MATH 113 - FINITE MATH FOR SOCIETY



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This text was compiled on 08/16/2025

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Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Introduction

1.1: About this Book

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1.1: About this Book

Many homes have a box or drawer with a few tools for simple household fixes — a hammer, a screwdriver, maybe a pair of pliers or a tape measure, etc. This book is like that box, but the tools are mathematical ones for simple tasks of everyday life. More specifically, these tools help you understand human or social aspects of the world: societal structures and their interactions, business and economic relationships, and so forth.

This book concentrates on two specific kinds of mathematical tool: linear functions and probability. Linear functions, and related linear equations and inequalities, are simple mathematical expressions that are nonetheless able to describe a wide variety of real-world situations. Because they are so simple, they are easy to work with and fairly intuitive. Probability is the mathematics of uncertainty, something with which the real world is filled. Using probability, we can reason about and identify likely or unlikely consequences of real-world uncertainty and make our own choices accordingly. Linear functions and probability aren't the only components of a complete mathematical toolkit, but they are good ones to start with. Students who study no mathematics beyond this book will still be well served, and those who wish to go further will have a good foundation for their studies.

The remainder of this book consists of two parts. Chapters 2 through 7 deal with linear functions, equations, and inequalities, while chapters 8 and 9 address probability.

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CHAPTER OVERVIEW

2: Linear Equations

Learning Objectives

By the end of this chapter, the student should be able to

- Solve linear equations (simple, dual-side variables, infinitely many solutions or no solution, rational coefficients)
- Solve literal equations with several variables for one of the variables
- Solve absolute value equations
- Apply solving techniques to application problems including perimeter, mark-up/discount, triangles, and interest problems

2.1: Linear Equations

2.1.1: Linear Equations (Exercises)

2.2: Absolute Value Equations

2.2.1: Absolute Value Equations (Exercises)

2.3: Literal Equations

2.3.1: Literal Equations (Exercises)

2.4: Word Problems

2.4.1: Word Problems (Exercises)

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2.1: Linear Equations

Definition: Linear Equation

A ***linear equation*** is an equation where the highest exponent on the given variables is one. A *linear equation in one variable* is an equation with one variable with exponent one, e.g.,

$$ax+b=c,$$

where a is called the *coefficient* of x, and b and c are *constant coefficients*.

Solving linear equations is an important and fundamental skill in algebra. In algebra, we are often presented with a problem where the answer is unknown. This is usually represented by a variable. There will be times when there are multiple unknowns and we use algebra techniques to solve for the variable.

Note

Be sure to understand the difference between an expression and an equation.

Recall. We simplify expressions and solve equations. Hence, expressions do not contain an equal sign, $=$, and we only distribute and combine like terms. Equations contain an equal sign, $=$, and we solve for the variable in addition to distributing and combining like terms.

The result to an expression is an expression and the result to an equation is a number (with the exception of when the result is infinitely many solutions or no solution).

Verifying Solutions

Example 2.1.1

Is $x=-5$ a solution to the equation $4x+16=-4$?

Solution

We substitute $x=-5$ into the equation and determine if the left side is equal to the right side.

$$\begin{aligned} 4(-5)+16 &\stackrel{?}{=} -4 && \text{Multiply } 4(-5) \\ -20+16 &\stackrel{?}{=} -4 && \text{Add } -20+16 \\ -4 &= -4 && \checkmark \text{ True} \end{aligned}$$

Hence, $x=-5$ is a solution to the equation $4x+16=-4$.

Example 2.1.2

Is $x=3$ a solution to the equation $4x+16=-4$?

Solution

We substitute $x=3$ into the equation and determine if the left side is equal to the right side.

$$\begin{aligned} 4(3)+16 &\stackrel{?}{=} -4 && \text{Multiply } 4(3) \\ 12+16 &\stackrel{?}{=} -4 && \text{Add } 12+16 \\ 28 &\neq -4 && X \text{ False} \end{aligned}$$

Hence, $x=3$ is not a solution to the equation $4x+16=-4$.

Note

Example 2.1.2 reiterates that the solution to each equation is unique. Once we see that one solution makes the equation true, then we look no further. The exception is when we have an identity, which we discuss later in this chapter.

One-Step Equations

Depending on the complexity of the problem, this "guess and check" method is not very efficient. Thus, we take a more algebraic approach for solving equations. Here we will focus on equations that only require one step to solve. While these equations often seem very fundamental, it is important to master the pattern for solving these problems so we can solve more complex problems.

Addition Property of Equations

Given an equation, $a = b$, the following is an equivalent statement:

$$a + c = b + c$$

i.e., we can add any number to one side of the equation as long as we add the same number to the other side. Often, we use this property to isolate the variable.

Example 2.1.3

Solve for x: $x + 7 = -5$

Solution

$$\begin{aligned} x + 7 &= -5 && \text{Isolate } y \text{ by adding the opposite of } 7 \\ x + 7 + (-7) &= -5 + (-7) && \text{Simplify} \\ x &= -12 && \text{Solution} \end{aligned}$$

Recall, it is encouraged for the student to check the obtained answer by verifying the solution:

$$\begin{aligned} (-12) + 7 &\stackrel{?}{=} -5 \\ -5 &= -5 \quad \checkmark \text{ True} \end{aligned}$$

Example 2.1.4

Solve for y: $4 + y = 8$

Solution

$$\begin{aligned} 4 + y &= 8 && \text{Isolate } y \text{ by adding the opposite of } 4 \\ 4 + y + (-4) &= 8 + (-4) && \text{Simplify} \\ y &= 4 && \text{Solution} \end{aligned}$$

Example 2.1.5

Solve for y: $7 = y + 9$

Solution

$$\begin{aligned} 7 &= y + 9 && \text{Isolate } y \text{ by adding the opposite of } 9 \\ 7 + (-9) &= y + 9 + (-9) && \text{Simplify} \\ -2 &= y && \text{Rewrite with } y \text{ on the left side} \\ y &= -2 && \text{Solution} \end{aligned}$$

Example 2.1.6

Solve for x: $5 = 8 + x$

Solution

$$\begin{aligned} 5 &= 8 + x && \text{Isolate } x \text{ by adding the opposite of } 8 \\ 5 + (-8) &= 8 + x + (-8) && \text{Simplify} \\ -3 &= x && \text{Rewrite with } x \text{ on the left side} \\ x &= -3 && \text{Solution} \end{aligned}$$

Note

Be sure to check your answer by verifying the solution! It only take a few seconds and will save you time and future common errors in the long run.

Example 2.1.7

Solve for y: $y - 5 = 4$

Solution

$$\begin{aligned} y - 5 &= 4 && \text{Isolate } y \text{ by adding the opposite of } -5 \\ y - 5 + (5) &= 4 + (5) && \text{Simplify} \\ y &= 9 && \text{Solution} \end{aligned}$$

Example 2.1.8

Solve for y: $-10 = y - 7$

Solution

$$\begin{aligned} -10 &= y - 7 && \text{Isolate } y \text{ by adding the opposite of } -7 \\ -10 + (7) &= y - 7 + (7) && \text{Simplify} \\ y &= -3 && \text{Solution} \end{aligned}$$

Multiplication Property of Equations

Given an equation, $a = b$, the following is an equivalent statement:

$$ac = bc,$$

where $c \neq 0$, i.e., we can multiply both sides of the equation by a nonzero number. Often, we use this property to isolate the variable when there is a coefficient in front of the variable.

Example 2.1.9

Solve for y: $4y = 20$

Solution

$$\begin{aligned} 4y &= 20 && \text{Isolate } y \text{ by multiplying by the reciprocal of } 4 \\ \tfrac{1}{4} \cdot 4y &= 20 \cdot \tfrac{1}{4} && \text{Simplify} \\ y &= 5 && \text{Solution} \end{aligned}$$

Note

In Example 2.1.9, we multiplied by the ***reciprocal***. The product of a number and its reciprocal is one, i.e., if c is a number, then its reciprocal is $\frac{1}{c}$ because

$$c \cdot \frac{1}{c} = 1$$

Example 2.1.10

Solve for y: $-5y = 30$

Solution

$$\begin{aligned} -5y &= 30 && \text{Isolate } y \text{ by multiplying by the reciprocal of } -5 \\ -\tfrac{1}{5} \cdot -5y &= 30 \cdot -\tfrac{1}{5} && \text{Simplify} \\ y &= -6 && \text{Solution} \end{aligned}$$

Example 2.1.11

Solve for x: $-42 = -7x$

Solution

$$\begin{aligned} -42 &= -7x && \text{Isolate } x \text{ by multiplying by the reciprocal of } -7 \\ -\tfrac{1}{7} \cdot -42 &= -7x \cdot -\tfrac{1}{7} && \text{Simplify} \\ 6 &= x && \text{Rewrite with } x \text{ on the left side} \\ x &= 6 && \text{Solution} \end{aligned}$$

Example 2.1.12

Solve for x: $\frac{x}{-7} = -2$

Solution

$$\begin{aligned} \tfrac{x}{-7} &= -2 && \text{Isolate } x \text{ by multiplying by the reciprocal of } -\tfrac{1}{7} \\ -7 \cdot \tfrac{x}{-7} &= -2 \cdot -7 && \text{Simplify} \\ x &= 14 && \text{Solution} \end{aligned}$$

Example 2.1.13

Solve for x: $\frac{x}{5} = -3$

Solution

$$\begin{aligned} \tfrac{x}{5} &= -3 && \text{Isolate } x \text{ by multiplying by the reciprocal of } \tfrac{1}{5} \\ 5 \cdot \tfrac{x}{5} &= -3 \cdot 5 && \text{Simplify} \\ x &= -15 && \text{Solution} \end{aligned}$$

The processes described above are fundamental for solving equations. Once these processes are mastered, we are given problems that involve multiple steps. These problems may seem more complex, but the processes and patterns are the same.

Note

The study of algebra was originally called "*Cossic Art*" from the Latin, meaning the study of "things," which we now call variables.

Two-Step Equations

After mastering the techniques for solving one-step equations, we are ready to consider two-step equations. As we solve two-step equations, the important thing to remember is that everything works in reverse. When working with one-step equations, we learned that in order to clear a "plus five" in the equation, we would subtract five (or add its opposite) from both sides. We learned that to clear "divided by seven," we multiply by seven (or multiply by its reciprocal) on both sides. When solving for our variable, we use order of operations in reverse. This means we will apply the addition property first, then the multiplication property second (then exponents, and, finally, any parentheses or grouping symbols, but that's another lesson).

Example 2.1.14

Solve for y: $4y - 20 = -8$

Solution

We have one extra term on the same side as the variable term, $4y$. We will first isolate the variable term, then isolate the variable by applying the properties of equations:

$$\begin{aligned} 4y - 20 &= -8 && \text{Add the opposite of } -20 \text{ to each side} \\ 4y - 20 + 20 &= -8 + 20 && \text{Simplify} \\ 4y &= 12 && \text{Multiply by the reciprocal of } 4 \\ \tfrac{1}{4} \cdot 4y &= 12 \cdot \tfrac{1}{4} && \text{Simplify} \\ y &= 3 && \text{Solution} \end{aligned}$$

Let's verify the solution:

$$\begin{aligned} 4(3) - 20 &\stackrel{?}{=} -8 \\ 12 - 20 &\stackrel{?}{=} -8 \\ -8 &= -8 \quad \checkmark \text{ True} \end{aligned}$$

Thus, $y = 3$ is the solution to the equation.

The same process is used to solve any two-step equations. Add or subtract first, then multiply or divide. Recall, the method never changes, just problems do.

Example 2.1.15

Solve for w: $5w + 7 = 7$

Solution

$$\begin{aligned} 5w + 7 &= 7 && \text{Add the opposite of } 7 \text{ to each side} \\ 5w + 7 + (-7) &= 7 + (-7) && \text{Simplify} \\ 5w &= 0 && \text{Multiply by the reciprocal of } 5 \\ \tfrac{1}{5} \cdot 5w &= 0 \cdot \tfrac{1}{5} && \text{Simplify} \\ w &= 0 && \text{Solution} \end{aligned}$$

Note

Notice in Example 2.1.15 the seven canceled out completely. Many students get stuck at this point. Let's not forget that the product of a number and zero is zero. Hence, zero is the solution to the equation.

Note

A common error students make with two-step equations is with negative signs. Remember, the sign always stays with the coefficient.

Example 2.1.16

Solve for t: $4-2t=10$

Solution

$$\begin{aligned} 4-2t &= 10 && \text{Add the opposite of 4 to each side} \\ 4-2t+(-4) &= 10+(-4) && \text{Simplify} \\ -2t &= 6 && \text{Multiply by the reciprocal of } -2 \\ -\tfrac{1}{2}\cdot -2t &= 6\cdot -\tfrac{1}{2} && \text{Simplify} \\ t &= -3 && \text{Solution} \end{aligned}$$

Example 2.1.17

Solve for n: $8-n=2$

Solution

$$\begin{aligned} 8-n &= 2 && \text{Add the opposite of 8 to each side} \\ 8-n+(-8) &= 2+(-8) && \text{Simplify} \\ -n &= -6 && \text{Rewrite } -n \text{ as } -1n \\ -1n &= -6 && \text{Multiply by the reciprocal of } -1 \\ -1\cdot -1n &= -6\cdot -1 && \text{Simplify} \\ n &= 6 && \text{Solution} \end{aligned}$$

Example 2.1.18

Solve for y: $-3y+7=-8$

Solution

$$\begin{aligned} -3y+7 &= -8 && \text{Add the opposite of 7 to each side} \\ -3y+7+(-7) &= -8+(-7) && \text{Simplify} \\ -3y &= -15 && \text{Multiply by the reciprocal of } -3 \\ -\tfrac{1}{3}\cdot -3y &= -15\cdot -\tfrac{1}{3} && \text{Simplify} \\ y &= 5 && \text{Solution} \end{aligned}$$

Example 2.1.19

Solve for x: $-3=\frac{x}{5}-4$

Solution

Notice the variable term is on the right side of the equation. However, we still follow the method as usual:

$$
\begin{aligned}
-3 &= \tfrac{x}{5} - 4 && \text{Add the opposite of } -4 \text{ to each side} \\
-3 + 4 &= \tfrac{x}{5} - 4 + 4 && \text{Simplify} \\
1 &= \tfrac{x}{5} && \text{Multiply by the reciprocal of } \tfrac{1}{5} \\
5 \cdot 1 &= \tfrac{x}{5} \cdot 5 && \text{Simplify} \\
5 &= x && \text{Rewrite the solution with the variable on the left side} \\
x &= 5 && \text{Solution}
\end{aligned}
$$

As problems in algebra become more complex, the method remains the same. In fact, as we solve problems like those in the next example, each one of them will have several steps to solve, but the last two steps are a two-step equation. This is a critical reason to master two-step equations.

Example 2.1.20

$$3x^2 + 4 = y + 6 \qquad \frac{1}{x-8} + \frac{1}{x} = \frac{1}{3} \qquad \sqrt{5y-5} + 1 = y \qquad \log_5(2y-4) = 1$$

Note

Persian mathematician *Omar Khayyam* would solve algebraic problems geometrically by intersecting graphs rather than solving them algebraically.

General Equations

Often as we are solving linear equations, we will need to do some preliminary work so that it is more familiar for us to solve. This section will focus on manipulating an equation in such a way that we can use our method for solving two-step equations to obtain the solution.

One such issue that needs to be addressed is parentheses. Sometimes parentheses can get in the way of solving an equation. As you might expect, we can rewrite an equation without parentheses by using the distributive property.

Distributive Property of Equations

The following is an equivalent statement:

$$c(a+b) = ca + cb$$

where $c \neq 0$, i.e., we can multiply each term in the parentheses by a nonzero number.

Example 2.1.21

Solve for y: $4(2y-6) = 16$

Solution

Notice the first step is distributing, then it is solved like any other two-step equation.

$$
\begin{aligned}
4(2y-6) &= 16 && \text{Distribute} \\
8y - 24 &= 16 && \text{Add the opposite of } -24 \text{ to each side} \\
8y - 24 + 24 &= 16 + 24 && \text{Simplify} \\
8y &= 40 && \text{Multiply by the reciprocal of } 8 \\
\tfrac{1}{8} \cdot 8y &= 40 \cdot \tfrac{1}{8} && \text{Simplify} \\
y &= 5 && \text{Solution}
\end{aligned}
$$

Example 2.1.22

Solve for p: $3(2p-4)+9=15$

Solution

Notice the first step is distributing, then it is solved like any other two-step equation.

$$\begin{aligned} 3(2p-4)+9&=15 && \text{Distribute}\\ 6p-12+9&=15 && \text{Combine like terms}\\ 6p-3&=15 && \text{Add the opposite of } -3 \text{ to each side}\\ 6p-3+3&=15+3 && \text{Simplify}\\ 6p&=18 && \text{Multiply by the reciprocal of } 6\\ \tfrac{1}{6}\cdot 6p&=18\cdot\tfrac{1}{6} && \text{Simplify}\\ p&=3 && \text{Solution} \end{aligned}$$

Example 2.1.23

Solve for a: $3(4a-5)=4(2a+1)+5$

Solution

Notice we have variable terms on each side of the equation. We will not only distribute first, but also isolate the variable term before applying the two-step method:

$$\begin{aligned} 3(4a-5)&=4(2a+1)+5 && \text{Distribute}\\ 12a-15&=8a+4+5 && \text{Combine like terms}\\ 12a-15&=8a+9 && \text{Isolate the variable term by adding the opposite of } 8a \text{ to each side}\\ 12a-15+(-8a)&=8a+9+(-8a) && \text{Simplify}\\ 4a-15&=9 && \text{Add the opposite of } -15 \text{ to each side}\\ 4a-15+15&=9+15 && \text{Simplify}\\ 4a&=24 && \text{Multiply by the reciprocal of } 4\\ \tfrac{1}{4}\cdot 4a&=24\cdot\tfrac{1}{4} && \text{Simplify}\\ a&=6 && \text{Solution} \end{aligned}$$

A general strategy to solving linear equations

In general, there is a 5-step process to solving any linear equation. While all five steps aren't always needed, this can serve as a guide for solving equations.

Step 1. Apply the distributive property to rewrite the equation without parentheses.

Step 2. Combine like terms on each side of the equation.

Step 3. Isolate the variable term by applying the addition property.

Step 4. Solve the equation by applying the multiplication property.

Step 5. ✓ Verify the solution.

Note

The Chinese developed a method for solving equations that involved finding each digit one at a time about 2,000 years ago.

There are two special cases when solving linear equations. The first is illustrated in the next two examples. Notice we start by distributing and moving the variables all to the same side.

Example 2.1.24

Solve for y: $3(2y-5)=6y-15$

Solution

$$\begin{aligned} 3(2y-5) &= 6y-15 && \text{Distribute} \\ 6y-15 &= 6y-15 && \text{Isolate the variable term by adding the opposite of } 6y \text{ to each side} \\ 6y-15+(-6y) &= 6y+15+(-6y) && \text{Simplify} \\ -15 &\stackrel{?}{=} -15 && \text{Is this true? Yes } \checkmark \\ -15 &= -15 \end{aligned}$$

Notice all variables canceled and we are left with only a statement with numbers. In this case, the statement we are left with is a true statement, i.e., -15 does equal -15, and so there are infinitely many solutions to this equation. In this special case, when we obtain infinitely many solutions, then the solution is *all real numbers*. When the solution is *all real numbers*, we call this equation an *identity*.

Example 2.1.25

Solve for n: $2(3n-5)-4n=2n+7$

Solution

$$\begin{aligned} 2(3n-5)-4n &= 2n+7 && \text{Distribute} \\ 6n-10-4n &= 2n+7 && \text{Combine like terms} \\ 2n-10 &= 2n+7 && \text{Isolate the variable term by adding the opposite of } 2n \text{ to each side} \\ -10 &\stackrel{?}{=} 7 && \text{Is this true? No } X \\ -10 &\neq 7 \end{aligned}$$

Notice all variables canceled and we are left with only a statement with numbers. In this case, the statement we are left with is a false statement, i.e., -10 does not equal 7, and so there is no solution to this equation. In this special case, when we obtain a false statement, then the solution is *no solution* and we call this equation a *contradiction*.

Conditional, Identity and Contradiction Equations

There are three types of equations we saw in the examples and in Examples 2.1.24 and 2.1.25:

- An equation is a ***conditional equation*** when there is *one unique solution*, i.e., $x=$ number.
- An equation is an ***identity*** when we obtain *infinitely many solutions*, as in Example 2.1.24, where the solution to the equation is *all real numbers*. Hence, we can choose any number for the variable and this number will make the statement true.
- An equation is a ***contradiction*** when the solution is *no solution*, as in Example 2.1.25. Hence, there is no number for the variable such that the statement is true.

Solving Equations with Fractions

Often when solving linear equations, we work with an equation with fraction coefficients. We can solve these problems as usual. Recall, the methods never change, just problems.

Example 2.1.26

Solve for y: $\frac{3}{4}y-\frac{7}{2}=\frac{5}{6}$

Solution

$$\begin{aligned} \frac{3}{4}y - \frac{7}{2} &= \frac{5}{6} && \text{Isolate the variable term by adding the opposite of } -\frac{7}{2} \\ \frac{3}{4}y - \frac{7}{2} + \frac{7}{2} &= \frac{5}{6} + \frac{7}{2} && \text{Simplify} \end{aligned}$$

Notice, in order to add $\frac{5}{6} + \frac{7}{2}$, we need to obtain the LCD. Since the $\text{LCD}(2, 6) = 6$, we can rewrite the right side with the LCD:

$$\begin{aligned} \frac{3}{4}y &= \frac{5}{6} + \frac{21}{6} && \text{Combine like terms} \\ \frac{3}{4}y &= \frac{26}{6} && \text{Simplify } \frac{26}{6} \\ \frac{3}{4}y &= \frac{13}{3} && \text{Solve by multiplying by the reciprocal of } \frac{3}{4} \\ \frac{4}{3} \cdot \frac{3}{4}y &= \frac{13}{3} \cdot \frac{4}{3} && \text{Simplify} \\ y &= \frac{52}{9} && \text{Solution} \end{aligned}$$

While this process does help us arrive at the correct solution, the fractions can make the process quite difficult and we are more inclined to make errors. Hence, we have an alternate method called *clearing denominators*.

Clearing Denominators

We can easily clear denominators in an equation by multiplying each term by the LCD. After completing this step, the fractions are cleared and we can work with a more familiar type of equation.

Let's try Example 2.1.26 again, but, now, by clearing denominators first, then solving.

Example 2.1.27

Solve for y: $\frac{3}{4}y - \frac{7}{2} = \frac{5}{6}$

Solution

$$\begin{aligned} \frac{3}{4}y - \frac{7}{2} &= \frac{5}{6} && \text{Multiply each term by the } \text{LCD}(2, 4, 6) = 12 \\ 12 \cdot \frac{3}{4}y - 12 \cdot \frac{7}{2} &= 12 \cdot \frac{5}{6} && \text{Simplify} \\ \frac{36}{4}y - \frac{84}{2} &= \frac{60}{6} && \text{Reduce the fractions} \\ 9y - 42 &= 10 && \text{Add the opposite of } -42 \text{ to each side} \\ 9y - 42 + 42 &= 10 + 42 && \text{Simplify} \\ 9y &= 52 && \text{Multiply by the reciprocal of } 9 \\ \frac{1}{9} \cdot 9y &= 52 \cdot \frac{1}{9} && \text{Simplify} \\ y &= \frac{52}{9} && \text{Solution} \end{aligned}$$

Thus, the solution is $y = \frac{52}{9}$ and Example 2.1.26 is a *conditional equation*. Also, we observe that as soon as we multiplied each term by the LCD, we cleared the denominators and the equation no longer contained fractions.

Example 2.1.28

Solve for t: $\frac{2}{3}t - 2 = \frac{3}{2}t + \frac{1}{6}$

Solution

$\frac{2}{3}t-2=\frac{3}{2}t+\frac{1}{6}$	Multiply each term by the $\text{LCD}(2,3,6)=6$
$6\cdot\frac{2}{3}t-6\cdot 2=6\cdot\frac{3}{2}t+6\cdot\frac{1}{6}$	Simplify
$\frac{12}{3}t-12=\frac{18}{2}t+\frac{6}{6}$	Reduce the fractions
$4t-12=9t+1$	Isolate the variable term by adding the opposite of $9t$ to each side
$4t-12+(-9t)=9t+1+(-9t)$	Simplify
$-5t-12=1$	Add the opposite of -12 to each side
$-5t-12+12=1+12$	Simplify
$-5t=13$	Multiply by the reciprocal of -5
$-\frac{1}{5}\cdot -5t=13\cdot -\frac{1}{5}$	Simplify
$t=-\frac{13}{5}$	Solution

Thus, the solution is $t=-\frac{13}{5}$ and Example 2.1.28 is a *conditional equation*.

Note

In Example 2.1.28, we could write the solution as -2.6 given we were using a calculator. A good rule of thumb is if you start with fractions, the solution should also be a fraction. Hence, since there were fractions in original problem, we will leave the solution as a fraction.

Note

The Egyptians were among the first to study fractions and linear equations. The most famous mathematical document from Ancient Egypt is the *Rhind Papyrus*, where the unknown variable was called "heap."

Solving Equations with Distributing Fractions

We can use this same method if parentheses are in the given problem. We will first distribute the coefficient in front of the parenthesis, then clear denominators.

Example 2.1.29

Solve for y: $\frac{3}{2}\left(\frac{5}{9}y+\frac{4}{27}\right)=3$

Solution

$\frac{3}{2}\left(\frac{5}{9}y+\frac{4}{27}\right)=3$	Distribute $\frac{3}{2}$ and reduce
$\frac{5}{6}y+\frac{2}{9}=3$	Multiply each term by the $\text{LCD}(6,9)=18$
$18\cdot\frac{5}{6}y+18\cdot\frac{2}{9}=18\cdot 3$	Multiply and simplify
$15y+4=54$	Add the opposite of 4 to each side
$15y+4+(-4)=54+(-4)$	Simplify
$15y=50$	Multiply by the reciprocal of 15
$\frac{1}{15}\cdot 15y=50\cdot\frac{1}{15}$	Simplify
$y=\frac{50}{15}$	Reduce
$y=\frac{10}{3}$	Solution

Thus, the solution is $y=\frac{10}{3}$ and Example 2.1.29 is a *conditional equation*.

Example 2.1.30

Solve for q: $\frac{1}{4}q-\frac{1}{2}=\frac{1}{3}\left(\frac{3}{4}q+6\right)-\frac{7}{2}$

Solution

$\frac{1}{4}q-\frac{1}{2}=\frac{1}{3}\left(\frac{3}{4}q+6\right)-\frac{7}{2}$	Distribute $\frac{1}{3}$ and reduce
$\frac{1}{4}q-\frac{1}{2}=\frac{1}{4}q+2-\frac{7}{2}$	Multiply each term by the LCD$(2,4)=4$
$4\cdot\frac{1}{4}q-4\cdot\frac{1}{2}=4\cdot\frac{1}{4}q+4\cdot 2-4\cdot\frac{7}{2}$	Multiply and reduce
$q-2=q+8-14$	Combine like terms
$q-2=q-6$	Isolate the variable term by adding the opposite of q to each side
$q-2+(-q)=q-6+(-q)$	Simplify
$-2\stackrel{?}{=}-6$	Is this true? No X
$-2\neq-6$	This implies there is no solution

Since we obtain a false statement, there is no solution and this equation is called a *contradiction*.

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2.1.1: Linear Equations (Exercises)

Solve the one-step equations.

Exercise 2.1.1

$v+9=16$

Exercise 2.1.2

$x-11=-16$

Exercise 2.1.3

$30=a+20$

Exercise 2.1.4

$x-7=-26$

Exercise 2.1.5

$13=n-5$

Exercise 2.1.6

$340=-17x$

Exercise 2.1.7

$-9=n12$

Exercise 2.1.8

$20v=-160$

Exercise 2.1.9

$340=20n$

Exercise 2.1.10

$16x=320$

Exercise 2.1.11

$-16+n=-13$

Exercise 2.1.12

$p-8=-21$

Exercise 2.1.13

$180=12x$

Exercise 2.1.14

$20b=-200$

Exercise 2.1.15

$\frac{r}{14}=\frac{5}{14}$

Exercise 2.1.16

$-7=a+4$

Exercise 2.1.17

$10=x-4$

Exercise 2.1.18

$13a=-143$

Exercise 2.1.19

$\frac{p}{20}=-12$

Exercise 2.1.20

$9+m=-7$

Exercise 2.1.21

$14=b+3$

Exercise 2.1.22

$-14=x-18$

Exercise 2.1.23

$-1+k=5$

Exercise 2.1.24

$-13+p=-19$

Exercise 2.1.25

$22=16+m$

Exercise 2.1.26

$4r=-28$

Exercise 2.1.27

$\frac{5}{9}=\frac{b}{9}$

Exercise 2.1.28

$-20x=-80$

Exercise 2.1.29

$\frac{1}{2}=\frac{a}{8}$

Exercise 2.1.30

$\frac{k}{13}=-16$

Exercise 2.1.31

$21=x+5$

Exercise 2.1.32

$m-4=-13$

Exercise 2.1.33

$3n=24$

Exercise 2.1.34

$-17=\frac{x}{12}$

Exercise 2.1.35

$n+8=10$

Exercise 2.1.36

$v-16=-30$

Exercise 2.1.37

$-15=x-16$

Exercise 2.1.38

$-8k=120$

Exercise 2.1.39

$-15=\frac{x}{9}$

Exercise 2.1.40

$-19=\frac{n}{20}$

Solve the two-step equations.

Exercise 2.1.41

$5+\frac{n}{4}=4$

Exercise 2.1.42

$102=-7r+4$

Exercise 2.1.43

$-8n+3=-77$

Exercise 2.1.44

$0=-6v$

Exercise 2.1.45

$-8=\frac{x}{5}-6$

Exercise 2.1.46

$0=-7+\frac{k}{2}$

Exercise 2.1.47

$-12+3x=0$

Exercise 2.1.48

$24=2n-8$

Exercise 2.1.49

$2=-12+2r$

Exercise 2.1.50

$\frac{b}{3}+7=10$

Exercise 2.1.51

$152=8n+64$

Exercise 2.1.52

$-16=8a+64$

Exercise 2.1.53

$56+8k=64$

Exercise 2.1.54

$-2x+4=22$

Exercise 2.1.55

$-20=4p+4$

Exercise 2.1.56

$-5=3+\frac{n}{2}$

Exercise 2.1.57

$\frac{r}{8}-6=-5$

Exercise 2.1.58

$-40=4n-32$

Exercise 2.1.59

$87=3-7v$

Exercise 2.1.60

$-x+1=-11$

Exercise 2.1.61

$-2=-2m+12$

Exercise 2.1.62

$27=21-3x$

Exercise 2.1.63

$-4-b=8$

Exercise 2.1.64

$-2+\frac{x}{2}=4$

Exercise 2.1.65

$-5=\frac{a}{4}-1$

Exercise 2.1.66

$-6=15+3p$

Exercise 2.1.67

$-5m+2=27$

Exercise 2.1.68

$-37=8+3x$

Exercise 2.1.69

$-8+\frac{n}{12}=-7$

Exercise 2.1.70

$\frac{x}{1} - 8 = -8$

Exercise 2.1.71

$-11 = -8 + \frac{v}{2}$

Exercise 2.1.72

$-2x-3=-29$

Exercise 2.1.73

$-4-3n=-16$

Exercise 2.1.74

$67=5m-8$

Exercise 2.1.75

$9 = 8 + \frac{x}{6}$

Exercise 2.1.76

$\frac{m}{4} - 1 = -2$

Exercise 2.1.77

$-80=4x-28$

Exercise 2.1.78

$33=3b+3$

Exercise 2.1.79

$3x-3=-3$

Exercise 2.1.80

$4 + \frac{a}{3} = 1$

Solve.

Exercise 2.1.81

$2-(-3a-8)=1$

Exercise 2.1.82

$-5(-4+2v)=-50$

Exercise 2.1.83

$66=6(6+5x)$

Exercise 2.1.84

$0=-8(p-5)$

Exercise 2.1.85

$-2+2(8x-7)=-16$

Exercise 2.1.86

$-21x+12=-6-3x$

Exercise 2.1.87

$-1-7m=-8m+7$

Exercise 2.1.88

$1-12r=29-8r$

Exercise 2.1.89

$20-7b=-12b+30$

Exercise 2.1.90

$-32-24v=34-2v$

Exercise 2.1.91

$-2-5(2-4m)=33+5m$

Exercise 2.1.92

$-4n+11=2(1-8n)+3n$

Exercise 2.1.93

$-6v-29=-4v-5(v+1)$

Exercise 2.1.94

$2(4x-4)=-20-4x$

Exercise 2.1.95

$-a-5(8a-1)=39-7a$

Exercise 2.1.96

$-57=-(-p+1)+2(6+8p)$

Exercise 2.1.97

$-2(m-2)+7(m-8)=-67$

Exercise 2.1.98

$50=8(7+7r)-(4r+6)$

Exercise 2.1.99

$-8(n-7)+3(3n-3)=41$

Exercise 2.1.100

$-61=-5(5r-4)+4(3r-4)$

Exercise 2.1.101

$-2(8n-4)=8(1-n)$

Exercise 2.1.102

$-3(-7v+3)+8v=5v-4(1-6v)$

Exercise 2.1.103

$-7(x-2)=-4-6(x-1)$

Exercise 2.1.104

$-6(8k+4)=-8(6k+3)-2$

Exercise 2.1.105

$-2(1-7p)=8(p-7)$

Exercise 2.1.106

$2(-3n+8)=-20$

Exercise 2.1.107

$2-8(-4+3x)=34$

Exercise 2.1.108

$32=2-5(-4n+6)$

Exercise 2.1.109

$-55=8+7(k-5)$

Exercise 2.1.110

$-(3-5n)=12$

Exercise 2.1.111

$-3n-27=-27-3n$

Exercise 2.1.112

$56p-48=6p+2$

Exercise 2.1.113

$4+3x=-12x+4$

Exercise 2.1.114

$-16n+12=39-7n$

Exercise 2.1.115

$17-2x=35-8x$

Exercise 2.1.116

$-25-7x=6(2x-1)$

Exercise 2.1.117

$-7(1+b)=-5-5b$

Exercise 2.1.118

$-8(8r-2)=3r+16$

Exercise 2.1.119

$-8n-19=-2(8n-3)+3n$

Exercise 2.1.120

$-4+4k=4(8k-8)$

Exercise 2.1.121

$16=-5(1-6x)+3(6x+7)$

Exercise 2.1.122

$7=4(n-7)+5(7n+7)$

Exercise 2.1.123

$-8(6+6x)+4(-3+6x)=-12$

Exercise 2.1.124

$-76=5(1+3b)+3(3b-3)$

Exercise 2.1.125

$-6(x-8)-4(x-2)=-4$

Exercise 2.1.126

$-4(1+a)=2a-8(5+3a)$

Exercise 2.1.127

$-6(x-3)+5=-2-5(x-5)$

Exercise 2.1.128

$-(n+8)+n=-8n+2(4n-4)$

Exercise 2.1.129

$-5(x+7)=4(-8x-2)$

Exercise 2.1.130

$8(-8n+4)=4(-7n+8)$

Solve.

Exercise 2.1.131

$\frac{3}{5}(1+p)=\frac{21}{20}$

Exercise 2.1.132

$0=\frac{-5}{4}(x-\frac{6}{5})$

Exercise 2.1.133

$\frac{3}{4}-\frac{5}{4}m=\frac{113}{24}$

Exercise 2.1.134

$\frac{635}{72}=-\frac{5}{2}(-\frac{11}{4}+x)$

Exercise 2.1.135

$2b+\frac{9}{5}=-\frac{11}{5}$

Exercise 2.1.136

$\frac{3}{2}(\frac{7}{3}n+1)=\frac{3}{2}$

Exercise 2.1.137

$-a-\frac{5}{4}(\frac{-8}{3}a+1)=\frac{-19}{4}$

Exercise 2.1.138

$\frac{55}{6}=\frac{-5}{2}(\frac{3}{2}p-\frac{5}{3})$

Exercise 2.1.139

$\frac{16}{9}=-\frac{4}{3}(-\frac{4}{3}n-\frac{4}{3})$

Exercise 2.1.140

$\frac{-5}{8} = \frac{5}{4}(r - \frac{3}{2})$

Exercise 2.1.141

$\frac{-11}{3} + \frac{3}{2}b = \frac{5}{2}(b - \frac{5}{3})$

Exercise 2.1.142

$-(-\frac{5}{2}x - \frac{3}{2}) = -\frac{3}{2} + x$

Exercise 2.1.143

$\frac{45}{16} + \frac{3}{2}n = \frac{7}{4}n - \frac{19}{16}$

Exercise 2.1.144

$\frac{3}{2}(v + \frac{3}{2}) = -\frac{7}{4}v - \frac{19}{6}$

Exercise 2.1.145

$\frac{47}{9} + \frac{3}{2}x = \frac{5}{3}(\frac{5}{2}x + 1)$

Exercise 2.1.146

$\frac{-1}{2} = \frac{3}{2}k + \frac{3}{2}$

Exercise 2.1.147

$\frac{3}{2}n - \frac{8}{3} = -\frac{29}{12}$

Exercise 2.1.148

$\frac{11}{4} + \frac{3}{4}r = \frac{163}{32}$

Exercise 2.1.149

$-\frac{16}{9} = -\frac{4}{3}(\frac{5}{3} + n)$

Exercise 2.1.150

$\frac{3}{2} - \frac{7}{4}v = -\frac{9}{8}$

Exercise 2.1.151

$\frac{41}{9} = \frac{5}{2}(x + \frac{2}{3}) - \frac{1}{3}x$

Exercise 2.1.152

$\frac{1}{3}(\frac{-7}{4}k + 1) - \frac{10}{3}k = \frac{-13}{8}$

Exercise 2.1.153

$-\frac{1}{2}(\frac{2}{3}x - \frac{3}{4}) - \frac{7}{2}x = -\frac{83}{24}$

Exercise 2.1.154

$\frac{2}{3}(m + \frac{9}{4}) - \frac{10}{3} = -\frac{53}{18}$

Exercise 2.1.155

$\frac{1}{12} = \frac{4}{3}x + \frac{5}{3}(x - \frac{7}{4})$

Exercise 2.1.156

$\frac{7}{6} - \frac{4}{3}n = -\frac{3}{2}n + 2(n + \frac{3}{2})$

Exercise 2.1.157

$-\frac{149}{16}-\frac{11}{3}r=-\frac{7}{4}r-\frac{5}{4}(-\frac{4}{3}r+1)$

Exercise 2.1.158

$\frac{-7}{2}(\frac{5}{3}a+\frac{1}{3})=\frac{11}{4}a+\frac{25}{8}$

Exercise 2.1.159

$\frac{-8}{3}-\frac{1}{2}x=-\frac{4}{3}x-\frac{2}{3}(\frac{-13}{4}x+1)$

Exercise 2.1.160

$\frac{1}{3}n+\frac{29}{6}=2(\frac{4}{3}n+\frac{2}{3})$

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2.2: Absolute Value Equations

When solving equations with absolute value, the solution may result in more than one possible answer because, recall, absolute value is just distance from zero. Since the integer -4 has distance 4 units from zero, and 4 has distance 4 units from zero, then there are *two* integers that have distance 4 from zero, $-4,\ 4$. We extend this concept to algebraic absolute value equations. This is illustrated in the following example.

Example 2.2.1

Solve for x: $|x| = 7$

Solution

$$\begin{aligned} |x| &= 7 && \text{Expression in the absolute value can be positive or negative} \\ x = 7 \text{ or } x &= -7 && \text{Solution} \end{aligned}$$

Let's think about the solution set. The equation is asking for all numbers in which the distance from zero is 7. Well, there are two integers that have a distance 7 from zero, -7 and 7. Hence, the solution set $\{-7, 7\}$.

Note

The first set of rules for working with negative numbers came from 7th century India. However, in 1758, more than a thousand years later, British mathematician Francis Maseres claimed that negatives "Darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple."

Definition: Absolute Value

Absolute value for linear equations in one variable is given by

$$\text{If } |x| = a, \text{ then } x = a \text{ or } x = -a$$

where a is a real number.

When we have an equation with absolute value, it is important to first isolate the absolute value, then remove the absolute value by applying the definition.

Example 2.2.2

Solve for x: $5 + |x| = 8$

Solution

$$\begin{aligned} 5 + |x| &= 8 && \text{Isolate the absolute value by subtracting 5 from each side} \\ |x| &= 3 && \text{Rewrite as two linear equations} \\ x = 3 \text{ or } x &= -3 && \text{Solution} \end{aligned}$$

Thus, the solution set is $\{-3, 3\}$.

Example 2.2.3

Solve for x: $-4|x| = -20$

Solution

$$\begin{aligned} -4|x| &= -20 && \text{Isolate the absolute value} \\ \tfrac{-4|x|}{-4} &= \tfrac{-20}{-4} && \text{Divide each side by } -4 \\ |x| &= 5 && \text{Rewrite as two linear equations} \\ x = 5 \text{ or } x &= -5 && \text{Solution} \end{aligned}$$

Thus, the solution set is $\{-5, 5\}$.

Note

Never combine the inside of the absolute value with factors or terms from outside the absolute value. We always have to isolate the absolute value first, then apply the definition to obtain two equations without the absolute value.

Example 2.2.4

Solve for y: $5|y| - 4 = 26$

Solution

$$\begin{aligned} 5|y| - 4 &= 26 && \text{Isolate the absolute value term by adding 4 to each side} \\ 5|y| &= 30 && \text{Divide each side by 5} \\ |y| &= 6 && \text{Rewrite as two linear equations} \\ y = 6 \quad \text{or} \quad y &= -6 && \text{Solution} \end{aligned}$$

Thus, the solution set is $\{-6, 6\}$.

Absolute Value Equations with Different Solutions

Often, we will have linear arguments inside the absolute value which changes the solution. Previously, all solution sets have been opposite integers, but in these cases, the solution sets contain different sized numbers.

Example 2.2.5

Solve for t: $|2t - 1| = 7$

Solution

$$\begin{aligned} |2t - 1| &= 7 && \text{The absolute value term is isolated. Rewrite as two linear equations.} \\ 2t - 1 = 7 \quad \text{or} \quad 2t - 1 &= -7 && \text{Solve each equation.} \end{aligned}$$

Notice we have two equations to solve where each equation results in a different solution. In any case, we solve as usual.

$$\begin{aligned} 2t - 1 &= 7 \quad & & 2t - 1 = -7 \\ 2t &= 8 \quad & \text{or} \quad & 2t = -6 \\ t &= 4 \quad & & t = -3 \end{aligned}$$

Thus, the solution set is $\{-3, 4\}$.

Multiple-Step Absolute Value Equations

Example 2.2.6

Solve for x: $2 - 4|2x + 3| = -18$

Solution

To isolate the absolute value, we first apply the addition rule for equations. Then apply the multiplication rule for equations.

$$\begin{aligned} 2 - 4|2x + 3| &= -18 && \text{Isolate the absolute value term by subtracting 2 from each side} \\ -4|2x + 3| &= -20 && \text{Divide each side by } -4 \\ |2x + 3| &= 5 && \text{Rewrite as two linear equations.} \\ 2x + 3 = 5 \quad \text{or} \quad 2x + 3 &= -5 \end{aligned}$$

Solve each equation.

$$\begin{aligned} 2x+3&=5 \qquad & 2x+3&=-5 \\ 2x&=2 \quad \text{or} & 2x&=-8 \\ x&=1 & x&=-4 \end{aligned}$$

We now have obtained two solutions, $x=1$ and $x=-4$. Thus, the solution set is $\{-4,1\}$.

Equations with Two Absolute Values

In this case, we have an absolute value on each side of the equals sign. However, even though there are two absolute values, we apply the same process. Recall, methods never change, only problems.

Example 2.2.7

Solve for m: $|2m-7|=|4m+6|$

Solution

In order to apply the definition, we rewrite this equation as two linear equations, but with the left side as its positive and negative value:

$$\begin{aligned} |2m-7|&=|4m+6| \quad \text{Rewrite as two linear equations} \\ 2m-7=4m+6 \quad &\text{or} \quad 2m-7=-(4m+6) \end{aligned}$$

Now, we can solve as usual. Be sure to distribute the negative for the equation on the right.

$$\begin{aligned} & & 2m-7&=-(4m+6) \\ 2m-7&=4m+6 & 2m-7&=-4m-6 \\ -13&=2m & 6m-7&=-6 \\ -\tfrac{13}{2}&=m \quad \text{or} & 6m&=1 \\ & & m&=\tfrac{1}{6} \end{aligned}$$

Thus gives the solutions, $m=-\frac{13}{2}$ or $m=\frac{1}{6}$. Thus, the solution set is $\left\{-\frac{13}{2},\frac{1}{6}\right\}$.

Note

In Example 2.2.7, because there are absolute value expressions on both sides of the equation, we could have easily applied the definition to the left side and obtained

$$2m-7=4m-6 \quad \text{or} \quad -(2m-7)=4m-6$$

Then solved each linear equation as usual and obtained the same results.

Special Cases

As we are solving absolute value equations, it is important to be aware of special cases. Remember, the result after evaluating absolute value must always be non-negative.

Example 2.2.8

Solve for x: $7+|2x-5|=4$

Solution

$$\begin{aligned} 7+|2x-5|&=4 \quad \text{Isolate the absolute value term by subtracting 7 from each side} \\ |2x-5|&=-3 \quad X \text{ False} \end{aligned}$$

Careful! Observe the absolute value of $2x-5$ is a negative number. This is impossible with absolute value because *the result after evaluating absolute value must always be non-negative*. Thus, we say this equation has **no solution**.

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2.2.1: Absolute Value Equations (Exercises)

Solve each equation.

Exercise 2.2.1

$|x|=8$

Exercise 2.2.2

$|b|=1$

Exercise 2.2.3

$|5+8a|=53$

Exercise 2.2.4

$|3k+8|=2$

Exercise 2.2.5

$|9+7x|=30$

Exercise 2.2.6

$|8+6m|=50$

Exercise 2.2.7

$|6-2x|=24$

Exercise 2.2.8

$-7|-3-3r|=-21$

Exercise 2.2.9

$7|-7x-3|=21$

Exercise 2.2.10

$\frac{|-4b-10|}{8}=3$

Exercise 2.2.11

$8|x+7|-3=5$

Exercise 2.2.12

$5|3+7m|+1=51$

Exercise 2.2.13

$3+5|8-2x|=63$

Exercise 2.2.14

$|6b-2|+10=44$

Exercise 2.2.15

$-7+8|-7x-3|=73$

Exercise 2.2.16

$|5x+3|=|2x-1|$

Exercise 2.2.17

$|3x-4|=|2x+3|$

Exercise 2.2.18

$\frac{|4x-2|}{5}=\frac{|6x+3|}{2}$

Exercise 2.2.19

$|n|=7$

Exercise 2.2.20

$|x|=2$

Exercise 2.2.21

$|9n+8|=46$

Exercise 2.2.22

$|3-x|=6$

Exercise 2.2.23

$|5n+7|=23$

Exercise 2.2.24

$|9p+6|=3$

Exercise 2.2.25

$|3n-2|=7$

Exercise 2.2.26

$|2+2b|+1=3$

Exercise 2.2.27

$\frac{|-4-3n|}{4}=2$

Exercise 2.2.28

$8|5p+8|-5=11$

Exercise 2.2.29

$3-|6n+7|=-40$

Exercise 2.2.30

$4|r+7|+3=59$

Exercise 2.2.31

$5+8|-10n-2|=101$

Exercise 2.2.32

$7|10v-2|-9=5$

Exercise 2.2.33

$8|3-3n|-5=91$

Exercise 2.2.34

$|2+3x|=|4-2x|$

Exercise 2.2.35

$\left|\frac{2x-5}{3}\right| = \left|\frac{3x+4}{2}\right|$

Exercise 2.2.36

$\frac{|-n+6|}{6} = 0$

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2.3: Literal Equations

A literal equation is synonymous with a formula and similar to solving general linear equations because we apply the same method. We say, methods never change, just the problems. The only difference is we have several variables in the equation and we will attempt to solve for one specific variable of the formula. For example, we may have a formula such as $A = \pi r^2 + \pi r s$, the formula for surface area of a right circular cone, and we may be interested in solving for the variable s. This means we want to isolate the variable s so the equation has s isolated on one side, and everything else on the other. This looks like

$$s = \frac{A - \pi r^2}{\pi r}$$

This second equation gives the same information as the first, meaning they are algebraically equivalent. However, the original formula gives area, while the other gives s, the slant height of the cone. In this section, we discuss the process in which we start from the first equation and result in the second equation.

Example 2.3.1

Let's take a look at these two examples below, side by side. The left equation is a familiar one-step equation and the right equation is also a one-step equation, this time a literal equation (or formula).

$3x = 12$	$wx = z$	Both have coefficients
		Multiply by the reciprocal of 3 and w, respectively
$\frac{1}{3} \cdot 3x = \frac{1}{3} \cdot 12$	$\frac{1}{w} \cdot wx = z \cdot \frac{1}{w}$	Simplify
$x = 4$	$x = \frac{z}{w}$	Solution

We used the same process for solving $3x = 12$ for x as we did for solving $wx = z$ for x. Because we are solving for x, we treat all the other variables the same way we would treat numbers or coefficients. Thus, we applied the multiplication property and multiplied by the reciprocal of 3 and w to isolate x.

Solving for a Variable with One and Two-Step Equations

Example 2.3.2

Solve the equation $m + n = p$ for n.

Solution

$m + n = p$	Add the opposite of m
$m + n + (-m) = p + (-m)$	Simplify
$n = p - m$	Solution

Since p and m are not like terms, they cannot be combined. Hence, $n = p - m$.

Example 2.3.3

Solve the equation $a(x - y) = b$ for x.

Solution

$a(x - y) = b$	Distribute
$ax - ay = b$	Add the opposite of ay
$ax + ay + (-ay) = b + (-ay)$	Simplify
$ax = b - ay$	Isolate x by multiplying by the reciprocal of a
$\frac{1}{a} \cdot ax = (b - ay) \cdot \frac{1}{a}$	Simplify
$x = \frac{b-ay}{a}$	Solution

Equivalently, x can be written as $\frac{b}{a} - y$ by simplifying the fraction. However, it is common practice to leave it as one fraction.

Example 2.3.4

Solve the equation $y = mx + b$ for m.

Solution

$$\begin{aligned} y &= mx + b && \text{Isolate the variable term by adding the opposite of } b \\ y + (-b) &= mx + b + (-b) && \text{Simplify} \\ y - b &= mx && \text{Isolate } m \text{ by multiplying by the reciprocal of } x \\ \tfrac{1}{x} \cdot (y - b) &= mx \cdot \tfrac{1}{x} && \text{Simplify} \\ \tfrac{y-b}{x} &= m && \text{Rewrite with } m \text{ on the left side} \\ m &= \tfrac{y-b}{x} && \text{Solution} \end{aligned}$$

Solving for a Variable in Multiple Steps

Example 2.3.5

Solve the equation $A = \pi r^2 + \pi rs$ for s. *This should remind you of the equation in the beginning of the section.*

Solution

$$\begin{aligned} A &= \pi r^2 + \pi rs && \text{Isolate the variable term by adding the opposite of } \pi r^2 \\ A + (-\pi r^2) &= \pi r^2 + \pi rs + (-\pi r^2) && \text{Simplify} \\ A - \pi r^2 &= \pi rs && \text{Isolate } s \text{ by multiplying by the reciprocal of } \pi r \\ \tfrac{1}{\pi r} \cdot (A - \pi r^2) &= \pi rs \cdot \tfrac{1}{\pi r} && \text{Simplify} \\ \tfrac{A-\pi r^2}{\pi r} &= s && \text{Rewrite with } s \text{ on the left side} \\ s &= \tfrac{A-\pi r^2}{\pi r} && \text{Solution} \end{aligned}$$

Solving for a Variable with Fractions

Formulas often include fractions and we can solve with the same method as used previously. First, identify the LCD, and then multiply each term by the LCD. After we clear denominators, we obtain a general equation and solve as usual.

Example 2.3.6

Solve the equation $h = \frac{2m}{n}$ for m.

Solution

$$\begin{aligned} h &= \tfrac{2m}{n} && \text{Multiply by the LCD}=n \\ n \cdot h &= \tfrac{2m}{n} \cdot n && \text{Simplify} \\ nh &= 2m && \text{Multiply by the reciprocal of } 2 \\ \tfrac{1}{2} \cdot nh &= 2m \cdot \tfrac{1}{2} && \text{Simplify} \\ \tfrac{nh}{2} &= m && \text{Rewrite with } m \text{ on the left side} \\ m &= \tfrac{nh}{2} && \text{Solution} \end{aligned}$$

Example 2.3.7

Solve the equation $\frac{a}{b} + \frac{c}{b} = e$ for a.

Solution

$\frac{a}{b}+\frac{c}{b}=e$	Multiply each term by the LCD=b
$b\cdot\frac{a}{b}+b\cdot\frac{c}{b}=e\cdot b$	Simplify
$a+c=eb$	Add the opposite of c
$a+c+(-c)=eb+(-c)$	Simplify
$a=eb-c$	Solution

Example 2.3.8

Solve the equation $a=\frac{A}{2-b}$ for b.

Solution

$a=\frac{A}{2-b}$	Multiply each term by the LCD $=(2-b)$
$(2-b)\cdot a=\frac{A}{2-b}\cdot(2-b)$	Simplify
$a(2-b)=A$	Distribute
$2a-2b=A$	Isolate the variable term by adding the opposite of $2a$
$2a-2b+(-2a)=A+(-2a)$	Simplify
$-2b=A-2a$	Multiply by the reciprocal of -2
$-\frac{1}{2}\cdot-2b=(A-2a)\cdot-\frac{1}{2}$	Simplify
$b=-\frac{(A-2a)}{2}$	Distribute the negative
$b=\frac{-A+2a}{2}$	Solution

Note, we could also write the solution as $b=\frac{2a-A}{2}$, where the positive term is written first in the numerator. It's not necessary, but for aesthetic reasons, we can write b this way.

Note

The father of algebra, Persian mathematician, *Muhammad ibn Musa Khwarizmi*, introduced the fundamental idea of balancing by subtracting the same term from the other side of the equation. He called this process *al-jabr*, which later became the world Algebra.

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2.3.1: Literal Equations (Exercises)

Solve each of the following equations for the indicated variable.

Exercise 2.3.1.1

$ab = c$ for b

Exercise 2.3.1.2

$\frac{f}{g}x = b$ for x

Exercise 2.3.1.3

$3x = \frac{a}{b}$ for x

Exercise 2.3.1.4

$E = mc^2$ for m

Exercise 2.3.1.5

$V = \frac{4}{3}\pi r^3$ for π

Exercise 2.3.1.6

$a + c = b$ for c

Exercise 2.3.1.7

$c = \frac{4y}{m+n}$ for y

Exercise 2.3.1.8

$V = \frac{\pi Dn}{12}$ for D

Exercise 2.3.1.9

$P = n(p - c)$ for n

Exercise 2.3.1.10

$T = \frac{D-d}{L}$ for D

Exercise 2.3.1.11

$L = L_0(1 + at)$ for L_0

Exercise 2.3.1.12

$2m + p = 4m + q$ for m

Exercise 2.3.1.13

$\frac{k-m}{r} = q$ for k

Exercise 2.3.1.14

$h = vt - 16t^2$ for v

Exercise 2.3.1.15

$Q_1 = P(Q_2 - Q_1)$ for Q_2

Exercise 2.3.1.16

$R = \frac{kA(T_1+T_2)}{d}$ for T_1

Exercise 2.3.1.17

$ax + b = c$ for a

Exercise 2.3.1.18

$lwh = V$ for w

Exercise 2.3.1.19

$\frac{1}{a} + b = \frac{c}{a}$ for a

Exercise 2.3.1.20

$at - bw = s$ for t

Exercise 2.3.1.21

$ax + bx = c$ for a

Exercise 2.3.1.22

$x + 5y = 3$ for y

Exercise 2.3.1.23

$3x + 2y = 7$ for y

Exercise 2.3.1.24

$5a - 7b = 4$ for b

Exercise 2.3.1.25

$4x - 5y = 8$ for y

Exercise 2.3.1.26

$g = \frac{h}{i}$ for h

Exercise 2.3.1.27

$p = \frac{3y}{q}$ for y

Exercise 2.3.1.28

$\frac{ym}{b} = \frac{c}{d}$ for y

Exercise 2.3.1.29

$DS = ds$ for D

Exercise 2.3.1.30

$E = \frac{mv^2}{2}$ for m

Exercise 2.3.1.31

$x - f = g$ for x

Exercise 2.3.1.32

$\frac{rs}{a-3} = k$ for r

Exercise 2.3.1.33

$F = k(R - L)$ for k

Exercise 2.3.1.34

$S = L + 2B$ for L

Exercise 2.3.1.35

$I = \frac{E_a - E_q}{R}$ for E_a

Exercise 2.3.1.36

$ax + b = c$ for x

Exercise 2.3.1.37

$q = 6(L - p)$ for L

Exercise 2.3.1.38

$R = aT + b$ for T

Exercise 2.3.1.39

$S = \pi rh + \pi r^2$ for h

Exercise 2.3.1.40

$L = \pi(r_1 + r_2) + 2d$ for r_1

Exercise 2.3.1.41

$P = \frac{V_1(V_2 - V_1)}{g}$ for V_2

Exercise 2.3.1.42

$rt = d$ for r

Exercise 2.3.1.43

$V = \frac{\pi r^2 h}{3}$ for h

Exercise 2.3.1.44

$\frac{1}{a} + b = \frac{c}{a}$ for b

Exercise 2.3.1.45

$at - bw = s$ for w

Exercise 2.3.1.46

$x + 5y = 3$ for x

Exercise 2.3.1.47

$3x + 2y = 7$ for x

Exercise 2.3.1.48

$5a - 7b = 4$ for a

Exercise 2.3.1.49

$4x - 5y = 8$ for x

Exercise 2.3.1.50

$C = \frac{5}{9}(F - 32)$ for F

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2.4: Word Problems

Now, let's apply the techniques from this chapter to some common word problems. Word problems can be tricky. The goal is becoming proficient in translating an English sentence into a mathematical sentence. In this section, we focus on word problems modeled by a linear equation and solve. We discuss geometry problems including perimeter and triangles, number, and distance problems.

Number Problems

Example 2.4.1

If 28 less than five times a number is 232, what is the number?

Solution

First, let n be the number. Now, translate the key words in the sentence:

$$\underbrace{\ldots 28 \text{ less than } \underbrace{\text{five times a number}}_{5n} \text{ is } \underbrace{232}_{232}\ldots}_{5n-28} \quad \underbrace{=}_{=}$$

Notice, after translating, we obtain the equation

$$5n-28=232$$

Let's solve:

$$\begin{aligned} 5n-8&=232 && \text{Isolate the variable term } 5n \\ 5n-28+28&=232+28 && \text{Simplify} \\ 5n&=260 && \text{Multiply by the reciprocal of } 5 \\ \tfrac{1}{5}\cdot 5n&=260\cdot\tfrac{1}{5} && \text{Simplify} \\ n&=52 && \text{Solution} \end{aligned}$$

Thus, the number is 52.

Example 2.4.2

Fifteen more than three times a number is the same as ten less than six times the number. What is the number?

Solution

Notice, this sentence is a bit more challenging than Example 2.4.1, but we still follow the method. Let n be the number.

$$\underbrace{\text{Fifteen more than } \underbrace{\text{three times a number}}_{3n}}_{3n+15} \quad \underbrace{\text{is the same as}}_{=} \quad \underbrace{\text{ten less than } \underbrace{\text{six times the number}}_{6n}}_{6n-10}$$

Notice, after translating, we obtain the equation

$$3n+15=6n-10$$

Let's solve:

$$\begin{aligned} 3n+15 &= 6n-10 && \text{Combine like terms} \\ 3n+15+(-6n) &= 6n-10+(-6n) && \text{Simplify} \\ -3n+15 &= -10 && \text{Isolate the variable term} \\ -3n+15+(-15) &= -10+(-15) && \text{Simplify} \\ -3n &= -25 && \text{Multiply by the reciprocal of } -3 \\ -\tfrac{1}{3}\cdot -3n &= -25\cdot -\tfrac{1}{3} && \text{Simplify} \\ n &= \tfrac{25}{3} && \text{Solution} \end{aligned}$$

Thus, the number is $\frac{25}{3}$.

Consecutive Integers

Another type of number problem involves *consecutive integers*.

Definition: Consecutive Integers

Consecutive integers are integers that come one after the other (such as $3,\ 4,\ 5,$ or $-3,\ -2,\ -1$).

- If we are trying to find several *consecutive integers*, it important to identify the first integer and then assign names to the following integers. E.g., if x is the first integer, then $x+1$ will be the next, and $x+2$ will be the following, and so on.
- If we are trying to find several *even or odd consecutive integers*, it important to identify the first integer and then assign names to the following even or odd integers. E.g., if x is the first integer, then $x+2$ will be the next odd or even integer, and $x+4$ will be the following, and so on.

Example 2.4.3

The sum of three consecutive positive integers is 93. What are the positive integers?

Solution

Since we want to obtain three consecutive positive integers, then we can assign each integer as the following:

$$\begin{aligned} x &\quad \text{is the first integer} \\ x+1 &\quad \text{is the second integer} \\ x+2 &\quad \text{is the third integer} \end{aligned}$$

The sum of these three integers is given to be 93. Translating this into an equation, we get

$$x+(x+1)+(x+2)=93$$

Let's solve this equation for x. Then we can obtain the other two integers.

$$\begin{aligned} x+(x+1)+(x+2) &= 93 && \text{Rewrite without the parenthesis} \\ x+x+1+x+2 &= 93 && \text{Combine like terms} \\ 3x+3 &= 93 && \text{Isolate the variable term} \\ 3x+3+(-3) &= 93+(-3) && \text{Simplify} \\ 3x &= 90 && \text{Multiply by the reciprocal of 3} \\ \tfrac{1}{3}\cdot 3x &= 90\cdot \tfrac{1}{3} && \text{Simplify} \\ x &= 30 && \text{First integer} \end{aligned}$$

Since the first integer is 30, the next two integers would be

$$\begin{aligned} 30+1 &= 31 \quad \text{is the second even integer} \\ 30+2 &= 32 \quad \text{is the third even integer} \end{aligned}$$

Thus, the integers are $30,\ 31,$ and 32.

Example 2.4.4

The sum of three consecutive even positive integers is 246. What are the numbers?

Solution

Since we want to obtain *three consecutive even positive integers*, then we can assign each integer as the following:

$$\begin{aligned} x &\quad \text{is the first odd integer} \\ x+2 &\quad \text{is the second odd integer} \\ x+4 &\quad \text{is the third odd integer} \end{aligned}$$

The sum of these three even integers is given to be 246. Translating this into an equation, we get

$$x+(x+2)+(x+4)=246$$

Let's solve this equation for x. Then we can obtain the other two integers.

$$\begin{aligned} x+(x+2)+(x+4)&=246 && \text{Rewrite without the parenthesis} \\ x+x+2+x+4&=246 && \text{Combine like terms} \\ 3x+6&=246 && \text{Isolate the variable term} \\ 3x+6+(-6)&=246+(-6) && \text{Simplify} \\ 3x&=240 && \text{Multiply by the reciprocal of 3} \\ \tfrac{1}{3}\cdot 3x&=240\cdot\tfrac{1}{3} && \text{Simplify} \\ x&=80 && \text{First integer} \end{aligned}$$

Since the first integer is 80, the next two even integers would be

$$\begin{aligned} 80+2&=82 \quad \text{is the second even integer} \\ 80+4&=32 \quad \text{is the third even integer} \end{aligned}$$

Thus, the integers are 80, 82, and 84.

Example 2.4.5

Find three consecutive odd positive integers so that the sum of twice the first integer, the second integer, and three times the third integer is 152.

Solution

Since we want to obtain *three consecutive odd positive integers*, then we can assign each integer as the following:

$$\begin{aligned} x &\quad \text{is the first odd integer} \\ x+2 &\quad \text{is the second integer} \\ x+4 &\quad \text{is the third odd integer} \end{aligned}$$

The sum of twice the first integer, the second integer, and three times the third integer is given to be 152. Translating this into an equation, we get

$$2\cdot x+(x+2)+3\cdot(x+4)=152$$

Let's solve this equation for x. Then we can obtain the other two integers.

$$\begin{aligned} 2\cdot x+(x+2)+3\cdot(x+4)&=152 && \text{Rewrite without the parenthesis} \\ 2x+x+2+3x+12&=152 && \text{Combine like terms} \\ 6x+14&=152 && \text{Isolate the variable term} \\ 6x+14+(-14)&=152+(-14) && \text{Simplify} \\ 6x&=138 && \text{Multiply by the reciprocal of 6} \\ \tfrac{1}{6}\cdot 6x&=138\cdot\tfrac{1}{6} && \text{Simplify} \\ x&=23 && \text{First integer} \end{aligned}$$

Since the first integer is 23, the next two odd integers would be

$$\begin{aligned} 23+2=25 &\quad \text{is the second odd integer} \\ 23+4=27 &\quad \text{is the third odd integer} \end{aligned}$$

Thus, the integers are $23,\ 25,$ and 27.

Perimeter Problems

Another problem from geometry involves perimeter or the distance around an object.

Definition: Perimeter of a Rectangle

The formula for the perimeter of a rectangle is given by

$$P=2w+2\ell,$$

where w is the width and ℓ is the length of the rectangle.

Example 2.4.6

The perimeter of a rectangle is 44 cm. The length is 5 less than double the width. Find the dimensions.

Solution

Let w be the width of the rectangle. Then the length is $2w-5$. Since the perimeter is 44 cm, the we can use the perimeter formula to obtain the dimensions.

$$\begin{aligned} P&=2w+2\ell && \text{Substitute in the width, length, and perimeter} \\ 44&=2(w)+2(2w-5) && \text{Rewrite with no parenthesis} \\ 44&=2w+4w-10 && \text{Combine like terms} \\ 44&=6w-10 && \text{Isolate the variable term} \\ 54&=6w && \text{Multiply by the reciprocal of 6} \\ 9&=w && \text{Length of the rectangle} \end{aligned}$$

Since the width is 9 cm, then the length is $(2(9)-5)=13$ cm.

Triangles

Sum of Angles in a Triangle

Given a triangle, the sum of the three angles is 180°. I.e., if the angles in a triangle are $a^\circ,\ b^\circ$, and c°, then

$$a^\circ+b^\circ+c^\circ=180^\circ$$

Note

German mathematician *Bernhart Thibaut* in 1809 tried to prove that the angles of a triangle add to 180 without using Euclid's parallel postulate (a point of much debate in math history). He created a proof, but it was later shown to have an error in the proof.

Example 2.4.7

The second angle of a triangle is double the first. The third angle is 40 less than the first. Find the three angles.

Solution

Let x be the measure of the first angle. Then

$2x$ is the measure of the second angle

$x-40$ is the measure of the third angle

Since the sum of these three angles is 180°, then we can write the equation

$$x+2x+(x-40)=180$$

Let's solve for the first angle x:

$$\begin{aligned} x+2x+(x-40)&=180 && \text{Rewrite without parenthesis}\\ x+2x+x-40&=180 && \text{Combine like terms}\\ 4x-40&=180 && \text{Isolate the variable term}\\ 4x&=220 && \text{Multiply by the reciprocal of 4}\\ x&=55 && \text{Measure of the first angle}\end{aligned}$$

Since the measure of the first angle is 55°, then the measures of the second and third angle are

$$\begin{aligned} 2(55)&=110^\circ && \text{is the measure of the second angle}\\ 55-40&=15^\circ && \text{is the measure of the third angle}\end{aligned}$$

Uniform Motion Problems

Another common application of linear equations is uniform motion problems. When solving uniform motion problems, we use the relationship $rt=d$ or

$$\text{rate (speed)}\cdot\text{time}=\text{distance}$$

For example, if a person were to travel 30 miles per hour (mph) for 4 hours, to find the total distance we would multiply rate and the time: $(30)(4)=120$. Hence, this person traveled a distance of 120 miles. The problems we solve in this section are just a few more steps than described. To keep the information in the problem organized, we use tables.

Opposite Directions

Example 2.4.8

Two joggers start from opposite ends of an 8 mile course running towards each other. One jogger is running at a rate of 4 miles per hour, and the other is running at a rate of 6 miles per hour. After how long will the joggers meet?

Solution

First, we can make a table to organize the given information and then create an equation. Let t represent the length of time until the joggers meet.

Table 2.4.1

Speed and distance data for two joggers

	rate	time	distance
Jogger 1	4	t	$4t$
Jogger 2	6	t	$6t$

Now we can set up the equation. If the total distance is 8 miles, then

$$4t=6t+8,$$

i.e., the sum of Jogger 1's distance and Jogger 2's distance is 8 miles. Let's solve.

$$\begin{aligned} 4t+6t&=8 && \text{Combine like terms}\\ 10t&=8 && \text{Multiply by the reciprocal of 10}\\ t&=\tfrac{4}{5} && \text{Hours until they meet}\end{aligned}$$

It will be $\frac{4}{5}$ hours (or 48 minutes) until they meet.

Example 2.4.9

Bob and Fred start from the same point and walk in opposite directions. Bob walks 2 miles per hour faster than Fred. After 3 hours they are 30 miles apart. How fast did each walk?

Solution

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of Fred.

Table 2.4.2

Speed and distance data for two walkers

	rate	time	distance
Bob	$r+2$	3	$3(r+2)$
Fred	r	3	$3r$

Now we can set up the equation. If the total distance is 30 miles, then

$$3(r+2)+3r=30,$$

i.e., the sum of Bob's distance and Fred's distance is 30 miles. Let's solve.

$$\begin{aligned} 3(r+2)+3r &= 30 && \text{Distribute} \\ 3r+6+3r &= 30 && \text{Combine like terms} \\ 6r+6 &= 30 && \text{Isolate the variable term} \\ 6r &= 24 && \text{Multiply by the reciprocal of } 6 \\ r &= 4 && \text{Rate of Fred} \end{aligned}$$

Since the rate of Fred is 4 mph, then Bob's rate is 6 mph $(4+2=6)$.

Example 2.4.10

Two campers left their campsite by canoe and paddled downstream at an average speed of 12 miles per hour. They turned around and paddled back upstream at an average rate of 4 miles per hour. The total trip took 1 hour. After how much time did the campers turn around downstream?

Solution

First, we can make a table to organize the given information and then create an equation. Let t represent the time it took to travel upstream.

Table 2.4.3

Speed and distance data for upstream vs downstream travel

	rate	time	distance
upstream	4	t	$4t$
downstream	12	$1-t$	$12(1-t)$

Now we can set up the equation. If the upstream and downstream routes' distances are the same, then

$$4t=12(1-t)$$

Let's solve.

$$\begin{aligned} 4t &= 12(1-t) && \text{Distribute} \\ 4t &= 12-12t && \text{Combine like terms} \\ 16t &= 12 && \text{Multiply by the reciprocal of } 16 \\ t &= \tfrac{12}{16} && \text{Reduce} \\ t &= \tfrac{3}{4} && \text{Time going upstream} \end{aligned}$$

Since the time going upstream is $\frac{3}{4}$ hours, then downstream's time is $\frac{1}{4}$ hours $\left(1-\frac{3}{4}=\frac{1}{4}\right)$. Thus, the campers spent 15 minutes going downstream.

Catch-Up

Example 2.4.11

Mike leaves his house traveling 2 miles per hour. Joy leaves 6 hours later to catch up with him traveling 8 miles per hour. How long will it take her to catch up with him?

Solution

First, we can make a table to organize the given information and then create an equation. Let t represent the time Joy traveled.

Table 2.4.4

Speed and distance data for two travelers

	rate	time	distance
Mike	2	$t+6$	$2(t+6)$
Joy	8	t	$8t$

Now we can set up the equation. If Joy catches up to Mike, then Mike and Joy would have traveled the same distance. Hence, giving the equation

$$2(t+6)=8t,$$

i.e., Mike's distance and Joy's distance are the same. Let's solve.

$$\begin{aligned} 2(t+6) &= 8t && \text{Distribute} \\ 2t+12 &= 8t && \text{Combine like terms} \\ 12 &= 6t && \text{Multiply by the reciprocal of } 6 \\ 2 &= t && \text{Time Joy traveled} \end{aligned}$$

Since the time Joy traveled was 2 hours, then Mike traveled 8 hours $(2+6=8)$. Thus, it took 2 hours for Joy to catch up with Mike.

Note

The $10,000$-meter race is the longest standard track event. Ten-thousand meters is approximately 6.2 miles. The current (at the time of printing) world record for this race is held by Ethiopian *Kenenisa Bekele* with a time of 26 minutes, 17.53 seconds. That is a rate of 12.7 miles per hour.

Total Time

Example 2.4.12

On a 130-mile trip, a car traveled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took 2.5 hours. For how long did the car travel 40 mph?

Solution

First, we can make a table to organize the given information and then create an equation. Let t represent the time the car traveled at the faster speed.

Table 2.4.5

Speed and distance data for a car trip

	rate	time	distance
First part	55	t	$55t$
Second part	40	$2.5-t$	$40(2.5-t)$

Now we can set up the equation. Since the total distance of the trip was 130 miles, then

$$55t+40(2.5-t)=130,$$

i.e., the sum of the first part's distance and the second part's distance is 130 miles. Let's solve.

$$\begin{aligned} 55t+40(2.5-t)&=130 && \text{Distribute}\\ 55t+100-40t&=130 && \text{Combine like terms}\\ 15t+100&=130 && \text{Isolate the variable term}\\ 15t&=30 && \text{Multiply by the reciprocal of } 15\\ t&=2 && \text{First part's travel time} \end{aligned}$$

Since the first part of the trip took 2 hours, then the car traveled 0.5 hours (or 30 minutes) at 40 mph.

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2.4.1: Word Problems (Exercises)

Exercise 2.4.1.1

When five is added to three more than a certain number, the result is 19. What is the number?

Exercise 2.4.1.2

If five is subtracted from three times a certain number, the result is 10. What is the number?

Exercise 2.4.1.3

When 18 is subtracted from six times a certain number, the result is -42. What is the number?

Exercise 2.4.1.4

A certain number added twice to itself equals 96. What is the number?

Exercise 2.4.1.5

A number plus itself, plus twice itself, plus 4 times itself, is equal to -104. What is the number?

Exercise 2.4.1.6

Sixty more than nine times a number is the same as two less than ten times the number. What is the number?

Exercise 2.4.1.7

Eleven less than seven times a number is five more than six times the number. Find the number.

Exercise 2.4.1.8

Fourteen less than eight times a number is three more than four times the number. What is the number?

Exercise 2.4.1.9

The sum of three consecutive integers is 108. What are the integers?

Exercise 2.4.1.10

The sum of three consecutive integers is -126. What are the integers?

Exercise 2.4.1.11

Find three consecutive integers such that the sum of the first, twice the second, and three times the third is -76.

Exercise 2.4.1.12

The sum of two consecutive even integers is 106. What are the integers?

Exercise 2.4.1.13

The sum of three consecutive odd integers is 189. What are the integers?

Exercise 2.4.1.14

The sum of three consecutive odd integers is 255. What are the integers?

Exercise 2.4.1.15

Find three consecutive odd integers such that the sum of the first, two times the second, and three times the third is 70.

Exercise 2.4.1.16

The second angle of a triangle is the same size as the first angle. The third angle is 12 degrees larger than the first angle. How large are the angles?

Exercise 2.4.1.17

Two angles of a triangle are the same size. The third angle is 12 degrees smaller than the first angle. Find the measure the angles.

Exercise 2.4.1.18

Two angles of a triangle are the same size. The third angle is 3 times as large as the first. How large are the angles?

Exercise 2.4.1.19

The third angle of a triangle is the same size as the first. The second angle is 4 times the third. Find the measure of the angles.

Exercise 2.4.1.20

The second angle of a triangle is 3 times as large as the first angle. The third angle is 30 degrees more than the first angle. Find the measure of the angles.

Exercise 2.4.1.21

The second angle of a triangle is twice as large as the first. The measure of the third angle is 20 degrees greater than the first. How large are the angles?

Exercise 2.4.1.22

The second angle of a triangle is three times as large as the first. The measure of the third angle is 40 degrees greater than that of the first angle. How large are the three angles?

Exercise 2.4.1.23

The second angle of a triangle is five times as large as the first. The measure of the third angle is 12 degrees greater than that of the first angle. How large are the angles?

Exercise 2.4.1.24

The second angle of a triangle is three times the first, and the third is 12 degrees less than twice the first. Find the measures of the angles.

Exercise 2.4.1.25

The second angle of a triangle is four times the first and the third is 5 degrees more than twice the first. Find the measures of the angles.

Exercise 2.4.1.26

The perimeter of a rectangle is 150 cm. The length is 15 cm greater than the width. Find the dimensions.

Exercise 2.4.1.27

The perimeter of a rectangle is 304 cm. The length is 40 cm longer than the width. Find the length and width.

Exercise 2.4.1.28

The perimeter of a rectangle is 152 meters. The width is 22 meters less than the length. Find the length and width.

Exercise 2.4.1.29

The perimeter of a rectangle is 280 meters. The width is 26 meters less than the length. Find the length and width.

Exercise 2.4.1.30

The perimeter of a college basketball court is 96 meters and the length is 14 meters more than the width. What are the dimensions?

Exercise 2.4.1.31

A is 60 miles from B. An automobile at A starts for B at the rate of 20 miles per hour at the same time that an automobile at B starts for A at the rate of 25 miles an hour. How long will it be before the automobiles meet?

Exercise 2.4.1.32

Two automobiles are 276 miles apart and start at the same time to travel toward each other. They travel at rates differing by 5 miles per hour. If they meet after 6 hours, find each rate.

Exercise 2.4.1.33

Two trains travel toward each other from points which are 195 miles apart. They travel at rate of 25 and 40 miles an hour, respectively. If they start traveling at the same time, how long before the trains will meet?

Exercise 2.4.1.34

Car A and Car B start traveling towards each other at the same time from points 150 miles apart. If Car A went at the rate of 20 miles an hour, at what rate must B travel if they meet in 5 hours?

Exercise 2.4.1.35

A passenger and a freight train start toward each other at the same time from two points 300 miles apart. If the rate of the passenger train exceeds the rate of the freight train by 15 miles per hour, and they meet after 4 hours, what are the rates of the passenger and train?

Exercise 2.4.1.36

Two automobiles started at the same time from a point, but traveled in opposite directions. Their rates were 25 and 35 miles per hour, respectively. After how many hours were they 180 miles apart?

Exercise 2.4.1.37

A man having ten hours at his disposal made an excursion, riding out at the rate of 10 miles an hour and returning on foot at the rate of 3 miles an hour. Find the distance he rode.

Exercise 2.4.1.38

A man walks at the rate of 4 miles per hour. How far can he walk into the country and ride back on a trolley that travels at the rate of 20 miles per hour if he must be back home 3 hours from the time he started?

Exercise 2.4.1.39

A boy rides away from home in an automobile at the rate of 28 miles an hour and walks back at the rate of 4 miles an hour. The round trip requires 2 hours. How far does he ride in the automobile?

Exercise 2.4.1.40

A motorboat leaves a harbor and travels at an average speed of 15 mph toward an island. The average speed on the return trip was 10 mph. How far was the island from the harbor if the total trip took 5 hours?

Exercise 2.4.1.41

A family drove to a resort at an average speed of 30 mph and later returned over the same road at an average speed of 50 mph. Find the distance to the resort if the total driving time was 8 hours.

Exercise 2.4.1.42

As part of his flight training, a student pilot was required to fly to an airport and then return. The average speed to the airport was 90 mph, and the average speed returning was 120 mph. Find the distance between the two airports if the total flying time was 7 hours.

Exercise 2.4.1.43

Annie, who travels 4 miles an hour starts from a certain place 2 hours in advance of Brandie, who travels 5 miles an hour in the same direction. How many hours must Brandie travel to overtake Annie?

Exercise 2.4.1.44

A man travels 5 miles an hour. After traveling for 6 hours another man starts at the same place following the first man at the rate of 8 miles an hour. When will the second man overtake the first man?

Exercise 2.4.1.45

A motorboat leaves a harbor and travels at an average speed of 8 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 16 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

Exercise 2.4.1.46

A long distance runner started on a course running at an average speed of 6 mph. One hour later, a second runner began the same course at an average speed of 8 mph. How long after the second runner started will the second runner overtake the first runner?

Exercise 2.4.1.47

A car traveling at 48 mph overtakes a cyclist who, riding at 12 mph, has had a 3-hour head start. How far from the starting point does the car overtake the cyclist?

Exercise 2.4.1.48

A jet plane traveling at 600 mph overtakes a propeller-driven plane which has had a 2-hour head start. The propeller-driven plane is traveling at 200 mph. How far from the starting point does the jet overtake the propeller-driven plane?

Exercise 2.4.1.49

Two men are traveling in opposite directions at the rate of 20 and 30 miles per hour at the same time and from the same place. In how many hours will they be 300 miles apart?

Exercise 2.4.1.50

Running at an average rate of 8 meters per second, a sprinter ran to the end of a track and then jogged back to the starting point at an average rate of 3 meters per second. The sprinter took 55 seconds to run to the end of the track and jog back. Find the length of the track.

Exercise 2.4.1.51

A motorboat leaves a harbor and travels at an average speed of 18 mph to an island. The average speed on the return trip was 12 mph. How far was the island from the harbor if the total trip took 5 hours?

Exercise 2.4.1.52

A motorboat leaves a harbor and travels at an average speed of 9 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 18 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

Exercise 2.4.1.53

A jet plane traveling at 570 mph overtakes a propeller-driven plane that has had a 2-hour head start. The propeller-driven plane is traveling at 190 mph. How far from the starting point does the jet overtake the propeller-driven plane?

Exercise 2.4.1.54

Two trains start at the same time from the same place and travel in opposite directions. If the rate of one is 6 miles per hour more than the rate of the other and they are 168 miles apart at the end of 4 hours, what is each rate?

Exercise 2.4.1.55

As part of flight training, a student pilot was required to fly to an airport and then return. The average speed on the way to the airport was 100 mph, and the average speed returning was 150 mph. Find the distance between the two airports if the total flight time was 5 hours.

Exercise 2.4.1.56

Two cyclists start from the same point and ride in opposite directions. One cyclist rides twice as fast as the other. In three hours they are 72 miles apart. Find the rate of each cyclist.

Exercise 2.4.1.57

A car traveling at 56 mph overtakes a cyclist who, riding at 14 mph, has had a 3-hour head start. How far from the starting point does the car overtake the cyclist?

Exercise 2.4.1.58

Two small planes start from the same point and fly in opposite directions. The first plane is flying 25 mph slower than the second plane. In two hours, the planes are 430 miles apart. Find the rate of each plane.

Exercise 2.4.1.59

A bus traveling at a rate of 60 mph overtakes a car traveling at a rate of 45 mph. If the car had a 1-hour head start, how far from the starting point does the bus overtake the car?

Exercise 2.4.1.60

Two small planes start from the same point and fly in opposite directions. The first plane is flying 25 mph slower than the second plane. In 2 hours, the planes are 470 mi apart. Find the rate of each plane.

Exercise 2.4.1.61

A truck leaves a depot at 11 a.m. and travels at a speed of 45 mph. At noon, a van leaves the same place and travels the same route at a speed of 65 mph. At what time does the van overtake the truck?

Exercise 2.4.1.62

A family drove to a resort at an average speed of 25 mph and later returned over the same road at an average speed of 40 mph. Find the distance to the resort if the total driving time was 13 hours.

Exercise 2.4.1.63

Three campers left their campsite by canoe and paddled downstream at an average rate of 10 mph. They then turned around and paddled back upstream at an average rate of 5 mph to return to their campsite. How long did it take the campers to canoe downstream if the total trip took 1 hour?

Exercise 2.4.1.64

A motorcycle breaks down and the rider has to walk the rest of the way to work. The motorcycle was being driven at 45 mph, and the rider walks at a speed of 6 mph. The distance from home to work is 25 miles, and the total time for the trip was 2 hours. How far did the motorcycle go before it broke down?

Exercise 2.4.1.65

A student walks and jogs to college each day. The student averages 5 kilometers per hour walking and 9 kilometers per hour jogging. The distance from home to college is 8 kilometers, and the student makes the trip in one hour. How far does the student jog?

Exercise 2.4.1.66

On a 130-mile trip, a car traveled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took a total of 2.5 hours. For how long did the car travel at 40 mph?

Exercise 2.4.1.67

On a 220-mile trip, a car traveled at an average speed of 50 mph and then reduced its average speed to 35 mph for the remainder of the trip. The trip took a total of 5 hours. How long did the car travel at each speed?

Exercise 2.4.1.68

An executive drove from home at an average speed of 40 mph to an airport where a helicopter was waiting. The executive boarded the helicopter and flew to the corporate offices at and average speed of 60 mph. The entire distance was 150 miles. The entire trip took 3 hours. Find the distance from the airport to the corporate offices.

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CHAPTER OVERVIEW

3: Graphing Linear Equations

Learning Objectives

By the end of this chapter, the student should be able to

- Find the slope of a line from two points or a graph
- Find the equation of a line from its graph, the standard form, two given points
- Obtain equations of parallel and perpendicular lines

3.1: Graphing and Slope

3.1.1: Graphing and Slope (Exercises)

3.2: Equations of Lines

3.2.1: Equations of Lines (Exercises)

3.3: Parallel and Perpendicular Lines

3.3.1: Parallel and Perpendicular Lines (Exercises)

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3.1: Graphing and Slope

Often, to get an idea of the behavior of an equation, we will make a picture that represents the solutions to the equations called a *graph*. Let's have an introduction to graphing so that later we can construct graphs.

Definition: Graph

A ***graph*** represents a visualization of the solutions to an equation.

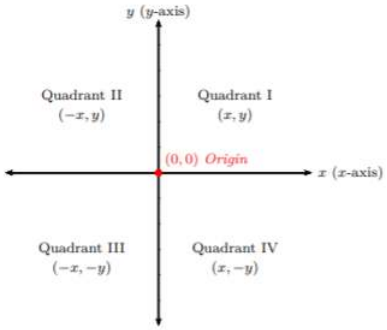


Figure 3.1.1: Coordinate plane

Coordinate Plane

- The ***coordinate plane*** is divided into four quadrants by a horizontal real number line, called the x-axis, and a vertical real number line, called the y-axis.
- The two axes meet at the ***origin***, where $x = 0$ and $y = 0$.
- The ***x-coordinate*** is the horizontal distance from the origin and the ***y-coordinate*** is the vertical distance from the origin.
- An ***ordered-pair*** is where a point in the coordinate plane is located and denoted by (x, y).
- The quadrants are labeled in counter-clockwise starting at the top right.

Note

Locations on the globe are similar to ordered-pairs. Each number is a distance from a central point, the origin, located where the prime meridian and the equator meet. This "origin" is just off the western coast of Africa.

Points and Lines

Example 3.1.1

Plot each ordered-pair and identify the quadrant in which lies the ordered pair:

$$A(-1, -5),\ B(3, -1),\ C(-2, 3),\ D(4, 2),\ E(0, 4),\ F(3, 0)$$

Solution

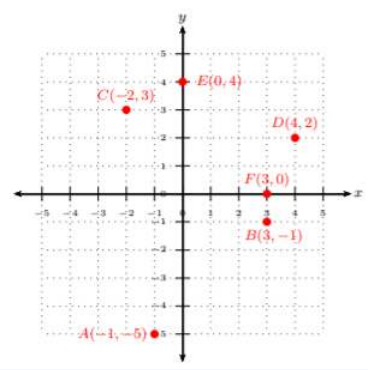


Figure 3.1.2

- For point $A(-1,-5)$, notice the x-coordinate is -1. Since the x-coordinate is the horizontal distance from the origin, then we move 1 unit to the left. Looking at the y-coordinate, -5, we see this will be the vertical distance. Hence, we will move 5 units downward from the origin. Starting at the origin, move one unit left, then 5 units down. Point A is in quadrant III.
- For point $B(3,-1)$, notice the x-coordinate is 3. Since the x-coordinate is the horizontal distance from the origin, then we move 3 units to the right. Looking at the y-coordinate, -1, we see this will be the vertical distance. Hence, we will move one unit downward from the origin. Starting at the origin, move 3 units right, then 1 unit down. Point B is in quadrant IV.
- For point $C(-2,3)$, notice the x-coordinate is -2. Since the x-coordinate is the horizontal distance from the origin, then we move 2 units to the left. Looking at the y-coordinate, 3, we see this will be the vertical distance. Hence, we will move 3 units upward from the origin. Starting at the origin, move 2 units left, then 3 units up. Point C is in quadrant III.
- For point $D(4,2)$, notice the x-coordinate is 4. Since the x-coordinate is the horizontal distance from the origin, then we move 4 units to the right. Looking at the y-coordinate, 2, we see this will be the vertical distance. Hence, we will move 2 units upward from the origin. Starting at the origin, move 4 units right, then 2 units up. Point D is in quadrant I.
- For point $E(0,4)$, notice the x-coordinate is 0. Since the x-coordinate is the horizontal distance from the origin, then we move no units horizontally from the origin. Looking at the y-coordinate, 4, we see this will be the vertical distance. Hence, we will move 4 units upward from the origin. Starting at the origin, move 4 units up. Point E is not in any quadrant as it lies on the y-axis.
- For point $F(3,0)$, notice the x-coordinate is 3. Since the x-coordinate is the horizontal distance from the origin, then we move 3 units to the right. Looking at the y-coordinate, 0, we see this will be the vertical distance. Hence, we will move no units vertically from the origin. Starting at the origin, move 3 units right. Point F is not in any quadrant as it lies on the x-axis.

Note

Notice, in points A, B, C, the negative coordinates didn't imply negative distance from the origin. The *negative on these coordinates implies the direction* in which we move: horizontal- we move left or right, vertical- we move up or down. If the x-coordinate is negative, then we move to the left. If the y-coordinate is negative, then we move downward.

Definition: Intercepts

From Example 3.1.1, with points E and F, we could see that these points did not lie in a quadrant, but on an axis. These are special points on graphs and are called ***intercepts***.

- The ***x-intercept*** of a graph is the point(s) where the graph crosses the x-axis, i.e., $y=0$.
- The ***y-intercept*** of a graph is the point(s) where the graph crosses the y-axis, i.e., $x=0$.

Example 3.1.2

Graph $y=1-x$ by plotting the intercepts.

Solution

To find the x and y-intercepts, we can follow the definition above and find where $y = 0$ and $x = 0$, respectively. Let's make a table.

Table 3.1.1

Finding intercepts for a line

x	$y = 1 - x$	(x, y)
$0 = 1 - x \implies x = 1$	0	$(1, 0)$
0	$y = 1 - 0 = 1$	$(0, 1)$

We can see when $y = 0,\ x = 1$ since $0 = 1 - x$ only when $x = 1$. Let's plot the two intercepts from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Draw the line to fill the grid and put arrows at the ends. It is recommended to purchase a small 6-inch ruler to make nice straight lines.

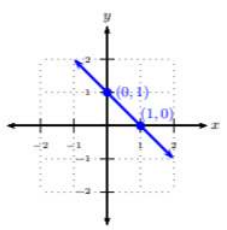


Figure 3.1.3

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$ and be interested in the type of solutions that are possible for this equation. We can visualize the solution by making a graph of possible x and y combinations that makes this equation a true statement. We have to start by finding possible x and y combinations. We do this by using a table of values.

Example 3.1.3

Graph $y = 2x - 3$ by plotting points, i.e., by making a T-table.

Solution

Usually, we pick three x-coordinates, and find corresponding y-values. Each x-value being positive, negative, and zero. This is common practice, but not required.

Table 3.1.2

Finding three points on a line

x	$y = 2x - 3$	(x, y)
-1	$y = 2(-1) - 3 = -2 - 3 = -5$	$(-1, -5)$
0	$y = 2(0) - 3 = 0 - 3 = -3$	$(0, -3)$
1	$y = 2(1) - 3 = 2 - 3 = -1$	$(1, -1)$

Plot the three ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Draw the line to fill the grid and put arrows at the ends. It is recommended to purchase a small 6-inch ruler to make nice straight lines.

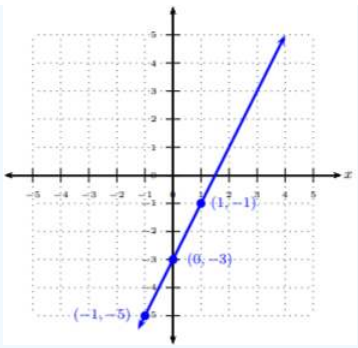


Figure 3.1.4

Example 3.1.4

Graph $2x - 3y = 6$ by plotting points, i.e., by making a T-table.

Solution

Let's begin by choosing x-values for the table. Notice this equation isn't as simple as the prior example, so we will have to do a bit of algebra to solve for the y-value. Then fill in the table.

Table 3.1.3

Values of x to begin plotting a line

x	y	(x, y)
-3		
0		
3		

Let's evaluate $2x - 3y = 6$ for each of the chosen x-values:

$x = -3$:

$$\begin{aligned} 2(-3) - 3y &= 6 \\ -6 - 3y + 6 &= 6 + 6 \\ -\frac{1}{3} \cdot -3y &= 12 \cdot -\frac{1}{3} \\ y &= -4 \end{aligned}$$

$x = 0$:

$$\begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -\frac{1}{3} \cdot -3y &= 6 \cdot -\frac{1}{3} \\ y &= -2 \end{aligned}$$

$x = 3$:

$$\begin{aligned} 2(3) - 3y &= 6 \\ 6 - 3y + (-6) &= 6 + (-6) \\ -\frac{1}{3} \cdot -3y &= 0 \cdot -\frac{1}{3} \\ y &= 0 \end{aligned}$$

Now, we fill in the table with y-values and ordered-pairs, and then graph $2x - 3y = 6$.

Table 3.1.4

Calculating y values from x to plot a line

x	y	(x, y)
-3	-4	$(-3, -4)$
0	-2	$(0, -2)$
3	0	$(3, 0)$

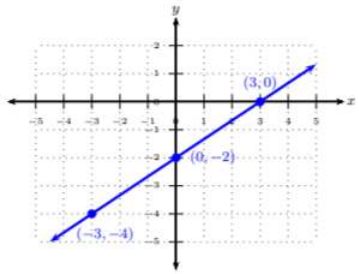


Figure 3.1.5

Obtaining the Slope of a Line from its Graph

As we graph lines, we want to identify different properties of lines. One of the most important properties of a line is its slope.

Definition: Slope

The ***slope*** of a line is the measure of the line's steepness.

- We denote slope with m. One theory from mathematicians that began working with slope was that it was called the *modular slope*.
- As $|m|$ increases, the line becomes steeper. As $|m|$ decreases, the line becomes flatter.
- A line that rises left to right has a positive slope and and a line that falls left to right has negative slope.
- m is the change of y divided by the change in x, i.e.,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{rise}{run}$$

Example 3.1.5

Find the slope from the graph given of the line.

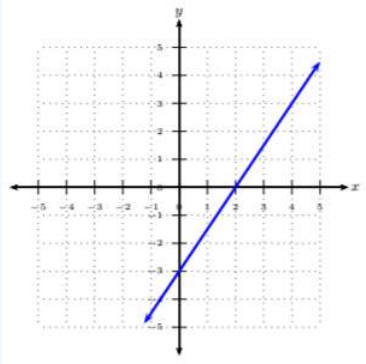


Figure 3.1.6

Solution

We start at a well-defined point, preferably a point on the y-axis, i.e., the y-intercept. Then count the number of units we rise (up/down) and run (left/right) to reach the next well-defined point. We will start at $(0, -3)$ and reach the next point, $(2, 0)$.

Notice we rise upward 3 units and run to the right 2 units. Using the ratio, $m = \frac{rise}{run}$, we get the rise to be 3 and the run to be 2:

$$m = \frac{rise}{run} = \frac{3}{2}$$

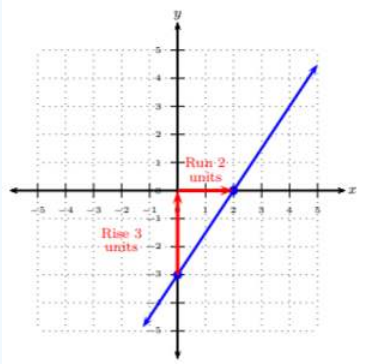


Figure 3.1.7

Thus, the slope is $\frac{3}{2}$.

Example 3.1.6

Find the slope from the graph given of the line.

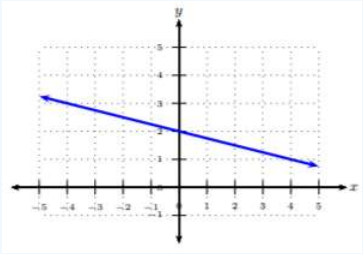


Figure 3.1.8

Solution

We start at a well-defined point, preferably a point on the y-axis, i.e., the y-intercept. Then count the number of units we rise (up/down) and run (left/right) to reach the next well-defined point. We will start at $(0, 2)$ and reach the next point, $(4, 1)$. Notice we rise downward 1 units and run to the right 4 units. Using the ratio, $m = \frac{rise}{run}$, we get the rise to be -1 and the run to be 4:

$$m = \frac{rise}{run} = \frac{-1}{4}$$

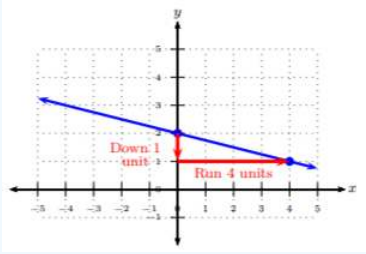


Figure 3.1.9

Thus, the slope is $-\frac{1}{4}$.

Note

Looking at Examples 3.1.5 and 3.1.6, notice the steepness. Since, in Example 3.1.5, the slope was $\frac{3}{2}$, which is larger than the slope in 3.1.6, Example 3.1.5 is a steeper line than Example 3.1.6. Also, the negative in Example 3.1.6 represents the line falling left to right; hence the negative slope.

Note

When French mathematicians *Rene Descartes* and *Pierre de Fermat* first developed the coordinate plane and the idea of graphing lines (and other functions), the y-axis was not a vertical line.

Let's look at two special cases with lines and their slope.

Example 3.1.7

Find the slope from the graph given of the line.

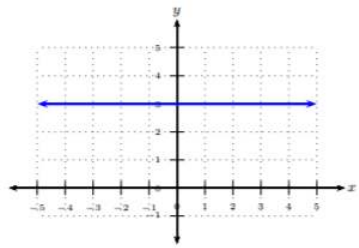


Figure 3.1.10

Solution

In this graph, there is no rise, but the run is 3 units. The slope is $\frac{0}{3} = 0$. When the slope of a line is zero, then we know the line is a horizontal line and vice versa.

Example 3.1.8

Find the slope from the graph given of the line.

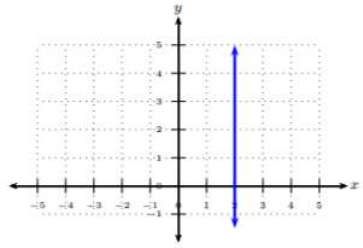


Figure 3.1.11

Solution

In this graph, there is no run, but the rise is 2 units. The slope is $\frac{2}{0} =$ undefined. When the slope of a line is undefined, then we know the line is a vertical line and vise versa.

Note

As you can see there is a big difference between having a zero slope and having undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all. In fact, it is flat. Therefore, it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express the steepness. Hence, being an undefined slope.

Obtaining the Slope of a Line from Two Points

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y-values, we can calculate this by subtracting the y-values of a point. Similarly, if run is a change in the x-values, we can calculate this by subtracting the x-values of a point.

Definition: Slope

Slope, m, is the change of y divided by the change in x, i.e.,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Example 3.1.9

Find the slope between the two points $(-4, 3)$ and $(2, -9)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2 - y_1}{x_2 - x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{-9-3}{2-(-4)} && \text{Simplify} \\ m &= \tfrac{-12}{6} && \text{Reduce} \\ m &= -2 && \text{Slope} \end{aligned}$$

Since this slope is -2, then the graph of this line would be falling left to right.

Example 3.1.10

Find the slope between the two points $(4, 6)$ and $(2, -1)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2 - y_1}{x_2 - x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{-1-6}{2-4} && \text{Simplify} \\ m &= \tfrac{-7}{-2} && \text{Reduce, dividing by } -1 \\ m &= \tfrac{7}{2} && \text{Slope} \end{aligned}$$

Since this slope is $\frac{7}{2}$, then the graph of this line would be rising left to right.

Example 3.1.11

Find the slope between the two points $(-4, -1)$ and $(-4, -5)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2 - y_1}{x_2 - x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{-5-(-1)}{-4-(-4)} && \text{Simplify} \\ m &= \tfrac{-4}{0} && \text{Undefined} \\ m &= \text{ undefined} && \text{Slope} \end{aligned}$$

Since the slope is undefined, then the graph of this line is a vertical line.

Example 3.1.12

Find the slope between the two points $(3,1)$ and $(-2,1)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{1-1}{-2-3} && \text{Simplify} \\ m &= \tfrac{0}{-5} && \text{Reduce} \\ m &= 0 && \text{Slope} \end{aligned}$$

Since the slope is zero, then the graph of this line is a horizontal line.

Example 3.1.13

Find the value of y between the points $(2,y)$ and $(5,-1)$ with slope -3.

Solution

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{We will plug values into slope formula} \\ -3 &= \tfrac{-1-y}{5-2} && \text{Simplify} \\ -3 &= \tfrac{-1-y}{3} && \text{Multiply both sides by 3} \\ 3\cdot -3 &= \tfrac{-1-y}{3}\cdot 3 && \text{Simplify} \\ -9 &= -1-y && \text{Isolate the variable term} \\ -9+1 &= -1-y+1 && \text{Simplify} \\ -8 &= -y && \text{Multiply each side by } -1 \\ -1\cdot -8 &= -y\cdot -1 && \\ 8 &= y && \text{Value of } y \end{aligned}$$

Example 3.1.14

Find the value of x such that the slope between the points $(-3,2)$ and $(x,6)$ is $\frac{2}{5}$.

Solution

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{We will plug values into slope formula} \\ \tfrac{2}{5} &= \tfrac{6-2}{x-(-3)} && \text{Simplify} \\ \tfrac{2}{5} &= \tfrac{4}{x+3} && \text{Multiply both sides by } x+3 \\ (x+3)\cdot\tfrac{2}{5} &= \tfrac{4}{x+3}\cdot(x+3) && \text{Simplify} \\ \tfrac{2}{5}(x+3) &= 4 && \text{Distribute} \\ \tfrac{2}{5}x+\tfrac{6}{5} &= 4 && \text{Multiply by the LCD } = 5 \\ 5\cdot\tfrac{2}{5}x+5\cdot\tfrac{6}{5} &= 4\cdot 5 && \text{Simplify} \\ 2x+6 &= 20 && \text{Isolate the variable term} \\ 2x+6+(-6) &= 20+(-6) && \text{Simplify} \\ 2x &= 14 && \text{Multiply each side by the reciprocal of 2} \\ \tfrac{1}{2}\cdot 2x &= 14\cdot\tfrac{1}{2} && \text{Simplify} \\ x &= 7 && \text{Value of } x \end{aligned}$$

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3.1.1: Graphing and Slope (Exercises)

Graphing and Slope Homework

Exercise 3.1.1.1

Find the slope of the line.

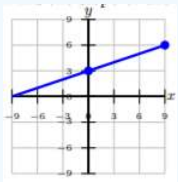


Figure 3.1.1.1

Exercise 3.1.1.2

Find the slope of the line.

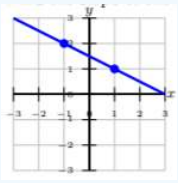


Figure 3.1.1.2

Exercise 3.1.1.3

Find the slope of the line.

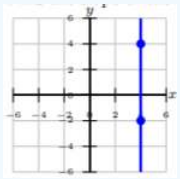


Figure 3.1.1.3

Exercise 3.1.1.4

Find the slope of the line.

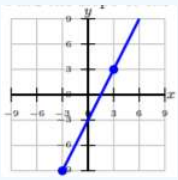


Figure 3.1.1.4

Exercise 3.1.1.5

Find the slope of the line.

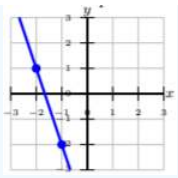


Figure 3.1.1.5

Exercise 3.1.1.6

Find the slope of the line.

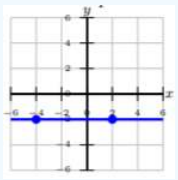


Figure 3.1.1.6

Find the slope of the line through each pair of points.

Exercise 3.1.1.7

$(-2, 10),\ (-2, -15)$

Exercise 3.1.1.8

$(-15, 10),\ (16, -7)$

Exercise 3.1.1.9

$(10, 18),\ (-11, -10)$

Exercise 3.1.1.10

$(-16, -14),\ (11, -14)$

Exercise 3.1.1.11

$(-4, 14),\ (-16, 8)$

Exercise 3.1.1.12

$(12, -19),\ (6, 14)$

Exercise 3.1.1.13

$(-5, -10),\ (-5, 20)$

Exercise 3.1.1.14

$(-17, 19),\ (10, -7)$

Exercise 3.1.1.15

$(7, -14),\ (-8, -9)$

Exercise 3.1.1.16

$(-5, 7),\ (-18, 14)$

Exercise 3.1.1.17

$(1, 2),\ (-6, -14)$

Exercise 3.1.1.18

$(13, -2),\ (7, 7)$

Exercise 3.1.1.19

$(-3, 6),\ (-20, 13)$

Exercise 3.1.1.20

$(13, 15),\ (2, 10)$

Exercise 3.1.1.21

$(9, -6),\ (-7, -7)$

Exercise 3.1.1.22

$(-16, 2),\ (15, -10)$

Exercise 3.1.1.23

$(8, 11),\ (-3, -13)$

Exercise 3.1.1.24

$(11, -2),\ (1, 17)$

Exercise 3.1.1.25

$(-18, -5),\ (14, -3)$

Exercise 3.1.1.26

$(19, 15),\ (5, 11)$

Find the value of x or y so that the line through the points has the given slope.

Exercise 3.1.1.27

$(2, 6)$ and $(x, 2)$; $m = \frac{4}{7}$

Exercise 3.1.1.28

$(-3, -2)$ and $(x, 6)$; $m = -\frac{8}{5}$

Exercise 3.1.1.29

$(-8, y)$ and $(-1, 1)$; $m = \frac{6}{7}$

Exercise 3.1.1.30

$(x, -7)$ and $(-9, -9)$; $m = \frac{2}{5}$

Exercise 3.1.1.31

$(x, 5)$ and $(8, 0)$; $m = -\frac{5}{6}$

Exercise 3.1.1.32

$(8, y)$ and $(-2, 4)$; $m = -\frac{1}{5}$

Exercise 3.1.1.33

$(-2, y)$ and $(2, 4)$; $m = \frac{1}{4}$

Exercise 3.1.1.34

$(x, -1)$ and $(-4, 6)$; $m = -\frac{7}{10}$

Exercise 3.1.1.35

$(2, -5)$ and $(3, y)$; $m = 6$

Exercise 3.1.1.36

$(6, 2)$ and $(x, 6)$; $m = -\frac{4}{5}$

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3.2: Equations of Lines

In this section, we discuss applying formulas to obtain equations of lines, graphing lines, and special cases. If we can identify some properties of the line, we may be able to graph the line much quicker and easier.

The Slope-Intercept Formula

One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m, and the y-intercept can be represented by $(0, b)$, where b is the y-value at which the graph crosses the y-axis. Any other point on the line can be represented by (x, y).

Example 3.2.1

Prove the slope-intercept formula by using the slope formula for the line that has slope m, y-intercept $(0, b)$, and passes through the point (x, y).

Solution

$$\begin{aligned} m &= \tfrac{y-b}{x-0} && \text{Simplify} \\ m &= \tfrac{y-b}{x} && \text{Multiply both sides by } x \\ mx &= y-b && \text{Add } b \text{ to both sides} \\ mx+b &= y \\ y &= mx+b && \text{Slope-intercept formula} \end{aligned}$$

Slope-Intercept Formula

The ***slope-intercept formula*** of a linear equation is given by $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.

Example 3.2.2

Find the equation of the line with slope $\frac{3}{4}$ and y-intercept $(0, -3)$.

Solution

$$\begin{aligned} y &= mx+b && m \text{ is the slope, } b \text{ is the } y\text{-intercept} \\ y &= \tfrac{3}{4}x-3 && \text{Equation of the line} \end{aligned}$$

Example 3.2.3

Find the equation of the line.

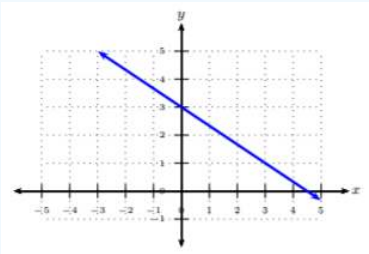


Figure 3.2.1

Solution

The y-intercept is where the graph crosses the y-axis. We can see, from the graph, that the line crosses the y-axis at $(0, 3)$. Hence, $b = 3$. To find the slope, we count the rise and run units. We can see that we rise downward 2 units and run to the right 3 units. Hence, the slope is $-\frac{2}{3}$. Now we can put the equation of the line together in slope-intercept form, where $m = -\frac{2}{3}$ and $b = 3$:

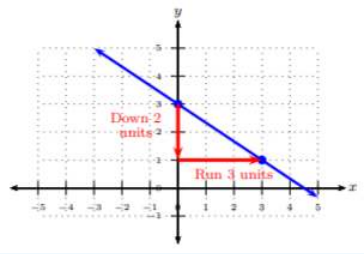


Figure 3.2.2

$$y = -\frac{2}{3}x + 3$$

Lines in Slope-Intercept Form

We can also identify the slope and y-intercept, and graph the equation from a given equation. However, we need to be sure the equation is in slope-intercept form. If it is not, we will have to rewrite the equation in slope-intercept form, i.e., solve the equation for y. Then we can easily identify the slope and the y-intercept.

Example 3.2.4

Write the equation $2x - 4y = 6$ in slope-intercept form. Find the slope and y-intercept of the line.

Solution

$$\begin{aligned} 2x - 4y &= 6 && \text{Isolate the variable term } -4y \\ 2x - 4y + (-2x) &= 6 + (-2x) && \text{Simplify} \\ -4y &= 6 - 2x && \text{Multiply by the reciprocal of } -4 \\ -\tfrac{1}{4} \cdot -4y &= -\tfrac{1}{4} \cdot 6 - 2x \cdot -\tfrac{1}{4} && \text{Simplify} \\ y &= \tfrac{1}{2}x - \tfrac{3}{2} && \text{Slope-intercept form} \end{aligned}$$

Next, we identify the slope and the y-intercept. If we line up the general slope-intercept form with the equation we obtained, we can easily see m and b:

$$\begin{aligned} y &= mx + b \\ y &= \tfrac{1}{2}x - \tfrac{3}{2} \end{aligned}$$

Hence, $m = \frac{1}{2}$ and $b = -\frac{3}{2}$, i.e., the slope is $\frac{1}{2}$ and the y-intercept is $-\frac{3}{2}$.

Graphing Lines

Once we have an equation in slope-intercept form, we can graph it by first plotting the y-intercept, then applying the slope to find a second point and even a third point. We connect these points to make a line. Let's look at Example 3.2.4 and graph the line.

Example 3.2.5

Graph $y = \frac{1}{2}x - \frac{3}{2}$ by using the slope and y-intercept.

Solution

The y-intercept, or b, is where the graph crosses the y-axis. We know from Example 3.2.4 that the y-intercept is $-\frac{3}{2}$ and the line will cross the y-axis at $\left(0, -\frac{3}{2}\right)$. The slope is $\frac{1}{2}$, and, using $\frac{rise}{run}$, we need to rise upward 1 unit and run to the right 2 units to reach the next point. We continue the pattern to obtain a third point. Now we can connect the dots and create a well-defined line. Be sure to draw it to fill the grid.

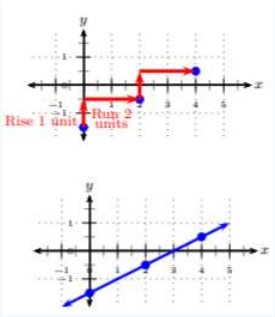


Figure 3.2.3

Note

When we graphed a line by plotting points from a table, we obtained three points. Hence, in Example 3.2.5, we obtained three points and then drew a line. Obtaining three points on a line is common practice and will help when drawing any line, even the special cases.

Note

Before our current system of graphing, French mathematician, *Nicole Oresme*, in 1323, suggested graphing lines that would look more like bar graphs with a constant slope.

Example 3.2.6

Write the equation $3x+4y=12$ in slope-intercept form. Find the slope and y-intercept of the line and then graph the line.

Solution

$$\begin{aligned} 3x+4y &= 12 && \text{Isolate the variable term } 4y \\ 3x+4y+(-3x) &= 12+(-3x) && \text{Simplify} \\ 4y &= 12-3x && \text{Multiply by the reciprocal of } 4 \\ \tfrac{1}{4}\cdot 4y &= \tfrac{1}{4}\cdot 12-3x\cdot\tfrac{1}{4} && \text{Simplify} \\ y &= -\tfrac{3}{4}x+3 && \text{Slope-intercept form} \end{aligned}$$

No we can graph the line. We see the y-intercept is 3 and the line will cross the y-axis at $(0,3)$. The slope is $-\frac{3}{4}$, and, using $\frac{rise}{run}$, we need to rise downward 3 units and run to the right 4 units to reach the next point. We continue the pattern to obtain a third point. Now we can connect the dots and create a well-defined line. Be sure to draw it to fill the grid.

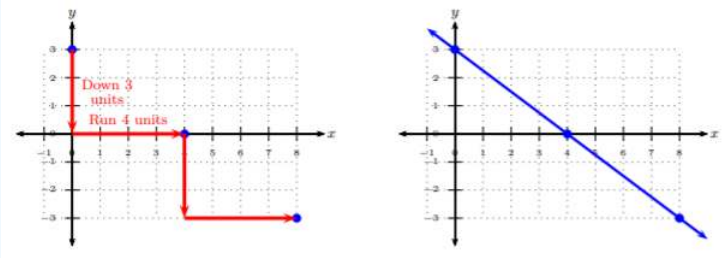


Figure 3.2.4

Vertical and Horizontal Lines

Lines with zero or undefined slope can make a problem seem very different. Zero slope, or a horizontal line, will simply have a slope of zero. So, the equation simply becomes $y=b$ or y equal to the y-coordinate of the graph. If we have undefined slope, or a vertical line, the equation can't be written in slope-intercept form because the slope is undefined. Hence, there is no y in these equations. We will simply make x equal to the x-coordinate of the graph.

Example 3.2.7

Graph the line $x = -4$.

Solution

Since $x = -4$ is a vertical line, then we know this line has no slope and the line is at its steepest. Every x-coordinate on this line is -4 and the line has no run. We can graph this line easily by plotting three points where the x-coordinate is -4. Let's plot $(-4, -1)$, $(-4, 0)$, and $(-4, 2)$; then connect the points with a well-defined line.

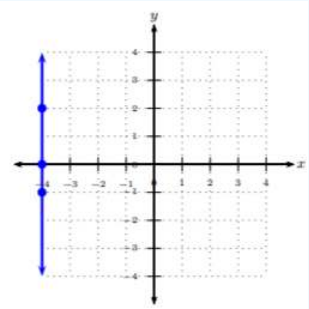


Figure 3.2.5

Example 3.2.8

Graph the line $y = 1$.

Solution

Since $y = 1$ is a horizontal line, then we know this line has zero slope and the line is at its flattest. Every y-coordinate on this line is 1 and the line has no rise. We can graph this line easily by plotting three points where the y-coordinate is 1. Let's plot $(3, 1)$, $(0, 1)$, and $(-2, 1)$; then connect the points with a well-defined line.

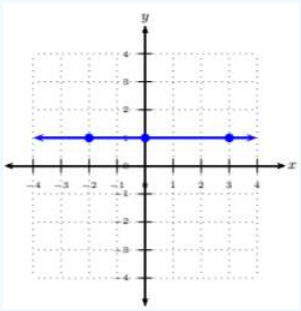


Figure 3.2.6

Point-Slope Formula

The slope-intercept form has the advantage of being simple to remember and use. However, it has one major disadvantage: we must know the y-intercept in order to graph the line. Generally, we do not know the y-intercept, but, usually, know one or more points on the line that are not the y-intercept. In these cases, we can't use the slope-intercept equation, so we will need a more general formula to assist us in graphing lines. If the slope of a line is m, and point (x_1, y_1) be a particular point on the line, and any other point on the line be (x, y), then we can use this to find this general formula.

$$\begin{aligned} m,\ (x_1, y_1),\ (x, y) &\quad \text{Recall slope formula} \\ \tfrac{y_2 - y_1}{x_2 - x_1} = m &\quad \text{Plug in values} \\ \tfrac{y - y_1}{x - x_1} = m &\quad \text{Multiply both sides by } (x - x_1) \\ y - y_1 = m(x - x_1) &\quad \text{New formula} \end{aligned}$$

Point-Slope Formula

The ***point-slope formula*** is given by

$$y - y_1 = m(x - x_1),$$

given the slope m and point (x_1, y_1) is on the line.

Example 3.2.9

Using the point-slope formula, write the equation of the line passing through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

Solution

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Plug values into point-slope formula} \\ y - (-4) &= \tfrac{3}{5}(x - 3) && \text{Simplify signs} \\ y + 4 &= \tfrac{3}{5}(x - 3) && \text{Equation in point-slope form} \end{aligned}$$

Note

Often, we will prefer final answers be written in slope-intercept form. If the directions prefer the equations of the line in slope-intercept form, we can distribute the slope, then solve for y.

Example 3.2.10

Let's rewrite Example 3.2.9 in slope-intercept form: $y + 4 = \frac{3}{5}(x - 3)$

Solution

$$\begin{aligned} y + 4 &= \tfrac{3}{5}(x - 3) && \text{Distribute} \\ y + 4 &= \tfrac{3}{5}x - \tfrac{9}{5} && \text{Isolate the variable term } y \\ y + 4 + (-4) &= \tfrac{3}{5}x - \tfrac{9}{5} + (-4) && \text{Simplify} \\ y &= \tfrac{3}{5}x - \tfrac{11}{5} && \text{Slope-intercept form} \end{aligned}$$

Example 3.2.11

Write the equation of the line passing through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

Solution

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute values into the point-slope formula} \\ y - 2 &= -\tfrac{2}{3}(x - (-6)) && \text{Simplify} \\ y - 2 &= -\tfrac{2}{3}(x + 6) && \text{Distribute} \\ y - 2 &= -\tfrac{2}{3}x - 4 && \text{Isolate the variable term } y \\ y - 2 + 2 &= -\tfrac{2}{3}x - 4 + 2 && \text{Simplify} \\ y &= -\tfrac{2}{3}x - 2 && \text{Slope-intercept form} \end{aligned}$$

Obtaining a Line Given Two Points

In order to find the equation of a line, we need to know the slope. If we aren't given the slope, but only two points on the line, then we complete some preliminary work to obtain the slope. Then we can use the point-slope formula as usual to obtain the equation of the line.

Example 3.2.12

Find the equation of the line passing through the points $(-3, 4)$ and $(-1, -2)$ in slope-intercept form

Solution

Since we are given two points, we can use the slope formula to obtain the slope:

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{-2-4}{-1-(-3)} && \text{Simplify} \\ m &= \tfrac{-6}{2} \\ m &= -3 && \text{Slope} \end{aligned}$$

Now that we have the slope, we can plug-n-chug the slope and one of the points into the pointslope formula. Notice we have two points and we can choose either one; the results will be the same. Let's choose $(-3, 4)$ with the slope $m = -3$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\ y - 4 &= -3(x - (-3)) && \text{Simplify} \\ y - 4 &= -3(x + 3) && \text{Distribute} \\ y - 4 &= -3x - 9 && \text{Isolate the variable term } y \\ y - 4 + 4 &= -3x - 9 + 4 && \text{Simplify} \\ y &= -3x - 5 && \text{Slope-intercept form} \end{aligned}$$

Example 3.2.13

Find the equation of the line through the points $(6, -2)$ and $(-4, 1)$ in slope-intercept form.

Solution

Since we are given two points, we can use the slope formula to obtain the slope:

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{1-(-2)}{-4-6} && \text{Simplify} \\ m &= -\tfrac{3}{10} && \text{Slope} \end{aligned}$$

Now that we have the slope, we can plug-n-chug the slope and one of the points into the pointslope formula. Notice we have two points and we can choose either one; the results will be the same. Let's choose $(-4, 1)$ with the slope $m = -\frac{3}{10}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\ y - 3 &= -\tfrac{3}{10}(x - (-4)) && \text{Simplify} \\ y - 1 &= -\tfrac{3}{10}(x + 4) && \text{Distribute} \\ y - 1 &= -\tfrac{3}{10}x - \tfrac{6}{5} && \text{Isolate the variable term } y \\ y - 1 + 1 &= -\tfrac{3}{10}x - \tfrac{6}{5} + 1 && \text{Simplify} \\ y &= -\tfrac{3}{10}x - \tfrac{1}{5} && \text{Slope-intercept form} \end{aligned}$$

Note

The city of *Konigsberg* (now Kaliningrad, Russia) had a river that flowed through the city breaking it into several parts. There were 7 bridges that connected the parts of the city. In 1735, Leonhard Euler considered the question of whether it was possible to cross each bridge exactly once and only once. It turned out that this problem was impossible, but the work laid the foundation of what would later become graph theory.

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3.2.1: Equations of Lines (Exercises)

Write the equation of the line in slope-intercept form given the slope and the y-intercept.

Exercise 3.2.1.1

$m = 2,\ y-intercept = 5$

Exercise 3.2.1.2

$m = 1,\ y-intercept = -4$

Exercise 3.2.1.3

$m = -\frac{3}{4},\ y-intercept = -1$

Exercise 3.2.1.4

$m = \frac{1}{3},\ y-intercept = 1$

Exercise 3.2.1.5

$m = -6,\ y-intercept = 4$

Exercise 3.2.1.6

$m = -1,\ y-intercept = -2$

Exercise 3.2.1.7

$m = -\frac{1}{4},\ y-intercept = 3$

Exercise 3.2.1.8

$m = \frac{2}{5},\ y-intercept = 5$

Write the equation of the line in slope-intercept form given the graph or equation.

Exercise 3.2.1.9

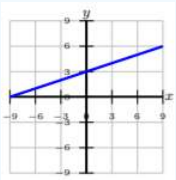


Figure 3.2.1.1

Exercise 3.2.1.10

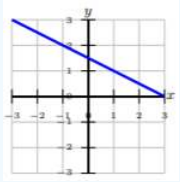


Figure 3.2.1.2

Exercise 3.2.1.11

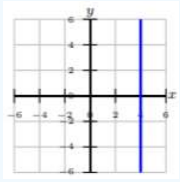


Figure 3.2.1.3

Exercise 3.2.1.12

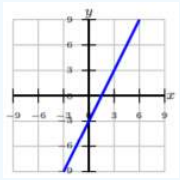


Figure 3.2.1.4

Exercise 3.2.1.13

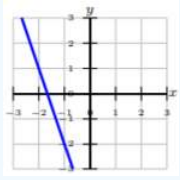


Figure 3.2.1.5

Exercise 3.2.1.14

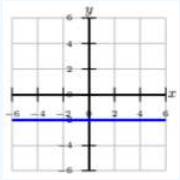


Figure 3.2.1.6

Exercise 3.2.1.15

$x+10y=-37$

Exercise 3.2.1.16

$2x+y=-1$

Exercise 3.2.1.17

$7x - 3y = 24$

Exercise 3.2.1.18

$x = -8$

Exercise 3.2.1.19

$y - 4 = -(x + 5)$

Exercise 3.2.1.20

$y - 4 = 4(x - 1)$

Exercise 3.2.1.21

$y + 5 = -4(x - 2)$

Exercise 3.2.1.22

$y + 1 = -\frac{1}{2}(x - 4)$

Exercise 3.2.1.23

$x - 10y = 3$

Exercise 3.2.1.24

$6x - 11y = -70$

Exercise 3.2.1.25

$4x + 7y = 28$

Exercise 3.2.1.26

$x - 7y = -42$

Exercise 3.2.1.27

$y - 5 = \frac{5}{2}(x - 2)$

Exercise 3.2.1.28

$y - 3 = -\frac{2}{3}(x + 3)$

Exercise 3.2.1.29

$0 = x - 4$

Exercise 3.2.1.30

$y+2=\frac{6}{5}(x+5)$

Sketch the graph of each line.

Exercise 3.2.1.31

$y=\frac{1}{3}x+4$

Exercise 3.2.1.32

$y=\frac{6}{5}x-5$

Exercise 3.2.1.33

$y=\frac{3}{2}x$

Exercise 3.2.1.34

$x-y+3=0$

Exercise 3.2.1.35

$-y-4+3x=0$

Exercise 3.2.1.36

$-3y=-5x+9$

Exercise 3.2.1.37

$y=-\frac{1}{5}x-4$

Exercise 3.2.1.38

$y=-\frac{3}{2}x-1$

Exercise 3.2.1.39

$y=-\frac{3}{4}x+1$

Exercise 3.2.1.40

$4x+5=5y$

Exercise 3.2.1.41

$-8=6x-2y$

Exercise 3.2.1.42

$-3y = 3 - \frac{3}{2}x$

Write the equation of the line in point-slope form given a point passing through the line and its slope.

Exercise 3.2.1.43

$(2, 3);\ m =$ undefined

Exercise 3.2.1.44

$(2, 2);\ m = \frac{1}{2}$

Exercise 3.2.1.45

$(-1, -5);\ m = 9$

Exercise 3.2.1.46

$(-4, 1);\ m = \frac{3}{4}$

Exercise 3.2.1.47

$(0, -2);\ m = -3$

Exercise 3.2.1.48

$(0, -5);\ m = -\frac{1}{4}$

Exercise 3.2.1.49

$(-5, -3);\ m = \frac{1}{5}$

Exercise 3.2.1.50

$(-1, 4);\ m = -\frac{5}{4}$

Exercise 3.2.1.51

$(1, 2);\ m = 0$

Exercise 3.2.1.52

$(2, 1);\ m = -\frac{1}{2}$

Exercise 3.2.1.53

$(2, -2);\ m = -2$

Exercise 3.2.1.54

$(4, -3);\ m = -2$

Exercise 3.2.1.55

$(-1, 1);\ m = 4$

Exercise 3.2.1.56

$(0, 2);\ m = -\frac{5}{4}$

Exercise 3.2.1.57

$(-1, -4);\ m = -\frac{2}{3}$

Write the equation of the line in slope-intercept form given a point passing through the line and its slope.

Exercise 3.2.1.58

$(-1, -5);\ m = 2$

Exercise 3.2.1.59

$(5, -1);\ m = -\frac{3}{5}$

Exercise 3.2.1.60

$(-4, 1);\ m = \frac{1}{2}$

Exercise 3.2.1.61

$(4, -2);\ m = -\frac{3}{2}$

Exercise 3.2.1.62

$(-5, -3);\ m = -\frac{2}{5}$

Exercise 3.2.1.63

$(2, -2);\ m = 1$

Exercise 3.2.1.64

$(-3, 4),\ m =$ undefined

Exercise 3.2.1.65

$(-4, 2);\ m = -\frac{1}{2}$

Exercise 3.2.1.66

$(2, -2);\ m = -2$

Exercise 3.2.1.67

$(-2, -2);\ m = -\frac{2}{3}$

Exercise 3.2.1.68

$(4, -3);\ m = -\frac{7}{4}$

Exercise 3.2.1.69

$(-2, 0);\ m = -\frac{5}{2}$

Exercise 3.2.1.70

$(3, 3);\ m = \frac{7}{3}$

Exercise 3.2.1.71

$(-4, -3);\ m = 0$

Exercise 3.2.1.72

$(-2, -5);\ m = 2$

Write the equation of the line in point-slope form given two points on the line.

Exercise 3.2.1.73

$(-4, 3)$ and $(-3, 1)$

Exercise 3.2.1.74

$(5, 1)$ and $(-3, 0)$

Exercise 3.2.1.75

$(-4, -2)$ and $(0, 4)$

Exercise 3.2.1.76

$(3, 5)$ and $(-5, 3)$

Exercise 3.2.1.77

$(3, -3)$ and $(-4, 5)$

Exercise 3.2.1.78

$(1, 3)$ and $(-3, 3)$

Exercise 3.2.1.79

$(-4, 5)$ and $(4, 4)$

Exercise 3.2.1.80

$(-4, 1)$ and $(4, 4)$

Exercise 3.2.1.81

$(-1, -4)$ and $(-5, 0)$

Write the equation of the line in slope-intercept form given two points on the line.

Exercise 3.2.1.82

$(-5, 1)$ and $(-1, -2)$

Exercise 3.2.1.83

$(-5, 5)$ and $(2, -3)$

Exercise 3.2.1.84

$(4, 1)$ and $(1, 4)$

Exercise 3.2.1.85

$(0, 2)$ and $(5, -3)$

Exercise 3.2.1.86

$(0, 3)$ and $(-1, -1)$

Exercise 3.2.1.87

$(-5, -1)$ and $(5, -2)$

Exercise 3.2.1.88

$(1, -1)$ and $(-5, -4)$

Exercise 3.2.1.89

$(0, 1)$ and $(-3, 0)$

Exercise 3.2.1.90

$(0, 2)$ and $(2, 4)$

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3.3: Parallel and Perpendicular Lines

In this section, we discuss parallel and perpendicular lines. The relationship between parallel lines and between perpendicular lines is unique, where the slope will be most interesting to us in this section.

The Slope of Parallel and Perpendicular Lines

Example 3.3.1

Find the slope of each line and compare. What is interesting about the slopes?

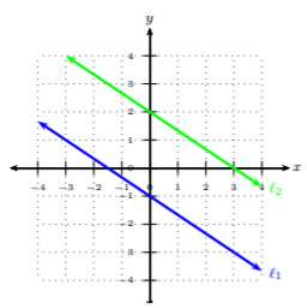


Figure 3.3.1

Solution

Looking at ℓ_1, we can start at $(-3, 1)$ and reach the next point at $(0, -1)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_1's slope is $-\frac{2}{3}$. Now let's look at ℓ_2 and obtain its slope. We will start at $(0, 2)$ and reach the next point at $(3, 0)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_2's slope is $-\frac{2}{3}$. The slopes of ℓ_1 and ℓ_2 are $-\frac{2}{3}$; they have the same exact slope but different y-intercepts.

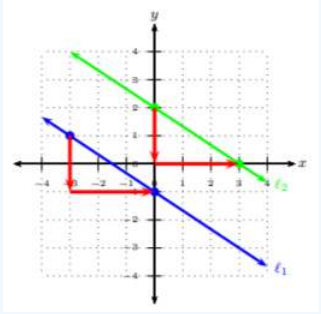


Figure 3.3.2

Definition: Parallel

Let m_1 and m_2 be slopes for lines ℓ_1 and ℓ_2, respectively. Lines ℓ_1 and ℓ_2 are ***parallel*** to each other if they have the same slope, but different y-intercepts, i.e., $m_1 = m_2$.

Example 3.3.2

Find the slope of each line and compare. What is interesting about the slopes?

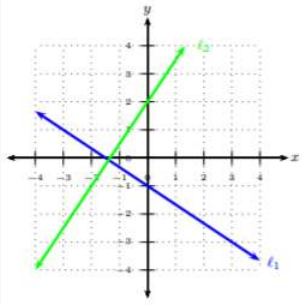


Figure 3.3.3

Solution

Looking at ℓ_1, we can start at $(-3, 1)$ and reach the next point at $(0, -1)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_1's slope is $-\frac{2}{3}$. Now let's look at ℓ_2 and obtain its slope. We will start at $(-2, -1)$ and reach the next point at $(0, 2)$. We see that we will move up three units and run to the right 2 units. Hence, ℓ_2's slope is $\frac{3}{2}$. The slopes of ℓ_1 and ℓ_2 are negative reciprocals, i.e., if one has slope m, then a line perpendicular to it will have slope $-\frac{1}{m}$. Also, note that if two lines are perpendicular, they create a right angle at the intersection.

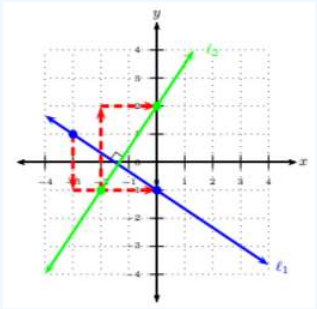


Figure 3.3.4

Definition: Perpendicular

Let m_1 and m_2 be slopes for lines ℓ_1 and ℓ_2, respectively. Lines ℓ_1 and ℓ_2 are ***perpendicular*** to each other if they have negative reciprocal slopes, i.e., ℓ_1 has slope m_1 and ℓ_2 has slope $m_2 = -\frac{1}{m_1}$.

Example 3.3.3

Find the slope of a line parallel to $5y - 2x = 7$.

Solution

We need to rewrite the equation in slope-intercept form. Then we can identify the slope and the slope for a line parallel to it.

$$\begin{aligned} 5y - 2x &= 7 && \text{Isolate the variable term } 5y \\ 5y - 2x + 2x &= 7 + 2x && \text{Simplify} \\ 5y &= 2x + 7 && \text{Multiply by the reciprocal of } 5 \\ \tfrac{1}{5} \cdot 5y &= \tfrac{1}{5} \cdot 2x + 7 \cdot \tfrac{1}{5} && \text{Simplify} \\ y &= \tfrac{2}{5}x + \tfrac{7}{5} \end{aligned}$$

We see the slope of the given line is $\frac{2}{5}$. By the definition, a line parallel will have the same slope $\frac{2}{5}$.

Example 3.3.4

Find the slope of a line perpendicular to $3x - 4y = 2$.

Solution

We need to rewrite the equation in slope-intercept form. Then we can identify the slope and the slope for a line perpendicular to it.

$$\begin{aligned} 3x-4y &= 2 && \text{Isolate the variable term } -4y \\ 3x-4y+(-3x) &= 2+(-3x) && \text{Simplify} \\ -4y &= -3x+2 && \text{Multiply by the reciprocal of } -4 \\ -\tfrac{1}{4}\cdot -4y &= -\tfrac{1}{4}\cdot -3x+2\cdot -\tfrac{1}{4} && \text{Simplify} \\ y &= \tfrac{3}{4}x-\tfrac{1}{2} \end{aligned}$$

We see the slope of the given line is $\frac{3}{4}$. By the definition, a line perpendicular will have a negative reciprocal slope $-\frac{4}{3}$.

Obtain Equations for Parallel and Perpendicular Lines

Once we have obtained the slope for a line perpendicular or parallel, it is possible to find the complete equation of the second line if we are given a point on the second line.

Example 3.3.5

Find the equation of a line passing through $(4,-5)$ and parallel to $2x-3y=6$.

Solution

First, we can rewrite the given line in slope-intercept form to obtain the slope for a line parallel to it:

$$\begin{aligned} 2x-3y &= 6 && \text{Isolate the variable term } -3y \\ 2x-3y+(-2x) &= 6+(-2x) && \text{Simplify} \\ -3y &= -2x+6 && \text{Multiply by the reciprocal of } -3 \\ -\tfrac{1}{3}\cdot -3y &= -\tfrac{1}{3}\cdot -2x+6\cdot -\tfrac{1}{3} && \text{Simplify} \\ y &= \tfrac{2}{3}x-2 \end{aligned}$$

We see the slope of the given line is $\frac{2}{3}$. By the definition, a line parallel will have the same slope $\frac{2}{3}$. Next, we can use the point-slope formula to obtain the equation of the line passing through $(4,-5)$ with slope $\frac{2}{3}$:

$$\begin{aligned} y-y_1 &= m(x-x_1) && \text{Substitute in the point and slope} \\ y-(-5) &= \tfrac{2}{3}(x-4) && \text{Simplify signs} \\ y+5 &= \tfrac{2}{3}(x-4) && \text{A line parallel to } 2x-3y=6 \text{ in point-slope form} \end{aligned}$$

Example 3.3.6

Find the equation of the line, in slope-intercept form, passing through $(6,-9)$ and perpendicular to $y=-\frac{3}{5}x+4$.

Solution

Since the given line is in slope-intercept form, we can easily observe the slope and the slope for a line perpendicular. We see the slope of the given line is $-\frac{3}{5}$. By the definition, a line perpendicular will have a negative reciprocal slope $\frac{5}{3}$. Next, we can use the point-slope formula to obtain the equation, in slope-intercept form, of the line passing through $(6,-9)$ with slope $\frac{5}{3}$:

$$\begin{aligned} y-y_1 &= m(x-x_1) && \text{Substitute in the point and slope} \\ y-(-9) &= \tfrac{5}{3}(x-6) && \text{Simplify signs} \\ y+9 &= \tfrac{5}{3}(x-6) && \text{Distribute} \\ y+9 &= \tfrac{5}{3}x-10 && \text{Isolate the variable term } y \\ y+9+(-9) &= \tfrac{5}{3}x-10+(-9) && \text{Simplify} \\ y &= \tfrac{5}{3}x-19 && \text{A line perpendicular to } y=-\tfrac{3}{5}x+4 \text{ in slope-intercept form} \end{aligned}$$

Note

Lines with zero slopes and undefined slopes may seem like opposites because a horizontal line has slope zero and a vertical line has slope that is undefined. Since a horizontal line is perpendicular to a vertical line, we can say, by definition, the slopes are negative reciprocals, i.e., $m_1 = 0$ would imply $m_2 = -\frac{1}{0}$, which is undefined.

Example 3.3.7

Find the equation of the line passing through $(3, 4)$ and perpendicular to $x = -2$.

Solution

Since $x = -2$ is a vertical line, then this line has slope that is undefined. Hence, a line perpendicular to it will have slope zero, i.e., $m = 0$. Next, we can use the point-slope formula to obtain the equation, in slope-intercept form, of the line passing through $(3, 4)$ with slope $m = 0$:

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\ y - 4 &= 0(x - 3) && \text{Distribute} \\ y - 4 &= 0 && \text{Isolate the variable term } y \\ y - 4 + 4 &= 0 + 4 && \text{Simplify} \\ y &= 4 && \text{A line perpendicular to } x = -2 \end{aligned}$$

Now, since we are aware that a line perpendicular to a vertical line is a horizontal line and we were given a point $(3, 4)$, we could have easily jumped to the equation, $y = 4$.

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3.3.1: Parallel and Perpendicular Lines (Exercises)

Given the line, find the slope of a line parallel.

Exercise 3.3.1.1

$y = 2x + 4$

Exercise 3.3.1.2

$y = 4x - 5$

Exercise 3.3.1.3

$x - y = 4$

Exercise 3.3.1.4

$7x + y = -2$

Exercise 3.3.1.5

$y = -\frac{2}{3}x + 5$

Exercise 3.3.1.6

$y = -\frac{10}{3}x - 5$

Exercise 3.3.1.7

$6x - 5y = 20$

Exercise 3.3.1.8

$3x + 4y = -8$

Given the line, find the slope of a line perpendicular.

Exercise 3.3.1.9

$x = 3$

Exercise 3.3.1.10

$y = -\frac{1}{3}x$

Exercise 3.3.1.11

$x - 3y = -6$

Exercise 3.3.1.12

$x+2y=8$

Exercise 3.3.1.13

$y=-\frac{1}{2}x-1$

Exercise 3.3.1.14

$y=\frac{4}{5}x$

Exercise 3.3.1.15

$3x-y=-3$

Exercise 3.3.1.16

$8x-3y=-9$

Find the equation of the line, in point-slope form, passing through the point and given the line to be parallel or perpendicular.

Exercise 3.3.1.17

$(2,5)$; parallel to $x=0$

Exercise 3.3.1.18

$(5,2)$; parallel to $y=\frac{7}{5}x+4$

Exercise 3.3.1.19

$(3,4)$; parallel to $y=\frac{9}{2}x-5$

Exercise 3.3.1.20

$(1,-1)$; parallel to $y=-\frac{3}{4}x+3$

Exercise 3.3.1.21

$(2,3)$; parallel to $y=\frac{7}{5}x+4$

Exercise 3.3.1.22

$(-1,3)$; parallel to $y=-3x-1$

Exercise 3.3.1.23

$(4,2)$; parallel to $x=0$

Exercise 3.3.1.24

$(1, 4)$; parallel to $y = \frac{7}{5}x + 2$

Exercise 3.3.1.25

$(1, -5)$; perpendicular to $-x + y = 1$

Exercise 3.3.1.26

$(1, -2)$; perpendicular to $-x + 2y = 2$

Exercise 3.3.1.27

$(5, 2)$; perpendicular to $5x + y = -3$

Exercise 3.3.1.28

$(1, 3)$; perpendicular to $-x + y = 1$

Exercise 3.3.1.29

$(4, 2)$; perpendicular to $-4x + y = 0$

Exercise 3.3.1.30

$(-3, -5)$; perpendicular to $3x + 7y = 0$

Exercise 3.3.1.31

$(2, -2)$; perpendicular to $3y - x = 0$

Exercise 3.3.1.32

$(-2, 5)$; perpendicular to $y - 2x = 0$

Find the equation of the line, in slope-intercept form, passing through the point and given the line to be parallel or perpendicular.

Exercise 3.3.1.33

$(4, -3)$; parallel to $y = -2x$

Exercise 3.3.1.34

$(-5, 2)$; parallel to $y = \frac{3}{5}x$

Exercise 3.3.1.35

$(-3, 1)$; parallel to $y = -\frac{4}{3}x - 1$

Exercise 3.3.1.36

$(-4, 0)$; parallel to $y = -\frac{5}{4}x + 4$

Exercise 3.3.1.37

$(-4, -1)$; parallel to $y = -\frac{1}{2}x + 1$

Exercise 3.3.1.38

$(2, 3)$; parallel to $y = \frac{5}{2}x - 1$

Exercise 3.3.1.39

$(-2, -1)$; parallel to $y = -\frac{1}{2}x - 2$

Exercise 3.3.1.40

$(-5, -4)$; parallel to $y = \frac{3}{5}x - 2$

Exercise 3.3.1.41

$(4, 3)$; perpendicular to $x + y = -1$

Exercise 3.3.1.42

$(-3, -5)$; perpendicular to $x + 2y = -4$

Exercise 3.3.1.43

$(5, 2)$; perpendicular to $x = 0$

Exercise 3.3.1.44

$(5, -1)$; perpendicular to $-5x + 2y = 10$

Exercise 3.3.1.45

$(-2, 5)$; perpendicular to $-x + y = -2$

Exercise 3.3.1.46

$(2, -3)$; perpendicular to $-2x + 5y = -10$

Exercise 3.3.1.47

$(4, -3)$; perpendicular to $-x + 2y = -6$

Exercise 3.3.1.48

$(-4, 1)$; perpendicular to $4x + 3y = -9$

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CHAPTER OVERVIEW

4: Systems of Linear Equations in Two and Three Variables

Learning Objectives

By the end of this chapter, the student should be able to

- Solve a system of equations with two and three linear equations in two and three variables by graphing, substitution, and elimination including infinitely many solutions or no solution
- Solve applications involving systems of equations including mixture, value, distance, and interest problems
- Graph and find the solutions for systems of two linear inequalities in two variables
- Use matrices to solve systems of two linear equations in two variables

We have solved linear equations like $3x-4=11$ by adding 4 to both sides and then dividing by 3 (solution is $x=5$). Notice, we only have one variable in this equation. What if we have 2 variables? Luckily, we have methods to solve equations with more than one variable. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables, such as x and y, we will need two equations with the same variables. When solving for more than one equation and one variable, we call the set of equations a *system of equations*. When solving a *system of equations*, we are looking for a solution that makes both equations true. Since we are solving for x and y, it should remind us of graphing lines, and the solution is an ordered pair (x,y). This ordered-pair is on both lines.

Definition: System of Two Linear Equations in Two Variables

A ***system of two linear equations in two variables*** is given in the form

$$\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$$

where $a,\ b,\ c,\ d,\ e$, and f are coefficients and x and y are variables. This system is represented in *standard form*.

4.1: System of Equations - Graphing

4.1.1: Systems of Equations - Graphing (Exercises)

4.2: Systems of Equations - The Substitution Method

4.2.1: Systems of Equations - Substitution (Exercises)

4.3: System of Equations - The Addition Method

4.3.1: Systems of Equations - Addition (Exercises)

4.4: Applications with Systems of Equations

4.4.1: Applications with Systems of Equations (Exercises)

4.5: Systems of Three Linear Equations in Three Variables

4.5.1: Systems of Three Linear Equations in Three Variables (Exercises)

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4.1: System of Equations - Graphing

Three Types of Solutions to a Linear System by Graphing

Given two linear equations and after graphing the lines,

Solution 1. If the two lines intersect, then the point of intersection is the solution to the system, i.e., the solution is an ordered-pair (x, y).

Solution 2. If the two lines do not intersect and are parallel, i.e., they have the same slope and different y-intercepts, then the system has *no solution*.

Solution 3. If the two lines are the same line, then the solution is *infinitely many solutions* on that line.

Verifying Solutions

Example 4.1.1

Is the ordered-pair $(2, 1)$ a solution to the system

$$\begin{cases} 3x - y = 5 \\ x + y = 3 \end{cases} \quad ?$$

Solution

To verify whether $(2, 1)$ is the solution to the system, we plug-n-chug $(2, 1)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for both equations in the system, then $(2, 1)$ will be the solution to the system.

$$\begin{aligned} 3x - y &= 5 && \text{Plug-n-chug } x = 2 \text{ and } y = 1 \\ 3(2) - (1) &\stackrel{?}{=} 5 && \text{Simplify} \\ 6 - 1 &\stackrel{?}{=} 5 && \text{Subtract} \\ 5 &= 5 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} x + y &= 3 && \text{Plug-n-chug } x = 2 \text{ and } y = 1 \\ (2) + (1) &\stackrel{?}{=} 3 && \text{Add} \\ 3 &= 3 && \checkmark \text{ True} \end{aligned}$$

Since the ordered-pair $(2, 1)$ makes both statements true, then $(2, 1)$ is a solution to the system. Hence, if we were to graph these lines, they would intersect at the point $(2, 1)$.

Example 4.1.2

Is the ordered-pair $(-3, -4)$ a solution to the system

$$\begin{cases} 5x + 4y = -31 \\ 3x + 6y = -36 \end{cases} \quad ?$$

Solution

To verify whether $(-3, -4)$ is the solution to the system, we plug-n-chug $(-3, -4)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for both equations in the system, then $(-3, -4)$ will be the solution to the system.

$$\begin{aligned} 5x+4y &= -31 && \text{Plug-n-chug } x=-3 \text{ and } y=-4 \\ 5(-3)+4(-4) &\stackrel{?}{=} -31 && \text{Simplify} \\ -15-16 &\stackrel{?}{=} 31 && \text{Subtract} \\ -31 &= -31 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} 3x+6y &= -36 && \text{Plug-n-chug } x=-3 \text{ and } y=-4 \\ 3(-3)+6(-4) &\stackrel{?}{=} -36 && \text{Simplify} \\ -9-24 &\stackrel{?}{=} -36 && \text{Subtract} \\ -33 &\neq -36 && X \text{ False} \end{aligned}$$

Since the ordered-pair $(-3,-4)$ makes only one of the statements true, then $(-3,-4)$ is not a solution to the system. Recall, the ordered-pair must make the statement true for both equations in order to be a solution to the system.

Solve a System by Graphing

Example4.1.3

Solve the system by graphing:

$$\begin{cases} y=-\frac{1}{2}x+3 \\ y=\frac{3}{4}x-2 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Notice both equations are given in slope-intercept form. Let's go ahead and use the slope-intercept form to graph the lines.

$$\begin{aligned} y&=-\frac{1}{2}x+3 \quad (1) \\ y&=\frac{3}{4}x-2 \quad (2) \end{aligned}$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.

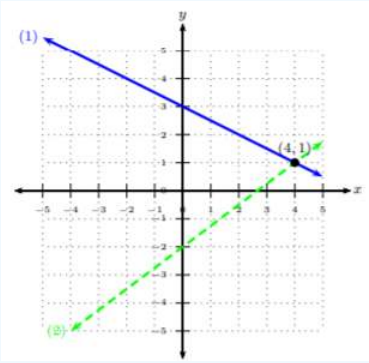


Figure 4.1.1

We can see after graphing the two lines that they intersect at the point $(4,1)$. Hence, the solution to the system is $(4,1)$.

Example 4.1.4

Solve the system by graphing:

$$\begin{cases} 6x-3y=-9 \\ 2x+2y=-6 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Since both equations are not given in slope-intercept form as in Example 4.1.3 , we can rewrite them in slope-intercept form, then graph. So, let's rewrite each equation in slope-intercept form:

$$\begin{aligned} 6x-3y&=-9 & 2x+2y&=-6 \\ -3y&=-6x-9 & 2y&=-2x-6 \\ y&=\frac{-6x}{-3}-\frac{9}{-3} & y&=\frac{-2}{2}x-\frac{6}{2} \\ y&=2x+3 & y&=-x-3 \end{aligned}$$

Let's go ahead and use the slope-intercept form to graph the lines.

$$\begin{aligned} y&=2x+3 \quad (1) \\ y&=-x-3 \quad (2) \end{aligned}$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.

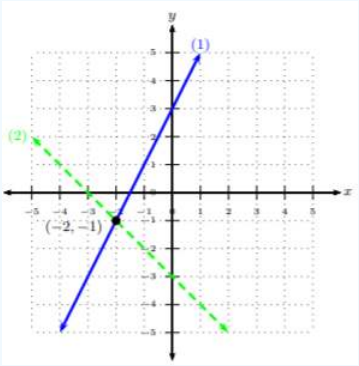


Figure 4.1.2

We can see after graphing the two lines that they intersect at the point $(-2,-1)$. Hence, the solution to the system is $(-2,-1)$.

Example 4.1.5

Solve the system by graphing:

$$\begin{cases} y=\frac{3}{2}x-4 \\ y=\frac{3}{2}x+1 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Notice both equations are given in slope-intercept form. Let's go ahead and use the slope-intercept form to graph the lines.

$$\begin{aligned} y&=\frac{3}{2}x-4 \quad (1) \\ y&=\frac{3}{2}x+1 \quad (2) \end{aligned}$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.

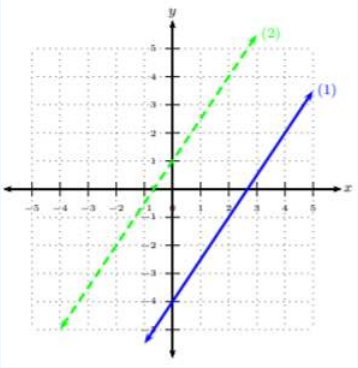


Figure 4.1.3

We can see after graphing the two lines that these two lines are parallel. Hence, there is no solution to the system (since they will never intersect). Note, we could see by the system that these lines shared the same slope, but had different y-intercepts. Without graphing, we could have seen that these lines were parallel, hence, having no solution.

Example 4.1.6

Solve the system by graphing:

$$\begin{cases} 2x-6y=12 \\ 3x-9y=18 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Since neither of the equations are written in slope-intercept form, let's try graphing by making a table for each equation. Start with equation (1):

$$\begin{aligned} x=-3 \qquad & 2(-3)-6y=12 \\ & -6-6y=12 \\ & -6y=18 \\ & y=-3 \\ x=0 \qquad & 2(0)-6y=12 \\ & -6y=12 \\ & y=-2 \\ x=3 \qquad & 2(3)-6y=12 \\ & 6-6y=12 \\ & -6y=6 \\ & y=-1 \end{aligned}$$

Table 4.1.1

x	y
-3	-3
0	-2
3	-1

Next, equation (2):

$$
\begin{aligned}
x=-3 \qquad 3(-3)-9y&=18\\
-9-9y&=18\\
-9y&=27\\
y&=-3\\
x=0 \qquad 3(0)-9y&=18\\
-9y&=18\\
y&=-2\\
x=3 \qquad 3(3)-9y&=18\\
9-9y&=18\\
-9y&=9\\
y&=-1
\end{aligned}
$$

Table 4.1.2

x	y
-3	-3
0	-2
3	-1

Now, let's graph the ordered-pairs. We will graph line (1) with a solid line and graph line (2) with a dashed line.

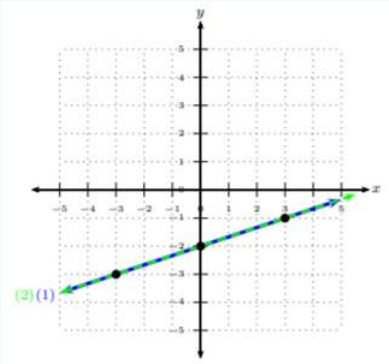


Figure 4.1.4

We can see after graphing the two lines that these two lines are the same. Hence, there are infinitely many solutions on the line $2x-6y=12$ (or the other equation) to the system (since they intersect at every point on the line). Note, we could see by the system that these lines shared the same slope and y-intercepts (after putting them in slope-intercept form). Without graphing, we could have seen that these lines were the same line, hence, having infinitely many solutions.

Note

The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks, around 300 AD, who would solve systems of equations with three or four variables and eventually developed methods for solving systems with any number of unknowns.

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4.1.1: Systems of Equations - Graphing (Exercises)

Determine whether the given ordered pair(s) is a solution to the system.

Exercise 4.1.1.1

$2x+8y=0$
$-8x+3y=38;\ (-4,1)$

Exercise 4.1.1.2

$-5x+6y=11$
$4x+2y=-4;\ (-1,1)$

Exercise 4.1.1.3

$6x+5y=49$
$-x-6y=-34;\ (4,5)$

Exercise 4.1.1.4

$-2x+2y=-8$
$-6x-3y=-6;\ (2,-2)$

Solve each system by graphing.

Exercise 4.1.1.5

$y=-x+1$
$y=-5x-3$

Exercise 4.1.1.6

$y=-3$
$y=-x-4$

Exercise 4.1.1.7

$y=-\frac{3}{4}x+1$
$y=-\frac{3}{4}x+2$

Exercise 4.1.1.8

$y=\frac{1}{3}x+2$
$y=-\frac{5}{3}x-4$

Exercise 4.1.1.9

$y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

Exercise 4.1.1.10

$x + 3y = -9$
$5x + 3y = 3$

Exercise 4.1.1.11

$x - y = 4$
$2x + y = -1$

Exercise 4.1.1.12

$2x + 3y = -6$
$2x + y = 2$

Exercise 4.1.1.13

$2x + y = 2$
$x - y = 4$

Exercise 4.1.1.14

$2x + y = -2$
$x + 3y = 9$

Exercise 4.1.1.15

$0 = -6x - 9y + 36$
$12 = 6x - 3y$

Exercise 4.1.1.16

$2x - y = -1$
$0 = -2x - y - 3$

Exercise 4.1.1.17

$3 + y = -x$
$-4 - 6x = -y$

Exercise 4.1.1.18

$-y + 7x = 4$
$-y - 3 + 7x = 0$

Exercise 4.1.1.19

$-12+x=4y$
$12-5x=4y$

Exercise 4.1.1.20

$y=-\frac{5}{4}x-2$
$y=-\frac{1}{4}x+2$

Exercise 4.1.1.21

$y=-x-2$
$y=\frac{2}{3}x+3$

Exercise 4.1.1.22

$y=2x+2$
$y=-x-4$

Exercise 4.1.1.23

$y=2x-4$
$y=\frac{1}{2}x+2$

Exercise 4.1.1.24

$y=\frac{1}{2}x+4$
$y=\frac{1}{2}x+1$

Exercise 4.1.1.25

$x+4y=-12$
$2x+y=4$

Exercise 4.1.1.26

$6x+y=-3$
$x+y=2$

Exercise 4.1.1.27

$3x+2y=2$
$3x+2y=-6$

Exercise 4.1.1.28

$x+2y=6$
$5x-4y=16$

Exercise 4.1.1.29

$x - y = 3$
$5x + 2y = 8$

Exercise 4.1.1.30

$-2y + x = 4$
$2 = -x + \frac{1}{2}y$

Exercise 4.1.1.31

$-2y = -4 - x$
$-2y = -5x + 4$

Exercise 4.1.1.32

$16 = -x - 4y$
$-2x = -4 - 4y$

Exercise 4.1.1.33

$-4 + y = x$
$x + 2 = -y$

Exercise 4.1.1.34

$-5x + 1 = -y$
$-y + x = -3$

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4.2: Systems of Equations - The Substitution Method

Solving a system by graphing has its limitations. First, it requires the graph to be precisely drawn. If the lines aren't drawn correctly, it is possible to obtain the incorrect point of intersection. Second, graphing is not a sufficient method to use if the point of intersection is far away from the origin, like the ordered-pair (12567, 9832), or if the answer is a decimal (or irrational number) that the graph will not display well. For these reasons, we rarely use graphing to solve systems. Instead, we use an algebraic approach. There are two approaches and the first approach is called substitution. We build the concepts of substitution through several examples and then conclude with a general four-step process to solve problems using this method.

Introduction to Substitution

Example 4.2.1

Solve the system of equations.

$$\begin{cases} x = 5 \\ y = 2x - 3 \end{cases}$$

Solution

We are given $x = 5$ in the first equation. Hence, we can plug-n-chug $x = 5$ into the second equation to find y because we know any ordered-pair with x coordinate 5 will satisfy the equation:

$$\begin{aligned} y &= 2x - 3 && \text{Plug-n-chug } x = 5 \\ y &= 2(5) - 3 && \text{Evaluate} \\ y &= 10 - 3 && \text{Subtract} \\ y &= 7 && y\text{-coordinate of the solution} \end{aligned}$$

Since it is given $x = 5$ and we obtained $y = 7$, then the solution to the system is the ordered-pair $(5, 7)$. Furthermore, if we were to graph these two lines, they would intersect at $(5, 7)$.

The Substitution Method

When we are given the solution to one of the two variables, we can easily plug-n-chug that value (or expression) in the other equation to obtain the value of the second variable. It is very important that when we substitute, we write parenthesis around the expression we are substituting; this will minimize errors with distribution.

Example 4.2.2

Solve the system by substitution.

$$\begin{cases} 2x - 3y = 7 \\ y = 3x - 7 \end{cases}$$

Solution

We can see that y is isolated in the second equation, $y = 3x - 7$, and we can substitute the right side for y in the first equation.

$$\begin{aligned} 2x - 3y &= 7 && \text{Plug-n-chug } y = 3x - 7 && \text{into the first equation} \\ 2x - 3(3x - 7) &= 7 && \text{Distribute} \\ 2x - 9x + 21 &= 7 && \text{Combine like terms} \\ -7x + 21 &= 7 && \text{Isolate the variable term} \\ -7x &= -14 && \text{Multiply by the reciprocal of } -7 \\ x &= 2 && x\text{-coordinate of the solution} \end{aligned}$$

Since $x = 2$, then we can plug-n-chug $x = 2$ into one of the equations to obtain y:

$$\begin{aligned} y &= 3x - 7 && \text{Plug-n-chug } x = 2 \\ y &= 3(2) - 7 && \text{Evaluate} \\ y &= 6 - 7 && \text{Subtract} \\ y &= -1 && y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(2, -1)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(2, -1)$. When we obtain a solution to a system, we call this system a *consistent system*. When we obtain one ordered-pair as the solution to the system, we call this solution an *independent solution*.

Definition: Systems

Given a system of equations,

- If there exists at least one solution to the system, then this is a ***consistent system***.
  - If the consistent system has one unique solution, i.e., the solution is an ordered-pair, then this is an ***independent system***.
  - If the consistent system has more than one solution, i.e., there are infinitely many solutions, then this is a ***dependent system***.
- If a solution to the system doesn't exist, i.e., there is no solution, then this is an ***inconsistent system***.

Solve for a Variable

Example 4.2.3

Solve the system by substitution:

$$\begin{cases} 3x + 2y = 1 \\ x - 5y = 6 \end{cases}$$

Solution

Notice neither of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation. We always want to work smarter, not harder, so let's be clever in choosing the equation and variable. Looking at the first equation, there is a coefficient in front of each variable. Looking at the second equation, we see only the y has a coefficient other than 1, and x's coefficient is one. Let's pick this because solving for x is a one-step equation while the other is a two-step equation.

$$\begin{aligned} x - 5y &= 6 \\ x &= 6 + 5y \end{aligned}$$

Now, we can substitute x into the first equation:

$$\begin{aligned} 3x + 2y &= 1 && \text{Plug-n-chug } x = 6 + 5y \\ 3(6 + 5y) + 2y &= 1 && \text{Distribute} \\ 18 + 15y + 2y &= 1 && \text{Combine like terms} \\ 18 + 17y &= 1 && \text{Isolate the variable term} \\ 17y &= -17 && \text{Multiply by the reciprocal of } 17 \\ y &= -1 && y\text{-coordinate of the solution} \end{aligned}$$

Since $y = -1$, then we can plug-n-chug $y = -1$ into one of the equations to obtain x:

$$\begin{aligned} x &= 6 + 5y && \text{Plug-n-chug } y = -1 \\ x &= 6 + 5(-1) && \text{Evaluate} \\ x &= 6 - 5 && \text{Subtract} \\ x &= 1 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(1, -1)$. Furthermore, if we were to graph these two lines, they would intersect at $(1, -1)$. Also, we know this system is a consistent system that is independent.

Note

When choosing the equation and variable to use in substitution, *we want to choose the one-step equation*, i.e., one equation with one of the variables with a coefficient of one. Even though we are used to always solving for y when graphing lines, with this method, it doesn't matter. We want to work smarter, not harder, and pick the equation that will give us the solution the quickest.

The Substitution Method

Given a system of two linear equations in two variables, we can use the following steps to solve by substitution.

- **Step 1.** Choose an equation and then solve for x or y. *(Choose the one-step equation when possible.)*
- **Step 2.** Substitute the expression for x or y in the other equation.
- **Step 3.** Solve the equation.
- **Step 4.** Substitute the value into one of the original equations to find the remaining variable.

It is common practice to write your answer as an ordered-pair of the form (x, y) since this is the point of intersection. Be sure to verify the solution.

Example 4.2.4

Solve the system by substitution:

$$\begin{cases} 4x-2y=2 \\ 2x+y=-5 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the second equation, we see only the y has a coefficient of one, and x's coefficient is two. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} 2x+y &= -5 \\ y &= -2x-5 \end{aligned}$$

Step 2. Now, we can substitute y into the first equation:

$$\begin{aligned} 4x-2y &= 2 \quad \text{Plug-n-chug } y=-2x-5 \\ 4x-2(-2x-5) &= 2 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 4x-2(-2x-5) &= 2 && \text{Distribute} \\ 4x+4x+10 &= 2 && \text{Combine like terms} \\ 8x+10 &= 2 && \text{Isolate the variable term} \\ 8x &= -8 && \text{Multiply by the reciprocal of } 8 \\ x &= -1 && x\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $x=-1$, then we can plug-n-chug $x=-1$ into one of the equations to obtain y:

$$\begin{aligned} y &= -2x-5 && \text{Plug-n-chug } x=-1 \\ y &= -2(-1)-5 && \text{Evaluate} \\ y &= 2-5 && \text{Subtract} \\ y &= -3 && y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(-1, -3)$. Furthermore, if we were to graph these two lines, they would intersect at $(-1, -3)$. Also, we know this system is a consistent system that is independent.

Substitution: Special Cases

Example 4.2.5

Solve the system by substitution:

$$\begin{cases} y+4=3x \\ 2y-6x=-8 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the first equation, we see only the y has a coefficient of one, and x's coefficient is three. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} y+4 &= 3x \\ y &= 3x-4 \end{aligned}$$

Step 2. Now, we can substitute y into the second equation:

$$\begin{aligned} 2y-6x &= -8 \quad \text{Plug-n-chug } y=3x-4 \\ 2(3x-4)-6x &= -8 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 2(3x-4)-6x &= -8 \quad \text{Distribute} \\ 6x-8-6x &= -8 \quad \text{Combine like terms} \\ -8 &= -8 \end{aligned}$$

Since all the variables cancel and we are left with a statement without variables, we ask," Is this statement true?"

$$\begin{aligned} -8 &\stackrel{?}{=} -8 \quad \text{Is this true?} \\ -8 &= -8 \quad \checkmark \text{ True} \end{aligned}$$

Since this statement is true, then there are infinitely many solutions on the line $y+4=3x$. Furthermore, if we were to graph these two lines, we know they would be the same line and intersect at every point on the line. Also, we know this system is a consistent system that is dependent.

Example 4.2.6

Solve the system by substitution:

$$\begin{cases} 6x-3y=-9 \\ -2x+y=5 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the second equation, we see only the y has a coefficient of one, and x's coefficient is -2. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} -2x+y &= 5 \\ y &= 2x+5 \end{aligned}$$

Step 2. Now, we can substitute y into the first equation:

$$\begin{aligned} 6x-3y &= -9 \quad \text{Plug-n-chug } y=2x+5 \\ 6x-3(2x+5) &= -9 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 6x-3(2x+5) &= -9 \quad \text{Distribute} \\ 6x-6x-15 &= -9 \quad \text{Combine like terms} \\ -15 &= -9 \end{aligned}$$

Since all the variables cancel and we are left with a statement without variables, we ask," Is this statement true?"

$$\begin{aligned} -15 &\stackrel{?}{=} -9 \quad \text{Is this true?} \\ -15 &\neq -9 \quad X \text{ False} \end{aligned}$$

Since this statement is false, then there is no solution to this system. Furthermore, if we were to graph these two lines, we know they would be parallel. Hence, this system is an inconsistent system.

Note

French mathematician *Rene Descartes* wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is the reason we are often solving for the variables x, y, and z.

Example 4.2.7

Solve the system by substitution:

$$\begin{cases} 5x-6y=-14 \\ -2x+4y=12 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at both equations, we see none of these are one-step equations. Hence, we can just pick one equation and solve for a variable. Notice in the second equation, all coefficients are divisible by 2. Let's pick this because solving for x would avoid fractions.

$$\begin{aligned} -2x+4y &= 12 \\ -2x &= -4y+12 \\ x &= 2y-6 \end{aligned}$$

Step 2. Now, we can substitute x into the first equation:

$$\begin{aligned} 5x-6y &= -14 \quad \text{Plug-n-chug } x=2y-6 \\ 5(2y-6)-6y &= -14 \end{aligned}$$

Step 3. Solve for y:

$$\begin{aligned} 5(2y-6)-6y &= -14 && \text{Distribute} \\ 10y-20-6y &= -14 && \text{Combine like terms} \\ 4y-30 &= -14 && \text{Isolate the variable term} \\ 4y &= 16 && \text{Multiply by the reciprocal of } 4 \\ y &= 4 && y\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $y=4$, then we can plug-n-chug $y=4$ into one of the equations to obtain x:

$$\begin{aligned} x &= 2y-6 && \text{Plug-n-chug } y=4 \\ x &= 2(4)-6 && \text{Evaluate} \\ x &= 8-6 && \text{Subtract} \\ x &= 2 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(2,4)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(2,4)$. Also, we know this system is a consistent system that is independent.

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4.2.1: Systems of Equations - Substitution (Exercises)

Solve each system by substitution. Determine if each system is consistent, independent or dependent, or inconsistent.

Exercise 4.2.1.1

$y = -3x$
$y = 6x - 9$

Exercise 4.2.1.2

$y = -2x - 9$
$y = 2x - 1$

Exercise 4.2.1.3

$y = 6x + 4$
$y = -3x - 5$

Exercise 4.2.1.4

$y = 3x + 2$
$y = -3x + 8$

Exercise 4.2.1.5

$y = 2x - 3$
$y = -2x + 9$

Exercise 4.2.1.6

$y = 6x - 6$
$-3x - 3y = -24$

Exercise 4.2.1.7

$y = -6$
$3x - 6y = 30$

Exercise 4.2.1.8

$y = -5$
$3x + 4y = -17$

Exercise 4.2.1.9

$-2x + 2y = 18$
$y = 7x + 15$

Exercise 4.2.1.10

$y = -8x + 19$
$-x + 6y = 16$

Exercise 4.2.1.11

$7x - 2y = -7$
$y = 7$

Exercise 4.2.1.12

$x - 5y = 7$
$2x + 7y = -20$

Exercise 4.2.1.13

$-2x - y = -5$
$x - 8y = -23$

Exercise 4.2.1.14

$-6x + y = 20$
$-3x - 3y = -18$

Exercise 4.2.1.15

$3x + y = 9$
$2x + 8y = -16$

Exercise 4.2.1.16

$y = x + 5$
$y = -2x - 4$

Exercise 4.2.1.17

$y = -6x + 3$
$y = 6x + 3$

Exercise 4.2.1.18

$y = 3x + 13$
$y = -2x - 22$

Exercise 4.2.1.19

$y = -2x - 9$
$y = -5x - 21$

Exercise 4.2.1.20

$y = 7x - 24$
$y = -3x + 16$

Exercise 4.2.1.21

$-x+3y=12$
$y=6x+21$

Exercise 4.2.1.22

$6x-4y=-8$
$y=-6x+2$

Exercise 4.2.1.23

$7x+2y=-7$
$y=5x+5$

Exercise 4.2.1.24

$y=x+4$
$3x-4y=-19$

Exercise 4.2.1.25

$y=-2x+8$
$-7x-6y=-8$

Exercise 4.2.1.26

$x-2y=-13$
$4x+2y=18$

Exercise 4.2.1.27

$3x-4y=15$
$7x+y=4$

Exercise 4.2.1.28

$6x+4y=16$
$-2x+y=-3$

Exercise 4.2.1.29

$7x+5y=-13$
$x-4y=-16$

Exercise 4.2.1.30

$-5x-5y=-20$
$-2x+y=7$

Exercise 4.2.1.31

$2x+y=2$
$3x+7y=14$

Exercise 4.2.1.32

$x + 5y = 15$
$-3x + 2y = 6$

Exercise 4.2.1.33

$-2x + 4y = -16$
$y = -2$

Exercise 4.2.1.34

$-6x + 6y = -12$
$8x - 3y = 16$

Exercise 4.2.1.35

$2x + 3y = 16$
$-7x - y = 20$

Exercise 4.2.1.36

$2x + y = -7$
$5x + 3y = -21$

Exercise 4.2.1.37

$2x + 3y = -10$
$7x + y = 3$

Exercise 4.2.1.38

$-2x + 2y = -22$
$-5x - 7y = -19$

Exercise 4.2.1.39

$-8x + 2y = -6$
$-2x + 3y = 11$

Exercise 4.2.1.40

$-x - 4y = -14$
$-6x + 8y = 12$

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4.3: System of Equations - The Addition Method

We first found that graphing isn't a sophisticated way for generally solving systems. We then considered a second method known as substitution. The substitution method is often used for solving systems in various areas of algebra. However, substitution can get quite involved, especially if there are fractions because this only allows more room for error. Hence, we need an even more sophisticated way for solving systems in general. We call this method the **addition method**, also called the elimination method. We will build the concept in the following examples, then define a four-step process we can use to solve by elimination.

The Addition Method

Example 4.3.1

Solve the system by addition (elimination).

$$\begin{cases} 3x-4y=8 \\ 5x+4y=-24 \end{cases}$$

Solution

We solve the system by addition because we do just that- add. We want to add the two equations together to obtain an equation of one variable. Hence, we cannot just add right away; we need to make sure that when we add, we will eliminate one of the variables. Looking at the y variable terms, we can see that the coefficients of y are the same but opposite signs. We can foresee that when we add these two equations, the y variable terms will cancel:

$$\begin{array}{rl} 3x-4y=8 & \\ +5x+4y=-24 & \text{Add the equations} \\ \hline 8x=-16 & \end{array}$$

Notice the y variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{array}{rl} 8x=-16 & \text{Multiply by the reciprocal of } 8 \\ x=-2 & x\text{-coordinate of the solution} \end{array}$$

Since $x=-2$, then we can plug-n-chug $x=-2$ into one of the equations to obtain y:

$$\begin{array}{rl} 3x-4y=8 & \text{Plug-n-chug } x=-2 \\ 3(-2)-4y=8 & \text{Evaluate} \\ -6-4y=8 & \text{Isolate the variable term} \\ -4y=14 & \text{Multiply by the reciprocal of } -4 \\ y=-\dfrac{14}{4} & \text{Reduce the fraction} \\ y=-\dfrac{7}{2} & y\text{-coordinate of the solution} \end{array}$$

The solution to the system is the ordered-pair $\left(-2,-\frac{7}{2}\right)$. Furthermore, if we were to graph these two lines, we know they would intersect at $\left(-2,-\frac{7}{2}\right)$. Also, we know this system is a consistent system with an independent solution.

The Addition Method with Multiplication

In Example 4.3.1 , one of the variable terms had the same coefficient, but opposite signs, and adding these together eliminated the variable terms completely, which allowed us to solve for the other variable. This is the idea behind the addition method. However, generally, we aren't given variable terms that have the same coefficient with opposite signs. So, we will manipulate the equations by multiplying one or both equations by the LCM of the coefficients for one of the variables. We want to work smarter, not harder, so we should be clever in which variable we choose.

Example 4.3.2

Solve the system by addition (elimination).

$$\begin{cases} -6x+5y=22 \\ 2x+3y=2 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient with opposite signs, we need to choose a variable and rewrite the equations so we can cancel the variable. Recall, the goal is to obtain an equation in one variable after adding. Looking at the x variable terms, we can see that the coefficients of x have opposite signs. So let's choose to eliminate x and we multiply the second equation by 3 to obtain the $\text{LCM}(2,6)=6$:

$$\begin{aligned} 3\cdot(2x+3y) &= (2)\cdot 3 \quad \text{Distribute} \\ 6x+9y &= 6 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs. Now we can add and eliminate x:

$$\begin{array}{rl} -6x+5y=22 & \\ +\quad 6x+9y=6 & \quad \text{Add the equations} \\ \hline 14y=28 & \end{array}$$

Notice the x variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{aligned} 14y &= 28 \quad \text{Multiply by the reciprocal of } 14 \\ y &= 2 \quad y\text{-coordinate of the solution} \end{aligned}$$

Since $y=2$, then we can plug-n-chug $y=2$ into one of the equations to obtain x:

$$\begin{aligned} 2x+3y &= 2 \quad \text{Plug-n-chug } y=2 \\ 2x+3(2) &= 2 \quad \text{Evaluate} \\ 2x+6 &= 2 \quad \text{Isolate the variable term} \\ 2x &= -4 \quad \text{Multiply by the reciprocal of } 2 \\ x &= -2 \quad x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(-2,2)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(-2,2)$. Also, we know this system is a consistent system with an independent solution.

Multiplying Two Equations

Example 4.3.3

Solve the system by addition (elimination).

$$\begin{cases} 3x+6y=-9 \\ 2x+9y=-26 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient with opposite signs, we need to choose a variable and rewrite the equations so we can cancel the variable. Recall, the goal is to obtain an equation in one variable after adding. Looking at the x and y variable terms, we can see that none of the coefficients are the same or with opposite signs. So we can choose any variable to eliminate. Let's choose to eliminate y and we multiply both equations by a factor to obtain the $\text{LCM}(9,6)=18$ with opposite signs:

$$\begin{aligned} -3\cdot(3x+6y) &= (-9)\cdot -3 \\ 2\cdot(2x+9y) &= (-26)\cdot 2 \end{aligned}$$

Notice the y variable terms have the same coefficients with opposite signs

$$-9x-18y=27$$
$$4x+18y=-52$$

Now we can add and eliminate y:

$$\begin{array}{rl} -9x-18y=27 & \\ +\quad 4x+18y=-52 & \text{Add the equations} \\ \hline -5x=-25 & \end{array}$$

Notice the y variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{array}{rl} -5x=-25 & \text{Multiply by the reciprocal of } -5 \\ x=5 & x\text{-coordinate of the solution} \end{array}$$

Since $x=5$, then we can plug-n-chug $x=5$ into one of the equations to obtain y:

$$\begin{array}{rl} 2x+9y=-26 & \text{Plug-n-chug } x=5 \\ 2(5)+9y=-26 & \text{Evaluate} \\ 10+9y=-26 & \text{Isolate the variable term} \\ 9y=-36 & \text{Multiply by the reciprocal of } 9 \\ y=-4 & y\text{-coordinate of the solution} \end{array}$$

The solution to the system is the ordered-pair $(5,-4)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(5,-4)$. Also, we know this system is a consistent system with an independent solution.

The Addition Method

Given a system of two linear equations in two variables, we can use the following steps to solve by addition (elimination).

- **Step 1.** Choose a variable to eliminate. *(Choose the variable with the variable terms with opposite signs, same coefficient, or both, if possible.)*
- **Step 2.** Multiply one or both equations so that the coefficients of this variable are the LCM of the coefficients with opposite signs.
- **Step 3.** Add the equations together, then solve.
- **Step 4.** Substitute the value into one of the original equations to find the remaining variable.

It is common practice to write your answer as an ordered-pair of the form (x,y) since this is the point of intersection. Be sure to verify the solution.

Example 4.3.4

Solve the system by addition (elimination):

$$\begin{cases} 2x-5y=-13 \\ 5x-3y=-4 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient, opposite signs, or both, we need to choose a variable and rewrite the equations so we can cancel the variable. We can choose any variable to eliminate.

Step 1. Let's choose to eliminate x and we multiply both equations by a factor to obtain the $\text{LCM}(2,5)=10$ with opposite signs.

Step 2. We can mulitply the first equation by a factor of 5 and the second equation by a factor of -2 so that we obtain variable terms with the same coefficient, 10, with opposite signs:

$$\begin{aligned} 5\cdot(2x-5y) &= (-13)\cdot 5 \\ -2\cdot(5x-3y) &= (-4)\cdot -2 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs:

$$\begin{aligned} 10x-25y &= -65 \\ -10x+6y &= 8 \end{aligned}$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rl} 10x-25y=-65 & \\ +\quad -10x+6y=8 & \text{Add the equations} \\ \hline -19y=-57 & \end{array}$$

Notice the x variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{aligned} -19y &= -57 && \text{Multiply by the reciprocal of } -19 \\ y &= 3 && y\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $y=3$, then we can plug-n-chug $y=3$ into one of the equations to obtain x:

$$\begin{aligned} 5x-3y &= -4 && \text{Plug-n-chug } y=3 \\ 5x-3(3) &= -4 && \text{Evaluate} \\ 5x-9 &= -4 && \text{Isolate the variable term} \\ 5x &= 5 && \text{Multiply by the reciprocal of } 5 \\ x &= 1 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(1,3)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(1,3)$. Also, we know this system is a consistent system with an independent solution.

Note

The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China, describes a process very similar to Gaussian elimination which is very similar to the addition method.

Addition: Special Cases

Just as with graphing and substitution, it is possible to have no solution or infinite solutions with elimination. Just as with substitution, if the variables eliminate, a true statement will indicate infinitely many solutions and a false statement will indicate there is no solution.

Example 4.3.5

Solve the system by addition (elimination):

$$\begin{cases} 2x-5y=3 \\ -6x+15y=-9 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient, but both have opposite signs, we can choose any variable to eliminate.

Step 1. Let's choose to eliminate y and we multiply both equations by a factor to obtain the $\text{LCM}(5,15)=15$ with opposite signs.

Step 2. We can mulitply the first equation by a factor of 3 and leave the second equation alone so that we obtain variable terms with the same coefficient, 15, with opposite signs:

$$3 \cdot (2x - 5y) = (3) \cdot 3 \quad \text{Distribute}$$
$$6x - 15y = 9$$

Notice the y variable terms have the same coefficients with opposite signs

$$6x - 5y = 9$$
$$-6x + 15y = -9$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rl} & 6x - 15y = 9 \\ + & -6x + 15y = -9 \\ \hline & 0 = 0 \end{array} \quad \text{Add the equations}$$

Since all the variables cancel and we are left with a statement without variables, we ask,“ Is this statement true?”

$$0 \stackrel{?}{=} 0 \quad \text{Is this true?}$$
$$0 = 0 \quad \checkmark \text{ True}$$

Since this statement is true, then there are infinitely many solutions to this system. Furthermore, if we were to graph these two lines, we know they would be the same line. Hence, this system is a consistent system with a dependent solution.

Example 4.3.6

Solve the system by addition (elimination):

$$\begin{cases} 4x - 6y = 8 \\ 6x - 9y = 15 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient, opposite signs, or both, we need to choose a variable and rewrite the equations so we can cancel the variable. We can choose any variable to eliminate.

Step 1. Let’s choose to eliminate x and we multiply both equations by a factor to obtain the $\text{LCM}(4, 6) = 12$ with opposite signs.

Step 2. We can multiply the first equation by a factor of 3 and the second equation by -2 so that we obtain variable terms with the same coefficient, 12, with opposite signs:

$$3 \cdot (4x - 6y) = (8) \cdot 3$$
$$-2 \cdot (6x - 9y) = (15) \cdot -2$$

Notice the y variable terms have the same coefficients with opposite signs:

$$12x - 18y = 24$$
$$-12x + 18y = -20$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rl} & 12x - 18y = 24 \\ + & -12x + 18y = -20 \\ \hline & 0 = 4 \end{array} \quad \text{Add the equations}$$

Since all the variables cancel and we are left with a statement without variables, we ask,“ Is this statement true?”

$$0 \stackrel{?}{=} 4 \quad \text{Is this true?}$$
$$0 \neq 4 \quad X \text{ False}$$

Since this statement is false, then there is no solution to this system. Furthermore, if we were to graph these two lines, we know they would be parallel. Hence, this system is an inconsistent system.

We discussed three different methods that can be used to solve a system of two equations in two variables. While all three can be used to solve any system, graphing works great for small integer solutions. Substitution works great when we have a given variable term with a coefficient of one, and addition works great for all other cases. As each method has its own strengths, it is important to be familiar with all three methods. Next, we use these methods to solve application problems.

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4.3.1: Systems of Equations - Addition (Exercises)

Solve each system by addition (elimination). Determine if each system is consistent, independent or dependent, or inconsistent.

Exercise 4.3.1.1

$4x+2y=0$
$-4x-9y=-28$

Exercise 4.3.1.2

$-9x+5y=-22$
$9x-5y=13$

Exercise 4.3.1.3

$-6x+9y=3$
$6x-9y=-9$

Exercise 4.3.1.4

$4x-6y=-10$
$4x-6y=-14$

Exercise 4.3.1.5

$-x-5y=28$
$-x+4y=-17$

Exercise 4.3.1.6

$2x-y=5$
$5x+2y=-28$

Exercise 4.3.1.7

$10x+6y=24$
$-6x+y=4$

Exercise 4.3.1.8

$2x+4y=24$
$4x-12y=8$

Exercise 4.3.1.9

$-7x+4y=-4$
$10x-8y=-8$

Exercise 4.3.1.10

$5x+10y=20$
$-6x-5y=-3$

Exercise 4.3.1.11

$-7x-3y=12$
$-6x-5y=20$

Exercise 4.3.1.12

$9x-2y=-18$
$5x-7y=-10$

Exercise 4.3.1.13

$9x+6y=-21$
$-10x-9y=28$

Exercise 4.3.1.14

$-7x+5y=-8$
$-3x-3y=12$

Exercise 4.3.1.15

$-8x-8y=-8$
$10x+9y=1$

Exercise 4.3.1.16

$9y=7-x$
$-18y+4x=-26$

Exercise 4.3.1.17

$0=9x+5y$
$y=\frac{2}{7}x$

Exercise 4.3.1.18

$-7x+y=-10$
$-9x-y=-22$

Exercise 4.3.1.19

$-x-2y=-7$
$x+2y=7$

Exercise 4.3.1.20

$5x-5y=-15$
$5x-5y=-15$

Exercise 4.3.1.21

$-3x+3y=-12$
$-3x+9y=-24$

Exercise 4.3.1.22

$-10x-5y=0$
$-10x-10y=-30$

Exercise 4.3.1.23

$-5x+6y=-17$
$x-2y=5$

Exercise 4.3.1.24

$x+3y=-1$
$10x+6y=-10$

Exercise 4.3.1.25

$-6x+4y=12$
$12x+6y=18$

Exercise 4.3.1.26

$-6x+4y=4$
$-3x-y=26$

Exercise 4.3.1.27

$-9x-5y=-19$
$3x-7y=-11$

Exercise 4.3.1.28

$-5x+4y=4$
$-7x-10y=-10$

Exercise 4.3.1.29

$3x+7y=-8$
$4x+6y=-4$

Exercise 4.3.1.30

$-4x-5y=12$
$-10x+6y=30$

Exercise 4.3.1.31

$8x+7y=-24$
$6x+3y=-18$

Exercise 4.3.1.32

$-7x+10y=13$
$4x+9y=22$

Exercise 4.3.1.33

$0=-9x-21+12y$
$1+\frac{4}{3}y+\frac{7}{3}x=0$

Exercise 4.3.1.34

$-6-42y=-12x$
$x-\frac{1}{2}-\frac{7}{2}y=0$

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4.4: Applications with Systems of Equations

We saw these types of examples in a previous chapter, but with one variable. In this section, we review the same types of applications, but solving in a more sophisticated way using systems of equations. Once we set up the system, we can solve using any method we choose. However, setting up the system may be the challenge, but as long as we follow the method we used before, we will be fine. We use tables to organize the parameters.

Value & Interest Problems

Example 4.4.1

There were 41 tickets sold for an event. Tickets for children cost $\$1.50$ and tickets for adults cost $\$2.00$. Total receipts for the event were $\$73.50$. How many of each type of ticket were sold?

Solution

First, we can make a table to organize the given information and then create an equation. Let c represent the number of children tickets sold and a represent the number of adult tickets sold.

Table 4.4.1

	Amount	Value (in $)	Total Value
Adult tickets	a	$\$2$	$2a$
Children tickets	c	$\$1.50$	$1.5c$
Total	41		$\$73.50$

Now let's set up the system. The total number of tickets sold was 41 and the total revenue from the tickets was $\$73.50$, and so we obtain the system

$$\begin{aligned} a+c &= 41 \\ 2a+1.5c &= 73.50 \end{aligned}$$

At this point, we can solve using any method we choose. Since the coefficient of a and c in the first equation are both one, then let's use the method of substitution. We will solve for a in the first equation:

$$\begin{aligned} a+c &= 41 \\ a &= 41-c \end{aligned}$$

Now, we can substitute a into the second equation and solve:

$$\begin{aligned} 2a+1.5c &= 73.50 && \text{Plug-n-chug } a=41-c \\ 2(41-c)+1.5c &= 73.50 && \text{Distribute} \\ 82-2c+1.5c &= 73.50 && \text{Combine like terms} \\ 82-0.5c &= 73.50 && \text{Isolate the variable term} \\ -0.5c &= -8.50 && \text{Multiply by the reciprocal of } -0.5 \\ c &= 17 && \text{Number of children tickets} \end{aligned}$$

Since $c=17$, then we can plug-n-chug $c=17$ into one of the equations to obtain a:

$$\begin{aligned} a &= 41-c && \text{Plug-n-chug } c=17 \\ a &= 41-(17) && \text{Evaluate} \\ a &= 24 && \text{Number of adult tickets} \end{aligned}$$

Thus, there were 17 children tickets and 24 adult tickets sold.

Example 4.4.2

Aaron invests $\$9,700$ in two different accounts. The first account paid 7%, the second account paid 11% in interest. At the end of the first year he had earned $\$863$ in interest. How much was in each account?

Solution

First, we can make a table to organize the given information and then create an equation. Let x represent the amount of investment in the first account and y represent the amount of investment in the second account.

Table 4.4.2

	Principal	rate	Total Interest
Account 1	x	0.07	$0.07x$
Account 2	y	0.11	$0.11y$
Total	$\$9,700$		$\$863$

Now let's set up the system. The total interest made in the one year was \$863 and the total invested was \$9,700, so we obtain the system

$$\begin{aligned} x+y &= 9700 \\ 0.07x+0.11y &= 863 \end{aligned}$$

At this point, we can solve using any method we choose. Since the coefficient of x and y in the first equation are both one, then let's use the method of substitution. We will solve for y in the first equation:

$$\begin{aligned} x+y &= 9700 \\ y &= 9700-x \end{aligned}$$

Now, we can substitute y into the second equation and solve:

$$\begin{aligned} 0.07x+0.11(9700-x) &= 863 && \text{Distribute} \\ 0.07x+1067-0.11x &= 863 && \text{Combine like terms} \\ -0.04x+1067 &= 863 && \text{Isolate the variable term} \\ -0.04x &= -204 && \text{Multiply by the reciprocal of } -0.04 \\ x &= 5100 && \text{Investment ammount for Account 1} \end{aligned}$$

Since the investment amount for Account 1 was $\$5,100$, then the investment amount for Account 2 was $\$4,600$ $(9700-5100=4600)$. Thus, the investment amounts for Account 1 and Account 2 was $\$5,100$ and $\$4,600$, respectively.

Mixture Problems

Example 4.4.3

A farmer has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 42 gallons of 20% butterfat?

Solution

First, we can make a table to organize the given information and then create a system. Let x represent the number of gallons of the 24% butterfat milk and y represent the number of gallons of the 18% butterfat milk.

Table 4.4.3

	Amount	Concentration	Total Butterfat
24% butterfat	x	0.24	$0.24x$
18% butterfat	y	0.18	$0.18y$
20% butterfat	42	0.20	$0.20(42)$

Now let's set up the system:

$$\begin{aligned} x+y &= 42 \\ 0.24x+0.18y &= 8.4 \end{aligned}$$

At this point, we can solve using any method we choose. Let's solve using elimination. We can choose to eliminate x and we will multiply the first equation by -0.24:

$$\begin{aligned} -0.24\cdot(x+y) &= (42)\cdot -0.24 \quad \text{Distribute} \\ -0.24x-0.24y &= -10.08 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs. Now we can add and eliminate x:

$$\begin{aligned} -0.24x-0.24y &= -10.08 \\ +\quad 0.24x+0.18y &= 8.4 \qquad \text{Add the equations} \\ \hline -0.06y &= -1.68 \end{aligned}$$

Now, we can easily solve as usual:

$$\begin{aligned} -0.06y &= -1.68 \quad \text{Multiply by the reciprocal of } -0.06 \\ y &= 28 \quad \text{Number of gallons from the } 18\% \text{ butterfat} \end{aligned}$$

Since the number of gallons from the 18% butterfat milk was 28, then the number of gallons from the 24% butterfat milk was $14(42-28=14)$. Thus, the farmer will need 14 gallons of the 24% butterfat milk and 28 gallons of the 18% butterfat milk to make 42 gallons of a 20% butterfat milk.

Example 4.4.4

A solution of pure antifreeze is mixed with water to make a 65% antifreeze solution. How much of each should be used to make 70 liters?

Solution

Mixture problems with a pure solution or water contains no other chemicals. For pure solutions, the percentage is 100% (or 1 in the table) and for water, the percentage is 0%. First, we can make a table to organize the given information and then create a system. Let a represent the number of liters of antifreeze and w represent the number of liters of water.

Table 4.4.4

	Amount	Concentration	Total Butterfat
Antifreeze	a	1	$1a$
Water	w	0	0
65% Solution	70	0.65	$0.65(70)$

Now let's set up the system:

$$\begin{aligned} a+w &= 70 \\ 1a &= 45.5 \end{aligned}$$

At this point, we can solve using any method we choose. Since we see from the system that $a=45.5$, then let's solve by substitution. We can put the second equation in for a in the first equation:

$$\begin{aligned} a+w &= 70 \quad \text{Plug-n-chug } a=45.5 \\ 45.5+w &= 70 \quad \text{Isolate the variable term} \\ w &= 24.5 \quad \text{The number of liters in water} \end{aligned}$$

Thus, the number of liters of water needed is 24.5 liters and antifreeze needed is 45.5 liters.

Example 4.4.5

In a candy shop, chocolate, which sells for $\$4$ per pound, is mixed with nuts, which are sold for $\$2.50$ per pound, to form a chocolate-nut candy which sells for $\$3.50$ a pound. How many pounds of each are used to make 30 pounds of the mixture?

Solution

First, we can make a table to organize the given information and then create an equation. Let c represent the number of pounds of chocolate and n represent the number of pounds of nuts.

Table 4.4.5

	Amount	Cost	Total Cost
Chocolate	c	$\$4$	$4c$
Nuts	n	$\$2.50$	$2.5n$
Mix	30	$\$3.50$	$3.5(30)$

Now let's set up the system:

$$\begin{aligned} c+n &= 30 \\ 4c+2.5n &= 105 \end{aligned}$$

At this point, we can solve using any method we choose. Let's solve using elimination. We can choose to eliminate c and we will multiply the first equation by -4:

$$\begin{aligned} -4\cdot(c+n) &= (3)\cdot -4 \quad \text{Distribute} \\ -4c-4n &= -120 \end{aligned}$$

Notice the c variable terms have the same coefficients with opposite signs. Now we can add and eliminate c:

$$\begin{array}{rl} -4c-4n=-120 & \\ +\quad 4c+2.5n=105 & \quad \text{Add the equations} \\ \hline -1.5n=-15 & \end{array}$$

Now, we can easily solve as usual:

$$\begin{aligned} -1.5n &= -15 \quad \text{Multiply by the reciprocal of } -1.5 \\ n &= 10 \quad \text{Number of pounds of nuts} \end{aligned}$$

Since we need 10 pounds of nuts, then this implies that we need 20 pounds of chocolate $(30-10=20)$.

Uniform Motion with Unknown Rates

When we looked at uniform motion in a prior chapter, we always were given something about the rate. However, now we discuss uniform motion with a wind force and a water current rates, where we know very little and will have to use a system to solve for the rates.

Example 4.4.6

Turkey the Pigeon travels the same distance of 72 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

Solution

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of Turkey in still air and w represent the rate of the wind. If Turkey travels with the wind, then Turkey is getting a little push from the wind, meaning traveling a little faster. If Turkey travels against the wind, then Turkey is getting a little push back from the wind, meaning traveling a little slower.

Table 4.4.6

	rate	time	distance
with the wind	$r+w$	3	$3(r+w)$
against the wind	$r-w$	4	$4(r-w)$

Since Turkey is traveling a distance of 72 miles, then this is the distance for both routes. Now let's set up the system:

$$\begin{aligned} 3(r+w) &= 72 \\ 4(r-w) &= 72 \end{aligned}$$

Since 3 is a factor of 72 and 4 is a factor of 72, let's divide each side of each equation:

$$\begin{aligned} \frac{3(r+w)}{3} &= \frac{72}{3} \\ r+w &= 24 \\ \frac{4(r-w)}{4} &= \frac{72}{4} \\ r-w &= 18 \end{aligned}$$

Notice the w variable terms have the same coefficients with opposite signs. Now we can add and eliminate w:

$$\begin{array}{rl} r+w=24 & \\ +\quad r-w=18 & \text{Add the equations} \\ \hline 2r=42 & \end{array}$$

Now, we can easily solve as usual:

$$\begin{aligned} 2r &= 42 \quad \text{Multiply by the reciprocal of } 2 \\ r &= 21 \quad \text{Rate of Turkey in still air} \end{aligned}$$

Since Turkey's rate is 21 miles per hour in still air, then this implies that the rate of the wind is 3 miles per hour $(24-21=3)$.

Example 4.4.7

A boat travels upstream for 156 miles in 3 hours and returns in 2 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Solution

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of boat in still water and c represent the rate of the current. If the boat travels with the current, then the boat is getting a little push from the current, meaning traveling a little faster. If the boat travels against the current, then the boat is getting a little push back from the current, meaning traveling a little slower.

Table 4.4.7

	rate	time	distance
with the current	$r+c$	2	$2(r+c)$
against the current	$r-c$	3	$3(r-c)$

Since the boat is traveled a distance of 156 miles, then this is the distance for both routes. Now let's set up the system:

$$\begin{aligned} 2(r+c) &= 156 \\ 3(r-c) &= 156 \end{aligned}$$

Since 2 is a factor of 156 and 3 is a factor of 156, let's divide each side of each equation:

$$\frac{2(r+c)=156}{2 \quad 2}$$
$$r+c=78$$
$$\frac{3(r-c)=156}{3 \quad 3}$$
$$r-c=32$$

Notice the c variable terms have the same coefficients with opposite signs. Now we can add and eliminate c:

$$\begin{array}{rl} r+c=78 & \\ +\quad r-c=52 & \text{Add the equations} \\ \hline 2r=130 & \end{array}$$

Now, we can easily solve as usual:

$$\begin{array}{rl} 2r=130 & \text{Multiply by the reciprocal of } 2 \\ r=65 & \text{Rate of the boat in still water} \end{array}$$

Since the rate of the boat in still water is 65 miles per hour, then this implies that the rate of the current is 13 miles per hour $(65-52=13)$.

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4.4.1: Applications with Systems of Equations (Exercises)

Exercise 4.4.1.1

The attendance at a school concert was 578. Admission was $2.00 for adults and $1.50 for children. The total receipts were $985.00. How many adults and how many children attended?

Exercise 4.4.1.2

There were 429 people at a play. Admission was $1 each for adults and 75 cents each for children. The receipts were $372.50. How many children and how many adults attended?

Exercise 4.4.1.3

There were 200 tickets sold for a women's basketball game. Tickets forstudents were 50 cents each and for adults 75 cents each. The total amount of money collected was $132.50. How many of each type of ticket was sold?

Exercise 4.4.1.4

There were 203 tickets sold for a volleyball game. For activity-card holders, the price was $1.25 each and for non-card holders the price was $2 each. The total amount of money collected was $310. How many of each type of ticket was sold?

Exercise 4.4.1.5

At a local ball game the hot dogs sold for $2.50 each and the hamburgers sold for $2.75 each. There were 131 total hamburgers and hot dogs sold for a total value of $342. How many of each was sold?

Exercise 4.4.1.6

At a recent Vikings game, there was $445 in admission tickets. The cost of a student ticket was $1.50 and the cost of a non-student ticket was $2.50. A total of 232 tickets were sold. How many students and how many non-students attended the game?

Exercise 4.4.1.7

A total of $27,000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is $3,385. How much was invested at each rate?

Exercise 4.4.1.8

A total of $50,000 is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is $3,250. How much was invested at each rate?

Exercise 4.4.1.9

A total of $9,000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is $1,030. How much was invested at each rate?

Exercise 4.4.1.10

A total of $18,000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is $1,248. How much was invested at each rate?

Exercise 4.4.1.11

An inheritance of $\$10,000$ is invested in 2 ways, part at 9.5 and the remainder at 11%. The combined annual interest was $\$1,038.50$ How much was invested at each rate?

Exercise 4.4.1.12

Kerry earned a total of $\$900$ last year on his investments. If $\$7,000$ was invested at a certain rate of return and $\$9,000$ was invested in a fund with a rate that was 2% higher, find the two rates of interest.

Exercise 4.4.1.13

Jason earned $\$256$ interest last year on his investments. If $\$1,600$ was invested at a certain rate of return and $\$2,400$ was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

Exercise 4.4.1.14

Millicent earned $\$435$ last year in interest. If $\$3,000$ was invested at a certain rate of return and $\$4,500$ was invested in a fund with a rate that was 2% lower, find the two rates of interest.

Exercise 4.4.1.15

A total of $\$8,500$ is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is $\$385$. How much was invested at each rate?

Exercise 4.4.1.16

A total of $\$12,000$ was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is $\$1,005$. How much was invested at each rate?

Exercise 4.4.1.17

A total of $\$15,000$ is invested, part of it at 8% and the rest at 11%. The total interest after one year is $\$1,455$. How much was invested at each rate?

Exercise 4.4.1.18

A total of $\$17,500$ is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is $\$1,227.50$ How much was invested at each rate?

Exercise 4.4.1.19

A total of $\$6,000$ is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is $\$300$. How much was invested at each rate?

Exercise 4.4.1.20

A total of $\$14,000$ is invested, part of it at 5.5% and the rest at 9%. The total interest after one year is $\$910$. How much was invested at each rate?

Exercise 4.4.1.21

A total of $\$11,000$ is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is $\$797$. How much was invested at each rate?

Exercise 4.4.1.22

An investment portfolio earned $\$2,010$ in interest last year. If $\$3,000$ was invested at a certain rate of return and $\$24,000$ was invested in a fund with a rate that was 4% lower, find the two rates of interest.

Exercise 4.4.1.23

Samantha earned $\$1,480$ in interest last year on her investments. If $\$5,000$ was invested at a certain rate of return and $\$11,000$ was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

Exercise 4.4.1.24

Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100 cc. of a solution that is 68% acid?

Exercise 4.4.1.25

A certain grade of milk contains 10% butterfat and a certain grade of cream 60% butterfat. How many quarts of each must be taken so as to obtain a mixture of 100 quarts that will be 45% butterfat?

Exercise 4.4.1.26

A farmer has some cream which is 21% butterfat and some which is 15% butterfat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

Exercise 4.4.1.27

A syrup manufacturer has some pure maple syrup and some which is 85% maple syrup. How many liters of each should be mixed to make 150 liters which is 96% maple syrup?

Exercise 4.4.1.28

A chemist wants to make 50 mL of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

Exercise 4.4.1.29

A hair dye is made by blending 7% hydrogen peroxide solution and a 4% hydrogen peroxide solution. How many milliliters of each are used to make a 300 mL solution that is 5% hydrogen peroxide?

Exercise 4.4.1.30

A paint that contains 21% green dye is mixed with a paint that contains 15% green dye. How many gallons of each must be used to make 60 gallons of paint that is 19% green dye?

Exercise 4.4.1.31

A candy mix sells for $\$2.20$ per kilogram. It contains chocolates worth $\$1.80$ per kilogram and other candy worth $\$3.00$ per kilogram. How much of each are in 15 kilograms of the mixture?

Exercise 4.4.1.32

To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth $4 per pound with a weed killer worth $8 per pound. The mixture will cost $6.00 per pound. How much of each should be used to prepare 500 pounds of the mixture?

Exercise 4.4.1.33

A grocer is mixing a 40-cent per pound coffee with a 60-cent per pound coffee to make a mixture worth 54¢ per pound How much of each kind of coffee should be used to make 70 pounds of the mixture?

Exercise 4.4.1.34

A grocer wishes to mix sugar at 9 cents per pound with sugar at 6 cents per pound to make 60 pounds at 7 cents per pound. What quantity of each must he take?

Exercise 4.4.1.35

A high-protein diet supplement that costs $6.75 per pound is mixed with a vitamin supplement that costs $3.25 per pound. How many pounds of each should be used to make 5 pounds of a mixture that costs $4.65 per pound?

Exercise 4.4.1.36

A goldsmith combined an alloy that costs $4.30 per ounce with an alloy that costs $1.80 per ounce. How many ounces of each were used to make a mixture of 200 ounces costing $2.50 per ounce?

Exercise 4.4.1.37

A grocery store offers a cheese and fruit sampler that combines cheddar cheese that costs $8 per kilogram with kiwis that cost $3 per kilogram. How many kilograms of each were used to make a 5 kilogram mixture that costs $4.50 per kilogram?

Exercise 4.4.1.38

A caterer made an ice cream punch by combining fruit juice that cost $2.25 per gallon with ice cream that costs $3.25 per gallon. How many gallons of each were used to make 100 gallons of punch costing $2.50 per pound?

Exercise 4.4.1.39

A clothing manufacturer has some pure silk thread and some thread that is 85% silk. How many kilograms of each must be woven together to make 75 kilograms of cloth that is 96% silk?

Exercise 4.4.1.40

A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 pounds of a fabric that is 28% wool?

Exercise 4.4.1.41

The manager of a specialty food store combined almonds that cost $4.50 per pound with walnuts that cost $2.50 per pound. How many pounds of each were used to make a 100-pound mixture that cost $3.24 per pound?

Exercise 4.4.1.42

A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 pounds of tea that is 18% jasmine?

Exercise 4.4.1.43

How many milliliters of pure chocolate must be added to 150 mL of chocolate topping that is 50% chocolate to make a topping that is 75% chocolate?

Exercise 4.4.1.44

How many ounces of pure bran flakes must be added to 50 ounces of cereal that is 40% bran flakes to produce a mixture that is 50% bran flakes?

Exercise 4.4.1.45

A ground meat mixture is formed by combining meat that costs $2.20 per pound with meat that costs $4.20 per pound. How many pounds of each were used to make a 50-pound mixture that costs $3 per pound?

Exercise 4.4.1.46

How many grams of pure water must be added to 50 grams of pure acid to make a solution that is 40% acid?

Exercise 4.4.1.47

A lumber company combined oak wood chips that cost $3.10 per pound with pine wood chips that cost $2.50 per pound. How many pounds of each were used to make an 80-pound mixture costing $2.65 per pound?

Exercise 4.4.1.48

How many ounces of pure water must be added to 50 ounces of a 15% saline solution to make a saline solution that is 10% salt?

Exercise 4.4.1.49

A boat travels upstream for 216 miles in 4 hours and returns in 3 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Exercise 4.4.1.50

A boat travels upstream for 12 miles in 3 hours and returns in 2 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Exercise 4.4.1.51

A boat travels upstream for 336 miles in 4 hours and returns in 3 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Exercise 4.4.1.52

Turkey the Pigeon travels the same distance of 280 miles in 5 hours against the wind as it does traveling 4 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

Exercise 4.4.1.53

Turkey the Pigeon travels the same distance of 24 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

Exercise 4.4.1.54

Turkey the Pigeon travels the same distance of 120 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

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4.5: Systems of Three Linear Equations in Three Variables

Solving systems of linear equations in three variables is very similar to the methods in which we solve linear systems in two variables. With linear systems in two variables, we reduced the system down to one linear equation in one variable (by substitution or addition). With linear systems in three variables, we apply the same method except we reduce the system down from three linear equations in three variables to two linear equations in two variables first, then to one linear equation in one variable.

Definition: System of Three Linear Equations in Three Variables

A ***system of three linear equations in three variables*** is given in the form

$$\begin{cases} ax+by+cz &= d \\ ex+fy+gz &= h \\ ix+jy+kz &= \ell \end{cases}$$

where a, b, c, d, e, f, g, h, i, j, k and ℓ are coefficients and x, y, and z are variables. This system is represented in *standard form*.

Verifying Solutions

As we did is the previous section for verifying solutions with two linear equations in two variables, we can verify whether an ***ordered triple*** is a solution to the system of linear equations in three variables.

Example 4.5.1

Determine whether $(5,-3,-4)$ is a solution to the system:

$$\begin{cases} x+y+z=-2 \\ x+2y-3z=12 \\ 2x-2y+z=-9 \end{cases}$$

Solution

To verify whether $(5,-3,-4)$ is the solution to the system, we plug-n-chug $(5,-3,-4)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for all equations in the system, then $(5,-3,-4)$ will be the solution to the system.

$$\begin{aligned} x+y+z&=-2 && \text{Plug-n-chug } x=5,\ y=-3 \text{ and } z=-4 \\ 5+(-3)+(-4) &\stackrel{?}{=} -2 && \text{Simplify} \\ 5-3-4 &\stackrel{?}{=} -2 && \text{Subtract} \\ -2&=-2 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} x+2y-3z&=12 && \text{Plug-n-chug } x=5,\ y=-3 \text{ and } z=-4 \\ 5+2(-3)-3(-4) &\stackrel{?}{=} 12 && \text{Simplify} \\ 11&=12 && X \text{ False} \end{aligned}$$

Since the ordered triple $(5,-3,-4)$ makes the second equation false, then $(5,-3,-4)$ is not the solution to the system.

Example4.5.2

Let's try a different ordered triple. Determine whether $(-1,2,-3)$ is a solution to the system:

$$\begin{cases} x+y+z=-2 \\ x+2y-3z=12 \\ 2x-2y+z=-9 \end{cases}$$

Solution

To verify whether $(-1, 2, -3)$ is the solution to the system, we plug-n-chug $(-1, 2, -3)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for all equations in the system, then $(-1, 2, -3)$ will be the solution to the system.

$$\begin{aligned} x+y+z &= -2 && \text{Plug-n-chug } x=-1,\ y=2 \text{ and } z=-3 \\ -1+2+(-3) &\stackrel{?}{=} -2 && \text{Simplify} \\ -2 &= -2 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} x+2y-3z &= 12 && \text{Plug-n-chug } x=-1,\ y=2 \text{ and } z=-3 \\ -1+2(2)-3(-3) &\stackrel{?}{=} 12 && \text{Simplify} \\ 12 &= 12 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the third equation:

$$\begin{aligned} 2x-2y+z &= -9 && \text{Plug-n-chug } x=-1,\ y=2 \text{ and } z=-3 \\ 2(-1)-2(2)+(-3) &\stackrel{?}{=} -9 && \text{Simplify} \\ -9 &= -9 && \checkmark \text{ True} \end{aligned}$$

Since the ordered triple $(-1, 2, -3)$ makes all equations in the system true, then $(-1, 2, -3)$ is a solution to the system.

Solving Systems of Three Linear Equations in Three Variables

In Example 4.5.2 , we determined that the ordered tripe $(-1, 2, -3)$ is a solution to the system

$$\begin{cases} x+y+z=-2 \\ x+2y-3z=12 \\ 2x-2y+z=-9 \end{cases}$$

What does this mean? With systems of two linear equations in two variables, we know that if the system is consistent with an independent solution, then the solution is an ordered pair and two lines intersect. However, with three linear equations in three variables, what does a system that is consistent with an independent solution look like? We do know the answer is an ***ordered triple*** of form (x, y, z), but what does the graph look like? Well, let's take a look.

If we obtain a consistent system with an independent solution, then the solution is an ordered triple. This unique ordered triple represents the *intersection of three planes*.

However, experience from the previous sections tells us that this isn't always the case, right? With two linear equations in two variables, we have two special cases: *no solution* and *infinitely many solutions on line* $y = mx + b$.

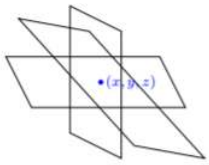
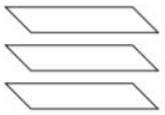


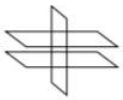
Figure 4.5.1 : In this figure, we see the intersection of three planes at point (x, y, z). This consistent system is independent with solution (x, y, z).

Recall. *No solution* meant the two lines were parallel and never intersected. *Infinitely many solutions on line* $y = mx + b$ meant that the two lines were, in fact, the same line $y = mx + b$.

Let us take look at two cases with systems that are inconsistent, but now with three linear equations in three variable and with planes.



(a) Inconsistent system



(b) Inconsistent system

Figure 4.5.2 : Here we can see two examples of inconsistent systems where in 4.2a, all three planes could be parallel, and in 4.2b, one plane intersects two parallel planes.

Lastly, let us take look at a case with a consistent system that is dependent, but now with three linear equations in three variable and with planes.



(a) Consistent system that is dependent

Figure 4.5.3 : In 4.3a, we see all three planes are the same plane and this example is called a consistent system that is dependent with solutions of form (x, y, z), where the solution is usually rewritten as $\{(x, y, z)|ax + by + cy = d\}$.

Example 4.5.3

Solve the system:

$$\begin{cases} 3x + 2y - z = -1 \\ -2x - 2y + 3z = 5 \\ 5x + 2y - z = 3 \end{cases}$$

Solution

Let's go ahead and number each equation so that we can identify each equation.

$$3x + 2y - z = -1 \quad (1)$$
$$-2x - 2y + 3z = 5 \quad (2)$$
$$5x + 2y - z = 3 \quad (3)$$

First, we choose a variable to eliminate. Then take two equations, say (1) and (2), and eliminate the chosen variable. Let's choose y since we can see that the coefficients of y are the same and we can easily eliminate it.

$$3x + 2y - z = -1 \quad (1)$$
$$-2x - 2y + 3z = 5 \quad (2)$$

Adding (1) and (2), we obtain

$$x + 2z = 4 \quad (4)$$

Now, let's take equations (2) and (3) and eliminate y again:

$$-2x - 2y + 3z = 5 \quad (2)$$
$$5x + 2y - z = 3 \quad (3)$$

Adding (2) and (3), we obtain

$$3x + 2z = 8 \quad (5)$$

Next, take equations (4) and (5). Notice, we have a system of two linear equations in two variables:

$$x + 2z = 4 \quad (4)$$
$$3x + 2z = 8 \quad (5)$$

We use the same process as we did in the previous section to obtain the solution for x and z. Then substitute those values into one of the original equations to obtain y. Thus, obtaining the ordered triple solution.

Let's choose to eliminate z and solve for x. First, let's multiply equation (4) by -1.

$$-1(x+2z) = -1(4) \quad (4)$$
$$3x+2z = 8 \quad (5)$$

Now, we can solve for x:

$$-x-2z = -4 \quad (4)$$
$$3x+2z = 8 \quad (5)$$

Adding these together, we get

$$2x = 4$$
$$x = 2$$

If $x = 2$, then this implies that $z = 1$, i.e.,

$$x+2z = 4 \quad (4)$$
$$2+2z = 4$$
$$2z = 2$$
$$z = 1$$

Lastly, we take $x = 2$ and $z = 1$ and substitute them into one of the original equations, like (1), and solve for y:

$$3x+2y-z = -1 \quad (1)$$
$$3(2)+2y-(1) = -1$$
$$6+2y-1 = -1$$
$$5+2y = -1$$
$$2y = -6$$
$$y = -3$$

Thus, the point of intersection of the three planes is the ordered triple $(2, -3, 1)$.

Note

Around 250 BCE, *The Nine Chapters on the Mathematical Art* were published in China. This book had 246 problems, and Chapter 8 was about solving systems of equations. One problem had four equations with five variables!

Special Cases

Looking back at Figure 1.2, we see that there are two special cases: Inconsistent systems and consistent systems with dependent solutions. Let's us see examples of these special cases.

Example 4.5.4

Solve the system:

$$\begin{cases} 5x-4y+3z = -4 \\ -10x+8y-6z = 8 \\ 15x-12y+9z = -12 \end{cases}$$

Solution

Let's go ahead and number each equation so that we can identify each equation.

$$5x-4y+3z = -4 \quad (1)$$
$$-10x+8y-6z = 8 \quad (2)$$
$$15x-12y+9z = -12 \quad (3)$$

First, we choose a variable to eliminate. Then take two equations, say (1) and (2), and eliminate the chosen variable. Let's choose z since we can see that the coefficients of z are almost the same and we can easily eliminate it. We first multiply equation (1) by 2 and then add.

$$\begin{aligned} 2(5x-4y+3z) &= 2(-4) && (1)\\ -10x+8y-6z &= 8 && (2)\end{aligned}$$

Now, we can add to eliminate z:

$$\begin{aligned} 10x-8y+6z &= -8 && (1)\\ -10x+8y-6z &= 8 && (2)\end{aligned}$$

Adding (1) and (2), we obtain

$$0=0 \quad \checkmark \text{ True}$$

Since all variables eliminate and we are left with a true statement, we know this is a consistent system with dependent solutions. However, what is the solution? We see from Figure 4.5.3 a that this means all three planes are the same plane. Hence, we should write the solution as $\{(x,y,z)|5x-4y+3z=-4\}$.

Example 4.5.5

Solve the system:

$$\begin{cases} 3x-4y+z=2 \\ -9x+12y-3z=-5 \\ 4x-2y-z=3 \end{cases}$$

Solution

Let's go ahead and number each equation so that we can identify each equation.

$$\begin{aligned} 3x-4y+z &= 2 && (1)\\ -9x+12y-3z &= -5 && (2)\\ 4x-2y-z &= 3 && (3)\end{aligned}$$

First, we choose a variable to eliminate. Then take two equations, say (1) and (2), and eliminate the chosen variable. Let's choose x since we can see that the coefficients of x are almost the same and we can easily eliminate it. We first multiply equation (1) by 3 and then add.

$$\begin{aligned} 3(3x-4y+z) &= 3(2) && (1)\\ -9x+12y-3z &= -5 && (2)\end{aligned}$$

Now, we can add to eliminate z:

$$\begin{aligned} 9x-12y+3z &= 6 && (1)\\ -9x+12y-3z &= -5 && (2)\end{aligned}$$

Adding (1) and (2), we obtain

$$0=1 \quad X \text{ False}$$

Since all variables eliminate and we are left with a false statement, we know this is an inconsistent system and there is no solution.

Systems of equations in three (or more) variables are not any more difficult than systems of equations in two variables as long as we are careful to keep organized. It is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in a different order.

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4.5.1: Systems of Three Linear Equations in Three Variables (Exercises)

Solve each system. Determine if each system is consistent, independent or dependent, or inconsistent. Write the solution as an ordered triple, if possible.

Exercise 4.5.1.1

$$a-2b+c=5$$
$$2a+b-c=-1$$
$$3a+3b-2c=-4$$

Exercise 4.5.1.2

$$3x+y-z=11$$
$$x+3y=z+13$$
$$x+y-3z=11$$

Exercise 4.5.1.3

$$x+6y+3z=4$$
$$2x+y+2z=3$$
$$3x-2y+z=0$$

Exercise 4.5.1.4

$$x+y+z=6$$
$$2x-y-z=-3$$
$$x-2y+3z=6$$

Exercise 4.5.1.5

$$x+y-z=0$$
$$x-y-z=0$$
$$x+y+2z=0$$

Exercise 4.5.1.6

$$-2x+y-3z=1$$
$$x-4y+z=6$$
$$4x+16y+4z=24$$

Exercise 4.5.1.7

$$2x+y-3z=0$$
$$x-4y+z=0$$
$$4x+16y+4z=0$$

Exercise 4.5.1.8

$$3x+2y+2z=3$$
$$x+2y-z=5$$
$$2x-4y+z=0$$

Exercise 4.5.1.9

$$x-2y+3z=4$$
$$2x-y+z=-1$$
$$4x+y+z=1$$

Exercise 4.5.1.10

$$x-y+2z=0$$
$$x-2y+3z=-1$$
$$2x-2y+z=-3$$

Exercise 4.5.1.11

$$4x-3y+2z=40$$
$$5x+9y-7z=47$$
$$9x+8y-3z=97$$

Exercise 4.5.1.12

$$2x+3y=z-1$$
$$3x=8z-1$$
$$5y+7z=-1$$

Exercise 4.5.1.13

$$x+y+z=2$$
$$6x-4y+5z=31$$
$$5x+2y+2z=13$$

Exercise 4.5.1.14

$$x-y+2z=-3$$
$$x+2y+3z=4$$
$$2x+y+z=-3$$

Exercise 4.5.1.15

$$x+y-z=0$$
$$x+2y-4z=0$$
$$2x+y+z=0$$

Exercise 4.5.1.16

$$x+2y-z=4$$
$$4x-3y+z=8$$
$$5x-y=12$$

Exercise 4.5.1.17

$4x+12y+16z=4$
$3x+4y+5z=3$
$x+8y+11z=1$

Exercise 4.5.1.18

$4x+12y+16z=0$
$3x+4y+5z=0$
$x+8y+11z=0$

Exercise 4.5.1.19

$p+q+r=1$
$p+2q+3r=4$
$4p+5q+6r=7$

Exercise 4.5.1.20

$x+2y-3z=9$
$2x-y+2z=-8$
$3x-y-4z=3$

Exercise 4.5.1.21

$4x-7y+3z=1$
$3x+y-2z=4$
$4x-7y+3z=6$

Exercise 4.5.1.22

$3x+y-z=10$
$8x-y-6z=-3$
$5x-2y-5z=1$

Exercise 4.5.1.23

$3x+3y-2z=13$
$6x+2y-5z=13$
$5x-2y-5z=-1$

Exercise 4.5.1.24

$3x-4y+2z=1$
$2x+3y-3z=-1$
$x+10y-8z=7$

Exercise 4.5.1.25

$m+6n+3p=8$
$3m+4n=-3$
$5m+7n=1$

Exercise 4.5.1.26

$$2x-3y+5z=1$$
$$3x+2y-z=4$$
$$4x+7y-7z=7$$

Exercise 4.5.1.27

$$2x+y=z$$
$$4x+z=4y$$
$$y=x+1$$

Exercise 4.5.1.28

$$3x+2y=z+2$$
$$y=1-2x$$
$$3z=-2y$$

Exercise 4.5.1.29

$$-2w+2x+2y-2z=-10$$
$$w+x+y+z=-5$$
$$3w+2x+2y+4z=-11$$
$$w+3x-2y+2z=-6$$

Exercise 4.5.1.30

$$w+x+y+z=2$$
$$w+2x+2y+4z=1$$
$$-w+x-y-z=-6$$
$$-w+3x+y-z=-2$$

Exercise 4.5.1.31

$$-w+2x-3y+z=-8$$
$$-w+x+y-z=-4$$
$$w+x+y+z=22$$
$$-w+x-y-z=-14$$

Exercise 4.5.1.32

$$w+x-y+z=0$$
$$-w+2x+2y+z=5$$
$$-w+3x+y-z=-4$$
$$-2w+x+y-3z=-7$$

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CHAPTER OVERVIEW

5: Matrices and Systems of Linear Equations

5.1: Introduction to Matrices

5.1.1: Introduction to Matrices (Exercises)

5.2: Systems of Linear Equations and the Gauss-Jordan Method

5.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)

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5.1: Introduction to Matrices

Learning Objectives

In this section, you will learn to:

1. Add and subtract matrices.
2. Multiply a matrix by a scalar.
3. Multiply two matrices.

A matrix is a 2 dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. Matrices have an abundance of applications and use in the real world. Matrices provide a useful tool for working with models based on systems of linear equations. We'll use matrices in sections 2.2, 2.3, and 2.4 to solve systems of linear equations with several variables in this chapter.

Matrices are used in encryption, which we will explore in section 2.5 and in economic modelling, explored in section 2.6. We use matrices again in chapter 4, in optimization problems such as maximizing profit or revenue, or minimizing cost. Matrices are used in business for scheduling, routing transportation and shipments, and managing inventory.

Just about any application that collects and manages data can apply matrices. Use of matrices has grown as the availability of data in many areas of life and business has increased. They are important tools for organizing data and solving problems in all fields of science, from physics and chemistry, to biology and genetics, to meteorology, and economics. In computer science, matrix mathematics lies behind animation of images in movies and video games.

Computer science analyzes diagrams of networks to understand how things are connected to each other, such as relationships between people on a social website, and relationships between results in line search and how people link from one website to another. The mathematics to work with network diagrams comprise the field of "graph theory"; it relies on matrices to organize the information in the graphs that diagram connections and associations in a network. For example, if you use Facebook or Linked-In, or other social media sites, these sites use network graphs and matrices to organize your relationships with other users.

Introduction to Matrices

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to get some idea of what matrices are all about, we will look at the following example.

Example 5.1.1

Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.

	2014		2015	
	CHAIRS	TABLES	CHAIRS	TABLES
SAN JOSE	30	18	36	20
HAYWARD	20	12	24	18
OAKLAND	16	10	20	12

a. Represent the production for the years 2014 and 2015 as the matrices A and B.
b. Find the difference in sales between the years 2014 and 2015.
c. The company predicts that in the year 2020 the production at these factories will be double that of the year 2014. What will the production be for the year 2020?

Solution

a) The matrices are as follows:

$$A = \begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix}$$

b) We are looking for the matrix $B - A$. When two matrices have the same number of rows and columns, the matrices can be added or subtracted entry by entry. Therefore, we get

$$\mathrm{B} - \mathrm{A} = \begin{bmatrix} 36-30 & 20-18 \\ 24-20 & 18-12 \\ 20-16 & 12-10 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

c) We would like a matrix that is twice the matrix of 2014, i.e., $2A$.

Whenever a matrix is multiplied by a number, each entry is multiplied by the number.

$$2\mathrm{A} = 2\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 60 & 36 \\ 40 & 24 \\ 32 & 20 \end{bmatrix}$$

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices. The numbers in a matrix are called the **entries** or the **elements** of a matrix.

Whenever we talk about a matrix, we need to know the **size** or the **dimension** of the matrix. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a "3 by 4 matrix", we are saying that it has 3 rows and 4 columns. The rows are always mentioned first and the columns second. This means that a 3×4 matrix does not have the same dimension as a 4×3 matrix.

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & -1 & 7 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 9 & 8 \\ -3 & 0 & 1 \\ 6 & 5 & -2 \\ -4 & 7 & 8 \end{bmatrix}$$

Matrix A has dimensions 3×4 and matrix B has dimensions 4×3.

A matrix that has the same number of rows as columns is called a **square matrix**. A matrix with all entries zero is called a **zero matrix.** A square matrix with 1's along the main diagonal and zeros everywhere else, is called an **identity matrix**. When a square matrix is multiplied by an identity matrix of same size, the matrix remains the same.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix I is a 3×3 identity matrix

A matrix with only one row is called a row matrix or a **row vector,** and a matrix with only one column is called a column matrix or a **column vector**. Two matrices are **equal** if they have the same size and the corresponding entries are equal.

We can perform arithmetic operations with matrices. Next we will define and give examples illustrating the operations of matrix addition and subtraction, scalar multiplication, and matrix multiplication. Note that matrix multiplication is quite different from what you would intuitively expect, so pay careful attention to the explanation. Note also that the ability to perform matrix

operations depends on the matrices involved being compatible in size, or dimensions, for that operation. The definition of compatible dimensions is different for different operations, so note the requirements carefully for each.

Matrix Addition and Subtraction

If two matrices have the same size, they can be added or subtracted. The operations are performed on corresponding entries.

Example 5.1.2

Given the matrices A, B, C and D, below

$$A=\begin{bmatrix}1&2&4\\2&3&1\\5&0&3\end{bmatrix}\quad B=\begin{bmatrix}2&-1&3\\2&4&2\\3&6&1\end{bmatrix}\quad C=\begin{bmatrix}4\\2\\3\end{bmatrix}\quad D=\begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

Find, if possible.

a. $A+B$
b. $C-D$
c. $A+D$.

Solution

As we mentioned earlier, matrix addition and subtraction involves performing these operations entry by entry.

a) We add each element of A to the corresponding entry of B.

$$A+B=\begin{bmatrix}3&1&7\\4&7&3\\8&6&4\end{bmatrix}$$

b) Just like the problem above, we perform the subtraction entry by entry.

$$\mathrm{C}-\mathrm{D}=\begin{bmatrix}6\\5\\-1\end{bmatrix}$$

c) The sum $A+D$ cannot be found because the two matrices have different sizes.

Note: Two matrices can *only* be added or subtracted if they have the same dimension.

Multiplying a Matrix by a Scalar

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar. We can consider scalar multiplication as multiplying a number and a matrix to obtain a new matrix as the product.

Example 5.1.3

Given the matrix A and C in the example above, find $2A$ and $-3C$.

Solution

To find $2A$, we multiply each entry of matrix A by 2, and to find $-3C$, we multiply each entry of C by -3. The results are given below.

a) We multiply each entry of A by 2.

$$2\mathrm{A}=\begin{bmatrix}2&4&8\\4&6&2\\10&0&6\end{bmatrix}$$

b) We multiply each entry of C by -3.

$$-3C = \begin{bmatrix} -12 \\ -6 \\ -9 \end{bmatrix}$$

Multiplication of Two Matrices

To multiply a matrix by another is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner. We first begin by finding a product of a row matrix and a column matrix.

Example 5.1.4

Find the product AB, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Solution

The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum.

$$\begin{aligned} \mathrm{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} \\ &= [(2\mathrm{a} + 3\mathrm{b} + 4\mathrm{c})] \end{aligned}$$

Note that AB is a 1×1 matrix, and its only entry is $2a + 3b + 4c$.

Example 5.1.5

Find the product AB, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Solution

Again, we multiply the corresponding entries and add.

$$\begin{aligned} \mathrm{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \\ &= [2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7] \\ &= [10 + 18 + 28] \\ &= [56] \end{aligned}$$

Note: In order for a product of a row matrix and a column matrix to exist, the number of entries in the row matrix must be the same as the number of entries in the column matrix.

Example 5.1.6

Find the product AB, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}.$$

Solution

We know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will multiply the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix.

$$\mathrm{AB} = \begin{bmatrix} 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 & 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We multiplied a 1×3 matrix by a matrix whose size is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions, if the number of entries in the rows of the first matrix is the same as the number of entries in the columns of the second matrix.

Example 5.1.7

Find the product AB, given:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

and

$$\mathrm{B} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$$

Solution

This time we are multiplying two rows of the matrix A with two columns of the matrix B. Since the number of entries in each row of A is the same as the number of entries in each column of B, the product is possible. We do exactly what we did in the last example. The only difference is that the matrix A has one more row.

We multiply the first row of the matrix A with the two columns of B, one at a time, and then repeat the process with the second row of A. We get

$$\mathrm{AB} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 & 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 \\ 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 & 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 \end{bmatrix}$$

$$\mathrm{AB} = \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

Example 5.1.8

Find, if possible:

a. EF
b. FE
c. FH
d. GH
e. HG

$$\mathrm{E}=\begin{bmatrix}1 & 2\\ 4 & 2\\ 3 & 1\end{bmatrix} \quad \mathrm{F}=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix} \quad \mathrm{G}=\begin{bmatrix}4 & 1\end{bmatrix} \quad \mathrm{H}=\begin{bmatrix}-3\\ -1\end{bmatrix}$$

Solution

a) To find EF, we multiply the first row $\begin{bmatrix}1 & 2\end{bmatrix}$ of E with the columns $\begin{bmatrix}2\\ 3\end{bmatrix}$ and $\begin{bmatrix}1\\ -2\end{bmatrix}$ of the matrix F, and then repeat the process by multiplying the other two rows of E with these columns of F. The result is as follows:

$$\mathrm{EF}=\begin{bmatrix}1 & 2\\ 4 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}=\begin{bmatrix}1\cdot 2+2\cdot 3 & 1\cdot -1+2\cdot 2\\ 4\cdot 2+2\cdot 3 & 4\cdot -1+2\cdot 2\\ 3\cdot 2+1\cdot 3 & 3\cdot -1+1\cdot 2\end{bmatrix}=\begin{bmatrix}8 & 3\\ 14 & 0\\ 9 & -1\end{bmatrix}$$

b) Product FE is not possible because the matrix F has two entries in each row, while the matrix E has three entries in each column. In other words, the matrix F has two columns, while the matrix E has three rows.

c)

$$\mathrm{FH}=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}-3\\ -1\end{bmatrix}=\begin{bmatrix}2\cdot -3+-1\cdot -1\\ 3\cdot -3+2\cdot -1\end{bmatrix}=\begin{bmatrix}-5\\ -11\end{bmatrix}$$

d)

$$\mathrm{GH}=\begin{bmatrix}4 & 1\end{bmatrix}\begin{bmatrix}-3\\ -1\end{bmatrix}=\begin{bmatrix}4\cdot -3+1\cdot -1\\ -1\end{bmatrix}=\begin{bmatrix}-13\end{bmatrix}$$

e)

$$\mathrm{HG}=\begin{bmatrix}-3\\ -1\end{bmatrix}\begin{bmatrix}4 & 1\end{bmatrix}=\begin{bmatrix}-3\cdot 4 & -3\cdot 1\\ -1\cdot 4 & -1\cdot 1\end{bmatrix}=\begin{bmatrix}-12 & -3\\ -4 & -1\end{bmatrix}$$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

In order for product $\mathbf{AB}$ to exist:

- **the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$**
- **if matrix $\mathbf{A}$ has dimension $\mathbf{m\times n}$ and matrix $\mathbf{B}$ has dimension $\mathbf{n\times p}$, then the product $\mathbf{AB}$ will be a matrix with dimension $\mathbf{m\times p}$.**

Matrix multiplication is not commutative: if both matrix products $\mathbf{AB}$ and $\mathbf{BA}$ exist, most of the time $\mathbf{AB}$ will not equal $\mathbf{BA}$.

Example 5.1.9

Given matrices R, S, and T below, find $2RS-3ST$.

$$R=\begin{bmatrix}1 & 0 & 2\\ 2 & 1 & 5\\ 2 & 3 & 1\end{bmatrix} \quad S=\begin{bmatrix}0 & -1 & 2\\ 3 & 1 & 0\\ 4 & 2 & 1\end{bmatrix} \quad T=\begin{bmatrix}-2 & 3 & 0\\ -3 & 2 & 2\\ -1 & 1 & 0\end{bmatrix}$$

Solution

We multiply the matrices R and S.

$$\mathrm{RS} = \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix}$$

$$2\mathrm{RS} = 2\begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix}$$

$$\mathrm{ST} = \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

$$3\mathrm{ST} = 3\begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix}$$

Thus

$$2\mathrm{RS} - 3\mathrm{ST} = \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix} = \begin{bmatrix} 13 & 6 & 14 \\ 73 & -15 & 12 \\ 71 & -45 & -2 \end{bmatrix}$$

Example 5.1.10

Find F^2 given matrix

$$\mathrm{F} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

Solution

F^2 is found by multiplying matrix F by itself, using matrix multiplication.

$$\mathrm{F}^2 = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 2+(-1)\cdot 3 & 2\cdot(-1)+(-1)\cdot 2 \\ 3\cdot 2+2\cdot 3 & 3\cdot(-1)+2\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix}$$

Note that F^2 is not found by squaring each entry of matrix F. The process of raising a matrix to a power, such as finding F^2, is only possible if the matrix is a square matrix.

USING MATRICES TO REPRESENT A SYSTEM OF LINEAR EQUATIONS

In this chapter, we will be using matrices to solve linear systems. In section 2.4, we will be asked to express linear systems as the **matrix equation $\mathbf{AX} = \mathbf{B}$**, where A, X, and B are matrices.

- Matrix A is called the **coefficient matrix.**
- Matrix X is a matrix with 1 column that contains the variables.
- Matrix B is a matrix with 1 column that contains the constants.

Example 5.1.11

Verify that the system of two linear equations with two unknowns:

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned} \tag{5.1.1}$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} h \\ k \end{bmatrix}$$

Solution

If we multiply the matrices A and X, we get

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

If $AX = B$ then

$$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$\begin{aligned} ax+by &= h \\ cx+dy &= k \end{aligned} \tag{5.1.2}$$

Example 5.1.12

Express the following system as a matrix equation in the form $AX = B$.

$$\begin{aligned} 2x+3y-4z &= 5 \\ 3x+4y-5z &= 6 \\ 5x \quad -6z &= 7 \end{aligned} \tag{5.1.3}$$

Solution

This system of equations can be expressed in the form $AX = B$ as shown below.

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

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5.1.1: Introduction to Matrices (Exercises)

A vendor sells hot dogs and corn dogs at three different locations. His total sales(in hundreds) for January and February from the three locations are given in the table below.

	JANUARY			FEBRUARY	
	HOT DOGS	CORN DOGS		HOT DOGS	CORN DOGS
PLACE I	10	8		8	7
PLACE II	8	6		6	7
PLACE III	6	4		6	5

Represent these tables as 3×2 matrices J and F, and answer problems 1 - 5.

1) Determine total sales for the two months, that is, find $J+F$.	2) Find the difference in sales, $J-F$.
3) If hot dogs sell for \$3 and corn dogs for \$2, find the revenue from the sale of hot dogs and corn dogs. *Hint: Let P be a 2×1 matrix. Find $(J+F)P$.*	4) If March sales will be up from February by 10%, 15%, and 20% at Place I, Place II, and Place III, respectively, find the expected number of hot dogs and corn dogs to be sold in March. *Hint: Let R be a 1×3 matrix with entries 1.10, 1.15, and 1.20. Find $M = RF$.*
5) Hots dogs sell for \$3 and corn dogs sell for \$2. Using matrix M that predicts the number of hot dogs and corn dogs expected to be sold in March from problem (4), find the 1×1 matrix that predicts total revenue in March. *Hint: Use 2×1 price matrix P from problem (3) and find MP.*	

Determine the sums and products in problems 6-13. Given the matrices A, B, C, and D as follows:

$$\mathrm{A} = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \mathrm{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathrm{D} = \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}$$

6) $3A-2B$	7) AB
8) BA	9) $AB+BA$
10) A^2	11) $2BC$
12) $2CD+3AB$	13) A^2B

14) Let $E = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ and $\mathrm{F} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$, find EF.	15) Let $E = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ and $\mathrm{F} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$, find FE.
16) Let $G = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find GH.	17) Let $G = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Explain why the product HG does not exist.

Express the following systems as $AX = B$, where A, X, and B are matrices.

18)

$$\begin{aligned} 4x - 5y &= 6 \\ 5x - 6y &= 7 \end{aligned} \qquad (5.1.1.1)$$

19)

$$\begin{aligned} x - 2y + 2z &= 3 \\ x - 3y + 4z &= 7 \\ x - 2y - 3z &= -12 \end{aligned} \qquad (5.1.1.2)$$

20)

$$\begin{aligned} 2x+3z &= 17 \\ 3x-2y &= 10 \\ 5y+2z &= 11 \end{aligned} \tag{5.1.1.3}$$

21)

$$\begin{aligned} x &+ 2y &+ 3z &+ 2w &&= 14 \\ x &- 2y &- z & &&= -5 \\ & y &- 2z &+ 4w &&= 9 \\ x & & + 3z &+ 3w &&= 15 \end{aligned} \tag{5.1.1.4}$$

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5.2: Systems of Linear Equations and the Gauss-Jordan Method

Learning Objectives

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix
2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the **augmented matrix**, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a **row operation**.

Example 5.2.1

Write the following system as an augmented matrix.

$$\begin{aligned} 2x+3y-4z&=5\\ 3x+4y-5z&=-6\\ 4x+5y-6z&=7 \end{aligned}$$

Solution

We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$

In the last section, we expressed the system of equations as $AX=B$, where A represented the coefficient matrix, and B the matrix of constant terms. As an augmented matrix, we write the matrix as $\left[\begin{array}{c|c} A & B \end{array}\right]$. It is clear that all of the information is maintained in this matrix form, and only the letters x, y and z are missing. A student may choose to write x, y and z on top of the first three columns to help ease the transition.

Example 5.2.2

For the following augmented matrix, write the system of equations it represents.

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 2 & 0 & -3 & -5 \\ 3 & 2 & -3 & -1 \end{array}\right]$$

Solution

The system is readily obtained as below.

$$\begin{aligned} x+3y-5z&=2\\ 2x-3z&=-5\\ 3x+2y-3z&=-1 \end{aligned}$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the **reduced row**

echelon form. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

Example 5.2.3

Solve the following system by the elimination method.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

Solution

We multiply the first equation by – 3, and add it to the second equation.

$$\begin{array}{r} -3x-9y = -21 \\ \underline{3x+4y = 11} \\ -5y = -10 \end{array}$$

By doing this we transformed our original system into an equivalent system:

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

We divide the second equation by – 5, and we get the next equivalent system.

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \end{aligned}$$

Now we multiply the second equation by – 3 and add to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

Example 5.2.4

Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned} \tag{5.2.1}$$

Solution

The augmented matrix for the system is as follows.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \quad \left[\begin{array}{c} x+3y=7 \\ 3x+4y=11 \end{array}\right]$$

We multiply the first row by – 3, and add to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right] \quad \left[\begin{array}{cc} x+3y & =7 \\ -5y & =-10 \end{array}\right]$$

We divide the second row by – 5, we get,

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{cc} x+3y & =7 \\ y & =2 \end{array}\right]$$

Finally, we multiply the second row by – 3 and add to the first row, and we get,

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{c} x=1 \\ y=2 \end{array}\right]$$

Now we list the three row operations the Gauss-Jordan method employs.

Row Operations

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operation may make the system look different, but they do not change the solution of the system.

The first row operation states that if any two rows of a system are interchanged, the new system obtained has the same solution as the old one. Let us look at an example in two equations with two unknowns. Consider the system

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We interchange the rows, and we get,

$$\begin{aligned} 3x+4y &= 11 \\ x+3y &= 7 \end{aligned}$$

Clearly, this system has the same solution as the one above.

The second operation states that if a row is multiplied by any non-zero constant, the new system obtained has the same solution as the old one. Consider the above system again,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We multiply the first row by –3, we get,

$$\begin{aligned} -3x-9y &= -21 \\ 3x+4y &= 11 \end{aligned}$$

Again, it is obvious that this new system has the same solution as the original.

The third row operation states that any constant multiple of one row added to another preserves the solution. Consider our system,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

If we multiply the first row by –3, and add it to the second row, we get,

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

And once again, the same solution is maintained.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations. As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the **reduced row echelon form**.

A matrix is in the **reduced row echelon form** if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. We state the Gauss-Jordan method as follows.

Gauss-Jordan Method

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row-echelon form.

Example 5.2.5

Solve the following system by the Gauss-Jordan method.

$$\begin{aligned} 2x+y+2z &= 10 \\ x+2y+z &= 8 \\ 3x+y-z &= 2 \end{aligned}$$

Solution

We write the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad \text{we interchanged row 1 (R1) and row 2 (R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by - 2 and add it to the second row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad -2R1+R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by - 3 and add it to the third row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad -3R1+R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need to make this entry(–3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by –3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad \text{R2} \div (-3)$$

Next, we make all other entries zeros in the second column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad -2R2+R1 \text{ and } 5R2+R3$$

We make the last diagonal entry a 1, by dividing row 3 by – 4.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R3 \div (-4)$$

Finally, we make all other entries zeros in column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -\mathrm{R3}+\mathrm{R1}$$

Clearly, the solution reads $x = 1$, $y = 2$, and $z = 3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called **pivoting**.

The number that is made a 1 is called the **pivot element,** and the row that contains the pivot element is called the **pivot row**.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the **target row**.

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5.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)

Solve the following by the Gauss-Jordan Method. Show all work.

1)

$$\begin{aligned} x+3y &= 1 \\ 2x-5y &= 13 \end{aligned}$$

2)

$$\begin{aligned} x-y-z &= -1 \\ x-3y+2z &= 7 \\ 2x-y+z &= 3 \end{aligned}$$

3)

$$\begin{aligned} x+2y+3z &= 9 \\ 3x+4y+z &= 5 \\ 2x-y+2z &= 11 \end{aligned}$$

4)

$$\begin{aligned} x+2y &= 0 \\ y+z &= 3 \\ x+3z &= 14 \end{aligned}$$

5) Two apples and four bananas cost $2.00 and three apples and five bananas cost $2.70. Find the price of each.

6) A bowl of corn flakes, a cup of milk, and an egg provide 16 grams of protein. A cup of milk and two eggs provide 21 grams of protein.
Two bowls of corn flakes with two cups of milk provide 16 grams of protein. How much protein is provided by one unit of food?

7)

$$\begin{aligned} x+2y &= 10 \\ y+z &= 5 \\ z+w &= 3 \\ x+w &= 5 \end{aligned}$$

8)

$$\begin{aligned} x+w &= 6 \\ 2x+y+w &= 16 \\ x-2z &= 0 \\ z+w &= 5 \end{aligned}$$

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CHAPTER OVERVIEW

6: Linear Inequalities in One and Two Variables

Learning Objectives

By the end of this chapter, the student should be able to

- Solve linear inequalities in one variable including compound and absolute value inequalities
- Graph the solutions to linear inequalities in one and two variables
- Write solutions to linear inequalities in one variable in interval notation

6.1: Linear Inequalities

6.1.1: Linear Inequalities (Exercises)

6.2: Compound Inequalities

6.2.1: Compound Inequalities (Exercises)

6.3: Absolute Value Inequalities

6.3.1: Absolute Value Inequalities (Exercises)

6.4: Linear Inequalities in Two Variables

6.4.1: Linear Inequalities in Two Variables (Exercises)

6.5: Systems of Linear Inequalities in Two Variables

6.5.1: Systems of Linear Inequalities in Two Variables (Exercises)

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6.1: Linear Inequalities

When there is a solution to an equation such as $x = 4$, this solution is unique and is the only solution that makes the statement true. However, with *inequalities*, the solution is an interval of numbers in which make the inequality true.

Definition: Inequality

An ***inequality*** is a relation between two numbers or two sets of numbers (or elements) in which are not strictly equal, but greater than (or equal to) or less than (or equal to).

$$\begin{aligned} &> \quad \text{greater than} \\ &\geq \quad \text{greater than or equal to} \\ &< \quad \text{less than} \\ &\leq \quad \text{less than or equal to} \end{aligned}$$

Note

English mathematician *Thomas Harriot* first used the inequality symbols in 1631. However, they were not immediately accepted, where symbols $\sqsubset$ and $\sqsupset$ were already coined by another English mathematician, *William Oughtred*.

Graphing Linear Inequalities

If we have an statement such as $x < 4$, this means a solution can be any number smaller than 4 such as $-2,\ 0,\ 3,\ 3.9$ or even 3.999999999as long as it is smaller than 4. If we have a statement such as $x \geq -2$, this means a solution can be any number greater than or equal to -2, such as $5,\ 0,\ -1,\ -1.9999$,or even -2. Because we don't have one value as the solution, it is often useful to draw a picture of the solutions to the inequality on a number line.

Definition: Linear Inequality

A ***linear inequality*** in one variable is an inequality of the form

$$ax + b < c,$$

where the inequality is written in the same form for $>,\ \leq,\ \geq$.

Interval Notation

We rewrite the $>,\ <,\ \leq,\ \geq$ symbols as parenthesis and brackets, i.e., $(,\ ),\ ],\ [$, respectively, when we write the inequality in interval notation.

- **Case 1.** If $x < a$, then the equivalent set of numbers in interval notation is $(-\infty, a)$; similarly, If $x \leq a$, then the equivalent set of numbers in interval notation is $(-\infty, a]$.
- **Case 2.** If $x > a$, then the equivalent set of numbers in interval notation is (a, ∞); similarly, If $x \geq a$, then the equivalent set of numbers in interval notation is $[a, \infty)$.

Example 6.1.1

Graph the inequality and rewrite the inequality in interval notation: $x < 2$

Solution

We will complete this example in steps and use this method for the remaining future examples involving inequalities.

Step 1. Draw a number line and mark the number in the inequality on the line.

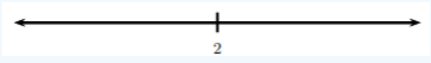


Figure 6.1.1

Step 2. Write a parenthesis or bracket, depending on the inequality sign, over the number on the number line. Note, we can easily use • for $\leq$ and $\geq$, and ◦ for $<$ and $>$. Since we have $x < 2$, then we use ◦ or) on the number line:

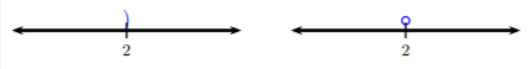


Figure 6.1.2

Step 3. Draw a line connecting the] or ◦ in the direction where lies the solution. Since $x < 2$, then we want all numbers less than 2; hence, we draw the line to the left to represent all numbers less than 2:

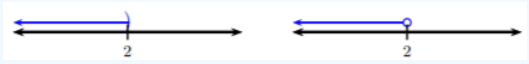


Figure 6.1.3

Note

Even though there are two graphs, one will suffice. At the discretion of the instructor, brackets and parenthesis, or closed and open circles will be used when graphing inequalities.

Step 4. Rewrite the inequality in interval notation. Since $x < 2$, we can see this is case 1 and so the interval is $(-\infty, 2)$. We use a parenthesis on the 2 since it is $<$.

Note

In Example 6.1.1, we use a parenthesis on $-\infty$ because $\pm\infty$ are not real numbers and symbolizes "some large (positive or negative) number beyond any real number." It is common practice to always use parentheses on infinities for intervals.

Note

The symbol for infinity was first used by the Romans even though, at the time, the number was used for $1,000$. The Greeks also used the symbol for $10,000$.

Example 6.1.2

Graph the inequality and rewrite the inequality in interval notation: $y \geq -1$

Solution

We start by labeling the number line with -1. Then draw a line to the right since all numbers greater than (or equal to) -1 are to the right:

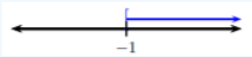


Figure 6.1.4

Next, we write $y \geq -1$ in interval notation. This is case 2 and the interval is $[-1, \infty)$; we use a bracket because of $\geq$.

Example 6.1.3

Given the graph, write the equivalent inequality and interval notation:

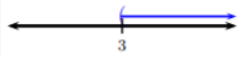


Figure 6.1.5

Solution

Since the graph shows all numbers (strictly) greater than 3, then the inequality is $x > 3$; we use $>$ because of the (on the number line. The equivalent interval would be, using case 2, $(3, \infty)$.

Example 6.1.4

Given the graph, write the equivalent inequality and interval notation:

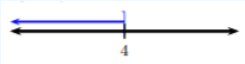


Figure 6.1.6

Solution

Since the graph shows all numbers less than (or equal to) 4, then the inequality is $x \leq 4$; we use $\leq$ because of the] on the number line. The equivalent interval would be, using case 1, $(-\infty, 4]$.

Solving Linear Inequalities

Properties of Inequalities

There are two properties are used with inequalities:

Property 1. *Addition Property of Inequalities:* For numbers $a,\ b,\ c$,

$$a < b \text{ is equivalent to } a + c < b + c,$$

where the form is the same for $>,\ \leq,\ \geq$. I.e., you can add a number to one side of the inequality as long as you add the same number to the other side. (This should remind you of the addition property for equations.)

Property 2. *Multiplication Property of Inequalities:* For numbers $a,\ b,\ c$ and $c \neq 0$,

$$a < b \text{ is equivalent to } a \cdot c < b \cdot c,$$

where the form is the same for $>,\ \leq,\ \geq$. I.e., you can multiply a nonzero number to one side of the inequality as long as you multiply the same nonzero number to the other side.
However, if $c < 0$, then the inequality reverses, i.e, ***if you multiply or divide by a negative, then reverse the inequality symbol.***

Example 6.1.5

Solve the inequality. Graph the solution and write the solution in interval notation.

$$5 - 2x \geq 11$$

Solution

$$\begin{aligned} 5 - 2x &\geq 11 && \text{Add the opposite of 5 to each side} \\ 5 - 2x + (-5) &\geq 11 + (-5) && \text{Simplify} \\ -2x &\geq 6 && \text{Multiply by the reciprocal of } -2 \\ \left(-\tfrac{1}{2}\right) \cdot -2x &\leq 6 \cdot \left(-\tfrac{1}{2}\right) && \text{Simplify and reverse the inequality sign} \\ x &\leq -3 && \text{Solution} \end{aligned}$$

Let's graph the solution $x \leq -3$:

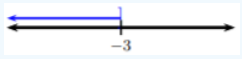


Figure 6.1.7

Next, we write $x \leq -3$ in interval notation. This is case 1 and the interval is $(-\infty, -3]$; we use a bracket because of $\leq$.

Example 6.1.6

Solve the inequality. Graph the solution and write the solution in interval notation.

$$3(2x - 4) + 4x < 4(3x - 7) + 8$$

Solution

$$\begin{aligned} 3(2x-4)+4x &< 4(3x-7)+8 && \text{Distribute} \\ 6x-12+4x &< 12x-28+8 && \text{Combine like terms} \\ 10x-12 &< 12x-20 && \text{Isolate the variable term} \\ 10x-12+(-10x) &< 12x-20+(-10x) && \text{Simplify} \\ -12 &< 2x-20 && \text{Add the opposite of } -20 \text{ to each side} \\ -12+20 &< 2x-20+20 && \text{Simplify} \\ 8 &< 2x && \text{Multiply by the reciprocal of 2} \\ \tfrac{1}{2}\cdot 8 &< 2x\cdot\tfrac{1}{2} && \text{Simplify} \\ 4 &< x && \text{Rewrite the inequality with } x \text{ on the left side} \\ x &> 4 && \text{Solution} \end{aligned}$$

Let's graph the solution $x > 4$:

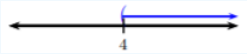


Figure 6.1.8

Next, we write $x > 4$ in interval notation. This is case 2 and the interval is $(4, \infty)$; we use a parenthesis because of $>$.

Note

It is important to be careful when the solution to the inequality has the isolated variable on the right side like in Example 6.1.6, i.e., $4 < x$ rather than $x > 4$. It is best practice to write the variable on the left side after isolating the variable. This will minimize confusion when graphing the solution and writing the solution in interval notation.

Tripartite Inequalities

A special type of compound inequality is called a *tripartite inequality*, when the variable (or expression containing the variable) is between two numbers. When solving these types of inequalities with three parts, we will apply the properties of inequalities to all three parts (rather than two sides) to isolate the variable in the middle.

Interval Notation for Tripartite Inequalities

Case 1. If $a < x < b$, then the equivalent set of numbers in interval notation is (a, b); similarly, If $a \le x \le b$, then the equivalent set of numbers in interval notation is $[a, b]$.

Case 2. If $a < x \le b$, then the equivalent set of numbers in interval notation is $(a, b]$; similarly, If $a \le x < b$, then the equivalent set of numbers in interval notation is $[a, b)$.

Example 6.1.7

Solve the inequality. Graph the solution and write the solution in interval notation.

$$-6 \le -4x+2 < 2$$

Solution

Solve the inequality, graph the solution, and give interval notation.

$$\begin{aligned} -6 &\le -4x+2 < 2 && \text{Add the opposite of 2 to each part of the inequality} \\ -6+(-2) &\le -4x+2+(-2) < 2+(-2) && \text{Simplify} \\ -8 &\le -4x < 0 && \text{Multiply by the reciprocal of } -4 \\ -\tfrac{1}{4}\cdot -8 &\ge -\tfrac{1}{4}\cdot -4x > -\tfrac{1}{4}\cdot 0 && \text{Simplify} \\ 2 &\ge x > 0 && \text{Rewrite with } \ge, > \text{ signs} \\ 0 &< x \le 2 && \text{Solution} \end{aligned}$$

Let's graph the solution $0 < x \leq 2$:

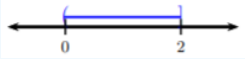


Figure 6.1.9

Next, we write $0 < x \leq 2$ in interval notation. This is case 2 and the interval is $(0, 2]$.

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6.1.1: Linear Inequalities (Exercises)

Graph the inequalities and rewrite the inequalities in interval notation.

Exercise 6.1.1.1

$n > -5$

Exercise 6.1.1.2

$-2 \geq k$

Exercise 6.1.1.3

$5 \geq x$

Exercise 6.1.1.4

$n > 4$

Exercise 6.1.1.5

$1 \geq k$

Exercise 6.1.1.6

$-5 < x$

Given the graph, write the equivalent inequality and interval notation.

Exercise 6.1.1.7

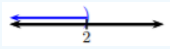


Figure 6.1.1.1

Exercise 6.1.1.8

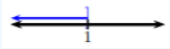


Figure 6.1.1.2

Exercise 6.1.1.9

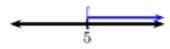


Figure 6.1.1.3

Exercise 6.1.1.10

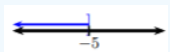


Figure 6.1.1.4

Exercise 6.1.1.11

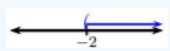


Figure 6.1.1.5

Exercise 6.1.1.12

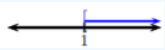


Figure 6.1.1.6

Solve the inequality. Graph the solution and write the solution in interval notation.

Exercise 6.1.1.13

$\frac{x}{11} \geq 10$

Exercise 6.1.1.14

$2 + r < 3$

Exercise 6.1.1.15

$8 + \frac{n}{3} \geq 6$

Exercise 6.1.1.16

$2 > \frac{a-2}{5}$

Exercise 6.1.1.17

$-47 \geq 8 - 5x$

Exercise 6.1.1.18

$-2(3 + k) < -44$

Exercise 6.1.1.19

$18 < -2(-8 + p)$

Exercise 6.1.1.20

$24 \geq -6(m - 6)$

Exercise 6.1.1.21

$-r - 5(r - 6) < -18$

Exercise 6.1.1.22

$24 + 4b < 4(1 + 6b)$

Exercise 6.1.1.23

$-5v - 5 < -5(4v + 1)$

Exercise 6.1.1.24

$4+2(a+5)<-2(-a-4)$

Exercise 6.1.1.25

$-(k-2)>-k-20$

Exercise 6.1.1.26

$-2\leq\frac{n}{13}$

Exercise 6.1.1.27

$\frac{m}{5}\leq-\frac{6}{5}$

Exercise 6.1.1.28

$11>8+\frac{x}{2}$

Exercise 6.1.1.29

$\frac{v-9}{-4}\leq 2$

Exercise 6.1.1.30

$\frac{6+x}{12}\leq-1$

Exercise 6.1.1.31

$-7n-10\geq 60$

Exercise 6.1.1.32

$5\geq\frac{x}{5}+1$

Exercise 6.1.1.33

$-8(n-5)\geq 0$

Exercise 6.1.1.34

$-60\geq-4(-6x-3)$

Exercise 6.1.1.35

$-8(2-2n)\geq-16+n$

Exercise 6.1.1.36

$-36+6x>-8(x+2)+4x$

Exercise 6.1.1.37

$3(n+3)+7(8-8n)<5n+5+2$

Exercise 6.1.1.38

$-(4-5p)+3\geq -2(8-5p)$

Exercise 6.1.1.39

$3\leq 9+x\leq 7$

Exercise 6.1.1.40

$11<8+k\leq 12$

Exercise 6.1.1.41

$-3<x-1<1$

Exercise 6.1.1.42

$-4<8-3m\leq 11$

Exercise 6.1.1.43

$-16\leq 2n-10\leq -22$

Exercise 6.1.1.44

$0\geq \frac{x}{9}\geq -1$

Exercise 6.1.1.45

$-11\leq n-9\leq -5$

Exercise 6.1.1.46

$1\leq \frac{p}{8}\leq 0$

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6.2: Compound Inequalities

Several inequalities are combined together to form ***compound inequalities***. There are three types of compound inequalities which we will investigate in this section: *and*, *or*, and *tripartite*.

Let A and B be sets.

- The ***union*** of two sets contains all the elements contained in either set (or both sets). The ***union*** is denoted $A \cup B$.
- The ***intersection*** of two sets contains only the elements that are in both sets. The ***intersection*** is denoted $A \cap B$.

Compound Inequalities with *or*

For *or* compound inequalities, the solution is a true statement from either one inequality, the other inequality, or both. This should remind you of the *union* of two sets except, in this case, we are going to determine the solution by graphing each inequality, and write the solution in interval notation. *Hint*: We will use the union symbol, $\cup$, in the solution.

Example 6.2.1

Solve the compound inequality. Graph the solution and write the solution in interval notation.

$$2x - 5 > 3 \quad \text{or} \quad 4 - x \geq 6$$

Solution

We begin the solution by solving for each inequality.

$$\begin{aligned} 2x - 5 > 3 \quad \text{or} \quad 4 - x \geq 6 \quad & \text{Solve each inequality} \\ 2x > 8 \quad \text{or} \quad -x \geq 2 \quad & \text{Divide by the coefficient of } x \\ x > 4 \quad \text{or} \quad x \leq -2 \quad & \text{Solution in inequality notation} \end{aligned}$$

Let's graph each of these inequalities to determine the union of the two sets.

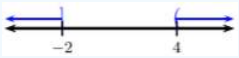


Figure 6.2.1

Looking for the union of these two sets, we see the solution is all the numbers to the left of -2 (inclusive), or to the right of 4, or in both. Hence, in interval notation, the solution is

$$(-\infty, -2] \cup (4, \infty)$$

Note

The symbol for infinity was first used by the Romans, although, at the time, the number was used for $1,000$. The Greeks also used the symbol for $10,000$.

Note

There are special cases with the *or* compound inequalities when obtaining a solution.

Case 1. If one graph is contained in the other, the union of the two sets will be the larger set.

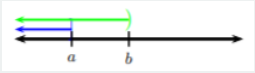


Figure 6.2.2

Hence, in this case, the solution is $(-\infty, b)$.



Figure 6.2.3

Hence, in this case, the solution is $(-\infty, \infty)$.

Compound Inequalities with *and*

For *and* compound inequalities, the solution is a true statement from *both* inequalities. This should remind you of the *intersection* of two sets except, in this case, we are going to determine the solution by graphing each inequality, and write the solution in interval notation. *Hint*: The solution is one interval, and no set symbols are used with *and* problems.

Example 6.2.2

Solve the compound inequality. Graph the solution and write the solution in interval notation.

$$2x+8 \geq 5x-7 \quad \text{and} \quad 5x-3 > 3x+1$$

Solution

We begin the solution by solving for each inequality.

$$\begin{aligned} 2x+8 \geq 5x-7 \quad &\text{and} \quad 5x-3 > 3x+1 && \text{Solve each inequality} \\ 8 \geq 3x-7 \quad &\text{and} \quad 2x-3 > 1 && \text{Isolate the variable term} \\ 15 \geq 3x \quad &\text{and} \quad 2x > 4 && \text{Divide by the coefficient of } x \\ 5 \geq x \quad &\text{and} \quad x > 2 && \text{Solution in inequality notation} \end{aligned}$$

Let's graph each of these inequalities to determine the intersection of the two sets.



Figure 6.2.4

Looking for the intersection of these two sets, we see the solution is all the numbers in between 2 and 5 (inclusive). Hence, in interval notation, the solution is $(2, 5]$.

Note

There are special cases with the *and* compound inequalities when obtaining a solution.

Case 1. If one graph in contained in the other, the intersection of the two sets will be the smaller set.

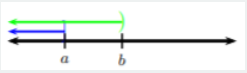


Figure 6.2.5

Hence, in this case, the solution is $(-\infty, a)$.

Case 2. If the graphs do not intersect, where there is a gap between sets, then there is no solution, i.e., $\emptyset$.



Figure 6.2.6

Hence, in this case, the solution is no solution or $\emptyset$.

Recall the "tripartite inequalities" from the previous section. Tripartite inequalities are compound inequalities, where we are find the intersection of the two sets. Notice that the solutions were one interval, as with "and" compound inequalities.

Example 6.2.3

Solve the compound inequality. Graph the solution and write the solution in interval notation.

$$-6 \leq -4x+2 < 2$$

Solution

We begin the solution by applying the properties of inequalities to every side of the inequality.

$$
\begin{aligned}
-6 \leq -4x + 2 < 2 \quad & \text{Subtract 2 from all three parts} \\
-8 \leq -4x < 0 \quad & \text{Divide all three parts by } -4 \\
2 \geq x > 0 \quad & \text{Flip entire statement so values increase from left to right} \\
0 < x \leq 2 \quad & \text{Solution in inequality notation}
\end{aligned}
$$

Let's graph this inequality to determine the intersection of the two sets.



Figure 6.2.7

We can see the intersection is all the numbers in between 0 and 2 (inclusive). Hence, in interval notation, the solution is $(0, 2]$.

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6.2.1: Compound Inequalities (Exercises)

Solve the compound inequalities. Graph the solution and write the solution in interval notation.

Exercise 6.2.1.1

$\frac{n}{3} \leq -3$ or $-5n \leq -10$

Exercise 6.2.1.2

$x + 7 \geq 12$ or $9x < -45$

Exercise 6.2.1.3

$x - 6 < -13$ or $6x \leq -60$

Exercise 6.2.1.4

$\frac{v}{8} > -1$ and $v - 2 < 1$

Exercise 6.2.1.5

$-8 + b < -3$ and $4b < 20$

Exercise 6.2.1.6

$a + 10 \geq 3$ and $8a \leq 48$

Exercise 6.2.1.7

$3 \leq 9 + x \leq 7$

Exercise 6.2.1.8

$11 < 8 + k \leq 12$

Exercise 6.2.1.9

$-3 < x - 1 < 1$

Exercise 6.2.1.10

$-4 < 8 - 3m \leq 11$

Exercise 6.2.1.11

$-16 \leq 2n - 10 \leq -22$

Exercise 6.2.1.12

$-5b + 10 \leq 30$ and $7b + 2 \leq -40$

Exercise 6.2.1.13

$3x-9<2x+10$ and $5+7x\leq 10x-10$

Exercise 6.2.1.14

$-8-6v\leq 8-8v$ and $7v+9\leq 6+10v$

Exercise 6.2.1.15

$1+5k\leq 7k-3$ or $k-10>2k+10$

Exercise 6.2.1.16

$2x+9\geq 10x+1$ and $3x-2<7x+2$

Exercise 6.2.1.17

$6m\geq -24$ or $m-7<-12$

Exercise 6.2.1.18

$10r>0$ or $r-5<-12$

Exercise 6.2.1.19

$9+n<2$ or $5n>40$

Exercise 6.2.1.20

$-9x<63$ and $\frac{x}{4}<1$

Exercise 6.2.1.21

$-6n\leq 12$ and $\frac{n}{3}\leq 2$

Exercise 6.2.1.22

$-6+v\geq 0$ and $2v>4$

Exercise 6.2.1.23

$0\geq \frac{x}{9}\geq -1$

Exercise 6.2.1.24

$-11\leq n-9\leq -5$

Exercise 6.2.1.25

$1\leq \frac{p}{8}\leq 0$

Exercise 6.2.1.26

$3+7r>59$ or $-6r-3>33$

Exercise 6.2.1.27

$-6-8x\geq -6$ or $2+10x>82$

Exercise 6.2.1.28

$n+10\geq 15$ or $4n-5<-1$

Exercise 6.2.1.29

$4n+8<3n-6$ or $10n-8\geq 9+9n$

Exercise 6.2.1.30

$5-2a\geq 2a+1$ or $10a-10\geq 9a+9$

Exercise 6.2.1.31

$8-10r\leq 8+4r$ or $-6+8r<2+8r$

Exercise 6.2.1.32

$-9m+2<-10-6m$ or $-m+5\geq 10+4m$

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6.3: Absolute Value Inequalities

When an inequality contains absolute value, we need to rewrite the inequality without the absolute value in order to solve the inequality. Let's consider $|x| < 2$. Recall, absolute value is defined as the distance from zero. The idea behind solving $|x| < 2$ is to find all numbers that have a distance from zero that is less than 2. Let's take a look at this graphically.

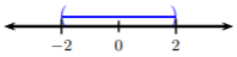


Figure 6.3.1

This graph should remind us of *tripartite (and) compound inequalities* and it is! Now let's consider $|x| > 2$. The idea behind solving $|x| > 2$ is to find all numbers that have a distance from zero that is more than 2. Let's take a look at this graphically.

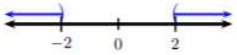


Figure 6.3.2

This graph should remind us of *or compound inequalities* and it is! Absolute inequalities will remind us of the concepts from compound inequalities.

Definition

Given below are the cases for absolute value for inequalities in one variable, where a is a real number.

Case 1. If $|x| < a$, then $-a < x < a$.

Case 2. If $|x| > a$, then $x < -a$ or $x > a$. Similar cases for $\leq$ and $\geq$, respectively.

Note

The phrase "absolute value" comes from German mathematician *Karl Weierstrass* in 1876, though he used the absolute value symbol for complex numbers. The first known use of the symbol for integers comes from a 1939 edition of a college algebra textbook.

Example 6.3.1

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

$$|4x - 5| \geq 6$$

Solution

We begin the solution by rewriting the absolute value inequality using the cases in the definition.

$$\begin{aligned} |4x-5| &\geq 6 && \text{Case 2. Rewrite as two inequalities using or} \\ 4x-5 \geq 6 \quad \text{or} \quad & 4x-5 \leq -6 && \text{Solve each inequality} \\ 4x \geq 11 \quad \text{or} \quad & 4x \leq -1 && \text{Divide by the coefficient of } x \\ x \geq \tfrac{11}{4} \quad \text{or} \quad & x \leq -\tfrac{1}{4} && \text{Solution in inequality notation} \end{aligned}$$

Let's graph each of these inequalities to determine the union of the two sets.

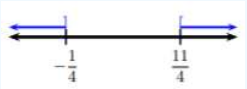


Figure 6.3.3

Looking for the union of these two sets, we see the solution is all the numbers to the left of $-\frac{1}{4}$ (inclusive), or to the right of $\frac{11}{4}$ (inclusive), or in both. Hence, in interval notation, the solution is

$$\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{11}{4}, \infty\right)$$

Isolate the Absolute Value

Recall. In the previous chapter, when solving absolute value equations, we were required to isolate the absolute value term before applying the definition. In this section, we will do the same method because *methods never change, only problems*.

Example 6.3.2

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

$$-4-3|x| \leq -16$$

Solution

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$$\begin{aligned} -4-3|x| &\leq -16 && \text{Isolate the absolute value term} \\ -3|x| &\leq -12 && \text{Divide both sides by } -3 \\ |x| &\geq 4 && \text{Case 2. Rewrite as two inequalities using or} \\ x \geq 4 \quad \text{or} \quad & x \leq -4 && \text{Solution in inequality notation} \end{aligned}$$

Let's graph each of these inequalities to determine the union of the two sets.

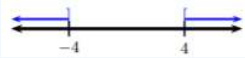


Figure 6.3.4

Looking for the union of these two sets, we see the solution is all the numbers to the left of -4 (inclusive), or to the right of 4 (inclusive), or in both. Hence, in interval notation, the solution is

$$(-\infty, -4) \cup [4, \infty)$$

Example 6.3.3

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

$$9-2|4x+1| > 3$$

Solution

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$$\begin{aligned} 9-2|4x+1| &> 3 && \text{Isolate the absolute value term} \\ -2|4x+1| &> -6 && \text{Divide both sides by } -2 \\ |4x+1| &< 3 && \text{Case 1. Rewrite as a tripartite inequality} \\ -3 < 4x+1 &< 3 && \text{Solve, start by subtracting 1 from all three parts} \\ -4 < 4x &< 2 && \text{Divide by the coefficient of } x \\ -1 < x &< \tfrac{1}{2} && \text{Solution in inequality notation} \end{aligned}$$

Let's graph this inequality to determine the intersection of the two sets.

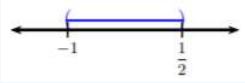


Figure 6.3.5

We can see the intersection is all the numbers in between -1 and $\frac{1}{2}$. Hence, in interval notation, the solution is $\left(-1, \frac{1}{2}\right)$.

Special Cases with Absolute Value Inequalities

There are special cases with the absolute value inequalities when obtaining a solution. We recall that after evaluating the absolute value of an expression, the result is positive. Hence, if we were to isolate the absolute value on the left side of the inequality, then that result is always positive, i.e., $|x| \geq 0$, no matter the case.

Example 6.3.4

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

$$12+4|6x-1|<4$$

Solution

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$$\begin{aligned} 12+4|6x-1| &< 4 && \text{Isolate the absolute value term} \\ 4|6x-1| &< -8 && \text{Divide by the coefficient } 4 \\ |6x-1| &< -2 && \text{STOP The absolute value is always non-negative} \end{aligned}$$

By definition of absolute value, $|6x-1| \geq 0$. Hence, $|6x-1|$ could never be less than zero, let alone less than -2. Thus, this inequality has no solution or $\emptyset$. To graph a $\emptyset$ solution, we leave a number line empty:

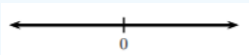


Figure 6.3.6

Note

Let's say, in Example 6.3.4 we didn't recognize that $|6x-1|$ was less than a negative number and we solved as usual. After applying Case 1. in the definition, we would obtain a false statement. In Example 6.3.4, we would have obtained

$$2<6x-1<-2,$$

but 2 isn't less than $-2 (2 \not< -2)$. Hence, a false statement. At this point, we would see that the inequality $|6x-1|<-2$ has no solution.

Example 6.3.5

Solve the absolute value inequality. Graph the solution and write the solution in interval notation.

$$5-6|x+7| \leq 17$$

Solution

We begin the solution by rewriting the absolute value inequality where the absolute value term is isolated on the left side. Then we can apply the cases in the definition.

$$\begin{aligned} 5-6|x+7| &\leq 17 && \text{Isolate the absolute value term} \\ -6|x+7| &\leq 12 && \text{Divide by the coefficient } -6 \\ |x+7| &\geq -2 && \text{STOP The absolute value is always non-negative} \end{aligned}$$

By definition of absolute value, $|x+7| \geq 0$. Hence, $|x+7|$ is always greater than -2. Thus, this inequality has every real number as a solution and the solution is all real numbers or $(-\infty, \infty)$. To graph a $(-\infty, \infty)$ solution, we draw a line including the entire number line:

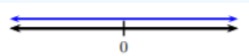


Figure 6.3.7

Note

Let's say, in Example 6.3.5 we didn't recognize that $|x+7|$ was always greater than a negative number and we solved as usual. After applying Case 2 in the definition, we would obtain the same result. In Example 6.3.5, we would have obtained

$$x+7 \geq -2 \quad \text{or} \quad x+7 \leq 2,$$

which results in, after solving,

$$x \geq -9 \quad \text{or} \quad x \leq -5$$

Graphing each inequality, we get



Figure 6.3.8

Hence, the solution is $(-\infty, \infty)$.

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6.3.1: Absolute Value Inequalities (Exercises)

Solve the absolute value inequalities. Graph the solution and write the solution in interval notation.

Exercise 6.3.1.1

$|x| < 3$

Exercise 6.3.1.2

$|2x| < 6$

Exercise 6.3.1.3

$|x - 2| < 6$

Exercise 6.3.1.4

$|x - 7| < 3$

Exercise 6.3.1.5

$|3x - 2| < 9$

Exercise 6.3.1.6

$1 + 2|x - 1| \leq 9$

Exercise 6.3.1.7

$6 - |2x - 5| \geq 3$

Exercise 6.3.1.8

$|3x| > 5$

Exercise 6.3.1.9

$|x - 3| \geq 3$

Exercise 6.3.1.10

$|3x - 5| > 3$

Exercise 6.3.1.11

$4 + 3|x - 1| \geq 10$

Exercise 6.3.1.12

$3 - 2|x - 5| \leq -15$

Exercise 6.3.1.13

$-2-3|4-2x| \geq -8$

Exercise 6.3.1.14

$4-5|-2x-7| < -1$

Exercise 6.3.1.15

$3-2|4x-5| \geq 1$

Exercise 6.3.1.16

$-5-2|3x-6| < -8$

Exercise 6.3.1.17

$4-4|-2x+6| > -4$

Exercise 6.3.1.18

$|-10+x| \geq 8$

Exercise 6.3.1.19

$|x| \leq 8$

Exercise 6.3.1.20

$|x+3| < 4$

Exercise 6.3.1.21

$|x-8| < 12$

Exercise 6.3.1.22

$|x+3| \leq 4$

Exercise 6.3.1.23

$|2x+5| < 9$

Exercise 6.3.1.24

$10-3|x-2| \geq 4$

Exercise 6.3.1.25

$|x| > 5$

Exercise 6.3.1.26

$|x-4|>5$

Exercise 6.3.1.27

$|2x-4|>6$

Exercise 6.3.1.28

$3-|2-x|<1$

Exercise 6.3.1.29

$3-2|3x-1|\geq -7$

Exercise 6.3.1.30

$4-6|-6-3x|\leq -5$

Exercise 6.3.1.31

$-3-2|4x-5|\geq 1$

Exercise 6.3.1.32

$-2+3|5-x|\leq 4$

Exercise 6.3.1.33

$-2-3|-3x-5\geq -5$

Exercise 6.3.1.34

$6-3|1-4x|<-3$

Exercise 6.3.1.35

$-3-4|-2x-5|\geq -7$

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6.4: Linear Inequalities in Two Variables

Previously, we graphed inequalities in one variable, but now we learn to graph inequalities in two variables. Although this section may seem similar to linear equations in two variables, linear inequalities in two variables have many applications. For example, business owners want to know when revenue is greater than cost so that their business makes a profit, e.g., revenue $>$ cost.

Definition: Linear Inequality

A ***linear inequality*** in two variables is an inequality of the form

$$ax + by < c,$$

where the inequality is written in the same form for $>$, $\leq$, $\geq$ and $a,\ b \neq 0$.

Recall. The solution to a linear inequality in one variable is an interval of numbers, e.g., $(-\infty, \infty)$, $[-2, 3)$, $(1, 9)$, $[-7, -3]$, etc.

Verifying Solutions

Solution to a Linear Inequality in Two Variables

An ordered pair (x, y) is a solution to a linear inequality in two variables, $ax + by < c$, if the ordered pair (x, y) makes the inequality true, where the the same is for $>$, $\leq$, $\geq$ and $a,\ b \neq 0$.

Example 6.4.1

Verify whether each ordered pair is a solution to the inequality $y > x + 4$.

a. $(0, 0)$
b. $(1, 6)$

Solution

We substitute the ordered pairs into the inequality and determine if the results are true.

a. Let's substitute $(0, 0)$ into the inequality and determine if the left side is greater than the right side.

$$\begin{aligned} y &\overset{?}{>} x + 4 && \text{Substitute } x = 0 \text{ and } y = 0 \\ 0 &\overset{?}{>} 0 + 4 && \text{Simplify} \\ 0 &\not> 4 && X \text{ False} \end{aligned}$$

Hence, $(0, 0)$ is not a solution to the inequality $y > x + 4$.

b. Let's substitute $(1, 6)$ into the inequality and determine if the left side is greater than the right side.

$$\begin{aligned} y &\overset{?}{>} x + 4 && \text{Substitute } x = 1 \text{ and } y = 6 \\ 6 &\overset{?}{>} 1 + 4 && \text{Simplify} \\ 6 &> 5 && \checkmark \text{ True} \end{aligned}$$

Hence, $(1, 6)$ is a solution to the inequality $y > x + 4$.

Boundary Lines

If we are given a linear inequality, $ax + by < c$, we could see from Example 6.4.1 that not all ordered pairs are a solution, only some. Why? Well, notice that $(0, 0)$ is below the line $y = x + 4$ and $(1, 6)$ is above the line $y = x + 4$. This implies that ordered pairs in certain regions are solutions to the inequality $y > x + 4$. Hence, the line $y = x + 4$ is critical when finding solutions to the inequality. We call the line $y = x + 4$ a ***boundary line***, a line that separates the ordered pairs that are solutions and the ordered pairs that are not solutions of the linear inequality in two variables $y > x + 4$.

Definition: Boundary Line

A linear equation in two variables $ax+by=c$ is called the ***boundary line***, the line that separates the region where $ax+by>c$ and from the region where $ax+by<c$.

Note

Since there are four inequality symbols: $>$, $<$, $\geq$, $\leq$, then we have linear inequalities in two variables that include the boundary, e.g., inequalities with $\leq$ and $\geq$, and linear inequalities in two variables that exclude the boundary, e.g., inequalities with $<$ and $>$.

We can use the table below to help identify the boundary line, determine whether to include the boundary line, and the way the boundary line looks graphically.

Table 6.4.1

Guidelines for interpreting boundary lines in inequalities

Case 1.	Case 2.
$ax+by<c$	$ax+by\leq c$
$ax+by>c$	$ax+by\geq c$
Boundary line: $ax+by=c$	Boundary line: $ax+by=c$
Boundary line is excluded in solution	Boundary line is included in solution
Boundary line is dashed	Boundary line is solid

Example 6.4.2

Let's revisit Example 6.4.1 and graph the boundary line and points $(0,0)$ and $(1,6)$.

Solution

Since we have $y>x+4$, we can see from the table above we have Case 1. and the boundary line is excluded. We represent this by graphing the line $y=x+4$ as a dashed line.

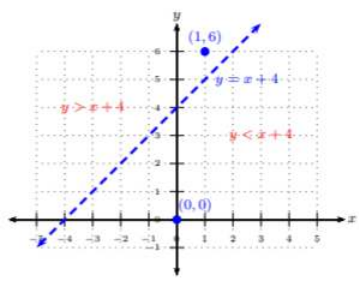


Figure 6.4.1

We can see from the graph that the point $(0,0)$ lies below the boundary line $y=x+4$ and point $(1,6)$ lies above the boundary line. Recall, point $(1,6)$ was verified as a solution of $y>x+4$ in Example 6.4.1. Furthermore, any ordered pair that lies above $y=x+4$ will verify as a solution, i.e., making the inequality true. We usually represent this area by *shading the region* where the set of ordered pairs make the inequality true.

Graphing Linear Inequalities

Example 6.4.3

Graph the inequality from Example 6.4.1.

Solution

Since we know that $(1, 6)$ is a solution to the inequality, then we shade above the dashed boundary line:

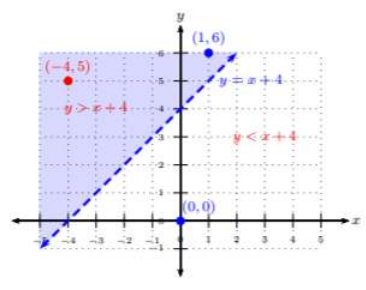


Figure 6.4.2

We see that any ordered pair in the shaded region is a solution to the inequality. For example, let's pick $(-4, 5)$ and verify this is a solution:

$$\begin{aligned} y &\overset{?}{>} x+4 && \text{Substitute } x=-4 \text{ and } y=5 \\ 5 &\overset{?}{>} -4+4 && \text{Simplify} 5>0 && \checkmark \text{ True} \end{aligned}$$

Hence, $(-4, 5)$ is a solution to the inequality $y > x + 4$.

Steps for Graphing Linear Inequalities in Two Variables

Given a linear inequality in two variables, $ax + by < c$, we use the steps below to graph $ax + by < c$, where the the same process is applied for $>$, $\leq$, $\geq$ and $a,\ b \neq 0$.

Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

Step 2. Graph the boundary line according to the two cases:

Case 1. If the inequality is $<$ or $>$, then the boundary line is dashed.

Case 2. If the inequality is $\geq$ or $\leq$, then the boundary line is solid.

Step 3. Select a test point that is *not on the boundary line.* **Ask:** *Does this ordered pair make the inequality true?*

Step 4. If the ordered pair is

- a solution to the inequality, i.e., makes the inequality true, then shade the side that includes the ordered pair.
- not a solution, then shade the opposite side of the boundary line.

Note

If we choose a test point on the boundary line, we will obtain an identity, where both sides of the inequality symbol are the same number. Hence, it is critical to choose a point not on the boundary line.

Example 6.4.4

Graph the inequality:

$$2x - y > 3.$$

Solution

Let's follow the steps given above to graph the inequality.

Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

$$\begin{aligned} 2x - y &> 3 \\ -y &> -2x + 3 \\ y &< 2x - 3 \end{aligned}$$

Step 2. Graph the boundary line according to the two cases. Since the given inequality is $<$, then we have Case 1.

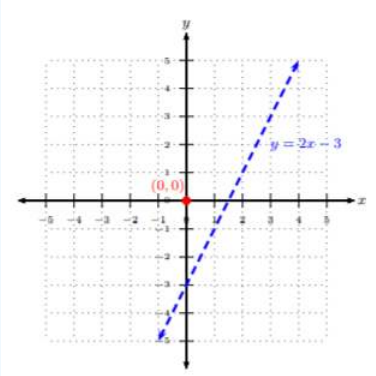


Figure 6.4.3

Step 3. Select a test point that is *not on the boundary line.* **Ask:** *Does this ordered pair make the inequality true?*
Let's pick the test point $(0, 0)$ as it is a great choice!

$$\begin{aligned} y &\overset{?}{<} 2x - 3 \\ 0 &\overset{?}{<} 2(0) - 3 \\ 0 &\not< -3 \end{aligned}$$

Hence, $(0, 0)$ doesn't make the inequality true.

Step 4. If the ordered pair is

- a solution to the inequality, i.e., makes the inequality true, then shade the side that includes the ordered pair.
- not a solution, then shade the opposite side of the boundary line.

Since the ordered pair $(0, 0)$ is not a solution to the inequality, then we shade on the opposite side of the boundary line from the location of the ordered pair.

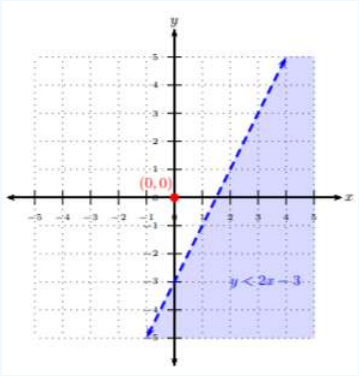


Figure 6.4.4

Note

Another way of graphing linear inequalities in two variables is to complete **Step 1.** and **Step 2.**, but instead of taking a test point in **Step 3.**, we can observe the inequality symbols. If the inequality has $<$ or $\leq$, then we easily shade below the boundary line, i.e., below the y-intercept. Similarly, if the inequality has $>$ or $\geq$, then we easily shade above the boundary line, i.e., above the y-intercept.

Example 6.4.5

Graph the inequality:

$$3x + 2y \geq -6.$$

Solution

Let's follow the steps given above to graph the inequality, but try skipping **Step 3.** and **Step 4.**

Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

$$\begin{aligned} 3x + 2y &\geq -6 \\ 2y &\geq -3x - 6 \\ y &\geq -\frac{3}{2}x - 3 \end{aligned}$$

Step 2. Graph the boundary line according to the two cases. Since the given inequality is $\geq$, then we have Case 2.

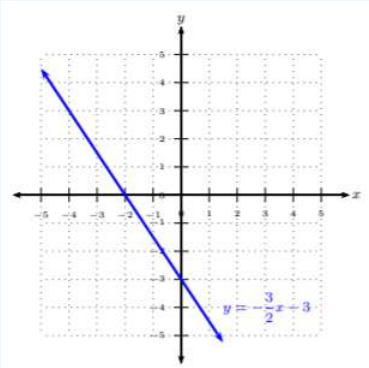


Figure 6.4.5

Since this inequality is $\geq$, where all ordered pairs above the boundary line are solutions to the inequality, we can easily shade above the y-intercept:

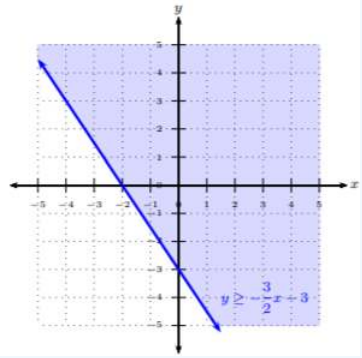


Figure \(\PageIndex{6}\

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6.4.1: Linear Inequalities in Two Variables (Exercises)

Determine whether the given ordered pairs are solutions to the inequality.

Exercise 6.4.1.1

$x+2y \geq -4;\ (0,-4);\ (1,1)$

Exercise 6.4.1.2

$2x-y \leq 2;\ (1,5);\ (3,1)$

Graph the following inequalities.

Exercise 6.4.1.3

$2x-y \leq 2$

Exercise 6.4.1.4

$x > 4y-8$

Exercise 6.4.1.5

$x+2y \geq -4$

Exercise 6.4.1.6

$3x+4y < 12$

Exercise 6.4.1.7

$6x+8y \leq 24$

Exercise 6.4.1.8

$5x+3y \leq 15$

Exercise 6.4.1.9

$y > 3x+1$

Exercise 6.4.1.10

$3x+2y \leq 12$

Exercise 6.4.1.11

$5x-2y < 10$

Exercise 6.4.1.12

$3x + 4y \geq 24$

Exercise 6.4.1.13

$y \leq 3x - 4$

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6.5: Systems of Linear Inequalities in Two Variables

Learning Objectives

After completing this section, you should be able to:

- Demonstrate whether an ordered pair is a solution to a system of linear inequalities.
- Solve systems of linear inequalities using graphical methods.
- Graph systems of linear inequalities.
- Interpret and solve applications of linear inequalities.

In this section, we will learn how to solve systems of linear inequalities in two variables. In Systems of Linear Equations in Two Variables, we learned how to solve for systems of linear equations in two variables and found a solution that would work in both equations. We can solve systems of inequalities by graphing each inequality (as discussed in Graphing Linear Equations and Inequalities) and putting these on the same coordinate system. The double-shaded part will be our solution to the system. There are many real-life examples for solving systems of linear inequalities.



Figure 6.5.4: Many college students find part-time jobs at places such as coffee shops to help pay for college. (credit: modification of work "TULLY's COFFEE" by MIKI Yoshihito/Flickr, CC BY 2.0)

Consider Ming who has two jobs to help her pay for college. She works at a local coffee shop for $7.50 per hour and at a research lab on campus for $12 per hour. Due to her busy class schedule, she cannot work more than 15 hours per week. If she needs to make at least $150 per week, can she work seven hours at the coffee shop and eight hours in the lab?

Determining If an Ordered Pair Is a Solution of a System of Linear Inequalities

The definition of a system of linear inequalities is similar to the definition of a system of linear equations. A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x+4y \geq 10 \\ 3x-2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the xy-coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

Example 6.5.1: Determining Whether an Ordered Pair Is a Solution to a System

Determine whether the ordered pair is a solution to the system:

$$\begin{cases} x+4y \geq 10 \\ 3x-2y < 12 \end{cases}$$

1. $(-2, 4)$
2. $(3, 1)$

Answer

1. Is the ordered pair $(-2, 4)$ a solution?

We substitute $\boldsymbol{x = -2}$ and $\boldsymbol{y = 4}$ into both inequalities.

$x+4y \geq 10$ $-\mathbf{2}+4(4) \overset{?}{\geq} 10$ $14 \geq 10$ true	$3\boldsymbol{x}-2\boldsymbol{y} < 12$ $3(-\mathbf{2})-2(4) \overset{?}{<} 12$ $-14 < 12$ true

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

2. Is the ordered pair $(3, 1)$ a solution?

We substitute $\boldsymbol{x = 3}$ and $\boldsymbol{y = 1}$ into both inequalities.

$\boldsymbol{x}+4\boldsymbol{y} \geq 10$ $\mathbf{3}+4(\mathbf{1}) \overset{?}{\geq} 10$ $7 \geq 10$ false	$3x-2y < 12$ $3(\mathbf{3})-2(\mathbf{1}) \overset{?}{<} 12$ $7 < 12$ true

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is not a solution to this system.

Your Turn 6.5.1

Determine whether the ordered pair is a solution to the system:

1. $(3, -1)$
2. $(6, -3)$

Solving Systems of Linear Inequalities Using Graphical Methods

The solution to a single linear inequality was the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. We will review graphs of linear inequalities and solve the linear inequality from its graph.

Example 6.5.2: Solving a System of Linear Inequalities by Graphing

Use Figure ✔ 6.5.2 to solve the system of linear inequalities:

$$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$$

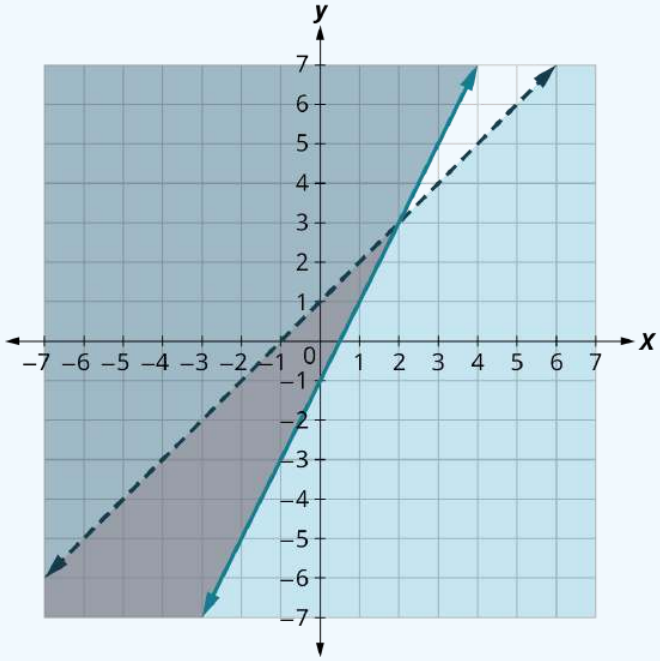


Figure 6.5.2

Answer

To solve the system of linear inequalities we look at the graph and find the region that satisfies BOTH inequalities. To do this we pick a test point and check. Let's us pick $(-1, -1)$.

Is $(-1, -1)$ a solution to $y \geq 2x - 1$?

$$-1 \stackrel{?}{\geq} 2(-1) - 1$$
$$-1 \geq -3 \text{ true}$$

Is $(\mathbf{-1}, \mathbf{-1})$ a solution to $y < x + 1$?

$$\mathbf{-1} \stackrel{?}{<} \mathbf{-1} + 1$$
$$-1 < 0 \text{ true}$$

The region containing $(-1, -1)$ is the solution to the system of linear inequalities. Notice that the solution is all the points in the area shaded twice, which appears as the darkest shaded region.

Your Turn 6.5.2

Use the graph shown to solve the system of linear inequalities:

$$\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$$

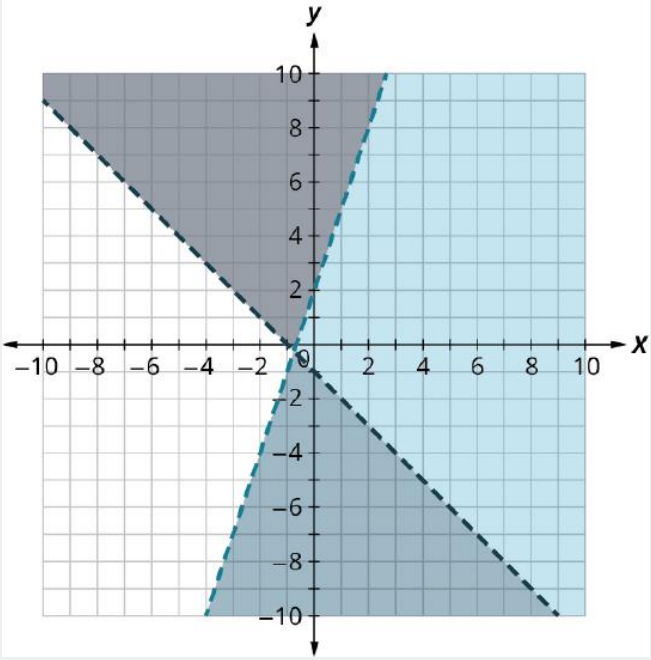


Figure 6.5.3

Graphing Systems of Linear Inequalities

We learned that the solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region by graphing, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

Step 1: Graph the first inequality.

Graph the boundary line.

Shade in the side of the boundary line where the inequality is true.

Step 2: On the same grid, graph the second inequality.

Graph the boundary line.

Shade in the side of that boundary line where the inequality is true.

Step 3: The solution is the region where the shading overlaps.

Step 4: Check by choosing a test point.

Example 6.5.3: Solving a System of Linear Inequalities by Graphing

Solve the system by graphing:

$$\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$$

Answer

Graph $x - y > 3$ by graphing $x - y = 3$ and testing a point (Figure

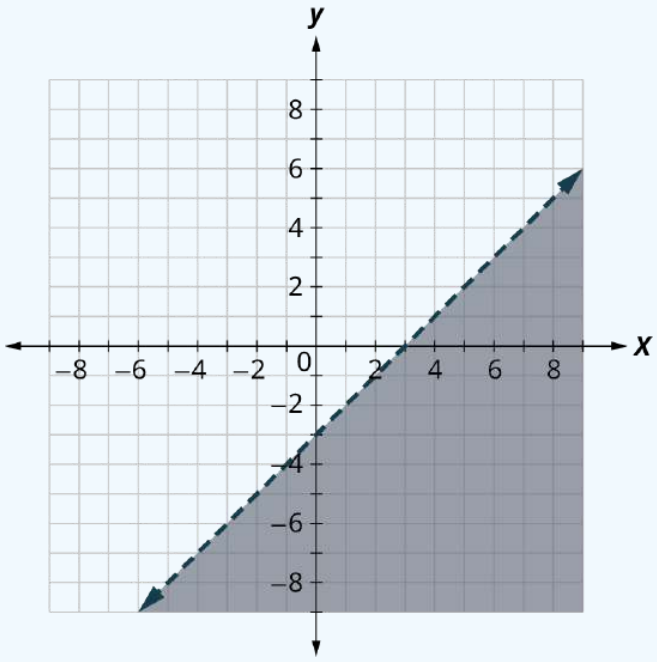


Figure ✔ 6.5.4

Graph
$y < -\frac{1}{5}x + 4$
by graphing
$y = -\frac{1}{5}x + 4$
using the slope
$m = -\frac{1}{5}$
and
y
-intercept
$b = 4$
(Figure
✔ 6.5.5
). The boundary line will be dashed. Test
$(0, 0)$
which makes the inequality true, so shade the side that contains
$(0, 0)$
.

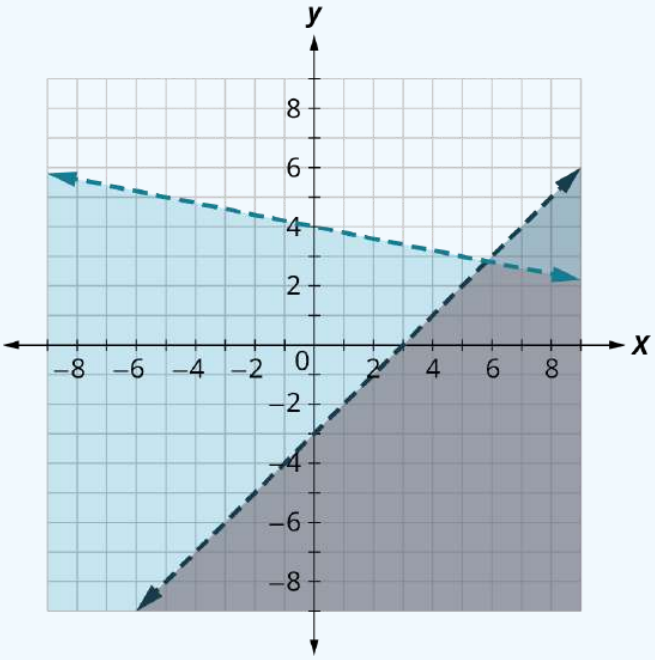


Figure ✔ 6.5.5

Choose a test point in the solution and verify that it is a solution to both inequalities. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

Your Turn 6.5.3

Solve the system by graphing:

$$\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$$

Example 6.5.4: Graphing a System of Linear Inequalities

Solve the system by graphing:

$$\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$$

Answer

Graph $x - 2y < 5$ by graphing $x - 2y = 5$ (Figure

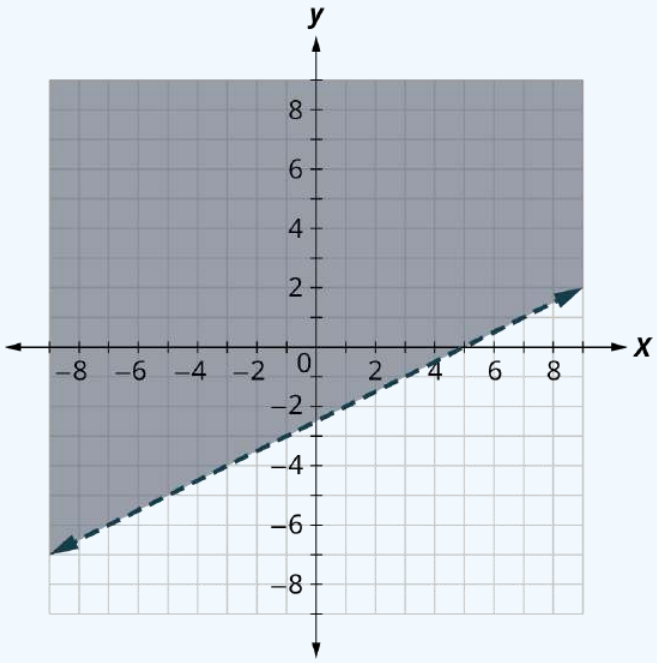


Figure 6.5.6

Graph
$y > -4$
by graphing
$y = -4$
and recognizing that it is a horizontal line through
$y = -4$
(Figure
✔ 6.5.7
). The boundary line will be dashed. Test
$(0, 0)$
, which makes the inequality true so shade the side that contains
$(0, 0)$
.

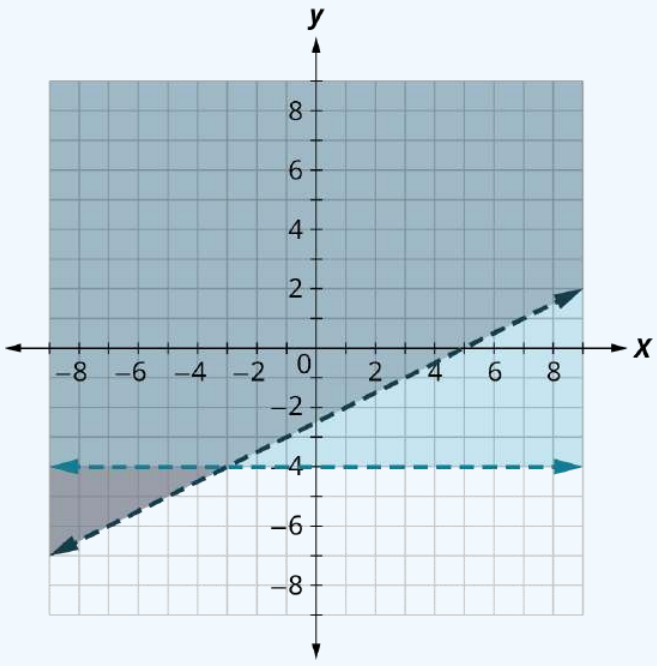


Figure ✔ 6.5.7

The point $(0, 0)$ is in the solution, and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice, which appears as the darkest shaded region.

Your Turn 6.5.4

Solve the system by graphing:

$$\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$$

Systems of linear inequalities where the boundary lines are parallel might have no solution. We will see this in the next example.

✔ Example 6.5.5: Graphing Parallel Boundary Lines with No Solution

Solve the system by graphing:

$$\begin{cases} 4x + 3y \geq 12 \\ y < -\frac{4}{3}x + 1 \end{cases}$$

Answer

Graph $4x+3y \geq 12$, by graphing $4x+3y=12$ (Figure ✔ $6.5.8$) and testing a point. The intercepts are $x=3$ and $y=4$ and the boundary line will be solid. Test $(0,0)$, which makes the inequality false, so shade the side that does not contain $(0,0)$.

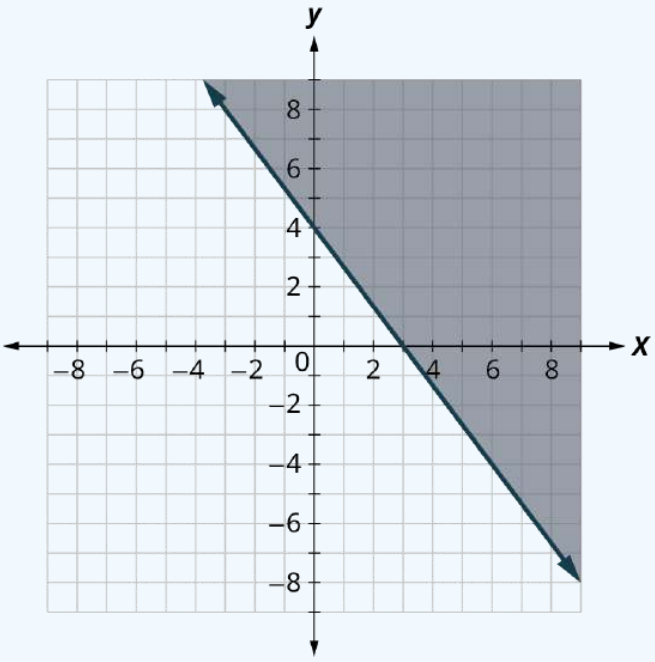
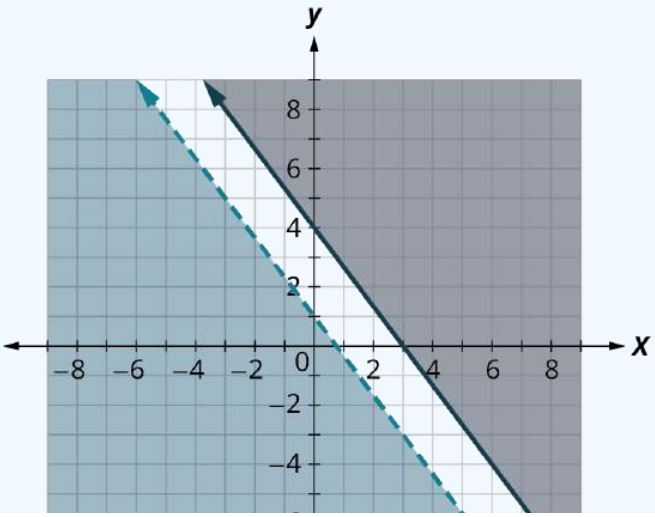


Figure ✔ 6.5.8

Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = -\frac{4}{3}$ and y-intercept $b = 1$ (Figure ✔ 6.5.9). The boundary line will be dashed. Test $(0, 0)$, which makes the inequality true, so shade the side that contains $(0, 0)$.



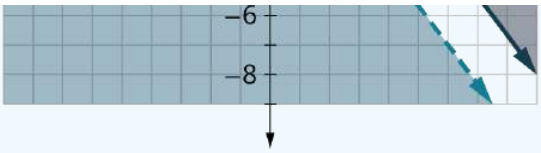


Figure ✔ 6.5.9

No shared point exists in both shaded regions, so the system has no solution.

Your Turn 6.5.5

Solve the system by graphing:

$$\begin{cases} 3x-2y \geq 12 \\ y \geq \frac{3}{2}x+1 \end{cases}$$

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We will see this in the next example.

✔ Example 6.5.6: Graphing Parallel Boundary Lines with a Solution

Solve the system by graphing:

$$\begin{cases} y > \frac{1}{2}x-4 \\ x-2y < -4 \end{cases}$$

Answer

Graph $y > \frac{1}{2}x-4$ by graphing $y = \frac{1}{2}x-4$ using the slope $m = \frac{1}{2}$ and the y-intercept $b = -4$ (Figure ✔ 6.5.10). The boundary line will be dashed. Test $(0,0)$, which makes the inequality true, so shade the side that contains $(0,0)$.

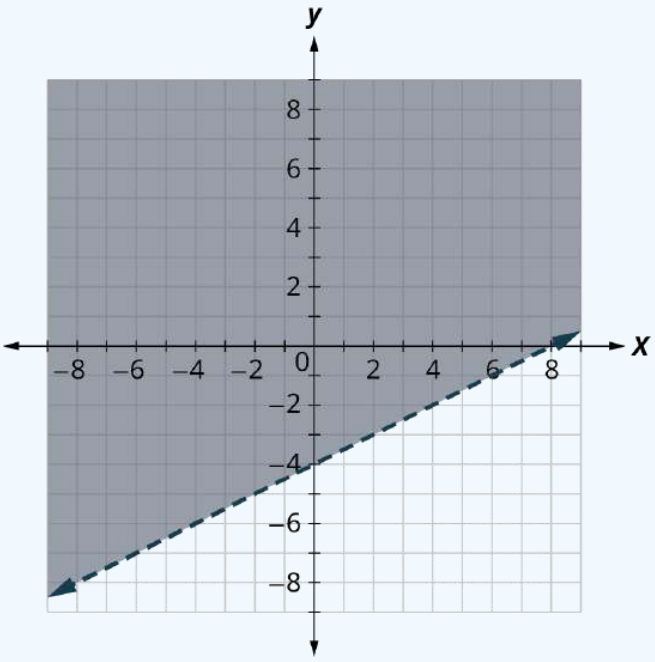


Figure ✔ 6.5.10

Graph $x - 2y < -4$ by graphing $x - 2y = -4$ (Figure

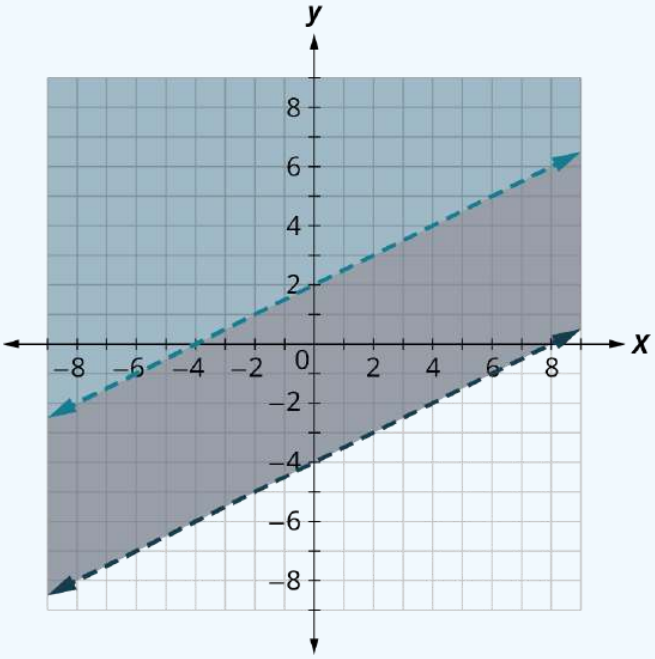


Figure ✔ 6.5.11

No point on the boundary lines is included in the solution as both lines are dashed. The solution is the region that is shaded twice which is also the solution to x − 2 y < − 4 x − 2 y < − 4 .

Your Turn 6.5.6

Solve the system by graphing:

$$\begin{cases} y \geq 3x+1 \\ -3x+y \geq -4 \end{cases}$$

Interpreting and Solving Applications of Linear Inequalities

When solving applications of systems of inequalities, first translate each condition into an inequality. Then graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so add inequalities to the system as additional requirements.

Example 6.5.7: Applying Linear Inequalities to Calculating Photo Costs

A photographer sells their prints at a booth at a street fair. At the start of the day, they want to have at least 25 photos to display at their booth. Each small photo they display costs $4 and each large photo costs $10. They do not want to spend more than $200 on photos to display.

1. Write a system of inequalities to model this situation.
2. Graph the system.
3. Could they display 10 small and 20 large photos?
4. Could they display 20 large and 10 small photos?

Answer

1. Let $x =$ the number of small photos and $y =$ the number of large photos. To find the system of equations translate the information. They want to have at least 25 photos.

The number of small plus the number of large should be at least 25.	$x+y \geq 25$
$4 for each small and $10 for each large must be no more than $200	$4x+10y \leq 200$
The number of small photos must be greater than or equal to 0.	$x \geq 0$
The number of large photos must be greater than or equal to 0.	$y \geq 0$
We have our system of equations.	$\begin{cases} x+y \geq 25 \\ 4x+10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$

2. Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only Quadrant I. To graph $x+y \geq 25$, graph $x+y=25$ as a solid line. Choose $(0,0)$ as a test point. Since it does not make the inequality true, shade the side that does not include the point $(0,0)$.

To graph $4x+10y \leq 200$, graph $4x+10y=200$ as a solid line. Choose $(0,0)$ as a test point. Since it does make the inequality true, shade (bottom left) the side that include the point $(0,0)$.

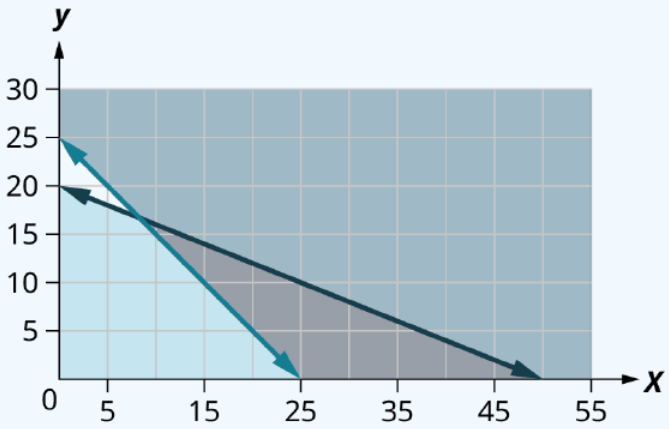


Figure 6.5.12:

The solution of the system is the region of Figure ✔ 6.5.12 that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(25, 0)$ to $(55, 0)$.

3. To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10, 20)$ is in the solution region. We could also test the point to see if it is a solution of both equations. It is not, so the photographer would not display 10 small and 20 large photos.

4. To determine if 20 small and 10 large photos would work, we look at the graph to see if the point $(20, 10)$ is in the solution region. We could also test the point to see if it is a solution of both equations. It is, so the photographer could choose to display 20 small and 10 large photos. Notice that we could also test the possible solutions by substituting the values into each inequality.

Your Turn 6.5.7

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than $5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs $1.40. Each cookie has 160 calories and costs $0.50.

Write a system of inequalities to model this situation.

Graph the system.

Could he eat 3 hamburgers and 2 cookies?

Could he eat 2 hamburgers and 4 cookies?

Video



Solving Systems of Linear Inequalities by Graphing



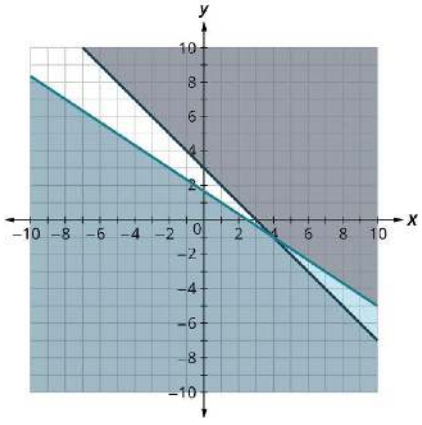
Systems of Linear Inequalities

Check Your Understanding

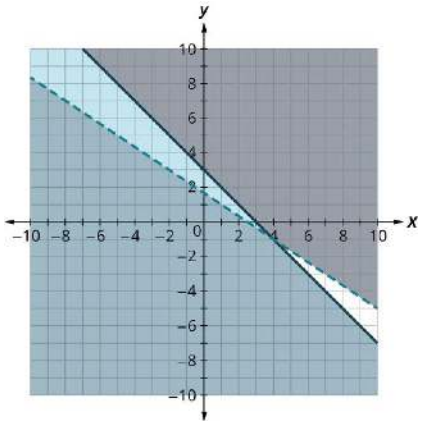
Match the correct graph to its system of inequalities.

1. $\begin{cases} 2x+3y<5 \\ x-y>3 \end{cases}$

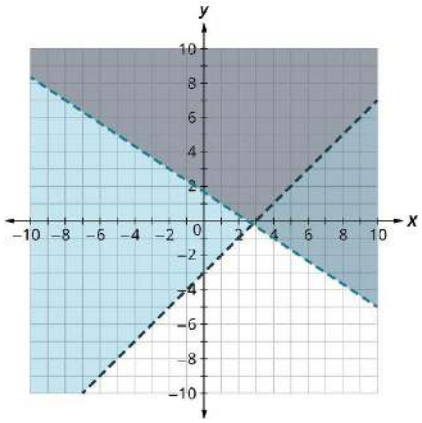
a.



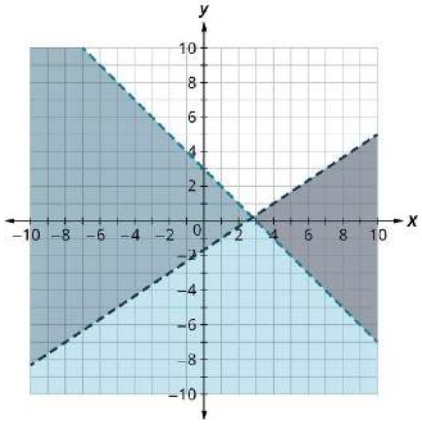
b.



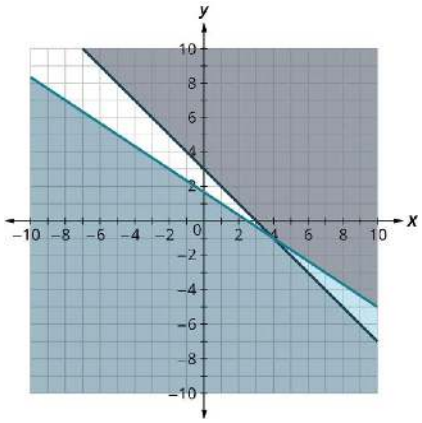
c.



d.

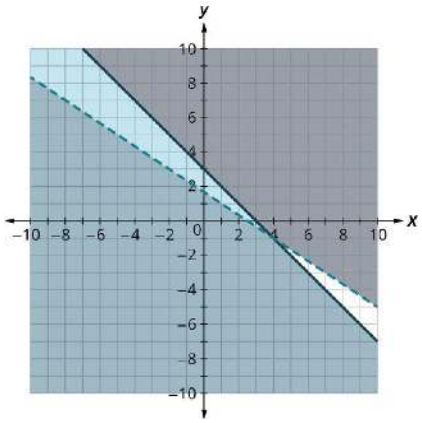


e.

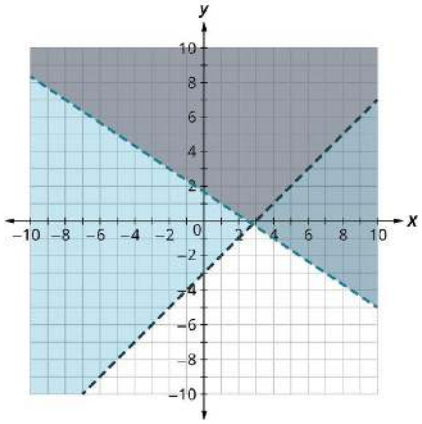


2. $\begin{cases} 2x - 3y \leq 5 \\ x - y \geq 3 \end{cases}$

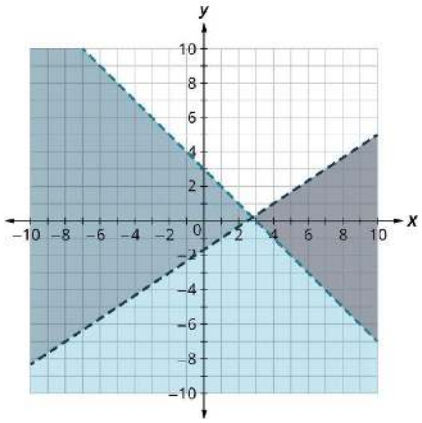
a.



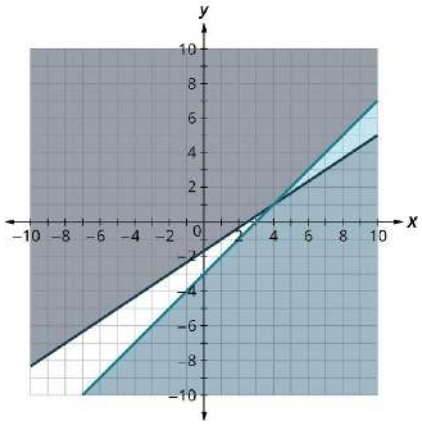
b.



c.

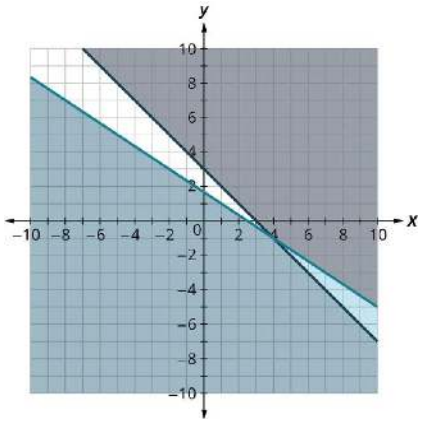


d.

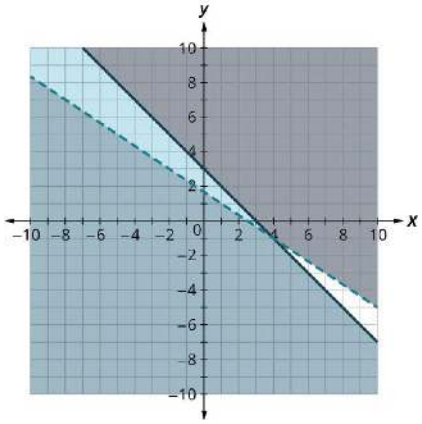


3. $\begin{cases} 2x-3y>5 \\ x+y<3 \end{cases}$

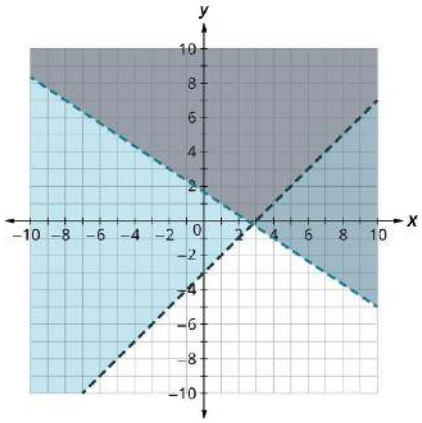
a.


y
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−8
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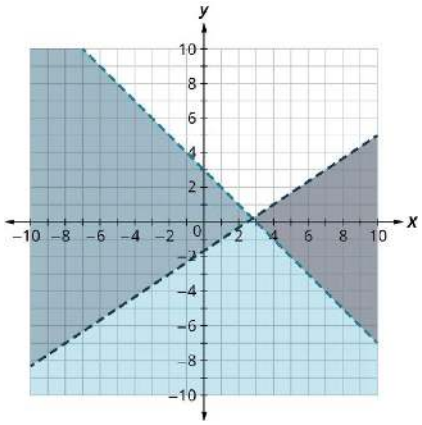
b.


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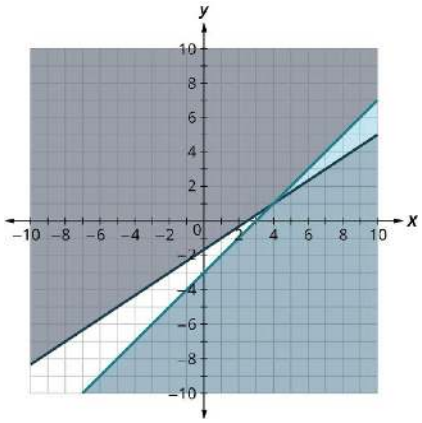
c.


y
x
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d.

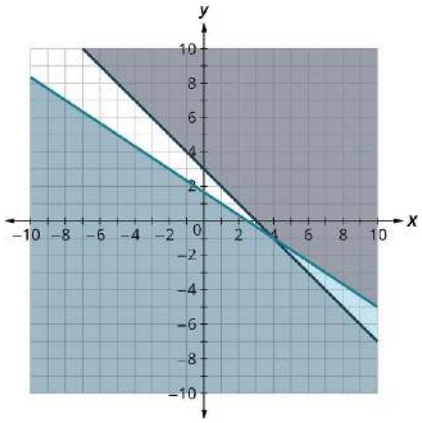

y
x
10
8
6
4
2
0
−2
−4
−6
−8
−10
−10 −8 −6 −4 −2 2 4 6 8 10

e.

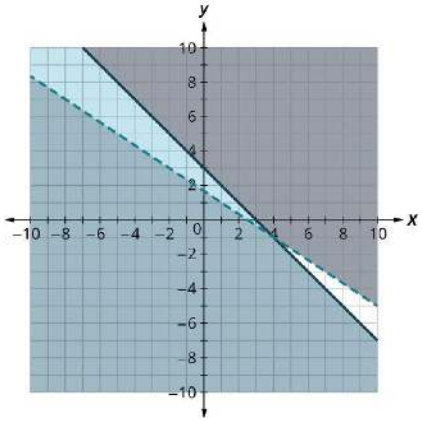


4. $\begin{cases} 2x+3y>5 \\ x+y \leq 3 \end{cases}$

a.

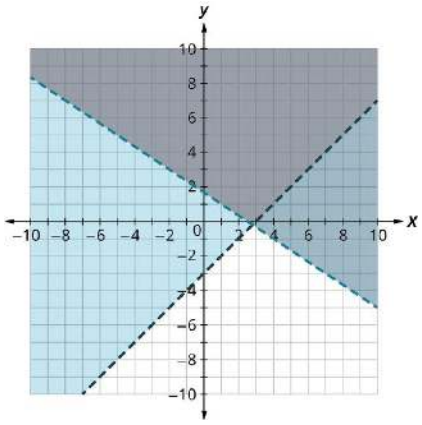


b.

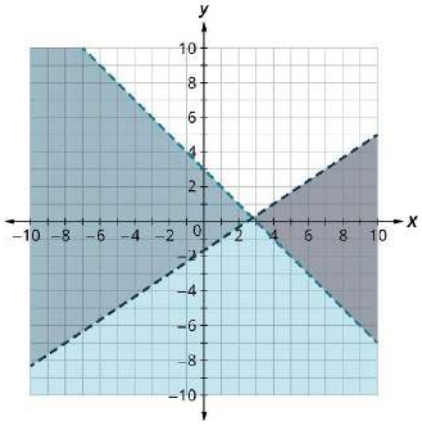


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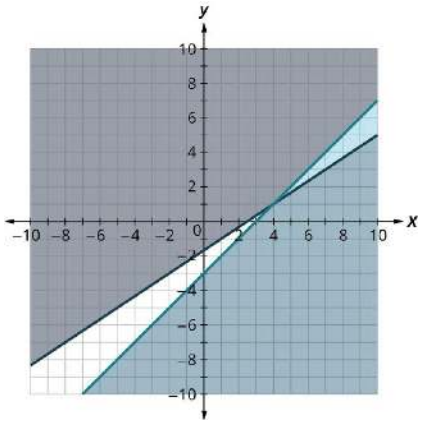
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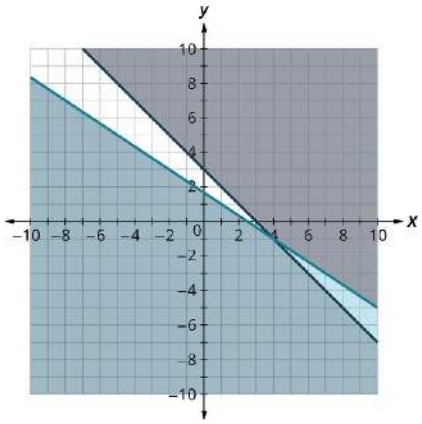
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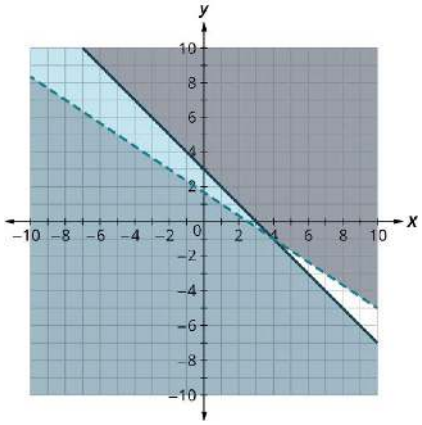
5. $\begin{cases} 2x+3y \leq 5 \\ x+y \geq 3 \end{cases}$

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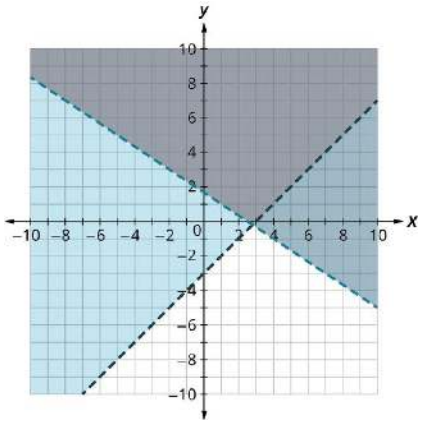
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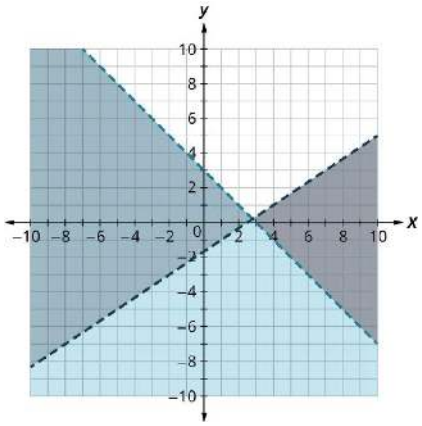
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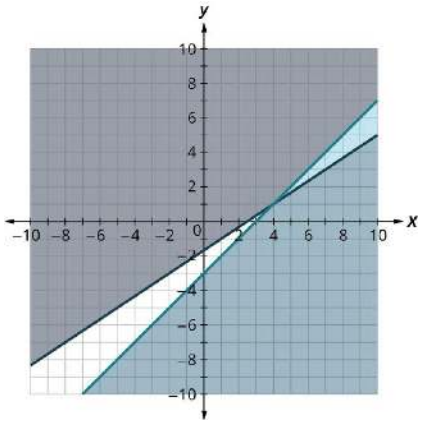

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e. 

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6.5.1: Systems of Linear Inequalities in Two Variables (Exercises)

For the following exercises, determine whether each ordered pair is a solution to the system.

Exercise 6.5.1.1

$$\begin{cases} 3x+y>5 \\ 2x-y\leq 10 \end{cases}$$

A. $(3,-3)$
B. $(7,1)$

Exercise 6.5.1.2

$$\begin{cases} y<\frac{3}{2}x+3 \\ \frac{3}{4}x-2y<5 \end{cases}$$

A. $(-4,-1)$
B. $(8,3)$

Exercise 6.5.1.3

$$\begin{cases} 4x-y<10 \\ -2x+2y>-8 \end{cases}$$

A. $(5,-2)$
B. $(-1,3)$

Exercise 6.5.1.4

$$\begin{cases} y>\frac{2}{3}x-5 \\ x+\frac{1}{2}y\leq 4 \end{cases}$$

A. $(6,-4)$
B. $(3,0)$

Exercise 6.5.1.5

$$\begin{cases} 6x-5y<20 \\ -2x+7y>-8 \end{cases}$$

A. $(1,-3)$
B. $(-4,4)$

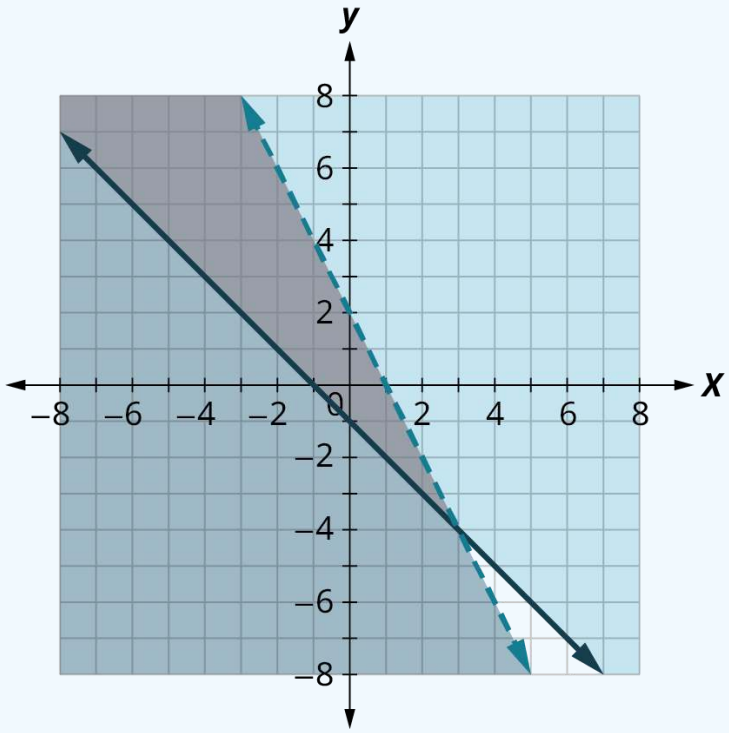
Exercise 6.5.1.6

$$\begin{cases} 7x+2y>14 \\ 5x-y\leq 8 \end{cases}$$

A. $(2,3)$
B. $(7,-1)$

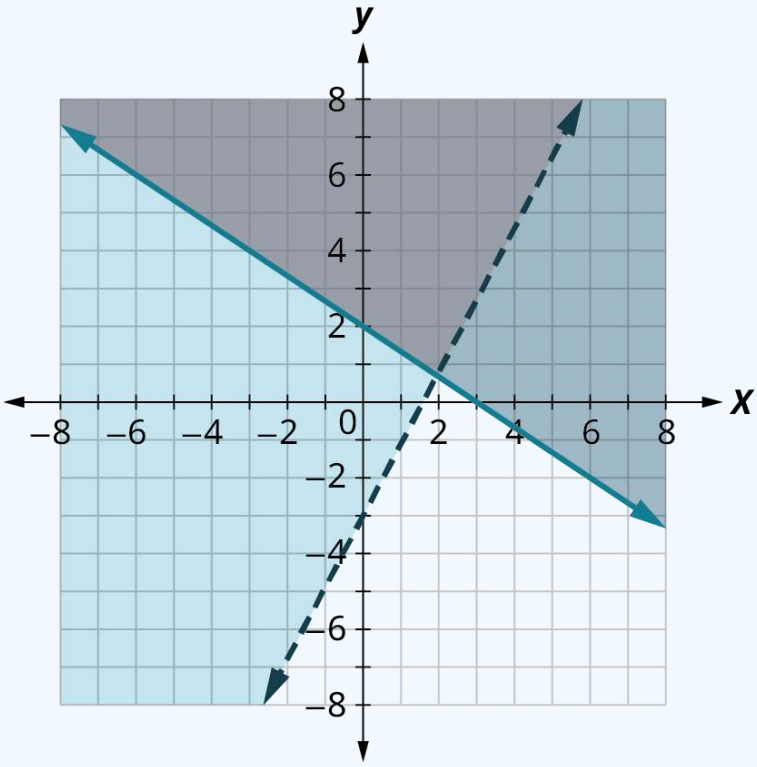
For the following exercises, determine whether each ordered pair is a solution to the darkest shaded region of the graph.

Exercise 6.5.1.7



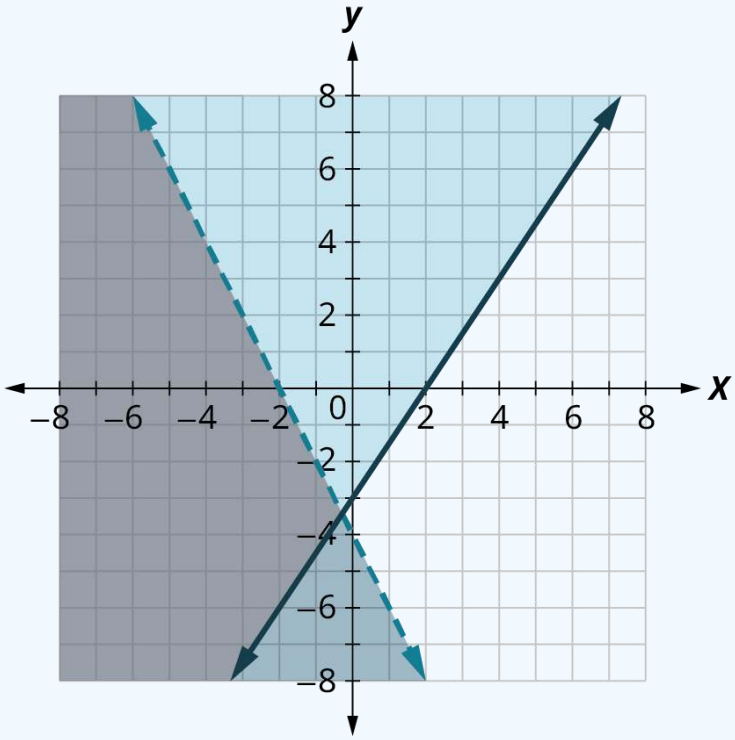
A. $(0, 0)$
B. $(6, -8)$

Exercise 6.5.1.8



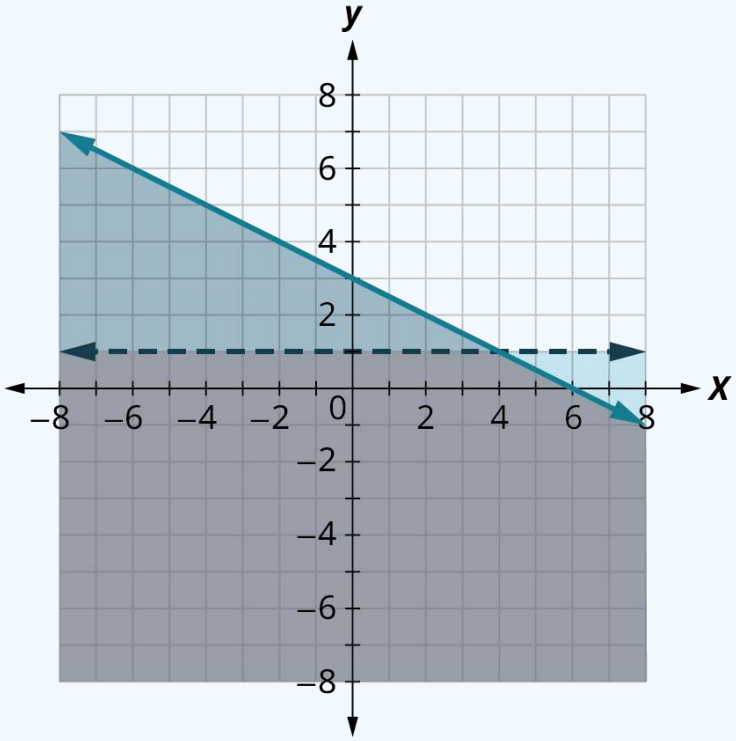
A. $(0, 0)$
B. $(2, -4)$

Exercise 6.5.1.9



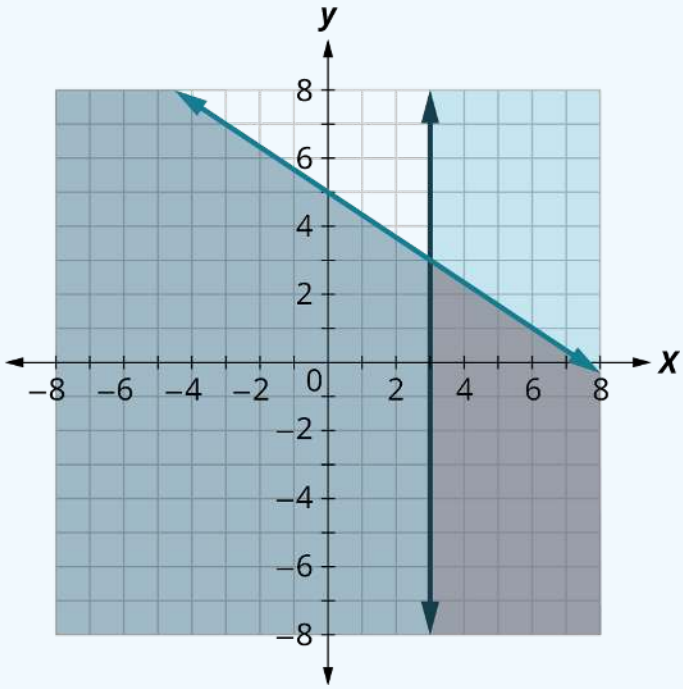
a. $(0, 0)$
b. $(2, 2)$

Exercise 6.5.1.10



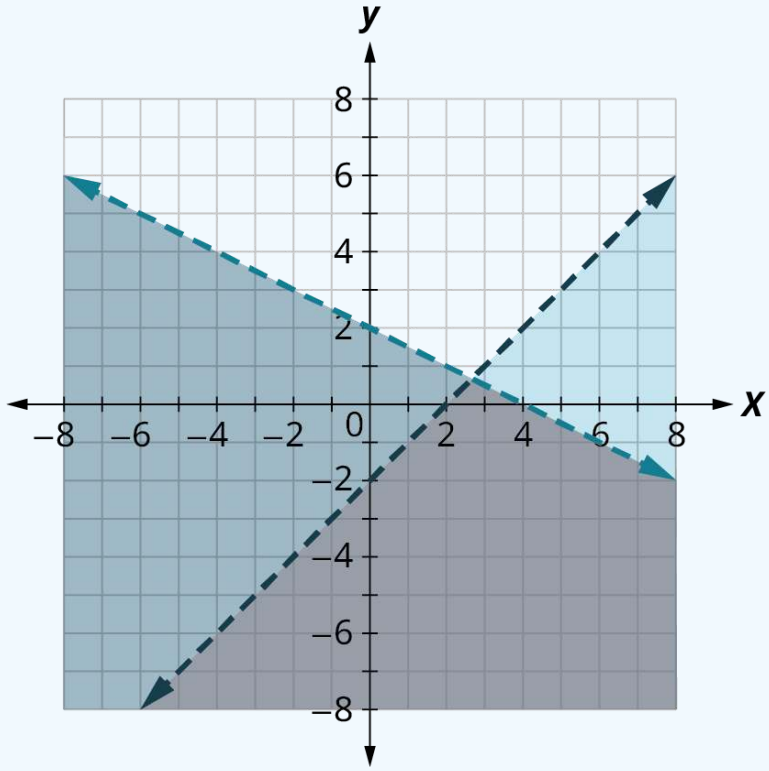
A. $(0, 0)$
B. $(2, 3)$

? Exercise 6.5.1.11



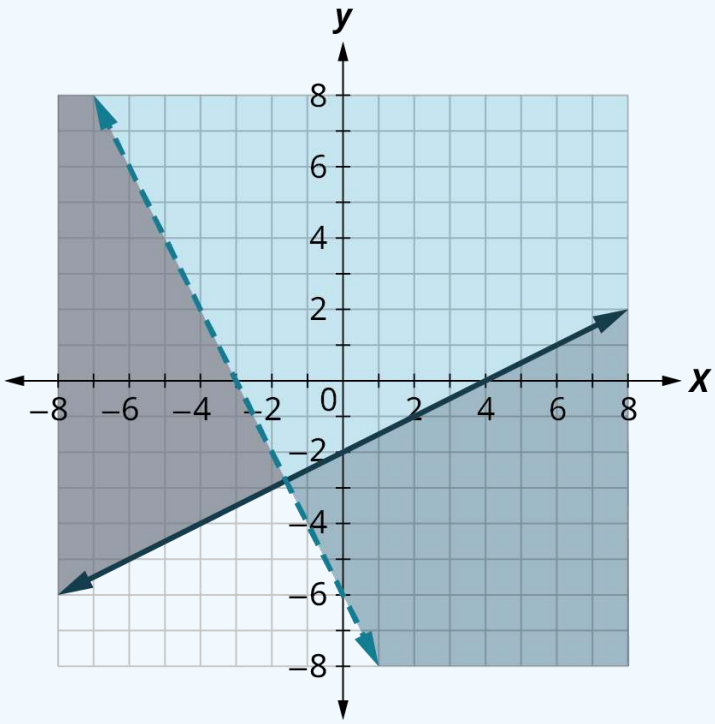
A. $(0, 0)$
B. $(6, 0)$

Exercise 6.5.1.12



A. $(0, 0)$
B. $(6, 2)$

Exercise 6.5.1.13



A. $(0,0)$
B. $(2,2)$

For the following exercises, solve the systems of linear equations by graphing.

Exercise 6.5.1.14

$$\begin{cases} y \leq 3x+2 \\ y > x-1 \end{cases}$$

Exercise 6.5.1.15

$$\begin{cases} y < 2x-1 \\ y \leq -\frac{1}{2}x+4 \end{cases}$$

Exercise 6.5.1.16

$$\begin{cases} x-y > 1 \\ y < -\frac{1}{4}x+3 \end{cases}$$

Exercise 6.5.1.17

$$\begin{cases} 3x-y \geq 6 \\ y \geq -\frac{1}{2}x \end{cases}$$

Exercise 6.5.1.18

$$\begin{cases} 2x+4y \geq 8 \\ y \leq \frac{3}{4}x \end{cases}$$

Exercise 6.5.1.19

$$\begin{cases} 2x-5y < 10 \\ 3x+4y \geq 12 \end{cases}$$

Exercise 6.5.1.20

$$\begin{cases} 2x+2y > -4 \\ -x+3y \geq 9 \end{cases}$$

Exercise 6.5.1.21

$$\begin{cases} x-2y < 3 \\ y \leq 1 \end{cases}$$

Exercise 6.5.1.22

$$\begin{cases} x-3y > 4 \\ y \leq -1 \end{cases}$$

Exercise 6.5.1.23

$$\begin{cases} y & \geq & -\frac{1}{2}x-3 \\ x & \leq & 2 \end{cases}$$

Exercise 6.5.1.24

$$\begin{cases} x-3y & \geq & 6 \\ y & > & \frac{1}{3}x+1 \end{cases}$$

Exercise 6.5.1.25

$$\begin{cases} y & \geq & \frac{3}{4}x-2 \\ y & < & 2 \end{cases}$$

Exercise 6.5.1.26

$$\begin{cases} 3x-4y & < & 8 \\ x & < & 1 \end{cases}$$

Exercise 6.5.1.27

$$\begin{cases} -3x+5y & > & 10 \\ x & > & -1 \end{cases}$$

Exercise 6.5.1.28

$$\begin{cases} x & \geq & 3 \\ y & \leq & 2 \end{cases}$$

Exercise 6.5.1.29

$$\begin{cases} x & \leq & -1 \\ y & \geq & 3 \end{cases}$$

Exercise 6.5.1.30

$$\begin{cases} 2x+4y & > & 4 \\ y & \leq & -\frac{1}{2}x-2 \end{cases}$$

For the following exercises, translate to a system of inequalities and solve.

A gardener does not want to spend more than $50 on bags of fertilizer and peat moss for their garden. Fertilizer costs $2 a bag and peat moss costs $5 a bag. The gardener's van can hold at most 20 bags.

Exercise 6.5.1.31

Write a system of inequalities to model this situation.

Exercise 6.5.1.32

Graph the system.

Exercise 6.5.1.33

Can they buy 15 bags of fertilizer and 4 bags of peat moss?

Exercise 6.5.1.34

Can they buy 10 bags of fertilizer and 10 bags of peat moss?

For the following exercises, translate to a system of inequalities and solve.

A student is studying for their final exams in chemistry and algebra. They only have 24 hours to study, and it will take them at least 3 times as long to study for algebra than chemistry.

Exercise 6.5.1.35

Write a system of inequalities to model this situation.

Exercise 6.5.1.36

Graph the system.

Exercise 6.5.1.37

Can they spend 4 hours on chemistry and 20 hours on algebra?

Exercise 6.5.1.38

Can they spend 6 hours on chemistry and 18 hours on algebra?

For the following exercises, translate to a system of inequalities and solve.

Mara is attempting to build muscle mass. To do this, she needs to eat an additional 80 grams of protein or more in a day. A bottle of protein water costs \$3.20 and a protein bar costs \$1.75. The protein water supplies 27 grams of protein and the bar supplies 16 grams. Let w be the number of water bottles Mara can buy, and let b be the number of protein bars she can buy. If Mara has \$10 dollars to spend:

Exercise 6.5.1.39

Write a system of inequalities to model this situation.

Exercise 6.5.1.40

Graph the system.

Exercise 6.5.1.41

Could she buy 3 bottles of protein water and 1 protein bar?

Exercise 6.5.1.42

Could she buy no bottles of protein water and 5 protein bars?

For the following exercises, translate to a system of inequalities and solve.

Mark is increasing his exercise routine by running and walking at least 4 miles each day. His goal is to burn a minimum of 1,500 calories from this exercise. Walking burns 270 calories/mile and running burns 650 calories/mile.

Exercise 6.5.1.43

Write a system of inequalities to model this situation.

Exercise 6.5.1.44

Graph the system.

Exercise 6.5.1.45

Could he meet his goal by walking 3 miles and running 1 mile?

Exercise 6.5.1.46

Could he meet his goal by walking 2 miles and running 2 miles?

For the following exercises, translate to a system of inequalities and solve.

Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts, which have 360 calories each, and $2 energy drinks, which have 110 calories.

Exercise 6.5.1.47

Write a system of inequalities that models this situation.

Exercise 6.5.1.48

Graph the system.

Exercise 6.5.1.49

Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?

Exercise 6.5.1.50

Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

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CHAPTER OVERVIEW

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7.5.1: Minimization By The Simplex Method (Exercises)

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7.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science

Learning Objectives

In this section, you will learn about real world applications of linear programming and related methods.

Many applied math problems across many different fields are "optimization" problems, i.e., problems that ask for the largest or smallest value some expression can have given some constraints on its variables. For example, imagine an ice cream parlor that's trying to finish their summer sales season by selling hot fudge sundaes. They sell two kinds of sundae, regular and deluxe. A regular sundae contains 2 scoops of ice cream and 1 ounce of fudge sauce; a deluxe sundae contains 3 scoops of ice cream and 2 ounces of fudge. The ice cream parlor makes a 50 cent profit on each regular sundae, and 75 cents on each deluxe one. Finally, they have enough ice cream on hand to serve 300 scoops, and 200 ounces of fudge sauce. The optimization problem is to figure out what mixture of regular and deluxe sundaes will produce the biggest profit without needing more ice cream or fudge than available.

To start modeling this problem mathematically, give names to the number of regular and deluxe sundaes sold, say R and D. Then the ice cream parlor's profit will be $50R+75D$. On the other hand, since each regular sundae needs 2 scoops of ice cream, and each deluxe 3, and there are 300 scoops available, there is a constraint that requires $2R+3D \leq 300$. The supply of fudge sauce dictates another constraint, namely $R+2D \leq 200$. Putting all of these ideas together, we could describe the ice cream parlor's optimization problem as:

"Maximize

$$50R+75D$$

"subject to

$$\begin{aligned} 2R+3D &\leq 300 \\ R+2D &\leq 200.\text{"} \end{aligned}$$

Notice that the quantity to optimize is a linear function of R and D, and all the constraints are linear inequalities. These features make this a linear optimization problem. Linear programming is the mathematics of solving such optimization problems. In this chapter we'll examine two approaches to linear programming, one based on graphing the constraints, and one based on something called the "Simplex Method." But first, we look in a little more depth at some of the situations in which linear optimization problems and linear programming occur.

Linear programming is used in business and industry in production planning, transportation and routing, and various types of scheduling. Airlines use linear programs to schedule their flights, taking into account both scheduling aircraft and scheduling staff. Delivery services use linear programs to schedule and route shipments to minimize shipment time or minimize cost. Retailers use linear programs to determine how to order products from manufacturers and organize deliveries with their stores. Manufacturing companies use linear programming to plan and schedule production. Financial institutions use linear programming to determine the mix of financial products they offer, or to schedule payments transferring funds between institutions. Health care institutions use linear programming to ensure the proper supplies are available when needed. And as we'll see below, linear programming has also been used to organize and coordinate life saving health care procedures.

In some of the applications, the techniques used are related to linear programming but are more sophisticated than the methods we study in this class. One such technique is called integer programming. In these situations, answers must be integers to make sense, and can not be fractions. Problems where solutions must be integers are more difficult to solve than the linear programs we've worked with. In fact, many of our problems have been very carefully constructed for learning purposes so that the answers just happen to turn out to be integers, but in the real world unless we specify that as a restriction, there is no guarantee that a linear program will produce integer solutions. There are also related techniques that are called non-linear programs, where the functions defining the objective function and/or some or all of the constraints may be non-linear rather than straight lines.

Many large businesses that use linear programming and related methods have analysts on their staff who can perform the analyses needed, including linear programming and other mathematical techniques. Consulting firms specializing in use of such techniques also aid businesses who need to apply these methods to their planning and scheduling processes.

When used in business, many different terms may be used to describe the use of techniques such as linear programming as part of mathematical business models. Optimization, operations research, business analytics, data science, industrial engineering, and management science are among the terms used to describe mathematical modelling techniques that may include linear programming and related methods.

In the rest of this section we'll explore six real world applications, and investigate what they are trying to accomplish using optimization, as well as what their constraints might represent.

Airline Scheduling

Airlines use techniques that include and are related to linear programming to schedule their aircrafts to flights on various routes, and to schedule crews to the flights. In addition, airlines also use linear programming to determine ticket pricing for various types of seats and levels of service or amenities, as well as the timing at which ticket prices change.

The process of scheduling aircraft and departure times on flight routes can be expressed as a model that minimizes cost, of which the largest component is generally fuel costs.

Constraints involve considerations such as:

- Each aircraft needs to complete a daily or weekly tour to return back to its point of origin.
- Scheduling sufficient flights to meet demand on each route.
- Scheduling the right type and size of aircraft on each route to be appropriate for the route and for the demand for number of passengers.
- Aircraft must be compatible with the airports it departs from and arrives at - not all airports can handle all types of planes.

A model to accomplish this could contain thousands of variables and constraints. Highly trained analysts determine ways to translate all the constraints into mathematical inequalities or equations to put into the model.

After aircraft are scheduled, crews need to be assigned to flights. Each flight needs a pilot, a co-pilot, and flight attendants. Each crew member needs to complete a daily or weekly tour to return back to his or her home base. Additional constraints on flight crew assignments take into account factors such as:

- Pilot and co-pilot qualifications to fly the particular type of aircraft they are assigned to
- Flight crew have restrictions on the maximum amount of flying time per day and the length of mandatory rest periods between flights or per day that must meet certain minimum rest time regulations.
- Numbers of crew members required for a particular type or size of aircraft.

When scheduling crews to flights, the objective function would seek to minimize total flight crew costs, determined by the number of people on the crew and pay rates of the crew members. However the cost for any particular route might not end up being the lowest possible for that route, depending on tradeoffs to the total cost of shifting different crews to different routes.

An airline can also use linear programming to revise schedules on short notice on an emergency basis when there is a schedule disruption, such as due to weather. In this case the considerations to be managed involve:

- Getting aircrafts and crews back on schedule as quickly as possible
- Moving aircraft from storm areas to areas with calm weather to keep the aircraft safe from damage and ready to come back into service as quickly and conveniently as possible
- Ensuring crews are available to operate the aircraft and that crews continue to meet mandatory rest period requirements and regulations.

Kidney Donation Chain

For patients who have kidney disease, a transplant of a healthy kidney from a living donor can often be a lifesaving procedure. Criteria for a kidney donation procedure include the availability of a donor who is healthy enough to donate a kidney, as well as a compatible match between the patient and donor for blood type and several other characteristics. Ideally, if a patient needs a kidney donation, a close relative may be a match and can be the kidney donor. However often there is not a relative who is a close enough match to be the donor. Considering donations from unrelated donor allows for a larger pool of potential donors. Kidney donations involving unrelated donors can sometimes be arranged through a chain of donations that pair patients with donors. For example a kidney donation chain with three donors might operate as follows:

- Donor A donates a kidney to Patient B.

- Donor B, who is related to Patient B, donates a kidney to Patient C.
- Donor C, who is related to Patient C, donates a kidney to Patient A, who is related to Donor A.

Linear programming is one of several mathematical tools that have been used to help efficiently identify a kidney donation chain. In this type of model, patient/donor pairs are assigned compatibility scores based on characteristics of patients and potential donors.

The objective is to maximize the total compatibility scores. Constraints ensure that donors and patients are paired only if compatibility scores are sufficiently high to indicate an acceptable match.

Advertisements in Online Marketing

Did you ever make a purchase online and then notice that as you browse websites, search, or use social media, you now see more ads related the item you purchased?

Marketing organizations use a variety of mathematical techniques, including linear programming, to determine individualized advertising placement purchases.

Instead of advertising randomly, online advertisers want to sell bundles of advertisements related to a particular product to batches of users who are more likely to purchase that product. Based on an individual's previous browsing and purchase selections, he or she is assigned a "propensity score" for making a purchase if shown an ad for a certain product. The company placing the ad generally does not know individual personal information based on the history of items viewed and purchased, but instead has aggregated information for groups of individuals based on what they view or purchase. However, the company may know more about an individual's history if he or she logged into a website making that information identifiable, within the privacy provisions and terms of use of the site.

The company's goal is to buy ads to present to specified size batches of people who are browsing. The linear program would assign ads and batches of people to view the ads using an objective function that seeks to maximize advertising response modelled using the propensity scores. The constraints are to stay within the restrictions of the advertising budget.

Loans

A car manufacturer sells its cars though dealers. Dealers can offer loan financing to customers who need to take out loans to purchase a car. Here we will consider how car manufacturers can use linear programming to determine the specific characteristics of the loan they offer to a customer who purchases a car. In a future chapter we will learn how to do the financial calculations related to loans.

A customer who applies for a car loan fills out an application. This provides the car dealer with information about that customer. In addition, the car dealer can access a credit bureau to obtain information about a customer's credit score.

Based on this information obtained about the customer, the car dealer offers a loan with certain characteristics, such as interest rate, loan amount, and length of loan repayment period.

Linear programming can be used as part of the process to determine the characteristics of the loan offer. The linear program seeks to maximize the profitability of its portfolio of loans. The constraints limit the risk that the customer will default and will not repay the loan. The constraints also seek to minimize the risk of losing the loan customer if the conditions of the loan are not favorable enough; otherwise the customer may find another lender, such as a bank, which can offer a more favorable loan.

Production Planning and Scheduling in Manufacturing

Consider the example of a company that produces yogurt. There are different varieties of yogurt products in a variety of flavors. Yogurt products have a short shelf life; it must be produced on a timely basis to meet demand, rather than drawing upon a stockpile of inventory as can be done with a product that is not perishable. Most ingredients in yogurt also have a short shelf life, so can not be ordered and stored for long periods of time before use; ingredients must be obtained in a timely manner to be available when needed but still be fresh. Linear programming can be used in both production planning and scheduling.

To start the process, sales forecasts are developed to determine demand to know how much of each type of product to make.

There are often various manufacturing plants at which the products may be produced. The appropriate ingredients need to be at the production facility to produce the products assigned to that facility. Transportation costs must be considered, both for obtaining and delivering ingredients to the correct facilities, and for transport of finished product to the sellers.

The linear program that monitors production planning and scheduling must be updated frequently - daily or even twice each day - to take into account variations from a master plan.

Bike Share Programs

Over 600 cities worldwide have bikeshare programs. Although bikeshare programs have been around for a long time, they have proliferated in the past decade as technology has developed new methods for tracking the bicycles.

Bikeshare programs vary in the details of how they work, but most typically people pay a fee to join and then can borrow a bicycle from a bike share station and return the bike to the same or a different bike share station. Over time the bikes tend to migrate; there may be more people who want to pick up a bike at station A and return it at station B than there are people who want to do the opposite. In chapter 9, we'll investigate a technique that can be used to predict the distribution of bikes among the stations.

Once other methods are used to predict the actual and desired distributions of bikes among the stations, bikes may need to be transported between stations to even out the distribution. Bikeshare programs in large cities have used methods related to linear programming to help determine the best routes and methods for redistributing bicycles to the desired stations once the desire distributions have been determined. The optimization model would seek to minimize transport costs and/or time subject to constraints of having sufficient bicycles at the various stations to meet demand.

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7.2: Maximization by Geometry

Learning Objectives

In this section, you will learn to:

1. Recognize the typical form of a linear programing problem
2. Formulate maximization linear programming problems
3. Graph feasibility regions for maximization linear programming problems
4. Determine optimal solutions for maximization linear programming problems.

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical **linear programming** problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The function we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing. In the next chapter, we'll learn an algorithm to find a solution numerically. That will provide us with a tool to solve problems with more than two variables. At that time, with a little more knowledge about linear programming, we'll also explore the many ways these techniques are used in business and wide variety of other fields.

We begin by solving a maximization problem.

Example 7.2.1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation.

If Nikki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

We start by choosing our variables.

- Let x = The number of hours per week Niki will work at Job I.
- Let y = The number of hours per week Niki will work at Job II.

Now we write the objective function. Since Niki gets paid $40 an hour at Job I, and $30 an hour at Job II, her total income I is given by the following equation.

$$I = 40x + 30y$$

Our next task is to find the constraints. The second sentence in the problem states, "She never wants to work more than a total of 12 hours a week." This translates into the following constraint:

$$x + y \leq 12$$

The third sentence states, "For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation." The translation follows.

$$2x + y \leq 16$$

The fact that x and y can never be negative is represented by the following two constraints:

$$x \geq 0, \text{ and } y \geq 0.$$

Well, good news! We have formulated the problem. We restate it as

$$\begin{aligned} &\textbf{Maximize} && \mathrm{I} = 40\mathrm{x} + 30\mathrm{y} \\ &\textbf{Subject to:} && \mathrm{x} + \mathrm{y} \leq 12 \\ &&& 2\mathrm{x} + \mathrm{y} \leq 16 \\ &&& \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{aligned} \tag{7.2.1}$$

In order to solve the problem, we graph the constraints and shade the region that satisfies **all** the inequality constraints.

Any appropriate method can be used to graph the lines for the constraints. However often the easiest method is to graph the line by plotting the x-intercept and y-intercept.

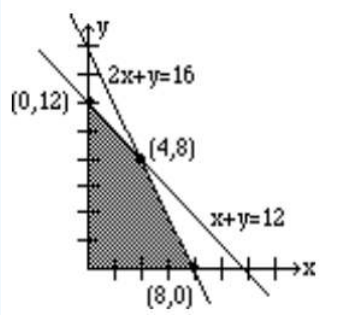
The line for a constraint will divide the plane into two region, one of which satisfies the inequality part of the constraint. A test point is used to determine which portion of the plane to shade to satisfy the inequality. Any point on the plane that is not on the line can be used as a test point.

- If the test point satisfies the inequality, then the region of the plane that satisfies the inequality is the region that contains the test point.
- If the test point does not satisfy the inequality, then the region that satisfies the inequality lies on the opposite side of the line from the test point.

In the graph below, after the lines representing the constraints were graphed using an appropriate method from Chapter 1, the point (0,0) was used as a test point to determine that

- (0,0) satisfies the constraint $x + y \leq 12$ because $0 + 0 < 12$
- (0,0) satisfies the constraint $2x + y \leq 16$ because $2(0) + 0 < 16$

Therefore, in this example, we shade the region that is below and to the left of **both** constraint lines, but also above the x axis and to the right of the y axis, in order to further satisfy the constraints $x \geq 0$ and $y \geq 0$.



The shaded region where all conditions are satisfied is called the **feasibility region** or the feasibility polygon.

The **Fundamental Theorem of Linear Programming** states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region.

Therefore, we will identify all the vertices (corner points) of the feasibility region. We call these points **critical points.** They are listed as (0, 0), (0, 12), (4, 8), (8, 0). To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below.

Critical Points	Income

Critical Points	Income
(0, 0)	40(0) + 30(0) = $0
(0, 12)	40(0) + 30(12) = $360
(4, 8)	40(4) + 30(8) = $400
(8, 0)	40(8) + 30(0) = $320

Clearly, the point (4, 8) gives the most profit: $400.

Therefore, we conclude that Niki should work 4 hours at Job I, and 8 hours at Job II.

Example 7.2.2

A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of $20 is realized for each regular gadget and $30 for a premium gadget, how many of each should be manufactured to maximize profit?

Solution

We choose our variables.

- Let x = The number of regular gadgets manufactured each day.
- and y = The number of premium gadgets manufactured each day.

The objective function is

$$P = 20x + 30y$$

We now write the constraints. The fourth sentence states that the company can make at most 7 gadgets a day. This translates as

$$x + y \leq 7$$

Since the regular gadget requires one hour of assembly and the premium gadget requires two hours of assembly, and there are at most 12 hours available for this operation, we get

$$x + 2y \leq 12$$

Similarly, the regular gadget requires two hours of finishing and the premium gadget one hour. Again, there are at most 12 hours available for finishing. This gives us the following constraint.

$$2x + y \leq 12$$

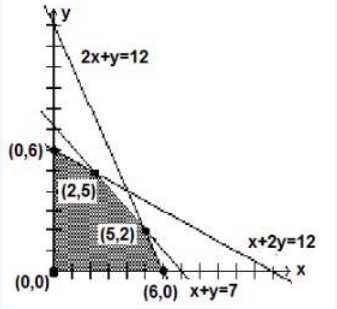
The fact that x and y can never be negative is represented by the following two constraints:

$$x \geq 0, \text{ and } y \geq 0.$$

We have formulated the problem as follows:

$$\begin{aligned} &\textbf{Maximize} && \mathrm{P = 20x + 30y} \\ &\textbf{Subject to:} && \mathrm{x + y \leq 7} \\ &&& \mathrm{x + 2y \leq 12} \\ &&& \mathrm{2x + y \leq 12} \\ &&& \mathrm{x \geq 0; y \geq 0} \end{aligned}$$

In order to solve the problem, we next graph the constraints and feasibility region.



Again, we have shaded the feasibility region, where all constraints are satisfied.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify all the critical points. They are listed as (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0). To maximize profit, we will substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below.

Critical Point	Income
(0, 0)	20(0) + 30(0) = $0
(0, 6)	20(0) + 30(6) = $180
(2, 5)	20(2) + 30(5) = $190
(5, 2)	20(5) + 30(2) = $160
(6,0)	20(6) + 30(0) = $120

The point (2, 5) gives the most profit, and that profit is $190.
Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of $190.

So far we have focused on "**standard maximization problems**" in which

1. The objective function is to be maximized
2. All constraints are of the form $ax + by \leq c$
3. All variables are constrained to by non-negative ($x \geq 0$, $y \geq 0$)

We will next consider an example where that is not the case. Our next problem is said to have "**mixed constraints**", since some of the inequality constraints are of the form $ax + by \leq c$ and some are of the form $ax + by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

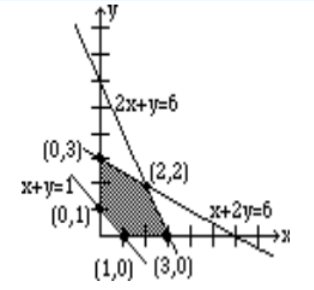
Example 7.2.3

Solve the following maximization problem graphically.

$$\begin{aligned} \textbf{Maximize} \quad & \mathrm{P} = 10\mathrm{x} + 15\mathrm{y} \\ \textbf{Subject to:} \quad & \mathrm{x} + \mathrm{y} \geq 1 \\ & \mathrm{x} + 2\mathrm{y} \leq 6 \\ & 2\mathrm{x} + \mathrm{y} \leq 6 \\ & \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{aligned}$$

Solution

The graph is shown below.



The five critical points are listed in the above figure. The reader should observe that the first constraint $x+y \geq 1$ requires that the feasibility region must be bounded below by the line $x+y=1$; the test point (0,0) does not satisfy $x+y \geq 1$, so we shade the region on the opposite side of the line from test point (0,0).

Critical point	Income
(1, 0)	10(1) + 15(0) = $10
(3, 0)	10(3) + 15(0) = $30
(2, 2)	10(2) + 15(2) = $50
(0, 3)	10(0) + 15(3) = $45
(0,1)	10(0) + 15(1) = $15

Clearly, the point (2, 2) maximizes the objective function to a maximum value of 50.

It is important to observe that that if the point (0,0) lies on the line for a constraint, then (0,0) could not be used as a test point. We would need to select any other point we want that does not lie on the line to use as a test point in that situation.

Finally, we address an important question. Is it possible to determine the point that gives the maximum value without calculating the value at each critical point?

The answer is yes.

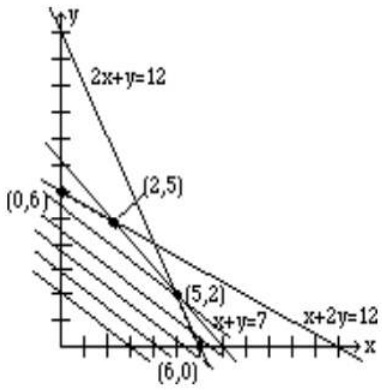
For example 7.2.2, we substituted the points (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0), in the objective function $P=20x+30y$, and we got the values $0, $180, $190, $160, $120, respectively.

Sometimes that is not the most efficient way of finding the optimum solution. Instead we could find the optimal value by also graphing the objective function.

To determine the largest P, we graph $P=20x+30y$ for any value P of our choice. Let us say, we choose $P=60$. We graph $20x+30y=60$.

Now we move the line parallel to itself, that is, keeping the same slope at all times. Since we are moving the line parallel to itself, the slope is kept the same, and the only thing that is changing is the P. As we move away from the origin, the value of P increases. The largest possible value of P is realized when the line touches the last corner point of the feasibility region.

The figure below shows the movements of the line, and the optimum solution is achieved at the point (2, 5). In maximization problems, as the line is being moved away from the origin, this optimum point is the farthest critical point.



We summarize:

The Maximization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
    1. For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$
    2. Since the variables are non-negative, we include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the maximum value.
    a. This is done by finding the value of the objective function at each corner point.
    b. This can also be done by moving the line associated with the objective function.

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7.2.1: Maximization by Geometry (Exercises)

For the following maximization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

2) Mr. Tran has $24,000 to invest, some in bonds and the rest in stocks. He has decided that the money invested in bonds must be at least twice as much as that in stocks. But the money invested in bonds must not be greater than $18,000. If the bonds earn 6%, and the stocks earn 8%, how much money should he invest in each to maximize profit?

3) A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize revenue?

4) The Silly Nut Company makes two mixtures of nuts: Mixture A and Mixture B. A pound of Mixture A contains 12 oz of peanuts, 3 oz of almonds and 1 oz of cashews and sells for $4. A pound of Mixture B contains 12 oz of peanuts, 2 oz of almonds and 2 oz of cashews and sells for $5. The company has 1080 lb. of peanuts, 240 lb. of almonds, 160 lb. of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit?
(Hint: Use consistent units. Work the entire problem in pounds by converting all values given in ounces into fractions of pounds).

5)

$$\begin{aligned} \text{Maximize:} \quad & Z = 4x + 10y \\ \text{Subject to:} \quad & x + y \leq 5 \\ & 2x + y \leq 8 \\ & x + 2y \leq 8 \\ & x \geq 0, y \geq 0 \end{aligned}$$

6) This maximization linear programming problem is not in “standard” form. It has mixed constraints, some involving $\leq$ inequalities and some involving $\geq$ inequalities. However with careful graphing, we can solve this using the techniques we have learned in this section.

$$\begin{aligned} \text{Maximize:} \quad & Z = 5x + 7y \\ \text{Subject to:} \quad & x + y \leq 30 \\ & 2x + y \leq 50 \\ & 4x + 3y \geq 60 \\ & 2x \geq y \\ & x \geq 0, y \geq 0 \end{aligned}$$

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7.3: Minimization by Geometry

Learning Objectives

In this section, you will learn to:

1. Formulate minimization linear programming problems
2. Graph feasibility regions for minimization linear programming problems
3. Determine optimal solutions for minimization linear programming problems.

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the **standard minimization linear program**, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasibility region, the possibility of no optimal solution exists.

Example 7.3.1

At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns $15 per hour for grading papers. Mary is an post-doctoral associate and can grade 30 papers per hour; Mary earns $25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment.
If Prof. Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Solution

We choose the variables as follows:

Let x = The number of hours per week John is employed.

and y = The number of hours per week Mary is employed.

The objective function is

$$C = 15x + 25y$$

The fact that each must work at least one hour each week results in the following two constraints:

$$x \geq 1$$
$$y \geq 1$$

Since John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, we get

$$20x + 30y \geq 110$$

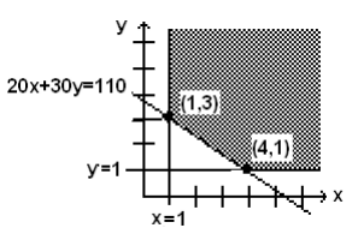
The fact that x and y are non-negative, we get

$$x \geq 0, \text{and } y \geq 0.$$

The problem has been formulated as follows.

$$\begin{aligned} &\textbf{Minimize} && \mathrm{C} = 15\mathrm{x} + 25\mathrm{y} \\ &\textbf{Subject to:} && \mathrm{x} \geq 1 \\ &&& \mathrm{y} \geq 1 \\ &&& 20\mathrm{x} + 30\mathrm{y} \geq 110 \\ &&& \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{aligned}$$

To solve the problem, we graph the constraints as follows:



Again, we have shaded the feasibility region, where all constraints are satisfied.

If we used test point (0,0) that does not lie on any of the constraints, we observe that (0, 0) **does not** satisfy any of the constraints $x \geq 1$, $y \geq 1$, $20x + 30y \geq 110$. Thus all the shading for the feasibility region lies on the opposite side of the constraint lines from the point (0,0).

Alternatively we could use test point (4,6), which also does not lie on any of the constraint lines. We'd find that (4,6) **does** satisfy all of the inequality constraints. Consequently all the shading for the feasibility region lies on the same side of the constraint lines as the point (4,6).

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, (1, 3) and (4, 1). To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below.

Critical points	Income
(1, 3)	15(1) + 25(3) = $90
(4, 1)	15(4) + 25(1) = $85

The point (4, 1) gives the least cost, and that cost is $85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of $85 per week.

Example 7.3.2

Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?

	PASTA	**TOFU**
PROTEIN	8g	16g
CARBOHYDRATES	60g	40g
VITAMIN C	2g	2g
CHOLESTEROL	60mg	50mg

Solution

We choose the variables as follows.

Let x = The number of days Mr. Hamer eats Pasta.

and y = The number of days Mr. Hamer eats Tofu.

Since he is trying to minimize his cholesterol intake, our objective function represents the total amount of cholesterol C provided by both meals.

$$C = 60x + 50y$$

The constraint associated with the total amount of protein provided by both meals is

$$8x + 16y \geq 200$$

Similarly, the two constraints associated with the total amount of carbohydrates and vitamins are obtained, and they are

$$60x + 40y \geq 960$$
$$2x + 2y \geq 40$$

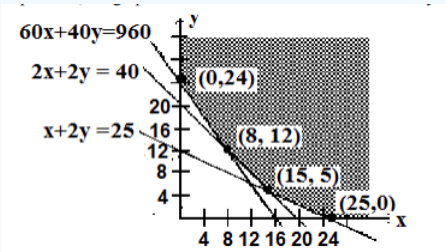
The constraints that state that x and y are non-negative are

$$x \geq 0, \text{ and } y \geq 0.$$

We summarize all information as follows:

$$\begin{array}{ll} \textbf{Minimize} & \mathrm{C} = 60\mathrm{x} + 50\mathrm{y} \\ \textbf{Subject to:} & 8\mathrm{x} + 16\mathrm{y} \geq 200 \\ & 60\mathrm{x} + 40\mathrm{y} \geq 960 \\ & 2\mathrm{x} + 2\mathrm{y} \geq 40 \\ & \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{array}$$

To solve the problem, we graph the constraints and shade the feasibility region.



We have shaded the unbounded feasibility region, where all constraints are satisfied.

To minimize the objective function, we find the vertices of the feasibility region. These vertices are (0, 24), (8, 12), (15, 5) and (25, 0). To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below.

Critical points	Income
(0, 24)	60(0) + 50(24) = 1200
(8, 12)	60(8) + 50(12) = 1080
(15, 5)	60(15) + 50(5) = 1150
(25, 0)	60(25) + 50(0) = 1500

The point (8, 12) gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta 8 days, and Tofu 12 days.

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution is if there is not a feasibility region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies **all** the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it can not have an optimal solution.
- A linear program can fail to have an optimal solution if the feasibility region is unbounded.

- The two minimization linear programs we examined had unbounded feasibility regions. The feasibility region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions.
- However, if we were to consider a maximization problem with a similar unbounded feasibility region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of the maximization problems, we still feel that we should summarize the steps involved.

Minimization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
   a. For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b. Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the minimum value.
   a. This can be done by finding the value of the objective function at each corner point.
   b. This can also be done by moving the line associated with the objective function.
   c. There is the possibility that the problem has no solution

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7.3.1: Minimization by Geometry (Exercises)

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. Food A costs $2 per unit and Food B cost $1 per unit. How many units of each food should be purchased to keep costs at a minimum?

2) A computer store sells two types of computers, laptops and desktops. The supplier demands that at least 150 computers be sold a month. Experience shows that most consumers prefer laptops, but some business customers require desktops. The result is that the number of laptops sold is at least twice of the number of desktops. The store pays its sales staff a $60 commission for each laptop, and a $40 commission for each desktop. Let x = the number of laptops and y = the number of desktop computers. How many of each type must be sold to minimize commission to its sales people?
What is the minimum commission?

3) An oil company has two refineries. Each day, Refinery A produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $12,000 to operate. Each day, Refinery B produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $10,000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil.
How many days should each refinery be operated to meet the goals at a minimum cost?

4) A print shop at a community college in Cupertino, California, employs two different contractors to maintain its copying machines. The print shop needs to have 12 IBM, 18 Xerox, and 20 Canon copying machines serviced. Contractor A can repair 2 IBM, 1 Xerox, and 2 Canon machines at a cost of $800 per month, while Contractor B can repair 1 IBM, 3 Xerox, and 2 Canon machines at a cost of $1000 per month. How many months should each of the two contractors be employed to minimize the cost?

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7.4: Maximization by the Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming maximization problems using the Simplex Method:

1. Identify and set up a linear program in standard maximization form
2. Convert inequality constraints to equations using slack variables
3. Set up the initial simplex tableau using the objective function and slack equations
4. Find the optimal simplex tableau by performing pivoting operations.
5. Identify the optimal solution from the optimal simplex tableau.

In the last chapter, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the **simplex method**.

The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm which was supposed to be revolutionary, but as it turned out it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

To learn the simplex method, we try a rather unconventional approach. We first list the algorithm, and then work a problem. We justify the reasoning behind each step during the process. A thorough justification is beyond the scope of this course.

We start out with an example we solved in the last chapter by the graphical method. This will provide us with some insight into the simplex method and at the same time give us the chance to compare a few of the feasible solutions we obtained previously by the graphical method. But first, we list the algorithm for the simplex method.

THE SIMPLEX METHOD

1. **Set up the problem.** That is, write the objective function and the inequality constraints.
2. **Convert the inequalities into equations.** This is done by adding one slack variable for each inequality.
3. **Construct the initial simplex tableau.** Write the objective function as the bottom row.
4. **The most negative entry in the bottom row identifies the pivot column.**
5. **Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.** The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
6. **Perform pivoting to make all other entries in this column zero.** This is done the same way as we did with the Gauss-Jordan method.
7. **When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.**
8. **Read off your answers.** Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.

Now, we use the simplex method to solve Example 7.4.1 solved geometrically in section 7.2.

Example 7.4.1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3, and so on.

Let

- x_1 = The number of hours per week Niki will work at Job I and
- x_2 = The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z.

The problem is formulated the same way as we did previouslyr.

$$\begin{aligned} &\textbf{Maximize} && \mathrm{Z} = 40\mathrm{x}_1 + 30\mathrm{x}_2 \\ &\textbf{Subject to:} && \mathrm{x}_1 + \mathrm{x}_2 \leq 12 \\ & && 2\mathrm{x}_1 + \mathrm{x}_2 \leq 16 \\ & && \mathrm{x}_1 \geq 0; \mathrm{x}_2 \geq 0 \end{aligned}$$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1 + x_2 \leq 12$ into an equation, we add a non-negative variable y_1, and we get

$$x_1 + x_2 + y_1 = 12$$

Here the variable y_1 picks up the slack, and it represents the amount by which $x_1 + x_2$ falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z = 40x_1 + 30x_2$ as $-40x_1 - 30x_2 + Z = 0$.

After adding the slack variables, our problem reads

$$\begin{aligned} &\text{Objective function} && -40x_1 - 30x_2 + Z = 0 \\ &\text{Subject to constraints:} && x_1 + x_2 + y_1 = 12 \\ & && 2x_1 + x_2 + y_2 = 16 \\ & && x1 \geq 0; x2 \geq 0 \end{aligned}$$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x1	x2	y1	y2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

The reader needs to observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\left[\begin{array}{cccc|c} y_1 & y_2 & Z & & C \\ 1 & 0 & 0 & | & 12 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 0 \end{array}\right]$$

which reads

$$y_1 = 12 \quad y_2 = 16 \quad Z = 0$$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the **basic solution** associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x1	x2	y1	y2	Z		
1	1	1	0	0	12	y1
2	1	0	1	0	16	y2
–40	–30	0	0	1	0	Z

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is -40; therefore the column 1 is identified.

x1	x2	y1	y2	Z		
1	1	1	0	0	12	y1
2	1	0	1	0	16	y2
–40	–30	0	0	1	0	Z
↑						

Question Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function $Z = 40x_1 + 30x2$, it will make more sense to increase the value of x_1 rather than x_2. The variable x_1 represents the number of hours per week Niki works at Job I. Since Job I pays $40 per hour as opposed to Job II which pays only $30, the variable x_1 will increase the objective function by $40 for a unit of increase in the variable x_1.

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x1	x2	y1	y2	Z			
1	1	1	0	0	12	y1	12 ÷ 1 = 12
[2]	1	0	1	0	16	y2	← 16 ÷ 2 = 8
–40	–30	0	0	1	0	Z	
↑							

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

Question Why do we find quotients, and why does the smallest quotient identify a row?

Answer When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1. But we cannot choose any value for x_1. Can we let $x_1 = 100$? Definitely not! That is because Niki never wants to work for more than 12 hours at both jobs combined: $x_1 + x_2 \leq 12$. Can we let $x_1 = 12$? Again,

the answer is no because the preparation time for Job I is two times the time spent on the job. Since Niki never wants to spend more than 16 hours for preparation, the maximum time she can work is 16 ÷ 2 = 8.

Now you see the purpose of computing the quotients; using the quotients to identify the pivot element guarantees that we do not violate the constraints.

Question Why do we identify the pivot element?

Answer As we have mentioned earlier, the simplex method begins with a corner point and then moves to the next corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables. We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the **entering variable,** and the variable whose units are being replaced is called the **departing variable**. The entering variable in the above table is x_1, and it was identified by the most negative entry in the bottom row. The departing variable y_2 was identified by the lowest of all quotients.

STEP 6. Perform pivoting to make all other entries in this column zero.

In chapter 5, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. So now our job is to make our pivot element a 1 by dividing the entire second row by 2. The result follows.

x1	x2	y1	y2	Z	
1	1	1	0	0	12
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by -1 and add it to row 1. We get

x1	x2	y1	y2	Z	
0	1/2	1	–1/2	0	4
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

x1	x2	y1	y2	Z		
0	1/2	1	–1/2	0	4	y1
[1]	1/2	0	1/2	0	8	x1
0	–10	0	20	1	320	Z

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ and $y_2 = 0$, we obtain $x_1 = 8$, $y_1 = 4$, and $z = 320$. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$$\left[\begin{array}{ccc|c} x_1 & y_1 & Z & C \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 320 \end{array}\right]$$

We can restate the solution associated with this matrix as $x_1 = 8$, $x_2 = 0$, $y_1 = 4$, $y_2 = 0$ and $z = 320$. At this stage of the game, it reads that if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be $320. Recall from Example 3.1.1 in section 3.1 that (8, 0) was one of our corner points. Here $y_1 = 4$ and $y_2 = 0$ mean that she will be left with 4 hours of working time and no preparation time.

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.

Since there is still a negative entry, -10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step, instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.

x1	x2	y1	y2	Z			
0	[1/2]	1	−1/2	0	4	y1	← 4 ÷ 1/2 = 8
1	1/2	0	1/2	0	8	x1	8 ÷ 1/2 = 16
0	−10	0	20	1	320	Z	
	↑						

We make the pivot element 1 by multiplying row 1 by 2, and we get

x1	x2	y1	y2	Z	
0	[1]	2	−1	0	8
1	1/2	0	1/2	0	8
0	−10	0	20	1	320

Now to make all other entries as zeros in this column, we first multiply row 1 by -1/2 and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.

x1	x2	y1	y2	Z		
0	1	2	−1	0	8	x2
1	0	−1	1	0	4	x1
0	0	20	10	1	400	Z

We no longer have negative entries in the bottom row, therefore we are finished.

Question Why are we finished when there are no negative entries in the bottom row?

Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400 \quad \text{or}$$
$$z = 400 - 20y1 - 10y2$$

Since all variables are non-negative, the highest value Z can ever achieve is 400, and that will happen only when y_1 and y_2 are zero.

STEP 8. Read off your answers.

We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a 1 and all other entries zeros. Since the columns labeled y_1 and y_2 are not such columns, we arbitrarily choose $y_1 = 0$, and $y_2 = 0$, and we get

$$\left[\begin{array}{ccc|c} x_1 & x_2 & Z & C \\ 0 & 1 & 0 & 8 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 400 \end{array}\right]$$

The matrix reads $x_1 = 4$, $x_2 = 8$ and $z = 400$.

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to $400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

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7.4.1: Maximization By The Simplex Method (Exercises)

Solve the following linear programming problems using the simplex method.

1)

$$\begin{array}{ll} \text{Maximize} & z = x_1 + 2x_2 + 3x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 + 3x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

2)

$$\begin{array}{ll} \text{Maximize} & z = x_1 + 2x_2 + x_3 \\ \text{subject to} & x_1 + x_2 \leq 3 \\ & x_2 + x_3 \leq 4 \\ & x_1 + x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

3) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

4) A factory manufactures chairs, tables and bookcases each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 600 hours; the second at most 500 hours; and the third at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing; a table needs 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; and a bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair, $30 for a table, and $25 for a bookcase, how many units of each should be manufactured to maximize profit?

5). The Acme Apple company sells its Pippin, Macintosh, and Fuji apples in mixes. Box I contains 4 apples of each kind; Box II contains 6 Pippin, 3 Macintosh, and 3 Fuji; and Box III contains no Pippin, 8 Macintosh and 4 Fuji apples. At the end of the season, the company has altogether 2800 Pippin, 2200 Macintosh, and 2300 Fuji apples left. Determine the maximum number of boxes that the company can make.

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7.5: Minimization by the Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming minimization problems using the simplex method.

1. Identify and set up a linear program in standard minimization form
2. Formulate a dual problem in standard maximization form
3. Use the simplex method to solve the dual maximization problem
4. Identify the optimal solution to the original minimization problem from the optimal simplex tableau.

In this section, we will solve the standard linear programming minimization problems using the simplex method. Once again, we remind the reader that in the standard minimization problems all constraints are of the form $ax + by \geq c$.

The procedure to solve these problems was developed by Dr. John Von Neuman. It involves solving an associated problem called the **dual problem.** To every minimization problem there corresponds a dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method.

From the final simplex tableau, we then extract the solution to the original minimization problem.

Before we go any further, however, we first learn to convert a minimization problem into its corresponding maximization problem called its **dual**.

Example 7.5.1

Convert the following minimization problem into its dual.

$$\begin{aligned} &\textbf{Minimize} && Z = 12x_1 + 16x_2 \\ &\textbf{Subject to:} && x_1 + 2x_2 \geq 40 \\ &&& x_1 + x_2 \geq 30 \\ &&& x_1 \geq 0; x_2 \geq 0 \end{aligned}$$

Solution

To achieve our goal, we first express our problem as the following matrix.

$$\begin{array}{cc|c} 1 & 2 & 40 \\ 1 & 1 & 30 \\ \hline 12 & 16 & 0 \end{array}$$

Observe that this table looks like an initial simplex tableau without the slack variables. Next, we write a matrix whose columns are the rows of this matrix, and the rows are the columns. Such a matrix is called a **transpose** of the original matrix. We get:

$$\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 1 & 16 \\ \hline 40 & 30 & 0 \end{array}$$

The following maximization problem associated with the above matrix is called its dual.

$$\begin{aligned} &\textbf{Maximize} && Z = 40y_1 + 30y_2 \\ &\textbf{Subject to:} && y_1 + y_2 \leq 12 \\ &&& 2y_1 + y_2 \leq 16 \\ &&& y_1 \geq 0; y_2 \geq 0 \end{aligned}$$

Note that we have chosen the variables as y's, instead of x's, to distinguish the two problems.

Example 7.5.2

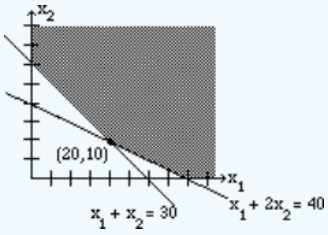
Solve graphically both the minimization problem and its dual maximization problem.

Solution

Our minimization problem is as follows.

$$\begin{aligned} \textbf{Minimize} \quad & Z = 12x_1 + 16x_2 \\ \textbf{Subject to:} \quad & x_1 + 2x_2 \geq 40 \\ & x_1 + x_2 \geq 30 \\ & x_1 \geq 0; x_2 \geq 0 \end{aligned}$$

We now graph the inequalities:

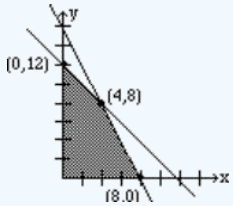


We have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (20, 10) gives the lowest value for the objective function and that value is 400.

Now its dual is:

$$\begin{aligned} \textbf{Maximize} \quad & Z = 40y_1 + 30y_2 \\ \textbf{Subject to:} \quad & y_1 + y_2 \leq 12 \\ & 2y_1 + y_2 \leq 16 \\ & y_1 \geq 0; y_2 \geq 0 \end{aligned}$$

We graph the inequalities:



Again, we have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (4, 8) gives the highest value for the objective function, with a value of 400.

The reader may recognize that Example 7.5.2 above is the same as Example 7.2.1, in section 7.2. It is also the same problem as Example 7.4.1 in section 7.4, where we solved it by the simplex method.

We observe that the minimum value of the minimization problem is the same as the maximum value of the maximization problem; in Example 7.5.2 the minimum and maximum are both 400. This is not a coincident. We state the duality principle.

The Duality Principle

The Duality Principle

The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.

Our next goal is to extract the solution for our minimization problem from the corresponding dual. To do this, we solve the dual by the simplex method.

Example 7.5.3

Find the solution to the minimization problem in Example 7.5.1 by solving its dual using the simplex method. We rewrite our problem.

$$\begin{aligned} &\textbf{Minimize} && Z = 12x_1 + 16x_2 \\ &\textbf{Subject to:} && x_1 + 2x_2 \geq 40 \\ & && x_1 + x_2 \geq 30 \\ & && x_1 \geq 0; x_2 \geq 0 \end{aligned}$$

Solution

$$\begin{aligned} &\textbf{Maximize} && Z = 40y_1 + 30y_2 \\ &\textbf{Subject to:} && y_1 + y_2 \leq 12 \\ & && 2y_1 + y_2 \leq 16 \\ & && y_1 \geq 0; y_2 \geq 0 \end{aligned}$$

Recall that we solved the above problem by the simplex method in Example 4.1.1, section 4.1. Therefore, we only show the initial and final simplex tableau.

The initial simplex tableau is

y_1	y_2	x_1	x_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
−40	−30	0	0	1	0

Observe an important change. Here our main variables are y_1 and y_2 and the slack variables are $x_1 and x_2$.

The final simplex tableau reads as follows:

y_1	y_2	x_1	x_2	Z	
0	1	2	−1	0	8
1	0	−1	1	0	4
0	0	20	10	1	400

A closer look at this table reveals that the x_1 and x_2 values along with the minimum value for the minimization problem can be obtained from the last row of the final tableau. We have highlighted these values by the arrows.

y_1	y_2	x_1	x_2	Z	
0	1	2	−1	0	8
1	0	−1	1	0	4
0	0	20	10	1	400
		↑	↑		↑

We restate the solution as follows:

The minimization problem has a minimum value of 400 at the corner point (20, 10)

We now summarize our discussion.

Minimization by the Simplex Method

1. Set up the problem.
2. Write a matrix whose rows represent each constraint with the objective function as its bottom row.
3. Write the transpose of this matrix by interchanging the rows and columns.
4. Now write the dual problem associated with the transpose.
5. Solve the dual problem by the simplex method learned in section 4.1.

6. The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables, and the minimum value of the objective function is the same as the maximum value of the dual.

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7.5.1: Minimization By The Simplex Method (Exercises)

Convert each of the following minimization problems into a maximization problem, the dual, and then solve by the simplex method.

1)

$$\begin{aligned} \text{Minimize } & z = 6x_1 + 8x_2 \\ \text{subject to } & 2x_1 + 3x_2 \geq 7 \\ & 4x_1 + 5x_2 \geq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

2)

$$\begin{aligned} \text{Minimize } & z = 5x_1 + 6x_2 + 7x_3 \\ \text{subject to } & 3x_1 + 2x_2 + 3x_3 \geq 10 \\ & 4x_1 + 3x_2 + 5x_3 \geq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

3)

$$\begin{aligned} \text{Minimize } & z = 4x_1 + 3x_2 \\ \text{subject to } & x_1 + x_2 \geq 10 \\ & 3x_1 + 2x_2 \geq 24 \\ & x_1, x_2 \geq 0 \end{aligned}$$

4) A diet is to contain at least 8 units of vitamins, 9 units of minerals, and 10 calories. Three foods, Food A, Food B, and Food C are to be purchased. Each unit of Food A provides 1 unit of vitamins, 1 unit of minerals, and 2 calories. Each unit of Food B provides 2 units of vitamins, 1 unit of minerals, and 1 calorie. Each unit of Food C provides 2 units of vitamins, 1 unit of minerals, and 2 calories. If Food A costs $3 per unit, Food B costs $2 per unit and Food C costs $3 per unit, how many units of each food should be purchased to keep costs at a minimum?

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CHAPTER OVERVIEW

8: Sets and Counting

Learning Objectives

In this chapter, you will learn to:

- Use set theory and Venn diagrams to solve counting problems.
- Use the Multiplication Axiom to solve counting problems.
- Use Permutations to solve counting problems.
- Use Combinations to solve counting problems.
- Use the Binomial Theorem to expand $(x+y)^n$

8.1: Sets and Counting

8.1.1: Sets and Counting (Exercises)

8.2: Tree Diagrams and the Multiplication Axiom

8.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)

8.3: Permutations

8.3.1: Permutations (Exercises)

8.4: Circular Permutations and Permutations with Similar Elements

8.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)

8.5: Combinations

8.5.1: Combinations (Exercises)

8.6: Combinations Involving Several Sets

8.6.1: Combinations Involving Several Sets (Exercises)

8.7: Binomial Theorem

8.7.1: Binomial Theorem (Exercises)

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8.1: Sets and Counting

Learning Objectives

In this section, you will learn to:

1. Use set notation to represent unions, intersections, and complements of sets
2. Use Venn diagrams to solve counting problems.

Introduction to Sets

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

Definition: Set and Elements

A **set** is a collection of objects, and its members are called the **elements** of the set.

We name the set by using capital letters, and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation.

C = { Ken, Bob, Tran, Shanti, Eric }

Definition: Empty Set

A set that has no members is called an **empty set**. The empty set is denoted by the symbol ∅.

Definition: Set Equality

Two sets are **equal** if they have the same elements.

Definition: Subset

Set A is a **subset** of a set B if every member of A is also a member of B.

Suppose C = { Al, Bob, Chris, David, Ed } and A = { Bob, David }. Then A is a subset of C, written as A ⊆ C.

Every set is a subset of itself, and the empty set is a subset of every set.

Definition: Union of Two Sets

Let A and B be two sets, then the **union** of A and B, written as $A \cup B$, is the set of all elements that are either in A or in B, or in both A and B.

Definition: Intersection of Two Sets

Let A and \(B\0 be two sets, then the **intersection** of A and B, written as $A \cap B$, is the set of all elements that are common to both sets A and B.

Definition: Universal Set

A **universal set** U is the set consisting of all elements under consideration.

Definition: Complement of a Set and Disjoint Sets

Let A be any set, then the **complement** of set A, written as $\bar{A}$, is the set consisting of elements in the universal set U that are not in A.

Two sets A and B are called **disjoint sets** if their intersection is an empty set. Clearly, a set and its complement are disjoint; however two sets can be disjoint and not be complements.

Example 8.1.1

List all the subsets of the set of primary colors { red, yellow, blue }.

Solution

The subsets are ∅, {red}, {yellow}, {blue}, {red, yellow}, {red, blue}, {yellow, blue}, {red, yellow, blue}

Note that the empty set is a subset of every set, and a set is a subset of itself.

Example 8.1.2

Let F = { Aikman, Jackson, Rice, Sanders, Young }, and let B = { Griffey, Jackson, Sanders, Thomas }.
Find the intersection of the sets F and B.

Solution

The intersection of the two sets is the set whose elements belong to both sets. Therefore, F ∩ B = { Jackson, Sanders }

Example 8.1.3

Find the union of the sets F and B given as follows.

- F = { Aikman, Jackson, Rice, Sanders, Young }
- B = { Griffey, Jackson, Sanders, Thomas }

Solution

The union of two sets is the set whose elements are either in A or in B or in both A and B. Observe that when writing the union of two sets, the repetitions are avoided.

$$F \cup B = \{ \text{Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young} \}$$

Example 8.1.4

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue }. Find the complement of P.

Solution

The complement of a set P is the set consisting of elements in the universal set U that are not in P. Therefore,

$$\bar{P} = \{ \text{orange, green, indigo, violet} \}$$

To achieve a better understanding, let us suppose that the universal set U represents the colors of the spectrum, and P the primary colors, then $\bar{P}$ represents those colors of the spectrum that are not primary colors.

Example 8.1.5

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue } . Find a set R so that R is not the complement of P but R and P are disjoint.

Solution

R = { orange, green } and P = { red, yellow, blue } are disjoint because the intersection of the two sets is the empty set. The sets have no elements in common. However they are not complements because their union P ∪ R = { red, yellow, blue, orange, green } is not equal to the universal set U.

Example 8.1.6

Let U = { red, orange, yellow, green, blue, indigo, violet }, P = { red, yellow, blue }, Q = { red, green }, and R = { orange, green, indigo }. Find $\overline{P \cup Q} \cap \overline{R}$.

Solution

We do the problems in steps:

P ∪ Q = { red, yellow, blue, green }

$\overline{P \cup Q}$ = { orange, indigo, violet }

$\overline{R}$ = { red, yellow, blue, violet }

$\overline{P \cup Q} \cap \overline{R}$ = {violet}

Venn Diagrams

We now use **Venn diagrams** to illustrate the relations between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationship between sets. He represented these relationships using diagrams, which are now known as Venn diagrams.

A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. To visualize an intersection or union of a set is easy. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

Example 8.1.7

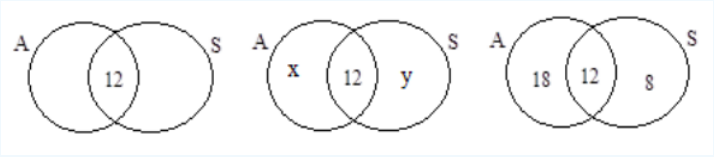
Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. If everyone in the survey drove cars with one of these transmissions, how many people participated in the survey?

Solution

We will use Venn diagrams to solve this problem.

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

Since 12 people drove both cars, we place the number 12 in the region common to both sets.



Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means that

x + 12 = 30, or x = 18.

Similarly, since 20 people drove cars with standard transmissions, the circle B must contain 20 elements.

Thus, y + 12 = 20 which in turn makes y = 8.

Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only.

Therefore, 18 + 12 + 8 = 38 people took part in the survey.

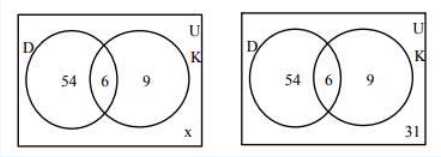
Example 8.1.8

A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. How many people have visited neither place?

Solution

The problem is similar to the one in the previous example.

Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.



We fill the three regions associated with the sets D and K in the same manner as before. Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects. Let x represent those people in the universal set that are neither in the set D nor in K. This means 54 + 6 + 9 + x = 100, or x = 31.

Therefore, there are 31 people in the survey who have visited neither place.

Example 8.1.9

A survey of 100 exercise conscious people resulted in the following information:

- 50 jog, 30 swim, and 35 cycle
- 14 jog and swim
- 7 swim and cycle
- 9 jog and cycle
- 3 people take part in all three activities

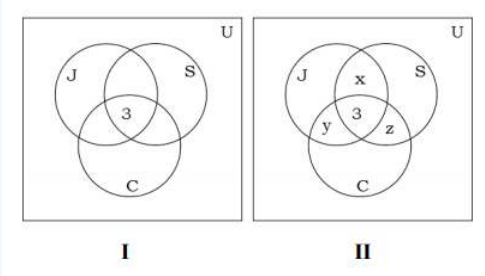
a. How many jog but do not swim or cycle?
b. How many take part in only one of the activities?
c. How many do not take part in any of these activities?

Solution

Let J represent the set of people who jog, S the set of people who swim, and C who cycle.

In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.

We'll show the solution step by step. As you practice working out such problems, you will find that with practice you will not need to draw multiple copies of the diagram.



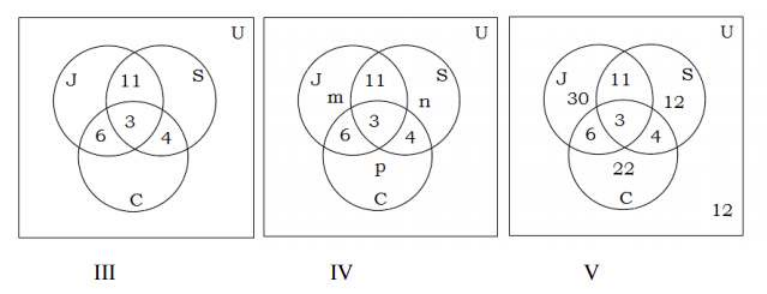
We place a 3 in the innermost region of figure I because it represents the number of people who participate in all three activities. Next we use figure II to compute x, y and z.

Since 14 people jog and swim, x + 3 = 14, or x = 11.

The fact that 9 people jog and cycle results in y + 3 = 9, or y = 6.

Since 7 people swim and cycle, z + 3 = 7, or z = 4.

This information is depicted in figure III.



Now we proceed to find the unknowns m, n and p, as shown in figure IV

Since 50 people jog, m + 11 + 6 + 3 = 50, or m = 30.

30 people swim, therefore, n + 11 + 4 + 3 = 30, or n = 12.

35 people cycle, therefore, p + 6 + 4 + 3 = 35, or p = 22.

By adding all the entries in all three sets, we get a sum of 88.
Since 100 people were surveyed, the number inside the universal set but outside of all three sets is 100 - 88, or 12.

In figure V, all the information is sorted out, and the questions can readily be answered.

a. The number of people who jog but do not swim or cycle is 30.
b. The number who take part in only one of these activities is 30 + 12 + 22 = 64.
c. The number of people who do not take part in any of these activities is 12.

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8.1.1: Sets and Counting (Exercises)

1. List all subsets of the following set: { Al, Bob }

2. List all subsets of the following set: { Al, Bob, Chris }

3. List the elements of the following set: { Al, Bob, Chris, Dave } ∩ { Bob, Chris, Dave, Ed }

4. List the elements of the following set: { Al, Bob, Chris, Dave } ∪ {Bob, Chris, Dave, Ed }

Problems 5 – 8: Let Universal set U = { a, b, c, d, e, f, g, h, i, j }, sets V = { a, e, i, f, h }, W = { a, c, e, g, i }. List the members of the following sets:

5. V ∪ W

6. V ∩ W

7. $\overline{V \cup W}$

8. $\overline{V} \cap \overline{W}$

Problems 9 – 12: Let Universal set U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 } and sets A = { 1, 2, 3, 4, 5 }, B = { 1, 3, 4, 6 }, and C = { 2, 4, 6 }. List the members of the following sets:

9. A ∪ B

10. A ∩ C

11. $\overline{A \cup B} \cap C$

12. $\bar{A} \cup \overline{B \cap C}$

13. In Mrs. Yamamoto's class of 35 students, 12 students are taking history, 18 are taking English, and 4 are taking both. Draw a Venn diagram and use it determine how many students are taking neither history nor English.

14. In a survey of 1200 college students, 700 used Spotify to listen to music and 400 used iTunes to listen to music; of these, 100 used both.

a. Draw a Venn Diagram and find the number of people in each region of the diagram.
b. How many used either Spotify or iTunes?

15. A survey of athletes revealed that for their minor aches and pains, 30 used aspirin, 50 used ibuprofen, and 15 used both. How many athletes were surveyed?

16. In 2016, 80 college students were surveyed about what video services they subscribed to. Suppose the survey showed that 50 use Amazon Prime, 30 use Netflix, 20 use Hulu. Of those, 13 use Amazon Prime and Netflix, 9 use Amazon Prime and Hulu, 7 use Netflix and Hulu. 3 students use all three services.

a. Draw a Venn Diagram and use it to determine the number of people in each region of the diagram.
b. How many use at least one of these?
c. How many use none of these?

17. A survey of 100 students at a college finds that 50 take math, 40 take Engish, and 30 take history. Of thee, 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below:

a. Find the number of students take math but not the other two subjects
b. The number of students taking Engish or math but not history
c. The number of students taking none of these subjects.

18. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.

a. Find the number of investors that participated in the survey.
b. How many invested in stocks or mutual funds but not in bonds?

c. How many invested in exactly one type of investment?

19. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. (This question relates back to question #17.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Students who take at least one of these classes
b. Students who take exactly one of these classes
c. Students who take at least two of these classes
d. Students who take exactly two of these classes
e. Students who take at most two of these classes
f. Students who take English or Math but not both
g. Students who take Math or History but not English
h. Students who take all of these classes

20. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. (This question relates back to question #18.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Investors who invested in mutual funds only
b. Investors who invested in stocks and bonds but not mutual funds
c. Investors who invested in exactly one of these investments
d. Investors who invested in exactly two of these investments
e. Investors who invested in at least two of these investments
f. Investors who invested in at most two of these investments
g. Investors who did not invest in bonds
h. Investors who invested in all three investments

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8.2: Tree Diagrams and the Multiplication Axiom

Learning Objectives

In this section you will learn to

1. Use trees to count possible outcomes in a multi-step process
2. Use the multiplication axiom to count possible outcomes in a multi-step process.

In this chapter, we are trying to develop counting techniques that will be used in the next chapter to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples.

Example 8.2.1

If a woman has two blouses and three skirts, how many different outfits consisting of a blouse and a skirt can she wear?

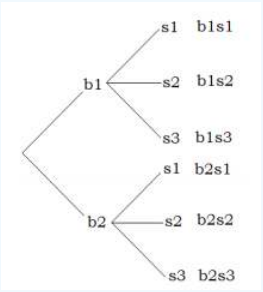
Solution

Suppose we call the blouses b_1 and b_2, and skirts s_1, s_2, and s_3.

We can have the following six outfits.

$$b_1s_1, b_1s_2, b_1s_3, b_2s_1, b_2s_2, b_2s_3$$

Alternatively, we can draw a tree diagram:



The tree diagram gives us all six possibilities. The method involves two steps. First the woman chooses a blouse. She has two choices: blouse one or blouse two. If she chooses blouse one, she has three skirts to match it with; skirt one, skirt two, or skirt three. Similarly if she chooses blouse two, she can match it with each of the three skirts, again. The tree diagram helps us visualize these possibilities.

The reader should note that the process involves two steps. For the first step of choosing a blouse, there are two choices, and for each choice of a blouse, there are three choices of choosing a skirt. So altogether there are $2 \cdot 3 = 6$ possibilities.

If, in the previous example, we add the shoes to the outfit, we have the following problem.

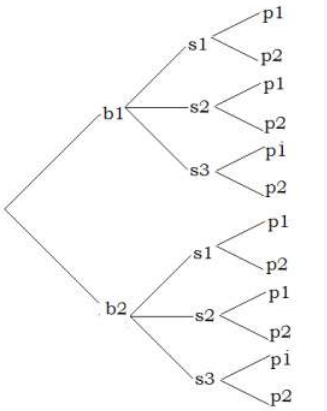
Example 8.2.2

If a woman has two blouses, three skirts, and two pumps, how many different outfits consisting of a blouse, a skirt, and a pair of pumps can she wear?

Solution

Suppose we call the blouses b_1 and b_2, the skirts s_1, s_2, and s_3, and the pumps p_1, and p_2.

The following tree diagram results.



We count the number of branches in the tree, and see that there are 12 different possibilities.

This time the method involves three steps. First, the woman chooses a blouse. She has two choices: blouse one or blouse two. Now suppose she chooses blouse one. This takes us to step two of the process which consists of choosing a skirt. She has three choices for a skirt, and let us suppose she chooses skirt two. Now that she has chosen a blouse and a skirt, we have moved to the third step of choosing a pair of pumps. Since she has two pairs of pumps, she has two choices for the last step. Let us suppose she chooses pumps two. She has chosen the outfit consisting of blouse one, skirt two, and pumps two, or $b_1 s_2 p_2$. By looking at the different branches on the tree, one can easily see the other possibilities.

The important thing to observe here, again, is that this is a three step process. There are two choices for the first step of choosing a blouse. For each choice of a blouse, there are three choices of choosing a skirt, and for each combination of a blouse and a skirt, there are two choices of selecting a pair of pumps.

All in all, we have $2 \cdot 3 \cdot 2 = 12$ different possibilities.

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example $8.2.2$ without a tree diagram. The problem involves three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below. By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

Number of ways to choose pieces of clothing

The number of ways of choosing a blouse	The number of ways of choosing a skirt	The number of ways of choosing pumps
2	3	2

The procedure we just employed is called the multiplication axiom.

The Multiplication Axiom

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

Example $8.2.3$

A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

Solution

Since there are 26 letters and 10 digits, we have the following choices for each.

Number of ways to choose elements of a license plate

Letter	Digit	Digit	Digit	Digit
26	10	10	10	10

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260,000$.

Example 8.2.4

In how many different ways can a 3-question true-false test be answered?

Solution

Since there are two choices for each question, we have

Number of ways to answer 3 true/false questions

Question 1	Question 2	Question 3
2	2	2

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

Example 8.2.5

In how many different ways can four people be seated in a row?

Solution

Suppose we put four chairs in a row, and proceed to put four people in these seats.

There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.

Number of people who can sit in each chair in a row

Chair 1	Chair 2	Chair 3	Chair 4
4	3	2	1

So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

Example 8.2.6

How many three-letter word sequences can be formed using the letters { A, B, C } if no letter is to be repeated?

Solution

The problem is very similar to the previous example.

Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.

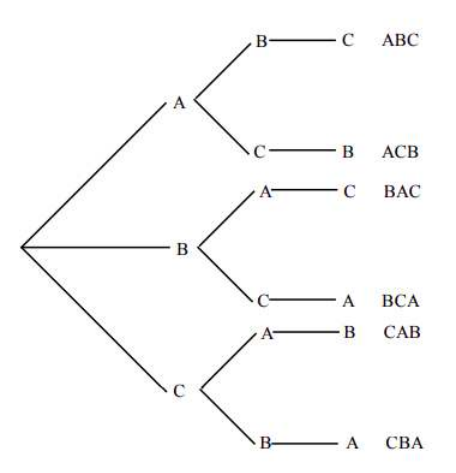
Number of choices for blocks in a stack

Bottom Block	Middle Block	Top Block

3	2	1

Therefore, 6 different word sequences can be formed.

Finally, we'd like to illustrate this with a tree diagram showing all six possibilities.



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8.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)

Do the following problems using a tree diagram or the multiplication axiom.

1. A man has 3 shirts, and 2 pairs of pants. Use a tree diagram to determine the number of possible outfits.
2. In a city election, there are 2 candidates for mayor, and 3 for supervisor. Use a tree diagram to find the number of ways to fill the two offices.
3. There are 4 roads from Town A to Town B, 2 roads from Town B to Town C. Use a tree diagram to find the number of ways one can travel from Town A to Town C.
4. Brown Home Construction offers a selection of 3 floor plans, 2 roof types, and 2 exterior wall types. Use a tree diagram to determine the number of possible homes available.
5. For lunch, a small restaurant offers 2 types of soups, three kinds of sandwiches, and two types of soft drinks. Use a tree diagram to determine the number of possible meals consisting of a soup, sandwich, and a soft drink.
6. A California license plate consists of a number from 1 to 5, then three letters followed by three digits. How many such plates are possible?

Do the following problems using the Multiplication Axiom

7. A license plate consists of three letters followed by three digits. How many license plates are possible if no letter may be repeated?
8. How many different 4-letter radio station call letters can be made if the first letter must be K or W and no letters can be repeated?
9. How many seven-digit telephone numbers are possible if the first two digits cannot be ones or zeros?
10. How many 3-letter word sequences can be formed using the letters {a, b, c, d} if no letter is to be repeated?

Use a tree diagram for questions 11 and 12:

11. A family has two children, use a tree diagram to determine all four possibilities of outcomes by gender.
12. A coin is tossed three times and the sequence of heads and tails is recorded. Use a tree diagram to list all the possible outcomes.

Do the following problems using the Multiplication Axiom

13. In how many ways can a 4-question true-false test be answered?
14. In how many ways can three people be arranged to stand in a straight line?
15. A combination lock is opened by first turning to the left, then to the right, and then to the left again. If there are 30 digits on the dial, how many possible combinations are there?
16. How many different answers are possible for a multiple-choice test with 10 questions and five possible answers for each question?
17. In the past, a college required students to use a 4 digit PIN (Personal Identification Number) as their password for its registration system. How many different PINs are possible if each must have 4 digits with no restrictions on selection or arrangement of the digits used?
18. The college decided that a more secure password system is needed. New passwords must have 3 numerical digits followed by 6 letters. There are no restrictions on the selection of the numerical digits. However, the letters I and O are not permitted. How many different passwords are possible?

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8.3: Permutations

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations (ordered arrangement) of n items taken r at a time
2. Count the number of possible permutations when there are conditions imposed on the arrangements
3. Perform calculations using factorials

In Example 7.2.6 of section 7.2, we were asked to find the word sequences formed by using the letters { A, B, C } if no letter is to be repeated. The tree diagram gave us the following six arrangements.

ABC, ACB, BAC, BCA, CAB, and CBA.

Arrangements like these, where order is important and no element is repeated, are called permutations.

Definition: Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

Example 8.3.1

How many three-letter word sequences can be formed using the letters { A, B, C, D }?

Solution

There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.

4	3	2

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ different arrangements.

Example 8.3.2

How many permutations of the letters of the word ARTICLE have consonants in the first and last positions?

Solution

In the word ARTICLE, there are 4 consonants.

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

4						3

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

4	5	4	3	2	1	3

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$.

Example 8.3.3

Given five letters { A, B, C, D, E }. Find the following:

a. The number of four-letter word sequences.
b. The number of three-letter word sequences.
c. The number of two-letter word sequences.

Solution

The problem is easily solved by the multiplication axiom, and answers are as follows:

a. The number of four-letter word sequences is $5 \cdot 4 \cdot 3 \cdot 2 = 120$.
b. The number of three-letter word sequences is $5 \cdot 4 \cdot 3 = 60$.
c. The number of two-letter word sequences is $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as **permutations of n objects taken r at a time**, and we write it as **nPr**.

Therefore, the above example can also be answered as listed below.

a. The number of four-letter word sequences is 5P4 = 120.
b. The number of three-letter word sequences is 5P3 = 60.
c. The number of two-letter word sequences is 5P2 = 20.

Before we give a formula for nPr, we'd like to introduce a symbol that we will use a great deal in this as well as in the next chapter.

Definition: Factorial

$$\mathrm{n!} = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots 3 \cdot 2 \cdot 1$$

where n is a natural number.

$$0! = 1$$

Now we define nPr.

Definition: nPr

The Number of Permutations of n Objects Taken r at a Time

$$\mathrm{nPr} = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots(\mathrm{n}-\mathrm{r}+1)$$

or

$$\mathrm{nPr} = \frac{\mathrm{n!}}{(\mathrm{n}-\mathrm{r})!}$$

where n and r are natural numbers.

The reader should become familiar with both formulas and should feel comfortable in applying either.

Example 8.3.4

Compute the following using both formulas.

a. 6P3
b. 7P2

Solution

We will identify n and r in each case and solve using the formulas provided.

a. $6P3 = 6 \cdot 5 \cdot 4 = 120$, alternately

$$6P3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$$

b. $7P2 = 7 \cdot 6 = 42$, or

$$7P2 = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 42$$

Next we consider some more permutation problems to get further insight into these concepts.

Example 8.3.5

In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?

Solution

Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we tie A and B together and treat them as one person.

The four people are $\boxed{AB}$ CD. Since $\boxed{AB}$ is treated as one person, we have the following possible arrangements.

$$\boxed{AB}CD, \boxed{AB}DC, C\boxed{AB}D, D\boxed{AB}C, CD\boxed{AB}, DC\boxed{AB}$$

Note that there are six more such permutations because A and B could also be tied in the order BA. And they are

$$\boxed{BA}CD, \boxed{BA}DC, C\boxed{BA}D, D\boxed{BA}C, CD\boxed{BA}, DC\boxed{BA}$$

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we tie two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in 3! ways. Since two people can be tied together 2! ways, there are 3! 2! = 12 different arrangements

Example 8.3.6

You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?

Solution

We first do the problem using the multiplication axiom.

Since the math books go in the first three slots, there are 4 choices for the first slot,
3 choices for the second and 2 choices for the third.

The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

4	3	2	5	4

Therefore, the number of permutations are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 480$.

Alternately, we can see that $4 \cdot 3 \cdot 2$ is really same as 4P3, and $5 \cdot 4$ is 5P2.

So the answer can be written as (4P3) (5P2) = 480.

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are 5P2 permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer (4P3) (5P2).

We summarize the concepts of this section:

Note

1. Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

2. Factorial

$$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

Where n is a natural number.

$$0! = 1$$

3. Permutations of n Objects Taken r at a Time

$$nPr = n(n-1)(n-2)(n-3)\cdots(n-r+1)$$

or

$$nPr = \frac{n!}{(n-r)!}$$

where n and r are natural numbers.

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8.3.1: Permutations (Exercises)

Do the following problems using permutations.

1. How many three-letter words can be made using the letters { a, b, c, d, e } if no repetitions are allowed?
2. A grocery store has five checkout counters, and seven clerks. How many different ways can the 7 clerks be assigned to the 5 counters?
3. A group of fifteen people who are members of an investment club wish to choose a president, and a secretary. How many different ways can this be done?

Compute the following:

4. 9P2
5. 6P4
6. 8P3
7. 7P4

Use permutations to answer the following:

8. In how many ways can the letters of the word CUPERTINO be arranged if each letter is used only once in each arrangement?
9. How many permutations of the letters of the word PROBLEM end in a vowel?
10. How many permutations of the letters of the word SECURITY end in a consonant?
11. How many permutations of the letters PRODUCT have consonants in the second and third positions?
12. How many three-digit numbers are there?
13. How many three-digit odd numbers are there?
14. In how many different ways can five people be seated in a row if two of them insist on sitting next to each other?
15. In how many different ways can five people be seated in a row if two of them insist on not sitting next to each other?
16. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if the English books are set on the left, history books in the middle, and math books on the right?
17. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if they are grouped by subject?
18. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with math books and the next three with history books?
19. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with the books of one subject and the next three slots are to be filled with the books of the other subject?
20. A bakery has 9 different fancy cakes. In how many ways can 5 of the 9 fancy cakes be lined up in a row in the bakery display case?
21. A landscaper has 6 different flowering plants. She needs to plant 4 of them in a row in a garden. How many different ways can 4 of the 6 plants be arranged in a row?
22. At an auction of used construction vehicles, there are 7 different vehicles for sale. In how many orders could these 7 vehicles be listed in the auction program?
23. A landscaper has 6 different flowering plants and 4 different non-flowering bushes. She needs to plant a row of 6 plants in a garden. There must be a bush at each end, and four flowering plants in a row in between the bushes. How many different arrangements in a row are possible?
24. In how many ways can all 7 letters of the word QUIETLY be arranged if the letters Q and U must be next to each other in the order QU?
25. In how many ways can the letters ABCDEXY be arranged if the X and Y must be next to each other in either order XY or YX?
26. In how many ways can the letters ABCDEXY be arranged if the X and Y can not be next to each other?

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8.4: Circular Permutations and Permutations with Similar Elements

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations of items arranged in a circle
2. Count the number of possible permutations when there are repeated items

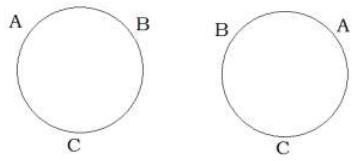
In this section we will address the following two problems.

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

Circular Permutations

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.



It happens that there are only two ways we can seat three people in a circle, relative to each other's positions. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. We are interested in the position of each person in relation to the others. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where he sits does not matter.

Definition: Circular Permutations

The number of permutations of n elements in a circle is $(n-1)!$

Example 8.4.1

In how many different ways can five people be seated at a circular table?

Solution

We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have

Number of possible occupants for seats around a table

Position 1	Position 2	Position 3	Position 4	Position 5
1	4	3	2	1

So the answer is 24.

Example 8.4.2

In how many ways can four couples be seated at a round table if the men and women want to sit alternately?

Solution

We again emphasize that the first person can sit anywhere without affecting the permutation.

So there is only one choice for the first spot. Suppose a man sat down first. The chair next to it must belong to a woman, and there are 4 choices. The next chair belongs to a man, so there are three choices and so on. We list the choices below.

Number of possible occupants for seats if men and women alternate

Position 1	Position 2	Position 3	Position 4	Position 5	Position 6	Position 7	Position 8
1	4	3	3	2	2	1	1

So the answer is 144.

PERMUTATIONS WITH SIMILAR ELEMENTS

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$$E_1LE_2ME_3NT$$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$$LE_1ME_2NE_3T$$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\begin{array}{l}\mathrm{LE_1ME_2NE_3}\\ \mathrm{LE_1ME_3NE_2}\\ \mathrm{LE_2ME_1NE_3T}\\ \mathrm{LE_2ME_3NE_1T}\\ \mathrm{LE_3ME_2NE_1T}\\ \mathrm{LE_3ME_INE_2T}\end{array}$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1LE_2ME_3NT$.

But we know there are 7! permutations of the letters $E_1LE_2ME_3NT$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

Definition: Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

Example 8.4.3

Find the number of different permutations of the letters of the word MISSISSIPPI.

Solution

The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!} = 34,650$.

Example 8.4.4

If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

Solution

Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!2!} = 15$.

Example 8.4.5

In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

Solution

Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!3!2!} = 1260$$

Example 8.4.6

A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

Solution

This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!5!5!5!} = 11,732,745,024$$

Example 8.4.7

A shopping mall has a straight row of 5 flagpoles at its main entrance plaza. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?

Solution

The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations:

GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG, YGGGY, YGGYG, YGYGY, YYGGG

We summarize.

Summary

1. Circular Permutations

The number of permutations of n elements in a circle is

$$(n-1)!$$

2. Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, such that $\mathrm{n} = \mathrm{r}_1 + \mathrm{r}_2 + \ldots + \mathrm{r}_\mathrm{k}$ is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

This is also referred to as **ordered partitions**.

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8.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)

Do the following problems using the techniques learned in this section.

1. In how many different ways can five children hold hands to play "Ring Around the Rosy"?
2. In how many ways can three people be made to sit at a round table?
3. In how many different ways can six children ride a "Merry Go Around" with six horses?
4. In how many ways can three couples be seated at a round table, so that men and women sit alternately?
5. In how many ways can six trinkets be arranged on a chain?
6. In how many ways can five keys be put on a key ring?
7. Find the number of different permutations of the letters of the word MASSACHUSETTS.
8. Find the number of different permutations of the letters of the word MATHEMATICS.
9. Seven flags are to be flown on seven poles: 3 flags are red, 2 are white, and 2 are blue,. How many different arrangements are possible?
10. How many different ways can 3 pennies, 2 nickels and 5 dimes be arranged in a row?
11. How many four-digit numbers can be made using two 2's and two 3's?
12. How many five-digit numbers can be made using two 6's and three 7's?
13. If a coin is tossed 5 times, how many different outcomes of 3 heads and 2 tails are possible?
14. If a coin is tossed 10 times, how many different outcomes of 7 heads and 3 tails are possible?
15. If a team plays ten games, how many different outcomes of 6 wins and 4 losses are possible?
16. If a team plays ten games, how many different ways can the team have a winning season?

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8.5: Combinations

Learning Objectives

In this section you will learn to

1. Count the number of combinations of r out of n items (selections without regard to arrangement)
2. Use factorials to perform calculations involving combinations

Suppose we have a set of three letters { A, B, C }, and we are asked to make two-letter word sequences. We have the following six permutations.

AB BA BC CB AC CA

Now suppose we have a group of three people { A, B, C } as Al, Bob, and Chris, respectively, and we are asked to form committees of two people each. This time we have only three committees, namely,

AB BC AC

When forming committees, the order is not important, because the committee that has Al and Bob is no different than the committee that has Bob and Al. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of **combinations** - the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In the above example, there were six permutations, but only three combinations.

Just as the symbol nPr represents the number of permutations of n objects taken r at a time, nCr represents the number of combinations of n objects taken r at a time.

So in the above example, 3P2 = 6, and 3C2 = 3.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In the above example, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

Example 8.5.1

Given the set of letters { A, B, C, D }. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.

Solution

We have the following four combinations.

ABC BCD CDA BDA

Since every combination has three letters, there are 3! permutations for every combination. We list them below.

ABC	BCD	CDA	BDA
ACB	BDC	CAD	BAD
BAC	CDB	DAC	DAB
BCA	CBD	DCA	DBA
CAB	DCB	ACD	ADB
CBA	DBC	ADC	ABD

The number of permutations are 3! times the number of combinations; that is

4P3 = 3! · 4C3

or

$$4C3 = \frac{4P3}{3!}$$

In general,

$$nCr = \frac{nPr}{r!}$$

Since

$$nPr = \frac{n!}{(n-r)!}$$

We have,

$$nCr = \frac{n!}{(n-r)!r!}$$

Summarizing,

Note

1. Combinations

A combination of a set of elements is an arrangement where each element is used once, and order is not important.

2. The Number of Combinations of n Objects Taken r at a Time

$$nCr = \frac{n!}{(n-r)!r!}$$

where n and r are natural numbers.

Example 8.5.2

Compute:

a. 5C3
b. 7C3

Solution

We use the above formula.

$$5C3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = 10$$

$$7C3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$$

Example 8.5.3

In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?

Solution

Since the order is not important, it is a combination problem, and the answer is

7C5 = 21

Example 8.5.4

How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

Solution

Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem.

It is a combination of 6 objects taken 2 at a time. Therefore, the answer is

$$6C2 = \frac{6!}{4!2!} = 15$$

Example 8.5.5

There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

Solution

Note that between any two people there is only one hand shake. Therefore, we have

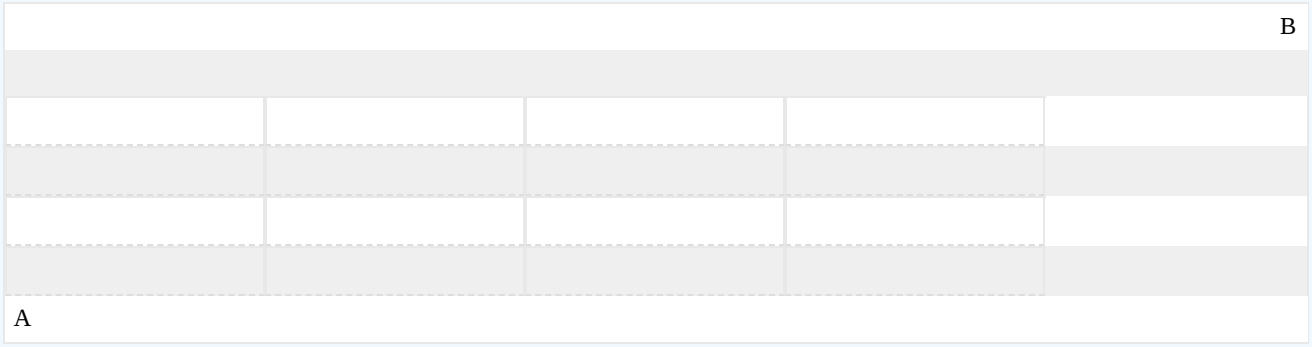
10C2 = 45 hand-shakes.

Example 8.5.6

The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?

Solution

Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.



To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa.

Therefore, all he has to do is to choose 5 out of ten to be the horizontal blocks

The answer is 10C5, or 252.

Alternately, the problem can be solved by permutations with similar elements.

The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, and a vertical block a V, then one possible route may be as follows.

HHHHHVVVVV

Clearly there are $\frac{10!}{5!5!} = 252$ permutations.

Further note that by definition 10C5 = $\frac{10!}{5!5!}$.

Example 8.5.7

If a coin is tossed six times, in how many ways can it fall four heads and two tails?

Solution

First we solve this problem using section 6.5 technique-permutations with similar elements.

We need 4 heads and 2 tails, that is

HHHHTT

There are $\frac{6!}{4!2!} = 15$ permutations.

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

6C4 = 15.

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

6C2 = 15.

Further observe that by definition

$$6C4 = \frac{6!}{2!4!}$$

and

$$6C2 = \frac{6!}{4!2!}$$

Which implies 6C4 = 6C2.

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8.5.1: Combinations (Exercises)

Compute the following:

1. 9C2
2. 6C4
3. 8C3
4. 7C4

Solve the following problems using combinations.

5. How many different 3-people committees can be chosen from ten people?
6. How many different 5-player teams can be chosen from eight players?
7. In how many ways can a person chose to vote for three out of five candidates on a ballot for a school board election?
8. How many 5-card hands can be chosen from a deck of cards?
9. How many 13-card bridge hands can be chosen from a deck of cards?
10. There are twelve people at a party. If they all shake hands, how many different hand-shakes are there?
11. In how many ways can a student choose to do four questions out of five on a test?
12. Five points lie on a circle. How many chords can be drawn through them?
13. How many diagonals does a hexagon have?
14. There are five team in a league. How many games are played if every team plays each other twice?
15. A team plays 15 games a season. In how many ways can it have 8 wins and 7 losses?
16. In how many different ways can a 4-child family have 2 boys and 2 girls?
17. A coin is tossed five times. In how many ways can it fall three heads and two tails?
18. The shopping area of a town is a square that is six blocks by six blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?
19. If the shopping area in the previous problem has a rectangular form of 5 blocks by 3 blocks, then how many different routes can a taxi driver take to drive from one end of the shopping area to the opposite kitty corner end?
20. A team of 7 workers is assigned to a project. In how many ways can 3 of the 7 workers be selected to make a presentation to the management about their progress on the project?
21. A real estate company has 12 houses listed for sale by their clients. In how many ways can 5 of the 12 houses be selected to be featured in an advertising brochures?
22. A frozen yogurt store has 9 toppings to choose from. In how many ways can 3 of the 9 toppings be selected?
23. A kindergarten teacher has 14 books about a holiday. In how many ways can she select 4 of the books to read to her class in the week before the holiday?

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8.6: Combinations Involving Several Sets

Learning Objectives

In this section you will learn to

1. count the number of items selected from more than one set
2. count the number of items selected when there are restrictions on the selections

So far we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

Example 8.6.1

How many five-people committees consisting of 2 men and 3 women can be chosen from a total of 4 men and 4 women?

Solution

We list 4 men and 4 women as follows:

$$M_1 M_2 M_3 M_4 W_1 W_2 W_3 W_4$$

Since we want 5-people committees consisting of 2 men and 3 women, we'll first form all possible two-man committees and all possible three-woman committees. Clearly there are 4C2 = 6 two-man committees, and 4C3 = 4 three-woman committees, we list them as follows:

Possible 2-man and 3-woman committees

2-Man Committees	3-Woman Committees
M_1M_2	$W_1W_2W_3$
M_1M_3	$W_1W_2W_4$
M_1M_4	$W_1W_3W_4$
M_2M_3	$W_2W_3W_4$
M_2M_4	
M_3M_4	

For every 2-man committee there are four 3-woman committees that can be chosen to make a 5-person committee. If we choose M_1M_2 as our 2-man committee, then we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees. As a result, we get

$$\boxed{M_1M_2}W_1W_2W_3, \boxed{M_1M_2}W_1W_2W_4, \boxed{M_1M_2}W_1W_3W_4, \boxed{M_1M_2}W_2W_3W_4$$

Similarly, if we choose M_1M_3 as our 2-man committee, then, again, we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees.

$$\boxed{M_1M_3}W_1W_2W_3, \boxed{M_1M_3}W_1W_2W_4, \boxed{M_1M_3}W_1W_3W_4, \boxed{M_1M_3}W_2W_3W_4$$

And so on.

Since there are six 2-man committees, and for every 2-man committee there are four 3-woman committees, there are altogether $6 \cdot 4 = 24$ five-people committees.

In essence, we are applying the multiplication axiom to the different combinations.

Example 8.6.2

A high school club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes

a. One student from each class?
b. All juniors?
c. Two freshmen and 2 seniors?
d. No freshmen?
e. At least three seniors?

Solution

a. Applying the multiplication axiom to the combinations involved, we get

$$(4C1) (5C1) (5C1) (6C1) = 600$$

b. We are choosing all 4 members from the 5 juniors, and none from the others.

$$5C4 = 5$$

c. $4C2 \cdot 6C2 = 90$

d. Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is

$$16C4 = 1820$$

e. Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.

$$(6C3) (14C1) + 6C4 = 295$$

Example 8.6.3

How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?

Solution

First we select a group of five letters consisting of 2 vowels and 3 consonants.
Since there are 4 vowels and 5 consonants, we have

$$(4C2) (5C3)$$

Since our next task is to make word sequences out of these letters, we multiply these by 5!.

$$(4C2) (5C3) (5!) = 7200.$$

Example 8.6.4

A standard deck of playing cards has 52 cards consisting of 4 suits each with 13 cards. In how many different ways can a 5-card hand consisting of four cards of one suit and one of another be drawn?

Solution

We will do the problem using the following steps.
Step 1. Select a suit.
Step 2. Select four cards from this suit.
Step 3. Select another suit.
Step 4. Select a card from that suit.

Applying the multiplication axiom, we have

Number of choices for suits and cards

Ways of selecting the first suit	Ways of selecting 4 cards from this suit	Ways of selecting the next suit	Ways of selecting a card from that suit
4C1	13C4	3C1	13C1

(4C1) (13C4) (3C1) (13C1) = 111,540.

A STANDARD DECK OF 52 PLAYING CARDS

As in the previous example, many examples and homework problems in this book refer to a standard deck of 52 playing cards. Before we end this section, we take a minute to describe a standard deck of playing cards, as some readers may not be familiar with this.

A standard deck of 52 playing cards has 4 suits with 13 cards in each suit.

♦ diamonds ♥ hearts ♠ spades ♣ clubs

Each suit is associated with a color, either black (spades, clubs) or red (diamonds, hearts)

Each suit contains 13 denominations (or values) for cards:

nine numbers 2, 3, 4, …., 10 and Jack(J), Queen (Q), King (K), Ace (A).

The Jack, Queen and King are called “face cards” because they have pictures on them. Therefore a standard deck has 12 face cards: (3 values JQK) x (4 suits ♦♥♠ ♣)

We can visualize the 52 cards by the following display

How playing cards divide by suits, colors, and values

Suit	Color	Values (Denominations)
♦ Diamonds	Red	2 3 4 5 6 7 8 9 10 J Q K A
♥ Hearts	Red	2 3 4 5 6 7 8 9 10 J Q K A
♠ Spades	Black	2 3 4 5 6 7 8 9 10 J Q K A
♣ Clubs	Black	2 3 4 5 6 7 8 9 10 J Q K A

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8.6.1: Combinations Involving Several Sets (Exercises)

The following problems involve combinations from several different sets.

1. How many 5-people committees consisting of three boys and two girls can be chosen from a group of four boys and four girls?
2. A club has 4 men, 5 women, 8 boys and 10 girls as members. In how many ways can a group of 2 men, 3 women, 4 boys and 4 girls be chosen?
3. How many 4-people committees chosen from 4 men and 6 women will have at least 3 men?
4. A batch contains 10 transistors of which three are defective. If three are chosen, in how many ways can they be selected with two defective?
5. In how many ways can five counters labeled A, B, C, D and E at a store be staffed by two men and three women chosen from a group of four men and six women?
6. How many 4-letter word sequences consisting of two vowels and two consonants can be made from the letters of the word PHOENIX if no letter is repeated?

Three marbles are chosen from an urn that contains 5 red, 4 white, and 3 blue marbles. How many samples of the following type are possible?

7. All three white.
8. Two blue and one white
9. One of each color.
10. All three of the same color.
11. At least two red.
12. None red.

Five coins are chosen from a bag that contains 4 dimes, 5 nickels, and 6 pennies. How many samples of five coins of the following types are possible?

13. At least four nickels.
14. No pennies.
15. Five of a kind.
16. Four of a kind.
17. Two of one kind and two of another kind.
18. Three of one kind and two of another kind.

Find the number of different ways to draw a 5-card hand from a deck to have the following combinations.

19. Three face cards.
20. A heart flush (all hearts).
21. Two hearts and three diamonds.
22. Two cards of one suit, and three of another suit.
23. Two kings and three queens.
24. 2 cards of one value and 3 of another value.

The party affiliation of the 100 United States Senators in the 114th Congress, January 2015, was: 44 Democrats, 54 Republicans, and 2 Independents.

25. In how many ways could a 10 person committee be selected if it is to contain 4 Democrats, 5 Republicans, and 1 Independent?
26. In how many different ways could a 10 person committee be selected with 6 or 7 Republicans and the Democrats (with no Independents)?

The 100 United States Senators in the 114th Congress, January 2015, included 80 men and 20 women. Suppose a committee of senators is working on legislation about wage discrimination by gender.

27. In how many ways could a 12 person committee be selected to contain equal numbers of men and women.
28. In how many ways could a 6 person committee be selected to contain fewer women than men?

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises. He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises. How many different playlists of 6 songs can be selected that

satisfy each of the following: (We care which songs are selected to be on the playlist, but not what order they are selected or listed in.)

29. Playlist has 2 songs of each type.
30. Playlist has no country songs.
31. Playlist has 3 rocks, 2 raps, and 1 country song.
32. Playlist has 3 or 4 rock songs and all the rest are rap songs.

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8.7: Binomial Theorem

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$\begin{aligned}(x+y)^2 &= x^2+2xy+y^2\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ (x+y)^4 &= x^4+4x^2y+6x^2y^2+4xy^3+y^4\\ (x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5\\ (x+y)^6 &= x6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6\end{aligned}$$

We make the following observations.

1. There are $n+1$ terms in the expansion $(x+y)^n$
2. The sum of the powers of x and y is n.
3. The powers of x begin with n and decrease by one with each successive term.
   The powers of y begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \square x^3+\square x^2y+\square xy^2+\square y^3 \tag{8.7.1}$$

Our next job is to replace each of the blanks in equation (8.7.1) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors $(x+y)(x+y)(x+y)$. For example, the product xxy is gotten by choosing x from the first factor, x from the second factor, and y from the third factor. There are three such products that simplify to x^2y, namely xxy, xyx, and yxx. These products take place when we choose an x from two of the factors and choose a y from the other factor. Clearly this can be done in 3C2, or 3 ways. Therefore, the coefficient of the term x^2y is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (8.7.1), and we get

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Example 8.7.1

Find the coefficient of the term x^2y^5 in the expansion $(x+y)^7$.

Solution

The expansion $(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$

In multiplying the right side, each product is gotten by picking an x or y from each of the seven factors $(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$.

The term x^2y^5 is obtained by choosing an x from two of the factors and a y from the other five factors. This can be done in 7C2, or 21 ways.

Therefore, the coefficient of the term x^2y^5 is 21.

Example 8.7.2

Expand $(x+y)^7$

Solution

We first write the expansion without the coefficients.

$$(x+y)^7 = \square x^7 + \square x^6 y + \square x^5 y^2 + \square x^4 y^3 + \square x^3 y^4 + \square x^2 y^5 + \square x y^6 + \square y^7$$

Now we determine the coefficient of each term as we did in Example 8.7.1.

The coefficient of the term x^7 is 7C7 or 7CO which equals 1.

The coefficient of the term $x^6 y$ is 7C6 or 7C1 which equals 7.

The coefficient of the term $x^5 y^2$ is 7C5 or 7C2 which equals 21.

The coefficient of the term $x^4 y^3$ is 7C4 or 7C3 which equals 35,

and so on.

Substituting, we get: $(x+y)^7 = x^7 + 7x^6 y + 21x^5 y^2 + 35x^4 y^3 + 35x^3 y^4 + 21x^2 y^5 + 7xy^6 + y^7$

We generalize the result.

Binomial Theorem

$$(x+y)^n = {}_n C_0 x^n + {}_n C_1 x^{n-1} y + {}_n C_2 x^{n-2} y^2 + \cdots + {}_n C_{n-1} x y^{n-1} + {}_n C_n y^n$$

Example 8.7.3

Expand $(3a-2b)^4$

Solution

If we let $x = 3a$ and $y = -2b$, and apply the Binomial Theorem, we get

$$\begin{aligned}(3a-2b)^4 &= 4\text{C}0(3a)^4 + 4\text{C}1(3a)^3(-2b) + 4\text{C}2(3a)^2(-2b)^2 + 4\text{C}3(3a)(-2b)^3 + 4\text{C}4(-2b)^4 \\ &= 1\left(81a^4\right) + 4\left(27a^3\right)(-2b) + 6\left(9a^2\right)\left(4b^2\right) + 4(3a)\left(-8b^3\right) + 1\left(16b^4\right) \\ &= 81a^4 - 216a^3 b + 216a^2 b^2 - 96ab^3 + 16b^4\end{aligned}$$

Example 8.7.4

Find the fifth term of the expansion $(3a-2b)^7$.

Solution

The Binomial theorem tells us that in the r-th term of an expansion, the exponent of the y term is always one less than r, and, the coefficient of the term is ${}_{\text{n}}\text{C}_{\text{r}-1}$.

$n = 7$ and $r - 1 = 5 - 1 = 4$, so the coefficient is $7\text{C}4 = 35$

Thus, the fifth term is $(7\text{C}4)(3\text{a})^3(-2\text{b})^4 = 35\left(27\text{a}^3\right)\left(16\text{b}^4\right) = 15120\text{a}^3\text{b}^4$

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8.7.1: Binomial Theorem (Exercises)

Use the Binomial Theorem to do the following problems.

1. Expand $(a+b)^5$.
2. Expand $(a-b)^6$.
3. Expand $(x-2y)^5$.
4. Expand $(2x-3y)^4$.
5. Find the third term of $(2x-3y)^6$.
6. Find the sixth term of $(5x+y)^8$.
7. Find the coefficient of the x^3y^4 term in the expansion of $(2x+y)^7$.
8. Find the coefficient of the a^4b^6 term in the expansion of $(3a-b)^{10}$.
9. A coin is tossed 5 times, in how many ways is it possible to get three heads and two tails?
10. A coin is tossed 10 times, in how many ways is it possible to get seven heads and three tails?
11. How many subsets are there of a set that has 6 elements?
12. How many subsets are there of a set that has n elements?

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CHAPTER OVERVIEW

9: Probability

Learning Objectives

In this chapter, you will learn to:

1. Write sample spaces.
2. Determine whether two events are mutually exclusive.
3. Use the Addition Rule.
4. Calculate probabilities using both tree diagrams and combinations.
5. Do problems involving conditional probability.
6. Determine whether two events are independent.

9.1: Sample Spaces and Probability

9.1.1: Sample Spaces and Probability (Exercises)

9.2: Mutually Exclusive Events and the Addition Rule

9.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)

9.3: Probability Using Tree Diagrams and Combinations

9.3.1: Probability Using Tree Diagrams and Combinations (Exercises)

9.4: Conditional Probability

9.4.1: Conditional Probability (Exercises)

9.5: Independent Events

9.5.1: Independent Events (Exercises)

9.6: Bayes' Formula

9.6.1: Bayes' Formula (Exercises)

9.7: Binomial Probability

9.7.1: Binomial Probability (Exercises)

9.8: Expected Value

9.8.1: Expected Value (Exercises)

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9.1: Sample Spaces and Probability

Learning Objectives

In this section, you will learn to:

1. Write sample spaces.
2. Calculate probabilities by examining simple events in sample spaces.

If two coins are tossed, what is the probability that both coins will fall heads? The problem seems simple enough, but it is not uncommon to hear the incorrect answer 1/3. A student may incorrectly reason that if two coins are tossed there are three possibilities, one head, two heads, or no heads. Therefore, the probability of two heads is one out of three. The answer is wrong because if we toss two coins there are four possibilities and not three. For clarity, assume that one coin is a penny and the other a nickel. Then we have the following four possibilities.

HH HT TH TT

The possibility HT, for example, indicates a head on the penny and a tail on the nickel, while TH represents a tail on the penny and a head on the nickel. It is for this reason, we emphasize the need for understanding sample spaces.

Sample Spaces

An act of flipping coins, rolling dice, drawing cards, or surveying people are referred to as a probability **experiment**. A **sample space** of an experiment is the set of all possible outcomes.

Example 9.1.1

If a die is rolled, write a sample space.

Solution

A die has six faces each having an equally likely chance of appearing. Therefore, the set of all possible outcomes S is

$$\{ 1, 2, 3, 4, 5, 6 \}.$$

Example 9.1.2

A family has three children. Write a sample space.

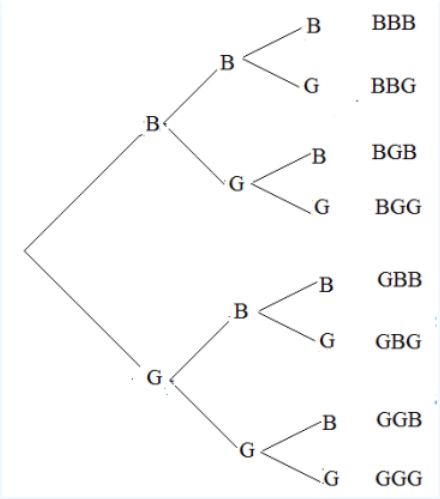
Solution

The sample space consists of eight possibilities.

$$\{ \text{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG} \}$$

The possibility BGB, for example, indicates that the first born is a boy, the second born a girl, and the third a boy.

We illustrate these possibilities with a tree diagram.



Example 9.1.3

Two dice are rolled. Write the sample space.

Solution

We assume one of the dice is red, and the other green. We have the following 36 possibilities.

Sample space for rolling 2 dice, one red and one green

6	Green					
Red	**1**	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The entry (2, 5), for example, indicates that the red die shows a 2, and the green a 5.

Probability

Now that we understand the concept of a sample space, we will define probability.

Definition: Probability

For a sample space S, and an outcome A of S, the following two properties are satisfied.

1. If A is an outcome of a sample space, then the probability of A, denoted by $P(A)$, is between 0 and 1, inclusive.

$$0 \leq P(A) \leq 1$$

2. The sum of the probabilities of all the outcomes in S equals 1.

The probability $P(A)$ of an event A describes the chance or likelihood of that event occurring.

- If $P(A) = 0$, event A is certain not to occur. If $P(A) = 1$, event A is certain to occur.
- If $P(A) = 0.5$, then event A is equally likely to occur or not occur.

- If we toss a fair coin that is equally likely to land on heads or tails, then P(Head) = 0.50.
- If the weather forecast says there is a 70% chance of rain today, then P(Rain) = 0.70, indicating is it more likely to rain than to not rain.

Example 9.1.4

If two dice, one red and one green, are rolled, find the probability that the red die shows a 3 and the green shows a six.

Solution

Since two dice are rolled, there are 36 possibilities. The probability of each outcome, listed in Example 9.1.3, is equally likely.

Since (3, 6) is one such outcome, the probability of obtaining (3, 6) is 1/36.

The example we just considered consisted of only one outcome of the sample space. We are often interested in finding probabilities of several outcomes represented by an event.

An **event** is a subset of a sample space. If an event consists of only one outcome, it is called a **simple event.**

Example 9.1.5

If two dice are rolled, find the probability that the sum of the faces of the dice is 7.

Solution

Let E represent the event that the sum of the faces of two dice is 7.

The possible cases for the sum to be equal to 7 are: (1, 6), (2,5), (3, 4), (4, 3), (5, 2), and (6, 1), so event E is

$$E = \{(1, 6), (2,5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

The probability of the event E is

$$P(E) = 6/36 \text{ or } 1/6.$$

Example 9.1.6

A jar contains 3 red, 4 white, and 3 blue marbles. If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

Solution

We assume the marbles are $r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3$. Let the event C represent that the marble is red or blue.

The sample space $S = \{r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3\}$.

And the event $C = \{r_1, r_2, r_3, b_1, b_2, b_3\}$

Therefore, the probability of C,

$$P(C) = 6/10 \text{ or } 3/5$$

Example 9.1.7

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **without replacement**. That means that after a marble is drawn it is not replaced in the jar, and therefore is no longer available to select on the second draw.

Solution

Since two marbles are drawn without replacement, the sample space consists of the following six possibilities.

$$S = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$$

Note that (1,1), (2,2) and (3,3) are not listed in the sample space. These outcomes are not possible when drawing without replacement, because once the first marble is drawn but not replaced into the jar, that marble is not available in the jar to be selected again on the second draw.

Let the event E represent that the sum of the numbers is five. Then

$$\mathrm{E} = \{(2,3),(3,2)\}$$

Therefore, the probability of E is

$$\mathrm{P}(\mathrm{E}) = 2/6 \text{ or } 1/3.$$

Example 9.1.8

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space, as in Example 9.1.7, consists of the following six possibilities.

$$\mathrm{S} = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$$

Let the event F represent that the sum of the numbers is at least four. Then

$$\mathrm{F} = \{(1,3),(3,1),(2,3),(3,2)\}$$

Therefore, the probability of F is

$$\mathrm{P}(\mathrm{F}) = 4/6 \text{ or } 2/3$$

Example 9.1.9

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

Solution

When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

$$\mathrm{S} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event E represent that the sum of the numbers is four. Then

$$\mathrm{E} = (2,3),(3,2)$$

Therefore, the probability of F is $\mathrm{P}(\mathrm{E}) = 2/9$

Note that in Example 9.1.9 when we selected marbles with replacement, the probability has changed from Example 9.1.7 where we selected marbles without replacement.

Example 9.1.10

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space when drawing with replacement consists of the following nine possibilities.

$$S = (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$$

Let the event F represent that the sum of the numbers is at least four. Then

$$F = (1,3), (3,1), (2,3), (3,2), (2,2), (3,3)$$

Therefore, the probability of F is

$$P(F) = 6/9 \text{ or } 2/3.$$

Note that in Example $9.1.10$ when we selected marbles with replacement, the probability is the same as in Example $9.1.8$ where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We'll re-examine the concepts of sampling with and without replacement in Section 8.3.

Example 9.1.11

One 6 sided die is rolled once. Find the probability that the result is greater than 4.

Solution

The sample space consists of the following six possibilities in set S: $S = 1, 2, 3, 4, 5, 6$

Let E be the event that the number rolled is greater than four: $E = 5, 6$

Therefore, the probability of E is: $P(E) = 2/6$ or $1/3$.

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9.1.1: Sample Spaces and Probability (Exercises)

In problems 1 - 6, write a sample space for the given experiment.

1. A die is rolled.
2. A penny and a nickel are tossed.
3. A die is rolled, and a coin is tossed.
4. Three coins are tossed.
5. Two dice are rolled.
6. A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn.

In problems 7 - 12, one card is randomly selected from a deck. Find the following probabilities.

7. P(an ace)
8. P(a red card)
9. P(a club)
10. P(a face card)
11. P(a jack or a spade)
12. P(a jack and a spade)

For problems 13 - 16: A jar contains 6 red, 7 white, and 7 blue marbles. If one marble is chosen at random, find the following probabilities.

13. P(red)
14. P(white)
15. P(red or blue)
16. P(red and blue)

For problems 17 - 22: Consider a family of three children. Find the following probabilities.

17. P(two boys and a girl)
18. P(at least one boy)
19. P(children of both sexes)
20. P(at most one girl)
21. P(first and third children are male)
22. P(all children are of the same gender)

For problems 23 - 27: Two dice are rolled. Find the following probabilities.

23. P(the sum of the dice is 5)
24. P(the sum of the dice is 8)
25. P(the sum is 3 or 6)
26. P(the sum is more than 10)
27. P(the result is a double) (Hint: a double means that both dice show the same value)

For problems 28-31: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

28. P(the sum of the numbers is 5)
29. P(the sum of the numbers is odd)
30. P(the sum of the numbers is 9)
31. P(one of the numbers is 3)

For problems 32-33: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITH REPLACEMENT. That means that after a marble is drawn it is replaced in the jar before the second marble is selected. Find the following probabilities.

32. P(the sum of the numbers is 5)
33. P(the sum of the numbers is 2)

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9.2: Mutually Exclusive Events and the Addition Rule

Learning Objectives

In this section, you will learn to:

1. Define compound events using union, intersection, and complement.
2. Identify mutually exclusive events
3. Use the Addition Rule to calculate probability for unions of events.

In the last chapter, we learned to find the union, intersection, and complement of a set. We will now use these set operations to describe events.

- The **union** of two events E and F, E ∪ F, is the set of outcomes that are in E or in F or in both.
- The **intersection** of two events E and F, E ∩ F, is the set of outcomes that are in both E and F.
- The **complement** of an event E, denoted by E^c, is the set of outcomes in the sample space S that are not in E.

It is worth noting that $P(E^c) = 1 - P(E)$. This follows from the fact that if the sample space has n elements and E has k elements, then E^c has $n-k$ elements. Therefore,

$$P(E^c) = \frac{n-k}{n} = 1 - \frac{k}{n} = 1 - P(E)$$

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: $E \cap F = \varnothing$.

Next we'll determine whether a given pair of events are mutually exclusive.

Example 9.2.1

A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

E = {The card drawn is an Ace}

F = {The card drawn is a heart}

Solution

Clearly the ace of hearts belongs to both sets. That is

$$E \cap F = \{\text{ Ace of hearts }\} \neq \varnothing$$

Therefore, the events E and F are not mutually exclusive.

Example 9.2.2

Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

G = {The sum of the faces is six}

H = {One die shows a four}

Solution

For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, $G \cap H = \{(2, 4), (4, 2)\} \neq \varnothing$.

Therefore, the two sets are not mutually exclusive.

Example 9.2.3

A family has three children. Determine whether the following pair of events are mutually exclusive.

$$M = \{\text{The family has at least one boy}\}$$

$$N = \{\text{The family has all girls}\}$$

Solution

Although the answer may be clear, we list both the sets.

$$M = \{ BBB, BBG, BGB, BGG, GBB, GBG, GGB \} \text{ and } N = \{ GGG \}$$

Clearly, $M \cap N = \varnothing$

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

Example 9.2.4

If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

Solution

Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = { 1, 2, 3, 4, 5, 6 }. The event E = { 2, 4, 6 }, and event F = { 5, 6 }

We need to find $P(E \cup F)$.

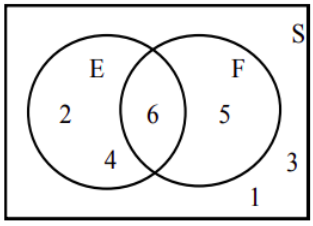
Since P(E) = 3/6, and P(F) = 2/6, a student may say $P(E \cup F) = 3/6 + 2/6$. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set $E \cup F$ has only four elements and not five: set $E \cup F$ = { 2,4,5,6 }

Therefore, $P(E \cup F) = 4/6$ and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example 9.2.4 and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and $E \cap F$ are listed below.

$$S = \{1, 2, 3, 4, 5, 6\}, E = \{2, 4, 6\}, F = \{5, 6\}, \text{ and } E \cap F = \{6\}.$$



The above figure shows S, E, F, and $E \cap F$.

Finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number $n(E \cap F)$, we remove the duplicity and get the correct answer. So as a rule,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

By dividing the entire equation by n(S), we get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Applying the above for Example 9.2.4, we get

$$P(E \cup F) = 3/6 + 2/6 - 1/6 = 4/6$$

This is because, when we add P(E) and P(F), we have added P(E ∩ F) twice. Therefore, we must subtract P(E ∩ F), once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event E ∪ F is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.** It states

Addition Rule

$$\mathbf{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

If, and only if, two events E and F are mutually exclusive, then $E \cap F = \varnothing$ and $P(E \cap F) = 0$, and we get $P(E \cup F) = P(E) + P(F)$

Example 9.2.5

If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

Solution

Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

P(A) = 4/52 and P(H) = 13/52.

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

P(A ∩ H) = 1/52

We need to find P(A ∪ H):

$$\begin{aligned} P(A \cup H) &= P(A) + P(H) - P(A \cap H) \\ &= 4/52 + 13/52 - 1/52 = 16/52 \end{aligned}$$

Example 9.2.6

Two dice are rolled, and the events F and T are as follows:

F = {The sum of the dice is four} and T = {At least one die shows a three}

Find P(F ∪ T).

Solution

We list F and T, and F ∩ T as follows:

F = {(1, 3), (2, 2), (3, 1)}

T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}

F ∩ T = {(1, 3), (3, 1)}

Since P(F ∪ T) = P(F) + P(T) - P(F ∩ T)

We have P(F ∪ T) = 3/36 + 11/36 - 2/36 = 12/36.

Example 9.2.7

Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

Solution

Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

P(A) = .80, P(S) = .70, and P(A ∪ S) = .90.

We need to find, P(A ∩ S).

The addition formula states that,

$$P(A \cup S) = P(A) + P(S) - P(A \cap S)$$

Substituting the known values, we get

$$.90 = .80 + .70 - P(\mathrm{A} \cap \mathrm{S})$$

Therefore, P(A ∩ S) = .60.

Example 9.2.8

The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

Solution

Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

$$\mathrm{P(C)} = .6, \quad \mathrm{P(R)} = .7, \quad \mathrm{P(C \cap R)} = .5$$

First we find P(C ∪ R) using the Addition Rule.

$$P(C \cup R) = P(C) + P(R) - P(C \cap R) = .6 + .7 - .5 = .8$$

Then we find P((C ∪ R)^c) using the Complement Rule.

$$P\left((C \cup R)^c\right) = 1 - P(C \cup R) = 1 - .8 = .2$$

We summarize this section by listing the important rules.

Summary

The Addition Rule

For Two Events E and F, $\mathrm{P(E \cup F)} = \mathrm{P(E)} + \mathrm{P(F)} - \mathrm{P(E \cap F)}$

The Addition Rule for Mutually Exclusive Events

If Two Events E and F are Mutually Exclusive, then $\mathrm{P(E \cup F)} = \mathrm{P(E)} + \mathrm{P(F)}$

The Complement Rule

If E^c is the Complement of Event E, then $\mathrm{P}(\mathrm{E}^c) = 1 - \mathrm{P(E)}$

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9.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)

Determine whether the following pair of events are mutually exclusive.

1. A = {A person earns more than $25,000}; B = {A person earns less than $20,000}.
2. A card is drawn from a deck and… C = {It is a King}; D = {It is a heart}.
3. A die is rolled and… E = {An even number shows}; F = {A number greater than 3 shows}
4. Two dice are rolled and… G = {The sum of dice is 8}; H = {One die shows a 6}
5. Three coins are tossed and… I = {Two heads come up}; J = {At least one tail comes up}
6. A family has three children and… K = {First born is a boy}; L = {The family has children of both sexes}

Use the Addition Rule to find the following probabilities.

7. A card is drawn from a deck. Events C and D are: C = {It is a king}, D = {It is a heart}. Find P(C or D).
8. A die is rolled. The events E and F are: E = {An even number shows}, F = {A number greater than 3 shows}. Find P(E or F).
9. Two dice are rolled. Events G and H are: G = {The sum of dice is 8}, H ={Exactly one die shows a 6}. Find P(G or H).
10. Three coins are tossed. Events I and J are: I = {Two heads come up}, J = {At least one tail comes up}. Find P(I or J).
11. At a college, 20% of the students take Finite Mathematics, 30% take Statistics and 10% take both. What percent of students take Finite Mathematics or Statistics?
12. This quarter, there is a 50% chance that Jason will pass Accounting, a 60% chance that he will pass English, and 80% chance that he will pass at least one of these two courses. What is the probability that he will pass both Accounting and English?

Questions 13 - 20 refer to the following: The table shows the distribution of Democratic and Republican U.S. senators by gender in the 114th Congress as of January 2015.

Distribution of senators by party and gender

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities.

13. P(M and D)
14. P(F and R)
15. P(M or D)
16. P(F or R)
17. P(M or R)
18. P(M or F)
19. Are the events F, R mutually exclusive? Use probabilities to support your conclusions.
20. Are the events F, T mutually exclusive? Use probabilities to support your conclusion.

Use the Addition Rule to find the following probabilities.

21. If P(E) = .5 , P(F) = .4 , E and F are mutually exclusive, find P(E and F).
22. If P(E) = .4 , P(F) = .2 , E and F are mutually exclusive, find P(E or F).
23. If P(E) = .3, P(E or F) = .6 , P(E and F) = .2, find P(F).
24. If P(E) = .4, P(F) = .5 , P(E or F) = .7, find P(E and F).
25. In a box of assorted cookies, 36% of cookies contain chocolate and 12% of cookies contain nuts. 8% of cookies have both chocolates and nuts. Sean is allergic to chocolate and nuts. Find the probability that a cookie has chocolate chips **or** nuts (he can't eat it).
26. At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper. Find the probability that a course requires a final exam **or** a research paper.

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9.3: Probability Using Tree Diagrams and Combinations

Learning Objectives

In this section, you will learn to:

1. Use probability tree diagrams to calculate probabilities
2. Use combinations to calculate probabilities

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree.
By multiplying probabilities along a path through the tree, we can find probabilities for “and” events, which are intersections of events.

We begin with an example.

Example 9.3.1

Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

Solution

Let E be the event that the first marble drawn is red, and let F be the event that the second marble drawn is red.

We need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

By the statement, "two marbles are drawn with replacement," we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 49 ordered pairs. Of the 49 ordered pairs, there are $3 \times 3 = 9$ ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

$$P(E \cap F) = \frac{9}{49}$$

Further note that in this particular case

$$P(E \cap F) = \frac{9}{49} = \frac{3}{7} \cdot \frac{3}{7}$$

giving us the result that in this example: $\mathbf{P}(\mathbf{E} \cap \mathbf{F}) = \mathbf{P}(\mathbf{E}) \cdot \mathbf{P}(\mathbf{F})$

Example 9.3.2

If in Example $9.3.1$, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

Solution

By the statement, "two marbles are drawn without replacement," we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 42 ordered pairs. Of the 42 ordered pairs, there are $3 \times 2 = 6$ ordered pairs that show red on the first draw and red on the second draw. Therefore,

$$P(E \cap F) = \frac{6}{42}$$

Note that we can break this calculation down as

$$P(E \cap F) = \frac{6}{42} = \frac{3}{7} \cdot \frac{2}{6}$$

.

Here 3/7 represents $\mathrm{P(E)}$, and 2/6 represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as P(red on the second | red on first) or $\mathrm{P(F|E)}$. The "|" represents the word "given" or “if”. This leads to the result that:

$$\mathbf{P(E \cap F) = P(E) \cdot P(F|E)}$$

The is an important result, called the **Multiplication Rule,** which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

Example 9.3.3

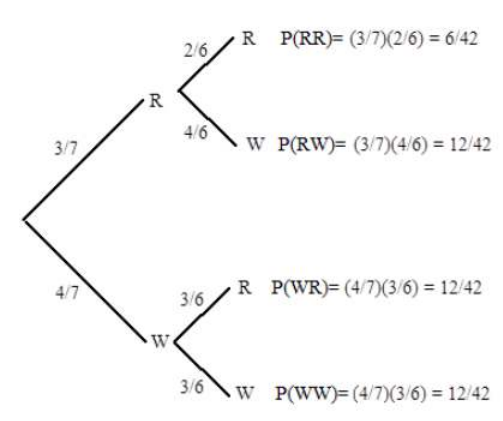
Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

a. The probability that both marbles are red.
b. The probability that the first marble is red and the second white.
c. The probability that one marble is red and the other white.

Solution

Let R be the event that the marble drawn is red, and let W be the event that the marble drawn is white.

We draw the following tree diagram.



a. The probability that both marbles are red is $\mathrm{P(RR)} = 6/42$
b. The probability that the first marble is red and the second is white is $\mathrm{P(RW)} = 12/42$
c. For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The “or” tells us we’ll be using the Addition Rule from Section 7.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

P(one marble is red and the other marble is white)

$$\begin{aligned} &= \mathrm{P}(\mathrm{RW} \text{ or } \mathrm{WR}) \\ &= \mathrm{P}(\mathrm{RW}) + \mathrm{P}(\mathrm{WR}) \\ &= 12/42 + 12/42 = 24/42 \end{aligned}$$

USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

Example 9.3.4

Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

a. P(Two red and one white)
b. P(One of each color)
c. P(None blue)
d. P(At least one blue)

Solution

Let us suppose the marbles are labeled as $R_1, R_2, R_3, W_1, W_2, B_1, B_2, B_3$.

a. P(Two red and one white)

Since we are choosing 3 marbles from a total of 8, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{C}2 \times 2\mathrm{C}1 = 6$ combinations consisting of 2 red and one white. Therefore,

$$P(\text{ Two red and one white }) = \frac{3\mathrm{C}2 \times 2\mathrm{C}1}{8\mathrm{C}3} = \frac{6}{56}.$$

b. P(One of each color)

Again, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{Cl} \times 2\mathrm{Cl} \times 3\mathrm{Cl} = 18$ combinations consisting of one red, one white, and one blue. Therefore,

$$P(\text{ One of each color }) = \frac{3\mathrm{C}1 \times 2\mathrm{C}1 \times 3\mathrm{C}1}{8\mathrm{C}3} = \frac{18}{56}$$

c. P(None blue)

There are 5 non-blue marbles, therefore

$$\mathrm{P}(\text{ None blue }) = \frac{5\mathrm{C}3}{8\mathrm{C}3} = \frac{10}{56} = \frac{5}{28}$$

d. P(At least one blue)

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, *OR* two blue marbles and one non-blue marble, *OR* all three blue marbles. So we have to find the sum of the probabilities of all three cases.

$$\mathrm{P}(\text{At least one blue }) = \mathrm{P}(1 \text{ blue, } 2 \text{ non-blue}) + \mathrm{P}(2 \text{ blue, l non-blue}) + \mathrm{P}(3 \text{ blue})$$

$$P(\text{ At least one blue }) = \frac{3\mathrm{C}1 \times 5\mathrm{C}2}{8\mathrm{C}3} + \frac{3\mathrm{C}2 \times 5\mathrm{C}1}{8\mathrm{C}3} + \frac{3\mathrm{C}3}{8\mathrm{C}3}$$

$$P(\text{ At least one blue }) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28$$

Alternately, we can use the fact that $\mathrm{P(E)} = 1 - \mathrm{P}(\mathrm{E}^c)$. If the event E = At least one blue, then E^c = None blue.

But from part c of this example, we have $(\mathrm{E}^c) = 5/28$, so $\mathrm{P(E)} = 1 - 5/28 = 23/28$.

Example 9.3.5

Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

Solution

Let us first do an easier problem-the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

$$\mathrm{P}(\text{A pair of kings and queens }) = \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5}$$

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are $13\mathrm{C}2$ different combinations of pairs.

$$P(\text{ Two pairs }) = 13\mathrm{C}2 \cdot \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5} = .04754$$

Example 9.3.6

A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment. Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

Solution

There are $8\mathrm{C}2$ ways of selecting 2 out of the 8 iPhones.

and $7\mathrm{C}4$ ways of selecting 4 out of the 7 Android phones

But altogether there are $15\mathrm{C}6$ ways of selecting 6 out of 15 cell phones.

Therefore we have

$$P(2 \text{ iPhones and } 4 \text{ Android phones }) = \frac{8\mathrm{C}2 \times 7\mathrm{C}4}{15\mathrm{C}6} = \frac{(28)(35)}{5005} = \frac{980}{5005} = 0.1958$$

Example 9.3.7

One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow's breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

Solution

There are $20\mathrm{C}4$ ways of selecting 4 out of the 20 plain bagels,

and $15\mathrm{C}3$ ways of selecting 3 out of the 15 poppyseed bagels,

and $18\mathrm{C}2$ ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are $53\mathrm{C}9$ ways of selecting 9 out of the 53 bagels.

$$\begin{aligned} \mathrm{P}(4 \text{ plain, } 3 \text{ poppyseed, and } 2 \text{ sesame seed}) &= \frac{20\mathrm{C}4 \times 15\mathrm{C}3 \times 18\mathrm{C}2}{53\mathrm{C}9} \\ &= \frac{(4845)(455)(153)}{4431613550} \\ &= 0.761 \end{aligned} \tag{9.3.1}$$

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We end the section by solving a famous problem called the **Birthday Problem**.

Example 9.3.8: Birthday Problem

If there are 25 people in a room, what is the probability that at least two people have the same birthday?

Solution

Let event E represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are 365^{25} possible birthdays for 25 people. Therefore, the sample space has 365^{25} elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

$$\mathrm{P}(\text{No two have the same birthday }) = \frac{365 \cdot 364 \cdot 363 \cdots 341}{365^{25}} = \frac{365\mathrm{P}25}{365^{25}}$$

Since P(at least two people have the same birthday) = 1 - P(No two have the same birthday),

$$\mathrm{P} \text{ at least two people have the same birthday }) = 1 - \frac{365\mathrm{P}25}{365^{25}} = .5687$$

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9.3.1: Probability Using Tree Diagrams and Combinations (Exercises)

Two apples are chosen from a basket containing five red and three yellow apples.
Draw a tree diagram below, and find the following probabilities.

1. P(both red)
2. P(one red, one yellow)
3. P(both yellow)
4. P(First red and second yellow)

A basket contains six red and four blue marbles. Three marbles are drawn at random. Find the following probabilities using the method shown in Example 9.3.2. Do not use combinations.

5. P(All three red)
6. P(two red, one blue)
7. P(one red, two blue)
8. P(first red, second blue, third red)

Three marbles are drawn from a jar containing five red, four white, and three blue marbles. Find the following probabilities using combinations.

9. P(all three red)
10. P(two white and 1 blue)
11. P(none white)
12. P(at least one red)

A committee of four is selected from a total of 4 freshmen, 5 sophomores, and 6 juniors. Find the probabilities for the following events.

13. At least three freshmen.
14. No sophomores.
15. All four of the same class.
16. Not all four from the same class.
17. Exactly three of the same class.
18. More juniors than freshmen and sophomores combined.

Five cards are drawn from a deck. Find the probabilities for the following events.

19. Two hearts, two spades, and one club.
20. A flush of any suit (*all cards of a single suit*).
21. A full house of nines and tens (*3 nines and 2 tens*).
22. Any full house.
23. A pair of nines and a pair of tens (*and the fifth card is not a nine or ten*).
24. Any two pairs (*two cards of one value, two more cards of another value, and the fifth card does not have the same value as either pair*).

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises. He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises. Find the following probabilities:

25. P(playlist has 2 songs of each type)
26. P(playlist has no country songs)
27. P(playlist has 3 rock, 2 rap, and 1 country song)
28. P(playlist has 3 or 4 rock songs and the rest are rap songs)

A project is staffed 12 people: 5 engineers, 4 salespeople, and 3 customer service representatives. A committee of 5 people is selected to make a presentation to senior management. Find the probabilities of the following events.

29. The committee has 2 engineers, 2 salespeople, and 1 customer service representative.
30. The committee contains 3 engineer and 2 salespeople.
31. The committee has no engineers.

32. The committee has all salespeople.

Do the following birthday problems.

33. If there are 5 people in a room, what is the probability that no two have the same birthday?
34. If there are 5 people in a room, find the probability that at least 2 have the same birthday.

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9.4: Conditional Probability

Learning Objectives

In this section, you will learn to:

1. recognize situations involving conditional probability
2. calculate conditional probabilities

Suppose a friend asks you the probability that it will snow today.

If you are in Boston, Massachusetts in the winter, the probability of snow today might be quite substantial. If you are in Cupertino, California in summer, the probability of snow today is very tiny, this probability is pretty much 0.

Let:

- A = the event that it will snow today
- B = the event that today you are in Boston in wintertime
- C = the event that today you are in Cupertino in summertime

Because the probability of snow is affected by the location and time of year, we can't just write $\mathrm{P}(\mathrm{A})$ for the probability of snow. We need to indicate the other information we know -location and time of year. We need to use **conditional probability**.

The event we are interested in is event A for snow. The other event is called the condition, representing location and time of year in this case.

We represent conditional probability using a vertical line | that means "if", or "given that", or "if we know that". The event of interest appears on the left of the |. The condition appears on the right side of the |.

The probability it will snow given that (if) you are in Boston in the winter is represented by $\mathbf{P}(\mathbf{A}|\mathbf{B})$. In this case, the condition is B.

The probability that it will snow given that (if) you are in Cupertino in the summer is represented by $\mathbf{P}(\mathbf{A}|\mathbf{C})$. In this case, the condition is C.

Now, let's examine a situation where we can calculate some probabilities.

Suppose you and a friend play a game that involves choosing a single card from a well-shuffled deck. Your friend deals you one card, face down, from the deck and offers you the following deal: If the card is a king, he will pay you $5, otherwise, you pay him $1. Should you play the game?

You reason in the following manner. Since there are four kings in the deck, the probability of obtaining a king is 4/52 or 1/13. So, probability of not obtaining a king is 12/13. This implies that the ratio of your winning to losing is 1 to 12, while the payoff ratio is only $1 to $5. Therefore, you determine that you should not play.

But consider the following scenario. While your friend was dealing the card, you happened to get a glance of it and noticed that the card was a face card. Should you, now, play the game?

Since there are 12 face cards in the deck, the total elements in the sample space are no longer 52, but just 12. This means the chance of obtaining a king is 4/12 or 1/3. So your chance of winning is 1/3 and of losing 2/3. This makes your winning to losing ratio 1 to 2 which fares much better with the payoff ratio of $1 to $5. This time, you determine that you should play.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event E under the condition that another event F has happened, we are finding conditional probability.

The symbol $\mathrm{P}(\mathrm{E}|\mathrm{F})$ denotes the problem of finding the probability of E given that F has occurred. We read $\mathrm{P}(\mathrm{E}|\mathrm{F})$ as "the probability of E, given F."

Example 9.4.1

A family has three children. Find the conditional probability of having two boys and a girl given that the first born is a boy.

Solution

Let event E be that the family has two boys and a girl, and F that the first born is a boy.

First, we the sample space for a family of three children as follows.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$

Since we know the first born is a boy, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, $P(E|F)$ = 2/4 or 1/2.

Example 9.4.2

One six sided die is rolled once.

a. Find the probability that the result is even.
b. Find the probability that the result is even given that the result is greater than three.

Solution

The sample space is $S = 1, 2, 3, 4, 5, 6$

Let event E be that the result is even and T be that the result is greater than 3.

a. $P(E)$ = 3/6 because $E = 2, 4, 6$

b. Because $T = 4, 5, 6$, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, $P(E|T)$ = 2/3

Example 9.4.3

A fair coin is tossed twice.

a. Find the probability that the result is is two heads.
b. Find the probability that the result is two heads given that at least one head is obtained.

Solution

The sample space is $S = HH, HT, TH, TT$

Let event E be that the two heads are obtained and F be at least one head is obtained

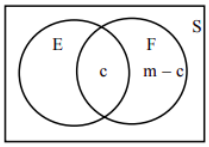
a. $P(E)$ = 1/4 because $E = HH$ and the sample space S has 4 outcomes.

b. $F = HH, HT, TH$. Since at least one head was obtained, TT did not occur.
We are interested in the probability event $E = HH$ out of the 3 outcomes in the reduced sample space F.

Therefore, $P(E|F)$ = 1/3

Let us now develop a formula for the conditional probability $P(E|F)$.

Suppose an experiment consists of n equally likely events. Further suppose that there are m elements in F, and c elements in $E \cap F$, as shown in the following Venn diagram.



If the event F has occurred, the set of all possible outcomes is no longer the entire sample space, but instead, the subset F. Therefore, we only look at the set F and at nothing outside of F. Since F has m elements, the denominator in the calculation of $\mathrm{P(E|F)}$ is m. We may think that the numerator for our conditional probability is the number of elements in E. But clearly we cannot consider the elements of E that are not in F. We can only count the elements of E that are in F, that is, the elements in $\mathrm{E \cap F}$. Therefore,

$$\mathrm{P(E|F) = \frac{c}{m}}$$

Dividing both the numerator and the denominator by n, we get

$$\mathrm{P(E|F) = \frac{c/n}{m/n}}$$

But $c/n = \mathrm{P(E \cap F)}$, and $m/n = \mathrm{P(F)}$.

Substituting, we derive the following formula for $\mathrm{P(E|F)}$.

Conditional Probability Rule

For two events E and F, the probability of "E Given F" is

$$\mathbf{P(E|F) = \frac{P(E \cap F)}{P(F)}}$$

Example 9.4.4

A single die is rolled. Use the above formula to find the conditional probability of obtaining an even number given that a number greater than three has shown.

Solution

Let E be the event that an even number shows, and F be the event that a number greater than three shows. We want $\mathrm{P(E|F)}$.

$\mathrm{E} = 2, 4, 6$ and $\mathrm{F} = 4, 5, 6$. Which implies, $\mathrm{E \cap F} = 4, 6$

Therefore, $\mathrm{P(F)}$ = 3/6, and $\mathrm{P(E \cap F)}$ = 2/6

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

Example 9.4.5

The following table shows the distribution by gender of students at a community college who take public transportation and the ones who drive to school.

Distribution of gender and transportation at a community college

	Male(M)	Female(F)	Total
Public Transportation(T)	8	13	21
Drive(D)	39	40	79
Total	47	53	100

The events M, F, T, and D are self explanatory. Find the following probabilities.

a. $\mathrm{P(D|M)}$
b. $\mathrm{P(F|D)}$
c. $\mathrm{P(M|T)}$

Solution 1

Conditional probabilities can often be found directly from a contingency table. If the condition corresponds to only one row or only one column in the table, then you can ignore the rest of the table and read the conditional probability directly from the row or column indicated by the condition.

a. The condition is event M; we can look at only the "Male" column of the table and ignore the rest of the table: $\mathrm{P(D|M)} = \frac{39}{47}$.
b. The condition is event D; we can look at only the "Drive" row of the table and ignore the rest of the table: $\mathrm{P(F|D)} = \frac{40}{79}$.
c. The condition is event T; we can look at only the "Public Transportation" row of the table and ignore the rest of the table: $\mathrm{P(M|T)} = \frac{8}{21}$.

Solution 2

We use the conditional probability formula $\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$.

a. $$P(D|M) = \frac{P(D \cap M)}{P(M)} = \frac{39/100}{47/100} = \frac{39}{47}.$$

b. $$P(F|D) = \frac{P(F \cap D)}{P(D)} = \frac{40/100}{79/100} = \frac{40}{79}.$$

c. $$\mathrm{P(M|T)} = \frac{P(M \cap T)}{P(T)} = \frac{8/100}{21/100} = \frac{8}{21}$$

Example 9.4.6

Given $\mathrm{P(E)}$ = .5, $\mathrm{P(F)}$ = .7, and $\mathrm{P(E \cap F)}$ = .3. Find the following:

a. $\mathrm{P(E|F)}$
b. $\mathrm{P(F|E)}$

Solution

We use the conditional probability formula.

a. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{3}{7} = \frac{3}{7}$
b. $P(F|E) = \frac{P(E \cap F)}{P(E)} = .3/.5 = 3/5$

Example 9.4.7

E and F are mutually exclusive events such that $\mathrm{P(E)}$ = .4, $\mathrm{P(F)}$ = .9. Find $\mathrm{P(E|F)}$.

Solution

E and F are mutually exclusive, so $\mathrm{P(E \cap F)}$ = 0.
Therefore $\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}} = \frac{0}{9} = 0$.

Example 9.4.8

Given $\mathrm{P(F|E)}$ = .5, and $\mathrm{P(E \cap F)}$ = .3. Find $\mathrm{P(E)}$.

Solution

Using the conditional probability formula $\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$, we get

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

Substituting and solving:

$$.5 = \frac{.3}{P(E)} \quad \text{or} \quad P(E) = 3/5$$

Example 9.4.9

In a family of three children, find the conditional probability of having two boys and a girl, given that the family has at least two boys.

Solution

Let event E be that the family has two boys and a girl, and let F be the probability that the family has at least two boys. We want $P(E|F)$.

We list the sample space along with the events E and F.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$
$$E = \{BBG, BGB, GBB\} \text{ and } F = \{BBB, BBG, BGB, GBB\}$$
$$E \cap F = \{BBG, BGB, GBB\}$$

Therefore, $P(F)$ = 4/8, and $P(E \cap F)$ = 3/8, and

$$P(E|F) = \frac{P(E \cap F)}{P(E)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Example 9.4.10

At a community college 65% of the students subscribe to Amazon Prime, 50% subscribe to Netflix, and 20% subscribe to both. If a student is chosen at random, find the following probabilities:

a. the student subscribes to Amazon Prime given that he subscribes to Netflix
b. the student subscribes to Netflix given that he subscribes to Amazon Prime

Solution

Let A be the event that the student subscribes to Amazon Prime, and N be the event that the student subscribes to Netflix.

First identify the probabilities and events given in the problem.

P(student subscribes to Amazon Prime) = $P(A)$ = 0.65

P(student subscribes to Netflix) = $P(N)$ = 0.50

P(student subscribes to both Amazon Prime and Netflix) = $P(A \cap N)$ = 0.20

Then use the conditional probability rule:

a. $P(A|N) = \frac{P(A \cap N)}{P(N)} = \frac{.20}{.50} = \frac{2}{5}$
b. $P(N|A) = \frac{P(A \cap N)}{P(A)} = \frac{.20}{.65} = \frac{4}{13}$

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9.4.1: Conditional Probability (Exercises)

Questions 1 - 4: Do these problems using the conditional probability formula: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

1. A card is drawn from a deck. Find the conditional probability of P(a queen | a face card).
2. A card is drawn from a deck. Find the conditional probability of P(a queen | a club).
3. A die is rolled. Find the conditional probability that it shows a three if it is known that an odd number has shown.
4. If $P(A)$ = .3 , $P(B)$ = .4, $P(A$ and $B)$ = .12, find:
   a. $P(A|B)$
   b. $P(B|A)$

Questions 5 - 8 refer to the following: The table shows the distribution of Democratic and Republican U.S. Senators by gender in the 114th Congress as of January 2015.

Distribution of senators by party and gender

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities:

5. $P(M|D)$
6. $P(D|M)$
7. $P(F|R)$
8. $P(R|F)$

Do the following conditional probability problems.

9. At a college, 20% of the students take Finite Math, 30% take History, and 5% take both Finite Math and History. If a student is chosen at random, find the following conditional probabilities.
   a. They are taking Finite Math given that they are taking History.
   b. They are taking History assuming that they are taking Finite Math.
10. At a college, 60% of the students pass Accounting, 70% pass English, and 30% pass both of these courses. If a student is selected at random, find the following conditional probabilities.
    a. They pass Accounting given that they passed English.
    b. They pass English assuming that they passed Accounting.
11. If $P(F) = .4$, $P(E|F) = .3$, find $P(E$ and $F)$.
12. $P(E) = .3$, $P(F) = .3$; E and F are mutually exclusive. Find $P(E|F)$.
13. If $P(E) = .6$, $P(E$ and $F)$ = .24, find $P(F|E)$.
14. If $P(E$ and $F)$ = .04, $P(E|F) = .1$, find $P(F)$.

At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper.

15. Find the probability that a course has a final exam given that it has a research paper.
16. Find the probability that a course has a research paper if it has a final exam.

Consider a family of three children. Find the following probabilities.

17. P(two boys | first born is a boy)
18. P(all girls | at least one girl is born)
19. P(children of both sexes | first born is a boy)
20. P(all boys | there are children of both sexes)

Questions 21 - 26 refer to the following: The table shows highest attained educational status for a sample of US residents age 25 or over:

Distribution educational achievement and age for U.S. residents

	(D) Did not Complete High School	(H) High School	(C) Some College	(A) Associate Degree	(B) Bachelor Degree	(G) Graduate Degree	TOTAL
25-44 (R)	95	228	143	81	188	61	796
45-64 (S)	83	256	136	80	150	67	772
65+ (T)	96	191	84	36	80	41	528
Total	274	675	363	197	418	169	2096

Use this table to determine the following probabilities:

21. $P(C|T)$
22. $P(S|A)$
23. $P(C \text{ and } T)$
24. $P(R|B)$
25. $P(B|R)$
26. $P(G|S)$

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9.5: Independent Events

Learning Objectives

In this section, you will:

1. Define independent events
2. Identify whether two events are independent or dependent

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.

In Example $9.5.1$ we revisit the discussion at the beginning of the previous section and then contrast that with Example $9.5.2$.

Example 9.5.1

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that the card is a face card.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

$$P(\text{The card is a king} \mid \text{The card is a face card}) = 4/12 = 1/3.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{The card is a face card}) \neq P(\text{The card is a king})$$

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

Example 9.5.2

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that a red card has shown.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = 2/26 = 1/13.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = P(\text{The card is a king})$$

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

Definition: Independent

Two Events E and F are **independent** if and only if at least one of the following two conditions is true.

1. $\mathbf{P(E|F) = P(E)}$ or
2. $\mathbf{P(F|E) = P(F)}$

If the events are not independent, then they are dependent.

If one of these conditions is true, then both are true.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$$

Multiplying both sides by $\mathrm{P(F)}$, we get

$$\mathrm{P(E \cap F) = P(E|F)P(F)}$$

Now if the two events are independent, then by definition

$$\mathrm{P(E|F) = P(E)}$$

Substituting, $P(E \cap F) = P(E)P(F)$

We state it formally as follows.

Test For Independence

Two events E and F are independent if and only if

$$\mathbf{P(E \cap F) = P(E)P(F)}$$

In the Examples $9.5.3$ and $9.5.4$, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F) = P(E)$ or $P(F|E) = P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

Example $9.5.3$

The table below shows the distribution of color-blind people by gender.

	Male(M)	Female(F)	Total
Color-Blind(C)	6	1	7
Not Color-Blind(N)	46	47	93
Total	52	48	100

where M represents male, F represents female, C represents color-blind, and N represents not color-blind. Are the events color-blind and male independent?

Solution 1: According to the test for independence, C and M are independent if and only if $\mathrm{P(C \cap M) = P(C)P(M)}$.

From the table: $P(C)$ = 7/100, $P(M)$ = 52/100 and $P(C \cap M)$ = 6/100

So $P(C)P(M)$ = (7/100)(52/100) = .0364

which is **not** equal to $P(C \cap M)$ = 6/100 = .06

Therefore, the two events are not independent. We may say they are dependent.

Solution 2: C and M are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C)$ = 7/100 = 0.07

From the male column $P(C|M)$ = 6/52= 0.1154

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

Example 9.5.4

In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.

- 45 flights in the survey departed from airport A; 9 of those flights departed late.
- 55 flights in the survey departed from airport B; 11 flights departed late.

Are the events "depart from airport A" and "departed late" independent?

Solution 1

Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$$P(A \cap L) = 9/100,\ P(A) = 45/100 \text{ and } P(L) = 20/100$$

In order for two events to be independent, we must have $P(A \cap L) = P(A)P(L)$

Since $P(A \cap L)$ = 9/100 = 0.09

and $P(A)P(L)$ = (45/100)(20/100) = 900/10000 = 0.09

the two events "departing from airport A" and "departing late" are independent.

Solution 2

The definition of independent events states that two events are independent if $P(E|F) = P(E)$.

In this problem we are given that

$$P(L|A) = 9/45 = 0.2 \text{ and } P(L) = 20/100 = 0.2$$

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

Example 9.5.5

A coin is tossed three times, and the events E, F and G are defined as follows:

E: The coin shows a head on the first toss.

F: At least two heads appear.

G: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. E and F
b. F and G
c. E and G

Solution

We list the sample space, the events, their intersections and the probabilities.

$$
\begin{aligned}
&S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\\
&E = \{HHH, HHT, HTH, HTT\}, \quad P(E) = 4/8 \text{ or } 1/2\\
&F = \{HHH, HHT, HTH, THH\}, \quad P(F) = 4/8 \text{ or } 1/2\\
&G = \{HHT, THH\}, \quad P(G) = 2/8 \text{ or } 1/4\\
&E \cap F = \{HHH, HHT, HTH\}, \quad P(E \cap F) = 3/8\\
&F \cap G = \{HHT, THH\}, \quad P(F \cap G) = 2/8 \text{ or } 1/4\\
&E \cap G = \{HHT\} \quad P(E \cap G) = 1/8
\end{aligned}
$$

a. E and F will be independent if and only if $P(E \cap F) = P(E)P(F)$

$$P(E \cap F) = 3/8 \text{ and } P(E)P(F) = 1/2 \cdot 1/2 = 1/4.$$

Since 3/8 ≠ 1/4, we have $P(E \cap F) \neq P(E)P(F)$.

Events E and F are not independent.

b. F and G will be independent if and only if $P(F \cap G) = P(F)P(G)$.

$$P(F \cap G) = 1/4 \text{ and } P(F)P(G) = 1/2 \cdot 1/4 = 1/8.$$

Since 3/8 ≠ 1/4, we have $P(F \cap G) \neq P(F)P(G)$.

Events F and G are not independent.

c. E and G will be independent if $P(E \cap G) = P(E)P(G)$

$$P(E \cap G) = 1/8 \text{ and } P(E)P(G) = 1/2 \cdot 1/4 = 1/8$$

Events E and G are independent events because $P(E \cap G) = P(E)P(G)$

Example 9.5.6

The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

Solution

Let A be the event that Jaime will visit his aunt this year, and R be the event that he will go river rafting.

We are given $P(A)$ = .30 and $P(R)$ = .50, and we want to find $P(A \cap R)$.

Since we are told that the events A and R are independent,

$$P(A \cap R) = P(A)P(R) = (.30)(.50) = .15$$

Example 9.5.7

Given $P(B|A) = .4$. If A and B are independent, find $P(B)$.

Solution

If A and B are independent, then by definition $P(B|A) = P(B)$

Therefore, $P(B) = .4$

Example 9.5.8

Given $P(A) = .7$, $P(B|A) = .5$. Find $P(A \cap B)$.

Solution 1

By definition $P(B|A) = \frac{P(A \cap B)}{P(A)}$

Substituting, we have

$$.5 = \frac{\mathrm{P(A \cap B)}}{.7}$$

Therefore, $P(A \cap B) = .35$

Solution 2

Again, start with $P(B|A) = \frac{P(A \cap B)}{P(A)}$

Multiplying both sides by $P(A)$ gives

$$P(A \cap B) = P(B|A)P(A) = (.5)(.7) = .35$$

Both solutions to Example $9.5.8$ are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for $P(A \cap B)$. That gives the following result:

Multiplication Rule for events that are NOT independent

If events E and F are not independent

$$\mathbf{P(E \cap F) = P(E|F)P(F)} \quad \text{and} \quad \mathbf{P(E \cap F) = P(F|E)P(E)}$$

Example $9.5.9$

Given $P(A) = .5$, $P(A \cup B) = .7$, if A and B are independent, find $P(B)$.

Solution

The addition rule states that

$$\mathrm{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Since A and B are independent, $P(A \cap B) = P(A)P(B)$

We substitute for $P(A \cap B)$ in the addition formula and get

$$\mathrm{P(A \cup B) = P(A) + P(B) - P(A)P(B)}$$

By letting $P(B) = x$, and substituting values, we get

$$\begin{aligned} .7 &= .5 + x - .5x \\ .7 &= .5 + .5x \\ .2 &= .5x \\ .4 &= x \end{aligned}$$

Therefore, $P(B) = .4$

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9.5.1: Independent Events (Exercises)

The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch on a given day is as follows.

Distribution of book loans by location and type

	MAIN (M)	BRANCH (B)	TOTAL
FICTION (F)	300	100	400
NON-FICTION (N)	150	50	200
TOTALS	450	150	600

Use this table to determine the following probabilities:

1. $P(F)$
2. $P(M|F)$
3. $P(N|B)$
4. Is the fact that a person checks out a fiction book independent of the main library? Use probabilities to justify your conclusion.

For a two-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

5. Find the following.
   a. $P(E)$
   b. $P(F)$
   c. $P(E \cap F)$
   d. Are E and F independent? Use probabilities to justify your conclusion.
6. Find the following.
   a. $P(F)$
   b. $P(G)$
   c. $P(F \cap G)$
   d. Are F and G independent? Use probabilities to justify your conclusion.

Do the following problems involving independence.

7. If $P(E) = .6$, $P(F) = .2$, and E and F are independent, find $P(E$ and $F)$.
8. If $P(E) = .6$, $P(F) = .2$, and E and F are independent, find $P(E$ or $F)$.
9. If $P(E) = .9$, $P(F|E) = .36$, and E and F are independent, find $P(F)$.
10. If $P(E) = .6$, $P(E$ or $F)$ = .8, and E and F are independent, find $P(F)$.
11. In a survey of 100 people, 40 were casual drinkers, and 60 did not drink. Of the ones who drank, 6 had minor headaches. Of the non-drinkers, 9 had minor headaches. Are the events "drinkers" and "had headaches" independent?
12. It is known that 80% of the people wear seat belts, and 5% of the people quit smoking last year. If 4% of the people who wear seat belts quit smoking, are the events, wearing a seat belt and quitting smoking, independent?
13. John's probability of passing statistics is 40%, and Linda's probability of passing the same course is 70%. If the two events are independent, find the following probabilities.
    a. P(both of them will pass statistics)
    b. P(at least one of them will pass statistics)
14. Jane is flying home for the Christmas holidays. She has to change planes twice. There is an 80% chance that she will make the first connection, and a 90% chance that she will make the second connection. If the two events are independent, find the probabilities:
    a. P(Jane will make both connections)

b. P(Jane will make at least one connection)

For a three-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

15. Find the following.
    a. $P(E)$
    b. $P(F)$
    c. $P(E \cap F)$
    d. Are E and F independent?
16. Find the following.
    a. $P(F)$
    b. $P(G)$
    c. $P(F \cap G)$
    d. Are F and G independent?

Answer the following:

17. $P(K|D) = 0.7$, $P(D) = 0.25$ and $P(K) = 0.7$
    a. Are events K and D independent? Use probabilities to justify your conclusion.
    b. Find $P(K \cap D)$
18. $P(R|S) = 0.4$, $P(S) = 0.2$ and $P(R) = 0.3$
    a. Are events R and S independent? Use probabilities to justify your conclusion.
    b. Find $P(R \cap S)$
19. At a college: 54% of students are female, 25% of students are majoring in engineering, 5% of female students are majoring in engineering.
    Event E = student is majoring in engineering. Event F = student is female.
    a. Are events E and F independent? Use probabilities to justify your conclusion.
    b. Find $P(E \cap F)$
20. At a college: 54% of all students are female, 60% of all students receive financial aid, 60% of female students receive financial aid.
    Event A = student receives financial aid. Event F = student is female
    a. Are events A and F independent? Use probabilities to justify your conclusion.
    b. Find $P(A \cap F)$

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9.6: Bayes' Formula

Learning Objectives

In this section, you will learn to:

1. Find probabilities using Bayes' formula
2. Use a probability tree to find and represent values needed when using Bayes' formula.

In this section, we will develop and use Bayes' Formula to solve an important type of probability problem. Bayes' formula is a method of calculating the conditional probability $P(F|E)$ from $P(E|F)$. The ideas involved here are not new, and most of these problems can be solved using a tree diagram. However, Bayes' formula does provide us with a tool with which we can solve these problems without a tree diagram.

We begin with an example.

Example 9.6.1

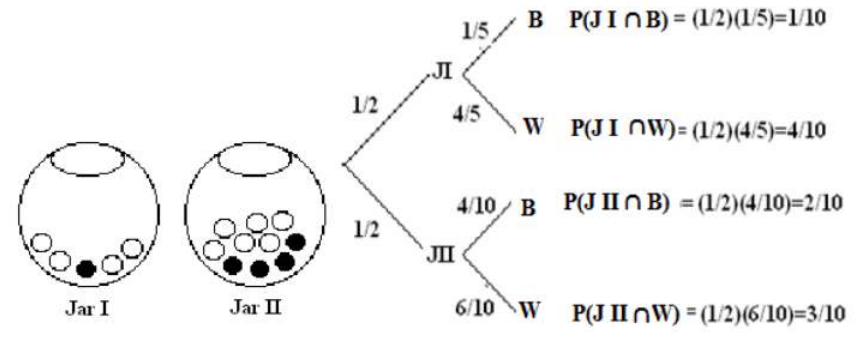
Suppose you are given two jars. Jar I contains one black and 4 white marbles, and Jar II contains 4 black and 6 white marbles. If a jar is selected at random and a marble is chosen,

a. What is the probability that the marble chosen is a black marble?
b. If the chosen marble is black, what is the probability that it came from Jar I?
c. If the chosen marble is black, what is the probability that it came from Jar II?

Solution

Let JI be the event that Jar I is chosen, JII be the event that Jar II is chosen, B be the event that a black marble is chosen and W the event that a white marble is chosen.

We illustrate using a tree diagram.



a. The probability that a black marble is chosen is $P(B)$ = 1/10 + 2/10 = 3/10.
b. To find $P(JI|B)$, we use the definition of conditional probability, and we get

$$P(JI|B) = \frac{P(JI \cap B)}{P(B)} = \frac{1/10}{3/10} = \frac{1}{3}$$

c. Similarly, $\mathrm{P(JII|B)} = \frac{\mathrm{P(JII \cap B)}}{\mathrm{P(B)}} = \frac{2/10}{3/10} = \frac{2}{3}$

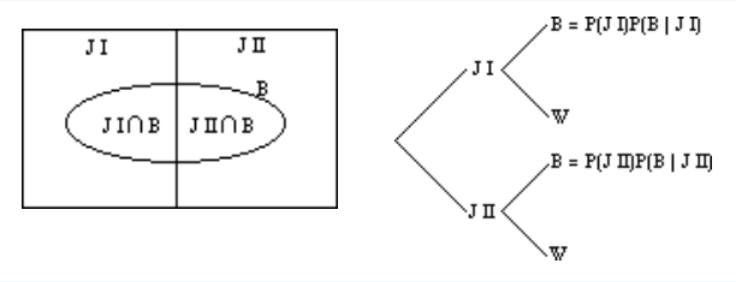
In parts b and c, the reader should note that the denominator is the sum of all probabilities of all branches of the tree that produce a black marble, while the numerator is the branch that is associated with the particular jar in question.

We will soon discover that this is a statement of Bayes' formula .

Let us first visualize the problem.

We are given a sample space S and two mutually exclusive events JI and JII. That is, the two events, JI and JII, divide the sample space into two parts such that $\mathrm{JI} \cup \mathrm{JII} = \mathrm{S}$. Furthermore, we are given an event B that has elements in both JI and

JII, as shown in the Venn diagram below.



From the Venn diagram, we can see that $\mathrm{B} = (\mathrm{B} \cap \mathrm{JI}) \cup (\mathrm{B} \cap \mathrm{JII})$ Therefore:

$$P(B) = P(B \cap JI) + P(B \cap JII) \tag{9.6.1}$$

But the product rule in chapter 7 gives us

$$P(B \cap JI) = P(JI) \cdot P(B|JI) \quad \text{and} \quad P(B \cap JII) = P(JII) \cdot P(B|JII)$$

Substituting in 9.6.1, we get

$$P(B) = P(JI) \cdot P(B|JI) + P(JII) \cdot P(B|JII)$$

The conditional probability formula gives us

$$P(JI|B) = \frac{P(JI \cap B)}{P(B)}$$

Therefore, $P(JI|B) = \frac{P(JI) \cdot P(B|JI)}{P(B)}$

or

$$P(JI|B) = \frac{P(JI) \cdot P(B|JI)}{P(JI) \cdot P(B|JI) + P(JII) \cdot P(B|JII)}$$

The last statement is Bayes' Formula for the case where the sample space is divided into two partitions.

The following is the generalization of Bayes' formula for n partitions.

Bayes' Formula for n partitions

Let S be a sample space that is divided into n partitions, $A_1, A_2, \ldots A_n$. If E is any event in S, then

$$\mathbf{P}(\mathbf{A_i}|\mathbf{E}) = \frac{\mathbf{P}(\mathbf{A_i})\,\mathbf{P}(\mathbf{E}|\mathbf{A_i})}{\mathbf{P}(\mathbf{A_1})\,\mathbf{P}(\mathbf{E}|\mathbf{A_1}) + \mathbf{P}(\mathbf{A_2})\,\mathbf{P}(\mathbf{E}|\mathbf{A_2}) + \cdots + \mathbf{P}(\mathbf{A_n})\,\mathbf{P}(\mathbf{E}|\mathbf{A_n})}$$

We begin with the following example.

Example 9.6.2

A department store buys 50% of its appliances from Manufacturer A, 30% from Manufacturer B, and 20% from Manufacturer C. It is estimated that 6% of Manufacturer A's appliances, 5% of Manufacturer B's appliances, and 4% of Manufacturer C's appliances need repair before the warranty expires. An appliance is chosen at random. If the appliance chosen needed repair before the warranty expired, what is the probability that the appliance was manufactured by Manufacturer A? Manufacturer B? Manufacturer C?

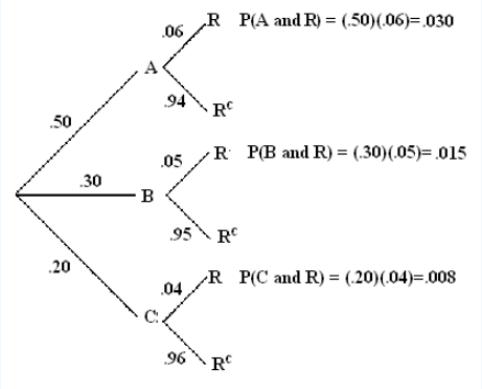
Solution

Let A, B and C be the events that the appliance is manufactured by Manufacturer A, Manufacturer B, and Manufacturer C, respectively. Further, suppose that the event R denotes that the appliance needs repair before the warranty expires.

We need to find P(A | R), P(B | R) and P(C | R).

We will do this problem both by using a tree diagram and by using Bayes' formula.

We draw a tree diagram.



The probability P(A | R), for example, is a fraction whose denominator is the sum of all probabilities of all branches of the tree that result in an appliance that needs repair before the warranty expires, and the numerator is the branch that is associated with Manufacturer A. P(B | R) and P(C | R) are found in the same way.

$$P(A|R) = \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566$$
$$P(B|R) = \frac{.015}{.053} = .283 \text{ and } P(C|R) = \frac{.008}{.053} = .151$$

Alternatively, using Bayes' formula,

$$\begin{aligned} \mathrm{P(A|R)} &= \frac{\mathrm{P(A)P(R|A)}}{\mathrm{P(A)P(R|A)+P(B)P(R|B)+P(C)P(R|C)}} \\ &= \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566 \end{aligned}$$

P(B | R) and P(C | R) can be determined in the same manner.

Example 9.6.3

There are five Jacy's department stores in San Jose. The distribution of number of employees by gender is given in the table below.

Employees and gender in Jacy's department stores

Store Number	Number of Employees	Percent of Women Employees
1	300	.40
2	150	.65
3	200	.60
4	250	.50
5	100	.70
	Total = 1000	

If an employee chosen at random is a woman, what is the probability that the employee works at store 3?

Solution

Let k = 1, 2, . . . , 5 be the event that the employee worked at store k, and W be the event that the employee is a woman. Since there are a total of 1000 employees at the five stores,

$$P(1) = .30 \quad P(2) = .15 \quad P(3) = .20 \quad P(4) = .25 \quad P(5) = .10$$

Using Bayes' formula,

$$\begin{aligned} \mathrm{P}(3|\mathrm{W}) &= \frac{\mathrm{P}(3)\mathrm{P}(\mathrm{W}|3)}{\mathrm{P}(1)\mathrm{P}(\mathrm{W}|1)+\mathrm{P}(2)\mathrm{P}(\mathrm{W}|2)+\mathrm{P}(3)\mathrm{P}(\mathrm{W}|3)+\mathrm{P}(4)\mathrm{P}(\mathrm{W}|4)+\mathrm{P}(5)\mathrm{P}(\mathrm{W}|5)} \\ &= \frac{(.20)(.60)}{(.30)(.40)+(.15)(.65)+(.20)(.60)+(.25)(.50)+(.10)(.70)} \\ &= .2254 \end{aligned}$$

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9.6.1: Bayes' Formula (Exercises)

Exercise 9.6.1.1

Jar I contains five red and three white marbles, and Jar II contains four red and two white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram below, and find the following probabilities.

a. P(marble is red)
b. P(It came from Jar II | marble is white)
c. P(Red | Jar I)

Exercise 9.6.1.2

In Mr. Symons' class, if a student does homework most days, the chance of passing the course is 90%. On the other hand, if a student does not do homework most days, the chance of passing the course is only 20%.
H = event that the student did homework
C = event that the student passed the course
Mr. Symons claims that 80% of his students do homework on a regular basis. If a student is chosen at random from Mr. Symons' class, find the following probabilities.

a. P(C)
b. P(H|C)
c. P(C|H)

Exercise 9.6.1.3

A city has 60% Democrats, and 40% Republicans. In the last mayoral election, 60% of the Democrats voted for their Democratic candidate while 95% of the Republicans voted for their candidate. Which party's mayor runs city hall?

Exercise 9.6.1.4

In a certain population of 48% men and 52% women, 56% of the men and 8% of the women are color-blind.

a. What percent of the people are color-blind?
b. If a person is found to be color-blind, what is the probability that the person is a male?

Exercise 9.6.1.5

A test for a certain disease gives a positive result 95% of the time if the person actually carries the disease. However, the test also gives a positive result 3% of the time when the individual is not carrying the disease. It is known that 10% of the population carries the disease. If a person tests positive, what is the probability that he or she has the disease?

Exercise 9.6.1.6

A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair?

Exercise 9.6.1.7

A computer company buys its chips from three different manufacturers. Manufacturer I provides 60% of the chips and is known to produce 5% defective; Manufacturer II supplies 30% of the chips and makes 4% defective; while the rest are supplied by Manufacturer III with 3% defective chips. If a chip is chosen at random, find the following probabilities:

a. P(the chip is defective)
b. P(chip is from Manufacturer II | defective)

c. P(defective |chip is from manufacturer III)

Exercise 9.6.1.8

Lincoln Union High School District is made up of three high schools: Monterey, Fremont, and Kennedy, with an enrollment of 500, 300, and 200, respectively. On a given day, the percentage of students absent at Monterey High School is 6%, at Fremont 4%, and at Kennedy 5%. If a student is chosen at random, find the probabilities below: *Hint: Convert the enrollments into percentages.*

a. P(the student is absent)
b. P(student is from Kennedy | student is absent)
c. P(student is absent | student is from Fremont)

Exercise 9.6.1.9

At a retail store, 20% of customers use the store's online app to assist them when shopping in the store ; 80% of store shoppers don't use the app.

Of those customers that use the online app while in the store, 50% are very satisfied with their purchases, 40% are moderately satisfied, and 10% are dissatisfied.

Of those customers that do not use the online app while in the store, 30% are very satisfied with their purchases, 50% are moderately satisfied and 20% are dissatisfied.

Indicate the events by the following:

A = shopper uses the app in the store
N = shopper does not use the app in the store
V = very satisfied with purchase
M = moderately satisfied
D = dissatisfied

a. Find P(A and D), the probability that a store customer uses the app and is dissatisfied

b. Find P(A|D), the probability that a store customer uses the app if the customer is dissatisfied.

Exercise 9.6.1.10

A medical clinic uses a pregnancy test to confirm pregnancy in patients who suspect they are pregnant. Historically data has shown that overall, 70% of the women at this clinic who are given the pregnancy test are pregnant, but 30% are not.

The test's manufacturer indicates that if a woman is pregnant, the test will be positive 92% of the time.

But if a woman is not pregnant, the test will be positive only 2% of the time and will be negative 98% of the time.

a. Find the probability that a woman at this clinic is pregnant and tests positive.

b. Find the probability that a woman at this clinic is actually pregnant given that she tests positive.

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9.7: Binomial Probability

Learning Objectives

In this section, you will learn to:

1. Recognize when to use the binomial probability distribution
2. Derive the formula for the binomial probability distribution
3. Calculate probabilities for a binomial probability experiment

In this section, we consider problems that involve a sequence of trials, where each trial has only two outcomes, a *success* or a *failure*. These trials are independent, that is, the outcome of one does not affect the outcome of any other trial. The probability of success, p, and the probability of failure, $(1-p)$, remains the same throughout the experiment. These problems are called **binomial probability** problems. Since these problems were researched by Swiss mathematician Jacques Bernoulli around 1700, they are also called **Bernoulli trials.**

We give the following definition:

Binomial Experiment

A binomial experiment satisfies the following four conditions:

1. There are only two outcomes, a success or a failure, for each trial.
2. The same experiment is repeated several times.
3. The trials are independent; that is, the outcome of a particular trial does not affect the outcome of any other trial.
4. The probability of success remains the same for every trial.

This probability model that will give us the tools to solve many real-life problems , such as:

1. If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?
2. If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 7 out of 10 free throws in a game?
3. If a medicine cures 80% of the people who take it, what is the probability that among the ten people who take the medicine, 6 will be cured?
4. If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?
5. If a telemarketing executive has determined that 15% of the people contacted will purchase the product, what is the probability that among the 12 people who are contacted, 2 will buy the product?

We now consider the following example to develop a formula for finding the probability of k successes in n Bernoulli trials.

Example 9.7.1

A baseball player has a batting average of .300. If he bats four times in a game, find the probability that he will have

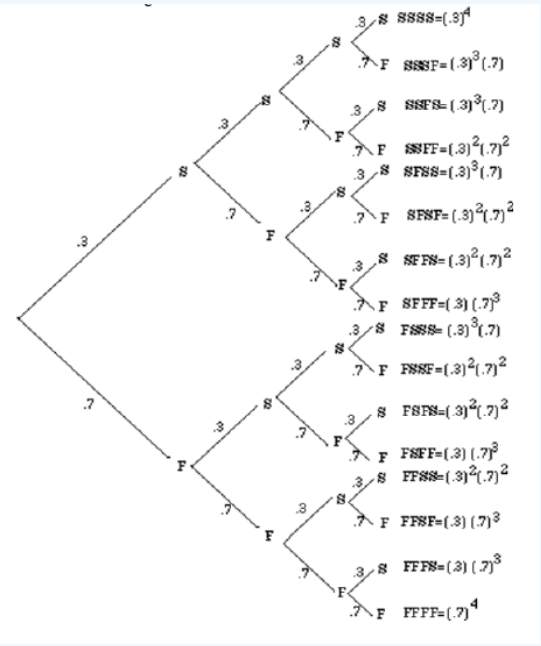
a. 4 hits
b. 3 hits
c. 2 hits
d. 1 hit
e. no hits.

Solution

Let S denote that the player gets a hit, and F denote that he does not get a hit.

This is a binomial experiment because it meets all four conditions. First, there are only two outcomes, S or F. Clearly the experiment is repeated four times. Lastly, if we assume that the player's skillfulness to get a hit does not change each time he comes to bat, the trials are independent with a probability of .3 of getting a hit during each trial.

We draw a tree diagram to show all situations.



Let us first find the probability of getting, for example, two hits. We will have to consider the six possibilities, SSFF, SFSF, SFFS, FSSF, FSFS, FFSS, as shown in the above tree diagram. We list the probabilities of each below.

$$\begin{aligned}
\mathrm{P(SSFF)} &= (.3)(.3)(.7)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFSF)} &= (.3)(.7)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFFS)} &= (.3)(.7)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FSSF)} &= (.7)(.3)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(FSFS)} &= (.7)(.3)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FFSS)} &= (.7)(.7)(.3)(.3) = (.3)^2(.7)^2
\end{aligned}$$

Since the probability of each of these six outcomes is $(.3)^2(.7)^2$, the probability of obtaining two successes is $6(.3)^2(.7)^2$.

The probability of getting one hit can be obtained in the same way. Since each permutation has one S and three F's, there are four such outcomes: SFFF, FSFF, FFSF, and FFFS.

And since the probability of each of the four outcomes is $(.3)(.7)^3$, the probability of getting one hit is $4(.3)(.7)^3$.

The table below lists the probabilities for all cases, and shows a comparison with the binomial expansion of fourth degree. Again, p denotes the probability of success, and $q = (1-p)$ the probability of failure.

Probabilities in binomial form

Outcome	Four Hits	Three Hits	Two Hits	One Hit	No Hits
Probability	$(.3)^4$	$4(.3)^3(.7)$	$6(.3)^2(.7)^2$	$4(.3)(.7)^3$	$(.7)^4$

$$\begin{aligned}
(.3+.7)^4 &= (.3)^4 + 4(.3)^3(.7) + 6(.3)^2(.7)^2 + 4(.3)(.7)^3 + (.7)^4 \\
(p+q)^4 &= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4
\end{aligned}$$

This gives us the following theorem:

Binomial Probability Theorem

The probability of obtaining k successes in n independent Bernoulli trials is given by

$$\mathbf{P(n,k;p) = nCk\ p^k q^{n-k}}$$

where p denotes the probability of success and $q = (1 - p)$ the probability of failure.

We use the binomial probability formula to solve the following examples.

Example 9.7.2

If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?

Solution

Let S denote the probability of obtaining a head, and F the probability of getting a tail.

Clearly, $n = 10$, $k = 3$, $p = 1/2$, and $q = 1/2$.

Therefore, $\mathrm{b}(10, 3; 1/2) = 10\mathrm{C}3\ (1/2)^3(1/2)^7 = .1172$

Example 9.7.3

If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 6 out of 10 free throws in a game?

Solution

The probability of making a free throw is 3/4.
Therefore, $p = 3/4$, $q = 1/4$, $n = 10$, and $k = 6$.

Therefore, $b(10, 6; 3/4) = 10\mathrm{C}6\ (3/4)^6(1/4)^4 = .1460$

Example 9.7.4

If a medicine cures 80% of the people who take it, what is the probability that of the eight people who take the medicine, 5 will be cured?

Solution

Here $p = .80$, $q = .20$, $n = 8$, and $k = 5$.

$$b(8, 5; .80) = 8\mathrm{C}5\ (.80)^5(.20)^3 = .1468$$

Example 9.7.5

If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?

Solution

If S denotes the probability that the chip is defective, and F the probability that the chip is not defective, then $p = .04$, $q = .96$, $n = 60$, and $k = 3$.

$$b(60, 3; .04) = 60\mathrm{C}3\ (.04)^3(.96)^{57} = .2138$$

Example 9.7.6

A telemarketing executive has determined that 15% of people contacted will purchase the product. 12 people are contacted about this product.

a. Find the probability that among 12 people contacted, 2 will buy the product.
b. Find the probability that among 12 people contacted, at most 2 will buy the product?

Solution

a. If S denoted the probability that a person will buy the product, and F the probability that the person will not buy the product, then $p = .15$, $q = .85$, $n = 12$, and $k = 2$.

$$b(12, 2; .15) = 12C2\ (.15)^2(.85)^{10} = .2924.$$

The probability that 2 people buy the product is 0.2924.

b. Again $p = .15$, $q = .85$, $n = 12$. But to find the probability that **at most 2** buy the product, we need to find the probabilities for $k = 0$, $k = 1$, $k = 2$ and add them together.

$$b(12, 0; .15) = 12C0\ (.15)^0(.85)^{12} = .1422$$
$$b(12, 1; .15) = 12C1\ (.15)^1(.85)^{11} = .3012$$

Adding all three probabilities gives: .1422 +0.3012 +.2924 = .7358

The probability that at most 2 people buy the product is 0.7358.

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9.7.1: Binomial Probability (Exercises)

Do the following problems using the binomial probability formula.

1. A coin is tossed ten times. Find the probability of getting six heads and four tails.
2. A family has three children. Find the probability of having one boy and two girls.
3. What is the probability of getting three aces(ones) if a die is rolled five times?
4. A baseball player has a .250 batting average. What is the probability that he will have three hits in five times at bat?
5. A basketball player has an 80% chance of sinking a basket on a free throw. What is the probability that he will sink at least three baskets in five free throws?
6. With a new flu vaccination, 85% of the people in the high risk group can go through the entire winter without contracting the flu. In a group of six people who were vaccinated with this drug, what is the probability that 1 does contract the flu?
7. A transistor manufacturer has known that 5% of the transistors produced are defective. What is the probability that a batch of twenty five transistors will have two defective?
8. It has been determined that only 80% of the people wear seat belts. If a police officer stops a car with four people, what is the probability that at least one person will not be wearing a seat belt?
9. What is the probability that a family of five children will have at least three boys?
10. What is the probability that a toss of four coins will yield at most two heads?
11. A telemarketing executive has determined that for a particular product, 20% of the people contacted will purchase the product. If 10 people are contacted, what is the probability that at most 2 will buy the product?
12. To the problem: "Five cards are dealt from a deck of cards, find the probability that three of them are kings," the following incorrect answer was offered by a student.

    $$5C3\ (1/13)^3(12/13)^2$$

    What change would you make in the wording of the problem for the given answer to be correct?
13. 63% of all registered voters in a large city voted in the last election. 20 registered voters from this city are randomly selected. Find the probability that
    a. exactly half of them voted in the last election
    b. all of them voted
14. 30% of customers at BigMart pay cash for their purchases. Suppose that 15 customers are randomly selected. Find the probability that
    a. 5 or 6 of them pay cash
    b. at most 1 pays cash
15. 12% of all cars on Brighton Expressway exceed the speed limit. If 10 vehicles on this road are randomly selected and their speed is recorded by radar, find the probability that
    a. none of them are exceeding the speed limit
    b. 1 or 2 are exceeding the speed limit
16. Suppose that 73% of all people taking a professional certification exam pass the exam. If 12 people who take this exam are randomly selected, find the probability that
    a. exactly half of them pass the exam
    b. all of them pass the exam
    c. 8 or 9 of them pass the exam

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9.8: Expected Value

Learning Objectives

In this section, you will learn to:

1. Find the expected value of a discrete probability distribution
2. Interpret expected value as a long-run average

An expected gain or loss in a game of chance is called **Expected Value**. The concept of expected value is closely related to a *weighted average*. Consider the following situations.

1. Suppose you and your friend play a game that consists of rolling a die. Your friend offers you the following deal: If the die shows any number from 1 to 5, he will pay you the face value of the die in dollars, that is, if the die shows a 4, he will pay you $4. But if the die shows a 6, you will have to pay him $18.

Before you play the game you decide to find the expected value. You analyze as follows.

Since a die will show a number from 1 to 6, with an equal probability of 1/6, your chance of winning $1 is 1/6, winning $2 is 1/6, and so on up to the face value of 5. But if the die shows a 6, you will lose $18. You write the expected value.

$$\mathrm{E} = \$1(1/6) + \$2(1/6) + \$3(1/6) + \$4(1/6) + \$5(1/6) - \$18(1/6) = -\$.50$$

This means that every time you play this game, you can expect to lose 50 cents. In other words, if you play this game 100 times, theoretically you will lose $50. Obviously, it is not to your interest to play.

2. Suppose of the ten quizzes you took in a course, on eight quizzes you scored 80, and on two you scored 90. You wish to find the average of the ten quizzes.
The average is

$$\mathrm{A} = \frac{(80)(8)+(90)(2)}{10} = (80)\frac{8}{10} + (90)\frac{2}{10} = 82$$

It should be observed that it would be incorrect to take the average of 80 and 90 because you scored 80 on eight quizzes, and 90 on only two of them. Therefore, you take a "weighted average" of 80 and 90. That is, the average of 8 parts of 80 and 2 parts of 90, which is 82.

In the first situation, to find the expected value, we multiplied each payoff by the probability of its occurrence, and then added up the amounts calculated for all possible cases. In the second example, if we consider our test score a payoff, we did the same. This leads us to the following definition.

Expected Value

If an experiment has the following probability distribution,

Payoff	x_1	x_2	x_3	$\ldots$	x_n
Probability	$p(x_1)$	$p(x_2)$	$p(x_3)$	$\ldots$	$p(x_n)$

then the expected value of the experiment is

$$\text{Expected Value } = x_1 p\left(x_1\right) + x_2 p\left(x_2\right) + x_3 p\left(x_3\right) + \cdots + x_n p\left(x_n\right)$$

Example 9.8.1

In a town, 10% of the families have three children, 60% of the families have two children, 20% of the families have one child, and 10% of the families have no children. What is the expected number of children to a family?

Solution

We list the information in the following table.

Number of children	3	2	1	0
Probability	0.10	0.60	0.20	0.10

Expected Value = $x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4)$

$E = 3(.10) + 2(.60) + 1(.20) + 0(.10) = 1.7$

So on average, there are 1.7 children to a family.

Example 9.8.2

To sell an average house, a real estate broker spends $1200 for advertisement expenses. If the house sells in three months, the broker makes $8,000. Otherwise, the broker loses the listing. If there is a 40% chance that the house will sell in three months, what is the expected payoff for the real estate broker?

Solution

The broker makes $8,000 with a probability of .40, but he loses $1200 whether the house sells or not.

$$E = (\$8000)(.40) - (\$1200) = \$2,000.$$

Alternatively, the broker makes $(8000 - 1200) with a probability of .40, but loses $1200 with a probability of .60. Therefore,

$$E = (\$6800)(.40) - (\$1200)(.60) = \$2,000.$$

Example 9.8.3

In a town, the attendance at a football game depends on the weather. On a sunny day the attendance is 60,000, on a cold day the attendance is 40,000, and on a stormy day the attendance is 30,000. If for the next football season, the weatherman has predicted that 30% of the days will be sunny, 50% of the days will be cold, and 20% days will be stormy, what is the expected attendance for a single game?

Solution

Using the expected value formula, we get

$$E = (60,000)(.30) + (40,000)(.50) + (30,000)(.20) = 44,000.$$

Example 9.8.4

A lottery consists of choosing 6 numbers from a total of 51 numbers. The person who matches all six numbers wins $2 million. If the lottery ticket costs $1, what is the expected payoff?

Solution

Since there are $51C6 = 18,009,460$combinations of six numbers from a total of 51 numbers, the chance of choosing the winning number is 1 out of 18,009,460.

So the expected payoff is: $\mathrm{E} = (\$2 \text{ million }) \left(\frac{1}{18009460}\right) - \$1 = -\$0.89$

This means that every time a person spends $1 to buy a ticket, he or she can expect to lose 89 cents.

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9.8.1: Expected Value (Exercises)

Solve the following problems using the expected value concepts learned in this section.

1. You are about to make an investment which gives you a 30% chance of making $60,000 and 70% chance of losing $ 30,000. Should you invest? Explain.
2. In a town, 40% of the men and 30% of the women are overweight. If the town has 46% men and 54% women, what percent of the people are overweight?
3. A game involves rolling a Korean die (4 faces). If a one, two, or three shows, the player receives the face value of the die in dollars, but if a four shows, the player is obligated to pay $4. What is the expected value of the game?
4. A game involves rolling a single die. One receives the face value of the die in dollars. How much should one be willing to pay to roll the die to make the game fair?
5. In a European country, 20% of the families have three children, 40% have two children, 30% have one child, and 10% have no children. On average, how many children are there to a family?
6. A game involves drawing a single card from a standard deck. One receives 60 cents for an ace, 30 cents for a king, and 5 cents for a red card that is neither an ace nor a king. If the cost of each draw is 10 cents, should one play? Explain.
7. Hillview Church plans to raise money by raffling a television worth $500. A total of 3000 tickets are sold at $1 each. Find the expected value of the winnings for a person who buys a ticket in the raffle.
8. During her four years at college, Niki received A's in 30% of her courses, B's in 60% of her courses, and C's in the remaining 10%. If A = 4, B = 3, and C = 2, find her grade point average.
9. Attendance at a Stanford football game depends upon which team Stanford is playing against. If the game is against U. C. Berkeley,attendance will be 70,000; if it is against another California team, it will be 40,000; and if it is against an out of state team, it will be 30,000. If the probability of playing against U. C. Berkeley is 10%, against a California team 50% and against an out of state team 40%, how many fans are expected to attend a game?
10. A Texas oil drilling company has determined that it costs $25,000 to sink a test well. If oil is hit, the revenue for the company will be $500,000. If natural gas is found, the revenue will be $150,000. If the probability of hitting oil is 3% and of hitting gas is 6%, find the expected value of sinking a test well.
11. A $1 lottery ticket offers a grand prize of $10,000; 10 runner-up prizes each pay $1000; 100 third-place prizes each pay $100; and 1,000 fourth-place prizes each pay $10. Find the expected value of entering this contest if 1 million tickets are sold.
12. Assume that for the next heavyweight fight the odds of current champion winning are 15 to 2. A gambler bets $10 that the current champion will lose. If current champion loses, how much can the gambler hope to receive?
13. In a housing development, 35% of households have no school age children, 20% of households have 1 school age child, 25% of households have 2 school age children, 15% have 3, and 5% have 4 school age children.
    a. Find the average number of children per household
    b. If there are 300 homes in this housing development, what is the total number of children expected to attend school?
14. At a large community college, 30% of students take one course, 15% take two courses, 25% take three courses and 20% take four courses. The rest of the students take five courses.
    a. What percent of students take 5 courses?
    b. Find the average number of courses that students take.

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