INTEGRATED MATH FOR TECHNICAL FIELDS

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Integrated Math for Technical Fields

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Instructor Guide

This book is a remix of several other texts available via Libretexts. The intended users of the book will be students pursuing certificates, degrees, or coursework in technical fields. At Imperial Valley College, semesters are usually 17 weeks, including a break (such as Spring Break or Thanksgiving) and finals week. The following pacing guide will include 16 weeks, and the respective break can be inserted as needed.

Week	Topics/Chapters	Assessments
Week 1	Chapter 1: The Whole Numbers	
Week 2	Chapter 2: Fractions	
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Week 15	Review for Final Exam	
Week 16	Final Exam	Final Exam

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CHAPTER OVERVIEW

1: The Whole Numbers

Welcome to the study of prealgebra. In this first chapter of study, we will introduce the set of natural numbers, then follow with the set of whole numbers. We will then follow with a quick review of addition, subtraction, multiplication, and division skills involving whole numbers that are prerequisite for success in the study of prealgebra. Along the way we will introduce a number of properties of the whole numbers and show how that can be used to evaluate expressions involving whole number operations. We will also define what is meant by *prime* and *composite* numbers, discuss a number of divisibility tests, then show how any composite number can be written uniquely as a product of prime numbers. This will lay the foundation for requisite skills with fractional numbers in later chapters. Finally, we will introduce the concept of a *variable*, then introduce equations and technique required for their solution. We will use equations to model and solve a number of real-world applications along the way.

Let's begin the journey.

1.1: An Introduction to the Whole Numbers
1.2: Adding and Subtracting Whole Numbers
1.3: Multiplication and Division of Whole Numbers
1.4: Prime Factorization
1.5: Order of Operations

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1.1: An Introduction to the Whole Numbers

A *set* is a collection of objects. If the set is finite, we can describe the set completely by simply listing all the objects in the set and enclosing the list in curly braces. For example, the set

$$S = \{dog,\ \text{cat},\ \text{parakeet}\}$$

is the set whose members are “dog”, “cat”, and “parakeet.” If the set is infinite, then we need to be more clever with our description. For example, the set of *natural numbers* (or *counting numbers*) is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}.$$

Because this set is infinite (there are an infinite number of natural numbers), we can’t list all of them. Instead, we list the first few then follow with “three dots,” which essentially mean “etcetera.” The implication is that the reader sees the intended pattern and can then intuit the remaining numbers in the set. Can you see that the next few numbers are 6, 7, 8, 9, etc.?

If we add the number zero to the set of natural numbers, then we have a set of numbers that are called the *whole numbers*.

The Whole Numbers

The set

$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

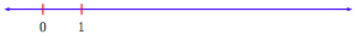
is called the set of *whole numbers*.

The whole numbers will be our focus in the remainder of this chapter.

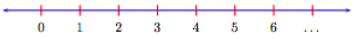
Graphing Numbers on the Number Line

It is a simple matter to set up a correspondence between the whole numbers and points on a **number line**. First, draw a number line, then set a tick mark at zero.

The next step is to declare a unit length.



The remainder of the whole numbers now fall easily in place on the number line.



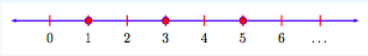
When asked to graph a whole number on a number line, shade in a solid dot at the position on the number line that corresponds to the given whole number.

Example 1.

Graph the whole numbers 1, 3, and 5 on the number line.

Solution

Shade the numbers 1, 3, and 5 on the number line as solid dots.



Exercise

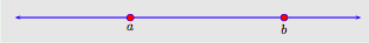
Graph the whole numbers 3, 4, and 6 on the number line.

Ordering the whole numbers

Now that we have a correspondence between the whole numbers and points on the number line, we can order the whole numbers in a natural way. Note that as you move to the left along the number line, the numbers get smaller; as you move to the right, the numbers get bigger. This inspires the following definition.

Ordering the Whole Numbers

Suppose that a and b are whole numbers located on the number line so that the point representing the whole number a lies to the left of the point representing the whole number b.



Then the whole number a is "less than" the whole number b and write

$$a < b.$$

Alternatively, we can also say that the whole number b is "greater than" the whole number a and write

$$b > a.$$

Comparison Property

When comparing two whole numbers a and b, only one of three possibilities is true:

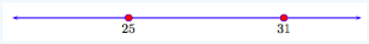
$$a < b \text{ or } a = b \text{ or } a > b.$$

Example 2.

Compare the whole numbers 25 and 31.

Solution

On the number line, 25 is located to the left of 31.



Therefore, 25 is less than 31 and write $25 < 31$. Alternatively, we could also note that 31 is located to the right of 25. Therefore, 31 is greater than 25 and write $31 > 25$.

Exercise

Compare the whole numbers 18 and 12.

Answer

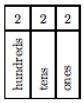
$18 > 12$.

Expanded Notation

The whole numbers

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

are called *digits* and are used to construct larger whole numbers. For example, consider the whole number 222 (pronounced "two hundred twenty two"). It is made up of three twos, but the position of each two describes a different meaning or value.



- The first two is in the "hundreds" position and represents two hundreds or 200.

- The second two is in the “tens” position and represents two tens or 20.
- The third two is in the “ones” position and represents two ones or 2.

Consider the larger number 123,456,789. The following table shows the place value of each digit.

1	2	3	4	5	6	7	8	9
hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones
millions			thousands			ones		

In “expanded notation,” we would write

1 hundred million + 2 ten millions + 3 millions + 4 hundred thousands + 5 ten thousands + 6 thousands + 7 hundreds + 8 tens + 9 ones.

We read the numeral 123,456,789 as “one hundred twenty three million, four hundred fifty six thousand, seven hundred eighty nine.”

Let’s look at another example.

Example 3.

Write the number 23,712 in expanded notation, then pronounce the result.

Solution

In expanded notation, 23,712 becomes

2 ten thousands + 3 thousands + 7 hundreds + 1 ten + 2 ones.

This is pronounced "twenty three thousand, seven hundred twelve."

Exercise

Write the number 54,615 in expanded notation. Pronounce the result.

Example 4.

Write the number 203,405 in expanded notation, then pronounce the result.

Solution

In expanded notation, 203,405 becomes

2 hundred thousands + 0 ten thousands + 3 thousands + 4 hundreds + 0 tens + 5 ones.

Since 0 ten thousands is zero and 0 tens is also zero, this can also be written

2 hundred thousands + 3 thousands + 4 hundreds + 5 ones.

This is pronounced “two hundred three thousand, four hundred five.”

Exercise

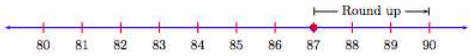
Write the number 430,705 in expanded notation. Pronounce the result.

Answer

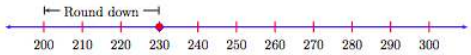
9,440

Rounding whole numbers

When less precision is needed, we round numbers to a particular place. For example, suppose a store owner needs approximately 87 boxes of ten-penny nails, but they can only be ordered in cartons containing ten boxes.



Note that 87 is located closer to 9 tens (or 90) than it is to 8 tens (or 80). Thus, rounded to the nearest ten, 87 ≈ 90 (87 approximately equals 90). The store owner decides that 90 boxes is probably a better fit for his needs. On the other hand, the same store owner estimates that he will need 230 bags of peatmoss for his garden section.



Note that 230 is closer to 2 hundreds (or 200) than it is to 3 hundreds (or 300). The store owner worries that might have overestimated his need, so he rounds down to the nearest hundred, 230 ≈ 200 (230 approximately equals 200). There is a simple set of rules to follow when rounding.

Rules for Rounding

To round a number to a particular place, follow these steps:

1. Mark the place you wish to round to. This is called the rounding digit.

2. Check the next digit to the right of your digit marked in step 1. This is called the test digit.

    a) If the test digit is greater than or equal to 5, add 1 to the rounding digit and replace all digits to the right of the rounding digit with zeros.

    b) If the test digit is less than 5, keep the rounding digit the same and replace all digits to the right of rounding digit with zero

Let's try these rules with an example or two.

Example 5

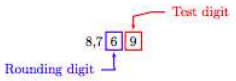
Round the number 8,769 to the nearest ten.

Solution

Mark the rounding and test digits.

Exercise

Round the number 9,443 to the nearest ten.



The test digit is greater than 5. The “Rules for Rounding” require that we add 1 to the rounding digit, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest ten,

$$8,769 \approx 8,770.$$

That is, 8,769 is *approximately* equal to 8,770.

Mathematical Notation.

The symbol

$$\approx$$

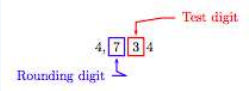
means *approximately equal*.

Example 6.

Round the number 4,734 to the nearest hundred.

Solution

Mark the rounding and test digits.



The test digit is less than 5. The "Rules for Rounding" require that we keep the rounding digit the same, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest hundred,

$$4,734 \approx 4,700.$$

Exercise

Round the number 6,656 to the nearest hundred.

Answer

6,700

Table 1.1: Atmospheric CO_2 values (ppmv) derived from in situ air samples collected at Mauna Loa, Hawaii, USA.

Year	1965	1975	1985	1995	2005
Atmospheric CO_2	319	330	344	359	378

Tables and Graphs

Reading data in graphical form is an important skill. The data in Table 1.1 provides measures of the carbon dioxide content (CO_2) in the atmosphere, gathered in the month of January at the observatory atop Mauna Loa in Hawaii.

In Figure 1.1(a), a *bar graph* is used to display the carbon dioxide measurements. The year the measurement was taken is placed on the horizontal axis, and the height of each bar equals the amount of carbon dioxide in the atmosphere during that year.

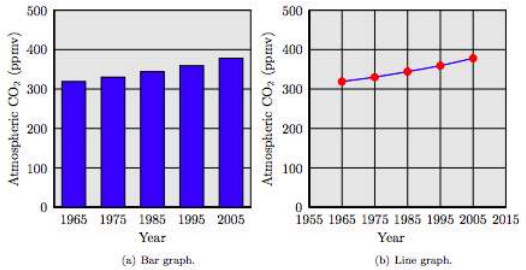


Figure 1.1: Using graphs to examine carbon dioxide data.

In Figure 1.1(b), a *line graph* is used to display the carbon dioxide measurements. Again, the dates of measurement are placed on the horizontal axis, and the amount of carbon dioxide in the atmosphere is placed on the vertical axis. Instead of using the height of a bar to represent the carbon dioxide measurement, we place a dot at a height that represents the carbon monoxide content. Once each data point is plotted, we connect consecutive data points with line segments.

Exercises

In Exercises 1-12, sketch the given whole numbers on a number line, then arrange them in order, from smallest to largest.

1. 2, 8, and 4
2. 2, 7, and 4
3. 1, 8, and 2
4. 0, 4, and 3
5. 0, 4, and 1
6. 3, 6, and 5
7. 4, 9, and 6
8. 2, 4, and 3
9. 0, 7, and 4
10. 2, 8, and 6
11. 1, 6, and 5
12. 0, 9, and 5

In Exercises 13-24, create a number line diagram to determine which of the two given statements is true.

13. $3 < 8$ or $3 > 8$
14. $44 < 80$ or $44 > 80$
15. $59 < 24$ or $59 > 24$
16. $15 < 11$ or $15 > 11$
17. $0 < 74$ or $0 > 74$
18. $11 < 18$ or $11 > 18$
19. $1 < 81$ or $1 > 81$
20. $65 < 83$ or $65 > 83$
21. $43 < 1$ or $43 > 1$
22. $62 < 2$ or $62 > 2$
23. $43 < 28$ or $43 > 28$
24. $73 < 21$ or $73 > 21$

---

25. Which digit is in the thousands column of the number 2,054,867,372?
26. Which digit is in the hundreds column of the number 2,318,999,087?
27. Which digit is in the hundred thousands column of the number 8,311,900,272?
28. Which digit is in the tens column of the number 1,143,676,212?
29. Which digit is in the hundred millions column of the number 9,482,616,000?
30. Which digit is in the hundreds column of the number 375,518,067?
31. Which digit is in the ten millions column of the number 5,840,596,473?
32. Which digit is in the hundred thousands column of the number 6,125,412,255?
33. Which digit is in the hundred millions column of the number 5,577,422,501?
34. Which digit is in the thousands column of the number 8,884,966,835?

35. Which digit is in the tens column of the number 2,461,717,362?

36. Which digit is in the ten millions column of the number 9,672,482,548?

---

37. Round the number 93, 857 to the nearest thousand.

38. Round the number 56, 872 to the nearest thousand.

39. Round the number 9, 725 to the nearest ten.

40. Round the number 6, 815 to the nearest ten.

41. Round the number 58, 739 to the nearest hundred.

42. Round the number 93, 146 to the nearest hundred.

43. Round the number 2, 358 to the nearest ten.

44. Round the number 8, 957 to the nearest ten.

45. Round the number 39, 756 to the nearest thousand.

46. Round the number 24, 965 to the nearest thousand.

47. Round the number 5, 894 to the nearest ten.

48. Round the number 3, 281 to the nearest ten.

49. Round the number 56, 123 to the nearest hundred.

50. Round the number 49, 635 to the nearest hundred.

51. Round the number 5, 483 to the nearest ten.

52. Round the number 9, 862 to the nearest ten.

---

53. According to the U.S. Census Bureau, the estimated population of the US is 304,059,724 as of July 2008. Round to the nearest hundred thousand.

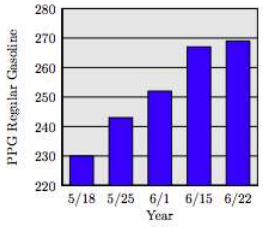
54. According to the U.S. Census Bureau, the estimated population of California is 36,756,666 as of July 2008. Round to the nearest hundred thousand.

55. According to the U.S. Census Bureau, the estimated population of Humboldt County is 129,000 as of July 2008. Round to the nearest ten thousand.

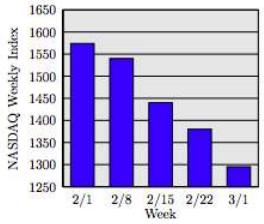
56. According to the U.S. Census Bureau, the estimated population of the state of Alaska was 686,293 as of July 2008. Round to the nearest ten thousand.

---

57. The following bar chart shows the average price (in cents) of one gallon of regular gasoline in the United States over five consecutive weeks in 2009, running from May 18 (5/18) through June 22 (6/22). What was the price (in cents) of one gallon of regular gasoline on June 1, 2009?



58. The following bar chart shows the average weekly NASDAQ index for five consecutive weeks in 2009, beginning with week starting February 1 (2/1) and ending with the week starting March 1 (3/1). What was the average NASDAQ index for the week starting February 8, 2009?



59. The population of Humboldt County is broken into age brackets in the following table. *Source: WolframAlpha*.

Age in years	Number
under 5	7,322
5 – 18	26,672
18 – 65	78,142
over 65	16,194

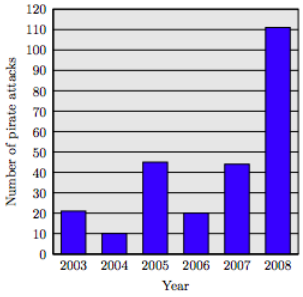
Create a bar chart for this data set with one bar for each age category.

60. The five cities with the largest number of reported violent crimes in the year 2007 are reported in the following table. *Source: Wikipedia*.

City	Violent Crimes
Setroit	2,289
St. Louis	2,196
Memphis	1,951
Oakland	1,918
Baltimore	1,631

Create a bar chart for this data set with one bar for each city.

61. The following bar chart tracks pirate attacks off the coast of Somalia.

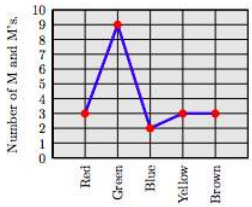


Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003?

b) How many pirate attacks were there in 2008?

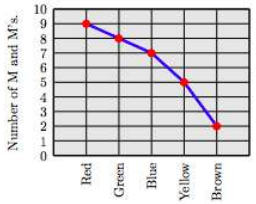
62. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



https://math.libretexts.org/@go/page/196378

How many red M and M's were in the bowl?

63. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



How many red M and M's were in the bowl?

---

64. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of each color.

Color	Number
Red	5
Green	9
Blue	7
Yellow	2
Brown	3

Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

65. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of every color.

Color	Number
Red	3
Green	7
Blue	2
Yellow	4
Brown	9

Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

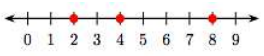
---

66. *Salmon count*. The table shows the number of adult coho salmon returning to the Shasta River over the past four years. Round the salmon count for each year to the nearest ten. *Times-Standard Shasta River coho rescue underway.*

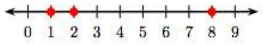
Year	Salmon count
2007	300
2008	31
2009	9
2010	4

Answers

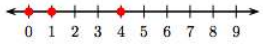
1. Smallest to largest: 2, 4, and 8.



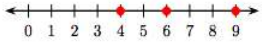
3. Smallest to largest: 1, 2, and 8.



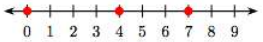
5. Smallest to largest: 0, 1, and 4.



7. Smallest to largest: 4, 6, and 9.



9. Smallest to largest: 0, 4, and 7.



11. Smallest to largest: 1, 5, and 6.

13.

Therefore, $3 < 8$.

15.

Therefore, $59 > 24$.

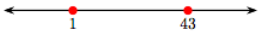
17.

Therefore, $0 < 74$.

19.

Therefore, $1 < 81$.

21.



Therefore, $43 > 1$.

23.

Therefore, $43 > 28$.

25. 7

27. 9

29. 4

31. 4

33. 5

35. 6

37. 94000

39. 9730

41. 58700

43. 2360

45. 40000

47. 5890

49. 56100

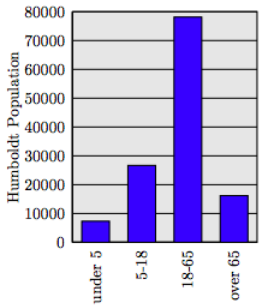
51. 5480

53. 304,100,000

55. 130,000

57. Approximately 252 cents

59.



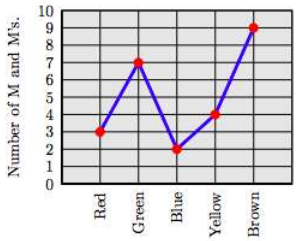
61.

a) Approximately 21

b) Approximately 111

63. 9

65.



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1.2: Adding and Subtracting Whole Numbers

In the expression 3 + 4, which shows the sum of two whole numbers, the whole numbers 3 and 4 are called *addends* or *terms*. We can use a visual approach to find the sum of 3 and 4. First, construct a number line as shown in Figure 1.2.

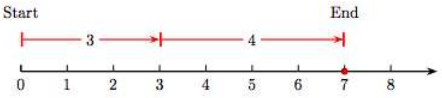


Figure 1.2: Adding whole numbers on the number line.

To add 3 and 4, proceed as follows.

1. Start at the number 0, then draw an arrow 3 units to the right, as shown in Figure 1.2. This arrow has magnitude (length) three and represents the whole number 3.
2. Draw a second arrow of length four, starting at the end of the first arrow representing the number 3. This arrow has magnitude (length) four and represents the whole number 4.
3. The sum of 3 and 4 could be represented by an arrow that starts at the number 0 and ends at the number 7. However, we prefer to mark this sum on the number line as a solid dot at the whole number 7. This number represents the sum of the whole numbers 3 and 4.

The Commutative Property of Addition

Let's change the order in which we add the whole numbers 3 and 4. That is, let's find the sum 4 + 3 instead.

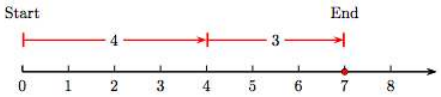


Figure 1.3: Addition is commutative; i.e., order doesn't matter.

As you can see in Figure 1.3, we start at zero then draw an arrow of length four, followed by an arrow of length three. However, the result is the same; i.e., 4 + 3 = 7.

Thus, the order in which we add three and four does not matter; that is,

$$3 + 4 = 4 + 3.$$

This property of addition of whole numbers is known as the *commutative property* of addition.

The Commutative Property of Addition

Let a and b represent two whole numbers. Then,

$$a + b = b + a.$$

Grouping Symbols

In mathematics, we use *grouping symbols* to affect the order in which an expression is evaluated. Whether we use parentheses, brackets, or curly braces, the expression inside any pair of grouping symbols must be evaluated first. For example, note how we first evaluate the sum in the parentheses in the following calculation.

$$(3 + 4) + 5 = 7 + 5$$

$$= 12$$

The rule is simple: Whatever is inside the parentheses is evaluated first.

Writing Mathematics

When writing mathematical statements, follow the mantra:

One equal sign per line.

We can use brackets instead of parentheses.

$$5 + [7 + 9] = 5 + 16$$
$$= 21$$

Again, note how the expression inside the brackets is evaluated first.

We can also use curly braces instead of parentheses or brackets.

$$\{2+3\} + 4 = 5 + 4$$
$$= 9$$

Again, note how the expression inside the curly braces is evaluated first.

If grouping symbols are nested, we evaluate the innermost parentheses first.

For example,

$$2 + [3 + (4 + 5)] = 2 + [3 + 9]$$
$$= 2 + 12$$
$$= 14.$$

Grouping Symbols

Use parentheses, brackets, or curly braces to delimit the part of an expression you want evaluated first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

The Associative Property of Addition

Consider the evaluation of the expression (2+3)+4. We evaluate the expression in parentheses first.

$$(2 + 3) + 4 = 5 + 4$$
$$= 9$$

Now, suppose we change the order of addition to 2 + (3 + 4). Then,

$$2 + (3 + 4) = 2 + 7$$
$$= 9.$$

Although the grouping has changed, the result is the same. That is,

$$(2 + 3) + 4 = 2 + (3 + 4).$$

This property of addition of whole numbers is called the *associate property* of addition.

Associate Property of Addition

Let a, b, and c represent whole numbers. Then,

$$(a + b) + c = a + (b + c).$$

Because of the associate property of addition, when presented with a sum of three numbers, whether you start by adding the first two numbers or the last two numbers, the resulting sum is the same.

The Additive Identity

Imagine a number line visualization of the sum of four and zero; i.e., 4 + 0.

In Figure 1.4, we start at zero, then draw an arrow of magnitude (length) four pointing to the right. Now, at the end of this arrow, attach a second arrow of length zero. Of course, that means that we remain right where we are, at 4. Hence the shaded dot at 4 is the sum. That is, 4 + 0 = 4.

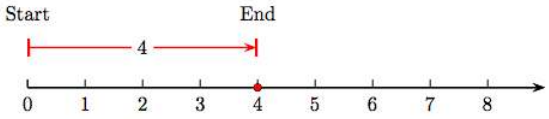


Figure 1.4: Adding zero to four.

The Additive Identity Property

The whole number zero is called the additive identity. If a is any whole number, the

$$a + 0 = a.$$

The number zero is called the additive identity because if you add zero to any number, you get the identical number back.

Adding Larger Whole Numbers

For completeness, we include two examples of adding larger whole numbers. Hopefully, the algorithm is familiar from previous coursework.

Example 1

Simplify: 1, 234 + 498.

Solution

Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, 4 + 8 = 12. Write the 2, then carry a 1 to the tens column. Next, add the digits in the tens column, 3 + 9 = 12, add the carry to get 13, then write the 3 and carry a 1 to the hundreds column. Continue in this manner, working from right to left

$$\begin{array}{r} {\scriptstyle 11} \\ 1234 \\ +\ \ 498 \\ \hline 1732 \end{array}$$

Therefore, 1, 234 + 498 = 1, 732

Exercise

Simplify: 1,286 + 349.

Answer

1635

Add three or more numbers in the same manner.

Example 2

Simplify: 256 + 322 + 418.

Solution

Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, 6 + 2 + 8 = 16. Write the 6, then carry a 1 to the tens column. Continue in this manner, working from right to left.

$$\begin{array}{r} 256 \\ 322 \\ +418 \\ \hline 996 \end{array}$$

Therefore, 256 + 322 + 418 = 996.

Exercise

Simplify: 256 + 342 + 283

Answer

881

Subtraction of Whole Numbers

The key idea is this: *Subtraction is the opposite of addition*. For example, consider the difference 7 - 4 depicted on the number line in Figure 1.5.

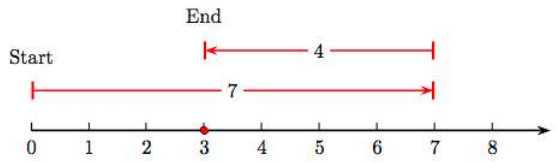


Figure 1.5: Subtraction means *add the opposite*.

If we were adding 7 and 4, we first draw an arrow starting at zero pointing to the right with magnitude (length) seven. Then, to add 4, we would draw a second arrow of magnitude (length) 4, attached to the end of the first arrow and pointing to the right.

However, because subtraction is the *opposite* of addition, in Figure 1.5 we attach an arrow of magnitude (length) four to the end of the first arrow, but *pointing in the opposite direction* (to the left). Note that this last arrow ends at the answer, which is a shaded dot on the number line at 3. That is, $7 - 4 = 3$.

Note that subtraction is **not commutative**; that is, it make no sense to say that $7 - 5$ is the same as $5 - 7$.

Subtraction is **not associative**. It is not the case that $(9 - 5) - 2$ is the same as $9 - (5 - 2)$. On the one hand,

$$(9 - 5) - 2 = 4 - 2$$
$$= 2,$$

but

$$9 - (5 - 2) = 9 - 3$$
$$= 6.$$

Subtracting Larger Whole Numbers

Much as we did with adding larger whole numbers, to subtract two large whole numbers, align them vertically then subtract, working from right to left. You may have to “borrow” to complete the subtraction at any step.

Example 3

Simplify: $1,755 - 328$.

Solution

Align the numbers vertically, then subtract, starting at the ones column, then working right to left. At the ones column, we cannot subtract 8 from 5, so we borrow from the previous column. Now, 8 from 15 is 7. Continue in this manner, working from right to left.

$$\begin{array}{rcccc} & & & {\scriptstyle 4} & \\ & 1 & 7 & \cancel{5} & {}^{1}5 \\ - & & 3 & 2 & 8 \\ \hline & 1 & 4 & 2 & 7 \end{array}$$

Therefore, 1, 755 − 328 = 1, 427.

Exercise

Simplify: 5,635 - 288.

Answer

5,347

Order of Operations

In the absence of grouping symbols, it is important to understand that addition holds no precedence over subtraction, and vice-versa.

> Perform all additions and subtractions in the order presented, moving left to right.

Let's look at an example.

Example 4

Simplify the expression $15-8+4$.

Solution

This example can be trickier than it seems. However, if we follow the rule (perform all additions and subtractions in the order presented, moving left to right), we should have no trouble. First comes fifteen minus eight, which is seven. Then seven plus four is eleven.

$$\begin{aligned} 15-8+4 &= 7+4 \\ &= 11. \end{aligned}$$

Exercise

Simplify: $25-10+8$.

Answer

23

Caution! Incorrect answer ahead!

Note that it is possible to arrive at a different (but incorrect) answer if we favor addition over subtraction in Example 4. If we first add eight and four, then 15 − 8 + 4 becomes 15 − 12, which is 3. However, note that **this is incorrect**, because it violates the rule "perform all additions and subtractions in the order presented, moving left to right."

Applications — Geometry

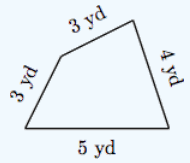
There are any number of applications that require a sum or difference of whole numbers. Let's examine a few from the world of geometry.

Perimeter of a Polygon

In geometry a polygon is a plane figure made up of a closed path of a finite sequence of segments. The segments are called the *edges* or *sides* of the polygon and the points where two edges meet are called the *vertices* of the polygon. The *perimeter* of any polygon is the sum of the lengths of its sides.

Example 5

A quadrilateral is a polygon with four sides. Find the perimeter of the quadrilateral shown below, where the sides are measured below.



Solution

To find the perimeter of the quadrilateral, find the sum of the lengths of the sides.

$$\text{Perimeter} = 3 + 3 + 4 + 5 = 15$$

Hence, the perimeter of the quadrilateral is 15 yards.

Exercise

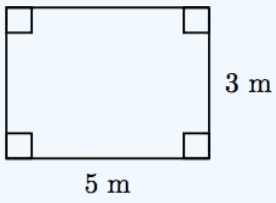
A quadrilateral has sides that measure in 4 in., 3 in., 5 in., and 5 in. Find the perimeter.

Answer

17 inches

Example 6

A quadrilateral (four sides) is a rectangle if all four of its angles are right angles. It can be shown that the opposite sides of a rectangle must be equal. Find the perimeter of the rectangle shown below, where the sides of the rectangle are measured in meters.



Solution

To find the perimeter of the rectangle, find the sum of the four sides. Because opposite sides have the same length, we have two sides of length 5 meters and two sides of length 3 meters. Hence,

$$\text{Perimeter} = 5 + 3 + 5 + 3 = 16.$$

Thus, the perimeter of the rectangle is 16 meters.

Exercise

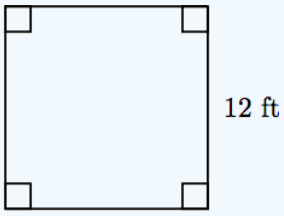
A rectangle has length 12 meters and width 8 meters. Find its perimeter.

Answer

40 meters

Example 7

A quadrilateral (four sides) is a *square* if all four of its sides measures 18 centimeters. Find its perimeter. are equal and all four of its angles are right angles. Pictured below is a square having a side of length 12 feet. Find the perimeter of the square.



Solution

Because the quadrilateral is a square, all four sides have the same length, namely 12 feet. To find the perimeter of the square, find the sum of the four sides. Perimeter = 12 + 12 + 12 + 12 = 48

Hence, the perimeter of the square is 48 feet.

Exercise

A square has a side that measures 18 centimeters. Find its perimeter.

Answer

72 centimeters

Application — Alternative Fuels

Automobiles that run on alternative fuels (other than gasoline) have increased in the United States over the years.

Example 8

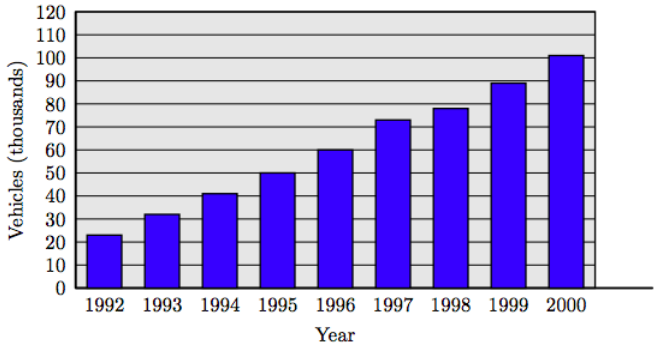
Table 1.2 show the number of cars (in thousands) running on compressed natural gas versus the year. Create a bar chart showing the number of cars running on compressed natural gas versus the year.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Number	23	32	4`	50	60	73	78	89	101

Table 1.2: Number of vehicles (in thousands) running on compressed natural gas.

Solution

Place the years on the horizontal axis. At each year, sketch a bar having height equal to the number of cars in that year that are running on compressed natural gas. Scale the vertical axis in thousands.



Exercise

The following table shows the number of hybrid cars (in thousands) by country.

Country	Number
U.S.	279
Japan	77
Canada	17
U.K.	14
Netherlands	11

Create a bar chart showing the number of cars versus the country of use.

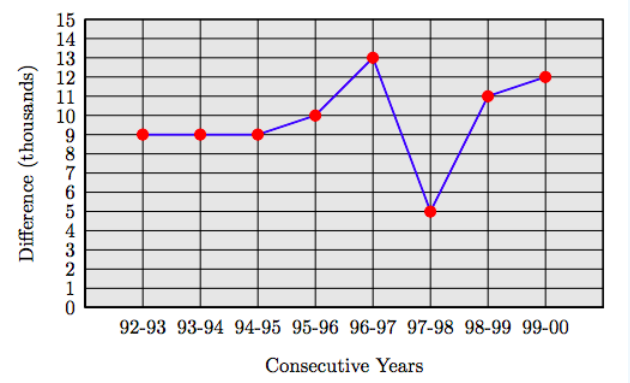
Example 9

Using the data in Table 1.2, crease a table that shows the differences in consecutive years, then create a line plot of the result. In what consecutive years did the United States see the greatest increase in cars powered by compressed natural gas?

Solution

Table 1.3 shows the differences in consecutive years.

Next, craft a line graph. Place consecutive years on the horizontal axis. At each consecutive year pair, plot a point at a height equal to the difference in alternative fuel vehicles. Connect the points with straight line segments.



Note how the line graph makes it completely clear that the greatest increase in vehicles powered by compressed natural gas occurred in the consecutive years 1996-1997, an increase of 13,000 vehicles.

Exercise

The following table show Alphonso's percentage scores on his examinations in mathematics.

Exam	Percentage
Exam #1	52
Exam #2	45
Exam #3	72
Exam #4	889
Exam #5	76

Construct a line graph of Alphonso's exam scores versus exam number.

Exercises

1. Sketch a number line diagram depicting the sum 3 + 2, as shown in Figure 1.2 in the narrative of this section.
2. Sketch a number line diagram depicting the sum 3 + 5, as shown in Figure 1.2 in the narrative of this section.
3. Sketch a number line diagram depicting the sum 3 + 4, as shown in Figure 1.2 in the narrative of this section.
4. Sketch a number line diagram depicting the sum 2 + 4, as shown in Figure 1.2 in the narrative of this section.
5. Sketch a number line diagram depicting the sum 4 + 2, as shown in Figure 1.2 in the narrative of this section.
6. Sketch a number line diagram depicting the sum 4 + 3, as shown in Figure 1.2 in the narrative of this section.
7. Sketch a number line diagram depicting the sum 2 + 5, as shown in Figure 1.2 in the narrative of this section.
8. Sketch a number line diagram depicting the sum 4 + 5, as shown in Figure 1.2 in the narrative of this section.
9. Sketch a number line diagram depicting the sum 4 + 4, as shown in Figure 1.2 in the narrative of this section.
10. Sketch a number line diagram depicting the sum 3 + 3, as shown in Figure 1.2 in the narrative of this section.

In Exercises 11-28, determine which property of addition is depicted by the given identity

11. 28 + 0 = 28
12. 53 + 0 = 53
13. 24 + 0 = 24
14. 93 + 0 = 93
15. (51 + 66) + 88 = 51 + (66 + 88)
16. (90 + 96) + 4 = 90 + (96 + 4)
17. 64 + 39 = 39 + 64
18. 68 + 73 = 73 + 68
19. (70 + 27) + 52 = 70 + (27 + 52)
20. (8 + 53) + 81 = 8 + (53 + 81)
21. 79 + 0 = 79
22. 42 + 0 = 42
23. 10 + 94 = 94 + 10
24. 55 + 86 = 86 + 55
25. 47 + 26 = 26 + 47

26. 62 + 26 = 26 + 62

27. (61 + 53) + 29 = 61 + (53 + 29)

28. (29 + 96) + 61 = 29 + (96 + 61)

---

29. Sketch a number line diagram depicting the difference 8−2, as shown in Figure 1.5 in the narrative of this section.

30. Sketch a number line diagram depicting the difference 8−4, as shown in Figure 1.5 in the narrative of this section.

31. Sketch a number line diagram depicting the difference 7−2, as shown in Figure 1.5 in the narrative of this section.

32. Sketch a number line diagram depicting the difference 9−5, as shown in Figure 1.5 in the narrative of this section.

33. Sketch a number line diagram depicting the difference 7−4, as shown in Figure 1.5 in the narrative of this section.

34. Sketch a number line diagram depicting the difference 6−4, as shown in Figure 1.5 in the narrative of this section.

35. Sketch a number line diagram depicting the difference 9−4, as shown in Figure 1.5 in the narrative of this section.

36. Sketch a number line diagram depicting the difference 6−5, as shown in Figure 1.5 in the narrative of this section.

37. Sketch a number line diagram depicting the difference 8−5, as shown in Figure 1.5 in the narrative of this section.

38. Sketch a number line diagram depicting the difference 9−3, as shown in Figure 1.5 in the narrative of this section.

In Exercises 39-50, simplify the given expression.

39. 16 − 8+2

40. 17 − 3+5

41. 20 − 5 + 14

42. 14 − 5+6

43. 15 − 2+5

44. 13 − 4+2

45. 12 − 5+4

46. 19 − 4 + 13

47. 12 − 6+4

48. 13 − 4 + 18

49. 15 − 5+8

50. 13 − 3 + 11

In Exercises 51-58, the width W and length L of a rectangle are given. Find the perimeter P of the rectangle.

51. W = 7 in, L = 9 in

52. W = 4 in, L = 6 in

53. W = 8 in, L = 9 in

54. W = 5 in, L = 9 in

55. W = 4 cm, L = 6 cm

56. W = 5 in, L = 8 in

57. W = 4 cm, L = 7 cm

58. W = 4 in, L = 9 in

In Exercises 59-66, the length s of a side of a square is given. Find the perimeter P of the square.

59. s = 25 cm

60. s = 21 in

61. s = 16 cm

62. s = 10 in

63. s = 18 in

64. s = 7 in

65. s = 3 in

66. s = 20 in

In Exercises 67-86, find the sum.

67. 3005 + 5217

68. 1870 + 5021

69. 575 + 354 + 759

70. 140 + 962 + 817

71. 472 + (520 + 575)

72. 318 + (397 + 437)

73. 274 + (764 + 690)

74. 638 + (310 + 447)

75. 8583 + 592

76. 5357 + 9936

77. 899 + 528 + 116

78. 841 + 368 + 919

79. (466 + 744) + 517

80. (899 + 996) + 295

81. 563 + 298 + 611 + 828

82. 789 + 328 + 887 + 729

83. 607 + 29 + 270 + 245

84. 738 + 471 + 876 + 469

85. (86 + 557) + 80

86. (435 + 124) + 132

In Exercises 87-104, find the difference.

87. 3493 − 2034 − 227

88. 3950 − 1530 − 2363

89. 8338 − 7366

90. 2157 − 1224

91. 2974 − 2374

92. 881 − 606

93. 3838 − (777 − 241)

94. 8695 − (6290 − 4233)

95. 5846 − 541 − 4577

96. 5738 − 280 − 4280

97. 3084 − (2882 − 614)

98. 1841 − (217 − 28)

99. 2103 − (1265 − 251)

100. 1471 − (640 − 50)

101. 9764 − 4837 − 150

102. 9626 − 8363 − 1052

103. 7095 − 226

104. 4826 − 1199

---

105. **Water Subsidies**. Since the drought began in 2007, California farms have received $79 million in water subsidies. California cotton and rice farmers received an additional $439 million. How much total water subsidies have farmers received? *Associated Press Times-Standard 4/15/09*

106. **War Budget**. The 2010 Federal budget allocates $534 billion for the Department of Defense base programs and an additional $130 billion for the nation's two wars. How much will the Department of Defense receive altogether? *Associated Press Times-Standard 5/8/09*

107. **Sun Frost**. Arcata, CA is home to Sun Frost, a manufacturer of highly efficient refrigerators and freezers. The AC model RF12 refrigerator/freezer costs $2,279 while an R16 model refrigerator/freezer costs $3,017. How much more does the R16 model cost? Source: www.sunfrost.com/retail pricelist.html

108. **Shuttle Orbit**. The space shuttle usually orbits at 250 miles above the surface of the earth. To service the Hubble Space Telescope, the shuttle had to go to 350 miles above the surface. How much higher did the shuttle have to orbit?

109. **Earth's Orbit**. Earth orbits the sun in an ellipse. When earth is at its closest to the sun, called *perihelion*, earth is about 147 million kilometers. When earth is at its furthest point from the sun, called *aphelion*, earth is about 152 million kilometers from the sun. What's the difference in millions of kilometers between aphelion and perihelion?

110. **Pluto's Orbit**. Pluto's orbit is highly eccentric. Find the difference between Pluto's closest approach to the sun and Pluto's furthest distance from the sun if Pluto's perihelion (closest point on its orbit about the sun) is about 7 billion kilometers and its aphelion (furthest point on its orbit about the sun) is about 30 billion kilometers.

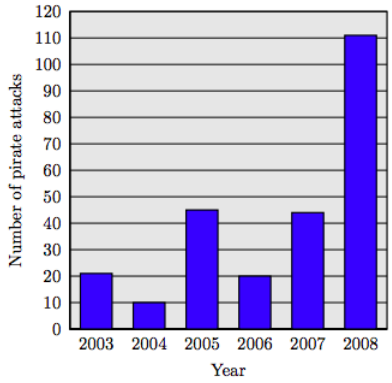
111. **Sunspot Temperature**. The surface of the sun is about 10,000 degrees Fahrenheit. Sunspots are darker regions on the surface of the sun that have a relatively cooler temperature of 6,300 degrees Fahrenheit. How many degrees cooler are sunspots?

112. **Jobs**. The Times-Standard reports that over the next year, the credit- and debitcard processing business Humboldt Merchant Services expects to cut 36 of its 80 jobs, but then turn around and hire another 21. How many people will be working for the company then? *Times-Standard 5/6/09*

113. **Wild tigers**. The chart shows the estimated wild tiger population, by region. According to this chart, what is the total wild tiger population worldwide? *Associated Press-Times-Standard 01/24/10 Pressure mounts to save the tiger*.

Region	Tiger population
India, Nepal and Bhutan	1650
China and Russia	450
Bangladesh	250
Sumatra (Indonesia)	400
Malaysia	500
other SE Asia	350

114. **Pirate Attacks**. The following bar chart tracks pirate attacks off the coast of Somalia.



Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003, 2004, and 2005 combined?

b) How many pirate attacks were there in 2006, 2007, and 2008 combined?

c) How many more pirate attacks were there in 2008 than in 2007?

115. Emily shows improvement on each successive examination throughout the term. Her exam scores are recorded in the following table.

Exam	**Score**
Exam #1	48
Exam #2	51
Exam #3	54
Exam #4	59
Exam #5	67
Exam #6	70

a) Create a bar plot for Emily's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Emily show the greatest improvement?

116. Jason shows improvement on each successive examination throughout the term. His exam scores are recorded in the following table.

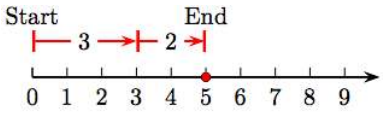
Exam	**Score**
Exam #1	34
Exam #2	42
Exam #3	45
Exam #4	50
Exam #5	57
Exam #6	62

a) Create a bar plot for Jason's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

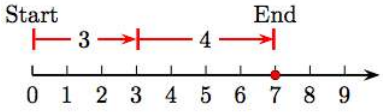
b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Jason show the greatest improvement?

Answers

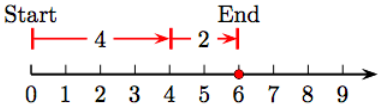
1. 3 + 2 = 5.



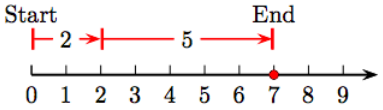
3. 3 + 4 = 7.



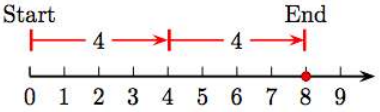
5. 4 + 2 = 6.



7. 2 + 5 = 7.



9. 4 + 4 = 8.



11. Additive identity property of addition.

13. Additive identity property of addition.

15. Associative property of addition

17. Commutative property of addition

19. Associative property of addition

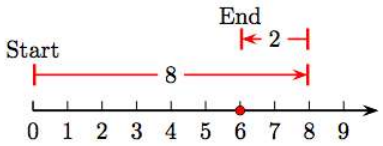
21. Additive identity property of addition.

23. Commutative property of addition

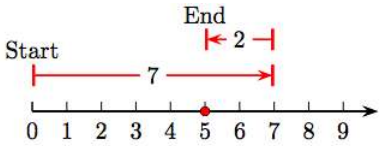
25. Commutative property of addition

27. Associative property of addition

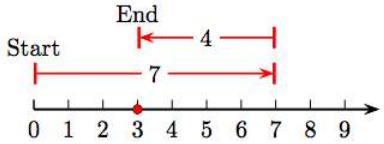
29. 8 − 2 = 6.



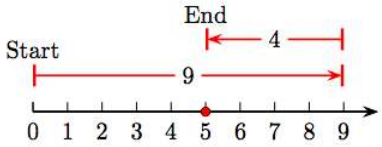
31. 7 − 2 = 5.



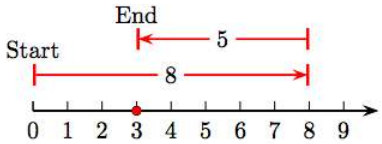
33. 7 − 4 = 3.



35. 9 − 4 = 5.



37. 8 − 5 = 3.



39. 10

41. 29

43. 18

45. 11

47. 10

49. 18

51. P = 32 in

53. P = 34 in

55. P = 20 cm

57. P = 22 cm

59. P = 100 cm

61. P = 64 cm

63. P = 72 in

65. P = 12 in

67. 8222

69. 1688

https://math.libretexts.org/@go/page/196379

71. 1567

73. 1728

75. 9175

77. 1543

79. 1727

81. 2300

83. 1151

85. 723

87. 1232

89. 972

91. 600

93. 3302

95. 728

97. 816

99. 1089

101. 4777

103. 6869

105. $518 million

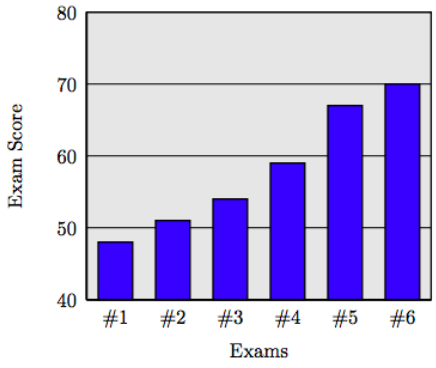
107. $738

109. 5 million kilometers

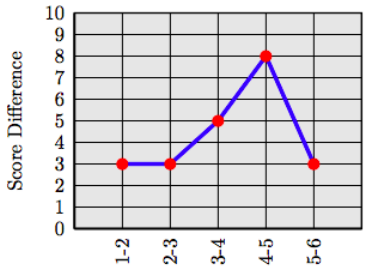
111. 3,700 degrees Fahrenheit

113. 3600

115. a) Bar chart.



b) Line plot of consecutive differences. The line plot of consecutive examination score differences.



The largest improvement was between Exam #4 and Exam #5, where Emily improved by 8 points.

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1.3: Multiplication and Division of Whole Numbers

We begin this section by discussing multiplication of whole numbers. The first order of business is to introduce the various symbols used to indicate multiplication of two whole numbers.

Mathematical symbols that indicate multiplication

Symbol		Example
×	times symbol	3 × 4
·	dot	3 · 4
( )	parentheses	(3)(4) or 3(4) of (3)4

Products and Factors

In the expression $3 \cdot 4$, the whole numbers 3 and 4 are called the **factors** and $3 \cdot 4$ is called the **product**.

The key to understanding multiplication is held in the following statement.

> Multiplication is equivalent to repeated addition.

Suppose, for example, that we would like to evaluate the product $3 \cdot 4$. Because multiplication is equivalent to repeated addition, $3 \cdot 4$ is equivalent to adding three fours. That is,

$$3 \cdot 4 = \underbrace{4+4+4}_{\text{three fours}}$$

Thus, $3 \cdot 4 = 12$. You can visualize the product $3 \cdot 4$ as the sum of three fours on a number line, as shown in Figure 1.6.

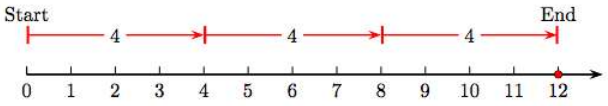


Figure 1.6: Note that 3 · 4 = 4 + 4 + 4. That is, 3 · 4 = 12.

Like addition, the order of the factors does not matter.

$$4 \cdot 3 = \underbrace{3+3+3+3}_{\text{four threes}}$$

Thus, $4 \cdot 3 = 12$. Consider the visualization of $4 \cdot 3$ in Figure 1.7.

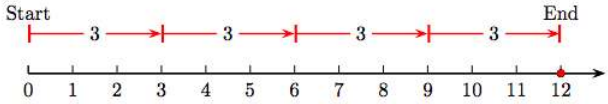


Figure 1.7: Note that 4 · 3 = 3 + 3 + 3 + 3. That is, 4 · 3 = 12.

The evidence in Figure 1.6 and Figure 1.7 show us that multiplication is commutative. That is,

$$3 \cdot 4 = 4 \cdot 3$$

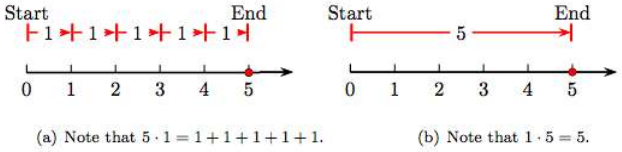
Commutative Property of Multiplication

If *a* and *b* are any whole numbers, then

$$a \cdot b = b \cdot a.$$

The Multiplicative Identity

In Figure 1.8(a), note that five ones equals 5; that is, $5 \cdot 1 = 5$. On the other hand, in Figure 1.8(b), we see that one five equals five; that is, $1 \cdot 5 = 5$.



(a) Note that $5 \cdot 1 = 1 + 1 + 1 + 1 + 1$. (b) Note that $1 \cdot 5 = 5$.

Figure 1.8: Note that 5 · 1 = 5 and 1 · 5 = 5.

Because multiplying a whole number by 1 equals that identical number, the whole number 1 is called the multiplicative identity.

The Multiplicative Identity Property

If a is any whole number, then

$$a \cdot 1 = a \text{ and } 1 \cdot a = a.$$

Multiplication by Zero

Because $3 \cdot 4 = 4 + 4 + 4$, we can say that the product $3 \cdot 4$ represents "3 sets of 4," as depicted in Figure 1.9, where three groups of four boxes are each enveloped in an oval.

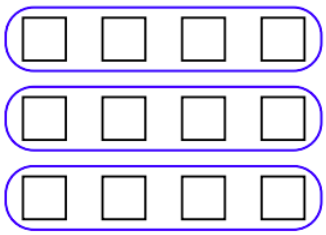


Figure 1.9: Three sets of four: 3 · 4 = 12.

Therefore, $0 \cdot 4$ would mean zero sets of four. Of course, zero sets of four is zero.

Multiplication by Zero.

If a represents any whole number, then

$$a \cdot 0 = 0 \text{ and } 0 \cdot a = 0.$$

The Associative Property of Multiplication

Like addition, multiplication of whole numbers is associative. Indeed,

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24 \end{aligned}$$

and

$$\begin{aligned} (2 \cdot 3) \cdot 4 &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

The Associative Property of Multiplication.

If a, b, and c are any whole numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Multiplying Larger Whole Numbers

Much like addition and subtraction of large whole numbers, we will also need to multiply large whole numbers. Again, we hope the algorithm is familiar from previous coursework.

Example 1

Simplify: $35 \cdot 127$.

Solution

Align the numbers vertically. The order of multiplication does not matter, but we'll put the larger of the two numbers on top of the smaller number. The first step is to multiply 5 times 127. Again, we proceed from right to left. So, 5 times 7 is 35. We write the 5, then carry the 3 to the tens column. Next, 5 times 2 is 10. Add the carry digit 3 to get 13. Write the 3 and carry the 1 to the hundreds column. Finally, 5 times 1 is 5. Add the carry digit to get 6.

$$\begin{array}{r} {\scriptstyle 1\ 3\ \ } \\ 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \end{array}$$

The next step is to multiply 3 times 127. However, because 3 is in the tens place, its value is 30, so we actually multiply 30 times 127. This is the same as multiplying 127 by 3 and placing a 0 at the end of the result.

$$\begin{array}{r} {\scriptstyle 2\ \ } \\ 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline \end{array}$$

After adding the 0, 3 times 7 is 21. We write the 1 and carry the 2 above the 2 in the tens column. Then, 3 times 2 is 6. Add the carry digit 2 to get 8. Finally, 3 times 1 is 1.

All that is left to do is to add the results.

$$\begin{array}{r} 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline 4\,4\,4\,5 \end{array}$$

Thus, $35 \cdot 127 = 4,445$.

Alternate Format

It does not hurt to omit the trailing zero in the second step of the multiplication, where we multiply 3 times 127. The result would look like this:

$$\begin{array}{r} 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline 4\,4\,4\,5 \end{array}$$

In this format, the zero is understood, so it is not necessary to have it physically present. The idea is that with each multiplication by a new digit, we indent the product one space from the right.

Exercise

Simplify: $56 \cdot 335$

Answer

18,760

Division of Whole Numbers

We now turn to the topic of division of whole numbers. We first introduce the various symbols used to indicate division of whole numbers.

Mathematical Symbols that Indicate Division

Symbol		Example
$\div$	division symbol	$12 \div 4$
-	fraction bar	$\frac{12}{4}$
$\longdiv{}$	division bar	$4\longdiv{12}$

Note that each of the following say the same thing; that is, "12 divided by 4 is 3."

$12 \div 4 = 3$ or $\frac{12}{4} = 3$ or $4\overline{)12}$

Quotients, Dividends, and Divisors

In the statement

$$\frac{3}{4)12}$$

the whole number 12 is called the *dividend*, the whole number 4 is called the *divisor*, and the whole number 3 is called the *quotient*. Note that this division bar notation is equivalent to

$$12 \div 4 = 3 \quad \text{and} \quad \frac{12}{4} = 3.$$

The expression a/b means "a divided by b," but this construct is also called a *fraction*.

Definition: Fraction

The expression

$$\frac{a}{b}$$

is called a **fraction**. The number a on top is called the **numerator** of the fraction; the number b on the bottom is called the **denominator** of the fraction.

The key to understanding division of whole numbers is contained in the following statement.

> Division is equivalent to repeated subtraction.

Suppose for example, that we would like to divide the whole number 12 by the whole number 4. This is equivalent to asking the question "how many fours can we subtract from 12?" This can be visualized in a number line diagram, such as the one in Figure 1.10.

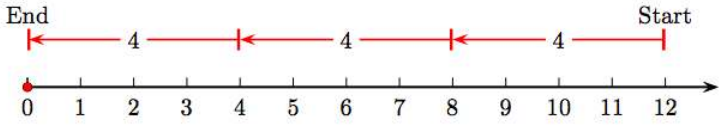


Figure 1.10: Division is repeated subtraction.

In Figure 1.10, note that we if we subtract three fours from twelve, the result is zero. In symbols,

$$12 - \underbrace{4 - 4 - 4}_{\text{three fours}} = 0.$$

Equivalently, we can also ask "How many groups of four are there in 12," and arrange our work as shown in Figure 1.11, where we can see that in an array of twelve objects, we can circle three groups of four ; i.e., 12 ÷ 4 = 3.

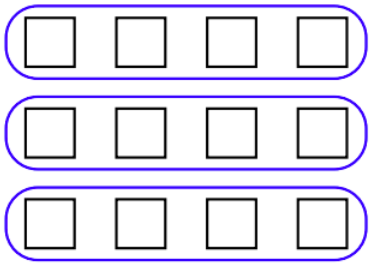


Figure 1.11: There are three groups of four in twelve.

In Figure 1.10 and Figure 1.11, note that the division (repeated subtraction) leaves no remainder. This is not always the case.

Example 2.

Divide 7 by 3.

Solution

In Figure 1.12, we can see that we can subtract two threes from seven, leaving a remainder of one.

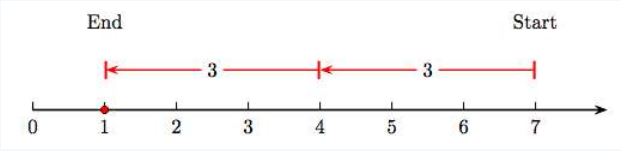


Figure 1.12: Division with a remainder.

Alternatively, in an array of seven objects, we can circle two groups of three, leaving a remainder of one.

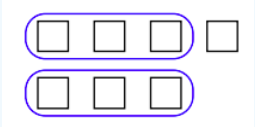


Figure 1.13: Dividing seven by three leaves a remainder of one.

Both Figure 1.12 and Figure1.13 show that there are two groups of three in seven, with one left over. We say "Seven divided by three is two, with a remainder of one.

Exercise

Use both the number line approach and the array of boxes approach to divide 12 by 5.

Division is not Commutative

When dividing whole numbers, the order matters. For example

$$12 \div 4 = 3,$$

but 4 ÷ 12 is not even a whole number. Thus, if a and b are whole numbers, then $a \div b$ does **not** have to be the same as $b \div a$.

Division is not Associative

When you divide three numbers, the order in which they are grouped will usually affect the answer. For example,

$$(48 \div 8) \div 2 = 6 \div 2$$
$$= 3,$$

but

$$48 \div (8 \div 2) = 48 \div 4$$
$$= 12.$$

Thus, if a, b, and c are whole numbers, $(a \div b) \div c$ does not have to be the same as $a \div (b \div c)$.

Division by Zero is Undefined

Suppose that we are asked to divide six by zero; that is, we are asked to calculate 6 ÷ 0. In Figure 1.14, we have an array of six objects.

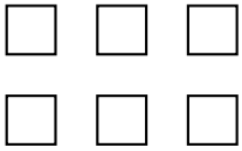


Figure 1.14: How many groups of zero do you see?

Now, to divide six by zero, we must answer the question "How many groups of zero can we circle in Figure 1.14?" Some thought will provide the answer: This is a meaningless request! It makes absolutely no sense to ask how many groups of zero can be circled in the array of six objects in Figure 1.14.

Division by Zero

Division by zero is **undefined**. Each of the expressions

$$6 \div 0 \quad \text{and} \quad \frac{6}{0} \quad \text{and} \quad 0\overline{)6}$$

is **undefined.**

On the other hand, it make sense to ask "What is zero divided by six?" If we create an array of zero objects, then ask how many groups of six we can circle, the answer is "zero groups of six." That is, zero divided by six is zero.

$$0 \div 6 = 0 \quad \text{and} \quad \frac{0}{6} = 0 \quad \text{and} \quad 6\frac{0}{0}$$

Dividing Larger Whole Numbers

We'll now provide a quick review of division of larger whole numbers, using an algorithm that is commonly called *long division*. This is not meant to be a thorough discussion, but a cursory one. We're counting on the fact that our readers have encountered this algorithm in previous courses and are familiar with the process.

Example 3

Simplify: 575/23.

Solution

We begin by estimating how many times 23 will divide into 57, guessing 1. We put the 1 in the quotient above the 7, multiply 1 times 23, place the answer underneath 57, then subtract.

$$\begin{array}{l} 23\overline{)\frac{1}{575}} \\ \quad\ \ \frac{23}{34} \end{array}$$

Because the remainder is larger than the divisor, our estimate is too small. We try again with an estimate of 2.

$$\begin{array}{r} 2 \\ 23\longdiv{575} \\ \underline{46} \\ 11 \end{array}$$

That's the algorithm. Divide, multiply, then subtract. You may continue only when the remainder is smaller than the divisor.

To continue, bring down the 5, estimate that 115 divided by 23 is 5, then multiply 5 times the divisor and subtract.

$$\begin{array}{r} 25 \\ 23\longdiv{575} \\ \underline{46} \\ 115 \\ \underline{115} \\ 0 \end{array}$$

Because the remainder is zero, 575/23 = 25.

? Exercise

Divide 980/35

Answer

28

Application — Counting Rectangular Arrays

Consider the rectangular array of stars in Figure 1.15. To count the number of stars in the array, we could use brute force, counting each star in the array one at a time, for a total of 20 stars. However, as we have four rows of five stars each, it is much faster to multiply: $4 \cdot 5 = 20$ stars.

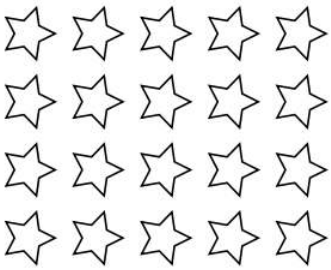


Figure 1.15: Four rows and five columns.

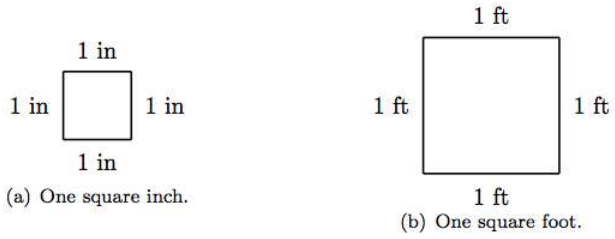


Figure 1.16: Measures of area are in square units.

Application — Area

In Figure 1.16(a), pictured is one square inch (1 in^2), a square with one inch on each side. In Figure 1.16(b), pictured is one square foot (1 ft^2), a square with one foot on each side. Both of these squares are measures of area. Now, consider the rectangle shown in Figure 1.17. The length of this rectangle is four inches (4 in) and the width is three inches (3 in).

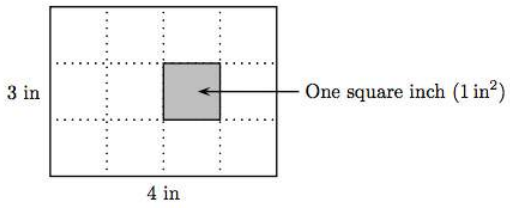


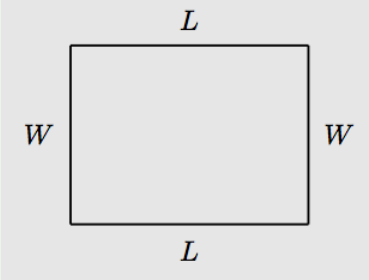
Figure 1.17: A rectangle with length 4 inches and width 3 inches.

To find the area of the figure, we can count the individual units of area that make up the area of the rectangle, twelve square inches (12 in^2) in all. However, as we did in counting the stars in the array in Figure 1.15, it is much faster to note that we have three rows of four square inches. Hence, it is much faster to multiply the number of squares in each row by the number of squares in each column: 4·3 = 12 square inches.

The argument presented above leads to the following rule for finding the area of a rectangle.

Area of a Rectangle

Let L and W represent the length and width of a rectangle, respectively.



To find the area of the rectangle, calculate the product of the length and width. That is, if A represents the area of the rectangle, then the area of the rectangle is given by the formula

$$A = LW.$$

Example 4

A rectangle has width 5 feet and length 12 feet. Find the area of the rectangle.

Solution

Substitute L = 12 ft and W = 5 ft into the area formula.

$$A = LW$$
$$= (12 \text{ ft})(5 \text{ ft})$$
$$= 60 \text{ ft}^2$$

Hence, the area of the rectangle is 60 square feet.

Exercise

A rectangle has width 17 inches and length 33 inches. Find the area of the rectangle.

Answer

561 square inches

Exercises

In Exercises 1-4 use number line diagrams as shown in Figure 1.6 to depict the multiplication.

1. $2 \cdot 4$.
2. $3 \cdot 4$.
3. $4 \cdot 2$.
4. $4 \cdot 3$.

In Exercises 5-16, state the property of multiplication depicted by the given identity.

5. $9 \cdot 8 = 8 \cdot 9$
6. $5 \cdot 8 = 8 \cdot 5$
7. $8 \cdot (5 \cdot 6) = (8 \cdot 5) \cdot 6$
8. $4 \cdot (6 \cdot 5) = (4 \cdot 6) \cdot 5$
9. $6 \cdot 2 = 2 \cdot 6$
10. $8 \cdot 7 = 7 \cdot 8$
11. $3 \cdot (5 \cdot 9) = (3 \cdot 5) \cdot 9$
12. $8 \cdot (6 \cdot 4) = (8 \cdot 6) \cdot 4$
13. $21 \cdot 1 = 21$
14. $39 \cdot 1 = 39$
15. $13 \cdot 1 = 13$
16. $44 \cdot 1 = 44$

In Exercises 17-28, multiply the given numbers.

17. $78 \cdot 3$
18. $58 \cdot 7$
19. $907 \cdot 6$
20. $434 \cdot 80$
21. $128 \cdot 30$
22. $454 \cdot 90$
23. $799 \cdot 60$
24. $907 \cdot 20$
25. $14 \cdot 70$
26. $94 \cdot 90$
27. $34 \cdot 90$
28. $87 \cdot 20$

In Exercises 29-40, multiply the given numbers.

29. $237 \cdot 54$
30. $893 \cdot 94$
31. $691 \cdot 12$
32. $823 \cdot 77$

33. 955 · 89

34. 714 · 41

35. 266 · 61

36. 366 · 31

37. 365 · 73

38. 291 · 47

39. 955 · 57

40. 199 · 33

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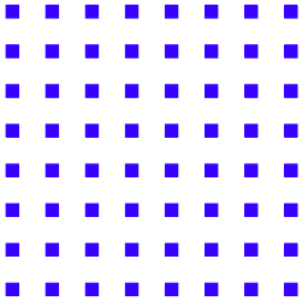
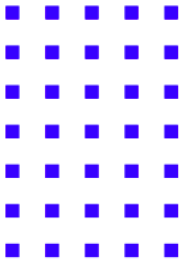
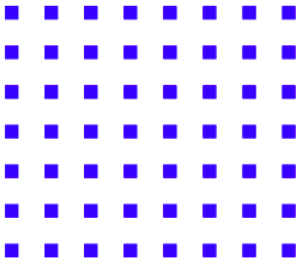
41. Count the number of objects in the array.



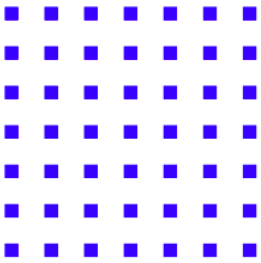
42. Count the number of objects in the array.



43. Count the number of objects in the array.



44. Count the number of objects in the array.



In Exercises 45-48, find the area of the rectangle having the given length and width.

45. L = 50 in, W = 25 in

46. L = 48 in, W = 24 in

47. L = 47 in, W = 13 in

48. L = 19 in, W = 10 in

In Exercises 49-52, find the perimeter of the rectangle having the given length and width.

49. L = 25 in, W = 16 in

50. L = 34 in, W = 18 in

51. L = 30 in, W = 28 in

52. L = 41 in, W = 25 in

---

53. A set of beads costs 50 cents per dozen. What is the cost (in dollars) of 19 dozen sets of beads?

54. A set of beads costs 60 cents per dozen. What is the cost (in dollars) of 7 dozen sets of beads?

55. If a math tutor worked for 47 hours and was paid $15 each hour, how much money would she have made?

56. If a math tutor worked for 46 hours and was paid $11 each hour, how much money would he have made?

57. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 24 gross?

58. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 11 gross?

59. If bricks weigh 4 kilograms each, what is the weight (in kilograms) of 5000 bricks?

60. If bricks weigh 4 pounds each, what is the weight (in pounds) of 2000 bricks?

In Exercises 61-68, which of the following four expressions differs from the remaining three?

61. $\frac{30}{5}$, $30 \div 5$, $5\longdiv{30}$, $5 \div 30$

62. $\frac{12}{2}$, $12 \div 2$, $2\longdiv{12}$, $2 \div 12$

63. $\frac{8}{2}$, $8 \div 2$, $2\longdiv{8}$, $8\longdiv{2}$

64. $\frac{8}{4}$, $8 \div 4$, $4\longdiv{8}$, $8\longdiv{4}$

65. $2\longdiv{14}$, $14\longdiv{2}$, $\frac{14}{2}$, $14 \div 2$

66. $9\longdiv{54}$, $54\longdiv{9}$, $\frac{54}{9}$, $54 \div 9$

67. $3\longdiv{24}$, $3 \div 24$, $\frac{24}{3}$, $24 \div 3$

68. $3\longdiv{15}$, $3 \div 15$, $\frac{15}{3}$, $15 \div 3$

In Exercises 69-82, simplify the given expression. If the answer doesn't exist or is undefined, write "undefined".

69. $0 \div 11$

70. $0 \div 5$

71. $17 \div 0$

72. $24 \div 0$

73. $10 \cdot 0$

74. $20 \cdot 0$

75. $\frac{7}{0}$

76. $\frac{23}{0}$

77. $16\longdiv{0}$

78. $25\longdiv{0}$

79. $\frac{0}{24}$

80. $\frac{0}{22}$

81. $0\longdiv{0}$

82. $0 \div 0$

In Exercises 83-94, divide the given numbers.

83. $\frac{2816}{44}$

84. $\frac{1998}{37}$

85. $\frac{2241}{83}$

86. $\frac{2716}{97}$

87. $\frac{3212}{73}$

88. $\frac{1326}{17}$

89. $\frac{8722}{98}$

90. $\frac{1547}{91}$

91. $\frac{1440}{96}$

92. $\frac{2079}{27}$

93. $\frac{8075}{85}$

94. $\frac{1587}{23}$

In Exercises 95-106, divide the given numbers.

95. $\frac{17756}{92}$

96. $\frac{46904}{82}$

97. $\frac{11951}{19}$

98. $\frac{22304}{41}$

99. $\frac{18048}{32}$

100. $\frac{59986}{89}$

101. $\frac{29047}{31}$

102. $\frac{33264}{86}$

103. $\frac{22578}{53}$

104. $\frac{18952}{46}$

105. $\frac{12894}{14}$

106. $\frac{18830}{35}$

---

107. A concrete sidewalk is laid in square blocks that measure 6 feet on each side. How many blocks will there be in a walk that is 132 feet long?

108. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 180 feet long?

109. One boat to the island can take 5 people. How many trips will the boat have to take in order to ferry 38 people to the island? (Hint: Round up your answer.)

110. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 46 people to the island? (Hint: Round up your answer.)

111. If street lights are placed at most 145 feet apart, how many street lights will be needed for a street that is 4 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

112. If street lights are placed at most 70 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

113. A concrete sidewalk is laid in square blocks that measure 4 feet on each side. How many blocks will there be in a walk that is 292 feet long?

114. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 445 feet long?

115. One boat to the island can take 3 people. How many trips will the boat have to take in order to ferry 32 people to the island? (Hint: Round up your answer.)

116. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 37 people to the island? (Hint: Round up your answer.)

117. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 2 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

118. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

---

119. **Writing articles**. Eli writes an average of 4 articles a day, five days a week, to support product sales. How many articles does Eli write in one week?

120. **Machine gun**. A 0.50-caliber antiaircraft machine gun can fire 800 rounds each minute. How many rounds could fire in three minutes? *Associated Press Times-Standard 4/15/09*

121. **Laps**. The swimming pool at CalCourts is 25 yards long. If one lap is up and back again, how many yards has Wendell swam doing 27 laps?

122. **Refrigerator wattage**. A conventional refrigerator will run about 12 hours each day can use 150 Watts of power each hour. How many Watts of power will a refrigerator use over the day?

123. **Horse hay**. A full-grown horse should eat a minimum of 12 pounds of hay each day and may eat much more depending on their weight. How many pounds minimum would a horse eat over a year?

124. **College costs**. After a $662 hike in fees, California residents who want to attend the University of California as an undergraduate should expect to pay $8,700 in for the upcoming academic year 2009- 2010. If the cost were to remain the same for the next several years, how much should a student expect to pay for a four-year degree program at a UC school?

125. **Non-resident costs**. Nonresident undergraduates who want to attend a University of California college should expect to pay about $22,000 for the upcoming academic year. Assuming costs remain the same, what can a four-year degree cost?

126. **Student tax**. The mayer of Providence, Rhode Island wants to tax its 25,000 Brown University students $150 each to contribute to tax receipts saying students should pay for the resources they use just like the town residents. How many dollars would the mayer generate?

127. **New iceberg**. A new iceberg, shaved off a glacier after a collision with another iceberg, measures about 48 miles long and 28 miles wide. What's the approximate area of the new iceberg? Associated PressTimes-Standard 02/27/10 2 Huge icebergs set loose off Antarctica's coast.

128. **Solar panels**. One of the solar panels on the International Space Station is 34 meters long and 11 meters wide. If there are eight of these, what's the total area for solar collection?

129. **Sidewalk**. A concrete sidewalk is to be 80 foot long and 4 foot wide. How much will it cost to lay the sidewalk at $8 per square foot?

130. **Hay bales**. An average bale of hay weighs about 60 pounds. If a horse eats 12 pounds of hay a day, how many days will one bale feed a horse?

131. **Sunspots**. Sunspots, where the sun's magnetic field is much higher, usually occur in pairs. If the total count of sunspots is 72, how many pairs of sunspots are there?

Answers

1. $2 \cdot 4 = \underbrace{4+4}_{2 \text{ times}} = 8$

3. $4 \cdot 2 = \underbrace{2+2+2+2}_{4 \text{ times}} = 8$

5. Commutative property of multiplication

7. Associative property of multiplication

9. Commutative property of multiplication

11. Associative property of multiplication

13. Multiplicative identity property

15. Multiplicative identity property

17. 234

19. 5442

21. 3840

23. 47940

25. 980

27. 3060

29. 12798

31. 8292

33. 84995

35. 16226

37. 26645

39. 54435

41. 64

43. 56

45. 1250 in^2

47. 611 in^2

49. 82 in

51. 116 in

53. 9.50

55. 705

57. 3456

59. 20000

61. $5 \div 30$

63. $8\sqrt{2}$

65. $14\sqrt{2}$

67. $3 \div 24$

69. 0

71. Undefined

73. 0

75. Undefined

77. 0

79. 0

81. Undefined

83. 64

85. 27

87. 44

89. 89

91. 15

93. 95

95. 193

97. 629

99. 564

101. 937

103. 426

105. 921

107. 22

109. 8

111. 147

113. 73

115. 11

117. 102

119. 20 articles

121. 1350 yards

123. 4380 pounds of hay

125. $88,000

127. 1344 mi^2

129. $2,560

131. 36

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1.4: Prime Factorization

In the statement $3 \cdot 4 = 12$, the number 12 is called the *product*, while 3 and 4 are called *factors*.

Example 1

Find all whole number factors of 18.

Solution

We need to find all whole number pairs whose product equals 18. The following pairs come to mind.

$$1 \cdot 18 = 18 \text{ and } 2 \cdot 9 = 18 \text{ and } 3 \cdot 6 = 18.$$

Hence, the factors of 18 are (in order) 1, 2, 3, 6, 9, and 18.

Exercise

Find all whole number factors of 21.

Answer

1, 3, 7, and 21

Divisibility

In Example 1, we saw $3 \cdot 6 = 18$, making 3 and 6 factors of 18. Because division is the inverse of multiplication, that is, division by a number undoes the multiplication of that number, this immediately provides

$$18 \div 6 = 3 \text{ and } 18 \div 3 = 6.$$

That is, 18 is divisible by 3 and 18 is divisible by 6. When we say that 18 is divisible by 3, we mean that when 18 is divided by 3, there is a zero remainder.

Divisible

Let a and b be whole numbers. Then a is divisible by b if and only if the remainder is zero when a is divided by b. In this case, we say that "b is a divisor of a."

Example 2

Find all whole number divisors of 18.

Solution

In Example 1, we saw that $3 \cdot 6 = 18$. Therefore, 18 is divisible by both 3 and 6 ($18 \div 3 = 6$ and $18 \div 6 = 3$). Hence, when 18 is divided by 3 or 6, the remainder is zero. Therefore, 3 and 6 are divisors of 18. Noting the other products in Example 1, the complete list of divisors of 18 is 1, 2, 3, 6, 9, and 18.

Exercise

Find all whole number divisors of 21.

Answer

1, 3, 7, and 21.

Example 1 and Example 2 show that when working with whole numbers, the words *factor* and *divisor* are interchangeable.

Factors and Divisors

If $c = a \cdot b$, then a and b are called *factors* of c. Both a and b are also called *divisors* of c.

Divisibility Tests

There are a number of very useful divisibility tests.

Divisible by 2. If a whole number ends in 0, 2, 4, 6, or 8, then the number is called an even number and is divisible by 2. Examples of even numbers are 238 and 1,246 (238 ÷ 2 = 119 and 1, 246 ÷ 2 = 623). A number that is not even is called an odd number. Examples of odd numbers are 113 and 2,339.

Divisible by 3. If the sum of the digits of a whole number is divisible by 3, then the number itself is divisible by 3. An example is 141. The sum of the digits is 1 + 4 + 1 = 6, which is divisible by 3. Therefore, 141 is also divisible by 3 (141 ÷ 3 = 47).

Divisible by 4. If the number represented by the last two digits of a whole number is divisible by 4, then the number itself is divisible by 4. An example is 11,524. The last two digits represent 24, which is divisible by 4 (24 ÷ 4 = 6). Therefore, 11,524 is divisible by 4 (11, 524 ÷ 4=2, 881).

Divisible by 5. If a whole number ends in a zero or a 5, then the number is divisible by 5. Examples are 715 and 120 (715÷5 = 143 and 120÷5 = 24).

Divisible by 6. If a whole number is divisible by 2 and by 3, then it is divisible by 6. An example is 738. First, 738 is even and divisible by 2. Second, 7+3+8=18, which is divisible by 3. Hence, 738 is divisible by 3. Because 738 is divisible by both 2 and 3, it is divisible by 6 (738 ÷ 6 = 123).

Divisible by 8. If the number represented by the last three digits of a whole number is divisible by 8, then the number itself is divisible by 8. An example is 73,024. The last three digits represent the number 24, which is divisible by 8 (24÷8 = 3). Thus, 73,024 is also divisible by 8 (73, 024÷8 = 9, 128).

Divisible by 9. If the sum of the digits of a whole number is divisible by 9, then the number itself is divisible by 9. An example is 117. The sum of the digits is 1 + 1 + 7 = 9, which is divisible by 9. Hence, 117 is divisible by 9 (117 ÷ 9 = 13).

Prime Numbers

We begin with the definition of a prime number.

Prime Number

A whole number (other than 1) is a prime number if its only factors (divisors) are 1 and itself. Equivalently, a number is prime if and only if it has exactly two factors (divisors).

Example 3

Which of the whole numbers 12, 13, 21, and 37 are prime numbers?

Solution

- The factors (divisors) of 12 are 1, 2, 3, 4, 6, and 12. Hence, 12 is not a prime number.
- The factors (divisors) of 13 are 1 and 13. Because its only divisors are 1 and itself, 13 is a prime number.
- The factors (divisors) of 21 are 1, 3, 7, and 21. Hence, 21 is not a prime number.
- The factors (divisors) of 37 are 1 and 37. Because its only divisors are 1 and itself, 37 is a prime number.

Exercise

Which of the whole numbers 15, 23, 51, and 59 are prime numbers?

Answer

23 and 59

Example 4

List all the prime numbers less than 20.

Solution

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19.

You try it!

List all the prime numbers less than 100.

Composite Numbers

If a whole number is not a prime number, then it is called a *composite number*.

Example 5

Is the whole number 1,179 prime or composite?

Solution

Note that 1 + 1 + 7 + 9 = 18, which is divisible by both 3 and 9. Hence, 3 and 9 are both divisors of 1,179. Therefore, 1,179 is a composite number.

Exercise

Is the whole number 2,571 prime or composite?

Answer

Composite

Factor Trees

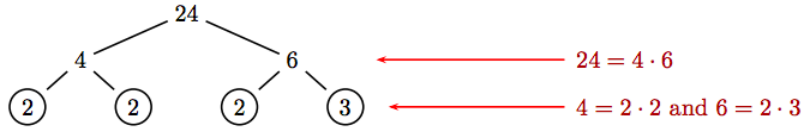
We will now learn how to express a composite number as a unique product of prime numbers. The most popular device for accomplishing this goal is the *factor tree*.

Example 6

Express 24 as a product of prime factors.

Solution

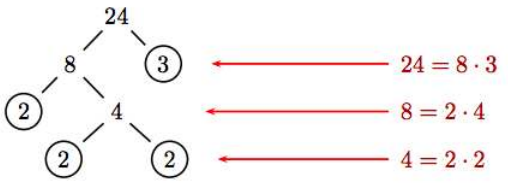
We use a factor tree to break 24 down into a product of primes.



At each level of the tree, break the current number into a product of two factors. The process is complete when all of the "circled leaves" at the bottom of the tree are prime numbers. Arranging the factors in the "circled leaves" in order,

$$24 = 2 \cdot 2 \cdot 2 \cdot 3.$$

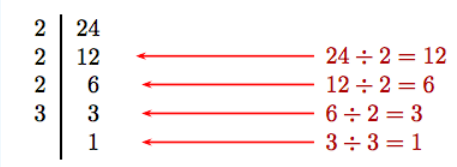
The final answer does not depend on product choices made at each level of the tree. Here is another approach.



The final answer is found by including all of the factors from the "circled leaves" at the end of each branch of the tree, which yields the same result, namely 24 = 2 · 2 · 2 · 3.

Alternate Approach

Some favor repeatedly dividing by 2 until the result is no longer divisible by 2. Then try repeatedly dividing by the next prime until the result is no longer divisible by that prime. The process terminates when the last resulting quotient is equal to the number 1.



The first column reveals the prime factorization; i.e., 24 = 2 · 2 · 2 · 3.

Exercise

Express 36 as a product of prime factors.

Answer

2 · 2 · 3 · 3.

The fact that the alternate approach in Example 6 yielded the same result is significant.

Unique Factorization Theorem

Every whole number can be **uniquely** factored as a product of primes.

This result guarantees that if the prime factors are ordered from smallest to largest, everyone will get the same result when breaking a number into a product of prime factors.

Exponents

We begin with the definition of an exponential expression.

Exponents

The expression a^m is defined to mean

$$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}$$

The number a is called the base of the exponential expression and the number m is called the exponent. The exponent m tells us to repeat the base a as a factor m times.

Example 7

Evaluate 2^5, 2^3, and 5^2.

Solution

- In the case of 2^5, we have

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$
$$= 32.$$

- In the case of 3^3, we have

$$3^3 = 3 \cdot 3 \cdot 3$$
$$= 27.$$

- In the case of 5^2, we have

$$5^2 = 5 \cdot 5$$
$$= 25.$$

Exercise

Evaluate: 3^5.

Answer

243.

Example 8

Express the solution to Example 6 in compact form using exponents.

Solution

In Example 6, we determined the prime factorization of 24.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

Because $2 \cdot 2 \cdot 2 = 23$, we can write this more compactly.

$$24 = 2^3 \cdot 3$$

Exercise

Prime factor 54.

Answer

$2 \cdot 3 \cdot 3 \cdot 3$

Example 9

Evaluate the expression $2^3 \cdot 3^2 \cdot 5^2$.

Solution

First raise each factor to the given exponent, then perform the multiplication in order (left to right).

$$2^3 \cdot 3^2 \cdot 5^2 = 8 \cdot 9 \cdot 25$$
$$= 72 \cdot 25$$
$$= 1800$$

Exercise

Evaluate: $3^3 \cdot 5^2$.

Answer

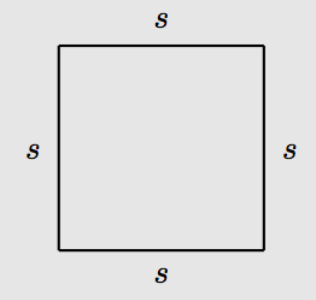
675

Application

A square is a rectangle with four equal sides.

Area of a Square

Let s represent the length of each side of a square.



Because a square is also a rectangle, we can find the area of the square by multiplying its length and width. However, in this case, the length and width both equal s, so $A = (s)(s) = s^2$. Hence, the formula for the area of a square is

$$A = s^2.$$

Example 10

The edge of a square is 13 centimeters. Find the area of a square.

Solution

Substitute $s = 13$ cm into the area formula.

$$A = s^2$$
$$= (13 \text{ cm})^2$$
$$= (13 \text{ cm})(13 \text{ cm})$$
$$= 169 \text{ cm}^2$$

Hence, the area of the square is 169 cm^2; i.e., 169 square centimeters.

Exercise

The edge of a square is 15 meters. Find the area of a square.

Answer

225 square meters

Exercises

In Exercises 1-12, find all divisors of the given number.

1. 30

2. 19

3. 83

4. 51

5. 91

6. 49

7. 75

8. 67

9. 64

10. 87

11. 14

12. 89

In Exercises 13-20, which of the following numbers is not divisible by 2?

13. 117, 120, 342, 230

14. 310, 157, 462, 160

15. 30, 22, 16, 13

16. 382, 570, 193, 196

17. 105, 206, 108, 306

18. 60, 26, 23, 42

19. 84, 34, 31, 58

20. 66, 122, 180, 63

In Exercises 21-28, which of the following numbers is not divisible by 3?

21. 561, 364, 846, 564

22. 711, 850, 633, 717

23. 186, 804, 315, 550

24. 783, 909, 504, 895

25. 789, 820, 414, 663

26. 325, 501, 945, 381

27. 600, 150, 330, 493

28. 396, 181, 351, 606

In Exercises 29-36, which of the following numbers is not divisible by 4?

29. 3797, 7648, 9944, 4048

30. 1012, 9928, 7177, 1592

31. 9336, 9701, 4184, 2460

32. 2716, 1685, 2260, 9788

33. 9816, 7517, 8332, 7408

34. 1788, 8157, 7368, 4900

35. 1916, 1244, 7312, 7033

36. 7740, 5844, 2545, 9368

In Exercises 37-44, which of the following numbers is not divisible by 5?

37. 8920, 4120, 5285, 9896
38. 3525, 7040, 2185, 2442
39. 8758, 3005, 8915, 3695
40. 3340, 1540, 2485, 2543
41. 2363, 5235, 4145, 4240
42. 9030, 8000, 5445, 1238
43. 1269, 5550, 4065, 5165
44. 7871, 9595, 3745, 4480

In Exercises 45-52, which of the following numbers is not divisible by 6?

45. 328, 372, 990, 528
46. 720, 288, 148, 966
47. 744, 174, 924, 538
48. 858, 964, 930, 330
49. 586, 234, 636, 474
50. 618, 372, 262, 558
51. 702, 168, 678, 658
52. 780, 336, 742, 312

In Exercises 53-60, which of the following numbers is not divisible by 8?

53. 1792, 8216, 2640, 5418
54. 2168, 2826, 1104, 2816
55. 8506, 3208, 9016, 2208
56. 2626, 5016, 1392, 1736
57. 4712, 3192, 2594, 7640
58. 9050, 9808, 8408, 7280
59. 9808, 1232, 7850, 7912
60. 3312, 1736, 9338, 3912

In Exercises 61-68, which of the following numbers is not divisible by 9?

61. 477, 297, 216, 991
62. 153, 981, 909, 919
63. 153, 234, 937, 675
64. 343, 756, 927, 891
65. 216, 783, 594, 928
66. 504, 279, 307, 432
67. 423, 801, 676, 936
68. 396, 684, 567, 388

In Exercises 69-80, identify the given number as prime, composite, or neither.

69. 19

70. 95

71. 41

72. 88

73. 27

74. 61

75. 91

76. 72

77. 21

78. 65

79. 23

80. 36

In Exercises 81-98, find the prime factorization of the natural number.

81. 224

82. 320

83. 108

84. 96

85. 243

86. 324

87. 160

88. 252

89. 32

90. 128

91. 360

92. 72

93. 144

94. 64

95. 48

96. 200

97. 216

98. 392

In Exercises 99-110, compute the exact value of the given exponential expression.

99. $5^2 \cdot 4^1$

100. $2^3 \cdot 4^1$

101. 0^1

102. 1^3

103. $3^3 \cdot 0^2$

104. $3^3 \cdot 2^2$

105. 4^1

106. 5^2

107. 4^3

108. 4^2

109. $3^3 \cdot 1^2$

110. $5^2 \cdot 2^3$

In Exercises 111-114, find the area of the square with the given side.

111. 28 inches

112. 31 inches

113. 22 inches

114. 13 inches

Create factor trees for each number in Exercises 115-122. Write the prime factorization for each number in compact form, using exponents.

115. 12

116. 18

117. 105

118. 70

119. 56

120. 56

121. 72

122. 270

---

123. **Sieve of Eratosthenes**. This exercise introduces the *Sieve of Eratosthenes*, an ancient algorithm for finding the primes less than a certain number n, first created by the Greek mathematician Eratosthenes. Consider the grid of integers from 2 through 100.

2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21
22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41
42	43	44	45	46	47	48	49	50	51
52	53	54	55	56	57	58	59	60	61
62	63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90	91
92	93	94	95	96	97	98	99	100	

To find the primes less than 100, proceed as follows.

i) Strike out all multiples of 2 (4, 6, 8, etc.)

ii) The list's next number that has not been struck out is a prime number.

iii) Strike out from the list all multiples of the number you identified in step (ii).

iv) Repeat steps (ii) and (iii) until you can no longer strike any more multiples.

v) All unstruck numbers in the list are primes.

Answers

1. 1, 2, 3, 5, 6, 10, 15, 30

3. 1, 83

5. 1, 7, 13, 91

7. 1, 3, 5, 15, 25, 75

9. 1, 2, 4, 8, 16, 32, 64

11. 1, 2, 7, 14

13. 117

15. 13

17. 105

19. 31

21. 364

23. 550

25. 820

27. 493

29. 3797

31. 9701

33. 7517

35. 7033

37. 9896

39. 8758

41. 2363

43. 1269

45. 328

47. 538

49. 586

51. 658

53. 5418

55. 8506

57. 2594

59. 7850

61. 991

63. 937

65. 928

67. 676

69. prime

71. prime

73. composite

75. composite

77. composite

79. prime

81. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7$

83. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

85. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

87. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

89. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

91. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

93. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

95. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

97. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

99. 100

101. 0

103. 0

105. 4

107. 64

109. 27

111. 784 in^2

113. 484 in^2

115. $12 = 22 \cdot 3$

117. $105 = 3 \cdot 5 \cdot 7$

119. $56 = 23 \cdot 7$

121. $72 = 23 \cdot 32$

123. Unstruck numbers are primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

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1.5: Order of Operations

The order in which we evaluate expressions can be ambiguous. Take for example, the expression $4 + 3 \cdot 2$. If we do the addition first, then

$$4+3 \cdot 2=7 \cdot 2$$
$$= 14.$$

On the other hand, if we do the multiplication first, then

$$4+3 \cdot 2=4+6$$
$$= 10.$$

So, what are we to do? Of course, grouping symbols can remove the ambiguity

Grouping Symbols

Parentheses, brackets, or curly braces can be used to group parts of an expression. Each of the following are equivalent:

$$(4 + 3) \cdot 2 \text{ or } [4 + 3] \cdot 2 \text{ or } \{4+3\} \cdot 2$$

In each case, the rule is "evaluate the expression inside the grouping symbols first." If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

Thus, for example,

$$(4 + 3) \cdot 2=7 \cdot 2$$
$$= 14.$$

Note how the expression contained in the parentheses was evaluated first. Another way to avoid ambiguities in evaluating expressions is to establish an order in which operations should be performed. The following guidelines should always be strictly enforced when evaluating expressions.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in

Example 1

Evaluate $4 + 3 \cdot 2$.

Solution

Because of the established *Rules Guiding Order of Operations*, this expression is no longer ambiguous. There are no grouping symbols or exponents, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right. After that we invoke rule four, performing all additions and subtractions in the order that they appear, moving left to right.

$$\begin{aligned} 4+3\dot{2} &= 4+6 \\ &= 10 \end{aligned}$$

Thus, $4 + 3 \cdot 2 = 10$.

? Exercise

Simplify: 8 + 2 · 5.

Answer

18

✓ Example 2

Evaluate 18 − 2 + 3.

Solution

Follow the *Rules Guiding Order of Operations*. Addition has no precedence over subtraction, nor does subtraction have precedence over addition. We are to perform additions and subtractions as they occur, moving left to right.

$$\begin{aligned} 18-2+3 &= 16+3 \quad \text{Subtract: } 18-2=16. \\ &= 19 \quad \text{Add: } 16+3=19. \end{aligned}$$

Thus, 18 − 2 + 3 = 19.

? Exercise

Simplify: 17 − 8 + 2.

Answer

11

✓ Example 3

Evaluate 54 ÷ 9 · 2.

Solution

Follow the *Rules Guiding Order of Operations*. Division has no precedence over multiplication, nor does multiplication have precedence over division. We are to perform divisions and multiplications as they occur, moving left to right.

$$\begin{aligned} 54 \div 9 \cdot 2 &= 6\dot{2} \quad \text{Divide: } 54 \div 9 = 6. \\ &= 12 \quad \text{Multiply: } 6 \cdot 2 = 12. \end{aligned}$$

Thus, 54 ÷ 9 · 2 = 12.

? Exercise

Simplify: 72 ÷ 9 · 2.

Answer

16

✓ Example 4

Evaluate $2 \cdot 3^2 - 12$.

Solution

Follow the *Rules Guiding Order of Operations*, exponents first, then multiplication, then subtraction.

$$\begin{aligned} 2 \cdot 3^2 - 12 &= 2\dot{9} - 12 && \text{Evaluate the exponent: } 3\hat{}2 = 9. \\ &= 18 - 12 && \text{Perform the multiplication: } 2 \cdot 9 = 18. \\ &= 6 && \text{Perform the subtraction: } 18 - 12 = 6. \end{aligned}$$

Thus, $2 \cdot 3^2 - 12 = 6$.

? Exercise

Simplify: $14 + 3 \cdot 4^2$

Answer

62

✓ Example 5

Evaluate $12 + 2(3 + 2 \cdot 5)^2$.

Solution

Follow the Rules Guiding Order of Operations, evaluate the expression inside the parentheses first, then exponents, then multiplication, then addition.

$$\begin{aligned} 12 + 2(3 + 5 \cdot 5)^2 &= 12 + 2(3 + 10)^2 && \text{Multiply inside parentheses: } 2 \cdot 5 = 10. \\ &= 12 + 2(13)^2 && \text{Add inside parentheses: } 3 + 10 = 13. \\ &= 12 + 2(169) && \text{Exponents are next: } (13)^2 = 169. \\ &= 12 + 338 && \text{Multiplication is next: } 2(169) = 338. \\ &= 350 && \text{Time to add: } 12 + 338 = 350. \end{aligned}$$

Thus, $12 + 2(3 + 2 \cdot 5)\, 2 = 350$.

? Exercise

Simplify: $3(2 + 3 \cdot 4)^2 - 11$.

Answer

577

✓ Example 6

Evaluate $2\{2 + 2[2 + 2]\}$.

Solution

When grouping symbols are nested, evaluate the expression between the pair of innermost grouping symbols first.

$$\begin{aligned} 2(2 + 2[2 + 2]) &= 2(2 + 2[4]) && \text{Innermost grouping first: } 2 + 2 = 4. \\ &= 2(2 + 8) && \text{Multiply next: } 2[4] = 8. \\ &= 2(10) && \text{Add inside braces: } 2 + 8 = 10. \\ &= 20 && \text{Multiply: } 2(10) = 20 \end{aligned}$$

Thus, $2(2 + 2[2 + 2]) = 20$.

? Exercise

Simplify: $2\{3 + 2[3 + 2]\}$.

Answer

26

Fraction Bars

Consider the expression

$$\frac{6^2+8^2}{(2+3)^2}$$

Because a fraction bar means division, the above expression is equivalent to

$$\left(6^2+8^2\right) \div (2+3)^2$$

The position of the grouping symbols signals how we should proceed. We should simplify the numerator, then the denominator, then divide.

Fractional Expressions

If a fractional expression is present, evaluate the numerator and denominator first, then divide.

Example 7

Evaluate the expression

$$\frac{6^2+8^2}{(2+3)^2}.$$

Solution

Simplify the numerator and denominator first, then divide.

$$\begin{aligned}
\frac{6^2+8^2}{(2+3)^2} &= \frac{6^2+8^2}{(5)^2} && \text{Parentheses in denominator first: } 2+3=5 \\
&= \frac{36+64}{25} && \text{Exponents are next: } 6^2=36,\ 8^2=64,\ 5^2=25. \\
&= \frac{100}{25} && \text{Add in numerator: } 36+64=100 \\
&= 4 && \text{Divide: } 100 \div 25 = 4.
\end{aligned}$$

Thus, $\frac{6^2+8^2}{(2+3)^2}=4$.

Exercise

Simplify: $\frac{12+3\cdot 2}{6}$ **Answer**

3

The Distributive Property

Consider the expression 2 · (3 + 4). If we follow the “Rules Guiding Order of Operations,” we would evaluate the expression inside the parentheses first. 2 · (3 + 4) = 2 · 7 Parentheses first: 3 + 4 = 7. = 14 Multiply: 2 · 7 = 14.

However, we could also choose to “distribute” the 2, first multiplying 2 times each addend in the parentheses.

$$\begin{aligned}
2\cdot(3+4) &= 2\cdot 3+2\cdot 4 && \text{Multiply 2 times both 3 and 4.} \\
&= 6+8 && \text{Multiply: } 2\cdot 3=6 \text{ and } 2\cdot 4=8. \\
&= 14 && \text{Add: } 6+8=14.
\end{aligned}$$

The fact that we get the same answer in the second approach is an illustration of an important property of whole numbers.[1]

The Distributive Property

Let a, b, and c be any whole numbers. Then,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Multiplication is distributive with respect to addition. If you are not computing the product of a number and a sum of numbers, the distributive property does not apply.

Caution! Wrong Answer Ahead!

If you are calculating the product of a number and the product of two numbers, the distributive property must not be used. For example, here is a common misapplication of the distributive property.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= (2 \cdot 3) \cdot (2 \cdot 4) \\ &= 6 \cdot 8 \\ &= 48 \end{aligned}$$

This result is quite distant from the correct answer, which is found by computing the product within the parentheses first.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24. \end{aligned}$$

In order to apply the distributive property, you must be multiplying times a sum.

Example 8

Use the distributive property to calculate $4 \cdot (5 + 11)$.

Solution

This is the product of a number and a sum, so the distributive property may be applied.

$$\begin{aligned} 4 \cdot (5+11) &= 4 \cdot 5 + 4 \cdot 11 && \text{Distribute the 4 times addend in the sum.} \\ &= 20 + 44 && \text{Multiply: } 4 \cdot 5 = 20 \text{ and } 4 \cdot 11 = 44. \\ &= 64 && \text{Add: } 20 + 44 = 64. \end{aligned}$$

Readers should check that the same answer is found by computing the sum within the parentheses first.

Exercise

Distribute: $5 \cdot (11 + 8)$.

Answer

95

The distributive property is the underpinning of the multiplication algorithm learned in our childhood years.

Example 9

Multiply: $6 \cdot 43$.

Solution

We'll express 43 as sum, then use the distributive property.

$$\begin{aligned} 6 \cdot 43 &= 6 \cdot (40+3) && \text{Express 43 as a sum: } 43 = 40+3 \\ &= 6 \cdot 40 + 6 \cdot 3 && \text{Distribute the 6.} \\ &= 240 + 18 && \text{Multiply: } 6 \cdot 40 = 240 \text{ and } 6 \cdot 3 = 18. \\ &= 258 && \text{Add: } 240 + 18 = 258. \end{aligned}$$

Readers should be able to see this application of the distributive property in the more familiar algorithmic form:

$$\begin{array}{r} 43 \\ \times 6 \\ \hline 18 \\ 240 \\ \hline 258 \end{array}$$

Or in the even more condensed form with "carrying:"

$$\begin{array}{r} {}^{1}43 \\ \times 6 \\ \hline 258 \end{array}$$

Exercise

Use the distributive property to evaluate $8 \cdot 92$.

Answer

736

Multiplication is also distributive with respect to subtraction.

The Distributive Property (Subtraction)

Let a, b, and c be any whole numbers. Then,

$$a \cdot (b - c) = a \cdot b - a \cdot c.$$

We say the multiplication is "distributive with respect to subtraction."

Example 10

Use the distributive property to simplify: $3 \cdot (12 - 8)$.

Solution

This is the product of a number and a difference, so the distributive property may be applied.

$$\begin{aligned} 3 \cdot (12-8) &= 3 \cdot 12 - 3 \cdot 8 && \text{Distribute the 3 times each term in the difference.} \\ &= 36 - 24 && \text{Multiply: } 3 \cdot 12 = 36 \text{ and } 3 \cdot 8 = 24. \\ &= 12 && \text{Subtract: } 36 - 24 = 12. \end{aligned}$$

Alternate solution

Note what happens if we use the usual "order of operations" to evaluate the expression.

$$\begin{aligned} 3 \cdot (12-8) &= 3 \cdot 4 && \text{Parentheses first: } 12 - 8 = 4. \\ &= 12 && \text{Multiply: } 3 \cdot 4 = 12. \end{aligned}$$

Same answer.

Exercise

Distribute: $8 \cdot (9 - 2)$.

Answer

56

Exercises

In Exercises 1-12, simplify the given expression.

1. $5+2 \cdot 2$
2. $5+2 \cdot 8$
3. $23-7 \cdot 2$
4. $37-3 \cdot 7$
5. $4 \cdot 3+2 \cdot 5$
6. $2 \cdot 5+9 \cdot 7$
7. $6 \cdot 5+4 \cdot 3$
8. $5 \cdot 2+9 \cdot 8$
9. $9+2 \cdot 3$
10. $3+6 \cdot 6$
11. $32-8 \cdot 2$
12. $24-2 \cdot 5$

In Exercises 13-28, simplify the given expression.

13. $45 \div 3 \cdot 5$
14. $20 \div 1 \cdot 4$
15. $2 \cdot 9 \div 3 \cdot 18$
16. $19 \cdot 20 \div 4 \cdot 16$
17. $30 \div 2 \cdot 3$
18. $27 \div 3 \cdot 3$
19. $8-6+1$
20. $15-5+10$
21. $14 \cdot 16 \div 16 \cdot 19$
22. $20 \cdot 17 \div 17 \cdot 14$
23. $15 \cdot 17+10 \div 10-12 \cdot 4$
24. $14 \cdot 18+9 \div 3-7 \cdot 13$
25. $22-10+7$
26. $29-11+1$
27. $20 \cdot 10+15 \div 5-7 \cdot 6$
28. $18 \cdot 19+18 \div 18-6 \cdot 7$

In Exercises 29-40, simplify the given expression.

29. $9+8 \div \{4+4\}$
30. $10+20 \div \{2+2\}$
31. $7 \cdot [8-5]-10$
32. $11 \cdot [12-4]-10$

33. $(18 + 10) \div (2 + 2)$

34. $(14 + 7) \div (2 + 5)$

35. $9 \cdot (10 + 7) - 3 \cdot (4 + 10)$

36. $9 \cdot (7 + 7) - 8 \cdot (3 + 8)$

37. $2 \cdot \{8 + 12\} \div 4$

38. $4 \cdot \{8+7\} \div 3$

39. $9+6 \cdot (12 + 3)$

40. $3+5 \cdot (10 + 12)$

In Exercises 41-56, simplify the given expression.

41. $2+9 \cdot [7 + 3 \cdot (9 + 5)]$

42. $6+3 \cdot [4 + 4 \cdot (5 + 8)]$

43. $7+3 \cdot [8 + 8 \cdot (5 + 9)]$

44. $4+9 \cdot [7 + 6 \cdot (3 + 3)]$

45. $6 - 5[11 - (2 + 8)]$

46. $15 - 1[19 - (7 + 3)]$

47. $11 - 1[19 - (2 + 15)]$

48. $9 - 8[6 - (2 + 3)]$

49. $4\{7[9 + 3] - 2[3 + 2]\}$

50. $4\{8[3 + 9] - 4[6 + 2]\}$

51. $9 \cdot [3 + 4 \cdot (5 + 2)]$

52. $3 \cdot [4 + 9 \cdot (8 + 5)]$

53. $3\{8[6 + 5] - 8[7 + 3]\}$

54. $2\{4[6 + 9] - 2[3 + 4]\}$

55. $3 \cdot [2 + 4 \cdot (9 + 6)]$

56. $8 \cdot [3 + 9 \cdot (5 + 2)]$

In Exercises 57-68, simplfiy the given expression.

57. $(5 - 2)^2$

58. $(5 - 3)^4$

59. $(4 + 2)^2$

60. $(3 + 5)^2$

61. $2^3 + 3^3$

62. $5^4 + 2^4$

63. $2^3 - 1^3$

64. $3^2 - 1^2$

65. $12 \cdot 5^2 + 8 \cdot 9+4$

66. $6 \cdot 3^2 + 7 \cdot 5 + 12$

67. $9 - 3 \cdot 2 + 12 \cdot 10^2$

68. $11 - 2 \cdot 3 + 12 \cdot 4^2$

In Exercises 69-80, simplify the given expression.

69. $4^2 - (13 + 2)$

70. $3^3 - (7 + 6)$

71. $3^3 - (7 + 12)$

72. $4^3 - (6 + 5)$

73. $19 + 3[12 - (2^3 + 1)]$

74. $13 + 12[14 - (2^2 + 1)]$

75. $17 + 7[13 - (2^2 + 6)]$

76. $10 + 1[16 - (2^2 + 9)]$

77. $4^3 - (12 + 1)$

78. $5^3 - (17 + 15)$

79. $5 + 7[11 - (2^2 + 1)]$

80. $10 + 11[20 - (2^2 + 1)]$

In Exercises 81-92, simplify the given expression.

81. $\frac{13+35}{3(4)}$

82. $\frac{35+28}{7(3)}$

83. $\frac{64-(8\cdot 6-3)}{4\cdot 7-9}$

84. $\frac{19-(4\cdot 3-2)}{6\cdot 3-9}$

85. $\frac{2+13}{4-1}$

86. $\frac{7+1}{8-4}$

87. $\frac{17+14}{9-8}$

88. $\frac{16+2}{13-11}$

89. $\frac{37+27}{8(2)}$

90. $\frac{16+38}{6(3)}$

91. $\frac{40-(3\cdot 7-9)}{8\cdot 2-2}$

92. $\frac{60-(8\cdot 6-3)}{5\cdot 4-5}$

In Exercises 93-100, use the distributive property to evaluate the given expression.

93. $5 \cdot (8 + 4)$

94. $8 \cdot (4 + 2)$

95. $7 \cdot (8 - 3)$

96. $8 \cdot (9 - 7)$

97. $6 \cdot (7 - 2)$

98. $4 \cdot (8 - 6)$

99. $4 \cdot (3 + 2)$

100. $4 \cdot (9 + 6)$

In Exercises 101-104, use the distributive property to evaluate the given expression using the technique shown in Example 9.

101. $9 \cdot 62$

102. $3 \cdot 76$

103. $3 \cdot 58$

104. $7 \cdot 57$

Answers

1. 9

3. 9

5. 22

7. 42

9. 15

11. 16

13. 75

15. 108

17. 45

19. 3

21. 266

23. 208

25. 19

27. 161

29. 10

31. 11

33. 7

35. 111

37. 10

39. 99

41. 443

43. 367

45. 1

47. 9

49. 296

51. 279

53. 24

55. 186

57. 9

59. 36

61. 35

63. 7

65. 376

67. 1203

69. 1

71. 8

73. 28

75. 38

77. 51

79. 47

81. 4

83. 1

85. 5

87. 31

89. 4

91. 2

93. 60

95. 35

97. 30

99. 20

101. 558

103. 174

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[1]Later, we'll see that this property applies to all numbers, not just whole numbers

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CHAPTER OVERVIEW

2: Fractions

In this chapter, you'll learn how we use fractions.

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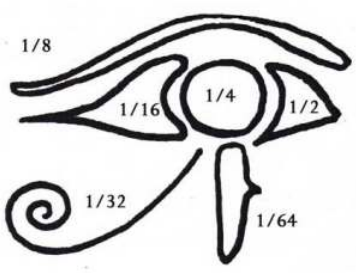
2.1: Prelude to Fractions - Egyptian Math

Around 3000 BC, Egyptians were carving hieroglyphs into stone monuments to their kings and queens. Hierglyphs are pictures that represent objects and they were used for words and numbers. Oddly, fractions were always written as sums of “unit fractions,” fractions whose numerator is always 1. For instance, instead of writing 3/5 , they would write a sum of unit fractions.

$$\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$$

Much of the ancient Egyptian math that we know of was in service to the agricultural and economic life of the people. in measuring dry goods such as grains, special glyphs were used to represent basic fractional amounts, glyphs that came together to represent the Eye of Horus.



Horus was a falcon-god whose father Osirus was murdered by his own brother Seth. When Horus attempted to avenge his father’s death, Seth ripped out Horus’ eye and cut it into six pieces, scattering them throughout Egypt.

Taking pity on Horus, Thot, the god of learning and magic, found the pieces and put them back together making Horus healthy and whole again. Each piece of the Eye of Horus represents a different fraction of a hekat, or volume of grain. It was written that an apprentice scribe added the fractions one day and got

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}=\frac{63}{64}.$$

Asking where the missing 1/64 was, he was told that Thot would make up the difference to anyone “who sought and accepted is protection.”

In this chapter, you’ll learn how we use fractions.

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2.2: Equivalent Fractions

In this section we deal with fractions, numbers or expressions of the form a/b.

Definition: Fractions

A number of the form

$$\frac{a}{b}$$

where a and b are numbers is called a *fraction*. The number a is called the *numerator* of the fraction, while the number b is called the *denominator* of the fraction.

Near the end of this section, we'll see that the numerator and denominator of a fraction can also be algebraic expressions, but for the moment we restrict our attention to fractions whose numerators and denominators are integers. We start our study of fractions with the definition of *equivalent fractions*.

Equivalent Fractions

Two fractions are *equivalent* if they represent the same numerical value.

But how can we tell if two fractions represent the same number? Well, one technique involves some simple visualizations. Consider the image shown in Figure 4.1, where the shaded region represents 1/3 of the total area of the figure (one of three equal regions is shaded).

Figure 4.1: The shaded region is 1/3 of the whole region.

In Figure 4.2, we've shaded 2/6 of the entire region (two of six equal regions are shaded).

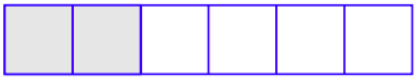


Figure 4.2: The shaded region is 2/6 of the whole region.

In Figure 4.3, we've shaded 4/12 of the entire region (four of twelve equal regions are shaded).

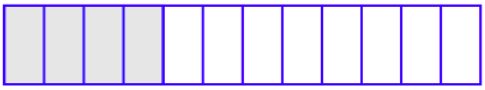


Figure 4.3: The shaded region is 4/12 of the whole region.

Let's take the diagrams from Figure 4.1, Figure 4.2, and Figure 4.3 and stack them one atop the other, as shown in Figure 4.4.

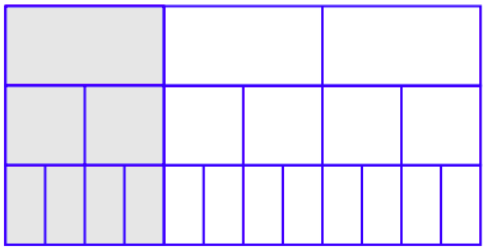


Figure 4.4: One of three equals two of six equals four of twelve.

Figure 4.4 provides solid visual evidence that the following fractions are equivalent.

$$\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$$

Key Observations

1. If we start with the fraction 1/3, then multiply both numerator and denominator by 2, we get the following result.

$$\begin{aligned} \frac{1}{3} &= \frac{1 \cdot 2}{3 \cdot 2} && \text{Multiply the numerator and denominator by 2.} \\ &= \frac{2}{6} && \text{Simplify numerator and denominator.} \end{aligned}$$

This is precisely the same thing that happens going from Figure 4.1 to 4.2, where we double the number of available boxes (going from 3 available to 6 available) and double the number of shaded boxes (going from 1 shaded to 2 shaded).

2. If we start with the fraction 1/3, then multiply both numerator and denominator by 4, we get the following result.

$$\begin{aligned} = \frac{1}{3} &= \frac{1 \cdot 4}{3 \cdot 4} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{4}{12} && \text{Simplify numerator and denominator.} \end{aligned}$$

This is precisely the same thing that happens going from Figure 4.1 to 4.3, where we multiply the number of available boxes by 4 (going from 3 available to 12 available) and multiply the number of shaded boxes by 4 (going from 1 shaded to 4 shaded).

The above discussion motivates the following fundamental result.

Creating Equivalent Fractions

If you start with a fraction, then multiply both its numerator and denominator by the same number, the resulting fraction is equivalent (has the same numerical value) to the original fraction. In symbols,

$$\frac{a}{b} = \frac{a \cdot x}{b \cdot x}$$

Arguing in Reverse

Reversing the above argument also holds true.

1. If we start with the fraction 2/6, then divide both numerator and denominator by 2, we get the following result.

$$\begin{aligned} \frac{2}{6} &= \frac{2 \div 2}{6 \div 2} && \text{Divide numerator and denominator by 2.} \\ &= \frac{1}{3} && \text{Simplify numerator and denominator.} \end{aligned}$$

This is precisely the same thing that happens going backwards from Figure 4.2 to 4.1, where we divide the number of available boxes by 2 (going from 6 available to 3 available) and dividing the number of shaded boxes by 2 (going from 2 shaded to 1 shaded).

2. If we start with the fraction 4/12, then divide both numerator and denominator by 4, we get the following result.

$$\begin{aligned} \frac{4}{12} &= \frac{4 \div 4}{12 \div 4} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{1}{3} && \text{Simplify numerator and denominator.} \end{aligned}$$

This is precisely the same thing that happens going backwards from Figure 4.3 to 4.1, where we divide the number of available boxes by 4 (going from 12 available to 3 available) and divide the number alignof shaded boxes by 4 (going from 4 shaded to 1 shaded).

The above discussion motivates the following fundamental result.

Creating Equivalent Fractions

If you start with a fraction, then divide both its numerator and denominator by the same number, the resulting fraction is equivalent (has the same numerical value) to the original fraction. In symbols,

$$\frac{a}{b} = \frac{a \div x}{b \div x}.$$

The Greatest Common Divisor

We need a little more terminology.

Divisor

If d and a are natural numbers, we say that "d divides a" if and only if when a is divided by d, the remainder is zero. In this case, we say that "d is a divisor of a."

For example, when 36 is divided by 4, the remainder is zero. In this case, we say that "4 is a divisor of 36." On the other hand, when 25 is divided by 4, the remainder is not zero. In this case, we say that "4 is not a divisor of 25."

Greatest Common Divisor

Let a and b be natural numbers. The common divisors of a and b are those natural numbers that divide both a and b. The *greatest common divisor* is the largest of these common divisors.

Example 1

Find the greatest common divisor of 18 and 24.

Solution

First list the divisors of each number, the numbers that divide each number with zero remainder.

Divisors of 18 : 1, 2, 3, 6, 9, and 18

Divisors of 24 : 1, 2, 3, 4, 6, 8, 12, and 24

The common divisors are:

Common Divisors : 1, 2, 3, and 6

The greatest common divisor is the largest of the common divisors. That is,

Greatest Common Divisor = 6.

That is, the largest number that divides both 18 and 24 is the number 6.

Exercise

Find the greatest common divisor of 12 and 18.

Answer

6

Reducing a Fraction to Lowest Terms

First, a definition.

Lowest Terms

A fraction is said to be *reduced to lowest terms* if the greatest common divisor of both numerator and denominator is 1.

Thus, for example, 2/3 is reduced to lowest terms because the greatest common divisior of 2 and 3 is 1. On the other hand, 4/6 is **not** reduced to lowest terms because the greatest common divisor of 4 and 6 is 2.

Example 2

Reduce the fraction 18/24 to lowest terms.

Solution

One technique that works well is dividing both numerator and denominator by the greatest common divisor of the numerator and denominator. In Example 1, we saw that the greatest common divisor of 18 and 24 is 6. We divide both numerator and denominator by 6 to get

$$\frac{18}{24} = \frac{18 \div 6}{24 \div 6} \quad \text{Divide numerator and denominator by 6.}$$
$$= \frac{3}{4} \quad \text{Simplify numerator and dice.}$$

Note that the greatest common divisor of 3 and 4 is now 1. Thus, 3/4 is reduced to lowest terms.

There is a second way we can show division of numerator and denominator by 6. First, factor both numerator and denominator as follows:

$$\frac{18}{24} = \frac{3 \cdot 6}{4 \cdot 6} \quad \text{Factor out a 6.}$$

You can then show “division” of both numerator and denominator by 6 by “crossing out” or “canceling” a 6 in the numerator for a 6 in the denominator, like this:

$$= \frac{3 \cdot \cancel{6}}{4 \cdot \cancel{6}} \quad \text{Cancel common factor.}$$
$$= \frac{3}{4}$$

Note that we get the same equivalent fraction, reduced to lowest terms, namely 3/4.

Exercise

Reduce the fraction 12/18 to lowest terms.

Answer

2/3

Important Point

In Example 2 we saw that 6 was both a *divisor* and a *factor* of 18. The words *divisor* and *factor* are equivalent.

We used the following technique in our second solution in Example 2.

Cancellation Rule

If you express numerator and denominator as a product, then you may cancel common factors from the numerator and denominator. The result will be an equivalent fraction.

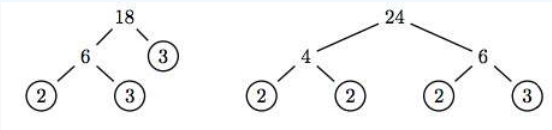
Because of the “Cancellation Rule,” one of the most effective ways to reduce a fraction to lowest terms is to first find prime factorizations for both numerator and denominator, then cancel all common factors.

Example 3

Reduce the fraction 18/24 to lowest terms.

Solution

Use factor trees to prime factor numerator and denominator.



Once we’ve factored the numerator and denominator, we cancel common factors.

$$\begin{aligned}\frac{18}{24} &= \frac{2\cdot 3\cdot 3}{2\cdot 2\cdot 2\cdot 3} && \text{Prime factor numerator and denominator.}\\ &= \frac{\cancel{2}\cdot\cancel{3}\cdot 3}{\cancel{2}\cdot 2\cdot 2\cdot\cancel{3}} && \text{Cancel common factors.}\\ &= \frac{3}{2\cdot 2} && \text{Remaining factors.}\\ &= \frac{3}{4} && \text{Simplify denominator.}\end{aligned}$$

Thus, 18/24 = 3/4.

Exercise 2.2.1

Reduce the fraction 28/35 to lowest terms.

Answer

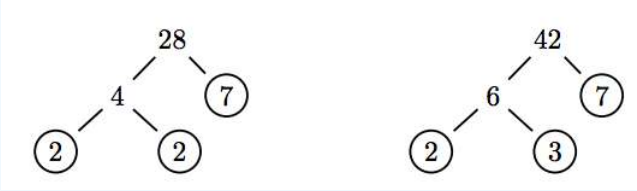
4/5

Example 4

Reduce the fraction 28/42 to lowest terms.

Solution

Use factor trees to prime factor numerator and denominator.



Now we can cancel common factors.

$$\begin{aligned}\frac{28}{42} &= \frac{2\cdot 2\cdot 7}{2\cdot 3\cdot 7} && \text{Prime factor numerator and denominator.}\\ &= \frac{\cancel{2}\cdot 2\cdot\cancel{7}}{\cancel{2}\cdot 3\cdot\cancel{7}} && \text{Cancel common factors.}\\ &= \frac{2}{3}\end{aligned}$$

Thus, 28/42 = 2/3.

Exercise

Reduce the fraction 36/60 to lowest terms.

Answer

3/5

Reducing Fractions with Variables

We use exactly the same technique to reduce fractions whose numerators and denominators contain variables.

Example 5

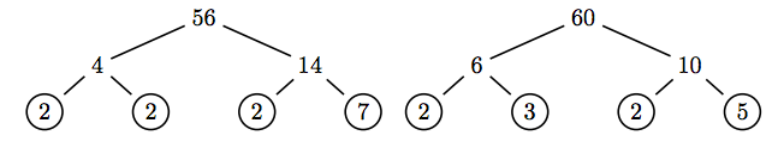
Reduce

$$\frac{56x^2y}{60xy^2}$$

to lowest terms.

Solution

Use factor trees to factor the coefficients of numerator and denominator.



Now cancel common factors.

$$\frac{56x^2y}{60xy^2} = \frac{2\cdot 2\cdot 2\cdot 7\cdot x\cdot x\cdot y}{2\cdot 2\cdot 3\cdot 5\cdot x\cdot y\cdot y} \quad \text{Prime factor numerator and denominator.}$$

$$= \frac{\cancel{2}\cdot\cancel{2}\cdot 2\cdot 7\cdot \cancel{x}\cdot x\cdot \cancel{y}}{\cancel{2}\cdot\cancel{2}\cdot 3\cdot 5\cdot \cancel{x}\cdot y\cdot \cancel{y}} \quad \text{Cancel cmmon factors.}$$

$$= \frac{2\cdot 7\cdot x}{3\cdot 5\cdot y} \quad \text{Remaining factors.}$$

$$= \frac{14x}{15y} \quad \text{Simplify numerator and denominator.}$$

Thus, $56x^2y/(60xy^2) = 14x/(15y)$.

Exercise

Reduce:

$$\frac{25a^3b}{40a^2b^3}$$

Answer

$$\frac{5a}{8b^2}$$

A Word on Mathematical Notation

There are two types of mathematical notation: (1) inline mathematical notation, and (2) displayed mathematical notation.

Inline Mathematical Notation

The notation 14*x*/(15*y*) is called **inline mathematical notation**. When the same expression is centered on its own line, as in

$$\frac{14x}{15y},$$

this type of notation is called displayed mathematical notation.

When you work a problem by hand, using pencil and paper calculations, the preferred format is displayed notation, like the displayed notation used to simplify the given expression in Example 5. However, computers and calculators require that you enter your expressions using inline mathematical notation. Therefore, it is extremely important that you are equally competent with either mathematical notation: displayed or inline.

By the way, order of operations, when applied to the inline expression 14x/(15y), requires that we perform the multiplication inside the parentheses first. Then we must perform multiplications and divisions as they occur, as we move from left to right through the expression. This is why the inline notation 14x/(15y) is equivalent to the displayed notation

$$\frac{14x}{15y}.$$

However, the expression 14x/15y is a different beast. There are no parentheses, so we perform multiplication and division as they occur, moving left to right through the expression. Thus, we must first take the product of 14 and x, divide the result by 15, then multiply by y. In displayed notation, this result is equivalent to

$$\frac{14x}{15} \cdot y,$$

which is a different result.

Some readers might wonder why we did not use the notation (14x)/(15y) to describe the solution in Example 5. After all, this inline notation is also equivalent to the displayed notation

$$\frac{14x}{15y}.$$

However, the point is that we do not need to, as order of operations already requires that we take the product of 14 and x before dividing by 15y. If this is hurting your head, know that it's quite acceptable to use the equivalent notation (14x)/(15y) instead of 14x/(15y). Both are correct.

Equivalent Fractions in Higher Terms

Sometimes the need arises to find an equivalent fraction with a different, larger denominator.

Example 6

Express 3/5 as an equivalent fraction having denominator 20.

Solution

The key here is to remember that multiplying numerator and denominator by the same number produces an equivalent fraction. To get an equivalent fraction with a denominator of 20, we'll have to multiply numerator and denominator of 3/5 by 4.

$$\begin{aligned} &\frac{3}{5} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{12}{20} && \text{Simplify numerator and denominator.} \end{aligned}$$

Therefore, 3/5 equals 12/20.

Exercise

Express 2/3 as an equivalent fraction having denominator 21.

Answer

14/21

Example 7

Express 8 as an equivalent fraction having denominator 5.

Solution

The key here is to note that

$$8 = \frac{8}{1} \quad \text{Understood denominator is 1.}$$

To get an equivalent fraction with a denominator of 5, we'll have to multiply numerator and denominator of 8/1 by 5.

$$= \frac{8 \cdot 5}{1 \cdot 5} \quad \text{Multiply numerator and denominator by 5.}$$
$$= \frac{40}{5} \quad \text{Simplify numerator and denominator.}$$

Therefore, 8 equals 40/5.

Exercise

Express 5 as an equivalent fraction having denominator 7.

Answer

35/7

Example 8

Express 2/9 as an equivalent fraction having denominator 18a.

Solution

To get an equivalent fraction with a denominator of 18a, we'll have to multiply numerator and denominator of 2/9 by 2a.

$$\frac{2}{9} = \frac{2 \cdot 2a}{9 \cdot 2a} \quad \text{Multiply numerator and denominator by } 2a.$$
$$= \frac{4a}{18a} \quad \text{Simplify numerator and denominator.}$$

Therefore, 2/9 equals 4a/(18a), or equivalently, (4a)/(18a).

Exercise

Express 3/8 as an equivalent fraction having denominator 24a.

Answer

$$\frac{9a}{24a}$$

Negative Fractions

We have to also deal with fractions that are negative. First, let's discuss placement of the negative sign.

- Positive divided by negative is negative, so

$$\frac{3}{-5} = -\frac{3}{5}.$$

- But it is also true that negative divided by positive is negative. Thus,

$$\frac{-3}{5} = \frac{-3}{5}.$$

These two observations imply that all three of the following fractions are equivalent (the same number):

$$\frac{3}{-5} = -\frac{3}{5} = \frac{-3}{5}.$$

Note that there are three possible placements for the negative sign: (1) the denominator, (2) the fraction bar, or (3) the numerator. Any one of these placements produces an equivalent fraction.

Fractions and Negative Signs

Let *a* and *b* be any integers. All three of the following fractions are equivalent (same number):

$$\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}.$$

Mathematicians prefer to place the negative sign either in the numerator or on the fraction bar. The use of a negative sign in the denominator is discouraged.

Example 9

Reduce:

$$\frac{50x^3}{-75x^5}$$

to lowest terms.

Solution

Prime factor numerator and denominator and cancel.

$$\begin{aligned}\frac{50x^3}{-75x^5} &= \frac{2\cdot 5\cdot 5\cdot x\cdot x\cdot x}{-3\cdot 5\cdot 5\cdot x\cdot x\cdot x\cdot x\cdot x} \\ &= \frac{2\cdot \cancel{5}\cdot \cancel{5}\cdot \cancel{x}\cdot \cancel{x}\cdot \cancel{x}}{-3\cdot \cancel{5}\cdot \cancel{5}\cdot \cancel{x}\cdot \cancel{x}\cdot \cancel{x}\cdot x\cdot x} \\ &= \frac{2}{-3\cdot x\cdot x} \\ &= \frac{2}{-3x^2}\end{aligned}$$

However, it is preferred that there be no negative signs in the denominator, so let's place the negative sign on the fraction bar (the numerator would suit as well). Thus,

$$\frac{50x^3}{-75x^5} = -\frac{2}{3x^2}$$

We also have the following result.

Fractions and Negative Signs

Let a and b be any integers. Then,

$$\frac{-a}{-b} = \frac{a}{b}.$$

Example 10

Reduce:

$$\frac{-12xy^2}{-18x^2y}$$

Solution

Unlike Example 9, some like to take care of the sign of the answer first.

$$\frac{-12xy^2}{-18x^2y} = \frac{12xy^2}{18x^2y}$$

Now we can factor numerator and denominator and cancel common factors.

$$= \frac{2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y}{2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot y}$$

$$= \frac{\cancel{2} \cdot 2 \cancel{3} \cdot \cancel{x} \cdot y \cdot \cancel{y}}{\cancel{2} \cdot \cancel{3} \cdot 3 \cdot \cancel{x} \cdot x \cdot \cancel{y}} \qquad = \frac{2y}{3x}$$

Thus,

$$\frac{-12xy^2}{-18x^2y} = \frac{2y}{3x}.$$

Exercise

Reduce:

$$\frac{-21a^2b^3}{-56a^3b}$$

Answer

$$\frac{3b^2}{8a}$$

Exercises

In Exercises 1-12, find the GCD of the given numbers.

1. 72, 8
2. 76, 52
3. 52, 20
4. 56, 96
5. 36, 63
6. 63, 21
7. 72, 44

8. 10, 40

9. 16, 56

10. 54, 66

11. 84, 24

12. 75, 45

In Exercises 13-28, reduce the given fraction to lowest terms.

13. $\frac{22}{98}$

14. $\frac{28}{56}$

15. $\frac{93}{15}$

16. $\frac{90}{39}$

17. $\frac{69}{21}$

18. $\frac{74}{62}$

19. $\frac{74}{12}$

20. $\frac{66}{10}$

21. $\frac{66}{57}$

22. $\frac{34}{30}$

23. $\frac{33}{99}$

24. $\frac{20}{58}$

25. $\frac{69}{24}$

26. $\frac{18}{96}$

27. $\frac{46}{44}$

28. $\frac{92}{24}$

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29. Express 3 as an equivalent fraction having denominator 24. 30. Express 3 as an equivalent fraction having denominator 8. 31. Express $\frac{25}{19}$ as an equivalent fraction having denominator 57. 32. Express $\frac{29}{22}$ as an equivalent fraction having denominator 44. 33. Express 2 as an equivalent fraction having denominator 2. 34. Express 2 as an equivalent fraction having denominator 8. 35. Express $\frac{18}{19}$ as an equivalent fraction having denominator 95. 36. Express $\frac{17}{22}$ as an equivalent fraction having denominator 44. 37. Express $\frac{1}{3}$ as an equivalent fraction having denominator 24. 38. Express $\frac{15}{19}$ as an equivalent fraction having denominator 95. 39. Express 16 as an equivalent fraction having denominator 4. 40. Express 5 as an equivalent fraction having denominator 2.

In Exercises 41-56, reduce the given fraction to lowest terms.

41. $\frac{34}{-86}$
42. $\frac{-48}{14}$
43. $\frac{-72}{-92}$
44. $\frac{27}{-75}$
45. $\frac{-92}{82}$
46. $\frac{-44}{-62}$
47. $\frac{-21}{33}$
48. $\frac{57}{-99}$
49. $\frac{22}{-98}$
50. $\frac{-33}{69}$
51. $\frac{42}{-88}$
52. $\frac{-100}{48}$
53. $\frac{94}{-6}$
54. $\frac{-36}{-38}$
55. $\frac{10}{-86}$
56. $\frac{-100}{-46}$

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57. Express $\frac{3}{2}$ as an equivalent fraction having denominator 62n.
58. Express $\frac{6}{25}$ as an equivalent fraction having denominator 50a.
59. Express $\frac{13}{10}$ as an equivalent fraction having denominator 60m.
60. Express $\frac{1}{16}$ as an equivalent fraction having denominator 80p.
61. Express $\frac{3}{2}$ as an equivalent fraction having denominator 50n.
62. Express $\frac{43}{38}$ as an equivalent fraction having denominator 76a.
63. Express 11 as an equivalent fraction having denominator 4m. 64. Express 13 as an equivalent fraction having denominator 6n.
65. Express 3 as an equivalent fraction having denominator 10m.
66. Express 10 as an equivalent fraction having denominator 8b.
67. Express 6 as an equivalent fraction having denominator 5n.

68. Express 16 as an equivalent fraction having denominator 2y.

In Exercises 69-84, reduce the given fraction to lowest terms.

69. $\frac{82y^5}{-48y}$
70. $\frac{-40y^5}{-55y}$
71. $\frac{-77x^5}{44x^4}$
72. $\frac{-34x^6}{-80x}$
73. $\frac{-14y^5}{54y^2}$
74. $\frac{96y^4}{-40y^2}$
75. $\frac{42x}{81x^3}$
76. $\frac{26x^2}{32x^6}$
77. $\frac{-12x^5}{14x^6}$
78. $\frac{-28y^4}{72y^6}$
79. $\frac{-74x}{22x^2}$
80. $\frac{56x^2}{26x^3}$
81. $\frac{-12y^5}{98y^6}$
82. $\frac{96x^2}{14x^4}$
83. $\frac{18x^6}{-54x^2}$
84. $\frac{32x^6}{62x^2}$

In Exercises 85-100, reduce the given fraction to lowest terms.

85. $\frac{26y^2x^4}{-62y^6x^2}$
86. $\frac{6x^2y^3}{40x^3y^2}$
87. $\frac{-2y^6x^4}{-94y^2x^5}$
88. $\frac{90y^6x^3}{39y^3x^5}$

89. $\frac{30y^5x^5}{-26yx^4}$

90. $\frac{74x^6y^4}{-52xy^3}$

91. $\frac{36x^3y^2}{-98x^4y^5}$

92. $\frac{84x^3y}{16x^4y^2}$

93. $\frac{-8x^6y^3}{54x^3y^5}$

94. $\frac{70y^5x^2}{16y^4x^5}$

95. $\frac{34yx^6}{-58y^5x^4}$

96. $\frac{99y^2x^3}{88y^6x}$

97. $\frac{-36y^3x^5}{51y^2x}$

98. $\frac{44y^5x^5}{-88y^4x}$

99. $\frac{91y^3x^2}{-28y^5x^5}$

100. $\frac{-76y^2x}{-57y^5x^6}$

---

101. **Hurricanes**. According to the National Atmospheric and Oceanic Administration, in 2008 there were 16 named storms, of which 8 grew into hurricanes and 5 were major.

i) What fraction of named storms grew into hurricanes? Reduce your answer to lowest terms.

ii) What fraction of named storms were major hurricanes? Reduce your answer to lowest terms.

iii) What fraction of hurricanes were major? Reduce your answer to lowest terms.

102. **Tigers**. Tigers are in critical decline because of human encroachment, the loss of more than nine-tenths of their habitat, and the growing trade in tiger skins and body parts. *Associated Press-Times-Standard 01/24/10 Pressure mounts to save the tiger.*

i) Write the loss of habitat as a fraction.

ii) Describe in words what the numerator and denominator of this fraction represent.

iii) If the fraction represents the loss of the whole original habitat, how much of the original habitat remains?

Answers

1. 8

3. 4

5. 9

7. 4

9. 8

11. 12
13. $\frac{11}{49}$
15. $\frac{31}{5}$
17. $\frac{23}{7}$
19. $\frac{37}{6}$
21. $\frac{22}{19}$
23. $\frac{1}{3}$
25. $\frac{23}{8}$
27. $\frac{23}{22}$
29. $\frac{72}{24}$
31. $\frac{75}{57}$
33. $\frac{4}{2}$
35. $\frac{90}{95}$
37. $\frac{8}{24}$
39. $\frac{64}{4}$
41. $\frac{-17}{43}$
43. $\frac{18}{23}$
45. $\frac{-46}{41}$
47. $\frac{-7}{11}$
49. $\frac{-11}{49}$
51. $\frac{-21}{44}$
53. $\frac{-47}{3}$
55. $\frac{-5}{43}$
57. $\frac{93n}{62n}$
59. $\frac{78m}{60m}$
61. $\frac{75n}{50n}$

63. $\frac{44m}{4m}$
65. $\frac{30m}{10m}$
67. $\frac{30n}{5n}$
69. $\frac{-41y^4}{24}$
71. $\frac{-7x}{4}$
73. $-\frac{7y^3}{27}$
75. $\frac{14}{27x^2}$
77. $-\frac{6}{7x}$
79. $-\frac{37}{11x}$
81. $-\frac{6}{49y}$
83. $-\frac{x^4}{3}$
85. $-\frac{13x^2}{31y^4}$
87. $\frac{y^4}{47x}$
89. $-\frac{15y^4x}{13}$
91. $-\frac{18}{49xy^3}$
93. $-\frac{4x^3}{27y^2}$
95. $-\frac{17x^2}{29y^4}$
97. $-\frac{12yx^4}{17}$
99. $-\frac{13}{4y^2x^3}$
101.
    i) $\frac{1}{2}$
    ii) $\frac{5}{16}$
    iii) $\frac{5}{8}$

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2.3: Multiplying Fractions

Consider the image in Figure 4.5, where the vertical lines divide the rectangular region into three equal pieces. If we shade one of the three equal pieces, the shaded area represents 1/3 of the whole rectangular region.

Figure 4.5: The shaded region is 1/3 of the whole region.

We'd like to visualize taking 1/2 of 1/3. To do that, we draw an additional horizontal line which divides the shaded region in half horizontally. This is shown in Figure 4.6. The shaded region that represented 1/3 is now divided into two smaller rectangular regions, one of which is shaded with a different color. This region represents 1/2 of 1/3.

Figure 4.6: Shading 1/2 of 1/3.

Next, extend the horizontal line the full width of the rectangular region, as shown in Figure 4.7.

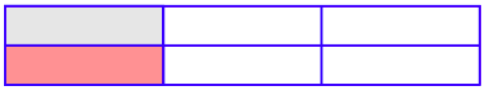


Figure 4.7: Shading 1/2 of 1/3.

Note that drawing the horizontal line, coupled with the three original vertical lines, has succeeded in dividing the full rectangular region into six smaller but equal pieces, only one of which (the one representing 1/2 of 1/3) is shaded in a new color. Hence, this newly shaded piece represents 1/6 of the whole region. The conclusion of our visual argument is the fact that 1/2 of 1/3 equals 1/6. In symbols,

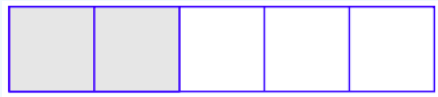
$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Example 1

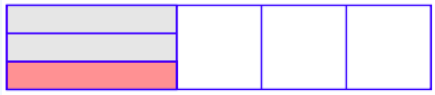
Create a visual argument showing that 1/3 of 2/5 is 2/15.

Solution

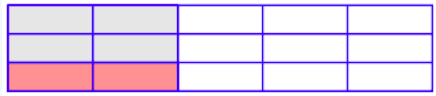
First, divide a rectangular region into five equal pieces and shade two of them. This represents the fraction 2/5.



Next, draw two horizontal lines that divide the shaded region into three equal pieces and shade 1 of the three equal pieces. This represents taking 1/3 of 2/5.



Next, extend the horizontal lines the full width of the region and return the original vertical line from the first image.



Note that the three horizontal lines, coupled with the five original vertical lines, have succeeded in dividing the whole region into 15 smaller but equal pieces, only two of which (the ones representing 1/3 of 2/5) are shaded in the new color. Hence, this

newly shaded piece represents 2/15 of the whole region. The conclusion of this visual argument is the fact that 1/3 of 2/5 equals 2/15. In symbols,

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}.$$

? Exercise

Create a visual argument showing that 1/2 of 1/4 is 1/8.

Answer

Diagram:

Multiplication Rule

In Figure 4.7, we saw that 1/2 of 1/3 equals 1/6. Note what happens when we multiply the numerators and multiply the denominators of the fractions 1/2 and 1/3.

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1 \cdot 1}{2 \cdot 3} \quad \text{Multiply numerators; multiply denominators.}$$
$$= \frac{1}{6} \quad \text{Simplify numerators and denominators.}$$

We get 1/6!

Could this be coincidence or luck? Let's try that again with the fractions from Example 1, where we saw that 1/3 of 2/5 equals 2/15. Again, multiply the numerators and denominators of 1/3 and 2/5.

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{1 \cdot 2}{3 \cdot 5} \quad \text{Multiply numerators; multiply denominators.}$$
$$= \frac{2}{15} \quad \text{Simplify numerators and denominators.}$$

Again, we get 2/15!

These two examples motivate the following definition.

Definition: Multiplication Rule

To find the product of the fractions a/b and c/d, multiply their numerators and denominators. In symbols,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

✓ Example 2

Multiply 1/5 and 7/9.

Solution

Multiply numerators and multiply denominators.

$$\frac{1}{5} \cdot \frac{7}{9} = \frac{1 \cdot 7}{5 \cdot 9} \quad \text{Multiply numerators; multiply denominators.}$$
$$= \frac{7}{45} \quad \text{Simplify numerators and denominators.}$$

Exercise

Multiply:

$$\frac{1}{3} \cdot \frac{2}{5}$$

Answer

Multiply: $\frac{6}{15}$

Example 3

Find the product of −2/3 and 7/9.

Solution

The usual rules of signs apply to products. Unlike signs yield a negative result.

$$-\frac{2}{3} \cdot \frac{7}{9} = -\frac{2 \cdot 7}{3 \cdot 9} \quad \text{Multiply numerators; multiply denominators.}$$
$$= -\frac{14}{27} \quad \text{Simplify numerators and denominators.}$$

It is not required that you physically show the middle step. If you want to do that mentally, then you can simply write

$$-\frac{2}{3} \cdot \frac{7}{9} = -\frac{14}{27}.$$

Exercise

Multiply:

$$-\frac{3}{5} \cdot \frac{2}{7}$$

Answer

$$-\frac{6}{35}$$

Multiply and Reduce

After multiplying two fractions, make sure your answer is reduced to lowest terms (see Section 4.1).

Example 4

Multiply 3/4 times 8/9.

Solution

After multiplying, divide numerator and denominator by the greatest common divisor of the numerator and denominator.

$$\frac{3}{4} \cdot \frac{8}{9} = \frac{3 \cdot 8}{4 \cdot 9} \quad \text{Multiply numerators and denominators.}$$
$$= \frac{24}{36} \quad \text{Simplify numerator and denominator.}$$
$$= \frac{24 \div 12}{36 \div 12} \quad \text{Divide numerator and denominator by GCD.}$$
$$= \frac{2}{3} \quad \text{Simplify numerator and denominator.}$$

Alternatively, after multiplying, you can prime factor both numerator and denominator, then cancel common factors.

$$\begin{aligned}
\frac{3}{4}\cdot\frac{8}{9} &= \frac{24}{36} && \text{Multiply numerators and denominators.}\\
&= \frac{2\cdot 2\cdot 2\cdot 3}{2\cdot 2\cdot 3\cdot 3} && \text{Prime factor numerator and denominator.}\\
&= \frac{\cancel{2}\cdot\cancel{2}\cdot 2\cdot\cancel{3}}{\cancel{2}\cdot\cancel{2}\cdot 3\cdot\cancel{3}} && \text{Cancel common factors.}\\
&= \frac{2}{3}
\end{aligned}$$

? Exercise

Multiply:

$$\frac{3}{7}\cdot\frac{14}{9}$$

Answer

$$\frac{2}{3}$$

✓ Example 5

Multiply $-7x/2$ and $5/(14x^2)$.

Solution

After multiplying, prime factor both numerator and denominator, then cancel common factors. Note that unlike signs yields a negative product.

$$\begin{aligned}
-\frac{7x}{2}\cdot\frac{5}{14x^2} &= -\frac{35x}{28x^2} && \text{Multiply numerators and denominators.}\\
&= -\frac{5\cdot 7\cdot x}{2\cdot 2\cdot 7\cdot x\cdot x} && \text{Prime factor numerator and denominator.}\\
&= -\frac{5\cdot\cancel{7}\cdot\cancel{x}}{2\cdot 2\cdot\cancel{7}\cdot\cancel{x}\cdot x} && \text{Cancel common factors.}\\
&= -\frac{5}{4x}
\end{aligned}$$

? Exercise

Multiply:

$$-\frac{3x}{2}\cdot\frac{6}{21x^3}$$

Answer

$$-\frac{3}{7x^2}$$

Multiply and Cancel or Cancel and Multiply

When you are working with larger numbers, it becomes a bit harder to multiply, factor, and cancel. Consider the following argument.

$$\begin{aligned}
\frac{18}{30}\cdot\frac{35}{6} &= \frac{630}{180} && \text{Multiply numerators; multiply denominators.}\\
&= \frac{2\cdot 3\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 3\cdot 3\cdot 5} && \text{Prime factor numerators and denominators.}\\
&= \frac{\cancel{2}\cdot \cancel{3}\cdot \cancel{3}\cdot \cancel{5}\cdot 7}{2\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{3}\cdot \cancel{5}} && \text{Cancel common factors.}\\
&= \frac{7}{2} && \text{Remaining factors.}
\end{aligned}$$

There are a number of difficulties with this approach. First, you have to multiply large numbers, and secondly, you have to prime factor the even larger results.

One possible workaround is to not bother multiplying numerators and denominators, leaving them in factored form.

$$\frac{18}{30}\cdot\frac{35}{6} = \frac{18\cdot 35}{30\cdot 6} \quad \text{Multiply numerators; multiply denominators.}$$

Finding the prime factorization of these smaller factors is easier.

$$= \frac{(2\cdot 3\cdot 3)\cdot(5\cdot 7)}{(2\cdot 3\cdot 5)\cdot(2\cdot 3)} \quad \text{Prime factor.}$$

Now we can cancel common factors. Parentheses are no longer needed in the numerator and denominator because both contain a product of prime factors, so order and grouping do not matter.

$$\begin{aligned}
&= \frac{\cancel{2}\cdot \cancel{3}\cdot \cancel{3}\cdot \cancel{5}\cdot \cancel{7}}{\cancel{2}\cdot \cancel{3}\cdot \cancel{5}\cdot 2\cdot \cancel{3}} && \text{Cancel common factors.}\\
&= \frac{7}{2} && \text{Remaining factors.}
\end{aligned}$$

Another approach is to factor numerators and denominators in place, cancel common factors, then multiply.

$$\begin{aligned}
\frac{18}{30}\cdot\frac{35}{6} &= \frac{2\cdot 3\cdot 3}{2\cdot 3\cdot 5}\cdot\frac{5\cdot 7}{2\cdot 3} && \text{Factor numerators and denominators.}\\
&= \frac{\cancel{2}\cdot \cancel{3}\cdot \cancel{3}\cdot \cancel{3}}{\cancel{2}\cdot \cancel{3}\cdot \cancel{5}}\cdot\frac{\cancel{5}\cdot 7}{2\cdot \cancel{3}} && \text{Cancel common factors.}\\
&= \frac{7}{2} && \text{Remaining factors.}
\end{aligned}$$

Note that this yields exactly the same result, 7/2.

Cancellation Rule

When multiplying fractions, cancel common factors according to the following rule: “Cancel a factor in a numerator for an identical factor in a denominator.”

Example 6

Find the product of 14/15 and 30/140.

Solution

Multiply numerators and multiply denominators. Prime factor, cancel common factors, then multiply.

$$\frac{14}{15}\cdot\frac{30}{140}=\frac{14\cdot 30}{15\cdot 140}$$ Multiply numerators; multiply denominators.

$$=\frac{(2\cdot 7)\cdot(2\cdot 3\cdot 5)}{(3\cdot 5)\cdot(2\cdot 2\cdot 5\cdot 7)}$$ Prime factor numerators and denominators.

$$=\frac{\cancel{2}\cdot\cancel{7}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{5}}{\cancel{3}\cdot 5cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{5}\cdot\cancel{7}}$$ Cancel common factors.

$$=\frac{1}{5}$$ Multiply.

Note: Everything in the numerator cancels because you've divided the numerator by itself. Hence, the answer has a 1 in its numerator.

Exercise

Multiply:

$$\frac{6}{35}\cdot\frac{70}{36}$$

Answer

$\frac{1}{3}$

When Everything Cancels

When all the factors in the numerator cancel, this means that you are dividing the numerator by itself. Hence, you are left with a 1 in the numerator. The same rule applies to the denominator. If everything in the denominator cancels, you're left with a 1 in the denominator.

Example 7

Simplify the product: \(- \frac{6x}{55y} \cdot \left(- \frac{110y^2}{105x^2} \right).\nonumber \]

Solution

The product of two negatives is positive.

$$-\frac{6x}{55y}\cdot\left(-\frac{110y^2}{105x^2}\right)=\frac{6x}{55y}\cdot\frac{110y^2}{105x^2}$$ Like signs gives a positive.

Prime factor numerators and denominators, then cancel common factors.

$$=\frac{2\cdot 3\cdot x}{5\cdot 11\cdot y}\cdot\frac{2\cdot 5\cdot 11\cdot y\cdot y}{3\cdot 5\cdot 7\cdot x\cdot x}$$ Prime factor numerators & denominators.

$$=\frac{2\cdot\cancel{3}\cdot\cancel{x}}{5\cdot\cancel{11}\cdot\cancel{y}}\cdot\frac{2\cdot\cancel{5}\cdot\cancel{11}\cdot\cancel{y}\cdot y}{\cancel{3}\cdot\cancel{5}\cdot 7\cdot\cancel{x}\cdot x}$$ Cancel common factors.

$$=\frac{2\cdot 2\cdot y}{5\cdot 7\cdot x}$$ Remaining factors.

$$=\frac{4y}{35x}$$ Multiply numerators; multiply denominators.

Exercise

Simplify:

$$\frac{6x}{15b}\cdot\left(-\frac{35b^2}{10a^2}\right)$$

Answer

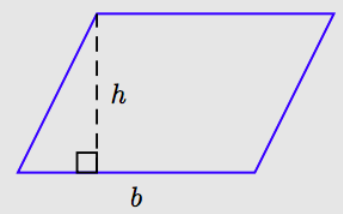
$$-\frac{21b}{5a}$$

Parallelograms

In this section, we are going to learn how to find the area of a parallelogram. Let's begin with the definition of a parallelogram. Recall that a quadrilateral is a polygon having four sides. A parallelogram is a very special type of quadrilateral.

Parallelogram

A parallelogram is a quadrilateral whose opposite sides are parallel.



The side on which the parallelogram rests is called its *base* (labeled *b* in the figure) and the distance from its base to the opposite side is called its *height* (labeled *h* in the figure). Note that the altitude is *perpendicular* to the base (meets the base at a 90◦ angle).

Figure 4.8 shows a *rectangle* having length *b* and width *h*. Therefore, the area of the rectangle in Figure 4.8 is *A* = *bh*, which is found by taking the product of the length and width. Take a pair of scissors and cut a triangle from the right end of the rectangle as shown in Figure 4.9(a), then paste the cut triangle to the left end as shown in Figure 4.9(b). The result, seen in Figure 4.9(b) is a parallelogram having base *b* and height *h*.

Because we've *thrown no material away* in creating the parallelogram from the rectangle, the parallelogram has the same area as the original rectangle. That is, the area of the parallelogram is *A* = *bh*.

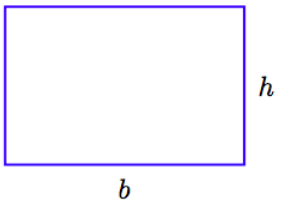
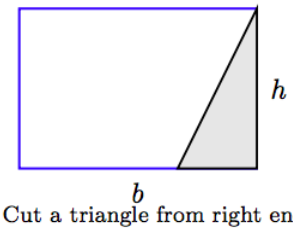
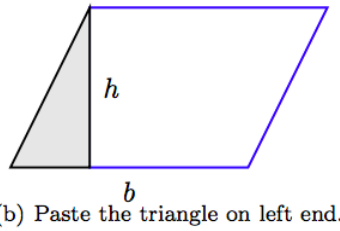


Figure 4.8: The area of the rectangle is A = bh.



(a) Cut a triangle from right end.



(b) Paste the triangle on left end.

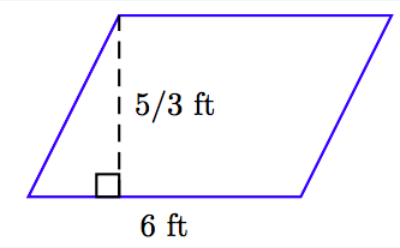
Figure 4.9: Creating a parallelogram from a rectangle.

Area of a Parallelogram

A parallelogram having base *b* and height *h* has area *A* = *bh*. That is, to find the area of a parallelogram, take the product of its base and height.

Example 8

Find the area of the parallelogram pictured below.



Solution

The area of the parallelogram is equal to the product of its base and height. That is,

$$\begin{aligned} A &= bh && \text{Area formula for a parallelogram.} \\ &= (6\text{ ft})\left(\frac{5}{3}\text{ ft}\right) && \text{Substitute: 6 ft for } b,\ 5/3 \text{ ft for } h. \\ &= \frac{30}{3}\text{ ft}^2. && \text{Multiply numerators and denominators.} \\ &= 10\text{ ft}^2. && \text{Divide.} \end{aligned}$$

Thus, the area of the parallelogram is 10 square feet.

Exercise

The base of a parallelogram measures 14 inches. The height is 8/7 of an inch. What is the area of the parallelogram?

Answer

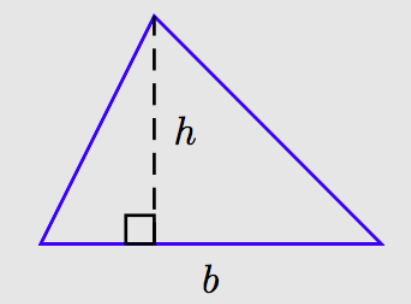
16 square inches

Triangles

Let's turn our attention to learning how to find the area of a *triangle*.

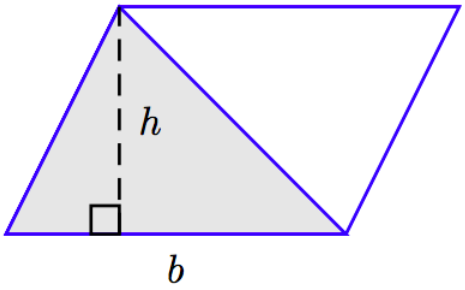
Triangle

A triangle is a three-sided polygon. It is formed by plotting three points and connecting them with three line segments. Each of the three points is called a *vertex* of the triangle and each of the three line segments is called a *side* of the triangle.



The side on which the triangle rests is called its *base*, and the distance between its base and opposite vertex is called its *height* of *altitude*. The altitude is always *perpendicular* to the base; that is, it forms a 90◦ angle with the base.

It's easily seen that a triangle has half the area of a parallelogram.



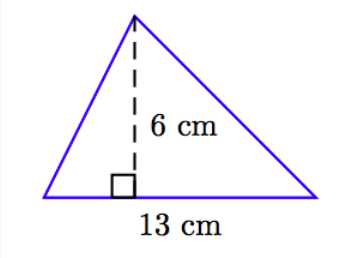
The parallelogram has area $A = bh$. Therefore, the triangle has one-half that area. That is, the area of the triangle is $A = (1/2)bh$.

Area of a Triangle

A triangle having base b and height h has area $A = (1/2)bh$. That is, to find the area of a triangle, take one-half the product of the base and height.

Example 9

Find the area of the triangle pictured below.



Solution

To find the area of the triangle, take one-half the product of the base and height.

$$A = \frac{1}{2}bh \quad \text{Area of a triangle formula.}$$
$$= \frac{1}{2}(13\text{ cm})(6\text{ cm}) \quad \text{Substitute: 13 cm for } b,\ 6 \text{ cm for } h.$$
$$= \frac{78\text{ cm}^2}{2} \quad \text{Multiply numerators; multiply denominators.}$$
$$= 39\text{ cm}^2. \quad \text{Simplify.}$$

Therefore, the area of the triangle is 39 square centimeters.

Exercise

The base of a triangle measures 15 meters. The height is 12 meters. What is the area of the triangle?

Answer

90 square meters

Identifying the Base and Altitude

Sometimes it can be a bit difficult to determine the base and altitude (height) of a triangle. For example, consider the triangle in Figure 4.10(a). Let's say we choose the bottom edge of the triangle as the base and denote its length with the variable b, as shown in Figure 4.10(a).

(a) The base is easy. (b) Adding the altitude (height)

Figure 4.10: Identifying the base and altitude (height) of a triangle.

The altitude (height) of the triangle is defined as the distance between the base of the triangle and its opposite vertex. To identify this altitude, we must first extend the base, as seen in the dashed extension in Figure 4.10(b), then drop a perpendicular dashed line from the opposite vertex to the extended base, also shown in Figure 4.10(b). This perpendicular is the altitude (height) of the triangle and we denote its length by *h*.

But we can go further. Any of the three sides of a triangle may be designated as the base of the triangle. Suppose, as shown in Figure 4.11(a), we identify a different side as the base, with length denoted by the variable *b*.

The altitude to this new base will be a segment from the opposite vertex, perpendicular to the base. Its length in Figure 4.11(b) is denoted by *h*.

In like manner, there is a third side of the triangle that could also be used as the base. The altitude to this third side is found by dropping a perpendicular from the vertex of the triangle directly opposite from this base. This would also require extending the base. We leave this to our readers to explore.

(a) A different base. (b) Adding the altitude (height).

Figure 4.11: Identifying the base and altitude (height) of a triangle.

Key Point

Any of the three sides of a triangle may be used as the base. The altitude is drawn by dropping a perpendicular from the opposite vertex to the chosen base. This sometimes requires that we extend the base. Regardless of which side we use for the base, the formula $A = bh/2$ will produce the same area result.

Exercises

1. Create a diagram, such as that shown in Figure 4.7, to show that 1/3 of 1/3 is 1/9.
2. Create a diagram, such as that shown in Figure 4.7, to show that 1/2 of 1/4 is 1/8.
3. Create a diagram, such as that shown in Figure 4.7, to show that 1/3 of 1/4 is 1/12.
4. Create a diagram, such as that shown in Figure 4.7, to show that 2/3 of 1/3 is 2/9.

In Exercises 1-28, multiply the fractions, and simplify your result.

5. $\frac{-21}{4} \cdot \frac{22}{19}$
6. $\frac{-4}{19} \cdot \frac{21}{8}$
7. $\frac{20}{11} \cdot \frac{-17}{22}$
8. $\frac{-9}{2} \cdot \frac{6}{7}$

9. $\frac{21}{8} \cdot \frac{-14}{15}$
10. $\frac{-17}{18} \cdot \frac{-3}{4}$
11. $\frac{-5}{11} \cdot \frac{7}{20}$
12. $\frac{-5}{2} \cdot \frac{-20}{19}$
13. $\frac{8}{13} \cdot \frac{-1}{6}$
14. $\frac{-12}{7} \cdot \frac{5}{9}$
15. $\frac{2}{15} \cdot \frac{-9}{8}$
16. $\frac{2}{11} \cdot \frac{-21}{8}$
17. $\frac{17}{12} \cdot \frac{3}{4}$
18. $\frac{7}{13} \cdot \frac{10}{21}$
19. $\frac{-6}{23} \cdot \frac{9}{10}$
20. $\frac{12}{11} \cdot \frac{-5}{2}$
21. $\frac{-23}{24} \cdot \frac{-6}{17}$
22. $\frac{4}{9} \cdot \frac{-21}{19}$
23. $\frac{24}{7} \cdot \frac{5}{2}$
24. $\frac{-20}{23} \cdot \frac{-1}{2}$
25. $\frac{1}{2} \cdot \frac{-8}{11}$
26. $\frac{-11}{18} \cdot \frac{-20}{3}$
27. $\frac{-24}{13} \cdot \frac{-7}{18}$
28. $\frac{21}{20} \cdot \frac{-4}{5}$

In Exercises 29-40, multiply the fractions, and simplify your result.

29. $\frac{-12y^3}{13} \cdot \frac{2}{9y^6}$
30. $\frac{-8x^3}{3} \cdot \frac{-6}{5x^5}$
31. $\frac{11y^3}{24} \cdot \frac{6}{5y^5}$
32. $\frac{11y}{18} \cdot \frac{21}{17y^6}$
33. $\frac{-8x^2}{21} \cdot \frac{-18}{19x}$
34. $\frac{2y^4}{11} \cdot \frac{-7}{18y}$
35. $\frac{13x^6}{15} \cdot \frac{9}{16x^2}$
36. $\frac{-22x^6}{15} \cdot \frac{17}{16x^3}$
37. $\frac{-6y^3}{5} \cdot \frac{-20}{7y^6}$
38. $\frac{-21y}{5} \cdot \frac{-8}{3y^2}$
39. $\frac{-3y^3}{4} \cdot \frac{23}{12y}$
40. $\frac{-16y^6}{15} \cdot \frac{-21}{13y^4}$

In Exercises 41-56, multiply the fractions, and simplify your result.

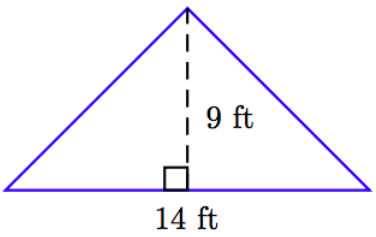
41. $\frac{13y^6}{20x^4} \cdot \frac{2x}{7y^2}$
42. $\frac{-8y^3}{13x^6} \cdot \frac{7x^2}{10y^2}$
43. $\frac{23y^4}{21x} \cdot \frac{-7x^6}{4y^2}$
44. $\frac{-2x^6}{9y^4} \cdot \frac{y^5}{20x}$
45. $\frac{11y^6}{12x^6} \cdot \frac{-2x^4}{7y^2}$
46. $\frac{16x^3}{13y^4} \cdot \frac{11y^2}{18x}$
47. $\frac{x^6}{21y^3} \cdot \frac{-7y^4}{9x^5}$
48. $\frac{-3y^3}{5x} \cdot \frac{14x^5}{15y^2}$
49. $\frac{19y^2}{18x} \cdot \frac{10x^3}{7y^3}$
50. $\frac{-20x}{9y^3} \cdot \frac{-y^6}{4x^3}$
51. $\frac{-4y^3}{5x^5} \cdot \frac{-10x}{21y^4}$
52. $\frac{11y^2}{14x^4} \cdot \frac{-22x}{21y^3}$
53. $\frac{-16x}{21y^2} \cdot \frac{-7y^3}{5x^2}$
54. $\frac{-4y}{5x} \cdot \frac{10x^3}{7y^6}$
55. $\frac{17x^3}{3y^6} \cdot \frac{-12y^2}{7x^4}$
56. $\frac{-6x^4}{11y^3} \cdot \frac{13y^2}{8x^5}$

In Exercises 57-62, find the area of the parallelogram having the given base and altitude.

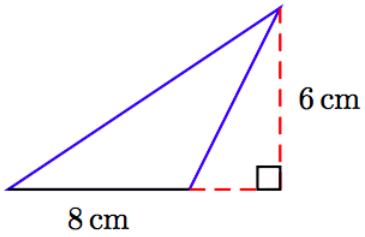
57. base = 8 cm, altitude = 7 cm
58. base = 2 cm, altitude = 11 cm
59. base = 6 cm, altitude = 13 cm
60. base = 2 cm, altitude = 6 cm
61. base = 18 cm, altitude = 14 cm
62. base = 20 cm, altitude = 2 cm

In Exercises 63-68, find the area of the triangle shown in the figure. (Note: Figures are not drawn to scale.)

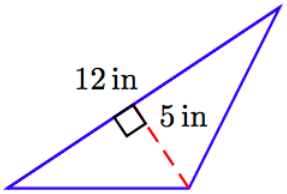
63.



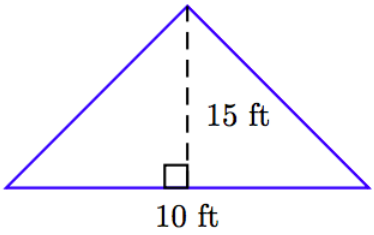
64.



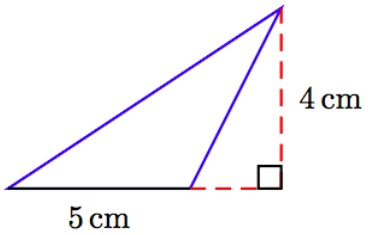
65.



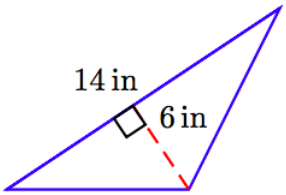
66.



67.



68.



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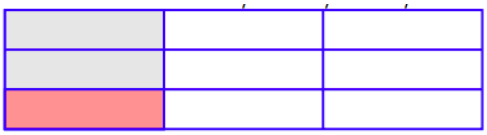
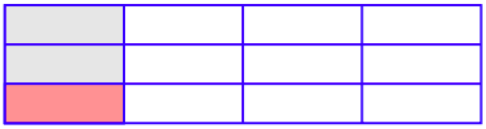
69. **Weight on the Moon.** On the moon, you would only weigh 1/6 of what you weigh on earth. If you weigh 138 pounds on earth, what would your weight on the moon be?

Answers

1. This shows that 1/3 of 1/3 is 1/9.



3. This shows that 1/3 of 1/4 is 1/12.



5. $\frac{-231}{38}$

7. $\frac{-170}{121}$

9. $\frac{-49}{20}$

11. $\frac{-7}{44}$

13. $\frac{-4}{39}$

15. $\frac{-3}{20}$

17. $\frac{17}{16}$

19. $\frac{-27}{115}$

21. $\frac{23}{68}$

23. $\frac{60}{7}$

25. $\frac{-4}{11}$

27. $\frac{28}{39}$

29. $-\frac{8}{39y^3}$

31. $\frac{11}{20y^2}$

33. $\frac{48x}{133}$

35. $\frac{39x^4}{80}$

37. $\frac{24}{7y^3}$

39. $-\frac{23y^2}{16}$

41. $\frac{13y^4}{70x^3}$

43. $-\frac{23y^2x^5}{12}$

45. $-\frac{11y^4}{42x^2}$

47. $-\frac{xy}{27}$

49. $\frac{95x^2}{63y}$

51. $\frac{8}{21yx^4}$

53. $\frac{16y}{15x}$

55. $-\frac{68}{7xy^4}$

57. 56 cm^2

59. 78 cm^2

61. 252 cm^2

63. 63 ft^2

65. 30 in^2

67. 10 cm^2

69. 23 pounds

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2.4: Dividing Fractions

Suppose that you have four pizzas and each of the pizzas has been sliced into eight equal slices. Therefore, each slice of pizza represents 1/8 of a whole pizza.

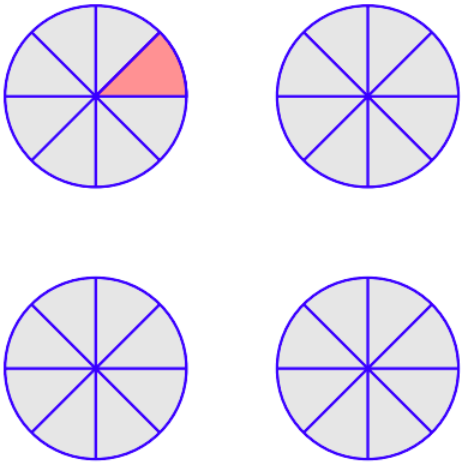

Figure 2.4.1: One slice of pizza is 1/8 of one whole pizza.

Now for the question: How many one-eighths are there in four? This is a division statement. To find how many one-eighths there are in 4, divide 4 by 1/8. That is,

$$\text{Number of one-eighths in four} = 4 \div \tfrac{1}{8}.$$

On the other hand, to find the number of one-eights in four, Figure 2.4.1 clearly demonstrates that this is equivalent to asking how many slices of pizza are there in four pizzas. Since there are 8 slices per pizza and four pizzas,

$$\text{Number of pizza slices} = 4 \cdot 8.$$

The conclusion is the fact that 4 ÷ (1/8) is equivalent to 4 · 8. That is,

$$\begin{aligned} 4 \div 1/8 &= 4 \cdot 8 \\ &= 32. \end{aligned}$$

Therefore, we conclude that there are 32 one-eighths in 4.

Reciprocals

The number 1 is still the *multiplicative identity* for fractions.

Multiplicative Identity Property

Let a/b be any fraction. Then,

$$\frac{a}{b} \cdot 1 = \frac{a}{b} \text{ and } 1 \cdot \frac{a}{b} = \frac{a}{b}.$$

The number 1 is called the *multiplicative identity* because the identical number is returned when you multiply by 1.

Next, if we invert 3/4, that is, if we turn 3/4 upside down, we get 4/3. Note what happens when we multiply 3/4 by 4/3.

The number 4/3 is called the *multiplicative inverse* or *reciprocal* of 3/4. The product of reciprocals is always 1.

Multiplicative Inverse Property

Let *a/b* be any fraction. The number *b/a* is called the *multiplicative inverse* or *reciprocal* of *a/b*. The product of reciprocals is 1.

$$\frac{a}{b} \cdot \frac{b}{a} = 1$$

Note: To find the multiplicative inverse (reciprocal) of a number, simply invert the number (turn it upside down).

For example, the number 1/8 is the multiplicative inverse (reciprocal) of 8 because

$$8 \cdot \frac{1}{8} = 1.$$

Note that 8 can be thought of as 8/1. Invert this number (turn it upside down) to find its multiplicative inverse (reciprocal) 1/8.

Example 2.4.1

Find the multiplicative inverses (reciprocals) of: (a) 2/3, (b) −3/5, and (c) −12.

Solution

a) Because

$$\frac{2}{3} \cdot \frac{3}{2} = 1,$$

the multiplicative inverse (reciprocal) of 2/3 is 3/2.

b) Because

$$-\frac{3}{5} \cdot \left(-\frac{5}{3}\right) = 1,$$

the multiplicative inverse (reciprocal) of −3/5 is −5/3. Again, note that we simply inverted the number −3/5 to get its reciprocal −5/3.

c) Because

$$-12 \cdot \left(-\frac{1}{12}\right) = 1,$$

the multiplicative inverse (reciprocal) of −12 is −1/12. Again, note that we simply inverted the number −12 (understood to equal −12/1) to get its reciprocal −1/12.

Exercise 2.4.1

Find the reciprocals of: (a) −3/7 and (b) 15

Answer

(a) −7/3, (b) 1/15

Division

Recall that we computed the number of one-eighths in four by doing this calculation:

$$\begin{aligned} 4 \div \frac{1}{8} &= 4 \cdot 8 \\ &= 32. \end{aligned}$$

Note how we inverted the divisor (second number), then changed the division to multiplication. This motivates the following definition of division.

Division Definition

If a/b and c/d are any fractions, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

That is, we invert the divisor (second number) and change the division to multiplication. *Note: We like to use the phrase "invert and multiply" as a memory aid for this definition.*

Example 2.4.2

Divide 1/2 by 3/5.

Solution

To divide 1/2 by 3/5, invert the divisor (second number), then multiply.

$$\frac{1}{2} \div \frac{3}{5} = \frac{1}{2} \cdot \frac{5}{3} \qquad \text{Invert the divisor (second number).}$$

$$= \frac{5}{6} \qquad \text{Multiply.}$$

Exercise 2.4.2

Divide:

$$\frac{2}{3} \div \frac{10}{3}$$

Answer

1/5

Example 2.4.3

Simplify the following expressions: (a) $3 \div \frac{2}{3}$ and (b) $\frac{4}{5} \div 5$.

Solution

In each case, invert the divisor (second number), then multiply.

a) Note that 3 is understood to be 3/1.

$$3 \div \frac{2}{3} = \frac{3}{1} \cdot \frac{3}{2} \qquad \text{Invert the divisor (second number).}$$

$$= \frac{9}{2} \qquad \text{Multiply numerators; multiply denominators.}$$

b) Note that 5 is understood to be 5/1.

$$\frac{4}{5} \div 5 = \frac{4}{5} \cdot \frac{1}{5} \qquad \text{Invert the divisor (second number).}$$

$$= \frac{4}{25} \qquad \text{Multiply numerators; multiply denominators.}$$

Exercise 2.4.3

Divide:

$$\frac{15}{7} \div 5$$

Answer

$\frac{3}{7}$

After inverting, you may need to factor and cancel, as we learned to do in Section 4.2.

Example 2.4.4

Divide −6/35 by 33/55.

Solution

Invert, multiply, factor, and cancel common factors.

$$\begin{aligned} -\frac{6}{35} \div \frac{33}{55} &= -\frac{6}{35} \cdot \frac{55}{33} && \text{Invert the divisor (second number).} \\ &= -\frac{6 \cdot 55}{35 \cdot 33} && \text{Multiply numerators; multiply denominators.} \\ &= -\frac{(2 \cdot 3) \cdot (5 \cdot 11)}{(5 \cdot 7) \cdot (3 \cdot 11)} && \text{Factor numerators and denominators.} \\ &= -\frac{2 \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{11}}{\cancel{5} \cdot 7 \cdot \cancel{3} \cdot \cancel{11}} && \text{Cancel common factors.} \\ &= -\frac{2}{7} && \text{Remaining factors.} \end{aligned}$$

Note that unlike signs produce a negative answer.

Exercise 2.4.4

Divide:

$$\frac{6}{15} \div \left(-\frac{42}{35}\right)$$

Answer

-1/3

Of course, you can also choose to factor numerators and denominators in place, then cancel common factors.

Example 2.4.5

Divide $-6/x$ by $-12/x^2$.

Solution

Invert, factor numerators and denominators, cancel common factors, then multiply.

$$\begin{aligned} -\frac{6}{x} \div \left(-\frac{12}{x^2}\right) &= -\frac{6}{x} \cdot \left(-\frac{x^2}{12}\right) && \text{Invert second number.} \\ &= -\frac{2 \cdot 3}{x} \cdot -\frac{x \cdot x}{2 \cdot 2 \cdot 3} && \text{Factor numerators and denominators.} \\ &= -\frac{\cancel{2} \cdot \cancel{3}}{\cancel{x}} \cdot -\frac{\cancel{x} \cdot x}{2 \cdot \cancel{2} \cdot \cancel{3}} && \text{Cancel common factors.} \\ &= \frac{x}{2} && \text{Multiply.} \end{aligned}$$

Note that like signs produce a positive answer.

Exercise 2.4.5

Divide:

$$-\frac{12}{a} \div \left(-\frac{15}{a^3}\right)$$

Answer

$$-\frac{4a^2}{5}$$

Exercises

In Exercises 1-16, find the reciprocal of the given number.

1. −16/5
2. −3/20
3. −17
4. −16
5. 15/16
6. 7/9
7. 30
8. 28
9. −46
10. −50
11. −9/19
12. −4/7
13. 3/17
14. 3/5
15. 11
16. 48

In Exercises 17-32, determine which property of multiplication is depicted by the given identity.

17. $\frac{2}{9} \cdot \frac{9}{2} = 1$
18. $\frac{12}{19} \cdot \frac{19}{12} = 1$
19. $\frac{-19}{12} \cdot 1 = \frac{-19}{12}$
20. $\frac{-19}{8} \cdot 1 = \frac{-19}{8}$
21. $-6 \cdot \left(-\frac{1}{6}\right) = 1$
22. $-19 \cdot \left(-\frac{1}{19}\right) = 1$
23. $\frac{-16}{11} \cdot 1 = \frac{-16}{11}$
24. $\frac{-7}{6} \cdot 1 = \frac{-7}{6}$
25. $-\frac{4}{1} \cdot \left(-\frac{1}{4}\right) = 1$
26. $-\frac{9}{10} \cdot \left(-\frac{10}{9}\right) = 1$

27. $\frac{8}{1} \cdot 1 = \frac{8}{1}$
28. $\frac{13}{15} \cdot 1 = \frac{13}{15}$
29. $14 \cdot \frac{1}{14} = 1$
30. $4 \cdot \frac{1}{4} = 1$
31. $\frac{13}{8} \cdot 1 = \frac{13}{8}$
32. $\frac{1}{13} \cdot 1 = \frac{1}{13}$

In Exercises 33-56, divide the fractions, and simplify your result.

33. $\frac{8}{23} \div \frac{-6}{11}$
34. $\frac{-10}{21} \div \frac{-6}{5}$
35. $\frac{18}{19} \div \frac{-16}{23}$
36. $\frac{13}{10} \div \frac{17}{18}$
37. $\frac{4}{21} \div \frac{-6}{5}$
38. $\frac{2}{9} \div \frac{-12}{19}$
39. $\frac{-1}{9} \div \frac{8}{3}$
40. $\frac{1}{2} \div \frac{-15}{8}$
41. $\frac{-21}{11} \div \frac{3}{10}$
42. $\frac{7}{24} \div \frac{-23}{2}$
43. $\frac{-12}{7} \div \frac{2}{3}$
44. $\frac{-9}{16} \div \frac{6}{7}$
45. $\frac{2}{19} \div \frac{24}{23}$
46. $\frac{7}{3} \div \frac{-10}{21}$
47. $\frac{-9}{5} \div \frac{-24}{19}$
48. $\frac{14}{17} \div \frac{-22}{21}$
49. $\frac{18}{11} \div \frac{14}{9}$
50. $\frac{5}{6} \div \frac{20}{19}$
51. $\frac{13}{18} \div \frac{4}{9}$
52. $\frac{-3}{2} \div \frac{-7}{12}$
53. $\frac{11}{2} \div \frac{-21}{10}$
54. $\frac{-9}{2} \div \frac{-13}{22}$
55. $\frac{3}{10} \div \frac{12}{5}$
56. $\frac{-22}{7} \div \frac{-18}{17}$

In Exercises 57-68, divide the fractions, and simplify your result.

57. $\frac{20}{17} \div 5$
58. $\frac{21}{8} \div 7$
59. $-7 \div \frac{21}{20}$

60. $-3 \div \frac{12}{17}$
61. $\frac{8}{21} \div 2$
62. $\frac{-3}{4} \div (-6)$
63. $8 \div \frac{-10}{17}$
64. $-6 \div \frac{20}{21}$
65. $-8 \div \frac{18}{5}$
66. $6 \div \frac{-21}{8}$
67. $\frac{3}{4} \div (-9)$
68. $\frac{2}{9} \div (-8)$

In Exercises 69-80, divide the fractions, and simplify your result.

69. $\frac{11x^2}{12} \div \frac{8x^4}{3}$
70. $\frac{-4x^2}{3} \div \frac{11x^6}{6}$
71. $\frac{17y}{9} \div \frac{10y^6}{3}$
72. $\frac{-5y}{12} \div \frac{-3y^5}{2}$
73. $\frac{-22x^4}{13} \div \frac{12x}{11}$
74. $\frac{-9y^6}{4} \div \frac{24y^5}{13}$
75. $\frac{-3x^4}{10} \div \frac{-4x}{5}$
76. $\frac{18y^4}{11} \div \frac{4y^2}{7}$
77. $\frac{-15y^2}{14} \div \frac{-10y^5}{13}$
78. $\frac{3x}{20} \div \frac{2x^3}{5}$
79. $\frac{-15x^5}{13} \div \frac{20x^2}{19}$
80. $\frac{18y^6}{7} \div \frac{14y^4}{9}$

In Exercises 81-96, divide the fractions, and simplify your result.

81. $\frac{11y^4}{14x^2} \div \frac{-9y^2}{7x^3}$
82. $\frac{-5x^2}{12y^3} \div \frac{-22x}{21y^5}$
83. $\frac{10x^4}{3y^4} \div \frac{7x^5}{24y^2}$
84. $\frac{20x^3}{11y^5} \div \frac{5x^5}{6y^3}$
85. $\frac{22y^4}{21x^5} \div \frac{-5y^2}{6x^4}$
86. $\frac{-7y^5}{8x^6} \div \frac{21y}{5x^5}$
87. $\frac{-22x^4}{21y^3} \div \frac{-17x^3}{3y^4}$
88. $\frac{-7y^4}{4x} \div \frac{-15y}{22x^4}$
89. $\frac{-16y^2}{3x^3} \div \frac{2y^6}{11x^5}$

90. $\frac{-20x}{21y^2} \div \frac{-22x^5}{y^6}$

91. $\frac{-x}{12y^4} \div \frac{-23x^3}{16y^3}$

92. $\frac{20x^2}{17y^3} \div \frac{8x^3}{15y}$

93. $\frac{y^2}{4x} \div \frac{-9y^5}{8x^3}$

94. $\frac{-10y^4}{13x^2} \div \frac{-5y^6}{6x^3}$

95. $\frac{-18x^6}{13y^4} \div \frac{3x}{y^2}$

96. $\frac{20x^4}{9y^6} \div \frac{14x^2}{17y^4}$

Answers

1. $-\frac{5}{16}$

3. $-\frac{1}{17}$

5. $\frac{16}{15}$

7. $\frac{1}{30}$

9. $-\frac{1}{46}$

11. $-\frac{19}{9}$

13. $\frac{17}{3}$

15. $\frac{1}{11}$

17. multiplicative inverse property

19. multiplicative identity property

21. multiplicative inverse property

23. multiplicative identity property

25. multiplicative inverse property

27. multiplicative identity property

29. multiplicative inverse property

31. multiplicative identity property

33. $-\frac{44}{69}$

35. $-\frac{207}{152}$

37. $-\frac{10}{63}$

39. $-\frac{1}{24}$

41. $-\frac{70}{11}$

43. $-\frac{18}{7}$

45. $\frac{23}{228}$

47. $\frac{57}{40}$

49. $\frac{81}{77}$

51. $\frac{13}{8}$

53. $-\frac{55}{21}$
55. $\frac{1}{8}$
57. $\frac{4}{17}$
59. $-\frac{20}{3}$
61. $\frac{4}{21}$
63. $-\frac{68}{5}$
65. $-\frac{20}{9}$
67. $-\frac{1}{12}$
69. $\frac{11}{32x^2}$
71. $\frac{17}{30y^5}$
73. $-\frac{121x^3}{78}$
75. $\frac{3x^3}{8}$
77. $\frac{39}{28y^3}$
79. $-\frac{57x^3}{52}$
81. $-\frac{11y^2x}{18}$
83. $\frac{80}{7xy^2}$
85. $-\frac{44y^2}{35x}$
87. $\frac{22xy}{119}$
89. $-\frac{88x^2}{3y^4}$
91. $\frac{4}{69x^2y}$
93. $-\frac{2x^2}{9y^3}$
95. $-\frac{6x^5}{13y^2}$

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2.5: Adding and Subtracting Fractions

Paul and Tony order a pizza which has been cut into eight equal slices. Thus, each slice is 1/8 of the whole pizza. Paul eats two slices (shaded in light gray in Figure 2.5.1), or 2/8 of the whole pizza. Tony eats three slices (shaded in light red (or a darker shade of gray in black-and-white printing) in Figure 2.5.1), or 3/8 of the whole pizza.

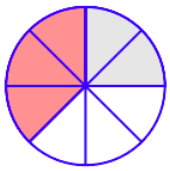


Figure 2.5.1: Paul eats two slices (2/8) and Tony eats three slices (3/8).

It should be clear that together Paul and Tony eat five slices, or 5/8 of the whole pizza. This reflects the fact that

$$\frac{2}{8}+\frac{3}{8}=\frac{5}{8}.$$

This demonstrates how to add two fractions with a common (same) denominator. Keep the common denominator and add the numerators. That is,

$$\begin{aligned}\frac{2}{8}+\frac{3}{8} &= \frac{2+3}{8} && \text{Keep denominator; add numerators.}\\ &= \frac{5}{8} && \text{Simplify numerator.}\end{aligned}$$

Adding Fractions with Common Denominators

Let *a/c* and *b/c* be two fractions with a common (same) denominator. Their sum is defined as

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$$

That is, to add two fractions having common denominators, keep the common denominator and add their numerators.

A similar rule holds for subtraction.

Subtracting Fractions with Common Denominators

Let *a/c* and *b/c* be two fractions with a common (same) denominator. Their difference is defined as

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

That is, to subtract two fractions having common denominators, keep the common denominator and subtract their numerators.

Example 2.5.1

Find the sum of 4/9 and 3/9.

Solution

Keep the common denominator and add the numerators.

$$\begin{aligned}\frac{4}{9}+\frac{3}{9} &= \frac{4+3}{9} && \text{Keep denominator; add numerators.}\\ &= \frac{7}{9} && \text{Simplify numerator.}\end{aligned}$$

? Exercise 2.5.1

Add:

$$\frac{1}{8}+\frac{2}{8}$$

Answer

3/8

✓ Example 2.5.2

Subtract 5/16 from 13/16.

Solution

Keep the common denominator and subtract the numerators.

$$\begin{aligned}\frac{13}{16}-\frac{5}{16}&=\frac{13-5}{16} && \text{Keep denominator; subtract numerators.}\\ &=\frac{8}{16} && \text{Simplify numerator.}\end{aligned}$$

Of course, as we learned in Section 4.1, we should always reduce our final answer to lowest terms. One way to accomplish that in this case is to divide numerator and denominator by 8, the greatest common divisor of 8 and 16.

$$\begin{aligned}&=\frac{8\div 8}{16\div 8} && \text{Divide numerator and denominator by 8.}\\ &=\frac{1}{2} && \text{Simplify numerator and denominator.}\end{aligned}$$

? Exercise 2.5.2

Subtract:

$$\frac{11}{12}-\frac{7}{12}$$

Answer

1/3

✓ Example 2.5.3

Simplify:

$$\frac{3}{x}-\left(-\frac{7}{x}\right).$$

Solution

Both fractions share a common denominator.

$$\begin{aligned}\frac{3}{x}-\left(-\frac{7}{x}\right)&=\frac{3}{x}+\frac{7}{x} && \text{Add the opposite.}\\ &=\frac{3+7}{x} && \text{Keep denominator, add numerators.}\\ &=\frac{10}{x} && \text{Simplify.}\end{aligned}$$

Adding Fractions with Different Denominators

Consider the sum

$$\frac{4}{9}+\frac{1}{6}.$$

We cannot add these fractions because they do not have a common denominator. So, what to do?

Goals

In order to add two fractions with different denominators, we need to:

1. Find a common denominator for the given fractions.
2. Make fractions with the common denominator that are equivalent to the original fractions.

If we accomplish the two items in the "Goal," we will be able to find the sum of the given fractions.

So, how to start? We need to find a common denominator, but not just any common denominator. Let's agree that we want to keep the numbers as small as possible and find a *least common denominator*.

Definition: Least Common Denominator

The *least common denominator* (LCD) for a set of fractions is the smallest number divisible by each of the denominators of the given fractions.

Consider again the sum we wish to find:

$$\frac{4}{9}+\frac{1}{6}.$$

The denominators are 9 and 6. We wish to find a least common denominator, the smallest number that is divisible by both 9 and 6. A number of candidates come to mind: 36, 54, and 72 are all divisible by 9 and 6, to name a few. But the smallest number that is divisible by both 9 and 6 is 18. This is the least common denominator for 9 and 6.

We now proceed to the second item in "Goal." We need to make fractions having 18 as a denominator that are equivalent to 4/9 and 1/6. In the case of 4/9, if we multiply both numerator and denominator by 2, we get

$$\begin{aligned}\frac{4}{9}&=\frac{4\cdot 2}{9\cdot 2} && \text{Multiply numerator and denominator by 2.}\\ &=\frac{8}{18}. && \text{Simplify numerator and denominator.}\end{aligned}$$

In the case of 1/6, if we multiply both numerator and denominator by 3, we get

$$\begin{aligned}\frac{1}{6}&=\frac{1\cdot 3}{6\cdot 3} && \text{Multiply numerator and denominator by 3.}\\ &=\frac{3}{18}. && \text{Simplify numerator and denominator.}\end{aligned}$$

Typically, we'll arrange our work as follows.

$$\begin{aligned}\frac{4}{+\,6}\frac{1}{}&=\frac{4\cdot 2}{9\cdot 2}+\frac{1\cdot 3}{6\cdot 3} && \text{Equivalent fractions with LCD} = 18.\\ &=\frac{8}{18}+\frac{3}{18} && \text{Simplify numerators and denominators.}\\ &=\frac{8+3}{18} && \text{Keep common denominator; add numerators.}\\ &=\frac{11}{18} && \text{Simplify numerator.}\end{aligned}$$

Let's summarize the procedure.

Adding or Subtracting Fractions with Different Denominators

1. Find the LCD, the smallest number divisible by all the denominators of the given fractions.
2. Create fractions using the LCD as the denominator that are equivalent to the original fractions.
3. Add or subtract the resulting equivalent fractions. Simplify, including reducing the final answer to lowest terms.

Example 2.5.4

Simplify: $\frac{3}{5}-\frac{2}{3}$.

Solution

The smallest number divisible by both 5 and 3 is 15.

$$\begin{aligned}\frac{3}{5}-\frac{2}{3} &= \frac{3\cdot 3}{5\cdot 3}-\frac{2\cdot 5}{3\cdot 5} && \text{Equivalent fractions with LCD} = 15.\\ &= \frac{9}{15}-\frac{10}{15} && \text{Simplify numerators and denominators.}\\ &= \frac{9-10}{15} && \text{Keep LCD; subtract numerators.}\\ &= \frac{-1}{15} && \text{Simplify numerator.}\end{aligned}$$

Although this answer is perfectly acceptable, negative divided by positive gives us a negative answer, so we could also write

$$= -\frac{1}{15}.$$

Exercise 2.5.4

Subtract:

$$\frac{3}{4}-\frac{7}{5}$$

Answer

-13/20

Example 2.5.5

Simplify: $-\frac{1}{4}-\frac{5}{6}$.

Solution

The smallest number divisible by both 4 and 6 is 12.

$$\begin{aligned}-\frac{1}{4}-\frac{5}{6} &= -\frac{1\cdot 3}{4\cdot 3}-\frac{5\cdot 2}{6\cdot 2} && \text{Equivalent fractions with LCD} = 12.\\ &= -\frac{3}{12}-\frac{10}{12} && \text{Simplify numerators and denominators.}\\ &= \frac{-3-10}{12} && \text{Keep LCD; subtract numerators.}\\ &= \frac{-13}{12} && \text{Simplify numerator.}\end{aligned}$$

Exercise 2.5.5

Subtract: $-\frac{3}{8}-\frac{1}{12}$

Answer

-11/24

Example 2.5.6

Simplify: $\frac{5}{x}+\frac{3}{4}$.

Solution

The smallest number divisible by both 4 and x is 4x.

$$\begin{aligned}\frac{5}{x}+\frac{3}{4} &= \frac{5\cdot 4}{x\cdot 4}+\frac{3\cdot x}{4\cdot x} && \text{Equivalent fractions with LCD} = 4x.\\ &== \frac{20}{4x}+\frac{3x}{4x} && \text{Simplify numerators and denominators.}\\ &= \frac{20+3x}{4x} && \text{Keep LCD; add numerators.}\end{aligned}$$

Exercise 2.5.6

Add:

$$\frac{5}{z}+\frac{2}{3}$$

Answer

$$\frac{15+2z}{3z}$$

Example 2.5.7

Simplify: $\frac{2}{3}-\frac{x}{5}$.

Solution

The smallest number divisible by both 3 and 5 is 15.

$$\begin{aligned}\frac{2}{3}-\frac{x}{5} &= \frac{2\cdot 5}{3\cdot 5}-\frac{x\cdot 3}{5\cdot 3} && \text{Equivalent fractions with LCD} = 15.\\ &= \frac{10}{15}-\frac{3x}{15} && \text{Simplify numerators and denominators.}\\ &= \frac{10-3x}{15} && \text{Keep LCD; subtract numerators.}\end{aligned}$$

Least Common Multiple

First we define the multiple of a number.

Definition: Multiples

The **multiples** of a number *d* are 1*d*, 2*d*, 3*d*, 4*d*, etc. That is, the multiples of *d* are the numbers *nd*, where n is a natural number.

For example, the multiples of 8 are $1\cdot 8$, $2\cdot 8$, $3\cdot 8$, $4\cdot 8$, etc., or equivalently, 8, 16, 24, 32, etc.

Definition: Least Common Multiple

The **least common multiple** (LCM) of a set of numbers is the smallest number that is a multiple of each number of the given set. The procedure for finding an LCM follows:

1. List all of the multiples of each number in the given set of numbers.
2. List the multiples that are in common.
3. Pick the least of the multiples that are in common.

Example 2.5.7

Find the least common multiple (LCM) of 12 and 16.

Solution

List the multiples of 12 and 16.

$$\text{Multiples of } 12 : 12, 24, 36, 48, 60, 72, 84, 96,...$$

$$\text{Multiples of } 16 : 16, 32, 48, 64, 80, 96, 112,...$$

Pick the common multiples.

$$\text{Common Multiples} : 48, 96,...$$

The LCM is the least of the common multiples.

$$\text{LCM}(12,16) = 48$$

Exercise 2.5.7

Find the least common denominator of 6 and 9.

Answer

18

Important Observation

The *least common denominator* is the *least common multiple* of the denominators.

For example, suppose your problem is 5/12 + 5/16. The LCD is the smallest number divisible by both 12 and 16. That number is 48, which is also the LCM of 12 and 16. Therefore, the procedure for finding the LCM can also be used to find the LCD.

Least Common Multiple Using Prime Factorization

You can also find the LCM using prime factorization.

LCM By Prime Factorization

To find an LCM for a set of numbers, follow this procedure:

1. Write down the prime factorization for each number in compact form using exponents.
2. The LCM is found by writing down every factor that appears in step 1 to the highest power of that factor that appears.

Example 2.5.8

Use prime factorization to find the least common multiple find the least common denominator of 18 and 24. (LCM) of 12 and 16.

Solution

Prime factor 12 and 16.

$$12 = 2 \cdot 2 \cdot 3$$
$$16 = 2 \cdot 2 \cdot 2 \cdot 2$$

Write the prime factorizations in compact form using exponents.

$$12 = 2^2 \cdot 3^1$$
$$16 = 2^4$$

To find the LCM, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2 and 3. The highest power of 2 that appears is 2^4. The highest power of 3 that appears is 3^1.

$$\text{LCM} = 2^4 \cdot 3^1 \quad \text{Keep highest power of each factor.}$$

Now we expand this last expression to get our LCM.

$$= 16 \cdot 3 \quad \text{Expand: } 2^4 = 16 \text{ and } 3^1 = 3.$$
$$= 48. \quad \text{Multiply.}$$

Note that this answer is identical to the LCM found in Example 8 that was found by listing multiples and choosing the smallest multiple in common.

Exercise 2.5.8

Use prime factorization to find the least common denominator of 18 and 24.

Answer

72

Example 2.5.10

Simplify: $\frac{5}{28} + \frac{11}{42}$.

Solution

Prime factor the denominators in compact form using exponents.

$$28 = 2 \cdot 2 \cdot 7 = 2^2 \cdot 7$$

$$42 = 2 \cdot 3 \cdot 7 = 2^1 \cdot 3^1 \cdot 7^1$$

To find the LCD, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2, 3, and 7. The highest power of 2 that appears is 2^2. The highest power of 3 that appears is 3^1. The highest power of 7 that appears is 7^1.

$$\text{LCM} = 2^2 \cdot 3^1 \cdot 7^1 \quad \text{Keep highest power of each factor.}$$
$$= 4 \cdot 3 \cdot 7 \quad \text{Expand: } 2^2 = 4,\ 3^1 = 3,\ 7^1 = 7.$$
$$= 84 \quad \text{Multiply.}$$

Create equivalent fractions with the new LCD, then add.

$$\frac{5}{28} + \frac{11}{42} = \frac{5 \cdot 3}{28 \cdot 3} + \frac{11 \cdot 2}{42 \cdot 2} \quad \text{Equivalent fractions with LCD} = 84.$$
$$= \frac{15}{84} + \frac{22}{84} \quad \text{Simplify numerators and denominators.}$$
$$= \frac{37}{84} \quad \text{Keep LCD; add numerators.}$$

Exercise 2.5.10

Simplify: $\frac{5}{24}+\frac{5}{36}$

Answer

25/72

Example 2.5.11

Simplify: $-\frac{11}{24}-\frac{1}{18}$.

Solution

Prime factor the denominators in compact form using exponents.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3^1$$

$$18 = 2 \cdot 3 \cdot 3 = 2^1 \cdot 3^2$$

To find the LCD, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2 and 3. The highest power of 2 that appears is 2^3. The highest power of 3 that appears is 3^2.

$$\begin{aligned} \text{LCM} &= 2^3 \cdot 3^2 && \text{Keep highest power of each factor.} \\ &= 8 \cdot 9 && \text{Expand: } 2^3 = 8 \text{ and } 3^2 = 9. \\ &= 72. && \text{Multiply.} \end{aligned}$$

Create equivalent fractions with the new LCD, then subtract.

$$\begin{aligned} -\frac{11}{24}-\frac{1}{18} &= -\frac{11 \cdot 3}{24 \cdot 3}-\frac{1 \cdot 4}{18 \cdot 4} && \text{Equivalent fractions with LCD} = 72. \\ &= -\frac{33}{72}-\frac{4}{72} && \text{Simplify numerators and denominators.} \\ &= \frac{-33-4}{72} && \text{Keep LCD; subtract numerators.} \\ &= \frac{-37}{72} && \text{Simplify numerator.} \end{aligned}$$

Of course, negative divided by positive yields a negative answer, so we can also write our answer in the form

$$-\frac{11}{24}-\frac{1}{18}=-\frac{37}{72}.$$

Exercise 2.5.11

Simplify: $-\frac{5}{24}-\frac{11}{36}$

Answer

−37/72

Comparing Fractions

The simplest way to compare fractions is to create equivalent fractions.

Example 2.5.12

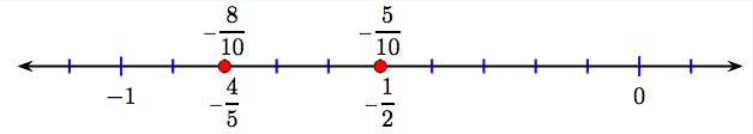
Arrange the fractions −1/2 and −4/5 on a number line, then compare them by using the appropriate inequality symbol.

Solution

The least common denominator for 2 and 5 is the number 10. First, make equivalent fractions with a LCD equal to 10.

$$-\frac{1}{2} = -\frac{1 \cdot 5}{2 \cdot 5} = -\frac{5}{10}$$
$$-\frac{4}{5} = -\frac{4 \cdot 2}{5 \cdot 2} = -\frac{8}{10}$$

To plot tenths, subdivide the interval between −1 and 0 into ten equal increments.



Because −4/5 lies to the left of −1/2, we have that −4/5 is less than −1/2, so we write

$$-\frac{4}{5} < -\frac{1}{2}.$$

? Exercise 2.5.12

Compare −3/8 and −1/2.

Answer

$$-\frac{1}{2} < -\frac{3}{8}$$

Exercises

In Exercises 1-10, list the multiples the given numbers, then list the common multiples. Select the LCM from the list of common multiples.

1. 9 and 15
2. 15 and 20
3. 20 and 8
4. 15 and 6
5. 16 and 20
6. 6 and 10
7. 20 and 12
8. 12 and 8
9. 10 and 6
10. 10 and 12

In Exercises 11-20, for the given numbers, calculate the LCM using prime factorization.

11. 54 and 12
12. 108 and 24
13. 18 and 24
14. 36 and 54
15. 72 and 108
16. 108 and 72
17. 36 and 24
18. 18 and 12

19. 12 and 18
20. 12 and 54

In Exercises 21-32, add or subtract the fractions, as indicated, and simplify your result.

21. $\frac{7}{12} - \frac{1}{12}$
22. $\frac{3}{7} - \frac{5}{7}$
23. $\frac{1}{9} + \frac{1}{9}$
24. $\frac{1}{7} + \frac{3}{7}$
25. $\frac{1}{5} - \frac{4}{5}$
26. $\frac{3}{5} - \frac{2}{5}$
27. $\frac{3}{7} - \frac{4}{7}$
28. $\frac{6}{7} - \frac{2}{7}$
29. $\frac{4}{11} + \frac{9}{11}$
30. $\frac{10}{11} + \frac{4}{11}$
31. $\frac{3}{11} + \frac{4}{11}$
32. $\frac{3}{7} + \frac{2}{7}$

In Exercises 33-56, add or subtract the fractions, as indicated, and simplify your result.

33. $\frac{1}{6} - \frac{1}{8}$
34. $\frac{7}{9} - \frac{2}{3}$
35. $\frac{1}{5} + \frac{2}{3}$
36. $\frac{7}{9} + \frac{2}{3}$
37. $\frac{2}{3} + \frac{5}{8}$
38. $\frac{3}{7} + \frac{5}{9}$
39. $\frac{4}{7} - \frac{5}{9}$
40. $\frac{3}{5} - \frac{7}{8}$
41. $\frac{2}{3} - \frac{3}{8}$
42. \(\frac{2}{5} − \frac{1}{8}\)
43. $\frac{6}{7} - \frac{1}{6}$
44. $\frac{1}{2} - \frac{1}{4}$
45. $\frac{1}{6} + \frac{2}{3}$
46. $\frac{4}{9} + \frac{7}{8}$
47. $\frac{7}{9} + \frac{1}{8}$
48. $\frac{1}{6} + \frac{1}{7}$
49. $\frac{1}{3} + \frac{1}{7}$
50. $\frac{5}{6} + \frac{1}{4}$
51. $\frac{1}{2} - \frac{2}{7}$

52. $\frac{1}{3}-\frac{1}{8}$
53. $\frac{5}{6}-\frac{4}{5}$
54. $\frac{1}{2}-\frac{1}{9}$
55. $\frac{1}{3}+\frac{1}{8}$
56. $\frac{1}{6}+\frac{7}{9}$

In Exercises 57-68, add or subtract the fractions, as indicated, by first using prime factorization to find the least common denominator.

57. $\frac{7}{36}+\frac{11}{54}$
58. $\frac{7}{54}+\frac{7}{24}$
59. $\frac{7}{18}-\frac{5}{12}$
60. $\frac{5}{54}-\frac{7}{12}$
61. $\frac{7}{36}+\frac{7}{54}$
62. $\frac{5}{72}+\frac{5}{108}$
63. $\frac{7}{24}-\frac{5}{36}$
64. $\frac{11}{54}+\frac{7}{72}$
65. $\frac{11}{12}+\frac{5}{18}$
66. $\frac{11}{24}+\frac{11}{108}$
67. $\frac{11}{54}-\frac{5}{24}$
68. $\frac{7}{54}-\frac{5}{24}$

In Exercises 69-80, add or subtract the fractions, as indicated, and simplify your result.

69. $\frac{-3}{7}+\left(\frac{-3}{7}\right)$
70. $\frac{-5}{9}+\left(\frac{-1}{9}\right)$
71. $\frac{7}{9}-\left(\frac{-1}{9}\right)$
72. $\frac{8}{9}-\left(\frac{-4}{9}\right)$
73. $\frac{7}{9}+\left(\frac{-2}{9}\right)$
74. $\frac{2}{3}+\left(\frac{-1}{3}\right)$
75. $\frac{-3}{5}-\frac{4}{5}$
76. $\frac{-7}{9}-\frac{1}{9}$
77. $\frac{-7}{8}+\frac{1}{8}$
78. $\frac{-2}{3}+\backslash(\frac{1}{3}$
79. $\frac{-1}{3}-\left(\frac{-2}{3}\right)$
80. $\frac{-7}{8}-\left(\frac{-5}{8}\right)$

In Exercises 81-104, add or subtract the fractions, as indicated, and simplify your result.

81. $\frac{-2}{7}$ + \frac{4}{5}\)
82. $\frac{-1}{4} + \frac{2}{7}$
83. $\frac{-1}{4} - \left(\frac{-4}{9}\right)$
84. \frac{−3}{4} −left(\frac{−1}{8} \right)
85. $\frac{-2}{7} + \frac{3}{4}$
86. $\frac{-1}{3} + \frac{5}{8}$
87. $\frac{-4}{9} - \frac{1}{3}$
88. $\frac{-5}{6} - \frac{1}{3}$
89. $\frac{-5}{7} - \left(\frac{-1}{5}\right)$
90. $\frac{-6}{7} - \left(\frac{-1}{8}\right)$
91. $\frac{1}{9} + \left(\frac{-1}{3}\right)$
92. $\frac{1}{8} + \left(\frac{-1}{2}\right)$
93. $\frac{2}{3} + \left(\frac{-1}{9}\right)$
94. $\frac{3}{4} + \left(\frac{-2}{3}\right)$
95. $\frac{-1}{2} + \left(\frac{-6}{7}\right)$
96. $\frac{-4}{5} + \left(\frac{-1}{2}\right)$
97. $\frac{-1}{2} + \left(\frac{-3}{4}\right)$
98. $\frac{-3}{5} + \left(\frac{-1}{2}\right)$
99. $\frac{-1}{4} - \frac{1}{2}$
100. $\frac{-8}{9} - \frac{2}{3}$
101. $\frac{5}{8} - \left(\frac{-3}{4}\right)$
102. $\frac{3}{4} - \left(\frac{-3}{8}\right)$
103. $\frac{1}{8} - \left(\frac{-1}{3}\right)$
104. $\frac{1}{2} - \left(\frac{-4}{9}\right)$

In Exercises 105-120, add or subtract the fractions, as indicated, and write your answer is lowest terms.

105. $\frac{1}{2} + \frac{3q}{5}$
106. $\frac{4}{7} - \frac{b}{3}$
107. $\frac{4}{9} - \frac{3a}{4}$
108. $\frac{4}{9} - \frac{b}{2}$
109. $\frac{2}{s} + \frac{1}{3}$
110. $\frac{2}{s} + \frac{3}{7}$

111. $\frac{1}{3} - \frac{7}{b}$

112. $\frac{1}{2} - \frac{9}{s}$

113. $\frac{4b}{7} + \frac{2}{3}$

114. $\frac{2a}{5} + \frac{5}{8}$

115. $\frac{2}{3} - \frac{9}{t}$

116. $\frac{4}{7} - \frac{1}{y}$

117. $\frac{9}{s} + \frac{7}{8}$

118. $\frac{6}{t} - \frac{1}{9}$

119. $\frac{7b}{8} - \frac{5}{9}$

120. $\frac{3p}{4} - \frac{1}{8}$

In Exercises 121-132, determine which of the two given statements is true.

121. $\frac{-2}{3} < \frac{-8}{7}$ or $\frac{-2}{3} > \frac{-8}{7}$

122. $\frac{-1}{7} < \frac{-8}{9}$ or $\frac{-1}{7} > \frac{-8}{9}$

123. $\frac{6}{7} < \frac{7}{3}$ or $\frac{6}{7} > \frac{7}{3}$

124. $\frac{1}{2} < \frac{2}{7}$ or $\frac{1}{2} > \frac{2}{7}$

125. $\frac{-9}{4} < \frac{-2}{3}$ or \frac{− 9}{4} > \frac{−2}{3}\)

126. $\frac{-3}{7} < \frac{-9}{2}$ or $\frac{-3}{7} > \frac{-9}{2}$

127. $\frac{5}{7} < \frac{5}{9}$ or \frac{5}{7} > \frac{5}{9}\)

128. $\frac{1}{2} < \frac{1}{3}$ or $\frac{1}{2} > \frac{1}{3}$

129. $\frac{-7}{2} < \frac{-1}{5}$ or $\frac{-7}{2} > \frac{-1}{5}$

130. $\frac{-3}{4} < \frac{-5}{9}$ or $\frac{-3}{4} > \frac{-5}{9}$

131. $\frac{5}{9} < \frac{6}{5}$ or $\frac{5}{9} > \frac{6}{5}$

132. $\frac{3}{2} < \frac{7}{9}$ or $\frac{3}{2} > \frac{7}{9}$

Answers

1. 45

3. 40

5. 80

7. 60

9. 30

11. 108

13. 72

15. 216

17. 72

19. 36

21. $\frac{1}{2}$

23. $\frac{2}{9}$
25. $\frac{-3}{5}$
27. $\frac{-1}{7}$
29. $\frac{13}{11}$
31. $\frac{7}{11}$
33. $\frac{1}{24}$
35. $\frac{13}{15}$
37. $\frac{31}{24}$
39. $\frac{1}{63}$
41. $\frac{7}{24}$
43. $\frac{29}{42}$
45. $\frac{5}{6}$
47. $\frac{65}{72}$
49. $\frac{10}{21}$
51. $\frac{3}{14}$
53. $\frac{1}{30}$
55. $\frac{11}{24}$
57. $\frac{43}{108}$
59. $\frac{-1}{36}$
61. $\frac{35}{108}$
63. $\frac{11}{72}$
65. $\frac{43}{36}$
67. $\frac{-1}{216}$
69. $\frac{-6}{7}$
71. $\frac{8}{9}$
73. $\frac{5}{9}$
75. $\frac{-7}{5}$
77. $\frac{-3}{4}$
79. $\frac{1}{3}$
81. $\frac{18}{35}$
83. $\frac{7}{36}$
85. $\frac{13}{28}$
87. $\frac{-7}{9}$
89. $\frac{-18}{35}$
91. $\frac{-2}{9}$

93. $\frac{5}{9}$

95. $\frac{-19}{14}$

97. $\frac{-5}{4}$

99. $\frac{-3}{4}$

101. $\frac{11}{8}$

103. $\frac{11}{24}$

105. $\frac{5+6q}{10}$

107. $\frac{16-27a}{36}$

109. $\frac{6+s}{3s}$

111. $\frac{b-21}{3b}$

113. $\frac{12b+14}{21}$

115. $\frac{2t-27}{3t}$

117. $\frac{72+7s}{8s}$

119. $\frac{63b-40}{72}$

121. $\frac{-2}{3} > \backslash(\frac{-8}{7}$

123. $\frac{6}{7} < \frac{7}{3}$

125. $\frac{-9}{4} < \frac{-2}{3}$

127. $\frac{5}{7} > \frac{5}{9}$

129. $\frac{-7}{2} < \frac{-1}{5}$

131. $\frac{5}{9} < \frac{6}{5}$

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2.6: Multiplying and Dividing Mixed Fractions

We begin with definitions of *proper* and *improper* fractions.

Proper and Improper Fractions

A *proper fraction* is a fraction whose numerator is smaller than its denominator. An *improper fraction* is a fraction whose numerator is larger than its denominator.

For example,

$$\frac{2}{3},\ -\frac{23}{39},\ \text{and}\ \frac{119}{127}$$

are all examples of proper fractions. On the other hand,

$$\frac{4}{3},\ -\frac{317}{123},\ \text{and}\ -\frac{233}{101}$$

are all examples of improper fractions.

A mixed fraction[1] is part whole number, part fraction.

Mixed Fractions

The number

$$5\frac{3}{4}$$

is called a *mixed fraction*. It is defined to mean

$$5\frac{3}{4} = 5 + \frac{3}{4}.$$

In the mixed fraction $5\frac{3}{4}$, the 5 is the *whole number part* and the 3/4 is the *fractional part*.

Changing Mixed Fractions to Improper Fractions

We have all the tools required to change a mixed fraction into an improper fraction. We begin with an example.

Example 1

Change the mixed fraction $4\frac{7}{8}$ into an improper fraction.

Solution

We employ the definition of a mixed fraction, make an equivalent fraction for the whole number part, then add.

$$\begin{aligned} 4\frac{7}{8} &= 4 + \frac{7}{8} && \text{By definition.} \\ &= \frac{4\cdot 8}{8} + \frac{7}{8} && \text{Equivalent fraction with LCD} = 8. \\ &= \frac{4\cdot 8 + 7}{8} && \text{Add numerators over common denominator.} \\ &= \frac{39}{8} && \text{Simplify the numerator.} \end{aligned}$$

Thus, $4\frac{7}{8}$ is equal to 39/8.

Exercise

Change $5\frac{3}{4}$ to an improper fraction.

Answer

23/4

There is a quick technique you can use to change a mixed fraction into an improper fraction.

Quick Way to Change a Mixed Fraction to an Improper Fraction

To change a mixed fraction to an improper fraction, multiply the whole number part by the denominator, add the numerator, then place the result over the denominator.

Thus, to quickly change $4\frac{7}{8}$ to an improper fraction, multiply the whole number 4 by the denominator 8, add the numerator 7, then place the result over the denominator. In symbols, this would look like this:

$$4\frac{7}{8} = \frac{4 \cdot 8 + 7}{8}.$$

This is precisely what the third step in Example 1 looks like; we're just eliminating a lot of the work.

Example 2

Change $4\frac{2}{3}$ to an improper fraction.

Solution

Take $4\frac{2}{3}$, multiply the whole number part by the denominator, add the numerator, then put the result over the denominator.

$$4\frac{2}{3} = \frac{4 \cdot 3 + 2}{3}$$

Thus, the result is

$$4\frac{2}{3} = \frac{14}{3}.$$

Exercise

Change \(7 \frac{3}{8}\) to an improper fraction.

Answer

59/8

It is very easy to do the intermediate step in Example 2 mentally, allowing you to skip the intermediate step and go directly from the mixed fraction to the improper fraction without writing down a single bit of work.

Example 3

Without writing down any work, use mental arithmetic to change $-2\frac{3}{5}$ to an improper fraction.

Solution

To change $-2\frac{3}{5}$ to an improper fraction, ignore the minus sign, proceed as before, then prefix the minus sign to the resulting improper fraction. So, multiply 5 times 2 and add 3. Put the result 13 over the denominator 5, then prefix the resulting improper fraction with a minus sign. That is,

$$-2\frac{3}{5} = -\frac{13}{5}.$$

? Exercise

Change $-3\frac{5}{12}$ to an improper fraction.

Answer

−41/12

Changing Improper Fractions to Mixed Fractions

The first step in changing the improper fraction 27/5 to a mixed fraction is to write the improper fraction as a sum.

$$\frac{27}{5} = \frac{25}{5} + \frac{2}{5}$$

Simplifying equation 4.1, we get

$$\begin{aligned}\frac{27}{5} &= 5 + \frac{2}{5} \\ &= 5\frac{2}{5}\end{aligned}$$

Comment. You can't just choose any sum. The sum used in equation 4.1 is constructed so that the first fraction will equal a whole number and the second fraction is proper. Any other sum will fail to produce the correct mixed fraction. For example, the sum

$$\frac{27}{5} = \frac{23}{5} + \frac{4}{5}$$

is useless, because 23/5 is not a whole number. Likewise, the sum

$$\frac{27}{5} = \frac{20}{5} + \frac{7}{5}$$

is no good. Even though 20/5 = 4 is a whole number, the second fraction 7/5 is still improper.

✓ Example 4

Change 25/9 to a mixed fraction.

Solution

Break 25/9 into the appropriate sum.

$$\begin{aligned}\frac{25}{9} &= \frac{18}{9} + \frac{7}{9} \\ &= 2 + \frac{7}{9} \\ &= 2\frac{7}{9}\end{aligned}$$

? Exercise

Change 25/7 to a mixed fraction.

Answer

$3\frac{4}{7}$.

Comment. A pattern is emerging. • In the case of 27/5, note that 27 divided by 5 is equal to 5 with a remainder of 2. Compare this with the mixed fraction result: 27/5=5 2 5 . • In the case of Example 4, note that 25 divided by 9 is 2 with a remainder of 7. Compare this with the mixed fraction result: 25/9=2 7 9 . These observations motivate the following technique.

Quick Way to Change an Improper Fraction to a Mixed Fraction

To change an improper fraction to a mixed fraction, divide the numerator by the denominator. The quotient will be the whole number part of the mixed fraction. If you place the remainder over the denominator, this will be the fractional part of the mixed fraction.

Example 5

Change 37/8 to a mixed fraction.

Solution

37 divided by 8 is 4, with a remainder of 5. That is:

$$\begin{array}{r} 4 \\ 8\overline{)37} \\ \underline{32} \\ 5 \end{array}$$

The quotient becomes the whole number part and we put the remainder over the divisor. Thus,

$$\frac{37}{8} = 4\frac{5}{8}.$$

Note: You can check your result with the "Quick Way to Change a Mixed Fraction to an Improper Fraction." 8 times 4 plus 5 is 37. Put this over 8 to get 37/8.

Exercise

Change 38/9 to a mixed fraction.

Answer

$4\frac{2}{9}$

Example 6

Change −43/5 to a mixed fraction.

Solution

Ignore the minus sign and proceed in the same manner as in Example 5. 43 divided by 5 is 8, with a remainder of 3.

$$\begin{array}{r} 8 \\ 5\overline{)43} \\ \underline{40} \\ 3 \end{array}$$

The quotient is the whole number part, then we put the remainder over the divisor. Finally, prefix the minus sign.

$$-\frac{43}{5} = -8\frac{3}{5}$$

Multiplying and Dividing Mixed Fractions

You have all the tools needed to multiply and divide mixed fractions. First, change the mixed fractions to improper fractions, then multiply or divide as you did in previous sections.

[1]A mixed fraction is sometimes called a *mixed number*.

Example 7

Simplify: $-2\frac{1}{12}\cdot 2\frac{4}{5}$.

Solution

Change to improper fractions, factor, cancel, and simplify.

$$\begin{aligned} -2\frac{1}{12}\cdot 2\frac{4}{5} &= -\frac{25}{12}\cdot\frac{14}{5} && \text{Change to improper fractions.}\\ &= -\frac{25\cdot 14}{12\cdot 5} && \text{Multiply numerators; multiply denominators.}\\ & && \text{Unlike signs; product is negative.}\\ &= -\frac{(5\cdot 5)\cdot(2\cdot 7)}{2\cdot 2\cdot 3)\cdot(5)} && \text{Prime factor.}\\ &= -\frac{\cancel{5}\cdot 5\cdot \cancel{2}\cdot 7}{\cancel{2}\cdot 2\cdot 3\cdot \cancel{5}} && \text{Cancel common factors.}\\ &= -\frac{35}{6} && \text{Multiply numerators and denominators.} \end{aligned}$$

This is a perfectly good answer, but if you want a mixed fraction answer, 35 divided by 6 is 5, with a remainder of 5. Hence,

$$-2\frac{1}{12}\cdot 2\frac{4}{5} = -5\frac{5}{6}.$$

Exercise 2.6.1

Simplify:

$$-3\frac{3}{4}\cdot 2\frac{2}{5}$$

Answer

-9

Example 8

Simplify:

$$-4\frac{4}{5}\div 5\frac{3}{5}.$$

Solution

Change to improper fractions, invert and multiply, factor, cancel, and simplify.

$$-4\frac{4}{5} \div 5\frac{3}{5} = -\frac{24}{5} \div \frac{28}{5}$$ Change to improper fractions.

$$= -\frac{24}{5} \cdot \frac{5}{28}$$ Invert and multiply.

$$= -\frac{2 \cdot 2 \cdot 2 \cdot 3}{5} \cdot \frac{5}{2 \cdot 2 \cdot 7}$$ Prime factor.

$$= -\frac{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 3}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{2} \cdot \cancel{2} \cdot 7}$$ Cancel common factors.

$$= -\frac{6}{7}$$ Multiply numerators and denominators.

? Exercise

Simplify:

$$-2\frac{4}{9} \cdot 3\frac{2}{3}$$

Answer

−2/3

Exercises

In Exercises 1-12, convert the mixed fraction to an improper fraction.

1. $2\frac{1}{3}$
2. $1\frac{8}{11}$
3. $1\frac{1}{19}$
4. $-1\frac{1}{5}$
5. $-1\frac{3}{7}$
6. $1\frac{3}{17}$
7. $1\frac{1}{9}$
8. $1\frac{5}{11}$
9. $-1\frac{1}{2}$
10. $-1\frac{5}{8}$
11. $1\frac{1}{3}$
12. $-1\frac{5}{7}$

In Exercises 13-24, convert the improper fraction to a mixed fraction.

13. $\frac{13}{7}$
14. $\frac{-17}{9}$
15. $\frac{-13}{5}$
16. $\frac{-10}{3}$
17. $\frac{-16}{5}$
18. $\frac{16}{13}$

19. $\frac{9}{8}$
20. $\frac{16}{5}$
21. $\frac{-6}{5}$
22. $\frac{-17}{10}$
23. $\frac{-3}{2}$
24. $\frac{-7}{4}$

In Exercises 25-48, multiply the numbers and express your answer as a mixed fraction.

25. $1\frac{1}{7} \cdot 2\frac{1}{2}$
26. $1\frac{1}{8} \cdot 1\frac{1}{6}$
27. $4 \cdot 1\frac{1}{6}$
28. $1\frac{7}{10} \cdot 4$
29. $\left(-1\frac{1}{12}\right)\left(3\frac{3}{4}\right)$
30. $\left(-3\frac{1}{2}\right)\left(3\frac{1}{3}\right)$
31. $7\frac{1}{2} \cdot 1\frac{1}{13}$
32. $2\frac{1}{4} \cdot 1\frac{5}{11}$
33. $\left(1\frac{2}{13}\right)\left(-4\frac{2}{3}\right)$
34. $\left(1\frac{1}{14}\right)\left(-2\frac{2}{5}\right)$
35. $\left(1\frac{3}{7}\right)\left(-3\frac{3}{4}\right)$
36. $\left(1\frac{4}{5}\right)\left(-3\frac{3}{4}\right)$
37. $9 \cdot \left(-1\frac{2}{15}\right)$
38. $4 \cdot \left(-2\frac{5}{6}\right)$
39. $\left(-2\frac{1}{8}\right)(-6)$
40. $(-9)\left(-3\frac{1}{6}\right)$
41. $\left(-4\frac{1}{2}\right)\left(-2\frac{2}{5}\right)$
42. $\left(-1\frac{3}{7}\right)\left(-3\frac{3}{4}\right)$
43. $\left(-2\frac{1}{6}\right) \cdot 4$
44. $(-6) \cdot \left(1\frac{1}{9}\right)$
45. $\left(-1\frac{4}{15}\right)\left(2\frac{1}{2}\right)$
46. $\left(-1\frac{1}{5}\right)\left(1\frac{5}{9}\right)$
47. $\left(-2\frac{1}{2}\right)\left(-1\frac{7}{11}\right)$
48. $\left(-1\frac{7}{11}\right)\left(-1\frac{7}{12}\right)$

In Exercises 49-72, divide the mixed fractions and express your answer as a mixed fraction.

49. $8 \div 2\frac{2}{9}$
50. $4\frac{2}{3} \div 4$
51. $\left(-3\frac{1}{2}\right) \div \left(1\frac{1}{16}\right)$

52. $\left(-1\frac{2}{5}\right) \div \left(1\frac{1}{15}\right)$

53. $6\frac{1}{2} \div 1\frac{7}{12}$

54. $5\frac{1}{2} \div 1\frac{9}{10}$

55. $(-4) \div \left(1\frac{5}{9}\right)$

56. $\left(-4\frac{2}{3}\right) \div 4$

57. $\left(-5\frac{2}{3}\right) \div \left(-2\frac{1}{6}\right)$

58. $\left(-2\frac{1}{2}\right) \div \left(-2\frac{2}{9}\right)$

59. $\left(-6\frac{1}{2}\right) \div \left(4\frac{1}{4}\right)$

60. $\left(-1\frac{1}{6}\right) \div \left(1\frac{1}{8}\right)$

61. $(-6) \div \left(-1\frac{3}{11}\right)$

62. $\left(-6\frac{2}{3}\right) \div (-6)$

63. $\left(4\frac{2}{3}\right) \div (-4)$

64. $\left(6\frac{2}{3}\right) \div (-6)$

65. $\left(1\frac{3}{4}\right) \div \left(-1\frac{1}{12}\right)$

66. $\left(2\frac{4}{7}\right) \div \left(-1\frac{1}{5}\right)$

67. $\left(5\frac{2}{3}\right) \div 1\frac{1}{9}$

68. $1\frac{2}{3} \div 1\frac{2}{9}$

69. $\left(-7\frac{1}{2}\right) \div \left(-2\frac{2}{5}\right)$

70. $\left(-5\frac{1}{3}\right) \div \left(-2\frac{5}{6}\right)$

71. $\left(3\frac{2}{3}\right) \div \left(-1\frac{1}{9}\right)$

72. $\left(8\frac{1}{2}\right) \div \left(-1\frac{3}{4}\right)$

---

73. **Small Lots**. How many quarter-acre lots can be made from $6\frac{1}{2}$ acres of land?

74. **Big Field.** A field was formed from $17\frac{1}{2}$ half-acre lots. How many acres was the resulting field ?

75. **Jewelry.** To make some jewelry, a bar of silver $4\frac{1}{2}$ inches long was cut into pieces $\frac{1}{12}$ inch long. How many pieces were made?

76. **Muffins.** This recipe will make 6 muffins: 1 cup milk, $1\frac{2}{3}$ cups flour, 2 eggs, 1/2 teaspoon salt, $1\frac{1}{2}$ teaspoons baking powder. Write the recipe for six dozen muffins.

Answers

1. $\frac{7}{3}$

3. $\frac{20}{19}$

5. $-\frac{10}{7}$

7. $\frac{10}{9}$

9. $-\frac{3}{2}$

11. $\frac{4}{3}$

13. $1\frac{6}{7}$

15. $-2\frac{3}{5}$

17. $-3\frac{1}{5}$
19. $1\frac{1}{8}$
21. $-1\frac{1}{5}$
23. $-1\frac{1}{2}$
25. $2\frac{6}{7}$
27. $4\frac{2}{3}$
29. $-4\frac{1}{16}$
31. $8\frac{1}{13}$
33. $-5\frac{5}{13}$
35. $-5\frac{5}{14}$
37. $-10\frac{1}{5}$
39. $12\frac{3}{4}$
41. $10\frac{4}{5}$
43. $-8\frac{2}{3}$
45. $-3\frac{1}{6}$
47. $4\frac{1}{11}$
49. $3\frac{3}{5}$
51. $-3\frac{5}{17}$
53. $4\frac{2}{19}$
55. $-2\frac{4}{7}$
57. $2\frac{8}{13}$
59. $-1\frac{9}{17}$
61. $4\frac{5}{7}$
63. $-1\frac{1}{6}$
65. $-1\frac{8}{13}$
67. $5\frac{1}{10}$
69. $3\frac{1}{8}$
71. $-3\frac{3}{10}$
73. 26 quarter-acre lots
75. 54 pieces

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2.7: Adding and Subtracting Mixed Fractions

In this section, we will learn how to add and subtract mixed fractions.

Adding Mixed Fractions

We can use tools we've already developed to add two or more mixed fractions.

Example 1

Simplify: $2\frac{7}{8}+1\frac{3}{4}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} 2\frac{7}{8}+1\frac{3}{4} &= \frac{23}{8}+\frac{7}{4} && \text{Change to equivalent fractions.}\\ &= \frac{23}{8}+\frac{7\cdot 2}{4\cdot 2} && \text{Equivalent fractions with LCD} = 8.\\ &= \frac{23}{8}+\frac{14}{8} && \text{Simplify numerators and denominators.}\\ &= \frac{37}{8} && \text{Add numerators over common denominator.} \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 37 divided by 8 is 4, with a remainder of 5. Thus,

$$2\frac{7}{8}+1\frac{3}{4}=4\frac{5}{8}.$$

Exercise

Simplify: $3\frac{2}{3}+4\frac{1}{8}$

Answer

$7\frac{19}{24}$

Example 2

Simplify: $3\frac{1}{4}+2\frac{1}{3}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} 3\frac{1}{4}+2\frac{1}{3} &= \frac{13}{4}+\frac{7}{3} && \text{Change to improper fractions.}\\ &= \frac{13\cdot 3}{4\cdot 3}+\frac{7\cdot 4}{3\cdot 4} && \text{Equivalent fractions with LCD} = 12.\\ &= \frac{39}{12}+\frac{28}{12} && \text{Simplify numerators and denominators.}\\ &= \frac{67}{12} && \text{Add numerators and denominators.} \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 67 divided by 12 is 5, with a remainder of 7. Thus,

$$3\frac{1}{4}+2\frac{1}{3}=5\frac{7}{12}.$$

Exercise

Simplify: $8\frac{1}{2}+2\frac{2}{3}$

Answer

$11\frac{1}{6}$

Mixed Fraction Approach

There is another possible approach, based on the fact that a mixed fraction is a sum. Let's revisit Example 2.

Example 3

Simplify: $3\frac{1}{4}+2\frac{1}{3}$.

Solution

Use the commutative and associative properties to change the order of addition, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} 3\frac{1}{4}+2\frac{1}{3} &= \left(3+\frac{1}{4}\right)+\left(2+\frac{1}{3}\right) && \text{Mixed fractions as sums.} \\ &= (3+2)+\left(\frac{1}{4}+\frac{1}{3}\right) && \text{Reorder and regroup.} \\ &= 5+\left(\frac{1\cdot 3}{4\cdot 3}+\frac{1\cdot 4}{3\cdot 4}\right) && \begin{array}{r}\text{Add whole numbers: } 3+2=5. \\ \text{Equivalent fractions; LCD} = 12.\end{array} \\ &= 5+\left(\frac{3}{12}+\frac{4}{12}\right) && \text{Simplify numerators and denominators.} \\ &= 5+\frac{7}{12} && \text{Add numerators over common denominators.} \end{aligned}$$

This result can be written in mixed fraction form. Thus,

$$3\frac{1}{4}+2\frac{1}{3}=5\frac{7}{12}.$$

Note that this solution is identical to the result found in Example 2.

Exercise

Simplify: $7\frac{2}{5}+3\frac{1}{8}$

Answer

$10\frac{21}{40}$

Example 3 leads us to the following result.

Adding Mixed Fractions

To add two mixed fractions, add the whole number parts, then add the fractional parts.

Working in Vertical Format

When adding mixed fractions, many prefer to work in a vertical format. For example, here is how we would arrange the solution from Example 2 and Example 3 in vertical format. We create equivalent fractions, then add the whole number parts and fractional parts.

$$\begin{array}{rcrcr} 3\frac{1}{4} & = & 3\frac{1\cdot 3}{4\cdot 3} & = & 3\frac{3}{12} \\ +2\frac{1}{3} & = & +2\frac{1\cdot 4}{3\cdot 4} & = & +2\frac{4}{12} \\ \hline & & & & 5\frac{7}{12} \end{array}$$

Note that the answer is identical to the answer found in Example 2 and Example 3. That is,

$$3\frac{1}{4}+2\frac{1}{3}=5\frac{7}{12}.$$

Example 4

Sarah is making window curtains for two rooms in her house. The kitchen will require $5\frac{2}{3}$ yards of material and the dining room will require $6\frac{5}{8}$ yards of material. How much total material is required?

Solution

To find the total material required for the two rooms, we must add $5\frac{2}{3}$ and $6\frac{5}{8}$. Create equivalent fractions with a common denominator, then add whole number parts and fractional parts.

$$\begin{array}{rcrcr} 5\frac{2}{3} & = & 5\frac{2\cdot 8}{3\cdot 8} & = & 5\frac{16}{24} \\ +6\frac{5}{8} & = & +6\frac{5\cdot 3}{8\cdot 3} & = & +6\frac{15}{24} \\ \hline & & & & 11\frac{31}{24} \end{array}$$

An answer that is part mixed fraction, part improper fraction, is **not** allowed. To finish, we need to change the improper fractional part to a mixed fraction, then add. 31 divided by 24 is 1, with a remainder of 7. That is, 31/24 = 1 7 24 . Now we can add whole number parts and fractional parts.

$$\begin{aligned} 11\frac{31}{24} &= 11+1\frac{7}{24} \\ &= 12\frac{7}{24}. \end{aligned}$$

Thus, the total material required is $12\frac{7}{24}$ yards.

Exercise

Jim is working on a project that requires two boards, the first cut to a length of $6\frac{1}{2}$ feet, the second cut to a length of $5\frac{7}{8}$ feet. How many total feet of board is required?

Answer

$12\frac{3}{8}$ feet.

Subtracting Mixed Fractions

Let's look at some examples that subtract two mixed fractions.

Example 5

Simplify: $4\frac{5}{8}-2\frac{1}{16}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then subtract.

$$
\begin{aligned}
4\frac{5}{8}-2\frac{1}{16} &= \frac{37}{8}-\frac{33}{16} && \text{Change to improper fractions.}\\
&= \frac{37\cdot 2}{8\cdot 2}-\frac{33}{16} && \text{Equivalent fractions with LCD} = 16.\\
&= \frac{74}{16}-\frac{33}{16} && \text{Simplify numerator and denominators.}\\
&= \frac{41}{16} && \text{Add numerators over common denominator.}
\end{aligned}
$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 41 divided by 16 is 2, with a remainder of 9. Thus,

$$4\frac{5}{8}-2\frac{1}{16}=2\frac{9}{16}.$$

? Exercise

Simplify: $5\frac{2}{3}-3\frac{1}{5}$

Answer

$2\frac{7}{15}$

✓ Example 6

Simplify: $5\frac{3}{4}-2\frac{1}{3}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then subtract.

$$
\begin{aligned}
5\frac{3}{4}-2\frac{1}{3} &= \frac{23}{4}-\frac{7}{3} && \text{Change to improper fractions.}\\
&= \frac{23\cdot 3}{4\cdot 3}-\frac{7\cdot 4}{3\cdot 4} && \text{Equivalent fractions with LCD} = 12.\\
&= \frac{69}{12}-\frac{28}{12} && \text{Simplify numerators and denominators.}\\
&= \frac{41}{12} && \text{Add numerators over common denominator.}
\end{aligned}
$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 41 divided by 12 is 3, with a remainder of 5. Thus,

$$5\frac{3}{4}-2\frac{1}{3}=3\frac{5}{12}.$$

? Exercise

Simplify: $4\frac{7}{9}-2\frac{3}{18}$

Answer

$2\frac{11}{18}$

Mixed Fraction Approach

There is another possible approach, based on the fact that a mixed fraction is a sum. Let's revisit Example 6.

Example 7

Simplify: $5\frac{3}{4}-2\frac{1}{3}$.

Solution

A mixed fraction is a sum.

$$5\frac{3}{4}-2\frac{1}{3}=\left(5+\frac{3}{4}\right)-\left(2+\frac{1}{3}\right)$$

Distribute the negative sign.

$$=5+\frac{3}{4}-2-\frac{1}{3}$$

We could change the subtraction to adding the opposite, change the order of addition, then change the adding of opposites back to subtraction. However, it is much easier if we look at this last line as a request to add four numbers, two of which are positive and two of which are negative. Changing the order does not affect the answer.

$$=(5-2)+\left(\frac{3}{4}-\frac{1}{3}\right)$$

Note that we did not change the signs of any of the four numbers. We just changed the order. Subtract the whole number parts. Make equivalent fractions with a common denominator, then subtract the fractional parts.

$$\begin{aligned}&=3+\left(\frac{3\cdot 3}{4\cdot 3}-\frac{1\cdot 4}{3\cdot 4}\right) && \text{Create equivalent fractions.}\\ &=3+\left(\frac{9}{12}-\frac{4}{12}\right) && \text{Simplify numerators and denominators.}\\ &=3+\frac{5}{12} && \text{Subtract fractional parts.}\end{aligned}$$

Thus,

$$5\frac{3}{4}-2\frac{1}{3}=3\frac{5}{12}.$$

Note that this is exactly the same answer as that found in Example 6.

Exercise

Simplify: $8\frac{5}{6}-4\frac{3}{8}$

Answer

$4\frac{11}{24}$

In Example 6, we see that we handle subtraction of mixed fractions in exactly the same manner that we handle addition of mixed fractions.

Subtracting Mixed Fractions

To subtract two mixed fractions, subtract their whole number parts, then subtract their fractional parts.

Working in Vertical Format

When subtracting mixed fractions, many prefer to work in a vertical format. For example, here is how we would arrange the solution from Example 6 and Example 7 in vertical format. We create equivalent fractions, then subtract the whole number parts and fractional parts.

$$\begin{array}{rcrcr} 5\frac{3}{4} & = & 5\frac{3\cdot 3}{4\cdot 3} & = & 5\frac{9}{12} \\ -2\frac{1}{3} & = & -3\frac{1\cdot 4}{3\cdot 4} & = & -2\frac{4}{12} \\ \hline & & & & 3\frac{5}{12} \end{array}$$

Note that the answer is identical to the answer found in Example 6 and Example 7. That is,

$$5\frac{3}{4}-2\frac{1}{3}=3\frac{5}{12}.$$

Borrowing in Vertical Format

Consider the following example.

Example 8

Simplify: $8\frac{1}{4}-5\frac{5}{6}$.

Solution

Create equivalent fractions with a common denominator.

$$\begin{array}{rcrcr} 8\frac{1}{4} & = & 8\frac{1\cdot 3}{4\cdot 3} & = & 8\frac{3}{12} \\ -5\frac{5}{6} & = & -5\frac{5\cdot 2}{6\cdot 2} & = & -5\frac{10}{12} \end{array}$$

You can see the difficulty. On the far right, we cannot subtract 10/12 from 3/12. The fix is to borrow 1 from 8 in the form of 12/12 and add it to the 3/12.

$$\begin{array}{rcrcr} 8\frac{3}{12} & = & 7+\frac{12}{12}+\frac{3}{12} & = & 7\frac{15}{12} \\ -5\frac{10}{12} & = & -5\frac{10}{12} & = & -5\frac{10}{12} \\ \hline & & & & 2\frac{5}{12} \end{array}$$

Now we can subtract. Hence, $8\frac{1}{4}-5\frac{5}{6}=2\frac{5}{12}$.

Exercise

Simplify: $7\frac{1}{14}-2\frac{5}{21}$

Answer

$4\frac{5}{6}$.

Example 9

Jim has a metal rod of length 10 inches. He cuts a length from the metal rod measuring $2\frac{7}{8}$ inches. What is the length of the remaining piece?

Solution

To find the length of the remaining piece, we must subtract $2\frac{7}{8}$ from 10. There is no fractional part on the first number. To remedy this absence, we borrow 1 from 10 in the form of 8/8. Then we can subtract.

$$\begin{array}{rcrcr} 10 & = & 9+\frac{8}{8} & = & 9\frac{8}{8} \\ -2\frac{7}{8} & = & -2\frac{7}{8} & = & -2\frac{7}{8} \\ \hline & & & & 7\frac{1}{8} \end{array}$$

Hence, the length of the remaining piece of the metal rod is $7\frac{1}{8}$ inches.

Exercise

Sarah has a length of curtain material that measures 12 feet. She cuts a length of $6\frac{2}{3}$ feet from her curtain material. What is the length of the remaining piece?

Answer

$5\frac{1}{3}$ feet

Exercises

In Exercises 1-24, add or subtract the mixed fractions, as indicated, by first converting each mixed fraction to an improper fraction. Express your answer as a mixed fraction.

1. $9\frac{1}{4}+9\frac{1}{2}$
2. $2\frac{1}{3}+9\frac{1}{2}$
3. $6\frac{1}{2}-1\frac{1}{3}$
4. $5\frac{1}{3}-1\frac{3}{4}$
5. $9\frac{1}{2}+7\frac{1}{4}$
6. $1\frac{1}{3}+9\frac{3}{4}$
7. $5\frac{2}{3}+4\frac{1}{2}$
8. $1\frac{9}{16}+2\frac{3}{4}$
9. $3\frac{1}{3}-1\frac{1}{4}$
10. $2\frac{1}{2}-1\frac{1}{4}$
11. $8\frac{1}{2}-1\frac{1}{3}$
12. $5\frac{1}{2}-1\frac{2}{3}$
13. $4\frac{1}{2}-1\frac{1}{8}$
14. $2\frac{1}{2}-1\frac{1}{3}$
15. $4\frac{7}{8}+1\frac{3}{4}$
16. $1\frac{1}{8}+5\frac{1}{2}$
17. $2\frac{1}{3}-1\frac{1}{4}$
18. $5\frac{1}{3}-1\frac{1}{4}$
19. $9\frac{1}{2}-1\frac{3}{4}$
20. $5\frac{1}{2}-1\frac{3}{16}$
21. $4\frac{2}{3}+1\frac{1}{4}$
22. $1\frac{1}{4}+1\frac{1}{3}$
23. $9\frac{1}{2}+3\frac{1}{8}$
24. $1\frac{1}{4}+1\frac{2}{3}$

In Exercises 25-48, add or subtract the mixed fractions, as indicated, by using vertical format. Express your answer as a mixed fraction.

25. $3\frac{1}{2}+3\frac{3}{4}$

26. $1\frac{1}{2}+2\frac{2}{3}$
27. $1\frac{3}{8}+1\frac{1}{4}$
28. $2\frac{1}{4}+1\frac{2}{3}$
29. $1\frac{7}{8}+1\frac{1}{2}$
30. $1\frac{3}{4}+4\frac{1}{2}$
31. $8\frac{1}{2}-5\frac{2}{3}$
32. $8\frac{1}{2}-1\frac{2}{3}$
33. $7\frac{1}{2}-1\frac{3}{16}$
34. $5\frac{1}{2}-1\frac{1}{3}$
35. $9\frac{1}{2}-1\frac{1}{3}$
36. $2\frac{1}{2}-1\frac{3}{16}$
37. $5\frac{1}{3}-2\frac{1}{2}$
38. $4\frac{1}{4}-1\frac{1}{2}$
39. $9\frac{1}{2}-2\frac{2}{3}$
40. $7\frac{1}{2}-4\frac{2}{3}$
41. $1\frac{1}{16}+1\frac{3}{4}$
42. $1\frac{1}{4}+1\frac{1}{3}$
43. $8\frac{1}{2}+3\frac{2}{3}$
44. $1\frac{2}{3}+2\frac{1}{2}$
45. $6\frac{1}{2}-1\frac{3}{16}$
46. $4\frac{1}{2}-1\frac{1}{3}$
47. $2\frac{2}{3}+1\frac{1}{4}$
48. $1\frac{1}{2}+1\frac{1}{16}$

Answers

1. $18\frac{3}{4}$
3. $5\frac{1}{6}$
5. $16\frac{3}{4}$
7. $10\frac{1}{6}$
9. $2\frac{1}{12}$
11. $7\frac{1}{6}$
13. $3\frac{3}{8}$
15. $6\frac{5}{8}$
17. $1\frac{1}{12}$
19. $7\frac{3}{4}$
21. $5\frac{11}{12}$

23. $12\frac{5}{8}$
25. $7\frac{1}{4}$
27. $2\frac{5}{8}$
29. $3\frac{3}{8}$
31. $2\frac{5}{6}$
33. $6\frac{5}{16}$
35. $8\frac{1}{6}$
37. $2\frac{5}{6}$
39. $6\frac{5}{6}$
41. $2\frac{13}{16}$
43. $12\frac{1}{6}$
45. $5\frac{5}{16}$
47. $3\frac{11}{12}$

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2.8: Order of Operations with Fractions

Let's begin by taking powers of fractions. Recall that

$$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}$$

Example 1

Simplify: $(-3/4)^2$.

Solution

By definition,

$$\begin{aligned} \left(-\frac{3}{4}\right)^2 &= \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right) && \text{Fact: } a^2 = a \cdot a. \\ &= \frac{3 \cdot 3}{4 \cdot 4} && \text{Multiply numerators and denominators.} \\ &= \frac{9}{16} && \text{Product of even number of negative factors is positive.} \\ & && \text{Simplify.} \end{aligned}$$

Exercise

Simplify:

$$\left(-\frac{2}{5}\right)^2$$

Answer

4/25

Example 2

Simplify: $(-2/3)^3$.

Solution

By definition,

$$\begin{aligned} \left(-\frac{2}{3}\right)^3 &= \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) && \text{Fact: } a^3 = a \cdot a \cdot a. \\ &= -\frac{2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3} && \text{Multiply numerators and denominators.} \\ & && \text{Product of odd number of negative factors is negative.} \\ &= -\frac{8}{27} && \text{Simplify.} \end{aligned}$$

Exercise

Simplify:

$$\left(-\frac{1}{6}\right)^3$$

Answer

−1/216

The last two examples reiterate a principle learned earlier.

Odd and Even

- The product of an even number of negative factors is positive.
- The product of an odd number of negative factors is negative.

Order of Operations

For convenience, we repeat here the rules guiding order of operations.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Example 3

Simplify: $-\frac{1}{2}+\frac{1}{4}\left(-\frac{1}{3}\right)$.

Solution

Multiply first, then add.

$$\begin{aligned} -\frac{1}{2}+\frac{1}{4}\left(-\frac{1}{3}\right) &= -\frac{1}{2}+\left(-\frac{1}{12}\right) && \text{Multiply: } \frac{1}{4}\left(-\frac{1}{3}\right)=-\frac{1}{12}. \\ &= -\frac{1\cdot 6}{2\cdot 6}+\left(-\frac{1}{12}\right) && \text{Equivalent fractions, LCD} = 12. \\ &= -\frac{6}{12}+\left(-\frac{1}{12}\right) && \text{Simplify numerator and denominator.} \\ &= -\frac{7}{12} && \text{Add over common denominator.} \end{aligned}$$

Exercise

Simplify: $-\frac{2}{3}+\frac{3}{4}\left(-\frac{1}{2}\right)$

Answer

−25/24

Example 4

Simplify: $2\left(-\frac{1}{2}\right)^2+4\left(-\frac{1}{2}\right)$.

Solution

Exponents first, then multiply, then add.

$$
\begin{aligned}
2\left(-\frac{1}{2}\right)^2+4\left(-\frac{1}{2}\right) &= 2\left(\frac{1}{4}\right)+4\left(-\frac{1}{2}\right) && \text{Exponent first: } \left(-\frac{1}{2}\right)^2=\frac{1}{4}.\\
&= \frac{1}{2}+\left(-\frac{2}{1}\right) && \text{Multiply: } 2\left(\frac{1}{4}\right)=\frac{1}{2} \text{ and } 4\left(-\frac{1}{2}\right)=-\frac{2}{1}.\\
&= \frac{1}{2}+\left(-\frac{2\cdot 2}{1\cdot 2}\right) && \text{Equivalent fractions, LCD} = 2.\\
&= \frac{1}{2}+\left(-\frac{4}{2}\right) && \text{Simplify numerator and denominator.}\\
&= -\frac{3}{2} && \text{Add over common denominator.}
\end{aligned}
$$

Exercise

Simplify: $3\left(-\frac{1}{3}\right)^2-2\left(-\frac{1}{3}\right)$

Answer

1

Example 5

Given $a = -3/4$, $b = 1/2$, $c = 1/3$, and $d = -1/4$, evaluate the expression $ab - cd$.

Solution

Recall that it is good practice to prepare parentheses before substituting.

$$ad-bc=(\,)(\,)-(\,)(\,)$$

Substitute the given values into the algebraic expression, then simplify using order of operations.

$$
\begin{aligned}
ab-cd &= \left(-\frac{3}{4}\right)\left(\frac{1}{2}\right)-\left(\frac{1}{3}\right)\left(-\frac{1}{4}\right) && \text{Substitute: } -3/4 \text{ for } a,\ 1/2 \text{ for } b,\ 1/3 \text{ for } c, \text{ and } -1/4 \text{ for } d.\\
&= -\frac{3}{8}-\left(-\frac{1}{12}\right) && \text{Multiply first: } \left(-\frac{3}{4}\right)\left(\frac{1}{2}\right)=-\frac{3}{8} \text{ and } \left(\frac{1}{3}\right)\left(-\frac{1}{4}\right)=-\frac{1}{12}.\\
&= -\frac{3}{8}+\frac{1}{12} && \text{Subtract by adding opposite.}\\
&= -\frac{3\cdot 3}{8\cdot 3}+\frac{1\cdot 2}{12\cdot 2} && \text{Equivalent fractions; LCD} = 24.\\
&= -\frac{9}{24}+\frac{2}{24} && \text{Simplify numerators and denominators.}\\
&= -\frac{7}{24} && \text{Add over common denominator.}
\end{aligned}
$$

Exercise

Given $a = -1/2$, $b = 1/3$, and $c = -1/5$, evaluate $a + bc$.

Answer

−17/30

Example 6

Given $a = -1/4$ and $b = 1/2$, evaluate $(a^2 - b^2) \div (a + b)$.

Solution

Recall that it is good practice to prepare parentheses before substituting.

$$(a^2-b^2)\div(a+b)-\left(\left(\ \right)^2-\left(\ \right)^2\right)\div\left(\left(\ \right)+\left(\ \right)\right)$$

Substitute the given values into the algebraic expression, then evaluate exponents first.

$$\begin{aligned}(a^2-b^2)\div(a+b) &= \left(\left(-\tfrac{1}{4}\right)^2-\left(\tfrac{1}{2}\right)^2\right)\div\left(\left(-\tfrac{1}{4}\right)+\left(\tfrac{1}{2}\right)\right)\\ &= \left(\tfrac{1}{16}-\tfrac{1}{4}\right)\div\left(-\tfrac{1}{4}+\tfrac{1}{2}\right)\end{aligned}$$

We must evaluate parentheses first. Inside each set of parentheses, create equivalent fractions and perform subtractions and additions next.

$$\begin{aligned}&= \left(\tfrac{1}{16}-\tfrac{1\cdot4}{4\cdot4}\right)\div\left(-\tfrac{1}{4}+\tfrac{1\cdot2}{2\cdot2}\right)\\ &= \left(\tfrac{1}{16}-\tfrac{4}{16}\right)\div\left(-\tfrac{1}{4}+\tfrac{2}{4}\right)\\ &= -\tfrac{3}{16}\div\tfrac{1}{4}\end{aligned}$$

Invert and multiply.

$$\begin{aligned}&= -\frac{3}{16}\cdot\frac{4}{1}\\ &= -\frac{12}{16}\end{aligned}$$

Reduce.

$$\begin{aligned}&= -\frac{12\div4}{16\div4}\\ &-\frac{3}{4}\end{aligned}$$

Note: In the last step, you could also reduce by prime factoring numerator and denominator and canceling common factors.

Exercise

Give $a = -1/2$ and $b = -1/3$, evaluate $ab \div (a + b)$.

Answer

$-1/5$

Complex Fractions

Complex Fractions

When the numerator and denominator of a fraction contain fractions themselves, such an expression is called a *complex fraction.*

You can use the standard order of operations to simplify a complex fraction. Recall the advice when a fraction is present.

Fractional Expressions

If a fractional expression is present, simplify the numerator and denominator separately, then divide.

Example 7

Simplify:

$$\frac{-\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{3}{2}}$$

Solution

We have addition in the numerator, subtraction in the denominator. In each case, we need equivalent fractions with a common denominator.

$$\begin{aligned}\frac{-\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{3}{2}} &= \frac{-\frac{1\cdot 3}{2\cdot 3}+\frac{1\cdot 2}{3\cdot 2}}{} && \text{Create equivalent fractions.}\\ &= \frac{-\frac{3}{6}+\frac{2}{6}}{\frac{3}{4}-\frac{6}{4}} && \text{Simplify numerator and denominator.}\\ &= \frac{-\frac{1}{6}}{-\frac{3}{4}} && \begin{matrix}\text{Numerator: } -\frac{3}{6}+\frac{2}{6}=-\frac{1}{6}.\\ \text{Denominator: } \frac{3}{4}-\frac{6}{4}=-\frac{3}{4}.\end{matrix}\end{aligned}$$

The last expression asks us to divide. Invert and multiply.

$$\begin{aligned}&= -\frac{1}{6}\div\left(-\frac{3}{4}\right) && \text{A complex fraction means divide.}\\ &= -\frac{1}{6}\cdot\left(-\frac{4}{3}\right) && \text{Invert and multiply.}\end{aligned}$$

Like signs (two negatives) give a positive product. Multiply numerators and denominators, then reduce.

$$\begin{aligned}&= \frac{4}{18} && \begin{matrix}\text{Like signs yields positive answer.}\\ \text{Multiply numerators and denominators.}\end{matrix}\\ &= \frac{4\div 2}{18\div 2} && \text{Divide both numerator and denominator by 2.}\\ &= \frac{2}{9} && \text{Simplify.}\end{aligned}$$

Alternatively, one could prime factor and cancel to reduce to lowest terms; that is,

$$\begin{aligned}\frac{4}{18} &= \frac{2\cdot 2}{2\cdot 3\cdot 3} && \text{Prime factor.}\\ &= \frac{\cancel{2}\cdot 2}{\cancel{2}\cdot 3\cdot 3} && \text{Cancel common factors.}\\ &= \frac{2}{9} && \text{Simplify.}\end{aligned}$$

Exercise

Simplify:

$$\frac{\frac{1}{4}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{3}}$$

Answer

−1/7

Clearing Fractions

An alternate technique for simplifying complex fractions is available.

Clearing Fractions from Complex Fractions

You can clear fractions from a complex fraction using the following algorithm:

1. Determine an LCD1 for the numerator.
2. Determine an LCD2 for the denominator.
3. Determine an LCD for both LCD1 and LCD2.
4. Multiply both numerator and denominator by this "combined" LCD.

Let's apply this technique to the complex fraction of Example 7.

Example 8

Simplify:

$$\frac{-\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{3}{2}}$$

Solution

As we saw in the solution in Example 7, common denominators of 6 and 4 were used for the numerator and denominator, respectively. Thus, a common denominator for both numerator and denominator would be 12. We begin the alternate solution technique by multiplying both numerator and denominator by 12.

$$\begin{aligned}
\frac{-\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{3}{2}} &= \frac{12\left(-\frac{1}{2}+\frac{1}{3}\right)}{12\left(\frac{3}{4}-\frac{3}{2}\right)} && \text{Multiply numerator and denominator by } 12.\\
&= \frac{12\left(-\frac{1}{2}\right)+12\left(\frac{1}{3}\right)}{12\left(\frac{3}{4}\right)-12\left(\frac{3}{2}\right)} && \text{Distribute the } 12.\\
&= \frac{-6+4}{9-18} && \text{Multiply: } 12(-1/2)=-6,\ 12(1/2)=4.\\
& && 12(3/4)=9, \text{ and } 12(3/2)=18.\\
&= \frac{-2}{-9} && \text{Simplify.}\\
&= \frac{2}{9} && \text{Like signs yield positive.}
\end{aligned}$$

Exercise

Simplify: $\frac{-\frac{2}{3}+\frac{1}{5}}{\frac{4}{5}-\frac{1}{2}}$

Answer

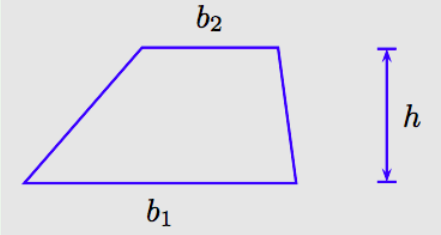
−14/9

Application — Trapezoid

A trapezoid is a special type of quadrilateral (four-sided polygon).

Trapezoid

A quadrilateral with one pair of parallel opposite sides is called a *trapezoid*.



The pair of parallel sides are called the *bases* of the trapezoid. Their lengths are marked by the variables b_1 and b_2 in the figure above. The distance between the parallel bases is called the *height* or *altitude* of the trapezoid. The height is marked by the variable h in the figure above.

Mathematicians use subscripts to create new variables. Thus, b_1 ("b sub 1") and b_2 ("b sub 2") are two distinct variables, used in this case to represent the length of the bases of the trapezoid.

By drawing in a diagonal, we can divide the trapezoid into two triangles (see Figure 4.14).

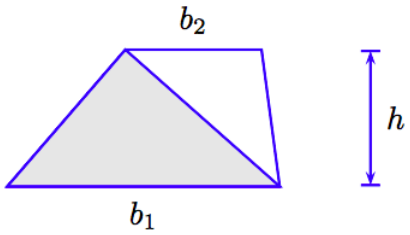


Figure 4.14: Dividing the trapezoid into two triangles.

We can find the area of the trapezoid by summing the areas of the two triangles.

- The shaded triangle in Figure 4.14 has base b_1 and height h. Hence, the area of the shaded triangle is $(1/2)b_1h$.
- The unshaded triangle in Figure 4.14 has base b_2 and height h. Hence, the area of the unshaded triangle is $(1/2)b_2h$.

Summing the areas, the area of the trapezoid is

$$\text{Area of Trapezoid} = \frac{1}{2}b_1h + \frac{1}{2}b_2h.$$

We can use the distributive property to factor out a $(1/2)h$.

Area of a Trapezoid

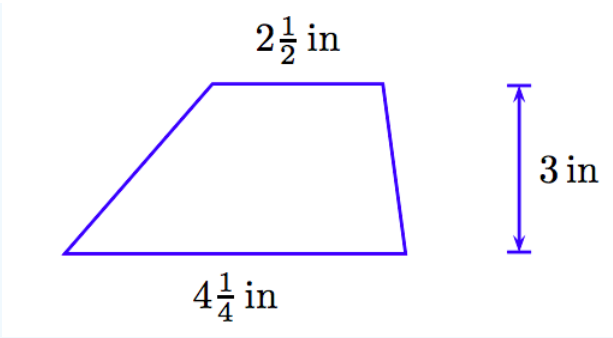
A trapezoid with bases b_1 and b_2 and height h has area

$$A = \frac{1}{2}h(b_1 + b_2).$$

That is, to find the area, sum the bases, multiply by the height, and take one-half of the result.

Example 9

Find the area of the trapezoid pictured below.



Solution

The formula for the area of a trapezoid is

$$A = \frac{1}{2}h(b_1 + b_2)$$

Substituting the given bases and height, we get

$$A = \frac{1}{2}(3)\left(4\frac{1}{4} + 2\frac{1}{2}\right).$$

Simplify the expression inside the parentheses first. Change mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} A &= \tfrac{1}{2}(3)\left(\tfrac{17}{4} + \tfrac{5}{2}\right) \\ &= \tfrac{1}{2}(3)\left(\tfrac{17}{4} + \tfrac{5\cdot 2}{2\cdot 2}\right) \\ &= \tfrac{1}{2}(3)\left(\tfrac{17}{4} + \tfrac{10}{4}\right) \\ &= \tfrac{1}{2}\left(\tfrac{3}{1}\right)\left(\tfrac{27}{4}\right) \end{aligned}$$

Multiply numerators and denominators.

$$= \frac{81}{8}$$

This improper fraction is a perfectly good answer, but let's change this result to a mixed fraction (81 divided by 8 is 10 with a remainder of 1). Thus, the area of the trapezoid is

$$A = 10\frac{1}{8} \text{ square inches.}$$

? Exercise

A trapezoid has bases measuring 6 and 15 feet, respectively. The height of the trapezoid is 5 feet. Find the area of the trapezoid.

Answer

$52\frac{1}{2}$ square feet

Exercises

In Exercises 1-8, simplify the expression.

1. $\left(-\frac{7}{3}\right)^3$
2. $\left(\frac{1}{2}\right)^3$
3. $\left(\frac{5}{3}\right)^4$

4. $\left(-\frac{3}{5}\right)^4$
5. $\left(\frac{1}{2}\right)^5$
6. $\left(\frac{3}{4}\right)^5$
7. $\left(\frac{4}{3}\right)^2$
8. $\left(-\frac{8}{5}\right)^2$

---

9. If $a = 7/6$, evaluate a^3.
10. If $e = 1/6$, evaluate e^3.
11. If $e = -2/3$, evaluate $-e^2$.
12. If $c = -1/5$, evaluate $-c^2$.
13. If $b = -5/9$, evaluate b^2.
14. If $c = 5/7$, evaluate c^2.
15. If $b = -1/2$, evaluate $-b^3$.
16. If $a = -2/9$, evaluate $-a^3$.

In Exercises 17-36, simplify the expression.

17. $\left(-\frac{1}{2}\right)\left(\frac{1}{6}\right)-\left(\frac{7}{8}\right)\left(-\frac{7}{9}\right)$
18. $\left(-\frac{3}{4}\right)\left(\frac{1}{2}\right)-\left(\frac{3}{5}\right)\left(\frac{1}{4}\right)$
19. $\left(-\frac{9}{8}\right)^2-\left(-\frac{3}{2}\right)\left(\frac{7}{3}\right)$
20. $\left(\frac{3}{2}\right)^2-\left(\frac{7}{8}\right)\left(-\frac{1}{2}\right)$
21. $\left(-\frac{1}{2}\right)\left(-\frac{7}{4}\right)-\left(-\frac{1}{2}\right)^2$
22. $\left(\frac{1}{5}\right)\left(-\frac{9}{4}\right)-\left(\frac{7}{4}\right)^2$
23. $-\frac{7}{6}-\frac{1}{7}\cdot\frac{7}{9}$
24. $-\frac{4}{9}-\frac{8}{5}\cdot\frac{8}{9}$
25. $\frac{3}{4}+\frac{9}{7}\left(-\frac{7}{6}\right)$
26. $\frac{3}{2}+\frac{1}{4}\left(-\frac{9}{8}\right)$
27. $\left(-\frac{1}{3}\right)^2+\left(\frac{7}{8}\right)\left(-\frac{1}{3}\right)$
28. $\left(-\frac{2}{9}\right)^2+\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)$
29. $\frac{5}{9}+\frac{5}{9}\cdot\frac{7}{9}$
30. $-\frac{1}{2}+\frac{9}{8}\cdot\frac{1}{3}$
31. $\left(-\frac{5}{6}\right)\left(\frac{3}{8}\right)+\left(-\frac{7}{9}\right)\left(-\frac{3}{4}\right)$
32. $\left(\frac{7}{4}\right)\left(\frac{6}{5}\right)+\left(-\frac{2}{5}\right)\left(\frac{8}{3}\right)$
33. $\frac{4}{3}-\frac{2}{9}\left(-\frac{3}{4}\right)$
34. $-\frac{1}{3}-\frac{1}{5}\left(-\frac{4}{3}\right)$
35. $\left(-\frac{5}{9}\right)\left(\frac{1}{2}\right)+\left(-\frac{1}{6}\right)^2$
36. $\left(\frac{1}{4}\right)\left(\frac{1}{6}\right)+\left(-\frac{5}{6}\right)^2$

---

37. Given $a = -5/4$, $b = 1/2$, and $c = 3/8$, evaluate $a + bc$.

38. Given $a = -3/5$, $b = 1/5$, and $c = 1/3$, evaluate $a + bc$.

39. Given $x = -1/8$, $y = 5/2$, and $z = -1/2$, evaluate the expression $x + yz$.

40. Given $x = -5/9$, $y = 1/4$, and $z = -2/3$, evaluate the expression $x + yz$.

41. Given $a = 3/4$, $b = 5/7$, and $c = 1/2$, evaluate the expression $a - bc$.

42. Given $a = 5/9$, $b = 2/3$, and $c = 2/9$, evaluate the expression $a - bc$.

43. Given $x = -3/2$, $y = 1/4$, and $z = -5/7$, evaluate $x^2 - yz$.

44. Given $x = -3/2$, $y = -1/2$, and $z = 5/3$, evaluate $x^2 - yz$.

45. Given $a = 6/7$, $b = 2/3$, $c = -8/9$, and $d = -6/7$, evaluate $ab + cd$.

46. Given $a = 4/9$, $b = -3/2$, $c = 7/3$, and $d = -8/9$, evaluate $ab + cd$.

47. Given $w = -1/8$, $x = -2/7$, $y = -1/2$, and $z = 8/7$, evaluate $wx - yz$.

48. Given $w = 2/7$, $x = -9/4$, $y = -3/4$, and $z = -9/2$, evaluate $wx - yz$.

49. Given $x = 3/8$, $y = 3/5$, and $z = -3/2$, evaluate $xy + z^2$.

50. Given $x = -1/2$, y = 7/5, and z = $-3/2$, evaluate $xy + z^2$.

51. Given $u = 9/7$, $v = 2/3$, and $w = -3/7$, evaluate $uv - w^2$.

52. Given $u = 8/7$, $v = -4/3$, and $w = 2/3$, evaluate $uv - w^2$.

53. Given $a = 7/8$, $b = -1/4$, and $c = -3/2$, evaluate $a^2 + bc$.

54. Given $a = -5/8$, $b = 3/2$, and $c = -3/2$, evaluate $a^2 + bc$.

55. Given $u = 1/3$, $v = 5/2$, and $w = -2/9$, evaluate the expression $u - vw$.

56. Given $u = -1/2$, $v = 1/4$, and $w = -1/4$, evaluate the expression $u - vw$.

In Exercises 57-68, simplify the complex rational expression.

57. $\dfrac{\frac{8}{3}+\frac{7}{6}}{-\frac{9}{2}-\frac{1}{4}}$

58. $\dfrac{\frac{7}{8}+\frac{1}{9}}{\frac{8}{9}-\frac{1}{6}}$

59. $\dfrac{\frac{3}{4}+\frac{4}{3}}{\frac{1}{9}+\frac{5}{3}}$

60. $\dfrac{-\frac{9}{8}-\frac{6}{5}}{\frac{7}{4}+\frac{1}{2}}$

61. $\dfrac{\frac{7}{5}+\frac{5}{2}}{-\frac{1}{4}+\frac{1}{2}}$

62. $\dfrac{\frac{5}{6}+\frac{2}{3}}{\frac{3}{5}+\frac{2}{3}}$

63. $\dfrac{-\frac{3}{2}-\frac{2}{3}}{-\frac{7}{4}-\frac{2}{3}}$

64. $\dfrac{\frac{8}{9}+\frac{3}{4}}{-\frac{2}{3}-\frac{1}{6}}$

65. $\dfrac{-\frac{1}{2}-\frac{4}{7}}{-\frac{5}{7}+\frac{1}{6}}$

66. $\dfrac{-\frac{3}{2}-\frac{5}{8}}{\frac{3}{4}-\frac{1}{2}}$

67. $\frac{-\frac{3}{7}-\frac{1}{3}}{\frac{1}{3}-\frac{6}{7}}$

68. $\frac{-\frac{5}{8}-\frac{6}{5}}{-\frac{5}{4}-\frac{3}{8}}$

---

69. A trapezoid has bases measuring $3\frac{3}{8}$ and $5\frac{1}{2}$ feet, respectively. The height of the trapezoid is 7 feet. Find the area of the trapezoid.

70. A trapezoid has bases measuring $2\frac{1}{2}$ and $6\frac{7}{8}$ feet, respectively. The height of the trapezoid is 3 feet. Find the area of the trapezoid.

71. A trapezoid has bases measuring $2\frac{1}{4}$ and $7\frac{3}{8}$ feet, respectively. The height of the trapezoid is 7 feet. Find the area of the trapezoid.

72. A trapezoid has bases measuring $3\frac{1}{8}$ and $6\frac{1}{2}$ feet, respectively. The height of the trapezoid is 3 feet. Find the area of the trapezoid.

73. A trapezoid has bases measuring $2\frac{3}{4}$ and $6\frac{5}{8}$ feet, respectively. The height of the trapezoid is 3 feet. Find the area of the trapezoid.

74. A trapezoid has bases measuring $2\frac{1}{4}$ and $7\frac{1}{8}$ feet, respectively. The height of the trapezoid is 5 feet. Find the area of the trapezoid.

Answers

1. $\frac{-343}{27}$
3. $\frac{625}{81}$
5. $\frac{1}{32}$
7. $\frac{16}{9}$
9. $\frac{343}{216}$
11. $\frac{-4}{9}$
13. $\frac{25}{81}$
15. $\frac{1}{8}$
17. $\frac{43}{72}$
19. $\frac{305}{64}$
21. $\frac{5}{8}$
23. $\frac{-23}{18}$
25. $\frac{-3}{4}$
27. $\frac{-13}{72}$
29. $\frac{80}{81}$
31. $\frac{13}{48}$
33. $\frac{3}{2}$
35. $\frac{-1}{4}$
37. $\frac{-17}{16}$
39. $\frac{-11}{8}$

41. $\frac{11}{28}$
43. $\frac{17}{7}$
45. $\frac{4}{3}$
47. $\frac{17}{28}$
49. $\frac{99}{40}$
51. $\frac{33}{49}$
53. $\frac{73}{64}$
55. $\frac{8}{9}$
57. $\frac{-46}{57}$
59. $\frac{75}{64}$
61. $\frac{78}{5}$
63. $\frac{26}{29}$
65. $\frac{45}{23}$
67. $\frac{16}{11}$
69. $31\frac{1}{16}$
71. $33\frac{11}{16}$
73. $14\frac{1}{16}$

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CHAPTER OVERVIEW

3: Decimals

Decimals are like fractions and represent non-whole number numbers.

3.1: Decimals
3.2: Introduction to Decimals
3.3: Adding and Subtracting Decimals
3.4: Multiplying Decimals
3.5: Dividing Decimals
3.6: Fractions and Decimals

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3.1: Decimals

On January 29, 2001, the New York Stock exchange ended its 200-year tradition of quoting stock prices in fractions and switched to decimals.

It was said that pricing stocks the same way other consumer items were priced would make it easier for investors to understand and compare stock prices. Foreign exchanges had been trading in decimals for decades. Supporters of the change claimed that trading volume, the number of shares of stock traded, would increase and improve efficiency.

But switching to decimals would have another effect of *narrowing the spread*. The *spread* is the difference between the best price offered by buyers, called the *bid*, and the price requested by sellers called the *ask*. Stock brokers make commissions as a percentage of the spread which, using fractions, could be anywhere upwards from 12 cents per share.

When the New York Stock Exchange began back in 1792, the dollar was based on the Spanish *real*, (pronounced ray-al), also called *pieces of eight* as these silver coins were often cut into quarters or eighths to make change. This is what led to stock prices first denominated in eighths. Thus, the smallest spread that could occur would be 1/8 of a dollar, or 12.5 cents. That may seem like small change, but buying 1000 shares for $1 per share with a $0.125 spread is a $125.00 commission. Not bad for a quick trade!

Decimalization of stock pricing allowed for spreads as small as 1 cent. Since the number of shares traded on stock market exchanges have skyrocketed, with trillions of shares traded daily, stock broker commissions have not suffered. And the ease with which investors can quickly grasp the price of stock shares has contributed to the opening of markets for all classes of people.

In this chapter, we'll learn about how to compute and solve problems with decimals, and see how they relate to fractions.

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3.2: Introduction to Decimals

Recall that whole numbers are constructed by using *digits*.

The Digits

The set

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

is called the set of *digits*.

As an example, the whole number 55,555 (“fifty-five thousand five hundred fifty-five”) is constructed by using a single digit. However, the position of the digit 5 determines its value in the number 55,555. The first occurrence of the

5	5	5	5	5
ten thousands	thousands	hundreds	tens	ones
10,000	1,000	100	10	1

Table 5.1: Place value.

digit 5 happens in the ten thousands place, so its value is 5 ten thousands, or 50,000. The next occurrence of the digit 5 is in the thousands place, so its value is 5 thousands, or 5,000. Indeed, the whole number 55,555 in expanded form is

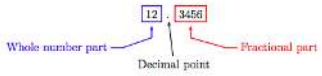
$$50000 + 5000 + 500 + 50 + 5,$$

which reflects the value of the digit 5 in each place.

Decimal Notation

In Table 5.1, each time you move one column to the left, the place value is 10 times larger than the place value of the preceding column. Vice-versa, each time you move one column to the right, the place value is 1/10 of the place value of the preceding column.

Now, consider the *decimal number* 12.3456, which consists of three parts: the whole number part, the decimal point, and the fractional part.



The whole number part of the decimal number is the part that lies strictly to the left of the decimal point, and the place value of each digit in the whole number part is given by the columns shown in Table 5.1.

The fractional part of the decimal number is the part that lies strictly to the right of the decimal point. As we saw in Table 5.1, each column has a value equal to 1/10 of the value of the column that lies to its immediate left. Thus, it should come as no surprise that:

- The first column to the right of the decimal point has place value 1/10 (tenths).
- The second column to the right of the decimal point has place value 1/100 (hundredths).
- The third column to the right of the decimal point has place value 1/1000 (thousandths).
- The fourth column to the right of the decimal point has place value 1/10000 (ten-thousandths).

These results are summarized for the decimal number 12.3456 in Table 5.2.

1	2	.	3	4	5	6
tens	ones	decimal point	tenths	hundredths	thousandths	ten-thousandths
10	1	.	1/10	1/100	1/1000	1/10000

Table 5.2: Place value.

Pronouncing Decimal Numbers

The decimal number 12.3456 is made up of 1 ten, 2 ones, 3 tenths, 4 hundredths, 5 thousandths, and 6 ten-thousandths (see Table 5.2), and can be written in *expanded form* as

$$12.3456 = 10 + 2 + \frac{3}{10} + \frac{4}{100} + \frac{5}{1000} + \frac{6}{10000}.$$

Note that the whole numbers can be combined and the fractions can be written with a common denominator and summed.

```
\begin{aligned} 12.3456 &= 12 + \frac{3 \cdot \textcolor{red}{1000}}{10 \cdot \textcolor{red}{1000}} + \frac{4 \cdot \textcolor{red}{100}}{100 \cdot \textcolor{red}{100}} + \frac{5 \cdot \textcolor{red}{10}}{1000 \c
```

The result tells us how to pronounce the number 12.3456. It is pronounced “twelve and three thousand, four hundred fifty-six ten-thousandths.”

Example 1

Place the decimal number 1,234.56 in expanded form, then combine the whole number part and sum the fractional part over a common denominator. Use the result to help pronounce the decimal number.

Solution

In expanded form,

$$1,234.56 = 1,000 + 200 + 30 + 4 + \frac{5}{10} + \frac{6}{100}$$

Sum the whole number parts. Express the fractional parts as equivalent fractions and combine over one common denominator.

$$\begin{aligned} &= 1,234 + \frac{5 \cdot \textcolor{10}{10}}{10 \cdot \textcolor{red}{10}} + \frac{6}{100} \\ &= 1,234 + \frac{50}{100} + \frac{6}{100} \\ &= 1, 234 + \frac{56}{100} \end{aligned} \nonumber$$

Hence, 1,234.56 is pronounced “one thousand, two hundred thirty-four and fifty-six hundreths."

Exercise

Place the decimal number 3,502.23 in expanded form, then combine the whole number part and sum the fractional part over a common denominator.

Answer

$3,502 + \frac{23}{100}$

Example 2

Place the decimal number 56.128 in expanded form, then combine the whole number part and sum the fractional part over a common denominator. Use the result to help pronounce the decimal number.

Solution

In expanded form,

$$56.128 = 50 + 6 + \frac{1}{10} + \frac{2}{100} + \frac{8}{1000}$$

Sum the whole number parts. Express the fractional parts as equivalent fractions and combine over one common denominator.

$$\begin{aligned} &= 56 + \frac{1 \cdot 100}{10 \cdot 100} + \frac{2 \cdot 10}{100 \cdot 10} + \frac{8}{1000} \\ &= 56 + \frac{100}{1000} + \frac{20}{1000} + \frac{8}{1000} \\ &= 56 + \frac{128}{1000} \end{aligned}$$

Thus, 56.128 is pronounced “fifty-six and one hundred twenty-eight thousandths.”

Exercise

Place the decimal number 235.568 in expanded form, then combine the whole number part and sum the fractional part over a common denominator.

Answer

$235 + \frac{568}{1000}$

The discussion and example leads to the following result.

How to Read a Decimal Number

1. Pronounce the whole number part to the left of the decimal as you would any whole number.
2. Say the word “and” for the decimal point.
3. State the fractional part to the right of the decimal as you would any whole number, followed by the place value of the digit in the rightmost column.

Example 3

Pronounce the decimal number 34.12.

Solution

The rightmost digit in the fractional part of 34.12 is in the hundredths column. Thus, 34.12 is pronounced “thirty-four and twelve hundredths.”

Exercise

Pronounce 28.73

Answer

“Twenty-eight and seventy-three hundredths”

Important Point

In pronouncing decimal numbers, the decimal point is read as “and.” No other instance of the word “and” should appear in the pronunciation.

Example 4

Explain why “four hundred and thirty-four and two tenths” is an incorrect pronunciation of the decimal number 434.2.

Solution

The decimal point is read as “and.” No other occurrence of the word “and” is allowed in the pronunciation. The correct pronunciation should be “four hundred thirty-four and two tenths.”

Exercise

Pronounce 286.9.

Answer

"Two hundred eighty six and nine tenths"

Example 5

Pronounce the decimal number 5,678.123.

Solution

The rightmost digit in the fractional part of 5,678.123 is in the thousandths column. Hence, 5,678.123 is pronounced “five thousand six hundred seventy-eight and one hundred twenty-three thousandths.”

Exercise

Pronounce 7, 002.207

Answer

Answer: “Seven thousand two and two hundred seven thousandths.”

Example 6

Pronounce the decimal number 995.4325.

Solution

The rightmost digit in the fractional part of 995.4325 is in the ten-thousandths column. Hence, 995.4325 is pronounced “nine hundred ninety-five and four thousand three hundred twenty-five ten-thoudandths.”

Exercise

Pronounce 500.1205.

Answer

Answer: “Five hundred and one thousand two hundred five ten-thousandths.”

Decimals to Fractions

Because we now have the ability to pronounce decimal numbers, it is a simple exercise to change a decimal to a fraction.[1] For example, 134.12 is pronounced “one hundred thirty-four and twelve hundredths,” so it can easily be written as a mixed fraction.

$$134.12 = 134\frac{12}{100}$$

But this mixed fraction can be changed to an improper fraction.

$$134\frac{12}{100} = \frac{100 \cdot 134 + 12}{100} = \frac{13400 + 12}{100}$$
$$* = \frac{13412}{100}$$

Note that the numerator is our original number without the decimal point. There are two decimal places in the original number and the denominator of the final improper fraction contains two zeros.

This discussion leads to the following result.

[1]Changing fractions to decimals will be covered in Section 5.5.

Changing Decimals to Improper Fractions

To change a decimal number to an improper fraction, proceed as follows:

1. Create a fraction.
2. Place the decimal number in the numerator without the decimal point.
3. Count the number of decimal places. Place an equal number of zeros in the denominator.

Example 7

Change the following decimal numbers to improper fractions: (a) 1.2345, and (b) 27.198.

Solution

In each case, place the number in the numerator without the decimal point. In the denominator, add a number of zeros equal to the number of decimal places.

a) The decimal number 1.2345 has four decimal places. Hence,

$$1.2345 = \frac{12345}{10000}$$

b) The decimal number 27.198 has three decimal places. Hence,

$$27.198 = \frac{27198}{1000}$$

? Exercise

Change 17.205 to an improper fraction.

Answer

$\frac{17205}{100}$

✔ Example 8

Change each of the following decimals to fractions reduced to lowest terms: (a) 0.35, and (b) 0.125.

Solution

Place each number in the numerator without the decimal point. Place a number of zeros in the denominator equal to the number of decimal places. Reduce to lowest terms.

a) First, place 35 over 100.

$$0.35 = \frac{35}{100}$$

We can divide both numerator and denominator by the greatest common divisor.

$$= \frac{35 \div 5}{100 \div 5} \quad \text{Divide numerator and denominator by 5.}$$
$$= \frac{7}{20} \quad \text{Simplify numerator and denominator.}$$

b) First, place 125 over 1000.

$$0.125 = \frac{125}{1000}$$

Prime factor and cancel common factors.

$$= \frac{5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} \quad \text{Prime factor numerator and denominator.}$$
$$= \frac{\cancel{5} \cdot \cancel{5} \cdot \cancel{5}}{2 \cdot 2 \cdot 2 \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5}} \quad \text{Cancel common factors.}$$
$$= \frac{1}{8} \quad \text{Simplify.}$$

? Exercise

Change 0.375 to a fraction, reduced to lowest terms.

Answer

3/8

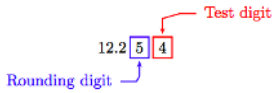
Rounding

The rules for rounding decimal numbers are almost identical to the rules for rounding whole numbers. First, a bit of terminology.

Rounding Digit and Test Digit

The digit in the place to which we wish to round is called the *rounding digit* and the digit that follows on its immediate right is called the *test* digit.

If we want to round the decimal number 12.254 to the nearest hundredth, then the rounding digit is 5 and the test digit is 4.



If we used the rules for rounding whole numbers, because the test digit 4 is less than 5, we would replace all digits to the right of the rounding digit with zeros to obtain the following approximation.

$$12.254 \approx 12.250$$

However, because

$$12.250 = 12\frac{250}{1000} = 12\frac{25}{100},$$

the trailing zero at the end of the fractional part is irrelevant. Hence, we *truncate* every digit after the rounding digit and use the following approximation.

$$12.254 \approx 12.25$$

Important Observation

Deleting trailing zeros from the end of the fractional part of a decimal number does not change its value.

The above discussion motivates the following algorithm for rounding decimal numbers.

Rounding Decimal Numbers

Locate the *rounding digit* and the *test digit*.

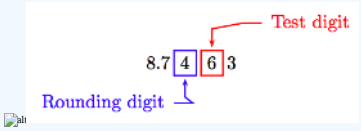
- If the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate all digits to the right of the rounding digit.
- If the test digit is less than 5, simply truncate all digits to the right of the rounding digit.

Example 9

Round 8.7463 to the nearest hundredth.

Solution

Locate the rounding digit in the hundredths place and the test digit to its immediate right.



Because the test digit is greater than 5, add 1 to the rounding digit and truncate all digits to the right of the rounding digit. Hence, to the nearest hundredth:

$$8.7463 \approx 8.75$$

Exercise

Round 9.2768 to the nearest hundredth.

Answer

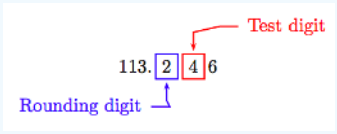
9.28

Example 10

Round 113.246 to the nearest tenth.

Solution

Locate the rounding digit in the tenths place and the test digit to its immediate right.



Because the test digit is less than 5, truncate all digits to the right of the rounding digit. Hence, to the nearest tenth:

$$113.246 \approx 113.2$$

Exercise

Round 58.748 to the nearest tenth.

Answer

58.7

Comparing Decimals

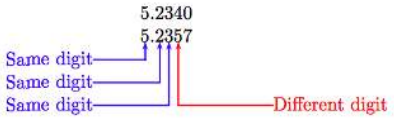
We can compare two positive decimals by comparing digits in each place as we move from left to right, place by place. For example, suppose we wish to compare the decimal numbers 5.234 and 5.2357. First, add enough trailing zeros to the decimal number with the fewer decimal places so that the numbers have the same number of decimal places. In this case, note that

$$5.234 = 5\frac{234}{1000} = 5\frac{2340}{10000} = 5.2340.$$

Important Observation

Adding trailing zeros to the end of the fractional part of a decimal number does not change its value.

Next, align the numbers as follows.



As you scan the columns, moving left to right, the first place that has different digits occurs in the thousandths place, where the digit 5 is the second number is greater than the digit 4 in the first number in the same place. Because 5 is greater than 4, the second number is larger than the first. That is:

$$5.234 < 5.2357$$

This discussion suggests the following algorithm.

Comparing Positive Decimal Numbers

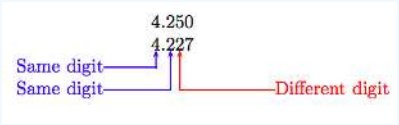
1. Add enough trailing zeros so that both numbers have the same number of decimal places.
2. Compare the digits in each place, moving left to right.
3. As soon as you find two digits in the same place that are different, the decimal number with the greatest digit in this place is the larger number.

Example 11

Compare 4.25 and 4.227.

Solution

Add a trailing zero to the first decimal number and align the numbers as follows.



The first difference is in the hundredths place, where the digit 5 in the first number is greater than the digit 2 in the same place of the second number. Hence, the first number is larger than the second; that is:

$$4.25 > 4.227$$

Exercise

Compare 8.34 and 8.348.

Answer

8.34 < 8.348

When comparing negative numbers, the number with the larger magnitude is the smaller number. Hence, we have to adjust our algorithm for comparing negative decimal numbers.

Comparing Negative Decimal Numbers

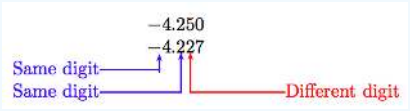
1. Add enough trailing zeros so that both numbers have the same number of decimal places.
2. Compare the digits in each place, moving left to right.
3. As soon as you find two digits in the same place that are different, the decimal number with the greatest digit in this place is the smaller number.

Example 12

Compare −4.25 and −4.227.

Solution

Add a trailing zero to the first decimal number and align the numbers as follows.



The first difference is in the hundredths place, where the digit 5 in the first number is greater than the digit 2 in the same place of the second number. Hence, the first number is **smaller** than the second; that is:

$$-4.25 < -4.227$$

Exercise

Compare −7.86 and −7.85.

Answer

−7.86 < −7.85

Exercises

1. Which digit is in the tenths column of the number 4,552.0908?
2. Which digit is in the thousandths column of the number 7,881.6127?
3. Which digit is in the tenths column of the number 4,408.2148?
4. Which digit is in the tenths column of the number 9,279.0075?
5. Which digit is in the ten-thousandths column of the number 2,709.5097?
6. Which digit is in the hundredths column of the number 1,743.1634?
7. Which digit is in the hundredths column of the number 3,501.4456?
8. Which digit is in the ten-thousandths column of the number 9,214.3625?
9. Which digit is in the hundredths column of the number 5,705.2193?
10. Which digit is in the hundredths column of the number 7,135.2755?
11. Which digit is in the tenths column of the number 8,129.3075?
12. Which digit is in the thousandths column of the number 6,971.4289?

In Exercises 13-20, write the given decimal number in expanded form.

13. 46.139
14. 68.392
15. 643.19
16. 815.64

17. 14.829
18. 45.913
19. 658.71
20. 619.38

In Exercises 21-28, follow the procedure shown in Examples 1 and 2 to write the decimal number in expanded form, then combine the whole number part and sum the fractional part over a common denominator.

21. 32.187
22. 35.491
23. 36.754
24. 89.357
25. 596.71
26. 754.23
27. 527.49
28. 496.15

In Exercises 29-40, pronounce the given decimal number. Write your answer out in words.

29. 0.9837
30. 0.6879
31. 0.2653
32. 0.8934
33. 925.47
34. 974.35
35. 83.427
36. 32.759
37. 63.729
38. 85.327
39. 826.57
40. 384.72

In Exercises 41-52, convert the given decimal to a mixed fraction. Do *not* simplify your answer.

41. 98.1
42. 625.591
43. 781.7
44. 219.999
45. 915.239
46. 676.037
47. 560.453
48. 710.9
49. 414.939
50. 120.58
51. 446.73
52. 653.877

In Exercises 53-60, convert the given decimal to an improper fraction. Do *not* simplify your answer.

53. 8.7
54. 3.1
55. 5.47
56. 5.27
57. 2.133
58. 2.893
59. 3.9
60. 1.271

In Exercises 61-68, convert the given decimal to a fraction. Reduce your answer to lowest terms.

61. 0.35
62. 0.38
63. 0.06
64. 0.84
65. 0.98
66. 0.88

67. 0.72

68. 0.78

---

69. Round 79.369 to the nearest hundredth.

70. Round 54.797 to the nearest hundredth.

71. Round 71.2427 to the nearest thousandth.

72. Round 59.2125 to the nearest thousandth.

73. Round 29.379 to the nearest tenth.

74. Round 42.841 to the nearest tenth.

75. Round 89.3033 to the nearest thousandth.

76. Round 9.0052 to the nearest thousandth.

77. Round 20.655 to the nearest tenth.

78. Round 53.967 to the nearest tenth.

79. Round 19.854 to the nearest hundredth.

80. Round 49.397 to the nearest hundredth.

In Exercises 81-92, determine which of the two given statements is true.

81. $0.30387617 < 0.3036562$ or $0.30387617 > 0.3036562$

82. $8.5934 < 8.554$ or $8.5934 > 8.554$

83. $-0.034 < -0.040493$ or $-0.034 > -0.040493$

84. $-0.081284 < -0.08118$ or $-0.081284 > -0.08118$

85. $-8.3527 < -8.36553$ or $-8.3527 > -8.36553$

86. $-0.00786 < -0.0051385$ or $-0.00786 > -0.0051385$

87. $18.62192 < 18.6293549$ or $18.62192 > 18.6293549$

88. $514.873553 < 514.86374$ or $514.873553 > 514.86374$

89. $36.8298 < 36.8266595$ or $36.8298 > 36.8266595$

90. $0.000681 < 0.00043174$ or $0.000681 > 0.00043174$

91. $-15.188392 < -15.187157$ or $-15.188392 > -15.187157$

92. $-0.049785 < -0.012916$ or $-0.049785 > -0.012916$

---

93. Write the decimal number in words.

i) A recently discovered 7.03-carat blue diamond auctioned at Sotheby's.

ii) The newly launched European Planck telescope will stay in orbit 1.75 years measuring radiation from the Big Bang.

iii) The sun composes 0.9985 of the mass in our solar system.

iv) Clay particles are small - only 0.0001 inch.

94. **Light speed**. The index of refraction for a given material is a value representing the number of times slower a light wave travels in that particular material than it travels in the vacuum of space.

i) Reorder the materials by their index of refraction from lowest to highest.

ii) How many times slower is a lightwave in a diamond compared with a vacuum?

Material	Index of Refraction
Diamond	2.417
Vacuum	1.0000
Plexiglass	1.51
Air	1.0003
Water	1.333
Zircon	1.923
Crown Glass	1.52
Ice	1.31

95. Shorter day? Scientists at NASA's Jet Propulsion Laboratory calculated that the earthquake in Chile may have shortened the length of a day on Earth by 1.26 millionths of a second.

i) Write that number completely as a decimal.

ii) Actual observations of the length of the day are accurate to five millionths of a second. Write this fraction as a decimal.

iii) Comparing the two decimals above and determine which is smaller. Do you think scientists can observe and measure the calculated slowing of the earth?

Answers

1. 0

3. 2

5. 7

7. 4

9. 1

11. 3
13. $40+6+110+\frac{3}{100}+\frac{9}{1000}$
15. $600+40+3+\frac{1}{10}+\frac{9}{100}$
17. $10+4+\frac{8}{10}+\frac{2}{100}+\frac{9}{1000}$
19. $600+50+8+\frac{7}{10}+\frac{1}{100}$
21. $32+\frac{187}{1000}$
23. $36+\frac{754}{1000}$
25. $596+\frac{71}{100}$
27. $527+\frac{49}{100}$
29. nine thousand eight hundred thirty-seven ten-thousandths
31. two thousand six hundred fifty-three ten-thousandths
33. nine hundred twenty-five and forty-seven hundredths
35. eighty-three and four hundred twenty-seven thousandths
37. sixty-three and seven hundred twenty-nine thousandths
39. eight hundred twenty-six and fifty-seven hundredths
41. $98\frac{1}{10}$
43. $781\frac{7}{10}$
45. $915\frac{239}{1000}$
47. $560\frac{453}{1000}$
49. $414\frac{939}{1000}$
51. $446\frac{73}{100}$
53. $\frac{87}{10}$
55. $\frac{547}{100}$
57. $\frac{2133}{1000}$
59. \(\frac{39}{10}|)
61. $\frac{7}{20}$
63. $\frac{3}{50}$
65. \(\frac{49}{50}|)
67. $\frac{18}{25}$
69. 79.37
71. 71.243
73. 29.4
75. 89.303
77. 20.7
79. 19.85
81. 0.30387617 > 0.3036562
83. −0.034 > −0.040493
85. −8.3527 > −8.36553
87. 18.62192 < 18.6293549
89. 36.8298 > 36.8266595
91. −15.188392 < −15.187157
93.
    i) seven and three hundredths
    ii) one and seventy-five hundredths
    iii) nine thousand nine hundred eighty-five ten-thousandths
    iv) one ten-thousandth of an inch
95.
    i) 0.00000126
    ii) 0.000005
    iii) 0.00000126 < 0.000005; scientists would be unable to measure the calculated change in the length of a day.

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3.3: Adding and Subtracting Decimals

Adding Decimals

Addition of decimal numbers is quite similar to addition of whole numbers. For example, suppose that we are asked to add 2.34 and 5.25. We could change these decimal numbers to **mixed fractions** and add.

$$\begin{aligned} 2.34+5.25 &= 2\frac{34}{100}+5\frac{25}{100} \\ &= 7\frac{59}{100} \end{aligned}$$

However, we can also line the decimal numbers on their decimal points and add vertically, as follows.

$$\begin{array}{r} 2.34 \\ +5.25 \\ \hline 7.59 \end{array}$$

Note that this alignment procedure produces the same result, “seven and fifty nine hundredths.” This motivates the following procedure for adding decimal numbers.

Adding Decimals

To add decimal numbers, proceed as follows:

1. Place the numbers to be added in vertical format, aligning the decimal points.
2. Add the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the same column as the decimal points above it.

Example 1

Add 3.125 and 4.814.

Solution

Place the numbers in vertical format, aligning on their decimal points. Add, then place the decimal point in the answer in the same column as the decimal points that appear above the answer.

$$\begin{array}{r} 3.125 \\ +4.814 \\ \hline 7.939 \end{array}$$

Thus, 3.125 + 4.814 = 7.939.

Exercise

Add: 2.864 + 3.029

Answer

5.893

Example 2

Jane has $4.35 in her purse. Jim has $5.62 in his wallet. If they sum their money, what is the total?

Solution

Arrange the numbers in vertical format, aligning decimal points, then add.

$$\begin{array}{r} \$4.35 \\ +\$5.62 \\ \hline \$9.97 \end{array}$$

Exercise

Alice has $8.63 in her purse and Joanna has $2.29. If they combine sum their money, what is the total?

Answer

$10.91

Before looking at another example, let's recall an important observation.

Important Observation

Adding zeros to the end of the fractional part of a decimal number does not change its value. Similarly, deleting trailing zeros from the end of a decimal number does not change its value.

For example, we could add two zeros to the end of the fractional part of 7.25 to obtain 7.2500. The numbers 7.25 and 7.2500 are identical as the following argument shows:

$$\begin{aligned} 7.2500 &= 7\frac{2500}{10000} \\ &= 7\frac{25}{100} \\ &= 7.25 \end{aligned}$$

Example 3

Add 7.5 and 12.23.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add a trailing zero to improve columnar alignment.

$$\begin{array}{r} 7.50 \\ +12.23 \\ \hline 19.73 \end{array}$$

Hence, 7.5 + 12.23 = 19.73.

Exercise

Add: 9.7 + 15.86

Answer

25.56

Example 4

Find the sum: 12.2+8.352 + 22.44.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add trailing zeros to improve the columnar alignment.

$$\begin{array}{r} 12.200 \\ 8.352 \\ +22.440 \\ \hline 42.992 \end{array}$$

Hence, 12.2+8.352 + 22.44 = 42.992.

Exercise

Add: 12.9+4.286 + 33.97

Answer

51.156

Subtracting Decimals

Subtraction of decimal numbers proceeds in much the same way as addition of decimal numbers.

Subtracting Decimals

To subtract decimal numbers, proceed as follows:

1. Place the numbers to be subtracted in vertical format, aligning the decimal points.
2. Subtract the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the same column as the decimal points above it.

Example 5

Subtract 12.23 from 33.57.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column, then subtract. Note that we subtract 12.23 **from** 33.57.

$$\begin{array}{r} 33.57 \\ -12.23 \\ \hline 21.34 \end{array}$$

Hence, 33.57 − 12.23 = 21.34.

Exercise

Subtract: 58.76 − 38.95

Answer

19.81

As with addition, we add trailing zeros to the fractional part of the decimal numbers to help columnar alignment.

Example 6

Find the difference: 13.3 − 8.572.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add trailing zeros to the fractional part of 13.3 to improve columnar alignment.

$$\begin{array}{r} 13.300 \\ -8.572 \\ \hline 4.728 \end{array}$$

Hence, 13.3 − 8.572 = 4.728.

Exercise

Subtract: 15.2 − 8.756

Answer

6.444

Adding and Subtracting Signed Decimal Numbers

We use the same rules for addition of signed decimal numbers as we did for the addition of integers.

Adding Two Decimals with Like Signs

To add two decimals with like signs, proceed as follows:

1. Add the magnitudes of the decimal numbers.
2. Prefix the common sign.

Example 7

Simplify: −3.2+(−18.95).

Solution

To add like signs, first add the magnitudes.

$$\begin{array}{r} 3.20 \\ +18.95 \\ \hline 22.15 \end{array}$$

Prefix the common sign. Hence, −3.2+(−18.95) = −22.15

Exercise

Simplify: −5.7 + (−83.85)

Answer

−89.55

We use the same rule as we did for integers when adding decimals with unlike signs.

Adding Two Decimals with Unlike Signs

To add two decimals with unlike signs, proceed as follows:

1. Subtract the smaller magnitude from the larger magnitude.
2. Prefix the sign of the decimal number with the larger magnitude.

Example 8

Simplify: −3 + 2.24.

Solution

To add unlike signs, first subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 3.00 \\ -2.24 \\ \hline 0.76 \end{array}$$

Prefix the sign of the decimal number with the larger magnitude. Hence, −3+2.24 = −0.76.

Exercise

Simplify: −8 + 5.74

Answer

−2.26

Subtraction still means *add the opposite*.

Example 9

Simplify: −8.567 − (−12.3).

Solution

Subtraction must first be changed to addition by adding the opposite.

$$-8.567-(-12.3)=-8.567+12.3$$

We have unlike signs. First, subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 12.300 \\ -8.567 \\ \hline 3.733 \end{array}$$

Prefix the sign of the decimal number with the larger magnitude. Hence:

$$\begin{aligned} -8.567-(-12.3) &= -8.567+12.3 \\ &= 3.733 \end{aligned}$$

Exercise

Simplify: −2.384 − (−15.2)

Answer

12.816

Order of operations demands that we simplify expressions contained in parentheses first.

Example 10

Simplify: −11.2 − (−8.45 + 2.7).

Solution

We need to add inside the parentheses first. Because we have unlike signs, subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 8.45 \\ -2.70 \\ \hline 5.75 \end{array}$$

Prefix the sign of the number with the larger magnitude. Therefore,

$$-11.2-(-8.45+2.7)=-11.2-(-5.75)$$

Subtraction means add the opposite.

$$-11.2-(-5.75)=-11.2+5.75$$

Again, we have unlike signs. Subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 11.20 \\ -5.75 \\ \hline 5.45 \end{array}$$

Prefix the sign of the number with the large magnitude.

$$-11.2+5.75=-5.45$$

Exercise

Simplify: −12.8 − (−7.44 + 3.7)

Answer

−9.06

Writing Mathematics

The solution to the previous example should be written as follows:

$$\begin{aligned} -11.2-(-8.45+2.7) &= -11.2-(-5.75) \\ &= -11.2+5.75 \\ &= -5.45 \end{aligned}$$

Any scratch work, such as the computations in vertical format in the previous example, should be done in the margin or on a scratch pad.

Example 11

Simplify: −12.3 −|− 4.6 − (−2.84)|.

Solution

We simplify the expression inside the absolute value bars first, take the absolute value of the result, then subtract.

$$
\begin{aligned}
&-12.3-|-4.6-(-2.84)| \\
&=-12.3-|-4.6+2.84| && \text{Add the opposite.} \\
&=-12.3-|-1.76| && \text{Add: } -4.6+2.84=-1.76. \\
&=-12.3-1.76 && |-1.76|=1.76. \\
&=-12.3+(-1.76) && \text{Add the opposite.} \\
&=-14.06 && \text{Add: } -12.3+(-1.76)=-14.06.
\end{aligned}
$$

? Exercise

Simplify: −8.6 −|− 5.5 − (−8.32)|

Answer

−11.42

Exercises

In Exercises 1-12, add the decimals.

1. $31.9+84.7$
2. $9.39+7.7$
3. $4+97.18$
4. $2.645+2.444$
5. $4+87.502$
6. $23.69+97.8$
7. $95.57+7.88$
8. $18.7+7$
9. $52.671+5.97$
10. $9.696+28.2$
11. $4.76+2.1$
12. $1.5+46.4$

In Exercises 13-24, subtract the decimals.

13. $9-2.261$
14. $98.14-7.27$
15. $80.9-6$
16. $9.126-6$
17. $55.672-3.3$
18. $4.717-1.637$
19. $60.575-6$
20. $8.91-2.68$
21. $39.8-4.5$
22. $8.210-3.7$
23. $8.1-2.12$
24. $7.675-1.1$

In Exercises 25-64, add or subtract the decimals, as indicated.

25. $-19.13-7$
26. $-8-79.8$
27. $6.08-76.8$
28. $5.76-36.8$
29. $-34.7+(-56.214)$
30. $-7.5+(-7.11)$
31. $8.4+(-6.757)$
32. $-1.94+72.85$
33. $-50.4+7.6$
34. $1.4+(-86.9)$
35. $-43.3+2.2$
36. $0.08+(-2.33)$
37. $0.19-0.7$
38. $9-18.01$
39. $-7-1.504$
40. $-4.28-2.6$
41. $-4.47+(-2)$
42. $-9+(-43.67)$
43. $71.72-(-6)$
44. $6-(-8.4)$
45. $-9.829-(-17.33)$
46. $-95.23-(-71.7)$
47. $2.001-4.202$
48. $4-11.421$
49. $2.6-2.99$
50. $3.57-84.21$
51. $-4.560-2.335$
52. $-4.95-96.89$
53. $-54.3-3.97$
54. $-2-29.285$
55. $-6.32+(-48.663)$
56. $-8.8+(-34.27)$
57. $-8-(-3.686)$
58. $-2.263-(-72.3)$
59. $9.365+(-5)$
60. $-0.12+6.973$
61. $2.762-(-7.3)$

62. $65.079 - (-52.6)$

63. $-96.1 + (-9.65)$

64. $-1.067 + (-4.4)$

In Exercises 65-80, simplify the given expression.

65. $-12.05 - |17.83 - (-17.16)|$

66. $15.88 - |-5.22 - (-19.94)|$

67. $-6.4 + |9.38 - (-9.39)|$

68. $-16.74 + |16.64 - 2.6|$

69. $-19.1 - (1.51 - (-17.35))$

70. $17.98 - (10.07 - (-10.1))$

71. $11.55 + (6.3 - (-1.9))$

72. $-8.14 + (16.6 - (-15.41))$

73. $-1.7 - (1.9 - (-16.25))$

74. $-4.06 - (4.4 - (-10.04))$

75. $1.2 + |8.74 - 16.5|$

76. $18.4 + |16.5 - 7.6|$

77. $-12.4 - |3.81 - 16.4|$

78. $13.65 - |11.55 - (-4.44)|$

79. $-11.15 + (11.6 - (-16.68))$

80. $8.5 + (3.9 - 6.98)$

---

81. **Big Banks**. Market capitalization of nation's four largest banks (as of April 23, 2009)

JPMorgan Chase & Co	$124.8 billion
Wells Fargo & Co	$85.3 billion
Goldman Sachs Group Inc.	$61.8 billion
Bank of America	$56.4 billion

What is the total value of the nation's four largest banks? *Associated Press Times-Standard 4/22/09*

82. **Telescope Mirror.** The newly launched Herschel Telescope has a mirror 11.5 feet in diameter while Hubble's mirror is 7.9 feet in diameter. How much larger is Herschel's mirror in diameter than Hubble's?

83. **Average Temperature**. The average temperatures in Sacramento, California in July are a high daytime temperature of 93.8 degrees Fahrenheit and a low nighttime temperature of 60.9 degrees Fahrenheit. What is the change in temperature from day to night? Hint: See Section 2.3 for the formula for comparing temperatures.

84. **Average Temperature**. The average temperatures in Redding, California in July are a high daytime temperature of 98.2 degrees Fahrenheit and a low nighttime temperature of 64.9 degrees Fahrenheit. What is the change in temperature from day to night? Hint: See Section 2.3 for the formula for comparing temperatures.

85. **Net Worth**. Net worth is defined as assets minus liabilities. Assets are everything of value that can be converted to cash while liabilities are the total of debts. General Growth Properties, the owners of the Bayshore Mall, have $29.6 billion in assets and $27 billion in liabilities, and have gone bankrupt. What was General Growth Properties net worth before bankruptcy? *Times-Standard 4/17/2009*

86. **Grape crush**. The California Department of Food and Agriculture's preliminary grape crush report shows that the state produced 3.69 million tons of wine grapes in 2009. That's just shy of the record 2005 crush of 3.76 million tons. By how many tons short of the record was the crush of 2009? *Associated Press-Times-Standard Calif. winegrapes harvest jumped 23% in '09.*

87. **Turnover**. The Labor Department's Job Openings and Labor Turnover Survey claims that employers hired about 4.08 million people in January 2010 while 4.12 million people were fired or otherwise left their jobs. How many more people lost jobs than were hired? Convert your answer to a whole number. *Associated Press-Times-Standard 03/10/10 Job openings up sharply in January to 2.7M.*

Answers

1. 116.6

3. 101.18

5. 91.502

7. 103.45

9. 58.641

11. 6.86

13. 6.739

15. 74.9

17. 52.372

19. 54.575

21. 35.3

23. 5.98

25. −26.13

27. −70.72

29. −90.914

31. 1.643

33. −42.8

35. −41.1

37. −0.51

39. −8.504

41. −6.47

43. 77.72

45. 7.501

47. −2.201

49. −0.39

51. −6.895

53. −58.27

55. −54.983

57. −4.314

59. 4.365

61. 10.062

63. −105.75

65. −47.04

67. 12.37

69. −37.96

71. 19.75

73. −19.85

75. 8.96

77. −24.99

79. 17.13

81. $328.3 billion

83. −32.9 degrees Fahrenheit

85. $2.6 billion

87. 40, 000

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3.4: Multiplying Decimals

Multiplying decimal numbers involves two steps: (1) multiplying the numbers as whole numbers, ignoring the decimal point, and (2) placing the decimal point in the correct position in the product or answer.

For example, consider (0.7)(0.08), which asks us to find the product of “seven tenths” and “eight hundredths.” We could change these decimal numbers to fractions, then multiply.

$$\begin{aligned}(0.7)(0.8) &= \frac{7}{10}\cdot\frac{8}{100}\\ &= \frac{56}{100}\\ &= 0.056\end{aligned}$$

The product is 56/1000, or “fifty six thousandths,” which as a decimal is written 0.056.

Important Observations

There are two very important observations to be made about the example (0.7)(0.08).

1. In fractional form

$$\frac{7}{10}\cdot\frac{8}{100}=\frac{56}{1000},$$

note that the numerator of the product is found by taking the product of the whole numbers 7 and 8. That is, you ignore the decimal points in 0.7 and 0.08 and multiply 7 and 8 as if they were whole numbers.

2. The first factor, 0.7, has one digit to the right of the decimal point. Its fractional equivalent, 7/10, has one zero in its denominator. The second factor, 0.08, has two digits to the right of the decimal point. Its fractional equivalent, 8/100, has two zeros in its denominator. Therefore, the product 56/1000 is forced to have three zeros in its denominator and its decimal equivalent, 0.056, must therefore have three digits to the right of the decimal point.

Let’s look at another example.

Example 1

Simplify: (2.34)(1.2).

Solution

Change the decimal numbers “two and thirty four hundredths” and “one and two tenths” to fractions, then multiply.

\[\begin{aligned} (2.34)(1.2) = 2 \frac{34}{100} \cdot 1 \frac{2}{10} ~ & \textcolor{red}{ \text{ Change decimals to fractions.}} \\ = \frac{234}{100 \cdot \frac{12}{10}} ~ & \textcolor{red}{ \text{ Change mixed to improper fractions.}} \\ = \frac{2808}{1000} ~ & \textcolor{red}{ \text{ Multiply numerators and denominators.}} \\ = 2 \frac{808}{1000} ~ & \textcolor{red}{ \text{ Change to mixed fraction.}} \\ = 2.808 ~ & \textcolor{red}{ \text{ Change back to decimal form.}} \end{aligned}\nonumber \]

Exercise

Multiply: (1.86)(9.5)

Answer

17.67

Important Observations

We make the same two observations as in the previous example.

1. If we treat the decimal numbers as whole numbers without decimal points, then (234)(12) = 2808, which is the numerator of the fraction 2808/1000 in the solution shown in Example 1. These are also the same digits shown in the answer 2.808.
2. There are two digits to the right of the decimal point in the first factor 2.34 and one digit to the right of the decimal point in the second factor 1.2. This is a total of three digits to the right of the decimal points in the factors, which is precisely the same number of digits that appear to the right of the decimal point in the answer 2.808.

The observations made at the end of the previous two examples lead us to the following method.

Multiplying Decimal Numbers

To multiply two decimal numbers, perform the following steps:

1. Ignore the decimal points in the factors and multiply the two factors as if they were whole numbers.
2. Count the number of digits to the right of the decimal point in each factor. Sum these two numbers.
3. Place the decimal point in the product so that the number of digits to the right of the decimal point equals the sum found in step 2.

Example 2

Use the steps outlined in Multiplying Decimal Numbers to find the product in Example 1.

Solution

We follow the steps outlined in *Multiplying Decimal Numbers*.

1. The first step is to multiply the factors 2.34 and 1.2 as whole numbers, ignoring the decimal points.

$$\begin{array}{r} 234 \\ \times 12 \\ \hline 468 \\ 234 \\ \hline 2808 \end{array}$$

2. The second step is to find the sum of the number of digits to the right of the decimal points in each factor. Note that 2.34 has two digits to the right of the decimal point, while 1.2 has one digit to the right of the decimal point. Thus, we have a total of three digits to the right of the decimal points in the factors.

3. The third and final step is to place the decimal point in the product or answer so that there are a total of three digits to the right of the decimal point. Thus,

$$(2.34)(1.2) = 2.808.$$

Note that this is precisely the same solution found in Example 1.

What follows is a convenient way to arrange your work in vertical format.

$$\begin{array}{r} 2.34 \\ \times 1.2 \\ \hline 468 \\ 2\ 34 \\ \hline 2.808 \end{array}$$

Exercise

Multiply: (5.98)(3.7)

Answer

22.126

Example 3

Simplify: (8.235)(2.3).

Solution

We use the convenient vertical format introduced at the end of Example 2.

$$\begin{array}{r} 8.235 \\ \times 2.3 \\ \hline 2\,4705 \\ 16\,470 \\ \hline 18.9405 \end{array}$$

The factor 8.235 has three digits to the right of the decimal point; the factor 2.3 has one digit to the right of the decimal point. Therefore, there must be a total of four digits to the right of the decimal point in the product or answer.

Exercise

Multiply: (9.582)(8.6)

Answer

82.4052

Multiplying Signed Decimal Numbers

The rules governing multiplication of signed decimal numbers are identical to the rules governing multiplication of integers.

Like Signs. The product of two decimal numbers with like signs is positive. That is:

$$(+)(+) = + \text{ and } (-)(-) = +$$

Unlike Signs. The product of two decimal numbers with unlike signs is negative. That is:

$$(+)(-) = - \text{ and } (-)(+) = -$$

Example 4

Simplify: (−2.22)(−1.23).

Solution

Ignore the signs to do the multiplication, then consider the signs in the final answer behind.

As each factor has two digits to the right of the decimal point, there should be a total of 4 decimals to the right of the decimal point in the product.

$$\begin{array}{r} 2.22 \\ \times 1.23 \\ \hline \\ 666 \\ 444 \\ 1\,11 \\ \hline 1.6206 \end{array}$$

Like signs give a positive product. Hence:

$$(-2.22)(-1.23) = 1.6206$$

Exercise

Multiply: $(-3.86)(-5.77)$

Answer

22.2722

Example 5

Simplify: $(5.68)(-0.012)$.

Solution

Ignore the signs to do the multiplication, then consider the signs in the final answer below.

The first factor has two digits to the right of the decimal point, the second factor has three. Therefore, there must be a total of five digits to the right of the decimal point in the product or answer. This necessitates prepending an extra zero in front of our product.

$$\begin{array}{r} 5.68 \\ \times 0.012 \\ \hline 1136 \\ 568 \\ \hline 0.06816 \end{array}$$

Unlike signs give a negative product. Hence:

$$(5.68)(-0.012) = -0.06816$$

Exercise

Multiply: $(9.23)(-0.018)$

Answer

-0.16614

Order of Operations

The same *Rules Guiding Order of Operations* also apply to decimal numbers.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Example 6

If $a = 3.1$ and $b = -2.4$, evaluate $a^2 - 3.2b^2$

Solution

Prepare the expression for substitution using parentheses.

$$a^2 - 3.2b^2 = (\ )^2 - 3.2(\ )^2$$

Substitute 3.1 for a and −2.4 for b and simplify.

$$\begin{aligned} a^2 - 3.2b^2 &= (3.1)^2 - 3.2(-2.4)^2 && \text{Substitute: } 3.1 \text{ for } a,\ -2.4 \text{ for } b. \\ &= 9.61 - 3.2(5.76) && \text{Exponents first: } (3.1)^2 = 9.61,\ (-2.4)^2 = 5.76 \\ &= 9.61 - 18.432 && \text{Multiply: } 3.2(5.76) = 18.432 \\ &= -8.822 && \text{Subtract: } 9.61 - 18.432 = -8.822 \end{aligned}$$

? Exercise

If a = 3.8 and b = −4.6, evaluate the expression: $2.5a^2 - b^2$

Answer

14.94

Powers of Ten

Consider:

$$\begin{aligned} 10^1 &= 10 \\ 10^2 &= 10 \cdot 10 = 100 \\ 10^3 &= 10 \cdot 10 \cdot 10 = 1,000 \\ 10^4 &= 10 \cdot 10 \cdot 10 \cdot 10 = 10,000 \end{aligned}$$

Note the answer for 10^4, a one followed by four zeros! Do you see the pattern?

Powers of Ten

In the expression 10^n, the exponent matches the number of zeros in the answer. Hence, 10^n will be a 1 followed by n zeros.

✓ Example 7

Simplify: 10^9.

Solution

10^9 should be a 1 followed by 9 zeros. That is,

$$10^9 = 1,000,000,000,$$

or “one billion.”

? Exercise

Simplify: 10^6

Answer

1,000,000

Multiplying Decimal Numbers by Powers of Ten

Let’s multiply 1.234567 by 10^3, or equivalently, by 1,000. Ignore the decimal point and multiply the numbers as whole numbers.

$$\begin{array}{r} 1.234567 \\ \times 1000 \\ \hline 1234.567000 \end{array}$$

The sum total of digits to the right of the decimal point in each factor is 6. Therefore, we place the decimal point in the product so that there are six digits to the right of the decimal point.

However, the trailing zeros may be removed without changing the value of the product. That is, 1.234567 times 1000 is 1234.567. Note that the decimal point in the product is three places further to the right than in the original factor. This observation leads to the following result.

Multiplying a Decimal Number by a Power of Ten

Multiplying a decimal number by 10^n will move the decimal point n places to the right.

Example 8

Simplify: $1.234567 \cdot 10^4$

Solution

Multiplying by 104 (or equivalently, by 10,000) moves the decimal 4 places to the right. Thus, $1.234567 \cdot 10,000 = 12345.67$.

Exercise

Simplify: $1.234567 \cdot 10^2$

Answer

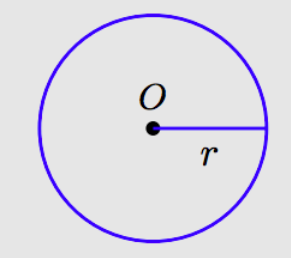
123.4567

The Circle

Let's begin with a definition.

The Circle

A *circle* is the collection of all points equidistant from a given point O, called the *center* of the circle.

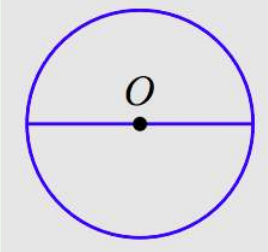


The segment joining any point on the circle to the center of the circle is called a *radius* of the circle. In the figure above, the variable r represents the length of the radius of the circle

We need another term, the diameter of a circle.

The Diameter of a Circle

If two points on a circle are connected with a line segment, then that segment is called a *chord* of the circle. If the chord passes through the center of the circle, then the chord is called the *diameter* of the circle.



In the figure above, the variable d represents the length of the diameter of the circle. Note that the diameter is twice the length of the radius; in symbols

$$d = 2r.$$

The Circumference of a Circle

When we work with polygons, the perimeter of the polygon is found by summing the lengths of its edges. The circle uses a different name for its perimeter.

The Circumference of a Circle

The length of the circle is called its *circumference*. We usually use the variable C to denote the circumference of a circle

That is, if one were to walk along the circle, the total distance traveled in one revolution is the circumference of the circle.

The ancient mathematicians of Egypt and Greece noted a striking relation between the circumference of a circle and its diameter. They discovered that whenever you divide a circle's circumference by its diameter, you get a constant. That is, if you take a very large circle and divide its circumference by its diameter, you get exactly the same number if you take a very small circle and divide its circumference by its diameter. This common constant was named π ("pi").

Relating the Circumference and Diameter

Whenever a circle's circumference is divided by its diameter, the answer is the constant π. That is, if C is the circumference of the circle and d is the circle's diameter, then

$$\frac{C}{d} = \pi.$$

In modern times, we usually multiply both sides of this equation by d to obtain the formula for the circumference of a circle.

$$C = \pi d$$

Because the diameter of a circle is twice the length of its radius, we can substitute $d = 2r$ in the last equation to get an alternate form of the circumference equation.

$$C = \pi(2r) = 2\pi r$$

The number π has a rich and storied history. Ancient geometers from Egypt, Babylonia, India, and Greece knew that π was slightly larger than 3. The earliest known approximations date from around 1900 BC (Wikipedia); they are 25/8 (Babylonia) and 256/81 (Egypt). The Indian text *Shatapatha Brahmana* gives π as 339/108 ≈ 3.139. Archimedes (287-212 BC) was the first to estimate π rigorously, approximating the circumference of a circle with inscribed and circumscribed polygons. He was able to prove that 223/71 < π < 22/7. Taking the average of these values yields π ≈ 3.1419. Modern mathematicians have proved that π is an *irrational* number, an infinite decimal that never repeats any pattern. Mathematicians, with the help of computers, routinely produce approximations of π with billions of digits after the decimal point.

Digits of Pi

Here is π, correct to the first fifty digits.

π = 3.14159265358979323846264338327950288419716939937510 ...

The number of digits of π used depends on the application. Working at very small scales, one might keep many digits of π, but if you are building a circular garden fence in your backyard, then fewer digits of π are needed.

Example 9

Find the circumference of a circle given its radius is 12 feet.

Solution

The circumference of the circle is given by the formula $C = \pi d$, or, because $d = 2r$,

$$C = 2\pi r.$$

Substitute 12 for r.

$$C = 2\pi r = 2\pi(12) = 24\pi$$

Therefore, the circumference of the circle is *exactly* $C = 24\pi$ feet.

We can approximate the circumference by entering an approximation for π. Let's use $\pi \approx 3.14$. *Note: The symbol ≈ is read "approximately equal to."*

$$C = 24\pi \approx 24(3.14) \approx 75.36 \text{ feet}$$

It is important to understand that the solution $C = 24\pi$ feet is the **exact** circumference, while $C \approx 75.36$ feet is only an approximation.

Exercise

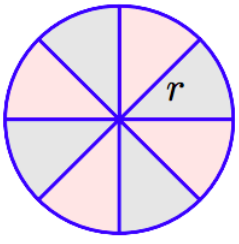
Find the radius of a circle having radius 14 inches. Use $\pi \approx 3.14$

Answer

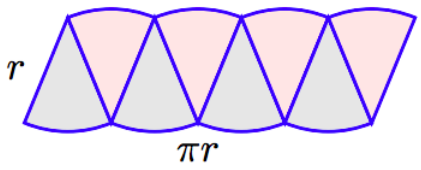
87.92 inches

The Area of a Circle

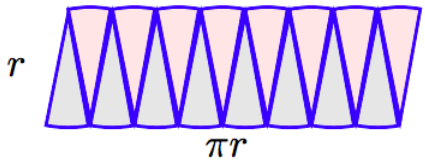
Here's a reasonable argument used to help develop a formula for the area of a circle. Start with a circle of radius r and divide it into 8 equal wedges, as shown in the figure that follows.



Rearrange the pieces as shown in the following figure.



Note that the rearranged pieces almost form a rectangle with length approximately half the circumference of the circle, πr, and width approximately r. The area would be approximately $A \approx$ (length)(width) $\approx (\pi r)(r) \approx \pi r^2$. This approximation would be even better if we doubled the number of wedges of the circle.



If we doubled the number of wedges again, the resulting figure would even more closely resemble a rectangle with length πr and width r. This leads to the following conclusion.

The Area of a Circle

The area of a circle of radius r is given by the formula

$$A = \pi r^2.$$

Example 10

Find the area of a circle having a diameter of 12.5 meters. Use 3.14 for π and round the answer for the area to the nearest tenth of a square meter.

Solution

The diameter is twice the radius.

$$d = 2r$$

Substitute 12.5 for d and solve for r.

$$\begin{aligned} 12.5 &= 2r && \text{Substitute } 12.5 \text{ for } d. \\ \frac{12.5}{2} &= \frac{2r}{2} && \text{Divide both sides by } 2. \\ 6.25 &= r && \text{Simplify.} \end{aligned}$$

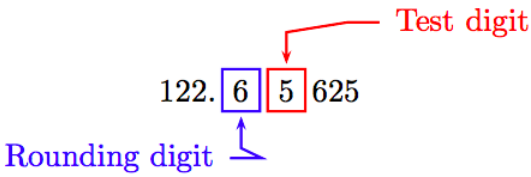
Hence, the radius is 6.25 meters. To find the area, use the formula

$$A = \pi r^2$$

and substitute: 3.14 for π and 6.25 for r.

$$\begin{aligned} A &= (3.14)(6.25)^2 && \text{Substitute: } 3.14 \text{ for } \pi,\ 6.25 \text{ for } r. \\ &= (3.14)(39.0625) && \text{Square first: } (6.25)^2 = 39.0625 \\ &= 122.65625 && \text{Multiply: } (3.14)(39.0625) = 122.65625 \end{aligned}$$

Hence, the approximate area of the circle is A = 122.65625 square meters. To round to the nearest tenth of a square meter, identify the rounding digit and the test digit.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus, correct to the nearest tenth of a square meter, the area of the circle is approximately

$$A \approx 122.7 \text{ m}^2.$$

? Exercise

Find the area of a circle having radius 12.2 centimeters. Use $\pi \approx 3.14$

Answer

467.3576 cm^2

Exercises

In Exercises 1-28, multiply the decimals.

1. (6.7)(0.03)
2. (2.4)(0.2)
3. (28.9)(5.9)
4. (33.5)(2.1)
5. (4.1)(4.6)
6. (2.6)(8.2)
7. (75.3)(0.4)
8. (21.4)(0.6)
9. (6.98)(0.9)
10. (2.11)(0.04)
11. (57.9)(3.29)
12. (3.58)(16.3)
13. (47.3)(0.9)
14. (30.7)(0.4)
15. (9.9)(6.7)
16. (7.2)(6.1)
17. (19.5)(7.9)
18. (43.4)(8.9)
19. (6.9)(0.3)
20. (7.7)(0.7)
21. (35.3)(3.81)
22. (5.44)(9.58)
23. (2.32)(0.03)

24. (4.48)(0.08)
25. (3.02)(6.7)
26. (1.26)(9.4)
27. (4.98)(6.2)
28. (3.53)(2.9)

In Exercises 29-56, multiply the decimals.

29. (−9.41)(0.07)
30. (4.45)(−0.4)
31. (−7.4)(−0.9)
32. (−6.9)(0.05)
33. (−8.2)(3.7)
34. (−7.5)(−6.6)
35. (9.72)(−9.1)
36. (6.22)(−9.4)
37. (−6.4)(2.6)
38. (2.3)(−4.4)
39. (−39.3)(−0.8)
40. (57.7)(−0.04)
41. (63.1)(−0.02)
42. (−51.1)(−0.8)
43. (−90.8)(3.1)
44. (−74.7)(2.9)
45. (47.5)(−82.1)
46. (−14.8)(−12.7)
47. (−31.1)(−4.8)
48. (−28.7)(−6.8)
49. (−2.5)(−0.07)
50. (−1.3)(−0.05)
51. (1.02)(−0.2)
52. (−7.48)(−0.1)
53. (7.81)(−5.5)
54. (−1.94)(4.2)
55. (−2.09)(37.9)
56. (20.6)(−15.2)

In Exercises 57-68, multiply the decimal by the given power of 10.

57. $24.264 \cdot 10$
58. $65.722 \cdot 100$
59. $53.867 \cdot 10^4$

60. $23.216 \cdot 10^4$

61. $5.096 \cdot 10^3$

62. $60.890 \cdot 10^3$

63. $37.968 \cdot 10^3$

64. $43.552 \cdot 10^3$

65. $61.303 \cdot 100$

66. $83.837 \cdot 1000$

67. $74.896 \cdot 1000$

68. $30.728 \cdot 100$

In Exercises 69-80, simplify the given expression.

69. $(0.36)(7.4) - (-2.8)^2$

70. $(-8.88)(-9.2) - (-2.3)^2$

71. $9.4 - (-7.7)(1.2)^2$

72. $0.7 - (-8.7)(-9.4)^2$

73. $5.94 - (-1.2)(-1.8)^2$

74. $-2.6 - (-9.8)(9.9)^2$

75. $6.3 - 4.2(9.3)^2$

76. $9.9 - (-4.1)(8.5)^2$

77. $(6.3)(1.88) - (-2.2)^2$

78. $(-4.98)(-1.7) - 3.52$

79. $(-8.1)(9.4) - 1.8^2$

80. $(-3.63)(5.2) - 0.8^2$

---

81. Given $a = -6.24$, $b = 0.4$, and $c = 7.2$, evaluate the expression $a - bc^2$.

82. Given $a = 4.1$, $b = -1.8$, and $c = -9.5$, evaluate the expression $a - bc^2$.

83. Given $a = -2.4$, $b = -2.1$, and $c = -4.6$, evaluate the expression $ab - c^2$.

84. Given $a = 3.3$, $b = 7.3$, and $c = 3.4$, evaluate the expression $ab - c^2$.

85. Given $a = -3.21$, $b = 3.5$, and $c = 8.3$, evaluate the expression $a - bc^2$.

86. Given $a = 7.45$, $b = -6.1$, and $c = -3.5$, evaluate the expression $a - bc^2$.

87. Given $a = -4.5$, $b = -6.9$, and $c = 4.6$, evaluate the expression $ab - c^2$.

88. Given $a = -8.3$, $b = 8.2$, and $c = 5.4$, evaluate the expression $ab - c^2$.

---

89. A circle has a diameter of 8.56 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

90. A circle has a diameter of 14.23 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

91. A circle has a diameter of 12.04 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

92. A circle has a diameter of 14.11 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

93. A circle has a diameter of 10.75 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

94. A circle has a diameter of 15.49 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

95. A circle has a diameter of 13.96 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

96. A circle has a diameter of 15.95 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

---

97. Sue has decided to build a circular fish pond near her patio. She wants it to be 15 feet in diameter and 1.5 feet deep. What is the volume of water it will hold? Use $\pi \approx 3.14$. *Hint: The volume of a cylinder is given by the formula $V = \pi r^2 h$, which is the area of the circular base times the height of the cylinder.*

98. John has a decision to make regarding his employment. He currently has a job at Taco Loco in Fortuna. After taxes, he makes about $9.20 per hour and works about 168 hours a month. He currently pays $400 per month for rent. He has an opportunity to move to Santa Rosa and take a job at Mi Ultimo Refugio which would pay $10.30 per hour after taxes for 168 hours a month, but his rent would cost $570 per month.

a) After paying for housing in Fortuna, how much does he have left over each month for other expenditures?

b) After paying for housing in Santa Rosa, how much would he have left over each month for other expenditures?

c) For which job would he have more money left after paying rent and how much would it be?

99. John decided to move to Santa Rosa and take the job at Mi Ultimo Refugio (see Exercise 98). He was able to increase his income because he could work 4 Sundays a month at time-and-a half. So now he worked 32 hours a month at time-and-a-half and 136 hours at the regular rate of $10.30 (all after taxes were removed). *Note: He previously had worked 168 hours per month at $10.30 per hour.*

a) What was his new monthly income?

b) How much did his monthly income increase?

100. **Electric Bill.** On a recent bill, PGE charged $0.11531 per Kwh for the first 333 Kwh of electrical power used. If a household used 312 Kwh of power, what was their electrical bill?

101. **Cabernet.** In Napa Valley, one acre of good land can produce about 3.5 tons of quality grapes. At an average price of $3,414 per ton for premium cabernet, how much money could you generate on one acre of cabernet farming? *Associated Press-Times-Standard 03/11/10 Grape moth threatens Napa Valley growing method.*

102. **Fertilizer.** Using the 2008 Ohio Farm Custom Rates, the average cost for spreading dry bulk fertilizer is about $4.50 per acre. What is the cost to fertilize 50 acres?

103. **Agribusiness.** Huge corporate agribarns house 1000 pigs each.

a) If each pig weighs approximately 100 pounds, how many pounds of pig is in each warehouse?

b) At an average $1.29 per pound, what is the total cash value for a corporate agribarn? *Associated Press-Times-Standard 12/29/09 Pressure rises to stop antibiotics in agriculture.*

104. **Shipwrecks.** A dozen centuries-old shipwrecks were found in the Baltic Sea by a gas company building an underwater pipeline between Russia and Germany. The 12 wrecks were found in a 30-mile-long and 1.2-mile-wide corridor at a depth of 430 feet. Model the corridor with a rectangle and find the approximate area of the region where the ships were found. *Associated Press-Times-Standard 03/10/10 Centuries-old shipwrecks found in Baltic Sea.*

105. **Radio dish.** The diameter of the “workhorse fleet” of radio telescopes, like the one in Goldstone, California, is 230 feet. What is the circumference of the radio telescope dish to the nearest tenth? *Associated Press-Times-Standard 03/09/2010 NASA will repair deep space antenna in California desert.*

Answers

1. 0.201

3. 170.51

5. 18.86

7. 30.12

9. 6.282

11. 190.491

13. 42.57

15. 66.33

17. 154.05

19. 2.07

21. 134.493

23. 0.0696

25. 20.234

27. 30.876

29. −0.6587

31. 6.66

33. −30.34

35. −88.452

37. −16.64

39. 31.44

41. −1.262

43. −281.48

45. −3899.75

47. 149.28

49. 0.175

51. −0.204

53. −42.955

55. −79.211

57. 242.64

59. 538670

61. 5096

63. 37968

65. 6130.3

67. 74896

69. −5.176

71. 20.488

73. 9.828

75. −356.958

77. 7.004

79. −79.38

81. −26.976

83. −16.12

85. −244.325

87. 9.89

89. 26.9 in.

91. 37.8 in.

93. 90.72 square inches

95. 152.98 square inches

97. 264.9375 cubic feet

99.

    a) $1895.20

    b) $164.80

101. $11, 949

103.

    a) 100, 000 pounds

    b) $129,000

105. 722.2 feet

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3.5: Dividing Decimals

In this and following sections we make use of the terms *divisor*, *dividend*, *quotient*, and *remainder*.

Divisor, Dividend, Quotient, and Remainder

This schematic reminds readers of the position of these terms in the division process.

$$\begin{array}{r} \text{quotient} \\ \text{divisor})\ \overline{\text{dividend}} \\ \underline{\quad\ldots\quad} \\ \text{remainder} \end{array}$$

Now that these terms are defined, we begin the discussion of division of decimal numbers. Suppose that we wish to divide 637 by 100. We could do this in fraction form, change the result to a mixed fraction, then the mixed fraction to decimal form.

$$\frac{637}{100} = 6\frac{37}{100} = 6.37$$

We can also arrange the division much as we would the division of two whole numbers.

$$\begin{array}{r} 6.37 \\ 100\overline{)637.00} \\ \underline{600} \\ 37\,0 \\ \underline{30\,0} \\ 7\,00 \\ \underline{7\,00} \\ 0 \end{array}$$

Note that adding two zeros after the decimal point in the dividend does not change the value of 637. Further, note that we proceed as if we are dividing two whole numbers, placing the decimal point in the quotient directly above the decimal point in the dividend.

These observations lead to the following algorithm.

Dividing a Decimal by a Whole Number

To divide a decimal number by a whole number, proceed as follows:

1. Set up the long division as you would the division of two whole numbers.
2. Perform the division as if the numbers were both whole numbers, adding zeros to the right of the decimal point in the dividend as necessary to complete the division.
3. Place the decimal point in the quotient immediately above the decimal point in the dividend.

Example 1

Divide 23 by 20.

Solution

Arrange as if using long division to divide whole numbers, adding enough zeros to the right of the decimal point in the dividend to complete the division.

$$\begin{array}{r} 1.15 \\ 20\overline{)23.00} \\ \underline{20} \\ 3\,0 \\ \underline{2\,0} \\ 1\,00 \\ \underline{1\,00} \\ 0 \end{array}$$

Hence, 23 divided by 20 is 1.15.

Adding Zeroes to the Right of the Decimal Point

Usually, one does not immediately see how many zeros to the right of the decimal point in the dividend are needed. These zeros are usually added at each step of the division, until the division is completed or the user is willing to terminate the process and accept only an estimate of the quotient.

Example 2

Divide: 155.2 ÷ 25.

Solution

Arrange as if using long division to divide whole numbers, and begin.

$$\begin{array}{r} 6.2 \\ 25\overline{)155.2} \\ \underline{150} \\ 5\,2 \\ \underline{5\,0} \\ 2 \end{array}$$

We still have a nonzero remainder. Adding another zero does no good.

$$\begin{array}{r} 6.20 \\ 25\overline{)155.20} \\ \underline{150} \\ 5\,2 \\ \underline{5\,0} \\ 20 \end{array}$$

However, if we add one more additional zero, the division completes with a zero remainder.

$$\begin{array}{r} 6.208 \\ 25\overline{)155.200} \\ \underline{150} \\ 5\,2 \\ \underline{5\,0} \\ 200 \\ \underline{200} \\ 0 \end{array}$$

Thus, 155.2 divided by 25 is 6.208.

? Exercise

Divide: 42.55 ÷ 23

Answer

1.85

Decimal Divisors When the divisor contains a decimal point, we have a little work to do before we begin the division process. Suppose that we wish to divide 1.25 by 2.5. In fraction form, we could start with

$$\frac{1.25}{2.5},$$

then clear the decimals from the denominator by multiplying both numerator and denominator by 10. *Note: Recall that multiplying by 10 moves the decimal point one place to the right.*

$$\begin{aligned} \frac{1.25}{2.5} &= \frac{1.25 \cdot 10}{2.5 \cdot 10} \\ &= \frac{12.5}{25} \end{aligned}$$

Thus, dividing 1.25 by 2.5 is equivalent to dividing 12.5 by 25. This we know how to do.

$$\begin{array}{r} 0.5 \\ 25\overline{)12.5} \\ \underline{12\,5} \\ 0 \end{array}$$

Thus, 1.25 divided by 2.5 is 0.5.

Writing Mathematics

Never write .5. Always add the leading zero in the ones place and write 0.5.

Instead of working in fraction form, we can take care of positioning the decimal point in the long division framework. Start with:

$$2.5\overline{)1.25}$$

Move the decimal point in the divisor to the end of the divisor, then move the decimal point in the dividend an equal number of places.

$$2.5\overline{)1.2\,5}$$

Thus, the division becomes

$$25\overline{)12.5}$$

and we proceed as above to find the quotient. This discussion motivates the following algorithm.

Dividing by a Decimal Divisor

If the divisor contains a decimal, proceed as follows:

1. Move the decimal to the end of the divisor.
2. Move the decimal in the dividend an equal number of places.

We can then complete the division using the rules for dividing a decimal by a whole number.

Example 3

Divide: $\overline{)4.392}$

Solution

Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (two places) to the right.

$$0.36\overline{)4.39\,2}$$

Now we can follow the algorithm for dividing a decimal number by a whole number.

$$\begin{array}{r} 12.2 \\ 36\overline{)439.2} \\ \underline{36} \\ 79 \\ \underline{72} \\ 7\,2 \\ \underline{7\,2} \\ 0 \end{array}$$

Thus, 4.392 divided by 0.36 is 12.2.

Exercise

Divide: $0.45\overline{)36.99}$

Answer

82.2

Dividing Signed Decimal Numbers

The rules governing division of signed decimal numbers are identical to the rules governing division of integers.

Like Signs. The quotient of two decimal numbers with like signs is positive. That is:

$$\frac{(+)}{(+)} = + \text{ and } \frac{(-)}{(-)} = +$$

Unlike Signs. The quotient of two decimal numbers with unlike signs is negative. That is:

$$\frac{(+)}{(-)} = - \text{ and } \frac{(-)}{(+)} = -$$

Example 4

Divide: −0.03 ÷ 0.024.

Solution

First, divide the magnitudes. Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (three places) to the right. Note that this requires an extra trailing zero in the dividend.

$$0.024\overline{)0.030}$$

Our problem then becomes:

\[24 \overline{24)30}\]

We can now follow the algorithm for dividing a decimal number by a whole number. Note that we have to add two trailing zeros in the dividend to complete the division with a zero remainder.

$$\begin{array}{r} 1.25 \\ 24\overline{)30.00} \\ \underline{24} \\ 60 \\ \underline{48} \\ 120 \\ \underline{120} \\ 0 \end{array}$$

Finally, because the quotient of unlike signs is negative, −0.03 divided by 0.024 is −1.25. That is,

$$\frac{-0.03}{0.024} = -1.25.$$

Exercise

Divide: −0.0113 ÷ 0.05

Answer

−0.226

Rounding

Sometimes an exact decimal representation of a fraction is not needed and is an approximation is more than adequate.

Example 5

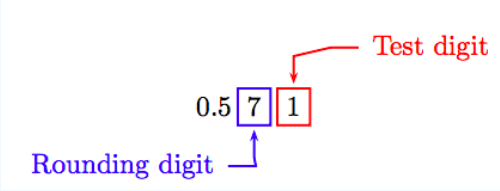
Convert 4/7 to a decimal. Round your answer to the nearest hundredth.

Solution

We need to carry the division one place beyond the hundredths place.

$$\begin{array}{r} 0.571 \\ 7\overline{)4.000} \\ \underline{3\,5} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 3 \end{array}$$

To round to the nearest hundredth, first identify the rounding and test digits.



Because the "test digit" is less 5, leave the rounding digit alone and truncate. Therefore, correct to the nearest hundredth, 4/7 ≈ 0.57.

Exercise

Convert 5/7 to a decimal. Round your answer to the nearest hundredth.

Answer

0.71

Dividing by Powers of Ten

Recall:

$$10^1 = 10$$
$$10^2 = 10 \cdot 10 = 100$$
$$10^3 = 10 \cdot 10 \cdot 10 = 1000$$

Powers of Ten

In the expression $10n$, the exponent matches the number of zeros in the answer. Hence, $10n$ will be a 1 followed by n zeros.

Thus, 10^4 = 10, 000, 10^5 = 100, 000, etc. The exponent tells us how many zeros will follow the 1.

Let's divide 123456.7 by 1000.

$$
\begin{array}{r}
123.4567 \\
1000\overline{)123456.7000} \\
\underline{1000} \\
2345 \\
\underline{2000} \\
3456 \\
\underline{3000} \\
456\,7 \\
\underline{400\,0} \\
56\,70 \\
\underline{50\,00} \\
6\,700 \\
\underline{6\,000} \\
7000 \\
\underline{7000} \\
0
\end{array}
$$

Note the result: 123456.7 divided by 1000 is 123.4567. Dividing by 1000 moves the decimal point 3 places to the left!

$$123456.7 \div 1000 = 123\,456.7 = 123.4567$$

This discussion leads to the following result.

Dividing a Decimal by a Power of Ten

Dividing a decimal number by $10n$ will move the decimal point n places to the left.

Example 6

Simplify: $123456.7 \div 10^4$

Solution

Dividing by 10^4 (or equivalently, 10,000) moves the decimal point four places to the left. Thus, $123456.7 \div 10^4 = 12.34567$.

Exercise

Simplify: $123456.7 \div 10^2$

Answer

1234.567

Order of Operations

We remind readers of the *Rules Guiding Order of Operations*.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

In addition, when fractions are present:

Fractional Expressions

If a fractional expression is present, simplify the numerator and denominator separately, then divide.

Example 7

Evaluate ab/(c + d), given that a = 2.1, b = −3.4, c = −1.3, and d = 1.1.

Solution

Recall that it is good practice to prepare an expression for substitution by using parentheses.

$$ab/(c+d) = (\,)(\,)/((\,)+(\,))$$

Substitute the given values for a, b, c, and d, then use the *Rules Guiding Order of Operations* to simplify the resulting expression.

$$\begin{aligned} ab/(c+d) &= (2.1)(-3.4)/((-1.3)+(1.1)) && 2.1, -3.4, -1.3, 1.1 \text{ for } a, b, c, d. \\ &= (2.1)(-3.4)/(-0.2) && \text{Parens: } (-1.3)+(1.1) = -0.2. \\ &= -7.14/(-0.2) && \text{Multiply: } (2.1)(-3.4) = -7.14. \\ &= 35.7 && \text{Divide: } -7.14/(-0.2) = 35.7 \end{aligned}$$

Exercise

If a = −2.1, b = 1.7, c = 4, and d = 0.05, evaluate:

$$\frac{a+b}{cd}$$

Answer

−2

Example 8

Given a = 0.1 and b = −0.3, evaluate the expression

$$\frac{a+2b}{2a+b}.$$

Solution

Substitute the given values, then use the *Rules Guiding Order of Operations* to simplify the resulting expression.

$$\frac{a+2b}{2a+b} = \frac{(0.1)+2(-0.3)}{2(0.1)+(-0.3)} \quad 0.1 \text{ for } a, \ -0.3 \text{ for } b.$$

Simplify the numerator, simplify the denominator, then divide.

$$\begin{aligned} &= \frac{0.1+(-0.6)}{0.2+(-0.3)} && \text{Numerator: } 2(-0.3) = -0.6. \\ &= \frac{-0.5}{-0.1} && \text{Denominator: } 2(0.1) = 0.2. \\ &= 5 && \text{Numerator: } 0.1+(-0.6) = -0.5 \\ & && \text{Denominator: } 0.2+(-0.3) = -0.1. \\ & && \text{Divide: } -0.5/(-0.1) = 5. \end{aligned}$$

Exercise

If a = 0.5 and b = −0.125, evaluate:

$$\frac{2a-b}{a+2b}$$

Answer

4.5

Exercises

In Exercises 1-16, divide the numbers.

1. $\frac{39}{52}$
2. $\frac{16}{25}$
3. $\frac{755.3}{83}$
4. $\frac{410.4}{76}$
5. $\frac{333}{74}$
6. $\frac{117}{65}$
7. $\frac{32.12}{73}$
8. $\frac{12.32}{44}$
9. $\frac{37.63}{71}$
10. $\frac{20.46}{31}$
11. $\frac{138}{92}$
12. $\frac{110}{25}$
13. $\frac{17}{25}$
14. $\frac{18}{75}$
15. $\frac{229.5}{51}$
16. $\frac{525.6}{72}$

In Exercises 17-40, divide the decimals.

17. $\frac{0.3478}{0.47}$
18. $\frac{0.4559}{0.97}$
19. $\frac{1.694}{2.2}$
20. $\frac{1.008}{1.8}$
21. $\frac{43.61}{4.9}$
22. $\frac{22.78}{3.4}$
23. $\frac{1.107}{0.41}$
24. $\frac{2.465}{0.29}$
25. $\frac{2.958}{0.51}$
26. $\frac{5.141}{0.53}$

27. $\frac{71.76}{7.8}$
28. $\frac{14.08}{8.8}$
29. $\frac{0.8649}{0.93}$
30. $\frac{0.3901}{0.83}$
31. $\frac{0.6958}{0.71}$
32. $\frac{0.1829}{0.31}$
33. $\frac{1.248}{0.52}$
34. $\frac{6.375}{0.85}$
35. $\frac{62.56}{9.2}$
36. $\frac{28.08}{7.8}$
37. $\frac{6.278}{8.6}$
38. $\frac{3.185}{4.9}$
39. $\frac{2.698}{7.1}$
40. $\frac{4.959}{8.7}$

In Exercises 41-64, divide the decimals.

41. $\frac{-11.04}{1.6}$
42. $\frac{-31.27}{5.3}$
43. $\frac{-3.024}{5.6}$
44. $\frac{-3.498}{5.3}$
45. $\frac{-0.1056}{0.22}$
46. $\frac{-0.2952}{-0.72}$
47. $\frac{0.3204}{-0.89}$
48. $\frac{0.3306}{-0.38}$
49. $\frac{-1.419}{0.43}$
50. $\frac{-1.625}{-0.25}$
51. $\frac{-16.72}{-2.2}$
52. $\frac{-66.24}{9.2}$
53. $\frac{-2.088}{-0.87}$
54. $\frac{-2.025}{-0.75}$
55. $\frac{-1.634}{-8.6}$
56. $\frac{-3.094}{3.4}$
57. $\frac{-0.119}{0.85}$
58. $\frac{0.5766}{-0.62}$
59. $\frac{-3.591}{-6.3}$
60. $\frac{-3.016}{5.8}$

61. $\frac{36.96}{-4.4}$
62. $\frac{-78.26}{-8.6}$
63. $\frac{-2.156}{-0.98}$
64. $\frac{-6.072}{0.66}$

In Exercises 65-76, divide the decimal by the given power of 10.

65. $\frac{524.35}{100}$
66. $\frac{849.39}{100}$
67. $\frac{563.94}{10^3}$
68. $\frac{884.15}{10^3}$
69. $\frac{116.81}{10^2}$
70. $\frac{578.01}{10^3}$
71. $\frac{694.55}{10}$
72. $\frac{578.68}{100}$
73. $\frac{341.16}{10^3}$
74. $\frac{46.63}{10^4}$
75. $\frac{113.02}{1000}$
76. $\frac{520.77}{1000}$

---

77. Compute the quotient 52/83, and round your answer to the nearest tenth.
78. Compute the quotient 43/82, and round your answer to the nearest tenth.
79. Compute the quotient 51/59, and round your answer to the nearest tenth.
80. Compute the quotient 17/69, and round your answer to the nearest tenth.
81. Compute the quotient 5/74, and round your answer to the nearest hundredth.
82. Compute the quotient 3/41, and round your answer to the nearest hundredth.
83. Compute the quotient 5/94, and round your answer to the nearest hundredth.
84. Compute the quotient 3/75, and round your answer to the nearest hundredth.
85. Compute the quotient 7/72, and round your answer to the nearest hundredth.
86. Compute the quotient 4/57, and round your answer to the nearest hundredth.
87. Compute the quotient 16/86, and round your answer to the nearest tenth.
88. Compute the quotient 21/38, and round your answer to the nearest tenth.

In Exercises 89-100, simplify the given expression.

89. $\frac{7.5 \cdot 7.1 - 19.5}{0.54}$
90. $\frac{1.5(-8.8) - (-18.6)}{1.8}$
91. $\frac{17.76 - (-11.7)}{0.52}$
92. $\frac{-14.8 - 2.1}{2.62}$
93. $\frac{-18.22 - 6.7}{14.75 - 7.75}$

94. $\frac{1.4-13.25}{-6.84-(-2.1)}$

95. $\frac{-12.9-(-10.98)}{0.5^2}$

96. $\frac{5.1-(-16.5)}{(-1.5)^2}$

97. $\frac{-9.5 \cdot 1.6-3.7}{-3.6}$

98. $\frac{6.5(-1.6)-3.35}{-2.75}$

99. $\frac{-14.98-9.6}{17.99-19.99}$

100. $\frac{-5.6-7.5}{-5.05-1.5}$

---

101. Given a = −2.21, c = 3.3, and d = 0.5, evaluate and simplify the following expression.

$$\frac{a-c}{d^2}$$

102. Given a = 2.8, c = −14.68, and d = 0.5, evaluate and simplify the following expression.

$$\frac{a-c}{d^2}$$

103. Given a = −5.8, b = 10.37, c = 4.8, and d = 5.64, evaluate and simplify the following expression:

$$\frac{a-b}{c-d}$$

104. Given a = −10.79, b = 3.94, c = −3.2, and d = −8.11, evaluate and simplify the following expression:

$$\frac{a-b}{c-d}$$

105. Given a = −1.5, b = 4.7, c = 18.8, and d = −11.75, evaluate and simplify the following expression.

$$\frac{ab-c}{d}$$

106. Given a = 9.3, b = 6.6, c = 14.27, and d = 0.2, evaluate and simplify the following expression.

$$\frac{ab-c}{d}$$

---

107. **Biodiesel plants.** There are about 180 biodiesel plants operating in about 40 states. Of the states that have them, what is the average number of biodiesel plants per state? *Associated Press-Times-Standard 01/02/10 Fledgling biofuel industry ends year on a dour note*.

108. **Bat fungus.** A fungus called ”white-nose syndrome” has killed an estimated 500,000 bats throughout the country. This means about 2,400,000 pounds of bugs aren’t eaten over the year, says Forest Service biologist Becky Ewing. How many pounds of insects does an average bat eat annually? *Associated Press-Times-Standard 5/2/09*

109. **Patent backlog.** In the U.S. Patent and Trademark Office, 6000 examiners have a backlog of 770,000 new, unexamined applications for patents. How many applications is that for each examiner to catch up on? Round your answer to the nearest tenth. *Associated Press-Times-Standard 5/5/09*

110. **Doing well.** The large health insurer Wellpoint, Inc., owner of Anthem Blue Cross, earned $536 million in the last three months of 2009. What was the average earnings per month for the insurer over that period? Round to the nearest million. *Associated Press-Times-Standard 02/09/10 HHS secretary asks Anthem Blue Cross to justify rate hike.*

111. **Cyber attacks.** The Pentagon has spent $100 million over a six-month period responding to and repairing damage from cyber-attacks and other computer network problems. What’s the average amount of money spent per month over that time? Round your answer to the nearest hundredth of a million. *Associated Press-Times-Standard 4/19/09*

112. **Daily milk.** The average California cow can produce 2,305 gallons of milk annually. How much milk can a cow produce each day? Round your answer to the nearest hundredth of a gallon. www.moomilk.com/faq.htm

113. **Media mail.** To promote her business, Theresa mails several packages via Media Mail. One package weighing 2 lbs. costs $2.77, another package weighing 3 lbs. costs $3.16, and the third package weighing 5 lbs. costs $3.94 to mail. What was the average cost per pound to mail the packages? Round your result to the nearest penny. www.usps.com/prices/media-mailprices.htm

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Answers

1. 0.75

3. 9.1

5. 4.5

7. 0.44

9. 0.53

11. 1.5

13. 0.68

15. 4.5

17. 0.74

19. 0.77

21. 8.9

23. 2.7

25. 5.8

27. 9.2

29. 0.93

31. 0.98

33. 2.4

35. 6.8

37. 0.73

39. 0.38

41. −6.9

43. −0.54

45. −0.48

47. −0.36

49. −3.3

51. 7.6

53. 2.4

55. 0.19

57. −0.14

59. 0.57

61. −8.4

63. 2.2

65. 5.2435

67. 0.56394

69. 1.1681

71. 69.455

73. 0.34116

75. 0.11302

77. 0.6

79. 0.9

81. 0.07

83. 0.05

85. 0.10

87. 0.2

89. 62.5

91. 117.84

93. −3.56

95. −7.68

97. 5.25

99. 12.29

101. −22.04

103. 19.25

105. 2.2

107. 4.5 biodiesel plants

109. 128.3

111. $16.67 million

113. $0.99 per pound

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3.6: Fractions and Decimals

When converting a fraction to a decimal, only one of two things can happen. Either the process will terminate or the decimal representation will begin to repeat a pattern of digits. In each case, the procedure for changing a fraction to a decimal is the same.

Changing a Fraction to a Decimal

To change a fraction to a decimal, divide the numerator by the denominator. *Hint: If you first reduce the fraction to lowest terms, the numbers will be smaller and the division will be a bit easier as a result.*

Terminating Decimals

Terminating Decimals

First reduce the fraction to lowest terms. If the denominator of the resulting fraction has a prime factorization consisting of strictly twos and/or fives, then the decimal representation will "terminate."

Example 1

Change 15/48 to a decimal.

Solution

First, reduce the fraction to lowest terms.

$$\frac{15}{48} = \frac{3 \cdot 5}{3 \cdot 16} = \frac{5}{16}$$

Next, note that the denominator of 5/16 has prime factorization 16 = 2·2·2·2. It consists only of twos. Hence, the decimal representation of 5/16 should terminate.

$$\begin{array}{r} 0.3125 \\ 16\overline{)5.0000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{32} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

The zero remainder terminates the process. Hence, 5/16 = 0.3125.

Exercise

Change 10/16 to a decimal.

Answer

0.625

Example 2

Change $3\frac{7}{20}$ to a decimal.

Solution

Note that 7/20 is reduced to lowest terms and its denominator has prime factorization 20 = 2 · 2 · 5. It consists only of twos and fives. Hence, the decimal representation of 7/20 should terminate.

$$\begin{array}{r} 0.35 \\ 20\overline{)7.00} \\ \underline{6\,0} \\ 1\,00 \\ \underline{1\,00} \\ 0 \end{array}$$

The zero remainder terminates the process. Hence, 7/20 = 0.35. Therefore, $3\frac{7}{20}$ = 3.35.

Exercise

Change $7\frac{11}{20}$ to a decimal.

Answer

7.55

Repeating Decimals

Repeating Decimals

First reduce the fraction to lowest terms. If the prime factorization of the resulting denominator does not consist strictly of twos and fives, then the division process will never have a remainder of zero. However, repeated patterns of digits must eventually reveal themselves.

Example 3

Change 1/12 to a decimal.

Solution

Note that 1/12 is reduced to lowest terms and the denominator has a prime factorization 12 = 2 · 2 · 3 that does not consist strictly of twos and fives. Hence, the decimal representation of 1/12 will not "terminate." We need to carry out the division until a remainder reappears for a second time. This will indicate repetition is beginning.

$$\begin{array}{r} .083 \\ 12\overline{)1.000} \\ \underline{96} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

Note the second appearance of 4 as a remainder in the division above. This is an indication that repetition is beginning. However, to be sure, let's carry the division out for a couple more places.

$$\begin{array}{r} .08333 \\ 12\overline{)1.00000} \\ \underline{96} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

Note how the remainder 4 repeats over and over. In the quotient, note how the digit 3 repeats over and over. It is pretty evident that if we were to carry out the division a few more places, we would get

$$\frac{1}{12} = 0.833333\cdots$$

The ellipsis is a symbolic way of saying that the threes will repeat forever. It is the mathematical equivalent of the word "etcetera."

? Exercise

Change 5/12 to a decimal.

Answer

0.41666...

There is an alternative notation to the ellipsis, namely

$$\frac{1}{12} = 0.08\bar{3}.$$

The bar over the 3 (called a "repeating bar") indicates that the 3 will repeat indefinitely. That is,

$$0.08\bar{3} = 0.083333\ldots.$$

Using the Repeating Bar

To use the repeating bar notation, take whatever block of digits are under the repeating bar and duplicate that block of digits infinitely to the right.

Thus, for example:

- $5.\overline{345} = 5.3454545\ldots.$
- $0.\overline{142857} = 0.142857142857142857\ldots.$

Important Observation

Although $0.8\overline{33}$ will also produce 0.8333333 ..., as a rule we should use as few digits as possible under the repeating bar. Thus, $0.8\bar{3}$ is preferred over $0.8\overline{33}$.

Example 4

Change 23/111 to a decimal.

Solution

The denominator of 23/111 has prime factorization 111 = 3 ·37 and does **not** consist strictly of twos and fives. Hence, the decimal representation will not "terminate." We need to perform the division until we spot a repeated remainder.

$$\begin{array}{r} 0.207 \\ 111\overline{)23.000} \\ 22\,2 \\ \hline 800 \\ 777 \\ \hline 23 \end{array}$$

Note the return of 23 as a remainder. Thus, the digit pattern in the quotient should start anew, but let's add a few places more to our division to be sure.

$$\begin{array}{r} 0.207207 \\ 111\overline{)23.000000} \\ 22\,2 \\ \hline 800 \\ 777 \\ \hline 230 \\ 222 \\ \hline 800 \\ 777 \\ \hline 23 \end{array}$$

Aha! Again a remainder of 23. Repetition! At this point, we are confident that

$$\frac{23}{111} = 0.207207\ldots.$$

Using a "repeating bar," this result can be written

$$\frac{23}{111} = 0.\overline{207}.$$

Exercise

Change 5/33 to a decimal.

Answer

0.151515...

Expressions Containing Both Decimals and Fractions

At this point we can convert fractions to decimals, and vice-versa, we can convert decimals to fractions. Therefore, we should be able to evaluate expressions that contain a mix of fraction and decimal numbers.

Example 5

Simplify: $-\frac{3}{8} - 1.25$.

Solution

Let's change 1.25 to an improper fraction.

\[\begin{aligned} 1.25 = \frac{125}{100} ~ & \textcolor{red}{ \text{ Two decimal places } \Rightarrow \text{ two zeroes.} } \\ = \frac{5}{4} ~ & \textcolor{red}{ \text{ Reduce to lowest terms.} } \end{aligned}\nonumber \]

In the original problem, replace 1.25 with 5/4, make equivalent fractions with a common denominator, then subtract.

$$\begin{aligned} -\frac{3}{8} - 1.25 &= -\frac{3}{8} - \frac{5}{4} && \text{Replace } 1.25 \text{ with } 5/4. \\ &= -\frac{3}{8} - \frac{5 \cdot 2}{4 \cdot 2} && \text{Equivalent fractions, LCD} = 8. \\ &= -\frac{3}{8} - \frac{10}{8} && \text{Simplify the numerator and denominator.} \\ &= -\frac{3}{8} + \left(-\frac{10}{8}\right) && \text{Add the opposite.} \\ &== \frac{13}{8} && \text{Add.} \end{aligned}$$

Thus, −3/8 − 1.25 = −13/8.

Alternate Solution

Because −3/8 is reduced to lowest terms and 8 = 2 ·2 ·2 consists only of twos, the decimal representation of −3/8 will terminate.

$$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{2\,4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \end{array}$$

Hence, −3/8 = −0.375. Now, replace −3/8 in the original problem with −0.375, then simplify.

$$\begin{aligned} -\frac{3}{8} - 1.25 &= -0.375 - 1.25 && \text{Replace } -3/8 \text{ with } -0.375. \\ &= -0.375 + (-1.25) && \text{Add the opposite.} \\ &= -1.625 && \text{Add.} \end{aligned}$$

Thus, −3/8 − 1.25 = −1.625.

Are They the Same?

The first method produced −13/8 as an answer; the second method produced −1.625. Are these the same results? One way to find out is to change −1.625 to an improper fraction.

$$\begin{aligned} -1.625 &= -\frac{1625}{1000} && \text{Three places } \Rightarrow \text{ three zeroes.} \\ &= -\frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 13}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} && \text{Prime factor.} \\ &= -\frac{13}{2 \cdot 2 \cdot 2} && \text{Cancel common factors.} \\ &= -\frac{13}{8} && \text{Simplify.} \end{aligned}$$

Thus, the two answers are the same.

Exercise

Simplify: $-\frac{7}{8} - 6.5$

Answer

$-7\frac{3}{8}$ or -7.375

Example 6

Simplify: $-\frac{2}{3}+0.35$.

Solution

Let's attack this expression by first changing 0.35 to a fraction.

$$-\frac{2}{3}+0.35=-\frac{2}{3}+\frac{35}{100} \quad \text{Change } 0.35 \text{ to a fraction.}$$
$$=-\frac{2}{3}+\frac{7}{20} \quad \text{Reduce } 35/100 \text{ to lowest terms.}$$

Find an LCD, make equivalent fractions, then add.

$$=-\frac{2\cdot 20}{3\cdot 20}+\frac{7\cdot 3}{20\cdot 3} \quad \text{Equivalent fractions with LCD} = 60.$$
$$=-\frac{40}{60}+\frac{21}{60} \quad \text{Simplify numerators and denominators.}$$
$$=-\frac{19}{60} \quad \text{Add.}$$

Then, $-\frac{2}{3}+0.35=-\frac{19}{60}$.

Exercise

Simplify: $-\frac{4}{9}+0.25$

Answer

−7/36

In Example 6, we run into trouble if we try to change −2/3 to a decimal. The decimal representation for −2/3 is a repeating decimal (the denominator is not made up of only twos and fives). Indeed, −2/3 = $-0.\bar{6}$. To add $-0.\bar{6}$ and 0.35, we have to align the decimal points, then begin adding at the right end. But $-0.\bar{6}$ has no right end! This observation leads to the following piece of advice.

Important Observation

When presented with a problem containing both decimals and fractions, if the decimal representation of any fraction repeats, its best to first change all numbers to fractions, then simplify.

Exercises

In Exercises 1-20, convert the given fraction to a terminating decimal.

1. $\frac{59}{16}$
2. $\frac{19}{5}$
3. $\frac{35}{4}$
4. $\frac{21}{4}$
5. $\frac{1}{16}$
6. $\frac{14}{5}$
7. $\frac{6}{8}$
8. $\frac{7}{175}$

9. $\frac{3}{2}$
10. $\frac{15}{16}$
11. $\frac{119}{175}$
12. $\frac{4}{8}$
13. $\frac{9}{8}$
14. $\frac{5}{2}$
15. $\frac{78}{240}$
16. $\frac{150}{96}$
17. $\frac{25}{10}$
18. $\frac{2}{4}$
19. $\frac{9}{24}$
20. $\frac{216}{150}$

In Exercises 21-44, convert the given fraction to a repeating decimal. Use the “repeating bar” notation.

21. $\frac{256}{180}$
22. $\frac{268}{180}$
23. $\frac{364}{12}$
24. $\frac{292}{36}$
25. $\frac{81}{110}$
26. $\frac{82}{99}$
27. $\frac{76}{15}$
28. $\frac{23}{9}$
29. $\frac{50}{99}$
30. $\frac{53}{99}$
31. $\frac{61}{15}$
32. $\frac{37}{18}$
33. $\frac{98}{66}$
34. $\frac{305}{330}$
35. $\frac{190}{495}$
36. $\frac{102}{396}$
37. $\frac{13}{15}$
38. $\frac{65}{36}$
39. $\frac{532}{21}$
40. $\frac{44}{60}$
41. $\frac{26}{198}$
42. $\frac{686}{231}$

43. $\frac{47}{66}$
44. $\frac{41}{198}$

In Exercises 45-52, simplify the given expression by first converting the fraction into a terminating decimal.

45. $\frac{7}{4} - 7.4$
46. $\frac{3}{2} - 2.73$
47. $\frac{7}{5} + 5.31$
48. $-\frac{7}{4} + 3.3$
49. $\frac{9}{10} - 8.61$
50. $\frac{3}{4} + 3.7$
51. $\frac{6}{5} - 7.65$
52. $-\frac{3}{10} + 8.1$

In Exercises 53-60, simplify the given expression by first converting the decimal into a fraction.

53. $\frac{7}{6} - 2.9$
54. $-\frac{11}{6} + 1.12$
55. $-\frac{4}{3} - 0.32$
56. $\frac{11}{6} - 0.375$
57. $-\frac{2}{3} + 0.9$
58. $\frac{2}{3} - 0.1$
59. $\frac{4}{3} - 2.6$
60. $-\frac{5}{6} + 2.3$

In Exercises 61-64, simplify the given expression.

61. $\frac{5}{6} + 2.375$
62. $\frac{5}{3} + 0.55$
63. $\frac{11}{8} + 8.2$
64. $\frac{13}{8} + 8.4$
65. $-\frac{7}{10} + 1.2$
66. $-\frac{7}{5} - 3.34$
67. $-\frac{11}{6} + 0.375$
68. $\frac{5}{3} - 1.1$

Answers

1. 3.6875
3. 8.75
5. 0.0625
7. 0.75
9. 1.5

11. 0.68

13. 1.125

15. 0.325

17. 2.5

19. 0.375

21. $1.4\overline{2}$

23. $30.\overline{3}$

25. $0.7\overline{36}$

27. $5.0\overline{6}$

29. $0.\overline{50}$

31. $4.0\overline{6}$

33. $1.\overline{48}$

35. $0.\overline{38}$

37. $0.8\overline{6}$

39. $25.\overline{3}$

41. $0.\overline{13}$

43. $0.7\overline{12}$

45. -5.65

47. 6.71

49. -7.71

51. -6.45

53. $-\frac{26}{15}$

55. $-\frac{124}{75}$

57. $\frac{7}{30}$

59. $-\frac{19}{15}$

61. $\frac{77}{24}$

63. 9.575

65. 0.5

67. $-\frac{35}{24}$

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CHAPTER OVERVIEW

4: Ratio, Proportion, Percent, and Fractions

4.1: Ratio and Proportion
4.2: Introduction to Ratios and Rates
4.3: Introduction to Proportion
4.4: Percent
4.5: Percent, Decimals, Fractions
4.6: Solving Basic Percent Problems
4.7: General Applicants of Percent
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4.9: Interest

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4.1: Ratio and Proportion

From the beginnings of the human race, we've long compared one quantity with another, a comparison that is called a ratio in mathematics. "Their tribe has twice as many cattle as ours" or "Two baskets of wheat cost 12 ducats" are examples of ratios that ring from distant times. Indeed, the concept of a ratio cannot be assigned to any one individual or class of individual. In his *History of Mathematics*, D. E. Smith writes:

> *It is rather profitless to speculate as to the domain in which the concept of ratio first appeared. The idea that one tribe is twice as large as another and the idea that one leather strap is only half as long as another both involve the notion of ratio; both are such as would develop early in the history of the race, and yet one has to do with ratio of numbers and the other with the ratio of geometric magnitudes. Indeed, when we come to the Greek writers we find Nicomachus including ratio in his arithmetic, Eudoxus in his geometry, and Theon of Smyrna in his chapter on music.*

Examples and applications of ratios are limitless: speed is a ratio that compares changes in distance with respect to time, acceleration is a ratio that compares changes in speed with respect to time, and percentages compare the part with the whole. We've already studied one classic ratio, the ratio of the circumference of a circle to its diameter, which gives us the definition of π.

One of the most famous ratios in history involves the division of a line segment AB into two segments AC and CB by selecting a point C on the segment AB.

The idea is to select a point C on the segment AB so that

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

This ratio has a special name, the **Golden Ratio**, and has an exact value equal to $(1+\sqrt{5})/2$. The Golden Ratio has been known since the time of Euclid. Ancient and modern architects have long held that the most pleasing rectangular shape is the one whose ratio of length to width is equal to the Golden Ratio.

The comparison of two ratios, such as $AB/AC = AC/CB$, is called a *proportion*. Proportions are used in a number of practical ways. For example, if 5 cans of tomato sauce cost 2 dollars, we can find the number of cans that can be purchased with 10 dollars by comparing two ratios in a proportion:

$$\frac{5 \text{ cans of tomato sauce}}{2 \text{ dollars}} = \frac{\text{x cans of tomato sauce}}{10 \text{ dollars}}$$

Any discussion of ratio involves comparing two quantities, so the units of each quantity become extremely important. Two different systems of units are used when measuring length, capacity, and time: the *American* system of units and the *metric* system of units. In this chapter we will discuss both systems and explain how to convert quantities measured in one system to quantities measured in the other system.

Let's begin the journey.

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4.2: Introduction to Ratios and Rates

We use *ratios* to compare two numeric quantities or quantities with the same units.

Ratio

A *ratio* is the quotient of two numerical quantities or two quantities with the same physical units.

For example, ancient Greek geometers believed that the most pleasing rectangle to the eye had length and width such that the ratio of length to width was a specific number, called the *Golden Ratio*, approximately equal to 1.6180339887.... Architects to this day use this ratio in their designs.

There are a number of equivalent ways of expressing ratios, three of which we will use in this text: fraction notation, “to” notation, and “colon” notation.

- 3/4 is a ratio, read as “the ratio of 3 to 4.”
- 3 to 4 is a ratio, read as “the ratio of 3 to 4.”
- 3:4 is a ratio, read as “the ratio of 3 to 4.”

Example 1

Express each of the following ratios as a fraction reduced to lowest terms: (a) 36 to 24, and (b) 0.12 : 0.18.

Solution

(a) To express the ratio “36 to 24” as a fraction, place 36 over 24 and reduce.

$$\begin{aligned}\frac{36}{24} &= \frac{3\cdot 12}{2\cdot 12} && \text{Factor.}\\ &= \frac{3\cdot \cancel{12}}{2\cdot \cancel{12}} && \text{Cancel common factor.}\\ &= \frac{3}{2}\end{aligned}$$

Thus, the ratio 36 to 24 equals 3/2.

(b) To express the ratio “0.12:0.18” as a fraction, place 0.12 over 0.18 and reduce.

$$\begin{aligned}\frac{0.12}{0.18} &= \frac{(0.18)(100)}{(0.18)(100)} && \text{Multiply numerator and denominator by 100.}\\ &= \frac{12}{18} && \text{Move each decimal 2 places right.}\\ &= \frac{2\cdot 6}{3\cdot 6} && \text{Factor.}\\ &= \frac{2\cdot \cancel{6}}{3\cdot \cancel{6}} && \text{Cancel.}\\ &= \frac{2}{3}\end{aligned}$$

Thus, the ratio 0.12:0.18 equals 2/3.

Exercise

Express 0.12 : 0.3 as a fraction reduced to lowest terms.

Answer

2/5

Example 2

For the rectangle that follows, express the ratio of length to width as a fraction reduced to lowest terms.

Solution.

The ratio length to width can be expressed as a fraction and reduced as follows.

$$\frac{\text{length}}{\text{width}} = \frac{3\frac{1}{4}\text{ ft}}{2\frac{1}{2}\text{ ft}} \quad \text{Length to width as a fraction.}$$

$$= \frac{3\frac{1}{4}\ \cancel{\text{ft}}}{2\frac{1}{2}\ \cancel{\text{ft}}} \quad \text{Cancel common units.}$$

$$= \frac{\frac{13}{4}}{\frac{5}{2}} \quad \text{Mixed to improper fractions.}$$

Invert and multiply, factor, and cancel common factors.

$$= \frac{13}{4} \cdot \frac{2}{5} \quad \text{Invert and multiply.}$$

$$= \frac{26}{20} \quad \text{Multiply numerators and denominators.}$$

$$= \frac{13 \cdot 2}{10 \cdot 2} \quad \text{Factor numerator and denominator.}$$

$$= \frac{13 \cdot \cancel{2}}{10 \cdot \cancel{2}} \quad \text{Cancel common factors.}$$

$$= \frac{13}{10}$$

Hence, the ratio length to width is 13/10.

Exercise

A rectangle has length $8\frac{1}{4}$ inches and width $3\frac{1}{2}$ inches. Express the ratio of length to width as a fraction reduced to lowest terms.

Answer

33/14

Rates

We now introduce the concept of *rate*, a special type of ratio.

Rate

A *rate* is a quotient of two measurements with different units.

The physical interpretation of a rate in terms of its units is an important skill.

Example 3

An automobile travels 224 miles on 12 gallons of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms. Write a short sentence explaining the physical significance of your solution. Include units in your description.

Solution

Place miles traveled over gallons of gasoline consumed and reduce.

$$\begin{aligned} \frac{224 \text{ mi}}{12 \text{ gal}} &= \frac{56 \cdot 4 \text{ mi}}{3 \cdot 4 \text{ gal}} && \text{Factor.} \\ &= \frac{56 \cdot \cancel{4} \text{ mi}}{3 \cdot \cancel{4} \text{ gal}} && \text{Cancel common factor.} \\ &= \frac{56 \text{ mi}}{3 \text{ gal}} \end{aligned}$$

Thus, the rate is 56 miles to 3 gallons of gasoline. In plain-speak, this means that the automobile travels 56 miles on 3 gallons of gasoline.

Exercise

Lanny travels 180 kilometers on 14 liters of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms.

Answer

90/7 kilometers per litre

Unit Rates

When making comparisons, it is helpful to have a rate in a form where the denominator is 1. Such rates are given a special name.

Unit Rate

A *unit rate* is a rate whose denominator is 1.

Example 4

Herman drives 120 miles in 4 hours. Find his average rate of speed.

Solution

Place the distance traveled over the time it takes to drive that distance.

$$\begin{aligned} \frac{120 \text{ miles}}{4 \text{ hours}} &= \frac{30 \text{ miles}}{1 \text{ hour}} && \text{Divide: } 120/4 = 30. \\ &= 30 \text{ miles/hour} \end{aligned}$$

Hence, Herman's average rate of speed is 30 miles per hour.

Exercise

Jacob drives 120 kilometers in 3 hours. Find his average rate of speed.

Answer

40 kilometers per hour

Example 5

Aditya works 8.5 hours and receives $95 for his efforts. What is his hourly salary rate?

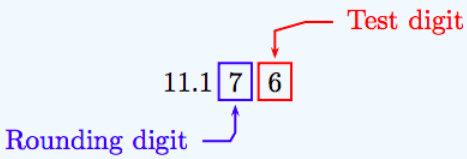
Solution

Let's place money earned over hours worked to get the following rate:

$$\frac{95 \text{ dollars}}{8.5 \text{ hours}}$$

We will get a much better idea of Aditya's salary rate if we express the rate with a denominator of 1. To do so, divide. Push the decimal in the divisor to the far right, then move the decimal an equal number of places in the dividend. As we are dealing with dollars and cents, we will round our answer to the nearest hundredth.

$$\begin{array}{r} 11.176 \\ 85\overline{)950.000} \\ \underline{85} \\ 100 \\ \underline{85} \\ 15\,0 \\ \underline{8\,5} \\ 6\,50 \\ \underline{5\,95} \\ 550 \\ \underline{510} \\ 40 \end{array}$$



Because the test digit is greater than or equal to 5, we add 1 to the rounding digit and truncate; i.e., 95/8.5 ≈ 11.18. Hence,

$$\begin{aligned} \frac{95 \text{ dollars}}{8.5 \text{ hours}} &= \frac{11.18 \text{ dollars}}{1 \text{ hour}} \quad \text{Divide: } 95/8.5 \approx 11.18. \\ &= 11.18 \text{ dollars/hour}. \end{aligned}$$

That is, his salary rate is 11.18 dollars per hour.

Exercise

Frannie works 5.5 hours and receives $120 for her efforts. What is her hourly salary rate? Round your answer to the nearest penny.

Answer

$21.82 per hour

Example 6

One automobile travels 422 miles on 15 gallons of gasoline. A second automobile travels 354 miles on 13 gallons of gasoline. Which automobile gets the better gas mileage?

Solution

Decimal division (rounded to the nearest tenth) reveals the better gas mileage.

In the case of the first automobile, we get the following rate:

$$\frac{422 \text{ mi}}{15 \text{gal}}$$

Divide.

$$\begin{array}{r} 28.13 \\ 15\overline{)422.00} \\ \underline{30} \\ 122 \\ \underline{120} \\ 20 \\ \underline{15} \\ 50 \\ \underline{45} \\ 5 \end{array}$$

To the nearest tenth, 28.1.

In the case of the second autombile, we get the following rate:

$$\frac{354 \text{ mi}}{13 \text{ gal}}$$

Divide.

$$\begin{array}{r} 27.23 \\ 13\overline{)354.00} \\ \underline{26} \\ 94 \\ \underline{91} \\ 30 \\ \underline{26} \\ 40 \\ \underline{39} \\ 1 \end{array}$$

To the nearest tenth, 27.2.

In the case of the first automobile, the mileage rate is 28.1 mi/1 gal, which can be read “28.1 miles per gallon.” In the case of the second automobile, the mileage rate is 27.2 mi/1 gal, which can be read “27.2 miles per gallon.” Therefore, the first automobile gets the better gas mileage.

? Exercise

Alicia works 8 hours and makes $100. Connie works 10 hours and makes $122. Which woman works at the larger hourly rate?

Answer

Alicia

Exercises

In Exercises 1-24, express the given ratio as a fraction reduced to lowest terms.

1. 0.14 : 0.44
2. 0.74 : 0.2
3. 0.05 : 0.75
4. 0.78 : 0.4
5. 0.1:0.95
6. 0.93 : 0.39
7. $2\frac{2}{9} : 1\frac{1}{3}$
8. $3\frac{2}{3} : 2\frac{4}{9}$
9. 0.36 : 0.6
10. 0.58 : 0.42
11. 15 : 21
12. 77 : 121
13. $2\frac{8}{9} : 2\frac{2}{3}$
14. $1\frac{2}{3} : 3\frac{8}{9}$

15. $3\frac{8}{9} : 2\frac{1}{3}$

16. $1\frac{5}{9} : 1\frac{1}{3}$

17. $2\frac{5}{8} : 1\frac{3}{4}$

18. $2\frac{4}{9} : 1\frac{1}{3}$

19. 10 : 35

20. 132 : 84

21. 9 : 33

22. 35 : 10

23. 27 : 99

24. 12 : 28

---

25. One automobile travels 271.8 miles on 10.1 gallons of gasoline. A second automobile travels 257.9 miles on 11.1 gallons of gasoline. Which automobile gets the better gas mileage?

26. One automobile travels 202.9 miles on 13.9 gallons of gasoline. A second automobile travels 221.6 miles on 11.8 gallons of gasoline. Which automobile gets the better gas mileage?

27. Todd is paid 183 dollars for 8.25 hours work. What is his hourly salary rate, rounded to the nearest penny?

28. David is paid 105 dollars for 8.5 hours work. What is his hourly salary rate, rounded to the nearest penny?

29. An automobile travels 140 miles in 4 hours. Find the average rate of speed.

30. An automobile travels 120 miles in 5 hours. Find the average rate of speed.

31. Judah is paid 187 dollars for 8 hours work. What is his hourly salary rate, rounded to the nearest penny?

32. Judah is paid 181 dollars for 8.75 hours work. What is his hourly salary rate, rounded to the nearest penny?

33. One automobile travels 234.2 miles on 10.8 gallons of gasoline. A second automobile travels 270.5 miles on 10.8 gallons of gasoline. Which automobile gets the better gas mileage?

34. One automobile travels 297.6 miles on 10.7 gallons of gasoline. A second automobile travels 298.1 miles on 12.6 gallons of gasoline. Which automobile gets the better gas mileage?

35. An automobile travels 180 miles in 5 hours. Find the average rate of speed.

36. An automobile travels 220 miles in 5 hours. Find the average rate of speed.

---

37. **Antarctic trek.** Seven women on a 562-mile Antarctic ski trek reached the South Pole 38 days after they began their adventure. What was the ladies' average rate of speed per day? Round your result to the nearest tenth of a mile. *Associated Press-Times-Standard 12/31/09 After 562-mile ski trek, seven women reach the South Pole.*

Answers

1. $\frac{7}{22}$
3. $\frac{1}{15}$
5. $\frac{2}{19}$
7. $\frac{5}{3}$
9. $\frac{3}{5}$
11. $\frac{5}{7}$
13. $\frac{13}{12}$
15. $\frac{5}{3}$

17. $\frac{3}{2}$
19. $\frac{2}{7}$
21. $\frac{3}{11}$
23. $\frac{3}{11}$
25. The first automobile has the better mileage per gallon.
27. 22.18 dollars/hr
29. 35 mi/hr
31. 23.38 dollars/hr
33. The second automobile has the better mileage per gallon.
35. 36 mi/hr
37. 14.8 miles per day

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4.3: Introduction to Proportion

In Section 6.1, we introduced the concepts or ratio and rate. In this section, we equate these ratios in a construct called a *proportion*.

Proportions

A *proportion* is a statement that equates two ratios or rates.

For example, each of the equations

$$\frac{1}{3}=\frac{2}{6},\ \frac{15\text{ miles}}{2\text{ hours}}=\frac{30\text{ miles}}{4\text{ hours}},\text{ and }\frac{a}{b}=\frac{c}{d}$$

compare two ratios or rates and is a proportion.

The proportion

$$\frac{1}{3}=\frac{2}{6}$$

is read “one is to three as two is to six.” The four numbers that make up this proportion are called the *terms* of the proportion and are ordered in a natural manner.

$$\begin{array}{r}\text{First term}\longrightarrow\\ \text{Second term}\longrightarrow\end{array}\begin{array}{l}1\\3\end{array}\ =\ \begin{array}{l}2\\6\end{array}\begin{array}{l}\longleftarrow\text{ Third term}\\ \longleftarrow\text{ Fourth term}\end{array}$$

Extremes and Means

The first and fourth terms are called the *extremes* of the proportion. The second and third terms are called the *means* of the proportion.

In the proportion

$$\frac{a}{b}=\frac{c}{d},$$

the terms *a* and *d* are the extremes; the terms *b* and *c* are the means.

If we multiply both sides of the proportion by the common denominator,

$$bd\left(\frac{a}{b}\right)=bd\left(\frac{c}{d}\right)$$

then cancel,

$$b\not{d}\left(\frac{a}{b\not{}}\right)=b\,d\not{}\left(\frac{c}{d\not{}}\right)$$

we get the following result.

$$ad=bc$$

This leads to the following observation.

Product of Extremes and Means

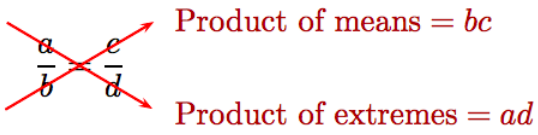
In the proportion

$$\frac{a}{b} = \frac{c}{d}$$

the product of the means equals the product of the extremes. That is,

$$ad = bc.$$

We can get an equivalent result using a technique called *cross multiplication*.



Example 1

Which of the following is a valid proportion: (a) $\frac{2}{3} = \frac{7}{12}$, or (b) $\frac{4}{9} = \frac{12}{27}$.

Solution

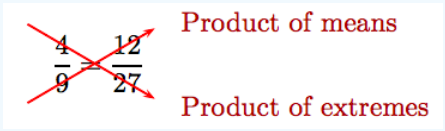
(a) Cross multiply

to get

$$24 = 21.$$

Hence, the product of the extremes does not equal the product of the means, so 2/3=7/12 is **not** a valid proportion.

(b) Cross multiply



to get

$$108 = 108.$$

Hence, the product of the extremes equals the product of the means, so 4/9 = 12/27 **is** a valid proportion.

Exercise

Is the following a valid proportion? $\frac{4}{3} = \frac{16}{11}$

Answer

No

Solving Proportions

We already have all the tools needed to solve proportions. Let's begin with the first example.

Example 2

Solve the proportion for x: $\frac{3}{4} = \frac{x}{12}$.

Solution

Cross multiply, then solve the resulting equation.

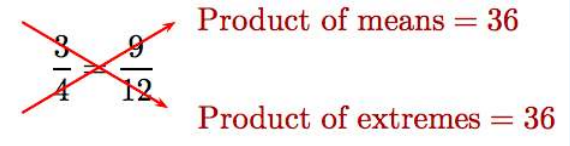
$$\begin{aligned} \frac{3}{4} &= \frac{x}{12} && \text{Original proportion.} \\ 4 \cdot x &= 3 \cdot 12 && \text{Products of means and extremes are equal.} \\ 4x &= 36 && \text{Simplify.} \\ \frac{4x}{4} &= \frac{36}{4} && \text{Divide both sides by 4.} \\ x &= 9 && \text{Simplify.} \end{aligned}$$

Check

Substitute 9 for x into the original proportion and check.

$$\begin{aligned} \frac{3}{4} &= \frac{x}{12} && \text{Original proportion.} \\ \frac{3}{4} &= \frac{9}{12} && \text{Substitute 9 for } x. \end{aligned}$$

Cross multiply.



Thus, the solution 9 checks.

Exercise

Solve the proportion for n: $\frac{2}{3} = \frac{n}{9}$

Answer

6

Example 3

Solve the proportion for n: $\frac{3}{2} = \frac{24}{n}$.

Solution

Cross multiply, then solve the resulting equation.

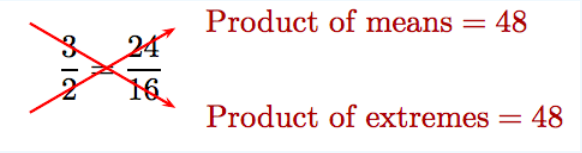
$$\begin{aligned} \frac{3}{2} &= \frac{24}{n} && \text{Original proportion.} \\ 3 \cdot n &= 2 \cdot 24 && \text{Products of means and extremes are equal.} \\ 3n &= 48 && \text{Simplify.} \\ \frac{3n}{3} &= \frac{48}{3} && \text{Divide both sides by 3.} \\ n &= 16 && \text{Simplify.} \end{aligned}$$

Check

Substitute 16 for n into the original proportion and check.

$$\frac{3}{2} = \frac{24}{n} \quad \text{Original proportion.}$$
$$\frac{3}{2} = \frac{24}{16} \quad \text{Substitute 16 for } n.$$

Cross multiply.



Thus, the solution 16 checks.

? Exercise

Solve the proportion for m: $\frac{9}{6} = \frac{m}{4}$

Answer

6

✔ Example 4

Solve the proportion for x: $\frac{2x+1}{15} = \frac{1}{3}$.

Solution

Cross multiply, then solve the resulting equation.

$$\frac{2x+1}{15} = \frac{1}{3} \quad \text{Original proportion.}$$
$$3(2x+1) = 15(1) \quad \text{Products f means and extremes are equal.}$$
$$6x+3 = 15 \quad \text{On the left, distribute. On the right, multiply.}$$
$$6x+3-3 = 15-3 \quad \text{Subtract 3 from both sides.}$$
$$6x = 12 \quad \text{Simplify.}$$
$$\frac{6x}{6} = \frac{12}{6} \quad \text{Divide both sides by 6.}$$
$$x = 2 \quad \text{Simplify both sides.}$$

Check

We'll leave it to our readers to check this solution.

? Exercise

Solve the proportion for y: $\frac{6+2y}{18} = \frac{8}{9}$

Answer

5

Applications

A number of practical applications involve solving a proportion.

Example 5

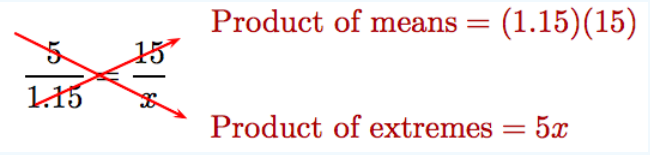
If 5 oranges cost \$1.15, what will be the cost for 15 oranges (assuming an equal rate)?

Solution

Let x represent the cost for 15 oranges. Assuming the rate for 5 oranges at \$1.15 equals the rate for 15 oranges at an unknown cost x, we set up the following proportion.

$$\frac{5}{1.15} = \frac{15}{x}$$

Cross multiply



to get

$$5x = 17.25.$$

Solve for x.

$$\frac{5x}{5} = \frac{17.25}{5}$$
$$x = 3.45$$

Thus, 15 oranges cost \$3.45.

Exercise

If 7 apples cost \$3.15, how much will 10 apples cost (assuming an equal rate)?

Answer

\$4.50

Checking Units is Extremely Important

When setting up a proportion, check to make sure that both numerators have the same units and both denominators have the same units.

For example, in Example 5, both numerators have “oranges” as units and both denominators have “dollars” as units.

$$\begin{array}{rcl} \text{Oranges} \longrightarrow & \frac{5}{1.15} = \frac{15}{x} & \longleftarrow \text{Oranges} \\ \text{Dollars} \longrightarrow & & \longleftarrow \text{Dollars} \end{array}$$

This proportion is set up correctly, because both numerators have the same units and both denominators have the same units. On the other hand, if we had set the proportion up incorrectly as follows,

$$\begin{array}{rcl} \text{Oranges} \longrightarrow & \frac{5}{1.15} = \frac{x}{15} & \longleftarrow \text{Dollars} \\ \text{Dollars} \longrightarrow & & \longleftarrow \text{Oranges} \end{array}$$

a quick check of the units reveals the error; i.e., the numerators have different units and the denominators have different units. Checking units helps us avoid errors!

Example 6

Dylan and David are planning a backpacking trip in Yosemite National Park. On their map, the legend indicates that 1.2 centimeters represents 2 miles. How long is their trip if the route measures 10.6 centimeters on the map? Round your answer to the nearest tenth of a mile.

Solution

Let's set up the proportion with units.

$$\frac{1.2 \text{ cm}}{2 \text{ mi}} = \frac{10.6 \text{ cm}}{x \text{ mi}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x.

$$\begin{aligned}
\frac{1.2}{2} &= \frac{10.6}{x} && \text{Original proportion.} \\
1.2x &= (2)(10.6) && \text{Cross multiply.} \\
1.2x &= 21.2 && \text{Simplify right-side hand.} \\
\frac{1.2x}{1.2} &= \frac{21.2}{1.2} && \text{Divide both sides by 1.2.} \\
x &\approx 17.66 && \text{On the right: Divide.}
\end{aligned}$$

We carried the division on the right one decimal place past the tenths place. The rounding digit is a 6 and the following test digit is a 6. Add 1 to the rounding digit and truncate.

To the nearest tenth of a mile, the backpacking route is approximately 17.7 miles.

Exercise

Eloise and Susannah are planning a trip is Sequoia National Park. On their map, 3 inches represents 50 miles. How long is their trip if the route measures $4\frac{1}{2}$ inches on the map?

Answer

75 miles

Example 7

A recipe making 2 dozen cookies requires $1\frac{3}{4}$ cups of flour, among other ingredients. If the baker wishes to make twice that number of cookies, how much flour is required?

Solution

Twice 2 dozen is 4 dozen cookies. Let x represent the amount of flour needed for 4 dozen cookies. Assuming an equal rate for 2 dozen cookies (2 dozen requires 1 3 4 cups of flour), we set up the following proportion. Again, using units helps us craft the correct proportion.

$$\frac{2 \text{ dozen}}{1\frac{3}{4} \text{ cups}} = \frac{4 \text{ dozen}}{x \text{ cups}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x.

$$\begin{aligned}
\frac{2}{1\frac{3}{4}} &= \frac{4}{x} && \text{Original proportion.} \\
2x &= 1\frac{3}{4} \cdot 4 && \text{Cross multiply.} \\
2x &= \frac{7}{4} \cdot 4 && \text{Change to improper fraction.} \\
2x &= 7 && \text{Multiply.}
\end{aligned}$$

Divide both sides of the equation by 2 and finish.

$$\frac{2x}{2} = \frac{7}{2} \quad \text{Divide both sides by 2.}$$
$$x = \frac{7}{2}$$

Change the improper fraction to a mixed fraction. Thus, it will take $3\frac{1}{2}$ cups of flour to make 4 dozen cookies.

? Exercise

Dough for 3 pizzas requires $8\frac{1}{2}$ cups of flour. If the baker wishes to make 9 pizzas, how many cups of flour are required?

Answer

$25\frac{1}{2}$ cups

Exercises

In Exercises 1-12, which of the following is a true proportion?

1. $\frac{9}{7} = \frac{27}{21}$, $\frac{4}{3} = \frac{9}{7}$, $\frac{7}{2} = \frac{8}{9}$, $\frac{4}{8} = \frac{9}{6}$
2. $\frac{6}{7} = \frac{18}{21}$, $\frac{2}{3} = \frac{8}{6}$, $\frac{4}{3} = \frac{3}{2}$, $\frac{8}{9} = \frac{3}{8}$
3. $\frac{7}{6} = \frac{28}{24}$, $\frac{5}{6} = \frac{5}{4}$, $\frac{9}{5} = \frac{7}{3}$, $\frac{9}{2} = \frac{8}{9}$
4. $\frac{7}{6} = \frac{2}{8}$, $\frac{4}{5} = \frac{5}{7}$, $\frac{3}{4} = \frac{15}{20}$, $\frac{8}{4} = \frac{8}{7}$
5. $\frac{6}{5} = \frac{24}{20}$, $\frac{7}{3} = \frac{2}{4}$, $\frac{2}{4} = \frac{2}{6}$, $\frac{5}{2} = \frac{2}{8}$
6. $\frac{9}{8} = \frac{4}{3}$, $\frac{5}{7} = \frac{10}{14}$, $\frac{8}{6} = \frac{5}{4}$, $\frac{8}{5} = \frac{2}{6}$
7. $\frac{3}{5} = \frac{2}{8}$, $\frac{3}{7} = \frac{6}{14}$, $\frac{5}{6} = \frac{2}{4}$, $\frac{7}{4} = \frac{5}{9}$
8. $\frac{7}{3} = \frac{7}{6}$, $\frac{4}{7} = \frac{8}{14}$, $\frac{5}{3} = \frac{7}{8}$, $\frac{5}{7} = \frac{6}{9}$
9. $\frac{5}{4} = \frac{25}{20}$, $\frac{9}{3} = \frac{9}{6}$, $\frac{7}{4} = \frac{3}{6}$, $\frac{3}{5} = \frac{9}{4}$
10. $\frac{7}{6} = \frac{6}{9}$, $\frac{7}{3} = \frac{2}{5}$, $\frac{6}{7} = \frac{30}{35}$, $\frac{4}{7} = \frac{2}{8}$
11. $\frac{9}{7} = \frac{4}{3}$, $\frac{9}{4} = \frac{7}{9}$, $\frac{3}{5} = \frac{6}{10}$, $\frac{3}{9} = \frac{9}{5}$
12. $\frac{4}{3} = \frac{8}{7}$, $\frac{2}{6} = \frac{5}{8}$, $\frac{7}{2} = \frac{3}{6}$, $\frac{9}{4} = \frac{36}{16}$

In Exercises 13-36, solve the given proportion.

13. $\frac{17}{3} = \frac{x}{18}$
14. $\frac{16}{5} = \frac{x}{20}$
15. $\frac{6x+10}{6} = \frac{11}{3}$
16. $\frac{4x+8}{12} = \frac{5}{3}$
17. $\frac{17}{9} = \frac{x}{18}$
18. $\frac{8}{9} = \frac{x}{18}$
19. $\frac{11}{2} = \frac{x}{8}$
20. $\frac{11}{4} = \frac{x}{8}$
21. $\frac{7x+15}{15} = \frac{10}{3}$
22. $\frac{7x+3}{8} = \frac{5}{4}$

23. $\frac{11}{2} = \frac{x}{10}$

24. $\frac{19}{6} = \frac{x}{18}$

25. $\frac{5x+8}{12} = \frac{2}{3}$

26. $\frac{3x+12}{6} = \frac{3}{2}$

27. $\frac{2}{15} = \frac{24}{x}$

28. $\frac{7}{8} = \frac{14}{x}$

29. $\frac{3}{16} = \frac{6}{x}$

30. $\frac{4}{21} = \frac{12}{x}$

31. $\frac{5}{22} = \frac{20}{x}$

32. $\frac{3}{22} = \frac{21}{x}$

33. $\frac{2x+10}{6} = \frac{14}{3}$

34. $\frac{2x+9}{9} = \frac{13}{3}$

35. $\frac{7}{2} = \frac{21}{x}$

36. $\frac{2}{15} = \frac{18}{x}$

---

37. If 13 dog bones cost $1.97, what will be the cost for 7 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

38. If 2 watermelons cost $3.89, what will be the cost for 11 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

39. If 7 bananas cost $2.55, what will be the cost for 14 bananas (assuming an equal rate)? Round your answer to the nearest penny.

40. If 2 apples cost $2.05, what will be the cost for 8 apples (assuming an equal rate)? Round your answer to the nearest penny.

41. If 13 oranges cost $3.61, what will be the cost for 11 oranges (assuming an equal rate)? Round your answer to the nearest penny.

42. If 3 watermelons cost $1.05, what will be the cost for 9 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

43. If 3 dog bones cost $1.23, what will be the cost for 13 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

44. If 3 watermelons cost $4.41, what will be the cost for 7 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

45. If 3 apples cost $3.24, what will be the cost for 13 apples (assuming an equal rate)? Round your answer to the nearest penny.

46. If 6 apples cost $3.43, what will be the cost for 7 apples (assuming an equal rate)? Round your answer to the nearest penny.

47. If 4 dog bones cost $1.03, what will be the cost for 8 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

48. If 4 oranges cost $4.28, what will be the cost for 3 oranges (assuming an equal rate)? Round your answer to the nearest penny.

---

49. **Two rolls.** In Haiti, two flat rolls cost 5 gourdes, about 12 cents. How many cents would 20 rolls cost? Associated Press-Times-Standard 02/18/10 Haiti's earthquake camps turning into shanty towns.

50. **Turbines.** As proposed, the Shell Wind Energy project consists of 25 ridge-top turbines that can generate up to 50 megawatts, or enough to supply electricity to about 1, 000 homes. Estimate the number of ridge-top turbines that would be needed to supply electricity to 70, 000 homes, the approximate number of properties in Humboldt County, CA. *John Driscoll Times-Standard 12/24/09 Wind power project goes under analysis.*

51. **Dumptrucks.** U.S. Highway 199 had a landslide where as much as 3, 000 cubic yards of material fell on the road, reportedly requiring about 200 large dumptrucks to remove. Only a week earlier, 40, 000 cubic yards of material fell on Highway 96. Estimate the number of dumptrucks needed for that slide rounded to the nearest whole number. *Associated Press-Times-Standard 03/09/10 Another highway closed by slide.*

52. **Timber sales.** Alaska's 26, 000 square mile Tongass National Forest plan allows for timber sales of up to 267 million board-feet per year – enough for nearly 27, 000 two-bedroom homes, but demand for timber is far short of that. Less than 25 million board-feet was logged in the forest in 2009. Forest Service officials have said they hope to increase logging in the Tongass to about 100 million board-feet per year. *Associated Press-Times-Standard 02/18/10 Industry loses lawsuit over logging in Alaska.*

i) Estimate the number of 2-bedroom homes that 25 million board-feet of timber would build.

ii) How many 2-bedroom homes would 100 million board-feet of timber build?

53. **Costly spill.** In Australia, penalties on ships causing oil spills are approximately $1.75 million Australian dollars, equivalent to about $1.64 million US dollars. After an oil tanker was grounded onto a coral reef, Australian officials are considering raising the fine to $10 million Australian dollars. What will the new fine be in US dollars? Round your answer to the nearest hundredth of a million dollars. *Associated Press-Times-Standard 04/13/10 Ship that leaked oil on Great Barrier Reef removed.*

Answers

1. $\frac{9}{7} = \frac{27}{21}$ is a proportion
3. $\frac{7}{6} = \frac{28}{24}$ is a proportion
5. $\frac{6}{5} = \frac{24}{20}$ is a proportion
7. $\frac{3}{7} = \frac{6}{14}$ is a proportion
9. $\frac{5}{4} = \frac{25}{20}$ is a proportion
11. $\frac{3}{5} = \frac{6}{10}$ is a proportion
13. 102
15. 2
17. 34
19. 44
21. 5
23. 55
25. 0
27. 180
29. 32
31. 88
33. 9
35. 6
37. $1.06
39. $5.10

41. $3.05

43. $5.33

45. $14.04

47. $2.06

49. $0.48

51. 2, 667 loads

53. $9.37 million US dollars

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4.4: Percent

When one hears the word “percent,” other words come immediately to mind, words such as “century,” “cents,” or “centimeters.” A century equals 100 years. There are one hundred cents in a dollar and there are 100 centimeters in a meter. Thus, it should come as no surprise that percent means “parts per hundred.”

In the world we live in we are constantly bombarded with phrases that contain the word “percent.” The sales tax in California is 8.25%. An employee is asking his boss for a 5% raise. A union has seen a 6.25% increase in union dues. The population of a town is increasing at a rate of 2.25% per year.

In this chapter we introduce the concept of percent, first addressing how to facilitate writing percents in fraction or decimal form and also performing the reverse operations, changing fractions and decimals to percents. Next we use our expertise in solving equations to solve the more common forms that involve percents, then we apply this ability to solving common applications from the real world that use percents. We’ll tackle applications of commission and sales tax, discount and marked price, percent increase or decrease, and simple interest.

Let’s begin the journey

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4.5: Percent, Decimals, Fractions

In the square shown in Figure 4.5.1, a large square has been partitioned into ten rows of ten little squares in each row. In Figure 7.1, we've shaded 20 of 100 possible little squares, or 20% of the total number of little squares.

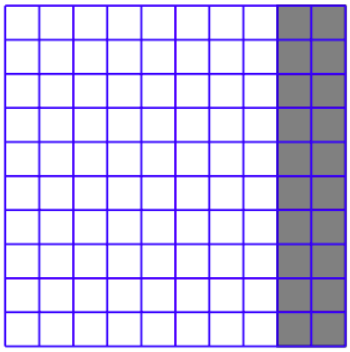


Figure 4.5.1: Shading 20 of 100 little squares, or 20% of the total number of little squares.

The Meaning of Percent

Percent means "parts per hundred."

In Figure 7.1, 80 out of a possible 100 squares are left unshaded. Thus, 80% of the little squares are unshaded. If instead we shaded 35 out of the 100 squares, then 35% of the little squares would be shaded. If we shaded all of the little squares, then 100% of the little squares would be shaded (100 out of 100).

So, when you hear the word "percent," think "parts per hundred."

Changing a Percent to a Fraction

Based on the discussion above, it is fairly straightforward to change a percent to a fraction.

Percent to Fraction

To change a percent to a fraction, drop the percent sign and put the number over 100.

Example 1

Change 24% to a fraction.

Solution

Drop the percent symbol and put 24 over 100.

$$\begin{aligned} 24\% &= \frac{24}{100} \quad \text{Percent: Parts per hundred.} \\ &= \frac{6}{25} \quad \text{Reduce.} \end{aligned}$$

Hence, 24% = 6/25.

Exercise

Change 36% to a fraction reduced to lowest terms.

Answer

9/25

Example 2

Change $14\frac{2}{7}\%$ to a fraction.

Solution

Drop the percent symbol and put $14\frac{2}{7}$ over 100.

$$\begin{aligned} 14\frac{2}{7}\% &= \frac{14\frac{2}{7}}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{\frac{100}{7}}{100} && \text{Mixed to improper fraction.} \\ &= \frac{100}{7} \cdot \frac{1}{100} && \text{Invert and multiply.} \\ &= \frac{\cancel{100}}{7} \cdot \frac{1}{\cancel{100}} && \text{Cancel.} \\ &= \frac{1}{7} \end{aligned}$$

Hence, $14\frac{2}{7}\% = 1/7$.

Exercise

Change $11\frac{1}{9}\%$ to a fraction reduced to lowest terms.

Answer

1/9

Example 3

Change 28.4% to a fraction.

Solution

Drop the percent symbol and put 28.4 over 100.

$$\begin{aligned} 28.4\% &= \frac{28.4}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{28.4 \cdot 10}{100 \cdot 10} && \text{Multiply numerator and denominator by 10.} \\ &= \frac{284}{1000} && \text{Multiplying by 10 moves decimal point one place right.} \\ &= \frac{71 \cdot 4}{250 \cdot 4} && \text{Factor.} \\ &= \frac{71}{250} && \text{Cancel common factor.} \end{aligned}$$

Exercise

Change 87.5% to a fraction reduced to lowest terms.

Answer

7/8

Changing a Percent to a Decimal

To change a percent to a decimal, we need only remember that percent means “parts per hundred.”

Example 4

Change 23.25% to a decimal.

Solution

Drop the percent symbol and put 23.25 over 100.

$$\begin{aligned} 23.25\% &= \frac{23.25}{100} && \text{Percent: Parts per hundred.} \\ &= 0.2325 && \text{Dividing by 100 moves decimal point 2 places left.} \end{aligned}$$

Therefore, 23.25% = 0.2325.

Exercise

Change 2.4% to a decimal.

Answer

0.024

This last example motivates the following simple rule.

Changing a Percent to a Decimal

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

Example 5

Change $5\frac{1}{2}\%$ to a decimal.

Solution

Note that 1/2=0.5, then move the decimal 2 places to the left.

$$\begin{aligned} 5\frac{1}{2}\% &= 5.5\% && 1/2 = 0.5. \\ &= 0.055 && \text{Drop \% symbol.} \\ &= 0.055 && \text{Move decimal point 2 places left.} \end{aligned}$$

Thus, $5\frac{1}{2}\% = 0.055$.

Exercise 4.5.1

Change $6\frac{3}{4}\%$ to a decimal.

Answer

0.0675

Changing a Decimal to a Percent

Changing a decimal to a percent is the exact opposite of changing a percent to a decimal. In the latter case, we drop the percent symbol and move the decimal point 2 places to the left. The following rule does just the opposite.

Changing a Decimal to a Percent

To change a decimal to a percent, move the decimal point two places to the right and add a percent symbol.

Example 6

Change 0.0725 to a percent.

Solution

Move the decimal point two places to the right and add a percent symbol.

$$\begin{aligned} 0.0725 &= 007.25\% \\ &= 7.25\% \end{aligned}$$

Exercise

Change to 0.0375 to a percent.

Answer

3.75%

Example 7

Change 1.025 to a percent.

Solution

Move the decimal point two places to the right and add a percent symbol.

$$\begin{aligned} 1.025 &= 102.5\% \\ &= 102.5\% \end{aligned}$$

Exercise

Change 0.525 to a percent.

Answer

52.5%

Changing a Fraction to a Percent

One way to proceed is to first change the fraction to a decimal, then change the resulting decimal to a percent.

Fractions to Percents: Technique #1

To change a fraction to a percent, follow these steps:

1. Divide numerator by the denominator to change the fraction to a decimal.
2. Move the decimal point in the result two places to the right and append a percent symbol.

Example 8

Use Technique #1 to change 5/8 to a percent.

Solution

Change 5/8 to a decimal, then change the decimal to a percent.

To change 5/8 to a decimal, divide 5 by 8. Since the denominator is a product of twos, the decimal should terminate.

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

To change 0.625 to a percent, move the decimal point 2 places to the right and append a percent symbol.

0.625 = 0 62.5% = 62.5%

? Exercise

Change 5/16 to a percent.

Answer

31.35%

A second technique is to create an equivalent fraction with a denominator of 100.

Fractions to Percents: Technique #2

To change a fraction to a percent, create an equivalent fraction with a denominator of 100.

✓ Example 9

Use Technique #2 to change 5/8 to a percent.

Solution

Create an equivalent fraction for 5/8 with a denominator of 100.

$$\frac{5}{8} = \frac{x}{100}$$

Solve this proportion for *x*.

$$\begin{aligned} 8x &= 500 && \text{Cross multiply.} \\ \frac{8x}{8} &= \frac{500}{8} && \text{Divide both sides by 8.} \\ x &= \frac{125}{2} && \text{Reduce: Divide numerator and denominator by 4.} \\ x &= 62.5 && \text{Divide.} \end{aligned}$$

Thus,

$$\frac{5}{8} = \frac{62.5}{100} = 62.5\%.$$

Alternate Ending

We could also change 125/2 to a mixed fraction; i.e., 125/2 = 62 1 2 . Then,

$$\frac{5}{8} = \frac{62\frac{1}{2}}{100} = 62\frac{1}{2}\%.$$

Same answer.

Exercise

Change 4/9 to a percent.

Answer

$44\frac{4}{9}\%$

Sometimes we will be content with an approximation.

Example 10

Change 4/13 to a percent. Round your answer to the nearest tenth of a percent.

Solution

We will use Technique #1.

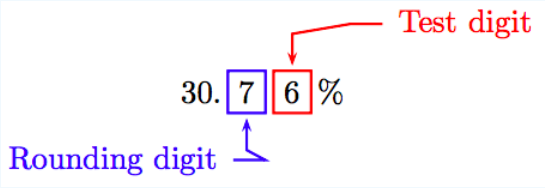
To change 4/13 to a decimal, divide 4 by 13. Since the denominator has factors other than 2's and 5's, the decimal will repeat. However, we intend to round to the nearest tenth of a percent, so we will carry the division to four decimal places only. (*Four places are necessary because we will be moving the decimal point two places to the right.*)

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

To change the decimal to a percent, move the decimal point two places to the right.

$$0.3076 \approx 0\ 30.76\% \approx 30.76\%$$

To round to the nearest tenth of a percent, identify the rounding and test digits.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus,

$$0.03076 \approx 30.8\%.$$

Exercise

Change 4/17 to a percent. Round your answer to the nearest tenth of a percent.

Answer

23.5%

Exercises

In Exercises 1-18, convert the given percent to a fraction, and simplify the result.

1. $4\frac{7}{10}\%$
2. $7\frac{1}{4}\%$
3. $7\frac{2}{9}\%$
4. $4\frac{9}{10}\%$
5. 11.76%
6. 15.2%
7. 13.99%
8. 18.66%
9. $4\frac{1}{2}\%$
10. $8\frac{5}{8}\%$
11. 192%
12. 5%
13. 86%
14. 177%
15. 130%
16. 80%
17. 4.07%
18. 6.5%

In Exercises 19-34, convert the given percent to a decimal.

19. 124%
20. 4%
21. 0.6379%
22. 0.21%
23. 28%
24. 5.4%
25. 0.83%
26. 0.3344%
27. 8%
28. 3%
29. 59.84%
30. 0.17%
31. 155%
32. 7%
33. 36.5%
34. 39.7%

In Exercises 35-50, convert the given decimal to a percent.

35. 8.888
36. 5.1
37. 0.85
38. 0.08
39. 1.681
40. 3.372
41. 0.14
42. 4.89
43. 8.7
44. 8.78
45. 0.38
46. 1.67
47. 0.02
48. 0.07
49. 0.044
50. 0.29

In Exercises 51-68, convert the given fraction to a percent.

51. $\frac{1}{2}$
52. $\frac{29}{8}$
53. $\frac{5}{2}$
54. $\frac{4}{5}$
55. $\frac{8}{5}$
56. $\frac{7}{20}$
57. $\frac{14}{5}$
58. $\frac{3}{2}$
59. $\frac{9}{2}$
60. \(\frac{18}{25}|)
61. $\frac{9}{4}$
62. $\frac{7}{8}$
63. $\frac{7}{5}$
64. $\frac{4}{25}$
65. $\frac{6}{5}$
66. $\frac{23}{8}$
67. $\frac{12}{5}$
68. $\frac{13}{2}$

---

69. Convert 24/29 to a percent, and round your answer to the nearest hundredth of a percent.

70. Convert 5/3 to a percent, and round your answer to the nearest hundredth of a percent.

71. Convert 15/7 to a percent, and round your answer to the nearest tenth of a percent.

72. Convert 10/7 to a percent, and round your answer to the nearest tenth of a percent.

73. Convert 7/24 to a percent, and round your answer to the nearest hundredth of a percent.

74. Convert 5/6 to a percent, and round your answer to the nearest hundredth of a percent.

75. Convert 8/3 to a percent, and round your answer to the nearest tenth of a percent.

76. Convert 22/21 to a percent, and round your answer to the nearest tenth of a percent.

77. Convert 9/23 to a percent, and round your answer to the nearest tenth of a percent.

78. Convert 11/9 to a percent, and round your answer to the nearest tenth of a percent.

79. Convert 17/27 to a percent, and round your answer to the nearest hundredth of a percent.

80. Convert 22/27 to a percent, and round your answer to the nearest hundredth of a percent.

---

81. **Crime rates.** Preliminary crime rates for the first six months of 2009 compared to the same period in 2008 are shown below. *Associated Press-Times-Standard 12/22/09 Despite recession, the national crime rates keep falling.*

Murder	-10.0%
Forcible rape	-3.3%
Robbery	-6.5%
Aggravated assault	-3.2%
Burglary	-2.5%
Larceny-theft	-5.3%
Motor vehicle theft	-18.75%
Arson	-8.2%

Source: Federal Bureau of Investigation

i) What do the negative signs indicate?

ii) Which type of crime decreased the most?

iii) Which type of crime decreased the least?

82. **Major Hurricanes.** 5 of the 8 hurricanes in 2008 were categorized as major. Write the fractional number of major hurricanes in 2008 as a percent. *NOAA Associated Press 5/22/09*

83. **Chance of flood.** These excerpts are from the story *Corps: Dam work lessens Seattle-area flood chance* published in the *Times-Standard* on Nov. 6, 2009. Write all four of the odds of flooding as a percent chance. Round to the nearest tenth of a percent if necessary.

i) Col. Anthony Wright, from the U.S. Army Corps of Engineers, speaking of the repairs to the Green River Dam, reported there was now a 1-in-25 chance that a storm would force the corps to release enough water from the dam's reservoir to cause a flood downstream in the Green River Valley.

ii) The odds of widespread flooding in the valley improve to 1-in-32 when all the sandbagging and flood-protection efforts are factored in.

iii) Previously, the Corps of Engineers said the chance of widespread flooding was 1-in-4.

iv) When the dam operates at capacity, there is a 1-in-140 chance of flooding.

Answers

1. $\frac{47}{1000}$

3. $\frac{13}{180}$

5. $\frac{147}{1250}$

7. $\frac{1399}{10000}$

9. $\frac{9}{200}$

11. $\frac{48}{25}$

13. $\frac{43}{50}$

15. $\frac{13}{10}$

17. $\frac{407}{10000}$

19. 1.24

21. 0.006379

23. 0.28

25. 0.0083

27. 0.08

29. 0.5984

31. 1.55

33. 0.365

35. 888.8%

37. 85%

39. 168.1%

41. 14%

43. 870%

45. 38%

47. 2%

49. 4.4%

51. 50%

53. 250%

55. 160%

57. 280%

59. 450%

61. 225%

63. 140%

65. 120%

67. 240%

69. 82.76%

71. 214.3%

73. 29.17%

75. 266.7%

77. 39.1%

79. 62.96%

81.

i) The negative signs indicate the crime rate has decreased from previous measures.

ii) Motor vehicle theft decreased the most with an 18.75% decrease.

iii) Burglary decreased the least with a 2.5% decrease.

83.

i) 4% chance of flood

ii) 3.1% chance of flood

iii) 25% chance of flood

iv) 0.7% chance of flood

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4.6: Solving Basic Percent Problems

There are three basic types of percent problems:

1. *Find a given percent of a given number.* For example, find 25% of 640.
2. *Find a percent given two numbers.* For example, 15 is what percent of 50?
3. *Find a number that is a given percent of another number.* For example, 10% of what number is 12?

Let's begin with the first of these types.

Find a Given Percent of a Given Number

Let's begin with our first example.

Example 1

What number is 25% of 640?

Solution

Let x represent the unknown number. Translate the words into an equation.

What number	is	25%	of	640
x	$=$	25%	$\cdot$	640

Now, solve the equation for x.

$$\begin{aligned} x &= 25\% \cdot 640 && \text{Original equation.} \\ x &= 0.25 \cdot 640 && \text{Change } 25\% \text{ to a decimal: } 25\% = 0.25. \\ x &= 160 && \text{Multiply: } 0.25 \cdot 640 = 160. \end{aligned}$$

Thus, 25% of 640 is 160.

Alternate Solution

We could also change 25% to a fraction.

$$\begin{aligned} x &= 25\% \cdot 640 && \text{Original equation.} \\ x &= \frac{1}{4} \cdot 640 && \text{Change } 25\% \text{ to a fraction: } 25\% = 25/100 = 1/4. \\ x &= \frac{640}{4} && \text{Multiply numerators and denominators.} \\ x &= 160 && \text{Divide: } 640/4 = 160. \end{aligned}$$

Same answer.

Exercise

What number is 36% of 120?

Answer

43.2

Example 2

What is number $8\frac{1}{3}\%$ of 120?

Solution

Let x represent the unknown number. Translate the words into an equation.

What number	is	$8\ (1/3)\%$	of	120
x	$=$	$8\frac{1}{3}\%$	$\cdot$	120

Now, solve the equation for *x*. Because

$$8\frac{1}{3}\% = 8.3\% = 0.08\overline{3},$$

working with decimals requires that we work with a repeating decimal. To do so, we would have to truncate the decimal representation of the percent at some place and satisfy ourselves with an approximate answer. Instead, let's change the percent to a fraction and seek an exact answer.

$$\begin{aligned} 8\frac{1}{3}\% &= \frac{8\frac{1}{3}}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{\frac{25}{3}}{100} && \text{Mixed to improper fraction.} \\ &= \frac{25}{3}\cdot\frac{1}{100} && \text{Invert and multiply.} \\ &= \frac{25}{300} && \text{Multiply numerators and denominators.} \\ &= \frac{1}{12} && \text{Reduce: Divide numerator and denominator by 25.} \end{aligned}$$

Now we can solve our equation for *x*.

$$\begin{aligned} &= 8\frac{1}{3}\%\cdot 120 && \text{Original equation.} \\ x &= \frac{1}{12}\cdot 120 && 8\frac{1}{3}\% = 1/12. \\ x &= \frac{120}{12} && \text{Multiply numerators and denominators.} \\ x &= 10 && \text{Divide: } 120/12 = 10. \end{aligned}$$

Thus, $8\frac{1}{3}\%$ of 120 is 10.

Exercise

What number is $4\frac{1}{6}\%$ of 1,200?

Answer

50

Example 3

What number is $105\frac{1}{4}\%$ of 18.2?

Solution

Let x represent the unknown number. Translate the words into an equation.

What number	is	$105\ (1/4)\ \%$	of	18.2
x	$=$	$105\frac{1}{4}\%$	$\cdot$	18.2

In this case, the fraction terminates as 1/4=0.25, so

$$105\frac{1}{4}\% = 105.25 \tag{4.6.1}$$

Now we can solve our equation for *x*.

$$x = 105\frac{1}{4}\% \cdot 18.2 \quad \text{Original equation.}$$
$$x = 1.0525 \cdot 18.2 \quad 5\frac{1}{4}\% = 1.0525.$$
$$x = 19.1555 \quad \text{Multiply.}$$

Thus, $105\frac{1}{4}\%$ of 18.2 is 19.1555.

? Exercise

What number is $105\frac{3}{4}\%$ of 222?

Answer

234.765

Find a Percent Given Two Numbers

Now we'll address our second item on the list at the beginning of the section.

✔ Example 4

15 is what percent of 50?

Solution

Let *x* represent the unknown percent. Translate the words into an equation.

15	is	what percent	of	50
15	=	x	·	50

The commutative property of multiplication allows us to change the order of multiplication on the right-hand side of this equation.

$$15 = 50x.$$

Now we can solve our equation for *x*.

$$15 = 50x \quad \text{Original equation.}$$
$$\frac{15}{50} = \frac{50x}{50} \quad \text{Divide both sides by 50.}$$
$$\frac{15}{50} = x \quad \text{Simplify right-hand side.}$$
$$x = 0.30 \quad \text{Divide: } 15/50 = 0.30.$$

But we must express our answer as a percent. To do this, move the decimal two places to the right and append a percent symbol.

$$0.30 = 0\,30.\% = 30\%$$

Thus, 15 is 30% of 50.

Alternative Conversion

At the third step of the equation solution, we had

$$x = \frac{15}{50}.$$

We can convert this to an equivalent fraction with a denominator of 100.

$$x = \frac{15 \cdot 2}{50 \cdot 2} = \frac{30}{100}$$

Thus, 15/50 = 30/100 = 30%.

Exercise

14 is what percent of 25?

Answer

56%

Example 5

10 is what percent of 80?

Solution

Let x represent the unknown percent. Translate the words into an equation.

10	is	what percent	of	80
10	=	x	·	80

The commutative property of multiplication allows us to write the right-hand side as

$$10 = 80x.$$

Now we can solve our equation for *x*.

$$\begin{aligned} 10 &= 80x && \text{Original equation.} \\ \frac{10}{80} &= \frac{80x}{80} && \text{Divide both sides by 80.} \\ \frac{1}{8} &= x && \text{Reduce: } 10/80 = 1/8. \\ 0.125 &= x && \text{Divide: } 1/8 = 0.125. \end{aligned}$$

But we must express our answer as a percent. To do this, move the decimal two places to the right and append a percent symbol.

$$0.125 = 0\,12.5\% = 12.5\%$$

Thus, 10 is 12.5% of 80.

Alternative Conversion

At the third step of the equation solution, we had

$$x = \frac{1}{8}.$$

We can convert this to an equivalent fraction with a denominator of 100 by setting up the proportion

$$\frac{1}{8} = \frac{n}{100}$$

Cross multiply and solve for *n*.

$$\begin{aligned} 8n &= 100 && \text{Cross multiply.} \\ \frac{8n}{8} &= \frac{100}{8} && \text{Divide both sides by 8.} \\ n &= \frac{25}{8} && \text{Reduce: Divide numerator and denominator by 4.} \\ n &= 12\frac{1}{2} && \text{Change } 25/2 \text{ to mixed fraction.} \end{aligned}$$

Hence,

$$\frac{1}{8} = \frac{12\frac{1}{2}}{100} = 12\frac{1}{2}\%.$$

Same answer.

? Exercise

10 is what percent of 200?

Answer

5%

Find a Number that is a Given Percent of Another Number

Let's address the third item on the list at the beginning of the section.

✔ Example 6

10% of what number is 12?

Solution

Let x represent the unknown number. Translate the words into an equation.

10%	of	what number	is	12
10%	·	x	=	12

Change 10% to a fraction: 10% = 10/100 = 1/10.

$$\frac{1}{10}x = 12$$

Now we can solve our equation for x.

$$\begin{aligned} 10\left(\frac{1}{10}x\right) &= 10(12) && \text{Multiply both sides by 10.} \\ x &= 120 && \text{Simplify.} \end{aligned}$$

Thus, 10% of 120 is 12.

Alternative Solution

We can also change 10% to a decimal: 10% = 0.10. Then our equation becomes

$$0.10x = 12$$

Now we can divide both sides of the equation by 0.10.

$$\begin{aligned} \frac{0.10x}{0.10} &= \frac{12}{0.10} && \text{Divide both sides by 0.10.} \\ x &= 120 && \text{Divide: } 12/0.10 = 120. \end{aligned}$$

Same answer.

Exercise

20% of what number is 45?

Answer

225

Example 7

$11\frac{1}{9}\%$ of what number is 20?

Solution

Let x represent the unknown number. Translate the words into an equation.

$11\ (1/9)\ \%$	of	what number	is	20
$11\frac{1}{9}\%$	$\cdot$	x	$=$	20

Change $11\frac{1}{9}\%$ to a fraction.

$$\begin{aligned} 11\frac{1}{9}\% &\quad \text{Percent: Parts per hundred.} \\ = \frac{\frac{100}{9}}{100} &\quad \text{Mixed to improper: } 11\frac{1}{9} = 100/9. \\ = \frac{100}{9} \cdot \frac{1}{100} &\quad \text{Invert and multiply.} \\ = \frac{\cancel{100}}{9} \cdot \frac{1}{\cancel{100}} &\quad \text{Cancel.} \\ = \frac{1}{9} &\quad \text{Simplify.} \end{aligned}$$

Replace $11\frac{1}{9}\%$ with 1/9 in the equation and solve for x.

$$\begin{aligned} \frac{1}{9}x = 20 &\quad 11\frac{1}{9}\% = 1/9/ \\ 9\left(\frac{1}{9}x\right) = 9(20) &\quad \text{Multiply both sides by } 9. \\ x = 180 & \end{aligned}$$

Thus, $11\frac{1}{9}\%$ of 180 is 20.

Exercise

$12\frac{2}{3}\%$ of what number is 760?

Answer

6,000

Exercises

1. What number is 22.4% of 125?
2. What number is 159.2% of 125?
3. 60% of what number is 90?
4. 25% of what number is 40?

5. 200% of what number is 132?

6. 200% of what number is 208?

7. 162.5% of what number is 195?

8. 187.5% of what number is 90?

9. 126.4% of what number is 158?

10. 132.5% of what number is 159?

11. 27 is what percent of 45?

12. 9 is what percent of 50?

13. 37.5% of what number is 57?

14. 162.5% of what number is 286?

15. What number is 85% of 100?

16. What number is 10% of 70?

17. What number is 200% of 15?

18. What number is 50% of 84?

19. 50% of what number is 58?

20. 132% of what number is 198?

21. 5.6 is what percent of 40?

22. 7.7 is what percent of 35?

23. What number is 18.4% of 125?

24. What number is 11.2% of 125?

25. 30.8 is what percent of 40?

26. 6.3 is what percent of 15?

27. 7.2 is what percent of 16?

28. 55.8 is what percent of 60?

29. What number is 89.6% of 125?

30. What number is 86.4% of 125?

31. 60 is what percent of 80?

32. 16 is what percent of 8?

33. What number is 200% of 11?

34. What number is 150% of 66?

35. 27 is what percent of 18?

36. 9 is what percent of 15?

37. $133\frac{1}{3}\%$ of what number is 80?

38. $121\frac{2}{3}\%$ of what number is 73?

39. What number is $54\frac{1}{3}\%$ of 6?

40. What number is $82\frac{2}{5}\%$ of 5?

41. What number is $62\frac{1}{2}\%$ of 32?

42. What number is $118\frac{3}{4}\%$ of 32?

43. $77\frac{1}{7}\%$ of what number is 27?

44. $82\frac{2}{3}\%$ of what number is 62?

45. What number is $142\frac{6}{7}\%$ of 77?

46. What number is $116\frac{2}{3}\%$ of 84?

47. $143\frac{1}{2}\%$ of what number is 5.74?

48. $77\frac{1}{2}\%$ of what number is 6.2?

49. $141\frac{2}{3}\%$ of what number is 68?

50. $108\frac{1}{3}\%$ of what number is 78?

51. What number is $66\frac{2}{3}\%$ of 96?

52. What number is $79\frac{1}{6}\%$ of 48?

53. $59\frac{1}{2}\%$ of what number is 2.38?

54. $140\frac{1}{5}\%$ of what number is 35.05?

55. $78\frac{1}{2}\%$ of what number is 7.85?

56. $73\frac{1}{2}\%$ of what number is 4.41?

57. What number is $56\frac{2}{3}\%$ of 51?

58. What number is $64\frac{1}{2}\%$ of 4?

59. What number is $87\frac{1}{2}\%$ of 70?

60. What number is $146\frac{1}{4}\%$ of 4?

---

61. It was reported that 80% of the retail price of milk was for packaging and distribution. The remaining 20% was paid to the dairy farmer. If a gallon of milk cost $3.80, how much of the retail price did the farmer receive?

62. At $1.689 per gallon of gas the cost is distributed as follows:

Crude oil supplies	$0.95
Oil Companies	$0.23
State and City taxes	$0.23
Federal tax	$0.19
Service Station	$0.10

Data is from Money, March 2009 p. 22, based on U. S. averages in December 2008. Answer the following questions rounded to the nearest whole percent.

a) What % of the cost is paid for crude oil supplies?

b) What % of the cost is paid to the service station?

Answers

1. 28

3. 150

5. 66

7. 120

9. 125

11. 60

13. 152

15. 85

17. 30

19. 116

21. 14

23. 23

25. 77

27. 45

29. 112

31. 75

33. 22

35. 150

37. 60

39. 3.26

41. 20

43. 35

45. 110

47. 4

49. 48

51. 64

53. 4

55. 10

57. 28.9

59. 61.25

61. $0.76

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4.7: General Applicants of Percent

In this section we will look at an assortment of practical problems involving percent.

Example 1

Myrna notes that 20% of her class is absent. If the class has 45 students, how many students are absent?

Solution

Let n represent the number of students that are absent. Then we can translate the problem statement into words and symbols.

Number absent	is	20%	of	total number of students in the class
n	$=$	20%	$\cdot$	45

Because 20% = 0.20,

$$\begin{aligned} n &= 0.20 \cdot 45 \quad 20 \\ n &= 9 \quad \text{Multiply: } 0.20 \cdot 45 = 9. \end{aligned}$$

Therefore, 9 students are absent.

Exercise

Aaron notes that 15% of his class is absent. If the class has 80 students, how many students are absent?

Answer

12

Example 2

Misty answered 90% of the questions on her mathematics examination correctly. If Misty had 27 correct answers, how many questions were on the exam?

Solution

Let *N* represent the number of questions on the examination.

Number of correct answers	is	90%	of	total number of questions
27	$=$	90%	$\cdot$	N

Because 90% = 0.90, this last equation can be written as

$$27 = 0.90N.$$

Solve for *N*.

$$\begin{aligned} \frac{27}{0.90} &= \frac{0.90N}{0.90} \quad \text{Divide both sides by } 0.90. \\ 30 &= N \quad \text{Divide: } 27/0.90 = 30. \end{aligned}$$

Hence, there were 30 questions on the examination.

Exercise

Erin asnwered 85% of the questions on her english examination correctly. If she had 34 correct answers, how many questions were on her exam?

Answer

40

Example 3

Misty answered 30 of 40 possible questions on her sociology examination correctly. What percent of the total number of questions did Misty mark correctly?

Solution

Let p represent the percent of the total number of questions marked correctly. Then we can translate the problem statement into words and symbols.

Number of correct answers	is	what percent	of	total number of questions
30	=	p	·	40

Because multiplication is commutative, we can write the last equation in the form

$$30 = 40p.$$

Solve for p.

$$\frac{30}{40} = \frac{40p}{40} \quad \text{Divide both sides by 40.}$$
$$\frac{3}{4} = p \quad \text{Reduce: } 30/40 = 3/4.$$

We need to change p = 3/4 to a percent. There are two ways to do this:

- We can divide 3 by 4 to get

$$p = \frac{3}{4}$$
$$= 0.75 \quad \text{Divide: } 3/4{=}0.75.$$
$$= 75\% \quad \text{Move decimal point 2 places right.}$$

- We can create an equivalent fraction with a denominator of 100; i.e.,

$$p = \frac{3}{4}$$
$$= \frac{3 \cdot 25}{4 \cdot 25} \quad \text{Multiply numerator and denominator by 25.}$$
$$= \frac{75}{100} \quad \text{Simplify numerator and denominator.}$$
$$= 75\%. \quad \text{Percent means parts per hundred.}$$

Either way, Misty got 75% of the questions on her sociology examination correct.

Exercise

Alphonso answered 19 of 25 questions on his biology test correctly. What percent of the questions did he mark correctly?

Answer

76%

Example 4

35 millilitres of a 60 millilitre solution is hydrochloric acid. What percent of the solution is hydrochloric acid?

Solution

Let p represent the percent of the percent of the solution that is hydrochloric acid. Then we can translate the problem statement into words and symbols.

Amount of hydrochloric acid	is	what percent	of	the tota amount of solution
35	$=$	p	$\cdot$	60

Because multiplication is commutative, we can write the right-hand side of the last equation as follows.

$$35 = 60p$$

Now we can solve for p.

$$\begin{aligned} \frac{35}{60} &= \frac{60p}{60} && \text{Divide both sides by 60.} \\ \frac{7}{12} &= p && \text{Reduce: Divide numerator and denominator by 5.} \end{aligned}$$

Now we must change p to a percent. We can do this exactly by creating an equivalent fraction with a denominator of 100.

$$\frac{7}{12} = \frac{n}{100}$$

Solve for n.

$$\begin{aligned} 12n &= 700 && \text{Cross multiply.} \\ \frac{12n}{12} &= \frac{700}{12} && \text{Divide both sides by 12.} \\ n &= \frac{175}{3} && \text{Reduce: Divide numerator and denominator by 4.} \\ n &= 58\frac{1}{3} && \text{Change improper to mixed fraction.} \end{aligned}$$

Hence,

$$p = \frac{7}{12} = \frac{58\frac{1}{3}}{100} = 58\frac{1}{3}\%.$$

Thus, $58\frac{1}{3}\%$ of the solution is hydrochloric acid.

Approximate Solution

If all that is needed is an approximate answer, say correct to the nearest tenth of a percent, then we would take a different approach starting with the line from above that has

$$\frac{35}{60} = p.$$

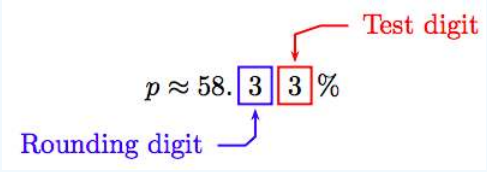
We would divide 35 by 60 to get

$$p \approx 0.5833.$$

Move the decimal two places to the right and append a percent symbol.

$$p \approx 0.5833 \approx 0\,58.33\% \approx 58.33\%.$$

Round to the nearest tenth of a percent.



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, correct to the nearest tenth of a percent,

$$p \approx 58.3\%.$$

Note that p ≈ 58.3% is approximate, but $p = 58\frac{1}{3}\%$ is exact.

? Exercise

25 millilitres of a 40 millilitre solution is sulfuric acid. What percent of the solution is sulfuric acid?

Answer

62.5%

Exercises

1. 31 millilitres of a 250 millilitre solution is sulphuric acid. What percent of the solution is sulphuric acid? Round your answer to the nearest tenth of a percent.

2. 34 millilitres of a 211 millilitre solution is phosphoric acid. What percent of the solution is phosphoric acid? Round your answer to the nearest tenth of a percent.

3. A family has completed 186 miles of a planned 346 mile trip. Find the percentage of the planned trip already traveled. Round your answer to the nearest percent.

4. A family has completed 153 miles of a planned 431 mile trip. Find the percentage of the planned trip already traveled. Round your answer to the nearest percent.

5. Erin takes roll in her fifth grade class and finds that 19 out of 34 total students on her roster are present. Find the percentage of the class that is present, correct to the nearest percent.

6. Barbara takes roll in her fifth grade class and finds that 15 out of 38 total students on her roster are present. Find the percentage of the class that is present, correct to the nearest percent.

7. Raven answered 135 of 150 possible questions on the meteorology examination correctly. What percent of the total number of questions did Raven mark correctly?

8. Liz answered 30 of 50 possible questions on the algebra examination correctly. What percent of the total number of questions did Liz mark correctly?

9. A family has traveled 114 miles of a planned trip. This is 37% of the total distance they must travel on the trip. Find, correct to the nearest mile, the total distance they will travel on their trip.

10. A family has traveled 102 miles of a planned trip. This is 23% of the total distance they must travel on the trip. Find, correct to the nearest mile, the total distance they will travel on their trip.

11. Trudy takes roll in her class at the university and finds that 65 students are present. If this is 50% of the total class enrollment, how many students are in the class?

12. Sandra takes roll in her class at the university and finds that 104 students are present. If this is 80% of the total class enrollment, how many students are in the class?

13. Bill earns a commission on all sales he makes. He sells a bed for $591 and earns a commission of $43. Find the percent commission, rounded to the nearest tenth of a percent.

14. Ira earns a commission on all sales he makes. He sells a sofa for $408 and earns a commission of $39. Find the percent commission, rounded to the nearest tenth of a percent.

15. Tami answered 70% of the questions on the physics examination correctly. If Tami had 98 correct answers, how many questions were on the exam?

16. Trinity answered 90% of the questions on the chemistry examination correctly. If Trinity had 99 correct answers, how many questions were on the exam?

17. A state charges 8% sales tax on all sales. If the sales tax on a computer is $20, find the sales price of the computer, correct to the nearest dollar.

18. A state charges 6.5% sales tax on all sales. If the sales tax on a bed is $33, find the sales price of the bed, correct to the nearest dollar.

19. Kenon earns 6% commission all his sales. If the sale of a computer earns him a $37 commission, find the sales price of the computer, correct to the nearest dollar.

20. Donald earns 4.5% commission all his sales. If the sale of a dryer earns him a $24 commission, find the sales price of the dryer, correct to the nearest dollar.

21. A 23% nitric acid solution contains 59 millilitres of nitric acid. How many total millilitres of solution are present? Round your answer to the nearest millilitre.

22. A 27% sulphuric acid solution contains 67 millilitres of sulphuric acid. How many total millilitres of solution are present? Round your answer to the nearest millilitre.

23. In a state, a television sold for $428 is assessed a sales tax of $45. Find the sales tax rate, rounded to the nearest tenth of a percent.

24. In a state, a refrigerator sold for $503 is assessed a sales tax of $44. Find the sales tax rate, rounded to the nearest tenth of a percent.

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25. **Mars gravity.** The force of gravity on Mars is only 38% of the force of gravity on earth. If you weigh 150 pounds on earth, how much will you weigh on Mars?

26. **Wiretaps.** In 2008, there were a total of 1,891 applications to federal and state judges to authorize the interception of wire, oral, or electronic communications. If 94% of all wiretap applications were for a portable device such as a cell phone or pager, how many applications were made to tap mobile devices? Round-off to the nearest application. *Associated Press Times-Standard 4/28/09*

27. **Seniors.** 13% of Humboldt County's population is age 65 and older, about 2% more than the state's average. If the population of Humboldt County is approximately 130,000, how many people in Humboldt County are age 65 and older? *Times-Standard 6/10/2009*

28. **Antibiotics.** "The U.S. used about 35 million pounds of antibiotics last year. 70 percent of the drugs went to pigs, chickens, and cows." How many million pounds of antibiotics went to the pigs, chickens, and cows? Associated Press-Times-Standard 12/29/09 Pressure rises to stop antibiotics in agriculture.

29. **Grow faster.** "Approximately 28 million pounds of antibiotics were fed to farm animals in the US during 2008. Thirteen percent of that was fed to healthy animals to make them grow faster." How many pounds of antibiotics were fed to healthy animals? *Associated Press-Times-Standard 12/29/09 Pressure rises to stop antibiotics in agriculture.*

30. **CO2 emissions.** The accord agreed to by the US at the Copenhagen climate talks had greenhouse gas emissions held to 3.5% of 1990 levels. If 1990 levels were 5022 MMT (millions of metric tons), how many millions of metric tons might greenhouse emissions be held to? Round the result to the nearest MMT. *Associated Press-Times-Standard 12/19/09 Elements of new Copenhagen accord.*

31. **Water supply.** A new water desalination plant, the largest in the Western hemisphere, could come online by 2012 in Carlsbad, California, providing 50 million gallons of drinking water per day, or 10% of the supply for San Diego County. What is the total amount of drinking water supplied to San Diego County daily? *Associated Press-Times-Standard*

32. **Earthquake damage.** After the recent earthquake in Chile, an estimated 33 million gallons of Chilean wine, or 13% of annual production, was lost. Estimate the total annual production of Chilean wine rounded to the nearest millions of gallons. *Associated Press-Times Standard 03/24/10 Hemorrhaging cabernet: Earthquake hits winemakers in Chile.*

33. **Snowpack.** At a meadow near Echo Summit in the northern Sierra Nevada, water officials measured the snow at 65.7 inches. The water content was 25.9 inches, which is 92% of the average for this time of year. Determine the average water content for this time of year rounded to the nearest tenth of an inch. *Associated Press-Times Standard 04/02/10 California's Sierra snowpack slightly above normal.*

34. **Storefronts.** According to the Times-Standard, as of April 2008 the Bayshore Mall had 55 occupied storefronts and 17 vacant storefronts. What percent of total storefronts are vacant? Round your answer to the nearest whole number. *Times-Standard 4/19/09*

35. **Recovered.** In Humboldt County, California, 427 of the 499 vehicles stolen between August 2008 and August 2009 were recovered. What percent of the stolen vehicles were recovered? Round your result to the nearest tenth of a percent. *Times-Standard CHP offers tips on avoiding vehicle theft.*

36. **Freshman admissions.** Stanford University sent acceptance letters to 2, 300 of 32, 022 freshman applicants. What percent of freshman applicants got acceptance letters, rounded to the nearest percent? *Associated Press-Times-Standard 03/30/10 Stanford U. reports record-low admission rate.*

37. **Reduce.** Each year, Americans throw out an average of about 1, 600 pounds of waste per person. Arcata, CA resident Michael Winkler only uses one trash bag every year – totaling at most 40 pounds. Find the percent of average annual waste per person Mr. Winkler throws out to a tenth of a percent. *Times-Standard Allison White 12/26/09 Waste not...*

38. **Population decrease.** The table below shows the population of Detroit, Michigan. *Associated Press-Times-Standard 03/09/10 Detroit wants to save itself by shrinking.*

Year	Population
1950	1,849,568
1990	1,027,974
2005	890,963

What is the population of Detroit in 2005 as a percent of the population in 1950? Round your result to the nearest percent.

Answers

1. 12.4

3. 54

5. 56

7. 90

9. 308 mi

11. 130 students

13. 7.3

15. 140

17. $250

19. $617

21. 257 ml

23. 10.5

25. 57 pounds

27. 16,900

29. 3.84 million pounds

31. 500 million gallons

33. The average water content is 28.2 inches.

35. 85.6% of the stolen vehicles were recovered.

37. Mr. Winkler throws out 2.5% of the average American's waste.

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4.8: Percent Increase or Decrease

A person's salary can increase by a percentage. A town's population can decrease by a percentage. A clothing firm can discount its apparel. These are the types of applications we will investigate in this section.

Percent Increase

Example 1

Add text here.

Solution

Let *x* represent the salesperson's salary increase. Then we can translate the problem into words and symbols.

Salary increase	is	5%	of	original salary
x	$=$	5%	$\cdot$	4000

Solve for *x*.

$$x = 0.05 \cdot 4000 \quad 5\% = 0.05.$$
$$x = 200 \quad \text{Multiply: } 0.05 \cdot 4000 = 200.$$

Therefore, the salary increase is $200. To compute the new salary *N*, we must add this increase to the original salary.

New salary	is	original salary	plus	increase
N	$=$	4000	$+$	200

Thus, the new salary is N = $4, 200 per month.

Alternative Solution

If the salesperson is to receive a 5% increase in his salary, then his new salary will be 105% of his original salary. Let N represent his new monthly salary. Then,

New salary	is	105%	of	original salary
N	$=$	105%	$\cdot$	4000

Solve for *N*.

$$N = 1.05 \cdot 4000 \quad 105\% = 1.05.$$
$$N = 4200 \quad \text{Multiply: } 1.05 \cdot 4000 = 4200.$$

Same answer.

Exercise

A computer technician is granted a 4% salary increase. If the salesperson's current salary is $2,800 per month, what will be his new salary?

Answer

$2,912

Example 2

A salesperson making a salary of $4,500 per month has his salary increased to $5,000 per month. What is the percent increase?

Solution

To find the increase in salary, first subtract the original salary from the new salary.

$$\begin{aligned}\text{Salary increase} &= \text{new salary} - \text{original salary}\\ &= 5000 - 4500\\ &= 500\end{aligned}$$

Hence, the salesperson sees an increase in salary of $500.

Next, let p represent the salesperson's percent salary increase. Then we can translate the problem into words and symbols.

Salary increase	is	what percent	of	original salary
500	=	p	·	4500

The commutative property of multiplication allows us to change the order of multiplication on the right-hand side of this last equation.

$$500 = 4500p$$

Solve for p.

$$\begin{aligned}\frac{500}{4500} &= \frac{4500p}{4500} && \text{Divide both sides by 4500.}\\ \frac{1}{9} &= p && \text{Reduce by dividing numerator and denominator of } 500/4500 \text{ by 500.}\end{aligned}$$

We need to change p = 1/9 to a percent. We can find an exact answer by creating an equivalent fraction with a denominator of 100.

$$\begin{aligned}\frac{1}{9} &= \frac{n}{100} && \text{Make equivalent fraction.}\\ 9n &= 100 && \text{Cross multiply.}\\ \frac{9n}{9} &= \frac{100}{9} && \text{Divide both sides by 9.}\\ n &= 11\frac{1}{9} && \text{Convert } 100/9 \text{ to mixed fraction.}\end{aligned}$$

Hence, the percent increase is

$$p = \frac{1}{9} = \frac{11\frac{1}{9}}{100} = 11\frac{1}{9}\%.$$

Alternative Solution

An alternative approach is to ask what percent of the original salary equals the new salary. In this approach, let p represent the percent of the original salary that equals the new salary.

New salary	is	what percent	of	original salary
5000	=	p	·	4500

Solve for p.

$$\begin{aligned}5000 &= 4500p && \text{Change the order of multiplication.}\\ \frac{5000}{4500} &= \frac{4500p}{4500} && \text{Divide both sides by 4500.}\\ \frac{10}{9} &= p && \text{Reduce: Divide numerator and denominator of } 5000/4500 \text{ by 500.}\end{aligned}$$

We need to change 10/9 to a percent. Again, create an equivalent fraction with a denominator of 100.

$$\begin{aligned} \frac{10}{9} &= \frac{n}{100} && \text{Make equivalent fraction.} \\ 9n &= 1000 && \text{Cross multiply.} \\ \frac{9n}{9} &= \frac{1000}{9} && \text{Divide both sides by 9.} \\ n &= 111\frac{1}{9} && \text{Convert } 1000/9 \text{ to a mixed fraction.} \end{aligned}$$

Thus,

$$p = \frac{10}{9} = \frac{111\frac{1}{9}}{100} = 111\frac{1}{9}\%.$$

Hence, the new salary is $111\frac{1}{9}\%$ of the original salary. To find the percent increase, subtract 100% from $111\frac{1}{9}\%$.

$$111\frac{1}{9}\% - 100\% = 11\frac{1}{9}\%$$

This represents an $11\frac{1}{9}\%$ increase in salary, which is the same answer garnered by the first solution technique.

? Exercise

A statistician making a salary of $3,200 per month has his salary increased to $3,368 per month. What is the percent increase?

Answer

$5\frac{1}{4}\%$

Percent Decrease

✓ Example 3

Due to a mill closure, the population of Silvertown decreases by 8.5%. If the original population was 10,200 hardy souls, what is the new population?

Solution

Let *x* represent the population decrease. Then we can translate the problem into words and symbols.

Population decrease	is	8.5%	of	original population
x	$=$	8.5%	$\cdot$	10200

Solve for *x*.

$$\begin{aligned} x &= 0.085 \cdot 10200 && 8.5\% = 0.085. \\ x &= 867 && \text{Multiply: } 0.085 \cdot 10200 = 867. \end{aligned}$$

Therefore, the population decrease is 867. To compute the new population P, we must subtract this decrease from the original population.

New population	is	original population	minus	population decrease
P	$=$	10200	$-$	867

Thus, the new population is P = 9,333 hardy souls.

Alternative Solution

Subtract 8.5% from 100% to obtain

$$100\% - 8.5\% = 91.5\%.$$

Thus, if 8.5% of the population leaves town, then 91.5% of the population stays. Thus, the new population P is calculated from the original as follows:

New population	is	91.5%	of	original population
P	$=$	91.5%	$\cdot$	10200

Solve for P.

$$\begin{aligned} P &= 0.915 \cdot 10200 && 91.5\% = 0.915. \\ P &= 9333 && \text{Multiply: } 0.915 \cdot 10200 = 9333. \end{aligned}$$

Same answer.

Exercise

Several retail stores close and the population of Athens decreases by 7.2% as a result. If the original population was 12,500, what is the new population?

Answer

11,600

Example 4

Millertown falls on hard times and its population decreases from 11,256 to 10,923 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

Solution

To find the decrease in population, first subtract the current population from the original population.

$$\begin{aligned} \text{Population decrease} &= \text{original population} - \text{current population} \\ &= 11256 - 10923 \\ &= 333 \end{aligned}$$

Hence, the population has decreased by 333 people.

Next, let p represent the percent population decrease. Then we can translate the problem into words and symbols.

Population decrease	is	what percent	of	original population
333	$=$	p	$\cdot$	11256

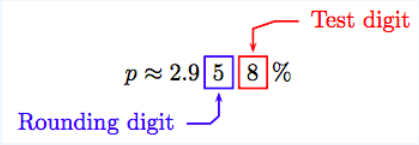
Solve for p.

$$\begin{aligned} \frac{333}{11256} &= \frac{11256p}{11256} && \text{Divide both sides by } 11256. \\ 0.02958 &\approx p && \text{Divide: } 333/11256 \approx 0.02958. \end{aligned}$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.02958 \approx 0\,02.958\% \approx 2.958\%.$$

We are asked to round to the nearest hundredth of a percent.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. That is,

$$p \approx 2.96\%.$$

Thus, the population of Millertown decreases approximately 2.96%.

Alternative Solution

An alternative approach is to ask what percent of the original population equals the new population.

New population	is	what percent	of	original population
10923	=	p	·	11256

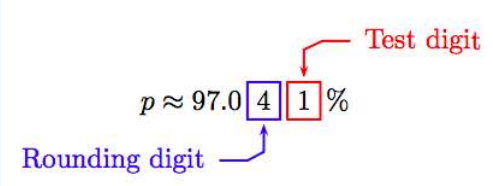
Solve for p.

$$\begin{aligned} 10923 &= 11256p && \text{Change the order of multiplication.} \\ \frac{10923}{11256} &= \frac{11256p}{11256} && \text{Divide both sides by } 11256. \\ 0.97041 &\approx p && \text{Divide: } 10923/11256 \approx 0.97041. \end{aligned}$$

To change p to a percent, move the decimal two places to the right and append a percent symbol.

$$p \approx 0.97041 \approx 0\,97.041\% \approx 97.041\%.$$

We are asked to round to the nearest hundredth of a percent.



Because the test digit is less than 5, leave the rounding digit alone and truncate. That is,

$$p \approx 97.04\%.$$

Thus, 97.04% of the Millertown population remains. To find the percent decrease (the percent who left), subtract 97.04% from 100%.

$$100\% - 97.04\% = 2.96\%$$

Hence, the population of Millertown decreases by 2.96%. Same answer.

? Exercise

A textile mill closure results in the population of the adjacent town decreasing from 8,956 to 7,800. What is the percent decrease in the population, rounded to the nearest tenth of a percent?

Answer

12.9%

Discount

Another important application of percent is the discounting of goods.

✔ Example 5

A pair of skis is marked at $310. However, a sign in the shop indicates that skis are being discounted at 15%. What will be the new selling price of the skis?

Solution

Let D represent the discount (in dollars) given for our pair of skis. Then, in words and symbols:

Discount	is	15%	of	original marked price
D	$=$	15%	$\cdot$	310

Solve for D.

$$D = 0.15 \cdot 310 \quad 15\% = 0.15.$$
$$D = 46.50 \quad \text{Multiply: } 0.15 \cdot 310 = 46.50.$$

Hence, the discount is \$46.50. To find the new selling price, subtract this discount from the original selling price.

$$\begin{aligned} \text{New selling price} &= \text{original selling price} - \text{discount} \\ &= 310 - 46.50 \\ &= 263.50 \end{aligned}$$

Hence, the new selling price is \$263.50.

Alternate Solution

Subtract 15% from 100% to obtain

$$100\% - 15\% = 85\%.$$

That is, if an item is discounted 15%, then its new selling price S is 85% of its original marked price.

New selling price	is	85%	of	original marked price
S	$=$	85%	$\cdot$	310

Solve for S.

$$S = 0.85 \cdot 310 \quad 85\% = 0.85.$$
$$S = 263.50 \quad \text{Multiply: } 0.85 \cdot 310 = 263.50.$$

Thus, the new selling price is \$263.50. Same answer.

? Exercise

A pair of hiking boots is marked at \$200. During a sale, the boots are discounted by 8%. What is the new price of the boots?

Answer

\$184

✔ Example 6

A pair of ski boots marked at \$210 is sold for \$180. Find the percent discount, correct to the nearest tenth of a percent.

Solution

We can find the discount (in dollars) by subtracting the sale price from the original marked price.

$$\begin{aligned} \text{Discount} &= \text{original marked price} - \text{sale price} \\ &= 210 - 180 \\ &= 30 \end{aligned}$$

Hence, the boots are discounted \$30.

Let p represent the percent discount. Then, in words and symbols:

Discount	is	percent discount	of	original marked price
30	$=$	p	$\cdot$	210

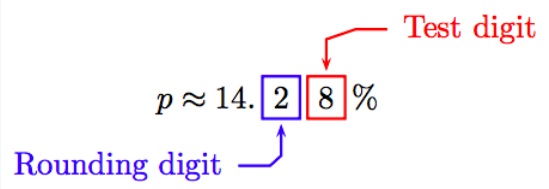
Solve for p.

$$\begin{aligned} 30 &= 210p && \text{Change order of multiplication.} \\ \frac{30}{210} &= \frac{210p}{210} && \text{Divide both sides by 210.} \\ \frac{1}{7} &= p && \text{Reduce: Divide numerator and denominator of } 30/210 \text{ by 30.} \\ p &\approx && \text{Divide: } 1/7 \approx 0.1428. \end{aligned}$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.1428 \approx 0\,14.28\% \approx 14.28\%.$$

Round to the nearest tenth of a percent.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus, correct to the nearest tenth of a percent, the percent discount is

$$p \approx 14.3\%.$$

Alternate Solution

An alternative approach is to ask what percent p of the original marked price equals the selling price.

New selling price	is	what percent	of	original marked price
180	=	p	·	210

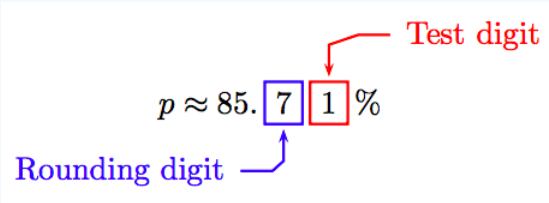
Solve for p.

$$\begin{aligned} 180 &= 210p && \text{Change the order of multiplication.} \\ \frac{180}{210} &= \frac{210p}{210} && \text{Divide both sides by 210.} \\ \frac{6}{7} &= p && \text{Reduce: Divide numerator and denominator of } 180/210 \text{ by 30.} \\ p &\approx && \text{Divide: } 6/7 \approx 0.8571. \end{aligned}$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.8571 \approx 0\,85.71\% \approx 85.71\%.$$

Round to the nearest tenth of a percent.



Because the test digit is less than 5, do nothing to the rounding digit and truncate. Thus, correct to the nearest tenth of a percent,

$$p \approx 85.7\%.$$

Thus, the new selling price is 85.7% of the original marked price. Subtract 85.7% from 100%.

$$100\% - 85.7\% = 14.3\%.$$

That is, if the new selling price is 85.7% of the original price, then the percent discount is 14.3%. This is the same answer found with the first method.

? Exercise

A computer marked at $2,000 is sold at a discount for $1,850. Find the percent discount, correct to the nearest tenth of a percent.

Answer

7.5%

Exercises

1. A television set is marked at $447. However, a sign in the shop indicates that the television set is being discounted at 20.5%. What will be the new selling price of the television set? Round your answer to the nearest penny.

2. A stereo set is marked at $380. However, a sign in the shop indicates that the stereo set is being discounted at 7.5%. What will be the new selling price of the stereo set? Round your answer to the nearest penny.

3. Due to a ball bearing plant closure, Anselm falls on hard times and its population decreases from 10,794 to 8,925 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

4. Due to a logging mill closure, Carlytown falls on hard times and its population decreases from 12,113 to 10,833 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

5. A car rack is marked at $500. However, a sign in the shop indicates that the car rack is being discounted at 3.5%. What will be the new selling price of the car rack? Round your answer to the nearest penny.

6. A car rack is marked at $405. However, a sign in the shop indicates that the car rack is being discounted at 17.5%. What will be the new selling price of the car rack? Round your answer to the nearest penny.

7. Due to a textile mill closure, the population of Silvertown decreases by 4.1%. If the original population was 14,678 hardy souls, what is the new population, correct to the nearest person?

8. Due to a department store closure, the population of Petroria decreases by 5.3%. If the original population was 14,034 hardy souls, what is the new population, correct to the nearest person?

9. A bartender is granted a 4.6% salary increase. If the bartender's current salary is $2,500 per month, find the bartender's new monthly salary, rounded to the nearest dollar.

10. A bartender is granted a 5.5% salary increase. If the bartender's current salary is $2,900 per month, find the bartender's new monthly salary, rounded to the nearest dollar.

11. A car rack marked at $358 is sold for $292. Find the percent discount, correct to the nearest tenth of a percent.

12. A bicycle marked at $328 is sold for $264. Find the percent discount, correct to the nearest tenth of a percent.

13. Due to a auto manufacturing plant closure, Carlytown falls on hard times and its population decreases from 14,393 to 12,623 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

14. Due to a ball bearing plant closure, Mayville falls on hard times and its population decreases from 8,494 to 6,609 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

15. Due to a auto manufacturing plant closure, the population of Silvertown decreases by 2.4%. If the original population was 8,780 hardy souls, what is the new population, correct to the nearest person?

16. Due to a textile mill closure, the population of Ghosttown decreases by 6.1%. If the original population was 14,320 hardy souls, what is the new population, correct to the nearest person?

17. A clerk making a salary of $2,600 per month has her salary increased to $2,950 per month. Find the percent increase correct to the nearest tenth of a percent.

18. A clerk making a salary of $3,600 per month has her salary increased to $4,100 per month. Find the percent increase correct to the nearest tenth of a percent.

19. A bartender making a salary of $4,200 per month has her salary increased to $4,300 per month. Find the percent increase correct to the nearest tenth of a percent.

20. A bartender making a salary of $3,200 per month has her salary increased to $3,550 per month. Find the percent increase correct to the nearest tenth of a percent.

21. A gardener is granted a 5.1% salary increase. If the gardener's current salary is $3,200 per month, find the gardener's new monthly salary, rounded to the nearest dollar.

22. A secretary is granted a 2.8% salary increase. If the secretary's current salary is $3,600 per month, find the secretary's new monthly salary, rounded to the nearest dollar.

23. A television set marked at $437 is sold for $347. Find the percent discount, correct to the nearest tenth of a percent.

24. A camera marked at $390 is sold for $328. Find the percent discount, correct to the nearest tenth of a percent.

---

25. Suppose that the price of an 8-ounce can of tomato sauce increased from $0.20 to $0.28.

    a) What was the amount of increase?

    b) What was the percent increase?

26. The following table summarizes summertime gasoline prices in San Francisco, CA. The price is the number of dollars required to purchase one gallon of unleaded gasoline. *Data from gasbuddy.com.*

Year	Price per gallon
2003	1.80
2004	2.28
2005	2.57
2006	3.20
2007	3.28
2008	4.61
2009	3.01

What is the percent increase or decrease from 2003 to 2005? Round your answer to the nearest whole percent.

27. Refer to the table of gas prices in Exercise 26. What is the percent increase or decrease from 2005 to 2007? Round your answer to the nearest whole percent.

28. Refer to the table of gas prices in Exercise 26. What is the percent increase or decrease from 2007 to 2009? Round your answer to the nearest whole percent.

29. **Rate hike.** A family paying $858 monthly for health coverage is faced with a 39% hike in rates. What will be their new monthly cost after the increase? *Associated Press-Times-Standard 02/09/10 HHS secretary asks Anthem Blue Cross to justify rate hike.*

30. **Parking fine.** San Francisco's Metropolitan Transportation Agency was expected to consider raising fines for the use of fake, stolen, or borrowed disabled parking tags from $100 to $825. What is the percent increase for this fine? *Associated Press-Times-Standard 01/06/10 Fines for fake disabled parking tags may go up in San Francisco.*

31. **Industrial move.** Regulations in California have caused factory farmers to move out of state. Idaho's industrial dairy went from 180, 000 cows in 1990 to 530, 000 cows in 2009. What is the percent increase for Idaho industrial dairy cows, rounded to the nearest percent? *Associated Press-Times-Standard 02/09/10 Idaho, others prepare for California egg farm exodus.*

32. **Whooping crane.** Whooping crane populations made a remarkable comeback from just 15 birds in 1941 to about 400 birds worldwide in 2010. What is the percent increase for the whooping crane population over the past seventy years, rounded to the nearest percent? *Associated Press-Times-Standard 03/17/10 Plucky whooping crane gives wildlife experts hope.*

33. **Underwater.** The loss of Arctic sea ice will allow for an underwater fiber optic cable that will cut the time it takes to send a message from London to Tokyo from a current 140 milliseconds down to 88 milliseconds. *Associated Press-Times-Standard 01/22/10 Global warming opens up Northwest Passage for underwater fiber optic cable.*

a) What is the estimated percent decrease in communication time?

b) At a cost of $1.2 billion, what is the cost per millisecond of saving (rounded to the nearest dollar)?

34. **Chinook salmon.** During the 2009 season in the Sacramento river basin, a record low 39,500 chinook salmon were recorded, way down from the more than 750,000 counted in 2002. What is the percent decrease in the salmon count from 2002 to 2009, rounded to the nearest percent? *Associated Press-Times-Standard 02/25/10 Feds predict better year for California salmon.*

35. **Daylight hours.** In middle latitudes, summer days can have as many as 14 hours of daylight, while winter days can have a few as 10 hours of daylight. What percent more daylight is there in summer than in winter?

36. **Cyber-experts.** Defense Secretary Robert Gates said the Pentagon will increase the number of cyber-experts it can train each year from 80 to 250 by 2011. What percent increase is this? Round your answer to the nearest percent. *Associated Press Times-Standard 4/19/09*

37. **Home prices.** Real estate data for Humboldt County, California, is given below. *Associated Press-Times-Standard 01/17/10 How is our local real estate market compared to other regions?*

Year	Number of homes sold	Average home price
2000	1,358	$152,257
2005	1,432	$344,500
2009	833	$285,000

a) What percent change in average home price occurred between 2000 and 2009?

b) What percent change in homes sold occurred between 2000 and 2009?

Answers

1. $355.36

3. 17.32%

5. $482.50

7. 14,076 people

9. $2,615

11. 18.4%

13. 12.30%

15. 8,569

17. 13.5%

19. 2.4%

21. $3,363

23. 20.6%

25.

a) $0.08

b) 40%

27. 28% increase

29. $1,193

31. 194% increase

33.

a) 37%

b) About $23,076,923 per millisecond

35. 40% more daylight

37.

a) 87% increase in average home price

b) 39% decrease in home sales

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4.9: Interest

One way of awarding interest is called simple interest. Before we provide the formula used in calculating simple interest, let's first define some basic terms.

Balance. The balance is the current amount in an account or the current amount owed on a loan.

Principal. The principal is the initial amount invested or borrowed.

Rate. This is the interest rate, usually given as a percent per year.

Time. This is the time duration of the loan or investment. If the interest rate is per year, then the time must be measured in years.

To calculate the simple interest on an account or loan, use the following formula.

Simple Interest

Simple interest is calculated with the formula

$$I = Prt,$$

where I is the interest, P is the principal, r is the interest rate, and t is the time.

Example 1

How much simple interest is earned if $1,200 is invested at 4% per year for 5 years?

Solution

Set up the formula for simple interest.

$$I = Prt$$

The principal is P = $1200, the interest rate is r = 4% = 0.04 per year, and the time or duration of the investment is t = 5 years. Substitute each of these numbers into the simple interest formula $I = Prt$.

$$\begin{aligned} I &= (1200)(0.04)(5) \quad \text{Substitute 1200 for } P, 0.04 \text{ for } r, \text{ and 5 for } t. \\ &= 240 \quad \text{Multiply.} \end{aligned}$$

Hence, the interest earned in 5 years is $240.

Exercise

How much simple interest is earned if $2,500 is invested at 5% per year for 8 years?

Answer

$1,000

To find the balance, we must add the interest to the principal.

Calculating the Balance

To find the balance, add the interest to the principal. That is,

$$\text{Balance} = \text{Principal} + \text{Interest}.$$

Example 2

A contractor borrows $5,000 at 4.5% per year. The interest accrued is simple interest. The duration of the loan is 6 months. How much will the contractor have to pay back at the end of the 6-month loan period?

Solution

Set up the formula for simple interest.

$$I = Prt$$

The principal is P = \$5000, the interest rate is r = 4.5% = 0.045 per year, and the time or duration of the loan is t = 6 months. Because the interest rate is per year, the time must be changed to years. That is,

$$\begin{aligned} 6 \text{ months} &= 6 \text{ months} \cdot \frac{1 \text{ yr}}{12 \text{ months}} && \text{Apply conversion factor.} \\ &= 6 \cancel{\text{ months}} \cdot \frac{1 \text{ yr}}{12 \cancel{\text{ months}}} && \text{Cancel common unit.} \\ &= \frac{6}{12} \text{ yr} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{1}{2} \text{ yr} && \text{Reduce.} \end{aligned}$$

Substitute these numbers into the simple interest formula $I = Prt$.

$$\begin{aligned} I &= (5000)(0.045)\left(\frac{1}{2}\right) && \text{Substitute 5000 for } P,\ 0.045 \text{ for } r,\ \text{and } 1/2 \text{ for } t. \\ &= 112.50 && \text{Multiply.} \end{aligned}$$

Hence, the interest accrued in 6 months is \$112.50. Therefore,

$$\begin{aligned} \text{Amount owed} &= \text{Principal} + \text{Interest} \\ &= 5000 + 112.50 \\ &= 5112.50 \end{aligned}$$

That is, the amount owed at the end of the 6-month loan period is \$5,112.50.

? Exercise

An accountant borrows \$8,000 at 5.5% per year. The interest accrued is simple interest. The duration of the loan is 3 months. How much will the aacountant have to pay back at the end of the 3-month loan period?

Answer

\$8,110

✓ Example 3

A business owner takes out a 4-month loan at 5.4% per year simple interest. At the end of the 4-month loan period, the interest owed is \$90. What was the principal amount borrowed?

Solution

Set up the formula for simple interest.

$$I = Prt$$

The interest owed is I = \$90, the interest rate is r = 5.4% = 0.054 per year, and the time or duration of the loan is t = 4 months. Because the interest rate is per year, the time must be changed to years. That is,

$$\begin{aligned} 4 \text{ months} &= 4 \text{ months} \cdot \frac{1 \text{ yr}}{12 \text{ months}} && \text{Apply conversion factor.} \\ &= 4 \cancel{\text{ months}} \cdot \frac{1 \text{ yr}}{12 \cancel{\text{ months}}} && \text{Cancel common unit.} \\ &= \frac{4}{12} \text{yr} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{1}{3} \text{yr} && \text{Reduce.} \end{aligned}$$

Substitute these numbers into the simple interest formula $I = Prt$.

$$\begin{aligned} 90 &= P(0.054)\left(\frac{1}{3}\right) && \text{Substitute 90 for } I,\ 0.054 \text{ for } r,\ \text{and } 1/3 \text{ for } t. \\ 90 &= \frac{0.054}{3}P && \text{Rearrange order of multiplication.} \\ 90 &= 0.018P && \text{Divide: } 0.054/3 = 0.018. \end{aligned}$$

Solve the equation for *P*.

$$\begin{aligned} \frac{90}{0.018} &= \frac{0.018P}{0.018} && \text{Divide both sides by } 0.018. \\ 5000 &= P && \text{Divide: } 90/0.018 = 5000. \end{aligned}$$

Thus, the principal amount borrowed was $5,000.

Exercise

The owner of Alioto Motors takes out a 8-month loan at 4% per year simple interest. At the end of the 8-month loan period, the interest owed is $80. What was the principal amount borrowed?

Answer

$3,000

Example 4

A pet shop owner borrows $8,000 for 6 months. At the end of the 6-month loan period, the interest owed is $200. What was the simple interest rate?

Solution

Set up the formula for simple interest.

$$I = Prt$$

The principal is *P* = $8, 000, the interest owed is *I* = $200, and the duration of the loan is *t* = 6 months. As we saw in Example 2, 6 months equals 1/2 year. Substitute these numbers into the simple interest formula $I = Prt$.

$$\begin{aligned} 200 &= (8000)(r)\left(\frac{1}{2}\right) && \text{Substitute } 8000 \text{ for } P,\ 200 \text{ for } I,\ \text{and } 1/2 \text{ for } t. \\ 200 &= \frac{8000}{2}r && \text{Rearrange order of multiplication.} \\ 200 &= 4000r && \text{Divide: } 8000/2 = 4000. \end{aligned}$$

Solve this last equation for *r*.

$$\begin{aligned} \frac{200}{4000} &= \frac{4000r}{4000} && \text{Divide both sides by } 4000. \\ \frac{1}{20} &= r && \text{Reduce: Divide numerator and denominator by } 200. \end{aligned}$$

We need to change r to a percent. This is easily accomplished by creating an equivalent fraction with a denominator of 100.

$$\begin{aligned}\frac{1}{20} &= \frac{1 \cdot 5}{20 \cdot 5} \\ &= \frac{5}{100} \\ &= 5\%\end{aligned}$$

Thus, the simple interest rate is 5%.

Exercise

A manufacturer borrows $10,000 for 4 years. At the end of the 4-year loan period, the interest owed is $3,200. What was the simple interest rate?

Answer

8%

Extending the Simple Interest Formula

In Example 2, we had to add the interest to the principal to discover the balance owed at the end of the loan. That is,

$$\text{Balance} = \text{Principal} + \text{Interest},$$

or in symbols,

$$A = P + I,$$

where A is the balance, P is the principal, and I is the simple interest. Because $I = Prt$, we substitute Prt for I in the last equation to get

$$A = P + Prt.$$

Use the distributive property to factor P from each term on the right.

$$\begin{aligned}A &= P \cdot 1 + P \cdot rt \\ A &= P(1 + rt).\end{aligned}$$

Balance Formula Using Simple Interest

If simple interest is applied, then the balance is given by the formula

$$A = P(1 + rt),$$

where A is the balance, P is the principal, r is the simple interest rate, and t is the duration of the loan or investment.

Example 5

If $4,000 is invested at 6.25% simple interest, what will be the balance after 2 years?

Solution

Start with the balance formula for simple interest.

$$A = P(1 + rt)$$

The principal is P = $4, 000, the rate is r = 6.25% = 0.0625 per year, and the time is t = 2 years. Substitute these numbers in the balance formula $A = P(1 + rt)$.

$$A = 40001 + (0.0625)(2) \quad \text{Substitute } 4000 \text{ for } P,\ 0.0625 \text{ for } r,\ \text{and } 2 \text{ for } t.$$
$$A = 4000(1 + 0.125) \quad \text{Order of Ops: } 0.0625 \cdot 2 = 0.125.$$
$$A = 4000(1.125) \quad \text{Order of Ops: } 1 + 0.125 = 1.125.$$
$$A = 4500 \quad \text{Multiply: } 4000 \cdot 1.125 = 4500.$$

Hence, the balance at the end of two years is A = $4, 500.

Exercise

If $8,000 is invested at 4.25% simple interest, what will be the balance after 4 years?

Answer

$9,360

Example 6

The balance due on a 2-year loan is $3,360. If the simple interest rate is 6%, what was the principal borrowed?

Solution

Start with the balance formula for simple interest.

$$A = P(1 + rt)$$

The balance is A = $3360, the rate is r = 6% = 0.06 per year, and the time is t = 2 years. Substitute these numbers in the balance formula $A = P(1 + rt)$.

$$3360 = P\,(1 + (0.06)(2)) \quad \text{Substitute } 3360 \text{ for } A,\ 0.06 \text{ for } r,\ \text{and } 2 \text{ for } t.$$
$$3360 = P(1 + 0.12) \quad \text{Order of Ops: } 0.06 \cdot 2 = 0.12.$$
$$3360 = P(1.12) \quad \text{Order of Ops: } 1 + 0.12 = 1.12.$$
$$3360 = 1.12P \quad \text{Change order of multiplication.}$$

Solve this last equation for P.

$$\frac{3360}{1.12} = \frac{1.12P}{1.12} \quad \text{Divide both sides by } 1.12.$$
$$3000 = P \quad \text{Divide: } 3360/1.12 = 3000.$$

Hence, the principal borrowed was P = $3, 000.

Exercise

The balance due on a 4-year loan is $6,300. If the simple interest rate is 10%, what was the principal borrowed?

Answer

$4,500

Example 7

The balance due on a 2-year loan is $2,200. If the principal borrowed was $2,000, what was the rate of simple interest?

Solution

Start with the balance formula for simple interest.

$$A = P(1 + rt)$$

The balance is A = $2, 200, the principal is P = $2, 000, and the time is t = 2 years. Substitute these numbers in the balance formula $A = P(1 + rt)$.

$$2200 = 2000(1+(r)(2)) \quad \text{Substitute 2200 for } A,\ 2000 \text{ for } P,\ \text{and } t = 2.$$
$$2200 = 2000(1+2r) \quad \text{Change the order of muliplication.}$$

Solve this last equation for *r*.

$$2200 = 2000 + 4000r \quad \text{Distribute 2000.}$$
$$2200 - 2000 = 2000 + 4000r - 2000 \quad \text{Subtract 2000 from both sides.}$$
$$200 = 4000r \quad \text{Simplify both sides.}$$
$$\frac{200}{4000} = \frac{4000r}{4000} \quad \text{Divide both sides by 4000.}$$
$$\frac{1}{20} = r \quad \text{Reduce: } 200/4000 = 1/20.$$

Of course, *r* must be changed to a percent. In Example 4, we encountered this same fraction.

$$r = \frac{1}{20} = \frac{5}{100} = 5\%$$

Hence, the rate of simple interest is r = 5%.

? Exercise

The balance due on a 2-year loan is $4,640. If the principal borrowed was $4,000, what was the rate of simple interest?

Answer

8%

Exercises

1. How much simple interest is earned if $7,600 is invested at 8% per year for 7 years?
2. How much simple interest is earned if $2,500 is invested at 5% per year for 6 years?
3. How much simple interest is earned if $5,800 is invested at 3.25% per year for 4 years?
4. How much simple interest is earned if $2,000 is invested at 8.5% per year for 6 years?
5. How much simple interest is earned if $2,400 is invested at 8.25% per year for 5 years?
6. How much simple interest is earned if $4,000 is invested at 6.5% per year for 6 years?
7. How much simple interest is earned if $4,000 is invested at 7.25% per year for 6 years?
8. How much simple interest is earned if $8,200 is invested at 8% per year for 6 years?

---

9. A business owner borrows $3,600 for 2 months at a 4.5% per year simple interest rate. At the end of the 2-month loan period, how much interest is owed?
10. A business owner borrows $3,200 for 4 months at a 9% per year simple interest rate. At the end of the 4-month loan period, how much interest is owed?
11. A business owner borrows $2,400 for 6 months at a 2% per year simple interest rate. At the end of the 6-month loan period, how much interest is owed?
12. A business owner borrows $2,200 for 4 months at a 3% per year simple interest rate. At the end of the 4-month loan period, how much interest is owed?
13. A business owner takes out a 6-month loan at a 8% per year simple interest rate. At the end of the 6-month loan period, the interest owed is $68. What was the principal amount borrowed?
14. A business owner takes out a 4-month loan at a 6% per year simple interest rate. At the end of the 4-month loan period, the interest owed is $194. What was the principal amount borrowed?

15. A business owner borrows $3,600 for 3 months at a 8% per year simple interest rate. At the end of the 3-month loan period, how much interest is owed?

16. A business owner borrows $2,400 for 4 months at a 8.25% per year simple interest rate. At the end of the 4-month loan period, how much interest is owed?

17. A business owner takes out a 2-month loan at a 8.5% per year simple interest rate. At the end of the 2-month loan period, the interest owed is $85. What was the principal amount borrowed?

18. A business owner takes out a 3-month loan at a 2% per year simple interest rate. At the end of the 3-month loan period, the interest owed is $45. What was the principal amount borrowed?

19. A business owner borrows $4,000 for 3 months. At the end of the 3-month loan period, the interest owed is $35. What was the simple yearly interest rate (as a percent)?

20. A business owner borrows $4,200 for 4 months. At the end of the 4-month loan period, the interest owed is $63. What was the simple yearly interest rate (as a percent)?

21. A business owner takes out a 6-month loan at a 7% per year simple interest rate. At the end of the 6-month loan period, the interest owed is $287. What was the principal amount borrowed?

22. A business owner takes out a 6-month loan at a 2% per year simple interest rate. At the end of the 6-month loan period, the interest owed is $40. What was the principal amount borrowed?

23. A business owner borrows $7,300 for 2 months. At the end of the 2-month loan period, the interest owed is $73. What was the simple yearly interest rate (as a percent)?

24. A business owner borrows $5,600 for 6 months. At the end of the 6-month loan period, the interest owed is $182. What was the simple yearly interest rate (as a percent)?

25. A business owner borrows $3,200 for 6 months. At the end of the 6-month loan period, the interest owed is $96. What was the simple yearly interest rate (as a percent)?

26. A business owner borrows $5,700 for 4 months. At the end of the 4-month loan period, the interest owed is $133. What was the simple yearly interest rate (as a percent)?

---

27. Suppose that $6,700 is invested at 9% simple interest per year. What will the balance be after 4 years?

28. Suppose that $5,200 is invested at 3.5% simple interest per year. What will the balance be after 2 years?

29. Suppose that $1,600 is invested at 2% simple interest per year. What will the balance be after 3 years?

30. Suppose that $8,100 is invested at 8.25% simple interest per year. What will the balance be after 4 years?

31. Suppose that $8,900 is invested at 2.5% simple interest per year. What will the balance be after 2 years?

32. Suppose that $9,800 is invested at 2.75% simple interest per year. What will the balance be after 6 years?

33. Suppose that $5,400 is invested at 4.25% simple interest per year. What will the balance be after 2 years?

34. Suppose that $8,400 is invested at 4.5% simple interest per year. What will the balance be after 4 years?

---

35. The balance on a 6-year loan is $10,222. If the principal borrowed was $7,600, what was the simple interest rate (as a percent)?

36. The balance on a 8-year loan is $12,264. If the principal borrowed was $8,400, what was the simple interest rate (as a percent)?

37. The balance on a 5-year loan is $4,640. If the simple interest rate is 9% per year, what was the principal borrowed?

38. The balance on a 6-year loan is $6,838. If the simple interest rate is 5.25% per year, what was the principal borrowed?

39. The balance on a 9-year loan is $9,593. If the simple interest rate is 9% per year, what was the principal borrowed?

40. The balance on a 8-year loan is $10,032. If the simple interest rate is 4% per year, what was the principal borrowed?

41. The balance on a 3-year loan is $5,941. If the principal borrowed was $5,200, what was the simple interest rate (as a percent)?

42. The balance on a 2-year loan is $9,589. If the principal borrowed was $8,600, what was the simple interest rate (as a percent)?

43. The balance on a 5-year loan is $5,400. If the principal borrowed was $4,000, what was the simple interest rate (as a percent)?

44. The balance on a 6-year loan is $12,635. If the principal borrowed was $9,500, what was the simple interest rate (as a percent)?

45. The balance on a 5-year loan is $11,550. If the simple interest rate is 7.5% per year, what was the principal borrowed?

46. The balance on a 8-year loan is $3,160. If the simple interest rate is 7.25% per year, what was the principal borrowed?

47. The balance on a 4-year loan is $5,720. If the principal borrowed was $4,400, what was the simple interest rate (as a percent)?

48. The balance on a 8-year loan is $4,422. If the principal borrowed was $3,300, what was the simple interest rate (as a percent)?

49. The balance on a 8-year loan is $9,768. If the simple interest rate is 4% per year, what was the principal borrowed?

50. The balance on a 2-year loan is $8,322. If the simple interest rate is 7% per year, what was the principal borrowed?

Answers

1. $4,256

3. $754

5. $990

7. $1,740

9. $27

11. $24

13. $1,700

15. $72

17. $6, 000

19. 3.5%

21. $8,200

23. 6%

25. 6%

27. $9,112

29. $1,696

31. $9,345

33. $5,859

35. 5.75%

37. $3,200

39. $5,300

41. 4.75%

43. 7%

45. $8,400

47. 7.5%

49. $7,400

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CHAPTER OVERVIEW

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5.1: Unit Conversion - American System

In this section we will develop a technique for converting units used in the American system. We begin with a discussion of common measurements of length in the United States.

Units of Length

The most common units of length are the inch, foot, yard, and mile. Our focus will be on the technique used to convert from one unit of length to another.

American Units of Length

Facts relating common units of length.

- 1 foot (ft) = 12 inches (in)
- 1 yard (yd) = 3 feet (ft)
- 1 mile (mi) = 5280 feet (ft)

Take for example, the fact that there are 3 feet in 1 yard, which can be stated as an equation, using the common abbreviations for feet (ft) and yards (yd).

$$3 \text{ ft} = 1 \text{ yd}$$

If we divide both sides of the equation by 3 ft,

$$\frac{3 \text{ ft}}{3 \text{ ft}} = \frac{1 \text{ yd}}{3 \text{ ft}},$$

or equivalently,

$$1 = \frac{1 \text{ yd}}{3 \text{ ft}}.$$

The key observation is the fact that the ratio 1 yd/3 ft equals the number 1. Consequently, multiplying by the "conversion factor" 1 yd/3 ft is equivalent to multiplying by 1. This can be used to change a measurement in feet to yards.

Example 1

Change 36 feet to yards.

Solution

Multiply by the conversion factor 1 yd/3 ft.

$$\begin{aligned} 36 \text{ ft} &= 36 \text{ ft} \cdot 1 && \text{Multiplicative Identity Property.} \\ &= 36 \text{ ft} \cdot \frac{1 \text{ yd}}{3 \text{ ft}} && \text{Replace 1 with 1 yd/3 ft.} \\ &= 36 \ \cancel{\text{ft}} \cdot \frac{1 \text{ yd}}{3 \ \cancel{\text{ft}}} && \text{Cancel common unit.} \\ &= \frac{36 \cdot 1}{3} \text{ yd} && \text{Multiply fractions.} \\ &= \frac{36}{3} \text{ yd} && \text{Simplify.} \\ &= 12 \text{ yd} && \text{Divide.} \end{aligned}$$

Hence, 36 feet equals 12 yards.

Exercise

Change 81 feet to yards.

Answer

27 yards

On the other hand, we can start again with

$$3 \text{ ft} = 1 \text{ yd}$$

and divide both sides of the equation by 1 yd.

$$\frac{3 \text{ ft}}{1 \text{ yd}} = \frac{1 \text{ yd}}{1 \text{ yd}}$$

This gives the conversion factor

$$\frac{3 \text{ ft}}{1 \text{ yd}} = 1.$$

The key observation is the fact that the ratio 3 ft/1 yd equals the number 1. Consequently, multiplying by the "conversion factor" 3 ft/1 yd is equivalent to multiplying by 1. This can be used to change a measurement in yards to feet.

Example 2

Change 18 yards to feet.

Solution

Multiply by the conversion factor 3 ft/1 yd.

\[\begin{aligned} 18 \text{ yd} = 18 \text{ yd} \cdot 1 ~ & \textcolor{red}{ \text{ Multiplicative Identity Property.}} \\ = 18 \text{ yd} \cdot \frac{1 \text{ ft}}{1 \text{ yd}} ~ & \textcolor{red}{ \text{ Replace 1 with 3 ft/1 yd.} } \\ = 18 \cancel{ \text{ yd}} \cdot \frac{3 \text{ ft}}{1 \cancel{ \text{ yd}}} ~ & \textcolor{red}{ \text{ Cancel common unit.}} \\ = \frac{18 \cdot 3}{1} \text{ ft} ~ & \textcolor{red}{ \text{ Multiply fractions.}} \\ = 54 \text{ ft} ~ & \textcolor{red}{ \text{ Simplify.}} \end{aligned}\nonumber \]

Hence, 18 yards equals 54 feet.

Exercise

Change 15 yards to feet.

Answer

45 feet

Another common comparison is the fact that there are 12 inches in 1 foot. This can be represented as an equation using the common abbreviation for inches (in) and feet (ft).

$$12 \text{ in} = 1 \text{ ft}$$

Dividing both sides by 12 in

$$\frac{12 \text{ in}}{12 \text{ in}} = \frac{1 \text{ ft}}{12 \text{ in}},$$

yields the conversion factor

$$1 = \frac{1 \text{ ft}}{12 \text{ in}}.$$

The key observation is the fact that the ratio 1 ft/12 in equals the number 1. Consequently, multiplying by the "conversion factor" 1 ft/12 in is equivalent to multiplying by 1. This can be used to change a measurement in inches to feet.

Example 3

Change 24 inches to feet.

Solution

Multiply by the conversion factor 1 ft/12 in.

$$\begin{aligned} 24 \text{ in} &= 24 \text{ in} \cdot 1 && \text{Multiplicative Identity Property.} \\ &= 24 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} && \text{Replace 1 with 1 ft/12 in.} \\ &= 24 \ \cancel{\text{in}} \cdot \frac{1 \text{ in}}{12 \ \cancel{\text{in}}} && \text{Cancel common unit.} \\ &= \frac{24 \cdot 1}{12} \text{ ft} && \text{Multiply fractions.} \\ &= 2 \text{ ft} && \text{Simplify.} \end{aligned}$$

Hence, 24 inches equals 2 feet.

Exercise

Change 48 inches to feet.

Answer

4 feet

We provide a summary of conversion factors for units of length in Table 6.1.

Table 6.1: Conversion factors for units of length.

Convert	Conversion Factor	Convert	Conversion Factor
feet to inches	12 in/1 ft	inches to feet	1 ft/12 in
yards to feet	3 ft/1 yd	feet to yards	1 yd/3 ft
miles to feet	5280 ft/1 mi	feet to miles	1 mi/5280 ft

Some conversions require more than one application of a conversion factor.

Example 4

Change 4 yards to inches.

Solution

We multiply by a chain of conversion factors, the first to change yards to feet, the second to change feet to inches.

$$\begin{aligned} 4 \text{ yd} &= 4 \text{ yd} \cdot \frac{3 \text{ ft}}{1 \text{ yd}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} && \text{Multiply by conversion factors.} \\ &= 4 \ \cancel{\text{yd}} \cdot \frac{3 \ \cancel{\text{ft}}}{1 \ \cancel{\text{yd}}} \cdot \frac{12 \text{ in}}{1 \ \cancel{\text{ft}}} && \text{Cancel common units.} \\ &= \frac{4 \cdot 3 \cdot 12}{1 \cdot 1} \text{ in} && \text{Multiply fractions.} \\ &= 144 \text{ in} && \text{Simplify.} \end{aligned}$$

Hence, 4 yards equals 144 inches.

Exercise

Change 8 yards to inches.

Answer

288 inches

Example 5

Change 2 miles to yards.

Solution

We multiply by a chain of conversion factors, the first to change miles to feet, the second to change feet to yards.

$$2 \text{ mi} = 2 \text{ mi} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ yd}}{3 \text{ ft}} \quad \text{Multiply by conversion factors.}$$

$$= 2 \ \cancel{\text{mi}} \cdot \frac{5280 \ \cancel{\text{ft}}}{1 \ \cancel{\text{mi}}} \cdot \frac{1 \text{ yd}}{3 \ \cancel{\text{ft}}} \quad \text{Cancel common units.}$$

$$= \frac{2 \cdot 5280 \cdot 1}{1 \cdot 3} \text{ yd} \quad \text{Multiply fractions.}$$

$$= 3520 \text{ yd} \quad \text{Simplify.}$$

Hence, 2 miles equals 3,520 yards.

Exercise

Change 5 miles to yards.

Answer

8,800 yards

Units of Weight

The most common units of weight are the ounce, pound, and ton. Our focus will remain on how to convert from one unit to another.

American Units of Weight

Facts relating common units of weight.

- 1 pound (lb) = 16 ounces (oz)
- 1 ton = 2000 pounds (lb)

The above facts lead to the conversion factors in Table 6.2.

Table 6.2: Conversion factors for units of weight.

Convert	Conversion Factor	Convert	Conversion Factor
pounds to ounces	16 oz/1 lb	ounces to pounds	1 lb/16 oz
tons to pounds	2000 lb/1 ton	pounds to tons	1 ton/2000 lb

Example 6

Change $2\frac{1}{2}$ pounds to ounces.

Solution

Multiply by the appropriate conversion factor.

$$\begin{aligned} 2\frac{1}{2}\text{ lb} &= 2\frac{1}{2}\text{ lb}\cdot\frac{16\text{ oz}}{1\text{ lb}} && \text{Multiply by conversion factor.}\\ &= 2\frac{1}{2}\ \cancel{\text{lb}}\cdot\frac{16\text{ oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.}\\ &= \left(2\frac{1}{2}\cdot 16\right)\text{ oz} && \text{Multiply fractions.}\\ &= \left(\frac{5}{2}\cdot 16\right)\text{ oz} && \text{Mixed to improper fraction.}\\ &= \frac{80}{2}\text{ oz} && \text{Multiply.}\\ &= 40\text{ oz} && \text{Divide.} \end{aligned}$$

Hence, $2\frac{1}{2}$ pounds equals 40 ounces.

Exercise

Change $6\frac{1}{4}$ pounds to ounces.

Answer

100 ounces

Example 7

Change 3.2 tons to ounces.

Solution

This exercise requires multiplying by a chain of conversion factors.

$$\begin{aligned} 3.2\text{ ton} &= 3.2\text{ ton}\cdot\frac{2000\text{ lb}}{1\text{ ton}}\cdot\frac{16\text{ oz}}{1\text{ lb}} && \text{Multiply by conversion factors.}\\ &= 3.2\ \cancel{\text{ton}}\cdot\frac{2000\ \cancel{\text{lb}}}{1\ \cancel{\text{ton}}}\cdot\frac{16\text{ oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.}\\ &= \frac{3.2\cdot 2000\cdot 16}{1\cdot 1}\text{ oz} && \text{Multiply fractions.}\\ &= 102,400\text{ oz} && \text{Simplify.} \end{aligned}$$

Hence, 3.2 tons equals 102,400 ounces.

Exercise

Change 4.1 tons to ounces.

Answer

128,000 ounces

Units of Volume

The most common units of volume are fluid ounces, cups, pints, quarts, and gallons. We will focus on converting from one unit to another.

American Units of Volume

Facts relating common units of volume.

- 1 cup (c) = 8 fluid ounces (fl oz)
- 1 pint (pt) = 2 cups (c)
- 1 quart (qt) = 2 pints (pt)
- 1 gallon (gal) = 4 quarts (qt)

These facts lead to the conversion factors listed in Table 6.3.

Table 6.3: Conversion factors for units of volume.

Convert	Conversion Factor	Convert	Conversion Factor
cups to ounces	8 fl oz/1 c	ounces to cups	1c/8 fl oz
pints to cups	2 c/1 pt	cups to pints	1 pt/2 c
quarts to pints	2 pt/1 qt	pints to quarts	1 qt/2 pt
gallons to quarts	4 qt/1 gal	quarts to gallons	1 gal/4 qt

Example 8

Change 5.6 gallons to pints.

Solution

This exercise requires multiplying by a chain of conversion factors.

$$\begin{aligned} 5.6\text{ gal} &= 5.6\text{ gal} \cdot \frac{4\text{ qt}}{1\text{ gal}} \cdot \frac{2\text{ pt}}{1\text{ qt}} && \text{Multiply by conversion factors.} \\ &= 5.6\ \cancel{\text{gal}} \cdot \frac{4\ \cancel{\text{qt}}}{1\ \cancel{\text{gal}}} \cdot \frac{2\text{ pt}}{1\ \cancel{\text{qt}}} && \text{Cancel common units.} \\ &= \frac{5.6 \cdot 4 \cdot 2}{1 \cdot 1}\text{ pt} && \text{Multiply fractions.} \\ &= 44.8\text{ pt} && \text{Simplify.} \end{aligned}$$

Hence, 5.6 gallons equals 44.8 pints.

Exercise

Change 3.2 gallons to pints.

Answer

25.6 pints

Units of Time

The most common units of time are seconds, minutes, hours, days, and years.

American Units of Time

Facts relating common units of time.

- 1 minute (min) = 60 seconds (s)
- 1 hour (hr) = 60 minutes (min)
- 1 day (day) = 24 hours (hr)
- 1 year (yr) = 365 days (day)

These facts lead to the conversion factors in Table 6.4.

Table 6.4: Conversion factors for units of time.

Convert	Conversion Factor	Convert	Conversion Factor
minutes to seconds	60 s/1 min	seconds to minutes	1 min/60 s
hours to minutes	60 min/1 hr	minutes to hours	1 hr/60 min
days to hours	24 hr/1 day	hours to days	1 day/24 hr
years to days	365 day/1 yr	days to years	1 yr/365 day

Example 9

How many seconds in a year?

Solution

A chain of conversion factors is needed.

$$1\text{ yr} = 1\text{ yr} \cdot \frac{365\text{ day}}{1\text{ yr}} \cdot \frac{24\text{ hr}}{1\text{ day}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{60\text{ s}}{1\text{ min}} \quad \text{Conversion factors.}$$

$$= 1\ \cancel{\text{yr}} \cdot \frac{365\ \cancel{\text{day}}}{1\ \cancel{\text{yr}}} \cdot \frac{24\ \cancel{\text{hr}}}{1\ \cancel{\text{day}}} \cdot \frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hr}}} \cdot \frac{60\text{ s}}{1\ \cancel{\text{min}}} \quad \text{Cancel common units.}$$

$$= \frac{1 \cdot 365 \cdot 24 \cdot 60 \cdot 60}{1 \cdot 1 \cdot 1 \cdot 1}\text{ s} \quad \text{Multiply fractions.}$$

$$= 31,536,000\text{ s} \quad \text{Simplify.}$$

Thus, 1 year equals 31,536,000 seconds.

Exercise

How many seconds in a day?

Answer

86,400 seconds

Converting Units of Speed

Ever wonder how fast a baseball is moving?

Example 10

A professional pitcher can throw a baseball at 95 miles per hour. How fast is this in feet per second? Round your answer to the nearest foot per second.

Solution

There are 5280 feet in a mile, 60 minutes in an hour, and 60 seconds in a minute.

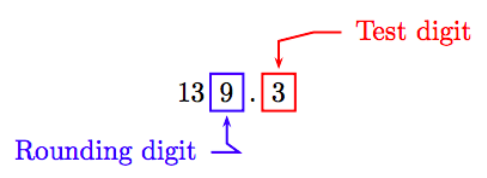
$$95\frac{\text{mi}}{\text{h}} \approx 95\frac{\text{mi}}{\text{h}} \cdot \frac{5280\ \text{ft}}{1\ \text{mi}} \cdot \frac{1\ \text{h}}{60\ \text{min}} \cdot \frac{1\ \text{min}}{60\ \text{s}} \quad \text{Conversion factors.}$$

$$\approx 95\frac{\cancel{\text{mi}}}{\cancel{\text{h}}} \cdot \frac{5280\ \text{ft}}{1\ \cancel{\text{mi}}} \cdot \frac{1\ \cancel{\text{h}}}{60\ \cancel{\text{min}}} \cdot \frac{1\ \cancel{\text{min}}}{60\ \text{s}} \quad \text{Cancel common units.}$$

$$\approx \frac{95 \cdot 5280 \cdot 1 \cdot 1}{1 \cdot 60 \cdot 60}\frac{\text{ft}}{\text{s}} \quad \text{Multiply fractions.}$$

$$\approx 139.3\frac{\text{ft}}{\text{s}} \quad \text{Multiply and divide.}$$

To round to the nearest foot per second, identify the rounding and test digits.



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, to the nearest foot per second, the speed is approximately 139 feet per second.

Whew! Since the batter stands at home plate, which is about 60 feet from where the pitch is delivered, the batter has less than 1/2 a second to react to the pitch!

? Exercise

A women's softball pitcher can throw her fastball at 60 miles per hour. How fast is this in feet per second? Round your answer to the nearest foot per second.

Answer

88 feet per second

Exercises

1. Change 8 yards to feet.
2. Change 60 yards to feet.
3. Change 261 feet to yards.
4. Change 126 feet to yards.
5. Change 235 inches to yards. Round your answer to the nearest tenth of a yard.
6. Change 244 inches to yards. Round your answer to the nearest tenth of a yard.
7. Change 141 feet to yards.
8. Change 78 feet to yards.
9. Change 2.8 miles to feet.
10. Change 4.9 miles to feet.
11. Change 104 inches to yards. Round your answer to the nearest tenth of a yard.
12. Change 101 inches to yards. Round your answer to the nearest tenth of a yard.
13. Change 168,372 inches to miles, correct to the nearest tenth of a mile.
14. Change 198,550 inches to miles, correct to the nearest tenth of a mile.
15. Change 82 feet to inches.
16. Change 80 feet to inches.

17. Change 2.9 yards to inches. Round your answer to the nearest inch.
18. Change 4.5 yards to inches. Round your answer to the nearest inch.
19. Change 25,756 feet to miles. Round your answer to the nearest tenth of a mile.
20. Change 19,257 feet to miles. Round your answer to the nearest tenth of a mile.
21. Change 5 yards to feet.
22. Change 20 yards to feet.
23. Change 169,312 inches to miles, correct to the nearest tenth of a mile.
24. Change 162,211 inches to miles, correct to the nearest tenth of a mile.
25. Change 1.5 yards to inches. Round your answer to the nearest inch.
26. Change 2.1 yards to inches. Round your answer to the nearest inch.
27. Change 360 inches to feet.
28. Change 768 inches to feet.
29. Change 48 inches to feet.
30. Change 528 inches to feet.
31. Change 15,363 feet to miles. Round your answer to the nearest tenth of a mile.
32. Change 8,540 feet to miles. Round your answer to the nearest tenth of a mile.
33. Change 1.7 miles to inches.
34. Change 4.7 miles to inches.
35. Change 3.1 miles to inches.
36. Change 1.8 miles to inches.
37. Change 3.6 miles to feet.
38. Change 3.1 miles to feet.
39. Change 18 feet to inches.
40. Change 33 feet to inches.

---

41. Change $5\frac{1}{8}$ pounds to ounces.
42. Change $3\frac{1}{16}$ pounds to ounces.
43. Change 2.4 tons to ounces.
44. Change 3.4 tons to ounces.
45. Change 34 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
46. Change 78 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
47. Change 2.2 tons to pounds.
48. Change 4.8 tons to pounds.
49. Change 70 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
50. Change 20 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
51. Change 9,560 pounds to tons. Round your answer to the nearest tenth of a ton.
52. Change 9,499 pounds to tons. Round your answer to the nearest tenth of a ton.
53. Change $2\frac{1}{2}$ pounds to ounces.
54. Change $7\frac{1}{16}$ pounds to ounces.

55. Change 5.9 tons to pounds.
56. Change 2.1 tons to pounds.
57. Change 2.5 tons to ounces.
58. Change 5.3 tons to ounces.
59. Change 8,111 pounds to tons. Round your answer to the nearest tenth of a ton.
60. Change 8,273 pounds to tons. Round your answer to the nearest tenth of a ton.

---

61. Change 4.5625 pints to fluid ounces.
62. Change 2.9375 pints to fluid ounces.
63. Change 32 fluid ounces to pints.
64. Change 160 fluid ounces to pints.
65. Change 3.7 gallons to pints.
66. Change 2.4 gallons to pints.
67. Change 216 pints to gallons.
68. Change 96 pints to gallons.
69. Change 544 fluid ounces to pints.
70. Change 432 fluid ounces to pints.
71. Change 112 pints to gallons.
72. Change 200 pints to gallons.
73. Change 7.7 gallons to pints.
74. Change 5.7 gallons to pints.
75. Change 3.875 pints to fluid ounces.
76. Change 3 pints to fluid ounces.

---

77. Change 7.8 years to hours.
78. Change 4.7 years to hours.
79. Change 7.6 years to hours.
80. Change 6.6 years to hours.
81. Change 4,025,005 seconds to days. Round your answer to the nearest tenth of a day.
82. Change 4,672,133 seconds to days. Round your answer to the nearest tenth of a day.
83. Change 37,668 hours to years.
84. Change 40,296 hours to years.
85. Change 22,776 hours to years.
86. Change 29,784 hours to years.
87. Change 96 days to seconds.
88. Change 50 days to seconds.
89. Change 40 days to seconds.
90. Change 10 days to seconds.
91. Change 3,750,580 seconds to days. Round your answer to the nearest tenth of a day.
92. Change 4,493,469 seconds to days. Round your answer to the nearest tenth of a day.

93. Change 367 feet per second to miles per hour. Round your answer to the nearest mile per hour.

94. Change 354 feet per second to miles per hour. Round your answer to the nearest mile per hour.

95. Change 442 feet per second to miles per hour. Round your answer to the nearest mile per hour.

96. Change 388 feet per second to miles per hour. Round your answer to the nearest mile per hour.

97. Change 30 miles per hour to feet per second. Round your answer to the nearest foot per second.

98. Change 99 miles per hour to feet per second. Round your answer to the nearest foot per second.

99. Change 106 miles per hour to feet per second. Round your answer to the nearest foot per second.

100. Change 119 miles per hour to feet per second. Round your answer to the nearest foot per second.

101. **Strong man.** Famed strongman Joe Rollino, who was still bending quarters with his fingers at age 104, once lifted 3, 200 pounds at Coney Island Amusement Park. How many tons did Joe lift that day? *Associated Press-Times-Standard 01/12/10 NYC amusement park strongman, 104, killed by van.*

102. **Earth day.** The amount of time it takes the Earth to rotate once around its axis is one day. How many seconds is that?

103. **Water break.** "The average age of Washington, DC's water pipes is 76 years, and they are not alone. Every two minutes, somewhere in the country, a pipe breaks." How many pipes break each year in the US? *New York Times 03/14/10 Saving U.S. water and sewer systems could be costly.*

Answers

1. 24 feet

3. 87 yards

5. 6.5 yards

7. 47 yards

9. 14,784 feet

11. 2.9 yards

13. 2.7 miles

15. 984 inches

17. 104 inches

19. 4.9 miles

21. 15 feet

23. 2.7 miles

25. 54 inches

27. 30 feet

29. 4 feet

31. 2.9 miles

33. 107,712 inches

35. 196,416 inches

37. 19,008 feet

39. 216 inches

41. 82 ounces

43. 76,800 ounces

45. $2\frac{1}{8}$ pounds

47. 4,400 pounds

49. $4\frac{3}{8}$ pounds

51. 4.8 tons

53. 40 ounces

55. 11,800 pounds

57. 80,000 ounces

59. 4.1 tons

61. 73 fluid ounces

63. 2 pints

65. 29.6 pints

67. 27 gallons

69. 34 pints

71. 14 gallons

73. 61.6 pints

75. 62 fluid ounces

77. 68,328 hours

79. 66,576 hours

81. 46.6 days

83. 4.3 years

85. 2.6 years

87. 8,294,400 seconds

89. 3,456,000 seconds

91. 43.4 days

93. 250 mi/hr

95. 301 mi/hr

97. 44 ft/s

99. 155 ft/s

101. 1.6 tons

103. 262, 800

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5.2: Unit Conversion- Metric System

The metric system of units is the standard system of units preferred by scientists. It is based on the base ten number system and its decimal format is more friendly to users of this system. There is a common set of prefixes adopted by the metric system to indicate a power of ten to apply to the base unit.

Metric System Preferences

This is a list of standard prefixes for the metric system and their meanings.

$$\begin{array}{ll} \text{deka} = 10 & \text{deci} = 1/10 \\ \text{hecto} = 100 & \text{centi} = 1/100 \\ \text{kilo} = 1000 & \text{milli} = 1/1000 \end{array}$$

Thus, for example, a decameter is 10 meters, a hectoliter is 100 liters, and a kilogram is 1000 grams. Similarly, a decimeter is 1/10 of a meter, a centiliter is 1/100 of a liter, and a milligram is 1/1000 of a gram.

Units of Length

The standard measure of length in the metric system is the meter.

> *Historically, the meter was defined by the French Academy of Sciences as the length between two marks on a platinum-iridium bar, which was designed to represent 110,000,000 of the distance from the equator to the north pole through Paris. In 1983, it was redefined by the International Bureau of Weights and Measures (BIPM) as the distance travelled by light in free space in 1299,792,458 of a second. (Wikipedia)*

We can apply the standard prefixes to get the following result.

Metric Units of Length

These units of length are used in the metric system.

Unit Length	Unit Abbreviation
1 kilometer = 1000 meters	km
1 hectometer = 100 meters	hm
1 dekameter = 10 meters	dam
1 meter	m
1 decimeter = $\frac{1}{10}$ meter	dm
1 centimeter = $\frac{1}{100}$ meter	cm
1 millimeter = $\frac{1}{1000}$ meter	mm

We can use these facts to build conversion factors as we did in Section 6.3. For example, because

$$1 \text{ km} = 1000 \text{ m},$$

we can divide both sides by 1000 m to produce the conversion factor

$$1 = \frac{1 \text{ km}}{1000 \text{ m}}.$$

This conversion factor can help change meters into kilometers.

Before using this conversion factor in an example, we repeat here the rules for multiplying and dividing by powers of ten. We will be making heavy use of these rules in this section.

Multiplying and Dividing by Powers of Ten

- Multiplying a decimal number by 10^n will move the decimal point n places to the right. For example, $3.2567 \cdot 10^2 = 3.2567 \cdot 100 = 325.67$.
- Dividing a decimal number by 10^n will move the decimal point n places to the left. For example, $3.2567/10^2 = 3.2567/100 = 0.032567$.

And now the example.

Example 1

Change 2,326 meters to kilometers.

Solution

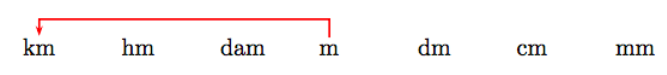
Multiply by the conversion factor 1 km/1000m.

$$\begin{aligned} 2326\ \text{m} &= 2326\ \text{m} \cdot \frac{1\ \text{km}}{1000\ \text{m}} && \text{Apply conversion factor.} \\ &= 2326\ \cancel{\text{m}} \cdot \frac{1\ \text{km}}{1000\ \cancel{\text{m}}} && \text{Cancel common units.} \\ &= \frac{2326 \cdot 1}{1000}\ \text{km} && \text{Multiply fractions.} \\ &= 2.326\ \text{km} && \text{Simplify.} \end{aligned}$$

In the last step, note that dividing by 1000 moves the decimal point three places to the left. Thus, 2326 meters is equal to 2.326 kilometers.

Alternate Solution

A second solution depends upon the fact that multiplying or dividing by a power of ten will move the decimal point right or left a number of places equal to the number of zeros present in the multiplier or divisor. Thus, as we saw above, dividing by 1000 moved the decimal point 3 places to the left. Suppose that we arrange the metric units of length in order, from largest to smallest, as shown below.



Note that we must move 3 places left to move from the meters (m) abbreviation to the kilometers (km) abbreviation. In like manner, if we write 2,326 meters as 2,326.0meters, then we can convert to kilometers by moving the decimal 3 places to the left.

Exercise

Change 1,156 meters to kilometers

Answer

1.156 kilometers

Example 2

Change 537 centimeters to meters.

Solution

We know that

$$1\text{ cm} = \frac{1}{100}\text{ m},$$

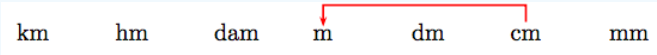
or multiplying both sides of this result by 100,

$$100\text{ cm} = 1\text{ m}.$$

Dividing both sides of this last result by 100 cm, we obtain the conversion factor 1 m/100 cm.

$$\begin{aligned} 537\text{ cm} &= 537\text{ cm} \cdot \frac{1\text{ m}}{100\text{ cm}} && \text{Apply conversion factor.} \\ &= 537\ \cancel{\text{cm}} \cdot \frac{1\text{ m}}{100\ \cancel{\text{cm}}} && \text{Cancel common units.} \\ &= \frac{537 \cdot 1}{100}\text{ m} && \text{Multiply fractions.} \\ &= 5.37\text{ m} \end{aligned}$$

In the last step, note that dividing by 100 moves the decimal point two places to the left. Alternately, we can set up our ordered list of units.



Note that we must move 2 places left to move from the centimeters (cm) abbreviation to the meters (m) abbreviation. In like manner, if we write 537 centimeters as 537.0 centimeters, then we can convert to meters by moving the decimal 2 places to the left.

Sometimes more than one conversion factor is needed.

Exercise

Change 276 centimeters to meters.

Answer

2.76 meters

Example 3

Change 10.2 dekameters to centimeters.

Solution

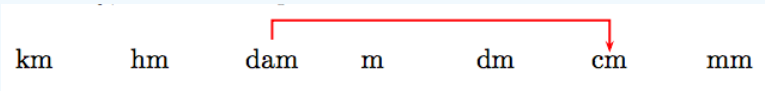
We have two facts:

- 1 dam=10 m, which yields the conversion factor 10 m/1 dam.
- 1 cm=(1/100) m or 100 cm=1 m, which yields the conversion factor 100 cm/1 m.

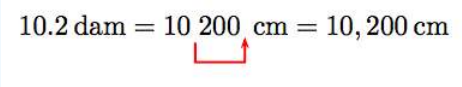
$$
\begin{aligned}
10.2\ \text{dam} &= 10.2\ \text{dam} \cdot \frac{10\ \text{m}}{1\ \text{dam}} \cdot \frac{100\ \text{cm}}{1\ \text{m}} && \text{Apply conversion factor.} \\
&= 10.2\ \cancel{\text{dam}} \cdot \frac{10\ \cancel{\text{m}}}{1\ \cancel{\text{dam}}} \cdot \frac{100\ \text{cm}}{1\ \cancel{\text{m}}} && \text{Cancel common units.} \\
&= \frac{10.2 \cdot 10 \cdot 100}{1}\ \text{cm} && \text{Multiply fractions.} \\
&= 10,200\ \text{cm}
\end{aligned}
$$

In the last step, note that multiplying by 10, then by 100, moves the decimal point three places to the right.

Alternately, we can set up our ordered list of units.



Note that we must move 3 places right to move from the dekameters (dam) abbreviation to the centimeters (cm) abbreviation. In like manner, we can convert 10.2 dekameters to centimeters by moving the decimal 3 places to the right.



Exercise

Change 13.5 dekameters to centimeters

Answer

13,500 centimeters

Units of Mass

The fundamental unit of mass in the metric system is called a gram. Originally, it was defined to be equal to one cubic centimeter of water measured at the temperature of melting ice. Now it is simply defined as 1/1000 of a kilogram, which is defined by a physical prototype preserved by the International Bureau of Weights and Measures (Wikipedia). The mass of an object is not the same as an object's weight, but rather a resistance to motion when an external force is applied.

The same metric system prefixes apply.

Metric Units of Mass

These units of mass are used in the metric system.

Units of Mass	Unit Abbreviation
1 kilogram = 1000 grams	kg
1 hectogram = 100 grams	hg
1 dekagram = 10 grams	dag
1 gram	g
1 decigram = $\frac{1}{10}$ gram	dg
1 centigram = $\frac{1}{100}$ gram	cg
1 milligram = $\frac{1}{1000}$ gram	mg

Example 4

Convert 0.025 dekagrams to milligrams.

Solution

We'll use two conversion factors:

- 1 dag=10 g, which yields the conversion factor 10 g/1 dag.
- 1 mg=(1/1000) g, which yields the conversion factor 1000 mg/1 g.

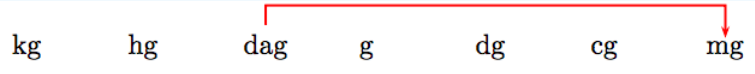
$$0.025\ \text{dag} = 0.025\ \text{dag} \cdot \frac{10\ \text{g}}{1\ \text{dag}} \cdot \frac{1000\ \text{mg}}{1\ \text{g}} \quad \text{Apply conversion factor.}$$

$$= 0.025\ \cancel{\text{dag}} \cdot \frac{10\ \cancel{\text{g}}}{1\ \cancel{\text{dag}}} \cdot \frac{1000\ \text{mg}}{1\ \cancel{\text{g}}} \quad \text{Cancel common units.}$$

$$= \frac{0.025 \cdot 10 \cdot 1000}{1}\ \text{mg} \quad \text{Multiply fractions.}$$

$$= 250\ \text{mg}$$

Alternately, we can set up our ordered list of units.



Note that we must move 4 places right to move from the dekagrams (dag) abbreviation to the milligrams (mg) abbreviation. In like manner, we can convert 0.025 dekagrams to milligrams by moving the decimal 4 places to the right.

$$0.0250\ \text{dag} = 0\ 0250.\ \text{mg} = 250\ \text{mg}$$

Exercise

Convert 0.05 dekagrams to milligrams.

Answer

500 milligrams

Units of Volume

The fundamental unit of volume in the metric system is called a litre. Originally, one litre was defined as the volume of one kilogram of water measured at 4° C at 760 millimeters of mercury (Wikipedia). Currently, 1 litre is defined as 1 cubic decimeter (imagine a cube with each edge 1/10 of a meter).

The same metric system prefixes apply.

Metric Units of Volume

These units of volume are used in the metric system.

Unit of Volume	Unit Abbreviation
1 kilolitre = 1000 litres	kL
1 hectolitre = 100 litres	hL
1 dekalitre = 10 litres	daL
1 litre	L
1 decilitre = $\frac{1}{10}$ litre	dL

1 centilitre = $\frac{1}{100}$ litre	cL
1 millilitre = $\frac{1}{1000}$ litre	mL

Example 5

Convert 11,725 millilitres to dekalitres.

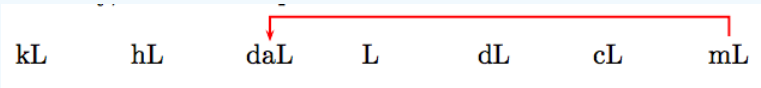
Solution

We'll use two conversion factors:

- 1 daL=10 L, which yields the conversion factor 1 daL/10 L.
- 1 mL=(1/1000) L, which yields the conversion factor 1 L/1000 mL.

$$\begin{aligned} 11,725\ \text{mL} &= 11,725\ \text{mL} \cdot \frac{1\ \text{L}}{1000\ \text{mL}} \cdot \frac{1\ \text{daL}}{10\ \text{L}} && \text{Apply conversion factors.} \\ &= 11,725\ \cancel{\text{mL}} \cdot \frac{1\ \cancel{\text{L}}}{1000\ \cancel{\text{mL}}} \cdot \frac{1\ \text{daL}}{10\ \cancel{\text{L}}} && \text{Cancel common units.} \\ &= \frac{11,725 \cdot 1 \cdot 1}{1000 \cdot 10}\ \text{daL} && \text{Multiply fractions.} \\ &= 1.1725\ \text{daL} \end{aligned}$$

Alternately, we can set up our ordered list of units.



Note that we must move 4 places left to move from the millitres (mL) abbreviation to the dekalitres (daL) abbreviation. In like manner, we can convert 11,725 millilitres to dekalitres by moving the decimal 4 places to the left.

$$11,725.0\ \text{mL} = 1.1725\ 0\ \text{daL} = 1.1725\ \text{daL}$$

Exercise

Convert 5,763 millilitres to dekalitres.

Answer

0.5763 dekalitres

Exercises

1. What is the meaning of the metric system prefix centi?
2. What is the meaning of the metric system prefix deka?
3. What is the meaning of the metric system prefix hecto?
4. What is the meaning of the metric system prefix kilo?
5. What is the meaning of the metric system prefix deci?
6. What is the meaning of the metric system prefix milli?

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7. What is the meaning of the metric system abbreviation mg?
8. What is the meaning of the metric system abbreviation g?
9. What is the meaning of the metric system abbreviation m?

10. What is the meaning of the metric system abbreviation km?
11. What is the meaning of the metric system abbreviation kL?
12. What is the meaning of the metric system abbreviation daL?
13. What is the meaning of the metric system abbreviation hm?
14. What is the meaning of the metric system abbreviation dm?
15. What is the meaning of the metric system abbreviation dam?
16. What is the meaning of the metric system abbreviation cm?
17. What is the meaning of the metric system abbreviation dL?
18. What is the meaning of the metric system abbreviation L?
19. What is the meaning of the metric system abbreviation hg?
20. What is the meaning of the metric system abbreviation kg?
21. What is the meaning of the metric system abbreviation dg?
22. What is the meaning of the metric system abbreviation dag?
23. What is the meaning of the metric system abbreviation hL?
24. What is the meaning of the metric system abbreviation cL?

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25. Change 5,490 millimeters to meters.
26. Change 8,528 millimeters to meters.
27. Change 64 meters to millimeters.
28. Change 65 meters to millimeters.
29. Change 4,571 millimeters to meters.
30. Change 8,209 millimeters to meters.
31. Change 15 meters to centimeters.
32. Change 12 meters to centimeters.
33. Change 569 centimeters to meters.
34. Change 380 centimeters to meters.
35. Change 79 meters to centimeters.
36. Change 60 meters to centimeters.
37. Change 7.6 kilometers to meters.
38. Change 4.9 kilometers to meters.
39. Change 861 centimeters to meters.
40. Change 427 centimeters to meters.
41. Change 4,826 meters to kilometers.
42. Change 1,929 meters to kilometers.
43. Change 4,724 meters to kilometers.
44. Change 1,629 meters to kilometers.
45. Change 6.5 kilometers to meters.
46. Change 7.9 kilometers to meters.
47. Change 17 meters to millimeters.

48. Change 53 meters to millimeters.

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49. Change 512 milligrams to centigrams.
50. Change 516 milligrams to centigrams.
51. Change 541 milligrams to centigrams.
52. Change 223 milligrams to centigrams.
53. Change 70 grams to centigrams.
54. Change 76 grams to centigrams.
55. Change 53 centigrams to milligrams.
56. Change 30 centigrams to milligrams.
57. Change 83 kilograms to grams.
58. Change 70 kilograms to grams.
59. Change 8,196 grams to kilograms.
60. Change 6,693 grams to kilograms.
61. Change 564 centigrams to grams.
62. Change 884 centigrams to grams.
63. Change 38 grams to centigrams.
64. Change 88 grams to centigrams.
65. Change 77 centigrams to milligrams.
66. Change 61 centigrams to milligrams.
67. Change 5,337 grams to kilograms.
68. Change 4,002 grams to kilograms.
69. Change 15 kilograms to grams.
70. Change 45 kilograms to grams.
71. Change 833 centigrams to grams.
72. Change 247 centigrams to grams.

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73. Change 619,560 centilitres to kilolitres.
74. Change 678,962 centilitres to kilolitres.
75. Change 15.2 litres to millilitres.
76. Change 9.7 litres to millilitres.
77. Change 10,850 centilitres to litres.
78. Change 15,198 centilitres to litres.
79. Change 10.7 litres to millilitres.
80. Change 17.3 litres to millilitres.
81. Change 15,665 millilitres to litres.
82. Change 12,157 millilitres to litres.
83. Change 6.3 kilolitres to centilitres.
84. Change 8.3 kilolitres to centilitres.
85. Change 4.5 kilolitres to centilitres.

86. Change 6.2 kilolitres to centilitres.
87. Change 10.6 litres to centilitres.
88. Change 16.6 litres to centilitres.
89. Change 14,383 centilitres to litres.
90. Change 11,557 centilitres to litres.
91. Change 9.9 litres to centilitres.
92. Change 19.5 litres to centilitres.
93. Change 407,331 centilitres to kilolitres.
94. Change 827,348 centilitres to kilolitres.
95. Change 14,968 millilitres to litres.
96. Change 18,439 millilitres to litres.

Answers

1. 1/100
3. 100
5. 1/10
7. milligram
9. meter
11. kilolitre or kiloliter
13. hectometer
15. dekameter
17. decilitre or deciliter
19. hectogram
21. decigram
23. hectolitre or hectoliter
25. 5.49 meters
27. 64,000 millimeters
29. 4.571 meters
31. 1,500 centimeters
33. 5.69 meters
35. 7,900 centimeters
37. 7,600 meters
39. 8.61 meters
41. 4.826 kilometers
43. 4.724 kilometers
45. 6,500 meters
47. 17,000 millimeters
49. 51.2 centigrams
51. 54.1 centigrams

53. 7,000 centigrams
55. 530 milligrams
57. 83,000 grams
59. 8.196 kilograms
61. 5.64 grams
63. 3,800 centigrams
65. 770 milligrams
67. 5.337 kilograms
69. 15,000 grams
71. 8.33 grams
73. 6.1956 kilolitres
75. 15,200 millilitres
77. 108.5 litres
79. 10,700 millilitres
81. 15.665 litres
83. 630,000 centilitres
85. 450,000 centilitres
87. 1,060 centilitres
89. 143.83 litres
91. 990 centilitres
93. 4.07331 kilolitres
95. 14.968 litres

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5.3: American Units to Metric Units and Vice-Versa

We often need to convert from the American system of units to the metric system of units or vice-versa (imagine traveling to a European country using the metric system). That will be our focus in this section.

Converting Units of Length

One meter is slightly longer than one yard. Indeed,

$$1\text{ m} \approx 1.0936\text{ yd}. \tag{5.3.1}$$

If we divide both sides of this equation by 1.0936, then

$$\frac{1\text{ m}}{1.0936} \approx \frac{1.0936\text{ yd}}{1.0936}$$
$$0.9144\text{ m} \approx 1\text{ yd}$$

Further conversions can be made. For example, to change meters to feet, we make the following conversions.

$$\begin{aligned} 1\text{ m} &\approx 1\text{ m} \cdot 1\frac{\text{yd}}{0.9144\text{ m}} \cdot \frac{3\text{ ft}}{1\text{ yd}} && \text{Apply conversion factors.} \\ &\approx 1\ \cancel{\text{m}} \cdot \frac{1\ \cancel{\text{yd}}}{0.9144\ \cancel{\text{m}}} \cdot \frac{3\text{ ft}}{1\ \cancel{\text{yd}}} && \text{Cancel common units.} \\ &\approx \frac{1 \cdot 1 \cdot 3}{0.9144 \cdot 1}\text{ ft} && \text{Multiply fractions.} \\ &\approx 3.2808\text{ ft} \end{aligned}$$

Table 6.5 shows some of the most common conversions between American units of length and metric units of length.

Table 6.5: Length conversions: American — Metric.

American to Metric	Metric to American
1 in ≈ 2.54 cm	1 cm ≈ 0.3937 in
1 ft ≈ 0.3048 m	1 m ≈ 3.2808 ft
1 yd ≈ 0.9144 m	1 m ≈ 1.0936 yd
1 mi ≈ 1.6093 km	1 km ≈ 0.6214 mi

Example 1

A car's speedometer shows that a family has currently traveled 154 miles in route to their vacation destination. Convert this distance to kilometers.

Solution

Choose 1 mi = 1.6093 km from Table 6.5.

$$\begin{aligned} 154\text{ mi} &\approx 154\text{ mi} \cdot \frac{1.6093\text{ km}}{1\text{ mi}} && \text{Apply conversion factor.} \\ &\approx 154\ \cancel{\text{mi}} \cdot \frac{1.6093\text{ km}}{1\ \cancel{\text{mi}}} && \text{Cancel common units.} \\ &\approx \frac{154 \cdot 1.6093}{1}\text{ km} && \text{Multiply fractions.} \\ &\approx 247.8\text{ km} \end{aligned}$$

Hence, 154 miles is approximately 247.8 kilometers.

Alternate Solution

Note that this would work equally well if we chose 1 km ≈ 0.6214 mi from Table 6.5.

$$\begin{aligned} 154 \text{ mi} &\approx 154 \text{ mi} \cdot \frac{1 \text{ km}}{0.6214 \text{ mi}} && \text{Apply conversion factor.} \\ &\approx 154 \ \cancel{\text{mi}} \cdot \frac{1 \text{ km}}{0.6214 \ \cancel{\text{mi}}} && \text{Cancel common units.} \\ &\approx \frac{154 \cdot 1}{0.6214} \text{ km} && \text{Multiply fractions.} \\ &\approx 247.8 \text{ km} \end{aligned}$$

? Exercise

Change 227 miles to kilometers. Round to the nearest tenth of a kilometer.

Answer

365.3 kilometers

Converting Units of Weight and Mass

It is known that

$$1 \text{ kg} \approx 2.2 \text{ lb}. \tag{5.3.2}$$

Dividing both sides of this equation by 2.2,

$$\frac{1 \text{ kg}}{2.2} \approx \frac{2.2 \text{ lb}}{2.2}$$

$$0.454 \text{ kg} \approx 1 \text{lb}$$

A summary of the more common conversion factors regarding mass and weight are given in Table 6.6.

Table 6.6: Mass—Weight conversions: American — Metric.

American to Metric	Metric to American
1 oz ≈ 28.35 g	1 g ≈ 0.035 oz
1 lb ≈ 0.454 kg	1 kg ≈ 2.2 lb

✔ Example 2

Change 2.3 kilograms to ounces.

Solution

One kilogram weighs 2.2 pounds and there are 16 ounces in a pound.

$$\begin{aligned} 2.3 \text{ kg} &\approx 2.3 \text{ kg} \cdot \frac{2.2 \text{ lb}}{1 \text{ kg}} \cdot \frac{16 \text{ oz}}{1 \text{ lb}} && \text{Apply conversion factors.} \\ &\approx 2.3 \ \cancel{\text{kg}} \cdot \frac{2.2 \ \cancel{\text{lb}}}{1 \ \cancel{\text{kg}}} \cdot \frac{16 \text{ oz}}{1 \ \cancel{\text{lb}}} && \text{Cancel common units.} \\ &\approx \frac{2.3 \cdot 2.2 \cdot 16}{1 \cdot 1} \text{ oz} && \text{Multiply fractions.} \\ &\approx 80.96 \text{ oz} \end{aligned}$$

Hence, 2.3 kilograms weighs 80.96 ounces.

Alternate Solution

Another approach uses the facts that 1 kilogram equals 1000 grams and 1 ounce equals 28.35 grams.

$$\begin{aligned} 2.3\text{ kg} &\approx 2.3\text{ kg} \cdot \frac{1000\text{ g}}{1\text{ kg}} \cdot \frac{1\text{ oz}}{28.35\text{ g}} && \text{Apply conversion factors.} \\ &\approx 2.3\ \cancel{\text{kg}} \cdot \frac{1000\ \cancel{\text{g}}}{1\ \cancel{\text{kg}}} \cdot \frac{1\text{ oz}}{28.35\ \cancel{\text{g}}} && \text{Cancel common units.} \\ &\approx \frac{2.3 \cdot 1000 \cdot 1}{1 \cdot 28.35}\text{ oz} && \text{Multiply fractions.} \\ &\approx 81.13\text{ oz} \end{aligned}$$

Roundoff Error

Why the discrepancy in answers? This difference in approximations is due to something called *round-off error*. Indeed, in the first calculation, we used a conversion factor that is rounded to the nearest tenth; i.e., 1 kg ≈ 2.2 lb. If we use a more accurate conversion factor to change kilograms to pounds, namely 1 kg ≈ 2.2046 lb, we get the following result.

$$\begin{aligned} 2.3\text{ kg} &\approx 2.3\text{ kg} \cdot \frac{2.2046\text{ lb}}{1\text{ kg}} \cdot \frac{16\text{ oz}}{1\text{ lb}} && \text{Apply conversion factors.} \\ &\approx 2.3\ \cancel{\text{kg}} \cdot \frac{2.2046\ \cancel{\text{lb}}}{1\ \cancel{\text{kg}}} \cdot \frac{16\text{oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.} \\ &\approx \frac{2.3 \cdot 2.2046 \cdot 16}{1 \cdot 1}\text{ oz} && \text{Multiply fractions.} \\ &\approx 81.13\text{ oz} \end{aligned}$$

Note that this result is in better agreement with the second result above.

? Exercise

Change 5.7 kilograms to ounces. Round to the nearest ounce.

Answer

201 ounces

Important Observation

To obtain better approximations, you need to use a conversion factor that is more accurate. Any time you feel you need more accuracy, you might try an online conversion utility, such as the one at: www.france-property-and-information.com/metric_conversion_table.htm

Converting Units of Volume

It is known that

$$1\text{ qt} \approx 0.946\text{ L}.$$

Dividing both sides of this equation by 0.946,

$$1\text{ L} \approx 1.06\text{ qt}.$$

Again, these conversion factors for volume have been well worked out by scientists. The more common conversion factors for volume are shown in Table 6.7.

American to Metric	Metric to American
1 fl oz ≈ 0.030 L	1 L ≈ 33.8 fl oz
1 pt ≈ 0.473 L	1 L ≈ 2.1 pt
1 qt ≈ 0.946 L	1 L ≈ 1.06 qt
1 gal ≈ 3.785 L	1 L ≈ 0.264 gal

Table 6.7: Volume conversions: American — Metric.

Example 3

Change 2.5 dekalitres to gallons.

Solution

Recall that 1 daL = 10 L and 1 L = 0.264 gal.

$$\begin{aligned} 2.5 \text{ daL} &\approx 2.5 \text{ daL} \cdot \frac{10 \text{ L}}{1 \text{ daL}} \cdot \frac{0.264 \text{ gal}}{1 \text{ L}} && \text{Apply conversion factors.} \\ &\approx 2.5 \ \cancel{\text{daL}} \cdot \frac{10 \ \cancel{\text{L}}}{1 \ \cancel{\text{daL}}} \cdot \frac{0.264 \text{ gal}}{1 \ \cancel{\text{L}}} && \text{Cancel common units.} \\ &\approx \frac{2.5 \cdot 10 \cdot 0.264}{1 \cdot 1} \text{ gal} && \text{Multiply fractions.} \\ &\approx 6.6 \text{ gal} \end{aligned}$$

Hence, 2.5 dekalitres is approximately equal to 6.6 gallons.

Exercise

Change 3.2 dekalitres to gallons.Round your answer to the nearest tenth of a gallon.

Answer

0.8 gallons

Converting Units of Speed

Modern speedometers often show a car's speed in both miles per hour and kilometers per hour.

Example 4

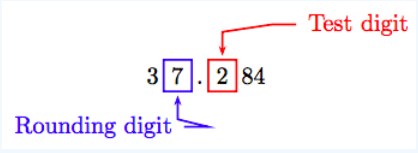
A car's speedometer is showing it speeding along at 60 kilometers per hour. How fast is it traveling in miles per hour? Round your answer to the nearest mile per hour.

Solution

From Table 6.5, 1 km ≈ 0.6214 mi.

$$\begin{aligned} 60 \frac{\text{km}}{\text{h}} &\approx 60 \frac{\text{km}}{\text{h}} \cdot \frac{0.6214 \text{ mi}}{1 \text{ km}} && \text{Apply conversion factor.} \\ &\approx 60 \frac{\cancel{\text{km}}}{\text{h}} \cdot \frac{0.6214 \text{ mi}}{1 \ \cancel{\text{km}}} && \text{Cancel common units.} \\ &\approx \frac{60 \cdot 0.6214}{1} \frac{\text{mi}}{\text{h}} && \text{Multiply fractions.} \\ &\approx 37.284 \frac{\text{mi}}{\text{h}} \end{aligned}$$

To round to the nearest mile per hour, identify the rounding and test digits.



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, to the nearest mile per hour, the speed is approximately 37 miles per hour.

Exercise

A car's speedometer registers 45 kilometers per hour. Change this speed to miles per hour. Round your answer to the nearest mile per hour.

Answer

28 miles per hour

Exercises

1. Use the conversion 1 in = 2.54 cm to convert 68 inches to centimeters, rounded to the nearest tenth of a centimeter.
2. Use the conversion 1 in = 2.54 cm to convert 42 inches to centimeters, rounded to the nearest tenth of a centimeter.
3. Use the conversion 1 in = 2.54 cm to convert 44 centimeters to inches, rounded to the nearest tenth of a inch.
4. Use the conversion 1 in = 2.54 cm to convert 22 centimeters to inches, rounded to the nearest tenth of a inch.
5. Use the conversion 1 mi = 1.6093 km to convert 79 miles to kilometers, rounded to the nearest tenth of a kilometer.
6. Use the conversion 1 mi = 1.6093 km to convert 39 miles to kilometers, rounded to the nearest tenth of a kilometer.
7. Use the conversion 1 yd = 0.9144 m to convert 1489 centimeters to yards, rounded to the nearest tenth of a yard.
8. Use the conversion 1 yd = 0.9144 m to convert 1522 centimeters to yards, rounded to the nearest tenth of a yard.
9. Use the conversion 1 yd = 0.9144 m to convert 28 yards to centimeters, rounded to the nearest tenth of a centimeter.
10. Use the conversion 1 yd = 0.9144 m to convert 34 yards to centimeters, rounded to the nearest tenth of a centimeter.
11. Use the conversion 1 m = 3.2808 ft to convert 8.6 meters to inches, rounded to the nearest tenth of a inch.
12. Use the conversion 1 m = 3.2808 ft to convert 8.3 meters to inches, rounded to the nearest tenth of a inch.
13. Use the conversion 1 in = 2.54 cm to convert 60 inches to centimeters, rounded to the nearest tenth of a centimeter.
14. Use the conversion 1 in = 2.54 cm to convert 75 inches to centimeters, rounded to the nearest tenth of a centimeter.
15. Use the conversion 1 m = 3.2808 ft to convert 208 inches to meters, rounded to the nearest tenth of a inch.
16. Use the conversion 1 m = 3.2808 ft to convert 228 inches to meters, rounded to the nearest tenth of a inch.
17. Use the conversion 1 m = 1.0936 yd to convert 20 yards to meters, rounded to the nearest tenth of a meter.
18. Use the conversion 1 m = 1.0936 yd to convert 44 yards to meters, rounded to the nearest tenth of a meter.
19. Use the conversion 1 mi = 1.6093 km to convert 29 miles to kilometers, rounded to the nearest tenth of a kilometer.
20. Use the conversion 1 mi = 1.6093 km to convert 15 miles to kilometers, rounded to the nearest tenth of a kilometer.
21. Use the conversion 1 m = 1.0936 yd to convert 8.2 meters to yards, rounded to the nearest tenth of a yard.
22. Use the conversion 1 m = 1.0936 yd to convert 6.9 meters to yards, rounded to the nearest tenth of a yard.
23. Use the conversion 1 mi = 1.6093 km to convert 4.9 kilometers to miles, rounded to the nearest tenth of a mile.
24. Use the conversion 1 mi = 1.6093 km to convert 4.2 kilometers to miles, rounded to the nearest tenth of a mile.
25. Use the conversion 1 m = 1.0936 yd to convert 25 yards to meters, rounded to the nearest tenth of a meter.
26. Use the conversion 1 m = 1.0936 yd to convert 2 yards to meters, rounded to the nearest tenth of a meter.
27. Use the conversion 1 in = 2.54 cm to convert 47 centimeters to inches, rounded to the nearest tenth of a inch.
28. Use the conversion 1 in = 2.54 cm to convert 19 centimeters to inches, rounded to the nearest tenth of a inch.
29. Use the conversion 1 mi = 1.6093 km to convert 8.3 kilometers to miles, rounded to the nearest tenth of a mile.
30. Use the conversion 1 mi = 1.6093 km to convert 4.8 kilometers to miles, rounded to the nearest tenth of a mile.

31. Use the conversion 1 yd = 0.9144 m to convert 41 yards to centimeters, rounded to the nearest tenth of a centimeter.
32. Use the conversion 1 yd = 0.9144 m to convert 20 yards to centimeters, rounded to the nearest tenth of a centimeter.
33. Use the conversion 1 m = 3.2808 ft to convert 3.7 meters to inches, rounded to the nearest tenth of a inch.
34. Use the conversion 1 m = 3.2808 ft to convert 7.9 meters to inches, rounded to the nearest tenth of a inch.
35. Use the conversion 1 yd = 0.9144 m to convert 1323 centimeters to yards, rounded to the nearest tenth of a yard.
36. Use the conversion 1 yd = 0.9144 m to convert 1715 centimeters to yards, rounded to the nearest tenth of a yard.
37. Use the conversion 1 m = 1.0936 yd to convert 8.4 meters to yards, rounded to the nearest tenth of a yard.
38. Use the conversion 1 m = 1.0936 yd to convert 7.3 meters to yards, rounded to the nearest tenth of a yard.
39. Use the conversion 1 m = 3.2808 ft to convert 289 inches to meters, rounded to the nearest tenth of a inch.
40. Use the conversion 1 m = 3.2808 ft to convert 251 inches to meters, rounded to the nearest tenth of a inch.

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41. Use the conversion 1 kg = 2.2 lb to convert 15.8 kilograms to pounds, rounded to the nearest tenth of a pound.
42. Use the conversion 1 kg = 2.2 lb to convert 19.4 kilograms to pounds, rounded to the nearest tenth of a pound.
43. Use the conversion 1 oz = 28.35 g to convert 35 ounces to grams, rounded to the nearest tenth of a gram.
44. Use the conversion 1 oz = 28.35 g to convert 33 ounces to grams, rounded to the nearest tenth of a gram.
45. Use the conversion 1 kg = 2.2 lb to convert 2.48 kilograms to ounces, rounded to the nearest tenth of a ounce.
46. Use the conversion 1 kg = 2.2 lb to convert 3.74 kilograms to ounces, rounded to the nearest tenth of a ounce.
47. Use the conversion 1 kg = 2.2 lb to convert 2.35 kilograms to ounces, rounded to the nearest tenth of a ounce.
48. Use the conversion 1 kg = 2.2 lb to convert 3.57 kilograms to ounces, rounded to the nearest tenth of a ounce.
49. Use the conversion 1 kg = 2.2 lb to convert 15 pounds to kilograms, rounded to the nearest tenth of a kilogram.
50. Use the conversion 1 kg = 2.2 lb to convert 27 pounds to kilograms, rounded to the nearest tenth of a kilogram.
51. Use the conversion 1 kg = 2.2 lb to convert 10.4 kilograms to pounds, rounded to the nearest tenth of a pound.
52. Use the conversion 1 kg = 2.2 lb to convert 17.7 kilograms to pounds, rounded to the nearest tenth of a pound.
53. Use the conversion 1 kg = 2.2 lb to convert 352 ounces to kilograms, rounded to the nearest tenth of a kilogram.
54. Use the conversion 1 kg = 2.2 lb to convert 326 ounces to kilograms, rounded to the nearest tenth of a kilogram.
55. Use the conversion 1 oz = 28.35 g to convert 96 grams to ounces, rounded to the nearest tenth of a ounce.
56. Use the conversion 1 oz = 28.35 g to convert 100 grams to ounces, rounded to the nearest tenth of a ounce.
57. Use the conversion 1 oz = 28.35 g to convert 14 ounces to grams, rounded to the nearest tenth of a gram.
58. Use the conversion 1 oz = 28.35 g to convert 29 ounces to grams, rounded to the nearest tenth of a gram.
59. Use the conversion 1 kg = 2.2 lb to convert 54 pounds to kilograms, rounded to the nearest tenth of a kilogram.
60. Use the conversion 1 kg = 2.2 lb to convert 82 pounds to kilograms, rounded to the nearest tenth of a kilogram.
61. Use the conversion 1 oz = 28.35 g to convert 92 grams to ounces, rounded to the nearest tenth of a ounce.
62. Use the conversion 1 oz = 28.35 g to convert 103 grams to ounces, rounded to the nearest tenth of a ounce.
63. Use the conversion 1 kg = 2.2 lb to convert 388 ounces to kilograms, rounded to the nearest tenth of a kilogram.
64. Use the conversion 1 kg = 2.2 lb to convert 395 ounces to kilograms, rounded to the nearest tenth of a kilogram.

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65. Use the conversion 1 qt = 0.946 L to convert 55.1 litres to quarts, rounded to the nearest tenth of a quart.
66. Use the conversion 1 qt = 0.946 L to convert 50.3 litres to quarts, rounded to the nearest tenth of a quart.
67. Use the conversion 1 L = 33.8 fl oz to convert 72073 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.
68. Use the conversion 1 L = 33.8 fl oz to convert 56279 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

69. Use the conversion 1 L = 33.8 fl oz to convert 2.5 kilolitres to fluid ounces.

70. Use the conversion 1 L = 33.8 fl oz to convert 4.5 kilolitres to fluid ounces.

71. Use the conversion 1 qt = 0.946 L to convert 24 quarts to litres, rounded to the nearest tenth of a litre.

72. Use the conversion 1 qt = 0.946 L to convert 21 quarts to litres, rounded to the nearest tenth of a litre.

73. Use the conversion 1 qt = 0.946 L to convert 30 quarts to litres, rounded to the nearest tenth of a litre.

74. Use the conversion 1 qt = 0.946 L to convert 22 quarts to litres, rounded to the nearest tenth of a litre.

75. Use the conversion 1 gal = 3.785 L to convert 11.8 gallons to litres, rounded to the nearest tenth of a litre.

76. Use the conversion 1 gal = 3.785 L to convert 13.5 gallons to litres, rounded to the nearest tenth of a litre.

77. Use the conversion 1 gal = 3.785 L to convert 50.5 litres to gallons, rounded to the nearest tenth of a gallon.

78. Use the conversion 1 gal = 3.785 L to convert 55.9 litres to gallons, rounded to the nearest tenth of a gallon.

79. Use the conversion 1 L = 33.8 fl oz to convert 8.3 kilolitres to fluid ounces.

80. Use the conversion 1 L = 33.8 fl oz to convert 5.3 kilolitres to fluid ounces.

81. Use the conversion 1 qt = 0.946 L to convert 42.4 litres to quarts, rounded to the nearest tenth of a quart.

82. Use the conversion 1 qt = 0.946 L to convert 55.4 litres to quarts, rounded to the nearest tenth of a quart.

83. Use the conversion 1 gal = 3.785 L to convert 17.2 gallons to litres, rounded to the nearest tenth of a litre.

84. Use the conversion 1 gal = 3.785 L to convert 19.6 gallons to litres, rounded to the nearest tenth of a litre.

85. Use the conversion 1 L = 33.8 fl oz to convert 51274 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

86. Use the conversion 1 L = 33.8 fl oz to convert 82164 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

87. Use the conversion 1 gal = 3.785 L to convert 55.6 litres to gallons, rounded to the nearest tenth of a gallon.

88. Use the conversion 1 gal = 3.785 L to convert 59.2 litres to gallons, rounded to the nearest tenth of a gallon.

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89. Change 60 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

90. Change 56 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

91. Change 77 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

92. Change 57 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

93. Change 42 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

94. Change 56 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

95. Change 62 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

96. Change 63 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

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97. **Tallest tower.** The world's tallest tower in Dubai has 160 floors at a height of 2, 717 feet. Convert the height of the tower to the nearest tenth of a meter. *Associated Press-Times-Standard 02/09/10 World's tallest tower closed a month after opening.*

98. **High peaks.** For the first time, foreigners will be allowed to climb nearly 100 high-altitude Himalayan peaks on the Indian side of Kashmir, peaks ranging from 9, 840 feet to nearly 26, 246 feet. Convert the highest of the peaks to the nearest tenth of a meter. *Associated Press-TimesStandard 04/11/10 India opens Himalayan peaks to foreigners.*

99. **Ancient find.** In the southern Egyptian town of Luxor, a 3, 400-year-old 4-meter statue of Thoth, the ancient Egyptian god of Wisdom and Magic, was unearthed. Convert the height of the statue to the nearest tenth of a foot. *Associated Press-Times-Standard 03/17/10 3,400-year-old statues unearthed in Egypt.*

100. **Arctic wind.** Blizzard condition winds in the Arctic blew 80 miles per hour. Find the wind speed to the nearest kilometer per hour. *Associated Press-Times-Standard 12/31/09 After 562-mile ski trek, seven women reach the South Pole.*

101. **Solar plane.** The Solar Impulse lifted off from a military airport at a speed no faster than 28 miles per hour. Convert the speed of the solar-powered plane to the nearest kilometer per hour. *Associated Press-Times-Standard 04/09/10 Solar-powered plane makes successful maiden flight.*

Answers

1. 172.7 centimeters
3. 17.3 inches
5. 127.1 kilometers
7. 16.3 yards
9. 2560.3 centimeters
11. 338.6 inches
13. 152.4 centimeters
15. 5.3 meters
17. 18.3 meters
19. 46.7 kilometers
21. 9.0 yards
23. 3.0 miles
25. 22.9 meters
27. 18.5 inches
29. 5.2 miles
31. 3749.0 centimeters
33. 145.7 inches
35. 14.5 yards
37. 9.2 yards
39. 7.3 meters
41. 34.8 pounds
43. 992.3 grams
45. 87.3 ounces
47. 82.7 ounces
49. 6.8 kilograms
51. 22.9 pounds
53. 10.0 kilograms
55. 3.4 ounces
57. 396.9 grams
59. 24.5 kilograms
61. 3.2 ounces
63. 11.0 kilograms
65. 58.2 quarts
67. 2.1 kilolitres

69. 84500 ounces

71. 22.7 litres

73. 28.4 litres

75. 44.7 litres

77. 13.3 gallons

79. 280540 ounces

81. 44.8 quarts

83. 65.1 litres

85. 1.5 kilolitres

87. 14.7 gallons

89. 97 km/hr

91. 124 km/hr

93. 26 mi/hr

95. 39 mi/hr

97. 828.2 meters

99. 13.1 feet

101. 45 km/hr

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5.4: How to Measure With Micrometer

A micrometer is an essential tool for taking precise measurements. It has a precision of 0.01 mm and if you ever aspire to make something with tight tolerance you will probably need to use it. We will have a look at how to use a traditional analog micrometer and how to read the dimension correctly.

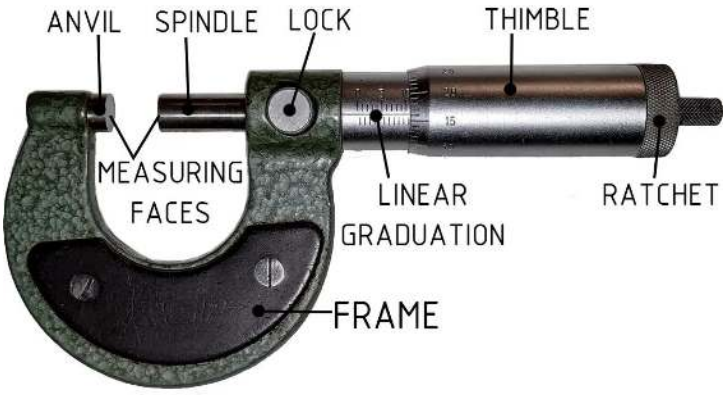
In the video, I am showing how to take the measurement and also talking a little bit about how the micrometer is made. So this could be helpful for understanding this device a little better.

Video: In this video, we are looking at a micrometer. This is a device for taking precise measurements. It has a precision of 0,01 mm and is essential in any machine shop. I am quickly showing how it is built and how it works as well as how to take a measurement with it. I hope this is helpful.

Supplies:

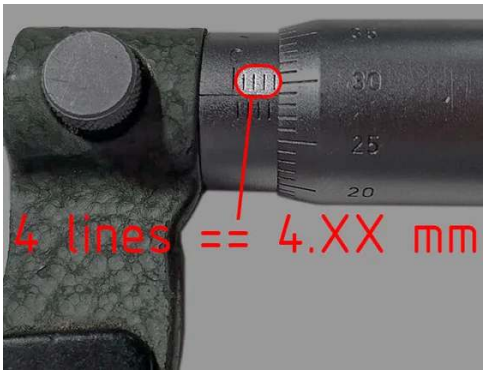
- Micrometer
- Part to measure

Step 1: Take the Measurement



1. Place your part between the measuring faces.
2. Bring the measuring face towards the part by rotating the spindle.
3. When you are close use rachet to press the measuring face against the part. You should achieve sufficient measuring force by rotating the rachet two or three times around.
4. Now you can lock the dimension using the locking nut.

Step 2: Reading the Linear Graduation

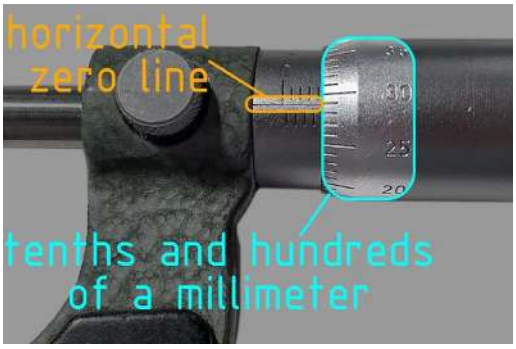


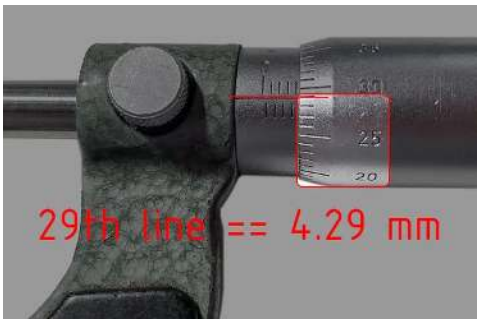
Read the linear graduation. :)

The number of lines on the top of the horizontal zero line tells you the millimeters. We can see 4 lines for this measurement. **Meaning our measurement is over 4 mm.**

Lines under the horizontal zero line show us the half-millimeters. When you **can not** see the half-millimeter line it means that your dimension is between 4.0 and 4.5 mm. If we **can** see the half mm line, it means that the dimension is between 4.5 and 5 mm.

Step 3: Reading the Thimble Graduation





This graduation shows you the tenths and hundreds of a millimeter. It is divided into 50 graduations. There fore each graduation equals 0.01 mm. We will simply have a look at which graduation lines up with the horizontal zero line. For this part, we can see that the number is 29. Because we **can not** see the half-millimeter line we just add this number to 4.0 mm. Therefore our final dimension is 4.29 mm.

If on the other hand, we **could** see the half mm line, we would add this number to 4.5 mm and the result would be 4.79 mm.

Step 4: Final Tips

Reading the micrometer can be tricky when you are not used to it. To mitigate any reading error, it is helpful to take a rougher measurement with calipers first to see what dimension you are aiming for.

If you want to practice reading a micrometer then here is a pretty sweet micrometer simulator.

Contributors

- JindraSykora https://www.instructables.com/id/How-to-Measure-With-Micrometer/

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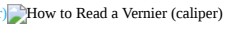
5.5: How to Read a Vernier (Caliper)

Looking at an early caliper, you'll notice that it looks like a tomahawk and has a scale down it's length. More about the scale later. On the bottom left, the "blade of the hatchet" are the jaws of the caliper for measuring **outside dimensions** - think of a tube. This would measure the outside diameter. The "pike" on top of the hatchet is used for measuring **inside dimensions**. Again, think of a tube's inside diameter. Looking all the way to the right...I know it's hard to see...but there's a little "tail" sticking out of the center of the handle. That will measure the **depth** of a step or some other similar feature. That little button on the top near the "pike" is a locking screw. The one just below it is a thumb button.



Step 1: Holding the Caliper

This is the proper way to hold a caliper. Use your thumb to slide the jaws open and closed. Yup, another tool that was totally designed for righties. Use the locking screw at the top if you want to set a dimension on the caliper and then check parts against your setting. Though it may not look it, the measurement between the inside and outside jaws are EXACTLY the same.

Step 2: Course Reading #1

So let's read a measurement off of the caliper.

First, a quick math refresher. Whole numbers (1,2,3, etc.) are followed by a decimal point and an endless number of zeros (2.000000). We are going to be working with the 3 places to the right of the decimal point. The first number to the right of the decimal point is a tenth of an inch (1/10), the second number signifies a hundreth of an inch (1/100), and the third number signifies a thousanth of an inch (1/1000). A human hair is about 3 thousands of an inch (0.003), give or take a thousanth. This number (0.456) is read as four hundred and fifty-six thousands.

If you noticed, the "handle" of the caliper has a scale that starts from zero on the left and goes up to 6 inches on the right, in this case. Other calipers can be much longer in length. The big number in the center of the scale is the inch number. In this case we are looking mostly at the area AFTER the first inch mark on the left, and then we see the number **2** on the right, which indicates the second full inch. Notice that each inch is divided into 10 equal parts with the numbers 1 through 9. That makes each one of those lengths 1/10 of an inch (0.100). And then each amount of space between each 1/10 of an inch has 3 lines - which breaks each 1/10 of an inch into 4 equal pieces. Each one of those little pieces, therefore, is twenty-five thousanths of an inch (0.025) long. There's also a scale on the bottom of the picture. It starts at zero, there's a 5, a 10 and a 15. It goes up to 25, but the image doesn't show that. THAT, is the vernier part of the caliper. Notice that the line for the 0 (zero) is to the right of the second line after the number 3. Since we know that there is a large number 1 to the left of the image that we can't see (which signifies 1 whole inch), let's write that down (1.) and we'll put the decimal point in. The 3 is to the left of the zero on the vernier, so we'll add that to our number (1.3). The zero line is to the right, of the second line after the 3. So there is at least fifty thousanths there (0.050), but there's more. Notice how the zero line is just a smidge to the right of that second line after the 3. Now look at the large 2 and notice that the vernier line of 13 is to the LEFT. So the trick is to find which lines on the vernier lines up exactly with the scale on the center. In this case, it looks like 6...but let's get a closer look.

Step 3: Fine Reading #1

Yup. It's 6 all right.

NOT the 6 on the top scale - the 6 on the bottom vernier scale.

So, we were up to 1.3, plus we knew we had at least another fifty thousanths (0.050), and the vernier tells us there is an addition six thousanths (0.006). Add them all up and we have 1.356 - read 1 inch, three hundred and fifty-six thousands.

Step 4: Course Reading #2

Let's try another one.

The zero on the vernier is to the left of the 1 whole inch mark...so the number is LESS THAN 1 full inch. Let's write that down (0.) and don't forget the decimal point.

We're between 6 and 7 on the 1/10 scale, so let's write that down too (0.6).

Hmmm, the vernier zero line is almost at the number 7; there are 3 lines...seventy-five thousanths (0.075)...to the left of the vernier zero line...and some more.

Let's look at the vernier scale and see what lines up to understand how much more.

16, 17, 18 - not sure. Let's get a better shot of that.

Step 5: Fine Reading #2

It looks like 15.

So, that's 0.6 plus 0.075 plus 0.015. That would be 0.690. Read, six hundred and ninty thousanths.

Contributors

- blackslax (https://www.instructables.com/id/How-to-read-a-vernier-caliper/)

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5.6: How to Read a Water Meter

The water meter shown here shows usage of 53,586.5 Gallons. However, if your water department bills in thousands, your bill may show 53,000, or 53,500 if they bill to 100 gallons. Follow the steps and we'll show you how this is determined and how to read your own meter accurately.

Figure 5.6.1: The water meter shown here shows usage of 53,586.5 Gallons

Step 1: The Printed Number

Figure 5.6.2

Figure 5.6.3

First, notice what the register increment is. In this case it is in GALLONS. It could be Cubic Feet, or outside the USA it may be Cubic Meters. You will notice on this register that one of the numerical readings of the odometer is not a moving wheel like the others, but is a printed zero. Almost no utility bills water to the actual gallon. If they bill to 10 gallon increments, they would read only the moving white and black wheels for a reading of 53580. Why 80 and not 90 is explained in the next step. On larger meters, there could be two or more printed zeros represented by the sweep hand.

The printed zero is actually replaced by the location of the sweep hand for actual usage, troubleshooting, or testing. If on the 2 circled, it would be replaced by 2. Each full number is one gallon in this case, so if the sweep hand moves from 3 to 4 as shown with the arrow, that is one gallon of usage. The numbers between are 10ths of gallons, so the number circled between 4 and 5 represents 4.7 gallons. In THIS meter, the sweep hand is on 6.5, so the fixed zero is replaced by 6.5 gallons for actual use.

Step 2: Moving Number Wheels

You will notice the first MOVING number wheel is moved between 8 and 9. It is not fully centered on 9 yet, and the sweep hand is on 6.5, so it is counted as 8. When the sweep hand reaches directly on "0" it will then count as 9. The odometer wheel will be centered and counted as whole. Reading the last TWO digits, one moving and one fixed is 86.5 gallons.

Figure 5.6.4: Moving Number Wheels

Step 3: Black Digits

You may have noticed that the last three digit are dark, and those to the left of them are white or light colored. This is an industry standard for manual reading. Most utilities that bill in gallons do so 1000 gallon increments. The amount used under the 1000 is 586.5 gallons in this case. When manually reading, most meter readers ignore the black/dark numbers and only read the white numbers which they enter on their meter book or handheld computer to bill by. This saves writing or entering insignificant numbers. A city billing in 1000 gallon increments would read ONLY the white digits and bill for 53,000 gallons. Meter reading almost never rounds up, but drops the last insignificant digits.

Figure 5.6.5A: Black Digits

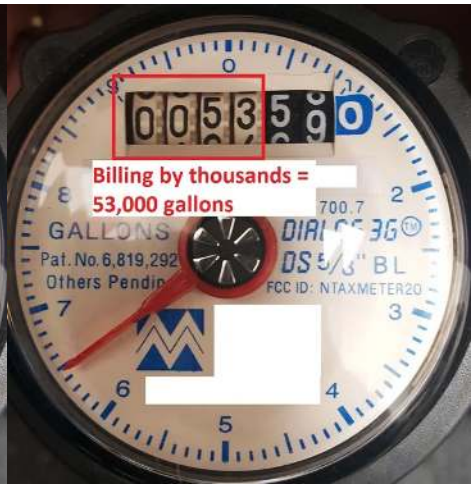

Figure 5.6.5B: white or light colored digits

Cubic Feet

If your meter registers in Cubic Feet instead of gallons, the black/dark digits may be 10 cubic feet instead of 100 gallons. Also in cubic feet, there may be no fixed zero and the sweep had represents 10ths and 100ths of gallons instead of gallons and tenths. Cubic feet is a larger increment. One cubic foot is 7.48 gallons

Cubic Meters

If you are billed in Cubic Meters, there may be no fixed printed zero, and the sweep represents 10ths (100 liters) and 100ths (10 liters), and maybe even 1000ths (1 liter) of a meter. Cubic Meters is an even larger increment than cubic feet with 1 cubic meter being approximately 264 gallons.

Step 4: The Final Reading

So now we have the explanation of how to read your water meter.

Figure 5.6.6: The final reading.

The meter in this photo is showing ACTUAL usage of 53,586.5 gallons. It is read If your billing resolutions is:

- Gallons = 53,586
- Tens of Gallons = 53,580
- Hundred Gallons = 53,500
- Thousand Gallons = 53,000

Step 5: Bonus - What Is That Other Moving Dial?

Besides the moving odometer wheels and sweep hand you may find another small moving dial. This is the leak indicator. It will move when water is flowing too slow to see the sweep hand move and allow you to know if you have a small leak or small water flow. In some meters, there is no sweep hand and the dial serves as both a leak indicator and to measure low flows not indicated by the odometer wheels.

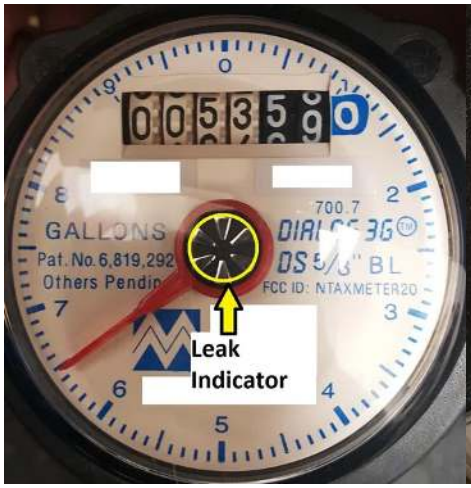

Figure 5.6.7A: Paste Caption Here

Figure 5.6.7B: Paste Caption Here

Contributors

- UtilityTechnologies (https://www.instructables.com/id/How-to-Read-a-Water-Meter/)

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5.7: How to Read Residential Electric and Natural Gas Meters

You can read your own meters to help monitor your electric or gas energy use. During the heating season, your energy use should be compared to the number of heating degree days for the same time period; during the cooling season, compare your energy use to the number of cooling degree days.

Heating and cooling degree days are a simple measure of the effect of weather on your energy needs: using the average temperature for each day, each degree Fahrenheit below 65°F is counted as one heating degree day, and each degree Fahrenheit above 65°F is counted as one cooling degree day. Your heating and cooling use should be proportional to the number of heating and cooling degree days for the time period in question.

You may also wish to contact your local utility companies for more information about reading your meter. If monthly information is good enough, your utility bills could have all the information you need. Just be sure the bills are based on actual, not estimated, meter readings, and be aware of when the meter was read, because the time period between readings can vary. Contact your local utility if you are uncertain about this.

Electric Meters

The basic unit of measure of electric power is the watt. One thousand watts are called a kilowatt. If you use one thousand watts of power in one hour you have used a kilowatt-hour (kWh). Your electric utility bills you by the kWh.

The standard electric power meter is a clock-like device driven by the electricity moving through it. As the home draws current from the power lines, a set of small gears inside the meter move. The number of revolutions is recorded by the dials that you can see on the face of the meter. The speed of the revolutions depends on the amount of current drawn -- the more power consumed at any one instant, the faster the gears will rotate.

When reading an electric meter, read and write down the numbers as shown on the dials from *right to left*. When the pointer is directly on a number, look at the dial to the right. If it has passed zero, use the next higher number. If it has not passed zero, use the lower number. Record the numbers shown by writing down the value of the dial to your extreme right first and the rest as you come to them. Should the hand of a dial fall between two numbers, use the smaller of the two numbers.

Natural Gas Meters

Natural gas is commonly measured by the cubic foot, and you are billed by the thousands of cubic feet (MCF) or hundreds of cubic feet (CCF). You may also be billed by the therm, which is about the same as a CCF or 100 cubic feet. To measure the amount of electricity or gas that you use, the utility installs a meter between the incoming electric power or gas lines and the point of distribution at the house.

A gas meter is driven by the force of the moving gas in the pipe, and also turns faster as the flow increases. Each time the dial with the lower value makes one complete revolution, the pointer on the next higher value dial moves ahead one digit.

When reading a gas meter, read and write down the numbers as shown on the dials from *left to right* (opposite of an electric meter). It is important to note that on both types of meters, the hands of adjacent dials turn in opposite directions to each other.

A natural gas meter on a house. | Photo courtesy of ©iStockphoto/fstockfoto

Contributors

US DOE: https://www.energy.gov/energysaver/how-read-residential-electric-and-natural-gas-meters

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CHAPTER OVERVIEW

6: The Integers

Today, much as we take for granted the fact that there exists a number zero, denoted by 0, such that $a+0 = a$ for any whole number a, we similarly take for granted that for any whole number a there exists a unique number $-a$, called the "negative" or "opposite" of a, so that $a + (-a) = 0$.

In a natural way, or so it seems to modern-day mathematicians, this easily introduces the concept of a *negative* number. However, history teaches us that the concept of negative numbers was not embraced wholeheartedly by mathematicians until somewhere around the 17th century.

In his work *Arithmetica* (c. 250 AD), the Greek mathematician Diophantus (c. 200-284 AD), who some call the "Father of Algebra," described the equation $4 = 4x + 20$ as "absurd," for how could one talk about an answer less than nothing? Girolamo Cardano (1501-1576), in his seminal work *Ars Magna* (c. 1545 AD) referred to negative numbers as "numeri ficti," while the German mathematician Michael Stifel (1487-1567) referred to them as "numeri absurdi." John Napier (1550-1617) (the creator of logarithms) called negative numbers "defectivi," and Rene Descartes (1596-1650) (the creator of analytic geometry) labeled negative solutions of algebraic equations as "false roots."

On the other hand, there were mathematicians whose treatment of negative numbers resembled somewhat our modern notions of the properties held by negative numbers. The Indian mathematician Brahmagupta described arithmetical rules in terms of fortunes (positive number) and debts (negative numbers). Indeed, in his work *Brahmasphutasiddhanta*, he writes "a fortune subtracted from zero is a debt," which in modern notation would resemble $0 - 4 = -4$.

If you find the study of the integers somewhat difficult, do not be discouraged, as centuries of mathematicians have struggled mightily with the topic. With this thought it mind, let's begin the study of the integers.

6.1: An Introduction to the Integers
6.2: Adding Integers
6.3: Subtracting Integers
6.4: Multiplication and Division of Integers
6.5: Introduction to Square Roots
6.6: Order of Operations

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6.1: An Introduction to the Integers

As we saw in the introduction to the chapter, negative numbers have a rich and storied history. One of the earliest applications of negative numbers had to do with credits and debits. For example, if \$5 represents a credit or profit, then −\$5 represents a debit or loss. Of course, the ancients had a different monetary system than ours, but you get the idea. Note that if a vendor experiences a profit of \$5 on a sale, then a loss of −\$5 on a second sale, the vendor breaks even. That is, the sum of \$5 and −\$5 is zero. In much the same way, every whole number has an *opposite* or *negative* counterpart.

The Opposite or Negative of a Whole Number

For every whole number a, there is a unique number $-a$, called the opposite or negative of a, such that $a + (-a) = 0$.

The opposite or negative of any whole number is easily located on the number line.

Number Line Locations

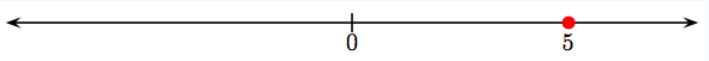
To locate the opposite (or negative) of any whole number, first locate the whole number on the number line. The opposite is the reflection of the whole number through the origin (zero).

Example 1

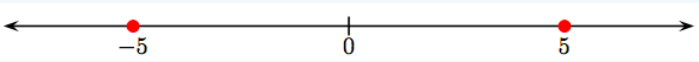
Locate the whole number 5 and its opposite (negative) on the number line.

Solution

Draw a number line, then plot the whole number 5 on the line as a shaded dot.



To find its opposite, reflect the number 5 through the origin. This will be the location of the opposite (negative) of the whole number 5, which we indicate by the symbol −5.



Note the symmetry. The whole number 5 is located five units to the right of zero. Its negative is located five units to the left of zero.

Exercise

Locate the number -7 and its opposite on the number line.

Important Pronunciation

The symbol −5 is pronounced in one of two ways: (1) “negative five,” or (2) “the opposite of five.”

In similar fashion, we can locate the opposite or negative of any whole number by reflecting the whole number through the origin (zero), which leads to the image shown in Figure 2.1.

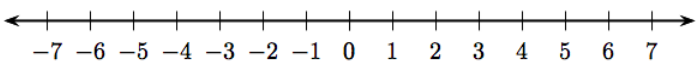


Figure 2.1: The opposite (negative) of any whole number is a reflection of that number through the origin (zero).

The Integers

The collection of numbers arranged on the number line in Figure 2.1 extend indefinitely to the right, and because the numbers on the left are reflections through the origin, the numbers also extend indefinitely to the left. This collection of numbers is called the set of *integers*.

The Integers

The infinite collection of numbers

$$\{..., -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, ...\}$$

is called the **set of integers.**

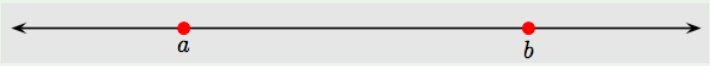
The ellipsis . . . at each end of this infinite collection means “etcetera,” as the integers continue indefinitely to the right and left. Thus, for example, both 23,456 and −117, 191 are elements of this set and are therefore integers.

Ordering the Integers

As we saw with the whole numbers, as you move to the right on the number line, the numbers get larger; as you move to the left, the numbers get smaller.

Order on the Number Line

Let a and b be integers located on the number line so that the point representing the integer a lies to the left of the point representing the integer b.



Then the integer a is “less than” the integer b and we write

$$a < b$$

Alternatively, we can also say that the integer b is “larger than” the integer a and write

$$b > a.$$

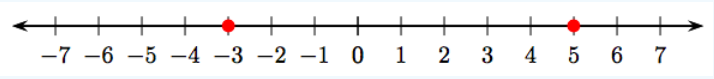
Example 3

Replace each shaded box with < (less than) or > (greater than) so the resulting inequality is a true statement.

$$-3 \quad \square \quad 5 \qquad\qquad -2 \quad \square \quad -4$$

Solution

For the first case, locate −3 and 5 on the number line as shaded dots.

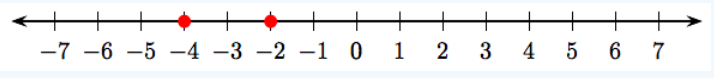


Note that −3 lies to the left of 5, so:

$$-3 \quad < \quad 5$$

That is, −3 is “less than” 5.

In the second case, locate −2 and −4 as shaded dots on the number line.



Note that −2 lies to the right of −4, so:

$$-2 \quad > \quad -4$$

That is, −2 is “greater than” −4.

Exercise

Compare −12 and −11.

Answer

−12 < −11

Important Observation

In Example 3, note that the "pointy end" of the inequality symbol always points towards the smaller number.

We stated earlier that every integer has a unique number called its "opposite" or "negative." Thus, the integer −5 is the opposite (negative) of the integer 5. Thus, we can say that the pair −5 and 5 are opposites. Each is the opposite of the other. Logically, this leads us to the conclusion that the opposite of −5 is 5. In symbols, we would write

$$-(-5) = 5.$$

Opposites of Opposites

Let a be an integer. Then the "opposite of the opposite of a is a." In symbols, we write

$$-(-a) = a.$$

We can also state that the "negative of a negative a is a.

Example 4

Simplify −(−13) and −(−119).

Solution

The opposite of the opposite of a number returns the original number. That is,

−(−13) = 13 and − (−119) = 11.

Exercise

Simplify: −(−50).

Answer

50

Positive and Negative

We now define the terms *positive integer* and *negative integer*.

Positive Integer

If a is an integer that lies to the right of zero (the origin) on the number line, then a is a **positive** integer. This means that a is a positive integer if and only if $a > 0$.

Thus, 2, 5, and 117 are positive integers.

Negative Integer

If a is an integer that lies to the left of zero (the origin) on the number line, then a is a **negative** integer. This means that a is a negative integer if and only if $a < 0$.

Thus, −4, −8, and −1, 123 are negative integers.

Zero

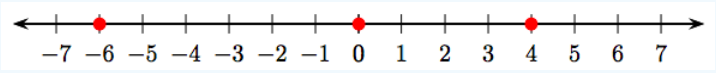
The integer zero is neither positive nor negative.

Example 5

Classify each of the following numbers as negative, positive, or neither: 4, −6, and 0.

Solution

Locate 4, −6, and 0 on the number line.



Thus:

- 4 lies to the right of zero. That is, $4 > 0$, making 4 a positive integer.
- −6 lies to the left of zero. That is, $-6 < 0$, making −6 a negative integer.
- The number 0 is neutral. It is neither negative nor positive.

Exercise

Classify −11 as positive, negative, or neither.

Answer

Negative

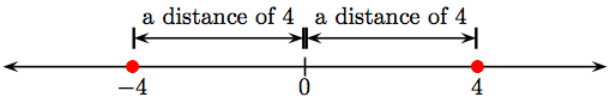
Absolute Value

We define the absolute value of an integer.

Absolute Value

The absolute value of an integer is defined as its distance from the origin (zero).

It is important to note that distance is always a nonnegative quantity (not negative); i.e., distance is either positive or zero. As an example, we've shaded the integers −4 and 4 on a number line.



The number line above shows two cases:

- The integer −4 is 4 units from zero. Because absolute value measures the distance from zero, $|-4| = 4$.
- The integer 4 is also 4 units from zero. Again, absolute value measures the distance from zero, so $|4| = 4$.

Let's look at another example.

Example 6

Determine the value of each expression: a) $|-7|$, b) $|3|$, and c) $|0|$.

Solution

The absolute value of any integer is equal to the distance that number is from the origin (zero) on the number line. Thus:

a) The integer -7 is 7 units from the origin; hence, $|-7| = 7$.

b) The integer 3 is 3 units from the origin; hence, $|3| = 3$.

c) The integer 0 is 0 units from the origin; hence, $|0| = 0$.

Exercise

Simplify: $|-33|$.

Answer

33

Example 7

Determine the value of each expression: a) $-(-8)$ and b) $-|-8|$

Solution

These are distinctly different problems.

a) The opposite of -8 is 8. That is, $-(-8) = 8$.

b) However, in this case, we take the absolute value of -8 first, which is 8, then the opposite of that result to get -8. That is,

$$\begin{aligned} -|-8| &= -(8) \quad \text{First: } |-8| = 8. \\ &= -8 \quad \text{Second: The opposite of } 8 \text{ is } -8. \end{aligned}$$

Exercise

Simplify: $-|-50|$.

Answer

50

Applications

There are a number of applications that benefit from the use of integers.

Example 8

Profits and losses for the first six months of the fiscal year for a small business are shown in Table 2.1. Profits and losses are measured in thousands of dollars. A positive number represents a profit, while a negative number represents a loss. Create a bar graph representing the profits and losses for this small business for each month of the first half of the fiscal year.

Month	Jan	Feb	Mar	Apr	May	Jun
Profit/Loss	10	12	7	-2	-4	5

Table 2.1: Profit and loss are measured in thousands of dollars.

Solution

Start by labeling the horizontal axis with the months of the fiscal year. Once you've completed that task, scale the vertical axis to accommodate the Profit/Loss values recorded in Table 2.1. Finally, starting at the 0 level on the horizontal axis, sketch bars having heights equal to the profit and loss for each month.

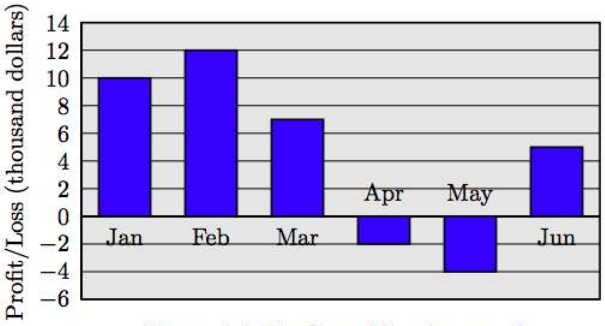


Figure 2.2: Profit and loss bar graph.

Figure 2.2 for the months January, February, March, and June have heights greater than zero, representing a profit in each of those months. The bars for the months April and May have heights that are less than zero, representing a loss for each of those months.

Exercise

The following table contains record low temperatures (degrees Fahrenheit) for Jackson Hole, Wyoming for the indicated months.

Month	**Temp**
Sept.	14
Oct.	2
Nov.	−27
Dec.	−49
Jan.	−50

Create a bar graphs of temperature versus months.

Example 9

Table 2.2 contains the low temperature recordings (degrees Fahrenheit) on five consecutive days in Fairbanks, Alaska, 1995. Create a line graph for the data in Table 2.2.

Date	Jan 21	Jan 22	Jan 23	Jan 24	Jan 25
Low Temp	1	10	5	−20	−28

Solution

Start by labeling the horizontal axis with the days in January that the temperatures occurred. Scale the vertical axis to accommodate the temperatures given in Table 2.2. Finally, plot points on each day at a height that equals the temperature for that given day. Connect consecutive pairs of points with line segments to produce the line graph shown in Figure 2.3.

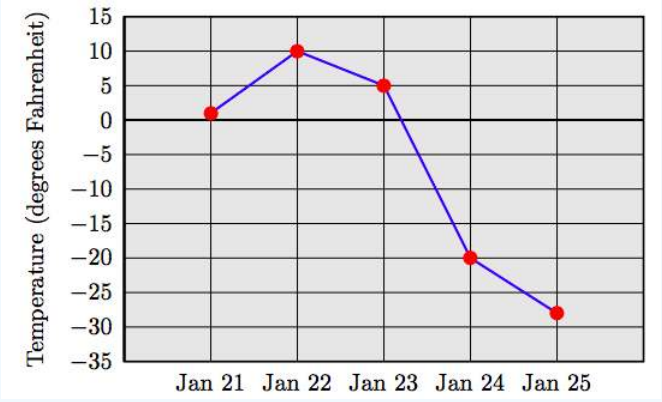


Figure 2.3: Plotting daily low temps versus the date they occurred.

Note that the points in Figure 2.3 have heights greater than zero for January 21-23, representing temperatures greater than zero. The points representing January 24-45 have negative heights corresponding to the negative temperatures of Table 2.2.

? Exercise

A man stands on the roof of a multistory building and throws a baseball vertically upward. The height (in feet) of the ball above the edge of the roof at measured times (in seconds) is given in the following table.

Time	Height
0	0
1	24
2	16
3	−24
4	−96

Create a line graph of the height of the ball versus time in the air.

Exercises

In Exercises 1-12, for each of the following questions, provide a number line sketch with your answer.

1. What number lies three units to the left of 4 on the number line?
2. What number lies three units to the left of 1 on the number line?
3. What number lies three units to the left of 6 on the number line?
4. What number lies four units to the left of −2 on the number line?
5. What number lies two units to the right of 0 on the number line?
6. What number lies four units to the right of −2 on the number line?
7. What number lies two units to the right of 1 on the number line?
8. What number lies two units to the right of −4 on the number line?
9. What number lies four units to the left of 6 on the number line?
10. What number lies two units to the left of 0 on the number line?
11. What number lies two units to the right of −5 on the number line?
12. What number lies three units to the right of −6 on the number line?

In Exercises 13-24, for each of the following sets of integers, perform the following tasks:

1. Plot each of the integers on a numberline.
2. List the numbers in order, from smallest to largest.

13. 6, 1, −3, and −5
14. 5, −3, −5, and 2
15. 5, −6, 0, and 2
16. 4, 2, 6, and −4
17. −3, −5, 3, and 5
18. −4, 5, 2, and −6
19. −5, 4, 2, and −3
20. 6, 1, −3, and −1
21. 3, 5, −5, and −1
22. −4, 6, −2, and 3
23. −2, −4, 3, and −6
24. 2, −6, −4, and 5

In Exercises 25-36, in each of the following exercises, enter the inequality symbol < or the symbol > in the box in order that the resulting inequality is a true statement.

25. $-4\square 0$
26. $-4\square 3$
27. $-2\square -1$
28. $3\square 0$
29. $-3\square -1$
30. $6\square 5$
31. $3\square 6$
32. $-4\square -2$
33. $-3\square -6$
34. $0\square -3$
35. $-1\square 4$
36. $1\square -4$

In Exercises 37-52, simplify each of the following expressions.

37. $-(-4)$.
38. $-(-6)$.
39. $|7|$.
40. $|1|$.
41. $|5|$.
42. $|3|$.
43. $-|-11|$.
44. $-|-1|$.

45. $|-5|$.

46. $|-1|$.

47. $-|-20|$.

48. $-|-8|$.

49. $|-4|$.

50. $|-3|$.

51. $-(-2)$.

52. $-(-17)$.

In Exercises 53-64, for each of the following exercises, provide a number line sketch with your answer.

53. Find two integers on the number line that are 2 units away from the integer 2.

54. Find two integers on the number line that are 2 units away from the integer -3.

55. Find two integers on the number line that are 4 units away from the integer -3.

56. Find two integers on the number line that are 2 units away from the integer -2.

57. Find two integers on the number line that are 3 units away from the integer -2.

58. Find two integers on the number line that are 4 units away from the integer 1.

59. Find two integers on the number line that are 2 units away from the integer 3.

60. Find two integers on the number line that are 3 units away from the integer 3.

61. Find two integers on the number line that are 3 units away from the integer 0.

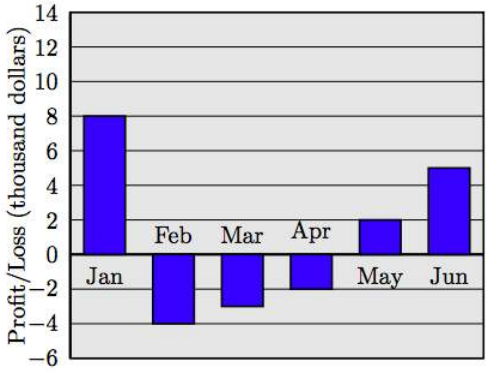
62. Find two integers on the number line that are 4 units away from the integer 2.

63. Find two integers on the number line that are 2 units away from the integer 0.

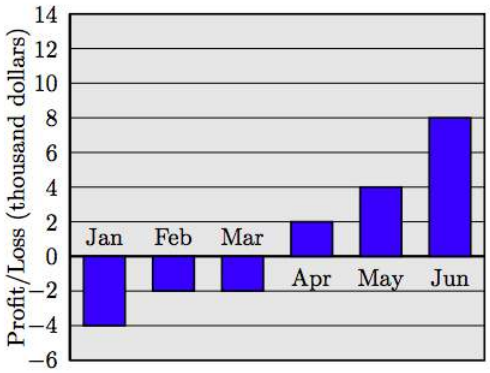
64. Find two integers on the number line that are 3 units away from the integer 1.

---

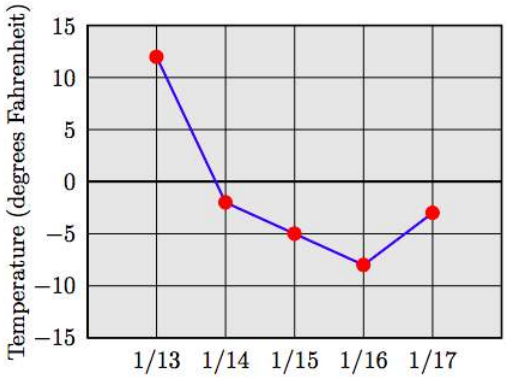
65. Dam. Utah's lowest point of elevation is 2,350 feet above sea level and occurs at Beaver Dam Wash. Express the height as an integer. 66. Underwater Glider. The National Oceanic and Atmospheric Administration's underwater glider samples the bottom of the Atlantic Ocean at 660 feet below sea level. Express that depth as an integer. Associated Press Times-Standard 4/19/09 67. Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart.



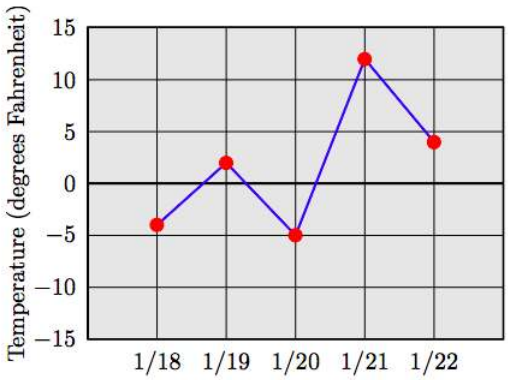
Create a table showing the profit and loss for each month. Use positive integers for the profit and negative integers for the loss. Create a line graph using the data in your table. 68. Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart.



Create a table showing the profit and loss for each month. Use positive integers for the profit and negative integers for the loss. Create a line graph using the data in your table. 69. The following line graph displays the low temperature recordings (degrees Fahrenheit) on five consecutive days in Big Bear, California. What was the lowest temperature reading for the week and on what date did it occur?



70. The following line graph displays the low temperature recordings (degrees Fahrenheit) on five consecutive days in Ogden, Utah. What was the lowest temperature reading for the week and on what date did it occur?



71. The following table contains the low temperature recordings (degrees Fahrenheit) on five consecutive days in Littletown, Ohio. Create a line graph for the data.

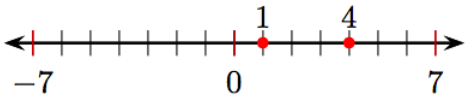
Date	Feb 11	Feb 12	Feb 13	Feb 14	Feb 15
Low Temp	10	-2	-5	-12	8

72. The following table contains the low temperature recordings (degrees Fahrenheit) on five consecutive days in MyTown, Ottawa. Create a line graph for the data.

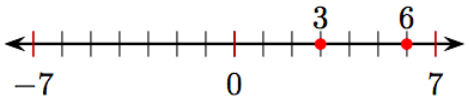
Date	Apr 20	Apr 21	Apr 22	Apr 23	Apr 24
Low Temp	-10	-2	8	5	-5

Answers

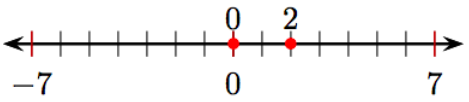
1.



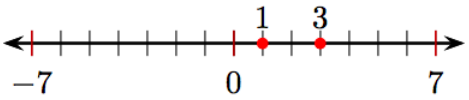
3.



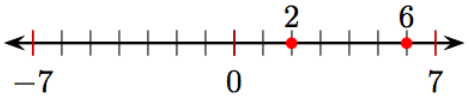
5.



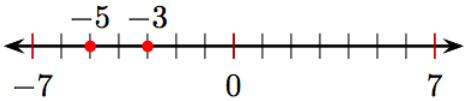
7.



9.

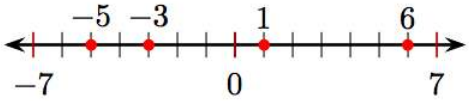


11.



13.

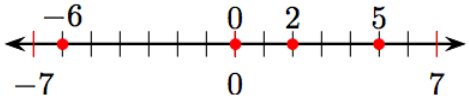
i) Arrange the integers 6, 1, −3, and −5 on a number line.



ii) −5, −3, 1, 6

15.

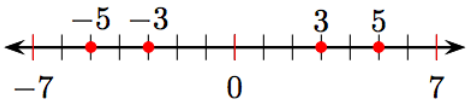
i) Arrange the integers 5, −6, 0, and 2 on a number line.



ii) −6, 0, 2, 5

17.

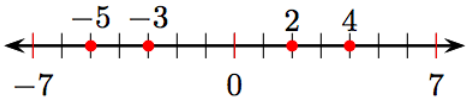
i) Arrange the integers −3, −5, 3, and 5 on a number line.



ii) −5, −3, 3, 5

19.

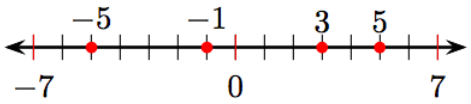
i) Arrange the integers −5, 4, 2, and −3 on a number line.



ii) −5, −3, 2, 4

21.

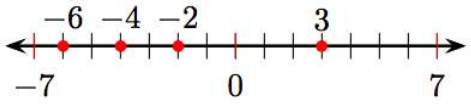
i) Arrange the integers 3, 5, −5 and −1 on a number line.



ii) −5, −1, 3, 5

23.

i) Arrange the integers −2, −4, 3, and −6 on a number line.



ii) −6, −4, −2, 3

25. −4 <0

27. −2 < −1

29. −3 < −1

31. 3 < 6

33. −3 > −6

35. −1 < 4

37. 4

39. 7

41. 5

43. −11

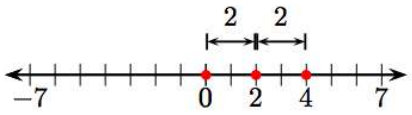
45. 5

47. −20

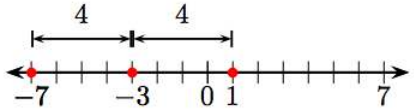
49. 4

51. 2

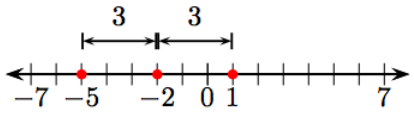
53. 0 and 4.



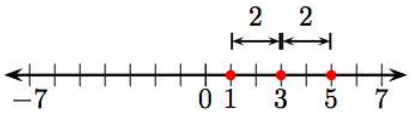
55. −7 and 1.



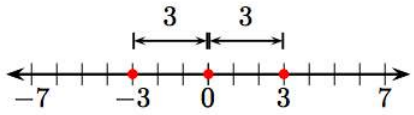
57. −5 and 1.



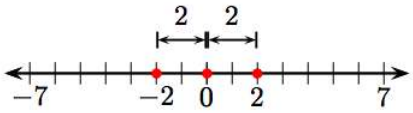
59. 1 and 5.



61. −3 and 3.



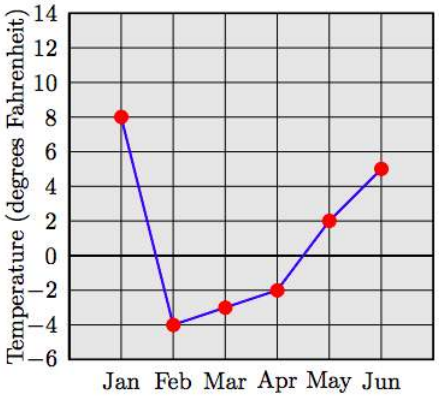
63. −2 and 2.



65. 2,350 feet

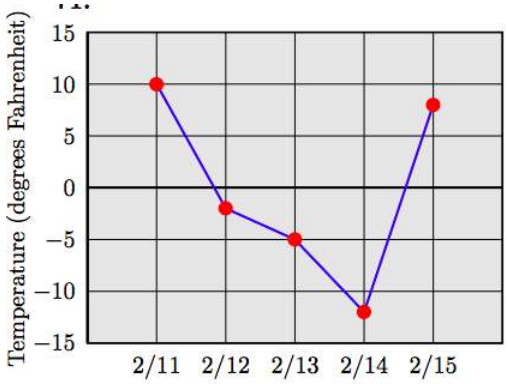
67.

Month	Profit/Loss
Jan	8
Feb	−4
Mar	−3
Apr	−2
May	2
Jun	5



69. Approximately -8° Fahrenheit on January 16.

71.



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6.2: Adding Integers

Like our work with the whole numbers, addition of integers is best explained through the use of number line diagrams. However, before we start, let's take a moment to discuss the concept of a vector.

Vectors

A *vector* is a mathematical object that possesses two import qualities: (1) magnitude or length, and (2) directon.

Vectors are a fundamental problem solving tool in mathematics, science, and engineering. In physics, vectors are used to represent forces, position, velocity, and acceleration, while engineers use vectors to represent both internal and external forces on structures, such as bridges and buildings. In this course, and in this particular section, we will concentrate on the use of vectors to help explain addition of integers.

Vectors on the Number Line

Consider the number line in Figure 6.2.1.

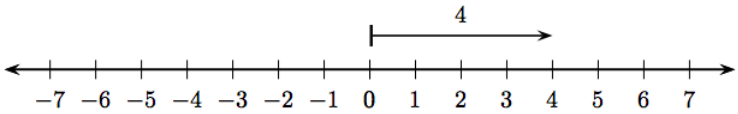


Figure 6.2.1: A vector representing positive four.

Above the line we've drawn a vector with tail starting at the integer 0 and arrowhead finishing at the integer 4. There are two important things to note about this vector:

1. The magnitude (length) of the vector in Figure 6.2.1 is four.
2. The vector in Figure 6.2.1 points to the right.

We will agree that the vector in Figure 6.2.1 represents positive four. It is not important that the vector start at the origin. Consider, for example, the vector pictured in Figure 6.2.2.

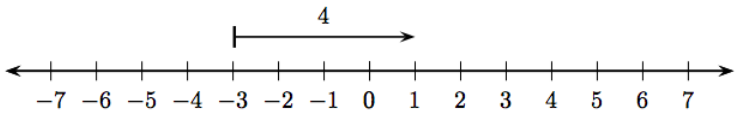


Figure 6.2.2: A vector representing positive four.

Again, there are two important observations to be made:

1. The magnitude (length) of the vector in Figure 6.2.2 is four.
2. The vector in Figure 6.2.2 points to the right.

Hopefully, you have the idea. Any vector that has length 4 and points to the right will represent positive four, regardless of its starting or finishing point. Conversely, consider the vector in Figure 6.2.3, which starts at the integer 4 and finishes at the integer −3.

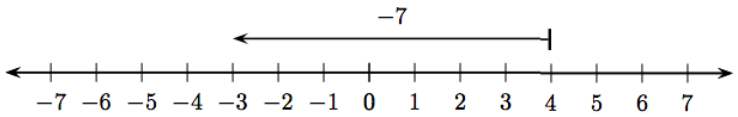


Figure 6.2.3: A vector representing negative seven.

Two observations:

1. The magnitude (length) of the vector in Figure 6.2.3 is seven.
2. The vector in Figure 6.2.3 points to the left.

We will agree that the vector in Figure 6.2.3 represents negative seven. We could select different starting and finishing points for our vector, but as long as the vector has length seven and points to the left, it represents the integer −7.

Important Observation

A vector pointing to the right represents a **positive** number. A vector pointing to the left represents a **negative** number.

Magnitude and Absolute Value

In Figure 6.2.1 and Figure 6.2.2, the vectors pictured represent the integer positive four. Note that the absolute value of four is four; that is, $|4| = 4$. Note also that this absolute value is the magnitude or length of the vectors representing the integer positive four in Figure 6.2.1 and Figure 6.2.2.

In Figure 6.2.3, the vector pictured represents the integer -7. Note that $|-7| = 7$. This shows that the absolute value represents the magnitude or length of the vector representing -7.

Magnitude and Absolute Value

If a is an integer, then $|a|$ gives the magnitude or length of the vector that represents the integer a.

Adding Integers with Like Signs

Because the positive integers are also whole numbers, we've already seen how to add to them in Section 1.2.

Example 1

Find the sum $3 + 4$.

Solution

To add the positive integers 3 and 4, proceed as follows.

1. Start at the integer 0, then draw a vector 3 units in length pointing to the right, as shown in Figure 6.2.4. This arrow has magnitude (length) three and represents the positive integer 3.
2. Draw a second vector of length four that points to the right, starting at the end of the first vector representing the positive integer 3. This arrow has magnitude (length) four and represents the positive integer 4.
3. The sum of the positive integers 3 and 4 could be represented by a vector that starts at the integer 0 and ends at the positive integer 7. However, we prefer to mark this sum on the number line as a solid dot at the positive integer 7. This integer represents the sum of the positive integers 3 and 4.

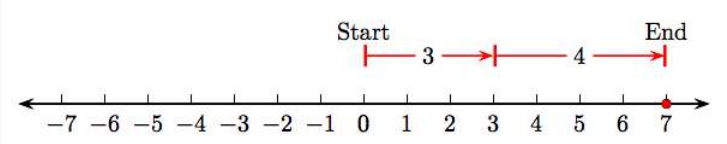


Figure 6.2.4: Adding two positive integers on the number line.

Thus, $3 + 4 = 7$

Exercise

Use a number line diagram to show the sum $5 + 7$.

Answer

12

Negative integers are added in a similar fashion.

Example 2

Find the sum −3+(−4).

Solution

To add the negative integers −3 and −4, proceed as follows.

1. Start at the integer 0, then draw a vector 3 units in length pointing to the left, as shown in Figure 6.2.5. This arrow has magnitude (length) three and represents the negative integer −3.
2. Draw a second vector of length four that points to the left, starting at the end of the first vector representing the negative integer −3. This arrow has magnitude (length) four and represents the negative integer −4.
3. The sum of the negative integers −3 and −4 could be represented by a vector that starts at the integer 0 and ends at the negative integer −7. However, we prefer to mark this sum on the number line as a solid dot at the negative integer −7. This integer represents the sum of the negative integers −3 and −4.

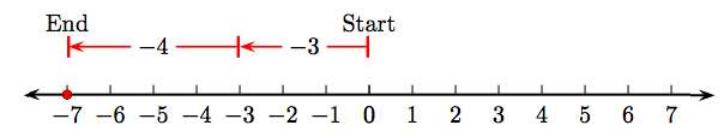


Figure 6.2.5: Adding two positive integers on the number line.

Thus, −3+(−4) = −7.

Drawing on Physical Intuition. Imagine that you are "walking the number line" in Figure 6.2.5. You start at the origin (zero) and take 3 paces *to the left*. Next, you walk an additional four paces *to the left*, landing at the number −7

Exercise

Use a number line diagram to show the sum −7+(−3).

Answer

−10

It should come as no surprise that the procedure used to add two negative integers comprises two steps.

Adding Two Negative Integers

To add two negative integers, proceed as follows:

1. Add the magnitudes of the integers.
2. Prefix the common negative sign

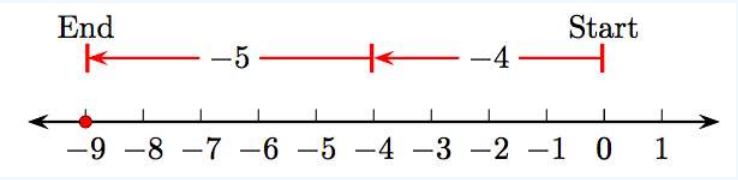
Example 3

Find the sums: (a) −4+(−5), (b) −12 + (−9), and (c) −2+(−16).

Solution

We'll examine three separate but equivalent approaches, as discussed in the narrative above.

a) The number line schematic



shows that (−4) + (−5) = −9.

b) Drawing on physical intuition, start at zero, walk 12 units to the left, then an additional 9 units to the left. You should find yourself 21 units to the left of the origin (zero). Hence, $-12 + (-9) = -21$.

c) Following the algorithm above in "Adding Two Negative Integers," first add the magnitudes of -2 and -16; that is, $2+16 = 18$. Now prefix the common sign. Hence, $-2+(-16) = -18$.

? Exercise

Find the sum: $-5+(-9)$.

Answer

-14

Adding Integers with Unlike Signs

Adding integers with unlike signs is no harder than adding integers with like signs.

✔ Example 4

Find the sum $-8 + 4$.

Solution

To find the sum $-8 + 4$, proceed as follows:

1. Start at the integer 0, then draw a vector eight units in length pointing to the left, as shown in Figure 6.2.6. This arrow has magnitude (length) eight and represents the negative integer -8.
2. Draw a second vector of length four that points to the right, starting at the end of the first vector representing the negative integer -8. This arrow (also shown in Figure 6.2.6) has magnitude (length) four and represents the positive integer 4.
3. The sum of the negative integers -8 and 4 could be represented by a vector that starts at the integer 0 and ends at the negative integer -4. However, we prefer to mark this sum on the number line as a solid dot at the negative integer -4. This integer represents the sum of the integers -8 and 4.

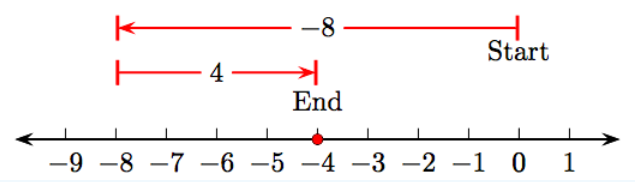


Figure 6.2.6: Adding -8 and 4 on the number line.

Thus, $-8+4= -4$.

Drawing on Physical Intuition. Imagine you are "walking the number line in Figure 6.2.6. You start at the origin (zero) and walk eight paces *to the left*. Next, turn around and walk four paces *to the right*, landing on the number -4.

? Exercise

Use a number line diagram to show the sum $-9 + 2$.

Answer

-7

Note that adding integers with unlike signs is a *subtractive process*. This is due to the reversal of direction experienced in drawing Figure 6.2.6 in Example 4.

Adding Two Integers with Unlike Signs

To add two integers with unlike signs, proceed as follows:

1. Subtract the smaller magnitude from the larger magnitude.
2. Prefix the sign of the number with the larger magnitude.

For example, to find the sum −8 + 4 of Example 4, we would note that the integers −8 and 4 have magnitudes 8 and 4, respectively. We would then apply the process outlined in "Adding Two Integers with Unlike Signs."

1. Subtract the smaller magnitude from the larger magnitude; that is, 8 − 4 = 4.
2. Prefix the sign of the number with the larger magnitude. Because −8 has the larger magnitude and its sign is negative, we prefix a negative sign to the difference of the magnitudes. Thus, −8+4= −4.

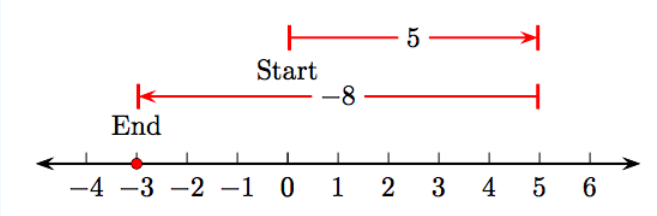
Example 5

Find the sums: (a) 5+ (−8), (b) −12+ 16, and (c) −117+115.

Solution

We'll examine three separate but equivalent approaches, as discussed in the narrative above.

a) The number line schematic



shows that 5 + (−8) = −3.

b) Drawing on physical intuition, start at zero, walk 12 units to the left, then turn around and walk 16 units to the right. You should find yourself 4 units to the right of the origin (zero). Hence, −12 + 16 = 4.

c) Following the algorithm in "Adding Two Integers with Unlike Signs," subtract the smaller magnitude from the larger magnitude, thus 117−115 = 2. Because −117 has the larger magnitude and its sign is negative, we prefix a negative sign to the difference of the magnitudes. Thus, −117 + 115 = −2.

Exercise

Use a number line diagram to show the sum 5 + (−11).

Answer

−6

Properties of Addition of Integers

You will be pleased to learn that the properties of addition for whole numbers also apply to addition of integers.

The Commutative Property of Addition

Let a and b represent two integers. Then,

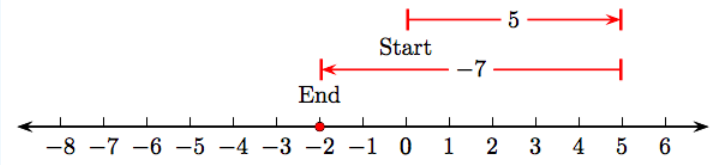
$$a + b = b + a.$$

Example 6

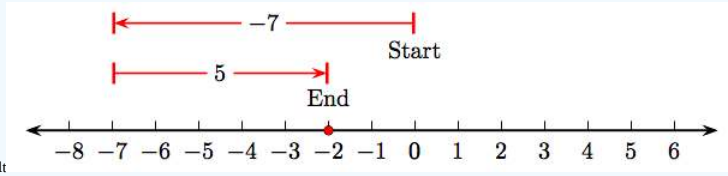
Show that 5 + (−7) = −7 + 5.

Solution

The number line schematic



shows that 5 + (−7) = −2. On the other hand, the number line schematic



shows that −7+5= −2. Therefore, 5 + (−7) = −7 + 5.

Exercise

Use a number line diagram to show that −8 + 6 is the same as 6 + (−8).

Addition of integers is also associative.

The Associative Property of Addition

Let a, b, and c represent integers. Then,

$$(a+b)+c=a+(b+c)$$

.

Example 7

Show that (−9 + 6) + 2 = −9 + (6 + 2).

Solution

On the left, the grouping symbols demand that we add −9 and 6 first. Thus,

$$\begin{aligned}(-9+6)+2 &= -3+2\\ &= -1.\end{aligned}$$

On the right, the grouping symbols demand that we add 6 and 2 first. Thus,

$$\begin{aligned}-9+(6+2) &= -9+8\\ &= -1.\end{aligned}$$

Both sides simplify to −1. Therefore, (−9 + 6) + 2 = −9 + (6 + 2).

Exercise

Show that the expression (−8 + 5) + 3 is the same as −8 + (5 + 3) by simplifying each of the two expressions independently.

The Additive Identity Property

The integer zero is called the *additive identity*. If a is any integer, then

$$a + 0 = a \text{ and } 0 + a = a.$$

Thus, for example, $-8+0 = -8$ and $0 + (-113) = -113$.

Finally, every integer has a unique opposite, called its *additive inverse*.

The Additive Inverse Property

Let a represent any integer. Then there is a unique integer $-a$, called the **opposite** or **additive inverse** of a, such that

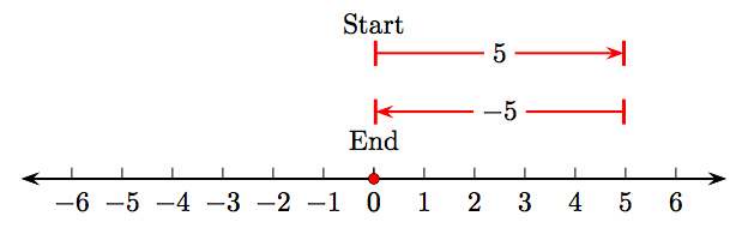
$$a + (-a) = 0 \text{ and } -a + a = 0.$$

Example 8

Show that $5 + (-5) = 0$.

Solution

The number line schematic



clearly shows that $5 + (-5) = 0$.

Exercise

Use a number line diagram to show that $9 + (-9) = 0$.

Important Observation

We have used several equivalent phrases to pronounce the integer $-a$. We've used "the opposite of a," "negative a," and "the additive inverse of a." All are equivalent pronunciations.

Grouping for Efficiency

Order of operations require that we perform all additions as they occur, working from left to right.

Example 9

Simplify $-7+8+(-9) + 12$.

Solution

We perform the additions as they occur, working left to right.

$$\begin{aligned} &= -7+8+(-9)+12 = 1+(-9)+12 && \text{Working left to right, } -7+8=1. \\ &= -8+12 && \text{Working left to right, } 1+(-9=-8. \\ &= 4 && -8+12=4 \end{aligned}$$

Thus, $-7+8+(-9) + 12 = 4$.

Exercise

Simplify: −8+9+(−4) + 2.

Answer

−1

The commutative property of addition tells us that changing the order of addition does not change the answer. The associative property of addition tells us that a sum is not affected by regrouping. Let's work Example 9 again, first grouping positive and negative numbers together.

Example 10

Simplify −7+8+(−9) + 12.

Solution

The commutative and associative properties allows us to change the order of addition and regroup.

$$\begin{aligned} -7+8+(-9)+12 &= -7+(-9)+8+12 && \text{Use the commutative property to change the order.} \\ &= [-7+(-9)]+[8+12] && \text{Use the associative property to regroup.} \\ &= -16+20 && \text{Add the negatives. Add the positives.} \\ &= 4 && \text{One final addition.} \end{aligned}$$

Thus, −7+8+(−9) + 12 = 4.

Exercise

Simplify: −11 + 7 + (−12) + 3.

Answer

−13

At first glance, there seems to be no advantage in using the technique in Example 10 over the technique used in Example 9. However, the technique in Example 10 is much quicker in practice, particularly if you eliminate some of the explanatory steps.

Efficient Grouping

When asked to find the sum of a number of integers, it is most efficient to first add all the positive integers, then add the negatives, then add the results.

Example 11

Simplify −7+8+(−9) + 12.

Solution

Add the positive integers first, then the negatives, then add the results.

$$\begin{aligned} -7+8+(-9)+12 &= 20+(-16) && \text{Add the positives: } 8+12=20. \\ & && \text{Add the negatives: } -7+(-9)=-16. \\ &= 4 && \text{Add the results: } 20+(-16)=4. \end{aligned}$$

Thus, −7+8+(−9) + 12 = 4.

Exercise

Simplify: −11 + 3 + (−2) + 7.

Answer

−7

Using Correct Notation

Never write +−! That is, the notation

$$9 + -4 \text{ and } -8 + -6$$

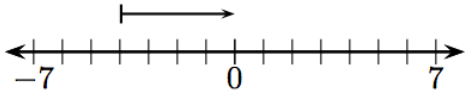
should not be used. Instead, use grouping symbols as follows:

$$9+(-4) \text{ and } -8+(-6)$$

Exercises

In Exercises 1-12, what integer is represented by the given vector?

1.



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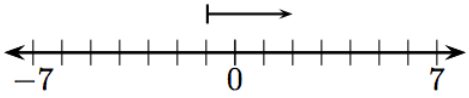

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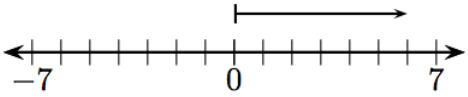

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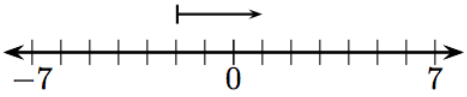

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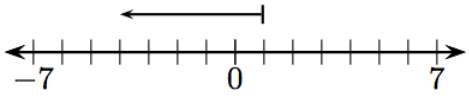

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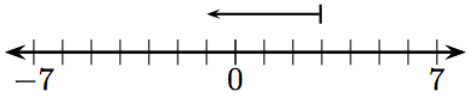

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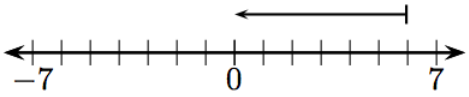

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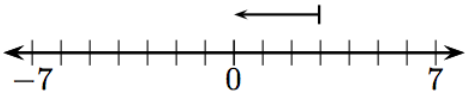
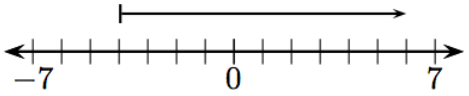
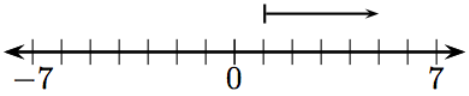


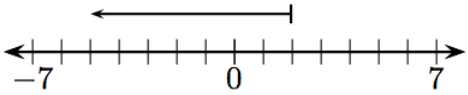
9.



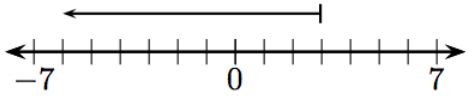
10.



11.



12.



In Exercises 13-36, find the sum of the given integers.

13. −15 + 1
14. −1 + 18
15. 18 + (−10)
16. 2+(−19)
17. −10 + (−12)
18. −1+(−7)
19. 5 + 10
20. 1 + 12

21. 2+5
22. 14 + 1
23. 19 + (−15)
24. 20 + (−17)
25. −2+(−7)
26. −14 + (−6)
27. −6 + 16
28. −2 + 14
29. −11 + (−6)
30. −7+(−8)
31. 14 + (−9)
32. 5+(−15)
33. 10 + 11
34. 14 + 11
35. −13 + 1
36. −8+2

In Exercises 37-52, state the property of addition depicted by the given identity.

37. −1 + (3 + (−8)) = (−1 + 3) + (−8)
38. −4 + (6 + (−5)) = (−4 + 6) + (−5)
39. 7+(−7) = 0
40. 14 + (−14) = 0
41. 15 + (−18) = −18 + 15
42. 14 + (−8) = −8 + 14
43. −15 + 0 = −15
44. −11 + 0 = −11
45. −7 + (1 + (−6)) = (−7 + 1) + (−6)
46. −4 + (8 + (−1)) = (−4 + 8) + (−1)
47. 17 + (−2) = −2 + 17
48. 5+(−13) = −13 + 5
49. −4+0= −4
50. −7+0= −7
51. 19 + (−19) = 0
52. 5+(−5) = 0

In Exercises 53-64, state the additive inverse of the given integer.

53. 18
54. 10
55. 12
56. 15

57. −16

58. −4

59. 11

60. 13

61. −15

62. −19

63. −18

64. −9

In Exercises 65-80, find the sum of the given integers.

65. 6+(−1) + 3 + (−4)

66. 6+(−3) + 2 + (−7)

67. 15 + (−1) + 2

68. 11 + (−16) + 16

69. −17 + 12 + 3

70. −5+(−3) + 2

71. 7 + 20 + 19

72. 14 + (−14) + (−20)

73. 4+(−8) + 2 + (−5)

74. 6+(−3) + 7 + (−2)

75. 7+(−8) + 2 + (−1)

76. 8+(−9) + 5 + (−3)

77. 9+(−3) + 4 + (−1)

78. 1+(−9) + 7 + (−6)

79. 9 + 10 + 2

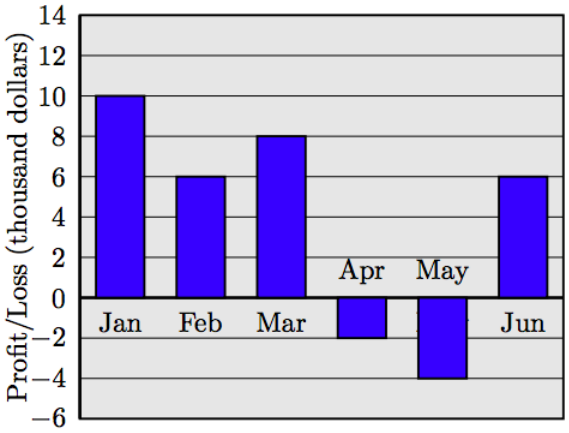
80. −6 + 15 + (−18)

---

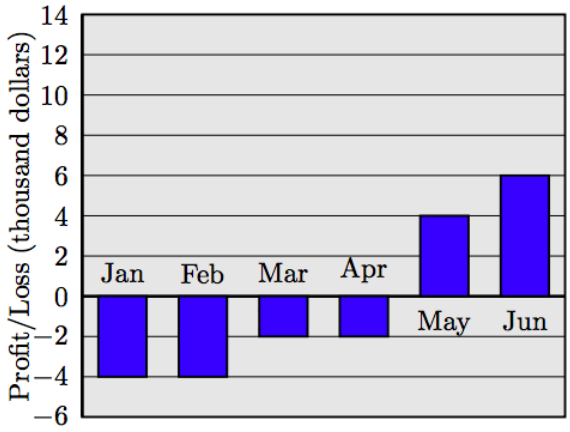
81. **Bank Account.** Gerry opened a new bank account, depositing a check for $215. He then made several withdraws of $40, $75, and $20 before depositing another check for $185. How much is in Gerry's account now?

82. **Dead Sea Sinking.** Due to tectonic plate movement, the Dead Sea is sinking about 1 meter each year. If it's currently −418 meters now, what will Dead Sea elevation be in 5 years? Write an expression that models this situation and compute the result.

83. **Profit and Loss.** Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart. Sum the profits and losses from each month. Was there a net profit or loss over the six-month period? How much?



84. **Profit and Loss.** Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart. Sum the profits and losses from each month. Was there a net profit or loss over the six-month period? How much?



Answers

1. 4
3. 6
5. −5
7. −6
9. 10
11. −7
13. −14
15. 8
17. −22
19. 15
21. 7
23. 4
25. −9
27. 10

29. −17

31. 5

33. 21

35. −12

37. Associative property of addition

39. Additive inverse property

41. Commutative property of addition

43. Additive identity property

45. Associative property of addition

47. Commutative property of addition

49. Additive identity property

51. Additive inverse property

53. −18

55. −12

57. 16

59. −11

61. 15

63. 18

65. 4

67. 16

69. −2

71. 46

73. −7

75. 0

77. 9

79. 21

81. $265

83. Net Profit: $24,000

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6.3: Subtracting Integers

In Section 1.2, we stated that "Subtraction is the opposite of addition." Thus, to subtract 4 from 7, we walked seven units to the right on the number line, but then walked 4 units in the opposite direction (to the left), as shown in Figure 6.3.1.

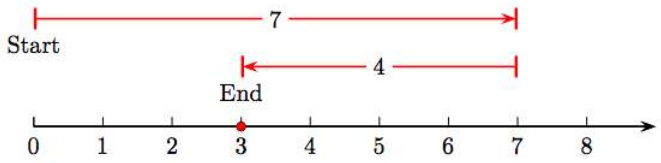


Figure 6.3.1: Subtraction requires that we reverse direction.

Thus, 7 − 4 = 3. The key phrase is "add the opposite." Thus, the subtraction 7 − 4 becomes the addition 7 + (−4), which we would picture on the number line as shown in Figure 6.3.2.

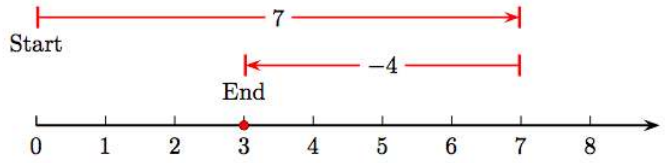


Figure 6.3.2: Subtraction means add the opposite.

Figure 6.3.1 and Figure 6.3.2 provide ample evidence that the subtraction 7−4 is identical to the addition 7+(−4). Again, subtraction means "add the opposite." That is, 7 − 4=7+(−4).

Defining Subtraction

Subtraction means "add the opposite." That is, if a and b are any integers, then

$$a - b = a + (-b).$$

Thus, for example, −123−150 = −123+(−150) and −57−(−91) = −57+91. In each case, subtraction means "add the opposite." In the first case, subtracting 150 is the same as adding −150. In the second case, subtracting−91 is the same as adding 91.

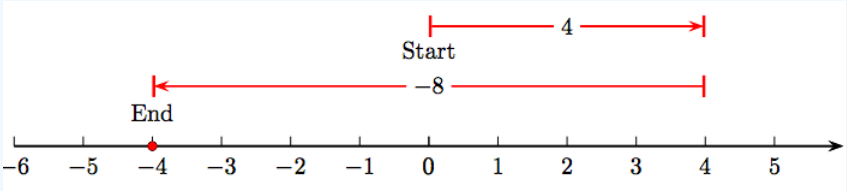
Example 1

Find the differences: (a) 4 − 8, (b) −15 − 13, and (c) −117 − (−115).

Solution

In each case, subtraction means "add the opposite."

a) Change the subtraction to addition with the phrase "subtraction means add the opposite." That is, 4−8 = 4+(−8). We can now perform this addition on the number line.



Thus, 4 − 8=4+(−8) = −4.

b) First change the subtraction into addition by "adding the opposite." That is, −15 − 13 = −15 + (−13). We can now use physical intuition to perform the addition. Start at the origin (zero), walk 15 units to the left, then an additional 13 units to the left, arriving at the answer −28. That is,

$$\begin{aligned} -15 - 13 &= -15 + (-13) \\ &= -28. \end{aligned}$$

c) First change the subtraction into addition by "adding the opposite." That is, −117 − (−115) = −117 + 115. Using "Adding Two Integers with Unlike Signs" from Section 2.2, first subtract the smaller magnitude from the larger magnitude; that is, 117 − 115 = 2. Because −117 has the larger magnitude and its sign is negative, prefix a negative sign to the difference in magnitudes. Thus,

$$\begin{aligned} -117-(-115) &= -117+115 \\ &= -2. \end{aligned}$$

? Exercise

Use each of the techniques in parts (a), (b), and (c) of Example 1 to evaluate the difference −11 − (−9).

Answer

−2

Order of Operations

We will now apply the "Rules Guiding Order of Operations" from Section 1.5 to a number of example exercises.

✔ Example 2

Simplify −5 − (−8) − 7.

Solution

We work from left to right, changing each subtraction by "adding the opposite."

$$\begin{aligned} -5-(-8)-7 &= -5+8+(-7) && \text{Add the opposite of } -8, \text{ which is } 8. \\ & && \text{Add the opposite of } 7, \text{ which is } -7. \\ &= 3+(-7) && \text{Working left to right, } -5+8=3. \\ &= -4 && 3+(-7)=-4. \end{aligned}$$

? Exercise

Simplify: −3 − (−9) − 11.

Answer

−5

Grouping symbols say "do me first."

✔ Example 6.3.1

Simplify −2 − (−2 − 4).

Solution

Parenthetical expressions must be evaluated first.

$$\begin{aligned} -2(-2-4) &= -2-(-2+(-4)) && \text{Simplify the parenthetical expression} \\ & && \text{first. Add the opposite of } 4, \text{ which is } -4. \\ &= -2-(-6) && \text{Inside the parentheses, } -2+(-4)=-6. \\ &= -2+6 && \text{Subtracting a } -6 \text{ is the same as adding a } 6. \\ &= 4 && \text{Add: } -2+6=4. \end{aligned}$$

Exercise

Simplify: $-3-(-3-3)$.

Answer

3

Change as a Difference

Suppose that when I leave my house in the early morning, the temperature outside is 40° Fahrenheit. Later in the day, the temperature measures 60° Fahrenheit. How do I measure the change in the temperature?

The Change in a Quantity

To measure the change in a quantity, always subtract the *former* measurement from the *latter* measurement. That is:

$$\text{Change in a Quantity} = \text{Latter Measurement} - \text{Former Measurement}$$

Thus, to measure the change in temperature, I perform a subtraction as follows:

$$\begin{aligned} \text{Change in Temperature} &= \text{Latter Measurement} && - \text{Former Measurement} \\ &= 60^\circ\text{F} && -40^\circ\text{F} \\ &= 20^\circ\text{F} \end{aligned}$$

Note that the positive answer is in accord with the fact that the temperature has increased.

Example 4

Suppose that in the afternoon, the temperature measures 65° Fahrenheit, then late evening the temperature drops to 44° Fahrenheit. Find the change in temperature.

Solution

To measure the change in temperature, we must subtract the former measurement from the latter measurement.

$$\begin{aligned} \text{Change in Temperature} &= \text{Latter Measurement} && - \text{Former Measurement} \\ &= 44^\circ\text{F} && -65^\circ\text{F} \\ &= -11^\circ\text{F} \end{aligned}$$

Note that the negative answer is in accord with the fact that the temperature has decreased. There has been a "change" of -11° Fahrenheit.

Exercise

Marianne awakes to a morning temperature of 54° Fahrenheit. A storm hits, dropping the temperature to 43° Fahrenheit. Find the change in temperature.

Answer

-11° Fahrenheit

Example 5

Sometimes a bar graph is not the most appropriate visualization for your data. For example, consider the bar graph in Figure 6.3.3 depicting the Dow Industrial Average for seven consecutive days in March of 2009. Because the bars are of almost equal height, it is difficult to detect fluctuation or change in the Dow Industrial Average.

Let's determine the change in the Dow Industrial average on a day-to-day basis. Remember to subtract the latter measurement minus the former (current day minus former day). This gives us the following changes.

Consecutive Days	Change in Dow Industrial Average
Sun-Mon	6900 - 7000 = -100
Mon-Tues	6800 - 6900 = -100
Tues-Wed	6800 - 6800 = 0
Wed-Thu	7000 - 6800 = 200
Thu-Fri	7100 - 7000 = 100
Fri-Sat	7200 - 7100 = 100

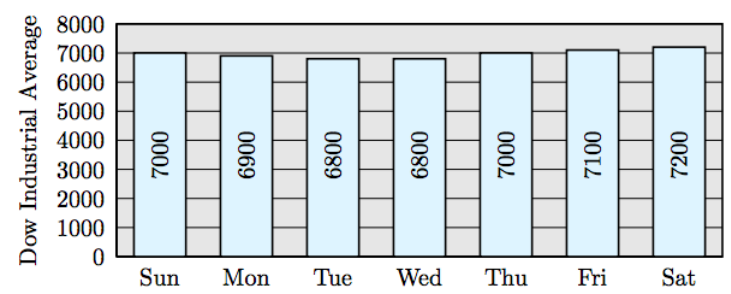


Figure 6.3.3: Profit and loss bar graph.

We will use the data in the table to construct a line graph. On the horizontal axis, we place the pairs of consecutive days (see Figure 6.3.4). On the vertical axis we place the Change in the Industrial Dow Average. At each pair of days we plot a point at a height equal to the change in Dow Industrial Average as calculated in our table.

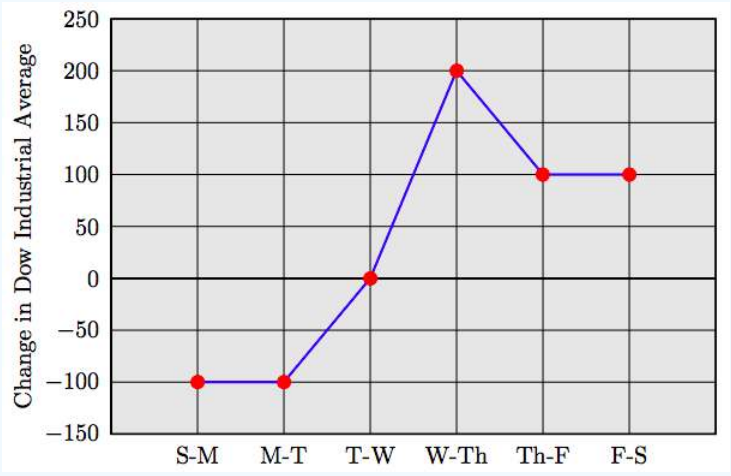


Figure 6.3.4: Plotting change in Dow versus consecutive days.

Note that the data as displayed by Figure 6.3.4 more readily shows the changes in the Dow Industrial Average on a day-to-day basis. For example, it is now easy to pick the day that saw the greatest increase in the Dow (from Wednesday to Thursday, the Dow rose 200 points).

Exercises

In Exercises 1-24, find the difference.

1. 16 − 20
2. 17 − 2

3. $10 - 12$

4. $16 - 8$

5. $14 - 11$

6. $5 - 8$

7. $7 - (-16)$

8. $20 - (-10)$

9. $-4 - (-9)$

10. $-13 - (-3)$

11. $8 - (-3)$

12. $14 - (-20)$

13. $2 - 11$

14. $16 - 2$

15. $-8 - (-10)$

16. $-14 - (-2)$

17. $13 - (-1)$

18. $12 - (-13)$

19. $-4 - (-2)$

20. $-6 - (-8)$

21. $7 - (-8)$

22. $13 - (-14)$

23. $-3 - (-10)$

24. $-13 - (-9)$

In Exercises 25-34, simplify the given expression.

25. $14 - 12 - 2$

26. $-19 - (-7) - 11$

27. $-20 - 11 - 18$

28. $7 - (-13) - (-1)$

29. $5 - (-10) - 20$

30. $-19 - 12 - (-8)$

31. $-14 - 12 - 19$

32. $-15 - 4 - (-6)$

33. $-11 - (-7) - (-6)$

34. $5 - (-5) - (-14)$

In Exercises 35-50, simplify the given expression.

35. $-2 - (-6 - (-5))$

36. $6 - (-14 - 9)$

37. $(-5 - (-8)) - (-3 - (-2))$

38. $(-6 - (-8)) - (-9 - 3)$

39. $(6 - (-9)) - (3 - (-6))$

40. $(-2 - (-3)) - (3 - (-6))$

41. $-1 - (10 - (-9))$

42. $7 - (14 - (-8))$

43. $3 - (-8 - 17)$

44. $1 - (-1 - 4)$

45. $13 - (16 - (-1))$

46. $-7 - (-3 - (-8))$

47. $(7 - (-8)) - (5 - (-2))$

48. $(6 - 5) - (7 - 3)$

49. $(6 - 4) - (-8 - 2)$

50. $(2 - (-6)) - (-9 - (-3))$

---

51. The first recorded temperature is 42°F. Four hours later, the second temperature is 65°F. What is the change in temperature?

52. The first recorded temperature is 79°F. Four hours later, the second temperature is 46°F. What is the change in temperature?

53. The first recorded temperature is 30°F. Four hours later, the second temperature is 51°F. What is the change in temperature?

54. The first recorded temperature is 109°F. Four hours later, the second temperature is 58°F. What is the change in temperature?

55. Typical temperatures in Fairbanks, Alaska in January are −2 degrees Fahrenheit in the daytime and −19 degrees Fahrenheit at night. What is the change in temperature from day to night?

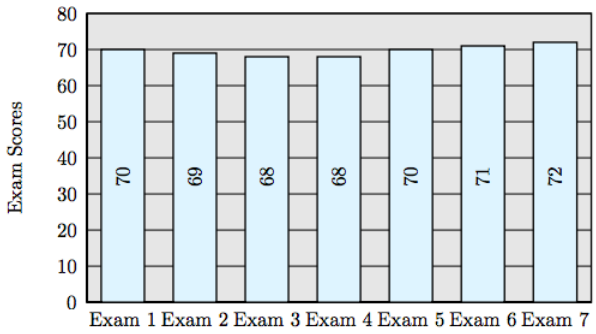
56. Typical summertime temperatures in Fairbanks, Alaska in July are 79 degrees Fahrenheit in the daytime and 53 degrees Fahrenheit at night. What is the change in temperature from day to night?

57. **Communication.** A submarine 1600 feet below sea level communicates with a pilot flying 22,500 feet in the air directly above the submarine. How far is the communique traveling?

58. **Highest to Lowest.** The highest spot on earth is on Mount Everest in Nepal-Tibet at 8,848 meters. The lowest point on the earth's crust is the Mariana's Trench in the North Pacific Ocean at 10,923 meters below sea level. What is the distance between the highest and the lowest points on earth? *Wikipedia* http://en.Wikipedia.org/wiki/Extremes_on_Earth

59. **Lowest Elevation.** The lowest point in North America is Death Valley, California at -282 feet. The lowest point on the entire earth's landmass is on the shores of the Dead Sea along the Israel-Jordan border with an elevation of -1,371 feet. How much lower is the Dead Sea shore from Death Valley?

60. **Exam Scores.** Freida's scores on her first seven mathematics exams are shown in the following bar chart. Calculate the differences between consecutive exams, then create a line graph of the differences on each pair of consecutive exams. Between which two pairs of consecutive exams did Freida show the most improvement?



Answers

1. −4

3. −2

5. 3

7. 23

9. 5 11. 11 13. −9 15. 2 17. 14 19. −2 21. 15 23. 7 25. 0 27. −49 29. −5 31. −45 33. 2 35. −1 37. 4 39. 6 41. −20 43. 28 45. −4 47. 8 49. 12 51. 23◦ F 53. 21◦ F 55. −17 degrees Fahrenheit 57. 24,100 feet 59. 1,089 feet lower

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6.4: Multiplication and Division of Integers

Before we begin, let it be known that the integers satisfy the same properties of multiplication as do the whole numbers.

Integer Properties of Multiplication

Commutative Property. If *a* and *b* are integers, then their product commutes. That is,

$$a \cdot b = b \cdot a.$$

Associative Property. If a, b, and c are integers, then their product is associative. That is,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Multiplicative Identity Property. If *a* is any integer, then

$$a \cdot 1 = a \text{ and } 1 \cdot a = a.$$

Because multiplying any integer by 1 returns the identical integer, the integer 1 is called the **multiplicative identity**.

In Section 1.3, we learned that multiplication is equivalent to *repeated addition*. For example,

$$3 \cdot 4 = \underbrace{4+4+4}_{\text{three fours}}$$

On the number line, three sets of four is equivalent to walking three sets of four units to the right, starting from zero, as shown in Figure 6.4.1.

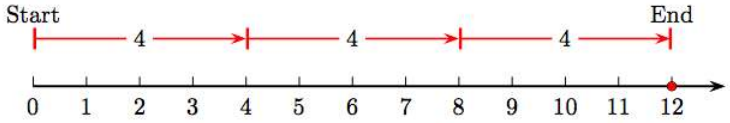

Figure 6.4.1: Note that 3 · 4 = 4 + 4 + 4. That is, 3 · 4 = 12.

This example and a little thought should convince readers that the product of two positive integers will always be a positive integer.

The Product of Two Positive Integers

If *a* and *b* are two positive integers, then their product *ab* is also a positive integer.

For example, 2 · 3 = 6 and 13 · 117 = 1521. In each case, the product of two positive integers is a positive integer.

The Product of a Positive Integer and a Negative Integer

If we continue with the idea that multiplication is equivalent to repeated addition, then it must be that

$$3 \cdot (-4) = \underbrace{-4+(-4)+(-4)}_{\text{three negative fours}}.$$

Pictured on the number line, 3 · (−4) would then be equivalent to walking three sets of negative four units (to the left), starting from zero, as shown in Figure 6.4.2.

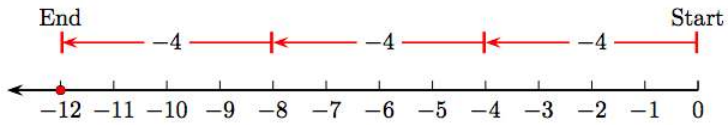

Figure 6.4.2: Note that 3 · (−4) = −4+(−4) + (−4). That is, 3 · (−4) = −12.

Note, at least in this particular case, that the product of a positive integer and a negative integer is a negative integer.

We've shown that 3 · (−4) = −12. However, integer multiplication is commutative, so it must also be true that −4 · 3 = −12. That is, the product of a negative integer and a positive integer is also a negative integer. Although not a proof, this argument motivates the

following fact about integer multiplication.

The Product of a Positive Integer and a Negative Integer

Two facts are true:

1. If a is a positive integer and b is a negative integer, then the product ab is a negative integer.
2. If a is a negative integer and b is a positive integer, then the product ab is a negative integer.

Thus, for example, $5 \cdot (-12) = -60$ and $-13 \cdot 2 = -26$. In each case the answer is negative because we are taking a product where one of the factors is positive and the other is negative.

The Distributive Property

The integers satisfy the distributive property.

The Distributive Property

Let a, b, and c be integers. Then,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Note how the a is "distributed." The a is multiplied times each term in the parentheses.

For example, consider the expression $3 \cdot (4 + 5)$. We can evaluate this expression according to the order of operations, simplifying the expression inside the parentheses first.

$$\begin{aligned} 3 \cdot (4+5) &= 3 \cdot 9 \\ &= 27 \end{aligned}$$

But we can also use the distributive property, multiplying each term inside the parentheses by three, then simplifying the result.

$$\begin{aligned} &\text{Distribute the 3: } 3(4+5) \\ &\text{Perform multiplications first: } = 3 \cdot 4 + 3 \cdot 5 \\ &\text{Add: } = 12 + 15 \\ &\text{Sum: } = 27 \end{aligned}$$

Note that evaluating $3 \cdot (4 + 5)$ using the distributive property provides the same result as the evaluation (2.1) using the order of operations.

The Multiplicative Property of Zero

The distributive property can be used to provide proofs of a number of important properties of integers. One important property is the fact that if you multiply an integer by zero, the product is zero. Here is a proof of that fact that uses the distributive property.

Let a be any integer. Then,

$$\begin{aligned} a \cdot 0 &= a \cdot (0+0) && \text{Additive Identity Property: } = 0 + 0 = 0. \\ a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{Distribute } a \text{ times each zero in the parentheses.} \end{aligned}$$

Next, to "undo" the effect of adding $a \cdot 0$, subtract $a \cdot 0$ from both sides of the equation.

$$\begin{aligned} a \cdot 0 - a \cdot 0 &= a \cdot 0 + a \cdot 0 + a \cdot 0 - a \cdot 0 && \text{Subtract } a \cdot 0 \text{ from both sides.} \\ 0 &= a \cdot 0 && a \cdot 0 - a \cdot 0 = 0 \text{ on each side.} \end{aligned}$$

Multiplicative Property of Zero

Let a represent any integer. Then

$$a \cdot 0 = 0 \text{ and } 0 \cdot a = 0.$$

Thus, for example, $-18 \cdot 0 = 0$ and $0 \cdot 122 = 0$.

Multiplying by Minus One

Here is another useful application of the distributive property.

$$\begin{aligned}(-1)a + a &= (-1)a + 1a && \text{Replace } a \text{ with } 1a.\\ &= (-1+1)a && \text{Use the distributive property to factor out } a.\\ &= 0a && \text{Replace } -1+1 \text{ with } 0.\\ &= 0 && \text{Replace } 0a \text{ with } 0.\end{aligned}$$

Thus, $(-1)a + a = 0$. That is, if you add $(-1)a$ to a you get zero. However, the Additive Inverse Property says that $-a$ is the *unique* number that you add to a to get zero. The conclusion must be that $(-1)a = -a$.

Multiplying by Minus One

If a is any integer, then

$$(-1)a = -a.$$

Thus, for example, $-1(4) = -4$ and $-1(-4) = -(-4) = 4$.

This property is rather important, as we will see in future work. Not only does it tell us that $(-1)a = -a$, but it also tells us that if we see $-a$, then it can be interpreted to mean $(-1)a$.

The Product of Two Negative Integers

We can employ the multiplicative property of -1, that is, $(-1)a = -a$ to find the product of two negative numbers.

$$\begin{aligned}(-4)(-3) &= [(-1)(4)](-3) && \text{Replace } -4 \text{ with } (-1)(4).\\ &= (-1)[(4)(-3)] && \text{Use the associative property to regroup.}\\ &= (-1)(-12) && \text{We know: } (4)(-3) = -12.\\ &= -(-12) && (-1)a = -a. \text{ Here } (-1)(-12) = -(-12).\\ &= 12 && -(-a) = a. \text{ Here } -(-12) = 12.\end{aligned}$$

Thus, at least in the case of $(-4)(-3)$, the product of two negative integers is a positive integer. This is true in general.

The Product of Two Negative Integers

If both a and b are negative integers, then their product ab is a positive integer.

Thus, for example, $(-5)(-7) = 35$ and $(-112)(-6) = 672$. In each case the answer is positive, because the product of two negative integers is a positive integer.

Memory Device

Here's a simple memory device to help remember the rules for finding the product of two integers.

Like and Unlike Signs

There are two cases:

Unlike Signs. The product of two integers with unlike signs is negative. That is:

(+)(−) = −

(−)(+) = −

Like Signs. The product of two integers with like signs is positive. That is:

(+)(+) = +

(−)(−)=+

Example 1

Simplify: (a) (−3)(−2), (b) (4)(−10), and (c) (12)(−3).

Solution

In each example, we use the “like” and “unlike” signs approach.

a) Like signs gives a positive result. Hence, (−3)(−2) = 6.

b) Unlike signs gives a negative result. Hence, (4)(−10) = −40.

c) Unlike signs gives a negative result. Hence, (12)(−3) = −36.

Exercise

Simplify: (a) (−12)(4) and (b) (−3)(−11).

Answer

(a) −48, (b) 33

Example 2

Simplify (−3)(2)(−4)(−2).

Solution

Order of operations demands that we work from left to right.

$$\begin{aligned}(-3)(2)(-4)(-2) &= (-6)(-4)(-2) && \text{Work left to right: } (-3)(2) = -6.\\ &= (24)(-2) && \text{Work left to right: } (-6)(-4) = 24.\\ &= -48 && \text{Multiply: } (24)(-2) = -48.\end{aligned}$$

Hence, (−3)(2)(−4)(−2) = −48.

Exercise

Simplify: (−2)(−3)(4)(−1).

Answer

−24.

Example 3

Simplify: (a)$(-2)^3$ and (c) $(-3)^4$.

Solution

In each example, use

$$a^m = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m \text{ times}},$$

then work left to right with the multiplication.

a) Use the definition of an exponent, then order of operations.

$$\begin{aligned}(-2)^3 &= (-2)(-2)(-2) && \text{Write } -2 \text{ as a factor three times.}\\ &= 4(-2) && \text{Work left to right: } (-2)(-2) = 4.\\ &= -8\end{aligned}$$

b) Use the definition of an exponent, then order of operations.

$$\begin{aligned}(-3)^4 &= (-3)(-3)(-3)(-3) && \text{Write } -3 \text{ as a factor four times.}\\ &= 9(-3)(-3) && \text{Work left to right: } (-3)(-3)=9.\\ &= -27(-3) && \text{Work left to right: } 9(-3)=-27.\\ &= 81\end{aligned}$$

? Exercise

Simplify: (a) $(-2)^2$ and (b) -2^2.

Answer

(a) 4 and (b) −4

Example 3 motivates the following fact.

Even and Odd

Two facts are apparent.

1. If a product contains an odd number of negative factors, then the product is negative.
2. If a product contains an even number of negative factors, then the product is positive.

Thus, for example,

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$$

quickly evaluates as −32 as it has an odd number of negative factors. On the other hand,

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = 64$$

quickly evaluates as 64 as it has an even number of negative factors.

Division of Integers

Consider that

$\frac{12}{3} = 4$ because $3(4) = 12$ and $\frac{-12}{-3} = 4$ because $-3(4) = -12$.

In like manner,

$\frac{12}{-3} = -4$ because -3(-4) = 12 and $\frac{-12}{3} = -4$ because $3(-4) = -12$.

Thus, the rules for dividing integers are the same as the rules for multiplying integers.

Like and Unlike Signs

There are two cases:

Unlike Signs. The quotient of two integers with *unlike* signs is negative. That is,

$$\begin{aligned}(+)(-) &= -\\ \frac{(-)}{(+)} &= -\end{aligned}$$

Like Signs. The quotient of two integers with like signs is positive. That is,

$$\begin{aligned}(+)(+) &= +\\ \frac{(-)}{(-)} &= +\end{aligned}$$

Thus, for example, 12/(−6) = −2 and −44/(−4) = 11. In the first case, unlike signs gives a negative quotient. In the second case, like signs gives a positive quotient.

One final reminder.

Division by Zero is Undefined

If a is any integer, the quotient

$$\frac{a}{0}$$

is undefined. Division by zero is meaningless.

See the discussion in Section 1.3 for a discussion on division by zero.

Example 4

Simplify: (a) −12/(−4), (b) 6/(−3), and (c) −15/0.

Solution

In each example, we use the “like” and “unlike” signs approach.

a) Like signs gives a positive result. Hence,

$$\frac{-12}{-4} = 3.$$

b) Unlike signs gives a negative result. Hence,

$$\frac{6}{-3} = -2.$$

c) Division by zero is undefined. Hence,

$$\frac{-15}{0}$$

is undefined.

Exercise

Simplify: (a) −24/4 and (b) −28/(−7).

Answer

(a) −6, (b) 4

Exercises

In Exercises 1-16, state the property of multiplication depicted by the given identity.

1. (−2) (−16)(13) = (−2)(−16) (13)
2. (10) (−15)(−6) = (10)(−15) (−6)
3. (−17)(−10) = (−10)(−17)
4. (−5)(3) = (3)(−5)
5. (4)(11) = (11)(4)
6. (−5)(−11) = (−11)(−5)
7. 16 · (8 + (−15) = 16 · 8 + 16 · (−15)

8. $1 \cdot (-16 + (-6) = 1 \cdot (-16) + 1 \cdot (-6)$

9. $(17)\ (20)(11) = (17)(20)\ (11)$

10. $(14)\ (-20)(-18) = (14)(-20)\ (-18)$

11. $-19 \cdot 1 = -19$

12. $-17 \cdot 1 = -17$

13. $8 \cdot 1=8$

14. $-20 \cdot 1 = -20$

15. $14 \cdot (-12 + 7 = 14 \cdot (-12) + 14 \cdot 7$

16. $-14 \cdot (-3+6 = -14 \cdot (-3) + (-14) \cdot 6$

In Exercises 17-36, simplify each given expression.

17. $4 \cdot 7$

18. $4 \cdot 2$

19. $3 \cdot (-3)$

20. $7 \cdot (-9)$

21. $-1 \cdot 10$

22. $-1 \cdot 11$

23. $-1 \cdot 0$

24. $-8 \cdot 0$

25. $-1 \cdot (-14)$

26. $-1 \cdot (-13)$

27. $-1 \cdot (-19)$

28. $-1 \cdot (-17)$

29. $2 \cdot 0$

30. $-6 \cdot 0$

31. $-3 \cdot 8$

32. $7 \cdot (-3)$

33. $7 \cdot 9$

34. $6 \cdot 3$

35. $-1 \cdot 5$

36. $-1 \cdot 2$

In Exercises 37-48, simplify each given expression.

37. $(-7)(-1)(3)$

38. $(10)(6)(3)$

39. $(-7)(9)(10)(-10)$

40. $(-8)(-5)(7)(-9)$

41. $(6)(5)(8)$

42. $(7)(-1)(-9)$

43. $(-10)(4)(-3)(8)$

44. $(8)(-2)(-5)(2)$

45. $(6)(-3)(-8)$

46. $(-5)(-4)(1)$

47. $(2)(1)(3)(4)$

48. $(7)(5)(1)(4)$

In Exercises 49-60, compute the exact value.

49. $(-4)^4$

50. $(-3)^4$

51. $(-5)^4$

52. $(-2)^2$

53. $(-5)^2$

54. $(-3)^3$

55. $(-6)^2$

56. $(-6)^4$

57. $(-4)^5$

58. $(-4)^2$

59. $(-5)^3$

60. $(-3)^2$

In Exercises 61-84, simplify each given expression.

61. $-16 \div (-8)$

62. $-33 \div (-3)$

63. $\frac{-8}{1}$

64. $\frac{40}{-20}$

65. $\frac{-1}{0}$

66. \(\frac{2}{0}\

67. $-3 \div 3$

68. $-58 \div 29$

69. $\frac{56}{-28}$

70. $\frac{60}{-12}$

71. $0 \div 15$

72. $0 \div (-4)$

73. $\frac{63}{21}$

74. $\frac{-6}{-1}$

75. $\frac{78}{13}$

76. $\frac{-84}{-14}$

77. $0 \div 5$

78. $0 \div (-16)$

79. $\frac{17}{0}$

80. $\frac{-20}{0}$

81. $-45 \div 15$

82. $-28 \div 28$

83. $12 \div 3$

84. $-22 \div (-22)$

---

85. **Scuba**. A diver goes down 25 feet. A second diver then dives down 5 times further than the first diver. Write the final depth of the second diver as an integer.

86. **Investing Loss**. An investing club of five friends has lost $4400 on a trade. If they share the loss equally, write each members' loss as an integer.

Answers

1. Associative property of multiplication

3. Commutative property of multiplication

5. Commutative property of multiplication

7. Distributive property

9. Associative property of multiplication

11. Multiplicative identity property

13. Multiplicative identity property

15. Distributive property

17. 28

19. −9

21. −10

23. 0

25. 14

27. 19

29. 0

31. −24

33. 63

35. −5

37. 21

39. 6300

41. 240

43. 960

45. 144

47. 24

49. 256

51. 625

53. 25

55. 36

57. −1024

59. −125

61. 2

63. −8

65. Division by zero is undefined.

67. −1

69. −2

71. 0

73. 3

75. 6

77. 0

79. Division by zero is undefined.

81. −3 83. 4

85. −125 feet

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6.5: Introduction to Square Roots

Recall that

$$x^2 = x \cdot x.$$

The Square of a Number

The number x^2 is called the square of the number x.

Thus, for example:

- $9^2 = 9 \cdot 9 = 81$. Therefore, the number 81 is the square of the number 9.
- $(-4)^2 = (-4)(-4) = 16$. Therefore, the number 16 is the square of the number -4.

In the margin, we've placed a "List of Squares" of the whole numbers ranging from 0 through 25, inclusive.

x	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

Square Roots

Once you've mastered the process of squaring a whole number, then you are ready for the inverse of the squaring process, taking the *square root* of a whole number.

- Above, we saw that $9^2 = 81$. We called the number 81 the square of the number 9. Conversely, we call the number 9 a *square root* of the number 81.
- Above, we saw that $(-4)^2 = 16$. We called the number 16 the *square* of the number -4. Conversely, we call the number -4 a *square root* of the number 16.

x	$\sqrt{x}$
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

Square Root

If $a^2 = b$, then a is called a square root of the number b.

Example 1

Find the square roots of the number 49.

Solution

To find a square root of 49, we must think of a number a such that $a^2 = 49$. Two numbers come to mind.

- $(-7)^2 = 49$. Therefore, -7 is a square root of 49.
- $7^2 = 49$. Therefore, 7 is a square root of 49.

Note that 49 has two square roots, one of which is positive and the other one is negative.

Exercise

Find the square roots of 256.

Answer

$-16, 16$

Example 2

Find the square roots of the number 196. Solution. To find a square root of 196, we must think of a number a such that a^2 = 196. With help from the "List of Squares," two numbers come to mind.

- $(-14)^2$ = 196. Therefore, −14 is a square root of 196.
- 14^2 = 196. Therefore, 14 is a square root of 196.

Note that 196 has two square roots, one of which is positive and the other one is negative.

Exercise

Find the square roots of 625.

Answer

−25, 25

Example 3

Find the square roots of the number 0.

Solution

To find a square root of 0, we must think of a number *a* such that a^2 = 0. There is only one such number, namely zero. Hence, 0 is the square root of 0.

Exercise

Find the square roots of 9.

Answer

−3, 3

Example 4

Find the square roots of the number −25.

Solution

To find a square root of −25, we must think of a number a such that a^2 = −25. This is impossible because no square of a real number (whole number, integer, fraction, or decimal) can be negative. Positive times positive is positive and negative times negative is also positive. You cannot square and get a negative answer. Therefore, −25 has no square roots[2].

Exercise

Find the square roots of −81.

Answer

There are none.

[2]At least not in Prealgebra. In later courses, you will be introduced to the set of complex numbers, where −25 will have two square roots.

Radical Notation

Because $(-3)^2$ = 9 and 3^2 = 9, both −3 and 3 are square roots of 9. Special notation, called *radical notation*, is used to request these square roots.

- The radical notation $\sqrt{9}$, pronounced "the nonnegative square root of 9," calls for the nonnegative[3] square root of 9. Hence,

$$\sqrt{9} = 3.$$

- The radical notation $-\sqrt{9}$, pronounced "the negative square root of 9," calls for the negative square root of 9. Hence,

$$-\sqrt{9} = -3.$$

Radical Notation

In the expression $\sqrt{9}$, the symbol $\sqrt{\ }$ is called a *radical* and the number within the radical, in this case the number 9, is called the *radicand*.

For example,

- In the expression $\sqrt{529}$, the number 529 is the radicand.
- In the expression $\sqrt{a^2 + b^2}$, the expression $a2^+b^2$ is the radicand.

Radical Notation and Square Root

If *b* is a positive number, then

1. $\sqrt{b}$ calls for the nonnegative square root of *b*.
2. $-\sqrt{b}$ calls for the negative square root of *b*.

Note: Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

Example 5

Simplify: (a) $\sqrt{121}$, (b) $-\sqrt{625}$, and (c) $\sqrt{0}$.

Solution

(a) Referring to the list of squares, we note that 11^2 = 121 and $(-11)^2$ = 121. Therefore, both 11 and −11 are square roots of 121. However, $\sqrt{121}$ calls for the nonnegative square root of 121. Thus,

$$\sqrt{121} = 11.$$

(b) Referring to the list of squares, we note that 25^2 = 625 and $(-25)^2$ = 625. Therefore, both 25 and −25 are square roots of 625. However, $-\sqrt{625}$ calls for the negative square root of 625. Thus,

$$-\sqrt{625} = -25.$$

(c) There is only one square root of zero. Therefore,

$$\sqrt{0} = 0.$$

Exercise

Simplify: a) $\sqrt{144}$ b) $-\sqrt{324}$

Answer

(a) 12 (b) −18

Example 6

Simplify: (a) $-\sqrt{25}$, and (b) $-\sqrt{25}$

Solution

(a) Because 5^2 = 25 and $(-5)^2$ = 25, both 5 and −5 are square roots of 25. However, the notation $-\sqrt{25}$ calls for the negative square root of 25. Thus, $-\sqrt{25} = -5$.

(b) It is not possible to square a real number (whole number, integer, fraction, or decimal) and get −25. Therefore, there is no real square root of −25. That is, $\sqrt{-25}$ is not a real number. It is undefined.[4]

Exercise

Simplify: a) $-\sqrt{36}$ b) $\sqrt{-36}$

Answer

(a) −6 (b) undefined

[3]Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

[4]At least in Prealgebra. In later courses you will be introduced to the set of complex numbers, where $\sqrt{-25}$ will take on a new meaning.

Order of Operations

With the addition of radical notation, the *Rules Guiding Order of Operations* change slightly.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents and radicals that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

The only change in the rules is in item #2, which says: "Evaluate all exponents and radicals that appear in the expression," putting radicals on the same level as exponents.

Example 7

Simplify: $-3\sqrt{9}+12\sqrt{4}$.

Solution

According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first.

$$\begin{aligned} -3\sqrt{9}+12\sqrt{4} &= -3(3)+12(2) && \text{Evaluate radicals first: } \sqrt{9}=3 \\ & && \text{and } \sqrt{4}=2. \\ &= -9+24 && \text{Multiply: } -3(3)=-9 \text{ and } 12(2)=24. \\ &= 15 && \text{Add: } -9+24=15. \end{aligned}$$

Exercise

Simplify: $2\sqrt{4}-3\sqrt{9}$

Answer

−5

Example 8

Simplify: $-2-3\sqrt{36}$.

Solution

According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first, moving left to right.

$$\begin{aligned} -2-3\sqrt{36} &= -2-36 && \text{Evaluate radicals first: } \sqrt{36}=6 \\ &= -2-18 && \text{Multiply: } 3(6)=18. \\ &= -20 && \text{Subtract: } -2-18=-2+(-18)=-20. \end{aligned}$$

? Exercise

Simplify: $5-8\sqrt{169}$

Answer

−99

✔ Example 9

Simplify: (a) $\sqrt{9+16}$ and (b) $\sqrt{9}+\sqrt{16}$.

Solution

Apply the *Rules Guiding Order of Operations*.

a) In this case, the radical acts like grouping symbols, so we must evaluate what is inside the radical first.

$$\begin{aligned} \sqrt{9+16} &= \sqrt{25} && \text{Add: } 9+16=25. \\ &= 5 && \text{Take nonnegative square root: } \sqrt{25}=5. \end{aligned}$$

b) In this example, we must evaluate the square roots first.

$$\begin{aligned} \sqrt{9}+\sqrt{16} &= 3+4 && \text{Square root: } \sqrt{9}=3 \text{ and } \sqrt{16}=4. \\ &= 7 && \text{Add: } 3+4=7. \end{aligned}$$

? Exercise

Simplify: a) $\sqrt{25+144}$ b) $\sqrt{25}+\sqrt{144}$

Answer

(a) 13 (b) 17

Fractions and Decimals

We can also find square roots of fractions and decimals.

✔ Example 10

Simplify: (a) $\sqrt{49}$, and (b) $-\sqrt{0.49}$.

Solution

(a) Because $\left(\frac{2}{3}\right)^2=\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=\frac{4}{9}$, then

$$\sqrt{\frac{4}{9}}=\frac{2}{3}.$$

(b) Because $(0.7)^2 = (0.7)(0.7) = 0.49$ and $(-0.7)^2 = (-0.7)(-0.7) = 0.49$, both 0.7 and −0.7 are square roots of 0.49. However, $-\sqrt{0.49}$ calls for the negative square root of 0.49. Hence,

$$-\sqrt{0.49}=-0.7$$

Exercise

Simplify: a) $\sqrt{\frac{25}{49}}$ b) $\sqrt{0.36}$

Answer

(a) 5/7 (b) 0.6

Estimating Square Roots

The squares in the "List of Squares" are called *perfect squares*. Each is the square of a whole number. Not all numbers are perfect squares. For example, in the case of $\sqrt{24}$, there is no whole number whose square is equal to 24. However, this does not prevent $\sqrt{24}$ from being a perfectly good number.

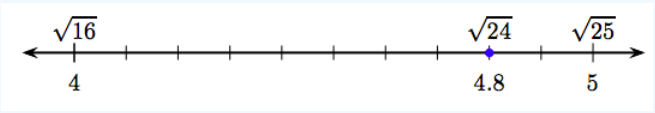
We can use the "List of Squares" to find decimal approximations when the radicand is not a perfect square.

Example 11

Estimate $\sqrt{24}$ by guessing. Use a calculator to find a more accurate result and compare this result with your guess.

Solution

From the "List of Squares," note that 24 lies betwen 16 and 25, so 24 will lie between 4 and 5, with √24 much closer to 5 than it is to 4.



Let's guess

$$\sqrt{24} \approx 4.8.$$

As a check, let's square 4.8.

$$(4.8)^2 = (4.8)(4.8) = 23.04$$

Not quite 24! Clearly, $\sqrt{24}$ must be a little bit bigger than 4.8.

Let's use a scientific calculator to get a better approximation. From our calculator, using the square root button, we find

$$\sqrt{24} \approx 4.89897948557.$$

Even though this is better than our estimate of 4.8, it is still only an approximation. Our calculator was only capable of providing 11 decimal places. However, the exact decimal representation of $\sqrt{24}$ is an infinite decimal that never terminates and never establishes a pattern of repetition.

Just for fun, here is a decimal approximation of $\sqrt{24}$ that is accurate to 1000 places, courtesy of http://www.wolframalpha.com/.

4.89897948556635619639456814941178278393189496131334025686538513450192075491463005307971886620928046963718920245322837824971773091967551468325156790247455710565782549505535314249526021054182354044696262135797338170726488670509120806761761787874917113569314944872260828854054043234840367660016317961567602617940145738798726167431618880160088747737509832902930787829002408945289626663258702188948362702657099008893234345326285099529663624900802313209072918018687172335863967331332533818263813071727532210516312358732472358220589344176709151025767105979664820111738041001283093224823470679882086211598579693467906510557472083659310343660782073560076724633259464660565809954782094852720141025275395093777354012819859118514346569290057761830288514926052059059264741510500684551198309085256259600612934415988485060457568524106813589572009319387995987119508123342717309306912496416512553772738561882612744867017729603144969267446489475909097628876958672740183948202955704655118212631969215662073401907064945 3

If you were to multiply this number by itself (square the number), you would get a number that is extremely close to 24, but it would not be exactly 24. There would still be a little discrepancy.

? Exercise

Estimate: $\sqrt{83}$.

Answer

9.1

Important Observation

A calculator can only produce a finite number of decimal places. If the decimal representation of your number does not terminate within this limited number of places, then the number in your calculator window is only an approximation.

- The decimal representation of 1/8 will terminate within three places, so most calculators will report the exact answer, 0.125.
- For contrast, 2/3 does not terminate. A calculator capable of reporting 11 places of accuracy produces the number 0.666666666667. However, the exact decimal representation of 2/3 is 0.6. Note that the calculator has rounded in the last place and only provides an approximation of 2/3. If your instructor asks for an exact answer on an exam or quiz then 0.666666666667, being an approximation, is not acceptable. You must give the exact answer 2/3.

Exercises

In Exercises 1-16, list all square roots of the given number. If the number has no square roots, write “none”.

1. 256
2. 361
3. −289
4. −400
5. 441
6. 36
7. 324
8. 0
9. 144
10. 100
11. −144
12. −100
13. 121
14. −196
15. 529
16. 400

In Exercises 17-32, compute the exact square root. If the square root is undefined, write “undefined”.

17. $\sqrt{-9}$
18. $-\sqrt{-196}$
19. $\sqrt{576}$
20. $\sqrt{289}$

21. $\sqrt{-529}$
22. $\sqrt{-256}$
23. $-\sqrt{25}$
24. $\sqrt{225}$
25. $-\sqrt{484}$
26. $-\sqrt{36}$
27. $-\sqrt{196}$
28. $-\sqrt{289}$
29. $\sqrt{441}$
30. $\sqrt{324}$
31. $-\sqrt{4}$
32. $\sqrt{100}$

In Exercises 33-52, compute the exact square root.

33. $\sqrt{0.81}$
34. $\sqrt{5.29}$
35. $\sqrt{3.61}$
36. $\sqrt{0.09}$
37. $\sqrt{\frac{225}{16}}$
38. $\sqrt{\frac{100}{81}}$
39. $\sqrt{3.24}$
40. $\sqrt{5.76}$
41. $\sqrt{\frac{121}{49}}$
42. $\sqrt{\frac{625}{324}}$
43. $\sqrt{\frac{529}{121}}$
44. $\sqrt{\frac{4}{121}}$
45. $\sqrt{2.89}$
46. $\sqrt{4.41}$
47. $\sqrt{\frac{144}{25}}$
48. $\sqrt{\frac{49}{36}}$
49. $\sqrt{\frac{256}{361}}$
50. $\sqrt{\frac{529}{16}}$
51. $\sqrt{0.49}$
52. $\sqrt{4.84}$

In Exercises 53-70, compute the exact value of the given expression.

53. $6-\sqrt{576}$
54. $-2-7\sqrt{576}$
55. $\sqrt{8^2+15^2}$
56. $\sqrt{7^2+24^2}$
57. $6\sqrt{16}-9\sqrt{49}$
58. $3\sqrt{441}+6\sqrt{484}$
59. $\sqrt{5^2+12^2}$
60. $\sqrt{15^2+20^2}$
61. $\sqrt{3^2+4^2}$
62. $\sqrt{6^2+8^2}$
63. $-2\sqrt{324}-6\sqrt{361}$
64. $-6\sqrt{576}-8\sqrt{121}$
65. $-4-3\sqrt{529}$
66. $-1+\sqrt{625}$
67. $-9\sqrt{484}+7\sqrt{81}$
68. $-\sqrt{625}-5\sqrt{576}$
69. $2-\sqrt{16}$
70. $8-6\sqrt{400}$

In Exercises 71-76, complete the following tasks to estimate the given square root.

a) Determine the two integers that the square root lies between.

b) Draw a number line, and locate the approximate location of the square root between the two integers found in part (a).

c) Without using a calculator, estimate the square root to the nearest tenth.

71. $\sqrt{58}$
72. $\sqrt{27}$
73. $\sqrt{79}$
74. $\sqrt{12}$
75. $\sqrt{44}$
76. $\sqrt{88}$

In Exercises 77-82, use a calculator to approximate the square root to the nearest tenth.

77. $\sqrt{469}$
78. $\sqrt{73}$
79. $\sqrt{615}$
80. $\sqrt{162}$
81. $\sqrt{444}$
82. $\sqrt{223}$

Answers

1. 16, −16
3. none
5. 21, −21
7. 18, −18
9. 12, −12
11. none
13. 11, −11
15. 23, −23
17. undefined
19. 24
21. undefined
23. −5
25. −22
27. −14
29. 21
31. −2
33. 0.9
35. 1.9
37. $\frac{15}{4}$
39. 1.8
41. 11 7
43. $\frac{23}{11}$
45. 1.7
47. $\frac{12}{5}$
49. $\frac{16}{19}$
51. 0.7
53. −18
55. 17
57. −39
59. 13
61. 5
63. −150
65. −73
67. −135
69. − 2
71. 7.6
73. 8.9

75. 6.6

77. 21.7

79. 24.8

81. 21.1

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6.6: Order of Operations

For convenience, we repeat the "Rules Guiding Order of Operations" first introduced in Section 1.5.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Let's look at a number of examples that require the use of these rules.

Example 1

Simplify: (a) $(-3)^2$ and (b) -3^2

Solution

Recall that for any integer a, we have $(-1)a = -a$. Because negating is equivalent to multiplying by -1, the "Rules Guiding Order of Operations" require that we address grouping symbols and exponents before negation.

a) Because of the grouping symbols, we negate first, then square. That is,

$$\begin{aligned}(-3)^2 &= (-3)(-3)\\ &= 9.\end{aligned}$$

b) There are no grouping symbols in this example. Thus, we must square first, then negate. That is,

$$\begin{aligned}-3^2 &= -(3 \cdot 3)\\ &= -9.\end{aligned}$$

Exercise

Simplify: -2^2.

Answer

-4

Example 2

Simplify: $-2 - 3(5 - 7)$.

Solution

Grouping symbols first, then multiplication, then subtraction.

$$\begin{aligned}-2-3(5-7) &= -2-3(-2) && \text{Perform subtraction within parentheses.}\\ &= 2-2-(-6) && \text{Multiply: } 3(-2) = -6.\\ &= -2+6 && \text{Add the opposite.}\\ &= 4\end{aligned}$$

Exercise

Simplify: $-3-2(6-8)$.

Answer

1

Example 3

Simplify: $-2(2-4)^2-3(3-5)^3$.

Solution

Grouping symbols first, then multiplication, and subtraction, in that order.

\[\begin{aligned} -2(2-4)^2 -3(3-5)^3 = -2(-2)^2 -3(-2)^3 ~ & \textcolor{red}{ \text{ Perform subtraction within parentheses first.} } \\ =2 (4) -3(-8) ~ & \textcolor{red}{ \text{ Exponents are next.} } \\ =-8-(-24) ~ & \textcolor{red}{ \text{ Multiplications are next.} } \\ =-8+24 ~ & \textcolor{red}{ \text{ Add the opposite.} } \\ =16 \end{aligned}\nonumber \]

Exercise

Simplify: $-2(5-6)^3-3(5-7)^2$

Answer

-10

Example 4

Simplify: $-24 \div 8(-3)$.

Solution

Division has no preference over multiplication, or vice versa. Divisions and multiplications must be performed in the order that they occur, moving left to right.

$$\begin{aligned} -24 \div 8(-3) &= -3(-3) \quad \text{Division first: } -24 \div 8 = -3. \\ &= 9 \end{aligned}$$

Note that if you multiply first, which would be incorrect, you would get a completely different answer.

Exercise

Simplify: $-48 \div 6(-2)$.

Answer

16

Example 5

Simplify: $(-2)(-3)(-2)^3$.

Solution

Exponents first, then multiplication in the order that it occurs, moving left to right.

$$\begin{aligned} (-2)(-3)(-2)^3 &= (-2)(-3)(-8) \quad \text{Exponent first: } (-2)^3 = -8. \\ &= 6(-8) \quad \text{Multiply from left to right: } (-2)(-3) = 6. \\ &= -48 \end{aligned}$$

? You try it!

Simplify: $(-4)(-2)^2(-1)^3$.

Answer

16

Evaluating Fractions

If a fraction bar is present, evaluate the numerator and denominator separately according to the "Rules Guiding Order of Operations," then perform the division in the final step.

✔ Example 6

Simplify:

$$\frac{-5-5(2-4)^3}{-22-3(-5)}$$

Solution

Evaluate numerator and denominator separately, then divide.

$$\begin{aligned}
\frac{-5-5(2-4)^3}{-22-3(-5)} &= \frac{-5-5(-2)^3}{-22-(-15)} && \text{Numerator: parentheses first.}\\
&= \frac{-5-5(-8)}{-22+15} && \text{Denominator: multiply } 3(-5)=-15.\\
&= \frac{-5-(-40)}{-7} && \text{Numerator: exponent } (-2)^3=-8.\\
&= \frac{-5+40}{-7} && \text{Denominator: add the opposite.}\\
&= \frac{35}{-7} && \text{Numerator: multiply } 5(-8)=-40.\\
&= -5 && \text{Denominator: add } -22+15=-7.\\
& && \text{Numerator: add the opposite.}\\
& && \text{Numerator: } -5+40=35.\\
& && \text{Divide: } 35/-7=-5.
\end{aligned}$$

? Exercise

Simplify:

$$\frac{6-2(-6)}{-2-(-2)^2}$$

Answer

-3

Absolute Value

Absolute value calculates the magnitude of the vector associated with an integer, which is equal to the distance between the number and the origin (zero) on the number line. Thus, for example, |4| = 4 and | − 5| = 5.

But absolute value bars also act as grouping symbols, and according to the "Rules Guiding Order of Operations," you should evaluate the expression inside a pair of grouping symbols first.

Example 7

Simplify: (a) $-(-3)$ and (b) $-|-3|$.

Solution

There is a huge difference between simple grouping symbols and absolute value.

a) This is a case of $-(-a) = a$. Thus, $-(-3) = 3$.

b) This case is much different. The absolute value of -3 is 3, and then the negative of that is -3. In symbols,

$$-|-3| = -3$$

Exercise

Simplify: $-|-8|$.

Answer

-8

Example 8

Simplify: $-3 - 2|5 - 7|$.

Solution

Evaluate the expression inside the absolute value bars first. Then multiply, then subtract.

$$-3-2|5-7| = -3-2|-2|$$

Exercise

Simplify: $-2 - 4|6 - 8|$.

Answer

-10

Exercises

In Exercises 1-40, compute the exact value of the given expression.

1. $7 - \frac{-14}{2}$
2. $-2 - \frac{-16}{4}$
3. $-7 - \frac{-18}{9}$
4. $-6 - \frac{-7}{7}$
5. -54
6. -33
7. $9 - 1(-7)$
8. $85 - 8(9)$
9. -63
10. -35
11. $3 + 9(4)$

12. $6 + 7(-1)$
13. $10 - 72 \div 6 \cdot 3+8$
14. $8 - 120 \div 5 \cdot 6+7$
15. $6 + \frac{14}{2}$
16. $16 + \frac{8}{2}$
17. -34
18. -22
19. $3 - 24 \div 4 \cdot 3+4$
20. $4 - 40 \div 5 \cdot 4+9$
21. $64 \div 4 \cdot 4$
22. $18 \div 6 \cdot 1$
23. $-2 - 3(-5)$
24. $64 - 7(7)$
25. $15 \div 1 \cdot 3$
26. $30 \div 3 \cdot 5$
27. $8 + 12 \div 6 \cdot 1 - 5$
28. $9 + 16 \div 2 \cdot 4 - 9$
29. $32 \div 4 \cdot 4$
30. $72 \div 4 \cdot 6$
31. $-11 + \frac{16}{16}$
32. $4 + \frac{-20}{4}$
33. -52
34. -43
35. $10 + 12(-5)$
36. $4 + 12(4)$
37. $2+6 \div 1 \cdot 6 - 1$
38. $1 + 12 \div 2 \cdot 2 - 6$
39. $40 \div 5 \cdot 4$
40. $30 \div 6 \cdot 5$

In Exercises 41-80, simplify the given expression.

41. $-11 + |-1 - (-6)^2|$
42. $13 + |-21 - (-4)^2|$
43. $|0(-4)| - 4(-4)$
44. $|10(-3)| - 3(-1)$
45. $(2 + 3 \cdot 4) - 8$
46. $(11 + 5 \cdot 2) - 10$
47. $(8 - 1 \cdot 12) + 4$
48. $(9 - 6 \cdot 1) + 3$

49. $(6 + 10 \cdot 4) - 6$

50. $(8 + 7 \cdot 6) - 12$

51. $10 + (6 - 4)^3 - 3$

52. $5 + (12 - 7)^2 - 6$

53. $(6 - 8)^2 - (4 - 7)^3$

54. $(3 - 8)^2 - (4 - 9)^3$

55. $|0(-10)| + 4(-4)$

56. $|12(-5)| + 7(-5)$

57. $|8(-1)| - 8(-7)$

58. $|6(-11)| - 7(-1)$

59. $3 + (3 - 8)^2 - 7$

60. $9 + (8 - 3)^3 - 6$

61. $(4 - 2)^2 - (7 - 2)^3$

62. $(1 - 4)^2 - (3 - 6)^3$

63. $8 - |- 25 - (-4)^2|$

64. $20 - |- 22 - 4^2|$

65. $-4 - |30 - (-5)^2|$

66. $-8 - |- 11 - (-6)^2|$

67. $(8 - 7)^2 - (2 - 6)^3$

68. $(2 - 7)^2 - (4 - 7)^3$

69. $4 - (3 - 6)^3 + 4$

70. $6 - (7 - 8)^3 + 2$

71. $-3 + | - 22 - 5^2|$

72. $12 + |23 - (-6)^2|$

73. $(3 - 4 \cdot 1) + 6$

74. $(12 - 1 \cdot 6) + 4$

75. $1 - (1 - 5)^2 + 11$

76. $9 - (3 - 1)^3 + 10$

77. $(2 - 6)^2 - (8 - 6)^3$

78. $(2 - 7)^2 - (2 - 4)^3$

79. $|9(-3)| + 12(-2)$

80. $|0(-3)| + 9(-7)$

In Exercises 81-104, simplify the given expression.

81. $\frac{4(-10)-5}{-9}$

82. $\frac{-4 \cdot 6-(-8)}{-4}$

83. $\frac{10^2-4^2}{2 \cdot 6-10}$

84. $\frac{3^2-9^2}{2 \cdot 7-5}$

85. $\frac{3^2+6^2}{5-1\cdot 8}$

86. $\frac{10^2+4^2}{1-6\cdot 5}$

87. $\frac{-8-4}{7-13}$

88. $\frac{13-1}{8-4}$

89. $\frac{2^2+6^2}{11-4\cdot 4}$

90. $\frac{7^2+3^2}{10-8\cdot 1}$

91. $\frac{1^2-5^2}{9\cdot 1-5}$

92. $\frac{5^2-7^2}{2\cdot 2-12}$

93. $\frac{4^2-8^2}{6\cdot 3-2}$

94. $\frac{7^2-6^2}{6\cdot 3-5}$

95. $\frac{10^2+2^2}{10-2\cdot 7}$

96. $\frac{2^2+10^2}{10-2\cdot 7}$

97. $\frac{16-(-2)}{19-1}$

98. $\frac{-8-20}{-15-(-17}$

99. $\frac{15-(-15)}{13-(-17)}$

100. $\frac{7-(-9)}{-1-1}$

101. $\frac{4\cdot 5-(-19)}{3}$

102. $\frac{10\cdot 7-(-11)}{-3}$

103. $\frac{-6\cdot 9-(-4)}{2}$

104. $\frac{-6\cdot 2-10}{-11}$

Answers

1. 14

3. −5

5. −625

7. 16

9. −216

11. 39

13. −18

15. 13

17. −81

19. −11

21. 64

23. 13

25. 45

27. 5

29. 32

31. −10

33. −25

35. −50

37. 37

39. 32

41. 26

43. 16

45. 6

47. 0

49. 40

51. 15

53. 31

55. −16

57. 64

59. 21

61. −121

63. −33

65. −9

67. 65

69. 35

71. 44

73. 5

75. −4

77. 8

79. 3

81. 5

83. 42

85. −15

87. 2

89. −8

91. −6

93. −3

95. −8

97. 1

99. 1

101. 13

103. −25

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CHAPTER OVERVIEW

7: The Fundamentals of Algebra

As his name portends, Abu Jafr Muhammad ibn Musa al-Khwarizmi was one of the greatest Arab mathematicians of his time. While living in Baghdad during the ninth century AD he became the Chief Librarian at the House of Wisdom, a library and major center of intellectual study. In the year 820AD, al-Khwarizmi wrote *Al-Kitab al-mukhtasar ti Hisab al-jabr w'al-muqabala*, translated to, *The Compendious Book on Calculation by Restoration and Reduction*, the first book to generalize solving equations using the principles of equality. In fact, the word algebra itself comes from *al-jabr*, meaning reunion or completion.

Many earlier cultures had employed what we might call algebra in the service of business, land management, inheritance, and trade. The Bablyonians were solving quadratic equations in 2000BC, but only specific equations, with specific numbers. Hindus on the Indian continent were also developing algebra along side their invention of the symbol for zero 0. But like al-Khwarizmi and the Moslem Arabs of the first millenium, to write equations these early cultures did not use symbols like x or y, or even equal signs = that we use today. al-Khwarizmi wrote absolutely everything in words! 42 would be forty-two!

Early algebra written all with words is called rhetorical algebra, and a thousand years ago, each mathematician had their own way of expressing it. Algebra was a language with many different dialects, and communicating it from one mathematician to another was difficult as they had to explain themselves with words. It wasn't until well after the Gutenberg printing press was invented in 1436 and print became standardized, that Rene Descartes, a Frenchman began to develop a modern symbolic algebra.

In this section we'll learn how to manipulate symbols in order to al-muqabalah (combine like terms) and al-jabr (restore and balance equations). But we'll use modern notation to make it easier!

7.1: Mathematical Expressions
7.2: Evaluating Algebraic Expressions
7.3: Simplifying Algebraic Expressions
7.4: Combining Like Terms
7.5: Solving Equations
7.6: More on Solving Equations
7.7: Applications

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7.1: Mathematical Expressions

Recall the definition of a *variable* presented in Section 1.6.

Definition: Variable

A variable is a symbol (usually a letter) that stands for a value that may vary.

Let's add the definition of a *mathematical expression*.

Definition: Mathematical Expression

When we combine numbers and variables in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, and other operations and functions as yet unlearned, the resulting combination of mathematical symbols is called a *mathematical expression*.

Thus,

$$2a,\ x + 5, \text{ and } y^2,$$

being formed by a combination of numbers, variables, and mathematical operators, are valid mathematical expressions. A mathematical expression must be *well-formed*. For example,

$$2 + \div 5x$$

is *not a valid expression* because there is no term following the plus sign (it is not valid to write +÷ with nothing between these operators). Similarly,

$$2 + 3(2$$

is not well-formed because parentheses are not balanced.

Translating Words into Mathematical Expressions

In this section we turn our attention to translating word phrases into mathematical expressions. We begin with phrases that translate into *sums*. There is a wide variety of word phrases that translate into sums. Some common examples are given in Table 7.1.1*a*, though the list is far from complete. In like manner, a number of phrases that translate into differences are shown in Table 7.1.1*b*.

Table 7.1.1: Translating words into symbols.

Phrase	Translates to:	Phrase	Translates to:
sum of x and 12	$x + 12$	difference of x and 12	$x - 12$
4 greater than b	$b + 4$	4 less than b	$b - 4$
6 more than y	$y + 6$	7 subtracted from y	$y - 7$
44 plus r	$44 + r$	44 minus r	$44 - r$
3 larger than z	$z + 3$	3 smaller than z	$z - 3$
a) Phrases that are sums		b) Phrases that are differences	

Let's look at some examples, some of which translate into expressions involving sums, and some of which translate into expressions involving differences.

Example 1

Translate the following phrases into mathematical expressions:

a. "12 larger than x,"
b. "11 less than y," and

c. "r decreased by 9."

Solution

Here are the translations.

a. “12 larger than x” becomes $x + 12$.
b. “11 less than y” becomes $y - 11$.
c. “r decreased by 9” becomes $r - 9$.

Exercise

Translate the following phrases into mathematical expressions:

a. "13 more than x" and
b. "12 fewer than y".

Answer

(a) $x + 13$ and

(b) $y - 12$

Example 2

Let W represent the width of the rectangle. The length of a rectangle is 4 feet longer than its width. Express the length of the rectangle in terms of its width W.

Solution

We know that the width of the rectangle is W. Because the length of the rectangle is 4 feet longer that the width, we must add 4 to the width to find the length.

Length	is	4	more than	the width
Length	$=$	4	$+$	W

Thus, the length of the rectangle, in terms of its width W, is $4 + W$.

Exercise

The width of a rectangle is 5 inches shorter than its length L. Express the width of the rectangle in terms of its length L.

Answer

$L - 5$

Example 3

A string measures 15 inches is cut into two pieces. Let x represent the length of one of the resulting pieces. Express the length of the second piece in terms of the length x of the first piece.

Solution

The string has original length 15 inches. It is cut into two pieces and the first piece has length x. To find the length of the second piece, we must subtract the length of the first piece from the total length.

Length of the second piece	is	Total length	minus	the length of the first piece
Length of the second piece	$=$	15	$-$	x

Thus, the length of the second piece, in terms of the length x of the first piece, Answer: $12 + x$ is $15 - x$.

Exercise

A string is cut into two pieces, the first of which measures 12 inches. Express the total length of the string as a function of x, where x represents the length of the second piece of string.

Answer

$12 + x$

There is also a wide variety of phrases that translate into products. Some examples are shown in Table 3.2(a), though again the list is far from complete. In like manner, a number of phrases translate into quotients, as shown in Table 3.2(b).

Table 7.1.2: Translating words into symbols.

Phrase	Translates to:	Phrase	Translates to:
product of x and 12	$12x$	quotient of x and 12	$x/12$
4 times b	$4b$	4 divided by b	$4/b$
twice r	$2r$	the ratio of 44 to r	$44/r$
a) Phrases that are products.		b) Phrases that are differences.	

Let's look at some examples, some of which translate into expressions involving products, and some of which translate into expressions involving quotients.

Example 4

Translate the following phrases into mathematical expressions: (a) "11 times x," (b) "quotient of y and 4," and (c) "twice a."

Solution

Here are the translations. a) "11 times x" becomes $11x$. b) "quotient of y and 4" becomes $y/4$, or equivalently, $\frac{y}{4}$. c) "twice a" becomes $2a$.

Exercise

Translate into mathematical symbols: (a) "the product of 5 and x" and (b) "12 divided by y."

Answer

(a) $5x$ and (b) $12/y$.

Example 5

A plumber has a pipe of unknown length x. He cuts it into 4 equal pieces. Find the length of each piece in terms of the unknown length x.

Solution

The total length is unknown and equal to x. The plumber divides it into 4 equal pieces. To find the length of each pieces, we must divide the total length by 4.

Length of each piece	is	Total length	divided by	4
Length of each piece	$=$	x	$\div$	4

Thus, the length of each piece, in terms of the unknown length x, is $x/4$, or equivalently, $\frac{x}{4}$.

Exercise

A carpenter cuts a board of unknown length L into three equal pieces. Express the length of each piece in terms of L.

Answer

L/3

Example 6

Mary invests A dollars in a savings account paying 2% interest per year. She invests five times this amount in a certificate of deposit paying 5% per year. How much does she invest in the certificate of deposit, in terms of the amount A in the savings account?

Solution

The amount in the savings account is A dollars. She invests five times this amount in a certificate of deposit.

$$\begin{array}{cccccc} \text{Amount in CD} & \text{is} & 5 & \text{times} & \text{Amount in savings} \\ \text{Amount in CD} & = & 5 & \cdot & A \end{array}$$

Thus, the amount invested in the certificate of deposit, in terms of the amount A in the savings account, is $5A$.

Exercise

David invests K dollars in a savings account paying 3% per year. He invests half this amount in a mutual fund paying 4% per year. Express the amount invested in the mutual fund in terms of K, the amount invested in the savings account.

Answer

$\frac{1}{2}K$

Combinations

Some phrases require combinations of the mathematical operations employed in previous examples.

Example 7

Let the first number equal x. The second number is 3 more than twice the first number. Express the second number in terms of the first number x.

Solution

The first number is x. The second number is 3 more than twice the first number.

$$\begin{array}{cccc} \text{Second number} & \text{is} & 3 \text{ more than} & \text{twice the first number} \\ \text{Second number} & = & 3+ & 2x \end{array}$$

Therefore, the second number, in terms of the first number x, is $3 + 2x$.

Exercise

A second number is 4 less than 3 times a first number. Express the second number in terms of the first number y.

Answer

$3y - 4$

Example 8

The length of a rectangle is L. The width is 15 feet less than 3 times the length. What is the width of the rectangle in terms of the length L?

Solution

The length of the rectangle is L. The width is 15 feet less than 3 times the length.

$$\begin{array}{lllll} \text{Width is} & \text{3 times the length} & \text{less} & 15 \\ \text{Width} = & 3L & - & 15 \end{array}$$

Therefore, the width, in terms of the length L, is $3L - 15$.

Exercise

The width of a rectangle is W. The length is 7 inches longer than twice the width. Express the length of the rectangle in terms of its length L.

Answer

$2W + 7$

Exercises

In Exercises 1-20, translate the phrase into a mathematical expression involving the given variable.

1. “8 times the width n ”
2. “2 times the length z ”
3. “6 times the sum of the number n and 3”
4. “10 times the sum of the number n and 8”
5. “the demand b quadrupled”
6. “the supply y quadrupled”
7. “the speed y decreased by 33”
8. “the speed u decreased by 30”
9. “10 times the width n ”
10. “10 times the length z ”
11. “9 times the sum of the number z and 2”
12. “14 times the sum of the number n and 10”
13. “the supply y doubled”
14. “the demand n quadrupled”
15. “13 more than 15 times the number p ”
16. “14 less than 5 times the number y ”
17. “4 less than 11 times the number x ”
18. “13 less than 5 times the number p ”
19. “the speed u decreased by 10”
20. “the speed w increased by 32”

---

21. Representing Numbers. Suppose n represents a whole number.

i) What does n + 1 represent?

ii) What does n + 2 represent?

iii) What does n − 1 represent?

22. Suppose 2n represents an even whole number. How could we represent the next even number after 2n?

23. Suppose 2n + 1 represents an odd whole number. How could we represent the next odd number after 2n + 1?

24. There are b bags of mulch produced each month. How many bags of mulch are produced each year?

25. Steve sells twice as many products as Mike. Choose a variable and write an expression for each man's sales.

26. Find a mathematical expression to represent the values.

i) How many quarters are in d dollars?

ii) How many minutes are in h hours?

iii) How many hours are in d days?

iv) How many days are in y years?

v) How many months are in y years?

vi) How many inches are in f feet?

vii) How many feet are in y yards?

Answers

1. 8n

3. 6(n + 3)

5. 4b

7. y − 33

9. 10n

11. 9(z + 2)

13. 2y

15. 15p + 13 17.

11x − 4

19. u − 10

21.

i) n+1 represents the next whole number after n.

ii) n+2 represents the next whole number after n + 1, or, two whole numbers after n.

iii) n − 1 represents the whole number before n.

23. 2n + 3

25. Let Mike sell p products. Then Steve sells 2p products.

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7.2: Evaluating Algebraic Expressions

In this section we will evaluate *algebraic expressions* for given values of the variables contained in the expressions. Here are some simple tips to help you be successful.

Tips for Evaluating Algebraic Expressions

1. Replace all occurrences of variables in the expression with open parentheses. Leave room between the parentheses to substitute the given value of the variable.
2. Substitute the given values of variables in the open parentheses prepared in the first step.
3. Evaluate the resulting expression according to the Rules Guiding Order of Operations.

Let's begin with an example.

Example 1

Evaluate the expression $x^2-2xy+y^2$ at $x=-3$ and $y=2$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $x^2 - 2xy + y^2$ with open parentheses.

$$x^2-2xy+y^2=(\,)^2-2(\,)(\,)+(\,)^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned} x^2-2xy+y^2 &\quad \text{Original expression.}\\ =(-3)^2-2(-3)(2)+(2)^2 &\quad \text{Substitute } -3 \text{ for } x \text{and } 2 \text{ for } y.\\ =9-2(-3)(2)+4 &\quad \text{Evaluate exponents first.}\\ =9-(-6)(2)+4 &\quad \text{Left to right, multiply } 2(-3)=-6.\\ =9-(-12)+4 &\quad \text{Left to right, multiply: } (-6)(2)=-12.\\ =9+12+4 &\quad \text{Add the opposite.}\\ =25 &\quad \text{Add.} \end{aligned}$$

Exercise

If $x = -2$ and $y = -1$, evaluate $x^3 - y^3$.

Answer

-7

Example 2

Evaluate the expression $(a - b)^2$ If $a = 3$ and $b = -5$, at $a = 3$ and $b = -5$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $(a - b)^2$ with open parentheses.

$$(a-b)^2=((\,)-(\,))^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}(a-b)^2 &= ((3)-(-5))^2 && \text{Substitute 3 for } a \text{ and } -5 \text{ for } b.\\ &= (3+5)^2 && \text{Add the opposite: } (3)-(-5)=3+5\\ &= 8^2 && \text{Simplify inside parentheses: } 3+5=8\\ &= 64 && \text{Evaluate exponent: } 8^2=64\end{aligned}$$

? Exercise

If $a = 3$ and $b = -5$, evaluate $a^2 - b^2$.

Answer

−16

✓ Example 3

Evaluate the expression |a|−|b| at a = 5 and b = −7.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression |a|−|b| with open parentheses.

$$|a|-|b| = |(\,)|-|(\,)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}|a|-|b| = |(5)| &= |(-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b.\\ &= 5-7 && \text{Absolute values first: } |(5)|=5 \text{ and } |(-7)|=7|\\ &= 5+(-7) && \text{Add the opposites: } 5-7=5+(-7).\\ &= -2 && \text{Add: } 5+(-7)=-2.\end{aligned}$$

? Exercise

If $a = 5$ and $b = -7$, evaluate $2|a| - 3|b|$.

Answer

−11

✓ Example 4

Evaluate the expression $|a - b|$ at $a = 5$ and $b = -7$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $|a - b|$ with open parentheses.

$$|a-b| = |(\,)-(\,)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}|a-b| &= |(5)-(-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b.\\ &= |5+7| && \text{Add the opposite: } 5-(-7)=5+7.\\ &= |12| && \text{Add: } 5+7=12.\\ &= 12 && \text{Take the absolute value: } |12|=12.\end{aligned}$$

Exercise

If $a = 5$ and $b = -7$, evaluate $|2a - 3b|$.

Answer

31

Example 5

Evaluate the expression

$$\frac{ad-bc}{a+b}$$

at $a = 5$, $b = -3$, $c = 2$, and $d = -4$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression with open parentheses.

$$\frac{ad-bc}{a+b} = \frac{(\,)(\,)-(\,)(\,)}{(\,)+(\,)}$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}
\frac{ad-bc}{a+b} &= \frac{(5)-(-3)(2)}{(5)+(-3)} && \text{Substitute: } 5 \text{ for } a,\ -3 \text{ for } b,\ 2 \text{ for } c,\ -4 \text{ for } d.\\
&= \frac{-20-(-6)}{2} && \text{Numerator: } (5)(=4) = -20,\ (-3)(2) = -6.\\
&&& \text{Denominator: } 5+(-3) = 2.\\
&= \frac{-20+6}{2} && \text{Numerator: Add the opposite.}\\
&= \frac{-14}{2} && \text{Numerator: } -20+6 = -14.\\
&= -7 && \text{Divide.}
\end{aligned}$$

Exercise

If a = −7, b = −3, c = −15, 15, and $d = -14$, evaluate:

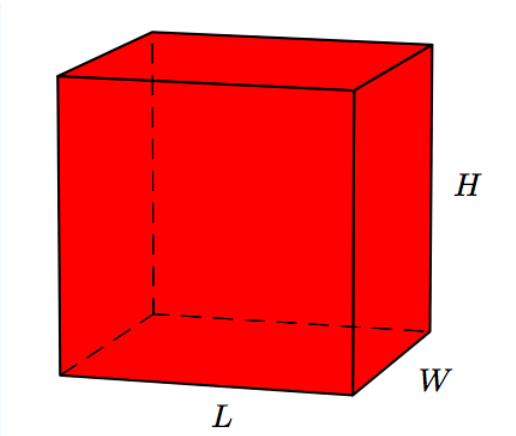
$$\frac{a^2+b^2}{c+d}$$

Answer

−2

Example 6

Pictured below is a rectangular prism.



The volume of the rectangular prism is given by the formula

$$V = LWH,$$

where L is the length, W is the width, and H is the height of the rectangular prism. Find the volume of a rectangular prism having length 12 feet, width 4 feet, and height 6 feet.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of of L, W, and H in the formula

$$V = LWH$$

with open parentheses.

$$V = (\,)(\,)(\,)$$

Next, substitute 12 ft for L, 4 ft for W, and 6 ft for H and simplify.

$$\begin{aligned} V &= (12\text{ft})(4\text{ft})(6\text{ft}) \\ &= 288\text{ft}^3 \end{aligned}$$

Hence, the volume of the rectangular prism is 288 cubic feet.

? Exercise

The surface area of the prism pictured in this example is given by the following formula:

$$S = 2(WH + LH + LW)$$

If L = 12, W = 4, and H = 6 feet, respectively, calculate the surface area.

Answer

288 square feet

Exercises

In Exercises 1-12, evaluate the expression at the given value of x.

1. $-3x^2 - 6x + 3$ at $x = 7$
2. $7x^2 - 7x + 1$ at $x = -8$
3. $-6x - 6$ at $x = 3$
4. $6x - 1$ at $x = -10$
5. $5x^2 + 2x + 4$ at $x = -1$

6. $4x^2 - 9x + 4$ at $x = -3$

7. $-9x - 5$ at $x = -2$

8. $-9x + 12$ at $x = 5$

9. $4x^2 + 2x + 6$ at $x = -6$

10. $-3x^2 + 7x + 4$ at $x = -7$

11. $12x + 10$ at $x = -12$

12. $-6x + 7$ at $x = 11$

In Exercises 13-28, evaluate the expression at the given values of x and y.

13. $|x|-|y|$ at $x = -5$ and $y = 4$

14. $|x|-|y|$ at $x = -1$ and $y = -2$

15. $-5x^2 + 2y^2$ at $x = 4$ and $y = 2$

16. $-5x^2 - 4y^2$ at $x = -2$ and $y = -5$

17. $|x|-|y|$ at $x = 0$ and $y = 2$

18. $|x|-|y|$ at $x = -2$ and $y = 0$

19. $|x - y|$ at $x = 4$ and $y = 5$

20. $|x - y|$ at $x = -1$ and $y = -4$

21. $5x^2 - 4xy + 3y^2$ at $x = 1$ and $y = -4$

22. $3x^2 + 5xy + 3y^2$ at $x = 2$ and $y = -1$

23. $|x - y|$ at $x = 4$ and $y = 4$

24. $|x - y|$ at $x = 3$ and $y = -5$

25. $-5x^2 - 3xy + 5y^2$ at $x = -1$ and $y = -2$

26. $3x^2 - 2xy - 5y^2$ at $x = 2$ and $y = 5$

27. $5x^2 + 4y^2$ at $x = -2$ and $y = -2$

28. $-4x^2 + 2y^2$ at $x = 4$ and $y = -5$

In Exercises 29-40, evaluate the expression at the given value of x.

29. $\frac{9+9x}{-x}$ at $x = -3$

30. $\frac{9-2x}{-x}$ at $x = -1$

31. $\frac{-8x+9}{-9+x}$ at $x = 10$

32. $\frac{2x+4}{1+x}$ at $x = 0$

33. $\frac{-4+9x}{7x}$ at $x = 2$

34. $\frac{-1-9x}{x}$ at $x = -1$

35. $\frac{-12-7x}{x}$ at $x = -1$

36. $\frac{12+11x}{3x}$ at $x = -6$

37. $\frac{6x-10}{5}$ + x at $x = -6$

38. $\frac{11x+11}{-4}$ + x at $x = 5$

39. $\frac{10x+11}{5}$ + x at $x = -4$

40. $\frac{6x+12}{-3}$ + x at x = 2

41. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in t = 4 seconds.

42. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in t = 24 seconds.

43. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature (◦ C) if the Fahrenheit temperature is F = 230◦ F.

44. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature (°C) if the Fahrenheit temperature is F = 95 °F.

45. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at K = 373° K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 28° K to Fahrenheit.

46. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at K = 373° K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 248° K to Fahrenheit.

47. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is g = 32 feet per second per second. If the initial velocity of the ball is v_0 = 272 feet per second, find the speed of the ball after t = 6 seconds.

48. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v_0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is g = 32 feet per second per second. If the initial velocity of the ball is v_0 = 470 feet per second, find the speed of the ball after t = 4 seconds.

49. **Even numbers**. Evaluate the expression 2n for the following values:

i) n = 1

ii) n = 2

iii) n = 3

iv) n = −4

v) n = −5

vi) Is the result always an even number? Explain.

50. **Odd numbers**. Evaluate the expression 2n + 1 for the following values:

i) n = 1

ii) n = 2

iii) n = 3

iv) n = −4

v) n = −5

vi) Is the result always an odd number? Explain.

Answers

1. −186

3. −24

5. 7

7. 13

9. 138

11. −134

13. 1

15. −72

17. −2

19. 1

21. 69

23. 0

25. 9

27. 36

29. −6

31. −71

33. 1

35. 5

37. 46

39. −29

41. 256 feet

43. 110 degrees

45. −409° F

47. 80 feet per second

49.

i) 2

ii) 4

iii) 6

iv) −8

v) −10

vi) Yes, the result will always be an even number because 2 will always be a factor of the product 2n.

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7.3: Simplifying Algebraic Expressions

Recall the commutative and associative properties of multiplication.

The Commutative Property of Multiplication. If a and b are any integers, then

$$a \cdot b = b \cdot a, \text{ or equivalently, } ab = ba.$$

The Associative Property of Multiplication. If a, b, and c are any integers, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \text{ or equivalently, } (ab)c = a(bc).$$

The commutative property allows us to change the order of multiplication without affecting the product or answer. The associative property allows us to regroup without affecting the product or answer.

Example 1

Simplify: $2(3x)$.

Solution

Use the associative property to regroup, then simplify.

$$\begin{aligned} 2(3x) &= (2 \cdot 3)x \quad \text{Regrouping with the associative property.} \\ &= 6x \quad \text{Simplify: } 2 \cdot 3 = 6. \end{aligned}$$

Exercise

Simplify: $-5(7y)$

Answer

$-35y$

The statement $2(3x)=6x$ is an *identity*. That is, the left-hand side and right-hand side of $2(3x)=6x$ are the same for all values of x. Although the derivation in Example 1 should be the proof of this statement, it helps the intuition to check the validity of the statement for one or two values of x.

If $x = 4$, then

$$\begin{aligned} 2(3x) &= 2(3(4)) \\ &= 2(12) \\ &= 24 \end{aligned} \quad \text{and} \quad \begin{aligned} 6x &= 6(4) \\ &= 24 \end{aligned}$$

If $x = -5$, then

$$\begin{aligned} 2(3x) &= 2(3(-5)) \\ &= 2(-15) \\ &= -30 \end{aligned} \quad \text{and} \quad \begin{aligned} 6x &= 6(-5) \\ &= -30 \end{aligned}$$

The above calculations show that $2(3x)=6x$ for both $x = 4$ and $x = -5$. Indeed, the statement $2(3x)=6x$ is true, regardless of what is substituted for x.

Example 2

Simplify: $(-3t)(-5)$.

Solution

In essence, we are multiplying three numbers, −3, *t*, and −5, but the grouping symbols ask us to multiply the −3 and the t first. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned}(-3t)(-5) &= ((-3)(-5))t \quad \text{Change the order and regroup.}\\ &= 15t \quad \text{Multiply: } (-3)(-5) = 15.\end{aligned}$$

? Exercise

Simplify: (−8*a*)(5)

Answer

−40*a*

✓ Example 3

Simplify: (−3*x*)(−2*y*)

Solution

In essence, we are multiplying four numbers, −3, x, −2, and y, but the grouping symbols specify a particular order. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned}(-3x)(-2y) &= ((-3)(-2))(xy) \quad \text{Change the order and regroup.}\\ &= 6xy \quad \text{Multiply: } (-3)(-2) = 6.\end{aligned}$$

? Exercise

Simplify: (−4*a*)(5*b*)

Answer

−20*ab*

Speeding Things Up

The meaning of the expression $2 \cdot 3 \cdot 4$ is clear. Parentheses and order of operations are really not needed, as the commutative and associative properties explain that it doesn't matter which of the three numbers you multiply together first.

- You can multiply 2 and 3 first:

$$\begin{aligned}2 \cdot 3 \cdot 4 &= (2 \cdot 3) \cdot 4\\ &= 6 \cdot 4\\ &= 24.\end{aligned}$$

- Or you can multiply 3 and 4 first:

$$\begin{aligned}2 \cdot 3 \cdot 4 &= 2 \cdot (3 \cdot 4)\\ &= 2 \cdot 12\\ &= 24.\end{aligned}$$

- Or you can multiply 2 and 4 first:

$$\begin{aligned}2 \cdot 3 \cdot 4 &= (2 \cdot 4) \cdot 3\\ &= 8 \cdot 3\\ &= 24.\end{aligned}$$

So, it does not matter which two factors you multiply first.

Of course, this would not be the case if there were a mixture of multiplication and other operators (division, addition, subtraction). Then we would have to strictly follow the "Rules Guiding Order of Operations." But if the only operator is multiplication, the order

of multiplication is irrelevant.

Thus, when we see 2(3x), as in Example 1, we should think "It's all multiplication and it doesn't matter which two numbers I multiply together first, so I'll multiply the 2 and the 3 and get 2(3x)=6x."

Our comments apply equally well to a product of four or more factors. It simply doesn't matter how you group the multiplication. So, in the case of (−3x)(−2y), as in Example 3, find the product of −2 and −3 and multiply the result by the product of x and y. That is, (−3x)(−2y)=6xy.

Example 4

Simplify: (2a)(3b)(4c).

Solution

The only operator is multiplication, so we can order and group as we please. So, we'll take the product of 2, 3, and 4, and multiply the result by the product of a, b, and c. That is,

$$(2a)(3b)(4c) = 24abc$$

Exercise

Simplify: (−3x)(−2y)(−4z)

Answer

−24xyz.

The Distributive Property

Multiplication is distributive with respect to addition.

The Distributive Property

If a, b, and c are any integers, then

$a \cdot (b + c) = a \cdot b + a \cdot c$, or equivalently, $a(b + c) = ab + ac$.

For example, if we follow the "Rules Guiding Order of Operations" and first evaluate the expression inside the parentheses, then

$$\begin{aligned} 3(4+5) &= 3(9) && \text{Parentheses first: } 4+5=9. \\ &= 27. && \text{Multiply: } 3(9)=27. \end{aligned}$$

But if we "distribute" the 3, we get the same answer.

$$\begin{aligned} 3(4+5) &= 3(4+5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 3(4)+3(5) \\ &= 12+15 && \text{Multiply first: } 3(4)=12,\ 3(5)=15. \\ &= 27 && \text{Add.} \end{aligned}$$

Example 5

Use the distributive property to simplify: 3(4x + 5).

Solution

Distribute the 3.

$$\begin{aligned} 3(4x+5) &= 3(4x)+3(5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 12x+15 && \text{Multiply first: } 3(4x)=12x,\ 3(5)=15. \end{aligned}$$

Exercise

Use the distributive property to simplify: 2(5z+7).

Answer

10z + 14

Multiplication is also distributive with respect to subtraction.

The Distributive Property

If a, b, and c are any integers, then

$$a \cdot (b - c) = a \cdot b - a \cdot c, \text{ or equivalently, } a(b - c) = ab - ac.$$

The application of this form of the distributive property is identical to the first, the only difference being the subtraction symbol.

Example 6

Use the distributive property to simplify: 5(3x − 2).

Solution

Distribute the 5.

$$\begin{aligned} 5(3x-2) &= 5(3x)-5(2) && \text{Each number in parentheses is multiplied by the number 5 outside the parentheses.} \\ &= 15x-10 && \text{Multiply first: } 5(3x)=15x, 5(2)=10. \end{aligned}$$

Exercise

Use the distributive property to simplify: 7(4a − 5).

Answer

28a − 35

Example 7

Remove parentheses: (a) −9(2t + 7), and (b) −5(4 − 3y).

Solution

a) Use the distributive property.

$$\begin{aligned} -9(2t+7) &= -9(2t)+(-9)(7) && \text{Distribute multiplication by } -9. \\ &= -18t+(-63) && \text{Multiply: } -9(2t)=-18t \text{ and } -9(7)=-63. \\ &= -18t-63 && \text{Write the answer in simpler form.} \\ & && \text{Adding } -63 \text{ is the same as subtracting } 63. \end{aligned}$$

b) Use the distributive property.

$$\begin{aligned} -5(4-3y) &= -5(4)-(-5)(3y) && \text{Distribute multiplication by } -5. \\ &= -20-(-15y) && \text{Multiply: } -5(4)=-20 \text{ and } -5(3y)=-15y. \\ &= -18t-63 && \text{Write the answer in simpler form.} \\ & && \text{Subtracting } -15y \text{ is the same as adding } 15y. \end{aligned}$$

Exercise

Remove parentheses: $-3(4t - 11)$.

Answer

$-12t + 33$

Writing Mathematics

Example 7 stresses the importance of using as few symbols as possible to write your final answer. Hence, $-18t - 63$ is favored over $-18t + (-63)$ and $-20 + 15y$ is favored over $-20 - (-15y)$. You should always make these final simplifications.

Moving a Bit Quicker

Once you've applied the distributive property to a number of problems, showing all the work as in Example 7, you should try to eliminate some of the steps. For example, consider again Example 7(a). It's not difficult to apply the distributive property without writing down a single step, getting:

$$-9(2t+7) = -18t - 63.$$

Here's the thinking behind this technique:

1. First, multiply -9 times $2t$, getting $-18t$.
2. Second, multiply -9 times $+7$, getting -63.

Note that this provides exactly the same solution found in Example 7(a).

Let try this same technique on Example 7(b).

$$-5(4 - 3y) = -20 + 15y$$

Here's the thinking behind this technique.

1. First, multiply -5 times 4, getting -20.
2. Second, multiply -5 times $-3y$, getting $+15y$.

Note that this provides exactly the same solution found in Example 7(b).

Extending the Distributive Property

Suppose that we add an extra term inside the parentheses.

Distributive Property

If a, b, c, and d are any integers, then

$$a(b + c + d) = ab + ac + ad.$$

Note that we "distributed" the a times each term inside the parentheses. Indeed, if we added still another term inside the parentheses, we would "distribute" a times that term as well.

Example 8

Remove parentheses: $-5(2x - 3y + 8)$.

Solution

We will use the "quicker" technique, "distributing" -5 times each term in the parentheses mentally.

$$-5(2x - 3y + 8) = -10x + 15y - 40$$

Here is our thought process:

1. First, multiply −5 times 2x, getting −10x.
2. Second, multiply −5 times −3y, getting +15y.
3. Third, multiply −5 times +8, getting −40.

? Exercise

Remove parentheses: −3(4a − 5b + 7)

Answer

−12a + 15b − 21

✓ Example 9

Remove parentheses: −4(−3a + 4b − 5c + 12).

Solution

We will use the "quicker" technique, "distributing" −4 times each term in the parentheses mentally.

$$-4(-3a+4b-5c+12)=12a-16b+20c-48$$

Here is our thought process:

1. First, multiply −4 times −3a, getting 12a.
2. Second, multiply −4 times +4b, getting −16b.
3. Third, multiply −4 times −5c, getting +20c.
4. Fourth, multiply −4 times +12, getting −48.

? Exercise

Remove parentheses: −2(−2x + 4y − 5z − 11).

Answer

4x − 8y + 10z + 22

Distributing a Negative

It is helpful to recall that negating is equivalent to multiplying by −1.

Multiplying by −1

Let a be any integer, then

$$(-1)a = -a \text{ and } -a = (-1)a.$$

We can use this fact, combined with the distributive property, to negate a sum.

✓ Example 10

Remove parentheses: −(a + b).

Solution

Change the negative symbol into multiplying by −1, then distribute the −1.

$$\begin{aligned} -(a+b) &= (-1)(a+b) && \text{Negating is equivalent to multiplying by } -1. \\ &= -a-b && \text{Distribute the } -1. \end{aligned}$$

We chose to use the "quicker" technique of "distributing" the −1. Here is our thinking:

1. Multiply -1 times a, getting $-a$.
2. Multiply -1 times $+b$, getting $-b$.

Exercise

Remove parentheses: $-(4a - 3c)$

Answer

$-4a + 3c$

The results in Example 10 and Example 11 show us how to negate a sum: Simply negate each term of the sum. Positive terms change to negative, negative terms turn to positive.

Negating a Sum

To negate a sum, simply negate each term of the sum. For example, if a and b are integers, then

$$-(a + b) = -a - b \text{ and } -(a - b) = -a + b.$$

Example 12

Remove parentheses: $-(5 - 7u + 3t)$.

Solution

Simply negate each term in the parentheses.

\[$-(5 - 7u + 3t) = -5+7u - 3t$\nonumber \]

Exercise

Remove parentheses: $-(5 - 2x + 4y - 5z)$

Answer

$-5+2x - 4y + 5z$

Exercises

In Exercises 1-20, use the associative and commutative properties of multiplication to simplify the expression.

1. 10(−4x)
2. 7(−8x)
3. (−10x)(−3)
4. (−5x)(−8)
5. −5(3x)
6. 9(6x)
7. (−4x)10
8. (−10x)(−6)
9. (5x)3
10. (3x)3
11. (5x)10
12. (−2x)(−10)

13. −9(−7x)
14. −10(5x)
15. 6(2x)
16. 3(−10x)
17. −8(−9x)
18. 3(−3x)
19. (6x)7
20. (−8x)(−5)

In Exercises 21-44, simplify the expression.

21. 8(7x + 8)
22. −2(5x + 5)
23. 9(−2 + 10x)
24. −9(4 + 9x)
25. −(−2x + 10y − 6)
26. −(−6y + 9x − 7)
27. 2(10 + x)
28. 2(10 − 6x)
29. 3(3 + 4x)
30. 3(4 + 6x)
31. −(−5 − 7x + 2y)
32. −(4x − 8 − 7y)
33. 4(−6x + 7)
34. 6(4x + 9)
35. 4(8x − 9)
36. 10(−10x + 1)
37. −(4 − 2x − 10y)
38. −(−4x + 6 − 8y)
39. −(−5x +1+9y)
40. −(−10 − 5x − 4y)
41. −(6x + 2 − 10y)
42. −(6x + 4 − 10y)
43. −(−3y − 4+4x)
44. −(−7 − 10x + 7y)

Answers

1. −40x
3. 30x
5. −15x

7. −40x

9. 15x

11. 50x

13. 63x

15. 12x

17. 72x

19. 42x

21. 56x + 64

23. −18 + 90x

25. 2x − 10y + 6

27. 20 + 2x

29. 9 + 12x

31. 5+7x − 2y

33. −24x + 28

35. 32x − 36

37. −4+2x + 10y

39. 5x − 1 − 9y

41. −6x − 2 + 10y

43. 3y + 4 − 4x

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7.4: Combining Like Terms

We begin our discussion with the definition of a *term*.

Definition: Term

A **term** is a single number or variable, or it can be the product of a number (called its coefficient) and one or more variables (called its variable part). The terms in an algebraic expression are separated by addition symbols.

Example 1

Identify the terms in the algebraic expression

$$3x^2 + 5xy + 9y^2 + 12$$

For each term, identify its coefficient and variable part.

Solution

In tabular form, we list each term of the expression 3x^2 + 5xy + 9y^2 + 12, its coefficient, and its variable part.

Term	Coefficient	Variable Part
$3x^2$	3	x^2
$5xy$	5	xy
$9y^2$	9	y^2
12	12	None

Exercise

How many terms are in the algebraic expression $3x^2 + 2xy - 3y^2$?

Answer

3

Example 2

Identify the terms in the algebraic expression

$$a^3 - 3a^2b + 3ab^2 - b^3$$

For each term, identify its coefficient and variable part.

Solution

The first step is to write each difference as a sum, because the terms of an expression are defined above to be those items separated by addition symbols.

$$a^3 + (-3a^2b) + 3ab^2 + (-b^3)$$

In tabular form, we list each term of the expression $a^3 + (-3a^2b) + 3ab^2 + (-b^3)$, its coefficient, and its variable part.

Term	Coefficient	Variable Part
a^3	1	a^3
$-3a^2b$	-3	a^2b

Term	Coefficient	Variable Part
$3ab^2$	3	ab^2
$-b^3$	-1	b^3

Exercise

How many terms are in the algebraic expression $11 - a^2 - 2ab + 3b^2a$?

Answer

4

Like Terms

We define what is meant by “like terms” and “unlike terms.”

Definition

Like and Unlike Terms. The variable parts of two terms determine whether the terms are like terms or unlike terms.

Like Terms. Two terms are called like terms if they have identical variable parts, which means that the terms must contain the same variables raised to the same exponential powers.

Unlike Terms. Two terms are called unlike terms if their variable parts are different.

Example 3

Classify each of the following pairs as either *like terms* or *unlike terms*: (a) $3x$ and $-7x$, (b) $2y$ and $3y^2$, (c) $-3t$ and $5u$, and (d) $-4a^3$ and $3a^3$.

Solution

Like terms must have *identical* variable parts.

a. $3x$ and $-7x$ have identical variable parts. They are “like terms.”
b. $2y$ and $3y^2$ do not have identical variable parts (the exponents differ). They are “unlike terms.”
c. $-3t$ and $5u$ do not have identical variable parts (different variables). They are “unlike terms.”
d. $-4a^3$ and $3a^3$ have identical variable parts. They are “like terms.”

Exercise

Are $-3xy$ and $11xy$ *like* or *unlike* terms?

Answer

Like terms

Combining Like Terms

When using the distributive property, it makes no difference whether the multiplication is on the left or the right, one still distributes the multiplication times each term in the parentheses.

Distributive Property

If a, b, and c are integers, then

$$a(b + c) = ab + ac \text{ and } (b + c)a = ba + ca.$$

In either case, you distribute a times each term of the sum.

"Like terms" can be combined and simplified. The tool used for combining like terms is the distributive property. For example, consider the expression $3y + 7y$, composed of two "like terms" with a common variable part. We can use the distributive property and write

$$3y + 7y = (3 + 7)y$$

Note that we are using the distributive property in reverse, "factoring out" the common variable part of each term. Checking our work, note that if we redistribute the variable part y times each term in the parentheses, we are returned to the original expression $3y + 7y$.

Example 4

Use the distributive property to combine like terms (if possible) in each of the following expressions: (a) $-5x^2 - 9x^2$, (b) $-5ab + 7ab$, (c) $4y^3 - 7y^2$, and (d) $3xy^2 - 7xy^2$.

Solution

If the terms are "like terms," you can use the distributive property to "factor out" the common variable part.

a) Factor out the common variable part x^2.

$$\begin{aligned} -5x^2 - 9x^2 &= (-5 - 9)x^2 \quad \text{Use the distributive property.} \\ &= -14x^2 \quad \text{Simplify: } -5 - 9 = -5 + (-9) = -14. \end{aligned}$$

b) Factor out the common variable part ab.

$$\begin{aligned} -5ab + 7ab &= (-5 + 7)ab \quad \text{Use the distributive property.} \\ &= 2ab \quad \text{Simplify: } -5 + 7 = 2. \end{aligned}$$

c) The terms in the expression $4y^3 - 7y^2$ have different variable parts (the exponents are different). These are "unlike terms" and cannot be combined.

d) Factor out the common variable part xy^2.

$$\begin{aligned} 3xy^2 - 7xy^2 &= (3 - 7)xy^2 \quad \text{Use the distributive property.} \\ &= -4xy^2 \quad \text{Simplify: } 3 - 7 = 3 + (-7) = -4. \end{aligned}$$

Exercise

Simplify: $-8z - 11z$

Answer

$-19z$

Speeding Things Up a Bit

Once you've written out all the steps for combining like terms, like those shown in Example 4, you can speed things up a bit by following this rule:

Combining Like Terms

To combine like terms, simply add their coefficients and keep the common variable part.

Thus for example, when presented with the sum of two like terms, such as in $5x + 8x$, simply add the coefficients and repeat the common variable part; that is, $5x + 8x = 13x$.

Example 5

Combine like terms:

a. $-9y - 8y$,
b. $-3y^5 + 4y^5$ and
c. $-3u^2 + 2u^2$.

Solution

a) Add the coefficients and repeat the common variable part. Therefore,

$$-9y - 8y = -17y.$$

b) Add the coefficients and repeat the common variable part. Therefore,

$$-3y^5 + 4y^5 = 1y^5.$$

However, note that $1y^5 = y^5$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3y^5 + 4y^5 = y^5$.

c) Add the coefficients and repeat the common variable part. Therefore,

$$-3u^2 + 2u^2 = (-1)u^2.$$

However, note that $(-1)u^2 = -u^2$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3u^2 + 2u^2 = -u^2$.

Exercise

Combine: $-3x$, and $-4x^2$

Answer

$-7x^2$

Simplify

A frequently occurring instruction asks the reader to *simplify* an expression.

Simplify

The instruction simplify is a generic term that means "try to write the expression in its most compact form, using the fewest symbols possible."

One way you can accomplish this goal is by combining like terms when they are present.

Example 6

Simplify: $2x + 3y - 5x + 8y$.

Solution

Use the commutative property to reorder terms and the associative and distributive properties to regroup and combine like terms.

$$\begin{aligned} 2x + 3y - 5x + 8y &= (2x - 5x) + (3y + 8y) && \text{Reorder and regroup.} \\ &= -3x + 11y && \text{Combine like terms:} \\ & && 2x - 5x = -3x \text{ and } 3y + 8y = 11y. \end{aligned}$$

Alternate solution

Of course, you do not need to show the regrouping step. If you are more comfortable combining like terms in your head, you are free to present your work as follows:

$$2x+3y-5x+8y=-3x+11y.$$

Exercise

Simplify: −3*a* + 4*b* − 7*a* − 9*b*

Answer

−10*a* − 5*b*

Example 7

Simplify: −2*x* − 3 − (3*x* + 4).

Solution

First, distribute the negative sign.

$$-2x-3-(3x+4)=-2x-3-3x-4 \quad -(3x+4)=-3x-4.$$

Next, use the commutative property to reorder, then the associative property to regroup. Then combine like terms.

$$\begin{aligned} &=(-2x-3x)+(-3-4) && \text{Reorder and regroup.}\\ &=-5x+(-7) && \text{Combine like terms:}\\ & && -2x-3x=-5x.\\ &=-5x-7 && \text{Simplify:}\\ & && -5x+(-7)=-5x-7. \end{aligned}$$

Alternate solution

You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} -2x-3-(3x+4)&=-2x-3-3x-4 && \text{Distribute negative sign.}\\ &=-5x-7 && \text{Combine like terms.} \end{aligned}$$

Exercise

Simplify: −9*a* − 4 − (4*a* − 8)

Answer

−13*a* + 4

Example 8

Simplify: 2(5 − 3*x*) − 4(*x* + 3).

Solution

Use the distributive property to expand, then use the commutative and associative properties to group the like terms and combine them.

\[\begin{aligned} 2(5-3x)-4(x+3) = 10-6x-4x-12 ~ & \textcolor{red}{ \text{ Use the distributive property.}} \\ =(-6x-4x)+(10-12) ~ & \textcolor{red}{ \text{ Group like terms.}} \\ =-10x-2 ~ & \textcolor{red}{ \text{ Combine like terms: }} \\ ~ & \textcolor{red}{-6x-4x=-10x \text{ and}} \\ ~ & \textcolor{red}{10-12=-2.} \end{aligned}\nonumber \]

Alternate solution

You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} 2(5-3x)-4(x+3) &= 10-6x-4x-12 \quad \text{Distribute.} \\ &= -10x-2 \quad \text{Combine like terms.} \end{aligned}$$

Exercise

Simplify: $-2(3a-4)-2(5-a)$

Answer

$-4a-2$

Example 9

Simplify: $-8(3x^2y-9xy)-8(-7x^2-8xy)$

Solution

We will proceed a bit quicker with this solution, using the distributive property to expand, then combining like terms mentally.

$$\begin{aligned} -8(3x^2y-9xy)-8(-7x^2y-8xy) &= -24x^2y+72xy+56x^2y+64xy \\ &= 32x^2y+136xy \end{aligned}$$

Exercise

Simplify: $(a^2-2ab)-2(3ab+a^2)$

Answer

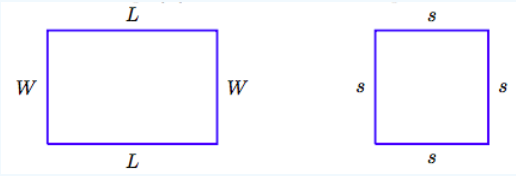
$-a^2-8ab$

Applications

We can simplify a number of useful formulas by combining like terms.

Example 10

Find the perimeter P of the (a) rectangle and (b) square pictured below. Simplify your answer as much as possible.



Solution

The perimeter of any polygonal figure is the sum of the lengths of its sides.

a) To find the perimeter P of the rectangle, sum its four sides.

$$P = L+W+L+W.$$

Combine like terms.

$$P = 2L+2W.$$

b) To find the perimeter P of the square, sum its four sides.

$$P = s + s + s + s.$$

Combine like terms.

$$P = 4s.$$

Exercise

A regular hexagon has six equal sides, each with length *x*. Find its perimeter in terms of *x*.

Answer

P = 6*x*

Sometimes it is useful to replace a variable with an expression containing another variable.

Example 11

The length of a rectangle is three feet longer than twice its width. Find the perimeter *P* of the rectangle in terms of its width alone.

Solution

From the previous problem, the perimeter of the rectangle is given by

$$P = 2L + 2W,$$

where *L* and *W* are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the width alone. However, we're also given the fact that the length is three feet longer than twice the width.

$$\begin{array}{rcrcr} cyanLength & \text{is} & \text{Three Feet} & \text{longer than} & \text{Twice the Width} \\ L & = & 3 & + & 2W \end{array}$$

Because *L* = 3+2*W*, we can replace *L* with 3+2*W* in the perimeter equation 3.1.

$$P = 2L + 2W$$

$$P = 2(3 + 2W) + 2W$$

Use the distributive property, then combine like terms.

$$P = 6 + 4W + 2W$$

$$P = 6 + 6W.$$

This last equation gives the perimeter *P* in terms of the width *W* alone.

Exercise

The length *L* of a rectangle is 5 meters longer than twice its width *W*. Find the perimeter *P* of the rectangle in terms of its width *W*.

Answer

Answer: *P* = 6*W* + 10

Example 12

The width of a rectangle is two feet less than its length. Find the perimeter P of the rectangle in terms of its length alone.

Solution

Again, the perimeter of a rectangle is given by the equation

$$P = 2L + 2W,$$

where L and W are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the length alone.

However, we're also given the fact that the width is two feet less than the length.

Width	is	Length	minus	Two feet
W	$=$	L	$-$	2

Because $W = L-2$, we can replace W with $L-2$ in the perimeter equation 3.2.

$$P = 2L + 2W$$

$$P = 2L + 2(L - 2)$$

Use the distributive property, then combine like terms.

$$P = 2L + 2L - 4$$

$$P = 4L - 4.$$

This last equation gives the perimeter P in terms of the length L alone.

Exercise

The width W of a rectangle is 5 feet less than twice its width L. Find the perimeter P of the rectangle in terms of its length L.

Answer

$P = 6L - 10$

Exercises

In Exercises 1-16, combine like terms by first using the distributive property to factor out the common variable part, and then simplifying.

1. $17xy^2 + 18xy^2 + 20xy^2$
2. $13xy - 3xy + xy$
3. $-8xy^2 - 3xy^2 - 10xy^2$
4. $-12xy - 2xy + 10xy$
5. $4xy - 20xy$
6. $-7y^3 + 15y^3$
7. $12r - 12r$
8. $16s - 5s$
9. $-11x - 13x + 8x$
10. $-9r - 10r + 3r$
11. $-5q + 7q$
12. $17n + 15n$

13. $r - 13r - 7r$

14. $19m + m + 15m$

15. $3x^3 - 18x^3$

16. $13x^2y + 2x^2y$

In Exercises 17-32, combine like terms by first rearranging the terms, then using the distributive property to factor out the common variable part, and then simplifying.

17. $-8 + 17n + 10 + 8n$

18. $11 + 16s - 14 - 6s$

19. $-2x^3 - 19x^2y - 15x^2y + 11x^3$

20. $-9x^2y - 10y^3 - 10y^3 + 17x^2y$

21. $-14xy - 2x^3 - 2x^3 - 4xy$

22. $-4x^3 + 12xy + 4xy - 12x^3$

23. $-13 + 16m + m + 16$

24. $9 - 11x - 8x + 15$

25. $-14x^2y - 2xy^2 + 8x^2y + 18xy^2$

26. $-19y^2 + 18y^3 - 5y^2 - 17y^3$

27. $-14x^3 + 16xy + 5x^3 + 8xy$

28. $-16xy + 16y^2 + 7xy + 17y^2$

29. $9n + 10 + 7 + 15n$

30. $-12r + 5 + 17 + 17r$

31. $3y + 1 + 6y + 3$

32. $19p + 6 + 8p + 13$

In Exercises 33-56, simplify the expression by first using the distributive property to expand the expression, and then rearranging and combining like terms mentally.

33. $-4(9x^2y + 8) + 6(10x^2y - 6)$

34. $-4(-4xy + 5y^3) + 6(-5xy - 9y^3)$

35. $3(-4x^2 + 10y^2) + 10(4y^2 - x^2)$

36. $-7(-7x^3 + 6x^2) - 7(-10x^2 - 7x^3)$

37. $-s + 7 - (-1 - 3s)$

38. $10y - 6 - (-10 - 10y)$

39. $-10q - 10 - (-3q + 5)$

40. $-2n + 10 - (7n - 1)$

41. $7(8y + 7) - 6(8 - 7y)$

42. $-6(-5n - 4) - 9(3 + 4n)$

43. $7(10x^2 - 8xy^2) - 7(9xy^2 + 9x^2)$

44. $10(8x^2y - 10xy^2) + 3(8xy^2 + 2x^2y)$

45. $-2(6 + 4n) + 4(-n - 7)$

46. $-6(-2 - 6m) + 5(-9m + 7)$

47. $8 - (4 + 8y)$

48. $-1 - (8 + s)$

49. $-8(-n + 4) - 10(-4n + 3)$

50. $3(8r - 7) - 3(2r - 2)$

51. $-5 - (10p + 5)$

52. $-1 - (2p - 8)$

53. $7(1 + 7r) + 2(4 - 5r)$

54. $(5 - s) + 10(9 + 5s)$

55. $-2(-5 - 8x^2) - 6(6)$

56. $8(10y^2 + 3x^3) - 5(-7y^2 - 7x^3)$

---

57. The length L of a rectangle is 2 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

58. The length L of a rectangle is 7 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

59. The width W of a rectangle is 8 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

60. The width W of a rectangle is 9 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

61. The length L of a rectangle is 9 feet shorter than 4 times its width W. Find the perimeter of the rectangle in terms of its width alone.

62. The length L of a rectangle is 2 feet shorter than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

Answers

1. $55xy^2$

3. $-21xy^2$

5. $-16xy$

7. 0

9. $-16x$

11. $2q$

13. $-19r$

15. $-15x^3$

17. $2 + 25n$

19. $9x^3 - 34x^2y$

21. $-18xy - 4x^3$

23. $3 + 17m$

25. $-6x^2y + 16xy^2$

27. $-9x^3 + 24xy$

29. $24n + 17$

31. $9y + 4$

33. $24x^2y - 68$

35. $-22x^2 + 70y^2$

37. $2s + 8$

39. $-7q - 15$

41. $98y + 1$

43. $7x^2 - 119xy^2$

45. $-40 - 12n$

47. $4 - 8y$

49. $48n - 62$

51. $-10 - 10p$

53. $15 + 39r$

55. $-26 + 16x^2$

57. $4 + 14W$

59. $4L - 16$

61. $10W - 18$

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7.5: Solving Equations

Recall (see Section 1.6) that a variable is a symbol (usually a letter) that stands for a value that varies. If a variable in an equation is replaced by a number and a true statement results, then that number is called a solution of the equation.

Example 1

Is −6 a solution of the equation 2x +5= −7?

Solution

Substitute −6 for x in the equation.

$$\begin{aligned} 2x+5 &= 7 && \text{Original equation.} \\ 2(-6)+5 &= -7 && \text{Substitute } -6 \text{ for } x. \\ -12+5 &= -7 && \text{On the left, multiply first.} \\ -7 &= -7 && \text{On the left, add.} \end{aligned}$$

Because the last statement is a true statement, −6 is a solution of the equation.

Exercise

Is −4 a solution of 8 − 2x = 5?

Answer

No

Adding or Subtracting the Same Amount

Two equations having the same set of solutions are *equivalent*. For example, 2x+5 = −7 and x = −6 have the same solutions. Therefore, they are equivalent equations. Certain algebraic operations produce equivalent equations.

Producing Equivalent Equations

Adding the Same Quantity to Both Sides of an Equation. If we start with the equation

$$a = b,$$

then adding c to both sides of the equation produces the equivalent equation

$$a + c = b + c.$$

Subtracting the Same Quantity from Both Sides of an Equation. If we start with the equation

$$a = b,$$

then subtracting c from both sides of the equation produces the equivalent equation

$$a - c = b - c.$$

That is, adding or subtracting the same amount from both sides of an equation will not change the solutions of the equation.

Example 2

Solve for x: x + 3= −7.

Solution

To undo the effect of adding 3, subtract 3 from both sides of the equation.

$$\begin{aligned} x+3 &= -7 && \text{Original equation.} \\ x+3-3 &= -7-3 && \text{Subtract 3 from both sides.} \\ x &= -7+(-3) && \text{Simplify the left hand side. On the right,} \\ & && \text{express subtractions as adding the opposite.} \\ x &= -10 \end{aligned}$$

To check the solution, substitute −10 for *x* in the original equation and simplify.

$$\begin{aligned} x+3 &= -7 && \text{Original equation.} \\ -10+3 &= -7 && \text{Substitute } -10 \text{ for } x. \\ =7 &= -7 && \text{Simplify both sides.} \end{aligned}$$

Since the last line of the check is a true statement, this confirms that −10 is a solution.

? Exercise

Solve for *x*: *x* + 9 = -11.

Answer

x = −20

✔ Example 3

Solve for *x*: *x* − 8 = −11.

Solution

To undo the effect of subtracting 8, add 8 to both sides of the equation.

$$\begin{aligned} x-8 &= -11 && \text{Original equation.} \\ x-8+8 &= -11+8 && \text{Add 8 to both sides.} \\ x &= -3 && \text{Simplify both equations.} \end{aligned}$$

To check the solution, substitute −3 for *x* in the original equation and simplify.

$$\begin{aligned} x-8 &= -11 && \text{Original equation.} \\ -3-8 &= -11 && \text{Substitute } -3 \text{ for } x. \\ -11 &= -11 && \text{Simplify both sides.} \end{aligned}$$

Since the last line of the check is a true statement, this confirms that −3 is a solution.

? Exercise

Solve for *x*: *x* − 2 = −7

Answer

x = −5

Sometimes a bit of simplification is in order before you start the solution process.

✔ Example 4

Solve for *y*: −8+2= *y* − 11(−4).

Solution

First, simplify both sides of the equation.

$$\begin{aligned} -8+2 &= y-11(-4) && \text{Original equation.} \\ -6 &= y-(-44) && \text{Simplify. On the left, } -8+2=-6. \text{ On the right, } 11(-4)=-44. \\ -6 &= y+44-44 && \text{Subtract 44 from both sides of the equation.} \\ -6+(-44) &= y && \text{Express subtraction as addition. Simplify on the right.} \\ -50 &= y \end{aligned}$$

To check the solution, substitute −50 for *y* in the original equation and simplify.

$$\begin{aligned} -8+2 &= y-11(-4) && \text{Original equation.} \\ -8+2 &= -50-11(-4) && \text{Substitute } -50 \text{ for } y. \\ -6 &= -50-(-44) && \text{Express subtraction on the right as addition.} \\ -6 &= -6 && \text{On the right, add: } -50+44=-6. \end{aligned}$$

Since the last line of the check is a true statement, this confirms that −50 is a solution.

Exercise

Solve for *y*: *y* + 2(−4) = −8+6

Answer

y = 6

Multiplying or Dividing by the Same Amount

Adding and subtracting is not the only way to produce an equivalent equation.

Producing Equivalent Equations

Multiplying Both Sides of an Equation by the Same Quantity. If we start with the equation

$$a = b,$$

then multiplying both sides of the equation by *c* produces the equivalent equation

$$a \cdot c = b \cdot c, \text{ or equivalently, } ac = bc,$$

provided c ≠ 0.

Dividing Both Sides of an Equation by the Same Quantity. If we start with the equation

$$a = b,$$

then dividing both sides of the equation by c produces the equivalent equation

$$\frac{a}{c} = \frac{b}{c},$$

provided c ≠ 0.

That is, multiplying or dividing both sides of an equation by the same amount will not change the solutions of the equation.

Example 5

Solve for *x*: −3*x* = 30.

Solution

To undo the effect of multiplying by −3, divide both sides of the equation by −3.

$$
\begin{aligned}
-3x &= 30 && \text{Original equation.} \\
\frac{-3x}{-3} &= \frac{30}{-3} && \text{Divide both sides by } -3. \\
x &= -10 && \text{On the left, } -3 \text{ times } x, \text{ divided by } -3 \text{ is } x. \\
& && \text{On the right, } 30/(-3) = -10.
\end{aligned}
$$

To check the solution, substitute −10 for x in the original equation and simplify.

$$
\begin{aligned}
-3x &= 30 && \text{Original equation.} \\
-3(-10) &= 30 && \text{Substitute } -10 \text{ for } x. \\
30 - 30 & && \text{Simplify.}
\end{aligned}
$$

Because the last line of the check is a true statement, this confirms that −10 is a solution.

? Exercise

Solve for z: −4z = −28

Answer

z = 7

✔ Example 6

Solve for x: $\frac{x}{-2} = -20$.

Solution

To undo the effect of dividing by −2, multiply both sides of the equation by −2.

$$
\begin{aligned}
\frac{x}{-2} &= -20 && \text{Original equation.} \\
-2\left(\frac{x}{-2}\right) &= -2(-20) && \text{Multiply both sides by } -2. \\
x &= 40 && \text{On the left, } x \text{ divided by } -2, \text{ multiplied by } -2, \\
& && \text{the result is } x. \text{ On the right, } -2(-20) = 40.
\end{aligned}
$$

To check the solution, substitute 40 for x in the original equation and simplify.

$$
\begin{aligned}
\frac{x}{-2} &= -20 && \text{Original equation.} \\
\frac{40}{-2} &= -20 && \text{Substitute 40 for } x. \\
-20 &= -20 && \text{Simplify both sides.}
\end{aligned}
$$

Because the last line of the check is a true statement, this confirms that 40 is a solution.

Combining Operations

Recall the “Wrap” and “Unwrap” discussion from Section 1.6. To wrap a present we: (1) put the gift paper on, (2) put the tape on, and (3) put the decorative bow on. To unwrap the gift, we must “undo” each of these steps in inverse order. Hence, to unwrap the gift we: (1) take off the decorative bow, (2) take off the tape, and (3) take off the gift paper.

Now, imagine a machine that takes its input, then: (1) multiplies the input by 2, and (2) adds 3 to the result. This machine is pictured on the left in Figure 2.16.

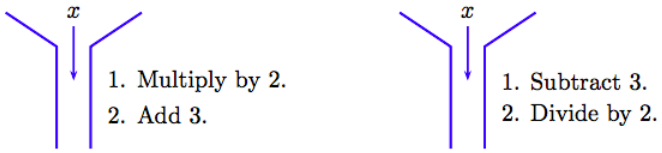

Figure 2.16: The second machine "unwraps" the first machine.

To "unwrap" the effect of the machine on the left, we need a machine that will "undo" each of the steps of the first machine, but in inverse order. The "unwrapping" machine is pictured on the right in Figure 2.16. It will first subtract three from its input, then divide the result by 2. Note that each of these operations "undoes" the corresponding operation of the first machine, but in inverse order.

For example, drop the integer 7 into the first machine on the left in Figure 2.16. First, we double 7, then add 3 to the result. The result is 2(7) + 3 = 17.

Now, to "unwrap" this result, we drop 17 into the second machine. We first subtract 3, then divide by 2. The result is (17 − 3)/2 = 7, the original integer input into the first machine.

Now, consider the equation

$$2x+3=7.$$

On the left, order of operations demands that we first multiply x by 2, then add 3. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will (1) subtract three from both sides of the equation, then (2) divide both sides of the resulting equation by 2.

$$\begin{aligned} 2x+3-3 &= 7-3 && \text{Subtract 3 from both sides.}\\ 2x &= 4 && \text{Simplify both sides.}\\ \frac{2x}{2} &= \frac{4}{2} && \text{Divide both sides by 2.}\\ x &= 2 && \text{Simplify both sides.} \end{aligned}$$

Readers should check this solution in the original equation.

Example 7

Solve for x: $\frac{x}{4}-3=-7$.

Solution

On the left, order of operations demands that we first divide x by 4, then subtract 3. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will (1) add 3 to both sides of the equation, then (2) multiply both sides of the resulting equation by 4.

$$\begin{aligned} \frac{x}{4}-3 &= -7 && \text{Original equation.}\\ \frac{x}{4}-3+3 &= -7+3 && \text{Add 3 to both sides.}\\ \frac{x}{4} &= -4 && \text{Simplify both sides.}\\ 4\left(\frac{x}{4}\right) &= 4(-4) && \text{Multiply both sides by 4.}\\ x &= -16 && \text{Simplify both sides.} \end{aligned}$$

Check

Substitute −16 for x in the original equation.

$$\begin{aligned} \frac{x}{4}-3 &= 7 && \text{Original equation.} \\ \frac{-16}{4}-3 &= -7 && \text{Substitute } -16 \text{ for } x. \\ -4-3 &= -7 && \text{Divide first: } -16/4 = -4. \\ -7 &= -7 && \text{Subtract: } -4-3 = -7. \end{aligned}$$

Because the last line of the check is a true statement, −16 is a solution of the original equation.

? Exercise

Solve for *x*:

$$\frac{x}{2}+6=4$$

Answer

x = -4

✔ Example 8

Solve for *t*: 0=8 − 2*t*.

Solution

On the right, order of operations demands that we first multiply t by −2, then add 8. To solve this equation for t, we must "undo" each of these operations in inverse order. Thus, we will (1) subtract 8 from both sides of the equation, then (2) divide both sides of the resulting equation by −2.

$$\begin{aligned} 0 &= 8-2t && \text{Original equation.} \\ 0-8 &= 8-2t-8 && \text{Subtract 8 from both sides.} \\ -8 &= -2t && \text{Simplify both sides.} \\ \frac{-8}{-2} &= \frac{-2t}{-2} && \text{Divide both sides by 2.} \\ 4 &= t && \text{Simplify both sides.} \end{aligned}$$

Check

Substitute 4 for t in the original equation.

$$\begin{aligned} 0 &= 8-2t && \text{Original equation.} \\ 0 &= 8-2(4) && \text{Substitute 4 for } t. \\ 0 &= 8-8 && \text{Multiply first: } 2(4)=8. \\ 0 &= 0 && \text{Subtract: } 8-8=0. \end{aligned}$$

Because the last line in the check is a true statement, 4 is a solution of the original equation.

? Exercise

Solve for *r*: 0 = 9 + 3*r*

Answer

r = -3

Example 9

Solve for p: $-12+3=-8+4+\frac{p}{-3}$.

Solution

Always simplify when possible.

$$\begin{aligned} -12+3 &= -8+4+\frac{p}{-3} && \text{Origianl equation.} \\ -9 &= -4+\frac{p}{-3} && \text{Simplify both sides.} \end{aligned}$$

On the right, order of operations demands that we first divide p by -3, then add -4. To solve this equation for p, we must "undo" each of these operations in inverse order. Thus, we will (1) add a positive 4 to both sides of the equation, then (2) multiply both sides of the resulting equation by -3.

$$\begin{aligned} -9+-4 &= -4+\frac{p}{-3}+4 && \text{Add 4 to both sides.} \\ -5 &= \frac{p}{-3} && \text{Simplify both sides.} \\ -3(-5) &= -3\left(\frac{p}{-3}\right) && \text{Multiply both sides by } -3. \\ 15 &= p && \text{Simplify both sides.} \end{aligned}$$

Check

Substitute 15 for p in the original equation.

$$\begin{aligned} -12+3 == 8+4+\frac{p}{-3} && \text{Original equation.} \\ -12+3 = -8+4+\frac{15}{-3} && \text{Substitute 15 for } p. \\ -9 = -8+4+(-5) && \text{On the left, add: } -12+3=-9. \text{ On the right, divide: } 15/(-3)=-5. \\ -9 = -4+(-5) && \text{On the right, } -8+4=-4. \\ -9 = -9 && \text{On the right, add: } -4+(-5)=-9. \end{aligned}$$

Because the last line in the check is a true statement, 15 is a solution of the original equation.

Exercise

Solve for q:

$$\frac{q}{-2}-9=-8+3$$

Answer

$q = -8$

Applications

Let's look at some applications of equations involving integers. First, we remind readers that a solution of a word problem must incorporate each of the following steps.

Requirements for Word Problem Solutions

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
    1. Statements such as "Let P represent the perimeter of the rectangle."

2. Labeling unknown values with variables in a table.
3. Labeling unknown quantities in a sketch or diagram.

2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.
4. **Answer the Question.** This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.
5. **Look Back.** It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Example 10

A student's bank account is overdrawn. After making a his account, Allen finds that he is overdrawn by $15. What was his account balance before his withdrawal? deposit of $120, he finds that his account is still overdrawn by an amount of $75. What was his balance before he made his deposit?

Solution

In our solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. In this case, the unknown is the original balance in the student's account. Let B represent this original balance.

2. *Set up an Equation.* A positive integer represents a healthy balance, while a negative number represents an account that is overdrawn. After the student's deposit, the account is still overdrawn by $75. We will say that this balance is −$75. Thus,

Original Balance	plus	Student Deposit	equals	Current Balance
B	$+$	$\$120$	$=$	$-\$75$

3. *Solve the Equation.* To "undo" the addition, subtract 120 from both sides of the equation.

$$\begin{aligned} B+120 &= -75 && \text{Original equation.} \\ B+120-120 &= -75-120 && \text{Subtract 120 from both sides.} \\ B &= -195 && \text{Simplify both sides.} \end{aligned}$$

4. *Answer the Question.* The original balance was overdrawn to the tune of $195.

5. *Look Back.* If the original balance was overdrawn by $195, then we let −$195 represent this balance. The student makes a deposit of $120. Add this to the original balance to get −$195 + $120 = −$75, the correct current balance.

Exercise

After withdrawing $125 from his account, Allen finds that he is overdrawn by $15. What was his account balance before his withdrawal?

Answer

$110

Example 11

Three more than twice a certain number is −11. Find the unknown number.

Solution

In our solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary.* Let x represent the unknown number. 2. Set up an Equation. “Three more than twice a certain number” becomes:

Three	more than	Twice a Certain Number	is	-11
3	$+$	$2x$	$=$	11

3. *Solve the Equation.* On the left, order of operations requires that we first multiply x by 2, then add 3. To solve this equation for x, we must “undo” each of these operations in inverse order. Thus, we will (1) subtract 3 from both sides of the equation, then (2) divide both sides of the resulting equation by 2.

$$\begin{aligned} 3+2x &= -11 && \text{Original equation.} \\ 3+2x-3 &= -11-3 && \text{Subtract 3 from both sides.} \\ 2x &= -14 && \text{Simplify both sides.} \\ \frac{2x}{2} &= \frac{-14}{2} && \text{Divide both sides by 2.} \\ x &= -7 && \text{Simplify both sides.} \end{aligned}$$

4. *Answer the Question.* The unknown number is −7.

5. *Look Back.* Does the answer satisfy the problem constraints? Three more than twice −7 is three more than −14, or −11. So the solution is correct.

? Exercise 7.5.1

Five less than twice a certain number is −7. Find the unknown number.

Answer

−1

Exercises

1. Is −11 a solution of 2x +3= −19?
2. Is −8 a solution of 2x +7= −9?
3. Is 6 a solution of 3x + 1 = 19?
4. Is −6 a solution of 2x +7= −5?
5. Is 12 a solution of 4x +5= −8?
6. Is −8 a solution of −3x + 8 = 18?
7. Is 15 a solution of 2x +6= −9?
8. Is 3 a solution of −4x +1= −20?
9. Is −15 a solution of −3x +6= −17?
10. Is −18 a solution of −3x +9= −9?
11. Is −6 a solution of −2x + 3 = 15?
12. Is 7 a solution of −3x +5= −16?

In Exercises 13-28, solve the given equation for x.

13. x − 13 = 11
14. x − 6 = 12
15. x − 3=6

16. $x - 3 = -19$
17. $x + 10 = 17$
18. $x + 3 = 9$
19. $x - 6 = 1$
20. $x - 10 = 12$
21. $x - 15 = -12$
22. $x - 2 = 13$
23. $x + 11 = -19$
24. $x + 3 = 17$
25. $x + 2 = 1$
26. $x + 2 = -20$
27. $x + 5 = -5$
28. $x + 14 = -15$

In Exercises 29-44, solve the given equation for x.

29. $-x = -20$
30. $5x = -35$
31. $\frac{x}{-7} = 10$
32. $\frac{x}{-6} = -20$
33. $\frac{x}{-10} = 12$
34. $\frac{x}{2} = 11$
35. $\frac{x}{9} = -16$
36. $\frac{x}{-3} = -7$
37. $-10x = 20$
38. $-17x = -85$
39. $14x = 84$
40. $-10x = -40$
41. $-2x = 28$
42. $-14x = 42$
43. $\frac{x}{-10} = 15$
44. $\frac{x}{-8} = -1$

In Exercises 45-68, solve the given equation for x.

45. $-4x - 4 = 16$
46. $-6x - 14 = 4$
47. $4x - 4 = 76$
48. $-5x - 15 = 45$
49. $5x - 14 = -79$
50. $15x - 2 = 43$

51. $-10x - 16 = 24$

52. $2x - 7 = -11$

53. $9x + 5 = -85$

54. $8x + 8 = -16$

55. $7x + 15 = -55$

56. $2x + 2 = -38$

57. $-x + 8 = 13$

58. $-5x + 20 = -50$

59. $12x - 15 = -3$

60. $-19x - 17 = -36$

61. $4x - 12 = -56$

62. $7x - 16 = 40$

63. $19x + 18 = 113$

64. $-6x + 20 = -64$

65. $-14x + 12 = -2$

66. $-9x + 5 = 104$

67. $14x + 16 = 44$

68. $-14x + 10 = -60$

---

69. Two less than eight times an unknown number is −74. Find the unknown number.

70. Six less than three times an unknown number is 21. Find the unknown number.

71. Eight more than two times an unknown number is 0. Find the unknown number.

72. Five more than eight times an unknown number is −35. Find the unknown number.

73. The number −6 is 2 more than an unknown number. Find the unknown number.

74. The number −4 is 7 more than an unknown number. Find the unknown number.

75. Three more than eight times an unknown number is −29. Find the unknown number.

76. Four more than nine times an unknown number is 85. Find the unknown number.

77. Alan's scores on his first three exams are 79, 61, and 54. What must Alan score on his next exam to average 71 for all four exams?

78. Benny's scores on his first three exams are 54, 68, and 54. What must Benny score on his next exam to average 61 for all four exams?

79. The quotient of two integers is 5. One of the integers is −2. Find the other integer.

80. The quotient of two integers is 3. One of the integers is −7. Find the other integer.

81. The quotient of two integers is 9. One of the integers is −8. Find the other integer.

82. The quotient of two integers is 9. One of the integers is −2. Find the other integer.

83. The number −5 is 8 more than an unknown number. Find the unknown number.

84. The number −6 is 8 more than an unknown number. Find the unknown number.

85. A student's bank account is overdrawn. After making a deposit of $260, he finds that his account is still overdrawn by an amount of $70. What was his balance before he made his deposit?

86. A student’s bank account is overdrawn. After making a deposit of $300, he finds that his account is still overdrawn by an amount of $70. What was his balance before he made his deposit?

87. A student’s bank account is overdrawn. After making a deposit of $360, he finds that his account is still overdrawn by an amount of $90. What was his balance before he made his deposit?

88. A student’s bank account is overdrawn. After making a deposit of $260, he finds that his account is still overdrawn by an amount of $50. What was his balance before he made his deposit?

89. The number −10 is −5 times larger than an unknown number. Find the unknown number.

90. The number −3 is −3 times larger than an unknown number. Find the unknown number. 91. The number −15 is −5 times larger than an unknown number. Find the unknown number.

92. The number −16 is 4 times larger than an unknown number. Find the unknown number.

93. Two less than nine times an unknown number is 7. Find the unknown number.

94. Four less than two times an unknown number is 8. Find the unknown number.

95. Mark’s scores on his first three exams are 79, 84, and 71. What must Mark score on his next exam to average 74 for all four exams?

96. Alan’s scores on his first three exams are 85, 90, and 61. What must Alan score on his next exam to average 77 for all four exams?

Answers

1. Yes

3. Yes

5. No

7. No

9. No

11. Yes

13. 24

15. 9

17. 7

19. 7

21. 3

23. −30

25. −1

27. −10

29. 20

31. −70

33. −120

35. −144

37. −2

39. 6

41. −14

43. −150

45. −5
47. 20
49. −13
51. −4
53. −10
55. −10
57. −5
59. 1
61. −11
63. 5
65. 1
67. 2
69. −9
71. −4
73. −8
75. −4
77. 90
79. −10
81. −72
83. −13
85. −$330
87. −$450
89. 2
91. 3
93. 1
95. 62

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7.6: More on Solving Equations

We return to solving equations involving integers, only this time the equations will be a bit more advanced, requiring the use of the distributive property and skill at combining like terms. Let's begin.

Example 1

Solve for x: $7x - 11x = 12$.

Solution

Combine like terms.

$$\begin{aligned} 7x - 11x &= 12 \quad \text{Original equation.} \\ = 4x &= 12 \quad \text{Combine like terms: } 7x - 11x = -4x. \end{aligned}$$

To undo the effect of multiplying by −4, divide both sides of the last equation by −4.

$$\begin{aligned} \frac{-4x}{-4} &= \frac{12}{-4} \quad \text{Divide both sides by } -4. \\ x &= -3 \quad \text{Simplify: } 12/(-4) = -3. \end{aligned}$$

Check

Substitute −3 for x in the original equation.

$$\begin{aligned} 7x - 11x &= 12 \quad \text{Original equation.} \\ 7(-3) - 11(-3) &= 12 \quad \text{Substitute } -3 \text{ for } x. \\ -21 + 33 &= 12 \quad \text{On the left, multiply first.} \\ 12 &= 12 \quad \text{On the left, add.} \end{aligned}$$

Because the last line of the check is a true statement, −3 is a solution of the original equation.

Exercise

Solve for x: $-6x - 5x = 22$.

Answer

$x = -2$

Example 2

Solve for x: $12 = 5x - (4 + x)$.

Solution

To take the negative of a sum, negate each term in the sum (change each term to its opposite). Thus, $-(4 + x) = -4 - x$.

$$\begin{aligned} 12 &= 5x - (4 + x) \quad \text{Original equation.} \\ 12 &= 5x - 4 - x \quad -(4 + x) = -4 - x. \\ 12 &= 4x - 4 \quad \text{Combine like terms: } 5x - x = 4x. \end{aligned}$$

To undo the effect of subtracting 4, add 4 to both sides of the last equation.

$$\begin{aligned} 12 + 4 &= 4x - 4 + 4 \quad \text{Add 4 to both sides.} \\ 16 &= 4x \quad \text{Simplify both sides.} \end{aligned}$$

To undo the effect of multiplying by 4, divide both sides of the last equation by 4.

\[\begin{aligned} \frac{16}{4} = \frac{4x}{4} ~ & \textcolor{red}{ \text{ Divide both sides by 4.}} \\ 4 = x ~ & \textcolor{red}{ \text{ Simplify: 16/4=4.} } \end{aligned}]

Check

Substitute 4 for x in the original equation.

$$\begin{aligned} 12 &= 5x - (4 + x) \quad \text{Original equation.} \\ 12 &= 5(4) - (4 + 4) \quad \text{Substitute 4 for } x. \\ 12 &= 20 - 8 \quad \text{On the right, } 5(4) = 20 \text{ and evaluate parentheses: } 4 + 4 = 8. \\ 12 &= 12 \quad \text{Simplify.} \end{aligned}$$

Because the last line of the check is a true statement, 4 is a solution of the original equation.

Exercise

Solve for x: $11 = 3x - (1 - x)$

Answer

$x = 3$

Variables on Both Sides

Variables can occur on both sides of the equation.

Goal

Isolate the terms containing the variable you are solving for on one side of the equation.

Example 3

Solve for x: $5x = 3x - 18$.

Solution

To isolate the variables on one side of the equation, subtract $3x$ from both sides of the equation and simplify.

$$\begin{aligned} 5x &= 3x - 18 && \text{Original equation.} \\ 5x - 3x &= 3x - 18 - 3x && \text{Subtract } 3x \text{ from both sides.} \\ 2x &= -18 && \text{Combine like terms: } 5x - 3x = 2x \\ & && \text{and } 3x - 3x = 0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. To undo the effect of multiplying by 2, divide both sides of the last equation by 2.

$$\begin{aligned} \frac{2x}{2} &= \frac{-18}{2} && \text{Divide both sides by 2.} \\ x &= -9 && \text{Simplify: } -18/2 == -9. \end{aligned}$$

Check

Substitute -9 for x in the original equation.

$$\begin{aligned} 5x &= 3x - 18 && \text{Original equation.} \\ 5(-9) &= 3(-9) - 18 && \text{Substitute } -9 \text{ for } x. \\ -45 &= -27 - 18 && \text{Multiply first on both sides.} \\ -45 &= -45 && \text{Subtract on the right: } -27 - 18 = -45. \end{aligned}$$

Because the last line of the check is a true statement, -9 is a solution of the original equation.

Exercise

Solve for x: $4x - 3 = x$

Answer

$x = 1$

Example 4

Solve for x: $5x = 3 + 6x$.

Solution

To isolate the variables on one side of the equation, subtract $6x$ from both sides of the equation and simplify.

$$\begin{aligned} 5x &= 3 + 6x && \text{Original equation.} \\ 5x - 6x &= 3 + 6x - 6x && \text{Subtract } 6x \text{ from both sides.} \\ -x &= 3 && \text{Combine like terms: } 5x - 6x = -x \\ & && \text{and } 6x - 6x = 0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation.

There are a couple of ways we can finish this solution. Remember, $-x$ is the same as $(-1)x$, so we could undo the effects of multiplying by -1 by dividing both sides of the equation by -1. Multiplying both sides of the equation by -1 will work equally well. But perhaps the easiest way to proceed is to simply negate both sides of the equation.

$$\begin{aligned} -(-x) &= -3 && \text{Negate both sides.} \\ x &= -3 && \text{Simplify: } -(-x) = x. \end{aligned}$$

Check

Substitute -3 for x in the original equation.

$$\begin{aligned} 5x &= 3 + 6x && \text{Original equation.} \\ 5(-3) &= 3 + 6(-3) && \text{Substitute } -3 \text{ for } x. \\ -15 &= 3 - 18 && \text{Multiply first on both sides.} \\ -15 &= -15 && \text{Subtract on the right: } 3 - 8 = -15. \end{aligned}$$

Because the last line of the check is a true statement, -3 is a solution of the original equation.

Exercise

Solve for x: $7x = 18 + 9x$

Answer

$x = -9$

Dealing with −x.

If your equation has the form

$$-x = c,$$

where c is some integer, note that this is equivalent to the equation $(-1)x = c$. Therefore, dividing both sides by -1 will produce a solution for x. Multiplying both sides by -1 works equally well. However, perhaps the easiest thing to do is negate each side, producing

$-(-x) = -c$, which is equivalent to $x = -c$.

Example 5

Solve for x: 6x − 5 = 12x + 19.

Solution

To isolate the variables on one side of the equation, subtract 12x from both sides of the equation and simplify.

$$\begin{aligned} 6x-5 &= 12x+19 && \text{Original equation.} \\ 6x-5-12x &= 12x+19-12x && \text{Subtract } 12x \text{ from both sides.} \\ -6x-5 &= 19 && \text{Combine like terms: } 6x-12x=-6x \\ & && \text{and } 12x-12x=0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. Next, to "undo" subtracting 5, add 5 to both sides of the equation.

$$\begin{aligned} -6x-5+5 &= 19+5 && \text{Add 5 to both sides.} \\ -6x &= 24 && \text{Simplify: } -5+5=0 \text{ and } 19+5=24. \end{aligned}$$

Finally, to "undo" multiplying by −6, divide both sides of the equation by −6.

$$\begin{aligned} \frac{-6x}{-6} &= \frac{24}{-6} && \text{Divide both sides by } -6. \\ x &= -4 && \text{Simplify: } 24/(-6)=-4. \end{aligned}$$

Check. Substitute −4 for x in the original equation

$$\begin{aligned} 6x-5 &= 12x+19 && \text{Original equation.} \\ 6(-4)-5 &= 12(-4)+19 && \text{Substitute } -4 \text{ for } x. \\ -24-5 &= -48+19 && \text{Multiply first on bo} \end{aligned}$$

Because the last line of the check is a true statement, −4 is a solution of the original equation.

Exercise

Solve for x: 2x + 3 = 18 − 3x

Answer

x = 3

Example 6

Solve for x: 2(3x + 2) − 3(4 − x) = x + 8.

Solution

Use the distributive property to remove parentheses on the left-hand side of the equation.

$$\begin{aligned} 2(3x+2)-3(4-x) &= x+8 && \text{Original equation.} \\ 6x+4-12+3x &= x+8 && \text{Use the distributive property.} \\ 9x-8 &= x+8 && \text{Combine like terms: } 6x+3x=9x \\ & && \text{and } 4-12=-8. \end{aligned}$$

Isolate the variables on the left by subtracting x from both sides of the equation

$$\begin{aligned} 9x-8-x &= x+8-x && \text{Subtract } x \text{ from both sides.} \\ 8x-8 &= 8 && \text{Combine like terms: } 9x-x=8x \\ & && \text{and } x-x=0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. Next, to "undo" subtracting 8, add 8 to both sides of the equation.

$$\begin{aligned} 8x-8+8 &= 8+8 && \text{Add 8 to both sides.} \\ 8x &= 16 && \text{Simplify: } -8+8=0 \text{ and } 8+8=16. \end{aligned}$$

Finally, to "undo" multiplying by 8, divide both sides of the equation by 8.

$$\begin{aligned} \frac{8x}{8} &= \frac{16}{8} && \text{Divide both sides by 8.} \\ x &= 2 && \text{Simplify: } 16/8=2. \end{aligned}$$

Check. Substitute 2 for x in the original equation.

Because the last line of the check is a true statement, 2 is a solution of the original equation.

Exercise

Solve for x: 3(2x − 4) − 2(5 − x) = 18

Answer

x = 5

Exercises

In Exercises 1-16, solve the equation.

1. −9x + x = −8
2. 4x − 5x = −3
3. −4=3x − 4x
4. −6 = −5x + 7x
5. 27x + 51 = −84
6. −20x + 46 = 26

7. $9=5x + 9 - 6x$

8. $-6 = x + 3 - 4x$

9. $0= -18x + 18$

10. $0= -x + 71$

11. $41 = 28x + 97$

12. $-65 = -x - 35$

13. $8x - 8 - 9x = -3$

14. $6x + 7 - 9x = 4$

15. $-85x + 85 = 0$

16. $17x - 17 = 0$

In Exercises 17-34, solve the equation.

17. $-6x = -5x - 9$

18. $-5x = -3x - 2$

19. $6x - 7=5x$

20. $3x +8= -5x$

21. $4x - 3=5x - 1$

22. $x - 2=9x - 2$

23. $-3x +5=3x - 1$

24. $-5x +9= -4x - 3$

25. $-5x = -3x + 6$

26. $3x = 4x - 6$

27. $2x - 2=4x$

28. $6x - 4=2x$

29. $-6x +8= -2x$

30. $4x - 9=3x$

31. $6x = 4x - 4$

32. $-8x = -6x + 8$

33. $-8x +2= -6x + 6$

34. $-3x +6= -2x - 5$

In Exercises 35-52, solve the equation.

35. $1 - (x - 2) = -3$

36. $1 - 8(x - 8) = 17$

37. $-7x + 6(x + 8) = -2$

38. $-8x + 4(x + 7) = -12$

39. $8(-6x - 1) = -8$

40. $-7(-2x - 4) = -14$

41. $-7(-4x - 6) = -14$

42. $-2(2x + 8) = -8$

43. $2 - 9(x - 5) = -16$

44. $7 - 2(x + 4) = -1$

45. $7x + 2(x + 9) = -9$

46. $-8x + 7(x - 2) = -14$

47. $2(-x + 8) = 10$

48. $2(-x - 2) = 10$

49. $8 + 2(x - 5) = -4$

50. $-5 + 2(x + 5) = -5$

51. $9x - 2(x + 5) = -10$

52. $-8x - 5(x - 3) = 15$

In Exercises 53-68, solve the equation.

53. $4(-7x + 5) + 8 = 3(-9x - 1) - 2$

54. $-4(-x + 9) + 5 = -(-5x - 4) - 2$

55. $-8(-2x - 6) = 7(5x - 1) - 2$

56. $5(-4x - 8) = -9(-6x + 4) - 4$

57. $2(2x - 9) + 5 = -7(-x - 8)$

58. $-6(-4x - 9) + 4 = -2(-9x - 8)$

59. $6(-3x + 4) - 6 = -8(2x + 2) - 8$

60. $-5(5x - 9) - 3 = -4(2x + 5) - 6$
61. $2(-2x - 3) = 3(-x + 2)$
62. $-2(7x + 1) = -2(3x - 7)$
63. $-5(-9x + 7) + 7 = -(-9x - 8)$
64. $7(-2x - 6) + 1 = 9(-2x + 7)$
65. $5(5x - 2) = 4(8x + 1)$
66. $5(-x - 4) = -(-x + 8)$
67. $-7(9x - 6) = 7(5x + 7) - 7$
68. $-8(2x + 1) = 2(-9x + 8) - 2$

Answers

1. 1
3. 4
5. −5
7. 0
9. 1
11. −2
13. −5
15. 1
17. 9
19. 7
21. −2
23. 1
25. −3
27. −1
29. 2
31. −2
33. −2
35. 6
37. 50
39. 0
41. −2
43. 7
45. −3
47. 3
49. −1
51. 0
53. 33
55. 3
57. −23
59. 21
61. −12
63. 1
65. −2
67. 0

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7.7: Applications

Because we've increased our fundamental ability to simplify algebraic expressions, we're now able to tackle a number of more advanced applications. Before we begin, we remind readers of required steps that must accompany solutions of applications.

Requirements for Word Problem Solutions

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
    1. Statements such as "Let P represent the perimeter of the rectangle."
    2. Labeling unknown values with variables in a table.
    3. Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.
4. **Answer the Question.** This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem.
5. **Look Back.** It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Consecutive Integers

The integers are *consecutive*, in the sense that one follows right after another. For example, 5 and 6 are a pair of consecutive integers. The important relation to notice is the fact that the second integer of this pair is one larger than its predecessor. That is, 6 = 5 + 1.

Definition: Consecutive Integers

Let k represent an integer. The next consecutive integer is the integer $k + 1$.

Thus, if k is an integer, then $k + 1$ is the next integer, $k + 2$ is the next integer after that, and so on.

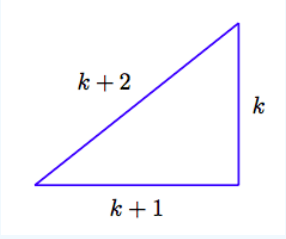
Example 1

The three sides of a triangle are consecutive integers and the perimeter is 72 inches. Find the measure of each side of the triangle.

Solution

We follow the Requirements for Word Problem Solutions.

1. *Set up a Variable Dictionary.* In this case, a carefully labeled diagram is the best way to indicate what the unknown variable represents.



In our schematic diagram, we've labeled the three sides of the triangle with expressions representing the consecutive integers k, k+ 1, and k+ 2.

2. *Set up an Equation.* To find the perimeter P of the triangle, sum the three sides.

$$P = k + (k+1) + (k+2)$$

However, we're given the fact that the perimeter is 72 inches. Thus,

$$72 = k + (k+1) + (k+2)$$

3. *Solve the Equation.* On the right, regroup and combine like terms.

$$72 = 3k + 3$$

Now, solve.

$$\begin{aligned} 72 - 3 &= 3k + 3 - 3 && \text{Subtract 3 from both sides.} \\ 69 &= 3k && \text{Simplify.} \\ \frac{69}{3} &= \frac{3k}{3} && \text{Divide both sides by 3.} \\ 23 &= k && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found one side, but the question asks for the measure of all three sides. However, the remaining two sides can be found by substituting 23 for k into the expressions k + 1 and k + 2.

k + 1 = 23 + 1 and k + 2 = 23 + 2

= 24 = 25

Hence, the three sides measure 23 inches, 24 inches, and 25 inches.

5. *Look Back.* Does our solution make sense? Well, the three sides are certainly consecutive integers, and their sum is 23 inches + 24 inches + 25 inches = 72 inches, which was the given perimeter. Therefore, our solution is correct.

? Exercise

The three sides of a triangle are consecutive integers and the perimeter is 57 centimeters. Find the measure of each side of the triangle.

Answer

18, 19, and 20 cm

Consecutive Odd Integers

The integer pair 19 and 21 are an example of a pair of *consecutive odd integers*. The important relation to notice is the fact that the second integer of this pair is two larger than its predecessor. That is, 21 = 19 + 2.

Definition: Consecutive Odd Integers

Let k represent an odd integer. The next consecutive odd integer is k + 2.

Thus, if k is an odd integer, then k + 2 is the next odd integer, k + 4 is the next odd integer after that, and so on.

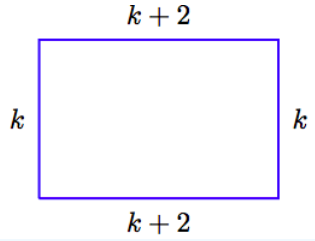
✓ Example 2

The length and width of a rectangle are consecutive odd integers and the perimeter is 168 centimeters. Find the length and width of the rectangle.

Solution

We follow the Requirements for Word Problem Solutions.

1. *Set up a Variable Dictionary.* In this case, a carefully labeled diagram is the best way to indicate what the unknown variable represents.



In our schematic diagram, if the width k is an odd integer, then the length k + 2 is the next consecutive odd integer.

2. *Set up an Equation*. To find the perimeter of the rectangle, sum the four sides.

$$P = k + (k+2) + k + (k+2)$$

However, we're given the fact that the perimeter is 168 centimeters. Thus,

$$168 = k + (k+2) + k + (k+2)$$

3. *Solve the Equation*. On the right, regroup and combine like terms.

$$168 = 4k + 4$$

Now, solve.

$$\begin{aligned} 168 - 4 &= 4k + 4 - 4 && \text{Subtract 4 from both sides.} \\ 164 &= 4k && \text{Simplify.} \\ \frac{164}{4} &= \frac{4k}{4} && \text{Divide both sides by 4.} \\ 41 &= k && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the width, but the question asks for the measure of both the width and the length. However, the length can be found by substituting 41 for k into the expression k + 2.

$$\text{k + 2 = 41 + 2}$$

$$\text{= 43}$$

Hence, the width is 41 centimeters and the length is 43 centimeters.

5. *Look Back.* Does our solution make sense? Well, the width is 41 cm and the length is 43 cm, certainly consecutive odd integers. Further, the perimeter would be 41 cm + 43 cm + 41 cm + 43 cm = 168 cm, so our solution is correct.

? Exercise

The length and width of a rectangle are consecutive odd integers and the perimeter is 120 meters. Find the length and width of the rectangle.

Answer

W = 29 cm, L = 31 cm

Tables

In the remaining applications in this section, we will strive to show how tables can be used to summarize information, define variables, and construct equations to help solve the application.

✔ Example 3

Hue inherits $10,000 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. He decides to invest $1,000 more in the certificate of deposit than in savings. Find the amount invested in each account.

Solution

We follow the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let S represent the amount Hue invests in the savings account. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting S represent the amount invested in savings is far better than letting x represent the amount invested in savings.*

Account Type	Amount Deposited
Savings Account (2%)	S
Certificate of Deposit (4%)	S + 1000
Totals	10000

Because S represents the investment in savings, and we're told that the investment in the certificate of deposit (CD) is $1,000 more than the investment in savings, the investment in the CD is therefore S + 1000, as indicated in the table.

2. *Set up an Equation.* The second column of the table reveals that the sum of the individual investments in the CD and savings totals $10,000. Hence, the equation that models this application is

$$(S+1000)+S=10000.$$

3. *Solve the Equation*. On the left, regroup and combine like terms.

$$2S+1000=10000$$

Now, solve.

$$\begin{aligned} 2S+1000-1000&=1000-1000 && \text{Subtract 1000 from both sides.}\\ 2S&=9000 && \text{Simplify.}\\ \frac{2S}{2}&=\frac{9000}{2} && \text{Divide both sides by 2.}\\ S&=4500 && \text{Simplify.}\end{aligned}$$

4. *Answer the Question*. We've only found the investment in savings, but the question also asks for the amount invested in the CD. However, the investment in the CD is easily found by substituting 4500 for S in the expression S + 1000.

S + 1000 = 4500 + 1000

= 5500.

Hence, the investment in savings is $4,500 and the investment in the CD is $5,500.

5. *Look Back*. Does our solution make sense? Well, the amount invested in the CD is $5,500, which is certainly $1,000 more than the $4,500 invested in savings. Secondly, the two investments total $5, 500 + $4, 500 = $10, 000, so our solution is correct.

? Exercise

Dylan invests a total of $2,750 in two accounts, a savings account paying 3% interest, and a mutual fund paying 5% interest. He invests $250 less in the mutual fund than in savings. Find the amount invested in each account.

Answer

$1,500 in savings, $1,250 in the mutual fund

Example 4

Jose cracks open his piggy bank and finds that he has $3.25 (325 cents), all in nickels and dimes. He has 10 more dimes than nickels. How many dimes and nickels does Jose have?

Solution

We follow the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let N represent the number of nickels from the piggy bank. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting N represent the number of nickels is far better than letting x represent the number of nickels.*

Coins	Number of Coins	Value (cents)
Nickels (5 cents apiece)	N	$5N$
Dimes (10 cents apiece)	$N + 10$	$10(N + 10)$
Totals	-	325

Because there are 10 more dimes than nickels, the number of dimes is N + 10, recorded in the second column. In the third column, N nickels, worth 5 cents apiece, have a value of 5N cents. Next, N + 10 dimes, worth 10 cents apiece, have a value of 10(N + 10) cents. The final entry in the column gives the total value of the coins as 325 cents.

2. *Set up an Equation.* The third column of the table reveals that the sum of the coin values is 325 cents. Hence, the equation that models this application is

$$5N + 10(N + 10) = 325,$$

which sums the value of the nickels and the value of the dimes to a total of 325 cents.

3. *Solve the Equation*. On the left, use the distributive property to remove parentheses.

$$5N + 10N + 100 = 325$$

Combine like terms.

$$15N + 100 = 325$$

Now, solve.

$$\begin{aligned} 15N + 100 - 100 &= 325 - 100 && \text{Subtract 100 from both sides.} \\ 15N &= 225 && \text{Simplify.} \\ \frac{15N}{15} &= \frac{225}{15} && \text{Divide both sides by 15.} \\ N &= 15 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the number of nickels, but the question also asks for the number of dimes. However, the number of dimes is easily found by substituting 15 for N in the expression N + 10.

$$\text{N} + 10 = 15 + 10$$

$$= 25.$$

Hence, Jose has 15 nickels and 25 dimes.

5. *Look Back.* Does our solution make sense? Well, the number of dimes is 25, which is certainly 10 more than 15 nickels. Also, the monetary value of 15 nickels is 75 cents and the monetary value of 25 dimes is 250 cents, a total of 325 cents, or $3.25, so our solution is correct.

Exercise

David keeps his change in a bowl made by his granddaughter. There is $1.95 in change in the bowl, all in dimes and quarters. There are two fewer quarters than dimes. How many dimes and quarters does he have in the bowl?

Answer

7 dimes, 5 quarters

Example 5

A large children's organization purchases tickets to the circus. The organization has a strict rule that every five children must be accompanied by one adult guardian. Hence, the organization orders five times as many child tickets as it does adult tickets. Child tickets are three dollars and adult tickets are six dollars. If the total cost of tickets is $4,200, how many child and adult tickets were purchased?

Solution

We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary*. We're going to use a table to summarize information and declare variables. In the table that follows, we let A represent the number of adult tickets purchased. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting A represent the number of adult tickets is far better than letting x represent the number of adult tickets.*

	Number of Tickets	Cost (dollars)
Adults ($6 apiece)	A	$6A$
Children ($3 apiece)	$5A$	$3(5A)$
Totals	-	4200

Because there are 5 times as many children's tickets purchased than adult tickets, the number of children's tickets purchased is 5A, recorded in the second column. In the third column, 5A children's tickets at $3 apiece will cost 3(5A) dollars, and A adult tickets at $6 apiece will cost 6A dollars. The final entry in the column gives the total cost of all tickets as $4,200.

2. *Set up an Equation*. The third column of the table reveals that the sum of the costs for both children and adult tickets is $4,200. Hence, the equation that models this application is

$$6A + 3(5A) = 4200$$

which sums the cost of children and adult tickets at $4,200.

3. *Solve the Equation*. On the left, use the associative property to remove parentheses.

$$6A + 15A = 4200$$

Combine like terms.

$$21A = 4200$$

Now, solve.

$$\frac{21A}{21} = \frac{4200}{21} \quad \text{Divide both sides by 21.}$$
$$A = 200 \quad \text{Simplify.}$$

4. *Answer the Question*. We've only found the number of adult tickets, but the question also asks for the number of children's tickets. However, the number of children's tickets is easily found by substituting 200 for A in the expression 5A.

$$5A = 5(200)$$

= 1000.

Hence, 1000 children's tickets and 200 adult tickets were purchased.

5. *Look Back*. Does our solution make sense? Well, the number of children's tickets purchased is 1000, which is certainly 5 times more than the 200 adult tickets purchased. Also, the monetary value of 1000 children's tickets at $3 apiece is $3,000, and the monetary value of 200 adult tickets at $6 apiece is $1,200, a total cost of $4,200. Our solution is correct.

? Exercise

Emily purchase tickets to the IMAX theater for her family. An adult ticket cost $12 and a child ticket costs $4. She buys two more child tickets than adult tickets and the total cost is $136. How many adult and child tickets did she buy?

Answer

8 adult and 10 child tickets

Exercises

1. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 39 inches, find the lengths of the sides of the triangle.

2. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 51 inches, find the lengths of the sides of the triangle.

3. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 142 inches, find the width and length of the rectangle.

4. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 166 inches, find the width and length of the rectangle.

5. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 240 inches, find the lengths of the sides of the triangle.

6. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 30 inches, find the lengths of the sides of the triangle.

7. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 374 inches, find the width and length of the rectangle.

8. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 318 inches, find the width and length of the rectangle.

9. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 208 inches, find the width and length of the rectangle.

10. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 152 inches, find the width and length of the rectangle.

11. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 76 inches, find the width and length of the rectangle.

12. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 300 inches, find the width and length of the rectangle.

13. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 144 inches, find the lengths of the sides of the triangle.

14. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 198 inches, find the lengths of the sides of the triangle.

15. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 228 inches, find the lengths of the sides of the triangle.

16. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 216 inches, find the lengths of the sides of the triangle.

17. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 92 inches, find the width and length of the rectangle.

18. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 228 inches, find the width and length of the rectangle.

19. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 105 inches, find the lengths of the sides of the triangle.

20. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 123 inches, find the lengths of the sides of the triangle.

21. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 288 inches, find the width and length of the rectangle.

22. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 352 inches, find the width and length of the rectangle.

23. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 165 inches, find the lengths of the sides of the triangle.

24. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 99 inches, find the lengths of the sides of the triangle.

25. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are $7 and adult tickets are $19. If the total cost of tickets is $975, how many adult tickets were purchased?

26. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are $6 and adult tickets are $16. If the total cost of tickets is $532, how many adult tickets were purchased?

27. Judah cracks open a piggy bank and finds $3.30 (330 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Judah have?

28. Texas cracks open a piggy bank and finds $4.90 (490 cents), all in nickels and dimes. There are 13 more dimes than nickels. How many nickels does Texas have?

29. Steve cracks open a piggy bank and finds $4.00 (400 cents), all in nickels and dimes. There are 7 more dimes than nickels. How many nickels does Steve have?

30. Liz cracks open a piggy bank and finds $4.50 (450 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Liz have?

31. Jason inherits $20,300 and decides to invest in two different types of accounts, a savings account paying 2.5% interest, and a certificate of deposit paying 5% interest. He decides to invest $7,300 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

32. Trinity inherits $24,300 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 5.75% interest. She decides to invest $8,500 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

33. Gina cracks open a piggy bank and finds $4.50 (450 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Gina have?

34. Dylan cracks open a piggy bank and finds $4.05 (405 cents), all in nickels and dimes. There are 6 more dimes than nickels. How many nickels does Dylan have?

35. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are $4 and adult tickets are $10. If the total cost of tickets is $216, how many adult tickets were purchased?

36. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are $7 and adult tickets are $11. If the total cost of tickets is $375, how many adult tickets were purchased?

37. Connie cracks open a piggy bank and finds $3.70 (370 cents), all in nickels and dimes. There are 7 more dimes than nickels. How many nickels does Connie have?

38. Don cracks open a piggy bank and finds $3.15 (315 cents), all in nickels and dimes. There are 3 more dimes than nickels. How many nickels does Don have?

39. Mary inherits $22,300 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. She decides to invest $7,300 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

40. Amber inherits $26,000 and decides to invest in two different types of accounts, a savings account paying 2.25% interest, and a certificate of deposit paying 4.25% interest. She decides to invest $6,200 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

41. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are $6 and adult tickets are $16. If the total cost of tickets is $1024, how many adult tickets were purchased?

42. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 3 children must be accompanied by one adult guardian. Hence, the organization orders 3 times as many child tickets as it does adult tickets. Child tickets are $3 and adult tickets are $18. If the total cost of tickets is $351, how many adult tickets were purchased?

43. Alan inherits $25,600 and decides to invest in two different types of accounts, a savings account paying 3.5% interest, and a certificate of deposit paying 6% interest. He decides to invest $6,400 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

44. Mercy inherits $27,100 and decides to invest in two different types of accounts, a savings account paying 3% interest, and a certificate of deposit paying 4% interest. She decides to invest $8,700 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

45. Tony inherits $20,600 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. He decides to invest $9,200 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

46. Connie inherits $17,100 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 5.5% interest. She decides to invest $6,100 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

47. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are $2 and adult tickets are $14. If the total cost of tickets is $234, how many adult tickets were purchased?

48. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are $8 and adult tickets are $13. If the total cost of tickets is $1078, how many adult tickets were purchased?

Answers

1. 11 in., 13 in., 15 in.
3. 35 in., 36 in.
5. 78 in., 80 in., 82 in.
7. 93 in., 94 in.
9. 51 in., 53 in.
11. 18 in., 20 in.
13. 46 in., 48 in., 50 in.
15. 75 in., 76 in., 77 in.
17. 22 in., 24 in.
19. 34 in., 35 in., 36 in.
21. 71 in., 73 in.
23. 53 in., 55 in., 57 in.
25. 13 adult tickets
27. 12 nickels
29. 22 nickels
31. $6, 500
33. 20 nickels
35. 12 child tickets
37. 20 nickels
39. $7, 500
41. 16 child tickets
43. $9, 600
45. $5, 700
47. 13 child tickets

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CHAPTER OVERVIEW

8: Plane Geometry

8.1: Angle Measurement
8.2: Perimeter and Circumference
8.3: Area of Polygons and Circles
8.4: Composite Figures

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8.1: Angle Measurement

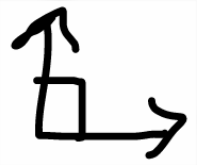
An angle is a measure of the size of the opening of two lines that intersect. The **VERTEX** is the point of intersection, while the lines that form the opening are called the **SIDES**.

The angle can be called by

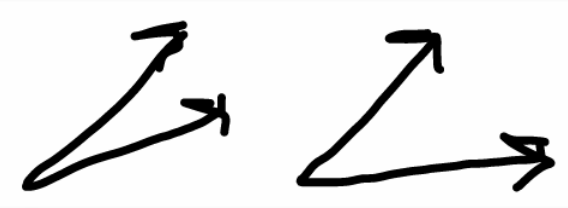
3 letters with the vertex in the middle: $\angle ABC$ or by the vertex only $\angle B$ or by a number or letter placed inside the angle.

There are 360 degrees in a circle. The angles are measured in degrees.

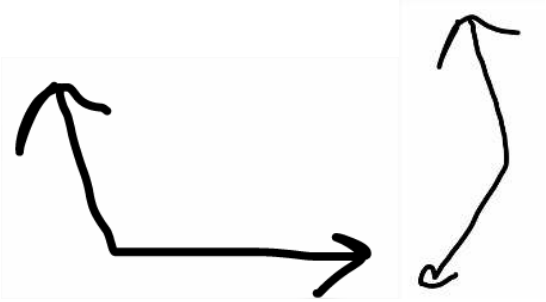
A **Right Angle** is 90 degrees or 1/4 of a circle. A **Right Angle** will look like the following.



An **Acute Angle** is an angle that is less than 90 degrees. The following are examples of Acute Angles



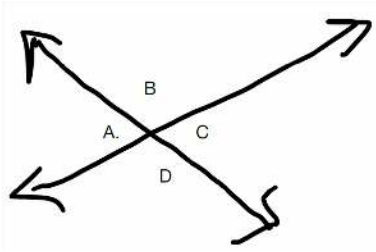
An **Obtuse Angle** is an angle that is greater than 90 degrees and less than 180 degrees. The following are examples of Obtuse Angles.



A **Straight Angle** is an angle equal to 180 degrees.

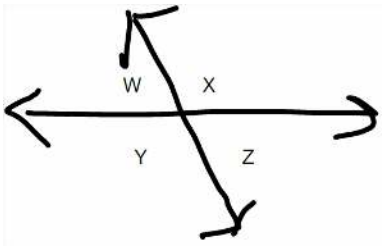
Vertical Angles

When two straight lines intersection they form four angles.



Let's say that $\angle A$ is 65 degrees, $\angle B$ is 115 degrees, $\angle C$ is 65 degrees, and $\angle D$ is 115 degrees

Did you notice that the opposite angles are equal in measurement? The opposite angles are also called **Vertical Angles**. When two straight lines cross or intersect, the **Vertical Angles** are **always equal**. A straight angle is 180 degrees.

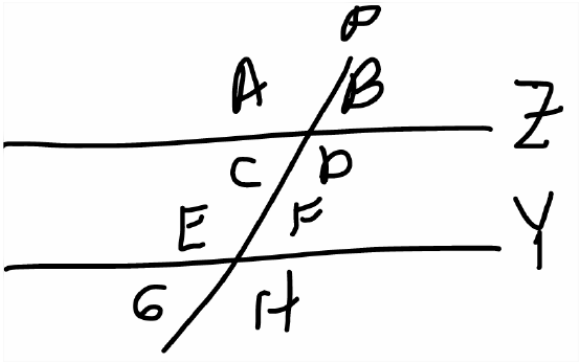


The angles W and X form a straight line, added together together they measure 180 degrees.

They are also know as **Adjacent Angles**. Adjacent Angles add up to 180 degrees. Adjacent Angles are also

- $\angle Y$ and $\angle Z$,
- $\angle W$ and $\angle Y$
- $\angle X$ and $\angle Z$.

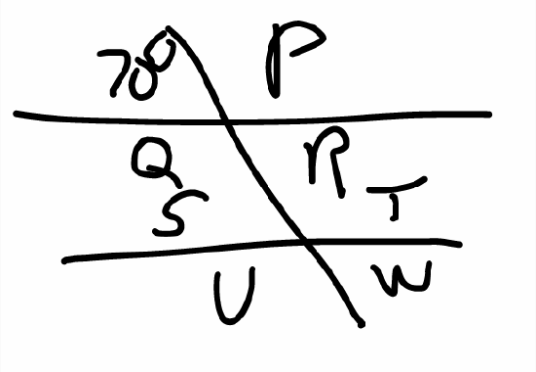
The three angles of a triangle will always add up to 180 degrees.



Lines Z and Y are parallel to each other. Line P the crosses both lines is called a **Transversal**.

$\angle C$ and $\angle F$ are called **Alternate Interior Angles**; They are equal in measurement.

$\angle D$ and $\angle E$ are also called **Alternate Interior Angles**.



With the angle measuring 70 degrees, $\angle P$ will equal 110 degrees, their total equals 180 degrees.

- $\angle P$ and $\angle Q$ are opposite angels so they equal 110 degrees because vertical angles are equal to each other.
- $\angle P$ and $\angle T$ and corresponding angles so the both equal 110 degrees.
- $\angle W$ equals 70 degrees because $\angle T$ plus $\angle W$ must equal a total of 180 degrees.

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8.2: Perimeter and Circumference

You may use a calculator throughout this module if needed.

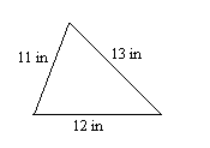
Perimeter

A polygon is a closed geometric figure with straight sides. Common polygons include triangles, squares, rectangles, parallelograms, trapezoids, pentagons, hexagons, octagons… The perimeter of a polygon is the distance around the outside. In general, to find the perimeter of a polygon, you can add up the lengths of all of its sides.

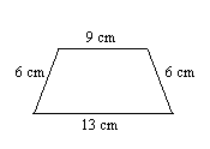
Also, if you haven't already, now is the time to get in the habit of including units in your answers.

Exercises 8.2.1

1. Find the perimeter of the triangle.



2. Find the perimeter of the trapezoid.



Answer

1. 36 in
2. 34 cm

If we know that some of the sides of a polygon are equal, we can use a formula as an alternative to adding up all of the lengths individually. The first formula shown below uses the variable s for the side of a square. The rectangle formulas use l for length and w for width, or b for base and h for height; these terms are interchangeable.

Perimeter Formulas

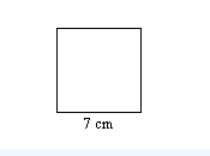
Square: $P = 4s$

Rectangle: $P = 2l + 2w$ or $P = 2b + 2h$

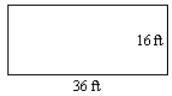
Rectangle: $P = 2(l + w)$ or $P = 2(b + h)$

Exercises 8.2.1

3. Find the perimeter of the square.



4. Find the perimeter of the rectangle.



5. A storage area, which is a rectangle that is 45 feet long and 20 feet wide, needs to be fenced around all four sides. How many feet of fencing is required? (To keep it simple, ignore any gates or other complications.)

6. Giancarlo is putting crown molding around the edge of the ceiling of his living room. If the room is a 12-foot by 16-foot rectangle, how much crown molding does he need?

Answer

3. 28 cm

4. 104 ft

5. 130 ft

6. 56 ft

The sides of a regular polygon are all equal in length. Therefore, multiplying the length of a side by the number of sides will give us the perimeter.

Perimeter Formula

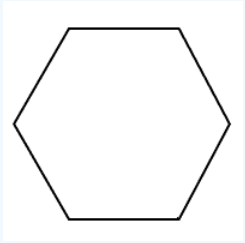
Regular Polygon with n sides of length s:

$$P = n \cdot s$$

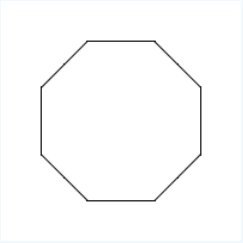
Exercises 8.2.1

Find the perimeter of each regular polygon.

7. Each side of the hexagon is 4 inches long.



8. Each side of the octagon is 2.5 centimeters long.



Answer

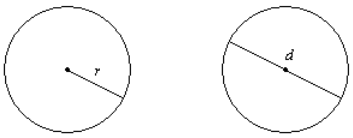
7. 24 in

8. 20 cm

Circumference

Instead of calling it the perimeter, the distance around the outside of circle is called the ***circumference***. Let's review some circle vocabulary before moving on.

Every point on a circle is the same distance from its center. This distance from the center to the edge of the circle is called the ***radius***. The distance from one edge to another, through the center of the circle, is called the ***diameter***. As you can see, the diameter is twice the length of the radius.



Throughout history, different civilizations have discovered that the circumference of a circle is slightly more than 3 times the length of its diameter. By the year 2000 BCE, the Babylonians were using the value $3\frac{1}{8} = 3.125$ and the Egyptians were using the value $3\frac{13}{81} \approx 3.1605$.[1] The value $3\frac{1}{7} \approx 3.1429$ is an even better approximation for the ratio of the circumference to the diameter. However, the actual value cannot be written as an exact fraction. It is the irrational number π, pronounced "pie", which is approximately 3.14159.

Circumference Formulas

$$C = \pi d$$

$$C = 2\pi r$$

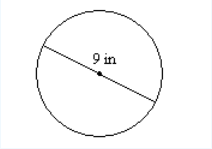
Any scientific calculator will have a π key; using this will give you the most accurate result, although you should be sure to round your answer appropriately. (See this module if you need a refresher on rounding with multiplying or dividing.) Many people use 3.14 as an approximation for π, but this can lead to round-off error; if you must use an approximation, 3.1416 is better than 3.14.

Sometimes we bend the rules in this textbook and ask you to round to a certain place value instead of rounding to a certain number of significant digits.

Exercises 8.2.1

Calculate the circumference of each circle. Round to the nearest tenth.

9.



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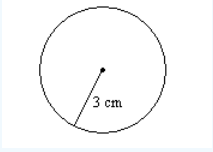

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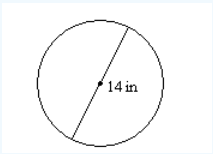

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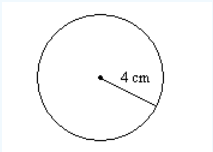

Answer

9. 28.3in
10. 18.8cm
11. 44.0in
12. 25.1cm

1. This information comes from Chapter 1 of the book *A History of Pi* by Petr Beckmann. It is a surprisingly interesting read. ↵

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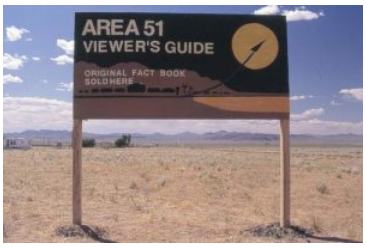
8.3: Area of Polygons and Circles

You may use a calculator for most of this module as needed.

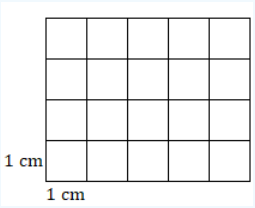


We have seen that the perimeter of a polygon is the distance around the outside. Perimeter is a length, which is one-dimensional, and so it is measured in linear units (feet, centimeters, miles, etc.). The area of a polygon is the amount of two-dimensional space inside the polygon, and it is measured in square units: square feet, square centimeters, square miles, etc.

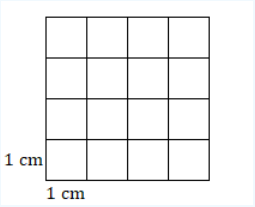
You can always think of area as the number of squares required to completely fill in the shape.

? Exercises 8.3.1

1. Find the area of this rectangle.



2. Find the area of this square.



Answer

1. 20 cm^2
2. 16 cm^2

Rectangles and Squares

There are of course formulas for finding the areas of rectangles and squares; we don't have to count little squares.

Area of a Rectangle

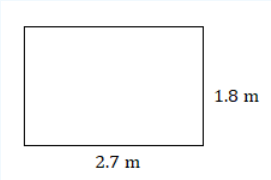
$A = lw$ [1] or $A = bh$

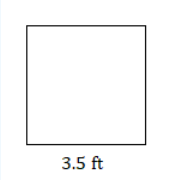
Area of a Square

$$A = s^2$$

Exercises 8.3.1

Find the area of each figure.

3. 

4. 

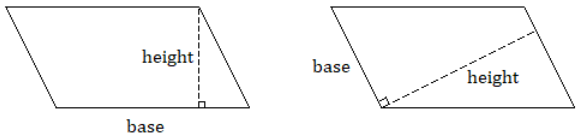
Answer

3. $4.86\ \text{m}^2$

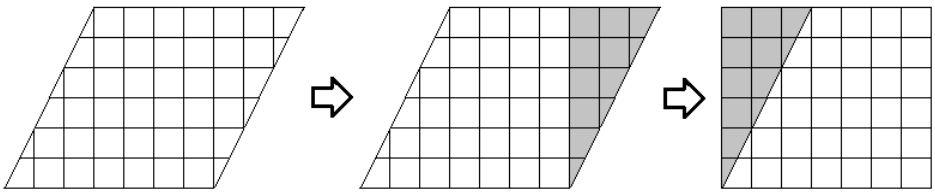
4. $12.25\ \text{ft}^2$

Parallelograms

Another common polygon is the ***parallelogram***, which looks like a tilted rectangle. As the name implies, the pairs of opposite sides are parallel and have the same length. Notice that, if we label one side as a base of the parallelogram, we have a perpendicular height which is *not* the length of the other sides.



The following set of diagrams shows that we can cut off part of a parallelogram and rearrange the pieces into a rectangle with the same base and height as the original parallelogram. A parallelogram with a base of 7 units and a vertical height of 6 units is transformed into a 7 by 6 rectangle, with an area of 42 square units.



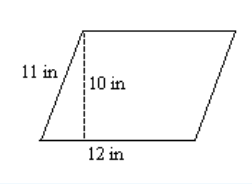
Therefore, the formula for the area of a parallelogram is identical to the formula for the area of a rectangle, provided that we are careful to use the base and the height, which must be perpendicular.

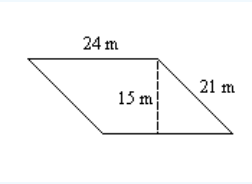
Area of a Parallelogram

$A = bh$

Exercises 8.3.1

Find the area of each parallelogram.

5. 

6. 

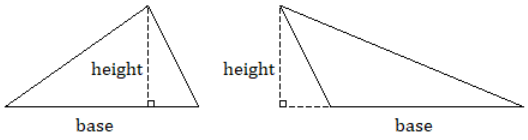
Answer

5. 120 in^2

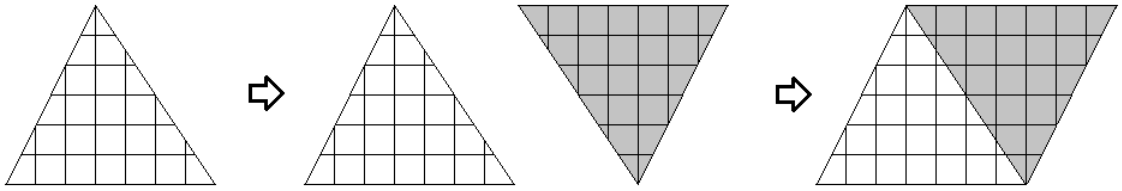
6. 360 m^2

Triangles

When we need to find the area of a triangle, we need to identify a base and a height that is perpendicular to that base. If the triangle is obtuse, you may have to imagine the height outside of the triangle and extend the base line to meet it.



As shown below, any triangle can be doubled to form a parallelogram. Therefore, the area of a triangle is one half the area of a parallelogram with the same base and height.



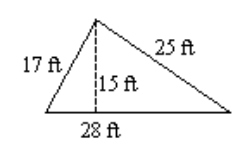
Area of a Triangle

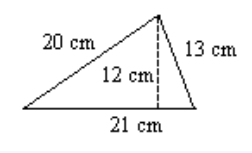
$A = \frac{1}{2}bh$ or $A = bh \div 2$

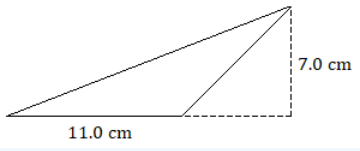
As with a parallelogram, remember that the height must be perpendicular to the base.

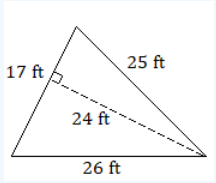
Exercises 8.3.1

Find the area of each triangle.

7. 

8. 

9. 

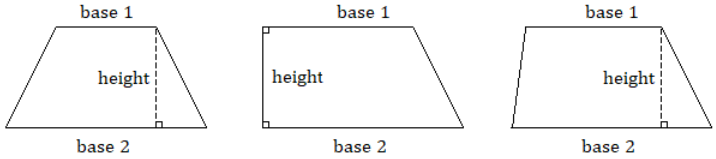
10. 

Answer

7. 210 ft^2
8. 126 cm^2
9. 38.5 cm^2
10. 204 ft^2

Trapezoids

A somewhat less common quadrilateral is the ***trapezoid***, which has exactly one pair of parallel sides, which we call the bases. The first example shown below is called an isosceles trapezoid because, like an isosceles triangle, its two nonparallel sides have equal lengths.



There are a number of ways to show where the area formula comes from, but the explanations are better in video because they can be animated.[2][3][4]

Area of a Trapezoid

$$A = \frac{1}{2}h(b_1 + b_2)$$

or

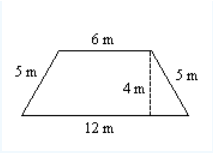
$$A = (b_1 + b_2)h \div 2$$

Don't be intimidated by the subscripts on b_1 and b_2; it's just a way to name two different measurements using the same letter for the variable. (Many people call the bases a and b instead; feel free to write it whichever way you prefer.) Whatever you call them, you just add the two bases, multiply by the height, and take half of that.

? Exercises 8.3.1

Find the area of each trapezoid.

11.



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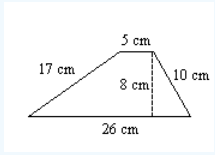

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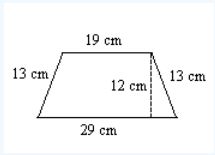


Answer

11. $36\ \text{m}^2$
12. $124\ \text{cm}^2$
13. $288\ \text{cm}^2$

Circles

The area of a circle is π times the square of the radius: $A = \pi r^2$. The units are still square units, even though a circle is round. (Think of the squares on a round waffle.) Because we can't fit a whole number of squares—or an exact fraction of squares—inside the circle, the area of a circle will be an approximation.

Area of a Circle

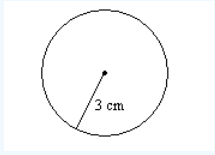
\[A=\pi{r^2}\]

Remember that $\pi \approx 3.1416$.

? Exercises 8.3.1

Find the area of each circle. Round to the nearest tenth or to three significant figures, whichever seems appropriate.

14.



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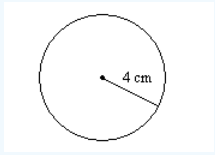
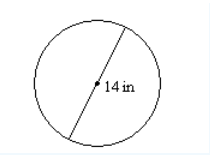
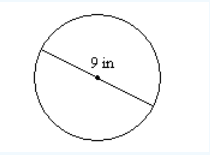
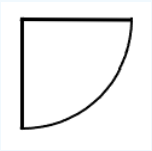


16. 

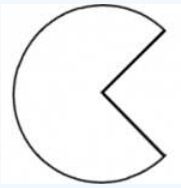
17. 

Each figure is a fraction of a circle. Calculate each area.

18. The radius of the quarter circle is 5 meters.



19. A quarter circle has been removed from a circle with a diameter of 7 feet.



Answer

14. $28.3\ \text{cm}^2$
15. $50.3\ \text{cm}^2$
16. $153.9\ \text{in}^2$
17. $63.6\ \text{in}^2$
18. $19.6\ \text{m}^2$
19. $28.9\ \text{ft}^2$

1. You might choose to use capital letters for the variables here because a lowercase letter "l" can easily be mistaken for a number "1". ↵
2. https://youtu.be/yTnYRpcZA9c ↵
3. https://youtu.be/WZtO3oERges ↵
4. https://youtu.be/uLHc6Br2veg ↵

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8.4: Composite Figures

This sunburst pattern is composed of 276 geometric tiles.

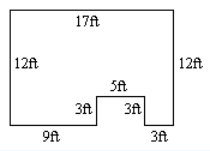
You may use a calculator throughout this module as needed.

Many objects have odd shapes made up of simpler shapes. A composite figure is a geometric figure which is formed by—or *composed* of—two or more basic geometric figures. We will look at a handful of fairly simple examples, but this concept can of course be extended to much more complicated figures.

To find the area of a composite figure, it is generally a good idea to divide it into simpler shapes and either add or subtract their areas as necessary.

? Exercises 8.4.1

A floor plan of a room is shown. The room is a 12-foot by 17-foot rectangle, with a 3-foot by 5-foot rectangle cut out of the south side.



1. Determine the amount of molding required to go around the perimeter of the room.

2. Determine the amount of flooring required to cover the entire area.

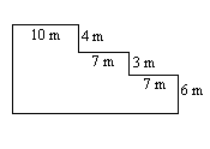
Answer

1. 64 ft
2. 189 ft^2

You may need to figure out some unmarked dimensions.

? Exercises 8.4.1

A plan for an irregular parking lot is shown.



3. Calculate the perimeter.

4. Calculate the area.

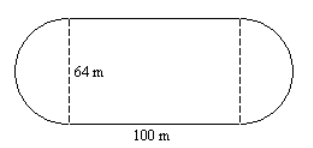
Answer

3. 74 m
4. 235 m^2

Sometimes, composite figures involve parts of circles coupled with polygons.

Exercises 8.4.1

A high school is building a track at its athletic fields. The track, which is formed by two straight sides and two semicircles as shown in the plans below, is supposed to have a total length of 400 meters.



5. Determine the distance around the track. Will the track be the right length?

6. After the new track is built, landscapers need to lay sod on the field inside the track. What is the area of the field inside the track?

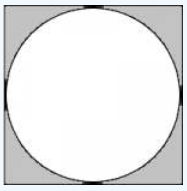
Answer

5. Based on the stated measurements, the distance around the track will be 401 meters, which appears to be 1 meter too long. In real life, precision would be very important here, and you might ask for the measurements to be given to the nearest tenth of a meter.

6. around $9,620\ \text{m}^2$

If we need to determine a fraction or percent out of the whole, we may be able to solve the problem without knowing any actual measurements.

Exercise 8.4.1



7. Circular disks are being cut from squares of sheet metal, with the remainder around the corners being discarded. Assuming that the circles are made as large as possible, what percent of the sheet metal will be discarded?

Answer

around 21.5%(Hint: Make up an easy number for the side of the square, like 2 or 10.)

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CHAPTER OVERVIEW

9: Triangle Trigonometry

Trigonometry is the study of the relations between the sides and angles of triangles. The word “trigonometry” is derived from the Greek words trigono (τρ´ιγωνο), meaning “triangle”, and metro (μϱτρω´), meaning “measure”. Though the ancient Greeks, such as Hipparchus and Ptolemy, used trigonometry in their study of astronomy between roughly 150 B.C. - A.D. 200, its history is much older. For example, the Egyptian scribe Ahmes recorded some rudimentary trigonometric calculations (concerning ratios of sides of pyramids) in the famous Rhind Papyrus sometime around 1650 B.C. Trigonometry is distinguished from elementary geometry in part by its extensive use of certain functions of angles, known as the trigonometric functions. Before discussing those functions, we will review some basic terminology about angles.

9.1: Angles
9.2: The Pythagorean Theorem
9.3: Trigonometric Functions of an Acute Angle
9.4: Applications and Solving Right Triangles
9.5: Trigonometric Functions of Any Angle
9.6: Rotations and Reflections of Angles

Contributors

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9.1: Angles

Recall the following definitions from elementary geometry:

a. An angle is **acute** if it is between 0° and 90°.
b. An angle is a **right angle** if it equals 90°.
c. An angle is **obtuse** if it is between 90° and 180°.
d. An angle is a **straight angle** if it equals 180°.

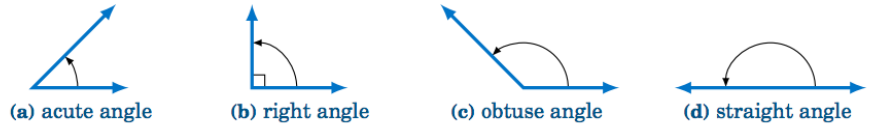


Figure 1.1.1 Types of angles

In elementary geometry, angles are always considered to be positive and not larger than 360°. For now we will only consider such angles. The following definitions will be used throughout the text:

a. Two acute angles are **complementary** if their sum equals 90°. In other words, if $0^\circ \leq \angle A, \angle B \leq 90^\circ$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90^\circ$.
b. Two angles between 0° and 180° are **supplementary** if their sum equals 180°. In other words, if $0^\circ \leq \angle A, \angle B \leq 180^\circ$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180^\circ$.
c. Two angles between 0° and 360° are **conjugate** (or **explementary**) if their sum equals 360°. In other words, if $0^\circ \leq \angle A, \angle B \leq 360^\circ$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360^\circ$.

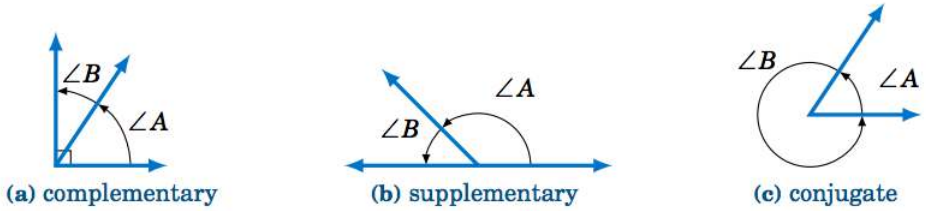


Figure 1.1.2 Types of pairs of angles

Instead of using the angle notation $\angle A$ to denote an angle, we will sometimes use just a capital letter by itself (e.g. A, B, C) or a lowercase variable name (e.g. x, y, t). It is also common to use letters (either uppercase or lowercase) from the Greek alphabet, shown in the table below, to represent angles:

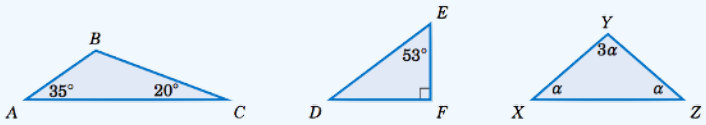
Table 1.1 The Greek alphabet

Letters		Name	Letters		Name	Letters		Name
A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

In elementary geometry you learned that the sum of the angles in a triangle equals 180°, and that an **isosceles triangle** is a triangle with two sides of equal length. Recall that in a **right triangle** one of the angles is a right angle. Thus, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

Example 1.1

For each triangle below, determine the unknown angle(s):



Note: We will sometimes refer to the angles of a triangle by their vertex points. For example, in the first triangle above we will simply refer to the angle $\angle BAC$ as angle A.

Solution:

For triangle $\triangle ABC$, $A = 35^\circ$ and $C = 20^\circ$, and we know that $A + B + C = 180^\circ$, so

$$35^\circ + B + 20^\circ = 180^\circ \Rightarrow B = 180^\circ - 35^\circ - 20^\circ \Rightarrow \boxed{B = 125^\circ}.$$

For the right triangle $\triangle DEF$, $E = 53^\circ$ and $F = 90^\circ$, and we know that the two acute angles D and E are complementary, so

$$D + E = 90^\circ \Rightarrow D = 90^\circ - 53^\circ \Rightarrow \boxed{D = 37^\circ}.$$

For triangle $\triangle XYZ$, the angles are in terms of an unknown number α, but we do know that $X + Y + Z = 180^\circ$, which we can use to solve for α and then use that to solve for $X, Y,$ and Z:

$$\alpha + 3\alpha + \alpha = 180^\circ \Rightarrow 5\alpha = 180^\circ \Rightarrow \alpha = 36^\circ \Rightarrow \boxed{X = 36^\circ,\ Y = 3 \times 36^\circ = 108^\circ,\ Z = 36^\circ}$$

Example 1.2: *Thales' Theorem*

Thales' Theorem states that if A, B, and C are (distinct) points on a circle such that the line segment $\overline{AB}$ is a diameter of the circle, then the angle $\angle ACB$ is a right angle (see Figure 1.1.3(a)). In other words, the triangle $\triangle ABC$ is a right triangle.

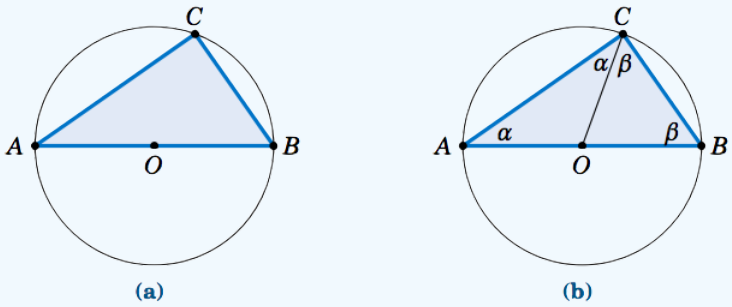


Figure 1.1.3 Thales' Theorem: $\angle ACB = 90^\circ$

To prove this, let O be the center of the circle and draw the line segment $\overline{OC}$, as in Figure 1.1.3(b). Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Since $\overline{AB}$ is a diameter of the circle, $\overline{OA}$ and $\overline{OC}$ have the same length (namely, the circle's radius). This means that $\triangle OAC$ is an isosceles triangle, and so $\angle OCA = \angle OAC = \alpha$. Likewise, $\triangle OBC$ is an isosceles triangle and $\angle OCB = \angle OBC = \beta$. So we see that $\angle ACB = \alpha + \beta$. And since the angles of $\triangle ABC$ must add up to 180°, we see that $180^\circ = \alpha + (\alpha + \beta) + \beta = 2(\alpha + \beta)$, so $\alpha + \beta = 90^\circ$. Thus, $\angle ACB = 90^\circ$. **QED**

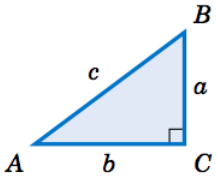


Figure 1.1.4

By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

? Theorem 1.1. Pythagorean Theorem

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

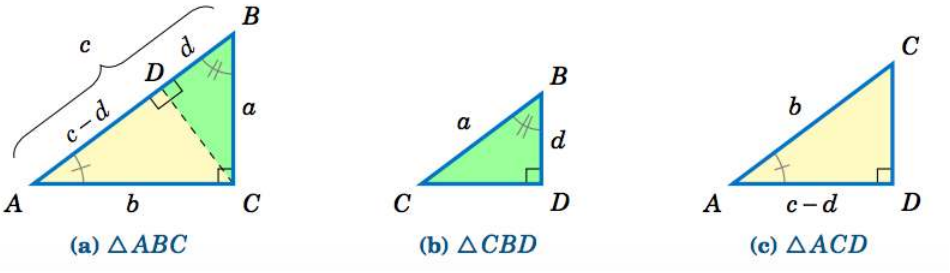


Figure 1.1.5 Similar triangles $\triangle ABC, \triangle CBD, \triangle ACD$

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle ABC$ is similar to $\triangle CBD$, by proportionality of corresponding sides we see that

$$\overline{AB} \text{ is to } \overline{CB} \text{ (hypotenuses) as } \overline{BC} \text{ is to } \overline{BD} \text{ (vertical legs)} \quad \Rightarrow \quad \frac{c}{a} = \frac{a}{d} \quad \Rightarrow \quad cd = a^2 \, .$$

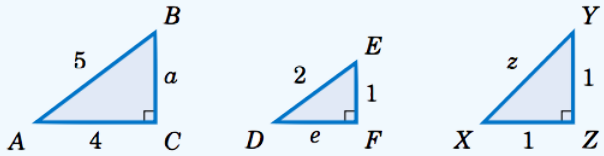
Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d} = \frac{c}{b} \quad \Rightarrow \quad b^2 = c^2 - cd = c^2 - a^2 \quad \Rightarrow \quad a^2 + b^2 = c^2 \, . \, \textbf{QED}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD} \perp \overline{AB}$ and $\triangle ABC \sim \triangle CBD \sim \triangle ACD$.

✔ Example 1.3

For each right triangle below, determine the length of the unknown side:



Solution:

For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad \Rightarrow \quad a^2 = 25 - 16 = 9 \quad \Rightarrow \quad \boxed{a = 3} \, .$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

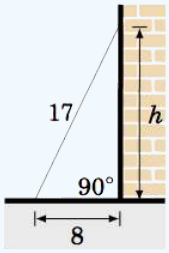
$$e^2 + 1^2 = 2^2 \quad \Rightarrow \quad e^2 = 4 - 1 = 3 \quad \Rightarrow \quad \boxed{e = \sqrt{3}} \, .$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad \Rightarrow \quad z^2 = 2 \quad \Rightarrow \quad \boxed{z = \sqrt{2}}.$$

Example 1.4

A 17 ft ladder leaning against a wall has its foot 8 ft from the base of the wall. At what height is the top of the ladder touching the wall?



Solution

Let h be the height at which the ladder touches the wall. We can assume that the ground makes a right angle with the wall, as in the picture on the right. Then we see that the ladder, ground, and wall form a right triangle with a hypotenuse of length 17 ft (the length of the ladder) and legs with lengths 8 ft and h ft. So by the Pythagorean Theorem, we have

$$h^2 + 8^2 = 17^2 \quad \Rightarrow \quad h^2 = 289 - 64 = 225 \quad \Rightarrow \quad \boxed{h = 15 \text{ ft}}.$$

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9.2: The Pythagorean Theorem

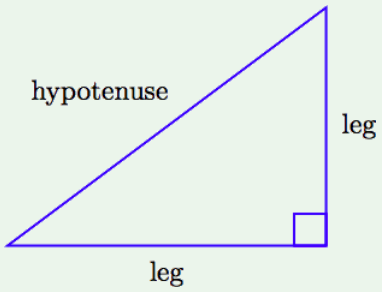
Pythagoras was a Greek mathematician and philosopher, born on the island of Samos (ca. 582 BC). He founded a number of schools, one in particular in a town in southern Italy called Crotone, whose members eventually became known as the Pythagoreans. The inner circle at the school, the Mathematikoi, lived at the school, rid themselves of all personal possessions, were vegetarians, and observed a strict vow of silence. They studied mathematics, philosophy, and music, and held the belief that numbers constitute the true nature of things, giving numbers a mystical or even spiritual quality.

Today, nothing is known of Pythagoras's writings, perhaps due to the secrecy and silence of the Pythagorean society. However, one of the most famous theorems in all of mathematics does bear his name, the Pythagorean Theorem.

Prior to revealing the contents of the Pythagorean Theorem, we pause to provide the definition of a right triangle and its constituent parts.

Right Triangle

A triangle with one right angle (90°) is called a **right triangle**. In the figure below, the right angle is marked with a little square.

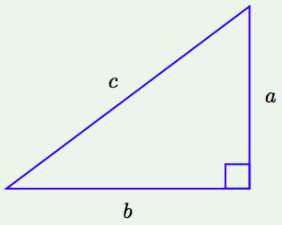


The side of the triangle that is directly opposite the right angle is called the **hypotenuse**. The sides of the triangle that include the right angle are called the *legs* of the right triangle.

Now we can state one of the most ancient theorems of mathematics, the *Pythagorean Theorem*.

Pythagorean Theorem

Let c represent the length of the *hypotenuse* of a right triangle, and let a and b represent the lengths of its legs, as pictured in the image that follows.



The relationship involving the legs and hypotenuse of the right triangle, given by

$$a^2 + b^2 = c^2,$$

is called the **Pythagorean Theorem**.

Here are two important observations.

Observations Regarding the Hypotenuse

Two important facts regarding the hypotenuse of the right triangle are:

1. The hypotenuse is the longest side of the triangle and lies directly opposite the right angle.
2. In the Pythagorean equation $a^2 + b^2 = c^2$, the hypotenuse lies by itself on one side of the equation.

The Pythagorean Theorem can **only** be applied to right triangles.

Let's look at a simple application of the Pythagorean Theorem.

Example 1

The legs of a right triangle measure 3 and 4 meters, respectively. Find the length of the hypotenuse.

Solution

Let's follow the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Let *c* represent the length of the hypotenuse, as pictured in the following sketch.

2. *Set up an Equation*. The Pythagorean Theorem says that

$$a^2 + b^2 = c^2.$$

In this example, the legs are known. Substitute 4 for *a* and 3 for *b* (3 for *a* and 4 for *b* works equally well) into the Pythagorean equation.

$$4^2 + 3^2 = c^2$$

3. *Solve the Equation*.

$$\begin{aligned} 4^2 + 3^2 &= c^2 && \text{The Pythagorean equation.} \\ 16 + 9 &= c^2 && \text{Exponents first: } 4^2 = 16 \text{ and } 3^2 = 9. \\ 25 &= c^2 && \text{Add: } 16 + 9 = 25. \\ 5 &= c && \text{Take the nonnegative square root.} \end{aligned}$$

Technically, there are two answers to $c^2 = 25$, i.e., $c = -5$ or $c = 5$. However, *c* represents the hypotenuse of the right triangle and must be nonnegative. Hence, we must choose c = 5.

4. *Answer the Question*. The hypotenuse has length 5 meters.

5. *Look Back*. Do the numbers satisfy the Pythagorean Theorem? The sum of the squares of the legs should equal the square of the hypotenuse. Let's check.

$$\begin{aligned} 4^2 + 3^2 &= 5^2 \\ 16 + 9 &= 25 \\ 25 &= 25 \end{aligned}$$

All is well!

Exercise

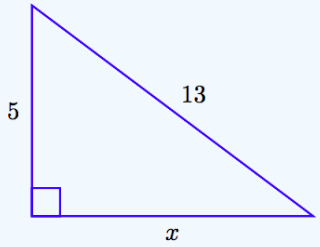
The legs of a right triangle measure 5 and 12 feet, respectively. Find the length of the hypotenuse.

Answer

13 feet

Example 2

Given the following right triangle, find the length of the missing side.



Solution

Note that the hypotenuse (across from the right angle) has length 13. This quantity should lie on one side of the Pythagorean equation all by itself. The sum of the squares of the legs go on the other side. Hence,

$$5^2 + x^2 = 13^2$$

Solve the equation for x.

$$\begin{aligned} 25 + x^2 &= 169 && \text{Exponents first: } 5^2 = 25 \text{ and } 13^2 = 169. \\ 25 + x^2 - 25 &= 169 - 25 && \text{Subtract 25 from both sides.} \\ x^2 &= 144 && \text{Simplify both sides.} \\ x &= 12 && \text{Take the nonnegative square root of 144.} \end{aligned}$$

Exercise

The hypotenuse of a right triangle measures 25 centimeters. One leg of the right triangle measures 24 centimeters. Find the length of the remaining leg.

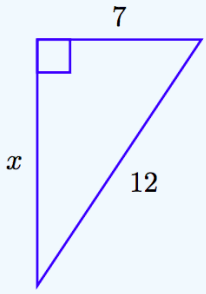
Answer

7 centimeters

Perfect squares are nice, but not required.

Example 3

Given the following right triangle, find the **exact** length of the missing side.



Solution

Note that the hypotenuse (across from the right angle) has length 12. This quantity should lie on one side of the Pythagorean equation all by itself. The sum of the squares of the legs go on the other side. Hence,

$$x^2 + 7^2 = 12^2$$

Solve the equation for x.

$$\begin{aligned} x^2+49 &= 144 && \text{Exponents first: } 7^2=49 \text{ and } 12^2=144. \\ x^2+49-49 &= 144-49 && \text{Subtract 49 from both sides.} \\ x^2 &= 95 && \text{Simplify both sides.} \\ x &= \sqrt{95} && \text{Take the nonnegative square root of 95.} \end{aligned}$$

Hence, the exact length of the missing side is $\sqrt{95}$.

? Exercise

The hypotenuse and one leg of a right triangle measure 9 and 7 inches, respectively. Find the length of the remaining leg.

Answer

Add texts here. Do not delete this text first.

Important Observation

Any attempt to use your calculator to approximate 95 in Example 3 would be an error as the instructions asked for an **exact** answer.

Sometimes an approximate answer is desired, particularly in applications.

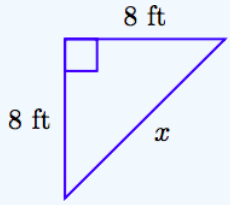
✓ Example 4

Ginny want to create a vegetable garden in the corner of her yard in the shape of a right triangle. She cuts two boards of length 8 feet which will form the legs of her garden. Find the length of board she should cut to form the hypotenuse of her garden, correct to the nearest tenth of a foot.

Solution

We follow the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. We begin with a labeled sketch. Let x represent the length of the unknown hypotenuse.



2. *Set Up an Equation*. The hypotenuse is isolated on one side of the Pythagorean equation.

$$x^2 = 8^2 + 8^2$$

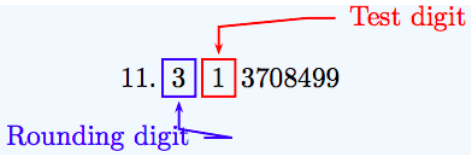
3. *Solve the Equation*.

$$\begin{aligned} x^2 &= 8^2+8^2 && \text{The Pythagorean equation.} \\ x^2 &= 64+64 && \text{Exponents first: } 8^2=64 \text{ and } 8^2=64. \\ x^2 &= 128 && \text{Add: } 64+64=128. \\ x &= \sqrt{128} && \text{Take the nonnegative square root.} \end{aligned}$$

4. *Answer the Question*. The exact length of the hypotenuse is $\sqrt{128}$ feet, but we're asked to find the hypotenuse to the nearest tenth of a foot. Using a calculator, we find an approximation for $\sqrt{128}$.

$$\sqrt{128} \approx 11.313708499$$

To round to the nearest tenth, first identify the rounding and test digits.



The test digit is less than five. So we leave the rounding digit alone and truncate. Therefore, correct to the nearest tenth of a foot, the length of the hypotenuse is approximately 11.3 feet.

5. *Look Back*. The sum of the squares of the legs is

$$\begin{aligned} 8^2+8^2 &= 64+64 \\ &= 128. \end{aligned}$$

The square of the hypotenuse is

$$(11.3)^2 = 127.69$$

These are almost the same, the discrepancy due to the fact that we rounded to find an approximation for the hypotenuse.

? Exercise

A 15 foot ladder leans against the wall of a building. The base of the ladder lies 5 feet from the base of the wall. How high up the wall does the top of the ladder reach? Round your answer to the nearest tenth of a foot.

Answer

14.1 feet

Exercises

In Exercises 1-16, your solutions should include a well-labeled sketch.

1. The length of one leg of a right triangle is 15 meters, and the length of the hypotenuse is 25 meters. Find the exact length of the other leg.

2. The length of one leg of a right triangle is 7 meters, and the length of the hypotenuse is 25 meters. Find the exact length of the other leg.

3. The lengths of two legs of a right triangle are 12 meters and 16 meters. Find the exact length of the hypotenuse.

4. The lengths of two legs of a right triangle are 9 meters and 12 meters. Find the exact length of the hypotenuse.

5. The length of one leg of a right triangle is 13 meters, and the length of the hypotenuse is 22 meters. Find the exact length of the other leg.

6. The length of one leg of a right triangle is 6 meters, and the length of the hypotenuse is 15 meters. Find the exact length of the other leg.

7. The lengths of two legs of a right triangle are 2 meters and 21 meters. Find the exact length of the hypotenuse.

8. The lengths of two legs of a right triangle are 7 meters and 8 meters. Find the exact length of the hypotenuse.

9. The length of one leg of a right triangle is 12 meters, and the length of the hypotenuse is 19 meters. Find the exact length of the other leg.

10. The length of one leg of a right triangle is 5 meters, and the length of the hypotenuse is 10 meters. Find the exact length of the other leg.

11. The lengths of two legs of a right triangle are 6 meters and 8 meters. Find the exact length of the hypotenuse.

12. The lengths of two legs of a right triangle are 5 meters and 12 meters. Find the exact length of the hypotenuse.

13. The length of one leg of a right triangle is 6 meters, and the length of the hypotenuse is 10 meters. Find the exact length of the other leg.

14. The length of one leg of a right triangle is 9 meters, and the length of the hypotenuse is 15 meters. Find the exact length of the other leg.

15. The lengths of two legs of a right triangle are 6 meters and 22 meters. Find the exact length of the hypotenuse.

16. The lengths of two legs of a right triangle are 9 meters and 19 meters. Find the exact length of the hypotenuse.

In Exercises 17-24, your solutions should include a well-labeled sketch.

17. The lengths of two legs of a right triangle are 3 meters and 18 meters. Find the length of the hypotenuse. Round your answer to the nearest hundredth.

18. The lengths of two legs of a right triangle are 10 feet and 16 feet. Find the length of the hypotenuse. Round your answer to the nearest tenth.

19. The length of one leg of a right triangle is 2 meters, and the length of the hypotenuse is 17 meters. Find the length of the other leg. Round your answer to the nearest tenth.

20. The length of one leg of a right triangle is 4 meters, and the length of the hypotenuse is 12 meters. Find the length of the other leg. Round your answer to the nearest hundredth.

21. The lengths of two legs of a right triangle are 15 feet and 18 feet. Find the length of the hypotenuse. Round your answer to the nearest hundredth.

22. The lengths of two legs of a right triangle are 6 feet and 13 feet. Find the length of the hypotenuse. Round your answer to the nearest tenth.

23. The length of one leg of a right triangle is 4 meters, and the length of the hypotenuse is 8 meters. Find the length of the other leg. Round your answer to the nearest hundredth.

24. The length of one leg of a right triangle is 3 meters, and the length of the hypotenuse is 15 meters. Find the length of the other leg. Round your answer to the nearest tenth.

---

25. Greta and Fritz are planting a 13-meter by 18-meter rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.

26. Markos and Angelina are planting an 11- meter by 19-meter rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Round your answer to the nearest tenth. Your solution should include a well-labeled sketch.

27. The base of a 24-meter long guy wire is located 10 meters from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.

28. The base of a 30-foot long guy wire is located 9 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.

---

29. **Hiking Trail.** A hiking trail runs due south for 8 kilometers, then turns west for about 15 kilometers, and then heads northeast on a direct path to the starting point. How long is the entire trail?

30. **Animal Trail.** An animal trail runs due east from a watering hole for 12 kilometers, then goes north for 5 kilometers. Then the trail turns southwest on a direct path back to the watering hole. How long is the entire trail?

31. **Upper Window.** A 10-foot ladder leans against the wall of a house. How close to the wall must the bottom of the ladder be in order to reach a window 8 feet above the ground?

32. **How high?** A 10-foot ladder leans against the wall of a house. How high will the ladder be if the bottom of the ladder is 4 feet from the wall? Round your answer to the nearest tenth.

Answers

1. 20 meters
3. 20 meters
5. $\sqrt{315}$ meters
7. $\sqrt{445}$ meters
9. $\sqrt{217}$ meters
11. 10 meters
13. 8 meters
15. $\sqrt{520}$ meters
17. 18.25 meters
19. 16.9 meters
21. 23.43 feet
23. 6.93 meters
25. 22.20 meters
27. 21.82 meters
29. 40 kilometers
31. 6 ft.

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9.3: Trigonometric Functions of an Acute Angle

Consider a right triangle $\triangle ABC$, with the right angle at C and with lengths a, b, and c, as in the figure on the right. For the acute angle A, call the leg $\overline{BC}$ its **opposite side**, and call the leg $\overline{AC}$ its **adjacent side**. Recall that the hypotenuse of the triangle is the side $\overline{AB}$. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we define the **six trigonometric functions** of A as follows:

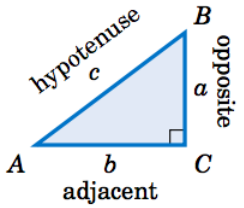


Table 1.2 The six trigonometric functions of A

Name of function	Abbreviation	Definition
sine A	$\sin A$	$= \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c}$
cosine A	$\cos A$	$= \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c}$
tangent A	$\tan A$	$= \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b}$
cosecant A	$\csc A$	$= \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{c}{a}$
secant A	$\sec A$	$= \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{c}{b}$
cotangent A	$\cot A$	$= \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{b}{a}$

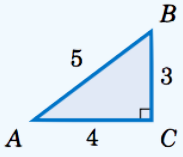
We will usually use the abbreviated names of the functions. Notice from Table 1.2 that the pairs $\sin A$ and $\csc A$, $\cos A$ and $\sec A$, and $\tan A$ and $\cot A$ are reciprocals:

$$\csc A = \frac{1}{\sin A} \qquad \sec A = \frac{1}{\cos A} \qquad \cot A = \frac{1}{\tan A}$$

$$\sin A = \frac{1}{\csc A} \qquad \cos A = \frac{1}{\sec A} \qquad \tan A = \frac{1}{\cot A}$$

Example 1.5

For the right triangle $\triangle ABC$ shown on the right, find the values of all six trigonometric functions of the acute angles A and B.



Solution:

The hypotenuse of $\triangle ABC$ has length 5. For angle A, the opposite side $\overline{BC}$ has length 3 and the adjacent side $\overline{AC}$ has length 4. Thus:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

For angle B, the opposite side $\overline{AC}$ has length 4 and the adjacent side $\overline{BC}$ has length 3. Thus:

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

$$\csc B = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} \qquad \sec B = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} \qquad \cot B = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

Notice in Example 1.5 that we did not specify the units for the lengths. This raises the possibility that our answers depended on a triangle of a specific physical size.

For example, suppose that two different students are reading this textbook: one in the United States and one in Germany. The American student thinks that the lengths 3, 4, and 5 in Example 1.5 are measured in inches, while the German student thinks that they are measured in centimeters. Since 1 in ≈ 2.54 cm, the students are using triangles of different physical sizes (see Figure 1.2.1 below, not drawn to scale).

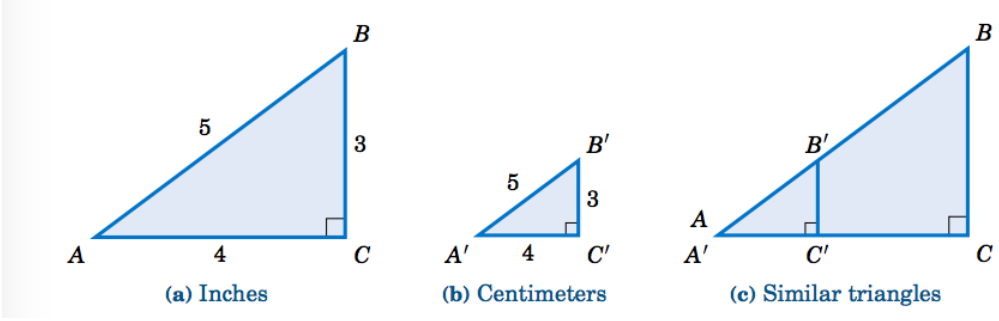


Figure 1.2.1: $\triangle ABC \sim \triangle A'B'C'$

If the American triangle is $\triangle ABC$ and the German triangle is $\triangle A'B'C'$, then we see from Figure 1.2.1 that $\triangle ABC$ is similar to $\triangle A'B'C'$, and hence the corresponding angles are equal and the ratios of the corresponding sides are equal. In fact, we know that common ratio: the sides of $\triangle ABC$ are approximately 2.54 times longer than the corresponding sides of $\triangle A'B'C'$. So when the American student calculates $\sin A$ and the German student calculates $\sin A'$, they get the same answer:

$$\triangle ABC \sim \triangle A'B'C' \quad \Rightarrow \quad \frac{BC}{B'C'} = \frac{AB}{A'B'} \quad \Rightarrow \quad \frac{BC}{AB} = \frac{B'C'}{A'B'} \quad \Rightarrow \quad \sin A = \sin A'$$

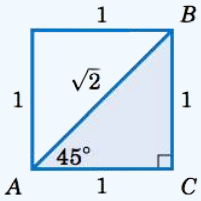
Likewise, the other values of the trigonometric functions of A and A' are the same. In fact, our argument was general enough to work with any similar right triangles. This leads us to the following conclusion:

When calculating the trigonometric functions of an acute angle A, you may use *any* right triangle which has A as one of the angles.

Since we defined the trigonometric functions in terms of ratios of sides, you can think of the units of measurement for those sides as canceling out in those ratios. This means that *the values of the trigonometric functions are unitless numbers*. So when the American student calculated $3/5$ as the value of $\sin A$ in Example 1.5, that is the same as the $3/5$ that the German student calculated, despite the different units for the lengths of the sides.

Example 1.6

Find the values of all six trigonometric functions of 45°.



Solution:

Since we may use any right triangle which has 45° as one of the angles, use the simplest one: take a square whose sides are all 1 unit long and divide it in half diagonally, as in the figure on the right. Since the two legs of the triangle $\triangle ABC$ have the same length, $\triangle ABC$ is an isosceles triangle, which means that the angles A and B are equal. So since $A+B=90^\circ$, this means that we must have $A=B=45^\circ$. By the Pythagorean Theorem, the length c of the hypotenuse is given by

$$c^2 = 1^2 + 1^2 = 2 \quad \Rightarrow \quad c = \sqrt{2}$$

Thus, using the angle A we get:

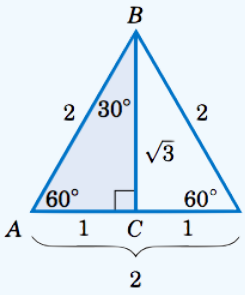
$$\sin 45^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \tan 45^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$

$$\csc 45^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \sqrt{2} \qquad \sec 45^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \sqrt{2} \qquad \cot 45^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1$$

Note that we would have obtained the same answers if we had used any right triangle similar to $\triangle ABC$. For example, if we multiply each side of $\triangle ABC$ by $\sqrt{2}$, then we would have a similar triangle with legs of length $\sqrt{2}$ and hypotenuse of length 2. This would give us $\sin 45^\circ = \frac{\sqrt{2}}{2}$, which equals $\frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$ as before. The same goes for the other functions.

Example 1.7

Find the values of all six trigonometric functions of 60°.



Solution:

Since we may use any right triangle which has 60° as one of the angles, we will use a simple one: take a triangle whose sides are all 2 units long and divide it in half by drawing the bisector from one vertex to the opposite side, as in the figure on the right. Since the original triangle was an *equilateral triangle* (i.e. all three sides had the same length), its three angles were all the same, namely 60°. Recall from elementary geometry that the bisector from the vertex angle of an equilateral triangle to its opposite side bisects both the vertex angle and the opposite side. So as in the figure on the right, the triangle $\triangle ABC$ has angle $A=60^\circ$ and angle $B=30^\circ$, which forces the angle C to be 90°. Thus, $\triangle ABC$ is a right triangle. We see that the hypotenuse has length $c=AB=2$ and the leg $\overline{AC}$ has length $b=AC=1$. By the Pythagorean Theorem, the length a of the leg $\overline{BC}$ is given by

$$a^2 + b^2 = c^2 \quad \Rightarrow \quad a^2 = 2^2 - 1^2 = 3 \quad \Rightarrow \quad a = \sqrt{3}\,.$$

Thus, using the angle A we get:

$$\sin 60^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \tan 60^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\csc 60^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{3}} \qquad \sec 60^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = 2 \qquad \cot 60^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\sqrt{3}}$$

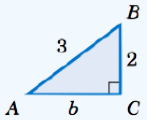
Notice that, as a bonus, we get the values of all six trigonometric functions of 30°, by using angle $B = 30^\circ$ in the same triangle $\triangle ABC$ above:

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}}$$

$$\csc 30^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = 2 \qquad \sec 30^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}} \qquad \cot 30^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Example 1.8

A is an acute angle such that $\sin A = \frac{2}{3}$. Find the values of the other trigonometric functions of A.



Solution:

In general it helps to draw a right triangle to solve problems of this type. The reason is that the trigonometric functions were defined in terms of ratios of sides of a right triangle, and you are given one such function (the sine, in this case) already in terms of a ratio: $\sin A = \frac{2}{3}$. Since $\sin A$ is defined as $\frac{\text{opposite}}{\text{hypotenuse}}$, use 2 as the length of the side opposite A and use 3 as the length of the hypotenuse in a right triangle $\triangle ABC$ (see the figure above), so that $\sin A = \frac{2}{3}$. The adjacent side to A has unknown length b, but we can use the Pythagorean Theorem to find it:

$$2^2 + b^2 = 3^2 \quad \Rightarrow \quad b^2 = 9 - 4 = 5 \quad \Rightarrow \quad b = \sqrt{5}$$

We now know the lengths of all sides of the triangle $\triangle ABC$, so we have:

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{\sqrt{5}}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{2} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{\sqrt{5}} \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{5}}{2}$$

You may have noticed the connections between the sine and cosine, secant and cosecant, and tangent and cotangent of the complementary angles in Examples 1.5 and 1.7. Generalizing those examples gives us the following theorem:

Theorem 1.2 Cofunction Theorem

If A and B are the complementary acute angles in a right triangle $\triangle ABC$, then the following relations hold:

$$\sin A = \cos B \qquad \sec A = \csc B \qquad \tan A = \cot B$$

$$\sin B = \cos A \qquad \sec B = \csc A \qquad \tan B = \cot A$$

We say that the pairs of functions $\{ \sin, \cos \}$, $\{ \sec, \csc \}$, and $\{ \tan, \cot \}$ are **cofunctions**

So sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. That is how the functions cosine, cosecant, and cotangent got the "co" in their names. The Cofunction Theorem says that any trigonometric

function of an acute angle is equal to its cofunction of the complementary angle.

Example 1.9

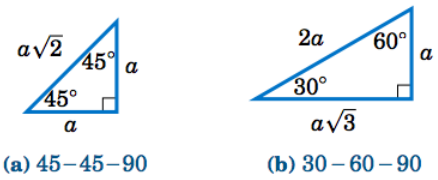
Write each of the following numbers as trigonometric functions of an angle less than 45° : **(a)** $\sin\ 65^\circ$; **(b)** $\cos\ 78^\circ$; **(c)** $\tan\ 59^\circ$.

Solution

(a) The complement of 65° is $90^\circ - 65^\circ = 25^\circ$ and the cofunction of sin is cos, so by the Cofunction Theorem we know that $\sin\ 65^\circ = \cos\ 25^\circ$.

(b) The complement of 78° is $90^\circ - 78^\circ = 12^\circ$ and the cofunction of cos is sin, so $\cos\ 78^\circ = \sin\ 12^\circ$.

(c) The complement of 59° is $90^\circ - 59^\circ = 31^\circ$ and the cofunction of tan is cot, so $\tan\ 59^\circ = \cot\ 31^\circ$.



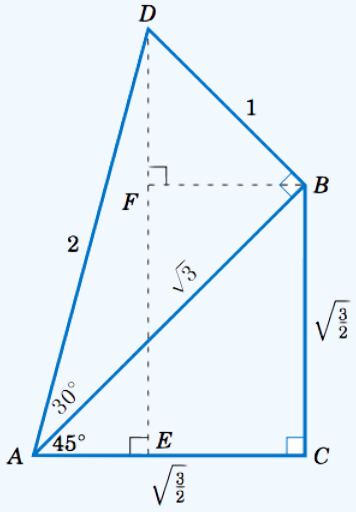
(a) 45−45−90 **(b)** 30−60−90

Figure 1.2.2 Two general right triangles (any $a > 0$)

The angles 30°, 45°, and 60° arise often in applications. We can use the Pythagorean Theorem to generalize the right triangles in Examples 1.6 and 1.7 and see what *any* $45-45-90$ and $30-60-90$ right triangles look like, as in Figure 1.2.2 above.

Example 1.10

Find the sine, cosine, and tangent of 75°.



Solution

Since $75^\circ = 45^\circ + 30^\circ$, place a $30-60-90$ right triangle $\triangle ADB$ with legs of length $\sqrt{3}$ and 1 on top of the hypotenuse of a $45-45-90$ right triangle $\triangle ABC$ whose hypotenuse has length $\sqrt{3}$, as in the figure on the right. From Figure 1.2.2(a) we know that the length of each leg of $\triangle ABC$ is the length of the hypotenuse divided by $\sqrt{2}$. So $AC = BC = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$. Draw $\overline{DE}$ perpendicular to $\overline{AC}$, so that $\triangle ADE$ is a right triangle. Since $\angle BAC = 45^\circ$ and $\angle DAB = 30^\circ$, we see that $\angle DAE = 75^\circ$ since it is the sum of those two angles. Thus, we need to find the sine, cosine, and tangent of $\angle DAE$.

Notice that $\angle ADE = 15^\circ$, since it is the complement of $\angle DAE$. And $\angle ADB = 60^\circ$, since it is the complement of $\angle DAB$. Draw $\overline{BF}$ perpendicular to $\overline{DE}$, so that $\triangle DFB$ is a right triangle. Then $\angle BDF = 45^\circ$, since it is the difference of

$\angle ADB = 60^\circ$ and $\angle ADE = 15^\circ$. Also, $\angle DBF = 45^\circ$ since it is the complement of $\angle BDF$. The hypotenuse $\overline{BD}$ of $\triangle DFB$ has length 1 and $\triangle DFB$ is a $45-45-90$ right triangle, so we know that $DF = FB = \frac{1}{\sqrt{2}}$.

Now, we know that $\overline{DE} \perp \overline{AC}$ and $\overline{BC} \perp \overline{AC}$, so $\overline{FE}$ and $\overline{BC}$ are parallel. Likewise, $\overline{FB}$ and $\overline{EC}$ are both perpendicular to $\overline{DE}$ and hence $\overline{FB}$ is parallel to $\overline{EC}$. Thus, $FBCE$ is a rectangle, since $\angle BCE$ is a right angle. So $EC = FB = \frac{1}{\sqrt{2}}$ and $FE = BC = \sqrt{\frac{3}{2}}$. Hence,

$$DE = DF + FE = \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}} = \frac{\sqrt{3}+1}{\sqrt{2}} \quad \text{and} \quad AE = AC - EC = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{2}}. \text{ Thus,}$$

$$\sin 75^\circ = \frac{DE}{AD} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}, \quad \cos 75^\circ = \frac{AE}{AD} = \frac{\frac{\sqrt{3}-1}{\sqrt{2}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}, \text{ and } \tan 75^\circ = \frac{DE}{AE} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{\frac{\sqrt{3}-1}{\sqrt{2}}} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}.$$

Note: Taking reciprocals, we get $\csc 75^\circ = \frac{4}{\sqrt{6}+\sqrt{2}}$, $\sec 75^\circ = \frac{4}{\sqrt{6}-\sqrt{2}}$, and $\cot 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.

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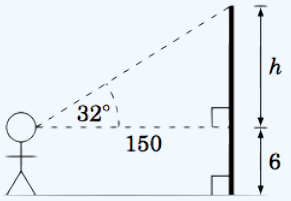
9.4: Applications and Solving Right Triangles

Throughout its early development, trigonometry was often used as a means of indirect measurement, e.g. determining large distances or lengths by using measurements of angles and small, known distances. Today, trigonometry is widely used in physics, astronomy, engineering, navigation, surveying, and various fields of mathematics and other disciplines. In this section we will see some of the ways in which trigonometry can be applied. Your calculator should be in degree mode for these examples.

Example 1.11

A person stands 150 ft away from a flagpole and measures an *angle of elevation* of 32° from his horizontal line of sight to the top of the flagpole. Assume that the person's eyes are a vertical distance of 6 ft from the ground. What is the height of the flagpole?



Solution:

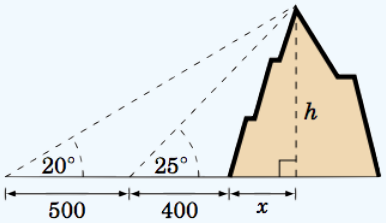
The picture on the right describes the situation. We see that the height of the flagpole is $h+6$ ft, where

$$\frac{h}{150} = \tan 32^\circ \quad \Rightarrow \quad h = 150 \tan 32^\circ = 150\,(0.6249) = 94\,.$$

How did we know that $\tan 32^\circ = 0.6249$? By using a calculator. And since none of the numbers we were given had decimal places, we rounded off the answer for h to the nearest integer. Thus, the height of the flagpole is $h+6 = 94+6 = \boxed{100 \text{ ft}}$.

Example 1.12

A person standing 400 ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be 25°. The person then walks 500 ft straight back and measures the angle of elevation to now be 20°. How tall is the mountain?



Solution:

We will assume that the ground is flat and not inclined relative to the base of the mountain. Let h be the height of the mountain, and let x be the distance from the base of the mountain to the point directly beneath the top of the mountain, as in the picture on the right. Then we see that

$$\frac{h}{x+400} = \tan 25^\circ \quad \Rightarrow \quad h = (x+400) \tan 25^\circ \text{ , and}$$

$$\frac{h}{x+400+500} = \tan 20^\circ \quad \Rightarrow \quad h = (x+900) \tan 20^\circ,$$

so

$(x+400)\ \tan 25^\circ = (x+900)\ \tan 20^\circ$, since they both equal h. Use that equation to solve for x:

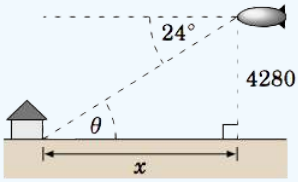
$$x\ \tan 25^\circ - x\ \tan 20^\circ = 900\ \tan 20^\circ - 400\ \tan 25^\circ \quad \Rightarrow \quad x = \frac{900\ \tan 20^\circ - 400\ \tan 25^\circ}{\tan 25^\circ - \tan 20^\circ} = 1378 \text{ ft}$$

Finally, substitute x into the first formula for h to get the height of the mountain:

$$h = (1378+400)\ \tan 25^\circ = 1778\ (0.4663) = \boxed{829 \text{ ft}}$$

Example 1.13

A blimp 4280 ft above the ground measures an *angle of depression* of 24° from its horizontal line of sight to the base of a house on the ground. Assuming the ground is flat, how far away along the ground is the house from the blimp?



Solution:

Let x be the distance along the ground from the blimp to the house, as in the picture to the right. Since the ground and the blimp's horizontal line of sight are parallel, we know from elementary geometry that the angle of elevation θ from the base of the house to the blimp is equal to the angle of depression from the blimp to the base of the house, i.e. $\theta = 24^\circ$. Hence,

$$\frac{4280}{x} = \tan 24^\circ \quad \Rightarrow \quad x = \frac{4280}{\tan 24^\circ} = \boxed{9613 \text{ ft}}.$$

Example 1.14

An observer at the top of a mountain 3 miles above sea level measures an angle of depression of 2.23° to the ocean horizon. Use this to estimate the radius of the earth.

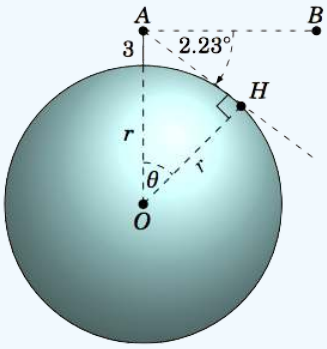


Figure 1.3.1

Solution:

We will assume that the earth is a sphere. Let r be the radius of the earth. Let the point A represent the top of the mountain, and let H be the ocean horizon in the line of sight from A, as in Figure 1.3.1. Let O be the center of the earth, and let B be a point on the horizontal line of sight from A (i.e. on the line perpendicular to $\overline{OA}$). Let θ be the angle $\angle AOH$.

Since A is 3 miles above sea level, we have $OA = r+3$. Also, $OH = r$. Now since $\overline{AB} \perp \overline{OA}$, we have $\angle OAB = 90^\circ$, so we see that $\angle OAH = 90^\circ - 2.23^\circ = 87.77^\circ$. We see that the line through A and H is a tangent line to the surface of the earth (considering the surface as the circle of radius r through H as in the picture). So by Exercise 14 in Section 1.1,

$\overline{AH} \perp \overline{OH}$ and hence $\angle OHA = 90^\circ$. Since the angles in the triangle $\triangle OAH$ add up to 180°, we have $\theta = 180^\circ - 90^\circ - 87.77^\circ = 2.23^\circ$. Thus,

$$\cos\,\theta = \frac{OH}{OA} = \frac{r}{r+3} \quad \Rightarrow \quad \frac{r}{r+3} = \cos\,2.23^\circ\,,$$

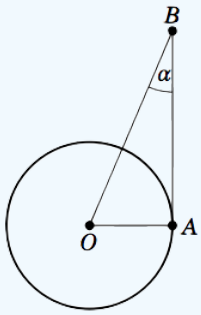
so solving for r we get

$$\begin{aligned} r = (r+3)\,\cos\,2.23^\circ \quad &\Rightarrow \quad r - r\,\cos\,2.23^\circ = 3\,\cos\,2.23^\circ \\ &\Rightarrow \quad r = \frac{3\,\cos\,2.23^\circ}{1\,-\,\cos\,2.23^\circ} \\ &\Rightarrow \quad \boxed{r = 3958.3 \text{ miles}}\,. \end{aligned}$$

Note: This answer is very close to the earth's actual (mean) radius of 3956.6miles.

Example 1.15

As another application of trigonometry to astronomy, we will find the distance from the earth to the sun. Let O be the center of the earth, let A be a point on the equator, and let B represent an object (e.g. a star) in space, as in the picture on the right. If the earth is positioned in such a way that the angle $\angle OAB = 90^\circ$, then we say that the angle $\alpha = \angle OBA$ is the *equatorial parallax* of the object. The equatorial parallax of the sun has been observed to be approximately $\alpha = 0.00244^\circ$. Use this to estimate the distance from the center of the earth to the sun.



Solution:

Let B be the position of the sun. We want to find the length of $\overline{OB}$. We will use the actual radius of the earth, mentioned at the end of Example 1.14, to get $OA = 3956.6$ miles. Since $\angle OAB = 90^\circ$, we have

$$\frac{OA}{OB} = \sin\,\alpha \quad \Rightarrow \quad OB = \frac{OA}{\sin\,\alpha} = \frac{3956.6}{\sin\,0.00244^\circ} = 92908394\,,$$

so the distance from the center of the earth to the sun is approximately $\boxed{93 \text{ million miles}}$.

Note: The earth's orbit around the sun is an ellipse, so the actual distance to the sun varies.

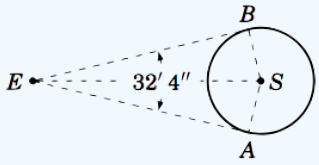
In the above example we used a very small angle (0.00244°). A degree can be divided into smaller units: a **minute** is one-sixtieth of a degree, and a **second** is one-sixtieth of a minute. The symbol for a minute is $'$ and the symbol for a second is $''$. For example, $4.5^\circ = 4^\circ\ 30'$. And $4.505^\circ = 4^\circ\ 30'\ 18''$:

$$4^\circ\ 30'\ 18'' = 4 + \frac{30}{60} + \frac{18}{3600}\ \text{degrees} = 4.505^\circ$$

In Example 1.15 we used $\alpha = 0.00244^\circ \approx 8.8''$, which we mention only because some angle measurement devices do use minutes and seconds.

Example 1.16

An observer on earth measures an angle of $32' \ 4''$ from one visible edge of the sun to the other (opposite) edge, as in the picture on the right. Use this to estimate the radius of the sun.



Solution:

Let the point E be the earth and let S be the center of the sun. The observer's lines of sight to the visible edges of the sun are tangent lines to the sun's surface at the points A and B. Thus, $\angle EAS = \angle EBS = 90^\circ$. The radius of the sun equals AS. Clearly $AS = BS$. So since $EB = EA$ (why?), the triangles $\triangle EAS$ and $\triangle EBS$ are similar. Thus, $\angle AES = \angle BES = \frac{1}{2} \angle AEB = \frac{1}{2} (32' \ 4'') = 16' \ 2'' = (16/60) + (2/3600) = 0.26722^\circ$.

Now, ES is the distance from the *surface* of the earth (where the observer stands) to the center of the sun. In Example 1.15 we found the distance from the *center* of the earth to the sun to be $92,908,394$miles. Since we treated the sun in that example as a point, then we are justified in treating that distance as the distance between the centers of the earth and sun. So $ES = 92908394 -$ radius of earth $= 92908394 - 3956.6 = 92904437.4$miles. Hence,

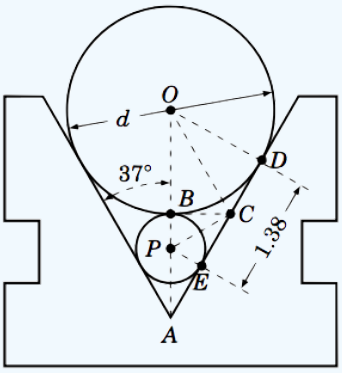
$$\sin(\angle AES) = \frac{AS}{ES} \quad \Rightarrow \quad AS = ES \sin 0.26722^\circ = (92904437.4) \sin 0.26722^\circ = \boxed{433,293 \text{ miles}}.$$

Note: This answer is close to the sun's actual (mean) radius of $432,200$miles.

You may have noticed that the solutions to the examples we have shown required at least one right triangle. In applied problems it is not always obvious which right triangle to use, which is why these sorts of problems can be difficult. Often no right triangle will be immediately evident, so you will have to create one. There is no general strategy for this, but remember that a right triangle requires a right angle, so look for places where you can form perpendicular line segments. When the problem contains a circle, you can create right angles by using the perpendicularity of the tangent line to the circle at a point with the line that joins that point to the center of the circle. We did exactly that in Examples 1.14, 1.15, and 1.16.

Example 1.17

The machine tool diagram on the right shows a symmetric *V-block*, in which one circular roller sits on top of a smaller circular roller. Each roller touches both slanted sides of the V-block. Find the diameter d of the large roller, given the information in the diagram.



Solution:

The diameter d of the large roller is twice the radius OB, so we need to find OB. To do this, we will show that $\triangle OBC$ is a right triangle, then find the angle $\angle BOC$, and then find BC. The length OB will then be simple to determine.

Since the slanted sides are tangent to each roller, $\angle ODA = \angle PEC = 90^\circ$. By symmetry, since the vertical line through the centers of the rollers makes a 37° angle with each slanted side, we have $\angle OAD = 37^\circ$. Hence, since $\triangle ODA$ is a right triangle, $\angle DOA$ is the complement of $\angle OAD$. So $\angle DOA = 53^\circ$.

Since the horizontal line segment $\overline{BC}$ is tangent to each roller, $\angle OBC = \angle PBC = 90^\circ$. Thus, $\triangle OBC$ is a right triangle. And since $\angle ODA = 90^\circ$, we know that $\triangle ODC$ is a right triangle. Now, $OB = OD$ (since they each equal the radius of the large roller), so by the Pythagorean Theorem we have $BC = DC$:

$$BC^2 = OC^2 - OB^2 = OC^2 - OD^2 = DC^2 \quad \Rightarrow \quad BC = DC$$

Thus, $\triangle OBC$ and $\triangle ODC$ are *congruent triangles* (which we denote by $\triangle OBC \cong \triangle ODC$), since their corresponding sides are equal. Thus, their corresponding angles are equal. So in particular, $\angle BOC = \angle DOC$. We know that $\angle DOB = \angle DOA = 53^\circ$. Thus,

$$53^\circ = \angle DOB = \angle BOC + \angle DOC = \angle BOC + \angle BOC = 2\angle BOC \quad \Rightarrow \quad \angle BOC = 26.5^\circ.$$

Likewise, since $BP = EP$ and $\angle PBC = \angle PEC = 90^\circ$, $\triangle BPC$ and $\triangle EPC$ are congruent right triangles. Thus, $BC = EC$. But we know that $BC = DC$, and we see from the diagram that $EC + DC = 1.38$. Thus, $BC + BC = 1.38$ and so $BC = 0.69$. We now have all we need to find OB:

$$\frac{BC}{OB} = \tan \angle BOC \quad \Rightarrow \quad OB = \frac{BC}{\tan \angle BOC} = \frac{0.69}{\tan 26.5^\circ} = 1.384$$

Hence, the diameter of the large roller is $d = 2 \times OB = 2\,(1.384) = \boxed{2.768}$ ~.

Example 1.18

A *slider-crank mechanism* is shown in Figure 1.3.2 below. As the piston moves downward the connecting rod rotates the crank in the clockwise direction, as indicated.

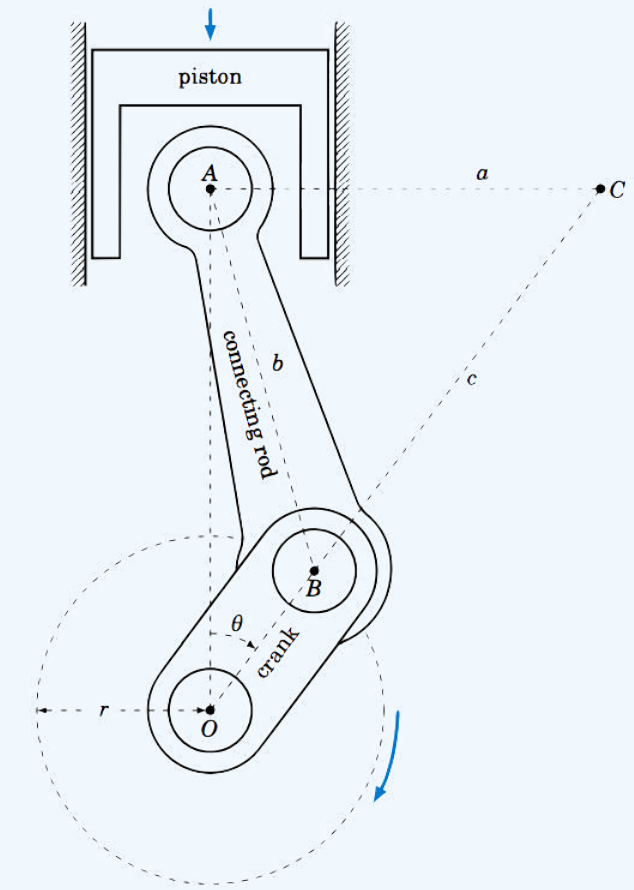
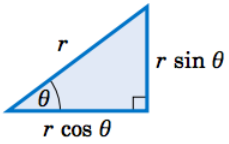


Figure 1.3.2 Slider-crank mechanism

The point A is the center of the connecting rod's *wrist pin* and only moves vertically. The point B is the center of the *crank pin* and moves around a circle of radius r centered at the point O, which is directly below A and does not move. As the crank rotates it makes an angle θ with the line $\overline{OA}$. The *instantaneous center of rotation* of the connecting rod at a given time is the point C where the horizontal line through A intersects the extended line through O and B. From Figure 1.3.2 we see that $\angle OAC = 90^\circ$, and we let $a = AC$, $b = AB$, and $c = BC$. In the exercises you will show that for $0^\circ < \theta < 90^\circ$,

$$c = \frac{\sqrt{b^2 - r^2\,(\sin\theta)^2}}{\cos\theta} \qquad \text{and} \qquad a = r\,\sin\theta + \sqrt{b^2 - r^2\,(\sin\theta)^2}\,\tan\theta\,.$$

For some problems it may help to remember that when a right triangle has a hypotenuse of length r and an acute angle θ, as in the picture below, the adjacent side will have length $r\,\cos\,\theta$ and the opposite side will have length $r\,\sin\,\theta$. You can think of those lengths as the horizontal and vertical ``components" of the hypotenuse.



Notice in the above right triangle that we were given two pieces of information: one of the acute angles and the length of the hypotenuse. From this we determined the lengths of the other two sides, and the other acute angle is just the complement of the known acute angle. In general, a triangle has six parts: three sides and three angles. **Solving a triangle** means finding the unknown parts based on the known parts. In the case of a right triangle, one part is always known: one of the angles is 90°.

Example 1.19

Solve the right triangle in Figure 1.3.3 using the given information:

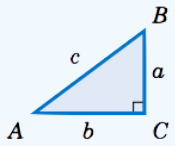


Figure 1.3.3

(a) $c = 10,\ A = 22^\circ$

Solution: The unknown parts are a, b, and B. Solving yields:

$$a = c \sin A = 10 \sin 22^\circ = 3.75 \quad (9.4.1)$$
$$b = c \cos A = 10 \cos 22^\circ = 9.27 \quad (9.4.2)$$
$$B = 90^\circ - A = 90^\circ - 22^\circ = 68^\circ \quad (9.4.3)$$

(b) $b = 8,\ A = 40^\circ$

Solution: The unknown parts are a, c, and B. Solving yields:

$$\frac{a}{b} = \tan A \quad \Rightarrow \quad a = b \tan A = 8 \tan 40^\circ = 6.71 \quad (9.4.4)$$
$$\frac{b}{c} = \cos A \quad \Rightarrow \quad c = \frac{b}{\cos A} = \frac{8}{\cos 40^\circ} = 10.44 \quad (9.4.5)$$

(c) $a = 3,\ b = 4$

Solution: The unknown parts are c, A, and B. By the Pythagorean Theorem,

$$c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5\,.$$

Now, $\tan A = \frac{a}{b} = \frac{3}{4} = 0.75$. So how do we find A? There should be a key labeled $\boxed{\tan^{-1}}$ on your calculator, which works like this: give it a number x and it will tell you the angle θ such that $\tan \theta = x$. In our case, we want the angle A such that $\tan A = 0.75$:

Enter: 0.75 Press: $\boxed{\tan^{-1}}$ Answer: 36.86989765

This tells us that $A = 36.87^\circ$, approximately. Thus $B = 90^\circ - A = 90^\circ - 36.87^\circ = 53.13^\circ$.

Note: The $\boxed{\sin^{-1}}$ and $\boxed{\cos^{-1}}$ keys work similarly for sine and cosine, respectively. These keys use the *inverse trigonometric functions*, which we will discuss in Chapter 5.

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9.5: Trigonometric Functions of Any Angle

To define the trigonometric functions of *any* angle - including angles less than 0° or greater than 360° - we need a more general definition of an angle. We say that an **angle** is formed by rotating a ray $\overrightarrow{OA}$ about the endpoint O (called the **vertex**), so that the ray is in a new position, denoted by the ray $\overrightarrow{OB}$. The ray $\overrightarrow{OA}$ is called the **initial side** of the angle, and $\overrightarrow{OB}$ is the **terminal side** of the angle (see Figure 1.4.1(a)).

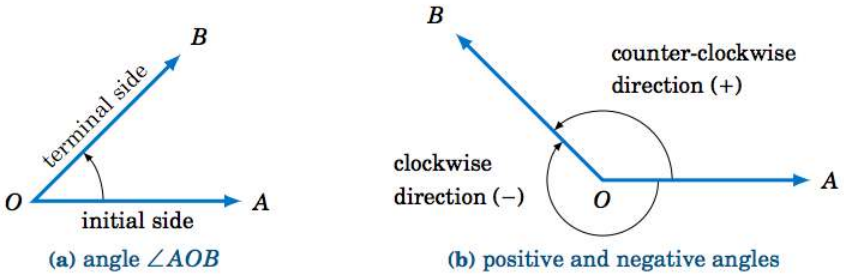


Figure 1.4.1 Definition of a general angle

We denote the angle formed by this rotation as $\angle AOB$, or simply $\angle O$, or even just O. If the rotation is counter-clockwise then we say that the angle is **positive**, and the angle is **negative** if the rotation is clockwise (see Figure 1.4.1(b)).

One full counter-clockwise rotation of $\overrightarrow{OA}$ back onto itself (called a **revolution**), so that the terminal side coincides with the initial side, is an angle of 360°; in the clockwise direction this would be -360°. Not rotating $\overrightarrow{OA}$ constitutes an angle of 0°. More than one full rotation creates an angle greater than 360°. For example, notice that 30° and 390° have the same terminal side in Figure 1.4.2, since $30 + 360 = 390$.

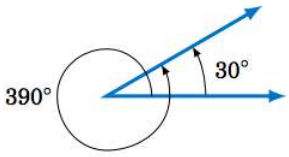


Figure 1.4.2 Angle greater than 360°

We can now define the trigonometric functions of any angle in terms of **Cartesian coordinates**. Recall that the **xy-coordinate plane** consists of points denoted by pairs (x, y) of real numbers. The first number, x, is the point's **x coordinate**, and the second number, y, is its **y coordinate**. The x and y coordinates are measured by their positions along the **x-axis** and **y-axis**, respectively, which determine the point's position in the plane. This divides the xy-coordinate plane into four **quadrants** (denoted by QI, QII, QIII, QIV), based on the signs of x and y (see Figure 1.4.3(a)-(b)).

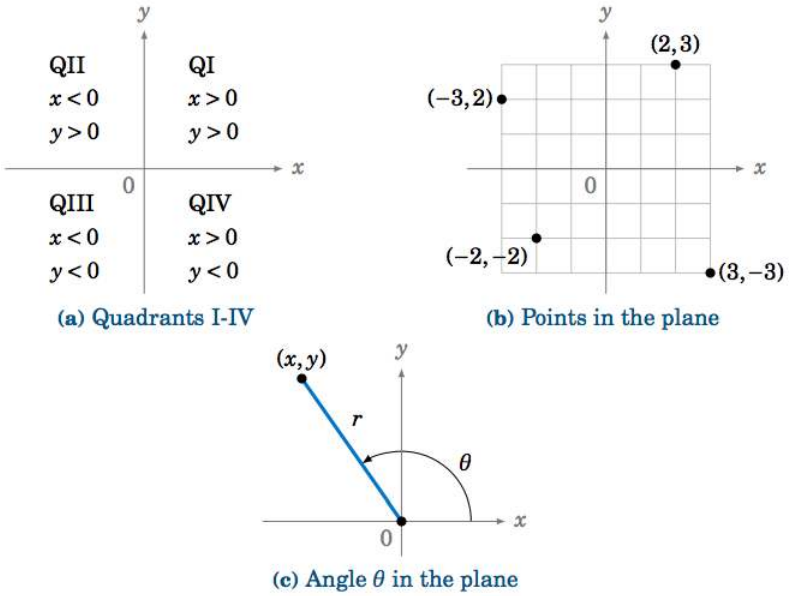


Figure 1.4.3 xy-coordinate plane

Now let θ be any angle. We say that θ is in **standard position** if its initial side is the positive x-axis and its vertex is the origin $(0,0)$. Pick *any* point (x,y) on the terminal side of θ a distance $r>0$ from the origin (see Figure 1.4.3(c)). (Note that $r=\sqrt{x^2+y^2}$. Why?) We then define the trigonometric functions of θ as follows:

$$\sin\theta=\frac{y}{r} \qquad \cos\theta=\frac{x}{r} \qquad \tan\theta=\frac{y}{x} \tag{9.5.1}$$

$$\csc\theta=\frac{r}{y} \qquad \sec\theta=\frac{r}{x} \qquad \cot\theta=\frac{x}{y} \tag{9.5.2}$$

As in the acute case, by the use of similar triangles these definitions are well-defined (i.e. they do not depend on which point (x,y) we choose on the terminal side of θ). Also, notice that $|\sin\theta|\le 1$ and $|\cos\theta|\le 1$, since $|y|\le r$ and $|x|\le r$ in the above definitions.

Notice that in the case of an acute angle these definitions are equivalent to our earlier definitions in terms of right triangles: draw a right triangle with angle θ such that $x=\text{adjacent side}$, $y=\text{opposite side}$, and $r=\text{hypotenuse}$. For example, this would give us $\sin\theta=\frac{y}{r}=\frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta=\frac{x}{r}=\frac{\text{adjacent}}{\text{hypotenuse}}$, just as before (see Figure 1.4.4(a)).

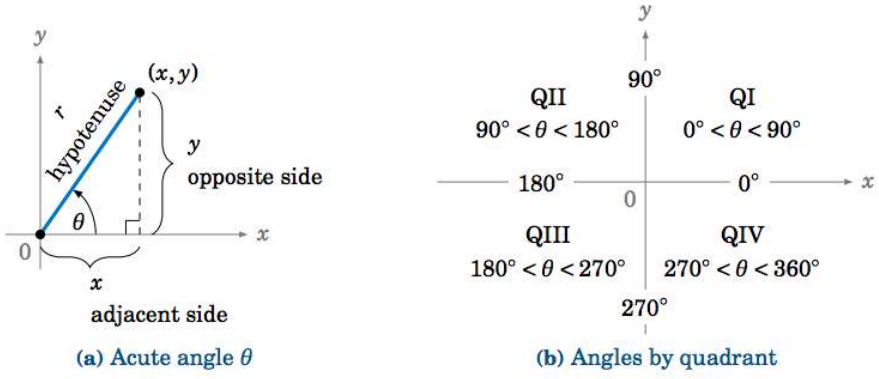


Figure 1.4.4

In Figure 1.4.4(b) we see in which quadrants or on which axes the terminal side of an angle $0^\circ\le\theta<360^\circ$ may fall. From Figure 1.4.3(a) and formulas Equation 9.5.1 and Equation 9.5.2, we see that we can get negative values for a trigonometric function. For example, $\sin\theta<0$ when $y<0$. Figure 1.4.5 summarizes the signs (positive or negative) for the trigonometric functions based on the angle's quadrant:

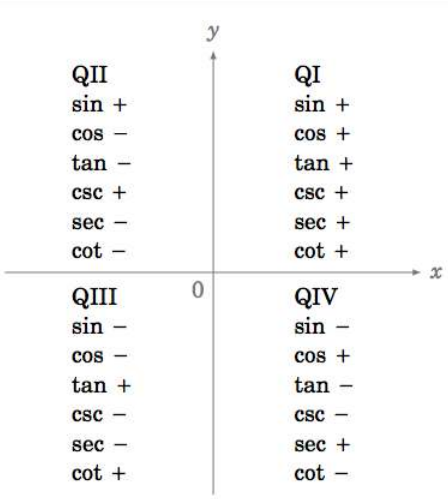
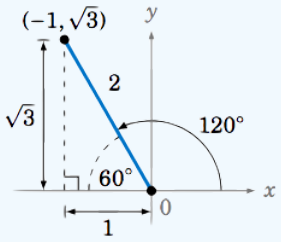


Figure 1.4.5 Signs of the trigonometric functions by quadrant

Example 1.20

Find the exact values of all six trigonometric functions of 120°.



Solution

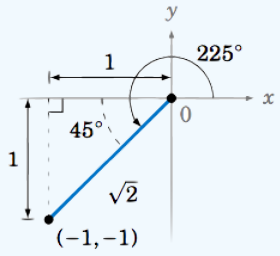
We know $120^\circ = 180^\circ - 60^\circ$. By Example 1.7 in Section 1.2, we see that we can use the point $(-1, \sqrt{3})$ on the terminal side of the angle 120° in QII, since we saw in that example that a basic right triangle with a 60° angle has adjacent side of length 1, opposite side of length $\sqrt{3}$, and hypotenuse of length 2, as in the figure on the right. Drawing that triangle in QII so that the hypotenuse is on the terminal side of 120° makes $r = 2$, $x = -1$, and $y = \sqrt{3}$. Hence:

$$\sin 120^\circ = \frac{y}{r} = \frac{\sqrt{3}}{2} \qquad \cos 120^\circ = \frac{x}{r} = \frac{-1}{2} \qquad \tan 120^\circ = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc 120^\circ = \frac{r}{y} = \frac{2}{\sqrt{3}} \qquad \sec 120^\circ = \frac{r}{x} = \frac{2}{-1} = -2 \qquad \cot 120^\circ = \frac{x}{y} = \frac{-1}{\sqrt{3}}$$

Example 1.21

Find the exact values of all six trigonometric functions of 225°.



Solution

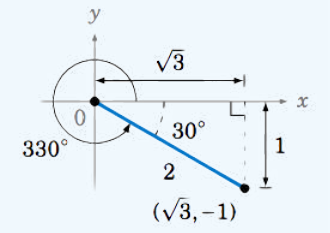
We know that $225^\circ = 180^\circ + 45^\circ$. By Example 1.6 in Section 1.2, we see that we can use the point $(-1, -1)$ on the terminal side of the angle 225° in QIII, since we saw in that example that a basic right triangle with a 45° angle has adjacent side of length 1, opposite side of length 1, and hypotenuse of length $\sqrt{2}$, as in the figure on the right. Drawing that triangle in QIII so that the hypotenuse is on the terminal side of 225° makes $r = \sqrt{2}$, $x = -1$, and $y = -1$. Hence:

$$\sin 225^\circ = \frac{y}{r} = \frac{-1}{\sqrt{2}} \qquad \cos 225^\circ = \frac{x}{r} = \frac{-1}{\sqrt{2}} \qquad \tan 225^\circ = \frac{y}{x} = \frac{-1}{-1} = 1$$

$$\csc 225^\circ = \frac{r}{y} = -\sqrt{2} \qquad \sec 225^\circ = \frac{r}{x} = -\sqrt{2} \qquad \cot 225^\circ = \frac{x}{y} = \frac{-1}{-1} = 1$$

Example 1.22

Find the exact values of all six trigonometric functions of 330°.



Solution

We know that $330^\circ = 360^\circ - 30^\circ$. By Example 1.7 in Section 1.2, we see that we can use the point $(\sqrt{3}, -1)$ on the terminal side of the angle 225° in QIV, since we saw in that example that a basic right triangle with a 30° angle has adjacent side of length $\sqrt{3}$, opposite side of length 1, and hypotenuse of length 2, as in the figure on the right. Drawing that triangle in QIV so that the hypotenuse is on the terminal side of 330° makes $r = 2$, $x = \sqrt{3}$, and $y = -1$. Hence:

$$\sin 330^\circ = \frac{y}{r} = \frac{-1}{2} \qquad \cos 330^\circ = \frac{x}{r} = \frac{\sqrt{3}}{2} \qquad \tan 330^\circ = \frac{y}{x} = \frac{-1}{\sqrt{3}}$$

$$\csc 330^\circ = \frac{r}{y} = -2 \qquad \sec 330^\circ = \frac{r}{x} = \frac{2}{\sqrt{3}} \qquad \cot 330^\circ = \frac{x}{y} = -\sqrt{3}$$

Example 1.23

Find the exact values of all six trigonometric functions of 0°, 90°, 180°, and 270°.

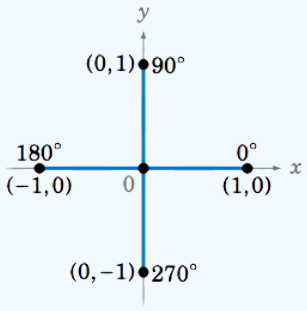


Figure 1.4.6

Solution

These angles are different from the angles we have considered so far, in that the terminal sides lie along either the x-axis or the y-axis. So unlike the previous examples, we do not have any right triangles to draw. However, the values of the trigonometric functions are easy to calculate by picking the simplest points on their terminal sides and then using the definitions in formulas Equation 9.5.1 and Equation 9.5.2.

For instance, for the angle 0° use the point $(1,0)$ on its terminal side (the positive x-axis), as in Figure 1.4.6. You could think of the line segment from the origin to the point $(1,0)$ as sort of a degenerate right triangle whose height is 0 and whose hypotenuse and base have the same length 1. Regardless, in the formulas we would use $r=1$, $x=1$, and $y=0$. Hence:

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0 \qquad \cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1 \qquad \tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0$$

$$\csc 0^\circ = \frac{r}{y} = \frac{1}{0} = \text{undefined} \qquad \sec 0^\circ = \frac{r}{x} = \frac{1}{1} = 1 \qquad \cot 0^\circ = \frac{x}{y} = \frac{1}{0} = \text{undefined}$$

Note that $\csc 0^\circ$ and $\cot 0^\circ$ are undefined, since division by 0 is not allowed.

Similarly, from Figure 1.4.6 we see that for 90° the terminal side is the positive y-axis, so use the point $(0,1)$. Again, you could think of the line segment from the origin to $(0,1)$ as a degenerate right triangle whose base has length 0 and whose height equals the length of the hypotenuse. We have $r=1$, $x=0$, and $y=1$, and hence:

$$\sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1 \qquad \cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan 90^\circ = \frac{y}{x} = \frac{1}{0} = \text{undefined}$$

$$\csc 90^\circ = \frac{r}{y} = \frac{1}{1} = 1 \qquad \sec 90^\circ = \frac{r}{x} = \frac{1}{0} = \text{undefined} \qquad \cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$$

Likewise, for 180° use the point $(-1,0)$ so that $r=1$, $x=-1$, and $y=0$. Hence:

$$\sin 180^\circ = \frac{y}{r} = \frac{0}{1} = 0 \qquad \cos 180^\circ = \frac{x}{r} = \frac{-1}{1} = -1 \qquad \tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0$$

$$\csc 180^\circ = \frac{r}{y} = \frac{1}{0} = \text{undefined} \qquad \sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$$

$$\cot 180^\circ = \frac{x}{y} = \frac{-1}{0} = \text{undefined}$$

Lastly, for 270° use the point $(0,-1)$ so that $r=1$, $x=0$, and $y=-1$. Hence:

$$\sin 270^\circ = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = \text{undefined}$$

$$\csc 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1 \qquad \sec 270^\circ = \frac{r}{x} = \frac{1}{0} = \text{undefined} \qquad \cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$$

The following table summarizes the values of the trigonometric functions of angles between 0° and 360° which are integer multiples of 30° or 45°:

Table 1.3 Table of trigonometric function values

Angle	sin	cos	tan	csc	sec	cot
$0°$	0	1	0	undefined	1	undefined
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$90°$	1	0	undefined	1	undefined	0
$120°$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	$-\frac{1}{\sqrt{3}}$
$135°$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$150°$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	$-\sqrt{3}$
$180°$	0	-1	0	undefined	-1	undefined
$210°$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$225°$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$240°$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2	$\frac{1}{\sqrt{3}}$
$270°$	-1	0	undefined	-1	undefined	0
$300°$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2	$-\frac{1}{\sqrt{3}}$
$315°$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$330°$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$	$-\sqrt{3}$

Since $360°$ represents one full revolution, the trigonometric function values repeat every $360°$. For example, $\sin\ 360° = \sin\ 0°$, $\cos\ 390° = \cos\ 30°$, $\tan\ 540° = \tan\ 180°$, $\sin\ (-45°) = \sin\ 315°$, etc. In general, if two angles differ by an integer multiple of $360°$ then each trigonometric function will have equal values at both angles. Angles such as these, which have the same initial and terminal sides, are called **coterminal**.

In Examples 1.20-1.22, we saw how the values of trigonometric functions of an angle θ larger than $90°$ were found by using a certain acute angle as part of a right triangle. That acute angle has a special name: if θ is a nonacute angle then we say that the **reference angle** for θ is the acute angle formed by the terminal side of θ and either the positive or negative x-axis. So in Example 1.20, we see that $60°$ is the reference angle for the nonacute angle $\theta = 120°$; in Example 1.21, $45°$ is the reference angle for $\theta = 225°$; and in Example 1.22, $30°$ is the reference angle for $\theta = 330°$.

Example 1.24

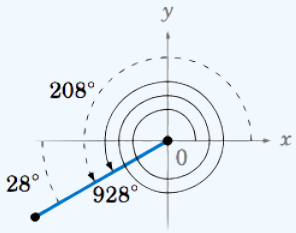


Figure 1.4.7

a. Which angle between $0°$ and $360°$ has the same terminal side (and hence the same trigonometric function values) as θ?
b. What is the reference angle for θ?

Solution

(a) Since $928° = 2 \times 360° + 208°$, then θ has the same terminal side as $208°$, as in Figure 1.4.7.

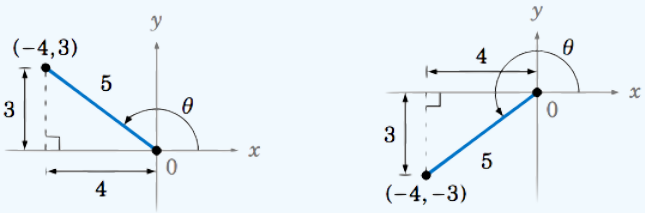
(b) 928° and 208° have the same terminal side in QIII, so the reference angle for $\theta = 928^\circ$ is $208^\circ - 180^\circ = 28^\circ$.

Example 1.25

Suppose that $\cos\ \theta = -\frac{4}{5}$. Find the exact values of $\sin\ \theta$ and $\tan\ \theta$.

Solution

We can use a method similar to the one used to solve Example 1.8 in Section 1.2. That is, draw a right triangle and interpret $\cos\ \theta$ as the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ of two of its sides. Since $\cos\ \theta = -\frac{4}{5}$, we can use 4 as the length of the adjacent side and 5 as the length of the hypotenuse. By the Pythagorean Theorem, the length of the opposite side must then be 3. Since $\cos\ \theta$ is negative, we know from Figure 1.4.5 that θ must be in either QII or QIII. Thus, we have *two* possibilities, as shown in Figure 1.4.8 below:



(a) θ in QII (b) θ in QIII

Figure 1.4.8 $\cos\theta = -\frac{4}{5}$

When θ is in QII, we see from Figure 1.4.8(a) that the point $(-4, 3)$ is on the terminal side of θ, and so we have $x = -4$, $y = 3$, and $r = 5$. Thus, $\sin\ \theta = \frac{y}{r} = \frac{3}{5}$ and $\tan\ \theta = \frac{y}{x} = \frac{3}{-4}$.

When θ is in QIII, we see from Figure 1.4.8(b) that the point $(-4, -3)$ is on the terminal side of θ, and so we have $x = -4$, $y = -3$, and $r = 5$. Thus, $\sin\ \theta = \frac{y}{r} = \frac{-3}{5}$ and $\tan\ \theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$.

Thus, either $\boxed{\sin\ \theta = \frac{3}{5} \text{ and } \tan\ \theta = -\frac{3}{4}}$ or $\boxed{\sin\ \theta = -\frac{3}{5} \text{ and } \tan\ \theta = \frac{3}{4}}$.

Since reciprocals have the same sign, $\csc\ \theta$ and $\sin\ \theta$ have the same sign, $\sec\ \theta$ and $\cos\ \theta$ have the same sign, and $\cot\ \theta$ and $\tan\ \theta$ have the same sign. So it suffices to remember the signs of $\sin\ \theta$, $\cos\ \theta$, and $\tan\ \theta$:

For an angle θ in standard position and a point (x, y) on its terminal side:

a. $\sin\ \theta$ has the same sign as y
b. $\cos\ \theta$ has the same sign as x
c. $\tan\ \theta$ is positive when x and y have the same sign
d. $\tan\ \theta$ is negative when x and y have opposite signs

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9.6: Rotations and Reflections of Angles

Now that we know how to deal with angles of any measure, we will take a look at how certain geometric operations can help simplify the use of trigonometric functions of any angle, and how some basic relations between those functions can be made. The two operations on which we will concentrate in this section are *rotation* and *reflection*.

To **rotate an angle** means to rotate its terminal side around the origin when the angle is in standard position. For example, suppose we rotate an angle θ around the origin by 90° in the counterclockwise direction. In Figure 1.5.1 we see an angle θ in QI which is rotated by 90°, resulting in the angle $\theta+90^\circ$ in QII. Notice that the complement of θ in the right triangle in QI is the same as the supplement of the angle $\theta+90^\circ$ in QII, since the sum of θ, its complement, and 90° equals 180°. This forces the other angle of the right triangle in QII to be θ.

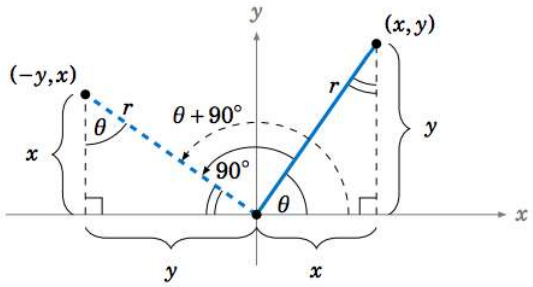


Figure 1.5.1 Rotation of an angle θ by 90°

Thus, the right triangle in QI is similar to the right triangle in QII, since the triangles have the same angles. The rotation of θ by 90° does not change the length r of its terminal side, so the hypotenuses of the similar right triangles are equal, and hence by similarity the remaining corresponding sides are also equal. Using Figure 1.5.1 to match up those corresponding sides shows that the point $(-y, x)$ is on the terminal side of $\theta+90^\circ$ when (x, y) is on the terminal side of θ. Hence, by definition,

$$\sin(\theta+90^\circ) = \frac{x}{r} = \cos\theta, \quad \cos(\theta+90^\circ) = \frac{-y}{r} = -\sin\theta, \quad \tan(\theta+90^\circ) = \frac{x}{-y} = -\cot\theta.$$

Though we showed this for θ in QI, it is easy (see Exercise 4) to use similar arguments for the other quadrants. In general, the following relations hold for all angles θ:

$$\sin(\theta+90^\circ) = \cos\theta \tag{9.6.1}$$

$$\cos(\theta+90^\circ) = -\sin\theta \tag{9.6.2}$$

$$\tan(\theta+90^\circ) = -\cot\theta \tag{9.6.3}$$

Example 1.26

Recall that any nonvertical line in the xy-coordinate plane can be written as $y = mx+b$, where m is the *slope* of the line (defined as $m = \frac{\text{rise}}{\text{run}}$) and b is the *y-intercept*}, i.e. where the line crosses the y-axis (see Figure 1.5.2(a)). We will show that the slopes of perpendicular lines are negative reciprocals. That is, if $y = m_1x+b_1$ and $y = m_2x+b_2$ are nonvertical and nonhorizontal perpendicular lines, then $m_2 = -\frac{1}{m_1}$ (see Figure 1.5.2(b)).

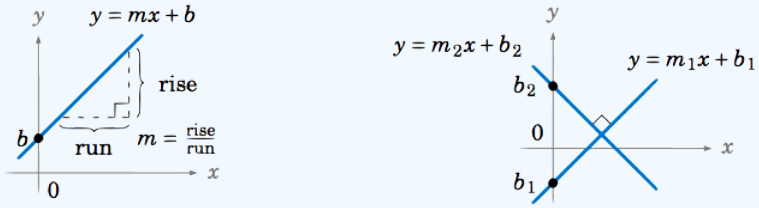
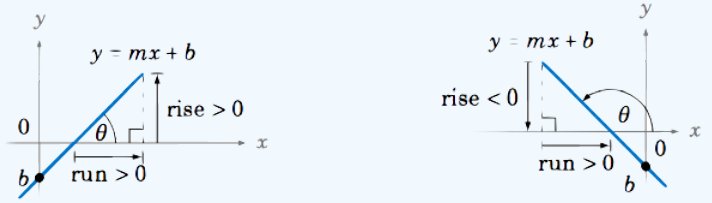


Figure 1.5.2

First, suppose that a line $y = mx + b$ has nonzero slope. The line crosses the x-axis somewhere, so let θ be the angle that the positive x-axis makes with the part of the line above the x-axis, as in Figure 1.5.3. For $m > 0$ we see that θ is acute and $\tan\,\theta = \frac{\text{rise}}{\text{run}} = m$.



(a) $m > 0$, θ acute, rise > 0 (b) $m < 0$, θ obtuse, rise < 0

Figure 1.5.3

If $m < 0$, then we see that θ is obtuse and the rise is negative. Since the run is always positive, our definition of $\tan\,\theta$ from Section 1.4 means that $\tan\,\theta = \frac{-\text{rise}}{-\text{run}} = \frac{\text{rise}}{\text{run}} = m$ (just imagine in Figure 1.5.3(b) the entire line being shifted horizontally to go through the origin, so that θ is unchanged and the point $(-\text{run}, -\text{rise})$ is on the terminal side of θ). Hence:

> *For a line $y = mx + b$ with $m \neq 0$, the slope is given by $m = \tan\,\theta$, where θ is the angle formed by the positive x-axis and the part of the line above the x-axis.}*

Now, in Figure 1.5.2(b) we see that if two lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are perpendicular then rotating one line counterclockwise by 90° around the point of intersection gives us the second line. So if θ is the angle that the line $y = m_1x + b_1$ makes with the positive x-axis, then $\theta + 90^\circ$ is the angle that the line $y = m_2x + b_2$ makes with the positive x-axis. So by what we just showed, $m_1 = \tan\,\theta$ and $m_2 = \tan\,(\theta + 90^\circ)$. But by formula Equation 9.6.3 we know that $\tan\,(\theta + 90^\circ) = -\cot\,\theta$. Hence, $m_2 = -\cot\,\theta = -\frac{1}{\tan\,\theta} = -\frac{1}{m_1}$. **QED**

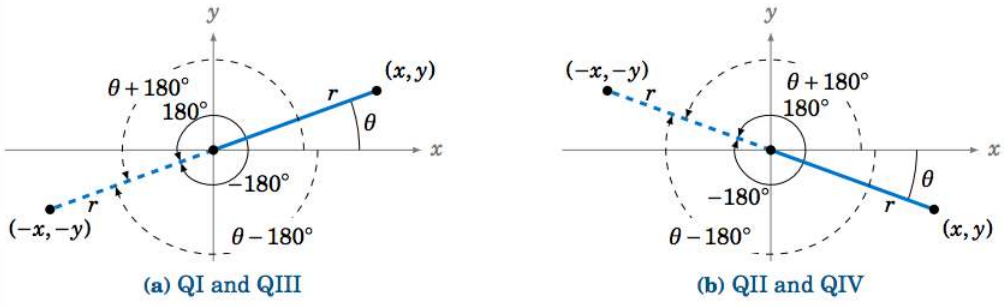
Rotating an angle θ by 90° in the clockwise direction results in the angle $\theta - 90^\circ$. We could use another geometric argument to derive trigonometric relations involving $\theta - 90^\circ$, but it is easier to use a simple trick: since Equations 9.6.1-9.6.3 hold for *any* angle θ, just replace θ by $\theta - 90^\circ$ in each formula. Since $(\theta - 90^\circ) + 90^\circ = \theta$, this gives us:

$$\sin\,(\theta - 90^\circ) = -\cos\,\theta \tag{9.6.4}$$

$$\cos\,(\theta - 90^\circ) = \sin\,\theta \tag{9.6.5}$$

$$\tan\,(\theta - 90^\circ) = -\cot\,\theta \tag{9.6.6}$$

We now consider rotating an angle θ by 180°. Notice from Figure 1.5.4 that the angles $\theta \pm 180^\circ$ have the same terminal side, and are in the quadrant opposite θ.



(a) QI and QIII (b) QII and QIV

Figure 1.5.4 Rotation of θ by $\pm 180^\circ$

Since $(-x, -y)$ is on the terminal side of $\theta \pm 180^\circ$ when (x, y) is on the terminal side of θ, we get the following relations, which hold for all θ:

$$\sin\,(\theta \pm 180^\circ) = -\sin\,\theta \tag{9.6.7}$$

$$\cos(\theta \pm 180^\circ) = -\cos\theta \tag{9.6.8}$$

$$\tan(\theta \pm 180^\circ) = \tan\theta \tag{9.6.9}$$

A **reflection** is simply the mirror image of an object. For example, in Figure 1.5.5 the original object is in QI, its reflection around the y-axis is in QII, and its reflection around the x-axis is in QIV. Notice that if we first reflect the object in QI around the y-axis and then follow that with a reflection around the x-axis, we get an image in QIII. That image is the *reflection around the origin* of the original object, and it is equivalent to a rotation of 180° around the origin. Notice also that a reflection around the y-axis is equivalent to a reflection around the x-axis followed by a rotation of 180° around the origin.

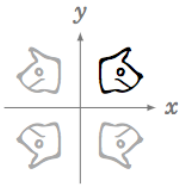


Figure 1.5.5

Applying this to angles, we see that the reflection of an angle θ around the x-axis is the angle $-\theta$, as in Figure 1.5.6.

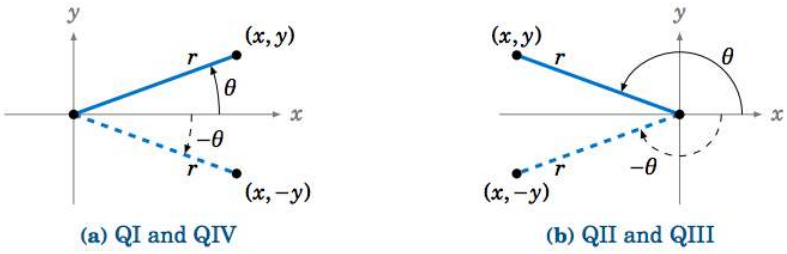


Figure 1.5.6 Reflection of θ around the x-axis

So we see that reflecting a point (x, y) around the x-axis just replaces y by $-y$. Hence:

$$\sin(-\theta) = -\sin\theta \tag{9.6.10}$$

$$\cos(-\theta) = \cos\theta \tag{9.6.11}$$

$$\tan(-\theta) = -\tan\theta \tag{9.6.12}$$

Notice that the cosine function does not change in Equation 9.6.11 because it depends on x, and not on y, for a point (x, y) on the terminal side of θ.

In general, a function $f(x)$ is an **even function** if $f(-x) = f(x)$ for all x, and it is called an **odd function** if $f(-x) = -f(x)$ for all x. Thus, the cosine function is even, while the sine and tangent functions are odd.

Replacing θ by $-\theta$ in Equations 9.6.1-9.6.3, then using Equations 9.6.10-9.6.12, gives:

$$\sin(90^\circ - \theta) = \cos\theta \tag{9.6.13}$$

$$\cos(90^\circ - \theta) = \sin\theta \tag{9.6.14}$$

$$\tan(90^\circ - \theta) = \cot\theta \tag{9.6.15}$$

Note that Equations 9.6.13-9.6.15 extend the Cofunction Theorem from Section 1.2 to *all* θ, not just acute angles. Similarly, Equations 9.6.7-9.6.9 and 9.6.10-9.6.12 give:

$$\sin(180^\circ - \theta) = \sin\theta \tag{9.6.16}$$

$$\cos(180^\circ - \theta) = -\cos\theta \tag{9.6.17}$$

$$\tan(180^\circ - \theta) = -\tan\theta \tag{9.6.18}$$

Notice that reflection around the y-axis is equivalent to reflection around the x-axis ($\theta \mapsto -\theta$) followed by a rotation of 180° ($-\theta \mapsto -\theta + 180^\circ = 180^\circ - \theta$), as in Figure 1.5.7.

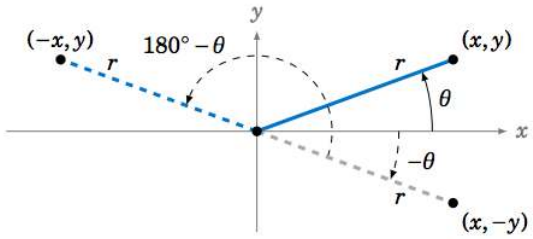


Figure 1.5.7 Reflection of θ around the y-axis = $180^\circ - \theta$

It may seem that these geometrical operations and formulas are not necessary for evaluating the trigonometric functions, since we could just use a calculator. However, there are two reasons for why they are useful. First, the formulas work for any angles, so they are often used to prove general formulas in mathematics and other fields, as we will see later in the text. Second, they can help in determining which angles have a given trigonometric function value.

Example 1.27

Find all angles $0^\circ \le \theta < 360^\circ$ such that $\sin\ \theta = -0.682$.

Solution

Using the $\boxed{\sin^{-1}}$ button on a calculator with -0.682 as the input, we get $\theta = -43^\circ$, which is not between 0° and 360°. Since $\theta = -43^\circ$ is in QIV, its reflection $180^\circ - \theta$ around the y-axis will be in QIII and have the same sine value. But $180^\circ - \theta = 180^\circ - (-43^\circ) = 223^\circ$ (see Figure 1.5.8). Also, we know that -43° and $-43^\circ + 360^\circ = 317^\circ$ have the same trigonometric function values. So since angles in QI and QII have positive sine values, we see that the only angles between 0° and 360° with a sine of -0.682 are $\boxed{\theta = 223^\circ \text{ and } 317^\circ}$.

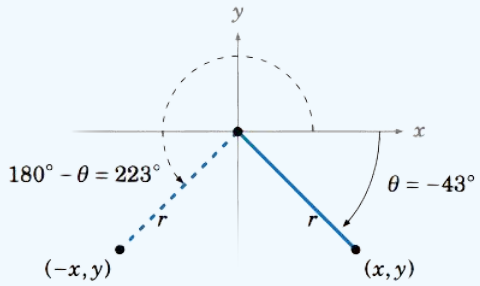


Figure 1.5.8 Reflection around the y-axis: -43° and 223°

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CHAPTER OVERVIEW

10: Systems of Equations

10.1: Solve Systems of Equations by Graphing
10.2: Solve Systems of Equations by Substitution
10.3: Solve Systems of Equations by Elimination
10.4: Solve Mixture Applications with Systems of Equations

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10.1: Solve Systems of Equations by Graphing

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Determine the number of solutions of linear system
- Solve applications of systems of equations by graphing

Note

Before you get started, take this readiness quiz.

1. For the equation $y = \frac{2}{3}x - 4$
   ⓐ is (6,0) a solution? ⓑ is (−3,−2) a solution?
   If you missed this problem, review Example 2.1.1.
2. Find the slope and y-intercept of the line 3x−y=12.
   If you missed this problem, review Example 4.5.7.
3. Find the x- and y-intercepts of the line 2x−3y=12.
   If you missed this problem, review Example 4.3.7.

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the section on Solving Linear Equations and Inequalities we learned how to solve linear equations with one variable. Remember that the solution of an equation is a value of the variable that makes a true statement when substituted into the equation. Now we will work with **systems of linear equations**, two or more linear equations grouped together.

Definition: SYSTEM OF LINEAR EQUATIONS

When two or more linear equations are grouped together, they form a system of linear equations.

We will focus our work here on systems of two linear equations in two unknowns. Later, you may solve larger systems of equations.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases} \tag{10.1.1}$$

A linear equation in two variables, like $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions to a system of equations**.

Definition: SolutionS OF A SYSTEM OF EQUATIONS

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Let's consider the system below:

$$\begin{cases} 3x-y=7 \\ x-2y=4 \end{cases} \tag{10.1.2}$$

Is the ordered pair (2,−1) a solution?

We substitute $x=2$ and $y=-1$ into both equations.

$$\begin{aligned} 3x-y &= 7 & \qquad x-2y &= 4 \\ 3(2)-(-1) &\stackrel{?}{=} 7 & \qquad 2-2(-1) &\stackrel{?}{=} 4 \\ 7 &= 7 \text{ true} & \qquad 4 &= 4 \text{ true} \end{aligned}$$

The ordered pair (2, −1) made both equations true. Therefore (2, −1) is a solution to this system.

Let's try another ordered pair. Is the ordered pair (3, 2) a solution?

We substitute $x=3$ and $y=2$ into both equations.

$$\begin{aligned} 3x-y &= 7 & \qquad x-2y &= 4 \\ 3(3)-2 &\stackrel{?}{=} 7 & \qquad 2-2(2) &\stackrel{?}{=} 4 \\ 7 &= 7 \text{ true} & \qquad -2 &= 4 \text{ false} \end{aligned}$$

The ordered pair (3, 2) made one equation true, but it made the other equation false. Since it is not a solution to **both** equations, it is not a solution to this system.

Example 10.1.1

Determine whether the ordered pair is a solution to the system: $\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$

1. (−2,−1)
2. (−4,−3)

Solution

1.

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

We substitute $x=-2$ and $y=-1$ into both equations

$$\begin{aligned} x-y &= 1 & \qquad 2x-y &= -5 \\ -2-(-1) &= -1 & \qquad 2(-2)-(-1) &\stackrel{?}{=} -5 \\ -1 &= -1 \checkmark & \qquad -3 &= -5 \times \end{aligned}$$

(−2, −1) does not make both equations true. (−2, −1) is not a solution.

2.

We substitute $x=-4$ and $y=-3$ into both equations.

$$\begin{aligned} x-y &= -1 & \qquad 2x-y &= -5 \\ -4-(-3) &\stackrel{?}{=} -1 & \qquad 2(-4)-(-3) &\stackrel{?}{=} -5 \\ -1 &= -1 \checkmark & \qquad -5 &= -5 \checkmark \end{aligned}$$

(−4, −3) makes both equations true. (−4, −3) is a solution.

Try It 10.1.2

Determine whether the ordered pair is a solution to the system: $\begin{cases} 3x+y=0 \\ x+2y=-5 \end{cases}$

1. (1,−3)

2. (0,0)

Answer

1. yes
2. no

? Try It 10.1.3

Determine whether the ordered pair is a solution to the system: $\begin{cases} x-3y=-8 \\ -3x-y=4 \end{cases}$

1. (2,−2)
2. (−2,2)

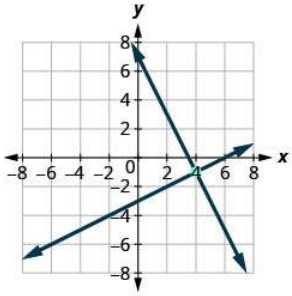
Answer

1. no
2. yes

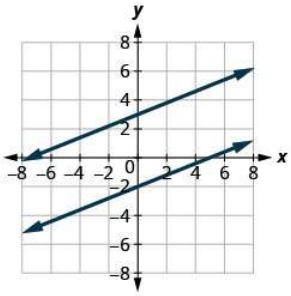
Solve a System of Linear Equations by Graphing

In this chapter we will use three methods to solve a system of linear equations. The first method we'll use is graphing. The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

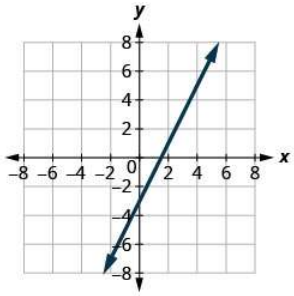
Most linear equations in one variable have one solution, but we saw that some equations, called **contradictions**, have no solutions and for other equations, called **identities**, all numbers are solutions. Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in Figure 10.1.1:



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

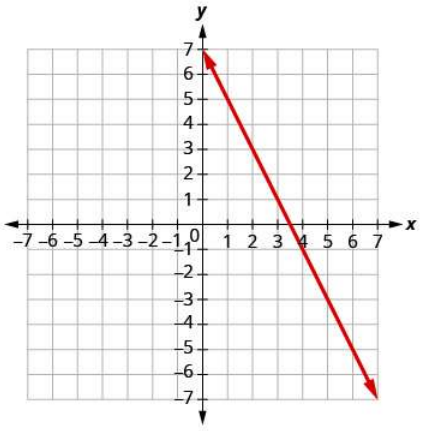
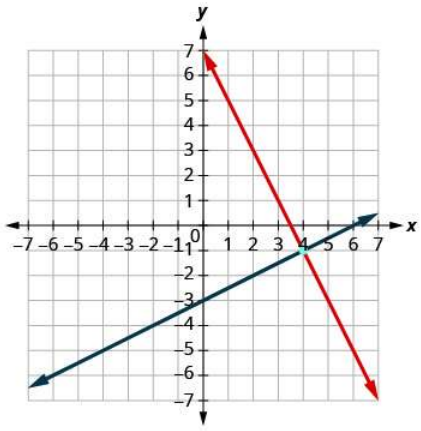
Figure 10.1.1

For the first example of solving a system of linear equations in this section and in the next two sections, we will solve the same system of two linear equations. But we'll use a different method in each section. After seeing the third method, you'll decide which method was the most convenient way to solve this system.

✔ Example 10.1.4: How to Solve a System of Linear Equations by Graphing

Solve the system by graphing: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Solution

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x + y = 7$ $y = -2x + 7$ $m = -2$ $b = 7$	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x - 2y = 6$ $(0, -3)$ $(6, 0)$	
Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4, -1)$. $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

? Try It 10.1.5

Solve each system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$

Answer

(3,2)

Try It 10.1.6

Solve each system by graphing: $\begin{cases} -x+y=1 \\ 3x+2y=12 \end{cases}$

Answer

(2,3)

The steps to use to solve a system of linear equations by graphing are shown below.

TO SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
   - If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
   - If the lines are parallel, the system has no solution.
   - If the lines are the same, the system has an infinite number of solutions.

Example 10.1.7

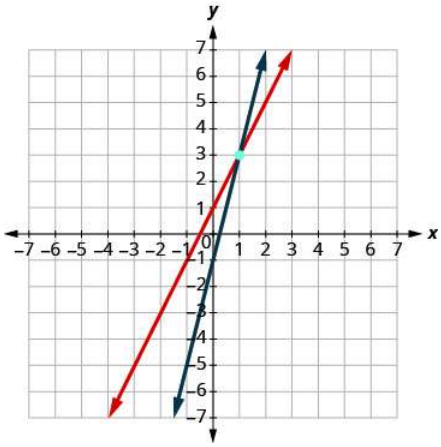
Solve the system by graphing: $\begin{cases} y=2x+1 \\ y=4x-1 \end{cases}$

Solution

Both of the equations in this system are in slope-intercept form, so we will use their slopes and y-intercepts to graph them.

$\begin{cases} y=2x+1 \\ y=4x-1 \end{cases}$

Find the slope and y-intercept of the first equation.	$y=2x+1$ $m=2$ $b=1$
Find the slope and y-intercept of the first equation.	$y=4x-1$ $m=4$ $b=-1$
Graph the two lines.	
Determine the point of intersection.	The lines intersect at (1, 3).

	
Check the solution in both equations.	$y = 2x + 1$ $\quad$ $y = 4x - 1$ $3 \stackrel{?}{=} 2 \cdot 1 + 1$ $\quad$ $3 \stackrel{?}{=} 4 \cdot 1 - 1$ $3 = 3\checkmark$ $\quad$ $3 = 3\checkmark$
	The solution is (1, 3).

? Try It 10.1.8

Solve the system by graphing: $\begin{cases} y = 2x + 2 \\ y = -x - 4 \end{cases}$

Answer

(−2,−2)

? Try It 10.1.9

Solve the system by graphing: $\begin{cases} y = 3x + 3 \\ y = -x + 7 \end{cases}$

Answer

(1,6)

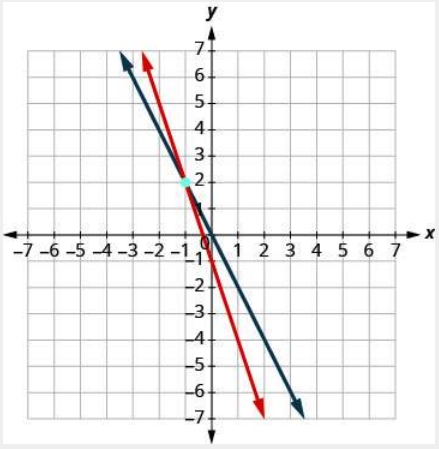
Both equations in Example 10.1.7 were given in slope–intercept form. This made it easy for us to quickly graph the lines. In the next example, we'll first re-write the equations into slope–intercept form.

✔ Example 10.1.10

Solve the system by graphing: $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Solution

We'll solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts. $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Solve the first equation for y.	
Find the slope and y-intercept.	$3x+y=-1$ $y=-3x-1$ $m=-3$ $b=-1$
Solve the second equation for y.	$2x+y=0$ $y=-2x$
Find the slope and y-intercept.	$b=0$
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1, 2)$.
Check the solution in both equations.	$3x+y=-1$ $\quad 3(-1)+2 \stackrel{?}{=} -1 \quad -1=-1\checkmark$ $2x+y=0$ $\quad 2(-1)+2 \stackrel{?}{=} 0 \quad 0=0\checkmark$
	The solution is $(-1, 2)$.

? Try It 10.1.11

Solve each system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$

Answer

(3,4)

? Try It 10.1.12

Solve each system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$

Answer

(5,−4)

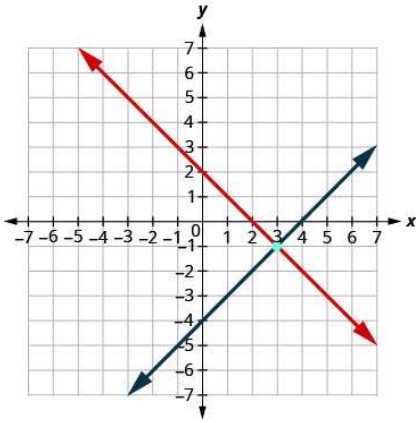
Usually when equations are given in standard form, the most convenient way to graph them is by using the intercepts. We'll do this in Example 10.1.13

Example 10.1.13

Solve the system by graphing: $\begin{cases} x+y=2 \\ x-y=4 \end{cases}$

Solution

We will find the *x*- and *y*-intercepts of both equations and use them to graph the lines.

	$x+y=2$	
To find the intercepts, let $x = 0$ and solve for y, then let $y = 0$ and solve for x.	$x+y=2 \quad x+y=2$ $0+y=2 \quad x+0=2$ $y=2 \quad x=2$	x | y 0 | 2 2 | 0
	$x-y=4$	
To find the intercepts, let $x = 0$ then let $y = 0$.	$x-y=4 \quad x-y=4$ $0-y=4 \quad x-0=4$ $-y=4 \quad x=4$ $y=-4$	x | y 0 | –4 4 | 0
Graph the line.		
Determine the point of intersection.	The lines intersect at (3, −1).	
Check the solution in both equations.	$x+y = 2 \quad x-y = 4$ $3+(-1) \stackrel{?}{=} 2 \quad 3-(-1) \stackrel{?}{=} 4$ $2 = 2\checkmark \quad 4 = 4\checkmark$ The solution is (3, −1).	

(10.1.3)

Try It 10.1.14

Solve each system by graphing: $\begin{cases} x+y=6 \\ x-y=2 \end{cases}$

Answer

(4,2)

Try It 10.1.15

Solve each system by graphing: $\begin{cases} x+y=2 \\ x-y=8 \end{cases}$

Answer

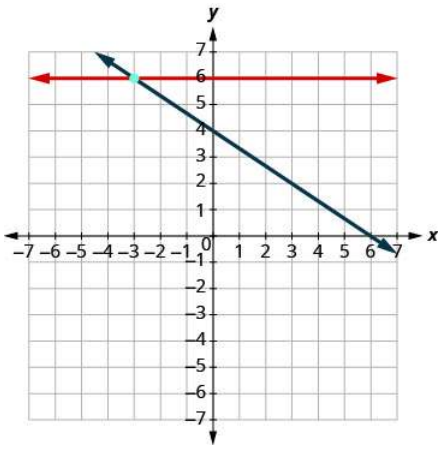
(5,−3)

Do you remember how to graph a linear equation with just one variable? It will be either a vertical or a horizontal line.

Example 10.1.16

Solve the system by graphing: $\begin{cases} y=6 \\ 2x+3y=12 \end{cases}$

Solution

	$y=6$ $2x+3y=12$
We know the first equation represents a horizontal line whose y-intercept is 6.	$y=6$
The second equation is most conveniently graphed using intercepts.	$2x+3y=12$
To find the intercepts, let $x=0$ and then $y=0$.	x | y 0 | 4 6 | 0
Graph the lines.	
Determine the point of intersection.	The lines intersect at (−3, 6).
Check the solution to both equations.	$y = 6$, $6 \stackrel{?}{=} 6$, $6 = 6\checkmark$ $2x+3y = 12$, $2(-3)+3(6) \stackrel{?}{=} 12$, $-6+18 \stackrel{?}{=} 12$, $12 = 12\checkmark$
	The solution is (−3, 6).

Try It 10.1.17

Solve each system by graphing: $\begin{cases} y = -1 \\ x + 3y = 6 \end{cases}$

Answer

$(9,-1)$

Try It 10.1.18

Solve each system by graphing: $\begin{cases} x = 4 \\ 3x - 2y = 24 \end{cases}$

Answer

$(4,-6)$

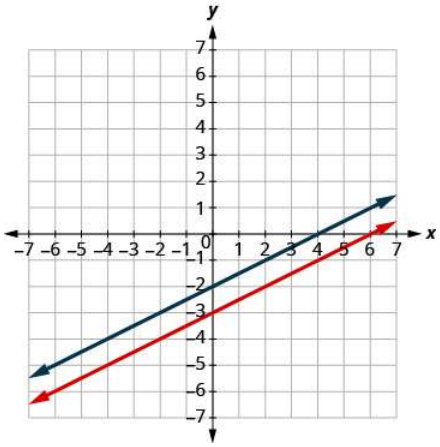
In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 10.1.19

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$

Solution

	$y = \frac{1}{2}x - 3$ $x - 2y = 4$
To graph the first equation, we will use its slope and y-intercept.	$y = \frac{1}{2}x - 3$
	$m = \frac{1}{2}$
	$b = -3$
To graph the second equation, we will use the intercepts.	$x - 2y = 4$
	x | y 0 | −2 4 | 0

Graph the lines.	
Determine the point of intersection.	The lines are parallel.
	Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Try It 10.1.20

Solve each system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = -8 \end{cases}$

Answer

no solution

Try It 10.1.21

Solve each system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$

Answer

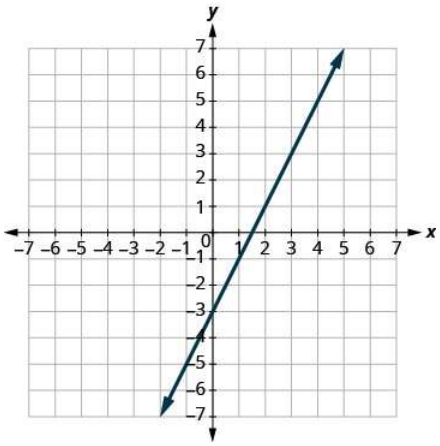
no solution

Example 10.1.22

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$

Solution

	$y = 2x - 3$ $-6x + 3y = -9$
Find the slope and y-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$

	x | y 0 | –3 $\frac{3}{2}$ | 0
Graph the lines.	
Determine the point of intersection.	The lines are the same!
	Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.
	There are infinitely many solutions to this system.

Try It 10.1.23

Solve each system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$

Answer

infinitely many solutions

Try It 10.1.24

Solve each system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$

Answer

infinitely many solutions

If you write the second equation in Example $10.1.22$ in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are coincident. Coincident lines have the same slope and same y-intercept.

COINCIDENT LINES

Coincident lines have the same slope and same y-intercept.

Determine the Number of Solutions of a Linear System

There will be times when we will want to know how many solutions there will be to a system of linear equations, but we might not actually have to find the solution. It will be helpful to determine this without graphing.

We have seen that two lines in the same plane must either intersect or are parallel. The systems of equations in Example 10.1.4 through Example 10.1.18all had two intersecting lines. Each system had one solution.

A system with parallel lines, like Example 10.1.19, has no solution. What happened in Example 10.1.22? The equations have **coincident lines**, and so the system had infinitely many solutions.

We'll organize these results in Figure 10.1.2below:

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

Figure 10.1.2

Parallel lines have the same slope but different *y*-intercepts. So, if we write both equations in a system of linear equations in slope–intercept form, we can see how many solutions there will be without graphing! Look at the system we solved in Example 10.1.19

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

The first line is in slope–intercept form. If we solve the second equation for y, we get

$$y = \frac{1}{2}x - 3 \qquad x - 2y = 4$$
$$x - 2y = -x + 4$$
$$y = \frac{1}{2}x - 2$$
$$m = \frac{1}{2}, b = -3 \qquad m = \frac{1}{2}, b = -2$$

The two lines have the same slope but different *y*-intercepts. They are parallel lines.

Figure 10.1.3shows how to determine the number of solutions of a linear system by looking at the slopes and intercepts.

Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

Figure 10.1.3

Let's take one more look at our equations in Example 10.1.19that gave us parallel lines.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases} \tag{10.1.4}$$

When both lines were in slope-intercept form we had:

$$y = \frac{1}{2}x - 3 \quad y = \frac{1}{2}x - 2 \tag{10.1.5}$$

Do you recognize that it is impossible to have a single ordered pair (x,y) that is a solution to both of those equations?

We call a system of equations like this **an inconsistent system**. It has no solution.

A system of equations that has at least one solution is called a **consistent system**.

CONSISTENT AND INCONSISTENT SYSTEMS

A **consistent system** of equations is a system of equations with at least one solution.

An **inconsistent system** of equations is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are **independent equations**, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two **dependent equations**, we get coincident lines.

INDEPENDENT AND DEPENDENT EQUATIONS

Two equations are **independent** if they have different solutions.

Two equations are **dependent** if all the solutions of one equation are also solutions of the other equation.

Let's sum this up by looking at the graphs of the three types of systems. See Figure 10.1.4and Figure 10.1.5

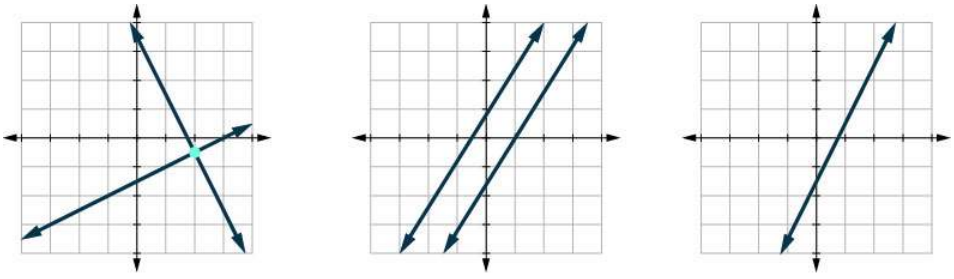


Figure 10.1.4

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

Figure 10.1.5

Example 10.1.25

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$

Solution

We will compare the slopes and intercepts of the two lines. $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$

The first equation is already in slope-intercept form.

$$y = 3x - 1$$

Write the second equation inslope–intercept form.

$$\begin{aligned} 6x - 2y &= 12 \\ -2y &= -6x - 12 \\ \tfrac{-2y}{-2} &= \tfrac{-6x+12}{-2} \\ y &= 3x - 6 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= 3x - 1 & y &= 3x - 6 \\ m &= 3 & m &= 3 \\ b &= -1 & b &= -6 \end{aligned}$$

Since the slopes are the same and y-intercepts are different, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

Try It 10.1.26

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$$

Answer

no solution, inconsistent, independent

Try It 10.1.27

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$$

Answer

no solution, inconsistent, independent

Example 10.1.28

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$

Solution

We will compare the slopes and intercepts of the two lines.

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$$

Write the second equation in slope–intercept form.

$$\begin{aligned} 2x + y &= -3 \\ y &= -2x - 3 \end{aligned} \qquad \begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \tfrac{-5y}{-5} &= \tfrac{-x+5}{-5} \\ y &= \tfrac{1}{5}x - 1 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= -2x - 3 \\ m &= -2 \\ b &= -3 \end{aligned} \qquad \begin{aligned} y &= \tfrac{1}{5}x - 1 \\ m &= \tfrac{1}{5} \\ b &= -1 \end{aligned}$$

Since the slopes are different, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

Try It 10.1.29

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 3x + 2y = 2 \\ 2x + y = 1 \end{cases}$$

Answer

one solution, consistent, independent

Try It 10.1.30

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} x+4y=12 \\ -x+y=3 \end{cases}$$

Answer

one solution, consistent, independent

Example 10.1.31

Without graphing, determine the number of solutions and then classify the system of equations. $\begin{cases} 3x-2y=4 \\ y=\frac{3}{2}x-2 \end{cases}$

Solution

We will compare the slopes and intercepts of the two lines. $\begin{cases} 3x-2y &= 4 \\ y &= \frac{3}{2}x-2 \end{cases}$

Write the second equation in slope–intercept form.

$$\begin{aligned} 3x-2y &= 4 \\ -2y &= -3x+4 \\ \frac{-2y}{-2} &= \frac{-3x+4}{-2} \\ y &= \frac{3}{2}x-2 \end{aligned}$$

Find the slope and intercept of each line.

$$y = \frac{3}{2}x-2$$

Since the equations are the same, they have the same slope and same y-intercept and so the lines are coincident.

A system of equations whose graphs are coincident lines has infinitely many solutions and is consistent and dependent.

Try It 10.1.32

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 4x-5y=20 \\ y=\frac{4}{5}x-4 \end{cases}$$

Answer

infinitely many solutions, consistent, dependent

Try It 10.1.33

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} -2x-4y=8 \\ y=-\frac{1}{2}x-2 \end{cases}$$

Answer

infinitely many solutions, consistent, dependent

Solve Applications of Systems of Equations by Graphing

We will use the same problem solving strategy we used in Math Models to set up and solve applications of systems of linear equations. We'll modify the strategy slightly here to make it appropriate for systems of equations.

USE A PROBLEM SOLVING STRATEGY FOR SYSTEMS OF LINEAR EQUATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Step 5 is where we will use the method introduced in this section. We will graph the equations and find the solution.

Example 10.1.34

Sondra is making 10 quarts of punch from fruit juice and club soda. The number of quarts of fruit juice is 4 times the number of quarts of club soda. How many quarts of fruit juice and how many quarts of club soda does Sondra need?

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.

We are looking for the number of quarts of fruit juice and the number of quarts of club soda that Sondra will need.

Step 3. Name what we are looking for. Choose variables to represent those quantities.

Let f= number of quarts of fruit juice.
c= number of quarts of club soda

Step 4. Translate into a system of equations.

The number of quarts of fruit juice and the number of quarts of club soda is 10

$$f + c = 10$$

The number of quarts of fruit juice is four times the number of quarts of club soda

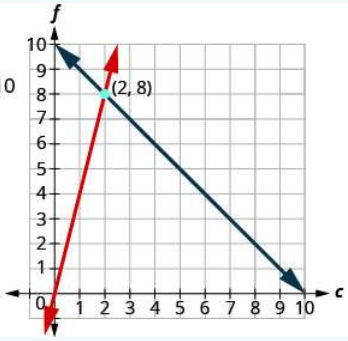
$$f = 4c$$

We now have the system. $\begin{cases} f + c = 10 \\ f = 4c \end{cases}$

Step 5. Solve the system of equations using good algebra techniques.

$f = 4c$ $\quad m = 4$ $\quad b = 0$

$f + c = 10$ $\quad f = -c + 10$ $\quad m = -1$ $\quad b = 10$



The point of intersection (2, 8) is the solution. This means Sondra needs 2 quarts of club soda and 8 quarts of fruit juice.

Step 6. Check the answer in the problem and make sure it makes sense.

Does this make sense in the problem?

Yes, the number of quarts of fruit juice, 8 is 4 times the number of quarts of club soda, 2.

Yes, 10 quarts of punch is 8 quarts of fruit juice plus 2 quarts of club soda.

Step 7. Answer the question with a complete sentence.

Sondra needs 8 quarts of fruit juice and 2 quarts of soda.

? Try It 10.1.35

Manny is making 12 quarts of orange juice from concentrate and water. The number of quarts of water is 3 times the number of quarts of concentrate. How many quarts of concentrate and how many quarts of water does Manny need?

Answer

Manny needs 3 quarts juice concentrate and 9 quarts water.

? Try It 10.1.36

Alisha is making an 18 ounce coffee beverage that is made from brewed coffee and milk. The number of ounces of brewed coffee is 5 times greater than the number of ounces of milk. How many ounces of coffee and how many ounces of milk does Alisha need?

Answer

Alisha needs 15 ounces of coffee and 3 ounces of milk.

Note

Access these online resources for additional instruction and practice with solving systems of equations by graphing.

- Instructional Video Solving Linear Systems by Graphing
- Instructional Video Solve by Graphing

Key Concepts

- **To solve a system of linear equations by graphing**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations.
- Determine the number of solutions from the graph of a linear system

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

- Determine the number of solutions of a linear system by looking at the slopes and intercepts

Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

- Determine the number of solutions and how to classify a system of equations

Lines	**Intersecting**	**Parallel**	**Coincident**
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

- **Problem Solving Strategy for Systems of Linear Equations**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose variables to represent those quantities.
  4. **Translate** into a system of equations.
  5. **Solve** the system of equations using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

coincident lines

Coincident lines are lines that have the same slope and same y-intercept.

consistent system

A consistent system of equations is a system of equations with at least one solution.

dependent equations

Two equations are dependent if all the solutions of one equation are also solutions of the other equation.

inconsistent system

An inconsistent system of equations is a system of equations with no solution.

independent equations

Two equations are independent if they have different solutions.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

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10.2: Solve Systems of Equations by Substitution

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of equations by substitution
- Solve applications of systems of equations by substitution

Note

Before you get started, take this readiness quiz.

1. Simplify −5(3−x).
   If you missed this problem, review Exercise 1.10.43.
2. Simplify 4−2(n+5).
   If you missed this problem, review Exercise 1.10.41.
3. Solve for y. 8y−8=32−2y
   If you missed this problem, review Exercise 2.3.22.
4. Solve for x. 3x−9y=−3
   If you missed this problem, review Exercise 2.6.22.

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between −10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

In this section, we will solve systems of linear equations by the substitution method.

Solve a System of Equations by Substitution

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

We'll fill in all these steps now in Exercise 10.2.1.

Exercise 10.2.1: How to Solve a System of Equations by Substitution

Solve the system by substitution. $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$
Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$
Step 4. Substitute the solution in Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into both equations and make sure they are both true.	$2x + y = 7$ $\quad$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $\quad$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $\quad$ $6 = 6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

Exercise 10.2.2

Solve the system by substitution. $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$

Answer

(6,1)

Exercise 10.2.3

Solve the system by substitution. $\begin{cases} x + 3y = 10 \\ 4x + y = 18 \end{cases}$

Answer

(4,2)

SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into one of the original equations to find the other variable.

5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

If one of the equations in the system is given in slope–intercept form, Step 1 is already done! We'll see this in Exercise 10.2.4

Exercise 10.2.4

Solve the system by substitution. $\begin{cases} x+y=-1 \\ y=x+5 \end{cases}$

Answer

The second equation is already solved for y. We will substitute the expression in place of y in the first equation.

	$x+y=-1$ $y=x+5$
The second equation is already solved for y. We will substitute into the first equation.	
Replace the y with $x+5$.	$y=x+5$ $x+y=-1$
Solve the resulting equation for x.	$x+x+5=-1$
	$2x+5=-1$
	$2x=-6$
Substitute $x=-3$ into $y=x+5$ to find y.	$x=-3$ $y=x+5$
	$y=-3+5$
The ordered pair is $(-3, 2)$.	$y=2$
Check the ordered pair in both equations: $x+y=-1 \quad y=x+5$ $-3+2 \stackrel{?}{=} -1 \quad 2 \stackrel{?}{=} -3+5$ $-1=-1\checkmark \quad 2=2\checkmark$	
	The solution is $(-3, 2)$.

Exercise 10.2.5

Solve the system by substitution. $\begin{cases} x+y=6 \\ y=3x-2 \end{cases}$

Answer

(2,4)

Exercise 10.2.6

Solve the system by substitution. $\begin{cases} 2x-y=1 \\ y=-3x-6 \end{cases}$

Answer

(−1,−3)

If the equations are given in standard form, we'll need to start by solving for one of the variables. In this next example, we'll solve the first equation for y.

Exercise 10.2.7

Solve the system by substitution. $\begin{cases} 3x+y=5 \\ 2x+4y=-10 \end{cases}$

Answer

We need to solve one equation for one variable. Then we will substitute that expression into the other equation.

Solve for y. Substitute into the other equation.	$3x+y=5$ $y=-3x+5$ $2x+4y=-10$
Replace the y with $-3x+5$.	$2x+4(-3x+5)=-10$
Solve the resulting equation for x.	$2x-12x+20=-10$
	$-10x+20=-10$ $-10x=-30$
Substitute $x=3$ into $3x+y=5$ to find y.	$x=3$ $3x+y=5$
	$3(3)+y=5$ $9+y=5$
The ordered pair is (3, −4).	$y=-4$
Check the ordered pair in both equations: $3x+y=5$: $3\cdot 3+(-4) \stackrel{?}{=} 5$, $9-4 \stackrel{?}{=} 5$, $5=5\checkmark$ $2x+4y=-10$: $2\cdot 3+4(-4) \stackrel{?}{=} -10$, $6-16 \stackrel{?}{=} -10$, $-10=-10\checkmark$	
	The solution is (3, −4).

Exercise 10.2.8

Solve the system by substitution. $\begin{cases} 4x+y=2 \\ 3x+2y=-1 \end{cases}$

Answer

(1,−2)

Exercise 10.2.9

Solve the system by substitution. $\begin{cases} -x+y=4 \\ 4x-y=2 \end{cases}$

Answer

(2,6)

In Exercise 10.2.7 it was easiest to solve for y in the first equation because it had a coefficient of 1. In Exercise 10.2.10 it will be easier to solve for x.

Exercise 10.2.10

Solve the system by substitution. $\begin{cases} x-2y=-2 \\ 3x+2y=34 \end{cases}$

Answer

We will solve the first equation for xx and then substitute the expression into the second equation.

	$x-2y=-2$
Solve for x. Substitute into the other equation.	$x=2y-2$ $3x+2y=34$
Replace the x with $2y-2$.	$3(2y-2)+2y=34$
Solve the resulting equation for y.	$6y-6+2y=34$
Substitute $y=5$ into $x-2y=-2$ to find x.	$8y-6=34$ $8y=40$ $y=5$ $x-2y=-2$ $x-2\cdot 5=-2$ $x-10=-2$ $x=8$
The ordered pair is (8, 5).	
Check the ordered pair in both equations: $x-2y=-2$ $8-2\cdot 5 \stackrel{?}{=} -2$ $8-10 \stackrel{?}{=} -2$ $-2=-2\checkmark$ $3x+2y=34$ $3\cdot 8+2\cdot 5 \stackrel{?}{=} 34$ $24+10 \stackrel{?}{=} 34$ $34=34\checkmark$	
	The solution is (8, 5).

Exercise 10.2.11

Solve the system by substitution. $\begin{cases} x-5y=13 \\ 4x-3y=1 \end{cases}$

Answer

$(-2,-3)$

Exercise 10.2.12

Solve the system by substitution. $\begin{cases} x-6y=-6 \\ 2x-4y=4 \end{cases}$

Answer

(6,2)

When both equations are already solved for the same variable, it is easy to substitute!

Exercise 10.2.13

Solve the system by substitution. $\begin{cases} y = -2x + 5 \\ y = \frac{1}{2}x \end{cases}$

Answer

Since both equations are solved for y, we can substitute one into the other.

Substitute $\frac{1}{2}x$ for y in the first equation.	$y = \frac{1}{2}x$ $y = -2x + 5$
Replace the y with $\frac{1}{2}x$	$\frac{1}{2}x = -2x + 5$
Solve the resulting equation. Start by clearing the fraction.	$2\left(\frac{1}{2}x\right) = 2(-2x + 5)$
Solve for x.	$x = -4x + 10$
	$5x = 10$
Substitute $x = 2$ into $y = \frac{1}{2}x$ to find y.	$x = 2$ $y = \frac{1}{2}x$ $y = \frac{1}{2} \cdot 2$ $y = 1$
The ordered pair is (2,1).	
Check the ordered pair in both equations: $\begin{array}{rcl} y & = & \frac{1}{2}x \\ 1 & \stackrel{?}{=} & \frac{1}{2} \cdot 2 \\ 1 & = & 1\checkmark \end{array}$ $\begin{array}{rcl} y & = & -2x + 5 \\ 1 & \stackrel{?}{=} & -2 \cdot 2 + 5 \\ 1 & = & -4 + 5 \\ 1 & = & 1\checkmark \end{array}$	
	The solution is (2,1).

Exercise 10.2.14

Solve the system by substitution. $\begin{cases} y = 3x - 16 \\ y = \frac{1}{3}x \end{cases}$

Answer

(6,2)

Exercise 10.2.15

Solve the system by substitution. $\begin{cases} y = -x + 10 \\ y = \frac{1}{4}x \end{cases}$

Answer

(8,2)

Be very careful with the signs in the next example.

Exercise 10.2.16

Solve the system by substitution. $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for y.

	$4x+2y=4$
Solve the first equation for y.	$2y=-4x+4$
Substitute $-2x+2$ for y in the second equation.	$y=-2x+2$ $6x-y=8$
Replace the y with $-2x+2$.	$6x-(-2x+2)=8$
Solve the equation for x.	$6x+2x-2=8$
	$8x-2=8$ $8x=10$
Substitute $x=\frac{5}{4}$ into $4x+2y=4$ to find y.	$x=\frac{5}{4}$ $4x+2y=4$ $4\left(\frac{5}{4}\right)+2y=4$ $5+2y=4$ $2y=-1$ $y=-\frac{1}{2}$
The ordered pair is $(\frac{5}{4},-\frac{1}{2})$.	
Check the ordered pair in both equations.	

$$\begin{aligned} 4x+2y &= 4 \\ 4(\tfrac{5}{4})+2(-\tfrac{1}{2}) &\stackrel{?}{=} 4 \\ 5-1 &\stackrel{?}{=} 4 \\ 4 &= 4\checkmark \end{aligned} \qquad \begin{aligned} 6x-y &= 8 \\ 6(\tfrac{5}{4})-(-\tfrac{1}{2}) &\stackrel{?}{=} 8 \\ \tfrac{15}{4}-(-\tfrac{1}{2}) &\stackrel{?}{=} 8 \\ \tfrac{16}{2} &\stackrel{?}{=} 8 \\ 8 &= 8\checkmark \end{aligned}$$

	The solution is (54,−12).

Exercise 10.2.17

Solve the system by substitution. $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$

Answer

$(2,\frac{3}{2})$

Exercise 10.2.18

Solve the system by substitution. $\begin{cases} 4x - y = 0 \\ 2x - 3y = 5 \end{cases}$

Answer

$(-\frac{1}{2}, -2)$

In Example, it will take a little more work to solve one equation for x or y.

Exercise 10.2.19

Solve the system by substitution. $\begin{cases} 4x - 3y = 6 \\ 15y - 20x = -30 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for x.

	$4x - 3y = 6$
Solve the first equation for x.	$4x = 3y + 6$
Substitute $\frac{3}{4}y + \frac{3}{2}$ for x in the second equation.	$x = \frac{3}{4}y + \frac{3}{2}$ $15y - 20x = -30$
Replace the x with $\frac{3}{4}y + \frac{3}{2}$	$15y - 20\left(\frac{3}{4}y + \frac{3}{2}\right) = -30$
Solve for y.	$15y - 15y - 30 = -30$
	$0 - 30 = -30$
	$0 = 0$

Since 0 = 0 is a true statement, the system is consistent. The equations are dependent. The graphs of these two equations would give the same line. The system has infinitely many solutions.

Exercise 10.2.20

Solve the system by substitution. $\begin{cases} 2x - 3y = 12 \\ -12y + 8x = 48 \end{cases}$

Answer

infinitely many solutions

Exercise 10.2.21

Solve the system by substitution. $\begin{cases} 5x + 2y = 12 \\ -4y - 10x = -24 \end{cases}$

Answer

infinitely many solutions

Look back at the equations in Exercise 10.2.22 Is there any way to recognize that they are the same line?

Let's see what happens in the next example.

Exercise 10.2.22

Solve the system by substitution. $\begin{cases} 5x-2y=-10 \\ y=\frac{5}{2}x \end{cases}$

Answer

The second equation is already solved for *y*, so we can substitute for *y* in the first equation.

Substitute *x* for *y* in the first equation.	$y=\left(\frac{5}{2}x\right)$ $5x-2y=-10$
Replace the *y* with $\frac{5}{2}x$.	$5x-2\left(\frac{5}{2}x\right)=-10$
Solve for *x*.	$5x-5x=-10$
	$0\neq-10$

Since 0 = −10 is a false statement the equations are inconsistent. The graphs of the two equation would be parallel lines. The system has no solutions.

Exercise 10.2.23

Solve the system by substitution. $\begin{cases} 3x+2y=9 \\ y=-\frac{3}{2}x+1 \end{cases}$

Answer

no solution

Exercise 10.2.24

Solve the system by substitution. $\begin{cases} 5x-3y=2 \\ y=\frac{5}{3}x-4 \end{cases}$

Answer

no solution

Solve Applications of Systems of Equations by Substitution

We'll copy here the problem solving strategy we used in the Solving Systems of Equations by Graphing section for solving systems of equations. Now that we know how to solve systems by substitution, that's what we'll do in Step 5.

HOW TO USE A PROBLEM SOLVING STRATEGY FOR SYSTEMS OF LINEAR EQUATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.

7. **Answer** the question with a complete sentence.

Some people find setting up word problems with two variables easier than setting them up with just one variable. Choosing the variable names is easier when all you need to do is write down two letters. Think about this in the next example—how would you have done it with just one variable?

Exercise 10.2.25

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let n= the first number Let m= the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n+m=0$
	One number is nine less than the other.
	$n=m-9$
The system is:	$\begin{aligned} n+m&=0\\ n&=m-9\end{aligned}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for n.	
Substitute $m-9$ for n in the first equation.	$n=m-9$ $n+m=0$
Solve for m.	$m-9+m=0$
	$2m-9=0$
	$2m=9$
Substitute $m=\frac{9}{2}$ into the second equation and then solve for n.	$m=\frac{9}{2}$ $n=m-9$
	$m=\frac{9}{2}-9$
	$m=\frac{9}{2}-\frac{18}{2}$
	$n=-\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

Exercise 10.2.26

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Answer

The numbers are 3 and 7.

Exercise 10.2.27

The sum of two number is −6. One number is 10 less than the other. Find the numbers.

Answer

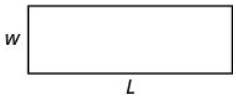
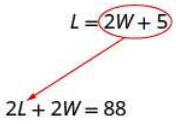
The numbers are 2 and −8.

In the Exercise 10.2.28 we'll use the formula for the perimeter of a rectangle, $P = 2L + 2W$.

Exercise 10.2.28

Find the dimensions of a rectangle in which the length is five more than twice the width.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let L= the length W= the width
Step 4. Translate into a system of equations.	The perimeter of a rectangle is 88.
	$2L + 2W = P$ $2L + 2W = 88$
	The length is five more than twice the width.
	$L = 2W + 5$
The system is:	$2L + 2W = 88$ $L = 2W + 5$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for L. Substitute $2W + 5$ for L in the first equation.	
Solve for W.	$2(2W + 5) + 2W = 88$
	$4W + 10 + 2W = 88$
	$6W + 10 = 88$
	$6W = 78$
Substitute $W = 13$ into the second equation and then solve for L.	$W = 13$ $L = 2W + 5$
	$L = 2 \cdot 13 + 5$

	$L = 31$
Step 6. Check the answer in the problem.	Does a rectangle with length 31 and width 13 have perimeter 88? Yes.
Step 7. Answer the equation.	The length is 31 and the width is 13.

Exercise 10.2.29

The perimeter of a rectangle is 40. The length is 4 more than the width. Find the length and width of the rectangle.

Answer

The length is 12 and the width is 8.

Exercise 10.2.30

The perimeter of a rectangle is 58. The length is 5 more than three times the width. Find the length and width of the rectangle.

Answer

The length is 23 and the width is 6.

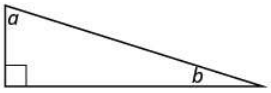
For Exercise 10.2.31we need to remember that the sum of the measures of the angles of a triangle is 180 degrees and that a right triangle has one 90 degree angle.

Exercise 10.2.31

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Answer

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let a= the measure of the 1st angle b= the measure of the 2nd angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is:	$\begin{aligned} a &= 3b + 10 \\ a + b + 90 &= 180 \end{aligned}$

Step 5. Solve the system of equations. We will use substitution since the first equation is solved for a.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute $3b + 10$ for a in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$
	$4b = 80$
	$b = 20$ $a = 3b + 10$
Substitute $b = 20$ into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70.

Exercise 10.2.32

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

The measure of the angles are 22 degrees and 68 degrees.

Exercise 10.2.33

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

The measure of the angles are 36 degrees and 54 degrees.

Exercise 10.2.34

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $25,000 plus $15 for each training session. Option B would pay her $10,000 + $40 for each training session. How many training sessions would make the salary options equal?

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let s= Heather's salary. n= the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her $25,000 plus $15 for each training session.

	$s = 25{,}000 + 15n$
	Option B would pay her \$10,000 + \$40 for each training session
	$s = 10{,}000 + 40n$
The system is:	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Step 5. Solve the system of equations. We will use substitution.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Substitute $25{,}000 + 15n$ for s in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for n.	$25{,}000 = 10{,}000 + 25n$
	$15{,}000 = 25n$
	$600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when $n = 600$?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

Exercise 10.2.35

Geraldine has been offered positions by two insurance companies. The first company pays a salary of \$12,000 plus a commission of \$100 for each policy sold. The second pays a salary of \$20,000 plus a commission of \$50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Answer

There would need to be 160 policies sold to make the total pay the same.

Exercise 10.2.36

Kenneth currently sells suits for company A at a salary of \$22,000 plus a \$10 commission for each suit sold. Company B offers him a position with a salary of \$28,000 plus a \$4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Answer

Kenneth would need to sell 1,000 suits.

Note

Access these online resources for additional instruction and practice with solving systems of equations by substitution.

- Instructional Video-Solve Linear Systems by Substitution
- Instructional Video-Solve by Substitution

Key Concepts

- **Solve a system of equations by substitution**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.

4. Substitute the solution in Step 3 into one of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to both original equations.

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10.3: Solve Systems of Equations by Elimination

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of equations by elimination
- Solve applications of systems of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Note

Before you get started, take this readiness quiz.

1. Simplify −5(6−3a).
   If you missed this problem, review Example 1.10.43.
2. Solve the equation $\frac{1}{3}x+\frac{5}{8}=\frac{31}{24}$.
   If you missed this problem, review Example 2.5.1.

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

Solve a System of Equations by Elimination

The **Elimination Method** is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*,

$$\begin{array}{ll}\text{if} & a=b\\ \text{and} & c=d\\ \text{then} & a+c=b+d\end{array} \tag{10.3.1}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{array}{r}3x+y=5\\ 2x-y=0\\ \hline 5x\quad\quad=5\end{array} \tag{10.3.2}$$

The *y*'s add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases}x+4y=2\\ 2x+5y=-2\end{cases} \tag{10.3.3}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by −2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by −2.

$$\begin{cases} -2(x+4y) = -2(2) \\ 2x+5y = -2 \end{cases}$$

$$\begin{cases} -2x-8y = -4 \\ 2x+5y = -2 \end{cases}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

Add the equations yourself—the result should be $-3y = -6$. And that looks easy to solve, doesn't it? Here is what it would look like.

$$\begin{array}{r} -2x-8y = -4 \\ 2x+5y = -2 \\ \hline -3y = -6 \end{array}$$

We'll do one more:

$$\begin{cases} 4x-3y = 10 \\ 3x+5y = -7 \end{cases} \tag{10.3.4}$$

It doesn't appear that we can get the coefficients of one variable to be opposites by multiplying one of the equations by a constant, unless we use fractions. So instead, we'll have to multiply both equations by a constant.

We can make the coefficients of x be opposites if we multiply the first equation by 3 and the second by -4, so we get $12x$ and $-12x$.

$$3(4x-3y) = 3(10)$$
$$-4(3x+5y) = -4(-7)$$

This gives us these two new equations:

$$\begin{cases} 12x-9y = 30 \\ -12x-20y = 28 \end{cases}$$

When we add these equations,

$$\begin{array}{r} 12x-9y = 30 \\ -12x-20y = 28 \\ \hline -29y = 58 \end{array} \tag{10.3.5}$$

the x's are eliminated and we just have $-29y = 58$.

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

Example 10.3.1: How to Solve a System of Equations by Elimination

Solve the system by elimination. $\begin{cases} 2x+y = 7 \\ x-2y = 6 \end{cases}$

Solution

https://math.libretexts.org/@go/page/196453

Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{array}{r} 4x + 2y = 14 \\ x - 2y = 6 \\ \hline 5x \quad\quad = 20 \end{array}$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into $2x + y = 7$ and $x - 2y = 6$ Do they make both equations true? Yes!	$2x + y = 7$: $2(4) + (-1) \stackrel{?}{=} 7$: $7 = 7$ ✓ $x - 2y = 6$: $4 - 2(-1) \stackrel{?}{=} 6$: $6 = 6$ ✓ The solution is $(4, -1)$.

Try It 10.3.2

Solve the system by elimination. $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Answer

$(2, -1)$

Try It 10.3.3

Solve the system by elimination. $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$

Answer

$(-2, 3)$

The steps are listed below for easy reference.

HOW TO SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.
   - Multiply one or both equations so that the coefficients of that variable are opposites.
3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

First we'll do an example where we can eliminate one variable right away.

Example 10.3.4

Solve the system by elimination. $\begin{cases} x+y=10 \\ x-y=12 \end{cases}$

Solution

	$\begin{array}{r} x+y=10 \\ x-y=12 \end{array}$
Both equations are in standard form.	
The coefficients of y are already opposites.	
Add the two equations to eliminate y. The resulting equation has only 1 variable, x.	$\begin{array}{r} x+y=10 \\ x-y=12 \\ \hline 2x \quad =22 \end{array}$
Solve for x, the remaining variable. Substitute $x = 11$ into one of the original equations.	$x=11$ $x+y=10$
	$11+y=10$
Solve for the other variable, y.	$y=-1$
Write the solution as an ordered pair.	The ordered pair is (11, −1).
Check that the ordered pair is a solution to **both** original equations. $\begin{array}{rcl} x+y & = & 10 \\ 11+(-1) & \stackrel{?}{=} & 10 \\ 10 & = & 10\checkmark \end{array}$ $\quad$ $\begin{array}{rcl} x-y & = & 12 \\ 11-(-1) & \stackrel{?}{=} & 12 \\ 12 & = & 12\checkmark \end{array}$	
	The solution is (11, −1).

Try It 10.3.5

Solve the system by elimination. $\begin{cases} 2x+y=5 \\ x-y=4 \end{cases}$

Answer

(3,−1)

Try It 10.3.6

Solve the system by elimination. $\begin{cases} x+y=3 \\ -2x-y=-1 \end{cases}$

Answer

$(-2,5)$

In Example 10.3.7, we will be able to make the coefficients of one variable opposites by multiplying one equation by a constant.

Example 10.3.7

Solve the system by elimination. $\begin{cases} 3x-2y=-2 \\ 5x-6y=10 \end{cases}$

Solution

	$\begin{aligned} 3x-2y&=-2\\ 5x-6y&=10 \end{aligned}$
Both equations are in standard form.	
None of the coefficients are opposites.	
We can make the coefficients of y opposites by multiplying the first equation by -3.	$\begin{aligned} -3(3x-2y)&=-3(-2)\\ 5x-6y&=10 \end{aligned}$
Simplify.	$\begin{aligned} -9x+6y&=6\\ 5x-6y&=10 \end{aligned}$
Add the two equations to eliminate y.	$\begin{array}{r} -9x+6y=6 \\ 5x-6y=10 \\ \hline -4x\quad\ =16 \end{array}$
Solve for the remaining variable, x. Substitute $x=-4$ into one of the original equations.	$x=-4$ $3x-2y=-2$
	$3(-4)-2y=-2$
Solve for y.	$\begin{aligned} -12-2y&=-2\\ -2y&=10\\ y&=-5 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(-4,-5)$.
Check that the ordered pair is a solution to both original equations.	$\begin{array}{rcl\|rcl} 3x-2y & = & -2 & 5x-6y & = & 10 \\ 3(-4)-2(-5) & \stackrel{?}{=} & -2 & 5(-4)-6(-5) & \stackrel{?}{=} & 10 \\ -12+10 & \stackrel{?}{=} & -2 & -20+30 & \stackrel{?}{=} & 10 \\ -2 & = & -2\checkmark & 10 & = & 10\checkmark \end{array}$	
	The solution is $(-4,-5)$.	

Try It 10.3.8

Solve the system by elimination. $\begin{cases} 4x-3y=1 \\ 5x-9y=-4 \end{cases}$

Answer

(1,1)

Try It 10.3.9

Solve the system by elimination. $\begin{cases} 3x+2y=2 \\ 6x+5y=8 \end{cases}$

Answer

(−2,4)

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Example 10.3.10

Solve the system by elimination. $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$

Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by a constant to get the opposites.

	$\begin{aligned} 4x-3y&=9 \\ 7x+2y&=-6 \end{aligned}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{aligned} 2(4x-3y)&=2(9) \\ 3(7x+2y)&=3(-6) \end{aligned}$
Simplify.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \end{aligned}$
Add the two equations to eliminate y.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \\ \hline 39x&=0 \end{aligned}$
Solve for x. Substitute $x = 0$ into one of the original equations.	$x=0$ $7x+2y=-6$
	$7\cdot 0+2y=-6$
Solve for y.	$2y=-6$
	$y=-3$
Write the solution as an ordered pair.	The ordered pair is (0, −3).
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} 4x-3y &= 9 \\ 4(0)-3(-3) &\stackrel{?}{=} 9 \\ 9 &= 9\checkmark \end{aligned}$ $\quad$ $\begin{aligned} 7x+2y &= -6 \\ 7(0)+2(-3) &\stackrel{?}{=} -6 \\ -6 &= -6\checkmark \end{aligned}$	
	The solution is (0, −3).

Try It 10.3.11

Solve the system by elimination. $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Answer

(1,3)

Try It 10.3.12

Solve the system by elimination. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Answer

(4,−3)

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by its LCD.

Example 10.3.13

Solve the system by elimination. $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$

Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by its LCD to clear the fractions.

	$x+\frac{1}{2}y=6$ $\frac{3}{2}x+\frac{2}{3}y=\frac{17}{2}$
To clear the fractions, multiply each equation by its LCD.	$2\left(x+\frac{1}{2}y\right)=2(6)$ $6\left(\frac{3}{2}x+\frac{2}{3}y\right)=6\left(\frac{17}{2}\right)$
Simplify.	$2x+y=12$ $9x+4y=51$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate y multiplying the top equation by −4.	$-4(2x+y)=-4(12)$ $9x+4y=51$
Simplify and add.	$-8x-4y=-48$ $9x+4y=51$ $x=3$
Substitute $x = 3$ into one of the original equations.	$x+\frac{1}{2}y=6$
Solve for y.	$3+\frac{1}{2}y=6$
	$\frac{1}{2}y=3$
	$y=6$
Write the solution as an ordered pair.	The ordered pair is (3, 6).

Check that the ordered pair is a solution to **both** original equations.

$$\begin{aligned} x+\tfrac{1}{2}y &= 6 \\ 3+\tfrac{1}{2}(6) &\stackrel{?}{=} 6 \\ 3+3 &\stackrel{?}{=} 6 \\ 6 &= 6\checkmark \end{aligned} \qquad \begin{aligned} \tfrac{3}{2}x+\tfrac{2}{3}y &= \tfrac{17}{2} \\ \tfrac{3}{2}(3)+\tfrac{2}{3}(6) &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{9}{2}+4 &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{9}{2}+\tfrac{8}{2} &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{17}{2} &= \tfrac{17}{2}\checkmark \end{aligned}$$

The solution is (3, 6).

Try It 10.3.14

Solve the system by elimination. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$

Answer

(6,2)

Try It 10.3.15

Solve the system by elimination. $\begin{cases} x+\frac{3}{5}y=-\frac{1}{5} \\ -\frac{1}{2}x-\frac{2}{3}y=\frac{5}{6} \end{cases}$

Answer

(1,−2)

In the Solving Systems of Equations by Graphing we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

Example 10.3.16

Solve the system by elimination. $\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$

Solution

$$\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$$

Write the second equation in standard form.

$$\begin{cases} 3x+4y=12 \\ \frac{3}{4}x+y=3 \end{cases}$$

Clear the fractions by multiplying thesecond equation by 4.

$$\begin{cases} 3x+4y=12 \\ 4\left(\frac{3}{4}x+y\right)=4(3) \end{cases}$$

Simplify.

$$\begin{cases} 3x+4y=12 \\ 3x+4y=12 \end{cases}$$

To eliminate a variable, we multiply thesecond equation by -1.

$$\begin{cases} 3x+4y=12 \\ \underline{-3x-4y=-12} \end{cases}$$

Simplify and add.

$$0=0$$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

? Try It 10.3.17

Solve the system by elimination. $\begin{cases} 5x-3y=15 \\ y=-5+\frac{5}{3}x \end{cases}$

Answer

infinitely many solutions

? Try It 10.3.18

Solve the system by elimination. $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

Answer

infinitely many solutions

✓ Example 10.3.19

Solve the system by elimination. $\begin{cases} -6x+15y=10 \\ 2x-5y=-5 \end{cases}$

Solution

The equations are in standard form. $\begin{cases} -6x+15y=10 \\ 2x-5y=-5 \end{cases}$

Multiply the second equation by 3 to eliminate a variable. $\begin{cases} -6x+15y=10 \\ 3(2x-5y)=3(-5) \end{cases}$

Simplify and add. $\begin{cases} -6x+15y=10 \\ 6x-15y=-15 \\ \hline 0 \neq 5 \end{cases}$

This statement is false. The equations are inconsistent and so their graphs would be parallel lines.

The system does not have a solution.

Try It 10.3.20

Solve the system by elimination. $\begin{cases} -3x+2y=8 \\ 9x-6y=13 \end{cases}$

Answer

no solution

Try It 10.3.21

Solve the system by elimination. $\begin{cases} 7x-3y=-2 \\ -14x+6y=8 \end{cases}$

Answer

no solution

Solve Applications of Systems of Equations by Elimination

Some applications problems translate directly into equations in standard form, so we will use the elimination method to solve them. As before, we use our Problem Solving Strategy to help us stay focused and organized.

Example 10.3.22

The sum of two numbers is 39. Their difference is 9. Find the numbers.

Solution

Step 1. Read the problem	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let n = the first number. m = the second number
Step 4. Translate into a system of equations.	
	The sum of two numbers is 39. $n+m=39$ Their difference is 9. $n-m=9$
The system is:	$\begin{cases} n+m=39 \\ n-m=9 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form and the coefficients of m are opposites. Add.	$\begin{cases} n+m=39 \\ \underline{n-m=9} \\ 2n \quad\quad =48 \end{cases}$
Solve for n.	$n=24$
Substitute n=24 into one of the original equations and solve form.	$n+m=39$ $24+m=39$ $m=15$
Step 6. Check the answer.	Since 24+15=39 and 24−15=9, the answers check.
Step 7. Answer the question.	The numbers are 24 and 15.

? Try It 10.3.23

The sum of two numbers is 42. Their difference is 8. Find the numbers.

Answer

The numbers are 25 and 17.

? Try It 10.3.24

The sum of two numbers is −15. Their difference is −35. Find the numbers.

Answer

The numbers are −25 and 10.

✓ Example 10.3.25

Joe stops at a burger restaurant every day on his way to work. Monday he had one order of medium fries and two small sodas, which had a total of 620 calories. Tuesday he had two orders of medium fries and one small soda, for a total of 820 calories. How many calories are there in one order of medium fries? How many calories in one small soda?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories in one order of medium fries and in one small soda.
Step 3. Name what we are looking for.	Let f = the number of calories in 1 order of medium fries. s = the number of calories in 1 small soda.
Step 4. Translate into a system of equations:	one medium fries and two small sodas had a total of 620 calories
	$f+2s=620$
	two medium fries and one small soda had a total of 820 calories.
	$2f+s=820$
Our system is:	$\begin{cases} f+2s=620 \\ 2f+s=820 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form. To get opposite coefficients of f, multiply the top equation by -2.	$\begin{cases} -2(f+2s)=-2(620) \\ 2f+s=820 \end{cases}$
Simplify and add.	$\begin{array}{r} -2f-4s=-1240 \\ 2f+s=820 \\ \hline -3s=-420 \end{array}$
Solve for s.	$s=140$
Substitute $s=140$ into one of the original equations and then solve for f.	$f+2s=620$
	$f+2\cdot 140=620$
	$f+280=620$
	$f=340$
Step 6. Check the answer.	Verify that these numbers make sense in the problem and that they are solutions to both equations. We leave this to you!
Step 7. Answer the question.	The small soda has 140 calories and the fries have 340 calories.

? Try It 10.3.26

Malik stops at the grocery store to buy a bag of diapers and 2 cans of formula. He spends a total of $37. The next week he stops and buys 2 bags of diapers and 5 cans of formula for a total of $87. How much does a bag of diapers cost? How much is one can of formula?

Answer

The bag of diapers costs $11 and the can of formula costs $13.

? Try It 10.3.27

To get her daily intake of fruit for the day, Sasha eats a banana and 8 strawberries on Wednesday for a calorie count of 145. On the following Wednesday, she eats two bananas and 5 strawberries for a total of 235 calories for the fruit. How many calories are there in a banana? How many calories are in a strawberry?

Answer

There are 105 calories in a banana and 5 calories in a strawberry.

Choose the Most Convenient Method to Solve a System of Linear Equations

When you will have to solve a system of linear equations in a later math class, you will usually not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved for one variable.	Use when the equations are in standard form.

✓ Example 10.3.28

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

1. $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$
2. $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Solution

1. $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$

Since both equations are in standard form, using elimination will be most convenient.

2. $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Since one equation is already solved for y, using substitution will be most convenient.

? Try It 10.3.29

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

1. $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$
2. $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

1. Since both equations are in standard form, using elimination will be most convenient.
2. Since one equation is already solved for xx, using substitution will be most convenient.

Try It 10.3.30

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

1. $\begin{cases} y = 2x - 1 \\ 3x - 4y = -6 \end{cases}$
2. $\begin{cases} 6x - 2y = 12 \\ 3x + 7y = -13 \end{cases}$

Answer

1. Since one equation is already solved for y, using substitution will be most convenient;
2. Since both equations are in standard form, using elimination will be most convenient.

Note

Access these online resources for additional instruction and practice with solving systems of linear equations by elimination.

- Instructional Video-Solving Systems of Equations by Elimination
- Instructional Video-Solving by Elimination
- Instructional Video-Solving Systems by Elimination

Key Concepts

- **To Solve a System of Equations by Elimination**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     - Decide which variable you will eliminate.
     - Multiply one or both equations so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.
  7. Check that the ordered pair is a solution to **both** original equations.

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10.4: Solve Mixture Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications

Note

Before you get started, take this readiness quiz.

1. Multiply 4.025(1,562).
   If you missed this problem, review Example 1.8.22.
2. Write 8.2% as a decimal.
   If you missed this problem, review Example 1.8.46.
3. Earl's dinner bill came to $32.50 and he wanted to leave an 18% tip. How much should the tip be?
   If you missed this problem, review Example 3.2.10.

Solve Mixture Applications

When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

Type	Number	• Value($)	= Total Value($)
nickels		0.05	
dimes		0.10	

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

Example 10.4.1

Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled $1,302. How many $11 adult and how many $8 child tickets were sold?

Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let a= the number of adult tickets. c= the number of child tickets
A table will help us organize the data. We have two types of tickets: adult and child.	Write a and c for the number of tickets.

Write the total number of tickets sold at the bottom of the Number column.	Altogether 147 were sold.
Write the value of each type of ticket in the Value column.	The value of each adult ticket is $11. The value of each child tickets is $8.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 11 = 11a$, and the total value of child tickets is $c \cdot 8 = 8c$.	<table><tr><th>Type</th><th>Number</th><th>•</th><th>Value ($)</th><th>=</th><th>Total Value ($)</th></tr><tr><td>adult</td><td>a</td><td></td><td>11</td><td></td><td>$11a$</td></tr><tr><td>child</td><td>c</td><td></td><td>8</td><td></td><td>$8c$</td></tr><tr><td></td><td>147</td><td></td><td></td><td></td><td>1302</td></tr></table>
Altogether the total value of the tickets was $1,302.	Fill in the Total Value column.
Step 4. Translate into a system of equations.	
The Number column and the Total Value column give us the system of equations. We will use the elimination method to solve this system.	$a + c = 147$ $11a + 8c = 1302$
Multiply the first equation by −8.	$-8(a + c) = -8(147)$ $11a + 8c = 1302$
Simplify and add, then solve for a.	$-8a - 8c = -1176$ $11a + 8c = 1302$ $3a = 126$
	$a = 42$
Substitute $a = 42$ into the first equation, then solve for c.	$42 + c = 147$
	$c = 105$
Step 5. Check the answer in the problem. 42 adult tickets at $11 per ticket makes $462 105 child tickets at $8 per ticket makes $840. The total receipts are $1,302.✓	
Step 6. Answer the question.	The movie theater sold 42 adult tickets and 105 child tickets.

Try It 10.4.2

Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled $3,936. How many $9 adult tickets and how many $6 child tickets were sold?

Answer

There were 206 adult tickets sold and 347 children tickets sold.

Try It 10.4.3

Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Answer

There were 521 adult tickets sold and 842 children tickets sold.

In Example 10.4.4 we'll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

Example 10.4.4

Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of nickels and the number of quarters.
Step 3. Name what we are looking for.	Let n= the number of nickels. q= the number of quarters
A table will help us organize the data. We have two types of coins, nickels and quarters.	Write n and q for the number of each type of coin.
Fill in the Value column with the value of each type of coin.	The value of each nickel is $0.05. The value of each quarter is $0.25.
The number times the value gives the total value, so, the total value of the nickels is n (0.05) = 0.05n and the total value of quarters is q(0.25) = 0.25q. Altogether the total value of the coins is $7.30.	

Type	Number • Value ($) = Total Value ($)		
nickels	n	0.05	$0.05n$
quarters	q	0.25	$0.25q$
			7.30

Step 4. Translate into a system of equations.	
The Total value column gives one equation.	$0.05n + 0.25q = 7.30$
We also know the number of nickels is six less than three times the number of quarters. Translate to get the second equation.	$n = 3q - 6$
Now we have the system to solve.	$0.05n + 0.25q = 7.30$ $n = 3q - 6$
Step 5. Solve the system of equations We will use the substitution method. Substitute $n = 3q - 6$ into the first equation. Simplify and solve for q.	$0.05n + 0.25q = 7.30$
	$0.05(3q - 6) + 0.25q = 7.3$

	$0.15q - 0.3 + 0.25q = 7.3$
	$0.4q - 0.3 = 7.3$
	$0.4q = 7.6$
	$q = 19$
To find the number of nickels, substitute $q = 19$ into the second equation.	$n = 3q - 6$
	$n = 3 \cdot 19 - 6$
	$n = 51$
Step 6. Check the answer in the problem. $19 \text{ quarters at } \$0.25 = \$4.75$ $51 \text{ ickels at } \$0.05 = \$2.55$ $\text{Total } = \$7.30\checkmark$ $3 \cdot 19 - 16 = 51\checkmark$	
Step 7. Answer the question.	Priam has 19 quarters and 51 nickels.

Try It 10.4.5

Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of $8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Answer

Matilda has 13 dimes and 29 quarters.

Try It 10.4.6

Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is $8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Answer

Juan has 36 nickels and 63 dimes.

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

Example 10.4.7

Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60 per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Solution

Step 1. Read the problem.	We will create a table to organize the information.

Step 2. Identify what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
Step 3. Name what we are looking for.	Let n= the number of pound of nuts. c= the number of pounds of chips
Carson will mix nuts and chocolate chips to get trail mix. Write in n and c for the number of pounds of nuts and chocolate chips. There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using	

Type	Number of pounds	• Value ($)	= Total Value ($)
nuts	n	9.00	$9n$
chocolate chips	c	2.00	$2c$
trail mix	20	7.60	$7.60(20) = 152$

Number · Value = Total Value	
Step 4. Translate into a system of equations. We get the equations from the Number and Total Value columns.	$\begin{aligned} n + c &= 20 \\ 9n + 2c &= 152 \end{aligned}$
Step 5. Solve the system of equations We will use elimination to solve the system.	
Multiply the first equation by −2 to eliminate c.	$\begin{aligned} -2(n + c) &= -2(20) \\ 9n + 2c &= 152 \end{aligned}$
Simplify and add. Solve for n.	$\begin{aligned} -2n - 2c &= -40 \\ 9n + 2c &= 152 \\ \hline 7n \quad &= 112 \end{aligned}$
	$n = 16$
To find the number of pounds of chocolate chips, substitute n = 16 into the first equation, then solve for c.	$\begin{aligned} n + c &= 20 \\ 16 + c &= 20 \end{aligned}$
	c=4
Step 6. Check the answer in the problem. $16 + 4 = 20$✓ $9 \cdot 16 + 2 \cdot 4 = 152$✓	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

? Try It 10.4.8

Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $6 per pound. Peanuts are $4 per pound and cashews are $9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Answer

Greta should use 3 pounds of peanuts and 2 pounds of cashews.

Try It 10.4.9

Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $3 per pound. The price of beans is $1 per pound and the price of ground beef is $5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Answer

Sammy should purchase 10 pounds of beans and 10 pounds of ground beef.

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

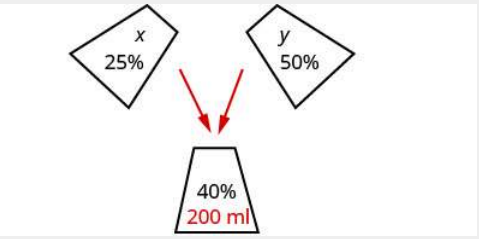
For these kinds of mixture problems, we'll use percent instead of value for one of the columns in our table.

Example 10.4.10

Translate to a system of equations and solve:

Sasheena is a lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Solution

Step 1. Read the problem.	A figure may help us visualize the situation, then we will create a table to organize the information.
Sasheena must mix some of the 25% solution and some of the 50% solution together to get 200 ml of the 40% solution.	
Step 2. Identify what we are looking for.	We are looking for how much of each solution she needs.
Step 3. Name what we are looking for.	Let x= number of ml of 25% solution. y= number of ml of 50% solution
A table will help us organize the data. She will mix x ml of 25% with y ml of 50% to get 200 ml of 40% solution. We write the percents as decimals in the chart. We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.	

Type	Number of units	• Concentration %	= Amount
25%	x	0.25	$0.25x$
50%	y	0.50	$0.50y$
40%	200	0.40	0.40(200)

Step 4. Translate into a system of equations. We get the equations from the Number column and the Amount column.	
Now we have the system.	$\begin{cases} x+y=200 \\ 0.25x+0.50y=0.40(200) \end{cases}$
Step 5. Solve the system of equations. We will solve the system by elimination. Multiply the first equation by −0.5 to eliminate y.	$\begin{cases} -0.5(x+y)=-0.5(200) \\ 0.25x+0.50y=80 \end{cases}$
Simplify and add to solve for x.	$\begin{cases} -0.5x-0.5y=-100 \\ 0.25x+0.5y=80 \end{cases}$ $-0.25x = -20$ $x = 80$
To solve for y, substitute $x = 80$ into the first equation.	$x+y=200$
	$80+y=200$
	$y=120$
Step 6. Check the answer in the problem. $80+120 = 120$✓ $0.25(80)+0.50(120) = 80$✓ Yes!	
Step 7. Answer the question.	Sasheena should mix 80 ml of the 25% solution with 120 ml of the 50% solution to get the 200 ml of the 40% solution.

Try It 10.4.11

Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Answer

LeBron needs 120 ml of the 25% solution and 30 ml of the 50% solution.

Try It 10.4.12

Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solutions should he mix to make the 25% solution?

Answer

Anatole should mix 125 ml of the 10% solution and 125 ml of the 40% solution.

Solve Interest Applications

The formula to model interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in Example 10.4.13

Example 10.4.13

Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1% interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the amount to invest in each fund.
Step 3. Name what we are looking for.	Let s= the amount invested in stocks. b= the amount invested in bonds.
Write the interest rate as a decimal for each fund. Multiply: Principal · Rate · Time to get the Interest.	<table><tr><th>Account</th><th>Principal</th><th>Rate</th><th>Time</th><th>Interest</th></tr><tr><td>stock fund</td><td>s</td><td>0.08</td><td>1</td><td>0.08s</td></tr><tr><td>bonds</td><td>b</td><td>0.03</td><td>1</td><td>0.03b</td></tr><tr><td>Total</td><td>40,000</td><td>0.071</td><td></td><td>0.071(40,000)</td></tr></table>
Step 4. Translate into a system of equations. We get our system of equations from the Principal column and the Interest column.	$\begin{cases} s + b = 40{,}000 \\ 0.08s + 0.03b = 0.071(40{,}000) \end{cases}$
Step 5. Solve the system of equations Solve by elimination. Multiply the top equation by −0.03.	$\begin{cases} -0.03(s + b) = -0.03(40{,}000) \\ 0.08s + 0.03b = 2{,}840 \end{cases}$
Simplify and add to solve for s.	$\begin{array}{r} -0.03s - 0.03b = -1{,}200 \\ 0.08s + 0.03b = 2{,}840 \\ \hline 0.05s = 1{,}640 \end{array}$
	$s = 32{,}800$
To find b, substitute s = 32,800 into the first equation.	$\begin{array}{r} s + b = 40{,}000 \\ 32{,}800 + b = 40{,}000 \end{array}$
	$b = 7{,}200$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	Adnan should invest $32,800 in stock and $7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s + b = 40{,}000$, represents the total amount of money invested and the second equation, $0.08s + 0.03b = 0.071(40{,}000)$, represents the interest earned.

Try It 10.4.14

Translate to a system of equations and solve:

Leon had $50,000 to invest and hopes to earn 6.2 % interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Answer

Leon should put $42,000 in the stock fund and $8000 in the savings account.

Try It 10.4.15

Translate to a system of equations and solve:

Julius invested $7,000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5% interest on the total investment. How much money did he put in each stock?

Answer

Julius invested $1,750 at 11% and $5,250 at 13%.

Example 10.4.16

Translate to a system of equations and solve:

Rosie owes $21,540 on her two student loans. The interest rate on her bank loan is 10.5% and the interest rate on the federal loan is 5.9%. The total amount of interest she paid last year was $1,669.68. What was the principal for each loan?

Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let b= the principal for the bank loan. f= the principal on the federal loan
The total loans are $21,540.	
Record the interest rates as decimals in the chart.	

Account	Principal •	Rate •	Time =	Interest
bank	b	0.105	1	$0.105b$
federal	f	0.059	1	$0.059f$
Total	21,540			1669.68

Multiply using the formula $I = Prt$ to get the Interest.	
Step 4. Translate into a system of equations. The system of equations comes from the Principal column and the Interest column.	$b + f = 21{,}540$ $0.105b + 0.059f = 1669.68$
Step 5. Solve the system of equations We will use substitution to solve. Solve the first equation for b.	$b + f = 21{,}540$ $b = -f + 21{,}540$
Substitute $b = -f + 21{,}540$ into the second equation.	$0.105b + 0.059f = 1669.68$ $0.105(-f + 21{,}540) + 0.059f = 1669.68$

Simplify and solve for f.	$-0.105f + 2261.70 + 0.059f = 1669.68$
	$-0.046f + 2261.70 = 1669.68$
	$-0.046f = -592.02$
	$f = 12{,}870$
To find b, substitute $f = 12{,}870$ into the first equation.	$b + f = 21{,}540$ $12{,}870 + f = 21{,}540$
	$f = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the bank loan is \$12,870 and the principal for the federal loan is \$8,670.

Try It 10.4.17

Translate to a system of equations and solve:

Laura owes \$18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9 %. The total amount of interest she paid last year was \$1,066. What was the principal for each loan?

Answer

The principal amount for the bank loan was \$4,000. The principal amount for the federal loan was \$14,000.

Try It 10.4.18

Translate to a system of equations and solve:

Jill's Sandwich Shoppe owes \$65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was \$3,582. What was the principal for each loan?

Answer

The principal amount for was \$41,200 at 4.5%. The principal amount was, \$24,000 at 7.2%.

Note

Access these online resources for additional instruction and practice with solving application problems with systems of linear equations.

- Cost and Mixture Word Problems
- Mixture Problems

Key Concepts

- **Table for coin and mixture applications**

Type	Number	• Value(\$)	= Total Value(\$)
Total			

- **Table for concentration applications**

Type	Number of units	•	Concentration %	=	Amount
Total					

- **Table for interest applications**

Account	Principal	•	Rate	•	Time	=	Interest
					1		
					1		
Total							

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CHAPTER OVERVIEW

11: Quadratic Equations

11.1: Quadratic Equations and Solving by Factoring
11.2: Solve Quadratic Equations Using the Square Root Property
11.3: Solve Quadratic Equations Using the Quadratic Formula
11.4: Solve Applications Modeled by Quadratic Equations

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11.1: Quadratic Equations and Solving by Factoring

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations by using the Zero Product Property
- Solve quadratic equations factoring
- Solve applications modeled by quadratic equations

Note

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
   If you missed this problem, review Example 2.3.1.
2. Solve: $10a=0$.
   If you missed this problem, review Example 2.2.1.
3. Combine like terms: $12x^2-6x+4x$.
   If you missed this problem, review Example 1.3.37.
4. Factor n^3-9n^2-22n completely.
   If you missed this problem, review Example 7.3.10.

We have already solved linear equations, equations of the form $ax+by=c$. In linear equations, the variables have no exponents. Quadratic equations are equations in which the variable is squared. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \quad 3y^2+4y=10 \quad 64u^2-81=0 \quad n(n+1)=42 \tag{11.1.1}$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$,with $a\neq 0$.

QUADRATIC EQUATION

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a, b, \text{ and } c \text{ are real numbers and } a\neq 0 \tag{11.1.2}$$

To solve **quadratic equations** we need methods different than the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

ZERO PRODUCT PROPERTY

If $a\cdot b=0$, then either $a=0$ or $b=0$ or both.

We will now use the **Zero Product Property**, to solve a quadratic equation.

Example 11.1.1: How to Use the Zero Product Property to Solve a Quadratic Equation

Solve: $(x+1)(x-4)=0$

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0$ or $x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1$ or $x=4$

Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0$ ✓ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0$ ✓

Try It 11.1.2

Solve: $(x-3)(x+5)=0$

Answer

$x=3, x=-5$

Try It 11.1.3

Solve: $(y-6)(y+9)=0$

Answer

$y=6, y=-9$

We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

Example 11.1.4

Solve: $(5n-2)(6n-1)=0$

Solution

	$(5n-2)(6n-1)=0$	
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n=\frac{2}{5}$	$n=\frac{1}{6}$
Check your answers.		
$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-\frac{5}{5}\right)\stackrel{?}{=}0$ $(0)\left(\frac{7}{5}\right)\stackrel{?}{=}0$ $0=0$ ✓	$n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0$ ✓	

Try It 11.1.5

Solve: $(3m-2)(2m+1)=0$

Answer

$m=\frac{2}{3}, m=-\frac{1}{2}$

Try It 11.1.6

Solve: $(4p+3)(4p-3)=0$

Answer

$p=-\frac{3}{4}, p=\frac{3}{4}$

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

Example 11.1.7

Solve: $3p(10p+7)=0$

Solution

	$3p(10p+7)=0$	
Use the Zero Product Property to set each factor to 0.	3p=0	10p+7=0
Solve the equations.	p=0	10p=−7
		$p=-\frac{7}{10}$
Check your answers.		

$p=0$	$p=-\frac{7}{10}$
$3p(10p+7)=0$	$3p(10p+7)=0$
$3\cdot 0(10\cdot 0+7)\stackrel{?}{=}0$	$3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right)+7\stackrel{?}{=}0$
$0(0+7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(-7+7)\stackrel{?}{=}0$
$0(7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(0)\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

Try It 11.1.8

Solve: $2u(5u-1)=0$

Answer

$u=0, u=\frac{1}{5}$

Try It 11.1.9

Solve: $w(2w+3)=0$

Answer

$w=0, w=-\frac{3}{2}$

It may appear that there is only one factor in the next example. Remember, however, that $(y-8)^2$ means $(y-8)(y-8)$.

Example 11.1.10

Solve: $(y-8)^2=0$

Solution

	$(y-8)^2=0$	
Rewrite the left side as a product.	(y−8)(y−8)=0	
Use the Zero Product Property and set each factor to 0.	y−8=0	y−8=0
Solve the equations.	y=8	y=8

When a solution repeats, we call it a double root.		
Check your answer.		
$y = 8$ $(y-8)^2 = 0$ $(8-8)^2 \stackrel{?}{=} 0$ $(0)^2 \stackrel{?}{=} 0$ $0 = 0$ ✓		

? Try It 11.1.11

Solve: $(x+1)^2 = 0$

Answer

$x = 1$

? Try It 11.1.12

Solve: $(v-2)^2 = 0$

Answer

$v = 2$

Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2 + bx + c = 0$. Then we factor the expression on the left.

✓ Example 11.1.13

Solve: $x^2 + 2x - 8 = 0$

Solution

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	The equation is already in standard form.	$x^2 + 2x - 8 = 0$
Step 2. Factor the quadratic expression.	Factor $x^2 + 2x - 8$ $(x+4)(x-2)$	$(x+4)(x-2) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x + 4 = 0$ or $x - 2 = 0$
Step 4. Solve the linear equations.	We have two linear equations.	$x = -4$ or $x = 2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2 + 2x - 8 = 0$ $x = -4$ $(-4)^2 - 2(-4) - 8 \stackrel{?}{=} 0$ $16 + (-8) - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 + 2x - 8 = 0$ $x = 2$ $2^2 - 2(2) - 8 \stackrel{?}{=} 0$ $4 + 4 - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓

Try It 11.1.14

Solve: $x^2 - x - 12 = 0$

Answer

$x = 4, x = -3$

Try It 11.1.15

Solve: $b^2 + 9b + 14 = 0$

Answer

$b = -2, b = -7$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check.

Before we factor, we must make sure the quadratic equation is in standard form.

Example 11.1.16

Solve: $2y^2 = 13y + 45$

Solution

	$2y^2 = 13y + 45$	
Write the quadratic equation in standard form.	$2y^2 - 13y - 45 = 0$	
Factor the quadratic expression.	$(2y + 5)(y - 9) = 0$	
Use the Zero Product Property to set each factor to 0.	$2y + 5 = 0$	$y - 9 = 0$
Solve each equation.	$y = -\frac{5}{2}$	$y = 9$
Check your answers.		

$y = -\frac{5}{2}$	$y = 9$
$2y^2 = 13y + 45$	$2y^2 = 13y + 45$
$2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$	$2(9)^2 \stackrel{?}{=} 13(9) + 45$
$2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$	$2(81) \stackrel{?}{=} 117 + 45$
$\frac{25}{2} = \frac{25}{2}$ ✓	$162 = 162$ ✓

Try It 11.1.17

Solve: $3c^2 = 10c - 8$

Answer

$c = 0, c = \frac{4}{3}$

Try It 11.1.18

Solve: $2d^2 - 5d = 3$

Answer

$d = 3, d = -\frac{1}{2}$

Example 11.1.19

Solve: $5x^2 - 13x = 7x$

Solution

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x-4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		

$x = 0$

$5x^2 - 13x = 7x$

$5(0)^2 - 13(0) \stackrel{?}{=} 7(0)$

$0 - 0 \stackrel{?}{=} 0$

$0 = 0$ ✓

$x = 4$

$5x^2 - 13x = 7x$

$5(4)^2 - 13(4) \stackrel{?}{=} 7(4)$

$5(16) - 52 \stackrel{?}{=} 28$

$28 = 28$ ✓

Try It 11.1.20

Solve: $6a^2 + 9a = 3a$

Answer

$a = 0, a = -1$

Try It 11.1.21

Solve: $45b^2 - 2b = -17b$

Answer

$b = 0, b = -\frac{1}{3}$

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example 11.1.22

Solve: $144q^2 = 25$

Solution

	$144q^2$	$=$	25			
Write the quadratic equation in standard form.	$144q^2 - 25$	$=$	0			
Factor. It is a difference of squares.	$(12q-5)(12q+5)$	$=$	0			
Use the Zero Product Property to set each factor to 0.	$12q - 5$	$=$	0	$12q + 5$	$=$	0
Solve each equation.	$12q$	$=$	5	$12q$	$=$	-5
	q	$=$	$\frac{5}{12}$	q	$=$	$-\frac{5}{12}$

Check your answers.

Try It 11.1.23

Solve: $25p^2 = 49$

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Try It 11.1.24

Solve: $36x^2 = 121$

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example 11.1.25

Solve: $(3x-8)(x-1) = 3x$

Solution

	$(3x-8)(x-1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x-2)(x-4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x-2=0 \quad x-4=0$
Solve each equation.	$3x = 2 \quad x = 4$
	$x = \frac{2}{3}$
Check your answers.	The check is left to you!

Try It 11.1.26

Solve: $(2m+1)(m+3) = 12m$

Answer

$m = 1, m = \frac{3}{2}$

Try It 11.1.27

Solve: $(k+1)(k-1) = 8$

Answer

$k = 3, k = -3$

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

Example 11.1.28

Solve: $9m^3 + 100m = 60m^2$

Solution

	$9m^3 + 100m$	$=$	$60m^2$					
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m$	$=$	0					
Factor the greatest common factor first.	$m\left(9m^2 - 60m + 100\right)$	$=$	0					
Factor the trinomial.	$m(3m-10)(3m-10)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	m	$=$	0	$3m-10$	$=$	0	$3m-10$	$=$	0
Solve each equation.	m	$=$	0	m	$=$	$\frac{10}{3}$	m	$=$	$\frac{10}{3}$
Check your answers.	The check is left to you.								

Try It 11.1.29

Solve: $8x^3 = 24x^2 - 18x$

Answer

$x = 0, x = \frac{3}{2}$

Try It 11.1.30

Solve: $16y^2 = 32y^3 + 2y$

Answer

$y = 0, y = \frac{1}{4}$

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example 11.1.31

Solve: $4x^2 = 16x + 84$

Solution

	$4x^2$	$=$	$16x+84$					
Write the quadratic equation in standard form.	$4x^2-16x-84$	$=$	0					
Factor the greatest common factor first.	$4\left(x^2-4x-21\right)$	$=$	0					
Factor the trinomial.	$4(x-7)(x+3)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	4	$\neq$	0	$x-7$	$=$	0	$x+3$	$=$ 0
Solve each equation.	4	$\neq$	0	x	$=$	7	x	$=$ -3
Check your answers.	The check is left to you.							

Try It 11.1.32

Solve: $18a^2 - 30 = -33a$

Answer

$a = -\frac{5}{2}, a = \frac{2}{3}$

Try It 11.1.33

Solve: $123b = -6 - 60b^2$

Answer

$b = 2, b = \frac{1}{20}$

Solve Applications Modeled by Quadratic Equations

The problem solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to quadratic equations. We will copy the problem solving strategy here so we can use it for reference.

USE A PROBLEM-SOLVING STRATEGY TO SOLVE WORD PROBLEMS

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a quadratic equation.

Example 11.1.34

The product of two consecutive integers is 132. Find the integers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive integers.
Step 3. Name what we are looking for.	Let $n =$ the first integer $n+1 =$ the next consecutive integer
Step 4. Translate into an equation. Restate the problem in a sentence.	The product of the two consecutive integers is 132.
Translate to an equation.	The first integer times the next integer is 132. $n(n+1) = 132$
Step 5. Solve the equation.	$n^2+n = 132$
Bring all the terms to one side.	$n^2+n-132 = 0$
Factor the trinomial.	$(n-11)(n+12) = 0$
Use the zero product property.	$n-11 = 0 \quad n+12 = 0$
Solve the equations.	$n = 11 \quad n = -12$

Try It 11.1.35

The product of two consecutive integers is 240. Find the integers.

Answer

$-15, -16$ and $15, 16$

Try It 11.1.36

The product of two consecutive integers is 420. Find the integers.

Answer

$-21, -20$ and $20, 21$

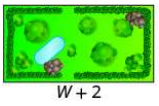
Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give 132.

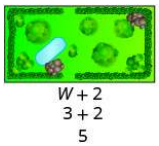
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

Example 11.1.37

A rectangular garden has an area of 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.		
Step 2. Identify what you are looking for.	We are looking for the length and width.	
Step 3. Name what you are looking for. The length is two feet more than width.	Let W = the width of the garden. $W + 2$ = the length of the garden	
Step 4. Translate into an equation. Restate the important information in a sentence.	The area of the rectangular garden is 15 square feet.	
Use the formula for the area of a rectangle.	$A = L \cdot W$	
Substitute in the variables.	$15 = (W+2)W$	
Step 5. Solve the equation. Distribute first.	$15 = W^2 + 2W$	
Get zero on one side.	$0 = W^2 + 2W - 15$	
Factor the trinomial.	$0 = (W+5)(W-3)$	
Use the Zero Product Property.	$0 = W+5$	$0 = W-3$
Solve each equation.	$-5 = W$	$3 = W$

Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.	$W = -5$ cannot be the width, since it's negative. $W = 3$	$3 = W$ Width is 3 feet.
Find the value of the length.	$\text{length} = W + 2$	
	$\text{length} = 3 + 2$	
	$\text{length} = 5$	Length is 5 feet.
Step 6. Check the answer. Does the answer make sense?		
 $A = L \cdot W$ $A = 3 \cdot 5$ $A = 15$		
	Yes, this makes sense.	
Step 7. Answer the question.	The width of the garden is 3 feet and the length is 5 feet.	

? Try It 11.1.38

A rectangular sign has an area of 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

55 feet and 66 feet

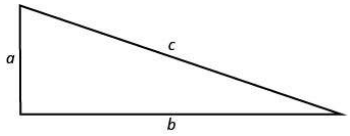
? Try It 11.1.39

A rectangular patio has an area of 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

12 feet and 15 feet

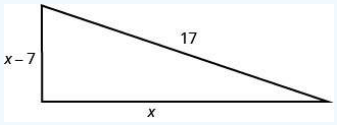
In an earlier chapter, we used the Pythagorean Theorem $\left(a^2 + b^2 = c^2\right)$. It gave the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

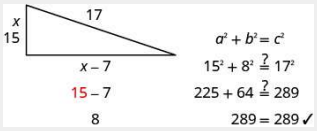
✓ Example 11.1.40

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown below. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



Solution

Step 1. Read the problem.			
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the deck.		
Step 3. Name what you are looking for. One side is 7 less than the other.	Let x = length of a side of the deck $x - 7$ = length of other side		

Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2+b^2=c^2$		
Substitute in the variables.	$x^2+(x-7)^2=17^2$		
Step 5. Solve the equation.	$x^2+x^2-14x+49=289$		
Simplify.	$2x^2-14x+49=289$		
It is a quadratic equation, so get zero on one side.	$2x^2-14x-240=0$		
Factor the greatest common factor.	$2\left(x^2-7x-120\right)=0$		
Factor the trinomial.	$2(x-15)(x+8)=0$		
Use the Zero Product Property.	$2\neq 0$	$x-15=0$	$x+8=0$
Solve.	$2\neq 0$	$x=15$	$x=-8$
Since x is a side of the triangle, $x=-8$ does not make sense.	$2\neq 0$	$x=15$	$x=-8$
Find the length of the other side.			
If the length of one side is		$x=15$	
then the length of the other side is		$x-7$	
		$15-7=8$	
	8 is the length of the other side.		
Step 6. Check the answer. Do these numbers make sense?			
			
Step 7. Answer the question.	The sides of the deck are 8, 15, and 17 feet.		

? Try It 11.1.41

A boat's sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.

Answer

5 feet and 12 feet

? Try It 11.1.42

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of one of the other legs. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

Key Concepts

- **Zero Product Property** If $a\cdot b=0$, then either a=0 or b=0 or both. See Example.
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See Example.
  1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.

4. Solve the linear equations.
5. Check.

- **Use a problem solving strategy to solve word problems** See Example.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

quadratic equations

are equations in which the variable is squared.

Zero Product Property

The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

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11.2: Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Quadratic equations are equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. They differ from linear equations by including a term with the variable raised to the second power. We use different methods to solve **quadratic equation**s than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will use three other methods to solve quadratic equations.

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the left side.	$(x-3)(x+3) = 0$
Use the Zero Product Property.	$(x-3) = 0, (x+3) = 0$
Solve each equation.	$x = 3, x = -3$
Combine the two solutions into $\pm$ form	$x = \pm 3$

(The solution is read 'x is equal to positive or negative 3.')

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

These equations are all of the form $x^2 = k$.
We defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

This leads to the **Square Root Property**.

Definition: SQUARE ROOT PROPERTY

If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Notice that the **Square Root Property** gives two solutions to an equation of the form $x^2 = k$: the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$

Now, we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm\sqrt{9}$
Simplify the radical.	$x = \pm 3$
Rewrite to show the two solutions.	$x = 3, x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

Use the Square Root Property.	$x = \pm\sqrt{7}$
Rewrite to show two solutions.	$x = \sqrt{7}, x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$ This is considered an exact answer.

We can use a calculator to get an approximate answer. For this example our answers would be 2.65, -2.65 rounded to the hundredth.

Example 11.2.1

Solve: $x^2 = 169$

Answer

	$x^2 = 169$
Use the Square Root Property.	$x = \pm\sqrt{169}$
Simplify the radical.	$x = \pm 13$
Rewrite to show two solutions.	$x = 13, x = -13$

Example 11.2.2

Solve: $x^2 = 81$

Answer

x=9, x=−9

Example 11.2.3

Solve: $y^2 = 121$

Answer

y = 11, y = −11

Definition:SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

To use the Square Root Property, the coefficient of the variable term must equal 1. In the next example, we must divide both sides of the equation by 5 before using the Square Root Property.

Example 11.2.7

Solve: $5m^2 = 80$

Answer

The quadratic term is isolated.	$5m^2 = 80$
Divide by 5 to make its cofficient 1.	$\frac{5m^2}{5} = \frac{80}{5}$
Simplify.	$m^2 = 16$
Use the Square Root Property.	$m = \pm\sqrt{16}$

Simplify the radical.	$m = \pm 4$
Rewrite to show two solutions.	m=4,m=−4
Check the solutions. $5m^2 = 80$ $\quad$ $5m^2 = 80$ $5(4)^2 \stackrel{?}{=} 80$ $\quad$ $5(-4)^2 \stackrel{?}{=} 80$ $5 \cdot 16 \stackrel{?}{=} 80$ $\quad$ $5 \cdot 16 \stackrel{?}{=} 80$ $80 = 80$ ✓ $\quad$ $80 = 80$ ✓	

Example 11.2.8

Solve: $2x^2 = 98$.

Answer

x=7, x=−7

Example 11.2.9

Solve: $3z^2 = 108$.

Answer

z=6, z=−6

The Square Root Property started by stating, 'If $x^2 = k$, and $k \geq 0$'. What will happen if $k < 0$? This will be the case in the next example.

Example 11.2.10

Solve: $q^2 + 24 = 0$.

Answer

	$q^2 = 24$
Isolate the quadratic term.	$q^2 = -24$
Use the Square Root Property.	$q = \pm\sqrt{-24}$
The $\sqrt{-24}$ is not a real number	There is no real solution

Example 11.2.11

Solve: $c^2 + 12 = 0$.

Answer

no real solution

Example 11.2.12

Solve: $d^2 + 81 = 0$.

Answer

no real solution

Remember, we first isolate the quadratic term and then make the coefficient equal to one.

Solve Quadratic Equations of the Form $a(x-h)^2=k$ Using the Square Root Property

We can use the Square Root Property to solve an equation like $(x-3)^2=16$, too. We will treat the whole binomial, (x−3), as the quadratic term.

Example 11.2.19

Solve: $(x-3)^2=16$.

Answer

	$(x-3)^2=16$
Use the Square Root Property.	$x-3=\pm\sqrt{16}$
Simplify.	$x-3=\pm 4$
Write as two equations.	$x-3=4$, $x-3=-4$
Solve.	x=7,x=−1
Check. $(7-3)^2=16$ $(4)^2=16$ $16=16$ ✓ $(-1-3)^2=16$ $(-4)^2=16$ $16=16$ ✓	

Example 11.2.20

Solve: $(q+5)^2=1$.

Answer

q=−6, q=−4

Example 11.2.21

Solve: $(r-3)^2=25$.

Answer

r=8, r=−2

We will start the solution to the next example by isolating the binomial.

Example 11.2.28

Solve: $(x-2)^2+3=30$.

Answer

	$(x-2)^2+3=30$
Isolate the binomial term.	$(x-2)^2=27$
Use the Square Root Property.	$x-2=\pm\sqrt{27}$
Calculate the radical.	$x-2=\pm 5.1961524\ldots$
Solve for x.	$x=2\pm 5.1961524$

Let's approximate our final answer to the thousandth place.	$x = 7.196, x = -3.196$
Check. We leave the check for you	

Example 11.2.29

Solve: $(a-5)^2+4=24$.

Answer

$a = 9.472, a = 0.528$

Example 11.2.30

Solve: $(b-3)^2-8=24$.

Answer

$b = 8.657, b = -2.657$

Example 11.2.31

Solve:$(3v-7)^2=-12$.

Answer

	$(3v-7)^2=-12$
Use the Square Root Property.	$3v-7=\pm\sqrt{-12}$
The $\sqrt{-12}$ is not a real number.	There is no real solution.

Example 11.2.32

Solve: $(3r+4)^2=-8$.

Answer

no real solution

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example 11.2.33

Solve: $p^2-10p+25=18$.

Answer

The left side of the equation is a perfect square trinomial. We will factor it first.

	$p^2-10p+25=18$
Factor the perfect square trinomial.	$(p-5)^2=18$
Use the Square Root Property.	$p-5=\pm\sqrt{18}$
Calculate the radical.	$p-5=\pm4.242640687$

Solve for p.	$p = 5 \pm 4.242640687$
Calculate to show the two solutions to the thousandth place.	$p = 9.243, p = 0.757$
Check. We leave the check for you.	

Example 11.2.34

Solve: $x^2 - 6x + 9 = 12$.

Answer

$x = 6.464, x = -0.464$

Example 11.2.36

Solve: $4n^2 + 4n + 1 = 16$.

Answer

Again, we notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n + 1)^2 = 16$
Use the Square Root Property.	$(2n + 1) = \pm\sqrt{16}$
Simplify the radical.	$(2n + 1) = \pm 4$
Solve for n.	$2n = -1 \pm 4$
Divide each side by 2.	$\frac{2n}{2} = \frac{-1 \pm 4}{2}$ $n = \frac{-1 \pm 4}{2}$
Rewrite to show two solutions.	$n = \frac{-1+4}{2}, n = \frac{-1-4}{2}$
Simplify each equation.	$n = \frac{3}{2}, n = -\frac{5}{2}$
Check. $4n^2 + 4n + 1 = 16$ $4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16$ $4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16$ $9 + 6 + 1 \stackrel{?}{=} 16$ $16 = 16$ ✓ $4n^2 + 4n + 1 = 16$ $4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$ $4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$ $25 - 10 + 1 \stackrel{?}{=} 16$ $16 = 16$ ✓	

Example 11.2.37

Solve: $9m^2 - 12m + 4 = 25$.

Answer

$m = \frac{7}{3}, m = -1$

? Example 11.2.38

Solve: $16n^2 + 40n + 25 = 4$.

Answer

$n = -\frac{3}{4}$, $n = -\frac{7}{4}$

Access these online resources for additional instruction and practice with solving quadratic equations:

- Solving Quadratic Equations: Solving by Taking Square Roots
- Using Square Roots to Solve Quadratic Equations
- Solving Quadratic Equations: The Square Root Method

Key Concepts

- Square Root Property
  If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Glossary

quadratic equation

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$.

Square Root Property

The Square Root Property states that, if $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

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11.3: Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the quadratic formula
- Use the discriminant to predict the number of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes.' In this section, we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general' so that we would do the algebraic steps only once and then use the new formula to find the value of the specific variable. Now, we will go through the steps of completing the square in general to solve a quadratic equation for *x*. It may be helpful to look at one of the examples at the end of the last section where we solved an equation of the form$ax^2+bx+c=0$ as you read through the algebraic steps below, so you see them with numbers as well as 'in general.'

We start with the standard form of a quadratic equation and solve it for x by completing the square.	$ax^2+bx+c=0$
Isolate the variable terms on one side.	$ax^2+bx=-c$
Make leading coefficient 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x=-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x=-\frac{c}{a}$
To complete the square, find$(\frac{1}{2}\cdot\frac{b}{a})^2$ and add it to both sides of the equation. $(\frac{1}{2}\frac{b}{a})^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$(x+\frac{b}{2a})^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and writeequivalent fractions with the common denominator.	$(x+\frac{b}{2a})^2=-\frac{c\cdot 4a}{a\cdot 4a}+\frac{b^2}{4a^2}$
Simplify.	$(x+\frac{b}{2a})^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$(x+\frac{b}{2a})=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify.	$(x+\frac{b}{2a})=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This last equation is the Quadratic Formula.

Definition: QUADRATIC FORMULA

The solutions to a quadratic equation of the form $ax^2+bx+c=0$, $a\geq 0$ are given by the formula:

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

To use the Quadratic Formula, we substitute the values of a, b, and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

How to Solve a Quadratic Equation Using the Quadratic Formula

? Example 11.3.1

Solve $2x^2 + 9x - 5 = 0$ by using the Quadratic Formula.

Answer

Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in $a = 2, b = 9, c = -5$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$
Step 3. Simplify the fraction, and solve for *x*.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4}$ $\quad x = \frac{-9 - 11}{4}$ $x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad x = -5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0$ ✓
	Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0$ ✓

Example 11.3.2

Solve $3y^2 - 5y + 2 = 0$ by using the Quadratic Formula.

Answer

$y = \frac{2}{3}, y = 1$

Example 11.3.3

Solve $4z^2 + 2z - 6 = 0$ by using the Quadratic Formula.

Answer

$z = -\frac{3}{2}, z = 1$

Definition: SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

1. Write the Quadratic Formula in standard form. Identify the a, b, and c values.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time. And remember, the Quadratic Formula is an equation. Be sure you start with '$x =$'.

Example 11.3.4

Solve $x^2 - 6x + 5 = 0$ by using the Quadratic Formula.

Answer

	$x^2 - 6x + 5 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the *a*, *b*, *c* values.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$ $x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, x = \frac{2}{2}$ $x = 5, x = 1$
Check. $x^2 - 6x + 5 = 0$ $\quad$ $x^2 - 6x + 5 = 0$ $5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$ $\quad$ $1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$ $25 - 30 + 5 \stackrel{?}{=} 0$ $\quad$ $1 - 6 + 5 \stackrel{?}{=} 0$ $0 = 0$ ✓ $\quad$ $0 = 0$ ✓	

Example 11.3.5

Solve $a^2 - 2a - 15 = 0$ by using the Quadratic Formula.

Answer

$a = -3, a = 5$

Example 11.3.6

Solve $b^2 + 10b + 24 = 0$ by using the Quadratic Formula.

Answer

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, this is the exact answer. We will get the approximate answer using our calculator to calculate the radical.

Example 11.3.7

Solve $4y^2 - 5y - 3 = 0$ by using the Quadratic Formula.

Answer

We can use the Quadratic Formula to solve for the variable in a quadratic equation, whether or not it is named '*x*'.

	$4y^2 - 5y - 3 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $4y^2 - 5y - 3 = 0$
Identify the *a*, *b*, *c* values.	$a = 4, b = -5, c = -3$
Write the Quadratic Formula.	$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4}$
Simplify.	$y = \frac{5 \pm \sqrt{25 + 48}}{8}$ $y = \frac{5 \pm \sqrt{73}}{8}$
Calculate the radical.	$y = \frac{5 \pm 8.544003745}{8}$
Calculate each answer, rounding to the nearest thousandth.	y = 1.693, -0.443
Check. We leave the check to you.	

Example 11.3.8

Solve $2p^2 + 8p + 5 = 0$ by using the Quadratic Formula.

Answer

$p = -0.775, -3.225$

Example 11.3.9

Solve $5q^2 - 11q + 3 = 0$ by using the Quadratic Formula.

Answer

$q = 1.881, 0.319$

We cannot take the square root of a negative number. So, when we substitute a, b, and c into the Quadratic Formula, if the quantity inside the radical is negative, the quadratic equation has no real solution. We will see this in the next example.

Example 11.3.13

Solve $3p^2 + 2p + 9 = 0$ by using the Quadratic Formula.

Answer

This equation is in standard form.	$ax^2 + bx + c = 0$ $3p^2 + 2p + 9 = 0$
Identify the *a*, *b*, *c* values.	$a = 3,\ b = 2,\ c = 9$
Write the Quadratic Formula.	$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
Simplify the radical.	$p = \frac{-2 \pm \sqrt{-104}}{6}$
We cannot take the square root of a negative number.	There is no real solution.

Example 11.3.14

Solve $4a^2 - 3a + 8 = 0$ by using the Quadratic Formula.

Answer

no real solution

Exeample 11.3.15

Solve $5b^2 + 2b + 4 = 0$ by using the Quadratic Formula.

Answer

no real solution

When we solved linear equations, if an equation had too many fractions we 'cleared the fractions' by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions—to solve. We can use the same strategy with quadratic equations.

Example 11.3.19

Solve $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$ by using the Quadratic Formula.

Answer

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$\underset{ax^2+bx+c=0}{3u^2+4u-2=0}$
Identify the *a*, *b*, *c* values.	$a=3,\ b=4,\ c=-2$
Write the Quadratic Formula.	$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot 3\cdot(-2)}}{2\cdot 3}$
Simplify.	$u=\frac{-4\pm\sqrt{16+24}}{6}$ $u=\frac{-4\pm\sqrt{40}}{6}$
Calculate the radical.	$u=\frac{-4\pm 6.32455532}{6}$
Calculate to show two solutions to the thousandth.	u = 0.387, -1.721
Check. We leave the check to you.	

? Example 11.3.20

Solve $\frac{1}{4}c^2-\frac{1}{3}c=\frac{1}{12}$ by using the Quadratic Formula.

Answer

$c=1.549,-0.215$

? Example 11.3.21

Solve $\frac{1}{9}d^2-\frac{1}{2}d=-\frac{1}{2}$ by using the Quadratic Formula.

Answer

$d=\frac{3}{2}$, $d=3$

Think about the equation $(x-3)^2=0$. We know from the Zero Products Principle that this equation has only one solution: $x=3$.

We will see in the next example how using the Quadratic Formula to solve an equation with a perfect square also gives just one solution.

? Example 11.3.22

Solve $4x^2-20x=-25$ by using the Quadratic Formula.

Answer

	$4x^2-20x=-25$
Add 25 to get the equation in standard form.	$\underset{ax^2+bx+c=0}{4x^2-20x+25=0}$
Identify the *a*, *b*, *c* values.	$a=4,\ b=-20,\ c=25$

Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$ $x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check. We leave the check to you.	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square?

? Example 11.3.23

Solve $r^2 + 10r + 25 = 0$ by using the Quadratic Formula.

Answer

$r = -5$

? Example 11.3.24

Solve $25t^2 - 40t = -16$ by using the Quadratic Formula.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two solutions, sometimes one solution, sometimes no real solutions. Is there a way to predict the number of solutions to a quadratic equation without actually solving the equation?

Yes, the quantity inside the radical of the Quadratic Formula makes it easy for us to determine the number of solutions. This quantity is called the **discriminant**.

Definition: DISCRIMINANT

In the Quadratic Formula$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the **discriminant**.

Let's look at the discriminant of the equations in Example, Example, and Example, and the number of solutions to those quadratic equations.

	Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Sign of the Discriminant	Number of real solution
Example	$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5) = 121$	+	2
Example	$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25 = 0$	0	1
Example	$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9 = -104$	−	0

When the discriminant is **positive** $x = \frac{-b \pm \sqrt{+}}{2a}$ the quadratic equation has **two solutions**.

When the discriminant is **zero** $x = \frac{-b \pm \sqrt{0}}{2a}$ the quadratic equation has **one solution**.

When the discriminant is **negative** $x = \frac{-b \pm \sqrt{-}}{2a}$ the quadratic equation has **no real solutions**.

Definition:USE THE DISCRIMINANT, $b^2 - 4ac$, TO DETERMINE THE NUMBER OF SolutionS OF A QUADRATIC EQUATION

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \geq 0$,

- if $b^2 - 4ac > 0$, the equation has two solutions.
- if $b^2 - 4ac = 0$, the equation has one solution.
- if $b^2 - 4ac < 0$, the equation has no real solutions.

Example 11.3.25

Determine the number of solutions to each quadratic equation:

1. $2v^2 - 3v + 6 = 0$
2. $3x^2 + 7x - 9 = 0$
3. $5n^2 + n + 4 = 0$
4. $9y^2 - 6y + 1 = 0$

Answer

1.

	$2v^2 - 3v + 6 = 0$
The equation is in standard form, identify a, b, c.	$a = 2, b = -3, c = 6$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(3)^2 - 4 \cdot 2 \cdot 6$
Simplify.	$9 - 48$ -39
Because the discriminant is negative, there are no real solutions to the equation.	

2.

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify a, b, c.	$a = 3, b = 7, c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$ 157
Because the discriminant is positive, there are two solutions to the equation.	

3.

	$5n^2 + n + 4 = 0$
The equation is in standard form, identify a, b, c.	$a = 5, b = 1, c = 4$
Write the discriminant.	$b^2 - 4ac$

Substitute in the values of a, b, c.	$(1)^2-4\cdot 5\cdot 4$
Simplify.	$1-80$ -79
Because the discriminant is negative, there are no real solutions to the equation.	

4.

	$9y^2-6y+1=0$
The equation is in standard form, identify a, b, c.	$a=9, b=-6, c=1$
Write the discriminant.	b^2-4ac
Substitute in the values of a, b, c.	$(-6)^2-4\cdot 9\cdot 1$
Simplify.	$36-36$ 0
Because the discriminant is 0, there is one solution to the equation.	

? Example 11.3.26

Determine the number of solutions to each quadratic equation:

1. $8m^2-3m+6=0$
2. $5z^2+6z-2=0$
3. $9w^2+24w+16=0$
4. $9u^2-2u+4=0$

Answer

1. no real solutions
2. 2
3. 1
4. no real solutions

? Example 11.3.27

Determine the number of solutions to each quadratic equation:

1. $b^2+7b-13=0$
2. $5a^2-6a+10=0$
3. $4r^2-20r+25=0$
4. $7t^2-11t+3=0$

Answer

1. 2
2. no real solutions
3. 1
4. 2

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We have used four methods to solve quadratic equations:

- Factoring

- Square Root Property
- Completing the Square
- Quadratic Formula

You can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method to use.

Definition: IDENTIFY THE MOST APPROPRIATE METHOD TO SOLVE A QUADRATIC EQUATION.

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any quadratic equation can be solved by using the Quadratic Formula.

What about the method of completing the square? Most people find that method cumbersome and prefer not to use it. We needed to include it in this chapter because we completed the square in general to derive the Quadratic Formula. You will also use the process of completing the square in other areas of algebra.

Example 11.3.28

Identify the most appropriate method to use to solve each quadratic equation:

1. $5z^2 = 17$
2. $4x^2 - 12x + 9 = 0$
3. $8u^2 + 6u = 11$

Answer

1. $5z^2 = 17$

   Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

2. $4x^2 - 12x + 9 = 0$

   We recognize that the left side of the equation is a perfect square trinomial, and so Factoring will be the most appropriate method.

3. $8u^2 + 6u = 11$

   Put the equation in standard form. $8u^2 + 6u - 11 = 0$

   While our first thought may be to try Factoring, thinking about all the possibilities for trial and error leads us to choose the Quadratic Formula as the most appropriate method.

Example 11.3.29

Identify the most appropriate method to use to solve each quadratic equation:

1. $x^2 + 6x + 8 = 0$
2. $(n-3)^2 = 16$
3. $5p^2 - 6p = 9$

Answer

1. factor
2. Square Root Property
3. Quadratic Formula

? Example 11.3.30

Identify the most appropriate method to use to solve each quadratic equation:

1. $8a^2+3a-9=0$
2. $4b^2+4b+1=0$
3. $5c2=125$

Answer

1. Quadratic Formula
2. factoring
3. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula:

- Solving Quadratic Equations: Solving with the Quadratic Formula
- How to solve a quadratic equation in standard form using the Quadratic Formula (example)
- Solving Quadratic Equations using the Quadratic Formula—Example 3
- Solve Quadratic Equations using Quadratic Formula

Key Concepts

- **Quadratic Formula** The solutions to a quadratic equation of the form $ax^2+bx+c=0$, $a\geq 0$ are given by the formula:

  $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- **Solve a Quadratic Equation Using the Quadratic Formula**
  To solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic formula in standard form. Identify the a, b, c values.
  2. Write the quadratic formula. Then substitute in the values of a, b, c.
  3. Simplify.
  4. Check the solutions.
- **Using the Discriminant, b^2-4ac, to Determine the Number of Solutions of a Quadratic Equation**
  For a quadratic equation of the form $ax^2+bx+c=0$, $a\geq 0$,
  - if $b^2-4ac>0$, the equation has 2 solutions.
  - if $b^2-4ac=0$, the equation has 1 solution.
  - if $b^2-4ac<0$, the equation has no real solutions.
- **To identify the most appropriate method to solve a quadratic equation:**
  1. Try Factoring first. If the quadratic factors easily this method is very quick.
  2. Try the Square Root Property next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
  3. Use the Quadratic Formula. Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

discriminant

In the Quadratic Formula,$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ the quantity b^2-4ac is called the discriminant.

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11.4: Solve Applications Modeled by Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by Quadratic Equations

Be Prepared

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is −100. Find the numbers.
   If you missed this problem, review [link].
2. The area of triangular mural is 64 square feet. The base is 16 feet. Find the height.
   If you missed this problem, review [link].
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review [link].

Solve Applications of the Quadratic Formula

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications. To get us started, we will copy our usual Problem Solving Strategy here so we can follow the steps.

Definition: USE THE PROBLEM SOLVING STRATEGY

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We have solved number applications that involved **consecutive even integers** and **consecutive odd integers** by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

Consecutive even integers		**Consecutive odd integers**	
	$64, 66, 68$		$77, 79, 81$
n	1st even number	n	1st odd number
$n+2$	2nd even number	$n+2$	2nd odd number
$n+4$	3rd even number	$n+4$	3rd odd number

(11.4.1)

Some applications of **consecutive odd integers** or **consecutive even integers** are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example 11.4.1

The product of two consecutive odd integers is 195. Find the integers.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive odd integers.
Step 3. Name what we are looking for.	Let $n =$ the first odd integer. $n + 2 =$ the next odd integer
Step 4. Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation.	$n(n + 2) = 195$
Step 5. Solve the equation. Distribute.	$n^2 + 2n = 195$
Subtract 195 to get the equation in standard form.	$ax^2 + bx + c = 0$ $n^2 + 2n - 195 = 0$
Identify the a, b, c values.	$a = 1,\ b = 2,\ c = -195$
Write the quadratic equation.	$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c..	$n = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-195)}}{2 \cdot 1}$
Simplify.	$n = \frac{-2 \pm \sqrt{4 + 780}}{2}$ $n = \frac{-2 \pm \sqrt{784}}{2}$
Simplify the radical.	$n = \frac{-2 \pm 28}{2}$
Rewrite to show two solutions.	$n = \frac{-2 + 28}{2}, n = \frac{-2 - 28}{2}$
Solve each equation.	$n = \frac{26}{2}, n = \frac{-30}{2}$ $n = 13\ \ n = -15$
There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.	First odd integer n=13 next odd integer n+2 13+2 15
	First odd integer n=−15 next odd integer n+2 −15+2 −13
Step 6. Check the answer. Do these pairs work? Are they consecutive odd integers? Is their product 195?	13, 15, yes −13, −15, yes 13·15=195, yes −13(−15)=195, yes
Step 7. Answer the question.	The two consecutive odd integers whose product is 195 are 13, 15, and −13, −15.

Example 11.4.2

The product of two consecutive odd integers is 99. Find the integers.

Answer

Two consecutive odd numbers whose product is 99 are 9 and 11, and −9 and −11.

Example 11.4.3

The product of two consecutive odd integers is 168. Find the integers.

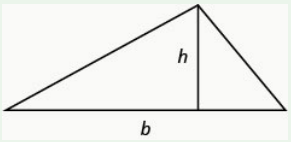
Answer

Two consecutive even numbers whose product is 168 are 12 and 14,and −12 and −14.

We will use the formula for the area of a triangle to solve the next example.

Definition: AREA OF A TRIANGLE

For a triangle with base b and height h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

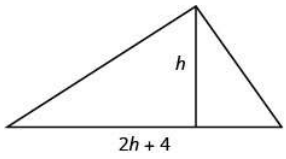


Recall that, when we solve geometry applications, it is helpful to draw the figure.

Example 11.4.4

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can have an area of 120 square feet and the architect wants the width to be 4 feet more than twice the height. Find the height and width of the window.

Answer

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height and width.
Step 3. Name what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the width of the triangle
Step 4. Translate.	We know the area. Write the formula for the area of a triangle. $A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$ax^2 + bx + c = 0$ $h^2 + 2h - 120 = 0$

Solve the equation using the Quadratic Formula. Identify the a, b, c values.	$a=1,\ b=2,\ c=-120$
Write the quadratic equation.	$h=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$h=\frac{-2\pm\sqrt{2^2-4\cdot 1\cdot(-120)}}{2\cdot 1}$
Simplify.	$h=\frac{-2\pm\sqrt{4+480}}{2}$ $h=\frac{-2\pm\sqrt{484}}{2}$
Simplify the radical.	$h=\frac{-2\pm 22}{2}$
Rewrite to show two solutions.	$h=\frac{-2+22}{2},\ h=\frac{-2-22}{2}$
Simplify.	$h=\frac{20}{2},\ h=\frac{-24}{2}$
Since h is the height of a window, a value of $h=-12$ does not make sense.	$h=10$ ~~$h=-12$~~
	The height of the triangle: $h=10$ The width of the triangle: $2h+4$ $2\cdot 10+4$ 24
Step 6. Check the answer. Does a triangle with a height 10 and width 24 have area 120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the width is 24 feet.

Notice that the solutions were integers. That tells us that we could have solved the equation by factoring.

When we wrote the equation in standard form, $h^2+2h-120=0$, we could have factored it. If we did, we would have solved the equation $(h+12)(h-10)=0$.

Example 11.4.5

Find the dimensions of a triangle whose width is four more than six times its height and has an area of 208 square inches.

Answer

The height of the triangle is 8 inches and the width is 52 inches.

Example 11.4.6

If a triangle that has an area of 110 square feet has a height that is two feet less than twice the width, what are its dimensions?

Answer

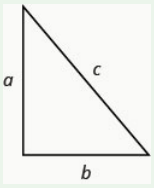
The height of the triangle is 20 feet and the width is 11 feet.

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Definition: PYTHAGOREAN THEOREM

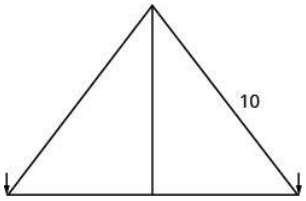
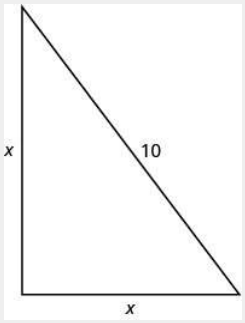
In any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse, $a^2 + b^2 = c^2$



Example 11.4.7

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Answer

Step 1. Read the problem. Draw a picture	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from the pole to stake
Each side is a right triangle. We draw a picture of one of them.	
Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*.	
Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5. Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$

Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}$ $x = -5\sqrt{2}$
Approximate this number to the nearest tenth with a calculator.	$x \approx 7.1$
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

Example 11.4.8

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth of a foot.

Answer

The length of the shadow is 6.3 feet and the length of the flag pole is 18.9 ft.

Example 11.4.9

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

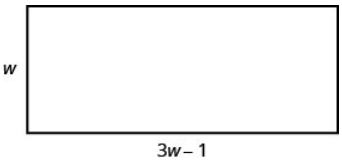
Answer

The distance to the opposite corner is 3.2.

Example 11.4.10

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than three times the width. Find the length and width. Round to the nearest tenth of a foot.

Answer

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5. Solve the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$

This is a quadratic equation, rewrite it in standard form.	$ax^2+bx+c=0$ $3w^2-w-150=0$
Solve the equation using the Quadratic Formula.	
Identify the a, b, c values.	$a=3,\ b=-1,\ c=-150$
Write the Quadratic Formula.	$w=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$w=\frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 3\cdot(-150)}}{2\cdot 3}$
Simplify.	$w=\frac{1\pm\sqrt{1+1800}}{6}$ $w=\frac{1\pm\sqrt{1801}}{6}$
Rewrite to show two solutions.	$w=\frac{1+\sqrt{1801}}{6},\ w=\frac{1-\sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w\approx 7.2,\quad w\approx -6.9$ Width $w\approx 7.2$ Length $\approx 3w-1$ $\approx 3(7.2)-1$ ≈ 20.6
Step 6. Check the answer. Make sure that the answers make sense.	
Step 7. Answer the question.	The width of the rectangle is approximately 7.2 feet and the length 20.6 feet.

Example 11.4.11

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden. Round to the nearest tenth of a foot..

Answer

The width of the garden is 11 feet and the length is 18 feet.

Example 11.4.12

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length. What are the length and width of the tablecloth? Round to the nearest tenth of a foot.

Answer

The width of the tablecloth is 6.8 feet and the length is 11.8 feet.

The height of a projectile shot upwards is modeled by a quadratic equation. The initial velocity, \(v_{0}\), propels the object up until gravity causes the object to fall back down.

Definition: PROJECTILE MOTION

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula:

$h = -16t^2 + v_0 t$

We can use the formula for projectile motion to find how many seconds it will take for a firework to reach a specific height.

Example 11.4.13

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation.	Use the formula.
	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second.	
The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form.	$\underset{ax^2}{16t^2} + \underset{bx}{130t} + \underset{c}{260} = \underset{0}{0}$
Solve the equation using the Quadratic Formula.	
Identify the a, b, c values.	$a = 16,\ b = -130,\ c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32},\ t = \frac{130 - \sqrt{260}}{32}$
Approximate the answers with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$
Step 6. Check the answer. The check is left to you.	

Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

? Example 11.4.14

An arrow is shot from the ground into the air at an initial speed of 108 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth of a second.

Answer

The arrow will reach 180 on its way up in 3 seconds, and on the way down in 3.8 seconds.

? Example 11.4.15

A man throws a ball into the air with a velocity of 96 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth of a second.

Answer

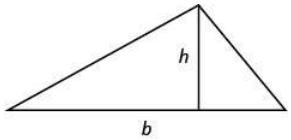
The ball will reach 48 feet on its way up in .6 seconds and on the way down in 5.5 seconds.

Access these online resources for additional instruction and practice with solving word problems using the quadratic equation:

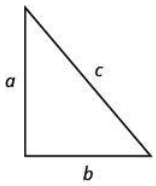
- General Quadratic Word Problems
- Word problem: Solve a projectile problem using a quadratic equation

Key Concepts

- **Area of a Triangle** For a triangle with base, b, and height, h, the area, A, is given by the formula: $A = \frac{1}{2}bh$



- **Pythagorean Theorem** In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypothenuse, $a^2 + b^2 = c^2$



- **Projectile motion** The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after tt seconds can be modeled by the formula

  $h = -16t^2 + v_0 t$

Glossary

consecutive even integers

Consecutive even integers are even integers that follow right after one another. If an even integer is represented by n, the next consecutive even integer is $n+2$, and the next after that is $n+4$.

consecutive odd integers

Consecutive odd integers are odd integers that follow right after one another. If an odd integer is represented by n, the next consecutive odd integer is $n+2$, and the next after that is $n+4$.

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CHAPTER OVERVIEW

12: Data and Statistics

12.1: Basics of Statistics
12.2: Describing Data
12.3: Pie Charts
12.4: Numerical Measures of Center and Variation

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12.1: Basics of Statistics

Like most people, you probably feel that it is important to "take control of your life." But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, statistics is one of the most important things that you can study.

Here are some claims that you may have heard on several occasions. (We are not saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 70% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

Before going further, however, two distinct meanings of the word “statistics” should be clarified. First, “statistics” could mean the academic discipline, a field of study, involving the collection, presentation, interpretation, and use of data. When used in this sense, the word is considered singular (though it looks plural). Second, in a narrower sense, a number collected in a survey or study is referred to as a “statistic.” For instance, the average height of a sample of 100 people would be a (one) statistic. This is a singular word (of course, two or more of these numbers would be referred to as “statistics”). Be aware of the difference when you encounter this word.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) This chapter will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, "There are three kinds of lies -- lies, damned lies, and statistics." This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950’s and 60’s to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

Populations and samples

Before we begin gathering and analyzing data, we need to characterize the population we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of California.
- All first-year students at public colleges and universities in the state of California.
- All first-year students at all colleges and universities in the state of California.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

Definition: Population

The population of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the target population since the collected data might not actually be representative of the intended population if we design our study badly.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at a large university might take slightly more diverse courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from a large university. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 12.1.1

A newspaper website contains a poll asking people their opinion on a recent news article. What is the population?

Solution

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define "average" more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Definition: Parameter

A parameter is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive unless the population is very small or we already have the data collected.

Definition: Census

A survey of an entire population is called a census.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a sample to study;

Definition: Sample

A sample is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a statistic.

Definition: Statistic

A statistic is a value (average, percentage, etc.) calculated using the data from a sample.

Example 12.1.2

A researcher wanted to know how citizens of Los Angeles felt about a voter initiative. To study this, she goes to a large shopping mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

Solution

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was LA citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all LA citizens.

The 60% value was based on the sample, so it is a statistic. (Note that this is singular).

Try it Now 1

The U.S. federal debt at the end of 2001 was $5.77 trillion and grew to $6.20 trillion by the end of 2002. At the end of 2005 it was $7.91 trillion and grew to $8.45 trillion by the end of 2006.[2] Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

Try it Now 2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

Categorizing data

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as categorical data or quantitative data.

Definition: Quantitative and Categorical Data

Categorical (qualitative) data are pieces of information that allow us to classify the objects under investigation into various categories.

Quantitative data are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

Example 12.1.3

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

Solution

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Avengers*, *or Dunkirk*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Finding Nemo*." This would be categorical data.

Example 12.1.4

A survey could ask the number of movies they have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, ...)

Solution

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, ...) or the amount of money you paid for a movie ticket the last time you went to a movie theater ($5.50, $7.75, $9, ...).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

Example 12.1.5

Suppose we gather respondents' ZIP codes in a survey to track their geographical location.

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example 12.1.6

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

1. it was awful
2. it was just OK
3. I liked it
4. it was great
5. best movie ever!

Solution

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

However, it DOES make sense to discuss the average rating and other statistics in this case. For instance, a movie whose average rating is a 1.8 is clearly different from one whose average is a 4.2. Also, it would mean something if the maximum rating of another movie was a 2. So, in a sense, this data can be considered quantitative.

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *The Avengers*. And 33-37 year-olds might prefer...well, *Finding Nemo*.

Try it Now 3

Classify each measurement as categorical or quantitative

a. Eye color of a group of people
b. Daily high temperature of a city over several weeks
c. Annual income

Sampling methods

As we mentioned in a previous section, the first thing we should do before conducting a survey is to identify the population that we want to study. Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote,

and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?

In November 1998, former professional wrestler Jesse "The Body" Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket, and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

Definition: Sampling Bias (selection bias)

A sampling bias is the bias caused when the sample chosen does not fairly represent the population being studied. The sampling method, therefore, must be carefully considered to avoid this type of bias.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

Returning to our hypothetical job as a political pollster, we would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Elks club would provide a useful picture of district-wide support for our candidate.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is simple random sampling.

Definition: Simple Random Sample

A random sample is one in which each member of the population has an equal probability of being chosen. A simple random sample is one in which every member of the population and any group of members has an equal probability of being chosen.

Example 12.1.7

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample.

Solution

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the state, some of these samples

might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

Definition: Sampling Variability

The natural variation of samples is called sampling variability.

This is unavoidable and expected in random sampling, and in most cases it is not an issue

To help account for variability, pollsters might instead use a **stratified sample**.

Definition: Stratified Sampling

In stratified sampling, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example 12.1.8

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans, and 24% independents. In a sample of 1000 people, they would then expect to get about 390 Democrats, 370 Republicans, and 240 independents. To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called quota sampling.

Definition: Quota Sampling

Quota sampling is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

Example 12.1.9

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat.

You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any "real" questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling but also does not ensure the same level of randomness.

Another sampling method is cluster sampling, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

Definition: Cluster Sampling

In cluster sampling, the population is divided into subgroups (clusters), and a set of subgroups is selected to be in the sample.

Example 12.1.10

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Other sampling methods include **systematic sampling**.

Definition: Systemic Sampling

In systematic sampling, every n^{th} member of the population is selected to be in the sample.

Example 12.1.11

To select a sample using systematic sampling, a pollster calls every 100th name in the phone book.

Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample), but it can yield acceptable samples.

Perhaps the worst types of sampling methods are convenience samples and voluntary response samples.

Definition: Convenience Sampling and Voluntary Response Sampling

Convenience sampling is the method of choosing a sample based on what is convenient or easy.

Voluntary response sampling is allowing the sample to volunteer.

Example 12.1.12

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is a convenience sample.

Example 12.1.13

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

Try it Now 4

In each case, indicate what sampling method was used:

a. Every 4th person in the class was selected
b. A sample was selected to contain 25 men and 35 women
c. Viewers of a new show are asked to vote on the show's website
d. A website randomly selects 50 of their customers to send a satisfaction survey to
e. To survey voters in a town, a polling company randomly selects 10 city blocks and interviews everyone who lives on those blocks.

How to mess things up before you start

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – sampling or selection bias, which is when the sample is not representative of the population. One example of this is voluntary response bias, which is bias introduced by collecting data from only those who volunteer to participate. This is not the only potential source of bias.

Note

Sampling bias – when the sample is not representative of the population

Voluntary response bias – the sampling bias that often occurs when the sample is volunteers. (This is also called "participation bias.")

Self-interest study – bias that can occur when the researchers have an interest in the outcome

Response bias – when the responder gives inaccurate responses for any reason

Perceived lack of anonymity – when the responder fears giving an honest answer might negatively affect them

Loaded questions – when the question wording influences the responses

Non-response bias – when people refusing to participate in the study can influence the validity of the outcome

Example 12.1.14

Consider a recent study which found that chewing gum may raise math grades in teenagers. This study was conducted by the Wrigley Science Institute, a branch of Wrigley, a chewing gum company. This is an example of a **self-interest study**; one in which the researches have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny.

Example 12.1.15

A survey asks people "when was the last time you visited your doctor?" This might suffer from response bias, since many people might not remember exactly when they last saw a doctor and give inaccurate responses.

Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

Example 12.1.16

A survey asks participants a question about their interactions with members of other races. Here, a perceived lack of anonymity could influence the outcome. The respondent might not want to be perceived as racist even if they are and give an untruthful answer.

Example 12.1.17

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Here, answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer.

Example 12.1.18

A survey asks "Do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?" This is an example of a loaded or leading question – questions whose wording leads the respondent towards an answer.

Loaded questions can occur intentionally by pollsters with an agenda or accidentally through poor question wording. Also a concern is question order, where the order of questions changes the results. A psychology researcher provides an example:

"My favorite finding is this: we did a study where we asked students, 'How satisfied are you with your life? How often do you have a date?' The two answers were not statistically related - you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, 'How often do you have a date? How satisfied are you with your life?' the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student's life as dating frequency."

Example 12.1.19

A telephone poll asks the question "Do you often have time to relax and read a book?", and 50% of the people called refused to answer the survey. It is unlikely that the results will be representative of the entire population. This is an example of non-response bias, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

Try it Now 5

In each situation, identify a potential source of bias:

1. A survey asks how many sexual partners a person has had in the last year.
2. A radio station asks readers to phone in their choice in a daily poll.
3. A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.
4. High school students are asked if they have consumed alcohol in the last two weeks.
5. The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.
6. A poll asks "Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?"

Experiments

So far, we have primarily discussed observational studies – studies in which conclusions would be drawn from observations of a sample or the population. In some cases these observations might be unsolicited, such as studying the percentage of cars that turn right at a red light even when there is a "no turn on red" sign. In other cases the observations are solicited, like in a survey or a poll.

In contrast, it is common to use experiments when exploring how subjects react to an outside influence. In an experiment, some kind of treatment is applied to the subjects, and the results are measured and recorded.

Definition: Observational Studies and Experiments

An observational study is a study based on observations or measurements.

An experiment is a study in which the effects of a treatment are measured.

Here are some examples of experiments:

Example 12.1.20

a. A pharmaceutical company tests a new medicine for treating Alzheimer's disease by administering the drug to 50 elderly patients with recent diagnoses. The treatment here is the new drug.
b. A gym tests out a new weight loss program by enlisting 30 volunteers to try out the program. The treatment here is the new program.
c. You test a new kitchen cleaner by buying a bottle and cleaning your kitchen. The new cleaner is the treatment.
d. A psychology researcher explores the effect of music on temperament by measuring people's temperament while listening to different types of music. The music is the treatment.

Try it Now 6

Is each scenario describing an observational study or an experiment?

a. The weights of 30 randomly selected people are measured
b. Subjects are asked to do 20 jumping jacks, and then their heart rates are measured
c. Twenty coffee drinkers and twenty tea drinkers are given a concentration test

When conducting experiments, it is essential to isolate the treatment being tested.

Example 12.1.21

Suppose a middle school (junior high) finds that their students are not scoring well on the state's standardized math test. They decide to run an experiment to see if an alternate curriculum would improve scores. To run the test, they hire a math specialist to come in and teach a class using the new curriculum. To their delight, they see an improvement in test scores.

The difficulty with this scenario is that it is not clear whether the curriculum is responsible for the improvement or the improvement is due to a math specialist teaching the class. This is called **confounding** – when it is not clear which factor or factors caused the observed effect. Confounding is the downfall of many experiments though sometimes it is hidden.

Definition: Confounding

Confounding occurs when there are two potential variables that could have caused the outcome and it is not possible to determine which actually caused the result.

Example 12.1.22

A drug company study about a weight loss pill might report that people lost an average of 8 pounds while using their new drug. However, in the fine print you find a statement saying that participants were encouraged to also diet and exercise. It is not clear in this case whether the weight loss is due to the pill, to diet and exercise, or a combination of both. In this case confounding has occurred.

Example 12.1.23

Researchers conduct an experiment to determine whether students will perform better on an arithmetic test if they listen to music during the test. They first give the student a test without music and then give a similar test while the student listens to music. In this case, the student might perform better on the second test, regardless of the music, simply because it was the second test and they were warmed up.

There are a number of measures that can be introduced to help reduce the likelihood of confounding. The primary measure is to use a control group.

Definition: Control Group

When using a control group, the participants are divided into two or more groups, typically a control group and a treatment group. The treatment group receives the treatment being tested; the control group does not receive the treatment.

Ideally, the groups are otherwise as similar as possible, isolating the treatment as the only potential source of difference between the groups. For this reason, the method of dividing groups is important. Some researchers attempt to ensure that the groups have similar characteristics (same number of females, same number of people over 50, etc.), but it is nearly impossible to control for every characteristic. Because of this, random assignment is very commonly used.

Example 12.1.24

To determine if a two-day prep course would help high school students improve their scores on the SAT test, a group of students was randomly divided into two subgroups. The first group, the treatment group, was given a two-day prep course. The second group, the control group, was not given the prep course. Afterwards, both groups were given the SAT.

Example 12.1.25

A company testing a new plant food grows two crops of plants in adjacent fields, the treatment group receiving the new plant food and the control group not. The crop yield would then be compared. By growing them at the same time in adjacent fields, they are controlling for weather and other confounding factors.

Sometimes not giving the control group anything does not completely control for confounding variables. For example, suppose a medicine study is testing a new headache pill by giving the treatment group the pill and the control group nothing. If the treatment group showed improvement, we would not know whether it was due to the medicine in the pill or a response to have taken any pill. This is called a **placebo effect.**

Definition: Placebo Effect

The placebo effect refers to the effectiveness of a treatment influenced by the patient's perception of how effective they think the treatment will be, so a result might be seen even if the treatment is ineffectual.

Example 12.1.26

A study found that when doing painful dental tooth extractions, patients who were told they were receiving a strong painkiller while actually receiving a saltwater injection found as much pain relief as patients receiving a dose of morphine.

To control for the placebo effect, a placebo, or dummy treatment, is often given to the control group. This way, both groups are truly identical except for the specific treatment given.

Definition: Placebo and Placebo controlled Experiments

A **placebo** is a dummy treatment given to control for the placebo effect.

An experiment that gives the control group a placebo is called a **placebo controlled experiment**.

Example 12.1.27

a. In a study for a new medicine that is dispensed in a pill form, a sugar pill could be used as a placebo.
b. In a study on the effect of alcohol on memory, a non-alcoholic beer might be given to the control group as a placebo.
c. In a study of a frozen meal diet plan, the treatment group would receive the diet food, and the control could be given standard frozen meals stripped of their original packaging.

In some cases, it is more appropriate to compare to a conventional treatment than a placebo. For example, in a cancer research study, it would not be ethical to deny any treatment to the control group or to give a placebo treatment. In this case, the currently acceptable medicine would be given to the control group, sometimes called a comparison group in this case. In our SAT test example, the non-treatment group would most likely be encouraged to study on their own, rather than be asked to not study at all, to provide a meaningful comparison.

When using a placebo, it would defeat the purpose if the participant knew they were receiving the placebo.

Definition: Blind studies

A **blind study** is one in which the participant does not know whether or not they are receiving the treatment or a placebo.

A **double-blind study** is one in which those interacting with the participants don't know who is in the treatment group and who is in the group receiving a placebo.

Example 12.1.28

In a study about anti-depression medicine, you would not want the psychological evaluator to know whether the patient is in the treatment or control group either, as it might influence their evaluation, so the experiment should be conducted as a double-blind study.

It should be noted that not every experiment needs a control group.

Example 12.1.29

If a researcher is testing whether a new fabric can withstand fire, she simply needs to torch multiple samples of the fabric – there is no need for a control group.

Try it Now 7

To test a new lie detector, two groups of subjects are given the new test. One group is asked to answer all the questions truthfully, and the second group is asked to lie on one set of questions. The person administering the lie detector test does not know what group each subject is in.

Does this experiment have a control group? Is it blind, double-blind, or neither?

Reference

1. References (10)
2. www.whitehouse.gov/sites/defa...s/hist07z1.xls

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12.2: Describing Data

Once we have collected data from surveys or experiments, we need to summarize and present the data in a way that will be meaningful to the reader. We will begin with graphical presentations of data and then explore numerical summaries of data.

Presenting Categorical Data Graphically

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table**.

Definition: Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 12.2.1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Table 12.2.1

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. Here, we will work with bar graphs that display categorical data; later, we will be devoted to bar graphs that display quantitative data.

Definition: Bar Graph

A bar graph is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

Example 12.2.2

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:

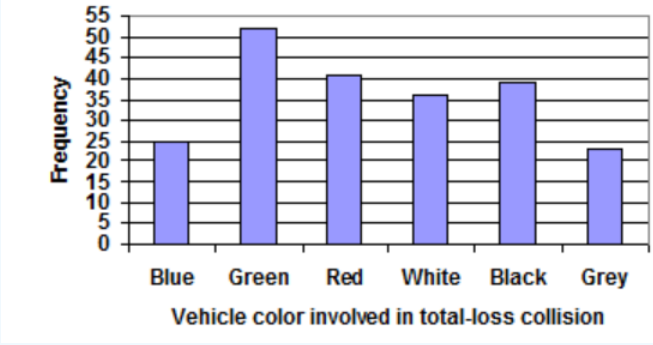

Figure 12.2.1: Car Data

Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:

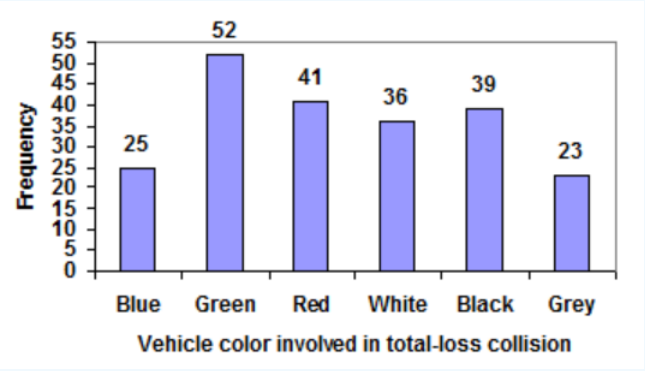

Figure 12.2.2: Car Data (Frequency listed at the top of each bar)

In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart.**

Definition: Pareto Chart

A pareto chart is a bar graph ordered from highest to lowest frequency.

Example 12.2.3

Transforming our bar graph from earlier into a Pareto chart, we get:

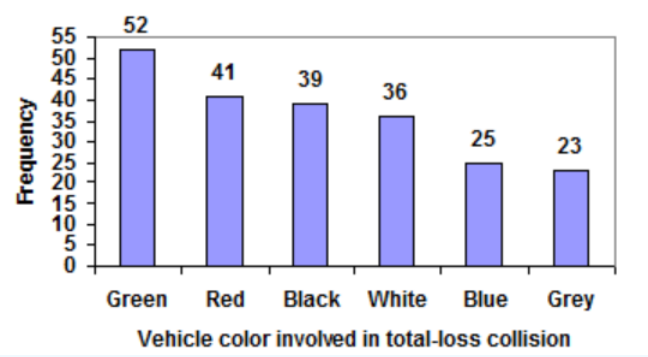

Figure 12.2.3: Car Data

Example 12.2.4

In a survey, adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried “a great deal” about some selected concerns are summarized below.

Table 12.2.2

Environmental Issue	Frequency
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

This data could be shown graphically in a bar graph:

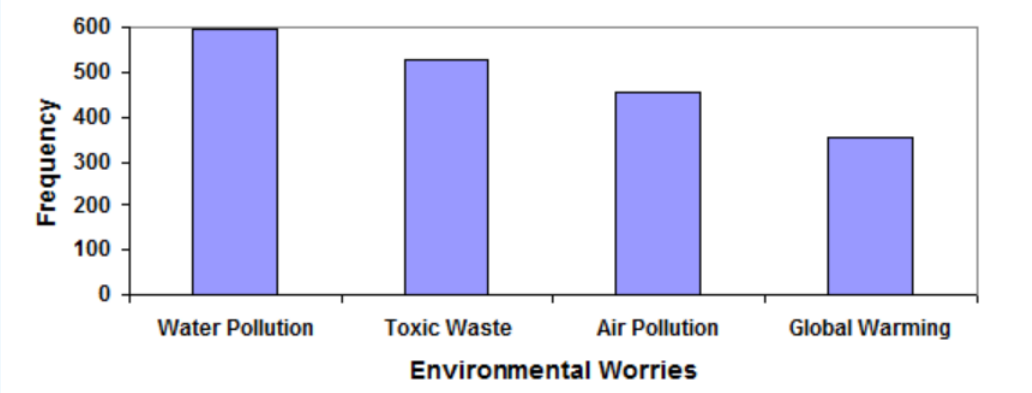

Figure 12.2.4: Environmental worries data

To show relative sizes, it is common to use a pie chart.

Definition: Pie Chart

A pie chart is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example 12.2.5

For our vehicle color data, a pie chart might look like this:

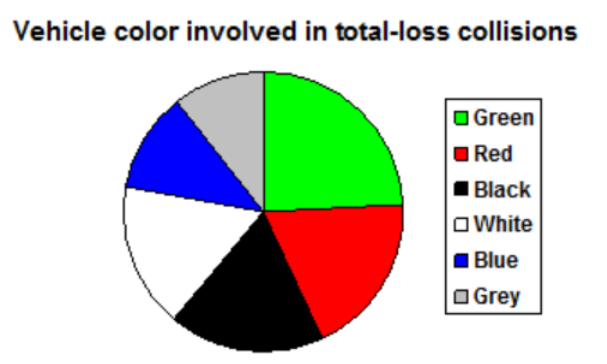

Figure 12.2.5: Pie chart of Vehicle color involved in total-loss collisions

Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.

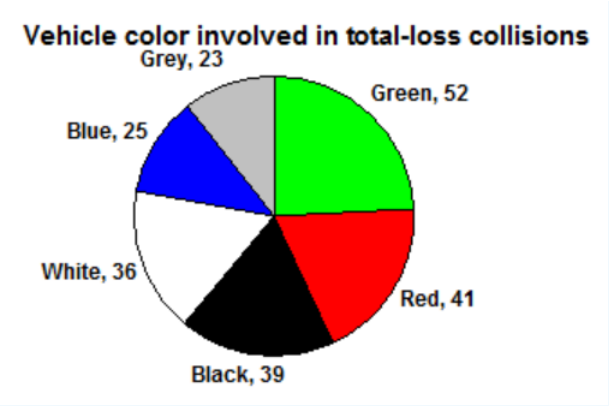

Figure 12.2.6: Frequencies (percents) in the chart next to the pie slices

Example 12.2.6

The pie chart below shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.

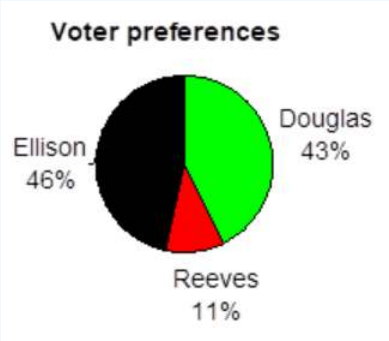

Figure 12.2.7: Voter preferences

Pie charts look nice but are harder to draw by hand than bar charts since we would need to compute the angle each wedge cuts out of the circle and then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs.

Try it Now 1

Create a bar graph and a pie chart to illustrate the grades on a history exam below. A: 12 student, B: 19 students, C: 14 students, D: 4 students, F: 5 students.

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.

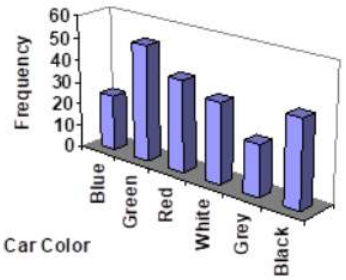

Figure 12.2.8: 3-dimensional bar chart

Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram.**

Definition: Pictogram

A pictogram is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example 12.2.7

A labor union might produce the graph below to show the difference between the average manager salary and the average worker salary.

Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.

Figure 12.2.9: Salary representations

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

Example 12.2.8

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008. The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like three times as many people oppose marriage rights as support it.

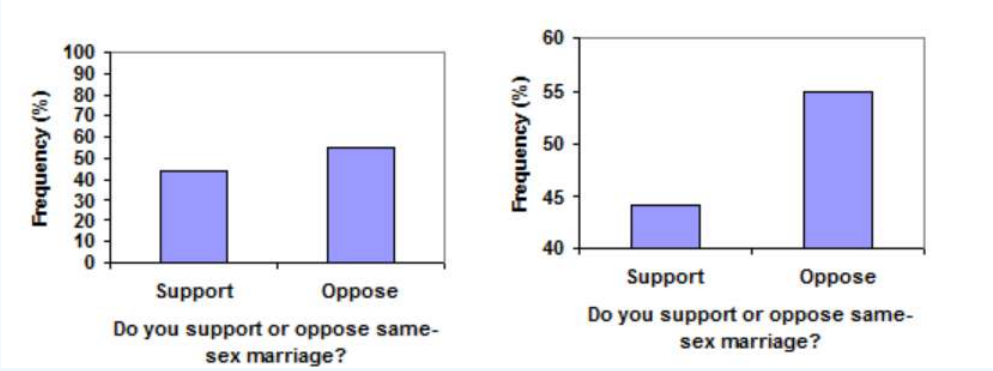

Figure 12.2.10: Same-sex marriage graphs

Try it Now 2

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.

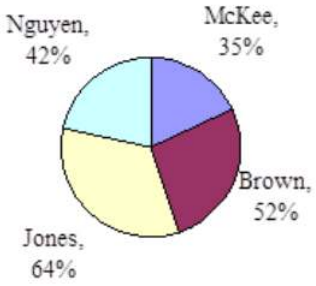

Figure 12.2.11: Pie chart

Presenting Quantitative Data Graphically

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 12.2.9

A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19, 20, 18, 18, 17, 18, 19, 17, 20, 18, 20, 16, 20, 15, 17, 12, 18, 19, 18, 19, 17, 20, 18, 16, 15, 18, 20, 5, 0, 0

Table 12.2.3

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary as we did for categorical data:

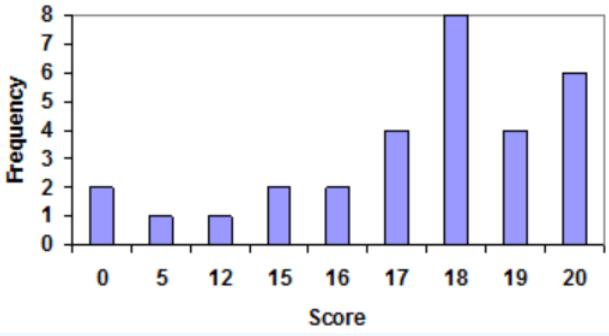

Figure 12.2.12: Standard bar chart from the above summary

However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be better to treat the horizontal axis as a number line. This type of graph is called a **histogram.**

Definition: Histogram

A histogram is like a bar graph, but the horizontal axis is a number line.

Example 12.2.10

For the values above, a histogram would look like:

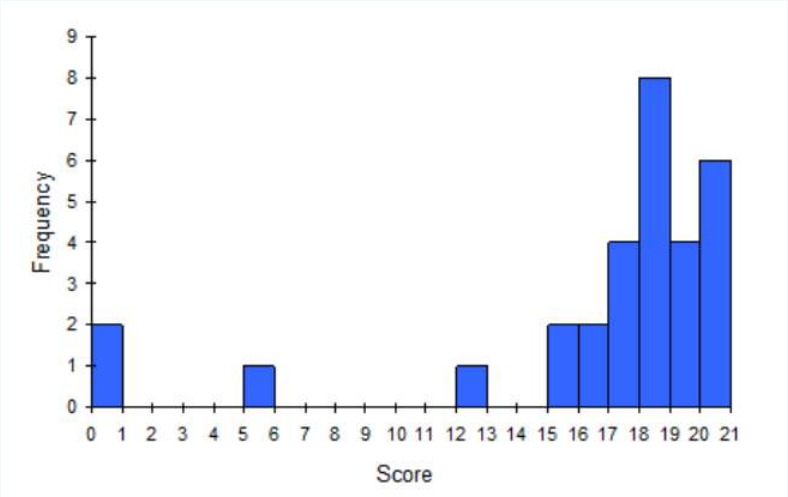

Figure 12.2.13: Histogram for above values

Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. So, for instance, the fact that there are four occurrences of 17 in the dataset is indicated by the bar, with height 4, between 17 and 18 in this histograph. Some people choose to have bars start at ½ values to avoid this ambiguity.

Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals.**

Definition: Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 12.2.11

Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20 or 14 intervals with a width of around 10, or somewhere in between. Often we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Table 12.2.4

Interval	Frequency
120-134	4

Interval	Frequency
135-149	14
150-164	16
165-179	28
180-194	12
195-209	8
210-224	7
225-239	6
240-254	2
255-269	3

A histogram of this data would look like:

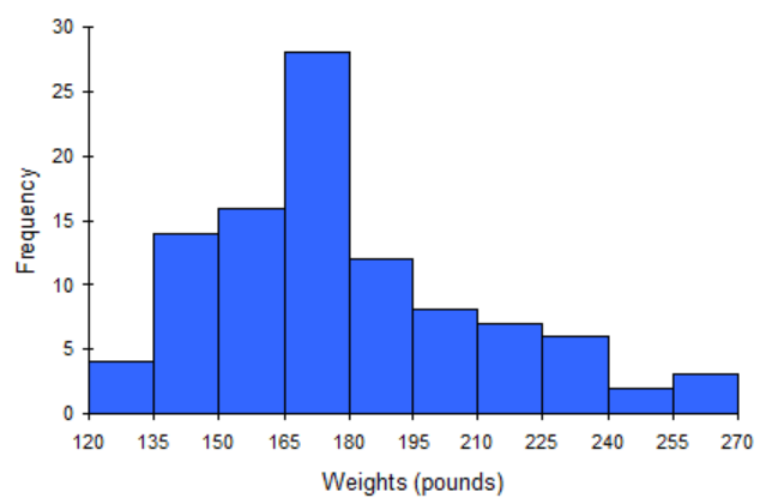

Figure 12.2.14: Histogram of the above data

In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.

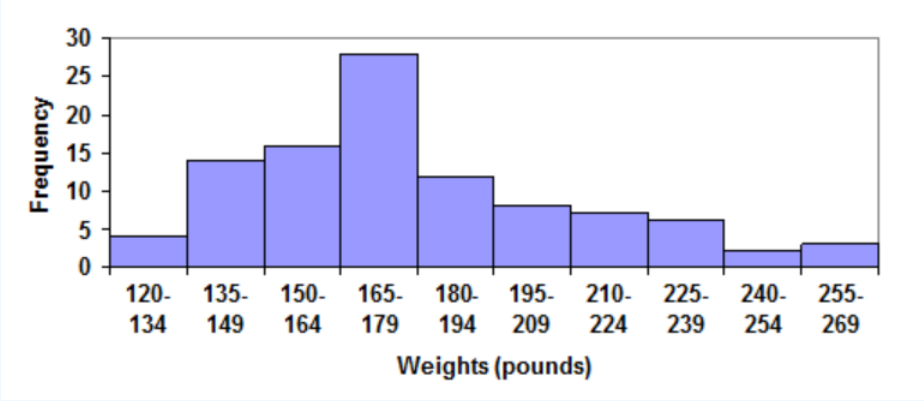

Figure 12.2.15: Histogram created by putting class intervals on bar chart

Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.

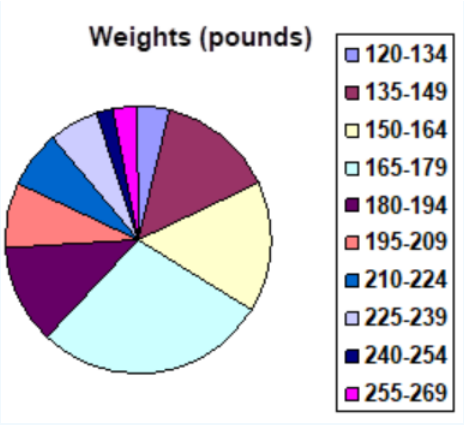

Figure 12.2.16: Pie chart of our weight data

Try it Now 3

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

Example 12.2.12

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Table 12.2.5

Interval (milliseconds)	Frequency (small target)	Frequency (large target)
300-399	0	0
400-499	1	5
500-599	3	10
600-699	6	5
700-799	5	0
800-899	4	0
900-999	0	0
1000-1099	1	0
1100-1199	0	0

One option to represent this data would be a comparative histogram or bar chart in which bars for the small target group and large target group are placed next to each other.

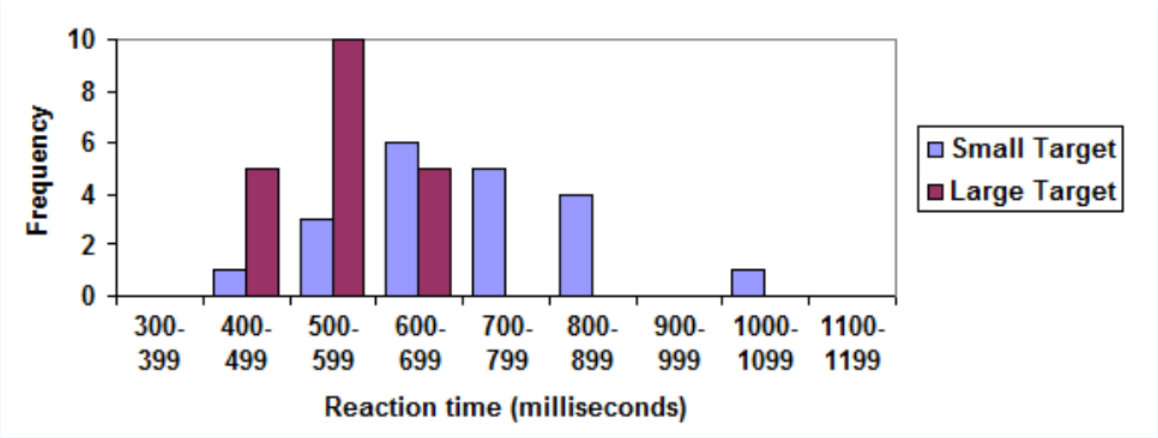

Figure 12.2.17: A comparative histogram or bar chart

Definition: Frequency Polygon

An alternative representation is a frequency polygon. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 12.2.13

This graph makes it easier to see that reaction times were generally shorter for the larger target and that the reaction times for the smaller target were more spread out.

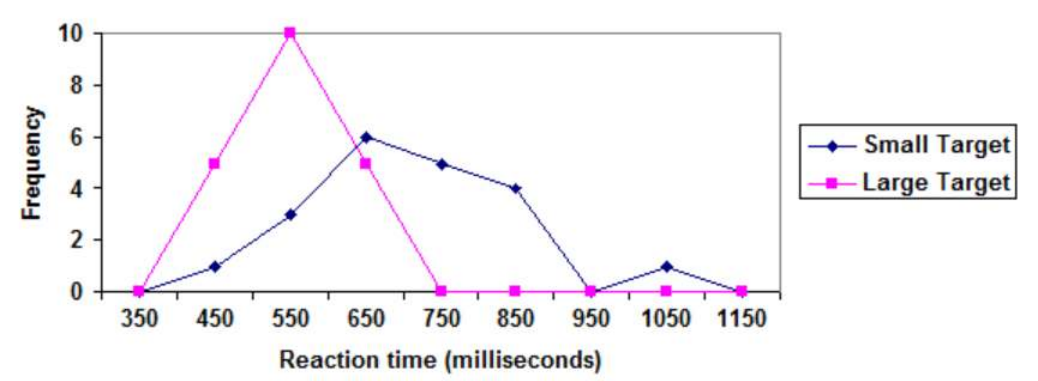

Figure 12.2.18: Graph

Reference

1. References (11)

Contributors and Attributions

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12.3: Pie Charts

In this section we turn our attention to pie charts, but before we do, we need to establish some fundamentals regarding measurement of angles.

If you take a circle and divide it into 360 equal increments, then each increment is called one degree (1°). See Figure 7.2.

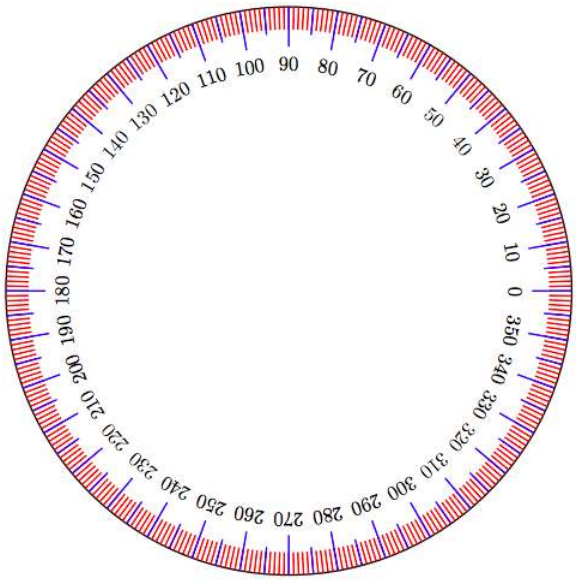


Figure 7.2: There are 360 degrees (360°) in a circle.

A *ray* is a line that starts at a point and then extends indefinitely in one direction. The starting point of the ray is called its *vertex*.

Figure 7.3: A ray with vertex V *extends indefinitely in one direction.*

If two rays have a common vertex, they form what is called an *angle*. In Figure 7.4 we've labeled the first ray as the "Initial Side" of the angle, and the second as the "Terminal Side" of the angle.

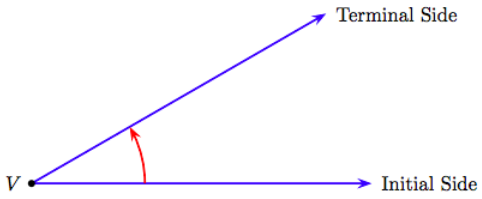


Figure 7.4: Two rays with a common vertex V *form an angle.*

We can find the degree measure of the angle by using a device called a *protractor*. Align the notch in the center of the base of the protractor with the vertex of the angle, then align the base of the protractor with the initial side of the angle. The terminal side of the angle will intersect the protractor edge where we can read the degree measure of the angle (see Figure 7.5). In Figure 7.5, note that the terminal side of the angle passes through the tick mark at the number 30, indicating that the degree measure of this angle is 30°.

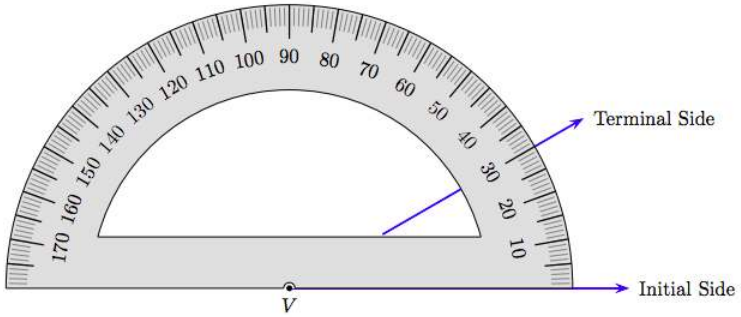


Figure 7.5: The degree measure is 30°.

Pie Charts

Now that we can measure angles, we can turn our attention to constructing *pie charts*.

Pie Chart

A pie chart is a circular chart that is divided into sectors, each sector representing a particular quantity. The area of each sector is a percentage of the area of the whole circle and is determined by calculating the quantity it represents as a percentage of the whole.

Example 1

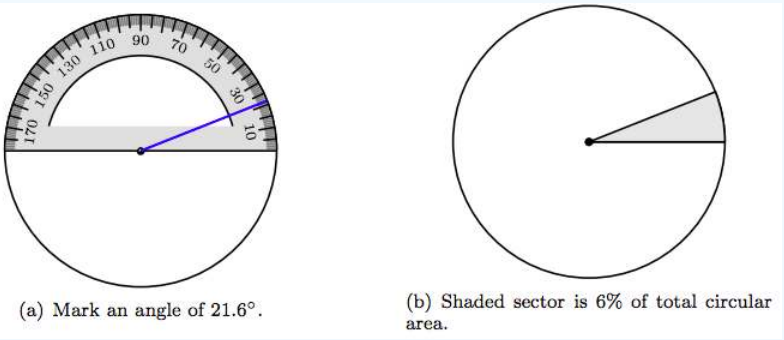
In a recent Gallup poll, 66% of those polled approved of the President's job performance, 28% disapproved, and 6% were undecided. Create a pie chart that displays these percentages.

Solution

Let's begin with 6% undecided. We want to create a sector that is 6% of the area of the whole circle. There are 360 degrees in a full circle, so 6% of this number is

$$\begin{aligned} 6\% \cdot 360^\circ &= 0.06 \cdot 360^\circ \\ &= 21.6^\circ. \end{aligned}$$

Start with a circle, set the baseline notch of the protractor on the center of the circle, then mark an angle of 21.6°, as seen in Figure 7.6(a). Shade the resulting region as shown in Figure 7.6(b), called a *sector*, which represents 6% of the total area of the circle.



(a) Mark an angle of 21.6°.

(b) Shaded sector is 6% of total circular area.

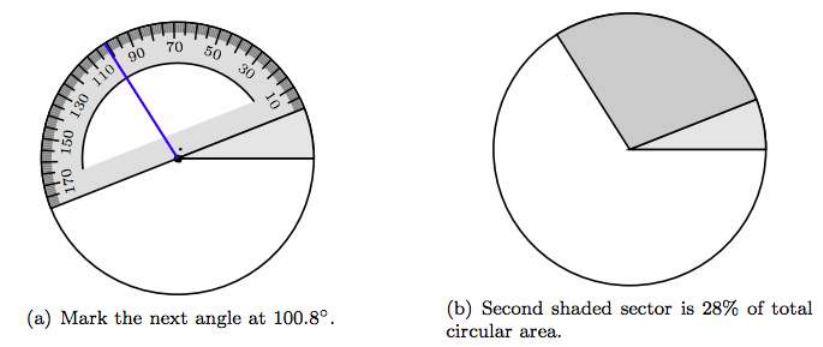
Figure 7.6: Sector with central angle 21.6° represents the 6% of the polling sample that were undecided about the president's performance.

Next, 28% disapproved of the President's job performance. Thus,

$$\begin{aligned} 28\% \cdot 360^\circ &= 0.28 \cdot 360^\circ \\ &= 100.8^\circ. \end{aligned}$$

Therefore, a sector with a central angle of 100.8° will represent 28% job disapproval.

Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the first angle, as shown in Figure 7.7(a). Mark a central angle of 100.8°, as shown in Figure 7.7(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.7(b). This sector contains 28% of the total area of the circle and represents the portion of the polling sample that disapproved of the president's job performance.



(a) Mark the next angle at 100.8°.

(b) Second shaded sector is 28% of total circular area.

Figure 7.7: Sector with central angle 100.8° represents the 28% of the polling sample that disapproved of the president's performance.

Finally, as we've shaded the sectors representing 6% and 28% of the polling data in Figure 7.7(b), the remaining sector in Figure 7.7(b), shaded in white, represents the 66% of the polling sample who approved of the president's job performance (and 66% of the area of the whole circle).

Once you have computed and plotted the correct central angles for each of the sectors, you will want to label your pie chart. One possible annotation method is shown in Figure 7.8.

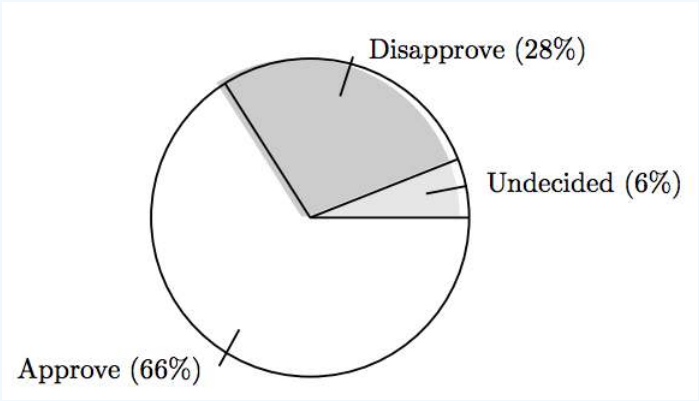
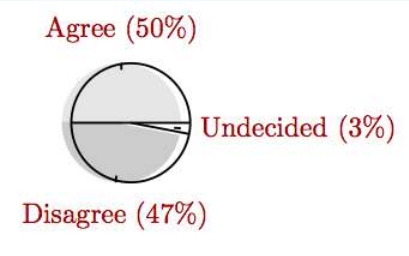


Figure 7.8: Annotating the final pie chart.

? Exercise

In a recent Gallup poll, 50% of the American public say it is the federal government's responsibility to make sure all Americans have health care, 47% disagree, and 3% were undecided. Create a pie chart that displays these percentages.

Answer



Example 2

One thousand people were polled with the question “Where does your dog sleep during the night?” The responses are shown in the following table.

Location	Number
Outside	30
Another Room	220
On the Bedroom Floor	330
On the Bed	420
Totals	1000

Create a pie chart showing the distribution of these responses.

Solution

The first step is to express the number in each location as a percentage of the totals. For example,

Outside	is	what percent	of	total
30	=	p	·	1000

Solving for p,

$$\frac{1000p}{1000} = \frac{30}{1000} \quad \text{Divide both sides by 1000.}$$
$$p = 0.03 \quad \text{Divide: } 30/1000 = 0.03.$$

Thus, p = 3%. In similar fashion, divide the number in each location by 1000 to find the following percents.

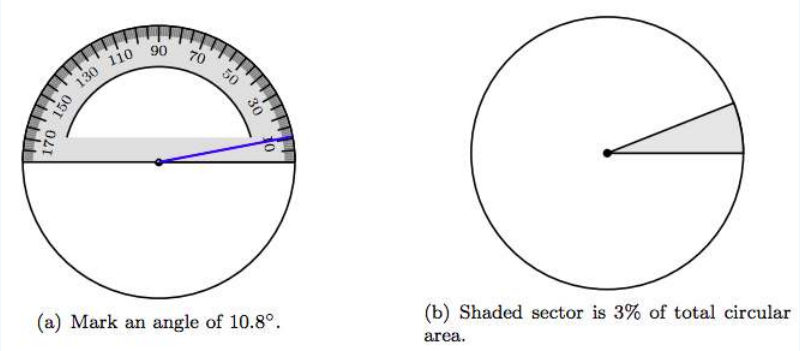
Location	Number	Percent
Outside	30	3%
Another Room	220	22%
On the Bedroom Floor	330	33%
On the Bed	420	42%
Totals	1000	100%

Note that the individual percents must total 100%.

Let’s begin with the fact that 3% of the dog owners have their dogs sleep outside. To find the portion of the full circle that represents 3%, we take 3% of 360°.

$$\begin{aligned} 3\% \cdot 360^\circ &= 0.03 \cdot 360^\circ \\ &= 10.8^\circ \end{aligned}$$

Start with a circle, set the baseline notch of the protractor on the center of the circle, then mark an angle of 10.8°, as seen in Figure 7.9(a).



(a) Mark an angle of 10.8°.

(b) Shaded sector is 3% of total circular area.

Figure 7.9: Sector with central angle 10.8° represents the 3% of the polling sample that have their dogs sleep outside.

Shade the resulting sector as shown in Figure 7.9(b), which represents 3% of the total area of the circle.

Next, 22% have their dog sleep in another room.

$$\begin{aligned} 22\% \cdot 360^\circ &= 0.22 \cdot 360^\circ \\ &= 79.2^\circ. \end{aligned}$$

Therefore, a sector with a central angle of 79.2° will represent the fact that 22% of the dog owners have their dog sleep in another room.

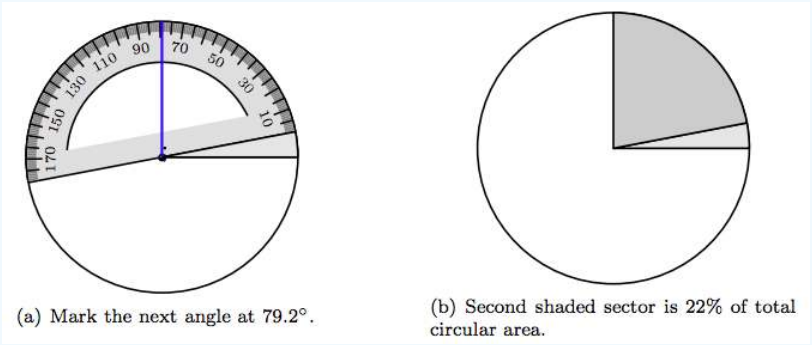
Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the first angle, as shown in Figure 7.10(a). Mark a central angle of 79.2◦, as shown in Figure 7.7(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.10(b). This sector contains 22% of the total area of the circle and represents the portion of the polling sample whose dog sleeps in another room.

Next, 33% allow their dog sleep on the bedroom floor.

$$\begin{aligned} 33\% \cdot 360^\circ &= 0.33 \cdot 360^\circ \\ &= 118.8^\circ. \end{aligned}$$

Therefore, a sector with a central angle of 118.8° will represent the fact that 33% of the dog owners allow their dog to sleep on the bedroom floor.

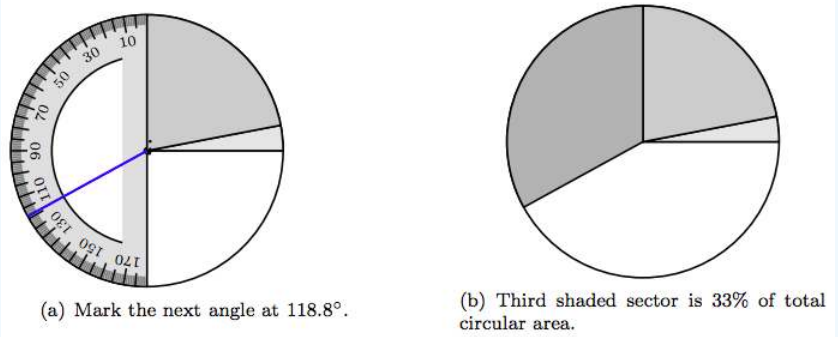
Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the second sector, as shown in Figure 7.11(a). Mark a central angle of 118.8°, as shown in Figure 7.11(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.11(b). This sector contains 33% of the total area of the circle and represents the portion of the polling sample whose dog sleeps on the bedroom floor.



(a) Mark the next angle at 79.2°.

(b) Second shaded sector is 22% of total circular area.

Figure 7.10: Sector with central angle 79.2° represents the 22% of the polling sample whose dogs sleep in another room.



(a) Mark the next angle at 118.8°.

(b) Third shaded sector is 33% of total circular area.

Figure 7.11: Sector with central angle 118.8° represents the 33% of the polling sample whose dogs sleep on the bedroom floor.

Because the first three sectors, shaded in various levels of gray, represent 3%, 22%, and 33% of the total circular area, respectively, the remaining sector (shaded in white) automatically represents

$$100\% - (3\% + 22\% + 33\%) = 42\%$$

of the total circular area. This region represents the percent of dog owners who allow their dogs to sleep on the bed. The final result, with annotations, is shown in Figure 7.1

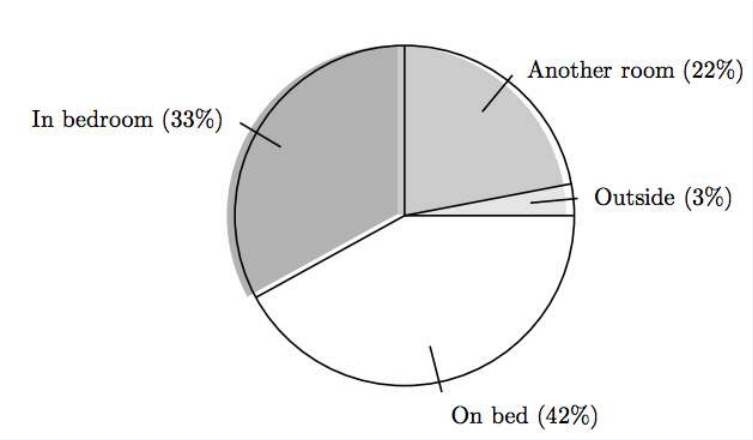
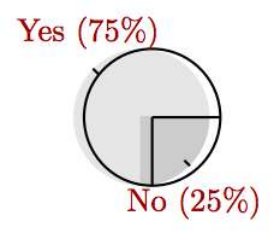


Figure 7.12: Annotating the final pie chart.

? Exercise

Two hundred people were asked whether they vote “Yes” or “No” on Proposition 8. There were 150 “Yes” votes and 50 “No” votes. Create a pie chart showing the distribution of these responses.

Answer



Exercises

1. In an election for class president, Raven received 21% of the votes, Anita received 27%, Jamal received 24% of the votes, and 28% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Raven’s share of the vote? Round your answer to the nearest degree.

2. In an election for class president, Fernando received 26% of the votes, Luisa received 26%, Ali received 26% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Fernando’s share of the vote? Round your answer to the nearest degree.

3. In an election for class president, Akbar received 23% of the votes, Ali received 27%, Juanita received 30% of the votes, and 20% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Akbar’s share of the vote? Round your answer to the nearest degree.

4. In an election for class president, Kamili received 21% of the votes, Bernardo received 22%, Fernando received 30% of the votes, and 27% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Kamili’s share of the vote? Round your answer to the nearest degree.

5. In an election for class president, Jamal received 30% of the votes, Luisa received 20%, Kamili received 28% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Jamal’s share of the vote? Round your answer to the nearest degree.

6. In an election for class president, Juanita received 30% of the votes, Ali received 24%, Estevan received 24% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree

measure of the central angle of the sector representing Juanita's share of the vote? Round your answer to the nearest degree.

7. In an election for class president, Chin received 5 votes, Mabel received 13 votes, and Juanita received the remaining 32 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Chin's share of the vote? Round your answer to the nearest degree.

8. In an election for class president, Anita received 11 votes, Jose received 9 votes, and Bernardo received the remaining 30 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Anita's share of the vote? Round your answer to the nearest degree.

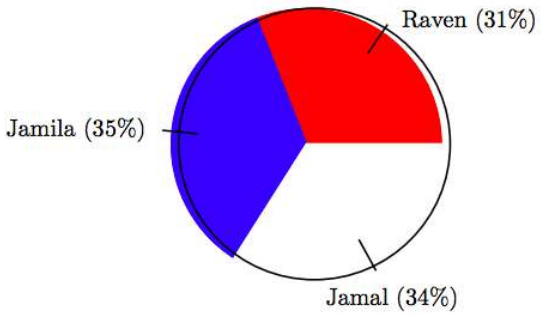
9. In an election for class president, Kamili received 14 votes, Jamal received 9 votes, and Jose received the remaining 27 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Kamili's share of the vote? Round your answer to the nearest degree.

10. In an election for class president, Jun received 13 votes, Abdul received 15 votes, and Raven received the remaining 22 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Jun's share of the vote? Round your answer to the nearest degree.

11. In an election for class president, Hue received 13 votes, Ali received 6 votes, and Henry received the remaining 31 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Hue's share of the vote? Round your answer to the nearest degree.

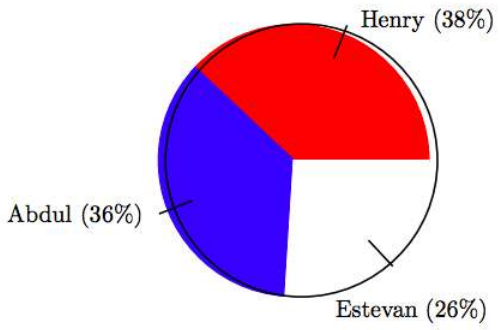
12. In an election for class president, Mercy received 9 votes, Bernardo received 7 votes, and Hans received the remaining 34 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Mercy's share of the vote? Round your answer to the nearest degree.

13. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



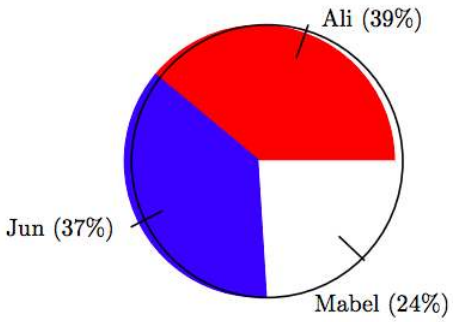
If there were a total of 95 votes cast in the election, find the number of votes that Raven received, correct to the nearest vote.

14. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



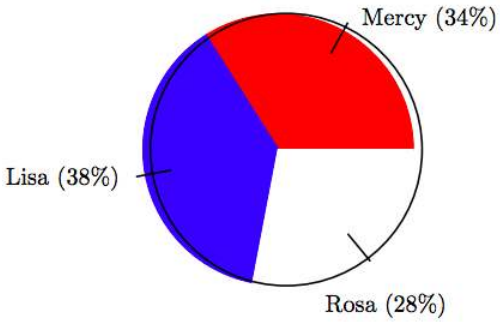
If there were a total of 79 votes cast in the election, find the number of votes that Henry received, correct to the nearest vote.

15. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



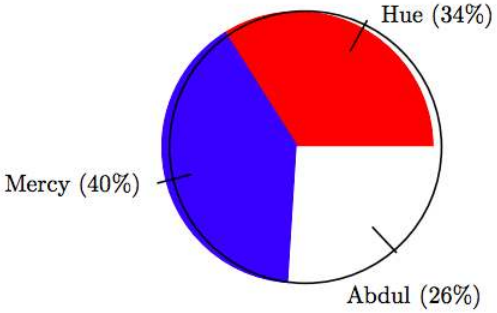
If there were a total of 58 votes cast in the election, find the number of votes that Ali received, correct to the nearest vote.

16. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



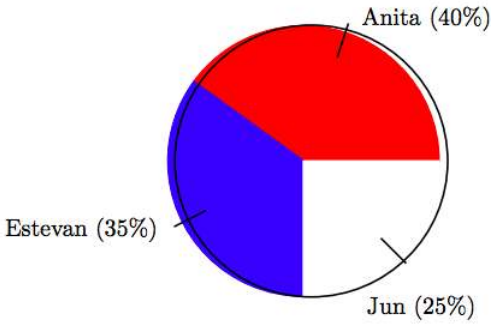
If there were a total of 65 votes cast in the election, find the number of votes that Mercy received, correct to the nearest vote.

17. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



If there were a total of 95 votes cast in the election, find the number of votes that Hue received, correct to the nearest vote.

18. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



If there were a total of 75 votes cast in the election, find the number of votes that Anita received, correct to the nearest vote.

---

19. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Ali	45
Jamal	34
Jun	52

Use a protractor to help create a pie chart showing the distribution of votes.

20. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Aisha	39
Akbar	31
Fernando	36

Use a protractor to help create a pie chart showing the distribution of votes.

21. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	44
Rosa	40
Abdul	58

Use a protractor to help create a pie chart showing the distribution of votes.

22. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Estevan	46
Ali	58
Henry	49

Use a protractor to help create a pie chart showing the distribution of votes.

23. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Mercy	56
Hans	53
Lisa	41

Use a protractor to help create a pie chart showing the distribution of votes.

24. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Estevan	60
Hue	33
Aisha	31

Use a protractor to help create a pie chart showing the distribution of votes.

25. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Raven	43
Mabel	40
Bernardo	52

Use a protractor to help create a pie chart showing the distribution of votes.

26. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Hue	48
Lisa	48
Akbar	47

Use a protractor to help create a pie chart showing the distribution of votes.

27. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Jun	57
Lisa	30
Aisha	58

Use a protractor to help create a pie chart showing the distribution of votes.

28. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	54
Mabel	38
Henry	49

Use a protractor to help create a pie chart showing the distribution of votes.

29. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Henry	35
Bernardo	32
Estevan	47

Use a protractor to help create a pie chart showing the distribution of votes.

30. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	38
Fernando	49
Aisha	44

Use a protractor to help create a pie chart showing the distribution of votes.

31. **Guard deployment.** The table shows the number of guard troop services since Sept. 11, 2001 (as of Dec. 2008; some troops have activated multiple times). *Associated Press-Times-Standard 02/18/10 Guard troops wait for promised pay.*

Mission	Troops
Operation Iraqi Freedom	193,598
Operation Enduring Freedom	29,212
Other missions	35,849

Use a protractor to help create a pie chart showing the distribution of National Guard troops.

Answers

1. 76°

3. 83°

5. 108°

7. 36°

9. 101°

11. 94°

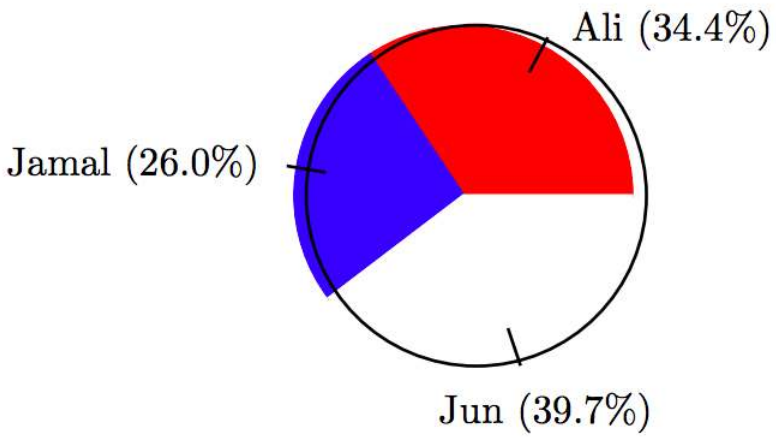
13. 29 votes

15. 23 votes

17. 32 votes

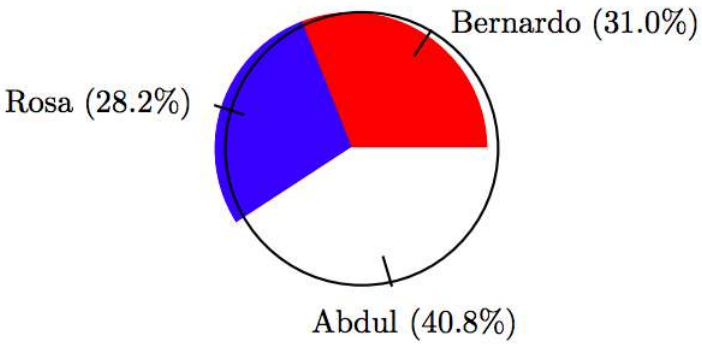
19.

Candidate	Votes	Percent	Degrees
Ali	45	34.4%	123.84°
Jamal	34	26.0%	93.6°
Jun	52	39.7%	142.92



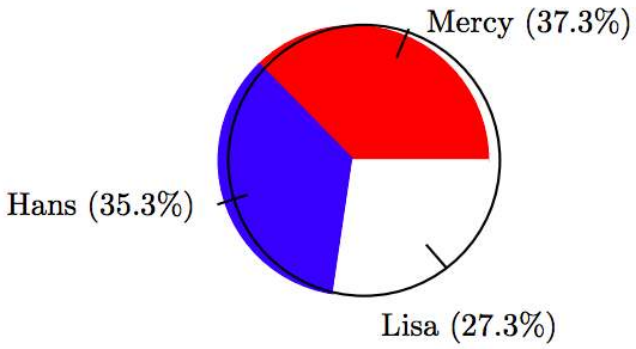
21.

Candidate	Votes	Percent	Degrees
Bernardo	44	31.0%	111.6°
Rosa	40	28.2%	101.52°
Abdul	58	40.8%	146.88°



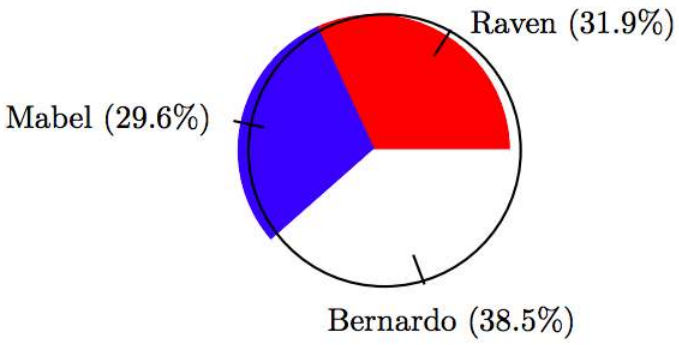
23.

Candidate	Votes	Percent	Degrees
Mercy	56	37.3%	134.28°
Hans	53	35.3%	127.08°
Lisa	41	27.3%	98.28°



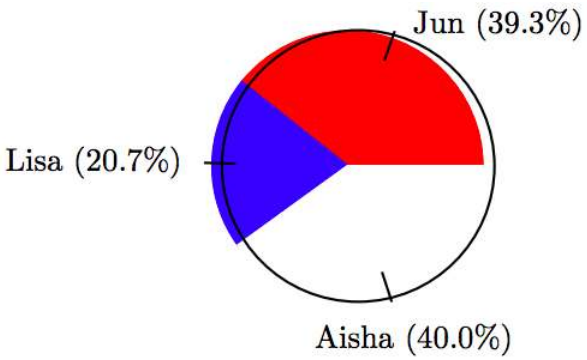
25.

Candidate	Votes	Percent	Degrees
Raven	43	31.9%	114.84°
Mabel	40	29.6%	106.56°
Bernardo	52	38.5%	138.6°



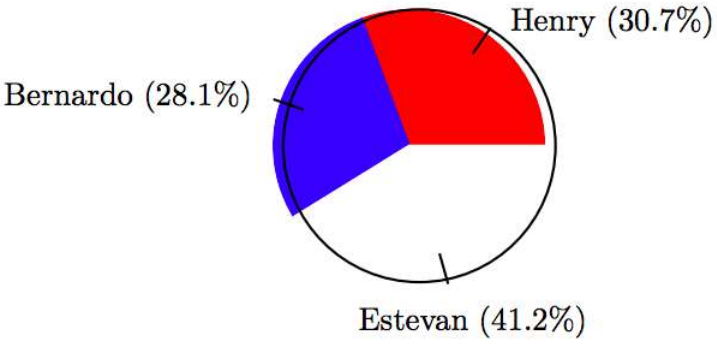
27.

Candidate	Votes	Percent	Degrees
Jun	57	39.3%	141.48°
Lisa	30	20.7%	74.52°
Aisha	58	40.0%	144°



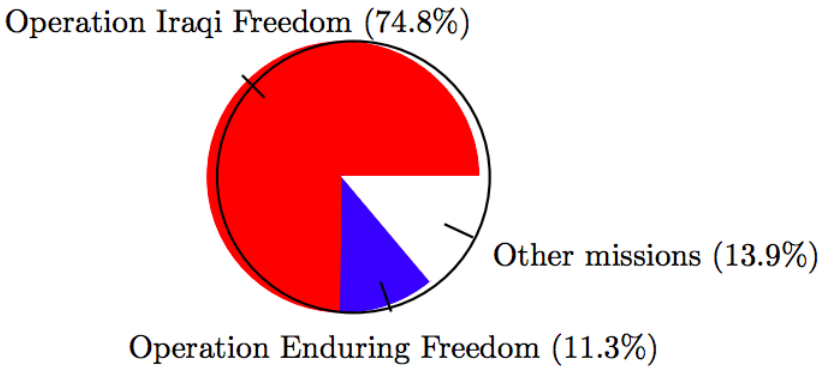
29.

Candidate	Votes	Percent	Degrees
Henry	35	30.7%	110.52°
Bernardo	32	28.1%	101.16°
Estevan	47	41.2%	148.32°



31.

Mission	Troops	Percent	Degrees
Operation Iraqi Freedom	193,598	74.8%	269.28°
Operation Enduring Freedom	29,212	11.3%	40.68°
Other missions	35,849	13.9%	50.04°



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12.4: Numerical Measures of Center and Variation

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second part of this section describes measures of variation.

Measures of Central Tendency

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the “arithmetic mean”, or just plain old “mean” for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average.”

Mean

The mean of a set of data is the sum of the data values divided by the number of values.

Example 12.4.1

Marci’s exam scores for her last math class were 79, 86, 82, 94. Find the mean.

Solution

The mean of these values would be:

79+86+82+944=85.25

Example 12.4.2

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20

20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6

What is the mean number of touchdown passes thrown in the NFL in the 2000 season?

Solution

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get 634/31 = 20.4516. It would be appropriate to round this to 20.5.

It would be correct for us to report that “The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes,” but it is not uncommon to see the more casual word “average” used in place of “mean.”

Try it Now 4

The prices of a jar of peanut butter at 5 stores were $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

Example 12.4.3

One hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below.

Table 12.4.1

Income (thousands of dollars)	Frequency
15	6

Income (thousands of dollars)	Frequency
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Calculate the mean.

Solution

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{\overbrace{15+\cdots 15}^{\text{6 terms}}+\overbrace{20+\cdots 20}^{\text{8 terms}}+\overbrace{25+\cdots 25}^{\text{11 terms}}+\cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as 90. Using this simplification, we get:

$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7}{100}=\frac{3390}{100}=33.9$$

The mean household income of our sample is 33.9 thousand dollars ($33,900).

Example 12.4.4

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5000 thousand).

Solution

Adding this to our sample, our mean is now:

$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7+5000\cdot 1}{101}=\frac{8390}{101}=83.069$$

While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a “typical” value.

An extreme value, much bigger or much smaller than most other values in a given dataset, is called an “outlier”. An outlier can affect the mean.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.

Figure 12.4.1

If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance.

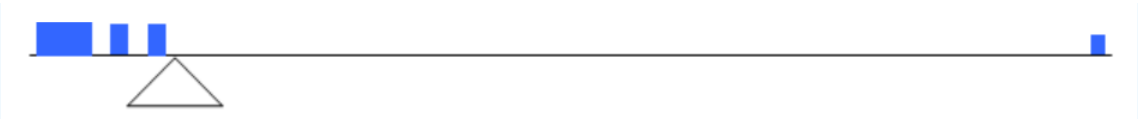

Figure 12.4.2

In situations like this, where one value is much bigger or smaller than most other value in the data, there is a better measure of center than the mean.

Median

The median of a set of data is the value in the middle when the data is in order.

To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, N, is odd, then the median is the middle data value. This can be found by rounding $N/2$ up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values $N/2$ and $N/2+1$).

Example 12.4.5

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20

20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6

Find the median touchdown pass.

Solution

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value (31/2 = 15.5, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 12.4.6

Find the median of these quiz scores: 5, 10, 8, 6, 4, 8, 2, 5, 7, 7

Solution

We start by listing the data in order: 2, 4, 5, 5, 6, 7, 7, 8, 8, 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get (6+7)/2 = 6.5.

The median quiz score was 6.5.

Try it Now 5

The prices of a jar of peanut butter at 5 stores were $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Example 12.4.7

Let us return now to our original household income data.

Table 12.4.2

Income (thousands of dollars)	Frequency
15	6

Income (thousands of dollars)	Frequency
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Find the mean and median income.

Solution

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values - the 50th and 51st data values. To find these, we start counting up from the bottom:

There are 6 data values of $15, so Values 1 to 6 are $15 thousand

The next 8 data values are $20, so Values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so Values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so Values 26 to (25+17)=42 are $30 thousand

The next 19 data values are $35, so Values 43 to (42+19)=61 are $35 thousand

From this we can tell that values 50 and 51 will be $35 thousand, and the mean of these two values is $35 thousand. The median income in this neighborhood is $35 thousand.

Example 12.4.8

If we add in the new neighbor with a $5 million household income, how does the median change?

Solution

If we add in the new neighbor with a $5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is $35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

This is the main reason the median, not the mean, is used to represent certain data, such as the average household income or home price.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode.**

Mode

The mode is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 12.4.9

In our vehicle color survey, we collected the data

Table 12.4.3

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

What is the mode?

Solution

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode (if several categories have the same frequency), or no modes (if every category occurs only once.)

Try it Now 6

Reviewers were asked to rate a product on a scale of 1 to 5. Find

a. The mean rating
b. The median rating
c. The mode rating

Table 12.4.4

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

Measures of Variation

Consider these three sets of quiz scores:

Section A: 5, 5, 5, 5, 5, 5, 5, 5, 5, 5

Section B: 0, 0, 0, 0, 0, 10, 10, 10, 10, 10

Section C: 4, 4, 4, 5, 5, 5, 5, 6, 6, 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range.**

Range

The range is the difference between the maximum value and the minimum value of the data.

Example 12.4.10

Using the quiz scores from above, find the range for each section.

Solution

For section A, the range is 0 since both maximum and minimum are 5 and 5 – 5 = 0.

For section B, the range is 10 since 10 – 0 = 10.

For section C, the range is 2 since 6 – 4 = 2.

In the last example, the range seems to be revealing how spread out the data is.

However, suppose we add a fourth section, Section D, with scores 0, 5, 5, 5, 5, 5, 5, 5, 5, 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different from Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Standard Deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero only if all the data values are equal and will get larger as the data spreads out.
- Standard deviation has the same unit as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Standard deviation is rather complicated. Let me explain below, step by step.

Using the data from section D (Example 12.4.10), we could compute for each data value the difference between the data value and the mean:

Table 12.4.5

Data value	Deviation: Data value - Mean
0	0-5 = -5
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
10	10-5 = 5

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column, the negative and positive values cancel each other out, resulting in the average deviation of 0 (this will always happen). To prevent this we square every value in the second column:

Table 12.4.6

Data value	Deviation: Data value - Mean	Deviation squared
0	0-5 = -5	$(-5)^2 = 25$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
10	10-5 = 5	$(5)^2 = 25$

We then add the squared deviations up to get 25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50. Ordinarily we would then divide by the number of scores, n (in this case, 10), to find the mean of the squares of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n-1$ (in this case, 10-1=9).

The quotient (by n or by $n-1$) is called the "variance" of the data set.

So in our example, we would have 50/10 = 5 if section D represents a population and 50/9 = about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

We are almost there—one more step, and we will find the standard deviation.

Variance can be a useful statistical concept, but note that the unit of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

Population standard deviation=$\sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$

or

sample standard deviation=$\sqrt{\frac{50}{9}} \approx 2.4$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data as we have done above.

As you can see, it takes multiple steps to calculate the standard deviation of a dataset. For most real-life applications, the calculations are performed using technology. Here is a brief summary.

To compute Standard Deviation

1. Find the deviation of each data value from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by n, the number of data values, if the data represents a whole population; divide by $n-1$ if the data is from a sample.
5. Compute the square root of the result.

Example 12.4.11

Compute the standard deviation for Section B above

Solution

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

Table 12.4.7

Data value	Deviation: Data value - Mean	Deviation squared
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}} = \sqrt{\frac{250}{10}} = 5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are as follows:

Table 12.4.8

Section	Standard deviation
Section A: 5, 5, 5, 5, 5, 5, 5, 5, 5, 5	Standard deviation: 0
Section B: 0, 0, 0, 0, 0, 10, 10, 10, 10, 10	Standard deviation: 5
Section C: 4, 4, 4, 5, 5, 5, 5, 6, 6, 6	Standard deviation: 0.8
Section D: 0, 5, 5, 5, 5, 5, 5, 5, 5, 10	Standard deviation: 2.2

Try it Now 7

The prices of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, quartiles are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.

The first quartile (Q1) is the value so that 25% of the data values are below it; the third quartile (Q3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q1, 25% is between Q1 and the median, 25% is between the median and Q3, and 25% is between Q3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a 5-number summary of the data.

Five-number Summary

The five-number summary takes this form:

Minimum, Q1, Median, Q3, Maximum.

To find the first quartile, we need to find the data value so that 25% of the data is below it.

But note that this is the same as the "median" of the first (lower) half of the dataset. So an easy way to find the first quartile is to split the original data set in half and find the median of the lower half. The third quartile is similar; find the median of the second (higher) half of the data set.

Warning: The precise definitions and method of finding quartiles vary slightly from book to book. You may see slightly different explanations in other resources.

Example 12.4.12

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59, 60, 62, 64, 66, 67, 69, 70, 72

Find Quartiles.

Solution

Note the median is 66.

The lower half, {59, 60, 62, 64}, has the median of 61, and the higher half, {67, 69, 70, 72}, has the median of 69.5.

So Q1 is 61, Q2 is 66, and Q3 is 69.5.

Example 12.4.13

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are:

59, 60, 62, 64, 66, 67, 69, 70

Find Quartiles.

Solution

Note the median in this case is 65, the mean of 64 and 66.

Then the lower half is the same as in Example 25 (its median is 61). The higher half now has the median 68.

So, Q1 is 61, Q2 is 65, and Q3 is 68.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 12.4.14

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5- number summary is: 59, 61, 66, 69.5, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5- number summary would be: 59, 61, 65, 68, 70.

Example 12.4.15

Returning to our quiz score data. In each case, there are 10 values, so the median is the mean of the 5th and 6th values. The first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Table 12.4.9

Section and data	5-number summary
Section A: 5, 5, 5, 5, 5, 5, 5, 5, 5, 5	5, 5, 5, 5, 5
Section B: 0, 0, 0, 0, 0, 10, 10, 10, 10, 10	0, 0, 5, 10, 10
Section C: 4, 4, 4, 5, 5, 5, 5, 6, 6, 6	4, 4, 5, 6, 6
Section D: 0, 5, 5, 5, 5, 5, 5, 5, 5, 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly since the summary contains almost as many values as the original data.

Try it Now 8

The total cost of textbooks for the term was collected from 36 students. Find the 5-number summary of this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

Example 12.4.16

Returning to the household income data from earlier, create the 5-number summary.

Table 12.4.10

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was $35 thousand. We can see in the table that the minimum income is $15 thousand, and the maximum is $50 thousand.

Counting up in the data as we did before,

There are 6 data values of $15, so Values 1 to 6 are $15 thousand

The next 8 data values are $20, so Values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so Values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so Values 26 to (25+17)=42 are $30 thousand

The lower half has 50 values, so Q1 is the mean of the 25th and the 26th values.

The 25th data value is $25 thousand, and the 26th data value is $30 thousand, so Q1 will be the mean of these: (25 + 30)/2 = $27.5 thousand.

Similarly, Q3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are $35, so Values 43 to (42+19)=61 are $35 thousand

The next 20 data values are $40, so Values 61 to (61+20)=81 are $40 thousand

Both the 75th and 76th data values lie in this group, so Q3 will be $40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50.

Note that the 5-number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between $40 thousand and $50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a box plot, or box and whisker graph.

Box Plot

A box plot is a graphical representation of a 5-number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

Example 12.4.17

Draw a box plot below based on the 5-number summary:

59, 62, 66, 69, 72.

Solution

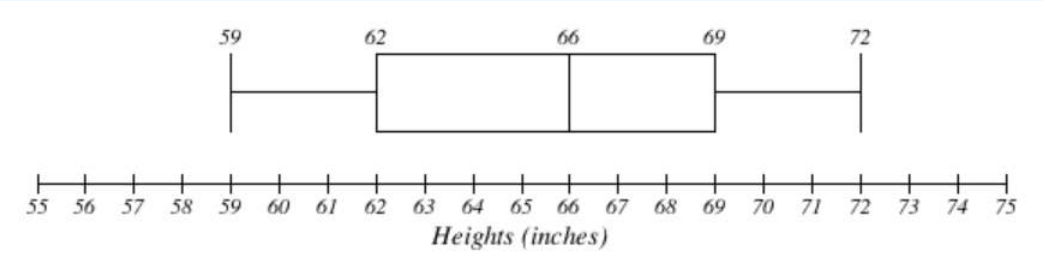

Figure 12.4.3: Box Plot

Example 12.4.18

Draw a box plot below based on the household income data with 5 number summary:

15, 27.5, 35, 40, 50

Solution

Figure 12.4.4: Box Plot

Try it Now 9

Create a boxplot based on the textbook price data from the last Try it Now.

Box plots are particularly useful for comparing data from two populations.

Example 12.4.19

The box plot of service times for two fast-food restaurants is shown below.

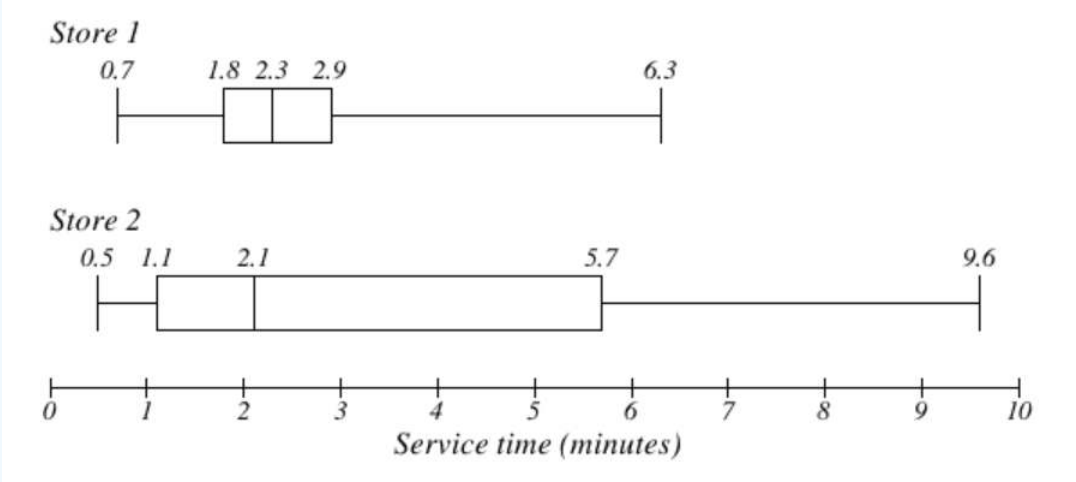

Figure 12.4.5: Box Plot

While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry?

Solution

That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 12.4.20

The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS). The boxplot is separated to show the birth weights of infants who survived and those that did not.

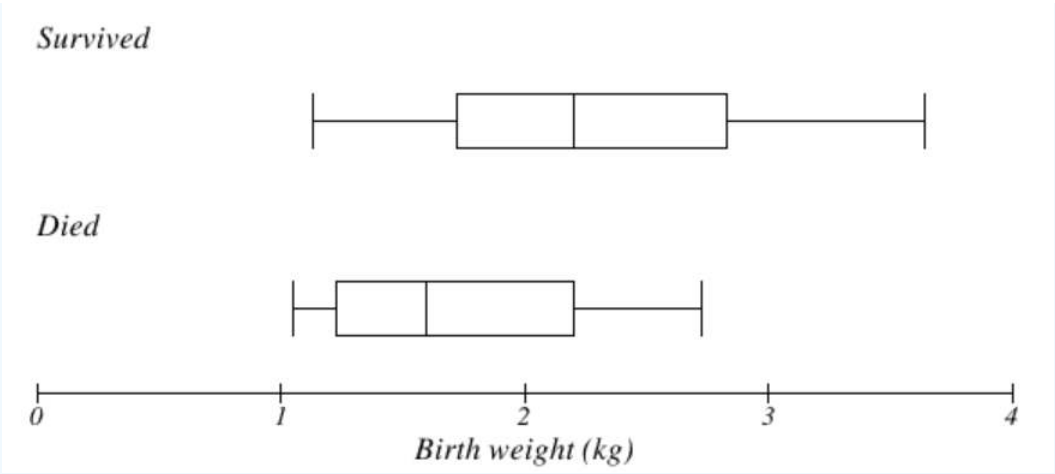

Figure 12.4.6: Box Plot

Compare the two groups to determine if birth weight is quite likely linked to survival of infants with SIRDS.

Solution

Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

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CHAPTER OVERVIEW

13: Growth and Decay

13.1: Growth Models

13.2: Logarithmic Scales in Natural Sciences

13.3: The pH Scale

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13.1: Growth Models

Populations of people, animals, and items are growing all around us. By understanding how things grow, we can better understand what to expect in the future. In this chapter, we focus on time-dependent change.

Linear Growth

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$\begin{aligned} P_0 &= 437 \\ P_n &= P_{n-1} + 32 \end{aligned}$$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1}. Using this relationship, we could calculate:

$$\begin{aligned} P_1 &= P_0 + 32 = 437 + 32 = 469 \\ P_2 &= P_1 + 32 = 469 + 32 = 501 \\ P_3 &= P_2 + 32 = 501 + 32 = 533 \\ P_4 &= P_3 + 32 = 533 + 32 = 565 \\ P_5 &= P_4 + 32 = 565 + 32 = 597 \end{aligned}$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly how a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An explicit equation allows us to calculate P_n directly, without needing to know P_{n-1}. While you may already be able to guess the **explicit equation**, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$\begin{aligned} P_1 &= 437 + 32 && = 437 + 1(32) \\ P_2 &= P_1 + 32 = 437 + 32 + 32 && = 437 + 2(32) \\ P_3 &= P_2 + 32 = (437 + 2(32)) + 32 && = 437 + 3(32) \\ P_4 &= P_3 + 32 = (437 + 3(32)) && = 437 + 4(32) \end{aligned}$$

You can probably see the pattern now, and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for P_n and solving for n

$$\begin{aligned} 1000 &= 437 + 32n \\ 563 &= 32n \\ n &= 563/32 = 17.59 \end{aligned}$$

So Marco will reach 1000 bottles in 18 years.

In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

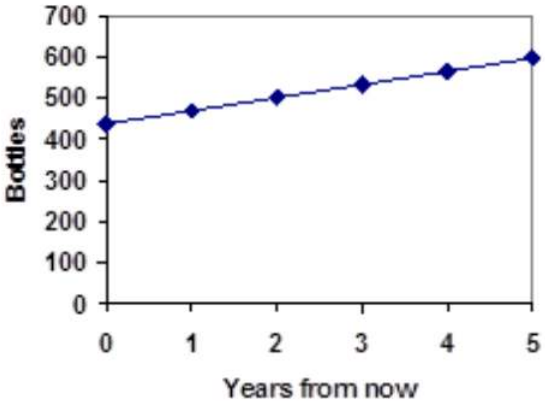

Figure 13.1.1: Marco's collection linear growth graph

Definition: Linear Growth

If a quantity starts at size P_0 and grows by d every time period, then the quantity after n time periods can be determined as follows:

$$P_n = P_0 + d \cdot n$$

In this equation, d represents the common difference, the amount that the population changes each time n increases by 1.

Connection to prior learning: slope and intercept

You may recognize the common difference, d, in our linear equation as slope. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$.

In the standard algebraic equation $y = mx + b$, recall$b$ was the y-intercept, or the y value when x was zero. In the form of the equation here, we are using P_0 to represent that initial amount.

In the $y = mx + b$ equation, recall that m was the slope. You might remember this as "rise over run" or the change in y divided by the change in x. Either way, it represents the same thing as the common difference, d, we are using, the amount the output P_n changes when the input n increases by 1.

The equations $y = mx + b$ and $P_n = P_0 + d \cdot n$ mean the same thing and can be used the same ways. We're just writing it somewhat differently.

Example 13.1.1

The population of elk in a national forest was measured to be 12,000 in 2003 and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

Solution

To begin, we need to define how we're going to measure n. Remember that P_0 is the population when $n = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $n = 0$ to be the year 2003. Then

$$P_0 = 12,000$$

Next we need to find d. Remember d is the growth per time period, in this case growth per year. Between the two measurements, the population grew by 15,000-12,000 = 3,000, but it took 2007-2003 = 4 years to grow that much. To find the growth per year, we can divide: 3000 elk / 4 years = 750 elk in 1 year.

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$$d = \text{slope} = \frac{15,000 - 12,000}{2007 - 2003} = \frac{3000}{4} = 750$$

We can now write our equation.

$$P_n = 12,000 + 750n$$

To answer the question, we need to first note that the year 2014 will be $n = 11$, since 2014 is 11 years after 2003.

$$P_{11} = 12,000 + 750(11) = 20,250 \text{ elk}$$

Example 13.1.2

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below. Find a model for this data and use it to predict consumption in 2023. If the trend continues, when will consumption reach 200 billion gallons?

Table 13.1.1

Year	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billion of gallons)	110	111	113	116	118	119	123	125	126	128	131	133	136

Solution

Plotting this data, it appears to have an approximately linear relationship:

While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

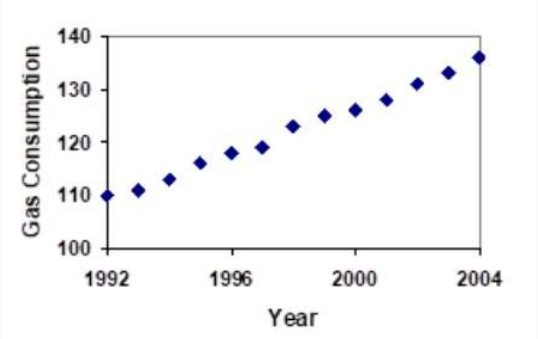

Figure 13.1.1: Gasoline consumption graph

Letting $n = 0$ correspond with 1993 would give $P_0 = 111$ billion gallons.

To find d, we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of 133 – 111 = 22 billion gallons, over 10 years.

This gives us an average change of 22 billion gallons / 10 year = 2.2 billion gallons per year.

Equivalently,

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{133-111}{10-0} = \frac{22}{10} = 2.2 \text{ billion gallons per year}$$

$$P_n = 111 + 2.2n$$

Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

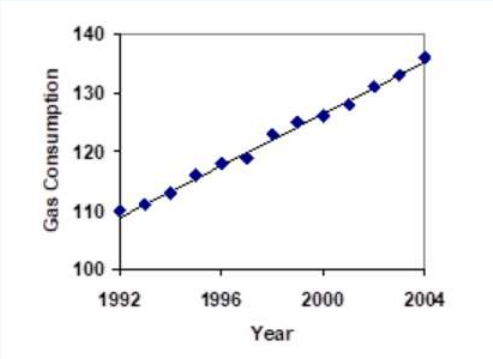

Figure 13.1.2: Calculating values using the explicit form and plotting them with the original data

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2023:

n = 30 (2023-1993=30 years later)

$$P_{23} = 111 + 2.2(30) = 177$$

Our model predicts that the US will consume 177 billion gallons of gasoline in 2023 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set $P_n = 200$, and solve for n:

$$\begin{aligned} P_n &= 200 && \text{Replace } P_n \text{ with our model} \\ 111 + 2.2n &= 200 && \text{Subtract 111 from both sides} \\ 2.2n &= 89 && \text{Divide both sides by 2.2} \\ n &= 40.4545 \end{aligned}$$

This tells us that consumption will reach 200 billion about 41 years after 1993, which would be in the year 2034.

By the way, the actual U.S. consumption in 2018 was 143 billion while this formula would give 166 billion. So, in reality, the growth rate appears to be going down a little.

Example 13.1.3

The cost, in dollars, of a gym membership for n months can be described by the explicit equation $P_n = 70 + 30n$. What does this equation tell us?

Solution

The value for P_0 in this equation is 70, so the initial starting cost is \$70. This tells us that there must be an initiation or start-up fee of \$70 to join the gym.

The value for d in the equation is 30, so the cost increases by \$30 each month. This tells us that the monthly membership fee for the gym is \$30 a month. This is a pretty good deal.

Try it Now 1

The number of stay-at-home fathers in Canada has been growing steadily. While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers; then use it to predict the number in 2020.

Table 13.1.2

Year	Number of stay-at-home fathers

Year	Number of stay-at-home fathers
1976	20,610
1984	28,725
1991	43,530
2000	47,665
2010	53,555

When good models go bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

Example 13.1.4

Suppose a four-year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year. Setup a growth model.

Solution

We can set up a growth model, with $n = 0$ corresponding to 4 years old.

$$P_n = 39 + 2.5n$$

So at 6 years old, we would expect him to be

$P_2 = 39 + 2.5(2) = 44$ inches tall

Most mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$P_{46} = 39 + 2.5(46) = 154$ inches tall = 12.8 feet tall!

Exponential Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. Assuming that none of the adult fish dies, if there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent of any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year. Whenever a quantity grows by a fixed percentage rather than a fixed amount, the growth is exponential, like a compound interest.

Suppose our lake began with 1000 fish and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1? The new population will be the old population plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0 \tag{13.1.1}$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = 1P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0 \tag{13.1.2}$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as "the original 100% plus an additional 10%", which is 110% (of the original population).

For our fish population,

$$P_0 = 1.10(1000) = 1100 \tag{13.1.3}$$

We could then calculate the population in later years:

$$P_2 = 1.10P_1 = 1.10(1100) = 1210 \tag{13.1.4}$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331 \tag{13.1.5}$$

Notice that in the first year, the population grew by 100 fish, in the second year, the population grew by 110 fish, and in the third year the population grew by 121 fish. While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

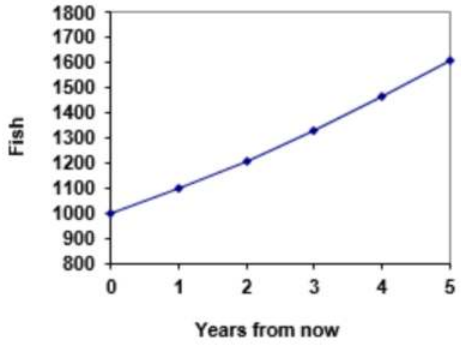

Figure 13.1.3: Fish growth graph

Graphing these values we see that this growth doesn't quite appear linear.

To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$$P_1 = 1.10P_0 = 1.10(1000)$$
$$P_2 = 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000)$$
$$P_3 = 1.10P_2 = 1.10\left(1.10^2(1000)\right) = 1.00^3(1000)$$
$$P_4 = 1.10P_3 = 1.10\left(1.10^3(1000)\right) = 1.10^4(1000)$$

Observing a pattern, we can generalize the explicit form to be:

$P_n = 1.10^n(1000)$ or equivalently, $P_n = 1000\left(1.10^n\right)$. Note that this is exactly like the compound interest formula with $r = 10\%$ annually.

From this, we can quickly calculate the number of fish in 10, 20, or 30 years:

$$P_{10} = 1.10^{10}(1000) = 2594$$
$$P_{20} = 1.10^{20}(1000) = 6727$$
$$P_{30} = 1.10^{30}(1000) = 17,449$$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years compared to almost 18,000 with this percent-based growth, called **exponential growth**.

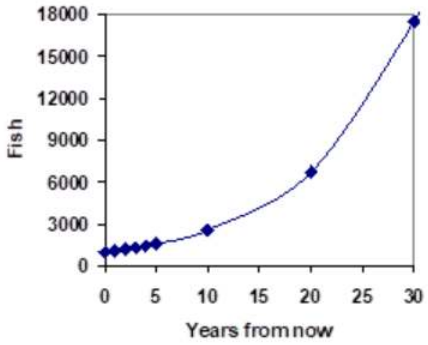

Figure 13.1.4: Fish population exponential growth graph

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

Definition: Exponential Growth

If a quantity starts at size P_0 and grows by $R\%$ (written as a decimal, r) every time period, then the quantity after n time periods can be determined as follows:

$$P_n = P_0(1+r)^n$$

We call r the growth rate.

Example 13.1.5

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

Solution

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245,000$. The year 2014 would then be $n = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$$P_6 = (1+0.03)^6(245,000) = 1.19405(245,000) = 292,542.25$$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Evaluating Exponents on the Calculator

To evaluate expressions like $(1.03)^6$, it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically labeled:

$$\wedge, \quad y^x, \text{ or } \quad x^y$$

To evaluate 1.036 we'd type 1.03 ^ 6, or 1.03 y^x 6.

Try it out. You should get an answer around 1.1940523.

Try it Now 2

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

Example 13.1.6

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

Solution

In the equation, $P_0 = 4600$, which is the starting value of the tuition when $n = 0$. This tells us that the tuition in 2010 was \$4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was \$4,600 and is expected to grow by 7.2% each year.

Example 13.1.7

In 1990, the residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons. If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Solution

Similar to before, we will correspond $n = 0$ with 1990, as that is the year for the first piece of data we have. That will make $P_0 = 962$ (million metric tons of CO_2). In this problem, we are not given the growth rate, but instead are given that $P_{10} = 1182$.

When $n = 10$, the explicit equation looks like:

$$P_{10} = (1 + r)^{10} P_0$$

We know the value for P_0, so we can put that into the equation:

$$P_{10} = (1 + r)^{10} 962$$

We also know that $P_0 = 1182$, so substituting that in, we get

$$1182 = (1 + r)^{10} 962$$

We can now solve this equation for the growth rate, r. Start by dividing by 962.

$$\frac{1182}{962} = (1 + r)^{10} \quad \text{Take the 10th root of both sides}$$

$$\sqrt[10]{\frac{1182}{962}} = 1 + r \quad \text{Subtract 1 from both sides}$$

$$r = \sqrt[10]{\frac{1182}{962}} - 1 = 0.0208 = 2.08\%$$

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding P_{60}.

$P_{60} = (1 + 0.0208)^{60} 962 = 3308.4$ million metric tons of CO_2 in 2050 (That's 3.3 billion metric tons!)

Rounding

As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

Evaluating Roots on the Calculator

In the previous example, we had to calculate the 10th root of a number. This is different from taking the basic square root, $\sqrt{}$. Many scientific calculators have a button for general roots. It is typically labeled like

$$\sqrt[n]{}, \sqrt[x]{}, \text{ or } \sqrt[y]{x}$$

To evaluate the 3rd root of 8, for example, we'd either type $3\sqrt[x]{}8$, or $8\sqrt[x]{}3$, depending on the calculator. Try it on yours to see which to use – you should get an answer of 2. If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that $\sqrt[n]{a} = a^1/n$. So, to compute the 3rd root of 8, you could use your calculator's exponent key to evaluate $8^{1/3}$. To do this, type: $8y^x(1 \div 3)$ The parentheses tell the calculator to divide 1/3 before doing the exponent.

Try it Now 3

The number of users on a social networking site was 45 thousand in February when they officially went public and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should they

expect two years after they went public?

Example 13.1.8

Looking back at the last example, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Solution

Again we will get $n = 0$ correspond with 1990, giving $P_0 = 962$. To find d, we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When $n = 10$, the explicit linear equation looks like:

$$P_{10} = P_0 + 10d$$

We know the value for P_0, so we can put that into the equation:

$$P_{10} = 962 + 10d$$

Since we know that $P_{10} = 1182$, substituting that in we get

$$1182 = 962 + 10d$$

We can now solve this equation for the common difference, d.

$$1182 - 962 = 10d$$
$$220 = 10d$$
$$d = 22$$

This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$P_{60} = 962 + 22(60) = 2282$ million metric tons.

You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.

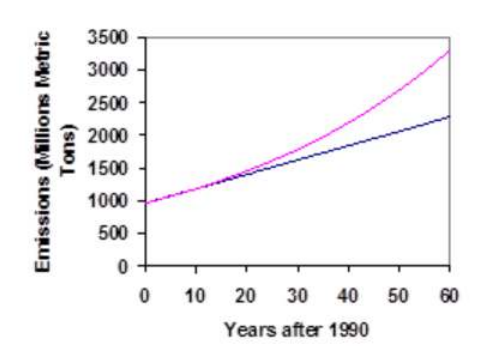

Figure 13.1.5: Predicting carbon emissions growth

So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

1. Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?
2. Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

Reference

1. References (13)

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13.2: Logarithmic Scales in Natural Sciences

Figure 13.2.1: Image by Daniel Pierce is licensed by CC 2.0

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[1]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[2] like those shown in the image above. Even though both caused substantial damage, the earthquake in Japan was 100 times as strong as the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale[3] whereas the Japanese earthquake registered a 9.0.[4]

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is 108-4=104=10,000 times as great! In this section, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ increasing. We can examine a graph, below, to better estimate the solution.

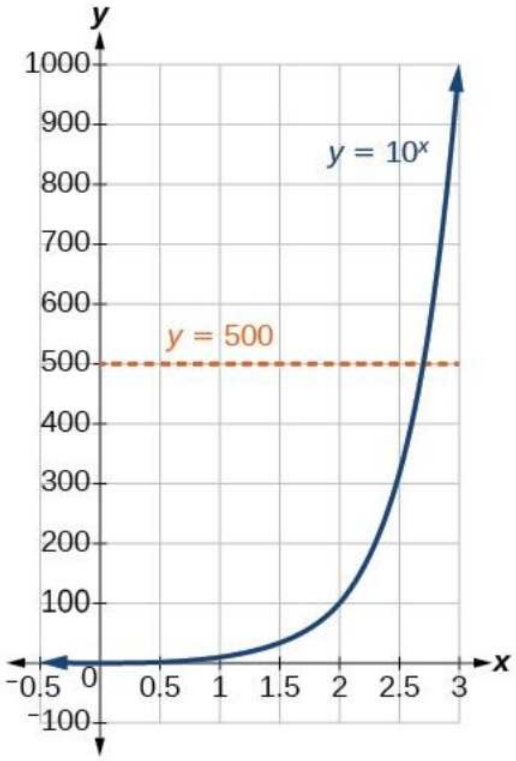

Figure 13.2.2: Graph to better estimate the solution.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Definition: Common Logarithm

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$

$y = \log(x)$ is equivalent to $10^y = x$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

How to

Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of 10 : $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get y?

Example 13.2.1: Finding value of a common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$
$$\therefore \log(1000) = 3$$

Try it Now 4

Evaluate $y = \log(1,000,000)$

How to

Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator.

1. Press [LOG].
2. Enter the value given for x followed by [ )].
3. Press [ENTER].

Example 13.2.2: Finding value of a common Logarithm using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$ This gives us the following:

$$100 < 321 < 1000$$
$$2 < 205065 < 3$$

Try it Now 5

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Example 13.2.3

The 1994 Northridge Earthquake in Southern California registered a 6.7 on the Richter scale. Exactly a year later, the 1995 Kobe Earthquake in Japan registered a 6.9. Compare the amount of energy released from these two quakes.

Solution

$$10^{6.9-6.7} = 10^{0.2} = 1.585$$

The Kobe Earthquake released about 1.56 times the energy (about 56% more) released by the Northridge quake.

Example 13.2.4: Rewriting and solving a real-world exponential model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$
$$\log(500) = x \quad \text{Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$

The difference in magnitudes was about 2.699. This means that the bigger earthquake has the magnitude about 2.7 more than the magnitude of the smaller earthquake.

Try it Now 6

The amount of energy released from one earthquake was 8,500 times as great as the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Reference

1. References (14)

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13.3: The pH Scale

You may have heard of pH levels in chemistry. pH is a measure of how acidic or basic a solution is. The pH scale is actually a logarithmic scale, similar to the Richter scale studied in the previous section. In chemistry, the level of acidity depends on the molar concentration of hydrogen ions in a given solution. The more hydrogen ions there are, the more acidic the solution is. But the concentrations are really, really small fractions, like 1/10,000,000 for pure water. Because this denominator ranges drastically, it is convenient to use a logarithmic scale.

pH is a logarithmic function of $[H^+]$, where $[H^+]$ is the concentration of the hydrogen ion $[H^+]$.

$$pH = -\log[H^+] \tag{13.3.1}$$

pH is usually between 0 and 14. Knowing the dependence of pH on $[\ H^+\ ]$, we can summarize as follows:

- If pH < 7, then the solution is acidic.
- If pH = 7, then the solution is neutral.
- If pH > 7, then the solution is basic.

This is known as the pH scale. You can use pH to make a quick determination of whether a given water based solution is acidic, basic, or neutral.

Example 13.3.1

Label each solution as acidic, basic, or neutral based only on the stated pH.

a. milk of magnesia, pH = 10.5
b. pure water, pH = 7
c. wine, pH = 3.0

Solution

a. With a pH greater than 7, milk of magnesia is basic. (Milk of magnesia is largely $Mg(OH)_2$.)
b. Pure water, with a pH of 7, is neutral.
c. With a pH of less than 7, wine is acidic.

Example 13.3.2

Identify each substance as acidic, basic, or neutral based only on the stated pH.

a. human blood, pH = 7.4
b. household ammonia, pH = 11.0
c. cherries, pH = 3.6

Solution

a. basic
b. basic (very)
c. acidic

The table below gives the typical pH values of some common substances. Note that several food items are on the list, and most of them are acidic.

Table 13.3.1: Typical pH Values of Various Substances*

Substance	pH
stomach acid	1.7
lemon juice	2.2
vinegar	2.9
soda	3.0

Substance	pH
wine	3.5
coffee, black	5.0
milk	6.9
pure water	7.0
blood	7.4
seawater	8.5
milk of magnesia	10.5
ammonia solution	12.5
1.0 M NaOH	14.0
*Actual values may vary depending on conditions.	

pH is a *logarithmic* scale, a solution that has a pH of 1.0 has 10 times the $[H^+]$ as a solution with a pH of 2.0, which in turn has 10 times the $[H^+]$ as a solution with a pH of 3.0 and so forth.

Using the definition of pH, it is also possible to calculate $[H^+]$ from pH and vice versa. The general formula for determining $[H^+]$ from pH is as follows:

$$\left[H^+\right] = 10^{-pH}$$

You need to determine how to evaluate the above expression on your calculator. Ask your instructor if you have any questions. The other issue that concerns us here is significant figures. Because the number(s) before the decimal point in a logarithm relate to the power on 10, the number of digits *after* the decimal point is what determines the number of significant figures in the final answer.

Example 13.3.2

What is $[H^+]$ for an acid solution whose pH is 4.88?

Solution

We need to evaluate the expression

$$\left[H^+\right] = 10^{-4.88}$$

Depending on the calculator you use, the method for solving this problem will vary. In some cases, the "−4.88" is entered and a "10^x" key is pressed; for other calculators, the sequence of keystrokes is reversed. In any case, the correct numerical answer is as follows:

$$\left[H^+\right] = 10^{-4.88} = 0.0001318\ldots \approx 1.32 \times 10^{-5}$$

Exercises

1. Marko currently has 20 tulips in his yard. Each year he plants 5 more.
   a. Write a recursive formula for the number of tulips Marko has
   b. Write an explicit formula for the number of tulips Marko has
2. Pam is a Disc Jockey. Every week she buys 3 new albums to keep her collection current. She currently owns 450 albums.
   1. Write a recursive formula for the number of albums Pam has
   2. Write an explicit formula for the number of albums Pam has
3. A store's sales (in thousands of dollars) grow according to the recursive rule $P_n = P_{n-1} + 15$, with initial population $P_0 = 40$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n

c. Use your formula to predict the store's sales in 10 years
d. When will the store's sales exceed $100,000?

4. The number of houses in a town has been growing according to the recursive rule $P_n = P_{n-1} + 30$, with initial population $P_0 = 200$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n
   c. Use your formula to predict the number of houses in 10 years
   d. When will the number of houses reach 400 houses?
5. A population of beetles is growing according to a linear growth model. The initial population (week 0) was $P_0 = 3$, and the population after 8 weeks is $P_8 = 67$.
   a. Find an explicit formula for the beetle population in week n.
   b. After how many weeks will the beetle population reach 187?
6. The number of streetlights in a town is growing linearly. Four months ago ($n = 0$) there were 130 lights. Now ($n = 4$) there are 146 lights. If this trend continues,
   a. Find an explicit formula for the number of lights in month n.
   b. How many months will it take to reach 200 lights?
7. Tacoma's population in 2000 was about 200 thousand, and had been growing by about 9% each year.
   a. Write a recursive formula for the population of Tacoma
   b. Write an explicit formula for the population of Tacoma
   c. If this trend continues, what will Tacoma's population be in 2016?
   d. When does this model predict Tacoma's population to exceed 400 thousand?
8. Portland's population in 2007 was about 568 thousand, and had been growing by about 1.1% each year.
   a. Write a recursive formula for the population of Portland
   b. Write an explicit formula for the population of Portland
   c. If this trend continues, what will Portland's population be in 2016?
   d. If this trend continues, when will Portland's population reach 700 thousand?
9. Diseases tend to spread according to the exponential growth model. In the early days of AIDS, the growth rate was around 190%. In 1983, about 1700 people in the U.S. died of AIDS. If the trend had continued unchecked, how many people would have died from AIDS in 2005?
10. The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2015.
11. A bacteria culture is started with 300 bacteria. After 4 hours, the population has grown to 500 bacteria. If the population grows exponentially,
    a. Write a recursive formula for the number of bacteria
    b. Write an explicit formula for the number of bacteria
    c. If this trend continues, how many bacteria will there be in 1 day?
    d. How long does it take for the culture to triple in size?
12. A native wolf species has been reintroduced into a national forest. Originally 200 wolves were transplanted. After 3 years, the population had grown to 270 wolves. If the population grows exponentially,
    a. Write a recursive formula for the number of wolves
    b. Write an explicit formula for the number of wolves
    c. If this trend continues, how many wolves will there be in 10 years?
    d. If this trend continues, how long will it take the population to grow to 1000 wolves?
13. One hundred trout are seeded into a lake. Absent constraint, their population will grow by 70% a year. The lake can sustain a maximum of 2000 trout. Using the logistic growth model,
    a. Write a recursive formula for the number of trout
    b. Calculate the number of trout after 1 year and after 2 years.
14. Ten blackberry plants started growing in my yard. Absent constraint, blackberries will spread by 200% a month. My yard can only sustain about 50 plants. Using the logistic growth model,

a. Write a recursive formula for the number of blackberry plants in my yard
b. Calculate the number of plants after 1, 2, and 3 months

15. In 1968, the U.S. minimum wage was $1.60 per hour. In 1976, the minimum wage was $2.30 per hour. Assume the minimum wage grows according to an exponential model where n represents the time in years after 1960.
    a. Find an explicit formula for the minimum wage.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was $5.15 in 1996, is this above, below or equal to what the model predicts?
16. The population of a small town can be described by the equation $P_n = 4000 + 70n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.
17. The population of a small town can be described by the equation $P_n = 4000(1.04)^n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.

Most of the examples in the text examined growing quantities, but linear and exponential equations can also describe decreasing quantities, as the next few problems will explore.

18. A new truck costs $32,000. The car's value will depreciate over time, which means it will lose value. For tax purposes, depreciation is usually calculated linearly. If the truck is worth $24,500 after three years, write an explicit formula for the value of the car after n years.
19. Inflation causes things to cost more, and for our money to buy less (hence your grandparents saying, "In my day, you could buy a cup of coffee for a nickel"). Suppose inflation decreases the value of money by 5% each year. In other words, if you have $1 this year, next year it will only buy you $0.95 worth of stuff. How much will $100 buy you in 20 years?
20. Suppose that you have a bowl of 500 M&M candies, and each day you eat ¼ of the candies you have. Is the number of candies left changing linearly or exponentially? Write an equation to model the number of candies left after n days.
21. A warm object in a cooler room will decrease in temperature exponentially, approaching the room temperature according to the formula $T_n = a(1-r)^n + T_r$ where T_n is the temperature after n minutes, r is the rate at which temperature is changing, a is a constant, and T_r is the temperature of the room. Forensic investigators can use this to predict the time of death of a homicide victim. Suppose that when a body was discovered ($n = 0$) it was 85 degrees. After 20 minutes, the temperature was measured again to be 80 degrees. The body was in a 70 degree room.
    a. Use the given information with the formula provided to find a formula for the temperature of the body.
    b. When did the victim die, if the body started at 98.6 degrees?
22. Recursive equations can be very handy for modeling complicated situations for which explicit equations would be hard to interpret. As an example, consider a lake in which 2000 fish currently reside. The fish population grows by 10% each year, but every year 100 fish are harvested from the lake by people fishing.
    a. Write a recursive equation for the number of fish in the lake after n years.
    b. Calculate the population after 1 and 2 years. Does the population appear to be increasing or decreasing?
    c. What is the maximum number of fish that could be harvested each year without causing the fish population to decrease in the long run?
23. The number of Starbucks stores grew after first opened. The number of stores from 1990-2007, as reported on their corporate website,[2] is shown below.
    a. Carefully plot the data. Does is appear to be changing linearly or exponentially?
    b. Try finding an equation to model the data by picking two points to work from. How well does the equation model the data?
    c. Try using an equation of the form $P_n = P_0 n^k$, where k is a constant, to model the data. This type of model is called a Power model. Compare your results to the results from part b. *Note: to use this model, you will need to have 1990 correspond with $n = 1$ rather than $n = 0$.*

Table 13.3.2: Starbucks stores 1997-2007

Year	Number of Starbucks stores
1990	84
1991	116
1992	165

Year	Number of Starbucks stores
1993	272
1994	425
1995	677
1996	1015
1997	1412
1998	1886
1999	2498
2000	3501
2001	4709
2002	5886
2003	7225
2004	8569
2005	10241
2006	12440
2007	15756

24. Thomas Malthus was an economist who put forth the principle that population grows based on an exponential growth model, while food and resources grow based on a linear growth model. Based on this, Malthus predicted that eventually demand for food and resources would out outgrow supply, with doom-and-gloom consequences. Do some research about Malthus to answer these questions.
    a. What societal changes did Malthus propose to avoid the doom-and-gloom outcome he was predicting?
    b. Why do you think his predictions did not occur?
    c. What are the similarities and differences between Malthus's theory and the logistic growth model?
25. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$
26. How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$ What is the result of composing these two functions?
    a. How can the logarithmic equation $\log_b x = y$ be solved using the properties of exponents?
27. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?
    a. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

For the following exercises, rewrite each equation in exponential form.

28. $\log_4(q) = m$
    a. $\log_a(b) = c$
29. $\log_{16}(y) = x$
    a. $\log_x(64) = y$
30. $\log_{16}(y) = x$
    a. $\log_x(64) = y$
31. $\log_y(x) = -11$
    a. $\log_{15}(a) = b$

32. $\log_y(137) = x$
    a. $\log_{13}(142) = a$
33. $\log(v) = t$
    a. $\ln(w) = n$

For the following exercises, rewrite each equation in logarithmic form.

34. The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?
    a. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?
35. Describe, using your own words, what happens to the energy of an earthquake each time the magnitude on the Richter scale increases by 1.
36. Remember that $\log(M)$ is the exponent so that 10 to that exponent is M. Given this fact, find
    a. $\log(1000)$
    b. $\log(10)$
    c. $\log(10,000,000)$
37. As mentioned in the text, the 1994 Northridge Earthquake registered a 6.7 on the Richter scale. In July, 2019, there was a major earthquake in Ridgecrest, California, with a magnitude of 7.1. How much bigger was the Ridgecrest quake compared to the Northridge Earthquake 25 years before (in terms of the energy released)? Use a calculator.
38. The pH level of lemon juice is 2.2 while the pH level of a typical wine is 3.5. They are both acidic, but one is much more acid than the other.
    a. Which one is more acidic?
    b. Compare the concentration of hydrogen ions between these two liquids. (How many times as much?) Hint: use the same method applied in the previous section to compare two earthquakes. The mathematics is the same.
39. Many orange farmers pay attention to the pH level of their oranges. If an orange produces a juice with a pH level of 4.0, how much is the concentration of hydrogen ions in that juice compared a lemon juice with a pH level of 2.2?

Reference

1. References (15)
2. www.starbucks.com/aboutus/Company_Timeline.pdf retrieved May 2009

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