $\csc^{-1}(-\frac{5}{2})$

Answer available at the end of this section.

Answers

Example 10.2.2

Example 10.2.3

10.3: Solving Trigonometric Equations - Algebraic Techniques

Solving Basic Trigonometric Equations

Example 10.3.1

Solve the equation $\sin(\theta) = -0.4$ for angles between 0° and 360° . Round your answers to the nearest tenth of a degree.

Example 10.3.2

Find all solutions to the given equation, then state the solutions occurring on the interval $[0, 2\pi]$. Round your answers to the nearest tenth.

- a. $\cos(x) = -0.9$
- b. $\tan(x) = -\frac{1}{\sqrt{3}}$
- c. $\csc(x) = -2.7$

Answer available at the end of this section.

Solving Trigonometric Equations Requiring Algebraic Techniques

Example 10.3.3

Solve the equation $5 \tan(\theta) + 7 = 2$ for $0 \leq \theta \leq 2\pi$.

Example 10.3.4

Solve the equation $3 \sec(C) - 6 = -2$, for $0^\circ \leq C \leq 360^\circ$. Round your solutions to three decimal places.

Example 10.3.5

Solve $2 \sec(\theta) - 5 = -4$ for $0^\circ \leq \theta \leq 360^\circ$.

Example 10.3.6

Solve the equation $2 \sin^2(\theta) = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

Example 10.3.7

Solve $9 \sin^2(\theta) - 3 \sin(\theta) = 2$ for $0^\circ \leq \theta \leq 360^\circ$. Round your answers to the nearest degree.

Solving Trigonometric Equations with Non-Standard Arguments

Example 10.3.8

Find exact values for all solutions of $\cos(2x) = -\frac{1}{\sqrt{2}}$ between 0 and 2π .

Example 10.3.9

Find all solutions of $\tan(2x) = \sqrt{3}$ between 0 and 2π .

Example 10.3.10

Solve $4 \cos(3x - 0.5) = -3.2$ for $0 \leq x \leq 2\pi$.

Answers

Example 10.3.2

10.4: Solving Trigonometric Equations Using Identities

Solving Equations Involving Different Trigonometric Functions Having the Same Arguments

Example 10.4.1

Solve each equation on the indicated interval.

- a. $1 - \sin^2 \theta + \cos \theta = 0$ for $0^\circ \leq \theta < 360^\circ$
- b. $2 \tan(\theta) \sin(\theta) + \tan(\theta) = 1 + 2 \sin(\theta)$ for $\theta \in [0, 2\pi)$

Answer available at the end of this section.

Solving Equations Involving Trigonometric Functions Having Different Angular Frequencies

Example 10.4.2

Solve the following equations and list the solutions in the interval $[0, 2\pi)$.

- a. $\sin(2x) = \sqrt{3} \sin(x)$
- b. $\cos(2x) = 3 \sin(x) - 2$
- c. $\sin(x) \cos\left(\frac{x}{2}\right) + \cos(x) \sin\left(\frac{x}{2}\right) = 1$

Answer available at the end of this section.

Solving Other Trigonometric Equations

Example 10.4.3

Solve the equation and list the solutions that lie in the interval $[0, 2\pi)$.

$$\cos(x) = \sin(x)$$

Answers

Example 10.4.1

Example 10.4.2

11: NON-RIGHT TRIANGLE TRIGONOMETRY

11.1: The Law of Sines

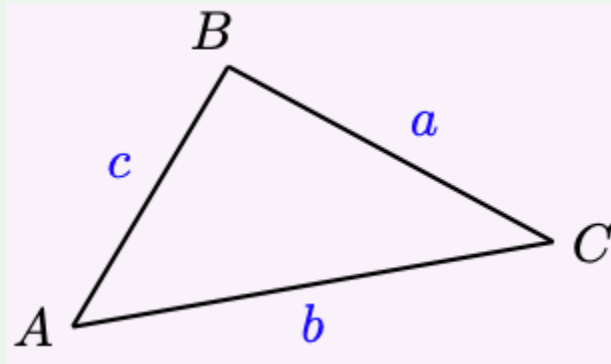
11.2: The Law of Cosines

This page titled [11: Non-Right Triangle Trigonometry](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by Katherine Yoshiwara & Roy Simpson.

11.1: The Law of Sines

The Law of Sines

Theorem: Law of Sines



If the angles of a triangle are A , B , and C , and the opposite sides are a , b , and c , respectively, then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},$$

or equivalently,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

Proof available at the end of this section.

Example 11.1.1: Our first non-right triangle problem!

Delbert and Francine are 40 feet apart on one side of a river. They make angle measurements to a pine tree on the opposite shore, as shown below. What is the distance from Francine to the pine tree?

Solving Triangles with the Law of Sines

Example 11.1.2

$A = 65^\circ$, $C = 42^\circ$, and $c = 16$. Solve the triangle.

Finding an Angle

Example 11.1.3

Sketch a triangle with $C = 93^\circ$, $a = 7$, and $c = 11$. Solve.

Example 11.1.4

Suppose that $C = 29.7^\circ$, $b = 8$, and $c = 5$. Solve.

Example 11.1.5

$B = 30^\circ$, $b = 7$, and $c = 15$. Solve.

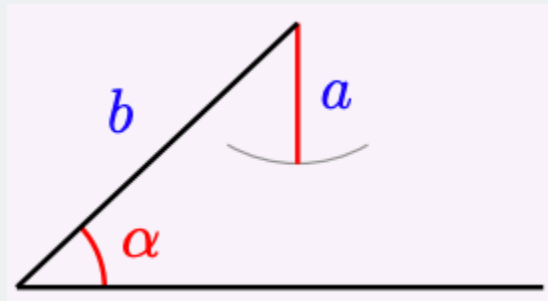
Visualizing the Law of Sines Cases

The Law of Sines - Ambiguous Case

In each of the following cases, $h = b \sin(\alpha)$ is the altitude of the triangle. Therefore, $b > h$.

The "No Solution" Case

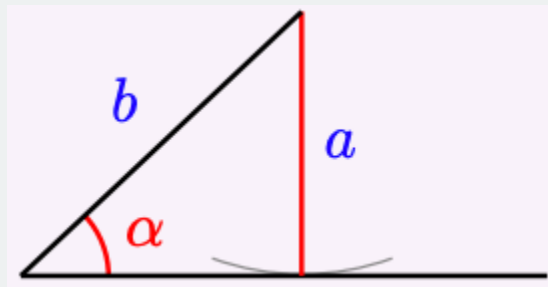
Condition: $a < h$



Reason: a is shorter than the altitude of the (non-existent) triangle. Therefore, it is impossible for side a to intersect the base of the "triangle."

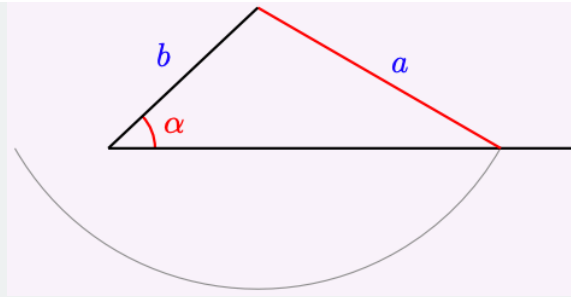
The "One Solution" Cases

Condition #1: $a = h$



Reason: a is exactly the right length to make a right triangle.

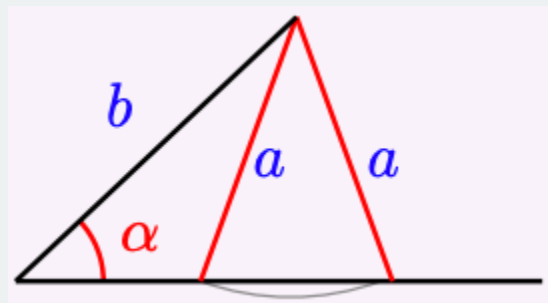
Condition #2: $a > b > h$



Reason: If $a > b$, then one triangle is created as side a swings downward to intersect the base; however, as that side continues to swing, it is too long to strike the base again. Hence, there can be only one triangle.

The "Two Solutions" Case

Condition: $b > a > h$



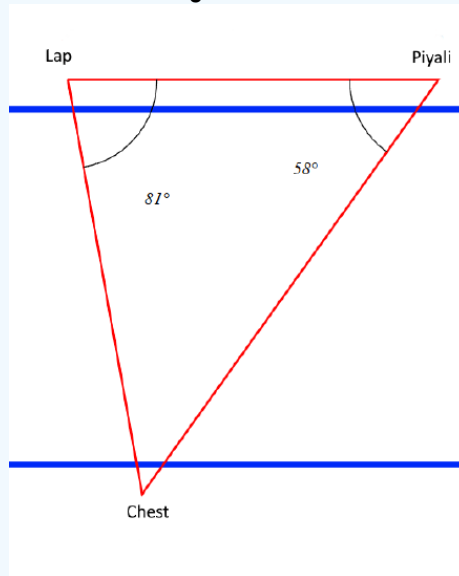
Reason: Since $a > h$, side a will strike the base as it swings downward, but before becoming vertical. Moreover, because $a < b$, side a will strike the base once more as it continues to swing clockwise.

Applications Requiring the Law of Sines

Example 11.1.6

Lap and Piyali are on the north side of a river that runs east to west. Lap is standing 225 meters due west of Piyali. They spot a treasure chest on the south shore of the river. For Lap, the angle between the chest and Piyali is 81° , and for Piyali, the angle between the chest and Lap is 58° (see Figure 11.1.14 below). How far is Lap from the chest? Round your answer to the nearest meter.

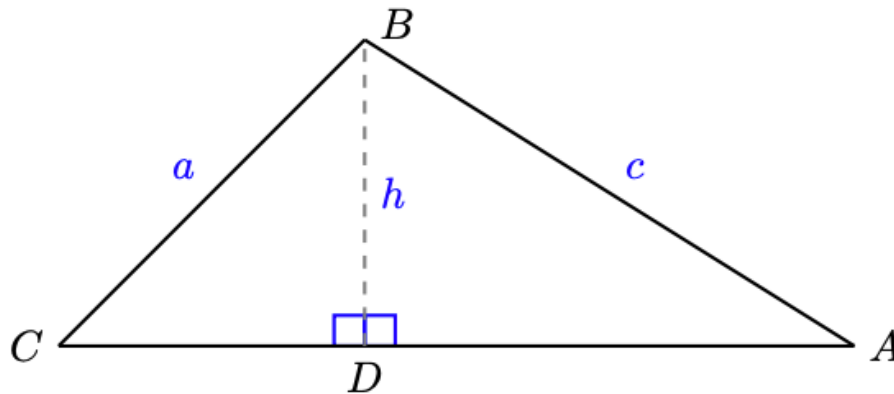
Figure 11.1.14



Answers

Theorem: Law of Sines

Consider the oblique triangle in the figure below.



By drawing in the altitude h of the triangle, we create two right triangles, $\triangle BCD$ and $\triangle ABD$. We can write expressions in terms of h for $\sin(A)$ and for $\sin(C)$. Looking at $\triangle BCD$, we see that

$$\frac{h}{a} = \sin(C).$$

Looking at $\triangle ABD$, we see that

$$\frac{h}{c} = \sin(A).$$

Now we solve each of these equations for h :

$$\frac{h}{a} = \sin(C) \implies h = a \sin(C)$$

$$\text{and } \frac{h}{c} = \sin(A) \implies h = c \sin(A)$$

Therefore,

$$\begin{aligned} a \sin(C) &= c \sin(A) \\ \implies \frac{\sin(C)}{c} &= \frac{\sin(A)}{a} \quad (\text{divide both sides by } ac) \end{aligned}$$

We have derived a relationship between the angles A and C and their opposite sides, a and c . If we know any three of these quantities, we can find the fourth. In a similar way, by drawing in the altitude from the vertex C , we can show that

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}.$$

Putting both results together, we have proved the Law of Sines.

11.2: The Law of Cosines

The Law of Cosines

Theorem: Law of Cosines

If the angles of a triangle are A , B , and C , and the opposite sides are respectively a , b , and c , then

$$a^2 = b^2 + c^2 - 2bc \cos(A)$$

$$b^2 = a^2 + c^2 - 2ac \cos(B)$$

$$c^2 = a^2 + b^2 - 2ab \cos(C)$$

Proof available at the end of this section.

Finding a Side

Example 11.2.1

In $\triangle ABC$, $a = 11$, $c = 23$, and $B = 87^\circ$. Find b , and round your answer to two decimal places.

Finding an Angle

Example 11.2.2

In $\triangle ABC$, $a = 5.3$, $b = 4.7$, and $c = 6.1$. Find angle B , and round your answer to two decimal places.

Applications Involving the Law of Cosines

Example 11.2.3

The hour hand on Kim's antique clock is 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.

Revisiting Navigation Problems

Example 11.2.4

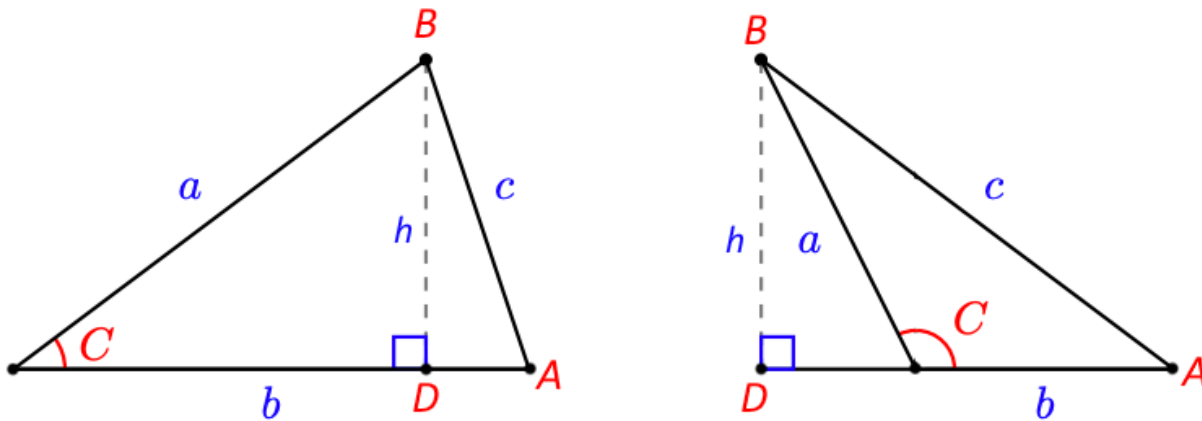
The sailing club leaves the marina on a heading of 15° and sails for 18 miles. They then change course, and after traveling for 12 miles on a heading of 35° , they experience engine trouble and radio for help. The marina sends a speed boat to rescue them. How far should the speed boat go, and on what heading?

Answers

Theorem: Law of Cosines

Start with general triangles as seen in the figures below.

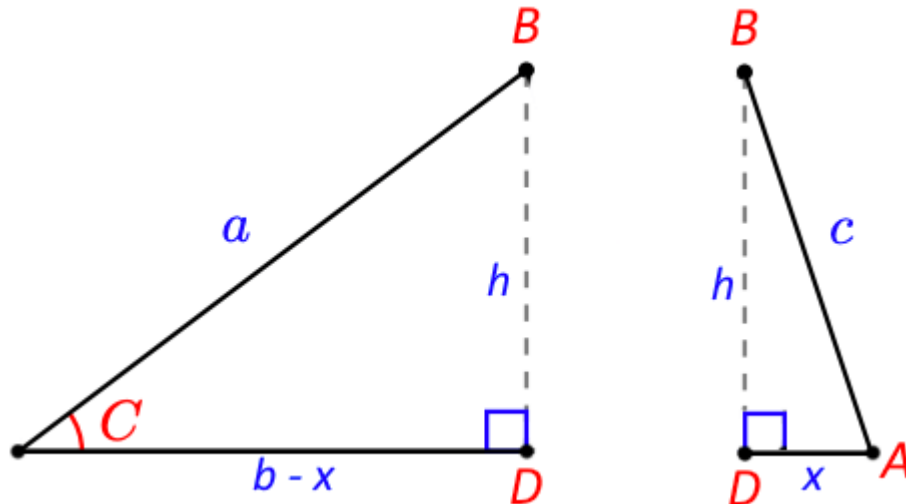
Two general triangles, one acute (left) and one obtuse (right), with altitudes labeled as h



We will perform the proof for the acute triangle (the proof associated with the obtuse triangle is similar).

The acute triangle is comprised of two right triangles - $\triangle ABD$ and $\triangle BCD$. Let x be the length of the line segment $\overline{AD}$. Then the length of the segment $\overline{CD}$ is $b - x$. We show these facts in the following figure, splitting the general right triangle into its two right triangle pieces.

"Splitting" the acute triangle into two right triangles



With the leftmost triangle, we have the following:

$$\begin{aligned} (b - x)^2 + h^2 &= a^2 && \text{(Pythagorean Theorem)} \\ \Rightarrow b^2 - 2bx + x^2 + h^2 &= a^2 && \text{(distributing)} \end{aligned}$$

With the rightmost triangle, we have

$$x^2 + h^2 = c^2 \quad \text{(Pythagorean Theorem)}$$

Therefore,

$$b^2 - 2bx + x^2 + h^2 = a^2 \quad \text{and} \quad x^2 + h^2 = c^2.$$

Replace $x^2 + h^2$ in the first equation with c^2 . Doing so yields the following:

$$b^2 - 2bx + c^2 = a^2 \Rightarrow b^2 + c^2 - 2bx = a^2.$$

From the rightmost triangle in the figure above, we see that $\cos(A) = \frac{x}{c}$, and so $c \cos(A) = x$. Hence, we have developed the following relationship between the sides of any triangle.

$$\begin{aligned} b^2 + c^2 - 2bx &= a^2 \\ \Rightarrow b^2 + c^2 - 2b(c \cos(A)) &= a^2 && \text{(substitution)} \\ \Rightarrow b^2 + c^2 - 2bc \cos(A) &= a^2 \end{aligned}$$

When A is a right angle, $\cos(A) = \cos(90^\circ) = 0$, so the equation reduces to the Pythagorean Theorem.

We can write similar equations involving the angles B or C . In all cases, the angle within the cosine is opposite the side playing the role of the "hypotenuse" (really, a **quasi-hypotenuse**).

This page titled [11.2: The Law of Cosines](#) is shared under a [CC BY-SA 1.2](#) license and was authored, remixed, and/or curated by [Roy Simpson](#).

12: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

12.1: Exponential Functions

12.2: Graphs of Exponential Functions

12.3: Logarithmic Functions

12.4: Graphs of Logarithmic Functions

12.5: Logarithmic Properties

12.6: Exponential and Logarithmic Equations

12.7: Exponential and Logarithmic Models

This page titled [12: Exponential and Logarithmic Functions](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by Roy Simpson via [source content](#) that was edited to the style and standards of the LibreTexts platform.

- [4: Exponential and Logarithmic Functions](#) by OpenStax is licensed [CC BY 4.0](#). Original source: <https://openstax.org/details/books/precalculus>.

12.1: Exponential Functions

Identifying Exponential Functions

Defining an Exponential Function

Example

Investigate $y = 2^x$ versus $y = 2x$.

Definition: Exponential Function

For any real number x , an **exponential function** is a function with the form

$$f(x) = ab^x,$$

where

- a is a non-zero real number called the **initial value**.
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The y -intercept is $(0, a)$, and the horizontal asymptote is $y = 0$.

Example

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1095.6^{-2x}$

Answer available at the end of this section.

Defining Exponential Growth

Definition: Exponential Growth

A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$$f(x) = ab^x,$$

where

- a is the initial or starting value of the function and
- b is the growth factor or growth multiplier per unit x .

Example

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031?

Finding Equations of Exponential Functions

Example

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t .

Example

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Applying the Compound-Interest Formula

Theorem: Compound Interest Formula

Compound interest can be calculated using the formula

$$A(t) = P \left(1 + \frac{r}{n} \right)^{nt},$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the **principal**, or more generally **present value**,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Example

An initial investment of \$100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Evaluating Functions with Base e

Definition: e

The letter e represents the irrational number

$$\left(1 + \frac{1}{n} \right)^n, \text{ as } n \text{ increases without bound.}$$

Example

Calculate $e^{-0.5}$. Round to five decimal places.

Investigating Continuous Growth

Definition: Continuous Growth/Decay Formula

For all real numbers t , and all positive numbers a and r , continuous growth or decay is represented by the formula

$$A(t) = ae^{rt},$$

where

- a is the initial value,
- r is the **continuous growth rate** per unit time,
- and t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt},$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- and t is the period or term of the investment.

Example

A person invests \$100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Example

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 30 days?

Answers

Example

12.2: Graphs of Exponential Functions

Graphing Exponential Functions

Characteristics of the Graph of the Parent Function $f(x) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x -intercept: none
- y -intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Example

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Graphing Transformations of Exponential Functions

Example

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Example

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Example

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Example

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y -axis. State its domain, range, and asymptote.

Example

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x -axis and then shifted down 2 units.

This page titled [12.2: Graphs of Exponential Functions](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by [Roy Simpson](#), Cosumnes River College via [source content](#) that was edited to the style and standards of the LibreTexts platform.

12.3: Logarithmic Functions

Converting from Logarithmic to Exponential Form

Definition: Logarithmic Function

A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x,$$

where,

- we read $\log_b(x)$ as, "the logarithm with base b of x ," or the "log base b of x ."
- the logarithm y is the exponent to which b must be raised to get x .

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Example

Write the following logarithmic equations in exponential form.

- $\log_{10}(1,000,000) = 6$
- $\log_5(25) = 2$

Answer available at the end of this section.

Converting from Exponential to Logarithmic Form

Example

Write the following exponential equations in logarithmic form.

- $3^2 = 9$
- $5^3 = 125$
- $2^{-1} = \frac{1}{2}$

Answer available at the end of this section.

Evaluating Logarithms

Example

Solve $y = \log_{121}(11)$ without using a calculator.

Example

Evaluate $y = \log_2 \left(\frac{1}{32} \right)$ without using a calculator.

Using Common Logarithms

Definition: Common Logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x.$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of x ."

The logarithm y is the exponent to which 10 must be raised to get x .

Example

Evaluate $y = \log(1,000,000)$.

Example

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Example

The amount of energy released from one earthquake was 8,500 times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Using Natural Logarithms

Definition: Natural Logarithm

A **natural logarithm** is a logarithm with base e . We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x.$$

We read $\ln(x)$ as, "the logarithm with base e of x " or "the natural logarithm of x ."

The logarithm y is the exponent to which e must be raised to get x .

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

Example

Evaluate $\ln(-500)$.

Answers

Example

Example

12.4: Graphs of Logarithmic Functions

Finding the Domain of a Logarithmic Function

Example

What is the domain of $f(x) = \log_5(x - 2) + 1$?

Example

What is the domain of $f(x) = \log(13 - 5x)$?

Example

State the domain of the function.

$$g(x) = -2 \ln \left(\frac{2x^3 - 3x^2 - 23x + 12}{2x^3 - 54} \right)$$

Graphing Logarithmic Functions

Characteristics of the Graph of the Parent Function, $f(x) = \log_b(x)$:

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x -intercept: $(1, 0)$ and key point $(b, 1)$
- y -intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

Example

Graph $f(x) = \log_2(x)$. State the domain, range, and asymptote.

Graphing Transformations of Logarithmic Functions

Example

Sketch $f(x) = -2 \log_3(-2x - 6) + 1$. State the domain, range, and asymptote.

Reflections of the Parent Function $y = \log_b(x)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the x -axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

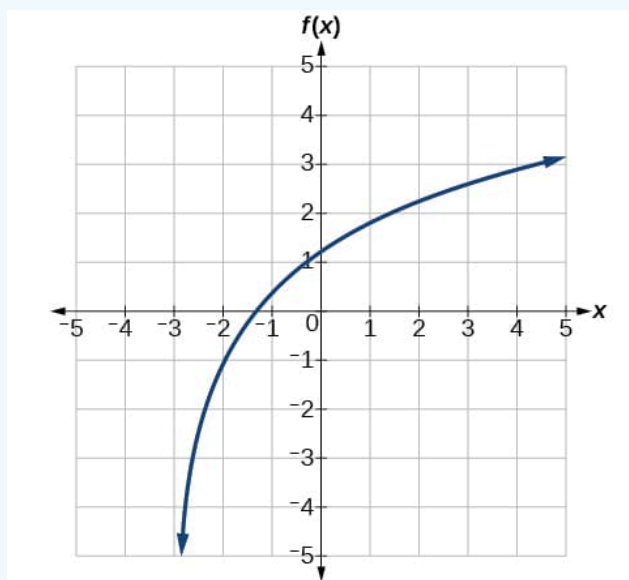
- reflects the parent function $y = \log_b(x)$ about the y -axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Example

Solve $5 \log(x + 2) = 4 - \log(x)$ using graphing technology. Round to the nearest thousandth.

Example

Give the equation of the natural logarithm graphed below.



Revisiting Graphs of Absolute Value Functions

Example

Graph

$$k(x) = \left| 3 \ln \left(-\frac{1}{3}x - 1 \right) + 3 \right|.$$

This page titled [12.4: Graphs of Logarithmic Functions](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by [Roy Simpson](#), [Cosumnes River College](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.

12.5: Logarithmic Properties

Using the Product Rule for Logarithms

Laws of Logarithms: Product Rule

The **Product Rule for Logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

for $b > 0$.

Example

Expand $\log_b(8k)$.

Using the Quotient Rule for Logarithms

Laws of Logarithms: Quotient Rule

The **Quotient Rule for Logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N.$$

Example

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Using the Power Rule for Logarithms

Laws of Logarithms: Power Rule

The **Power Rule for Logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n \log_b M$$

Example

Expand $\ln(x^2)$.

Example

Expand $\ln\left(\frac{1}{x^2}\right)$.

Example

Rewrite $2\log_3 4$, using the Power Rule for logs to a single logarithm with a leading coefficient of 1.

Expanding Logarithmic Expressions

Example

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Example

Expand $\ln(\sqrt[3]{x^2})$.

Example

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{x^2-9}\right)$.

Condensing Logarithmic Expressions

Example 12.5.9: Using the Product and Quotient Rules to Combine Logarithms

Condense $\log 3 - \log 4 + \log 5 - \log 6$.

Example

Rewrite $\log(5) + 0.5\log(x) - \log(7x - 1) + 3\log(x - 1)$ as a single logarithm.

Example

Condense $4(3\log(x) + \log(x + 5) - \log(2x + 3))$.

Example

In chemistry, $\text{pH} = -\log[H^+]$. If the concentration of hydrogen ions in a liquid is halved, what is the effect on pH?

Using the Change-of-Base Formula for Logarithms

Theorem: Change-of-Base Formula

For any positive real numbers M , b , and n , where $n \neq 1$, and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the Change-of-Base Formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \quad \text{and} \quad \log_b M = \frac{\log M}{\log b}.$$

Example

Change $\log_{0.5} 8$ to a quotient of natural logarithms.

Example

Evaluate $\log_5(100)$, using the Change-of-Base Formula.

12.6: Exponential and Logarithmic Equations

Using Like Bases to Solve Exponential Equations

Theorem: One-to-One Property of Exponents

For any algebraic expressions S and T , and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T.$$

Example

Solve $5^{2x} = 5^{3x+2}$.

Rewriting Equations So All Powers Have the Same Base

Example

Solve $5^{2x} = 25^{3x+2}$.

Example

Solve $5^x = \sqrt{5}$.

Example

Solve $2^x = -100$.

Solving Exponential Equations Using Logarithms

Example

Solve $2^x = 3^{x+1}$.

Equations Containing e

Example

Solve $3e^{0.5t} = 11$.

Example

Solve $3 + e^{2t} = 7e^{2t}$.

Extraneous Solutions**Example**

Solve $e^{2x} = e^x + 2$.

Using the Definition of a Logarithm to Solve Logarithmic Equations**Example**

Solve $6 + \ln x = 10$.

Example

Solve $2\ln(x + 1) = 10$.

Example

Solve $\log_7(1 - 2x) = 1 - \log_7(3 - x)$.

Example

Use graphing technology to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations**Example**

Solve $\ln(x^2) = \ln 1$.

12.7: Exponential and Logarithmic Models

Modeling Exponential Growth and Decay

Example

A population of bacteria doubles every three hours. If the culture started with 100 bacteria, graph the population as a function of time. Write this as a function with the natural base.

Half-Life

Table 12.7.1

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

Example

The half-life of plutonium-244 is 80,000,000 years. Find a function that gives the amount of plutonium-244 remaining as a function of time, measured in years.

Radiocarbon Dating

Example

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Calculating Doubling Time

Example

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Newton's Law of Cooling

Theorem: Newton's Law of Cooling

The temperature of an object, T , in a surrounding medium having temperature T_s will behave according to the formula

$$T(t) = Ae^{kt} + T_s,$$

where

- t is time,
- A is the difference between the initial temperature of the object and the surroundings, and
- k is a constant, the continuous rate of cooling of the object.

Example

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Using Logistic Growth Models

Definition: Logistic Growth

The **logistic growth model** is

$$f(t) = \frac{c}{1 + ae^{-bt}},$$

where

- $\frac{c}{1+a}$ is the initial value,
- c is the **carrying capacity**, or **limiting value**, and
- b is a constant determined by the rate of growth.

Example

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after fifteen days. Predict how many people in this community will have had this flu after a long period of time has passed.

Expressing an Exponential Model in Base e

Example

Change the function $y = 3(0.5)^x$ to one having e as the base.

Logarithmic Scales

Earthquakes

Definition: Richter Scale

For an earthquake with **seismic trace amplitude** A , a measurement from the baseline of the seismograph, the **Richter value**, or **magnitude** of the earthquake, is

$$M = \log \left(\frac{A}{A_0} \right),$$

where A_0 is a baseline measure for the seismograph.

Example

Lookup and compare.

INDEX

A

absolute value equation
2.6: Absolute Value Functions

Absolute Value Function
2.1: Functions and Function Notation
2.7: Functions and Function Notation (1)

absolute value inequality
2.6: Absolute Value Functions

acute angle
1.1: Angles and Basic Geometry

acute triangle
1.2: Triangles

Alternate Interior Angles
1.1: Angles and Basic Geometry

altitude
1.3: Similar Triangles

altitude (of triangle)
1.2: Triangles

Ambiguous case
11.1: The Law of Sines

amplitude
8.1: Base Graphs of the Fundamental Trigonometric Functions

angle
1.1: Angles and Basic Geometry

Angle of Declination
6.4: Applications Involving Right Triangles

angle of depression
6.4: Applications Involving Right Triangles

angle of displacement
7.2: Radian Measure, Arc Length, and Sector Area

angle of elevation
6.4: Applications Involving Right Triangles

angle of inclination
6.4: Applications Involving Right Triangles

arc
7.2: Radian Measure, Arc Length, and Sector Area

arc length
7.2: Radian Measure, Arc Length, and Sector Area

arccosecant
10.2: The Remaining Inverse Trigonometric Functions

arccosine
10.1: The Fundamental Inverse Trigonometric Functions

arccotangent
10.2: The Remaining Inverse Trigonometric Functions

arclength
7.2: Radian Measure, Arc Length, and Sector Area

arcsecant
10.2: The Remaining Inverse Trigonometric Functions

arcsine
10.1: The Fundamental Inverse Trigonometric Functions

arctangent

10.1: The Fundamental Inverse Trigonometric Functions

Area of a Circle

1.1: Angles and Basic Geometry

Arrow Notation

3.5: Rational Functions

average rate of change

2.4: Behavior of Graphs of Functions

axial angle

1.4: Circles and Angles in the Rectangular Coordinate System

axis of symmetry

3.1: Quadratic Functions

B

base angle

1.2: Triangles

bearing

6.4: Applications Involving Right Triangles

C

Cartesian coordinate plane

1.4: Circles and Angles in the Rectangular Coordinate System

Cartesian coordinate system

1.4: Circles and Angles in the Rectangular Coordinate System

Cartesian coordinates

1.4: Circles and Angles in the Rectangular Coordinate System

center of a circle

1.1: Angles and Basic Geometry

circle

1.1: Angles and Basic Geometry

circular functions

7.3: The Trigonometric Functions - Unit Circle Definition

circumference

1.1: Angles and Basic Geometry

coefficient

3.2: Power Functions and Polynomial Functions

Cofunction Identities

6.1: The Trigonometric Functions - Right Triangle Definition

cofunction theorem

6.1: The Trigonometric Functions - Right Triangle Definition

cofunctions

6.1: The Trigonometric Functions - Right Triangle Definition

common logarithms

12.3: Logarithmic Functions

complementary angle

1.1: Angles and Basic Geometry

composite function

2.3: The Arithmetic and Composition of Functions

compound interest

12.1: Exponential Functions

compound right triangle

6.4: Applications Involving Right Triangles

compressing

8.2: Non-Rigid Transformations

conditional equation

5.3: Introduction to Trigonometric Identities

congruent

1.1: Angles and Basic Geometry
1.3: Similar Triangles

constant function

1.4: Circles and Angles in the Rectangular Coordinate System
2.1: Functions and Function Notation
2.7: Functions and Function Notation (1)

continuous function

3.2: Power Functions and Polynomial Functions

continuous growth formula

12.1: Exponential Functions

coordinate definition of trigonometric functions

5.1: The Trigonometric Functions - Coordinate Definition

Coordinate Trigonometry

5.1: The Trigonometric Functions - Coordinate Definition

corresponding angles

1.1: Angles and Basic Geometry

cosine curve

8.1: Base Graphs of the Fundamental Trigonometric Functions

coterminal angles

1.4: Circles and Angles in the Rectangular Coordinate System

Cube root function

2.1: Functions and Function Notation
2.7: Functions and Function Notation (1)

cubic function

2.1: Functions and Function Notation
2.7: Functions and Function Notation (1)

D

DD

6.2: Technology and Right Triangle Trigonometry

Decimal Degrees

6.2: Technology and Right Triangle Trigonometry

decreasing

1.4: Circles and Angles in the Rectangular Coordinate System

definition of a logarithm

12.6: Exponential and Logarithmic Equations

Definition of exponential functions

12.1: Exponential Functions

Definition of exponential growth

12.1: Exponential Functions

degree

3.2: Power Functions and Polynomial Functions

degree measure

1.1: Angles and Basic Geometry

degree mode

[6.2: Technology and Right Triangle Trigonometry](#)

dependent variable

[2.1: Functions and Function Notation](#)
[2.7: Functions and Function Notation \(1\)](#)
[5.1: The Trigonometric Functions - Coordinate Definition](#)

diameter

[1.1: Angles and Basic Geometry](#)

difference of cubes

[5.2: Algebra with Trigonometric Functions](#)

difference of squares

[5.2: Algebra with Trigonometric Functions](#)

Distance Formula

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

Division Algorithm

[3.4: Dividing Polynomials](#)

DMS

[6.2: Technology and Right Triangle Trigonometry](#)

domain

[2.1: Functions and Function Notation](#)
[2.2: Domain and Range](#)
[2.7: Functions and Function Notation \(1\)](#)
[3.5: Rational Functions](#)
[4.1: Inverse Functions](#)

Double Angle Formulas

[9.3: Double-Angle Identities](#)

doubling time

[12.7: Exponential and Logarithmic Models](#)

E

end behavior

[3.2: Power Functions and Polynomial Functions](#)
[3.3: Graphs of Polynomial Functions](#)

equation of a circle

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

equilateral triangle

[1.2: Triangles](#)

equivalent expressions

[5.3: Introduction to Trigonometric Identities](#)

evaluate

[5.2: Algebra with Trigonometric Functions](#)

even function

[2.5: Transformations of Functions](#)

expression

[5.2: Algebra with Trigonometric Functions](#)

extraneous solution

[10.4: Solving Trigonometric Equations Using Identities](#)

extrema

[2.4: Behavior of Graphs of Functions](#)

F

factoring by grouping

[5.2: Algebra with Trigonometric Functions](#)

function

[2.1: Functions and Function Notation](#)
[2.7: Functions and Function Notation \(1\)](#)
[5.1: The Trigonometric Functions - Coordinate Definition](#)

fundamental trigonometric functions

[8.1: Base Graphs of the Fundamental Trigonometric Functions](#)

G

General form for the translation of the parent function

[12.2: Graphs of Exponential Functions](#)

General form for the translation of the parent logarithmic function

[12.4: Graphs of Logarithmic Functions](#)

general form of a polynomial function

[3.2: Power Functions and Polynomial Functions](#)

general form of a quadratic function

[3.1: Quadratic Functions](#)

global maximum

[3.3: Graphs of Polynomial Functions](#)

global minimum

[3.3: Graphs of Polynomial Functions](#)

H

heading

[6.4: Applications Involving Right Triangles](#)

horizontal asymptote

[3.5: Rational Functions](#)

horizontal line test

[2.1: Functions and Function Notation](#)
[2.7: Functions and Function Notation \(1\)](#)

horizontal reflection

[2.5: Transformations of Functions](#)

horizontal shifts

[12.2: Graphs of Exponential Functions](#)
[2.5: Transformations of Functions](#)

horizontal stretch

[12.2: Graphs of Exponential Functions](#)
[2.5: Transformations of Functions](#)

I

identities

[5.3: Introduction to Trigonometric Identities](#)

identity

[5.3: Introduction to Trigonometric Identities](#)

identity function

[2.1: Functions and Function Notation](#)
[2.7: Functions and Function Notation \(1\)](#)

increasing

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

independent variable

[2.1: Functions and Function Notation](#)
[2.7: Functions and Function Notation \(1\)](#)
[5.1: The Trigonometric Functions - Coordinate Definition](#)

initial point

[1.1: Angles and Basic Geometry](#)

initial side

[1.1: Angles and Basic Geometry](#)

initial value

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

input

[2.1: Functions and Function Notation](#)
[2.7: Functions and Function Notation \(1\)](#)
[5.1: The Trigonometric Functions - Coordinate Definition](#)

intermediate value theorem

[3.3: Graphs of Polynomial Functions](#)

interval notation

[2.2: Domain and Range](#)

invariant

[6.1: The Trigonometric Functions - Right Triangle Definition](#)

inverse cosine

[6.2: Technology and Right Triangle Trigonometry](#)

inverse function

[4.1: Inverse Functions](#)

Inverse of a polynomial function

[4.1: Inverse Functions](#)

inverse sine

[6.2: Technology and Right Triangle Trigonometry](#)

inverse tangent

[6.2: Technology and Right Triangle Trigonometry](#)

inverse trigonometric functions

[10.1: The Fundamental Inverse Trigonometric Functions](#)
[6.2: Technology and Right Triangle Trigonometry](#)

isosceles right triangle

[1.2: Triangles](#)

isosceles triangle

[1.2: Triangles](#)

K

key number

[8.2: Non-Rigid Transformations](#)

L

Law of Cosines

[11.2: The Law of Cosines](#)

Law of Sines

[11.1: The Law of Sines](#)

leading coefficient

[3.2: Power Functions and Polynomial Functions](#)

leading term

[3.2: Power Functions and Polynomial Functions](#)

local maximum

[2.4: Behavior of Graphs of Functions](#)

local minimum

[2.4: Behavior of Graphs of Functions](#)

logarithmic function

[12.3: Logarithmic Functions](#)

logistic growth

[12.7: Exponential and Logarithmic Models](#)

long division

[3.4: Dividing Polynomials](#)

M

maxima

[2.4: Behavior of Graphs of Functions](#)

midline

[8.3: Rigid Transformations](#)

minima

[2.4: Behavior of Graphs of Functions](#)

minutes

[6.2: Technology and Right Triangle Trigonometry](#)

multiplicity

[3.3: Graphs of Polynomial Functions](#)

N

natural logarithm

[12.3: Logarithmic Functions](#)

natural period

[8.2: Non-Rigid Transformations](#)

negative angle

[1.1: Angles and Basic Geometry](#)

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

Newton's Law of Cooling

[12.7: Exponential and Logarithmic Models](#)

nonrigid transformations

[8.2: Non-Rigid Transformations](#)

O

obtuse angle

[1.1: Angles and Basic Geometry](#)

obtuse triangle

[1.2: Triangles](#)

odd function

[2.5: Transformations of Functions](#)

oriented angle

[1.1: Angles and Basic Geometry](#)

output

[2.1: Functions and Function Notation](#)

[2.7: Functions and Function Notation \(1\)](#)

[5.1: The Trigonometric Functions - Coordinate Definition](#)

P

parabola

[3.1: Quadratic Functions](#)

parallax

[11.1: The Law of Sines](#)

parallelogram

[1.3: Similar Triangles](#)

period

[8.1: Base Graphs of the Fundamental Trigonometric Functions](#)

[8.2: Non-Rigid Transformations](#)

periodic function

[8.1: Base Graphs of the Fundamental Trigonometric Functions](#)

periodic motion

[8.1: Base Graphs of the Fundamental Trigonometric Functions](#)

phase shift

[8.3: Rigid Transformations](#)

polynomial division

[3.4: Dividing Polynomials](#)

polynomial function

[3: Polynomial and Rational Functions](#)

[3.2: Power Functions and Polynomial Functions](#)

population growth

[12.1: Exponential Functions](#)

positive angle

[1.1: Angles and Basic Geometry](#)

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

power functions

[3.2: Power Functions and Polynomial Functions](#)

power rule

[12.5: Logarithmic Properties](#)

product rule for logarithms

[12.5: Logarithmic Properties](#)

proportional

[1.3: Similar Triangles](#)

Pythagorean identities

[5.3: Introduction to Trigonometric Identities](#)

Pythagorean Identity

[5.3: Introduction to Trigonometric Identities](#)

Pythagorean Theorem

[1.2: Triangles](#)

Pythagorean Triangle

[6.1: The Trigonometric Functions - Right Triangle Definition](#)

Pythagorean triples

[6.1: The Trigonometric Functions - Right Triangle Definition](#)

Q

Quadrant

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

quadrantal angle

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

quadrants

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

quadratic formula

[3.1: Quadratic Functions](#)

quadratic function

[2.1: Functions and Function Notation](#)

[2.7: Functions and Function Notation \(1\)](#)

quotient rule

[12.5: Logarithmic Properties](#)

R

radian

[7.2: Radian Measure, Arc Length, and Sector Area](#)

radian measure

[7.2: Radian Measure, Arc Length, and Sector Area](#)

radian mode

[6.2: Technology and Right Triangle Trigonometry](#)

radians

[7.2: Radian Measure, Arc Length, and Sector Area](#)

radical function

[4.2: Radical Functions](#)

radicals

[4.2: Radical Functions](#)

radius

[1.1: Angles and Basic Geometry](#)

range

[2.1: Functions and Function Notation](#)

[2.2: Domain and Range](#)

[2.7: Functions and Function Notation \(1\)](#)

rate of change

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

Ratio Identities

[5.3: Introduction to Trigonometric Identities](#)

rational function

[3: Polynomial and Rational Functions](#)

[3.5: Rational Functions](#)

ray

[1.1: Angles and Basic Geometry](#)

Reciprocal function

[2.1: Functions and Function Notation](#)

[2.7: Functions and Function Notation \(1\)](#)

Reciprocal Identities

[5.3: Introduction to Trigonometric Identities](#)

Reciprocal squared function

[2.1: Functions and Function Notation](#)

[2.7: Functions and Function Notation \(1\)](#)

rectangular coordinate system

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

reference angle

[7.1: Reference Angles](#)

reference triangle

[7.1: Reference Angles](#)

reflecting

[8.2: Non-Rigid Transformations](#)

reflection

[12.2: Graphs of Exponential Functions](#)

relation

[2.1: Functions and Function Notation](#)

[2.7: Functions and Function Notation \(1\)](#)

right angle

[1.1: Angles and Basic Geometry](#)

right triangle

[1.2: Triangles](#)

Right Triangle Trigonometry

[6.1: The Trigonometric Functions - Right Triangle Definition](#)

rigid transformations

[8.2: Non-Rigid Transformations](#)

S

scalene triangle

[1.2: Triangles](#)

scaling

[8.2: Non-Rigid Transformations](#)

seconds

[6.2: Technology and Right Triangle Trigonometry](#)

shift

[8.2: Non-Rigid Transformations](#)

similar triangles

[1.3: Similar Triangles](#)

sine curve

[8.1: Base Graphs of the Fundamental Trigonometric Functions](#)

sinusoidal

[8.1: Base Graphs of the Fundamental Trigonometric Functions](#)

slope

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

smooth curve

[3.2: Power Functions and Polynomial Functions](#)

solve a triangle

[6.3: Solving Right Triangles](#)

special angles

[5.1: The Trigonometric Functions - Coordinate Definition](#)

special triangle

[1.2: Triangles](#)

square root function

[2.1: Functions and Function Notation](#)

[2.7: Functions and Function Notation \(1\)](#)

standard form of a quadratic function

[3.1: Quadratic Functions](#)

standard position

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

step size

[8.2: Non-Rigid Transformations](#)

straight angle

[1.1: Angles and Basic Geometry](#)

stretching

[8.2: Non-Rigid Transformations](#)

subtend

[7.2: Radian Measure, Arc Length, and Sector Area](#)

Sum and Difference Formulas

[9.2: Sum and Difference Identities](#)

sum and difference formulas for cosine

[9.2: Sum and Difference Identities](#)

Sum and Difference Identities

[9.2: Sum and Difference Identities](#)

Sum and Difference Identities for Sine

[9.2: Sum and Difference Identities](#)

sum of cubes

[5.2: Algebra with Trigonometric Functions](#)

supplementary angle

[1.1: Angles and Basic Geometry](#)

surface area

[4.1: Inverse Functions](#)

Symmetric Identities

[8.1: Base Graphs of the Fundamental Trigonometric Functions](#)

synthetic division

[3.4: Dividing Polynomials](#)

T**term of a polynomial function**

[3.2: Power Functions and Polynomial Functions](#)

terminal side

[1.1: Angles and Basic Geometry](#)

transformation

[8.2: Non-Rigid Transformations](#)

transversal

[1.1: Angles and Basic Geometry](#)

triangle

[1.2: Triangles](#)

triangle inequality

[1.2: Triangles](#)

Triangle Sum

[1.2: Triangles](#)

trigonometric equation

[10.3: Solving Trigonometric Equations - Algebraic Techniques](#)

trigonometric expression

[5.4: Introduction to Proofs in Trigonometry](#)

Trigonometric Functions

[5.1: The Trigonometric Functions - Coordinate Definition](#)

turning point

[3.2: Power Functions and Polynomial Functions](#)

U**undefined**

[1.4: Circles and Angles in the Rectangular Coordinate System](#)

unit circle

[1.4: Circles and Angles in the Rectangular Coordinate System](#)
[7.3: The Trigonometric Functions - Unit Circle Definition](#)

unitless measure

[7.2: Radian Measure, Arc Length, and Sector Area](#)

V**vertex**

[1.1: Angles and Basic Geometry](#)
[3.1: Quadratic Functions](#)

vertex angle

[1.2: Triangles](#)

vertex form of a quadratic function

[3.1: Quadratic Functions](#)

vertex of a triangle

[1.2: Triangles](#)

vertical angles

[1.1: Angles and Basic Geometry](#)

vertical asymptote

[3.5: Rational Functions](#)

vertical compression

[2.5: Transformations of Functions](#)

vertical line test

[2.1: Functions and Function Notation](#)
[2.7: Functions and Function Notation \(1\)](#)

vertical reflection

[2.5: Transformations of Functions](#)

vertical shift

[8.3: Rigid Transformations](#)

vertical shifts

[12.2: Graphs of Exponential Functions](#)
[2.5: Transformations of Functions](#)

vertical stretch

[12.2: Graphs of Exponential Functions](#)
[2.5: Transformations of Functions](#)

vertices

[1.2: Triangles](#)

Z**zeros of a function**

[3.1: Quadratic Functions](#)

GLOSSARY

S

Sample Word 1 | Sample Definition 1

DETAILED LICENSING

Overview

Title: Math 384: Lecture Notes

Webpages: 76

All licenses found:

- [CC BY-SA 1.2](#): 46.1% (35 pages)
- [CC BY 4.0](#): 23.7% (18 pages)
- [CC BY-SA 4.0](#): 14.5% (11 pages)
- [Undeclared](#): 14.5% (11 pages)
- [CC BY-SA 3.0](#): 1.3% (1 page)

By Page

- Math 384: Lecture Notes - [CC BY-SA 4.0](#)
 - Front Matter - [Undeclared](#)
 - [TitlePage](#) - [Undeclared](#)
 - [InfoPage](#) - [Undeclared](#)
 - [Table of Contents](#) - [Undeclared](#)
 - [Licensing](#) - [Undeclared](#)
 - 1: Triangles and Circles - [CC BY-SA 1.2](#)
 - [1.1: Angles and Basic Geometry](#) - [CC BY-SA 1.2](#)
 - [1.2: Triangles](#) - [CC BY-SA 1.2](#)
 - [1.3: Similar Triangles](#) - [CC BY-SA 1.2](#)
 - [1.4: Circles and Angles in the Rectangular Coordinate System](#) - [CC BY-SA 1.2](#)
 - 2: Functions - [CC BY-SA 4.0](#)
 - [2.1: Functions and Function Notation](#) - [CC BY-SA 4.0](#)
 - [2.2: Domain and Range](#) - [CC BY-SA 4.0](#)
 - [2.3: The Arithmetic and Composition of Functions](#) - [CC BY-SA 4.0](#)
 - [2.4: Behavior of Graphs of Functions](#) - [CC BY-SA 4.0](#)
 - [2.5: Transformations of Functions](#) - [CC BY-SA 4.0](#)
 - [2.6: Absolute Value Functions](#) - [CC BY-SA 4.0](#)
 - [2.7: Functions and Function Notation \(1\)](#) - [CC BY-SA 4.0](#)
 - [2.7.1: Resources and Key Concepts](#) - [CC BY-SA 4.0](#)
 - [2.7.2: Homework](#) - [CC BY-SA 4.0](#)
 - 3: Polynomial and Rational Functions - [CC BY 4.0](#)
 - [3.1: Quadratic Functions](#) - [CC BY 4.0](#)
 - [3.2: Power Functions and Polynomial Functions](#) - [CC BY 4.0](#)
 - [3.3: Graphs of Polynomial Functions](#) - [CC BY 4.0](#)
 - [3.4: Dividing Polynomials](#) - [CC BY 4.0](#)
 - [3.5: Rational Functions](#) - [CC BY 4.0](#)
 - 4: Inverse and Radical Functions - [Undeclared](#)
 - [4.1: Inverse Functions](#) - [CC BY 4.0](#)
 - [4.2: Radical Functions](#) - [CC BY 4.0](#)
 - [4.3: Radical Equations](#) - [CC BY 4.0](#)
 - [4.4: Rational Exponent Expressions Encountered in Calculus](#) - [CC BY 4.0](#)
 - 5: Coordinate Trigonometry - [CC BY-SA 1.2](#)
 - [5.1: The Trigonometric Functions - Coordinate Definition](#) - [CC BY-SA 1.2](#)
 - [5.2: Algebra with Trigonometric Functions](#) - [CC BY-SA 1.2](#)
 - [5.3: Introduction to Trigonometric Identities](#) - [CC BY-SA 1.2](#)
 - [5.4: Introduction to Proofs in Trigonometry](#) - [CC BY-SA 1.2](#)
 - 6: Right Triangle Trigonometry - [CC BY-SA 1.2](#)
 - [6.1: The Trigonometric Functions - Right Triangle Definition](#) - [CC BY-SA 1.2](#)
 - [6.2: Technology and Right Triangle Trigonometry](#) - [CC BY-SA 1.2](#)
 - [6.3: Solving Right Triangles](#) - [CC BY-SA 1.2](#)
 - [6.4: Applications Involving Right Triangles](#) - [CC BY-SA 1.2](#)
 - 7: Radian Measure and the Circular Functions - [CC BY-SA 1.2](#)
 - [7.1: Reference Angles](#) - [CC BY-SA 1.2](#)
 - [7.2: Radian Measure, Arc Length, and Sector Area](#) - [CC BY-SA 1.2](#)
 - [7.3: The Trigonometric Functions - Unit Circle Definition](#) - [CC BY-SA 1.2](#)
 - 8: Graphs of the Trigonometric Functions - [CC BY-SA 1.2](#)
 - [8.1: Base Graphs of the Fundamental Trigonometric Functions](#) - [CC BY-SA 1.2](#)
 - [8.2: Non-Rigid Transformations](#) - [CC BY-SA 1.2](#)
 - [8.3: Rigid Transformations](#) - [CC BY-SA 1.2](#)
 - [8.4: Graphs of the Remaining Trigonometric Functions](#) - [CC BY-SA 1.2](#)

- 9: Analytic Trigonometry - CC BY-SA 1.2
 - 9.1: Basic Trigonometric Identities and Proof Techniques - CC BY-SA 3.0
 - 9.2: Sum and Difference Identities - CC BY-SA 1.2
 - 9.3: Double-Angle Identities - CC BY-SA 1.2
- 10: Trigonometric Equations - CC BY-SA 1.2
 - 10.1: The Fundamental Inverse Trigonometric Functions - CC BY-SA 1.2
 - 10.2: The Remaining Inverse Trigonometric Functions - CC BY-SA 1.2
 - 10.3: Solving Trigonometric Equations - Algebraic Techniques - CC BY-SA 1.2
 - 10.4: Solving Trigonometric Equations Using Identities - CC BY-SA 1.2
- 11: Non-Right Triangle Trigonometry - CC BY-SA 1.2
 - 11.1: The Law of Sines - CC BY-SA 1.2
 - 11.2: The Law of Cosines - CC BY-SA 1.2
- 12: Exponential and Logarithmic Functions - CC BY 4.0
 - 12.1: Exponential Functions - CC BY 4.0
 - 12.2: Graphs of Exponential Functions - CC BY 4.0
 - 12.3: Logarithmic Functions - CC BY 4.0
 - 12.4: Graphs of Logarithmic Functions - CC BY 4.0
 - 12.5: Logarithmic Properties - CC BY 4.0
 - 12.6: Exponential and Logarithmic Equations - CC BY 4.0
 - 12.7: Exponential and Logarithmic Models - CC BY 4.0
- Back Matter - *Undeclared*
 - Index - *Undeclared*
 - Glossary - *Undeclared*
 - Detailed Licensing - *Undeclared*
 - Detailed Licensing - *Undeclared*

DETAILED LICENSING

Overview

Title: Math 384: Lecture Notes

Webpages: 76

All licenses found:

- [CC BY-SA 1.2](#): 46.1% (35 pages)
- [CC BY 4.0](#): 23.7% (18 pages)
- [CC BY-SA 4.0](#): 14.5% (11 pages)
- [Undeclared](#): 14.5% (11 pages)
- [CC BY-SA 3.0](#): 1.3% (1 page)

By Page

- Math 384: Lecture Notes - [CC BY-SA 4.0](#)
 - Front Matter - [Undeclared](#)
 - [TitlePage](#) - [Undeclared](#)
 - [InfoPage](#) - [Undeclared](#)
 - [Table of Contents](#) - [Undeclared](#)
 - [Licensing](#) - [Undeclared](#)
 - 1: Triangles and Circles - [CC BY-SA 1.2](#)
 - [1.1: Angles and Basic Geometry](#) - [CC BY-SA 1.2](#)
 - [1.2: Triangles](#) - [CC BY-SA 1.2](#)
 - [1.3: Similar Triangles](#) - [CC BY-SA 1.2](#)
 - [1.4: Circles and Angles in the Rectangular Coordinate System](#) - [CC BY-SA 1.2](#)
 - 2: Functions - [CC BY-SA 4.0](#)
 - [2.1: Functions and Function Notation](#) - [CC BY-SA 4.0](#)
 - [2.2: Domain and Range](#) - [CC BY-SA 4.0](#)
 - [2.3: The Arithmetic and Composition of Functions](#) - [CC BY-SA 4.0](#)
 - [2.4: Behavior of Graphs of Functions](#) - [CC BY-SA 4.0](#)
 - [2.5: Transformations of Functions](#) - [CC BY-SA 4.0](#)
 - [2.6: Absolute Value Functions](#) - [CC BY-SA 4.0](#)
 - [2.7: Functions and Function Notation \(1\)](#) - [CC BY-SA 4.0](#)
 - [2.7.1: Resources and Key Concepts](#) - [CC BY-SA 4.0](#)
 - [2.7.2: Homework](#) - [CC BY-SA 4.0](#)
 - 3: Polynomial and Rational Functions - [CC BY 4.0](#)
 - [3.1: Quadratic Functions](#) - [CC BY 4.0](#)
 - [3.2: Power Functions and Polynomial Functions](#) - [CC BY 4.0](#)
 - [3.3: Graphs of Polynomial Functions](#) - [CC BY 4.0](#)
 - [3.4: Dividing Polynomials](#) - [CC BY 4.0](#)
 - [3.5: Rational Functions](#) - [CC BY 4.0](#)
 - 4: Inverse and Radical Functions - [Undeclared](#)
 - [4.1: Inverse Functions](#) - [CC BY 4.0](#)
 - [4.2: Radical Functions](#) - [CC BY 4.0](#)
 - [4.3: Radical Equations](#) - [CC BY 4.0](#)
 - [4.4: Rational Exponent Expressions Encountered in Calculus](#) - [CC BY 4.0](#)
 - 5: Coordinate Trigonometry - [CC BY-SA 1.2](#)
 - [5.1: The Trigonometric Functions - Coordinate Definition](#) - [CC BY-SA 1.2](#)
 - [5.2: Algebra with Trigonometric Functions](#) - [CC BY-SA 1.2](#)
 - [5.3: Introduction to Trigonometric Identities](#) - [CC BY-SA 1.2](#)
 - [5.4: Introduction to Proofs in Trigonometry](#) - [CC BY-SA 1.2](#)
 - 6: Right Triangle Trigonometry - [CC BY-SA 1.2](#)
 - [6.1: The Trigonometric Functions - Right Triangle Definition](#) - [CC BY-SA 1.2](#)
 - [6.2: Technology and Right Triangle Trigonometry](#) - [CC BY-SA 1.2](#)
 - [6.3: Solving Right Triangles](#) - [CC BY-SA 1.2](#)
 - [6.4: Applications Involving Right Triangles](#) - [CC BY-SA 1.2](#)
 - 7: Radian Measure and the Circular Functions - [CC BY-SA 1.2](#)
 - [7.1: Reference Angles](#) - [CC BY-SA 1.2](#)
 - [7.2: Radian Measure, Arc Length, and Sector Area](#) - [CC BY-SA 1.2](#)
 - [7.3: The Trigonometric Functions - Unit Circle Definition](#) - [CC BY-SA 1.2](#)
 - 8: Graphs of the Trigonometric Functions - [CC BY-SA 1.2](#)
 - [8.1: Base Graphs of the Fundamental Trigonometric Functions](#) - [CC BY-SA 1.2](#)
 - [8.2: Non-Rigid Transformations](#) - [CC BY-SA 1.2](#)
 - [8.3: Rigid Transformations](#) - [CC BY-SA 1.2](#)
 - [8.4: Graphs of the Remaining Trigonometric Functions](#) - [CC BY-SA 1.2](#)

- 9: Analytic Trigonometry - CC BY-SA 1.2
 - 9.1: Basic Trigonometric Identities and Proof Techniques - CC BY-SA 3.0
 - 9.2: Sum and Difference Identities - CC BY-SA 1.2
 - 9.3: Double-Angle Identities - CC BY-SA 1.2
- 10: Trigonometric Equations - CC BY-SA 1.2
 - 10.1: The Fundamental Inverse Trigonometric Functions - CC BY-SA 1.2
 - 10.2: The Remaining Inverse Trigonometric Functions - CC BY-SA 1.2
 - 10.3: Solving Trigonometric Equations - Algebraic Techniques - CC BY-SA 1.2
 - 10.4: Solving Trigonometric Equations Using Identities - CC BY-SA 1.2
- 11: Non-Right Triangle Trigonometry - CC BY-SA 1.2
 - 11.1: The Law of Sines - CC BY-SA 1.2
 - 11.2: The Law of Cosines - CC BY-SA 1.2
- 12: Exponential and Logarithmic Functions - CC BY 4.0
 - 12.1: Exponential Functions - CC BY 4.0
 - 12.2: Graphs of Exponential Functions - CC BY 4.0
 - 12.3: Logarithmic Functions - CC BY 4.0
 - 12.4: Graphs of Logarithmic Functions - CC BY 4.0
 - 12.5: Logarithmic Properties - CC BY 4.0
 - 12.6: Exponential and Logarithmic Equations - CC BY 4.0
 - 12.7: Exponential and Logarithmic Models - CC BY 4.0
- Back Matter - *Undeclared*
 - Index - *Undeclared*
 - Glossary - *Undeclared*
 - Detailed Licensing - *Undeclared*
 - Detailed Licensing - *Undeclared*