MATH 4A: MULTIVARIABLE CALCULUS



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Multivariable Calculus (IVC Math 4A)

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Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Multivariable Precalculus

Calculus in multiple variables requires the ability to draw and manipulate points and vectors on the 3D axes as well as recognize commonly used curves and surfaces in $\mathbb{R}^3$.

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1.1: The Third Dimension

Learning Objectives

- Plot points in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.
- Recognize axes, octants, and coordinate planes in three dimensional space.
- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Calculate distance in two and three dimensions.
- Solve for equations of relationships based on distances.
- Graph circles, ellipses, and hyperbolas in two dimensions.
- Write the equations for simple planes and spheres.
- Describe a plane vector, using correct notation.
- Perform basic vector operations (scalar multiplication, addition, subtraction).
- Express a vector in component form.
- Explain the formula for the magnitude of a vector.
- Express a vector in terms of unit vectors.

The study of multivariable calculus requires visualization in more than two dimensions. While students are not expected to be artistically impressive to solve questions, they will still need to be able to at least mentally see many complicated pictures beyond what we see in the Cartesian plane. It is also important that they are comfortable with the popular two-dimensional relationships in order to generalize them into three dimensional surfaces and solids.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed. When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities (for example force) are defined in terms of both size (also called magnitude) and direction. A quantity that has magnitude and direction is called a **vector.**

In textbooks, vectors are often denoted by boldface letters, such as **v** . Since it is difficult to write in clear boldface when we write vectors by hand, we will also include an arrow or harpoon above the letter representing the vector. To make the vectors clearer in this textbook (and to reinforce the way you should write them by hand), we will generally use arrows or harpoons above boldface (or italic) letters to represent vectors, giving us $\vec{v}$ or $\vec{\mathrm{v}}$. Note that some figures will still only use boldface letters to denote vectors.

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1.1: The 3D Axes

Mathematics and calculus involve the study of relationships between numbers. Initially, we solve equations involving one variable, and visualize the solutions on the classic **number line**. Solutions to equations or inequalities involving only the variable x are visualized on this number line, commonly referred to as $\mathbb{R}$:

Example 1.1.1

Find and visualize all solutions to the following:

(a) $3x + 4 = -11$

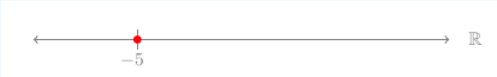
(b) $x^2 \leq 1$

(c) $|2x + 1| > 3$

Solution

(a) We solve for x here easily: $3x + 4 = -11 \longrightarrow 3x = -15 \longrightarrow \boxed{x = -5}$

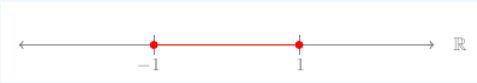
This represents one specific point on the number line:



(b) Since this is quadratic, we typically compare to zero: $x^2 \leq 1 \longrightarrow x^2 - 1 \leq 0 \longrightarrow (x - 1)(x + 1) \leq 0$

This is satisfied when $x - 1 \leq 0$ and $x + 1 \geq 0$, (no other combinations will work, since the product is otherwise positive).

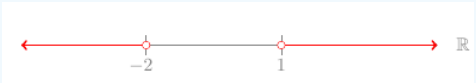
This is the closed interval $\boxed{[-1, 1]}$ in the number line:



(c) For absolute values, we know $|2x + 1| > 3 \longleftrightarrow 2x + 1 > 3$ or $-(2x + 1) > 3$

These give two separate possible solution sets: $x > 1$ or $x < -2$.

This gives a solution set of $\boxed{(-\infty, -2) \cup (1, \infty)}$ which we can visualize as:



Once we master these algebraic topics, we move on to studying how two variables can be related. By convention, we often refer to them as x and y and in order to visualize these relationships between these variables we use the **Cartesian plane**, denoted $\mathbb{R}^2$. In the study of functions, this leads to graphs of the form $y = f(x)$.

Example 1.1.2

Graph the relations below in the Cartesian plane:

(a) $y = x^2$

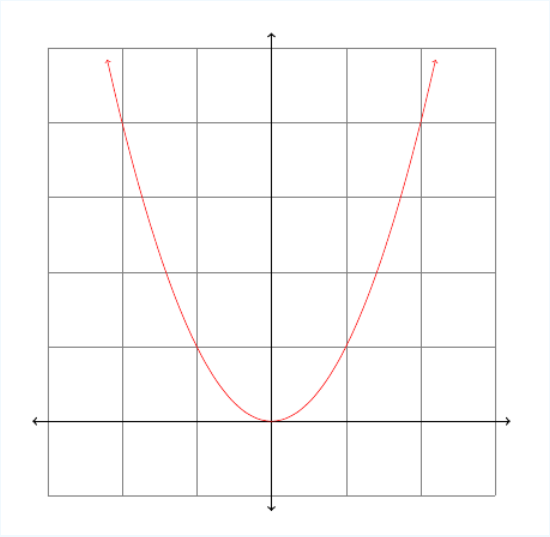
(b) $(x - 1)^2 + (y + 2)^2 = 4$

(c) $2x + y = 4$

Solution

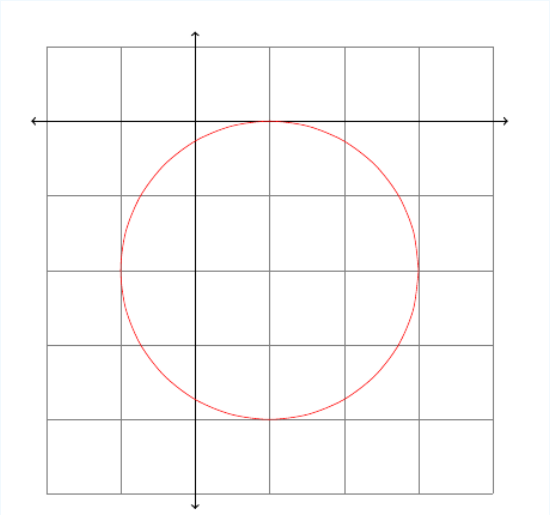
(a) The function $f(x) = x^2$ is famously quadratic, and has a graph which is a parabola with vertex at $(0, 0)$.

The collection of all of the infinitely many points satisfying $y = x^2$ is:



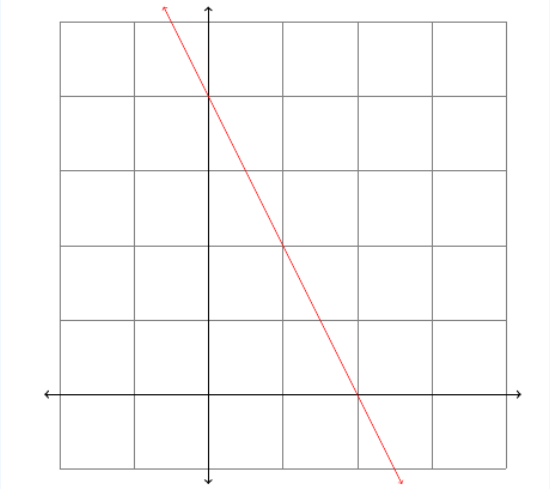
(b) This relationship is not a function, but is still a famous one. This represents a circle with center $(1, -2)$ and radius 2.

Again, we can graph the points satisfying this relationship on the xy-plane:



(c) Note that $2x + y = 4 \longleftrightarrow y = -2x + 4$ is a typical linear relationship.

The graph of all points satisfying it forms a line in the plane:



Multivariable calculus will often involve the relationship between three variables. As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal x-axis and the vertical y-axis. We can add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition: Three-dimensional Rectangular Coordinate System

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In the figure below, the positive z-axis is shown above the plane containing the x- and y-axes. The positive x-axis appears to the left and the positive y-axis is to the right. A natural question to ask is: How was this arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis. In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard convention.

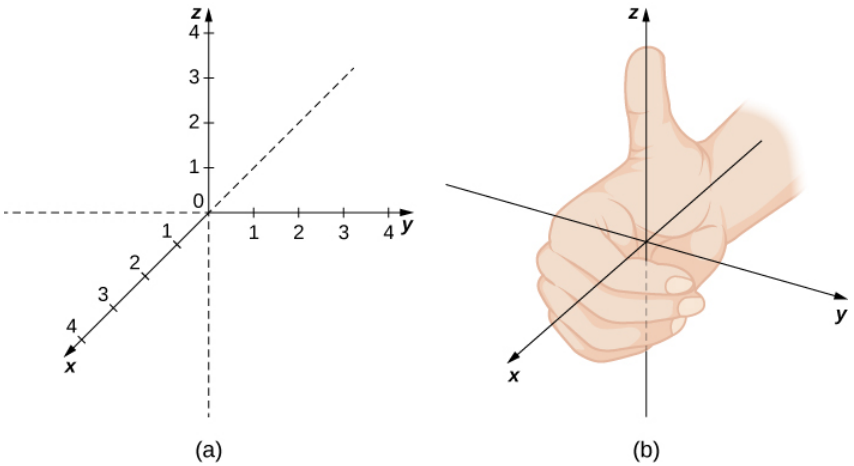


Figure 1.1.1: (a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the z-axis, that is perpendicular to both the x-axis and the y-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z). A point

in space is identified by all three coordinates. To plot the point (x, y, z), move x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

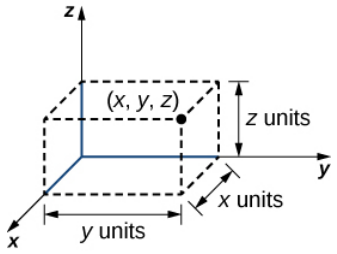


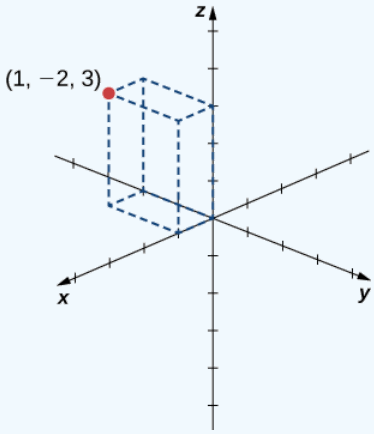
Figure 1.1.2

Example 1.1.3: Locating Points in Space

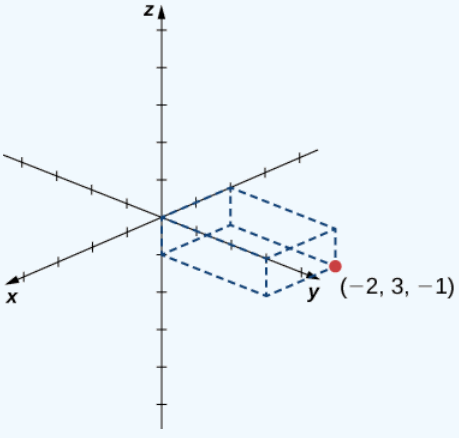
Sketch the points (a) $(1, -2, 3)$ and (b) $(-2, 3, -1)$ in three-dimensional space.

Solution

(a) To sketch this point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point:



(b) To sketch this point, start by sketching three sides of a rectangular prism along the coordinate axes: 2 units in the negative x direction, 3 units in the positive y direction, and one unit in the negative z direction. Complete the prism to plot the point:



In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the xy-plane, the

xz-plane, and the yz-plane. We define the xy-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the xz-plane and the yz-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the xy-plane, the wall to your right is the xz-plane, and the wall to your left is the yz-plane.

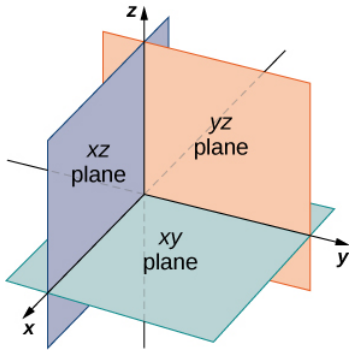


Figure 1.1.3: The plane containing the x- *and* y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants.** The octants fill $\mathbb{R}^3$ in the same way that quadrants fill $\mathbb{R}^2$:

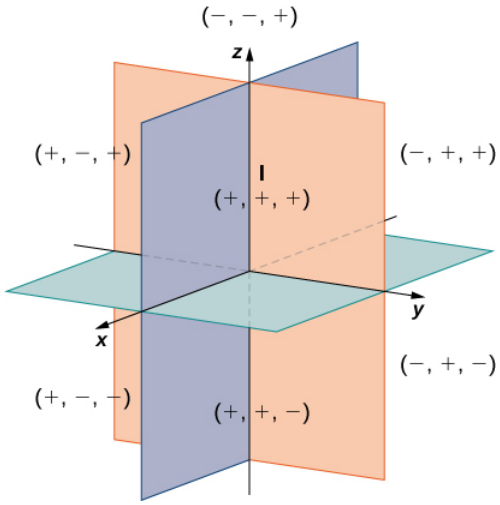


Figure 1.1.4: Points that lie in octants have three nonzero coordinates.

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In the next subsection, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We know that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The formula for the distance between two points in space is a natural extension of this formula.

The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The proof of this theorem follows from two applications of the Pythagorean Theorem, as shown below:

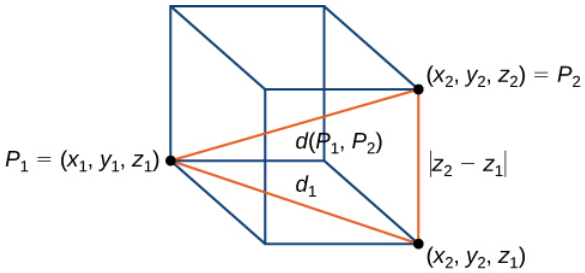
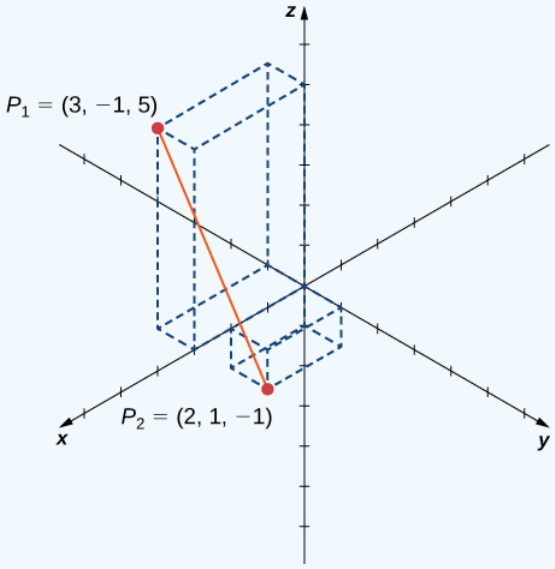


Figure 1.1.5: The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

Example 1.1.4: Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.



Solution

Substitute values directly into the distance formula:

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} \\ &= \boxed{\sqrt{41}} \end{aligned}$$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$,and $\mathbb{R}^3$. From these graphs, we can see the same equation can describe a point, a line, or a plane.

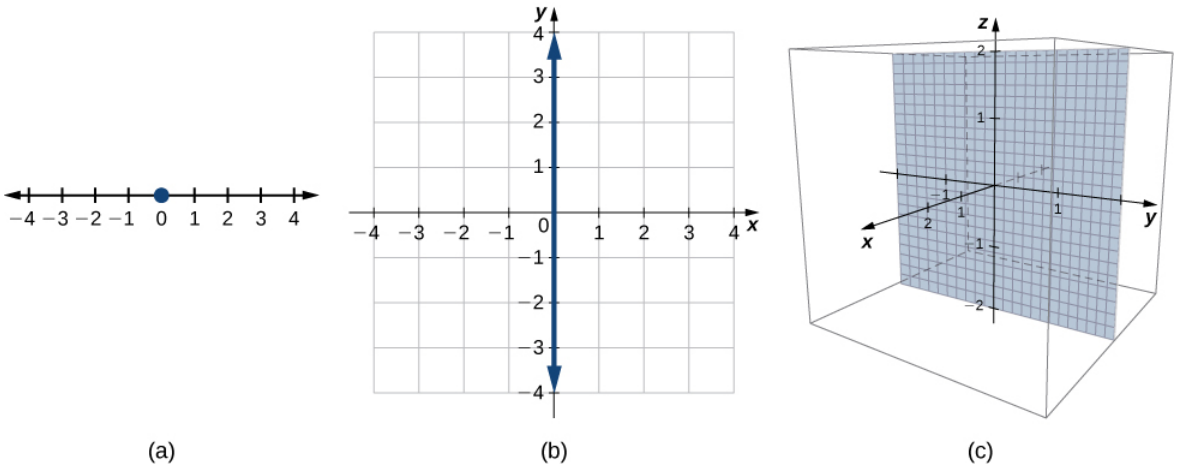


Figure 1.1.6: (a) In $\mathbb{R}$, the equation $x = 0$ describes a single point.
(b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the y-axis.
(c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the yz-plane.

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the yz-plane. Similarly, the xy-plane contains all points of the form $(x, y, 0)$. The equation $z = 0$ defines the xy-plane and the equation $y = 0$ describes the xz-plane.

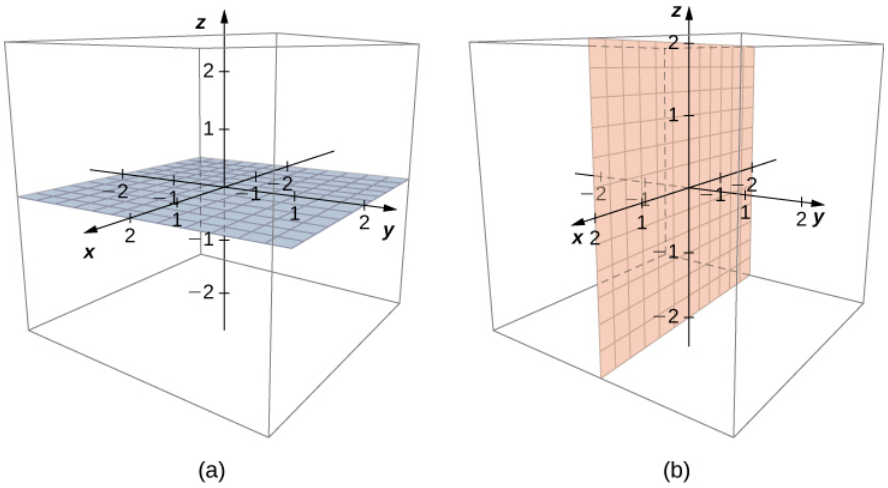


Figure 1.1.7: (a) $z = 0$ and (b) $y = 0$

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the xy-plane, for example, the z-coordinate of each point in the plane has the same constant value. Only the x- and y-coordinates of points in that plane vary from point to point.

Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the xy-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the xz-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the yz-plane and contains point (a, b, c) can be represented by the equation $x = a$.

As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the y-axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the yz-plane.

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1.1: Distance Based Objects

The circle in two dimensions is defined as a **locus**: a collection of points that follow a particular rule. The circle $(x-h)^2+(y-k)^2=r^2$ is the collection of points which are a distance of exactly r away from the center (h,k). A natural extension of this type of equation is found in the equation of a sphere.

Definition: Sphere

A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere, just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius*.

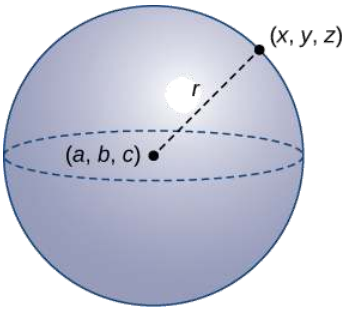


Figure 1.1.8: Each point (x,y,z) on the surface of a sphere is r units away from the center (a,b,c).

The equation of a sphere is therefore based on the three-dimensional formula for distance:

Standard Equation of a Sphere

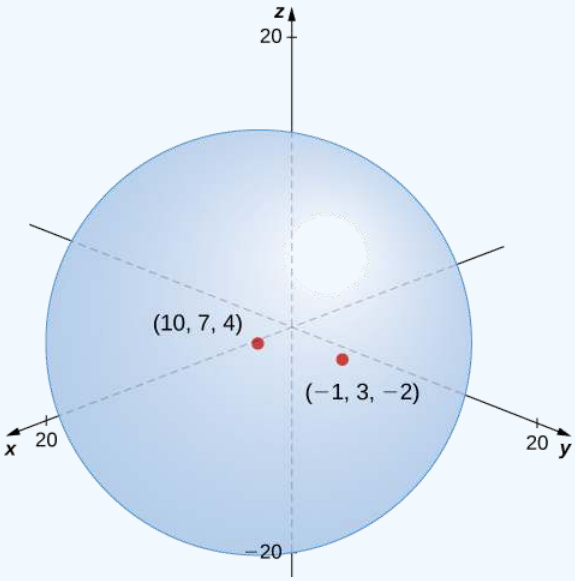
The sphere with center (a,b,c) and radius r can be represented by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

This equation is known as the *standard equation of a sphere*.

Example 1.1.5: Finding an Equation of a Sphere

Find the standard equation of the sphere with center $(10,7,4)$ passing through the point $(-1,3,-2)$.



Solution

Use the distance formula to find the radius r of the sphere:

$$\begin{aligned} r &= \sqrt{(-1-10)^2+(3-7)^2+(-2-4)^2} \\ &= \sqrt{(-11)^2+(-4)^2+(-6)^2} \\ &= \sqrt{173} \end{aligned}$$

The standard equation of the sphere is

$$\boxed{(x-10)^2+(y-7)^2+(z-4)^2=173}$$

Example 1.1.6: Finding the Equation of a Sphere

Let $P=(-5,2,3)$ and $Q=(3,4,-1)$, and suppose line segment $\overline{PQ}$ forms the diameter of a sphere. Find the equation of the sphere.

Solution:

Since $\overline{PQ}$ is a diameter of the sphere, we know the center of the sphere is the midpoint of $\overline{PQ}$. Then,

$$C=\left(\frac{-5+3}{2},\frac{2+4}{2},\frac{3+(-1)}{2}\right)=(-1,3,1)$$

Furthermore, we know the radius of the sphere is half the length of the diameter. This gives

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-5-3)^2+(2-4)^2+(3-(-1))^2} \\ &= \frac{1}{2}\sqrt{64+4+16} \\ &= \sqrt{21} \end{aligned}$$

Then, the equation of the sphere is $\boxed{(x+1)^2+(y-3)^2+(z-1)^2=21}$

Instead of finding the set of points equidistant from one point, what happens if we want the collection of points which are equidistant from *two* points? We start our investigation in $\mathbb{R}^2$.

Example 1.1.7: Equidistance from two points in $\mathbb{R}^2$

Find the relationship which defines the points which are equidistant from the points $(0,1)$ and $(4,3)$.

Solution:

We begin by finding the distance from a general point (x,y) to each of $P_1=(0,1)$ and $P_2=(4,3)$:

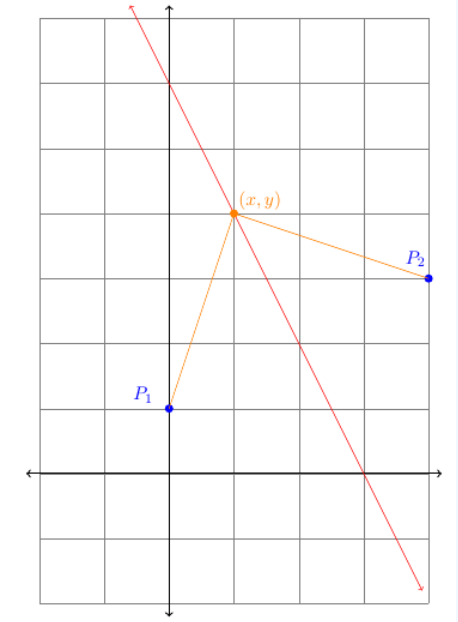
$$d_1=\sqrt{(x-0)^2+(y-1)^2}=\sqrt{x^2+y^2-2y+1}$$

$$d_2=\sqrt{(x-4)^2+(y-3)^2}=\sqrt{x^2-8x+y^2-6y+25}$$

Now we set these two distances equal to each other and solve:

$$\begin{aligned} d_1=d_2 &\longrightarrow \sqrt{x^2+y^2-2y+1}=\sqrt{x^2-8x+y^2-6y+25} \\ &\longrightarrow x^2+y^2-2y+1=x^2-8x+y^2-6y+25 \\ &\longrightarrow -2y+1=-8x-6y+25 \\ &\longrightarrow 4y=-8x+24 \\ &\longrightarrow y=-2x+6 \end{aligned}$$

Therefore the relationship defined here is the line $\boxed{y=-2x+6}$. This is consistent with the graph below:



Now we try the same type of question in three dimensional space:

Example 1.1.8: Equidistance from two points in $\mathbb{R}^3$

Find the relationship which defines the points which are equidistant from $(0, 2, -1)$ and $(4, 0, 3)$.

Solution:

We begin by finding the distance from a general point (x, y, z) to each of $P_1 = (0, 2, -1)$ and $P_2 = (4, 0, 3)$:

$$d_1 = \sqrt{(x-0)^2 + (y-2)^2 + (z+1)^2} = \sqrt{x^2 + y^2 - 4y + z^2 + 2z + 5}$$

$$d_2 = \sqrt{(x-4)^2 + (y-0)^2 + (z-3)^2} = \sqrt{x^2 - 8x + y^2 + z^2 - 6z + 25}$$

Now we set these two distances equal to each other and solve:

$$\begin{aligned} d_1 = d_2 &\longrightarrow \sqrt{x^2 + y^2 - 4y + z^2 + 2z + 5} = \sqrt{x^2 - 8x + y^2 + z^2 - 6z + 25} \\ &\longrightarrow x^2 + y^2 - 4y + z^2 + 2z + 5 = x^2 - 8x + y^2 + z^2 - 6z + 25 \\ &\longrightarrow -4y + 2z + 5 = -8x - 6z + 25 \\ &\longrightarrow 8x - 4y + 8z = 20 \\ &\longrightarrow \boxed{2x - y + 2z = 5} \end{aligned}$$

This relationship defines a plane in $\mathbb{R}^3$, which we will discuss later in this chapter.

Many famous shapes in the Cartesian plane are defined in terms of distances. For example, we know an ellipse is the collection of points for which the *sum of the distances* from (x, y) to two separate points is always constant.

Illustration 1.1.1: Sums of distances from two points in $\mathbb{R}^2$

We will find the relationship which defines the points for which the sum of the distances to $(0, c)$ and $(0, -c)$ is always $2D$ (where $D > c$).

We begin by finding the distance from a general point (x, y) to each of $P_1 = (0, c)$ and $P_2 = (0, -c)$:

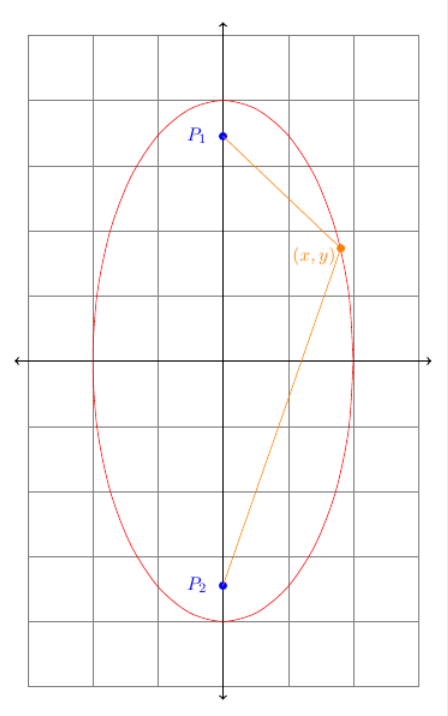
$$d_1 = \sqrt{(x-0)^2 + (y-c)^2} = \sqrt{x^2 + y^2 - 2cy + c^2}$$

$$d_2 = \sqrt{(x-0)^2 + (y+c)^2} = \sqrt{x^2 + y^2 + 2cy + c^2}$$

Now we set the sum of these distances equal to $2D$ and solve:

$$\begin{aligned}
d_1 + d_2 = 2D &\longrightarrow \sqrt{x^2 + y^2 - 2cy + c^2} + \sqrt{x^2 + y^2 + 2cy + c^2} = 2D \\
&\longrightarrow \left(\sqrt{x^2 + y^2 - 2cy + c^2} + \sqrt{x^2 + y^2 + 2cy + c^2}\right)^2 = 4D^2 \\
&\longrightarrow x^2 + y^2 - 2cy + c^2 + 2\sqrt{(x^2 + y^2 - 2cy + c^2)(x^2 + y^2 + 2cy + c^2)} \\
&\quad + x^2 + y^2 + 2cy + c^2 = 4D^2 \\
&\longrightarrow 2x^2 + 2y^2 + 2c^2 - 4D^2 = -2\sqrt{(x^2 + y^2 - 2cy + c^2)(x^2 + y^2 + 2cy + c^2)} \\
&\longrightarrow x^2 + y^2 + c^2 - 2D^2 = -\sqrt{(x^2 + y^2 + c^2 - 2cy)(x^2 + y^2 + c^2 + 2cy)} \\
&\longrightarrow (x^2 + y^2 + c^2 - 2D^2)^2 = \left(-\sqrt{(x^2 + y^2 + c^2 - 2cy)(x^2 + y^2 + c^2 + 2cy)}\right)^2 \\
&\longrightarrow (x^2 + y^2 + c^2)^2 - 4D^2(x^2 + y^2 + c^2) + 4D^4 = (x^2 + y^2 + c^2)^2 - 4c^2y^2 \\
&\longrightarrow D^2(x^2 + y^2 + c^2) - D^4 = c^2y^2 \\
&\longrightarrow D^2x^2 + (D^2 - c^2)y^2 = D^4 - D^2c^2 \\
&\longrightarrow \frac{x^2}{D^2 - c^2} + \frac{y^2}{D^2} = 1
\end{aligned}$$

This matches the standard form of an ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ where $A^2 = D^2 - c^2$ and $B^2 = D^2$.



Example 1.1.9: Graphing ellipses

Sketch the graph of each shape below in $\mathbb{R}^2$.

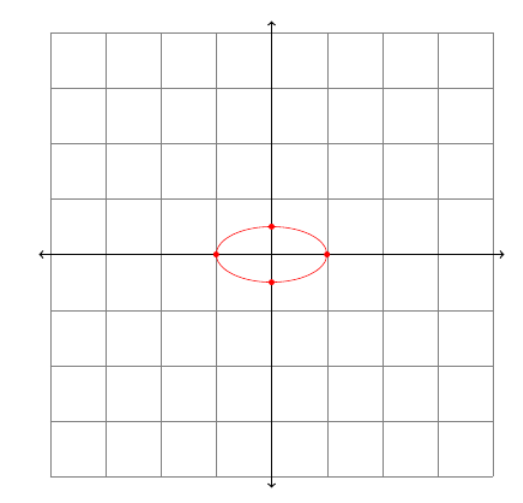
(a) $x^2 + 4y^2 = 1$

(b) $x^2 + 4y^2 = 4$

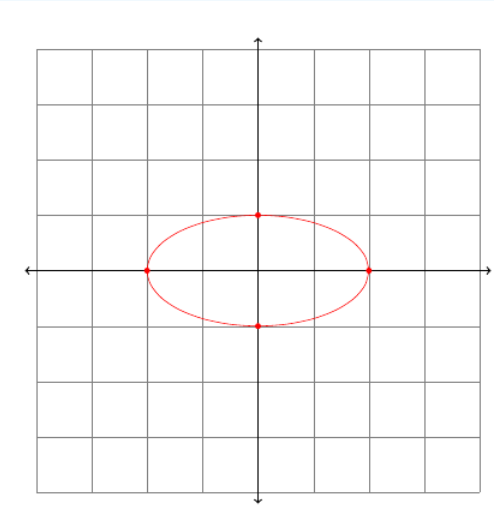
(c) $x^2 + 4y^2 = 16$

Solution:

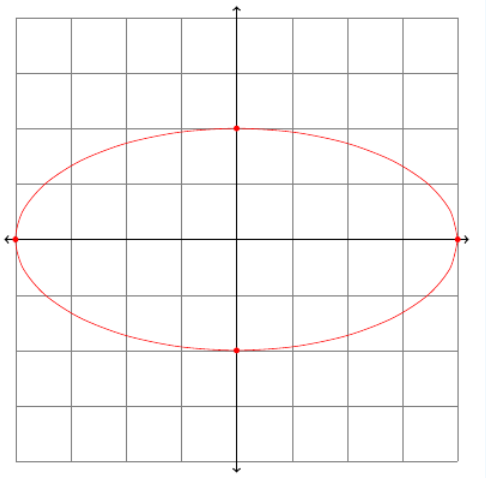
(a) In general form this becomes $\frac{x^2}{1^2}+\frac{y^2}{\left(\frac{1}{2}\right)^2}=1$, so its graph is:



(b) In general form this becomes $\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$, so its graph is:



(c) In general form this becomes $\frac{x^2}{4^2}+\frac{y^2}{2^2}=1$, so its graph is:



Another common conic section is the set of points for which the difference of the distances to two points is constant: $\|d_1 - d_2\| = 2D$.

It is possible (but not pleasant) to show that this leads to the general forms for hyperbola: $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ or $\frac{y^2}{A^2} - \frac{x^2}{B^2} = 1$These shapes will occur frequently in this chapter, so students are expected to be comfortable with their graphs.

Example 1.1.10: Graphing hyperbolas

Sketch the graph of each shape below in $\mathbb{R}^2$.

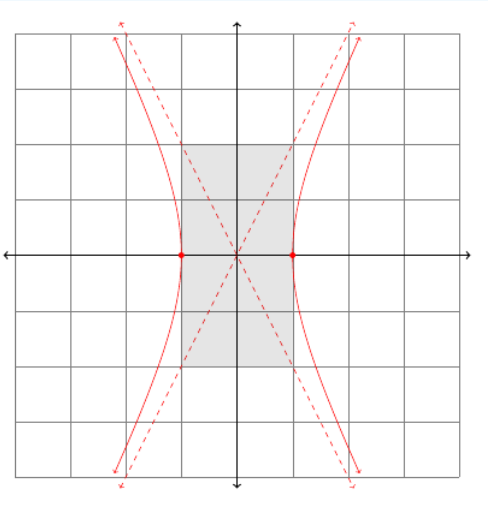
(a) $4x^2 - y^2 = 4$

(b) $4x^2 - y^2 = 1$

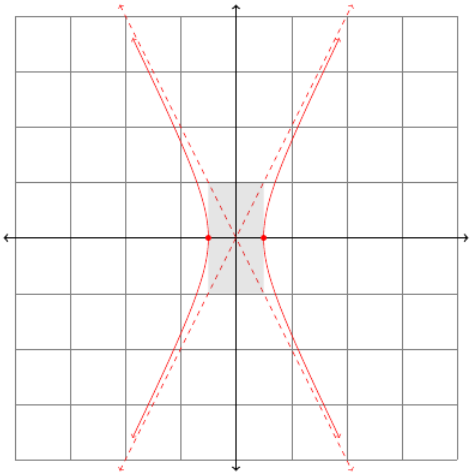
(c) $4x^2 - y^2 = -4$

Solution:

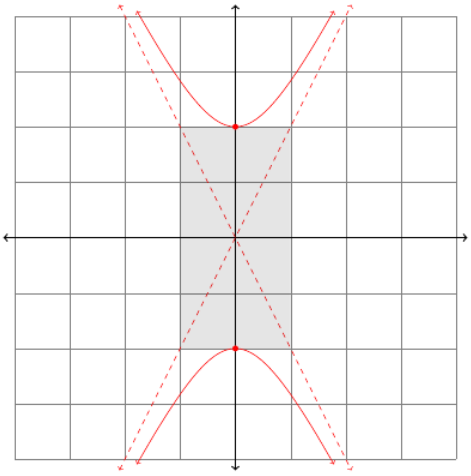
(a) In general form this becomes $\frac{x^2}{1^2} - \frac{y^2}{2^2} = 1$, so its graph is:



(b) In general form this becomes $\frac{x^2}{\left(\frac{1}{2}\right)^2} - \frac{y^2}{1^2} = 1$, so its graph is:



(c) In general form this becomes $\frac{y^2}{2^2} - \frac{x^2}{1^2} = 1$, so its graph is:



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- **12.2: Vectors in Three Dimensions** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

1.1: Vectors in the Plane and Space

A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\vec{v}\|$ to denote the magnitude of the vector $\vec{v}$. A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\vec{0}$. The zero vector is the only vector without a fixed direction, and by convention can be considered to have any direction convenient to the problem at hand.

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\vec{v}$ and $\vec{w}$ are equivalent, we write

$$\vec{v} = \vec{w}$$

Definition: Equivalent Vectors

Vectors are said to be *equivalent vectors* if they have the same magnitude and direction.

The arrows in Figure 1.1.9 are equivalent. Each arrow has the same length and direction. A closely related concept is the idea of parallel vectors. Two vectors are said to be parallel if they have identical or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.

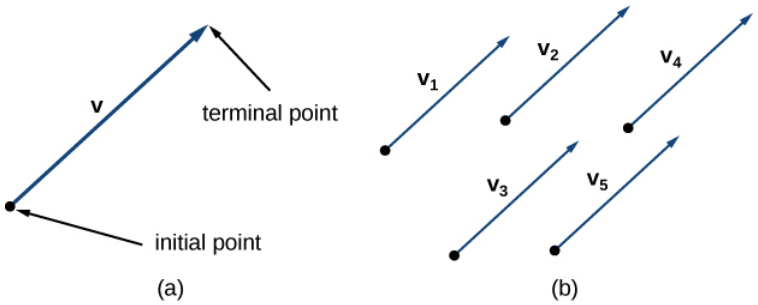


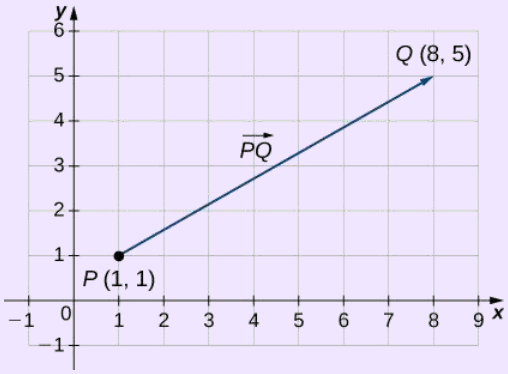
Figure 1.1.9: (a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\vec{v}_1$ through $\vec{v}_5$ are equivalent.

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable than to show it is a vector: $\vec{v}$. When a vector has initial point P and terminal point Q, the notation $\overrightarrow{PQ}$ is useful because it indicates the direction and location of the vector.

Illustration 1.1.2: Sketching Vectors

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

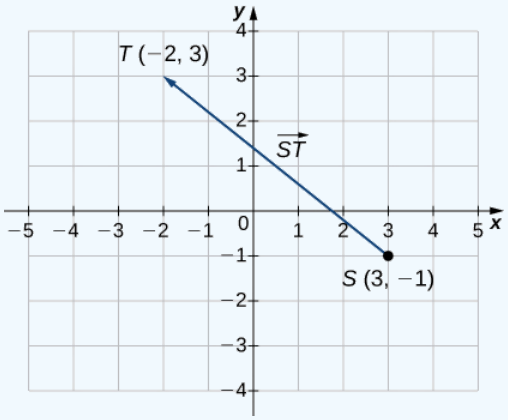
Because the vector goes from point P to point Q, we name it $\overrightarrow{PQ}$.



Example 1.1.11

Sketch the vector $\overrightarrow{ST}$ where S is point $(3,-1)$ and T is point $(-2,3)$.

Solution



Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Definition: Scalar Multiplication

The product $k\vec{\mathrm{v}}$ of a vector $\vec{\mathrm{v}}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\vec{\mathrm{v}}$, and with a direction that is the same as the direction of $\vec{\mathrm{v}}$ if $k>0$, and opposite the direction of $\vec{\mathrm{v}}$ if $k<0$. This is called *scalar multiplication*. If $k=0$ or $\vec{\mathrm{v}}=\vec{0}$, then $k\vec{\mathrm{v}}=\vec{0}$.

As you might expect, if $k=-1$, we denote the product $k\vec{\mathrm{v}}$ as

$$k\vec{\mathrm{v}}=(-1)\vec{\mathrm{v}}=-\vec{\mathrm{v}}$$

Note that $-\vec{\mathrm{v}}$ has the same magnitude as $\vec{\mathrm{v}}$, but has the opposite direction.

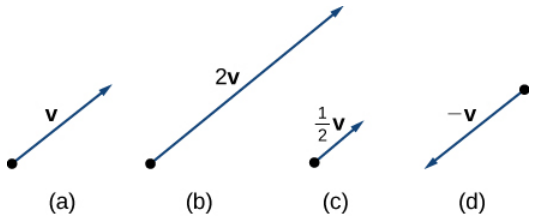


Figure 1.1.10: (a) The original vector $\vec{\mathrm{v}}$ has length n units. (b) The length of $2\vec{\mathrm{v}}$ equals $2n$ units. (c) The length of $\vec{\mathrm{v}}/2$ is $n/2$ units. (d) The vectors $\vec{\mathrm{v}}$ and $-\vec{\mathrm{v}}$ have the same length but opposite directions.

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place

the initial point of the second vector at the terminal point of the first, as in Figure 1.1.11a.

Definition: Vector Addition

The sum of two vectors $\vec{v}$ and $\vec{w}$ can be constructed graphically by placing the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. Then, the vector sum, $\vec{v}+\vec{w}$, is the vector with an initial point that coincides with the initial point of $\vec{v}$ and has a terminal point that coincides with the terminal point of $\vec{w}$. This operation is known as *vector addition*.

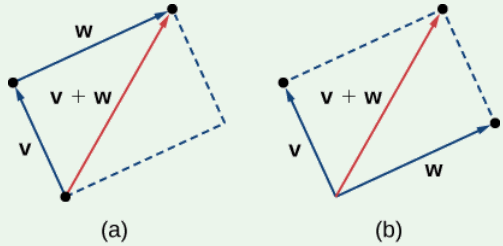


Figure 1.1.11: (a) When adding vectors by the triangle method, the initial point of $\vec{w}$ is the terminal point of $\vec{v}$. (b) When adding vectors by the parallelogram method, the vectors $\vec{v}$ and $\vec{w}$ have the same initial point.

To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\vec{v}$, then from the initial point to the terminal point of vector $\vec{w}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\vec{v}+\vec{w}$. For obvious reasons, this approach is called the triangle method. Notice that if we had switched the order, so that $\vec{w}$ was our first vector and $\vec{v}$ was our second vector, we would have ended up in the same place. Thus,

$$\vec{v}+\vec{w}=\vec{w}+\vec{v}$$

A second method for adding vectors is called the parallelogram method. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in Figure 1.1.11b. The length of the diagonal of the parallelogram is the sum. Comparing Figure 1.1.11b and Figure 1.1.11a, we can see that we get the same answer using either method. The vector $\vec{v}+\vec{w}$ is called the vector sum.

It is also appropriate here to discuss vector subtraction. We define $\vec{v}-\vec{w}$ as $\vec{v}+(-\vec{w})=\vec{v}+(-1)\vec{w}$. The vector $\vec{v}-\vec{w}$ is called the **vector difference**. Graphically, the vector $\vec{v}-\vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$.

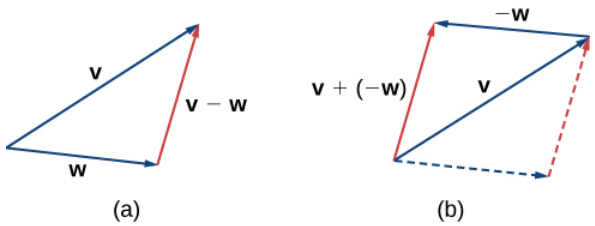


Figure 1.1.12: (a) The vector difference $\vec{v}-\vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$. (b) The vector $\vec{v}-\vec{w}$ is equivalent to the vector $\vec{v}+(-\vec{w})$.

In Figure 1.1.3, the initial point of $\vec{v}+\vec{w}$ is the initial point of $\vec{v}$. The terminal point of $\vec{v}+\vec{w}$ is the terminal point of $\vec{w}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

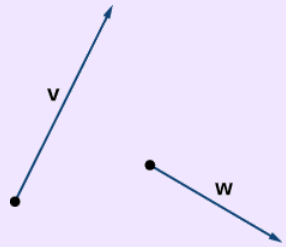
$$\|\vec{v}+\vec{w}\|\leq\|\vec{v}\|+\|\vec{w}\|.$$

This is known more generally as the **triangle inequality**. There is one case, however, when the resultant vector $\vec{v}+\vec{w}$ has the same magnitude as the sum of the magnitudes of $\vec{v}$ and $\vec{w}$. This happens only when $\vec{v}$ and $\vec{w}$ have the same direction.

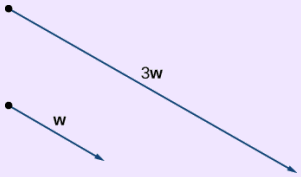
Illustration 1.1.3: Combining Vectors

Given the vectors $\vec{v}$ and $\vec{w}$ shown below, we will sketch the vectors

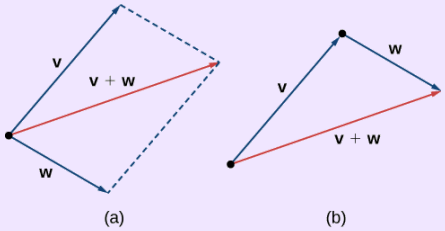
(a) $3\vec{w}$, (b) $\vec{v}+\vec{w}$, (c) $2\vec{v}-\vec{w}$, and (d) $2\vec{w}-\vec{v}$



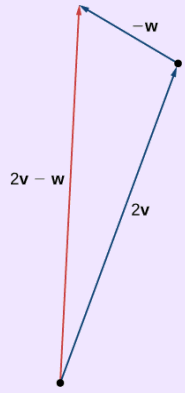
(a) The vector $3\vec{w}$ has the same direction as $\vec{w}$; it is three times as long as $\vec{w}$.



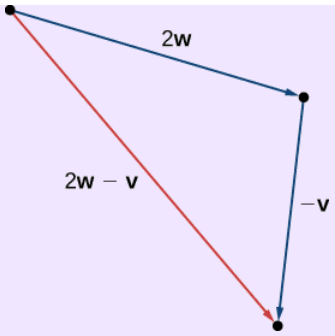
(b) Use either addition method to find $\vec{v}+\vec{w}$.



(c) To find $2\vec{v}-\vec{w}$, we can first rewrite the expression as $2\vec{v}+(-\vec{w})$. Then we can draw the vector $-\vec{w}$, then add it to the vector $2\vec{v}$.



(d) To find $2\vec{w}-\vec{v}$, we can first rewrite the expression as $2\vec{w}+(-\vec{v})$. Then we can draw the vector $-\vec{v}$, then add it to the vector $2\vec{w}$.



Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example 1.1.12: Comparing Vectors

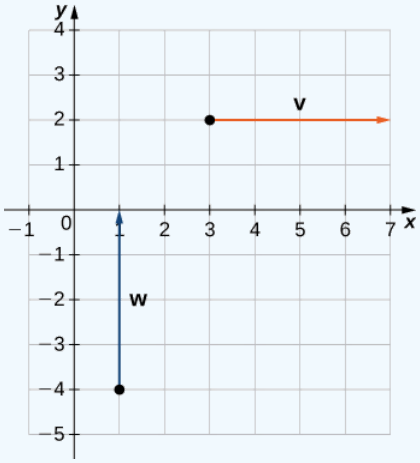
Are $\vec{v}$ and $\vec{w}$ equivalent vectors?

(a) $\vec{v}$ has initial point $(3, 2)$ and terminal point $(7, 2)$ while $\vec{w}$ has initial point $(1, -4)$ and terminal point $(1, 0)$

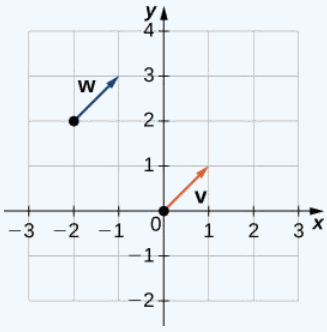
(b) $\vec{v}$ has initial point $(0, 0)$ and terminal point $(1, 1)$ while $\vec{w}$ has initial point $(-2, 2)$ and terminal point $(-1, 3)$

Solution

(a) The vectors are each 4 units long, but they are oriented in different directions. So $\boxed{\vec{v} \text{ and } \vec{w} \text{ are not equivalent}}$

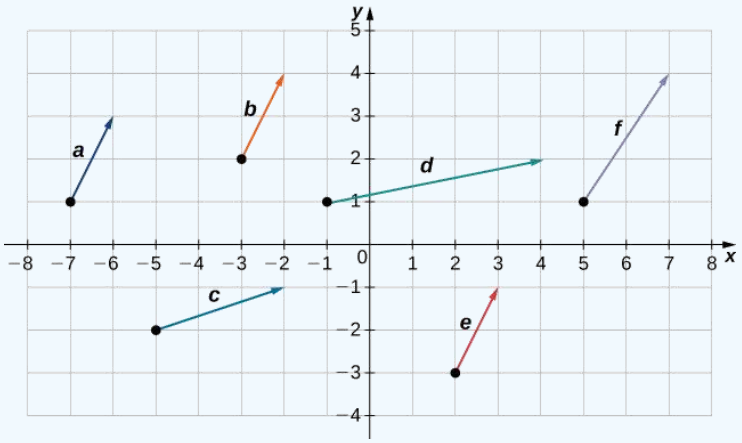


(b) It is clear these vectors have the same length and the same direction, so $\boxed{\vec{v} \text{ and } \vec{w} \text{ are equivalent}}$



Example 1.1.13

Which of the following vectors are equivalent?



Solution

Vectors $\vec{a}$, $\vec{b}$, and $\vec{e}$ are equivalent, but no other pairs match in both direction and length.

We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector.** Because the initial point of any vector in standard position is known to be $(0, 0)$, we can describe the vector by looking at the coordinates of its terminal point.

Definition: Vector components

The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as

$$\vec{v} = \langle x, y \rangle$$

The scalars x (horizontal) and y (vertical) are the **components** of $\vec{v}$.

Recall that vectors are named with lowercase letters in bold type or by drawing an arrow over their name. We have also learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets. However, when writing the component form of a vector, it is important to distinguish between $\langle x, y \rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane. The initial point of $\langle x, y \rangle$ is $(0, 0)$; the terminal point of $\langle x, y \rangle$ is (x, y).

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector. Alternatively, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x-component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y-component.

Component Form of a Vector

Let $\vec{v}$ be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express $\vec{v}$ in component form as $\vec{v} = \langle x_t - x_i, y_t - y_i \rangle$

Illustration 1.1.4: Expressing Vectors in Component Form

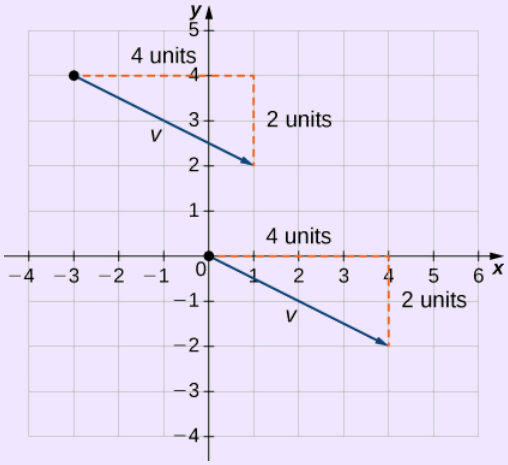
Express vector $\vec{v}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Solution:

This can be done geometrically:

1. Sketch the vector in the coordinate plane.
2. The terminal point is 4 units to the right and 2 units down from the initial point.
3. Find the point that is 4 units to the right and 2 units down from the origin.
4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$:

$$\vec{v} = \langle 4, -2 \rangle$$



Or we can solve for $\vec{v}$ algebraically using the above formula:

$$\vec{v} = \langle x_t - x_i, y_t - y_i \rangle = \langle 1 - (-3), 2 - 4 \rangle = \langle 4, -2 \rangle$$

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The **magnitude** of vector $\vec{v} = \langle x, y \rangle$ is denoted $\|\vec{v}\|$, or $|\vec{v}|$, and can be computed using the formula

$$\|\vec{v}\| = \sqrt{x^2 + y^2}$$

Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x, y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0, 0)$ and terminal point (x, y), we have

$$\|\vec{v}\| = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$$

Based on this formula, it is clear that for any vector $\vec{v}$, $\|\vec{v}\| \geq 0$, and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$.

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.

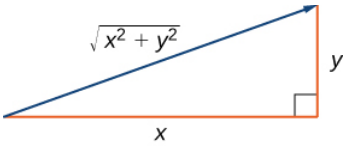


Figure 1.1.13: If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

Rule: Scalar multiplication and Vector addition

Let $\vec{\mathbf{v}} = \langle x_1, y_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.

- *Scalar multiplication:*

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1 \rangle$$

- *Vector addition:*

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$$

Example 1.1.14: Performing Operations in Component Form

Let $\vec{\mathbf{v}}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$, and let $\vec{\mathbf{w}} = \langle -2, 4 \rangle$.

(a) Express $\vec{\mathbf{v}}$ in component form and find $\|\vec{\mathbf{v}}\|$

(b) Find $\vec{\mathbf{v}} + \vec{\mathbf{w}}$

(c) Find $3\vec{\mathbf{v}}$

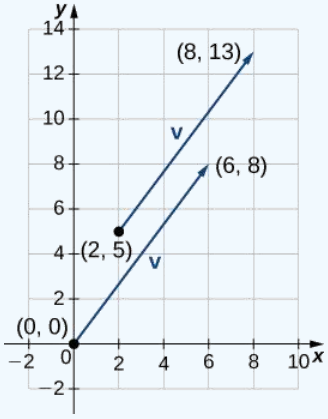
(d) Find $\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$

Solution

(a) To place the initial point of $\vec{\mathbf{v}}$ at the origin, we must translate the vector **2** units to the left and **5** units down.

Using the algebraic method, we can express $\vec{\mathbf{v}}$ as $\vec{\mathbf{v}} = \langle 8 - 2, 13 - 5 \rangle = \boxed{\langle 6, 8 \rangle}$:

$$\|\vec{\mathbf{v}}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = \boxed{10}$$



(b) To find $\vec{\mathbf{v}} + \vec{\mathbf{w}}$, add the x-components and the y-components separately:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \boxed{\langle 4, 12 \rangle}$$

(c) To find $3\vec{\mathbf{v}}$, multiply $\vec{\mathbf{v}}$ by the scalar $k = 3$:

$$3\vec{\mathbf{v}} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \boxed{\langle 18, 24 \rangle}$$

(d) To find $\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$, find $-2\vec{\mathbf{w}}$ and add it to $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}} - 2\vec{\mathbf{w}} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \boxed{\langle 10, 0 \rangle}$$

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

Rule: Properties of Vector Operations

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in a plane. Let r and s be scalars.

i. *Commutative property*

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

ii. *Associative property*

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

iii. *Additive identity property*

$$\vec{u} + \vec{0} = \vec{u}$$

iv. *Additive inverse property*

$$\vec{u} + (-\vec{u}) = \vec{0}$$

v. *Associativity of scalar multiplication*

$$r(s\vec{u}) = (rs)\vec{u}$$

vi. *Distributive property*

$$(r + s)\vec{u} = r\vec{u} + s\vec{u}$$

vii. *Distributive property*

$$r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$$

viii. *Identity and zero properties*

$$1\vec{u} = \vec{u},\ 0\vec{u} = \vec{0}$$

Proof of Commutative Property

Let $\vec{u} = \langle x_1, y_1 \rangle$ and $\vec{v} = \langle x_2, y_2 \rangle$. Apply the commutative property for real numbers:

$$\begin{aligned} \vec{u} + \vec{v} &= \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle x_2 + x_1, y_2 + y_1 \rangle \\ &= \vec{v} + \vec{u} \end{aligned}$$

Proof of Distributive Property

Apply the distributive property for real numbers:

$$\begin{aligned} r(\vec{u} + \vec{v}) &= r \cdot \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle r(x_1 + x_2), r(y_1 + y_2) \rangle \\ &= \langle rx_1 + rx_2, ry_1 + ry_2 \rangle \\ &= \langle rx_1, ry_1 \rangle + \langle rx_2, ry_2 \rangle \\ &= r\vec{u} + r\vec{v} \end{aligned}$$

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry.

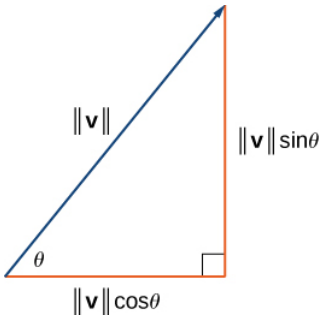


Figure 1.1.14: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

Consider the angle θ formed by the vector $\vec{v}$ and the positive x-axis. We can see from the triangle that the components of vector $\vec{v}$ are $\langle \|\vec{v}\|\cos(\theta),\ \|\vec{v}\|\sin(\theta)\rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

A **unit vector** is a vector with magnitude 1. For any nonzero vector $\vec{v}$, we can use scalar multiplication to find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

Recall that when we defined scalar multiplication, we noted that $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$. For $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$, it follows that $\|\vec{u}\| = \frac{1}{\|\vec{v}\|}(\|\vec{v}\|) = 1$. We say that $\vec{u}$ is the *unit vector in the direction of* $\vec{v}$. The process of using scalar multiplication to find a unit vector with a given direction is called **normalization.**



Figure 1.1.15: The vector $\vec{v}$ and associated unit vector $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$. In this case, $\|\vec{v}\| > 1$

Example 1.1.15: Finding a Unit Vector

Let $\vec{v} = \langle 1, 2\rangle$

(a) Find a unit vector with the same direction as $\vec{v}$

(b) Find a vector $\vec{w}$ with the same direction as $\vec{v}$ such that $\|\vec{w}\| = 7$

Solution:

(a) First, find the magnitude of $\vec{v}$, then divide the components of $\vec{v}$ by the magnitude:

$$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \boxed{\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle}$$

(b) The vector $\vec{u}$ is in the same direction as $\vec{v}$ and $\|\vec{u}\| = 1$. Use scalar multiplication to increase the length of $\vec{u}$ without changing direction:

$$\vec{w} = 7\vec{u} = 7\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \boxed{\left\langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}} \right\rangle}$$

We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors. The **standard unit vectors** are the vectors $\hat{i} = \langle 1, 0\rangle$ and $\hat{j} = \langle 0, 1\rangle$

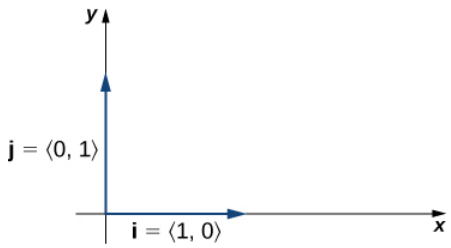


Figure 1.1.16: The standard unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

By applying the properties of vectors, it is possible to express any vector in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in what we call a *linear combination:*

$$\vec{\mathbf{v}} = \langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle = x\langle 1, 0\rangle + y\langle 0, 1\rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Thus, $\vec{\mathbf{v}}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in Figure 1.1.17.

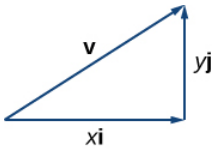


Figure 1.1.17: The vector $\vec{\mathbf{v}}$ is the sum of $x\hat{\mathbf{i}}$ and $y\hat{\mathbf{j}}$.

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\vec{\mathbf{v}} = \langle x, y, z\rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\vec{\mathbf{0}} = \langle 0, 0, 0\rangle$. So, for example, the three dimensional vector $\vec{\mathbf{v}} = \langle 2, 4, 1\rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

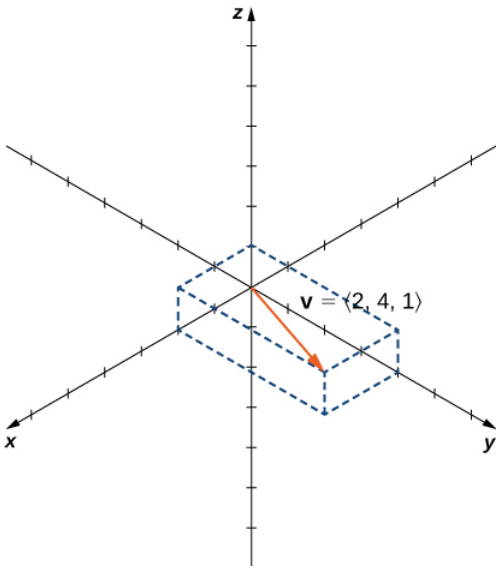


Figure 1.1.18: Vector $\vec{\mathbf{v}} = \langle 2, 4, 1\rangle$

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\vec{\mathbf{v}} = \langle x_1, y_1, z_1\rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2\rangle$ are vectors, and k is a scalar, then

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle$$

and

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1\rangle$$

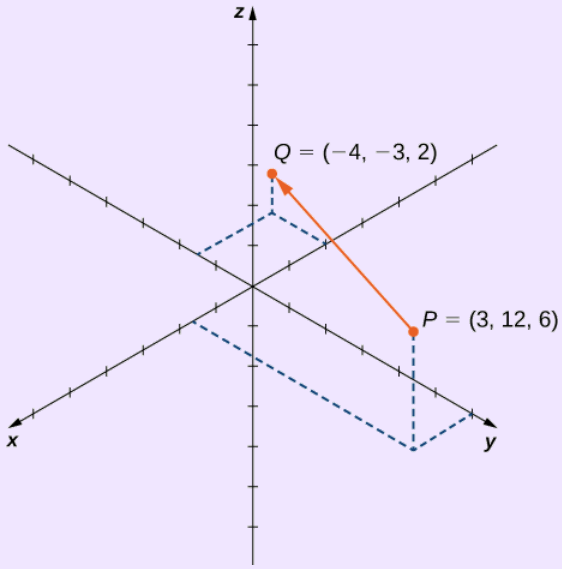
If $k = -1$, then $k\vec{\mathbf{v}} = (-1)\vec{\mathbf{v}}$ is written as $-\vec{\mathbf{v}}$, and vector subtraction is defined by $\vec{\mathbf{v}} - \vec{\mathbf{w}} = \vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$.

The standard unit vectors extend easily into three dimensions as well, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$, and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\vec{\mathbf{v}} = \langle x, y, z \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Illustration 1.1.5: Vector Representations

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown below. Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.



Solution

In component form,

$$\begin{aligned} \overrightarrow{PQ} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\ &= \langle -4 - 3, -3 - 12, 2 - 6 \rangle \\ &= \langle -7, -15, -4 \rangle \end{aligned}$$

In standard unit form,

$$\overrightarrow{PQ} = -7\hat{\mathbf{i}} - 15\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space.

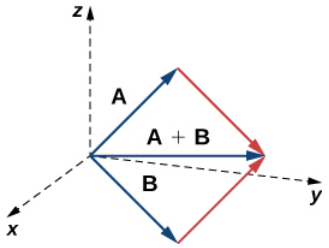


Figure 1.1.19: The geometric interpretation of sums of two vectors in $\mathbb{R}^3$.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Properties of Vectors in Space

Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle$$

- **Vector addition**:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

- **Vector subtraction**:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$

- **Vector magnitude**:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

- **Unit vector in the direction of $\vec{\mathbf{v}}$**:

$$\frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \left\langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|} \right\rangle, \quad \text{if } \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example 1.1.16: Vector Operations in Three Dimensions

Let $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$. Find the following vectors.

(a) $3\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$

(b) $5\|\vec{\mathbf{w}}\|$

(c) $\|5\vec{\mathbf{w}}\|$

(d) A unit vector in the direction of $\vec{\mathbf{v}}$

Solution

(a) First, use scalar multiplication of each vector, then subtract:

$$\begin{aligned} 3\vec{\mathbf{v}} - 2\vec{\mathbf{w}} &= 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle \\ &= \langle -6, 27, 15 \rangle - \langle 2, -2, 0 \rangle \\ &= \langle -6 - 2, 27 - (-2), 15 - 0 \rangle \\ &= \boxed{\langle -8, 29, 15 \rangle} \end{aligned}$$

(b) Write the equation for the magnitude of the vector, then use scalar multiplication:

$$5\|\vec{\mathbf{w}}\| = 5\sqrt{1^2 + (-1)^2 + 0^2} = \boxed{5\sqrt{2}}$$

(c) First, use scalar multiplication, then find the magnitude of the new vector. Note that the result is the same as for part b:

$$\|5\vec{\mathbf{w}}\| = \| \langle 5, -5, 0 \rangle \| = \sqrt{5^2 + (-5)^2 + 0^2} = \sqrt{50} = \boxed{5\sqrt{2}}$$

(d) Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:

$$\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle -2, 9, 5\rangle$$
$$= \frac{1}{\sqrt{(-2)^2 + 9^2 + 5^2}}\langle -2, 9, 5\rangle$$
$$= \frac{1}{\sqrt{110}}\langle -2, 9, 5\rangle$$
$$= \boxed{\left\langle \frac{-2}{\sqrt{110}}, \frac{9}{\sqrt{110}}, \frac{5}{\sqrt{110}} \right\rangle}$$

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1.1: Practice Set

Exercise 1.1.1

Plot the following points in $\mathbb{R}^3$:

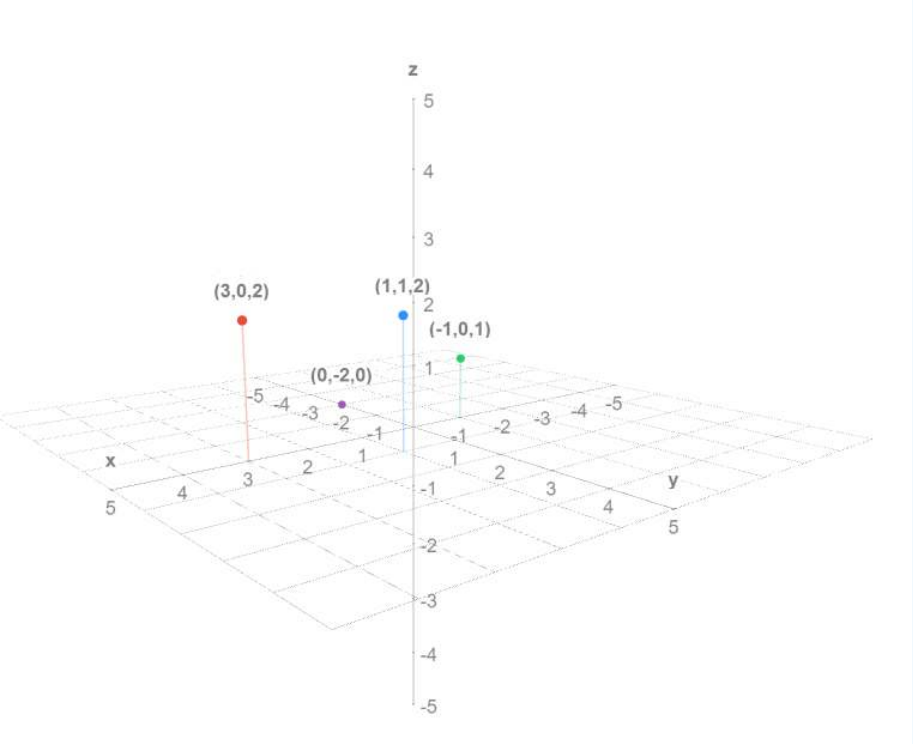
(a) $(1, 1, 2)$

(b) $(-1, 0, 1)$

(c) $(0, -2, 0)$

(d) $(3, 0, 2)$

Answer



Exercise 1.1.2

Find the distance between each of the following pairs of points:

(a) $(2, 5)$ and $(-4, -3)$

(b) $(1, 2, 4)$ and $(2, 4, -1)$

(c) $(-1, -1, 2)$ and $(1, 3, -2)$

Answer

(a) $\text{dist} = \sqrt{(2-(-4))^2 + (5-(-3))^2} = \sqrt{36+64} = \sqrt{100} = 10$

(b) $\text{dist} = \sqrt{(1-2)^2 + (2-4)^2 + (4-(-1))^2} = \sqrt{1+4+25} = \sqrt{30}$

(c) $\text{dist} = \sqrt{(-1-1)^2 + (-1-3)^2 + (2-(-2))^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

Exercise 1.1.3

Find the distance from the origin to the point:

(a) $\left(\frac{10}{13}, \frac{24}{13}\right)$

(b) $(4\cos(2), 4\sin(2))$

(c) $(100, -100, 50)$

(d) $\left(3\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{7}\right), 3\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{7}\right), 3\sin\left(\frac{\pi}{5}\right)\right)$

Answer

(a) $\text{dist} = \sqrt{\left(\frac{10}{13}-0\right)^2 + \left(\frac{24}{13}-0\right)^2} = \sqrt{\frac{100}{169}+\frac{576}{169}} = \sqrt{\frac{676}{169}} = \sqrt{4} = 2$

(b) $\text{dist} = \sqrt{(4\cos(2)-0)^2 + (4\sin(2)-0)^2} = \sqrt{16\cos^2(2)+16\sin^2(2)} = \sqrt{16} = 4$

(c) $\text{dist} = \sqrt{(100-0)^2+(-100-0)^2+(50-0)^2} = \sqrt{10000+10000+2500} = \sqrt{22500} = 150$

(d)
$$\begin{aligned}\text{dist} &= \sqrt{\left(3\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{7}\right)-0\right)^2 + \left(3\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{7}\right)-0\right)^2 + \left(3\sin\left(\frac{\pi}{5}\right)-0\right)^2} \\ &= \sqrt{9\cos^2\left(\frac{\pi}{5}\right)\sin^2\left(\frac{\pi}{7}\right) + 9\cos^2\left(\frac{\pi}{5}\right)\cos^2\left(\frac{\pi}{7}\right) + 9\sin^2\left(\frac{\pi}{5}\right)} \\ &= \sqrt{9\cos^2\left(\frac{\pi}{5}\right)\left[\sin^2\left(\frac{\pi}{7}\right)+\cos^2\left(\frac{\pi}{7}\right)\right] + 9\sin^2\left(\frac{\pi}{5}\right)} \\ &= \sqrt{9\cos^2\left(\frac{\pi}{5}\right) + 9\sin^2\left(\frac{\pi}{5}\right)} \\ &= \sqrt{9} = 3\end{aligned}$$

Exercise 1.1.4

What is the equation of the collection of points in $\mathbb{R}^2$ which:

(a) are all exactly 10 units away from the point $(4, -3)$

(b) are equidistant from the points $(2, 1)$ and $(-4, -3)$

(c) have the sum of the distance to $(-1, -1)$ and distance to $(1, 1)$ total to 4

Answer

(a) $d(x, y) = \sqrt{(x-4)^2+(y-(-3))^2} = \sqrt{x^2-8x+y^2+6y+25}$

Setting $d = 10$ gives the equation

$$\sqrt{x^2-8x+y^2+6y+25} = 10$$

If we square both sides, this looks like

$$x^2-8x+y^2+6y+25 = 100$$

This simplifies to our final answer $x^2-8x+y^2+6y = 75$

(This is more recognizable as a circle if we do not multiply out the perfect squares initially: $(x-4)^2+(y+3)^2 = 100$)

(b) $d_1(x, y) = \sqrt{(x-2)^2+(y-1)^2} = \sqrt{x^2-4x+y^2-2y+5}$ and $d_2(x, y) = \sqrt{(x-(-4))^2+(y-(-3))^2} = \sqrt{x^2+8x+y^2+6y+25}$

Setting these equal gives the equation

$$\sqrt{x^2-4x+y^2-2y+5} = \sqrt{x^2+8x+y^2+6y+25}$$

If we square both sides, this becomes

$$x^2 - 4x + y^2 - 2y + 5 = x^2 + 8x + y^2 + 6y + 25$$

This simplifies to our final answer $3x + 2y = -5$, which is a line.

(c) $d_1(x,y) = \sqrt{(x-(-1))^2 + (y-(-1))^2} = \sqrt{x^2 + 2x + y^2 + 2y + 2}$ and $d_2(x,y) = \sqrt{(x-1)^2 + (y-1)^2} = \sqrt{x^2 - 2x + y^2 - 2y + 2}$

Setting $d_1 + d_2 = 4$ this becomes:

$$\sqrt{x^2 + 2x + y^2 + 2y + 2} + \sqrt{x^2 - 2x + y^2 - 2y + 2} = 4$$

Squaring both sides gives:

$$(x^2 + 2x + y^2 + 2y + 2) + 2\sqrt{(x^2 + 2x + y^2 + 2y + 2)(x^2 - 2x + y^2 - 2y + 2)} + (x^2 - 2x + y^2 - 2y + 2) = 16$$

$$2\sqrt{(x^2 + y^2 + 2 - (2x + 2y))(x^2 + y^2 + 2 + (2x + 2y))} = 12 - 2x^2 - 2y^2$$

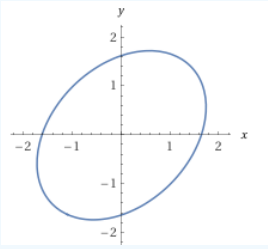
Squaring both sides again:

$$4\left((x^2 + y^2 + 2)^2 - (2x + 2y)^2\right) = (12 - 2x^2 - 2y^2)^2$$

$$4x^4 + 8x^2y^2 - 32xy + 4y^4 + 16 = 4x^4 + 8x^2y^2 - 48x^2 + 4y^4 - 48y^2 + 144$$

$$48x^2 + 48y^2 - 32xy = 128$$

This simplifies to $3x^2 - 2xy + 3y^2 = 8$, which is a tilted ellipse (pictured below)



? Exercise 1.1.5

Let $A = (-c, 0)$ and $B = (c, 0)$ be fixed points for some $c > 0$. If $d_A(x,y)$ and $d_B(x,y)$ are the distance between (x,y) and A and B respectively, find the equation for (x,y) such that $d_A - d_B = 2D$ when:

(a) $D = c$

(b) $0 < D < c$

Answer

$d_A(x,y) = \sqrt{(x-(-c))^2 + (y-0)^2} = \sqrt{x^2 + 2cx + c^2 + y^2}$ and $d_B(x,y) = \sqrt{(x-c)^2 + (y-0)^2} = \sqrt{x^2 - 2cx + c^2 + y^2}$

Setting $|d_A - d_B| = 2D$ is the same as saying $(d_A - d_B)^2 = 4D^2$ which gives

$$\left(\sqrt{x^2 + 2cx + c^2 + y^2} - \sqrt{x^2 - 2cx + c^2 + y^2}\right)^2 = 4D^2$$

$$(x^2 + 2cx + c^2 + y^2) - 2\sqrt{(x^2 + 2cx + c^2 + y^2)(x^2 - 2cx + c^2 + y^2)} + (x^2 - 2cx + c^2 + y^2) = 4D^2$$

$$2x^2 + 2c^2 + 2y^2 - 4D^2 = 2\sqrt{(x^2 + 2cx + c^2 + y^2)(x^2 - 2cx + c^2 + y^2)}$$

Squaring both sides again gives:

$$(2x^2 + 2c^2 + 2y^2 - 4D^2)^2 = 4\left((x^2 + c^2 + y^2) + 2cx\right)\left((x^2 + c^2 + y^2) - 2cx\right)$$

$$4(x^2 + c^2 + y^2)^2 - 16D^2(x^2 + c^2 + y^2) + 16D^4 = 4\left((x^2 + c^2 + y^2)^2 - 4c^2x^2\right)$$

$$-16D^2(x^2+c^2+y^2)+16D^4=-16c^2x^2$$

$$(16D^2-16c^2)x^2+16D^2y^2=16D^4-16D^2c^2$$

(a) When $D=c$ the coefficient of x^2 and the right-hand side are zero so we get:

$16c^2y^2=0$, which simplifies to the line $y=0$.

(b) When $0<D<c$ the right side is non-zero so we divide by it to get:

$$\frac{(D^2-c^2)x^2}{D^2(D^2-c^2)}+\frac{D^2y^2}{D^2(D^2-c^2)}=1$$

Reducing gives our final answer: $\frac{x^2}{D^2}-\frac{y^2}{c^2-D^2}=1$ which is a hyperbola.

Exercise 1.1.6

What is the equation of the collection of points in $\mathbb{R}^3$ which:

(a) are all exactly 6 units away from the point $(1,2,-3)$

(b) are equidistant from the points $(1,1,-4)$ and $(3,1,2)$

(c) have the sum of the distance to $(0,0,-1)$ and distance to $(0,0,1)$ total to 4

Answer

(a) $d(x,y,z)=\sqrt{(x-1)^2+(y-2)^2+(z+3)^2}$

Setting this equal to 6 and squaring both sides we immediately get $(x-1)^2+(y-2)^2+(z+3)^2=36$ which is a sphere.

(b) $d_1(x,y,z)=\sqrt{(x-1)^2+(y-1)^2+(z-(-4))^2}=\sqrt{x^2-2x+y^2-2y+z^2+8z+18}$

$d_2(x,y,z)=\sqrt{(x-3)^2+(y-1)^2+(z-2)^2}=\sqrt{x^2-6x+y^2-2y+z^2-4z+14}$

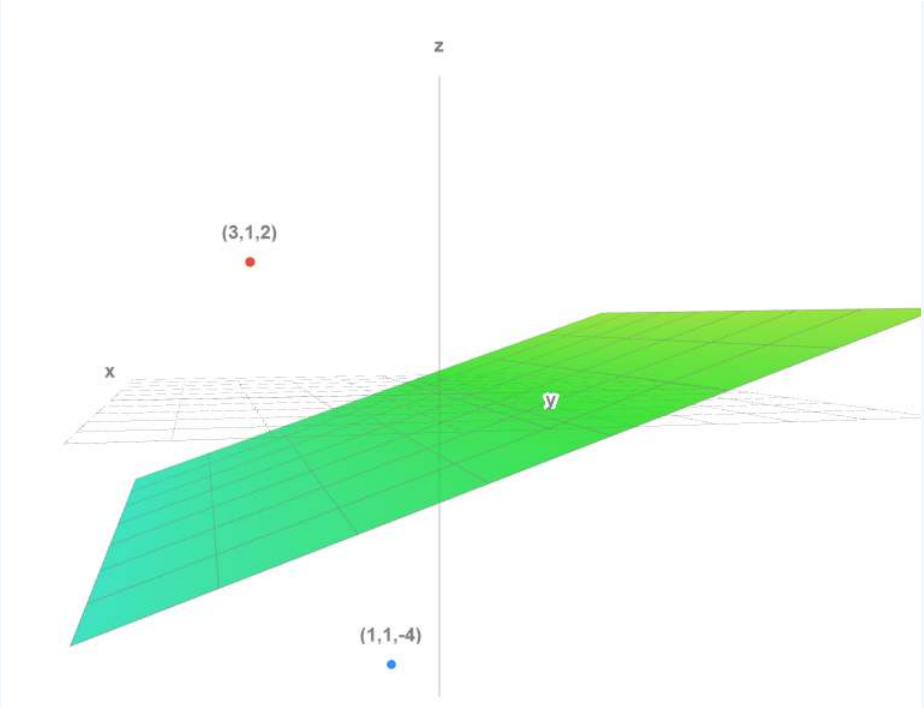
Setting these equal and squaring both sides we get:

$$\left(\sqrt{x^2-2x+y^2-2y+z^2+8z+18}\right)^2=\left(\sqrt{x^2-6x+y^2-2y+z^2-4z+14}\right)^2$$

$$x^2-2x+y^2-2y+z^2+8z+18=x^2-6x+y^2-2y+z^2-4z+14$$

$$4x+12z=-4$$

This simplifies down to $x+3z=-1$, which is a plane (we will discuss these soon, but for now a picture is provided below)



(c) $d_1(x,y,z) = \sqrt{(x-0)^2+(y-0)^2+(z-(-1))^2} = \sqrt{x^2+y^2+z^2+2z+1}$

$d_2(x,y,z) = \sqrt{(x-0)^2+(y-0)^2+(z-1)^2} = \sqrt{x^2+y^2+z^2-2z+1}$

Setting $d_1 + d_2 = 4$ gives:

$$\sqrt{x^2+y^2+z^2+2z+1} + \sqrt{x^2+y^2+z^2-2z+1} = 4$$

Square both sides:

$$\left(\sqrt{x^2+y^2+z^2+2z+1} + \sqrt{x^2+y^2+z^2-2z+1}\right)^2 = 4^2$$

$$(x^2+y^2+z^2+2z+1) + 2\sqrt{(x^2+y^2+z^2+2z+1)(x^2+y^2+z^2-2z+1)} + (x^2+y^2+z^2-2z+1) = 16$$

$$2\sqrt{(x^2+y^2+z^2+1+2z)(x^2+y^2+z^2+1-2z)} = 16 - 2(x^2+y^2+z^2+1)$$

$$\sqrt{(x^2+y^2+z^2+1+2z)(x^2+y^2+z^2+1-2z)} = 8 - (x^2+y^2+z^2+1)$$

Square both sides again:

$$\left(\sqrt{(x^2+y^2+z^2+1+2z)(x^2+y^2+z^2+1-2z)}\right)^2 = \left(8-(x^2+y^2+z^2+1)\right)^2$$

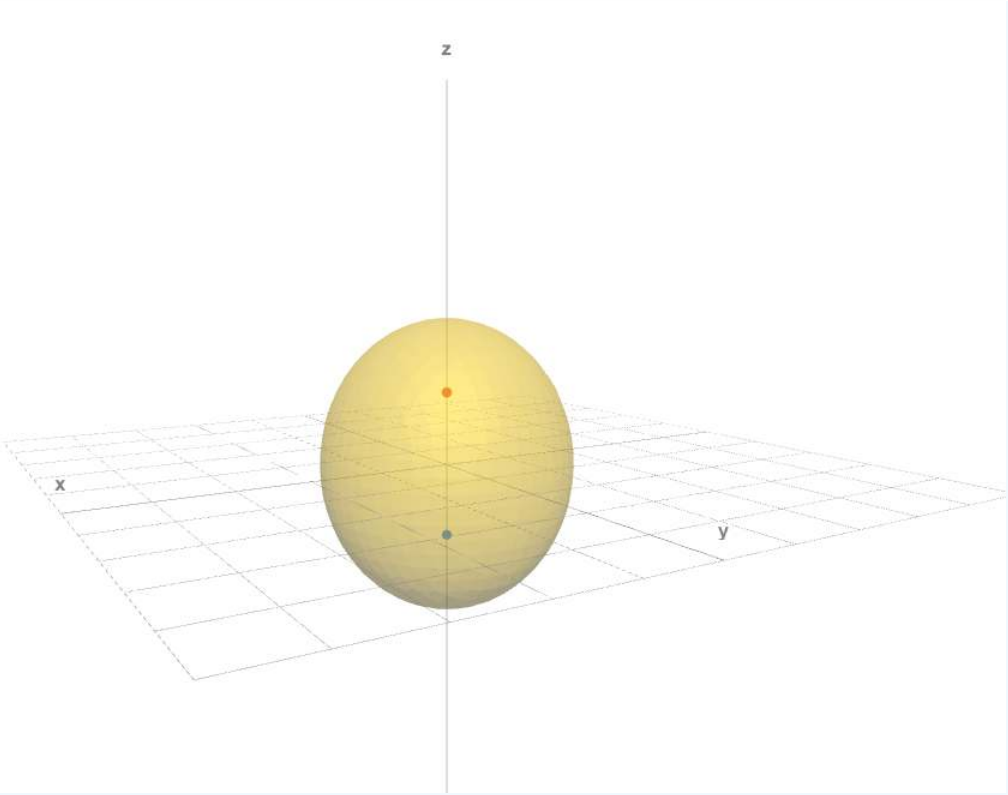
$$(x^2+y^2+z^2+1)^2 - (2z)^2 = 64 - 16(x^2+y^2+z^2+1) + (x^2+y^2+z^2+1)^2$$

$$-4z^2 = 64 - 16(x^2+y^2+z^2+1)$$

$$16x^2+16y^2+12z^2 = 48$$

Dividing everything by 48 makes this into its final form: $\frac{x^2}{3} + \frac{y^2}{3} + \frac{z^2}{4} = 1$

This shape is known as an *ellipsoid*, which we will also discuss soon and is pictured below.



Exercise 1.1.7

Consider the vectors $\vec{v} = \langle 2, -2 \rangle$ and $\vec{w} = \langle 1, 3 \rangle$

(a) Sketch $\vec{v}$

(b) Sketch $\vec{w}$

(c) Sketch $\vec{v} + \vec{w}$

(d) Sketch $\vec{v} - \vec{w}$

(e) Sketch $\frac{1}{2}\vec{v}$

(f) Find $\|\vec{v}\|$

(g) Find $\|\vec{w}\|$

(h) Find a unit vector in the direction of $\vec{w}$

Answer

(a-e)

(f) $\|\vec{v}\| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

(g) $\|\vec{w}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$

(h) $\frac{1}{\|\vec{w}\|}\vec{w} = \frac{1}{\sqrt{10}}\langle 1, 3 \rangle = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$

Exercise 1.1.8

Consider the vectors $\vec{a} = \langle 1, 2, 0 \rangle$ and $\vec{b} = \langle 2, 2, 1 \rangle$

(a) Sketch $\vec{a}$

(b) Sketch $\vec{b}$

(c) Sketch $\vec{a} + \vec{b}$

(d) Sketch $2\vec{a} - \vec{b}$

(e) Find $\|\vec{a}\|$

(f) Find $\|3\vec{a} + 4\vec{b}\|$

(g) Find a unit vector in the direction of $\vec{b}$

Answer

(a-d)

(e) $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$

(f) $3\vec{a} + 4\vec{b} = 3\langle 1, 2, 0 \rangle + 4\langle 2, 2, 1 \rangle = \langle 11, 14, 4 \rangle \longrightarrow \|3\vec{a} + 4\vec{b}\| = \|\langle 11, 14, 4 \rangle\| = \sqrt{11^2 + 14^2 + 4^2} = \sqrt{333} = 3\sqrt{37}$

(g) $\|\vec{b}\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$ so $\dfrac{1}{\|\vec{b}\|}\vec{b} = \dfrac{1}{3}\langle 2, 2, 1 \rangle = \left\langle \dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3} \right\rangle$

Exercise 1.1.9

Consider the vectors $\vec{r} = \langle 1, 1, 1 \rangle$, $\vec{s} = \langle 1, -2, -1 \rangle$ and $\vec{t} = \langle 0, 1, 2 \rangle$:

(a) Evaluate $10\vec{r}$

(b) Evaluate $-\vec{s}$

(c) Evaluate $3\vec{r} - 2\vec{t}$

(d) Evaluate $2\vec{r} + \vec{s} + 3\vec{t}$

(e) Evaluate $\|\vec{r} - \vec{s}\| + \|\vec{r} + \vec{t}\|$

(f) For what α does $\alpha\vec{r} + 3\vec{s} + 2\vec{t} = \langle 7, 0, 5 \rangle$

(g) For what β does $\beta\vec{r} - \vec{s} + 4\vec{t} = \langle 0, 7, 9 \rangle$

(h) For what γ does $\|\vec{r} + \gamma\vec{s}\| = 3$

Answer

(a) $10\vec{r} = 10\langle 1, 1, 1 \rangle = \langle 10, 10, 10 \rangle$

(b) $-\vec{s} = -\langle 1, -2, -1 \rangle = \langle -1, 2, 1 \rangle$

(c) $3\vec{r} - 2\vec{t} = 3\langle 1, 1, 1 \rangle - 2\langle 0, 1, 2 \rangle = \langle 3, 1, -1 \rangle$

(d) $2\vec{r} + \vec{s} + 3\vec{t} = 2\langle 1, 1, 1 \rangle + \langle 1, -2, -1 \rangle + 3\langle 0, 1, 2 \rangle = \langle 3, 3, 7 \rangle$

(e) $\vec{r} - \vec{s} = \langle 1, 1, 1 \rangle - \langle 1, -2, -1 \rangle = \langle 0, 3, 2 \rangle \longrightarrow \|\vec{r} - \vec{s}\| = \|\langle 0, 3, 2 \rangle\| = \sqrt{0^2 + 3^2 + 2^2} = \sqrt{13}$

$\vec{r} + \vec{t} = \langle 1, 1, 1 \rangle + \langle 0, 1, 2 \rangle = \langle 1, 2, 3 \rangle \longrightarrow \|\vec{r} + \vec{t}\| = \|\langle 1, 2, 3 \rangle\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

Therefore $\|\vec{r} - \vec{s}\| + \|\vec{r} + \vec{t}\| = \sqrt{13} + \sqrt{14}$

(f) $\alpha\vec{r} + 3\vec{s} + 2\vec{t} = \alpha\langle 1, 1, 1 \rangle + 3\langle 1, -2, -1 \rangle + 2\langle 0, 1, 2 \rangle = \langle \alpha + 3, \alpha - 4, \alpha + 1 \rangle$

$$\langle \alpha+3, \alpha-4, \alpha+1\rangle = \langle 7,0,5\rangle \implies \begin{cases} \alpha+3=7 \\ \alpha-4=0 \\ \alpha+1=5 \end{cases} \implies \alpha = 4$$

(g) $\beta\vec{\mathbf{r}} - \vec{\mathbf{s}} + 4\vec{\mathbf{t}} = \beta\langle 1,1,1\rangle - \langle 1,-2,-1\rangle + 4\langle 0,1,2\rangle = \langle \beta-1, \beta+6, \beta+9\rangle$

$$\langle \beta-1, \beta+6, \beta+9\rangle = \langle 0,7,9\rangle \implies \begin{cases} \beta-1=0 \\ \beta+6=7 \\ \beta+9=9 \end{cases}$$ which cannot be satisfied by any β

(h) $\vec{\mathbf{r}} + \gamma\vec{\mathbf{s}} = \langle 1,1,1\rangle + \gamma\langle 1,-2,-1\rangle = \langle \gamma+1, -2\gamma+1, -\gamma+1\rangle$

$\|\vec{\mathbf{r}} + \gamma\vec{\mathbf{s}}\| = \|\langle \gamma+1, -2\gamma+1, -\gamma+1\rangle\| = \sqrt{(\gamma+1)^2 + (-2\gamma+1)^2 + (-\gamma+1)^2} = \sqrt{6\gamma^2 - 4\gamma + 3}$

Setting this equal to three, we square both sides to get: $6\gamma^2 - 4\gamma + 3 = 9 \longrightarrow 6\gamma^2 - 4\gamma - 6 = 0$

The quadratic formula gives us the solutions $\gamma = \dfrac{1 \pm \sqrt{10}}{3}$

? Exercise 1.1.10

Prove the following Properties of Vector Operations for vectors in $\mathbb{R}^2$:

(a) Associative property: $\vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}}$

(b) Additive Identity property: $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$

(c) Additive Inverse property: $\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}}$

(d) Associativity of Scalar Multiplication: $(rs)\vec{\mathbf{u}} = r(s\vec{\mathbf{u}})$

(e) Distributive Property: $(a+b)\vec{\mathbf{u}} = a\vec{\mathbf{u}} + b\vec{\mathbf{u}}$

(f) Identity Property: $1\vec{\mathbf{u}} = \vec{\mathbf{u}}$

(g) Zero Property: $0\vec{\mathbf{u}} = \vec{\mathbf{0}}$

Answer

(a)

$$\begin{aligned} \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) &= \langle u_1, u_2\rangle + (\langle v_1, v_2\rangle + \langle w_1, w_2\rangle) \\ &= \langle u_1, u_2\rangle + \langle v_1 + w_1, v_2 + w_2\rangle \\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle \\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle \\ &= \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle \\ &= (\langle u_1, u_2\rangle + \langle v_1, v_2\rangle) + \langle w_1, w_2\rangle \\ &= (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} \end{aligned}$$

(b)

$$\begin{aligned} \vec{\mathbf{u}} + \vec{\mathbf{0}} &= \langle u_1, u_2\rangle + \langle 0, 0\rangle \\ &= \langle u_1 + 0, u_2 + 0\rangle \\ &= \langle u_1, u_2\rangle \\ &= \vec{\mathbf{u}} \end{aligned}$$

(c)

$$\begin{aligned} \vec{\mathbf{u}} + (-\vec{\mathbf{u}}) &= \langle u_1, u_2\rangle + \langle -u_1, -u_2\rangle \\ &= \langle u_1 + (-u_1), u_2 + (-u_2)\rangle \\ &= \langle 0, 0\rangle \\ &= \vec{\mathbf{0}} \end{aligned}$$

(d)

$$\begin{aligned}(rs)\vec{\mathbf{u}} &= (rs)\langle u_1, u_2\rangle \\ &= \langle (rs)u_1, (rs)u_2\rangle \\ &= \langle r(su_1), r(su_2)\rangle \\ &= r\langle su_1, su_2\rangle \\ &= r\left(s\langle u_1, u_2\rangle\right) \\ &= r(s\vec{\mathbf{u}})\end{aligned}$$

(e)

$$\begin{aligned}(a+b)\vec{\mathbf{u}} &= (a+b)\langle u_1, u_2\rangle \\ &= \langle (a+b)u_1, (a+b)u_2\rangle \\ &= \langle au_1 + bu_1, au_2 + bu_2\rangle \\ &= \langle au_1, au_2\rangle + \langle bu_1, bu_2\rangle \\ &= a\langle u_1, u_2\rangle + b\langle u_1, u_2\rangle \\ &= a\vec{\mathbf{u}} + b\vec{\mathbf{u}}\end{aligned}$$

(f)

$$\begin{aligned}1\vec{\mathbf{u}} &= 1\langle u_1, u_2\rangle \\ &= \langle 1u_1, 1u_2\rangle \\ &= \langle u_1, u_2\rangle \\ &= \vec{\mathbf{u}}\end{aligned}$$

(g)

$$\begin{aligned}0\vec{\mathbf{u}} &= 0\langle u_1, u_2\rangle \\ &= \langle 0u_1, 0u_2\rangle \\ &= \langle 0, 0\rangle \\ &= \vec{\mathbf{0}}\end{aligned}$$

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1.2: Vector Products

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.
- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas and volumes by using the cross product.

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, while the second is called the cross product. These two products are wildly different and have different applications, but both will be used very often in multivariable calculus.

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1.2: The Dot Product

In this section, we develop an operation called the dot product, which allows us to quantify how much two vectors have in common even when they initially have different directions. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

Definition: Dot Product

The *dot product* of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar.

Example 1.2.1: Calculating Dot Products

(a) Evaluate $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ for $\vec{\mathbf{u}} = \langle 3, 5, 2 \rangle$ and $\vec{\mathbf{v}} = \langle -1, 3, 0 \rangle$

(b) Find the dot product of $\vec{\mathbf{p}} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{q}} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

(c) Evaluate $\langle 2, 9, -1 \rangle \cdot \langle -3, 1, -4 \rangle$

Solution:

(a) $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 3(-1) + 5(3) + 2(0) = \boxed{12}$

(b) $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 10(-2) + (-4)(1) + (7)(6) = \boxed{18}$

(c) $\langle 2, 9, -1 \rangle \cdot \langle -3, 1, -4 \rangle = 2(-3) + 9(1) + (-1)(-4) = \boxed{7}$

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Rule: Properties of the Dot Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors, and let c be a scalar.

i. *Commutative property*

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$$

ii. *Distributive property*

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$$

iii. *Associative property*

$$c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$$

iv. *Property of magnitude*

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$$

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= v_1 u_1 + v_2 u_2 + v_3 u_3 \\ &= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} \end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned} c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) &= c(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= c(u_1 v_1) + c(u_2 v_2) + c(u_3 v_3) \\ &= (cu_1) v_1 + (cu_2) v_2 + (cu_3) v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} \end{aligned}$$

The proof that $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned} \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} &= \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (v_1)^2 + (v_2)^2 + (v_3)^2 \\ &= \left[\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} \right]^2 \\ &= \|\vec{\mathbf{v}}\|^2 \end{aligned}$$

Note that the definition of the dot product yields $\vec{\mathbf{0}} \cdot \vec{\mathbf{v}} = 0$. By property iv. if $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Example 1.2.2: Using Properties of the Dot Product

Let $\vec{\mathbf{a}} = \langle 1, 2, -3 \rangle$, $\vec{\mathbf{b}} = \langle 0, 2, 4 \rangle$, and $\vec{\mathbf{c}} = \langle 5, -1, 3 \rangle$.

Find each of the following products.

(a) $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$

(b) $\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}})$

(c) $\|\vec{\mathbf{b}}\|^2$

Solution

(a) Note that this expression asks for the scalar multiple of $\vec{\mathbf{c}}$ by $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$:

$$\begin{aligned} (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\ &= -8\langle 5, -1, 3 \rangle \\ &= \boxed{\langle -40, 8, -24 \rangle} \end{aligned}$$

(b) This expression is a dot product of vector $\vec{\mathbf{a}}$ and scalar multiple $2\vec{\mathbf{c}}$:

$$\begin{aligned}\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}}) &= 2(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) \\ &= 2(\langle 1, 2, -3\rangle \cdot \langle 5, -1, 3\rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = \boxed{-12}\end{aligned}$$

(c) Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned}\|\vec{\mathbf{b}}\|^2 &= \vec{\mathbf{b}} \cdot \vec{\mathbf{b}} \\ &= \langle 0, 2, 4\rangle \cdot \langle 0, 2, 4\rangle \\ &= 0^2 + 2^2 + 4^2 \\ &= 0 + 4 + 16 \\ &= \boxed{20}\end{aligned}$$

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them. The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

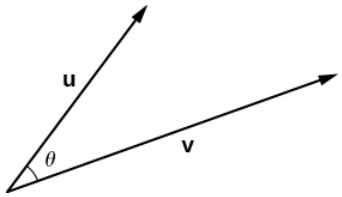


Figure 1.2.1: Let θ be the angle between two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that $0 \leq \theta \leq \pi$

Rule: Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos(\theta)$$

Proof

Place vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in standard position and consider the vector $\vec{\mathbf{v}} - \vec{\mathbf{u}}$ pictured below. These three vectors form a triangle with side lengths $\|\vec{\mathbf{u}}\|$, $\|\vec{\mathbf{v}}\|$, and $\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|$.

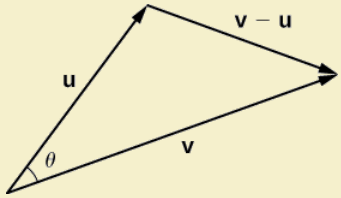


Figure 1.2.2: The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 = \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2 - 2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos(\theta)$$

The dot product provides a way to rewrite the left side of this equation:

$$\begin{aligned}\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 &= (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \\ &= (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} - (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot \vec{\mathbf{u}} \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \\ &= \|\vec{\mathbf{v}}\|^2 - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \|\vec{\mathbf{u}}\|^2\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}\|\vec{v}-\vec{u}\|^2 &= \|\vec{u}\|^2+\|\vec{v}\|^2-2\|\vec{u}\|\|\vec{v}\|\cos(\theta)\\ \|\vec{v}\|^2-2\vec{u}\cdot\vec{v}+\|\vec{u}\|^2 &= \|\vec{u}\|^2+\|\vec{v}\|^2-2\|\vec{u}\|\|\vec{v}\|\cos(\theta)\\ -2\vec{u}\cdot\vec{v} &= -2\|\vec{u}\|\|\vec{v}\|\cos(\theta)\\ \vec{u}\cdot\vec{v} &= \|\vec{u}\|\|\vec{v}\|\cos(\theta)\end{aligned}$$

We can use the dot product to find the measure of the angle between two nonzero vectors by rearranging this equation to solve for the cosine of the angle:

$$\cos(\theta)=\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0°\leq\theta\leq 180°$ (or $0\leq\theta\leq\pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the angle θ.

Example 1.2.3: Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

(a) $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}-3\hat{k}$

(b) $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

(c) $\vec{a}=\langle 1,1,1\rangle$ and $\vec{b}=\langle 2\sqrt{3}+\sqrt{2},2\sqrt{3}+\sqrt{2},2\sqrt{3}-2\sqrt{2}\rangle$

Solution

(a) We know that:

$$\begin{aligned}\cos(\theta) &= \frac{(\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}-3\hat{k})}{\|\,\hat{i}+\hat{j}+\hat{k}\,\|\cdot\|\,2\hat{i}-\hat{j}-3\hat{k}\,\|}\\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}}\\ &= \frac{-2}{\sqrt{3}\sqrt{14}}=\frac{-2}{\sqrt{42}}\end{aligned}$$

Therefore, $\boxed{\theta=\cos^{-1}\left(\frac{-2}{\sqrt{42}}\right)\text{ radians}}$

(b) Once more:

$$\begin{aligned}\cos(\theta) &= \frac{\langle 2,5,6\rangle\cdot\langle -2,-4,4\rangle}{\|\,\langle 2,5,6\rangle\,\|\cdot\|\,\langle -2,-4,4\rangle\,\|}\\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}}\\ &= \frac{0}{\sqrt{65}\sqrt{36}}=0\end{aligned}$$

Now, $\cos(\theta)=0$ and $0\leq\theta\leq\pi$, so $\boxed{\theta=\frac{\pi}{2}\text{ radians}}$

(c) Again:

$$\cos(\theta) = \frac{\langle 1,1,1\rangle \cdot \langle 2\sqrt{3}+\sqrt{2}, 2\sqrt{3}+\sqrt{2}, 2\sqrt{3}-2\sqrt{2}\rangle}{\| \langle 1,1,1\rangle \| \cdot \| \langle 2\sqrt{3}+\sqrt{2}, 2\sqrt{3}+\sqrt{2}, 2\sqrt{3}-2\sqrt{2}\rangle \|}$$
$$= \frac{1(2\sqrt{3}+\sqrt{2}) + 1(2\sqrt{3}+\sqrt{2}) + 1(2\sqrt{3}-2\sqrt{2})}{\sqrt{1^2+1^2+1^2}\sqrt{(2\sqrt{3}+\sqrt{2})^2 + (2\sqrt{3}+\sqrt{2})^2 + (2\sqrt{3}-2\sqrt{2})^2}}$$
$$= \frac{6\sqrt{3}}{\sqrt{3}\sqrt{48}} = \frac{\sqrt{3}}{2}$$

Therefore, $\boxed{\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \text{ radians}}$

The angle between two vectors can be acute $(0 < \cos(\theta) < 1)$, obtuse $(-1 < \cos(\theta) < 0)$, or straight $(\cos(\theta) = -1)$. If $\cos(\theta) = 1$, then both vectors have the same direction. If $\cos(\theta) = 0$, then the vectors, when placed in standard position, form a right angle. We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

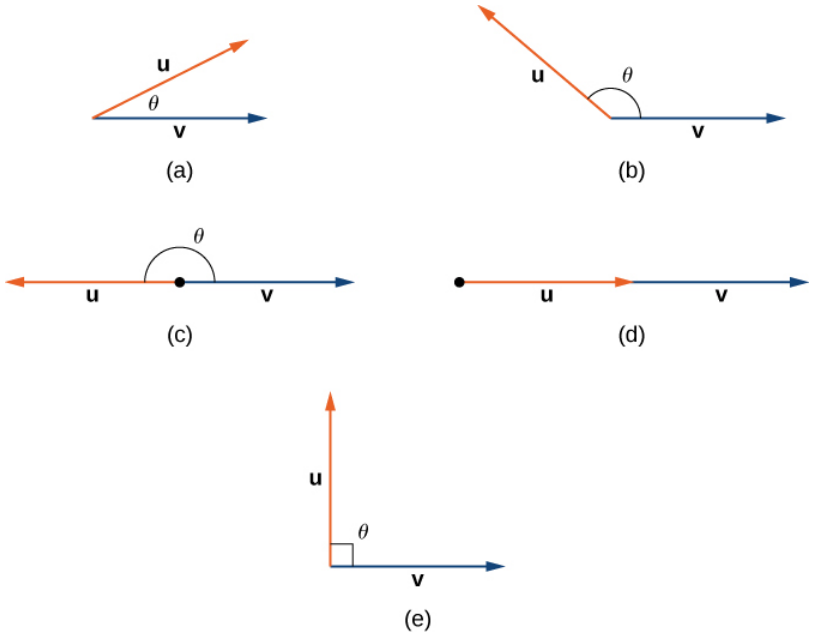


Figure 1.2.3: (a) An acute angle has $0 < \cos(\theta) < 1$
(b) An obtuse angle has $-1 < \cos(\theta) < 0$
(c) A straight line has $\cos(\theta) = -1$
(d) If the vectors have the same direction, $\cos(\theta) = 1$
(e) If the vectors are orthogonal (perpendicular), $\cos(\theta) = 0$

Rule: Orthogonal Vectors

The nonzero vectors $\vec{u}$ and $\vec{v}$ are **orthogonal vectors** if and only if $\vec{u} \cdot \vec{v} = 0$.

Proof

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\vec{u} \cdot \vec{v} = 0$.Then

$$\|\vec{u}\|\|\vec{v}\| \cos(\theta) = 0$$

However, $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$, so we must have $\cos(\theta) = 0$. Hence, $\theta = 90°$, and the vectors are orthogonal.

Now assume $\vec{u}$ and $\vec{v}$ are orthogonal. Then $\theta = 90°$ and we have

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta \\ &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos(90°) \\ &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|(0) \\ &= 0\end{aligned}$$

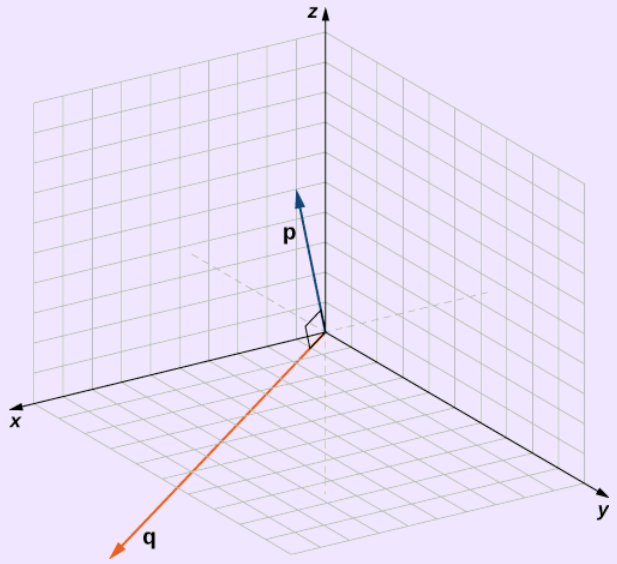
The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Illustration 1.2.1: Identifying Orthogonal Vectors

To determine whether $\vec{\mathbf{p}} = \langle 1, 0, 5\rangle$ and $\vec{\mathbf{q}} = \langle 10, 3, -2\rangle$ are orthogonal vectors, we need only check the dot product of the vectors:

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0$$

Because $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0$, the vectors are orthogonal.



Example 1.2.4

For which value(s) of x is $\vec{\mathbf{p}} = \langle 2, 8, -1\rangle$ orthogonal to $\vec{\mathbf{q}} = \langle x, -1, 2\rangle$?

Solution

We calculate the dot product of the vectors:

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 2(x) + 8(-1) + (-1)(2) = 2x - 10$$

Setting this equal to zero gives $2x - 10 = 0 \longrightarrow \boxed{x = 5}$ as our only solution.

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward. We return to this example and learn how to solve it after we see how to calculate projections.

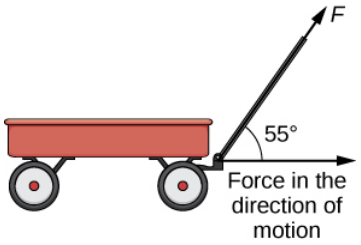


Figure 1.2.4: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector and Projection

The **vector projection** of $\vec{v}$ onto $\vec{u}$ is the vector labeled $\text{proj}_{\vec{u}}\vec{v}$ below. It has the same initial point as $\vec{u}$ and $\vec{v}$ and the same direction as $\vec{u}$, and represents the component of $\vec{v}$ that acts in the direction of $\vec{u}$. If θ represents the angle between $\vec{u}$ and $\vec{v}$, then, by properties of triangles, we know the length of $\text{proj}_{\vec{u}}\vec{v}$ is $\|\text{proj}_{\vec{u}}\vec{v}\| = \|\vec{v}\|\cos(\theta)$. When expressing $\cos(\theta)$ in terms of the dot product, this becomes

$$\|\text{proj}_{\vec{u}}\vec{v}\| = \|\vec{v}\|\cos(\theta) = \|\vec{v}\|\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$$

We now multiply by a unit vector in the direction of $\vec{u}$ to get $\text{proj}_{\vec{u}}\vec{v}$:

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\left(\frac{1}{\|\vec{u}\|}\vec{u}\right) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}$$

The length of this vector is also known as the **scalar projection** of $\vec{v}$ onto $\vec{u}$ and is denoted by

$$\|\text{proj}_{\vec{u}}\vec{v}\| = \text{comp}_{\vec{u}}\vec{v} = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|}$$

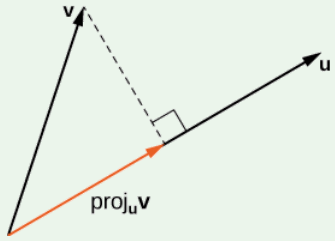


Figure 1.2.5: The projection of $\vec{v}$ onto $\vec{u}$ shows the component of vector $\vec{v}$ in the direction of $\vec{u}$.

Example 1.2.5: Finding Projections

Find the projection of $\vec{v}$ onto $\vec{u}$.

(a) $\vec{v} = \langle 3, 5, 1\rangle$ and $\vec{u} = \langle -1, 4, 3\rangle$

(b) $\vec{v} = 3\hat{i} - 2\hat{j}$ and $\vec{u} = \hat{i} + 6\hat{j}$

Solution

(a) Substitute the components of $\vec{v}$ and $\vec{u}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\langle -1,4,3\rangle \\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle \\ &= \frac{20}{26}\langle -1,4,3\rangle \\ &= \boxed{\left\langle -\frac{10}{13},\frac{40}{13},\frac{30}{13}\right\rangle}\end{aligned}$$

(b) To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{(\hat{\mathbf{i}}+6\hat{\mathbf{j}})\cdot(3\hat{\mathbf{i}}-2\hat{\mathbf{j}})}{\|\hat{\mathbf{i}}+6\hat{\mathbf{j}}\|^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= \boxed{-\frac{9}{37}\hat{\mathbf{i}}-\frac{54}{37}\hat{\mathbf{j}}}\end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution or decomposition of a vector** into components. Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{\mathbf{v}}=\langle 6,-4\rangle$ and let $\vec{\mathbf{u}}=\langle 3,1\rangle$. We want to decompose the vector $\vec{\mathbf{v}}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{\mathbf{u}}$.

We first find the component that has the same direction as $\vec{\mathbf{u}}$ by projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$. Let $\vec{\mathbf{p}}=\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. Then, we have

$$\begin{aligned}\vec{\mathbf{p}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{18-4}{9+1}\vec{\mathbf{u}} \\ = \frac{7}{5}\vec{\mathbf{u}} = \frac{7}{5}\langle 3,1\rangle &= \left\langle\frac{21}{5},\frac{7}{5}\right\rangle\end{aligned}$$

Now consider the vector $\vec{\mathbf{q}}=\vec{\mathbf{v}}-\vec{\mathbf{p}}$. We have

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}}-\vec{\mathbf{p}} \\ &= \langle 6,-4\rangle-\left\langle\frac{21}{5},\frac{7}{5}\right\rangle \\ &= \left\langle\frac{9}{5},-\frac{27}{5}\right\rangle\end{aligned}$$

Clearly, by the way we defined $\vec{\mathbf{q}}$, we have $\vec{\mathbf{v}}=\vec{\mathbf{q}}+\vec{\mathbf{p}}$, and

$$\begin{aligned}\vec{\mathbf{q}}\cdot\vec{\mathbf{p}} &= \left\langle\frac{9}{5},-\frac{27}{5}\right\rangle\cdot\left\langle\frac{21}{5},\frac{7}{5}\right\rangle \\ &= \frac{9(21)}{25}+-\frac{27(7)}{25} \\ &= \frac{189}{25}-\frac{189}{25}=0\end{aligned}$$

Therefore, $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are orthogonal.

Illustration 1.2.2: Resolving Vectors into Components

We will express $\vec{\mathbf{v}} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}} = \langle 2, 3, 2 \rangle$:

Let $\vec{\mathbf{p}}$ represent the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\begin{aligned}\vec{\mathbf{p}} &= \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} \\ &= \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle 2,3,2 \rangle \cdot \langle 8,-3,-3 \rangle}{\| \langle 2,3,2 \rangle \|^2}\langle 2,3,2 \rangle \\ &= \frac{16-9-6}{2^2+3^2+2^2}\langle 2,3,2 \rangle \\ &= \frac{1}{17}\langle 2,3,2 \rangle \\ &= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle\end{aligned}$$

Then,

$$\begin{aligned}\vec{\mathbf{q}} = \vec{\mathbf{v}} - \vec{\mathbf{p}} &= \langle 8, -3, -3 \rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \\ &= \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle\end{aligned}$$

To check our work, we can use the dot product to verify that $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal vectors:

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \cdot \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle \\ &= \frac{268}{289} - \frac{162}{289} - \frac{106}{289} = 0\end{aligned}$$

Then,

$$\vec{\mathbf{v}} = \vec{\mathbf{p}} + \vec{\mathbf{q}} = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle + \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle$$

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the **work done by the force** is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{\mathbf{F}}$, acting at an angle θ from the line of motion, is given by

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos(\theta)$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force.

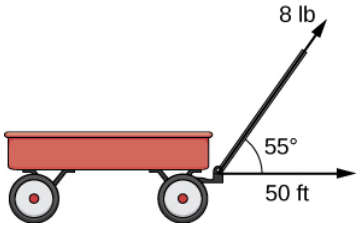


Figure 1.2.6: The horizontal component of the force is the projection of $\vec{\mathbf{F}}$ onto the positive x-axis.

We have

$$W = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos(\theta) = (8\,\text{lb})\,(50\,\text{ft.})\,\cos(55°) \approx 229\,\text{ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\| \vec{\mathbf{F}} \|$ in pounds. The magnitude of the displacement vector $\| \overrightarrow{PQ} \|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 1.2.6: Calculating Work

A conveyor belt generates a force $\vec{\mathbf{F}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

$$\overrightarrow{PQ} = \langle 9 - 1, 4 - 1, 7 - 1 \rangle = \langle 8, 3, 6 \rangle = 8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$$

Work is the dot product of force and displacement:

$$\begin{aligned} W &= \vec{\mathbf{F}} \cdot \overrightarrow{PQ} \\ &= (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37\,\text{N·m} \\ &= \boxed{37\,\text{J}} \end{aligned}$$

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1.2: The Cross Product

In this section, we develop an operation called the *cross product*, which allows us to find a vector perpendicular to two given vectors.

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$—that is, we want to find $\vec{\mathbf{w}}$ such that $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0$$

If we select

$$w_1 = u_2 v_3 - u_3 v_2$$
$$w_2 = -(u_1 v_3 - u_3 v_1)$$

we get a possible solution vector. Substituting these values back into the original equations gives

$$w_3 = u_1 v_2 - u_2 v_1$$

That is, vector

$$\vec{\mathbf{w}} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, which leads us to define the following operation, called the *cross product*.

Definition: Cross Product

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then, the ***cross product*** $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is vector

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle \end{aligned}$$

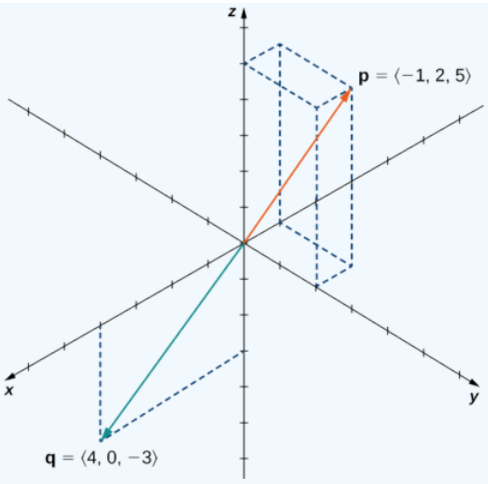
From the way we have developed $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, it should be clear that the cross product is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. However, it never hurts to check. To show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}}$, we calculate the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1 \rangle \\ &= u_1(u_2 v_3 - u_3 v_2) + u_2(-u_1 v_3 + u_3 v_1) + u_3(u_1 v_2 - u_2 v_1) \\ &= u_1 u_2 v_3 - u_1 u_3 v_2 - u_1 u_2 v_3 + u_2 u_3 v_1 + u_1 u_3 v_2 - u_2 u_3 v_1 \\ &= (u_1 u_2 v_3 - u_1 u_2 v_3) + (-u_1 u_3 v_2 + u_1 u_3 v_2) + (u_2 u_3 v_1 - u_2 u_3 v_1) \\ &= 0 \end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\vec{\mathbf{v}}$.

Example 1.2.7: Finding a Cross Product

Let $\vec{\mathbf{p}} = \langle -1, 2, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3 \rangle$. Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.



Solution

Substitute the components of the vectors into the cross product formula:

$$\begin{aligned}\vec{\mathbf{p}} \times \vec{\mathbf{q}} &= \langle -1, 2, 5\rangle \times \langle 4, 0, -3\rangle \\ &= \langle p_2q_3 - p_3q_2, -(p_1q_3 - p_3q_1), p_1q_2 - p_2q_1\rangle \\ &= \langle 2(-3) - 5(0), -(-1)(-3) + 5(4), (-1)(0) - 2(4)\rangle \\ &= \boxed{\langle -6, 17, -8\rangle}\end{aligned}$$

Example 1.2.8

Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$ for $\vec{\mathbf{p}} = \langle 5, 1, 2\rangle$ and $\vec{\mathbf{q}} = \langle -2, 0, 1\rangle$. Express the answer using standard unit vectors.

Solution

Substitute the components of the vectors into the cross product formula:

$$\begin{aligned}\vec{\mathbf{p}} \times \vec{\mathbf{q}} &= \langle 5, 1, 2\rangle \times \langle -2, 0, 1\rangle \\ &= \langle 1(1) - 2(0), -(5)(1) + 2(-2), 5(0) - 1(-2)\rangle \\ &= \langle 1, -9, 2\rangle\end{aligned}$$

Thus our final answer is $\vec{\mathbf{p}} \times \vec{\mathbf{q}} = \boxed{1\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}$

Although it may not be obvious from the formula, the direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\vec{\mathbf{u}}$, then curl the fingers toward vector $\vec{\mathbf{v}}$, the thumb points in the direction of the cross product, as shown below.

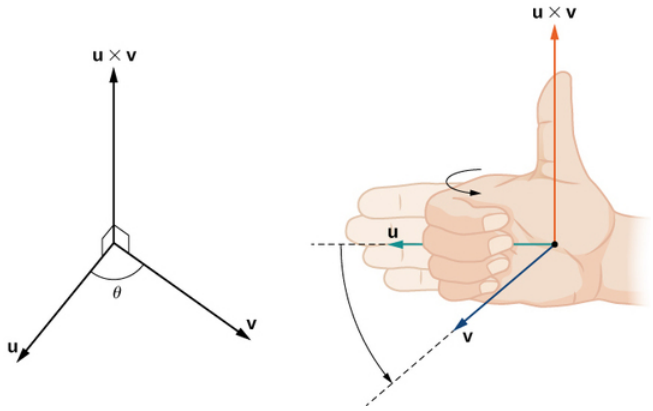


Figure 1.2.7: The direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is determined by the right-hand rule.

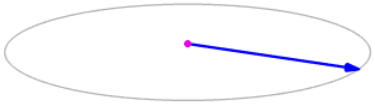
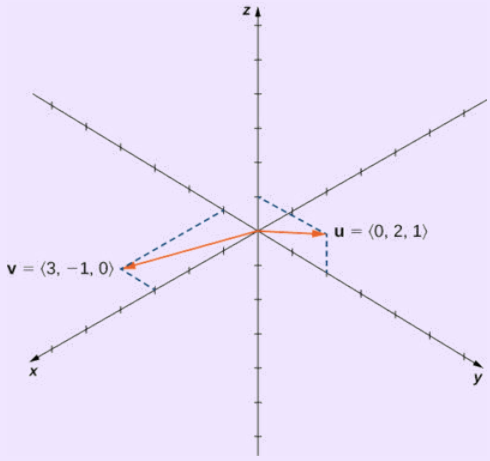


Figure 1.2.8: The cross product $\vec{a} \times \vec{b}$ (vertical, in pink) changes as the angle between the vectors $\vec{a}$ (blue) and $\vec{b}$ (red) changes. The cross product (purple) is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude $\|\vec{a}\|\|\vec{b}\|$ when they are perpendicular. (Public Domain; LucasVB).

Notice what this means for the direction of $\vec{v} \times \vec{u}$. If we apply the right-hand rule to $\vec{v} \times \vec{u}$, we start with our fingers pointed in the direction of $\vec{v}$, then curl our fingers toward the vector $\vec{u}$. In this case, the thumb points in the opposite direction of $\vec{u} \times \vec{v}$. (Try it!)

Illustration 1.2.3: Anticommutativity of the Cross Product

Let $\vec{u} = \langle 0, 2, 1 \rangle$ and $\vec{v} = \langle 3, -1, 0 \rangle$. We calculate $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ and graph them.



We have

$$\vec{u} \times \vec{v} = \langle (0+1), -(0-3), (0-6) \rangle = \langle 1, 3, -6 \rangle$$

$$\vec{v} \times \vec{u} = \langle (-1-0), -(3-0), (6-0) \rangle = \langle -1, -3, 6 \rangle$$

We see that, in this case, $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$. We prove this in general later in this section.

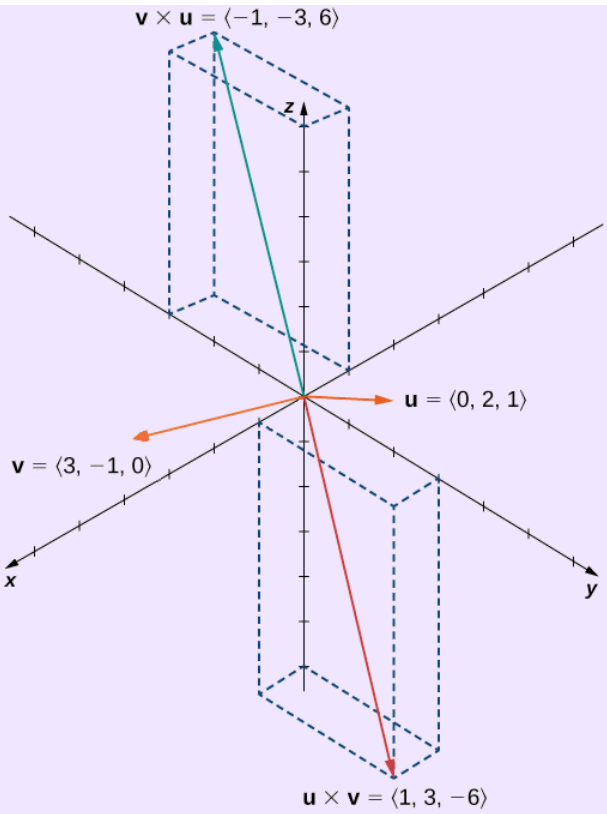


Figure 1.2.9: The cross products $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ are both orthogonal to $\vec{u}$ and $\vec{v}$, but in opposite directions.

The cross products of the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ can be useful for simplifying some calculations, so let's consider these cross products. A straightforward application of the definition shows that

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{0}$$

(The cross product of two vectors is a vector, so each of these products results in the zero vector, not the scalar 0.) Feel free verify these calculations.

Furthermore, because the cross product of two vectors is orthogonal to each of these vectors, we know that the cross product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is parallel to $\hat{\mathbf{k}}$. Similarly, the vector product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{j}}$, and the vector product of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{i}}$.

We can use the right-hand rule to determine the direction of each product. Then we have

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

These formulas come in handy later.

Example 1.2.9: Cross Product of Standard Unit Vectors

Evaluate each expression.

(a) $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}})$

(b) $(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{i}})$

Solution

(a) We know that $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$. Therefore, $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \boxed{\vec{0}}$

(b) $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$

Therefore $(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{i}}) = \hat{\mathbf{k}} \times \hat{\mathbf{j}} = \boxed{-\hat{\mathbf{i}}}$

As we have seen, the dot product results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product.** These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Rule: Properties of the Cross Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in space, and let c be a scalar.

i. Anticommutative property:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$$

ii. Distributive property:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$$

iii. Multiplication by a constant:

$$c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$$

iv. Cross product of the zero vector:

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$$

v. Cross product of a vector with itself:

$$\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

vi. Triple scalar product:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}$$

vii. Triple cross product:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\,\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\,\vec{\mathbf{w}}$$

Proof

For property i, we want to show $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$. We have

$$\begin{aligned}
\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle \\
&= \langle u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1 \rangle \\
&= -\langle u_3 v_2 - u_2 v_3, -u_3 v_1 + u_1 v_3, u_2 v_1 - u_1 v_2 \rangle \\
&= -\langle v_1, v_2, v_3 \rangle \times \langle u_1, u_2, u_3 \rangle \\
&= -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})
\end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property iv., this follows directly from the definition of the cross product. We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \langle u_2(0) - u_3(0), -(u_1(0) - u_3(0)), u_1(0) - u_2(0) \rangle = \langle 0, 0, 0 \rangle = \vec{\mathbf{0}}$$

Then, by property i., $\vec{0} \times \vec{u} = \vec{0}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* 0, whereas the cross product of a vector with the zero vector is the *vector* $\vec{0}$.

Property vi. looks like the associative property, but note the change in operations:

$$\begin{aligned}\vec{u} \cdot (\vec{v} \times \vec{w}) &= u \cdot \langle v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1 \rangle \\ &= u_1(v_2 w_3 - v_3 w_2) + u_2(-v_1 w_3 + v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1 \\ &= (u_2 v_3 - u_3 v_2) w_1 + (u_3 v_1 - u_1 v_3) w_2 + (u_1 v_2 - u_2 v_1) w_3 \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \cdot \langle w_1, w_2, w_3 \rangle = (\vec{u} \times \vec{v}) \cdot \vec{w}\end{aligned}$$

Example 1.2.10: Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\hat{i} \times 3\hat{j}) \times \hat{j}$.

Solution

$$\begin{aligned}(2\hat{i} \times 3\hat{j}) \times \hat{j} &= 2(\hat{i} \times 3\hat{j}) \times \hat{j} \\ &= 2(3)(\hat{i} \times \hat{j}) \times \hat{j} \\ &= (6\hat{k}) \times \hat{j} \\ &= 6(\hat{k} \times \hat{j}) \\ &= 6(-\hat{i}) = \boxed{-6\hat{i}}\end{aligned}$$

So far in this section, we have been concerned with the direction of the vector $\vec{u} \times \vec{v}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\vec{u} \times \vec{v}$ involving the magnitudes of $\vec{u}$ and $\vec{v}$, and the sine of the angle between them.

Rule: Magnitude of the Cross Product

Let $\vec{u}$ and $\vec{v}$ be vectors, and let θ be the angle between them. Then, $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin(\theta)$

Proof

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{aligned}\|\vec{u} \times \vec{v}\|^2 &= (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 \\ &= u_2^2 v_3^2 - 2u_2 u_3 v_2 v_3 + u_3^2 v_2^2 + u_3^2 v_1^2 - 2u_1 u_3 v_1 v_3 + u_1^2 v_3^2 + u_1^2 v_2^2 - 2u_1 u_2 v_1 v_2 + u_2^2 v_1^2 \\ &= u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2 \\ &\quad - (u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 + 2u_1 u_2 v_1 v_2 + 2u_1 u_3 v_1 v_3 + 2u_2 u_3 v_2 v_3) \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\ &= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 \\ &= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2(\theta) \\ &= \|\vec{u}\|^2 \|\vec{v}\|^2 \left(1 - \cos^2(\theta)\right) \\ &= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2(\theta)\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2(\theta)} = \sin(\theta)$ for $0 \leq \theta \leq 180°$, we have the desired result:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin(\theta)$$

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

The formula to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using determinant notation.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

This is referred to as the cofactor expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example 1.2.11: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinants

(a) $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$

Solution

(a) Expanding on the first row gives:

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4-9) - 5(-4+6) - 1(-3+2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= \boxed{-19} \end{aligned}$$

(b) Once more:

$$\begin{aligned} \begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix} &= 1 \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} - (-2) \begin{vmatrix} 3 & -3 \\ 1 & 4 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} \\ &= 1(8 - (-15)) - (-2)(12 - (-3)) + (-1)(15 - 2) \\ &= 1(23) + 2(15) - 1(13) \\ &= \boxed{40} \end{aligned}$$

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Rule: Cross Product Calculated by a Determinant

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}}$$

Example 1.2.12: Using Determinant Notation to find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$

Let $\vec{\mathbf{p}} = \langle -1, 2, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3 \rangle$. Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\vec{\mathbf{u}}$ in the second row, and the components of $\vec{\mathbf{v}}$ in the third row. Then, we have

$$\begin{aligned} \vec{\mathbf{p}} \times \vec{\mathbf{q}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} \hat{\mathbf{k}} \\ &= (-6 - 0)\hat{\mathbf{i}} - (3 - 20)\hat{\mathbf{j}} + (0 - 8)\hat{\mathbf{k}} \\ &= \boxed{-6\hat{\mathbf{i}} + 17\hat{\mathbf{j}} - 8\hat{\mathbf{k}}} \end{aligned}$$

Notice that this answer confirms the calculation of the cross product in Example 1.2.7.

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 1.2.13: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\vec{\mathbf{a}} = \langle 5, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Solution

The cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to both vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We can calculate it with a determinant:

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} \hat{\mathbf{k}} \\ &= (8 - 1)\hat{\mathbf{i}} - (20 - 0)\hat{\mathbf{j}} + (-5 - 0)\hat{\mathbf{k}} \\ &= 7\hat{\mathbf{i}} - 20\hat{\mathbf{j}} - 5\hat{\mathbf{k}} \end{aligned}$$

Normalize this vector to find a unit vector in the same direction:

$$\|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}$$

Thus, $\boxed{\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle}$ is a unit vector orthogonal to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

To use the cross product for calculating areas, we state and prove the following theorem.

Rule: Area of a Parallelogram

If we locate vectors $\vec{u}$ and $\vec{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{u} \times \vec{v}\|$

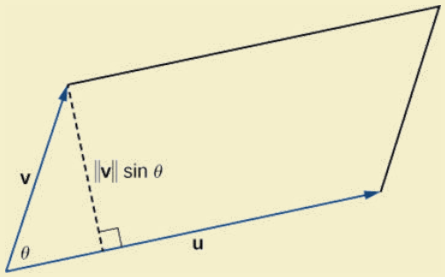


Figure 1.2.10: The parallelogram with adjacent sides $\vec{u}$ and $\vec{v}$ has base $\|\vec{u}\|$ and height $\|\vec{v}\| \sin(\theta)$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned} \text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\vec{u}\| \left(\|\vec{v}\| \sin(\theta)\right) \\ &= \|\vec{u} \times \vec{v}\| \end{aligned}$$

Example 1.2.14

Find the area of the parallelogram $PQRS$ with vertices $P(1,1,0)$, $Q(7,1,0)$, $R(9,4,2)$, and $S(3,4,2)$

Solution

We have $\overrightarrow{PQ} = \langle 7-1, 1-1, 0-0 \rangle = \langle 6,0,0 \rangle$ and $\overrightarrow{PS} = \langle 3-1, 4-1, 2-0 \rangle = \langle 2,3,2 \rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is given by $\| \overrightarrow{PQ} \times \overrightarrow{PS} \|$:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PS} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 0 & 0 \\ 2 & 3 & 2 \end{vmatrix} \\ &= (0-0)\hat{\mathbf{i}} - (12-0)\hat{\mathbf{j}} + (18-0)\hat{\mathbf{k}} \\ &= -12\hat{\mathbf{j}} + 18\hat{\mathbf{k}} \end{aligned}$$

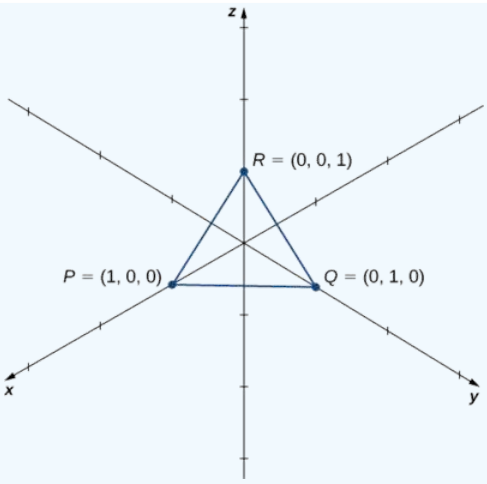
$$\begin{aligned} \| \overrightarrow{PQ} \times \overrightarrow{PS} \| &= \| \langle 0, -12, 18 \rangle \| \\ &= \sqrt{0^2 + (-12)^2 + 18^2} \\ &= 6\sqrt{13} \end{aligned}$$

So the area of parallelogram $PQRS$ is $\boxed{6\sqrt{13}\,\text{units}^2}$

Notice that if you used $\| \overrightarrow{PQ} \times \overrightarrow{PR} \|$ or $\| \overrightarrow{PR} \times \overrightarrow{PS} \|$ you would still get the same answer. Can you visualize why?

Example 1.2.15: Finding the Area of a Triangle

Let $P = (1,0,0), Q = (0,1,0)$, and $R = (0,0,1)$ be the vertices of a triangle. Find its area.



Solution

We have $\overrightarrow{PQ} = \langle 0-1, 1-0, 0-0\rangle = \langle -1, 1, 0\rangle$ and $\overrightarrow{PR} = \langle 0-1, 0-0, 1-0\rangle = \langle -1, 0, 1\rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by $\| \overrightarrow{PQ} \times \overrightarrow{PR} \|$:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= (1-0)\hat{\mathbf{i}} - (-1-0)\hat{\mathbf{j}} + (0-(-1))\hat{\mathbf{k}} \\ &= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \end{aligned}$$

$$\begin{aligned} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| &= \| \langle 1, 1, 1\rangle \| \\ &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3} \end{aligned}$$

The area of ΔPQR is half the area of the parallelogram or $\boxed{\frac{\sqrt{3}}{2} \text{ units}^2}$

Because the cross product of two vectors is a vector, it is possible to combine the dot product and the cross product. The dot product of a vector with the cross product of two other vectors is called the **triple scalar product** because the result is a scalar.

Definition: Triple Scalar Product

The triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$$

Rule: Calculating a Triple Scalar Product

The triple scalar product of vectors

$$\vec{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$$

$$\vec{\mathbf{v}} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{w}} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$$

is the determinant of the 3×3 matrix formed by the components of the vectors:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof

The calculation is straightforward.

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1 \rangle \\ &= u_1(v_2 w_3 - v_3 w_2) + u_2(-v_1 w_3 + v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= u_1(v_2 w_3 - v_3 w_2) - u_2(v_1 w_3 - v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

Example 1.2.16: Calculating the Triple Scalar Product

Let $\vec{\mathbf{u}} = \langle 1, 3, 5 \rangle$, $\vec{\mathbf{v}} = \langle 2, -1, 0 \rangle$ and $\vec{\mathbf{w}} = \langle -3, 0, -1 \rangle$. Calculate the triple scalar product $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$

Solution

Apply the triple product equation directly:

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \begin{vmatrix} 1 & 3 & 5 \\ 2 & -1 & 0 \\ -3 & 0 & -1 \end{vmatrix} \\ &= 1 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} + 5 \begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} \\ &= (1 - 0) - 3(-2 - 0) + 5(0 - 3) \\ &= 1 + 6 - 15 = \boxed{-8} \end{aligned}$$

When we create a matrix from three vectors, we must be careful about the order in which we list the vectors. If we list them in a matrix in one order and then rearrange the rows, the absolute value of the determinant remains unchanged. However, each time two rows switch places, the determinant changes sign:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = d \qquad \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -d \qquad \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = d \qquad \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -d$$

Verifying this fact is straightforward, but rather messy. Let's take a look at this with an example:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ -2 & 0 & 3 \\ 4 & 1 & -1 \end{vmatrix} &= \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ 4 & 1 \end{vmatrix} \\ &= (0 - 3) - 2(2 - 12) + (-2 - 0) \\ &= -3 + 20 - 2 = 15 \end{aligned}$$

Switching the top two rows we have

$$\begin{aligned} \begin{vmatrix} -2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & -1 \end{vmatrix} &= -2 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\ &= -2(-2 - 1) + 3(1 - 8) \\ &= 6 - 21 = -15 \end{aligned}$$

Rearranging vectors in the triple products is equivalent to reordering the rows in the matrix of the determinant. Let $\vec{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$, $\vec{\mathbf{v}} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$, and $\vec{\mathbf{w}} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$. Applying Calculating a Triple Scalar Product, we have

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

and

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

We can obtain the determinant for calculating $\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$ by switching the bottom two rows of $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$. Therefore, $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$.

Following this reasoning and exploring the different ways we can interchange variables in the triple scalar product lead to the following identities:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$$

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$$

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two vectors in standard position. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram. We saw in Area of a Parallelogram that the area of this parallelogram is $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$. Now suppose we add a third vector $\vec{\mathbf{w}}$ that does not lie in the same plane as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ but still shares the same initial point. Then these vectors form three edges of a **parallelepiped**, a three-dimensional prism with six faces that are each parallelograms, as shown below. The volume of this prism is the product of the figure's height and the area of its base. The triple scalar product of $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ provides a simple method for calculating the volume of the parallelepiped defined by these vectors.

Rule: Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is the absolute value of the triple scalar product:

$$V = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$$

Note that, as the name indicates, the triple scalar product produces a scalar. The volume formula just presented uses the absolute value of a scalar quantity.

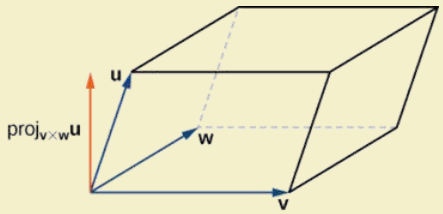


Figure 1.2.11: The height of the parallelepiped is given by $\|\text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}} \vec{\mathbf{u}}\|$

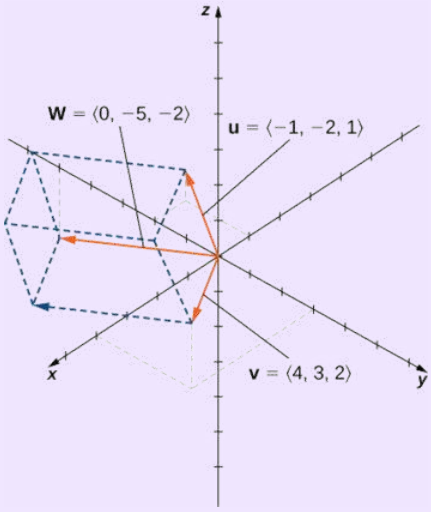
Proof

The area of the base of the parallelepiped is given by $\|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\|$. The height of the figure is given by $\|\text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}} \vec{\mathbf{u}}\|$. The volume of the parallelepiped is the product of the height and the area of the base, so we have

$$\begin{aligned} V &= \| \text{ proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}} \vec{\mathbf{u}} \, \| \, \|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\| \\ &= \left| \frac{\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})}{\|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\|} \right| \|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\| \\ &= |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})| \end{aligned}$$

Illustration 1.2.4: Calculating the Volume of a Parallelepiped

Let $\vec{\mathbf{u}} = \langle -1, -2, 1\rangle, \vec{\mathbf{v}} = \langle 4, 3, 2\rangle$, and $\vec{\mathbf{w}} = \langle 0, -5, -2\rangle$. Find the volume of the parallelepiped with adjacent edges $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.



We have

$$\begin{aligned}
\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \begin{vmatrix} -1 & -2 & 1 \\ 4 & 3 & 2 \\ 0 & -5 & -2 \end{vmatrix} \\
&= (-1)\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 0 & -5 \end{vmatrix} \\
&= (-1)(-6+10) + 2(-8-0) + (-20-0) \\
&= -4 - 16 - 20 \\
&= -40
\end{aligned}$$

Thus, the volume of the parallelepiped is $|-40| = 40$ units^3

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1.2: Practice Set

Exercise 1.2.1

Calculate the following dot and cross products:

(a) $\langle 1, 9\rangle \cdot \langle 8, -1\rangle$

(b) $(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}) \cdot (7\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$

(c) $\langle 2, 6, -3\rangle \cdot \langle 1, -2, 4\rangle$

(d) $(\hat{\mathbf{i}} - 3\hat{\mathbf{k}}) \cdot (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 7\hat{\mathbf{k}})$

(e) $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \times (3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$

(f) $(6\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \times (-3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

(g) $\langle -3, 2, 2\rangle \times \langle 3, 2, 2\rangle$

(h) $\langle 6, 7, 8\rangle \times \langle 3, 5, 7\rangle$

Answer

(a) $\langle 1, 9\rangle \cdot \langle 8, -1\rangle = 1(8) + 9(-1) = 8 - 9 = -1$

(b) $\langle 3, -2\rangle \cdot \langle 7, 4\rangle = 3(7) + (-2)(4) = 21 - 8 = 13$

(c) $\langle 2, 6, -3\rangle \cdot \langle 1, -2, 4\rangle = 2(1) + 6(-2) + (-3)(4) = 2 - 12 - 12 = -22$

(d) $\langle 1, 0, -3\rangle \cdot \langle 4, 2, 7\rangle = 1(4) + 0(2) + (-3)(7) = 4 + 0 - 21 = -17$

(e) $\langle 1, 1, 2\rangle \times \langle 3, 1, -1\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = (-1 - 2)\hat{\mathbf{i}} + (1 + 6)\hat{\mathbf{j}} + (1 - 3)\hat{\mathbf{k}} = \langle -3, 7, -2\rangle$

(f) $\langle 6, 4, -2\rangle \times \langle -3, -2, 1\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{vmatrix} = (4 - 4)\hat{\mathbf{i}} + (-6 + 6)\hat{\mathbf{j}} + (-12 + 12)\hat{\mathbf{k}} = \langle 0, 0, 0\rangle$

(g) $\langle -3, 2, 2\rangle \times \langle 3, 2, 2\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 2 & 2 \\ 3 & 2 & 2 \end{vmatrix} = (4 - 4)\hat{\mathbf{i}} + (6 + 6)\hat{\mathbf{j}} + (-6 - 6)\hat{\mathbf{k}} = \langle 0, 12, -12\rangle$

(h) $\langle 6, 7, 8\rangle \times \langle 3, 5, 7\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 7 & 8 \\ 3 & 5 & 7 \end{vmatrix} = (49 - 40)\hat{\mathbf{i}} + (-42 + 24)\hat{\mathbf{j}} + (30 - 21)\hat{\mathbf{k}} = \langle 9, -18, 9\rangle$

Exercise 1.2.2

Find the angle between each pair of given vectors:

(a) $\langle 2, -2\rangle$ and $\langle 0, -4\rangle$

(b) $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

(c) $\langle 5, -1, 1\rangle$ and $\langle 1, 1, -1\rangle$

(d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} + \hat{\mathbf{k}}$

(e) The unit vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ with $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \left\langle \frac{1}{3}, \frac{1}{4}, 0 \right\rangle$

Answer

(a) Letting $\vec{\mathbf{u}} = \langle 2, -2\rangle$ and $\vec{\mathbf{v}} = \langle 0, -4\rangle$,

we have $\|\vec{u}\| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$, $\|\vec{v}\| = \sqrt{0^2 + (-4)^2} = 4$ and $\vec{u} \cdot \vec{v} = 2(0) + (-2)(-4) = 8$

Therefore the angle between is $\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) = \cos^{-1}\left(\dfrac{8}{(2\sqrt{2})(4)}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$

(b) Letting $\vec{u} = \langle 3, -2\rangle$ and $\vec{v} = \langle 2, 3\rangle$,

we have $\|\vec{u}\| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$, $\|\vec{v}\| = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $\vec{u} \cdot \vec{v} = 3(2) + (-2)(3) = 0$

Therefore the angle between is $\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) = \cos^{-1}\left(\dfrac{0}{(\sqrt{13})(\sqrt{13})}\right) = \cos^{-1}(0) = \dfrac{\pi}{2}$

(c) Letting $\vec{u} = \langle 5, -1, 1\rangle$ and $\vec{v} = \langle 1, 1, -1\rangle$,

we have $\|\vec{u}\| = \sqrt{5^2 + (-1)^2 + 1^2} = 3\sqrt{3}$, $\|\vec{v}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$ and $\vec{u} \cdot \vec{v} = 5(1) + (-1)(1) + 1(-1) = 3$

Therefore the angle between is $\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) = \cos^{-1}\left(\dfrac{3}{(3\sqrt{3})(\sqrt{3})}\right) = \cos^{-1}\left(\dfrac{1}{3}\right)$

(d) Letting $\vec{u} = \langle 1, 1, 1\rangle$ and $\vec{v} = \langle 2, 0, 1\rangle$,

we have $\|\vec{u}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, $\|\vec{v}\| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$ and $\vec{u} \cdot \vec{v} = 1(2) + 1(0) + 1(1) = 3$

Therefore the angle between is $\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) = \cos^{-1}\left(\dfrac{3}{(\sqrt{3})(\sqrt{5})}\right) = \cos^{-1}\left(\dfrac{3}{\sqrt{15}}\right)$

(e) We know $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin(\theta)$, and $\|\vec{a}\| = \|\vec{b}\| = 1$ since they are unit vectors.

$\|\vec{a} \times \vec{b}\| = \left\|\left\langle \dfrac{1}{3}, \dfrac{1}{4}, 0\right\rangle\right\| = \sqrt{\left(\dfrac{1}{3}\right)^2 + \left(\dfrac{1}{4}\right)^2 + 0^2} = \sqrt{\dfrac{25}{144}} = \dfrac{5}{12}$, so $\theta = \sin^{-1}\left(\dfrac{5}{12}\right)$

? Exercise 1.2.3

Find the indicated projections below:

(a) $\langle 5, 1\rangle$ projected onto $\langle 1, -1\rangle$

(b) $\langle 3, 7\rangle$ projected onto $\langle 1, 1\rangle$

(c) $\langle 5, -1, 1\rangle$ projected onto $\langle 1, 1, -1\rangle$

(d) $\langle 1, 1, -1\rangle$ projected onto $\langle 5, -1, 1\rangle$

Answer

(a) Letting $\vec{u} = \langle 5, 1\rangle$ and $\vec{v} = \langle 1, -1\rangle$, we have $\vec{u} \cdot \vec{v} = 5(1) + 1(-1) = 4$ and $\vec{v} \cdot \vec{v} = 1^2 + (-1)^2 = 2$

Therefore $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \dfrac{4}{2}\vec{v} = 2\langle 1, -1\rangle = \langle 2, -2\rangle$

(b) Letting $\vec{u} = \langle 3, 7\rangle$ and $\vec{v} = \langle 1, 1\rangle$, we have $\vec{u} \cdot \vec{v} = 3(1) + 7(1) = 10$ and $\vec{v} \cdot \vec{v} = 1^2 + 1^2 = 2$

Therefore $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \dfrac{10}{2}\vec{v} = 5\langle 1, 1\rangle = \langle 5, 5\rangle$

(c) Letting $\vec{u} = \langle 5, -1, 1\rangle$ and $\vec{v} = \langle 1, 1, -1\rangle$, we have $\vec{u} \cdot \vec{v} = 5(1) + (-1)(1) + 1(-1) = 3$ and $\vec{v} \cdot \vec{v} = 1^2 + 1^2 + (-1)^2 = 3$

Therefore $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \dfrac{3}{3}\vec{v} = 1\langle 1, 1, -1\rangle = \langle 1, 1, -1\rangle$

(d) Letting $\vec{u} = \langle 1, 1, -1\rangle$ and $\vec{v} = \langle 5, -1, 1\rangle$, we have $\vec{u} \cdot \vec{v} = 1(5) + 1(-1) + (-1)(1) = 3$ and $\vec{v} \cdot \vec{v} = 5^2 + (-1)^2 + 1^2 = 27$

Therefore $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \dfrac{3}{27}\vec{v} = \dfrac{1}{9}\langle 5, -1, 1\rangle = \left\langle \dfrac{5}{9}, -\dfrac{1}{9}, \dfrac{1}{9}\right\rangle$

Exercise 1.2.4

Find the area of the described object:

(a) The parallelogram with vertices $(0,1,0), (2,2,1), (1,2,3)$, and $(3,3,4)$

(b) The triangle with vertices $(0,0,0), (5,3,1)$, and $(4,-1,3)$

(c) If the parallelogram with two adjacent sides defined by $\vec{v}$ and $\vec{w}$ has area 12, what is the area of the parallelogram with sides $\vec{v}$ and $2\vec{v}+3\vec{w}$?

Answer

(a) This parallelogram has adjacent sides defined by the vectors $\vec{\mathbf{u}} = \langle 2-0, 2-1, 1-0\rangle = \langle 2,1,1\rangle$ and $\vec{\mathbf{v}} = \langle 1-0, 2-1, 3-0\rangle = \langle 1,1,3\rangle$

We calculate $\vec{\mathbf{u}}\times\vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = (3-1)\hat{\mathbf{i}} + (-6+1)\hat{\mathbf{j}} + (2-1)\hat{\mathbf{k}} = \langle 2,-5,1\rangle$

Therefore, the area is $\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\| = \|\langle 2,-5,1\rangle\| = \sqrt{2^2+(-5)^2+1^2} = \sqrt{30}$

(b) This triangle is half of the parallelogram that has adjacent sides defined by the vectors $\vec{\mathbf{u}} = \langle 5,3,1\rangle$ and $\vec{\mathbf{v}} = \langle 4,-1,3\rangle$

We calculate $\vec{\mathbf{u}}\times\vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 3 & 1 \\ 4 & -1 & 3 \end{vmatrix} = (9+1)\hat{\mathbf{i}} + (-15+4)\hat{\mathbf{j}} + (-5-12)\hat{\mathbf{k}} = \langle 10,-11,-17\rangle$

Therefore, the area is $\frac{1}{2}\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\| = \frac{1}{2}\|\langle 10,-11,-17\rangle\| = \frac{1}{2}\sqrt{10^2+(-11)^2+(-17)^2} = \frac{\sqrt{510}}{2}$

(c) We are given that $\|\vec{\mathbf{v}}\times\vec{\mathbf{w}}\| = 12$, and want the area given by $\|\vec{\mathbf{v}}\times(2\vec{\mathbf{v}}+3\vec{\mathbf{w}})\|$

Since $\vec{\mathbf{v}}\times(2\vec{\mathbf{v}}+3\vec{\mathbf{w}}) = 2(\vec{\mathbf{v}}\times\vec{\mathbf{v}}) + 3(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = 2(\vec{\mathbf{0}}) + 3(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = 3(\vec{\mathbf{v}}\times\vec{\mathbf{w}})$

We must have area $\|\vec{\mathbf{v}}\times(2\vec{\mathbf{v}}+3\vec{\mathbf{w}})\| = 3\|\vec{\mathbf{v}}\times\vec{\mathbf{w}}\| = 3(12) = 36$

Exercise 1.2.5

Solve for the t that satisfies each description below (if possible):

(a) $(t\hat{\mathbf{i}}+2\hat{\mathbf{j}})\cdot(3\hat{\mathbf{i}}-4\hat{\mathbf{j}}) = 1$

(b) $\langle 4,1,3\rangle\cdot\langle t,-t,-t\rangle = 4$

(c) $\langle t,t,t\rangle\cdot\langle 2,3,-1\rangle = \langle 4,-3,1\rangle$

(d) $\text{proj}_{\hat{\mathbf{j}}}(\langle t,2t,3t\rangle) = \hat{\mathbf{j}}$

(e) For $\vec{\mathbf{x}} = \langle t,2,1\rangle$, $\text{proj}_{\vec{\mathbf{x}}}\left(15\hat{\mathbf{j}}+24\hat{\mathbf{k}}\right) = \langle -12,12,6\rangle$

(f) The angle between $\vec{\mathbf{a}} = \langle t,t\rangle$ and $\vec{\mathbf{b}} = \langle 4,0\rangle$ is exactly $\frac{\pi}{4}$ radians

(g) The angle between $\vec{\mathbf{u}} = 4\hat{\mathbf{i}}+4\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = \hat{\mathbf{j}}+t\hat{\mathbf{k}}$ is exactly $\frac{\pi}{3}$ radians

(h) The angle between $\vec{\eta} = \langle -1,t,2\rangle$ and $\vec{\Omega} = \langle t,t,-3\rangle$ is exactly $\frac{\pi}{2}$ radians

(i) $(t\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})\times(t\hat{\mathbf{j}}+\hat{\mathbf{k}}) = -2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

(j) The parallelogram with vertices $(0,0,0), (0,4,1), (t,t,0)$, and $(t,t+4,1)$ has area $27\sqrt{2}$

(k) The parallelogram with opposing vertices $(0,0)$ and $(5,3)$, both connecting to $(t,2)$ and some other point has an area of 14

Answer

(a) $(t\hat{\mathbf{i}}+2\hat{\mathbf{j}})\cdot(3\hat{\mathbf{i}}-4\hat{\mathbf{j}}) = \langle t,2\rangle\cdot\langle 3,-4\rangle = t(3)+2(-4) = 3t-8$

Setting this equal to 1 gives $3t-8=1 \implies 3t=9 \implies t=3$

(b) $\langle 4,1,3\rangle \cdot \langle t,-t,-t\rangle = 4(t)+1(-t)+3(-t) = 0$

Setting this equal to 4 gives $0 = 4$ which has no solution, so this is **impossible**.

(c) $\langle t,t,t\rangle \cdot \langle 2,3,-1\rangle = \langle 4,-3,1\rangle$ is immediately **impossible** since the dot product is a scalar, not a vector!

(d) $\text{proj}_{\hat{\mathbf{j}}}(\langle t,2t,3t\rangle) = \dfrac{\langle 0,1,0\rangle \cdot \langle t,2t,3t\rangle}{\langle 0,1,0\rangle \cdot \langle 0,1,0\rangle}\hat{\mathbf{j}} = \dfrac{2t}{1}\hat{\mathbf{j}}$

Setting this equal to $\hat{\mathbf{j}}$ is the same as setting $2t = 1$ which has solution $t = \dfrac{1}{2}$

(e) Define $\vec{\mathbf{u}} = \langle 0,15,24\rangle$ and $\vec{\mathbf{x}} = \langle t,2,1\rangle$

We have $\vec{\mathbf{u}} \cdot \vec{\mathbf{x}} = 0(t)+15(2)+24(1) = 54$ and $\vec{\mathbf{x}} \cdot \vec{\mathbf{x}} = t^2+2^2+1^2 = t^2+5$

Thus $\text{proj}_{\vec{x}}(\vec{\mathbf{u}}) = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{x}}}{\vec{\mathbf{x}} \cdot \vec{\mathbf{x}}}\vec{\mathbf{x}} = \dfrac{54}{t^2+5}\langle t,2,1\rangle = \left\langle \dfrac{54t}{t^2+5}, \dfrac{108}{t^2+5}, \dfrac{54}{t^2+5}\right\rangle$

Setting $\left\langle \dfrac{54t}{t^2+5}, \dfrac{108}{t^2+5}, \dfrac{54}{t^2+5}\right\rangle = \langle -12,12,6\rangle \implies \begin{cases} \dfrac{54t}{t^2+5} = -12 \\ \dfrac{108}{t^2+5} = 12 \\ \dfrac{54}{t^2+5} = 6 \end{cases}$ which has solution $t = -2$.

(f) For $\vec{\mathbf{a}} = \langle t,t\rangle$ and $\vec{\mathbf{b}} = \langle 4,0\rangle$ we have $\|\vec{\mathbf{a}}\| = \sqrt{2t^2}, \|\vec{\mathbf{b}}\| = 4$, and $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 4t$

The angle between them must satisfy $\cos(\theta) = \dfrac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\|\vec{\mathbf{a}}\|\,\|\vec{\mathbf{b}}\|} = \dfrac{4t}{\sqrt{2t^2}\,4} = \dfrac{t}{|t|\sqrt{2}}$

Therefore we need $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ to equal $\dfrac{t}{|t|\sqrt{2}}$

This is equivalent to saying $\dfrac{t}{|t|} = 1$ which is true for all $t > 0$.

(g) For $\vec{\mathbf{u}} = \langle 4,4,0\rangle$ and $\vec{\mathbf{v}} = \langle 0,1,t\rangle$ we have $\|\vec{\mathbf{u}}\| = 4\sqrt{2}, \|\vec{\mathbf{v}}\| = \sqrt{t^2+1}$ and $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 4$.

The angle between them must satisfy $\cos(\theta) = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\|} = \dfrac{4}{4\sqrt{2}\sqrt{t^2+1}} = \dfrac{1}{\sqrt{2t^2+2}}$

Therefore we need $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$ to equal $\dfrac{1}{\sqrt{2t^2+2}}$

This is equivalent to saying $2t^2+2 = 4$ which has solutions $t = \pm 1$.

(h) For $\vec{\eta} = \langle -1,t,2\rangle$ and $\vec{\Omega} = \langle t,t,-3\rangle$ we have $\|\vec{\eta}\| = \sqrt{t^2+5}, \|\vec{\Omega}\| = \sqrt{2t^2+9}$ and $\vec{\eta} \cdot \vec{\Omega} = t^2-t-6$

The angle between them must satisfy $\cos(\theta) = \dfrac{\vec{\eta} \cdot \vec{\Omega}}{\|\vec{\eta}\|\,\|\vec{\Omega}\|} = \dfrac{t^2-t-6}{\sqrt{t^2+5}\sqrt{2t^2+9}}$

Therefore we need $\cos\left(\dfrac{\pi}{2}\right) = 0$ to equal $\dfrac{t^2-t-6}{\sqrt{t^2+5}\sqrt{2t^2+9}}$

This is equivalent to saying $t^2-t-6 = 0$, which has solutions $t = 3, -2$.

(i) $\langle t,2,1\rangle \times \langle 0,t,1\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & 2 & 1 \\ 0 & t & 1 \end{vmatrix} = (2-t)\hat{\mathbf{i}} + (-t+0)\hat{\mathbf{j}} + (t^2-0)\hat{\mathbf{k}} = \langle 2-t,-t,t^2\rangle$

For this to equal $-2\hat{\mathbf{j}} + 4\hat{\mathbf{k}} = \langle 0,-2,4\rangle$ we need $\begin{cases} 2-t = 0 \\ -t = -2 \\ t^2 = 4 \end{cases}$

The only value of t which satisfies all three equations is $t = 2$.

(j) The parallelogram with vertices $(0,0,0), (0,4,1), (t,t,0)$, and $(t,t+4,1)$ has adjacent sides $\vec{\mathbf{u}} = \langle 0,4,1\rangle$ and $\vec{\mathbf{v}} = \langle t,t,0\rangle$

The cross product is $\langle 0,4,1\rangle \times \langle t,t,0\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 4 & 1 \\ t & t & 0 \end{vmatrix} = (0-t)\hat{\mathbf{i}} + (0+t)\hat{\mathbf{j}} + (0-4t)\hat{\mathbf{k}} = \langle -t,t,-4t\rangle$

Therefore the area is $\|\langle -t, t, -4t\rangle\| = \sqrt{18t^2}$. Setting this equal to $27\sqrt{2}$ and squaring both sides gives

$18t^2 = 27^2 \cdot 2 \longrightarrow t^2 = 81$ which has solutions $t = \pm 9$

(k) The parallelogram with opposing vertices $(0,0)$ and $(5,3)$, both connecting to $(t,2)$ is in the xy-plane.

In $\mathbb{R}^3$ the adjacent sides are $\vec{\mathbf{u}} = \langle t, 2, 0\rangle$ and $\vec{\mathbf{v}} = \langle 5-t, 1, 0\rangle$.

The cross product is $\langle t,2,0\rangle \times \langle 5-t,1,0\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & 2 & 0 \\ 5-t & 1 & 0 \end{vmatrix} = (0-0)\hat{\mathbf{i}} + (0+0)\hat{\mathbf{j}} + (t-(10-2t))\hat{\mathbf{k}} = \langle 0, 0, 3t-10\rangle$

Therefore the area is $\|\langle 0,0,3t-10\rangle\| = |3t-10|$. Setting this equal to 14 and gives

$|3t-10| = 14 \longrightarrow 3t-10 = \pm 14 \longrightarrow t = \dfrac{10 \pm 14}{3}$, so our solutions are $t = 8, -\dfrac{4}{3}$.

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1.3: Lines and Planes

Learning Objectives

- Write the vector, parametric, and symmetric of a line through a given point in a given direction, and a line through two given points.
- Find the distance from a point to a given line.
- Write the vector and scalar equations of a plane through a given point with a given normal.
- Find the distance from a point to a given plane.
- Find the angle between two planes.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

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1.3: Lines

Let's first explore what it means for two vectors to be **parallel**. Recall that parallel vectors must have identical or opposite directions. If two nonzero vectors, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, are parallel, we claim there must be a scalar, k, such that $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction, simply choose

$$k = \frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}$$

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions, choose

$$k = -\frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}$$

Note that the converse holds as well. If $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k, then either $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction $(k > 0)$ or opposite directions $(k < 0)$, so $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel. Therefore, two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if and only if $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k. By convention, the zero vector $\vec{\mathbf{0}}$ is considered to be parallel to all vectors.

As in three dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the **direction vector** or **slope vector**. Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\vec{\mathbf{v}} = \langle a, b, c\rangle$ be a vector parallel to L. Then, for any point on line $Q(x, y, z)$, we know that $\overrightarrow{PQ}$ is parallel to $\vec{\mathbf{v}}$.

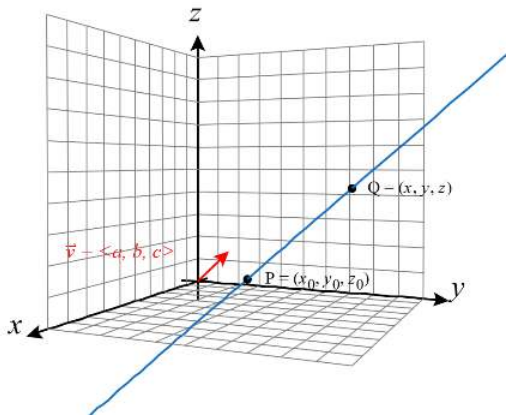


Figure 1.3.1: Vector $\vec{\mathbf{v}}$ is the direction vector for $\overrightarrow{PQ}$

Thus, as we just discussed, there is a scalar, t, such that $\overrightarrow{PQ} = t\vec{\mathbf{v}}$, which gives

$$\begin{aligned} \overrightarrow{PQ} &= t\vec{\mathbf{v}} \\ \langle x - x_0, y - y_0, z - z_0\rangle &= t\langle a, b, c\rangle \\ \langle x - x_0, y - y_0, z - z_0\rangle &= \langle ta, tb, tc\rangle \end{aligned}$$

Using vector operations, we can rewrite this as

$$\begin{aligned} \langle x - x_0, y - y_0, z - z_0\rangle &= \langle ta, tb, tc\rangle \\ \langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle &= t\langle a, b, c\rangle \\ \underbrace{\langle x, y, z\rangle}_{\vec{\mathbf{r}}} &= \underbrace{\langle x_0, y_0, z_0\rangle}_{\vec{\mathbf{r}}_0} + t\underbrace{\langle a, b, c\rangle}_{\vec{\mathbf{v}}} \end{aligned}$$

Setting $\vec{\mathbf{r}} = \langle x, y, z\rangle$ and $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0\rangle$, we now have the **vector equation of a line:**

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$$

Equating components, this shows that the following equations are simultaneously true: $x - x_0 = ta, y - y_0 = tb$, and $z - z_0 = tc$. If we solve each of these equations for the component variables x, y, and z, we get a set of equations in which each variable is defined

in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line:**

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t$$

$$\frac{y - y_0}{b} = t$$

$$\frac{z - z_0}{c} = t$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line:**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

We summarize the results in the following rule.

Rule: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, y = y_0 + tb, \text{ and } z = z_0 + tc$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Illustration 1.3.1: Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points

(a) $(1, 4, -2)$ and $(-3, 5, 0)$ and (b) $(1, -3, 2)$ and $(5, -2, 8)$

For the points in (a), first identify a vector parallel to the line:

$$\vec{\mathbf{v}} = \langle -3 - 1, 5 - 4, 0 - (-2) \rangle = \langle -4, 1, 2 \rangle$$

Use either of the given points on the line to complete the parametric equations:

$$x = 1 - 4t$$

$$y = 4 + t,$$

$$z = -2 + 2t$$

Solve each equation for t to create the symmetric equation of the line:

$$\frac{x - 1}{-4} = y - 4 = \frac{z + 2}{2}$$

For (b), we again identify a vector parallel to the line:

$$\vec{\mathbf{v}} = \langle 5-1, -2-(-3), 8-2\rangle = \langle 4, 1, 6\rangle$$

Use either of the given points on the line to complete the parametric equations:

$$x = 1 + 4t$$
$$y = -3 + t$$
$$z = 2 + 6t$$

Solve each equation for t to create the symmetric equation of the line:

$$\frac{x-1}{4} = y + 3 = \frac{z-2}{6}$$

Sometimes we don't want the equation of a whole line, just a line segment. In this case, we limit the values of our parameter t. For example, let $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ be points on a line, and let $\vec{\mathbf{p}} = \langle x_0, y_0, z_0\rangle$ and $\vec{\mathbf{q}} = \langle x_1, y_1, z_1\rangle$ be the associated position vectors. In addition, let $\vec{\mathbf{r}} = \langle x, y, z\rangle$. We want to find a vector equation for the line segment between P and Q. Using P as our known point on the line, and $\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$ as the direction vector this yields:

$$\vec{\mathbf{r}} = \vec{\mathbf{p}} + t(\overrightarrow{PQ})$$

This equation can be expanded using properties of vectors:

$$\begin{aligned}\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\ &= \langle x_0, y_0, z_0\rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle \\ &= \langle x_0, y_0, z_0\rangle + t(\langle x_1, y_1, z_1\rangle - \langle x_0, y_0, z_0\rangle) \\ &= \langle x_0, y_0, z_0\rangle + t\langle x_1, y_1, z_1\rangle - t\langle x_0, y_0, z_0\rangle \\ &= (1-t)\langle x_0, y_0, z_0\rangle + t\langle x_1, y_1, z_1\rangle \\ &= (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}\end{aligned}$$

Thus, the vector equation of the line passing through P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}$$

Remember that we did not want the equation of the whole line, just the line segment between P and Q. Notice that when $t = 0$, we have $\vec{\mathbf{r}} = \vec{\mathbf{p}}$, and when $t = 1$, we have $\vec{\mathbf{r}} = \vec{\mathbf{q}}$. Therefore, the vector equation of the line segment between P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}, \quad 0 \le t \le 1$$

We can also find parametric equations for this line segment. We have

$$\begin{aligned}\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\ \langle x, y, z\rangle &= \langle x_0, y_0, z_0\rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle \\ &= \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0)\rangle\end{aligned}$$

Then, the parametric equations are (for $0 \le t \le 1$):

$$x = x_0 + t(x_1 - x_0)$$
$$y = y_0 + t(y_1 - y_0)$$
$$z = z_0 + t(z_1 - z_0)$$

Example 1.3.1: Parametric Equations of a Line Segment

Find parametric equations of the line segment between the points

(a) $P(2, 1, 4)$ and $Q(3, -1, 3)$

(b) $P(-1, 3, 6)$ and $Q(-8, 2, 4)$

Solution

(a) Start with the parametric equations for a line and work with each component separately:

$$x = x_0 + t(x_1 - x_0) = 2 + t(3 - 2) = 2 + t$$
$$y = y_0 + t(y_1 - y_0) = 1 + t(-1 - 1) = 1 - 2t$$
$$z = z_0 + t(z_1 - z_0) = 4 + t(3 - 4) = 4 - t$$

Therefore, the parametric equations for the line segment are:

$$\boxed{x = 2 + t,\ y = 1 - 2t,\ z = 4 - t, \quad 0 \le t \le 1}$$

(b) Start with the parametric equations for a line and work with each component separately:

$$x = x_0 + t(x_1 - x_0) = -1 + t(-8 - (-1)) = -1 - 7t$$
$$y = y_0 + t(y_1 - y_0) = 3 + t(2 - 3) = 3 - t$$
$$z = z_0 + t(z_1 - z_0) = 6 + t(4 - 6) = 6 - 2t$$

Therefore, the parametric equations for the line segment are:

$$\boxed{x = -1 - 7t,\ y = 3 - t,\ z = 6 - 2t, \quad 0 \le t \le 1}$$

We already know how to calculate the distance between two points in space. We now expand this definition to describe the distance between a point and a line in space. Several real-world contexts exist when it is important to be able to calculate these distances. When building a home, for example, builders must consider "setback" requirements, when structures or fixtures have to be a certain distance from the property line. Air travel offers another example. Airlines are concerned about the distances between populated areas and proposed flight paths.

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of line segment $\overline{MP}$, where P is a point on L such that $\overline{MP}$ is perpendicular to L.

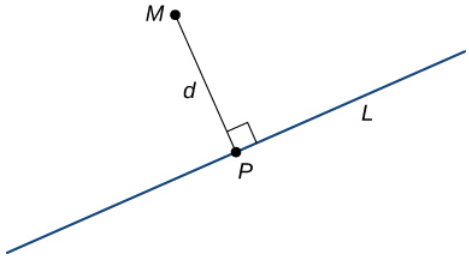


Figure 1.3.2: The distance from point M to line L is the length of $\overline{MP}$.

When we're looking for the distance between a line and a point in space, Figure 1.3.2 still applies. We still define the distance as the length of the perpendicular line segment connecting the point to the line. In space, however, there is no clear way to know which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance. Therefore, let P be an arbitrary point on line L and let $\vec{\mathbf{v}}$ be a direction vector for L as below.

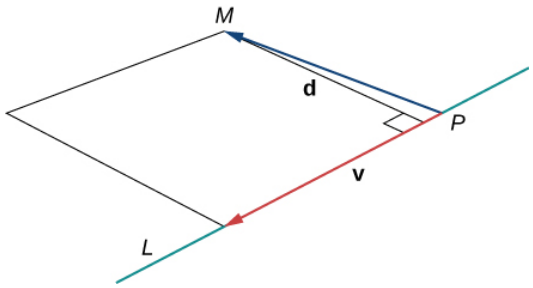


Figure 1.3.3: Vectors $\overrightarrow{PM}$ *and* $\vec{\mathbf{v}}$ *form two sides of a parallelogram with base* $\|\vec{\mathbf{v}}\|$ *and height* d*, which is the distance between a line and a point in space.*

Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ form two sides of a parallelogram with area $\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|$. Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height:

$$\|\overrightarrow{PM} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\| d$$

We can use this formula to find a general formula for the distance between a line in space and any point not on the line.

Rule: Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$$

Example 1.3.2: Calculating the Distance from a Point to a Line

Find the distance between the point $M = (1, 1, 3)$ and line $\frac{x-3}{4} = \frac{y+1}{2} = z - 3$

Solution:

From the symmetric equations of the line, we know that vector $\vec{\mathbf{v}} = \langle 4, 2, 1 \rangle$ is a direction vector for the line. Setting the symmetric equations of the line equal to zero, we see that point $P(3, -1, 3)$ lies on the line. Then,

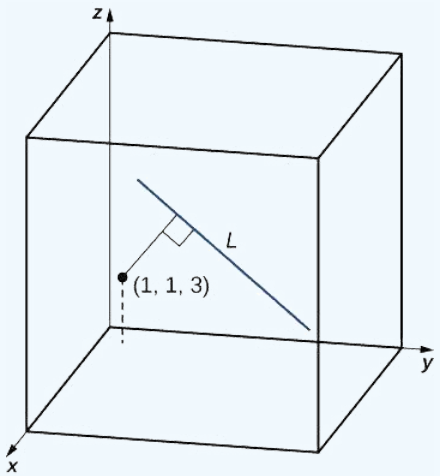
$$\overrightarrow{PM} = \langle 1 - 3, 1 - (-1), 3 - 3 \rangle = \langle -2, 2, 0 \rangle$$

To calculate the distance, we need to find $\overrightarrow{PM} \times \vec{\mathbf{v}}$:

$$\overrightarrow{PM} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$$

Therefore, the distance between the point and the line is

$$d = \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|} = \frac{\sqrt{2^2 + 2^2 + 12^2}}{\sqrt{4^2 + 2^2 + 1^2}} = \boxed{\frac{2\sqrt{38}}{\sqrt{21}} \text{ units}}$$



Example 1.3.3

Find the distance between point $\vec{\mathbf{A}} = (0, 3, 6)$ and the line with parametric equations $x = 1 - t, y = 1 + 2t, z = 5 + 3t$

Solution:

We can write the line as $\langle x, y, z \rangle = \langle 1, 1, 5 \rangle + \langle -1, 2, 3 \rangle t$ so it contains the point $\vec{\mathbf{P}} = (1, 1, 5)$ and direction vector $\vec{v} = \langle -1, 2, 3 \rangle$.

The vector $\overrightarrow{PA} = \langle 0 - 1, 3 - 1, 6 - 5 \rangle = \langle -1, 2, 1 \rangle$, so we find the cross product $\overrightarrow{PA} \times \vec{\mathbf{v}}$:

$$\overrightarrow{PA} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ -1 & 2 & 3 \end{vmatrix} = 4\hat{i} + 2\hat{j} + 0\hat{k}$$

Therefore, the distance between the point and the line is

$$d = \frac{\|\overrightarrow{PA} \times \vec{v}\|}{\|\vec{v}\|} = \frac{\sqrt{4^2 + 2^2 + 0^2}}{\sqrt{(-1)^2 + 2^2 + 3^2}} = \boxed{\frac{2\sqrt{5}}{\sqrt{14}} \text{ units}}$$

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines**.

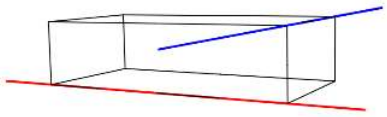


Figure 1.3.4: In three dimensions, it is possible that two lines do not cross, even when they have different directions.

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point.

Direction Vectors Are Parallel? \ Lines Share A Common Point?	Yes	No
Yes	Equal	Parallel but not equal
No	Intersecting	Skew

Figure 1.3.5: Determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point.

Illustration 1.3.2: Classifying Lines in Space

For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

(a) $L_1(s) = \langle 2s - 1, s - 1, s - 4 \rangle$ and $L_2(t) = \langle t - 3, 3t + 8, 5 - 2t \rangle$

(b) $L_1 : x = -y = z$ and $L_2 : \dfrac{x-3}{2} = y = z - 2$

(c) $L_1(s) = \langle 6s - 1, -2s, 3s + 1 \rangle$ and $L_2 : \dfrac{x-4}{6} = \dfrac{y+3}{-2} = \dfrac{z-1}{3}$

Solution

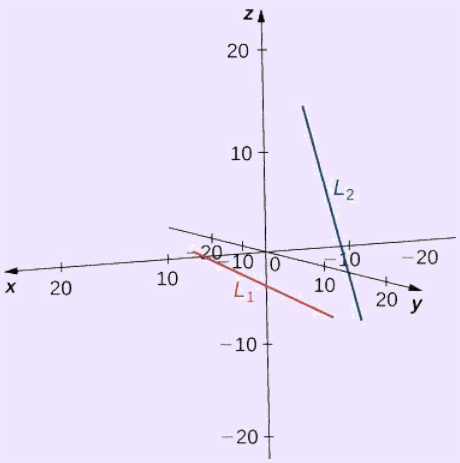
(a) Line L_1 has direction vector $\vec{v}_1 = \langle 2, 1, 1 \rangle$; line L_2 has direction vector $\vec{v}_2 = \langle 1, 3, -2 \rangle$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To determine whether the lines intersect, we see if there is a point, (x, y, z), that lies on both lines. To find this point, we use the parametric equations to create a system of equalities:

$$\begin{cases} 2s-1=t-3 \\ s-1=3t+8 \\ s-4=5-2t \end{cases}$$

By the first equation, $t=2s+2$. Substituting into the second and third equations yield:

$$\begin{cases} s-1=3(2s+2)+8 \\ s-4=5-2(2s+2) \end{cases} \longrightarrow \begin{cases} s=-3 \\ s=1 \end{cases}$$

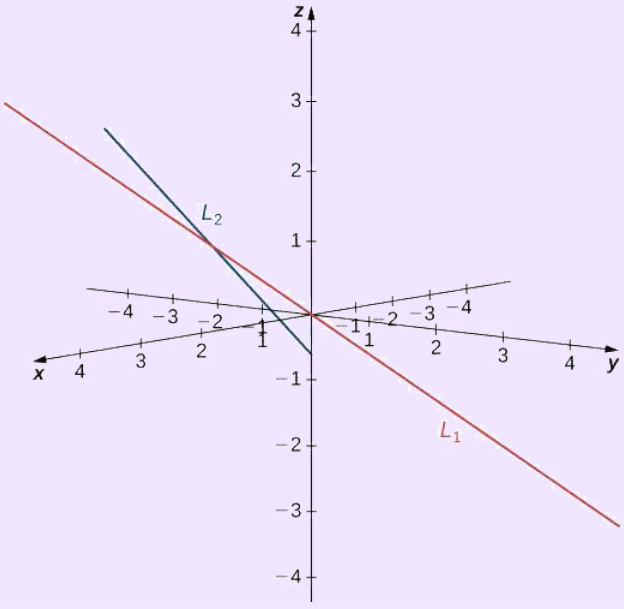
There is no single point that satisfies the parametric equations for L_1 and L_2 simultaneously. These lines do not intersect, so they are skew:



(b) Line L_1 has direction vector $\vec{\mathbf{v}}_1=\langle 1,-1,1\rangle$ and passes through the origin, $(0,0,0)$. Line L_2 has a different direction vector, $\vec{\mathbf{v}}_2=\langle 2,1,1\rangle$, so these lines are not parallel or equal. Let r represent the parameter for line L_1 and let s represent the parameter for L_2:

$$\begin{cases} r=x=2s+3 \\ -r=y=s \\ r=z=s+2 \end{cases}$$

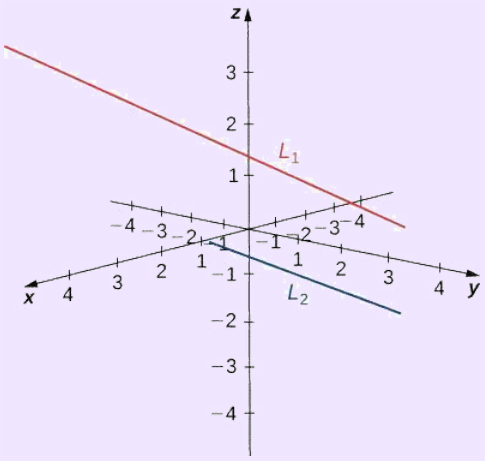
Solve the system of equations to find $r=1$ and $s=-1$. If we need to find the point of intersection, we can substitute these parameters into the original equations to get $(1,-1,1)$:



(c) Lines L_1 and L_2 have equivalent direction vectors: $\vec{v} = \langle 6, -2, 3 \rangle$. Let s represent the parameter for line L_1 and let r represent the parameter for L_2:

$$\begin{cases} 6s - 1 = x = 6r + 4 \\ -2s = y = -2r - 3 \\ 3s + 1 = z = 3r + 1 \end{cases}$$

The bottom equation simplifies to $s = r$ which does not satisfy the other two equations, so there is no intersection point. These two lines are parallel:



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1.3: Planes

We know that a line is determined by two points. In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three dimensions. Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three points.

This may be the simplest way to characterize a plane, but we can use other descriptions as well. For example, given two distinct, intersecting lines, there is exactly one plane containing both lines. A plane is also determined by a line and any point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\vec{\mathbf{n}} = \langle a, b, c \rangle$ be a vector and $P = (x_0, y_0, z_0)$ be a point. Then the set of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is orthogonal to $\vec{\mathbf{n}}$ forms a plane. We say that $\vec{\mathbf{n}}$ is a **normal vector**, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the **vector equation of a plane**:

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$$

Rewriting this equation provides additional ways to describe the plane:

$$\begin{aligned} \vec{\mathbf{n}} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \end{aligned}$$

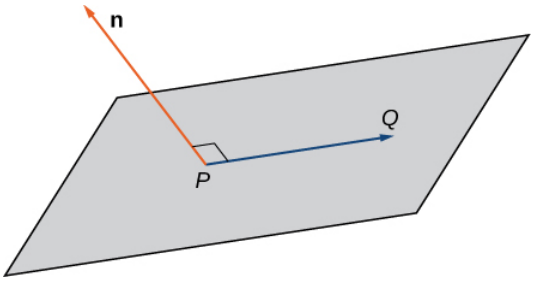


Figure 1.3.6: Given a point P and vector $\vec{\mathbf{n}}$, *the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\vec{\mathbf{n}}$ forms a plane.*

Definition: Scalar equation of a plane

Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying the equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$$

is known as the *vector equation of a plane*.

The *scalar equation of a plane* containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the *general form of the equation of a plane.*

As described earlier in this section, any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.

Example 1.3.4: Writing an Equation of a Plane Given Three Points in the Plane

Write an equation for the plane containing points $P=(1,1,-2), Q=(0,2,1)$, and $R=(-1,-1,0)$ in both standard and general forms.

Solution

To write an equation for a plane, we must find a normal vector for the plane. We start by identifying two vectors in the plane:

$$\overrightarrow{PQ} = \langle 0-1, 2-1, 1-(-2)\rangle = \langle -1,1,3\rangle$$
$$\overrightarrow{QR} = \langle -1-0, -1-2, 0-1\rangle = \langle -1,-3,-1\rangle$$

The cross product $\overrightarrow{PQ}\times\overrightarrow{QR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, so it is normal to the plane that contains these two vectors:

$$\begin{aligned}\vec{\mathbf{n}} &= \overrightarrow{PQ}\times\overrightarrow{QR}\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -1 & 1 & 3\\ -1 & -3 & -1\end{vmatrix}\\ &= (-1+9)\hat{\mathbf{i}} - (1+3)\hat{\mathbf{j}} + (3+1)\hat{\mathbf{k}}\\ &= 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\end{aligned}$$

Thus, $n=\langle 8,-4,4\rangle$, and we can choose any of the three given points to write an equation of the plane:

$$\begin{aligned}8(x-1)-4(y-1)+4(z+2) &= 0\\ 8x-4y+4z+4 &= 0\\ \boxed{2x-y+z+1=0}\end{aligned}$$

The scalar equations of a plane vary depending on the normal vector and point chosen.

Example 1.3.5: Writing an Equation for a Plane Given a Point and a Line

Find an equation of the plane that passes through point $P=(1,4,3)$ and contains the line given by $x=\dfrac{y-1}{2}=z+1$

Solution

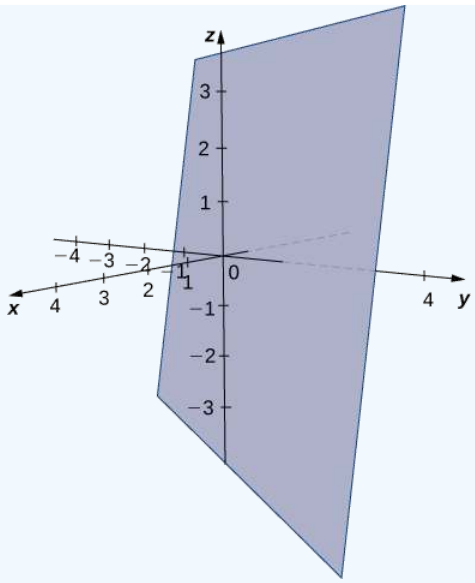
Symmetric equations describe the line that passes through point $Q=(0,1,-1)$ parallel to vector $\vec{\mathbf{v}}_1=\langle 1,2,1\rangle$. Use this point and the given point, $(1,4,3)$, to identify a second vector parallel to the plane:

$$\overrightarrow{QP} = \langle 1-0, 4-1, 3-(-1)\rangle = \langle 1,3,4\rangle$$

Use the cross product of these vectors to identify a normal vector for the plane:

$$\begin{aligned}\vec{\mathbf{n}} &= \vec{\mathbf{v}}_1\times\overrightarrow{QP}\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 2 & 1\\ 1 & 3 & 4\end{vmatrix}\\ &= (8-3)\hat{\mathbf{i}} - (4-1)\hat{\mathbf{j}} + (3-2)\hat{\mathbf{k}}\\ &= 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}\end{aligned}$$

The scalar equation for the plane is $5x-3(y-1)+(z+1)=0$, which simplifies to $\boxed{5x-3y+z+4=0}$



Now that we can write an equation for a plane, we can use the equation to find the distance d between a point P and the plane. It is defined as the shortest possible distance from P to a point on the plane.

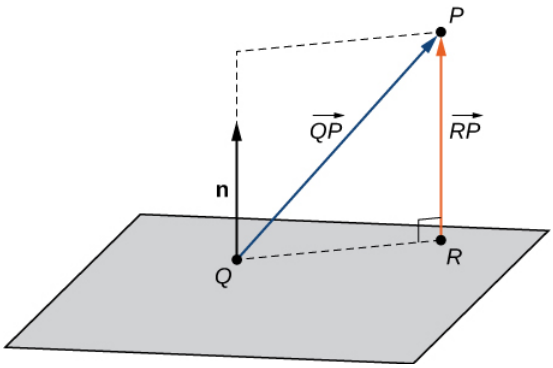


Figure 1.3.7: We want to find the shortest distance from point P to the plane.
Let point R be the point in the plane such that, for any point in the plane Q, we have $\|\overrightarrow{RP}\| \leq \|\overrightarrow{QP}\|$

Just as we find the two-dimensional distance between a point and a line by calculating the length of a line segment perpendicular to the line, we find the three-dimensional distance between a point and a plane by calculating the length of a line segment perpendicular to the plane. Let R be the point in the plane such that $\overrightarrow{RP}$ is orthogonal to the plane, and let Q be an arbitrary point in the plane. Then the projection of vector $\overrightarrow{QP}$ onto the normal vector describes vector $\overrightarrow{RP}$.

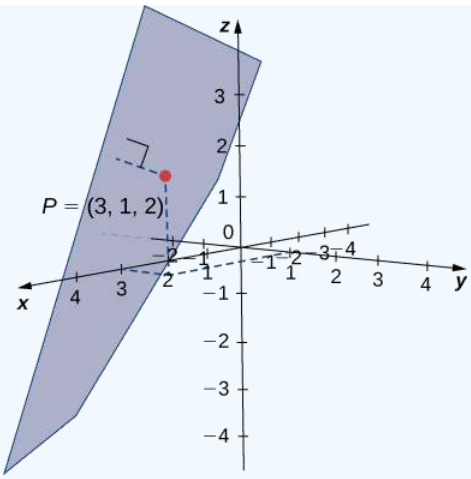
Rule: The Distance Between a Plane and a Point

Suppose a plane with normal vector $\vec{\mathbf{n}}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\text{proj}_{\vec{\mathbf{n}}} \overrightarrow{QP}\| = \frac{|\overrightarrow{QP} \cdot \vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}$$

Example 1.3.6: Distance between a Point and a Plane

Find the distance between point $P = (3, 1, 2)$ and the plane given by $x - 2y + z = 5$.



Solution

The coefficients of the plane's equation provide a normal vector for the plane: $\vec{\mathbf{n}} = \langle 1, -2, 1 \rangle$. To find vector $\overrightarrow{QP}$, we need a point in the plane. Any point will work, so set $y = z = 0$ to see that point $Q = (5, 0, 0)$ lies in the plane. Find the component form of the vector from Q to P:

$$\overrightarrow{QP} = \langle 3 - 5, 1 - 0, 2 - 0 \rangle = \langle -2, 1, 2 \rangle$$

Apply the distance formula above:

$$d = \frac{|\overrightarrow{QP} \cdot \vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|} = \frac{|\langle -2, 1, 2 \rangle \cdot \langle 1, -2, 1 \rangle|}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{|-2 - 2 + 2|}{\sqrt{6}} = \boxed{\frac{2}{\sqrt{6}} \text{ units}}$$

We have discussed the various possible relationships between two lines in two dimensions and three dimensions. When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line.

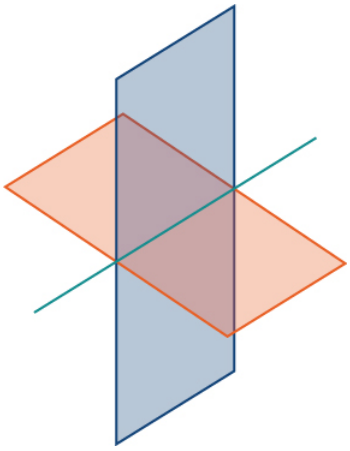
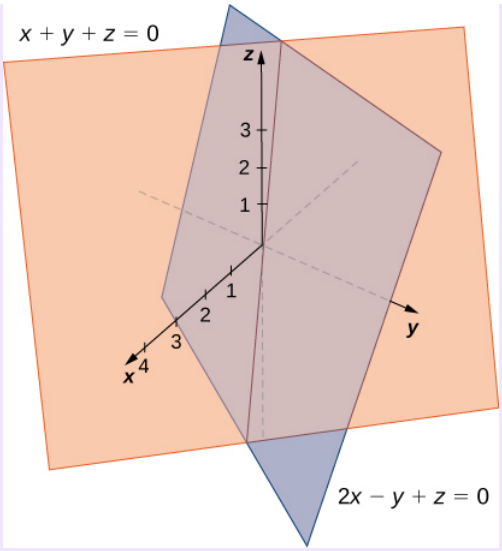


Figure 1.3.8: The intersection of two nonparallel planes is always a line.

We can use the equations of the two planes to find parametric equations for the line of intersection.

Illustration 1.3.3: Finding the Line of Intersection for Two Planes

Find parametric and symmetric equations for the line formed by the intersection of the planes given by $x + y + z = 0$ and $2x - y + z = 0$



Solution

Note that the two planes have nonparallel normals, so the planes intersect. Further, the origin satisfies each equation, so we know the line of intersection passes through the origin. Add the plane equations so we can eliminate one of the variables, in this case, y:

$$(x+y+z=0)+(2x-y+z=0)=(3x+2z=0)$$

This gives us $x=-\frac{2}{3}z$. We substitute this value into the first equation to express y in terms of z:

$$\begin{aligned} x+y+z&=0 \\ -\frac{2}{3}z+y+z&=0 \\ y+\frac{1}{3}z&=0 \\ y&=-\frac{1}{3}z \end{aligned}$$

We now have the first two variables, x and y, in terms of the third variable, z. Now we define z in terms of t. To eliminate the need for fractions, we choose to define the parameter t to satisfy $z=3t$. Then, $x=-\frac{2}{3}(3t)=-2t$ and $y=-\frac{1}{3}(3t)=-t$.

This gives the parametric equations for the line of intersection as $x=-2t, y=-t, z=3t$

Alternatively, the symmetric equations for the line are $\frac{x}{-2}=-y=\frac{z}{3}$

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections of the roof meet to know whether the roof will look good and drain properly. We can use normal vectors to calculate the angle between the two planes. We can do this because the angle between the normal vectors is the same as the angle between the planes.

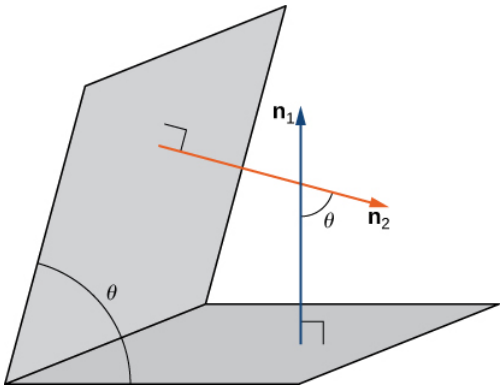


Figure 1.3.9: The angle between two planes has the same measure as the angle between the normal vectors for the planes.

We can find the measure of the angle θ between two intersecting planes by first finding the cosine of the angle, using the following equation:

$$\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}$$

We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

Example 1.3.7: Finding the Angle between Two Planes

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them. Give the answer in radians and round to two decimal places.

(a) $x + 2y - z = 8$ and $2x + 4y - 2z = 10$

(b) $2x - 3y + 2z = 3$ and $6x + 2y - 3z = 1$

(c) $3x + y - 2z = 4$ and $x - 2y - 3z = 1$

Solution:

(a) The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1, 2, -1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 2, 4, -2\rangle$. These two vectors are scalar multiples of each other.

The normal vectors are parallel, so the $\boxed{\text{planes are parallel}}$

(b) The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 2, -3, 2\rangle$ and $\vec{\mathbf{n}}_2 = \langle 6, 2, -3\rangle$. Taking the dot product of these vectors, we have

$$\begin{aligned}\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2 &= \langle 2, -3, 2\rangle \cdot \langle 6, 2, -3\rangle \\ &= 2(6) - 3(2) + 2(-3) = 0\end{aligned}$$

The normal vectors are orthogonal, so the corresponding $\boxed{\text{planes are orthogonal}}$ as well.

(c) The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 3, 1, -2\rangle$ and $\vec{\mathbf{n}}_2 = \langle 1, -2, -3\rangle$:

$$\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|} = \frac{|\langle 3, 1, -2\rangle \cdot \langle 1, -2, -3\rangle|}{\sqrt{3^2 + 1^2 + (-2)^2}\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{1}{2}$$

Then $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ rad.

Thus $\boxed{\text{the angle between the two planes is about } \frac{\pi}{3} \text{ radians, or } 60°}$

When we find that two planes are parallel, we may need to find the distance between them. To find this distance, we simply select a point in one of the planes. The distance from this point to the other plane is the distance between the planes.

Previously, we introduced the formula for calculating this distance:

$$d = \frac{\overrightarrow{QP} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|}$$

where Q is a point on the plane, P is a point not on the plane, and $\vec{n}$ is the normal vector that passes through point Q.

If the planes are $ax + by + cz = D_1$ and $ax + by + cz = D_2$, this simplifies to

$$d = \frac{|D_1 - D_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 1.3.8:

Find the distance between parallel planes

(a) $5x - 2y + z = 6$ and $5x - 2y + z = -3$

(b) $2x + y - 2z = 8$ and $-6x - 3y + 6z = 3$

Solution:

(a) The normal vector for these planes is $\vec{\mathbf{n}} = \langle 5, -2, 1\rangle$, so we use the formula above to get:

$$d = \frac{|6 - (-3)|}{\sqrt{5^2 + (-2)^2 + 1^2}} = \boxed{\frac{9}{\sqrt{30}} \text{ units}}$$

(b) The normal vectors for these planes are parallel, but not equal yet. We scale one of them to match the other:

$$2x + y - 2z = 8 \longrightarrow -6x - 3y + 6z = -24$$

Now they share the normal vector $\vec{\mathbf{n}} = \langle -6, -3, 6\rangle$, so we get:

$$d = \frac{|-24 - 3|}{\sqrt{(-6)^2 + (-3)^2 + 6^2}} = \frac{27}{\sqrt{81}} = \boxed{3 \text{ units}}$$

Example 1.3.9: Finding the Distance Between Skew Lines

Find the distance between the skew lines $L_1(s) = \langle 2s - 1, s - 1, s - 4\rangle$ and $L_2(t) = \langle t - 3, 3t + 8, 5 - 2t\rangle$

Solution:

While it is not possible to find a single plane containing these two skew lines, it is possible to find two *parallel* planes which each contain one of the lines. The distance between these two planes is equal to the distance between the closest points on the skew lines.

The slope vectors are $\vec{\mathbf{v}}_1 = \langle 2, 1, 1\rangle$ and $\vec{\mathbf{v}}_2 = \langle 1, 3, -2\rangle$, so the parallel planes will have normal vector:

$$\vec{\mathbf{n}} = \vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = -5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

Therefore each line is on a plane of the form $-5x + 5y + 5z = D$ for some D.

Plugging in any point on Line 1 (like $L_1(0) = (-1, -1, -4)$) gives $D_1 = -5(-1) + 5(-1) + 5(-4) = -20$

Plugging in any point on Line 2 (like $L_2(0) = (-3, 8, 5)$) gives $D_2 = -5(-3) + 5(8) + 5(5) = 80$

Finally the distance between these lines will equal

$$d = \frac{|-20 - 80|}{\sqrt{(-5)^2 + (5)^2 + 5^2}} = \boxed{\frac{20}{\sqrt{3}} \text{ units}}$$

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1.3: Practice Set

Exercise 1.3.1

Determine if each pair of lines is parallel, intersecting, or skew. If parallel, determine if they are the same line or not. If intersecting, determine if the lines are orthogonal or not. If the lines are skew, determine their distance apart.

(a) $L(t) = \langle 3t, 2t, t+1\rangle$ and $\ell(s) = \langle s+2, 2, 3-s\rangle$

(b) $L: \dfrac{x}{2} = \dfrac{y-3}{2} = \dfrac{z-4}{-2}$ and $\ell(s) = \langle 5-3s, 3-3s, 3s\rangle$

(c) $L(t) = \langle 1, 1+t, 1+2t\rangle$ and $\ell(s) = \langle 1, 4-2s, 7-4s\rangle$

(d) $L(t) = \langle t, 0, 3t+1\rangle$ and $\ell(s) = \langle s, 1, 2s-1\rangle$

(e) $L: \dfrac{x+1}{4} = \dfrac{y-2}{5} = \dfrac{z}{3}$ and $\ell(s) = \langle 2s+14, 40s+2, 9\rangle$

Answer

(a) $L(t) = \langle 0,0,1\rangle + t\langle 3,2,1\rangle$ and $\ell(s) = \langle 2,2,3\rangle + s\langle 1,0,-1\rangle$

The slope vectors are not parallel, so the lines cannot be parallel.

$$\begin{cases} 3t = s+2 \\ 2t = 2 \\ t+1 = 3-s \end{cases} \longrightarrow \begin{cases} t = 1 \\ s = 1 \end{cases}$$

The two lines are **intersecting** at $L(1) = \ell(1) = (3,2,2)$. The slopes satisfy $\langle 3,2,1\rangle \cdot \langle 1,0,-1\rangle = 2 \neq 0$ so they are not orthogonal.

(b) $L(t) = \langle 0,3,4\rangle + t\langle 2,2,-2\rangle$ and $\ell(s) = \langle 5,3,0\rangle + s\langle -3,-3,3\rangle$

These slope vectors are parallel, so the lines are parallel.

$$\begin{cases} 2t = 5-3s \\ 2t+3 = 3-3s \\ -2t+4 = 3s \end{cases} \longrightarrow \begin{cases} t = \frac{5}{2} - \frac{3}{2}s \\ t = -\frac{3}{2}s \\ t = 2 - \frac{3}{2}s \end{cases} \longrightarrow \text{no solution exists.}$$

These lines are **parallel**, but not the same line.

(c) $L(t) = \langle 1,1,1\rangle + t\langle 0,1,2\rangle$ and $\ell(s) = \langle 1,4,7\rangle + s\langle 0,-2,-4\rangle$

These slope vectors are parallel, so the lines are parallel.

$$\begin{cases} 1 = 1 \\ 1+t = 4-2s \\ 1+2t = 7-4s \end{cases} \longrightarrow \begin{cases} 0 = 0 \\ t = 3-2s \\ t = 3-2s \end{cases} \longrightarrow t = 3-2s$$

These parallel lines meet at infinitely many points, so they must be the **same line**.

(d) $L(t) = \langle 0,0,1\rangle + t\langle 1,0,3\rangle$ and $\ell(s) = \langle 0,1,-1\rangle + s\langle 1,0,2\rangle$

The slope vectors are not parallel, so the lines cannot be parallel.

$$\begin{cases} t = s \\ 0 = 1 \\ 3t+1 = 2s-1 \end{cases} \longrightarrow \text{no solution exists.}$$

The two lines are not parallel or intersecting, so they must be **skew**. They lie in parallel planes with normal vector $\vec{\mathbf{n}} = \langle 1,0,3\rangle \times \langle 1,0,2\rangle = \langle 0,1,0\rangle$, so they are of the form $0x + 1y + 0z = D$

Plugging in any point from each line, these two planes are $y = 0$ and $y = 1$, so their distance apart is $\text{dist} = \dfrac{|1-0|}{\|\vec{\mathbf{n}}\|} = \dfrac{1}{1} = 1$.

(e) $L(t) = \langle -1,2,0\rangle + t\langle 4,5,3\rangle$ and $\ell(s) = \langle 14,2,9\rangle + s\langle 2,40,0\rangle$

The slope vectors are not parallel, so the lines cannot be parallel.

$$\begin{cases} 4t-1=2s+14 \\ 5t+2=40s+2 \\ 3t=9 \end{cases} \longrightarrow \begin{cases} t=\frac{1}{2}s+\frac{15}{4} \\ t=8s \\ t=3 \end{cases} \longrightarrow \text{no solution exists.}$$

The two lines are not parallel or intersecting, so they must be **skew**. They reside in parallel planes with normal vector $\vec{\mathbf{n}}=\langle 4,5,3\rangle \times \langle 2,40,0\rangle = \langle -120,6,150\rangle = 6\langle -20,1,25\rangle$, so they are of the form $-20x+1y+25z=D$

Plugging in any point from each line, these two planes are $-20x+y+25z=22$ and $-20x+y+25z=-53$, so their distance apart is $\text{dist}=\dfrac{|22-(-53)|}{\|\vec{\mathbf{n}}\|}=\dfrac{75}{\sqrt{1026}}$

Exercise 1.3.2

Find the equation of the line described.

(a) Containing the points $(1,3,2)$ and $(2,0,1)$

(b) Containing the points $(2,0,-1)$ and $(0,-2,2)$

(c) Containing the point $(5,2,2)$ and parallel to $\ell: \dfrac{x-1}{2}=\dfrac{y+1}{3}=z+5$

(d) Containing the point $(1,0,6)$ and perpendicular to $2x+5y-3z=5$

(e) Contained in both $2x+y+z=5$ and $x+2y-z=1$

Answer

(a) Let $\vec{\mathbf{m}}=\langle 2-1,0-3,1-2\rangle=\langle 1,-3,-1\rangle$, so $L(t)=\langle 1,3,2\rangle+t\langle 1,-3,-1\rangle=\langle 1+t,3-3t,2-t\rangle$

(b) Let $\vec{\mathbf{m}}=\langle 0-2,-2-0,2-(-1)\rangle=\langle -2,-2,3\rangle$ so $L(t)=\langle 2,0,-1\rangle+t\langle -2,-2,3\rangle=\langle 2-2t,-2t,-1+3t\rangle$

(c) We see the given $\ell(s)=\langle 1+2s,-1+3s,-5+s\rangle=\langle 1,-1,-5\rangle+s\langle 2,3,1\rangle$, so $\vec{\mathbf{m}}=\langle 2,3,1\rangle$

Therefore we want $L(t)=\langle 5,2,2\rangle+t\langle 2,3,1\rangle=\langle 5+2t,2+3t,2+t\rangle$

(d) The plane $2x+5y-3z=5$ has normal vector $\vec{\mathbf{n}}=\langle 2,5,-3\rangle$, so $\vec{\mathbf{m}}=\langle 2,5,-3\rangle$

Therefore we want $L(t)=\langle 1,0,6\rangle+t\langle 2,5,-3\rangle=\langle 1+2t,5t,6-3t\rangle$

(e) The normal vectors to these planes are $\vec{\mathbf{n}}_1=\langle 2,1,1\rangle$ and $\vec{\mathbf{n}}_2=\langle 1,2,-1\rangle$

Therefore our slope must be $\vec{\mathbf{m}}=\vec{\mathbf{n}}_1\times\vec{\mathbf{n}}_2=\langle -3,3,3\rangle$.

We can find a single point they share in common by setting $x=0$ (arbitrary choice) to get the system $\begin{cases} y+z=5 \\ 2y-z=1 \end{cases}$ which has solution $y=2, z=3$,

Therefore $L(t)=\langle 0,2,3\rangle+t\langle -3,3,3\rangle=\langle -3t,2+3t,3+3t\rangle$

Exercise 1.3.3

Find the equation of the plane described.

(a) Containing the points $(1,0,0),(0,3,0)$ and $(0,0,-1)$

(b) Containing the points $(2,3,1),(-1,1,1)$ and $(5,5,5)$

(c) Containing the point $(3,2,-3)$ and the line $\vec{L}(t)=\langle t,1,t\rangle$

(d) Containing the point $(2,-5,1)$ and parallel to the plane $5x-7y+3z=1$

(e) Containing the lines $L(t)=\langle t,2t,1\rangle$ and $\ell: \dfrac{x}{3}=\dfrac{y-4}{2}=z$

(f) Perpendicular to line $L(t)=\langle t+1,t,2-t\rangle$ containing the origin

Answer

(a) The plane must contain the two vectors $\langle 0-1, 3-0, 0-0\rangle = \langle -1, 3, 0\rangle$ and $\langle 0-1, 0-0, -1-0\rangle = \langle -1, 0, -1\rangle$

Thus the normal vector is $\vec{\mathbf{n}} = \langle -1, 3, 0\rangle \times \langle -1, 0, -1\rangle = \langle -3, -1, 3\rangle$

Plugging in the point $(1, 0, 0)$ into $-3x - y + 3z = D$ gives $D = -3$, so the plane is $-3x - y + 3z = -3$

(b) The plane must contain the two vectors $\langle -1-2, 1-3, 1-1\rangle = \langle -3, -2, 0\rangle$ and $\langle 5-2, 5-3, 5-1\rangle = \langle 3, 2, 4\rangle$.

Thus the normal vector is $\vec{\mathbf{n}} = \langle -3, -2, 0\rangle \times \langle 3, 2, 4\rangle = \langle -8, 12, 0\rangle$

Plugging in the point $(5, 5, 5)$ into $-8x + 12y + 0z = D$ gives $D = 20$, so the plane is $-8x + 12y = 20$ or equivalently $-2x + 3y = 5$

(c) The line $L(t) = \langle 0, 1, 0\rangle + t\langle 1, 0, 1\rangle$ has slope $\vec{\mathbf{m}} = \langle 1, 0, 1\rangle$

The vector connecting $(3, 2, -3)$ and $L(0) = (0, 1, 0)$ is $\langle 3-0, 2-1, -3-0\rangle = \langle 3, 1, -3\rangle$

Thus the normal vector is $\vec{\mathbf{n}} = \langle 1, 0, 1\rangle \times \langle 3, 1, -3\rangle = \langle -1, 6, 1\rangle$

Plugging in the point $(3, 2, -3)$ into $-x + 6y + z = D$ gives $D = 6$, so the plane is $-x + 6y + z = 6$

(d) The given plane has normal vector $\vec{\mathbf{n}} = \langle 5, -7, 3\rangle$, and to be parallel we will use the same normal vector.

Plugging in the point $(2, -5, 1)$ into $5x - 7y + 3z = D$ gives $D = 48$, so the plane is $5x - 7y + 3z = 48$

(e) The lines $L(t) = \langle 0, 0, 1\rangle + t\langle 1, 2, 0\rangle$ and $\ell(s) = \langle 0, 4, 0\rangle + s\langle 3, 2, 1\rangle$

are in the plane with normal vector $\vec{\mathbf{n}} = \langle 1, 2, 0\rangle \times \langle 3, 2, 1\rangle = \langle 2, -1, -4\rangle$

and intersect at the point $\begin{cases} t = 3s \\ 2t = 4 + 2s \\ 1 = s \end{cases} \longrightarrow t = 3, s = 1 \longrightarrow L(3) = \ell(1) = (3, 6, 1)$

Plugging in the point $(3, 6, 1)$ into $2x - y - 4z = D$ gives $D = -4$, so the plane is $2x - y - 4z = -4$

(f) The line $L(t) = \langle 1, 0, 2\rangle + t\langle 1, 1, -1\rangle$ is perpendicular, so $\vec{\mathbf{n}} = \langle 1, 1, -1\rangle$.

Plugging in the point $(0, 0, 0)$ into $x + y - z = D$ gives $D = 0$, so the plane is $x + y - z = 0$

Exercise 1.3.4

Find the angle between the planes:

(a) $3x - 2y + z = 9$ and $2x + y + 3z = 13$

(b) $3x + 2y - 4z = 12$ and $\frac{1}{4}x + \frac{1}{6}y - \frac{1}{3}z = 1$

(c) $x - y = 2$ and $y + z = 43$

Answer

(a) These planes have normal vectors $\vec{\mathbf{n}}_1 = \langle 3, -2, 1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 2, 1, 3\rangle$

The angle between them is $\cos^{-1}\left(\dfrac{\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2}{\|\vec{\mathbf{n}}_1\| \, \|\vec{\mathbf{n}}_2\|}\right) = \cos^{-1}\left(\dfrac{7}{\sqrt{14}\sqrt{14}}\right) = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$ radians

(b) These planes have normal vectors $\vec{\mathbf{n}}_1 = \langle 3, 2, -4\rangle$ and $\vec{\mathbf{n}}_2 = \left\langle \frac{1}{4}, \frac{1}{6}, -\frac{1}{3}\right\rangle$

The angle between them is $\cos^{-1}\left(\dfrac{\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2}{\|\vec{\mathbf{n}}_1\| \, \|\vec{\mathbf{n}}_2\|}\right) = \cos^{-1}\left(\dfrac{\frac{29}{12}}{\sqrt{29}\sqrt{\frac{29}{144}}}\right) = \cos^{-1}(1) = 0$, so they are parallel.

(these are in fact the same plane)

(c) These planes have normal vectors $\vec{\mathbf{n}}_1 = \langle 1, -1, 0\rangle$ and $\vec{\mathbf{n}}_2 = \langle 0, 1, 1\rangle$

The angle between them is $\cos^{-1}\left(\dfrac{\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2}{\|\vec{\mathbf{n}}_1\| \, \|\vec{\mathbf{n}}_2\|}\right) = \cos^{-1}\left(\dfrac{-1}{\sqrt{2}\sqrt{2}}\right) = \cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$ radians

Exercise 1.3.5

Find the distance or closest distance between the two objects:

(a) the plane $-3x+2y-6z=2$ and the plane $3x-2y+6z=3$

(b) the plane $5x+5y+5z=10$ and the plane $3x+3y+3z=6$

(c) the plane $5x-4y+3z=11$ and the plane $15x-12y+9z=3$

(d) the point $(1,4,-2)$ and the plane $2x+y-6z=1$

(e) the point $(2,2,2)$ and the plane $x+2y+2z=10$

(f) the lines $L(t)=\langle t,3,t\rangle$ and $\ell(s)=\langle s,s,2s-1\rangle$

Answer

(a) Rewriting them as $3x-2y+6z=-2$ and $3x-2y+6z=3$ we have $\vec{\mathbf{n}}=\langle 3,-2,6\rangle$

So, $\text{dist}=\dfrac{|3-(-2)|}{\|\vec{\mathbf{n}}\|}=\dfrac{5}{\sqrt{49}}=\dfrac{5}{7}$

(b) Rewriting them as $x+y+z=2$ and $x+y+z=2$ we see these are the same plane, so $\text{dist}=0$

(c) Rewriting them as $15x-12y+9z=33$ and $15x-12y+9z=3$ we have $\vec{\mathbf{n}}=\langle 15,-12,9\rangle$

So, $\text{dist}=\dfrac{|33-3|}{\|\vec{\mathbf{n}}\|}=\dfrac{30}{\sqrt{450}}=\dfrac{30}{15\sqrt{2}}=\sqrt{2}$

(d) Plugging in the point $(1,4,-2)$ into $2x+y-6z=D$ gives $D=18$ and $\vec{\mathbf{n}}=\langle 2,1,-6\rangle$

So, $\text{dist}=\dfrac{|18-1|}{\|\vec{\mathbf{n}}\|}=\dfrac{17}{\sqrt{41}}$

(e) Plugging in the point $(2,2,2)$ into $x+2y+2z=D$ gives $D=10$ and $\vec{\mathbf{n}}=\langle 1,2,2\rangle$

So, $\text{dist}=\dfrac{|10-10|}{\|\vec{\mathbf{n}}\|}=\dfrac{0}{3}=0$

This is because the point is already on the given plane!

(f) These lines have slope vectors $\vec{\mathbf{m}}_1=\langle 1,0,1\rangle$ and $\vec{\mathbf{m}}_2=\langle 1,1,2\rangle$

Therefore we will use $\vec{\mathbf{n}}=\langle 1,0,1\rangle\times\langle 1,1,2\rangle=\langle -1,-1,1\rangle$

Plugging in $L(0)=(0,3,0)$ into $-x-y+z=D_1$ gives $D_1=-3$, plugging in $\ell(0)=(0,0,-1)$ into $-x-y+z=D_2$ gives $D_2=-1$

So, $\text{dist}=\dfrac{|-3-(-1)|}{\|\vec{\mathbf{n}}\|}=\dfrac{2}{\sqrt{3}}$

Exercise 1.3.6

Find the point closest to the first object on the second object.

(a) Closest to $(-9,5,15)$ on $-3x+y+4z=-12$

(b) Closest to the origin on $x+2y+3z=6$

(c) Closest to $(2,1,-2)$ on $\vec{L}(t)=\langle t,2t+1,3t+2\rangle$

(d) Closest to $\vec{L}(t)=\langle t,-2t,t\rangle$ on $\vec{\ell}(s)=\langle s+2,s,s+1\rangle$

(e) Closest to $\vec{\ell}(s)=\langle s+2,s,s+1\rangle$ on $\vec{L}(t)=\langle t,-2t,t\rangle$

Answer

(a) The normal vector to the plane is $\vec{\mathbf{n}}=\langle -3,1,4\rangle$, so the straight line from $(-9,5,15)$ to this plane is

$L(t) = \langle -9, 5, 15\rangle + t\langle -3, 1, 4\rangle = \langle -9-3t, 5+t, 15+4t\rangle$, which intercepts the plane when

$-3(-9-3t) + (5+t) + 4(15+4t) = -12 \longrightarrow 92 + 26t = -12 \longrightarrow t = -4$

Thus the closest point is $L(-4) = (3, 1, -1)$

(b) The normal vector to the plane is $\vec{\mathbf{n}} = \langle 1, 2, 3\rangle$, so the straight line from the origin to this plane is

$L(t) = \langle 0, 0, 0\rangle + t\langle 1, 2, 3\rangle = \langle t, 2t, 3t\rangle$, which intercepts the plane when

$(t) + 2(2t) + 3(3t) = 6 \longrightarrow 14t = 6 \longrightarrow t = \dfrac{3}{7}$

Thus the closest point is $L\left(\dfrac{3}{7}\right) = \left(\dfrac{3}{7}, \dfrac{6}{7}, \dfrac{9}{7}\right)$

(c) The distance between $L(t)$ and $(2, 1, -2)$ is

$d(t) = \sqrt{(t-2)^2 + ((2t+1)-1)^2 + ((3t+2)-(-2))^2} = \sqrt{14t^2 + 20t + 20}$

We take a derivative of this distance function: $d'(t) = \dfrac{14t + 10}{\sqrt{14t^2 + 20t + 20}}$

Setting this to zero gives us the critical point $t = -\dfrac{5}{7}$ which yields a minimum for $d(t)$.

Thus the closest point is $L\left(-\dfrac{5}{7}\right) = \left(-\dfrac{5}{7}, -\dfrac{3}{7}, -\dfrac{1}{7}\right)$

(d) We want to find the s which minimizes the distance between $\ell(s)$ and some point on $L(t)$.

$d(s) = \sqrt{((s+2)-t)^2 + (s-(-2t))^2 + ((s+1)-t)^2}$ for some constant value of t.

We take the derivative with respect to s to get: $d'(s) = \dfrac{(s+2-t) + (s+2t) + (s+1-t)}{\sqrt{((s+2)-t)^2 + (s-(-2t))^2 + ((s+1)-t)^2}}$

$= \dfrac{3s+3}{\sqrt{((s+2)-t)^2 + (s-(-2t))^2 + ((s+1)-t)^2}}$

Setting this to zero gives us the critical point $s = -1$ which yields a minimum for $d(s)$.

Thus the closest point is $\ell(s) = (1, -1, 0)$

(e) We want to find the t which minimizes the distance between $L(t)$ and some point on $\ell(s)$.

$d(t) = \sqrt{((s+2)-t)^2 + (s-(-2t))^2 + ((s+1)-t)^2}$ for some constant value of t.

We take the derivative with respect to t to get: $d'(t) = \dfrac{-(s+2-t) + 2(s+2t) - (s+1-t)}{\sqrt{((s+2)-t)^2 + (s-(-2t))^2 + ((s+1)-t)^2}}$

$= \dfrac{6t-3}{\sqrt{((s+2)-t)^2 + (s-(-2t))^2 + ((s+1)-t)^2}}$

Setting this to zero gives us the critical point $t = \dfrac{1}{2}$ which yields a minimum for $d(t)$.

Thus the closest point is $L\left(\dfrac{1}{2}\right) = \left(\dfrac{1}{2}, -1, \dfrac{1}{2}\right)$

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1.4: Famous Surfaces

Learning Objectives

- Identify a cylinder as a type of three-dimensional surface.
- Recognize the main features of ellipsoids, paraboloids, and hyperboloids.
- Use traces to draw the intersections of quadric surfaces with the coordinate planes.

We have been exploring vectors and vector operations in three-dimensional space, and we have developed equations to describe lines, planes, and spheres. In this section, we use our knowledge of planes and spheres, which are examples of three-dimensional figures called surfaces, to explore a variety of other surfaces that can be graphed in a three-dimensional coordinate system.

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1.4: Cylindrical and Quadric Surfaces

The first surface we'll examine is the cylinder. Although most people immediately think of a hollow pipe or a soda straw when they hear the word cylinder, here we use the broad mathematical meaning of the term. Cylindrical surfaces don't have to be circular. A rectangular heating duct is a cylinder, as is a rolled-up yoga mat, the cross-section of which is a spiral shape.

In one dimension, we know that the set $x = 0$ represents a single point on the number line. However, when we discuss that same equation $x = 0$ in $\mathbb{R}^2$, it becomes a line. This is because the second variable y is not restricted by the equation, and is *free* to be any number, giving us all points of the form $(0, y)$. In the context of $\mathbb{R}^3$, we know that $x = 0$ defines an entire plane (all points of the form $(0, y, z)$)! Adding an extra variable essentially extends the graph from lower dimensional space into higher dimensional space.

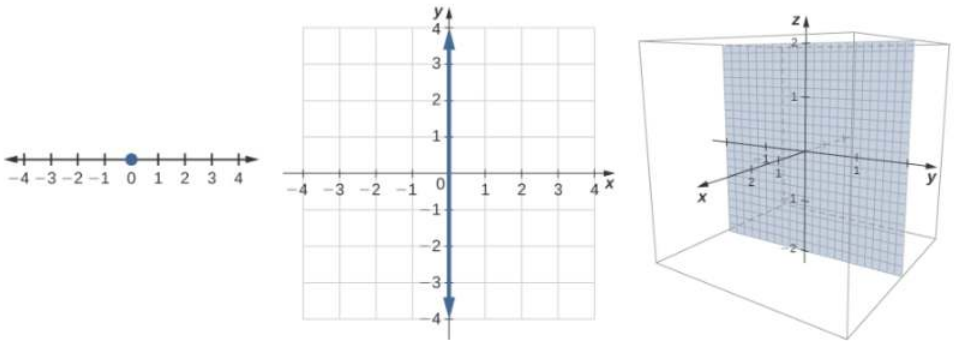


Figure 1.4.1: In one-, two-, and three-dimensional space, the graph of equation $x = 1$ is different.

In the two-dimensional coordinate plane, the equation $x^2 + y^2 = 9$ describes a circle centered at the origin with radius 3. In three-dimensional space, this same equation represents a surface. Imagine copies of a circle stacked on top of each other centered on the z-axis, forming a hollow tube. We can then construct a cylinder from the set of lines parallel to the z-axis passing through the circle $x^2 + y^2 = 9$ in the xy-plane, as shown in the figure. In this way, any curve in one of the coordinate planes can be extended to become a surface.

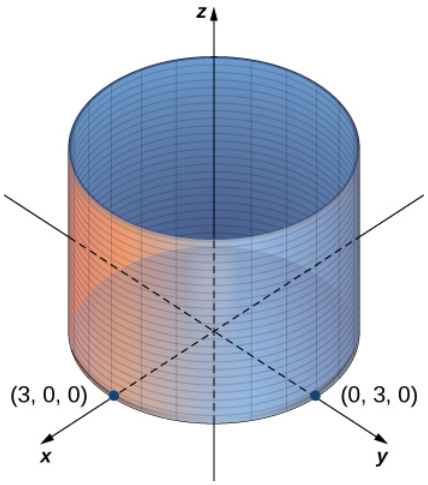


Figure 1.4.2: In three-dimensional space, the graph of equation $x^2 + y^2 = 9$ is a cylinder with radius 3 centered on the z-axis. It continues indefinitely in the positive and negative directions.

Definition: Cylinders and Rulings

A set of lines parallel to a given line passing through a given curve is known as a cylindrical surface, or ***cylinder.*** The parallel lines are called ***rulings.***

From this definition, we can see that we still have a cylinder in three-dimensional space, even if the curve is not a circle. Any curve can form a cylinder, and the rulings that compose the cylinder may be parallel to any given line.

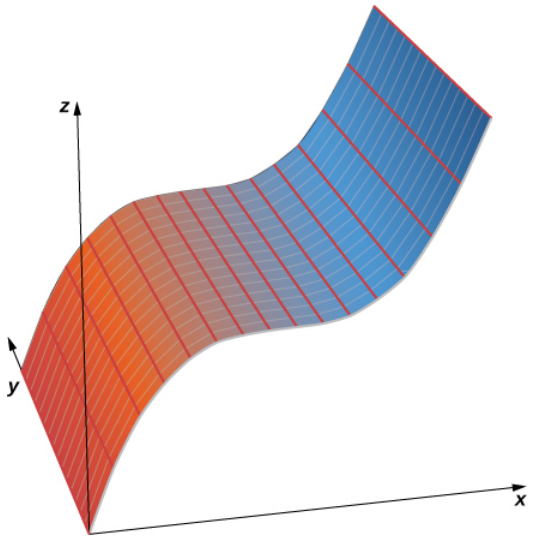


Figure 1.4.3: In three-dimensional space, the graph of equation $z = x^3$ is a cylinder, or a cylindrical surface with rulings parallel to the y-axis.

Illustration 1.4.1: Graphing Cylindrical Surfaces

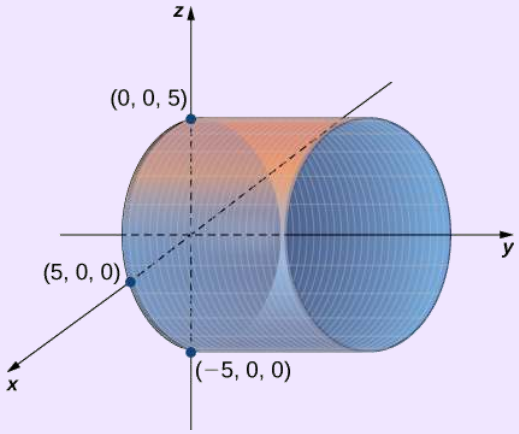
Sketch the graphs of the following cylindrical surfaces.

(a) $x^2 + z^2 = 25$

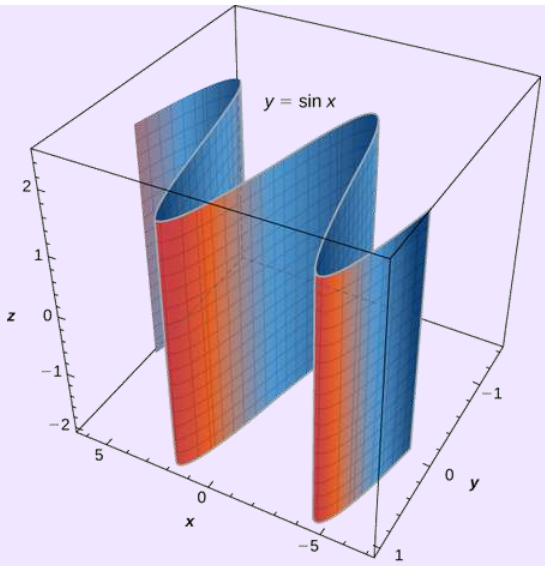
(b) $y = \sin x$

(c) $z = y^2$

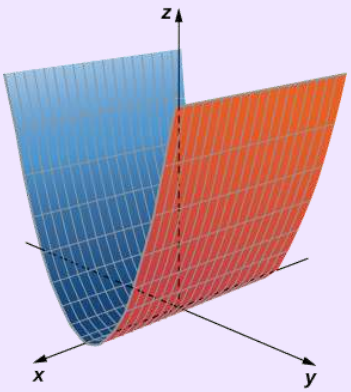
(a) The variable y can take on any value without limit. Therefore, the lines ruling this surface are parallel to the y-axis. The intersection of this surface with the xz-plane forms a circle centered at the origin with radius 5.



(b) In this equation, the variable z can take on any value without limit. Therefore, the lines composing this surface are parallel to the z-axis. The intersection of this surface with the xy-plane outlines curve $y = \sin x$.



(c) In this equation, the variable x can take on any value without limit. Therefore, the lines composing this surface are parallel to the x-axis. The intersection of this surface with the yz-plane outlines curve $z = y^2$.



When sketching surfaces, we have seen that it is useful to sketch the intersection of the surface with a plane parallel to one of the coordinate planes. These curves are called traces.

Definition: Traces

The ***traces*** of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes.

Traces are useful in sketching cylindrical surfaces. For a cylinder in three dimensions, though, only one set of traces is useful. Notice that the trace of the graph of $z = \sin(x)$ in the xz-plane is useful in constructing the graph. The trace in the xy-plane, though, is just a series of parallel lines, and the trace in the yz-plane is simply one line.

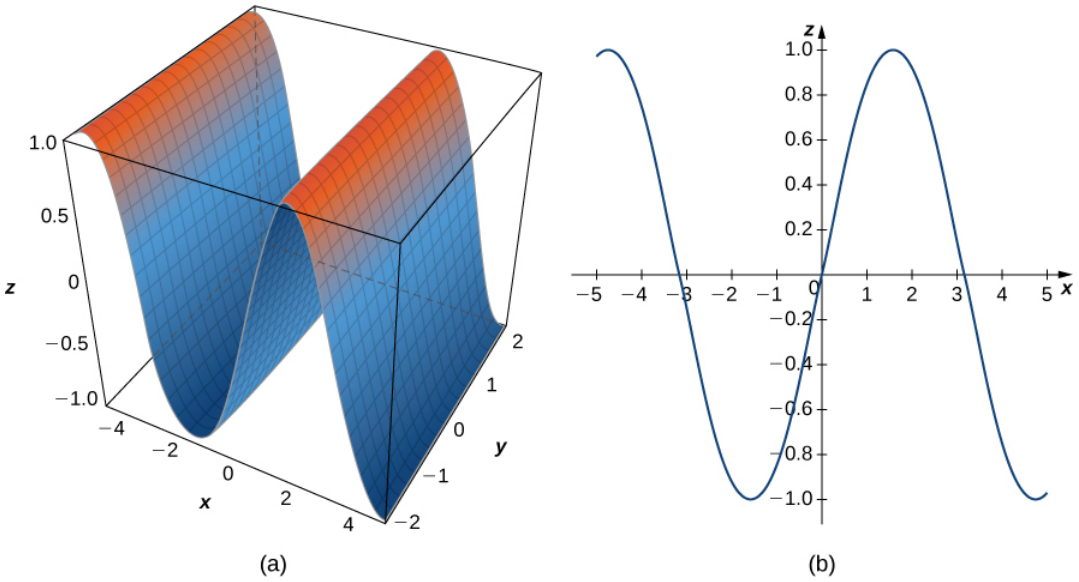


Figure 1.4.4: (a) This is one view of the graph of equation $z = \sin(x)$.
(b) To find the trace of the graph in the xz-plane, set $y = 0$. The trace is simply a two-dimensional sine wave.

Cylindrical surfaces are formed by a set of parallel lines. Not all surfaces in three dimensions are constructed so simply, however. We now explore more complex surfaces, and traces are an important tool in this investigation.

We have learned about surfaces in three dimensions described by first-order equations; these are planes. Some other common types of surfaces can be described by second-order equations. We can view these surfaces as three-dimensional extensions of the conic sections we discussed earlier: the ellipse, the parabola, and the hyperbola. We call these graphs quadric surfaces.

Definition: Quadric Surfaces and Conic Sections

Quadric surfaces are the graphs of equations that can be expressed in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0$$

When a quadric surface intersects a coordinate plane, the trace is a conic section.

An *ellipsoid* is a surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Set $x = 0$ to see the trace of the ellipsoid in the yz -plane. To see the traces in the xy- and xz-planes, set $z = 0$ and $y = 0$, respectively. Notice that if $a = b$, the trace in the xy-plane is a circle. Similarly, if $a = c$, the trace in the xz-plane is a circle and, if $b = c$, then the trace in the yz-plane is a circle. A sphere is thus an ellipsoid with $a = b = c$.

Illustration 1.4.2: Sketching an Ellipsoid

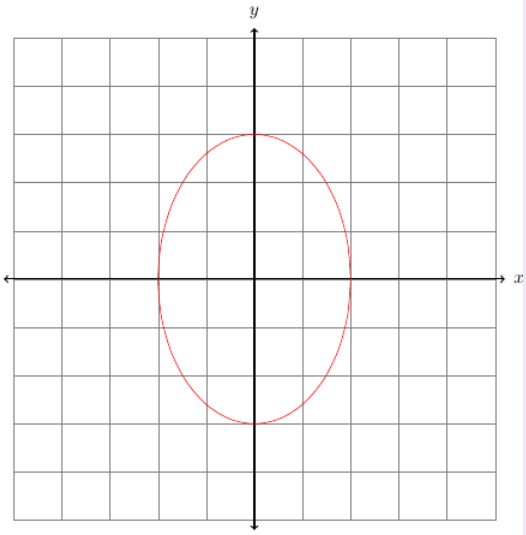
Sketch the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1$$

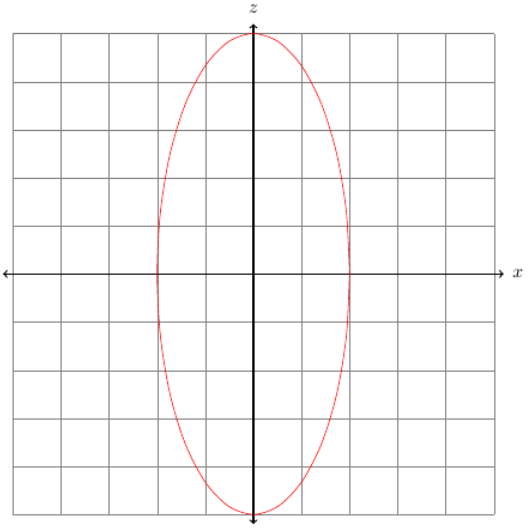
Solution

Start by sketching the traces.

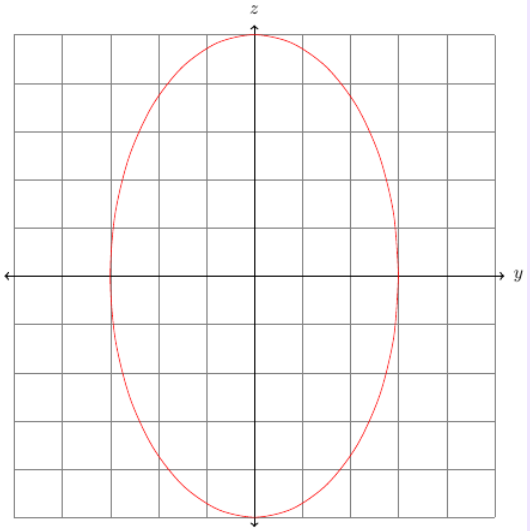
To find the trace in the xy-plane, set $z = 0$: $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$



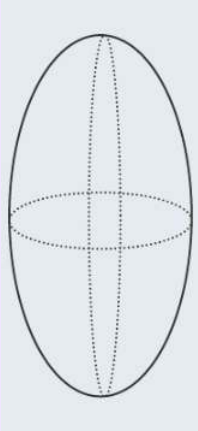
To find the trace in the xz-plane, set $y = 0$: $\frac{x^2}{2^2} + \frac{z^2}{5^2} = 1$



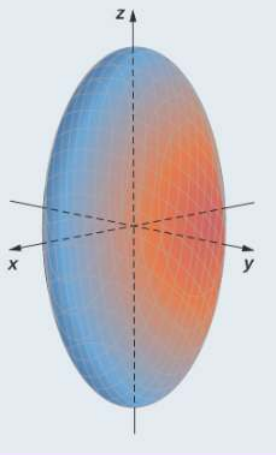
To find the trace in the yz-plane, set $x = 0$: $\frac{y^2}{3^2} + \frac{z^2}{5^2} = 1$



Now that we know what traces of this solid look like, we can sketch the skeleton of this surface in three dimensions:



Thus, the final 3D sketch looks like:



The trace of an ellipsoid is an ellipse in each of the coordinate planes. However, this does not have to be the case for all quadric surfaces. Many quadric surfaces have traces that are different kinds of conic sections, and this is usually indicated by the name of the surface. For example, if a surface can be described by an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

then we call that surface an **elliptic paraboloid**. The trace in the xy-plane is an ellipse, but the traces in the xz-plane and yz-plane are parabolas. Other elliptic paraboloids can have other orientations simply by interchanging the variables, resulting in a different variable in the linear term of the equation:

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{y}{b} \text{ or } \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}$$

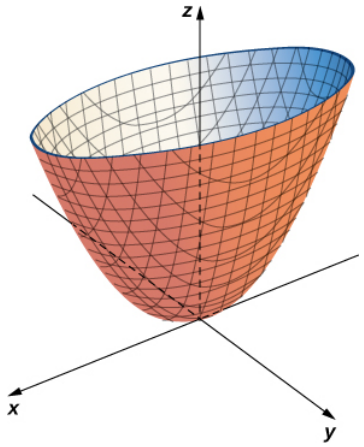


Figure 1.4.5: This quadric surface is called an elliptic paraboloid with elliptic cross sections in planes of the form $z = \#$

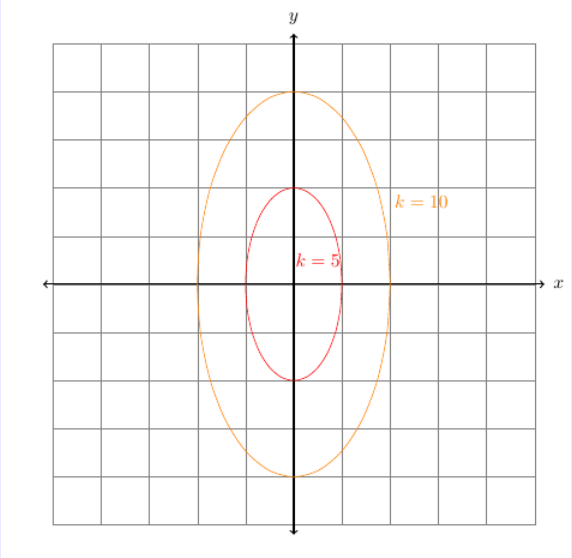
Illustration 1.4.3: Identifying Traces of Quadric Surfaces

Describe the traces of and sketch the elliptic paraboloid $x^2 + \frac{y^2}{2^2} = \frac{z}{5}$

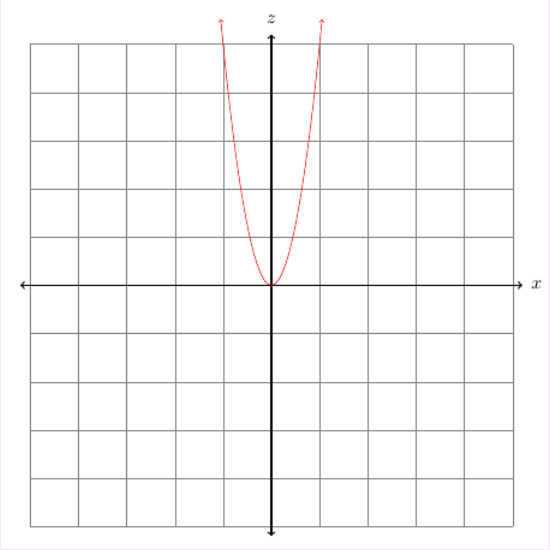
Solution

To find the trace in the xy-plane, set $z = 0 : x^2 + \frac{y^2}{2^2} = 0$. The trace in the plane $z = 0$ is reduced to a single point: the origin. Since a single point does not tell us what the shape is, we consider higher planes $z = k$ to examine the shape of other traces of the figure.

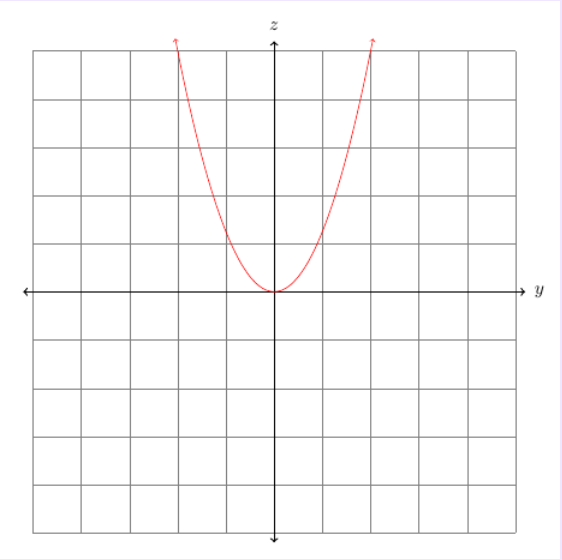
The trace in plane $z = k$ is the graph of equation $\frac{x^2}{1^2 \cdot \frac{k}{5}} + \frac{y^2}{2^2 \cdot \frac{k}{5}} = 1$, which is an ellipse:



In the xz-plane, the equation becomes $z = 5x^2$



In the yz-plane, the equation becomes $z = \frac{5}{4}y^2$



Thus, the final 3D sketch looks like:

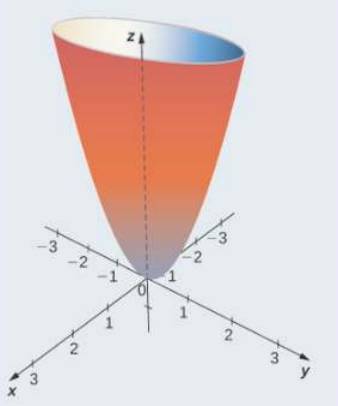


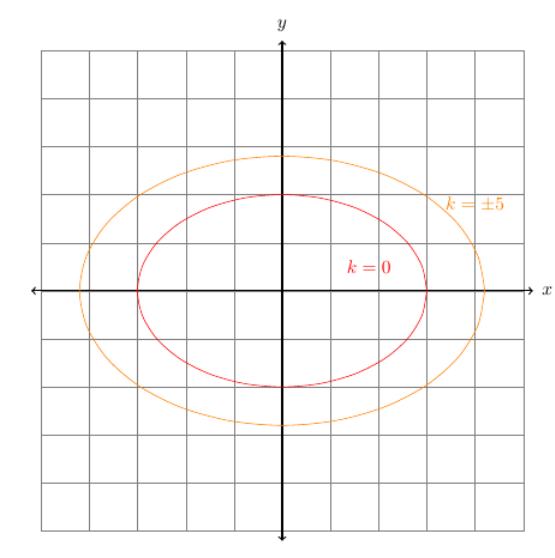
Illustration 1.4.4: Graphing a Hyperboloid of One Sheet

A hyperboloid of one sheet is any surface that can be described with an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. Sketch the hyperboloid of one sheet given by the equation $\frac{x^2}{3^2}+\frac{y^2}{2^2}-\frac{z^2}{5^2}=1$

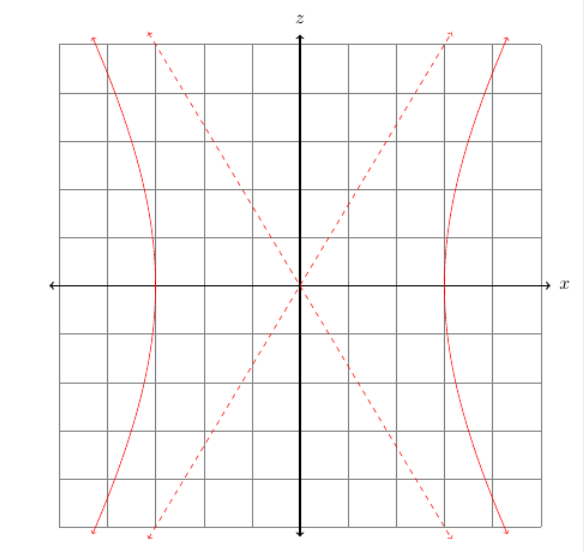
Solution

The traces parallel to the xy-plane are ellipses, and the traces parallel to the xz- and yz-planes are hyperbolas. Specifically:

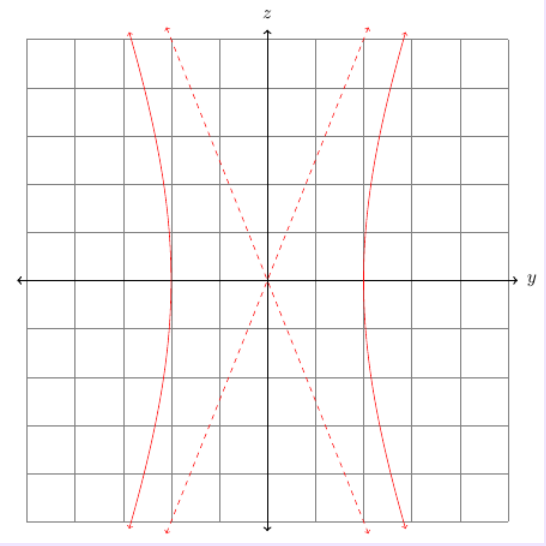
The trace in the $z=k$-plane is the ellipse $\frac{x^2}{3^2\left(\frac{k^2}{5^2}+1\right)}+\frac{y^2}{2^2\left(\frac{k^2}{5^2}+1\right)}=1$



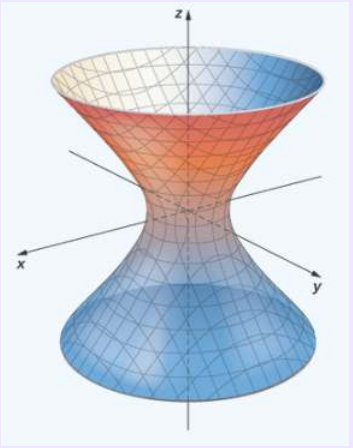
The trace in the xz-plane is the hyperbola $\frac{x^2}{3^2}-\frac{z^2}{5^2}=1$



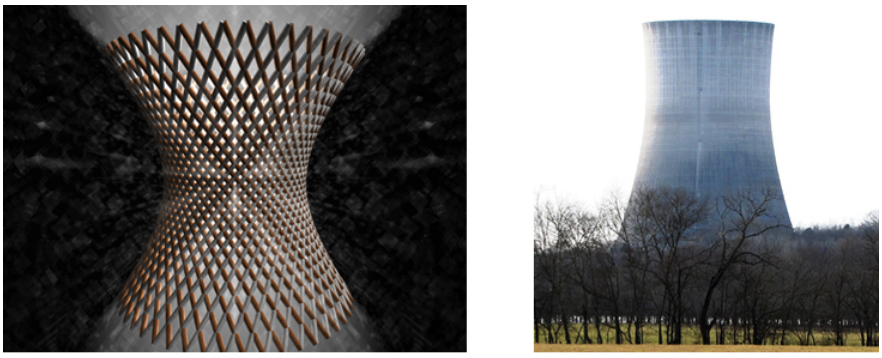
The trace in the yz-plane is the hyperbola $\frac{y^2}{2^2}-\frac{z^2}{5^2}=1$



Thus, the final 3D sketch looks like:



Hyperboloids of one sheet have some fascinating properties. For example, they can be constructed using straight lines, such as in the sculpture below. In fact, cooling towers for nuclear power plants are often constructed in the shape of a hyperboloid. The builders are able to use straight steel beams in the construction, which makes the towers very strong while using relatively little material.



(a) (b)

Figure 1.4.7: (a) A sculpture in the shape of a hyperboloid can be constructed of straight lines. (b) Cooling towers for nuclear power plants are often built in the shape of a hyperboloid.

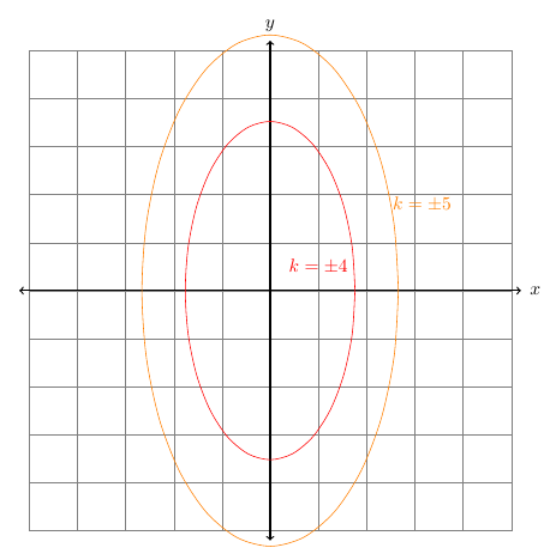
Illustration 1.4.5: Graphing a Hyperboloid of Two Sheets

A hyperboloid of two sheets is any surface that can be described with an equation of the form $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Describe the traces of the hyperboloid of two sheets given by the equation $-\frac{x^2}{2^2}-\frac{y^2}{4^2}+\frac{z^2}{3^2}=1$

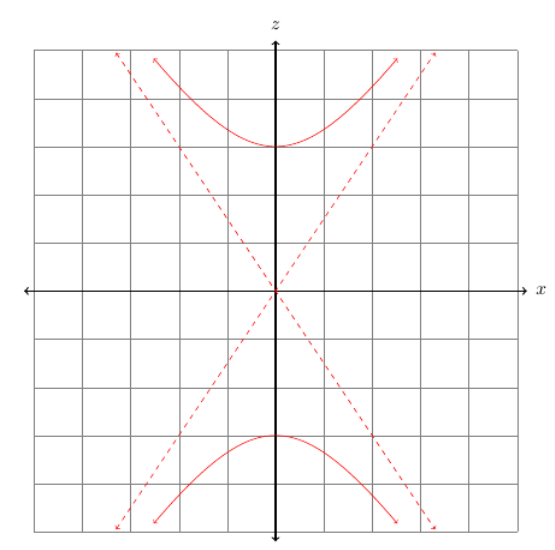
Solution

The traces parallel to the xy-plane are ellipses, and the traces parallel to the xz- and yz-planes are hyperbolas. Specifically:

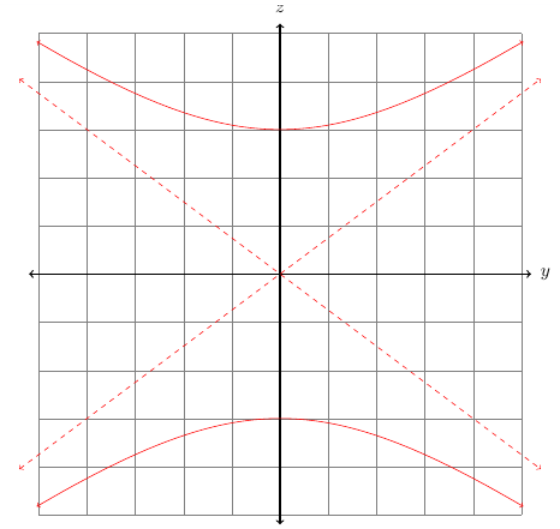
(a) The trace in the $z=k$-plane (for $|k|\geq 3$ is the ellipse $\frac{x^2}{2^2\left(\frac{k^2}{3^2}-1\right)}+\frac{y^2}{4^2\left(\frac{k^2}{3^2}-1\right)}=1$



The trace in the xz-plane is the hyperbola $\frac{z^2}{3^2}-\frac{x^2}{2^2}=1$:



The trace in the yz-plane is the hyperbola $\frac{z^2}{3^2}-\frac{y^2}{4^2}=1$



Seventeen standard quadric surfaces can be derived from the general equation

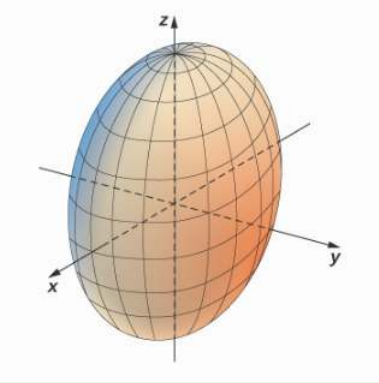
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0$$

The following definitions summarize the most important ones.

Definition: Ellipsoid

The relationship $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ defines an ellipsoid.

The traces in $x = p$ form ellipses, in $y = q$ form ellipses, and in $z = r$ form ellipses.

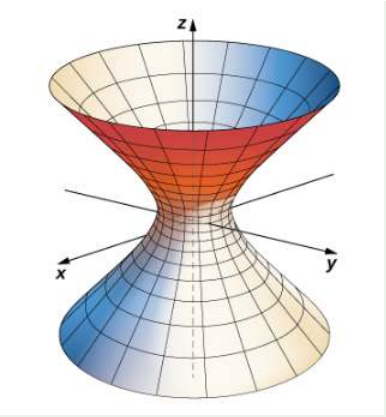


When $a = b = c$ an ellipsoid is a sphere.

Definition: Hyperboloid of One Sheet

The relationship $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ describes a hyperboloid of one sheet.

The traces in $x = p$ and $y = q$ form ellipses, but those in $z = r$ form hyperbolas.

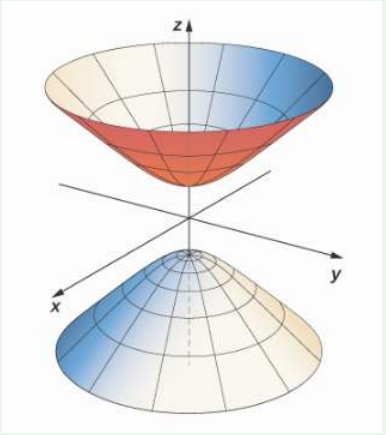


This shape can be rotated in other directions, as long as two squared terms are positive and one is negative.
The hyperbola cross sections will occur only for the traces involving the one variable for which the square is negative in the equation.

Definition: Hyperboloid of Two Sheets

The relationship $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ defines a hyperboloid of two sheets.

The traces in $x=p$ and $y=q$ form hyperbola, and in $z=r$ form ellipses (for sufficiently large values of r).

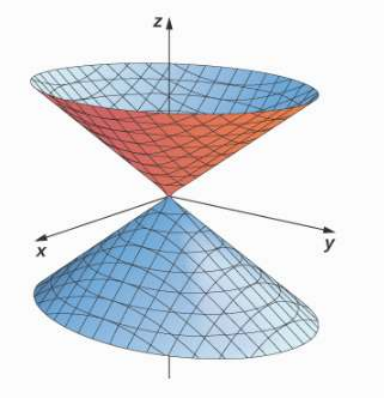


This shape can be rotated in other directions, as long as two of the variables squared are negative and the third is positive.
The elliptical cross sections will occur only for the traces involving the one variable for which the square is positive in the equation.

Definition: Elliptic Cone

The relationship $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ defines a cone.

The traces in $x=0$ and $y=0$ form crossing lines (or absolute value functions), and in $z=r$ form ellipses.



Only the top side of the cone exists when the equation is written as $z = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$

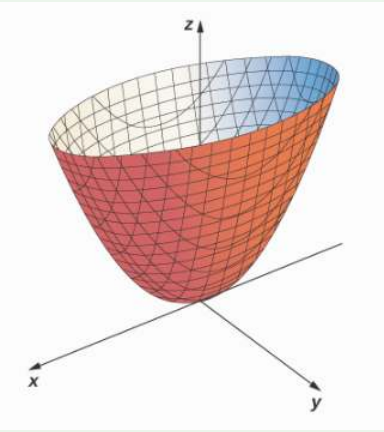
Similarly, the bottom half exists when it is written as $z = -\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$

This shape can be rotated in other directions, as long as two of the variables squared are positive and the third is negative.
The hyperbola cross sections will occur only for the traces involving the one variable for which the square is negative in the equation.

Definition: Elliptic Paraboloid

The relationship $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ defines an elliptic paraboloid.

The traces in $x = p$ and $y = q$ form parabolas, and in $z = r$ form ellipses.

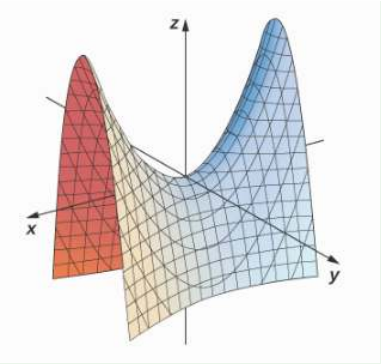


This shape can be rotated in other directions, as long as two of the variables squared are both positive or both negative and the third is linear.
These always open in the positive or negative direction of the axis of the linear term.

Definition: Hyperbolic Paraboloid

The relationship $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ defines a hyperbolic paraboloid.

The traces in $x = p$ and $y = q$ form opposing parabolas, and in $z = r$ form hyperbolas.



This shape can be rotated in other directions, as long as one of the variables squared and positive, one is squared and negative, and the third is linear.
This shape is commonly referred to as a "saddle" by equestrian fans or "pringle" by chip fans.

Example 1.4.1: Identifying Equations of Quadric Surfaces

Identify the surfaces represented by the given equations.

(a) $16x^2 + 9y^2 + 16z^2 = 144$

(b) $9x^2 - 18x + 4y^2 + 16y - 36z + 25 = 0$

Solution

(a) The x, y, and z terms are all squared, and are all positive, so this is most likely an ellipsoid. However, let's put the equation into the standard form for an ellipsoid just to be sure. We have

$$16x^2 + 9y^2 + 16z^2 = 144$$

Dividing through by 144 gives

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

So, this is $\boxed{\text{an ellipsoid centered at the origin}}$

(b) We first notice that the z term is raised only to the first power, so this is either an elliptic paraboloid or a hyperbolic paraboloid. We also note there are x terms and y terms that are not squared, so this quadric surface is not centered at the origin. We need to complete the square to put this equation in one of the standard forms. We have

$$\begin{aligned} 9x^2 - 18x + 4y^2 + 16y - 36z + 25 &= 0 \\ 9x^2 - 18x + 4y^2 + 16y + 25 &= 36z \\ 9(x^2 - 2x) + 4(y^2 + 4y) + 25 &= 36z \\ 9(x^2 - 2x + 1 - 1) + 4(y^2 + 4y + 4 - 4) + 25 &= 36z \\ 9(x-1)^2 - 9 + 4(y+2)^2 - 16 + 25 &= 36z \\ 9(x-1)^2 + 4(y+2)^2 &= 36z \\ \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} &= z \end{aligned}$$

This is $\boxed{\text{an elliptic paraboloid centered at } (1, -2, 0)}$

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1.4: Practice Set

Exercise 1.4.1

Try to find an equation to match the described surfaces:

(a) when x is constant, cross sections look like parabolas; when y is constant, cross sections look like ellipses; when z is constant, cross sections look like parabolas.

(b) when x is constant, cross sections look like circles; when y is constant, cross sections look like parabolas; when z is constant, cross sections look like parabolas.

(c) when x is constant, cross sections look like circles; when y is constant, cross sections look like circles; when z is constant, cross sections look like circles.

(d) when x is constant, cross sections look like circles; when y is constant, cross sections look like ellipses; when z is constant, cross sections look like ellipses.

(e) when x is constant, cross sections look like hyperbolas; when y is constant, cross sections look like ellipses; when z is constant, cross sections look like hyperbolas.

Answer

(a) For constant y we have $ax^2 + bz^2 = 1$, so x and z must be squared with different coefficients.
For constant x we have $y = \pm cz^2$ or $z = \pm cy^2$, since z must be squared, it's $y = \pm cz^2$.
For constant z we have $x = \pm dy^2$ or $y = \pm dx^2$, since x must be squared, it's $y = \pm dx^2$.

Putting these together, we see that $y = \pm(ax^2 + bz^2)$ defines an **elliptic paraboloid**.

(b) For constant x we have $y^2 + z^2 = a^2$, so y and z must be squared with equal coefficients.
For constant y we have $x = \pm bz^2$ or $z = \pm bx^2$, since z must be squared, it's $x = \pm bz^2$.
For constant z we have $x = \pm cy^2$ or $y = \pm cx^2$, since y must be squared, it's $x = \pm cy^2$.

Putting these together, we see that $x = \pm a(y^2 + z^2)$ defines a **circular paraboloid**.

(c) For constant x we have $y^2 + z^2 = a^2$, so y and z must be squared with equal coefficients.
For constant y we have $x^2 + z^2 = b^2$, so x and z must be squared with equal coefficients.
For constant z we have $x^2 + y^2 = c^2$, so x and y must be squared with equal coefficients.

Putting these together, we see that $x^2 + y^2 + z^2 = a^2$ defines a **sphere**.

(d) For constant x we have $y^2 + z^2 = a^2$, so y and z must be squared with equal coefficients.
For constant y we have $bx^2 + cz^2 = 1$, so x and z must be squared with different coefficients.
For constant z we have $dx^2 + ey^2 = 1$, so x and y must be squared with different coefficients.

Putting these together, we see that $ax^2 + by^2 + bz^2 = 1$ defines an **ellipsoid**.

(e) For constant y we have $ax^2 + bz^2 = 1$, so x and z must be squared with different coefficients.
For constant x we have $cy^2 - dz^2 = 1$ or $cz^2 - dy^2 = 1$.
For constant z we have $ex^2 - fy^2 = 1$ or $ey^2 - fx^2 = 1$.

Putting these together, we see that $ax^2 - by^2 + cz^2 = 1$ or $-ax^2 + by^2 - cz^2 = 1$ which defines a **hyperboloid of one or two sheets**.

Exercise 1.4.2

Describe the cross sections on the xy-plane as z increases from $-\infty$ to ∞:

(a) $x^2 + y^2 = 1$

(b) $z = 2\sqrt{x^2 + y^2}$

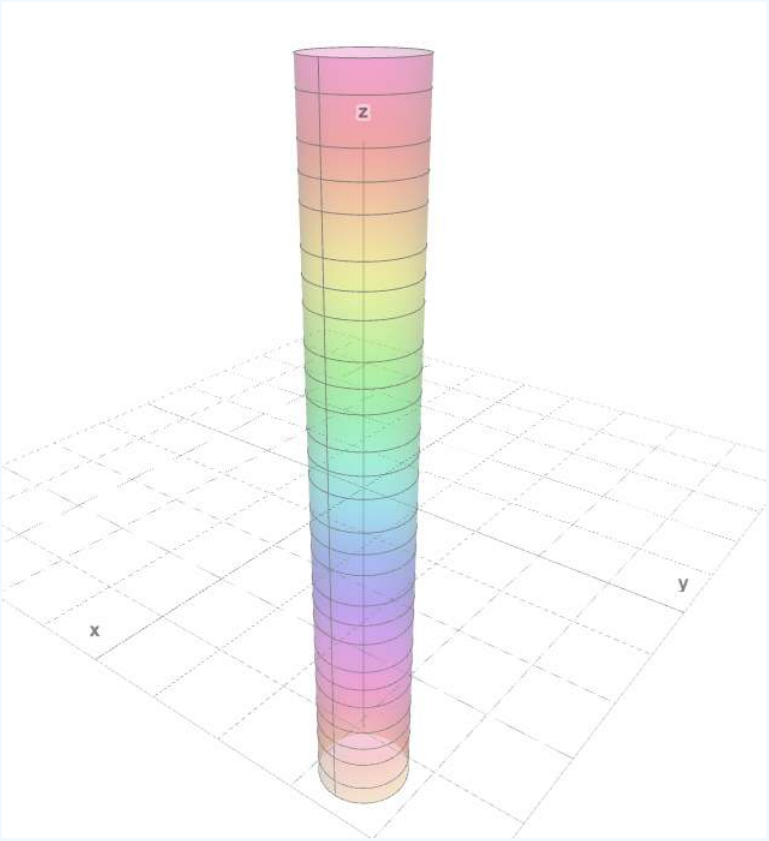
(c) $z = 4x^2 + y^2$

(d) $x^2 + 2y^2 - z^2 = 1$

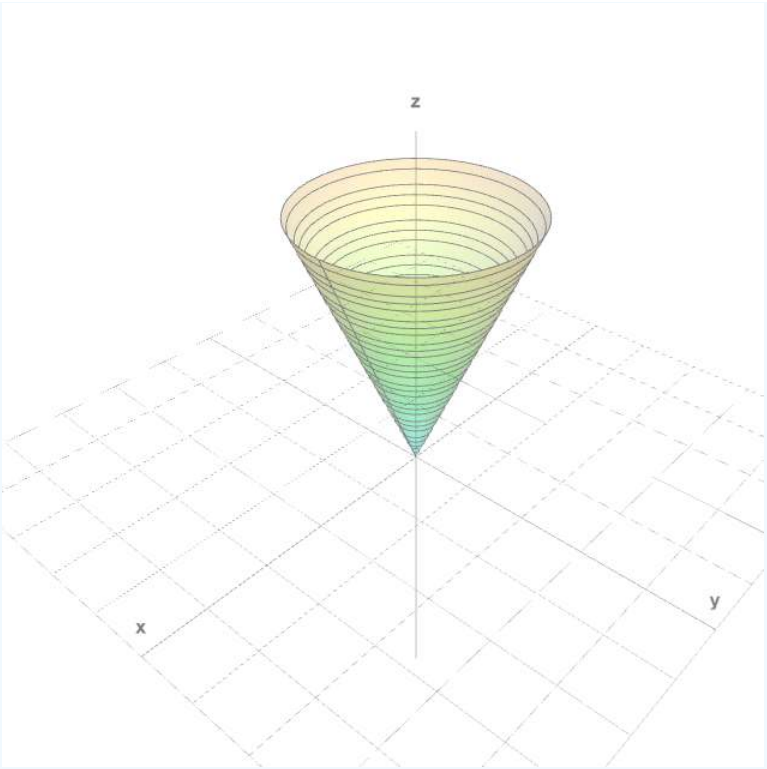
(e) $1 + x^2 + y^2 = z^2$

Answer

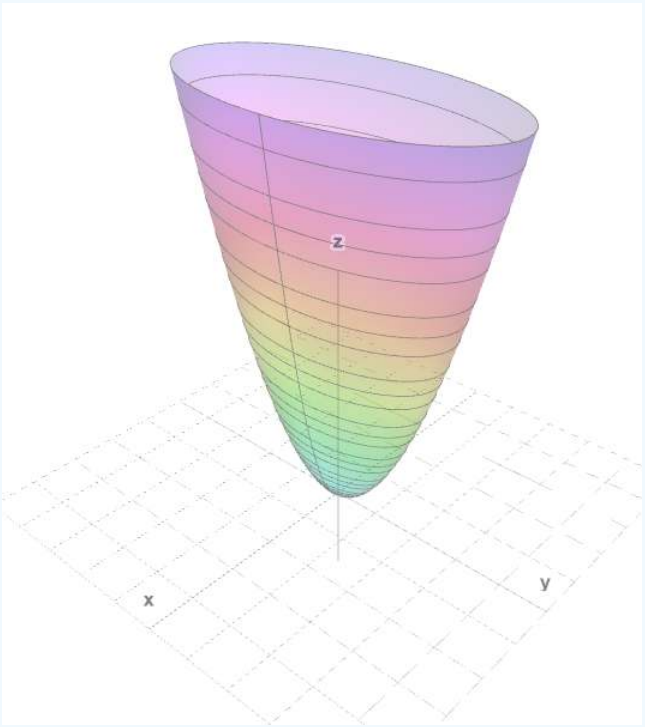
(a) This is a cylinder with constant cross sections that are always the unit circle.



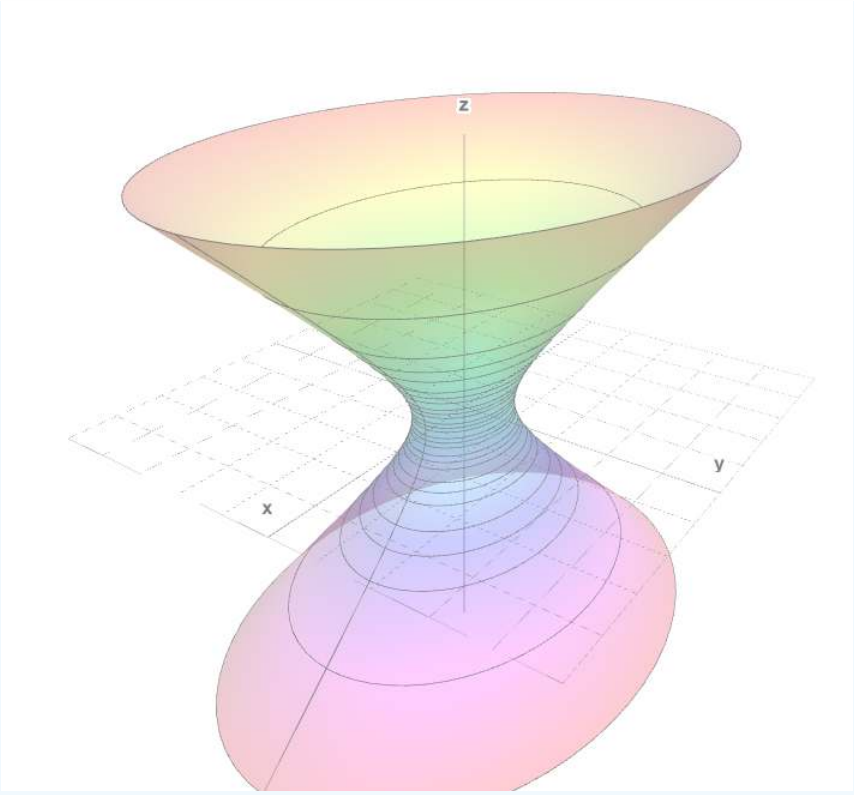
(b) This is the top half of a cone with circular cross sections with increasing radius half the size of z.



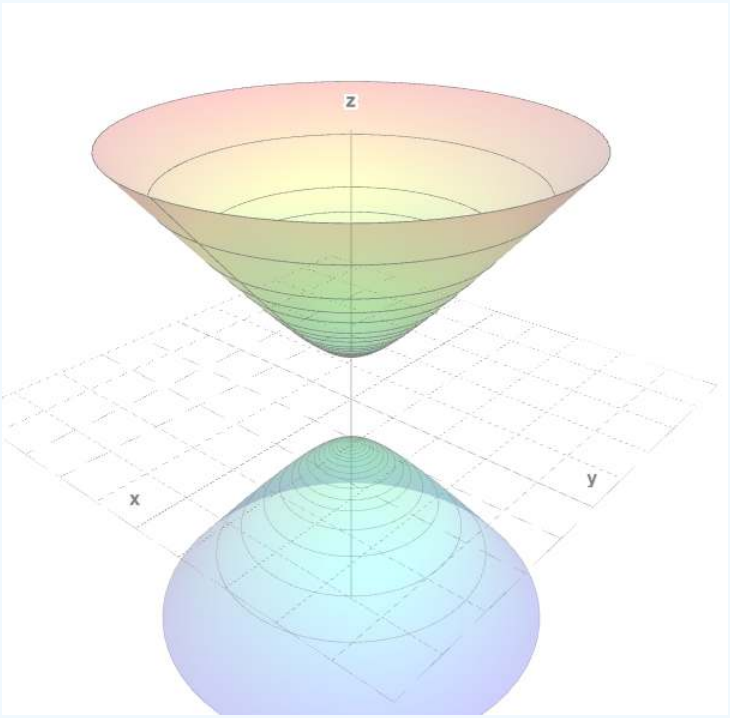
(c) This is an elliptic paraboloid with cross sections that are ellipses that are twice as wide in the y direction as they are in the x direction.



(d) This is a hyperboloid of one sheet with elliptic cross sections that are $\sqrt{2}$ times as wide in the x direction as they are in the y direction. The cross sections start out large, then decrease down as z approaches zero from the negative side to a minimum ellipse $x^2 + 2y^2 = 1$ when $z = 0$, then expand again as z becomes a larger positive value.



(e) This is a hyperboloid of two sheets with circular cross sections. The circles start out large, then decrease down to a single point as z increases to $z = -1$. There is no cross section for $-1 < z < 1$, then at $z = 1$ they begin as a point and expand as z becomes a larger positive number.

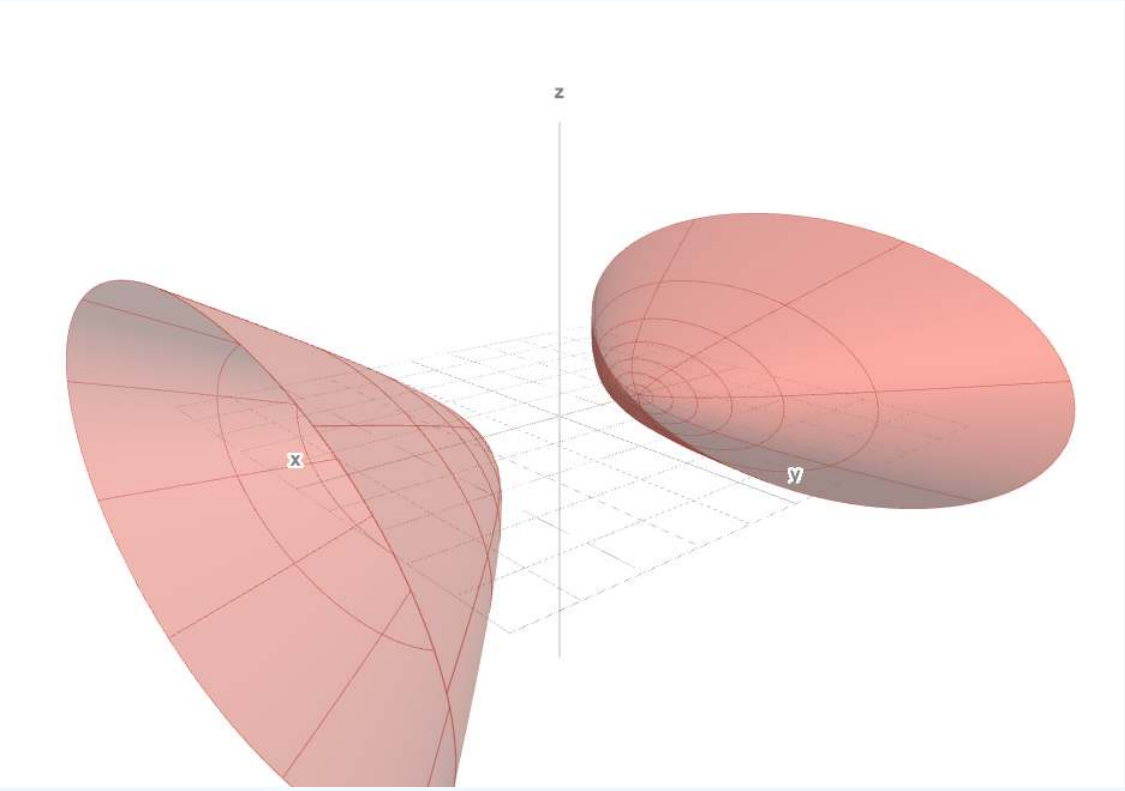


Exercise 1.4.3a

Sketch and classify $x^2 - y^2 - 4z^2 = 4$

Answer

$\frac{x^2}{4} - \frac{y^2}{4} - z^2 = 1$ is a **hyperboloid of two sheets**, opening in the x directions.

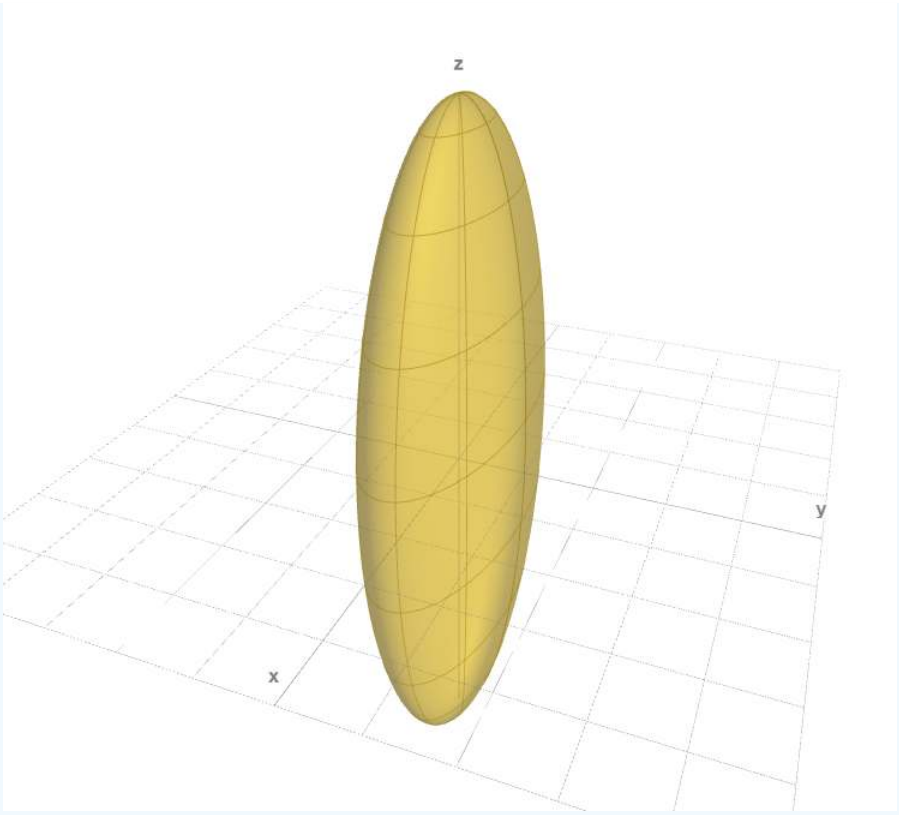


Exercise 1.4.3b

Sketch and classify $4x^2 + 25y^2 + z^2 = 100$

Answer

$\frac{x^2}{25} + \frac{y^2}{4} + \frac{z^2}{100} = 1$ is an **ellipsoid** with x-radius 5, y-radius 2, and z-radius 10.



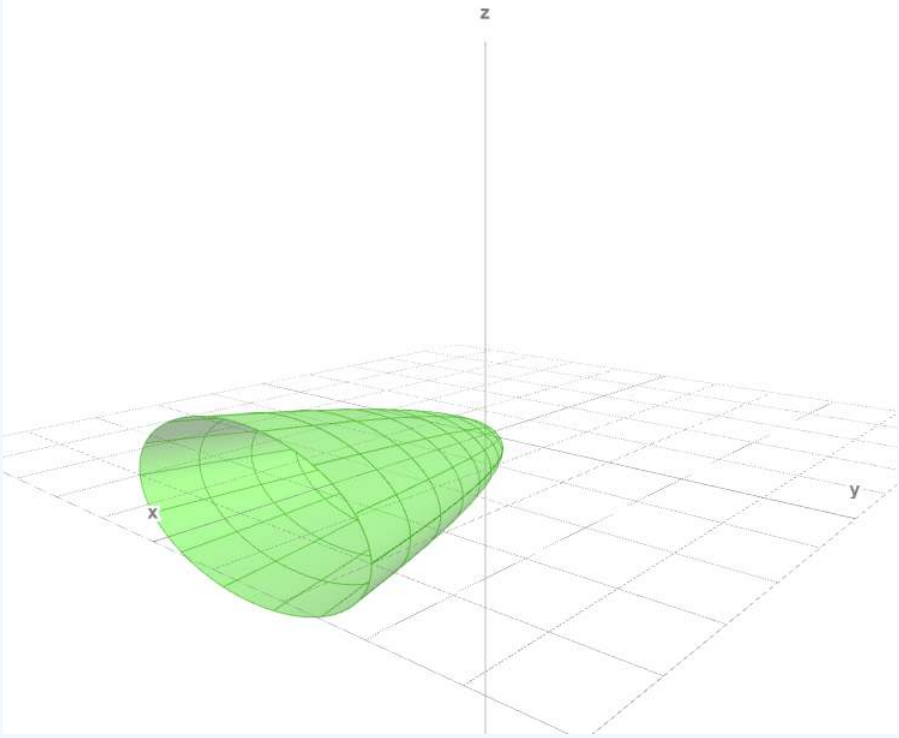
Exercise $1.4.3c$

Sketch and classify $x - y^2 = 4z^2$

Answer

$x = y^2 + 4z^2$ is an **elliptical paraboloid** that opens along the positive x-axis.

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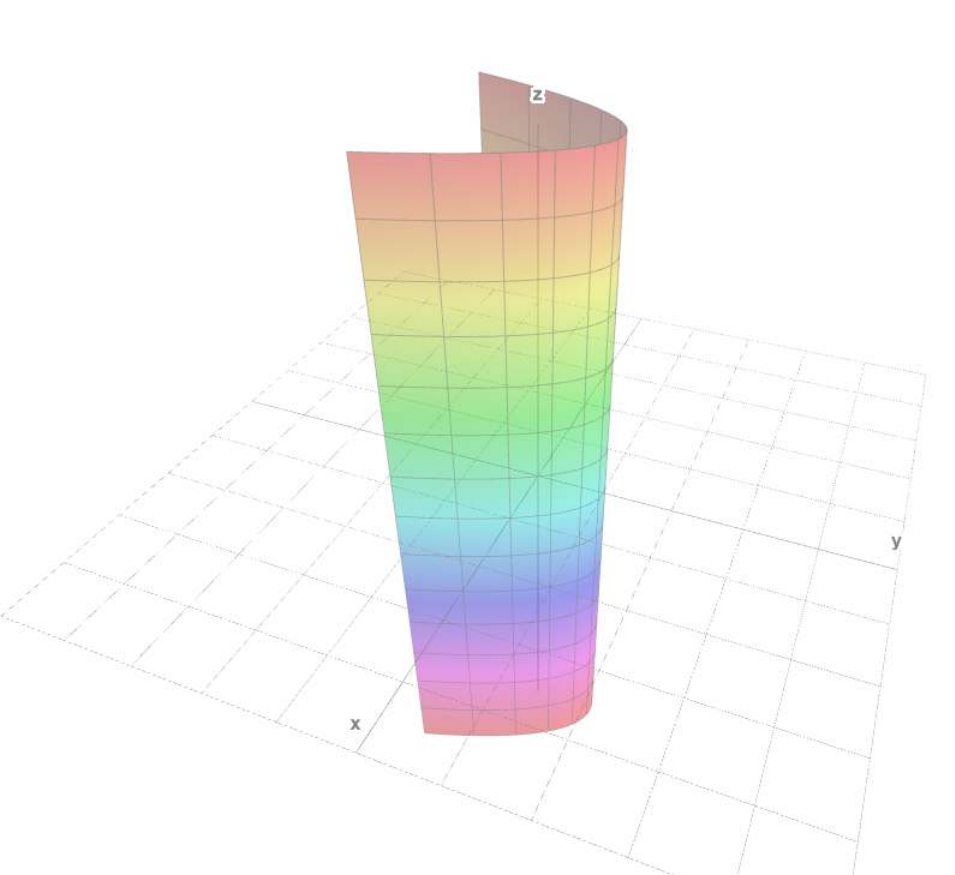


? Exercise 1.4.3d

Sketch and classify $x = 1 - y^2$

Answer

$x = 1 - y^2$ is a **parabolic cylinder** which is repeated along the z-axis.



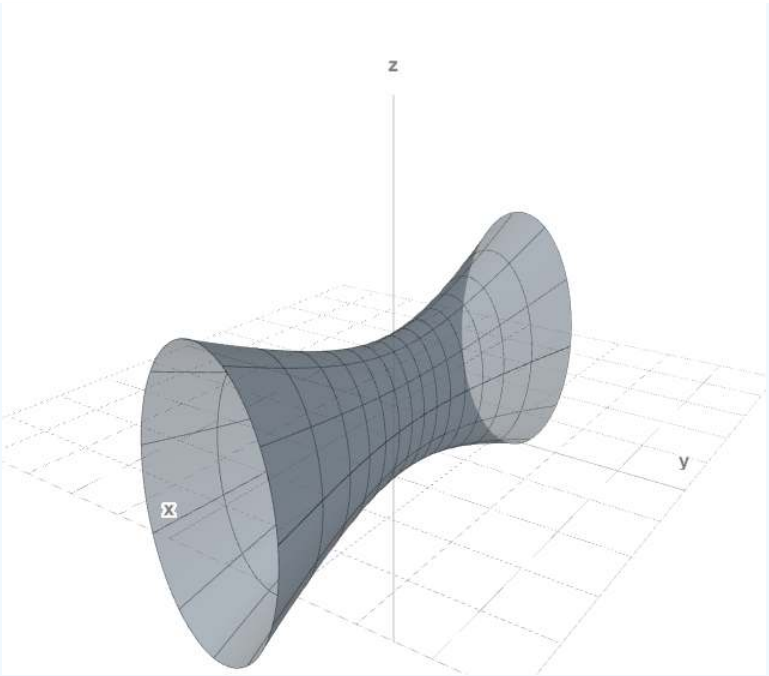
Exercise $1.4.3e$

Sketch and classify $16y^2 + 4z^2 = 16 + x^2$

Answer

$-\frac{x^2}{16} + y^2 + \frac{z^2}{4} = 1$ is a **hyperboloid of one sheet** with elliptical cross sections that opens along the x-axis.

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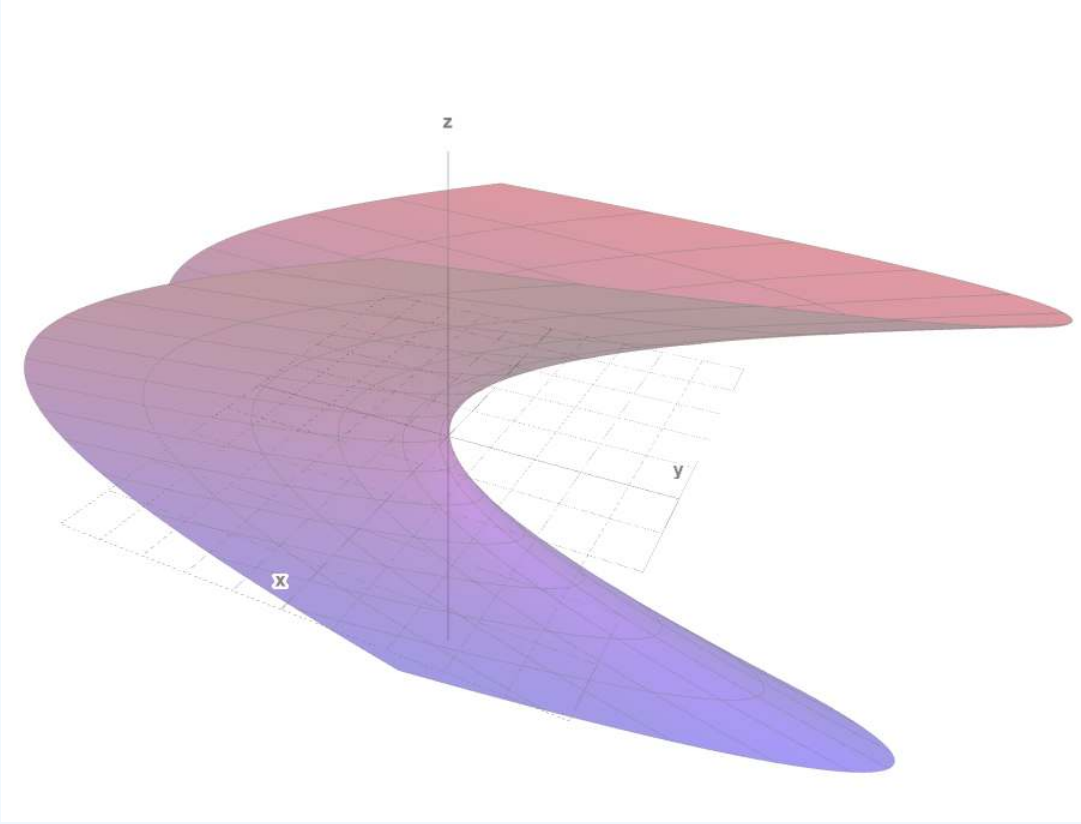


? Exercise 1.4.3f

Sketch and classify $y = z^2 - 4x^2$

Answer

$y = z^2 - 4x^2$ is a **hyperbolic paraboloid** with upward parabolas in the yz-plane and downward parabolas in the xy-plane.

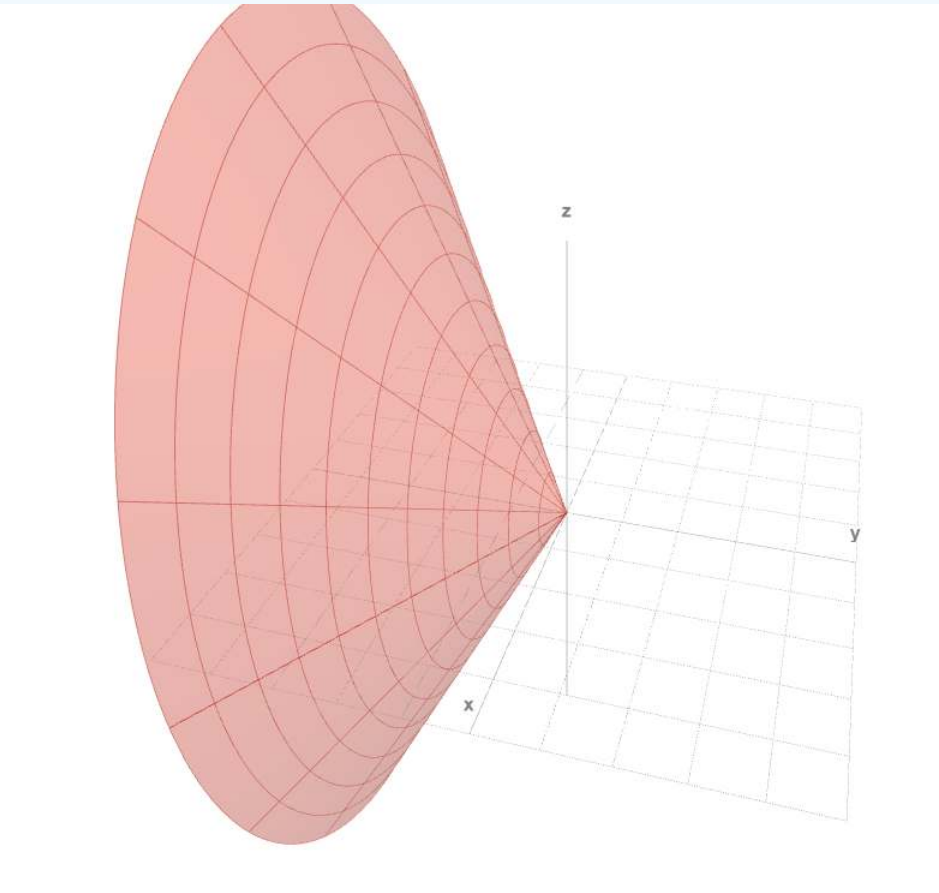


Exercise $1.4.3g$

Sketch and classify $y = -2\sqrt{\frac{x^2}{9} + \frac{z^2}{16}}$

Answer

$y = -\sqrt{\frac{x^2}{\frac{9}{4}} + \frac{z^2}{4}}$ is an **elliptical cone** which opens along the negative y-axis.



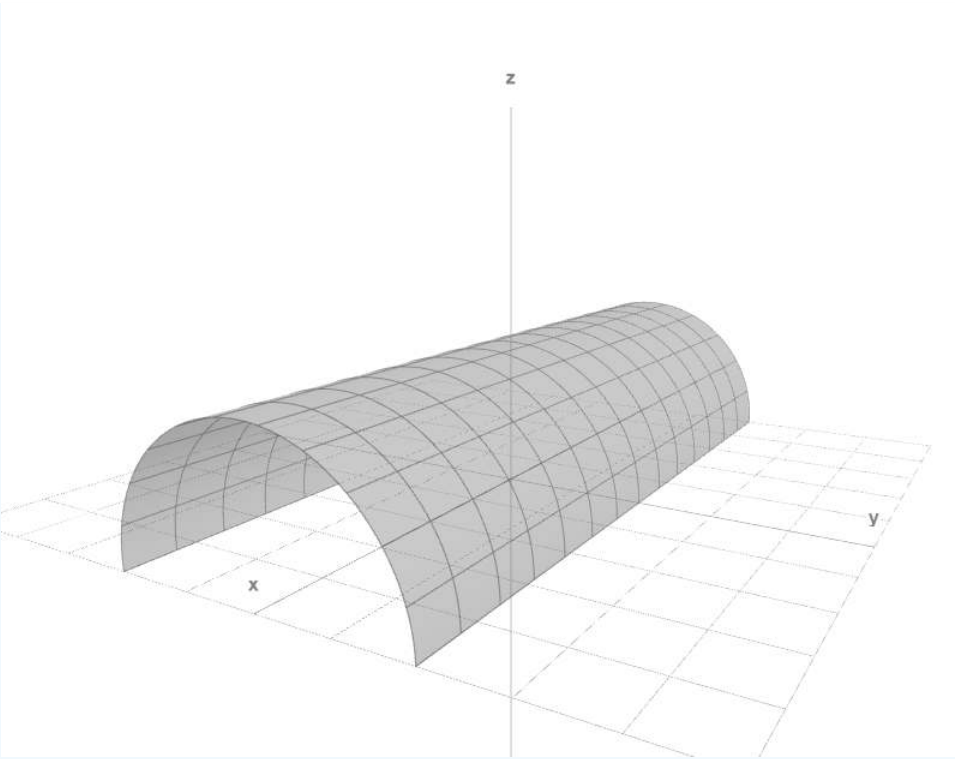
Exercise $1.4.3h$

Sketch and classify $z = \sqrt{2 - y^2}$

Answer

$z = \sqrt{2 - y^2}$ is the top half of a **circular cylinder** that is repeated along the x-axis.

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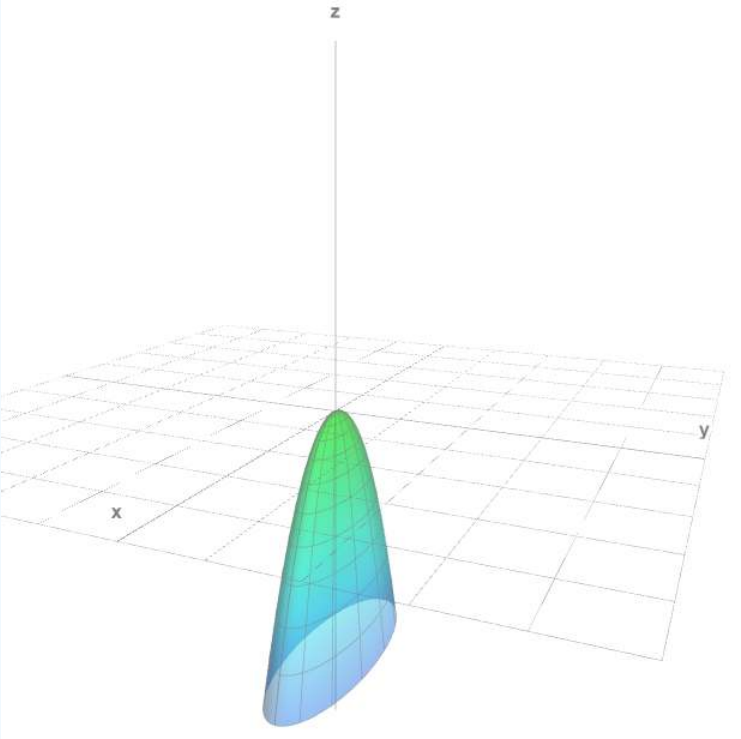


Exercise $1.4.3i$

Sketch and classify $z = -x^2 - 9y^2$

Answer

$z = -x^2 - 9y^2$ is an **elliptical paraboloid** which opens along the negative z-axis.



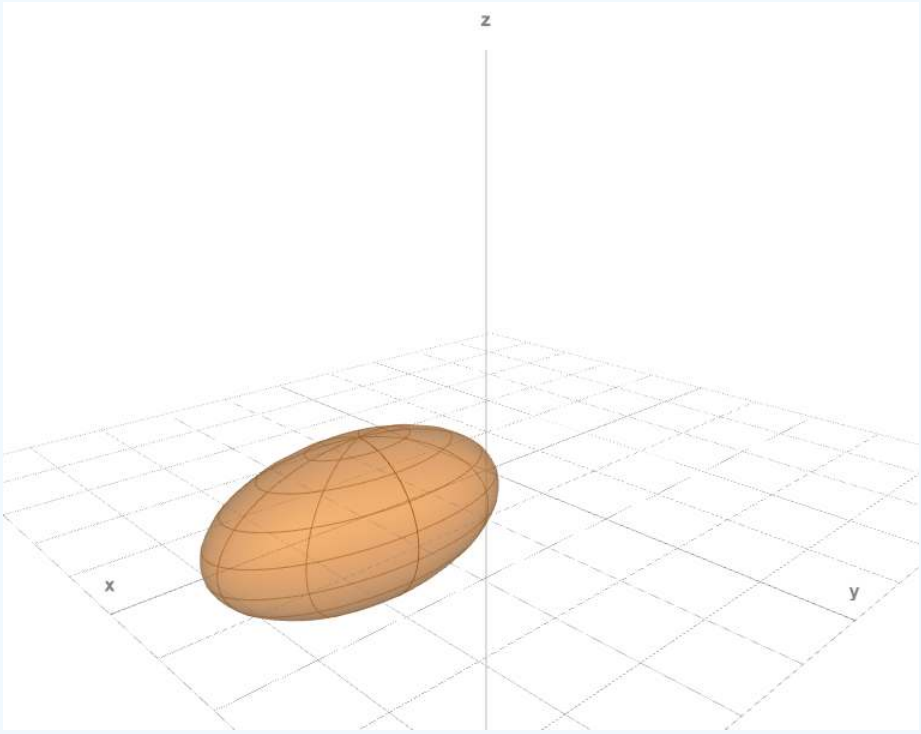
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Exercise $1.4.3j$

Sketch and classify $x = \frac{1}{4}x^2 + y^2 + z^2$

Answer

$1 = \frac{(x-2)^2}{4} + y^2 + z^2$ is an **ellipsoid** centered at $(2, 0, 0)$ with x-radius 2, y-radius 1, and z-radius 1.

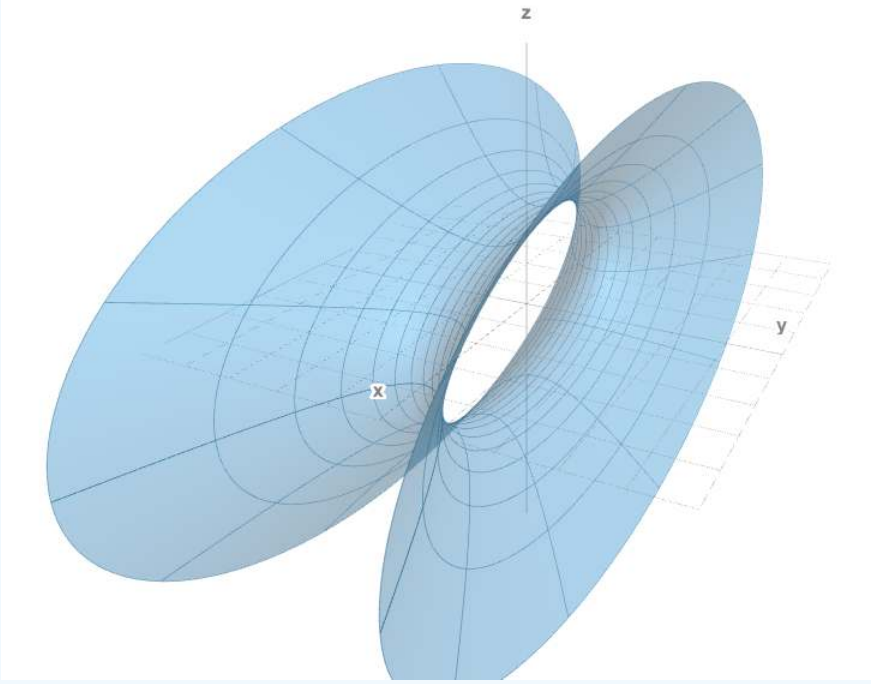


Exercise $1.4.3k$

Sketch and classify $x^2 + 4z^2 = 36 + 9y^2$

Answer

$\frac{x^2}{36} - \frac{y^2}{4} + \frac{z^2}{9} = 1$ is a **hyperboloid of one sheet** that opens along the y-axis.



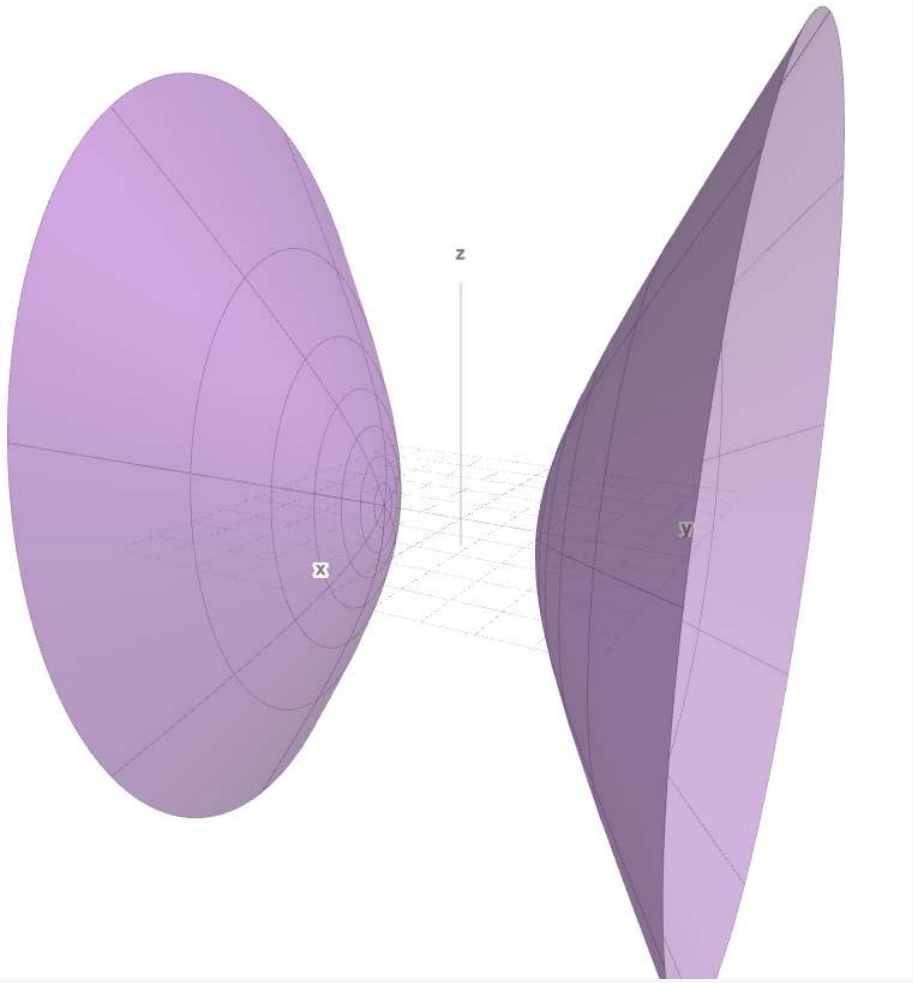
Exercise $1.4.3l$

Sketch and classify $9(y^2 - x^2) = 4(9 + z^2)$

Answer

$-\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{9} = 1$ is a **hyperboloid of two sheets** that opens along the y-axis.

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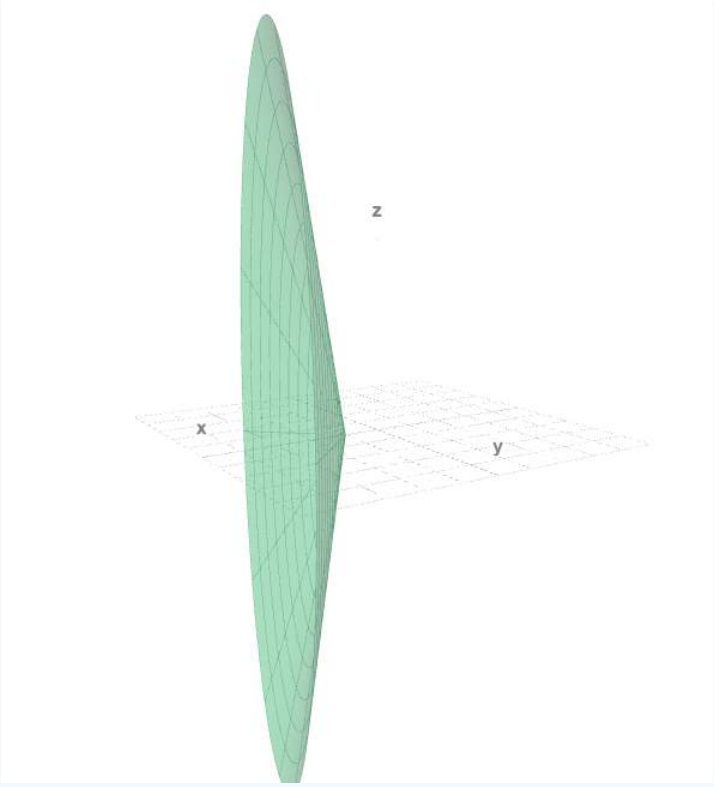
Exercise $1.4.3m$

Sketch and classify $5x = 5 + \sqrt{25y^2 + z^2}$

Answer

$x - 1 = \sqrt{y^2 + \frac{z^2}{25}}$ is an **elliptical cone** that begins at $(1, 0, 0)$ and expands along the positive x-axis.

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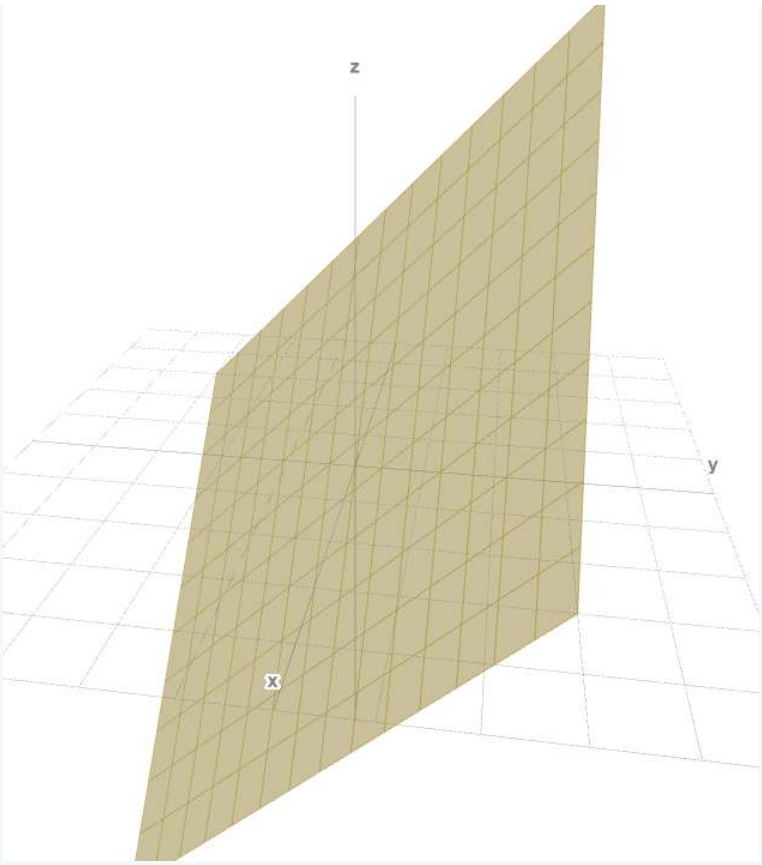
Exercise $1.4.3n$

Sketch and classify $(x-2)^2+(y+1)^2+z^2=x^2+(y-1)^2+(z+2)^2$

Answer

$x-y+z=0$ is a **plane**.

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1.5: Polar, Cylindrical, and Spherical Coordinates

Learning Objectives

- Convert between rectangular and polar coordinates in $\mathbb{R}^2$.
- Convert between cylindrical and rectangular coordinates in $\mathbb{R}^3$.
- Convert between spherical and rectangular coordinates in $\mathbb{R}^3$.
- Convert between cylindrical and spherical coordinates in $\mathbb{R}^3$.

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we refresh on polar coordinates in two dimensions and then look at two different ways of describing the location of points in three dimensional space based on extensions of polar coordinates.

As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

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1.5: Cylindrical and Spherical Coordinates

In two dimensions, we use the plane $\mathbb{R}^2$ to graph relationships between x and y. Some relationships involving circles (or curves related to circles) are difficult to describe using these two variables, and instead we describe the plane using **polar coordinates**. We translate each point (x, y) into polar form and describe them as (r, θ) where r and θ are defined as in the picture below.

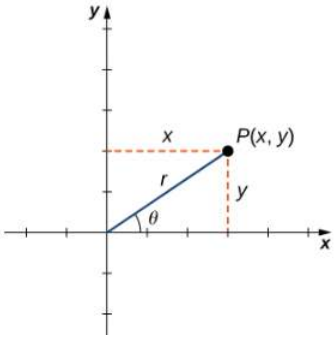


Figure 1.5.1: The relationship between polar and rectangular coordinates in two dimensions.

Switching between these two coordinate systems is generally easy for a single point, as the right triangle pictured above clearly gives:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \qquad \text{and} \qquad \begin{aligned} r^2 &= x^2 + y^2 \\ \tan(\theta) &= \frac{y}{x} \end{aligned}$$

It is worth noting that the same point in rectangular coordinates can be written infinitely many different ways in polar coordinates. For example, the point $(x, y) = \left(1, \sqrt{3}\right)$ can be written as $(r, \theta) = \left(2, \frac{\pi}{6} + 2\pi k\right)$ or $(r, \theta) = \left(-2, \frac{7\pi}{6} + 2\pi k\right)$ for any integer k. For this reason, the "default" choice of values of r is positive, and of θ is in $[0, 2\pi)$, but these rules are not firm.

Relationships and equations that are complicated in rectangular form can often be simpler in polar form.

Example 1.5.1

Write each equation below in polar form.

(a) $x^2 + y^2 = 100$

(b) $(x-1)^2 + (y-1)^2 = 2$

(c) $(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$

Solution

(a) We substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into the rectangular form and simplify:

$$\begin{aligned} x^2 + y^2 &= 100 \\ (r\cos(\theta))^2 + (r\sin(\theta))^2 &= 100 \\ r^2(\cos^2(\theta) + \sin^2(\theta)) &= 100 \\ r^2 &= 100 \\ r &= 10 \end{aligned}$$

Therefore, the circle defined by $x^2 + y^2 = 100$ is represented in polar form as $\boxed{r = 10}$

(b) We substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into the rectangular form and simplify:

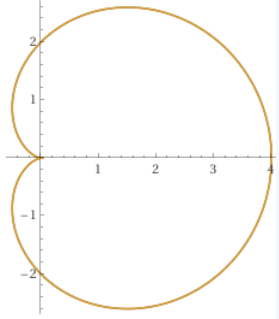
$$
\begin{aligned}
(x-1)^2+(y-1)^2 &= 2 \\
(r\cos(\theta)-1)^2+(r\sin(\theta)-1)^2 &= 2 \\
r^2\cos^2(\theta)-2r\cos(\theta)+r^2\sin^2(\theta)-2r\sin(\theta)+2 &= 2 \\
r^2(\cos^2(\theta)+\sin^2(\theta))-2r(\cos(\theta)+\sin(\theta)) &= 0 \\
r^2-2r\left(\cos(\theta)+\sin(\theta)\right) &= 0 \\
r\left(r-2\cos(\theta)-2\sin(\theta)\right) &= 0 \\
r=0 \text{ or } r &= 2\cos(\theta)+2\sin(\theta)
\end{aligned}
$$

Clearly $r=0$ is just a single point, so the circle defined by $(x-1)^2+(y-1)^2=2$ is represented in polar form as $\boxed{r=2\cos(\theta)+2\sin(\theta)}$

(c) We substitute $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into the rectangular form and simplify:

$$
\begin{aligned}
(x^2+y^2-2x)^2 &= 4(x^2+y^2) \\
\left(r^2\cos^2(\theta)+r^2\sin^2(\theta)-2r\cos(\theta)\right)^2 &= 4\left(r^2\cos^2(\theta)+r^2\sin^2(\theta)\right) \\
\left(r^2-2r\cos(\theta)\right)^2 &= 4r^2 \\
r^4-4r^3\cos(\theta)+4r^2\cos^2(\theta) &= 4r^2 \\
r^2(r^2-4r\cos(\theta)+4\cos^2(\theta)-4) &= 0 \\
r^2\left((r-2\cos(\theta))^2-4\right) &= 0 \\
r=0 \text{ or } r &= 2+2\cos(\theta)
\end{aligned}
$$

Clearly $r=0$ is just a single point, so the cardioid defined by $(x^2+y^2-2x)^2=4(x^2+y^2)$ is represented in polar form as $\boxed{r=2+2\cos(\theta)}$



When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Definition: The Cylindrical Coordinate System

In the cylindrical coordinate system**,** a point in space is represented by the ordered triple (r,θ,z), where

- (r,θ) is the polar coordinates of the point's projection in the xy-plane
- z is the usual z-**coordinate** in the Cartesian coordinate system

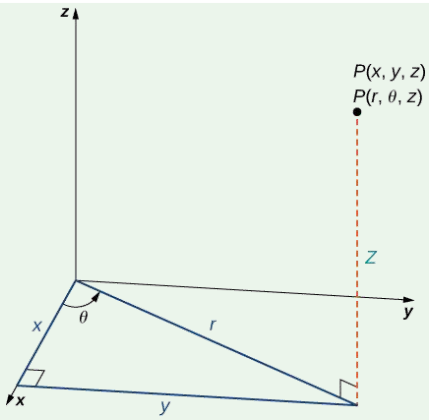


Figure 1.5.2: The right triangle lies in the xy-plane. The length of the hypotenuse is r, and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the xy-plane.

In the xy-plane, the right triangle shown above provides the key to the transformation between cylindrical and Cartesian, or rectangular, coordinates.

Rule: Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related just like in polar form:

These equations are used to convert from cylindrical coordinates to rectangular coordinates.

- $x = r\cos(\theta)$
- $y = r\sin(\theta)$
- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

- $r^2 = x^2 + y^2$
- $\tan(\theta) = \dfrac{y}{x}$
- $z = z$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan(\theta) = \dfrac{y}{x}$ has an infinite number of solutions. Again, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the xy-plane in which the original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is $\dfrac{\pi}{2}$ when $y > 0$, or $\dfrac{3\pi}{2}$ when $y < 0$, or is 0 when $y = 0$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c, y = c$, or $z = c$ are all planes. Planes of these forms are parallel to the yz-plane, the xz-plane, and the xy-plane, respectively. When we convert to cylindrical coordinates, the z-coordinate does not change.

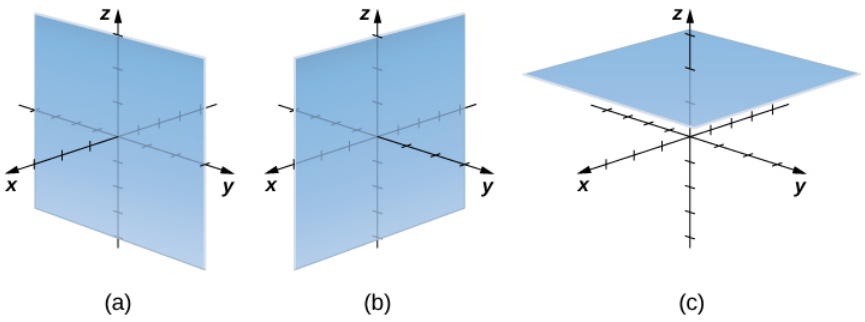


Figure 1.5.3: In rectangular coordinates,
(a) surfaces of the form $x = c$ are planes parallel to the yz-plane
(b) surfaces of the form $y = c$ are planes parallel to the xz-plane
(c) surfaces of the form $z = c$ are planes parallel to the xy-plane

Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the xy-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the z-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the x-axis, which gives us a half-plane that starts at the z-axis.

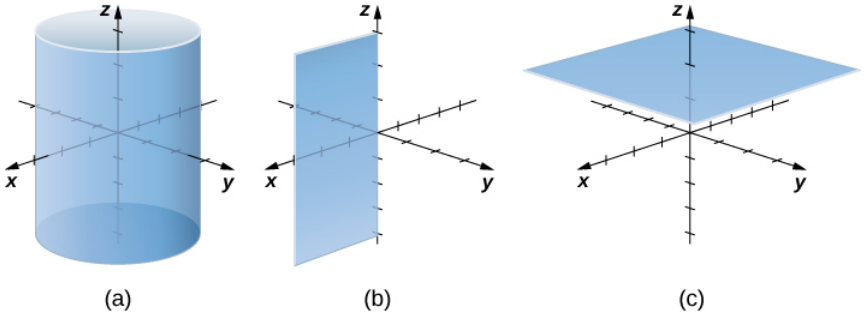


Figure 1.5.4: In cylindrical coordinates,
(a) surfaces of the form $r = c$ are vertical cylinders of radius r
(b) surfaces of the form $\theta = c$ are half-planes at angle θ from the x-axis
(c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

Example 1.5.2: Converting from Cylindrical to Rectangular Coordinates

Plot the points given in cylindrical coordinates and express its location in rectangular coordinates.

(a) $\left(4, \frac{2\pi}{3}, -2\right)$

(b) $\left(5, \frac{\pi}{6}, 4\right)$

Solution

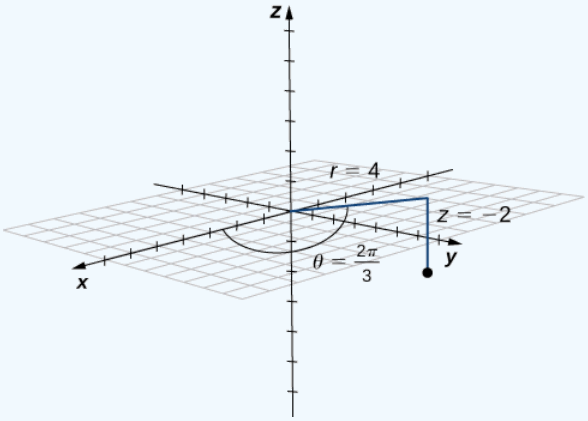
(a) Conversion from cylindrical to rectangular coordinates requires a simple application of the equations listed above:

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = -2$$

$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3}$$

$$z = -2$$

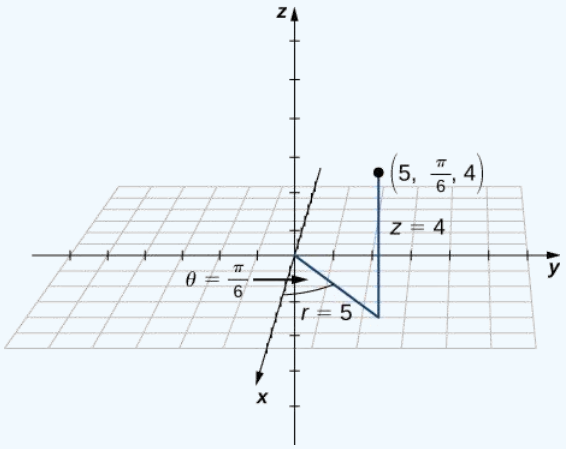
The point with cylindrical coordinates $\left(4, \frac{2\pi}{3}, -2\right)$ has rectangular coordinates $\boxed{(-2, 2\sqrt{3}, -2)}$



(b) Again, converting from cylindrical to rectangular coordinates:

$$x = r\cos\theta = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2}$$
$$y = r\sin\theta = 5\sin\frac{\pi}{6} = \frac{5}{2}$$
$$z = 4$$

The point with cylindrical coordinates $\left(5, \frac{\pi}{6}, 4\right)$ has rectangular coordinates $\boxed{\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4\right)}$



Example 1.5.3: Converting from Rectangular to Cylindrical Coordinates

Convert the point given in rectangular coordinates to cylindrical coordinates.

(a) $(1, -3, 5)$

(b) $(-8, 8, -7)$

Solution

(a) Using the second set of equations above to translate from rectangular to cylindrical coordinates:

$$r^2 = x^2 + y^2$$
$$r = \pm\sqrt{1^2 + (-3)^2}$$
$$= \pm\sqrt{10}$$

We choose the positive square root, so $r = \sqrt{10}$. Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\frac{3\pi}{2}$ and 2π:

$$\tan(\theta) = \frac{y}{x} = \frac{-3}{1} = -3$$
$$\theta = 2\pi + \tan^{-1}(-3)$$

In this case, the *z*-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates equal to $\boxed{\left(\sqrt{10}, 2\pi + \tan^{-1}(-3), 5\right)}$

Note that it is often acceptable to write this answer as $(\sqrt{10}, \tan^{-1}(-3), 5)$, even though this places θ outside of $[0, 2\pi)$

(b) Once again from rectangular to cylindrical coordinates:

$$r^2 = x^2 + y^2$$
$$r = \sqrt{(-8)^2 + 8^2} = \sqrt{128} = 8\sqrt{2}$$

So $r = 8\sqrt{2}$. Now, we apply the formula to find θ. In this case, x is negative and y is positive, which means we must select the value of θ between $\frac{\pi}{2}$ and π:

$$\tan(\theta) = \frac{y}{x} = \frac{8}{-8} = -1$$
$$\theta = \pi + \tan^{-1}(-1) = \frac{3\pi}{4}$$

Once more the *z*-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = -7$$

The point with rectangular coordinates $(-8, 8, -7)$ has cylindrical coordinates equal to $\boxed{\left(8\sqrt{2}, \frac{3\pi}{4}, -7\right)}$

Please notice that this answer is *not the same* as the point $\left(8\sqrt{2}, \tan^{-1}(-1), -7\right) = \left(8\sqrt{2}, -\frac{\pi}{4}, -7\right)$

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2 + y^2 = 25$ in the Cartesian system can be represented by cylindrical equation $r = 5$.

Illustration 1.5.1: Identifying Surfaces in the Cylindrical Coordinate System

We will describe the surfaces with the given cylindrical equations below:

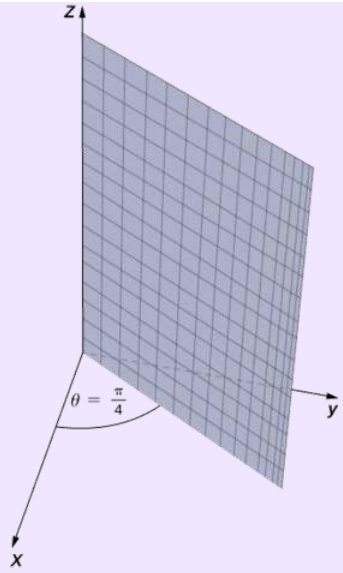
(a) $\theta = \frac{\pi}{4}$

(b) $r^2 + z^2 = 9$

(c) $z^2 = r^2$

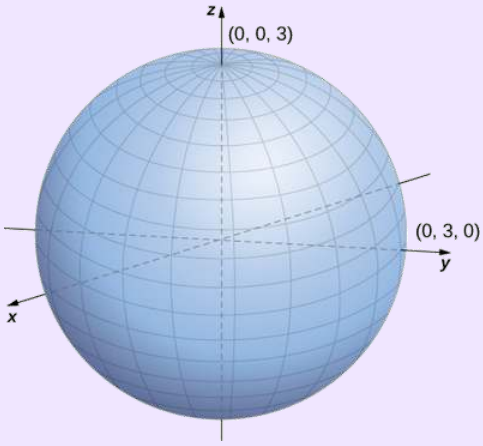
(d) $r = 6$

For (a), when the angle θ is held constant while r and z are allowed to vary, the result is a half-plane.

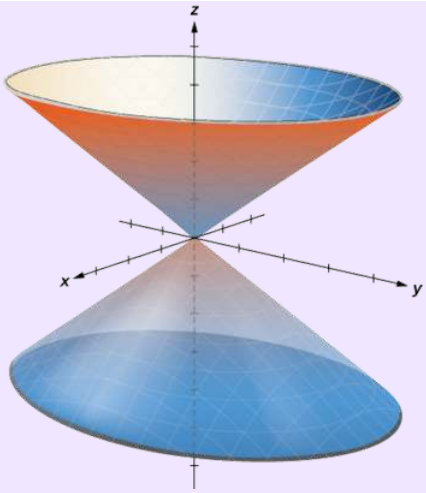


(b) Substitute $r^2 = x^2 + y^2$ into equation $r^2 + z^2 = 9$ to express the rectangular form of the equation: $x^2 + y^2 + z^2 = 9$

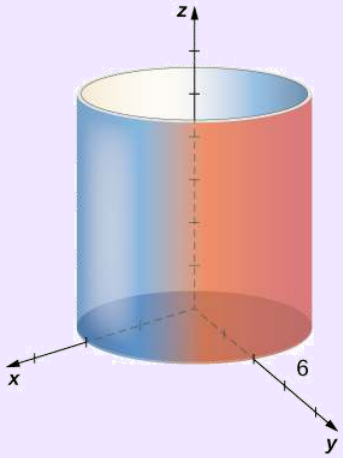
This equation describes a sphere centered at the origin with radius **3**.



(c) To describe the surface defined by equation $z^2 = r^2$, it is useful to examine traces parallel to the xy-plane.
For example, the trace in plane $z = 1$ is circle $r = 1$, the trace in plane $z = 3$ is circle $r = 3$, and so on. Each trace is a circle.
As the value of $|z|$ increases, the radius of the circle also increases. The resulting surface is a cone.



(d) To describe the surface defined by equation $r = 6$, notice that the value of z does not change the trace in any plane $z = k$. Each trace is a circle with radius 6. The resulting surface is a cylinder with radius 6.



In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, the location of a point in space is described using two distances (r and z) and an angle measure (θ). We can now define a third system, called the spherical coordinate system, in which we again use an ordered triple to describe the location of a point in space. In this new system, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

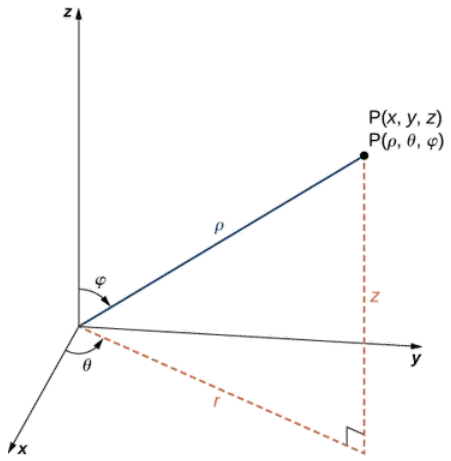


Figure 1.5.5: The relationship among spherical, rectangular, and cylindrical coordinates.

Definition: The Spherical Coordinate System

In the *spherical coordinate system*, a point P in space is represented by the ordered triple (ρ, θ, ϕ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- ϕ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \phi \leq \pi$.

By convention, the origin is represented as $(0, 0, 0)$ in spherical coordinates.

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at the figure below, it is easy to see that $r = \rho \sin \varphi$. Then, looking at the triangle in the xy-plane with r as its hypotenuse, we have $x = r\cos(\theta) = \rho \sin(\phi)\cos(\theta)$. The derivation of the formula for y is similar. We also see that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho\cos(\phi)$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2 + z^2}$ (from the first equation) yields $\phi = \cos^{-1}\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$. Alternatively, taking cotangent of ϕ yields $\frac{z}{r}$, so we can write $\phi = \cot^{-1}\left(\frac{z}{r}\right)$. Both of these inverse functions have a range of $[0, \pi]$, which is the default interval we use when solving for ϕ in spherical coordinates. Unfortunately, as before, we must be careful when using the formula $\tan(\theta) = \frac{y}{x}$ to choose the correct value of θ since the range of the inverse tangent is not a perfect match for our default range of $[0, 2\pi)$.

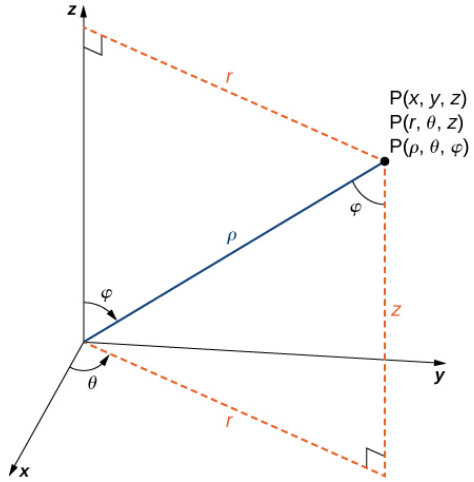


Figure 1.5.6: The equations that convert from one system to another are derived from right-triangle relationships.

Rule: Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z), and spherical coordinates (ρ, θ, ϕ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho \sin(\phi) \cos(\theta)$
- $y = \rho \sin(\phi) \sin(\theta)$
- $z = \rho \cos(\phi)$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan(\theta) = \dfrac{y}{x}$
- $\phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \cot^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2}}\right)$

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho \sin(\phi)$
- $\theta = \theta$
- $z = \rho \cos(\phi)$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\phi = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right) = \cot^{-1}\left(\dfrac{z}{r}\right)$

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\phi = c$. The points on these surfaces are at a fixed angle from the z-axis and form a half-cone.

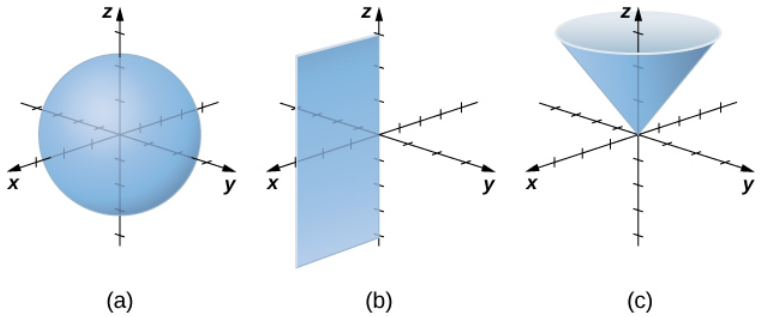


Figure 1.5.7: In spherical coordinates,
(a) surfaces of the form $\rho = c$ are spheres of radius ρ
(b) surfaces of the form $\theta = c$ are half-planes at an angle θ from the x-axis
(c) surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the z-axis

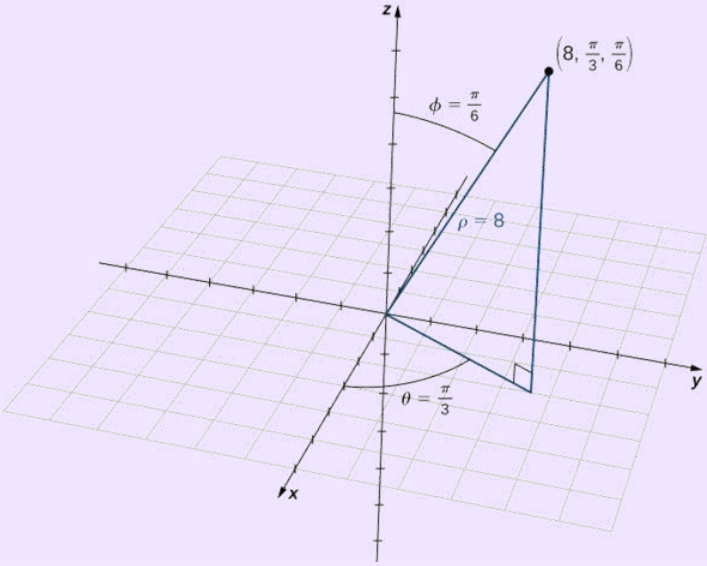
Illustration 1.5.2: Converting from Spherical Coordinates

We will plot each point given in spherical coordinates and describe its location in both rectangular and cylindrical coordinates for

(a) $\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$

(b) $\left(2, -\frac{5\pi}{6}, \frac{\pi}{6}\right)$

(a)



$$x = \rho\sin(\phi)\cos(\theta) = 8\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) = 8\left(\frac{1}{2}\right)\frac{1}{2} = 2$$

$$y = \rho\sin(\phi)\sin(\theta) = 8\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{3}\right) = 8\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$z = \rho\cos(\phi) = 8\cos\left(\frac{\pi}{6}\right) = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$$

The point with spherical coordinates $\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$ has rectangular coordinates $\left(2, 2\sqrt{3}, 4\sqrt{3}\right)$

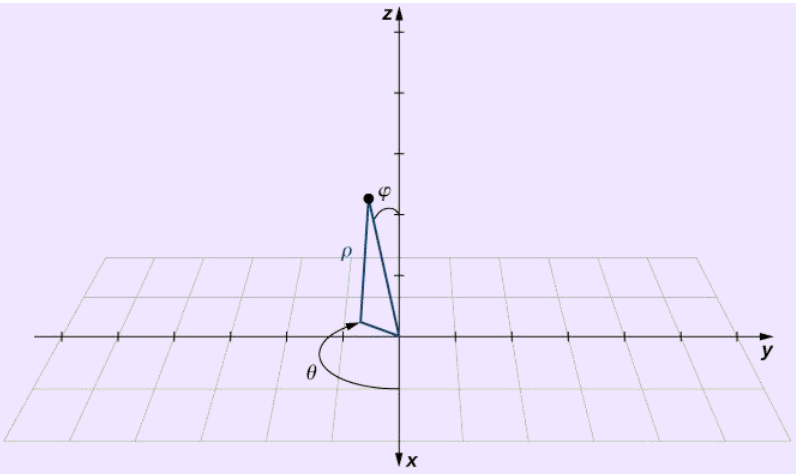
$$r = \rho\sin(\phi) = 8\sin\left(\frac{\pi}{6}\right) = 4$$

$$\theta = \theta$$

$$z = \rho\cos(\phi) = 8\cos\left(\frac{\pi}{6}\right) = 4\sqrt{3}$$

Thus, cylindrical coordinates for the point are $\left(4, \frac{\pi}{3}, 4\sqrt{3}\right)$

(b)



$$x = \rho\sin(\phi)\cos(\theta) = 2\sin\left(-\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) = 2\left(-\frac{1}{2}\right)\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

$$y = \rho\sin(\phi)\sin(\theta) = 2\sin\left(-\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) = 2\left(-\frac{1}{2}\right)\frac{1}{2} = -\frac{1}{2}$$

$$z = \rho\cos(\phi) = 2\cos\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

The point with spherical coordinates $\left(2, -\frac{5\pi}{6}, \frac{\pi}{6}\right)$ has rectangular coordinates $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}\right)$

$$r = \rho\sin(\phi) = 2\sin\left(\frac{\pi}{6}\right) = 1$$

$$\theta = \theta$$

$$z = \rho\cos(\phi) = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$$

Thus, cylindrical coordinates for the point are $\left(1, -\frac{5\pi}{6}, \sqrt{3}\right)$

Example 1.5.4: Converting from Rectangular Coordinates

Convert the rectangular coordinates $(-1, 1, \sqrt{6})$ to both spherical and cylindrical coordinates.

Solution

Start by converting from rectangular to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + 1^2 + (\sqrt{6})^2 = 8 \longrightarrow \rho = 2\sqrt{2}$$

$$\tan\theta = \frac{1}{-1} = -1 \text{ and } x < 0, y > 0 \longrightarrow \theta = \frac{3\pi}{4}$$

There are actually three equivalent methods to identify ϕ. We can use the equation $\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ or $\phi = \cot^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$. A more simple approach, however, is to use equation $z = \rho\cos(\phi)$.

We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so

$$\cos(\phi) = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \longrightarrow \phi = \frac{\pi}{6}$$

The spherical coordinates of the point are $\boxed{\left(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right)}$

To find the cylindrical coordinates for the point, we already know θ and z so we need only to find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

The cylindrical coordinates for the point are $\boxed{\left(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6}\right)}$

Illustration 1.5.3: Identifying Surfaces in the Spherical Coordinate System

We describe the surfaces with the given spherical equations.

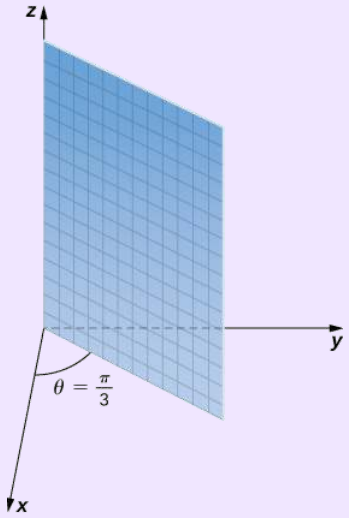
(a) $\theta = \frac{\pi}{3}$

(b) $\phi = \frac{\pi}{3}$

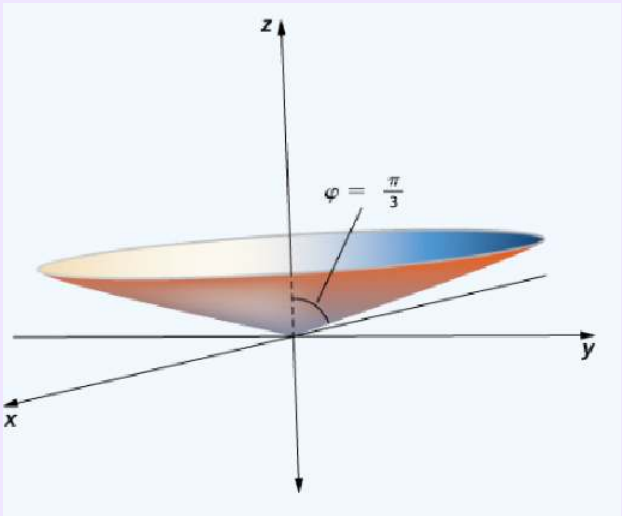
(c) $\rho = 6$

(d) $\rho = \sin(\theta)\sin(\phi)$

(a) The variable θ represents the measure of the same angle in both the cylindrical and spherical coordinate systems. Points with coordinates $\left(\rho, \frac{\pi}{3}, \phi\right)$ lie on the plane that forms angle $\theta = \frac{\pi}{3}$ with the positive x-axis. Because $\rho > 0$, the surface described by equation $\theta = \frac{\pi}{3}$ is the half-plane shown below.



(b) Equation $\phi = \frac{\pi}{3}$ describes all points in the spherical coordinate system that lie on a line from the origin forming an angle measuring $\frac{\pi}{3}$ rad with the positive z-axis. These points form the half-cone pictured below. Because there is only one value for ϕ that is measured from the positive z-axis, we do not get the full cone (with two pieces).

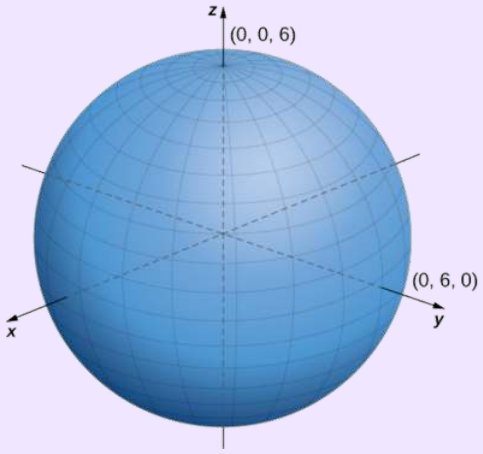


To find the equation in rectangular coordinates, use equation $\phi = \cot^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$:

$$\frac{\pi}{3} = \cot^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$$
$$\cot\frac{\pi}{3} = \frac{z}{\sqrt{x^2+y^2}}$$
$$\frac{1}{\sqrt{3}} = \frac{z}{\sqrt{x^2+y^2}}$$
$$z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$$
$$z = \sqrt{\frac{x^2}{3}+\frac{y^2}{3}}$$

This is the equation of a cone centered on the z-axis.

(c) Equation $\rho = 6$ describes the set of all points 6 units away from the origin — a sphere with radius 6



(d) To identify this surface, convert the equation from spherical to rectangular coordinates $\rho^2 = x^2 + y^2 + z^2$:

$$\rho = \sin(\theta)\sin(\phi) \xrightarrow{\text{Multiply both sides of the equation by } \rho} \rho^2 = \rho\sin(\theta)\sin(\phi)$$
$$\xrightarrow{\text{Substitute into terms of x, y, and z}} x^2+y^2+z^2 = y$$
$$\xrightarrow{\text{Subtract y from both sides}} x^2+y^2-y+z^2 = 0$$
$$\xrightarrow{\text{Complete the square}} x^2+y^2-y+\frac{1}{4}+z^2 = \frac{1}{4}$$
$$\xrightarrow{\text{Rewrite as perfect squares}} x^2+\left(y-\frac{1}{2}\right)^2+z^2 = \left(\frac{1}{2}\right)^2$$

The equation describes a sphere centered at point $\left(0, \frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$.

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. In geography, latitude and longitude are used to describe locations on Earth's surface. Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth:

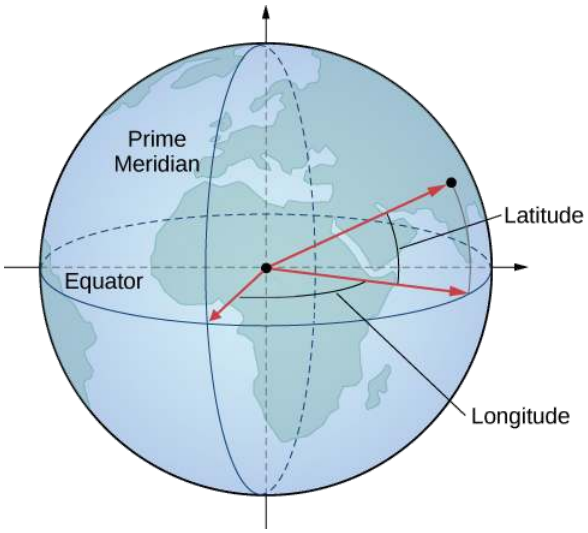


Figure 1.5.8: Navigating the Earth using spherical coordinates.

Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees. Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive z-axis. The prime meridian represents the trace of the surface as it intersects the xz -plane. The equator is the trace of the sphere intersecting the xy-plane.

Points in $\mathbb{R}^3$ can be described in many different ways, but the three most common are rectangular form, cylindrical form, and spherical form. Between these three, we generally have at least one way to describe a surface in a relatively simple way.

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1.5: Practice Set

Exercise 1.5.1a

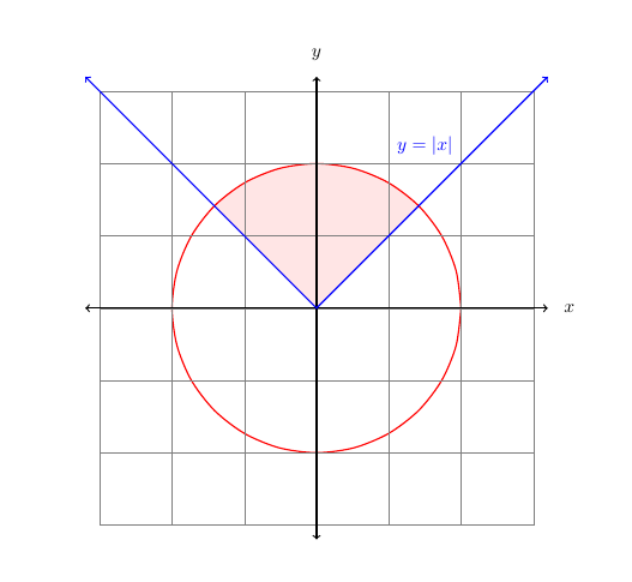
Write the region $A = \{(x, y, z) \mid x^2 + y^2 \leq 4,\ y \geq |x|,\ 0 \leq z \leq (x^2 + y^2)^2\}$ in cylindrical coordinates

Answer

For z, we are given $0 \leq z \leq (x^2 + y^2)^2 \longleftrightarrow 0 \leq z \leq r^4$

The radius r is defined by the circle $x^2 + y^2 \leq 4 \longleftrightarrow 0 \leq r \leq 2$

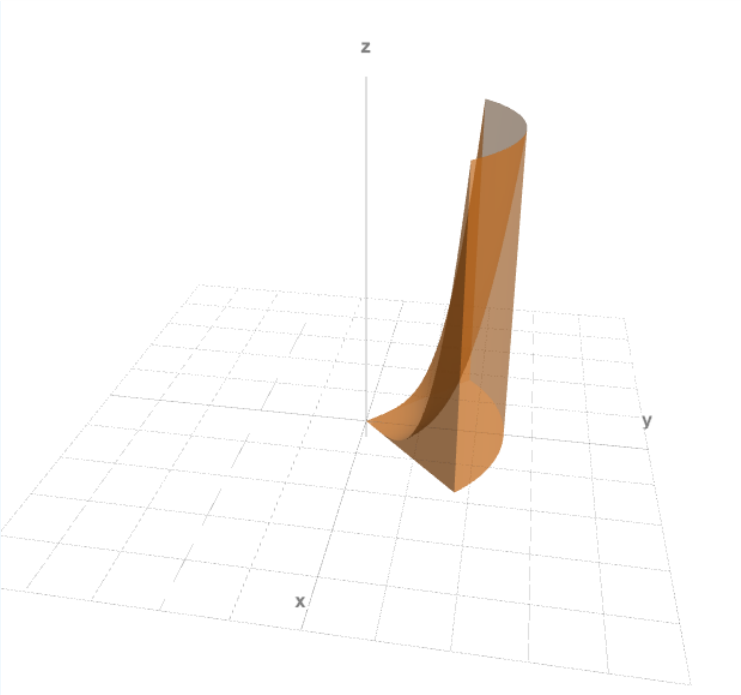
Lastly, the values of θ are defined by $|x| \geq y$, which we can visualize within the circle below:



One choice of such θ is $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

Putting this together gives our region in cylindrical coordinates:

$$A = \left\{(r\cos(\theta), r\sin(\theta), z) \;\middle|\; 0 \leq r \leq 2,\ \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4},\ 0 \leq z \leq r^4\right\}$$



Exercise 1.5.1b

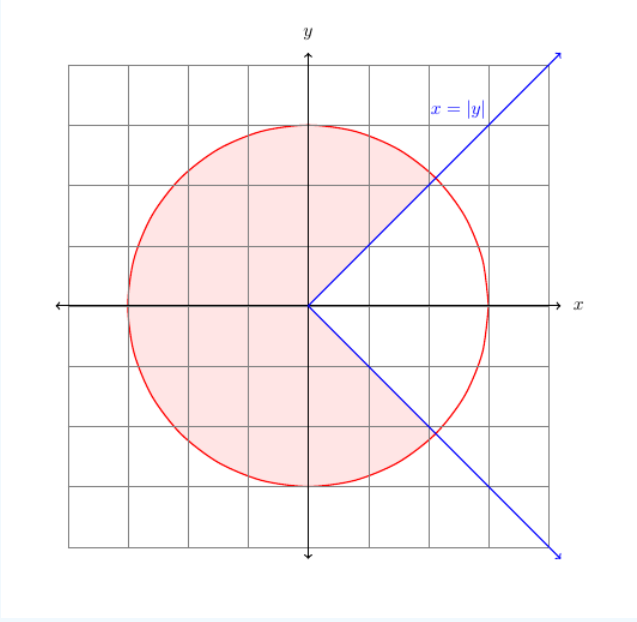
Write the region $B = \{(x, y, z) \mid x \leq |y|,\ x^2 + y^2 \leq 9,\ 2 \leq z \leq 4\}$ in cylindrical coordinates

Answer

For z, we are given $2 \leq z \leq 4$

The radius r is defined by the circle $x^2 + y^2 \leq 9 \longleftrightarrow 0 \leq r \leq 3$

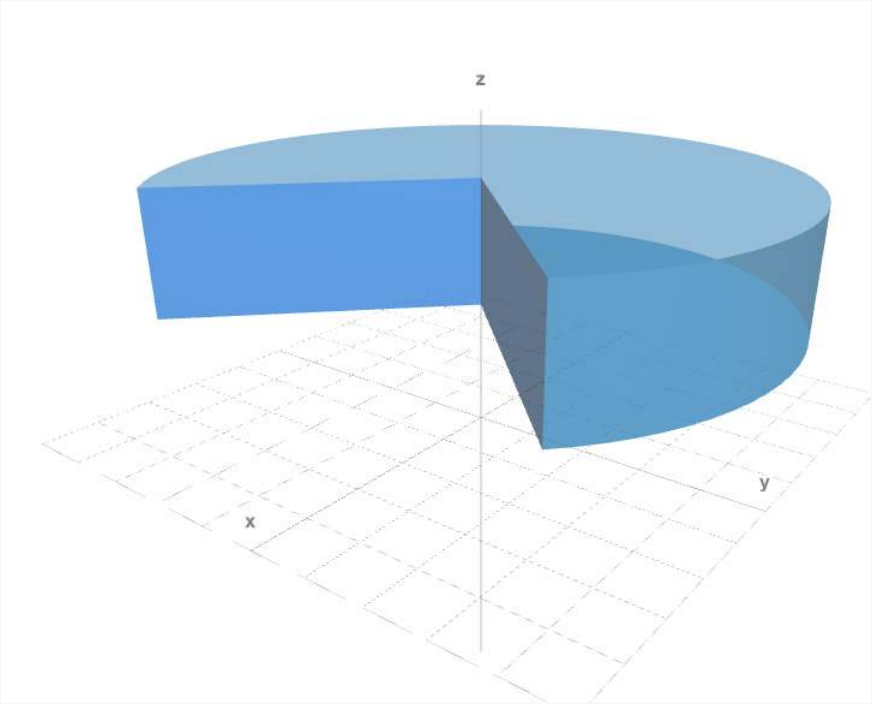
Lastly, the values of θ are defined by $x \leq |y|$, which we can visualize within the circle below:



One choice of such θ is $\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$

Putting this together gives our region in cylindrical coordinates:

$$B = \left\{ (r\cos(\theta), r\sin(\theta), z) \;\middle|\; 0 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{7\pi}{4},\ 2 \le z \le 4 \right\}$$



Exercise 1.5.1c

Write the region $C = \left\{ (x, y, z) \mid \sqrt{x^2 + y^2} \le z \le \sqrt{100 - x^2 - y^2} \right\}$ in cylindrical coordinates

Answer

For z, we are given $\sqrt{x^2 + y^2} \le z \le \sqrt{100 - x^2 - y^2} \longleftrightarrow r \le z \le \sqrt{100 - r^2}$

The radius r is defined by the intersection of the cone $z = r$ and sphere $z = \sqrt{100 - r^2}$:

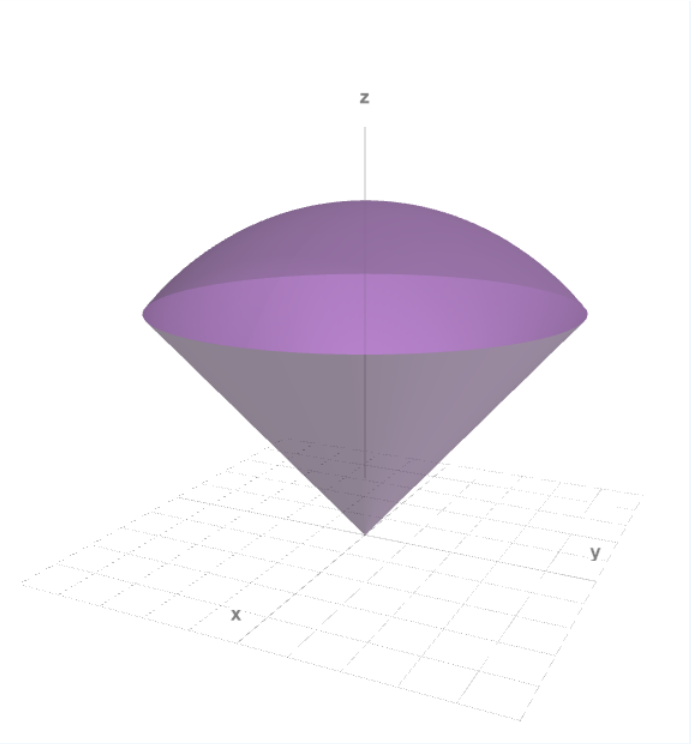
$r = \sqrt{100 - r^2} \longleftrightarrow r^2 = 50 \longleftrightarrow r = 5\sqrt{2}$

Therefore we have $0 \le r \le 5\sqrt{2}$

Lastly, the values of θ are not restricted at all, so we use $0 \le \theta \le 2\pi$

Putting this together gives our region in cylindrical coordinates:

$$C = \left\{ (r\cos(\theta), r\sin(\theta), z) \mid 0 \le r \le 5\sqrt{2},\ 0 \le \theta \le 2\pi,\ r \le z \le \sqrt{100 - r^2} \right\}$$



Exercise $1.5.1d$

Write the region $D = \{(x, y, z) \mid 4 \leq x^2 + y^2 \leq 9,\ 2x - y + 1 \leq z \leq 10\}$ in cylindrical coordinates

Answer

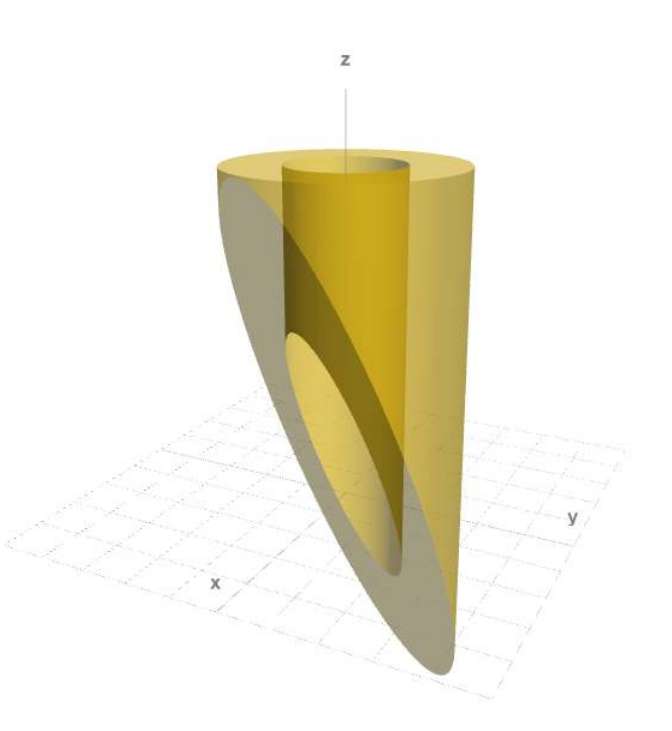
For z, we are given $2x - y + 1 \leq z \leq 10 \longleftrightarrow 2r\cos(\theta) - r\sin(\theta) + 1 \leq z \leq 10$

The radius r is defined by the annulus $4 \leq x^2 + y^2 \leq 9 \longleftrightarrow 2 \leq r \leq 3$

Lastly, the values of θ are not restricted at all, so we use $0 \leq \theta \leq 2\pi$

Putting this together gives our region in cylindrical coordinates:

$$D = \{(r\cos(\theta), r\sin(\theta), z) \mid 2 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi,\ 2r\cos(\theta) - r\sin(\theta) + 1 \leq z \leq 10\}$$



Exercise 1.5.1e

Write the region $E = \left\{(x, y, z) \mid |y| \leq \sqrt{2 - x^2},\ |z| \leq e^{2-x^2-y^2}\right\}$ in cylindrical coordinates

Answer

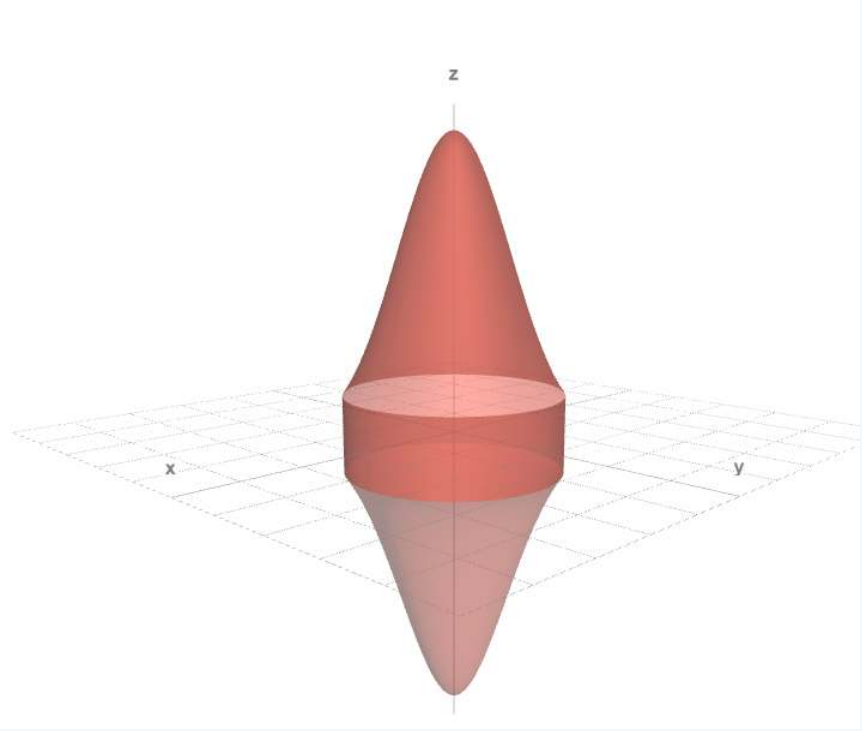
For z, we are given $-e^{2-x^2-y^2} \leq z \leq e^{2-x^2-y^2} \longleftrightarrow -e^{2-r^2} \leq z \leq e^{2-r^2}$

The radius r is defined by the circle $-\sqrt{2-x^2} \leq y \leq \sqrt{2-x^2} \longleftrightarrow 0 \leq r \leq \sqrt{2}$

Lastly, the values of θ are not restricted at all, so we use $0 \leq \theta \leq 2\pi$

Putting this together gives our region in cylindrical coordinates:

$$E = \left\{(r\cos(\theta), r\sin(\theta), z) \mid 0 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq 2\pi,\ -e^{2-r^2} \leq z \leq e^{2-r^2}\right\}$$



? Exercise 1.5.1f

Write the region $F = \{(x, y, z) \mid x^2 + y^2 \leq \alpha^2,\ x^2 + z^2 \leq \alpha^2\}$ in cylindrical coordinates

Answer

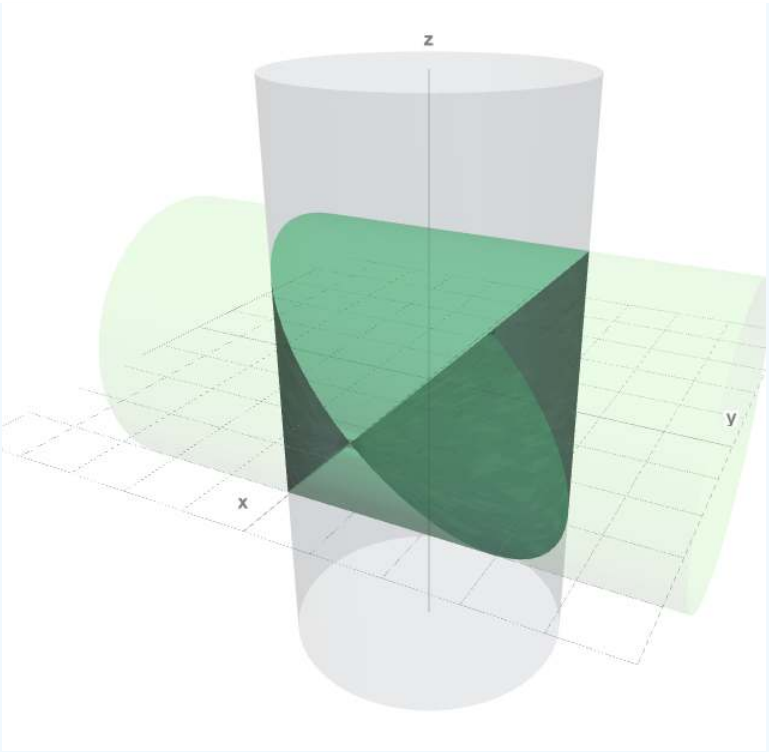
For z, we are given $x^2 + z^2 \leq \alpha^2 \longleftrightarrow -\sqrt{\alpha^2 - x^2} \leq z \leq \sqrt{\alpha^2 - x^2} \longleftrightarrow -\sqrt{\alpha^2 - r^2\cos^2(\theta)} \leq z \leq \sqrt{\alpha^2 - r^2\cos^2(\theta)}$

The radius r is defined by the circle $x^2 + y^2 \leq \alpha^2 \longleftrightarrow 0 \leq r \leq \alpha$

Lastly, the values of θ are not restricted at all, so we use $0 \leq \theta \leq 2\pi$

Putting this together gives our region in cylindrical coordinates:

$$F = \left\{(r\cos(\theta), r\sin(\theta), z) \mid 0 \leq r \leq \alpha,\ 0 \leq \theta \leq 2\pi,\ -\sqrt{\alpha^2 - r^2\cos^2(\theta)} \leq z \leq \sqrt{\alpha^2 - r^2\cos^2(\theta)}\right\}$$



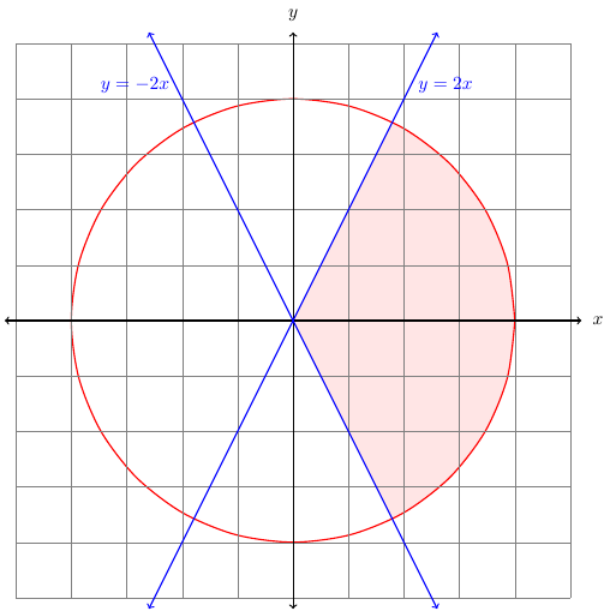
Exercise $1.5.2a$

Write the region $A = \left\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 16,\ |y| \leq 2x,\ z \geq \sqrt{2x^2 + 2y^2}\right\}$ in spherical coordinates

Answer

For ρ, we are given $x^2 + y^2 + z^2 \leq 16 \longleftrightarrow 0 \leq \rho \leq 4$

The angle θ is defined by the restriction $|y| \leq 2x \longleftrightarrow -2x \leq y \leq 2x$, which we can visualize within the circle below:



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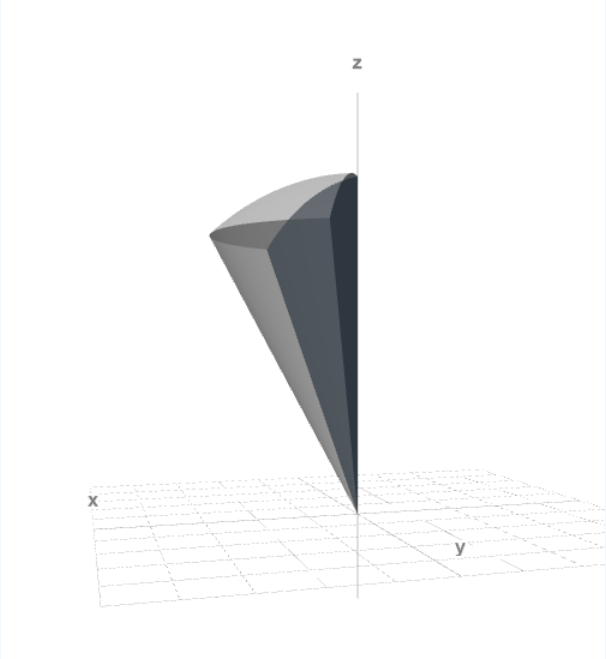
The line $y = 2x \longleftrightarrow \theta = \tan^{-1}(2)$ and $y = -2x \longleftrightarrow \theta = \tan^{-1}(-2)$, so $\tan^{-1}(-2) \leq \theta \leq \tan^{-1}(2)$

The values of ϕ are restricted by the cone $z = \sqrt{2x^2 + 2y^2} = \sqrt{2} \cdot \sqrt{x^2 + y^2}$, which means $\cot(\phi) = \sqrt{2}$

Since z is *above* this cone, we have $0 \leq \phi \leq \cot^{-1}(\sqrt{2})$

Putting this together gives our region in spherical coordinates:

$$A = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \mid 0 \leq \rho \leq 4,\ \tan^{-1}(-2) \leq \theta \leq \tan^{-1}(2),\ 0 \leq \phi \leq \cot^{-1}(\sqrt{2})\right\}$$



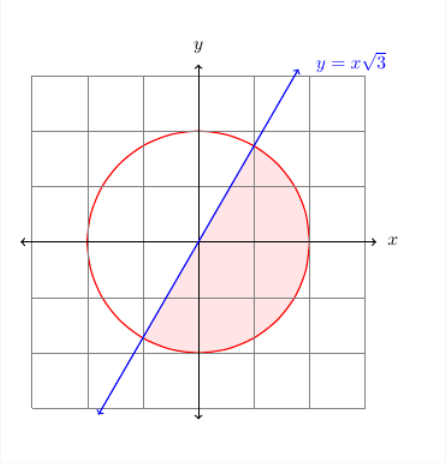
Exercise 1.5.2*b*

Write the region $B = \left\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 4,\ y \leq x\sqrt{3}\right\}$ in spherical coordinates

Answer

For ρ, we are given $x^2 + y^2 + z^2 \leq 4 \longleftrightarrow 0 \leq \rho \leq 2$

The angle θ is defined by the restriction $y \leq x\sqrt{3}$, which we can visualize within the circle below:

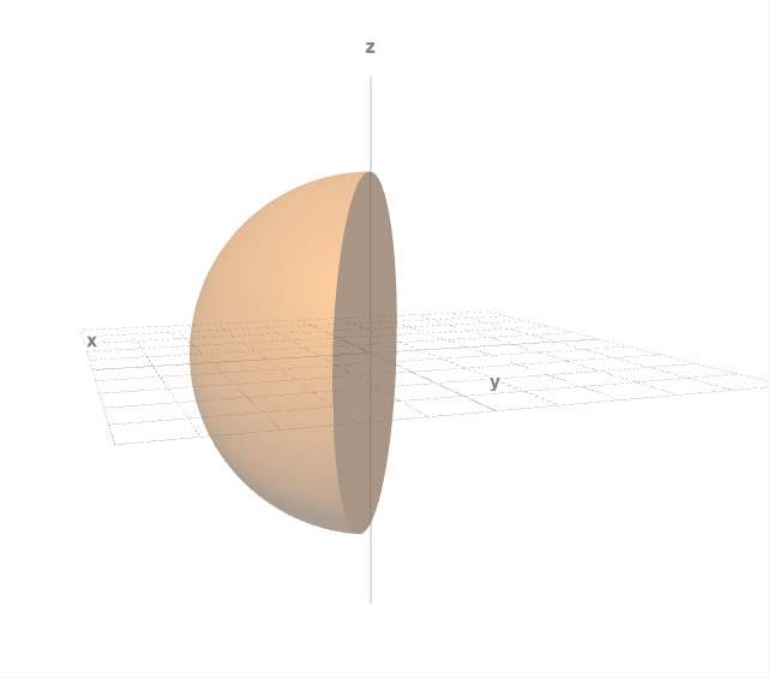


Since $y = x\sqrt{3} \longleftrightarrow \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ we can use $-\frac{2\pi}{3} \leq \theta \leq \frac{\pi}{3}$

Since there is no restriction on z within the sphere, we have the full range for ϕ, which is $0 \leq \phi \leq \pi$

Putting this together gives our region in spherical coordinates:

$$B = \left\{ (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \;\middle|\; 0 \leq \rho \leq 2,\ -\frac{2\pi}{3} \leq \theta \leq \frac{\pi}{3},\ 0 \leq \phi \leq \pi \right\}$$



? Exercise $1.5.2c$

Write the region $C = \left\{ (x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq \sqrt{64 - x^2 - y^2} \right\}$ in spherical coordinates

Answer

For ρ, we are given $z \leq \sqrt{64 - x^2 - y^2} \longleftrightarrow 0 \leq \rho \leq 8$

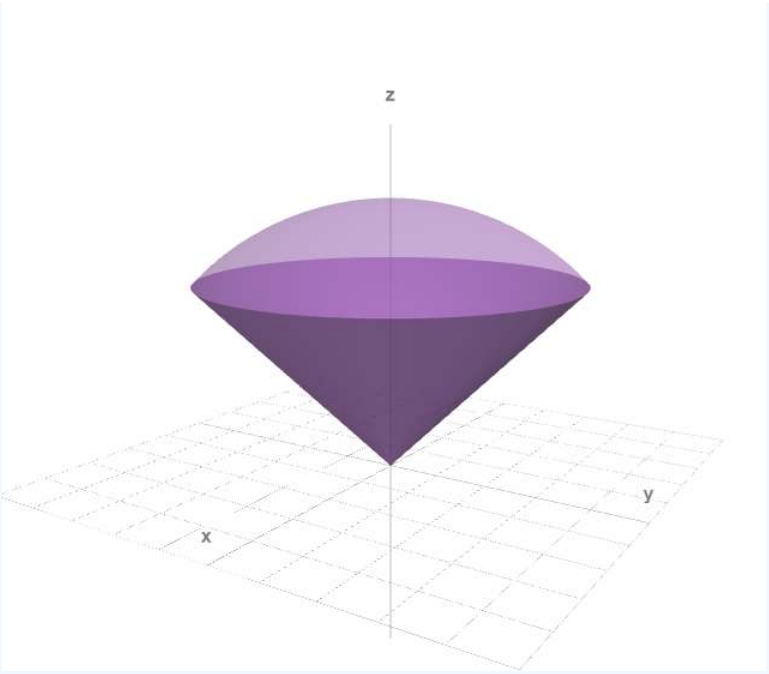
The angle θ is not restricted since there are no relationships between only x and y, so $0 \leq \theta \leq 2\pi$

The values of ϕ are restricted by the cone $z = \sqrt{x^2 + y^2} = 1 \cdot \sqrt{x^2 + y^2}$, which means $\cot(\phi) = 1 \longleftrightarrow \phi = \frac{\pi}{4}$

Since z is *above* this cone, we have $0 \leq \phi \leq \frac{\pi}{4}$

Putting this together gives our region in spherical coordinates:

$$C = \left\{ (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \mid 0 \leq \rho \leq 8,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{4} \right\}$$



Exercise $1.5.2d$

Write the region $D = \left\{(x, y, z) \mid -\sqrt{x^2 + y^2} \leq z \leq \sqrt{3x^2 + 3y^2},\ 1 \leq x^2 + y^2 + z^2 \leq 49\right\}$ in spherical coordinates

Answer

For ρ, we are given $1 \leq x^2 + y^2 + z^2 \leq 49 \longleftrightarrow 1 \leq \rho \leq 7$

The angle θ is not restricted since there are no relationships between only x and y, so $0 \leq \theta \leq 2\pi$

The values of ϕ are restricted by two cones:

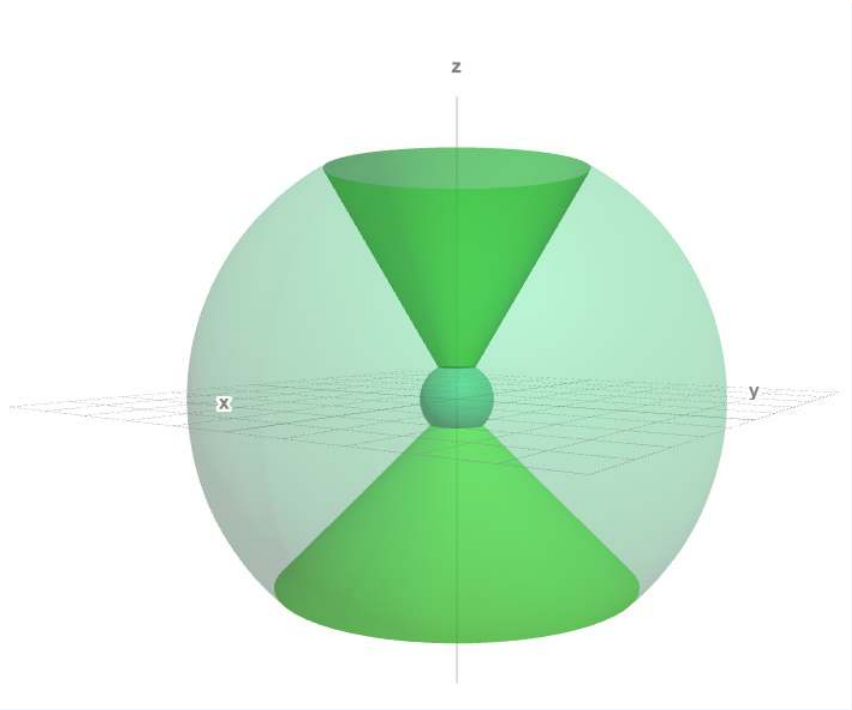
$z = \sqrt{3x^2 + 3y^2} = \sqrt{3} \cdot \sqrt{x^2 + y^2}$, which means $\cot(\phi) = \sqrt{3} \longleftrightarrow \phi = \dfrac{\pi}{6}$

$z = -\sqrt{x^2 + y^2} = -1 \cdot \sqrt{x^2 + y^2}$, which means $\cot(\phi) = -1 \longleftrightarrow \phi = \dfrac{3\pi}{4}$

Since z is *between* these cones, we have $\dfrac{\pi}{6} \leq \phi \leq \dfrac{3\pi}{4}$

Putting this together gives our region in spherical coordinates:

$$D = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \mid 1 \leq \rho \leq 7,\ 0 \leq \theta \leq 2\pi,\ \frac{\pi}{6} \leq \phi \leq \frac{3\pi}{4}\right\}$$



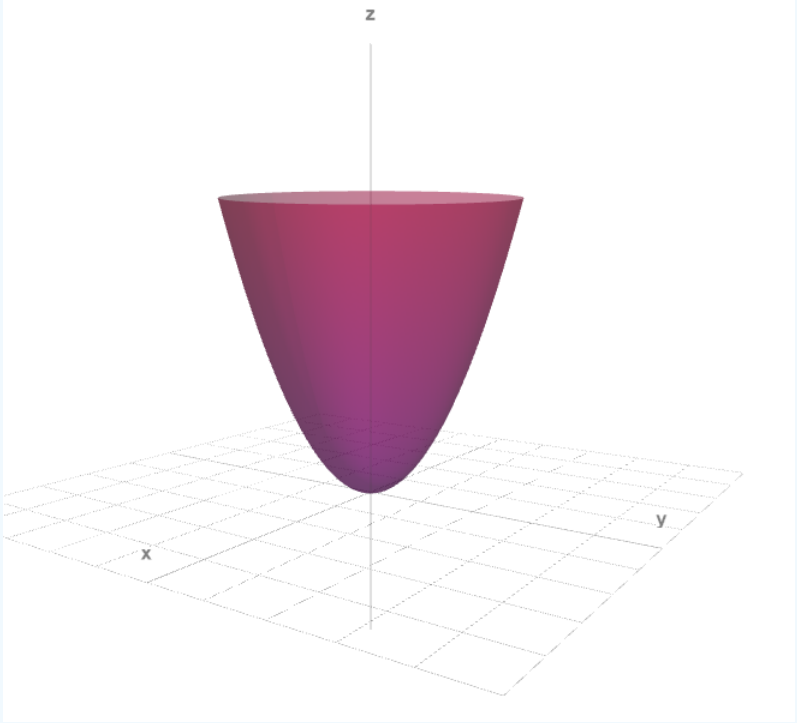
Exercise $1.5.3a$

Determine the shape of and sketch the set $A = \{(r\cos(\theta), r\sin(\theta), z) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi,\ z = r^2\}$

Answer

The relationship $z = r^2 \longleftrightarrow z = x^2 + y^2$ represents a **paraboloid.**

Since $0 \leq r \leq 2$ and θ does a full rotation, it is restricted to the inside of the cylinder $x^2 + y^2 \leq 4$:



Exercise $1.5.3b$

Determine the shape of and sketch the set $B = \{(r\cos(\theta), r\sin(\theta), z) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi,\ z = r\sin(\theta) - r\cos(\theta)\}$

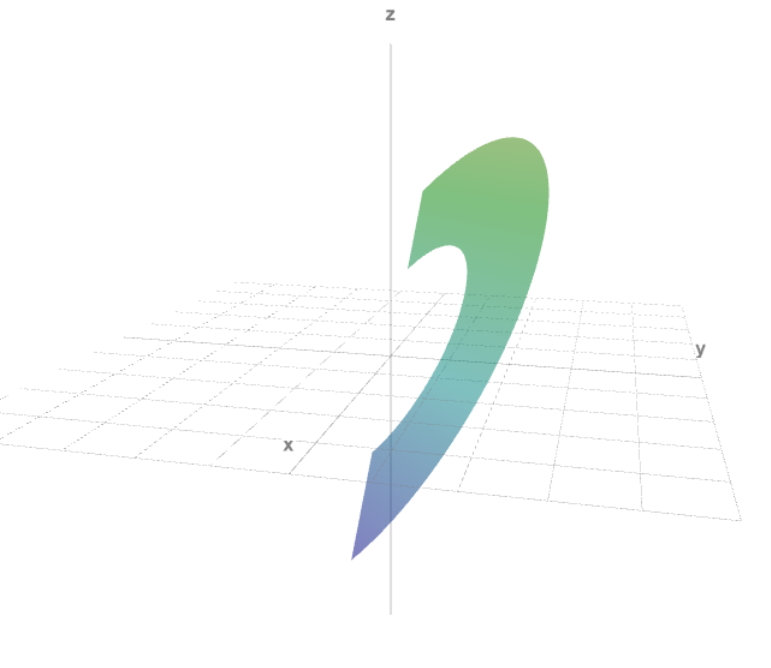
Answer

The relationship $z = r\sin(\theta) - r\cos(\theta) \longleftrightarrow z = y - x$ represents a **plane.**

Since $1 \leq r \leq 2$, it is restricted to the inside of the annulus $1 \leq x^2 + y^2 \leq 4$

Since $0 \leq \theta \leq \pi$, we only get the values of $y \geq 0$ within this annulus:

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Exercise $1.5.3c$

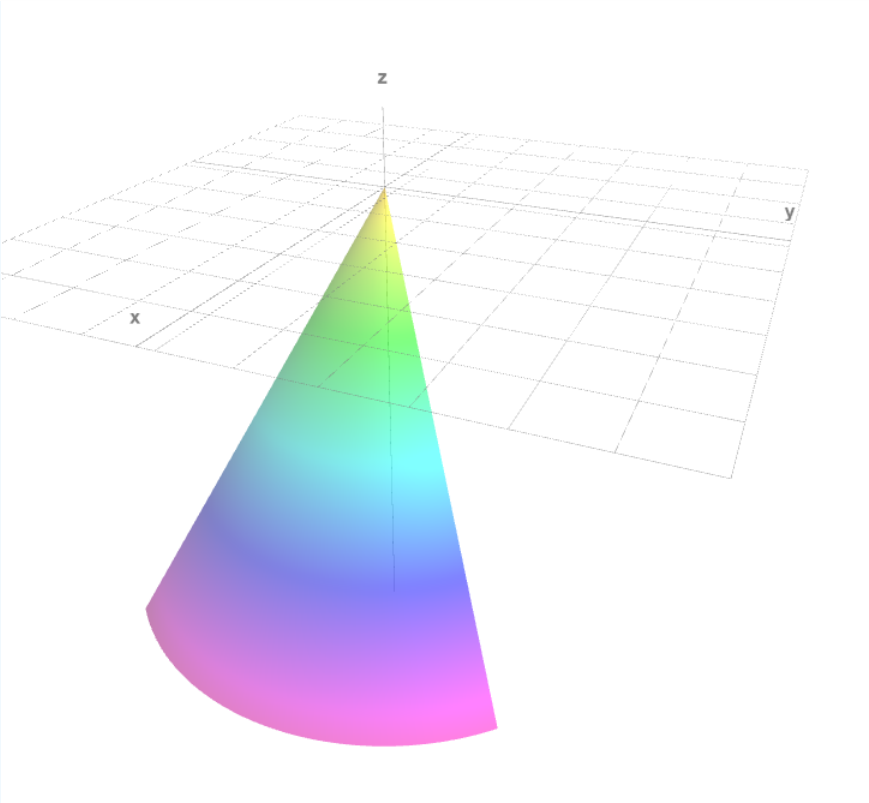
Determine the shape of and sketch the set $C = \left\{(r\cos(\theta), r\sin(\theta), z) \mid 0 \leq r \leq 4, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, 3r = -2z\right\}$

Answer

The relationship $3r = -2z \leftrightarrow z = -\frac{3}{2}\sqrt{x^2+y^2}$ represents a **cone.**

Since $0 \leq r \leq 4$, it is restricted to the inside of the circle $x^2 + y^2 \leq 16$

Since $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, we only get the region with $-x \leq y \leq x$ within this circle:



Exercise $1.5.3d$

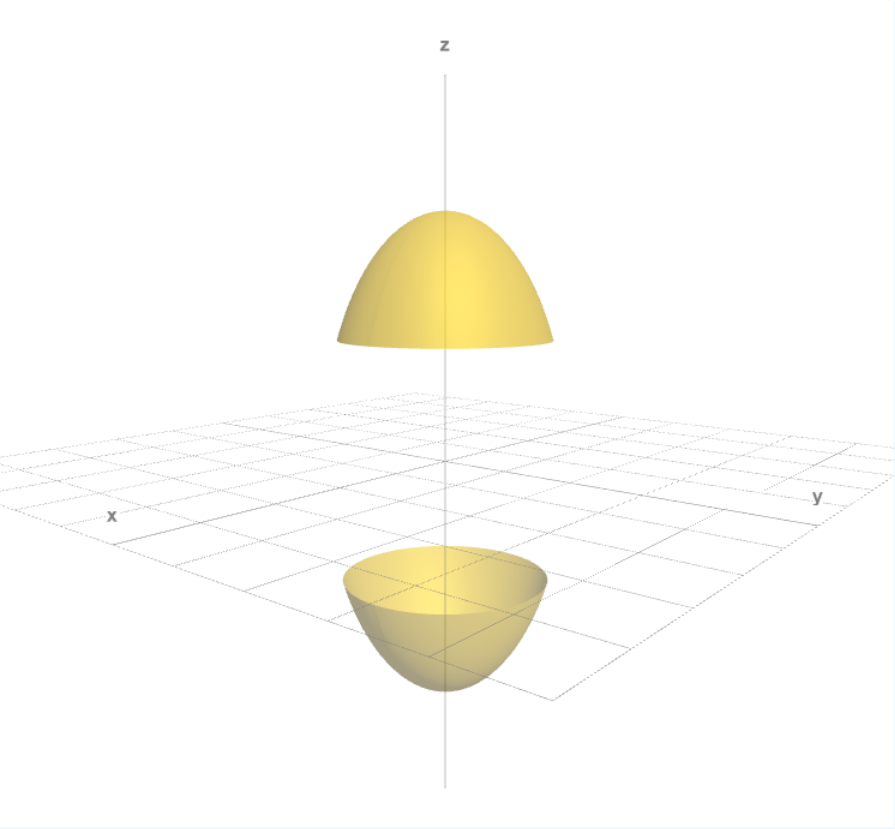
Determine the shape of and sketch the set $D = \{(r\cos(\theta), r\sin(\theta), z) \mid 0 \leq r \leq 1,\ \pi \leq \theta \leq 3\pi,\ 4r^2 + z^2 = 4\}$

Answer

The relationship $4r^2 + z^2 = 4 \longleftrightarrow 4x^2 + 4y^2 + z^2 = 4 \longleftrightarrow x^2 + y^2 + \frac{z^2}{4} = 1$ represents an **ellipsoid.**

Since $0 \leq r \leq 1$ and θ does a full rotation, it is restricted to the inside of the cylinder $x^2 + y^2 \leq 1$:

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? Exercise 1.5.4a

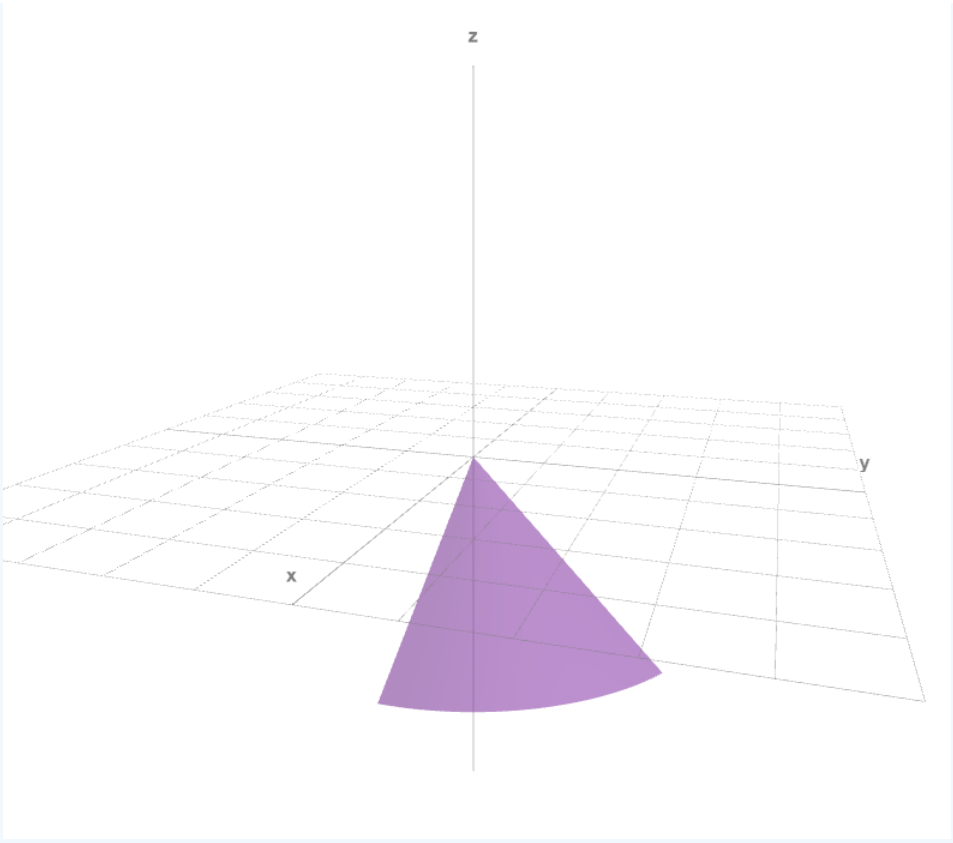
Determine the shape of and sketch the set $A = \left\{ (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \;\middle|\; 0 \le \rho \le 2,\ 0 \le \theta \le \frac{\pi}{3},\ \phi = \frac{2\pi}{3} \right\}$

Answer

The relationship $\phi = \frac{2\pi}{3} \longleftrightarrow z = \cot\left(\frac{2\pi}{3}\right) \cdot \sqrt{x^2+y^2} \longleftrightarrow z = -\sqrt{\frac{x^2+y^2}{3}}$ represents a **cone.**

Since $0 \le \rho \le 2$, it is restricted to the inside of the sphere $x^2 + y^2 + z^2 \le 4$

Since $0 \le \theta \le \frac{\pi}{3}$, we only get the region with $0 \le y \le x\sqrt{3}$ within this sphere:



Exercise $1.5.4b$

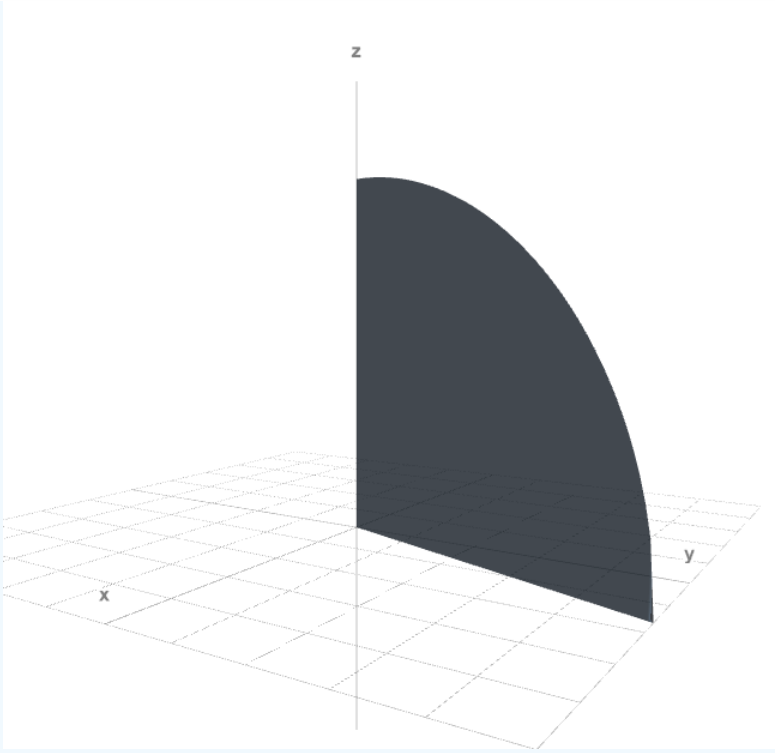
Determine the shape of and sketch the set $B = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \;\middle|\; 0 \le \rho \le 5,\ 0 \le \phi \le \frac{\pi}{2},\ \theta = \tan^{-1}(3)\right\}$

Answer

The relationship $\theta = \tan^{-1}(3) \longleftrightarrow y = \tan\left(\tan^{-1}(3)\right) x \ =\longleftrightarrow y = 3x$ represents a **plane.**

Since $0 \le \rho \le 5$, it is restricted to the inside of the sphere $x^2 + y^2 + z^2 \le 25$

Since $0 \le \phi \le \frac{\pi}{2}$, we only get the region with $z \ge 0$ within this sphere:



Exercise $1.5.4c$

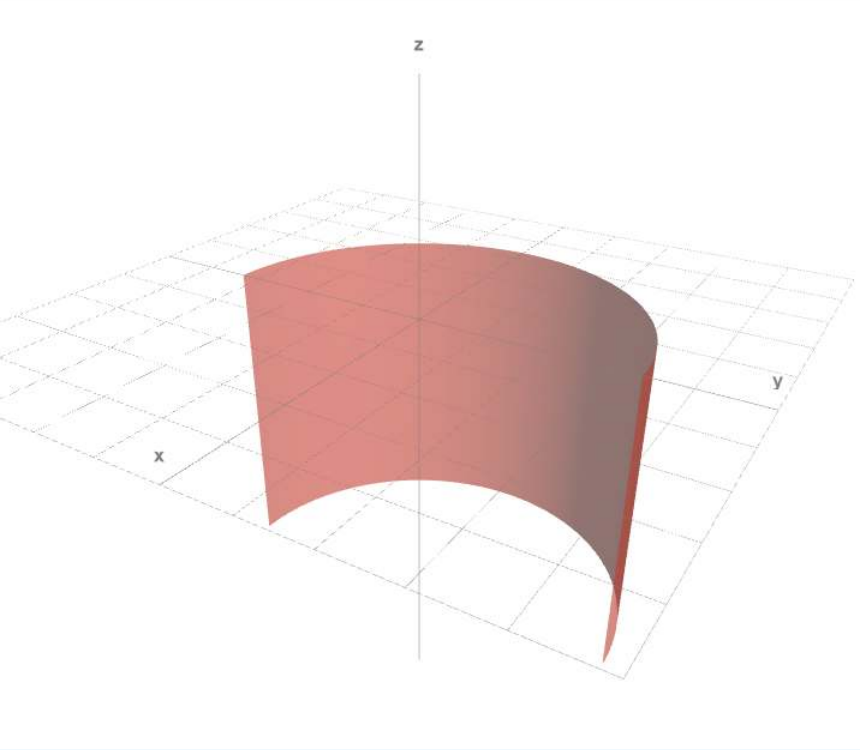
Determine the shape of and sketch the set $C = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \;\middle|\; \frac{\pi}{2} \le \theta \le \frac{3\pi}{2},\ \frac{\pi}{2} \le \phi \le \frac{3\pi}{4},\ \rho = 3\csc(\phi)\right\}$

Answer

The relationship $\rho = 3\csc(\phi) \longleftrightarrow \rho\sin(\phi) = 3 \longleftrightarrow r = 3$ in cylindrical coordinates, which is a **cylinder.**

Since $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, it is restricted to points with $x \le 0$

Since $\frac{\pi}{2} \le \phi \le \frac{3\pi}{4} \longleftrightarrow \cot\left(\frac{3\pi}{4}\right) \cdot \sqrt{x^2+y^2} \le z \le \cot\left(\frac{\pi}{2}\right) \cdot \sqrt{x^2+y^2}$, plugging in $r = 3$ gives $-3 \le z \le 0$:



Exercise $1.5.4d$

Determine the shape of and sketch the set $D = \left\{ (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \dfrac{\theta}{2},\ \rho = 2 \right\}$

Answer

Since $\rho = 2$ this surface is on the sphere of radius 2, $x^2 + y^2 + z^2 = 4$

Since $0 \le \theta \le 2\pi$, there are no restrictions on the x or y values on the surface of this sphere.

The restriction $0 \le \phi \le \dfrac{\theta}{2}$ means that the sphere ranges over different z values at each angle θ

As we rotate counterclockwise the sphere is slowly allowed to reach further from the top point $(0, 0, 2)$ down the arc to $(0, 0, -2)$.

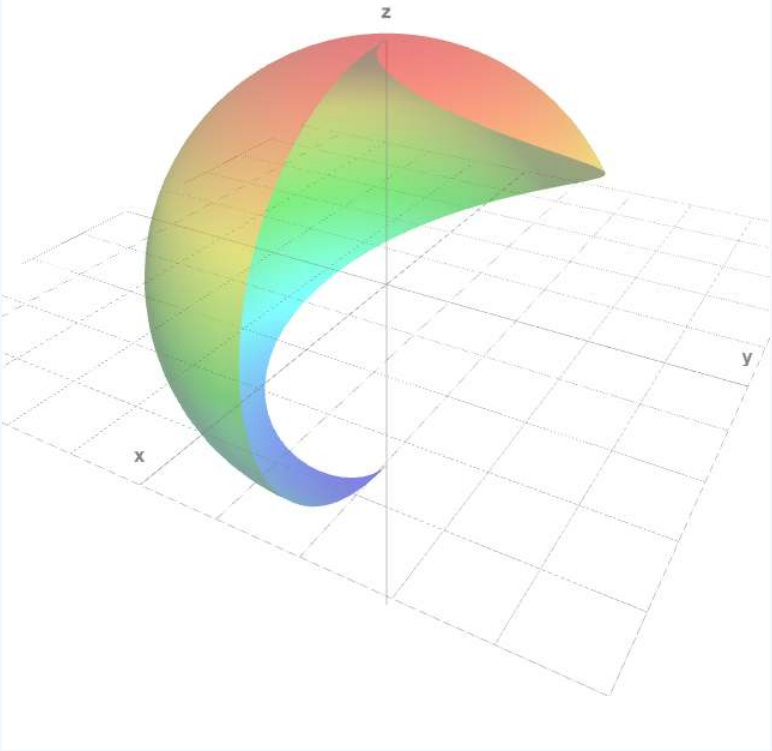
At the positive x-axis (when $\theta = 0$), this is simply one point at the top of the sphere.

By the positive y-axis (which is $\theta = \dfrac{\pi}{2}$), it reaches down to a height of $z = 2\cos\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

By the negative x-axis (which is $\theta = \pi$), it reaches down to a height of $z = 2\cos\left(\dfrac{\pi}{2}\right) = 0$

By the negative y-axis (which is $\theta = \dfrac{3\pi}{2}$), it reaches down to a height of $z = 2\cos\left(\dfrac{3\pi}{4}\right) = -\sqrt{2}$

When it reaches back to the positive x-axis (at $\theta = 2\pi$), it reaches all the way down to the bottom of the sphere:



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1.6: Space Curves

Learning Objectives

- Write the general equation of a vector-valued function in component form and unit-vector form.
- Recognize parametric equations for a space curve.
- Describe the shape of a helix and write its equation.
- Define the limit of a vector-valued function.
- Write an expression for the derivative of a vector-valued function.
- Find the tangent vector at a point for a given position vector.
- Find the unit tangent vector at a point for a given position vector and explain its significance.
- Calculate the definite integral of a vector-valued function.

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section, we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

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- **13.2: Calculus of Vector-Valued Functions** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

1.6: Vector-Valued Functions

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Definition: Vector-Valued Functions

A vector-valued function is a function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \quad \text{or} \quad \vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

where the component functions f, g, and h are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\vec{\mathbf{r}}(t) = \langle f(t),\, g(t)\rangle \quad \text{or} \quad \vec{\mathbf{r}}(t) = \langle f(t),\, g(t),\, h(t)\rangle$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function. The parameter t might take on all real numbers or t can lie between two real numbers: $a \leq t \leq b$. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of t. We often use t as a parameter because t can represent time.

Illustration 1.6.1: Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$, and $\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right)$. Do any of these functions have domain restrictions?

(a) $\vec{\mathbf{r}}(t) = 4\cos(t)\,\hat{\mathbf{i}} + 3\sin(t)\,\hat{\mathbf{j}}$

(b) $\vec{\mathbf{r}}(t) = 3\tan(t)\,\hat{\mathbf{i}} + 4\sec(t)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$

(a) To calculate each of the function values, substitute the appropriate value of t into the function:

$$\vec{\mathbf{r}}(0) = 4\cos(0)\hat{\mathbf{i}} + 3\sin(0)\hat{\mathbf{j}} = 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}} = 4\hat{\mathbf{i}}$$

$$\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) = 4\cos\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} = 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = 3\hat{\mathbf{j}}$$

$$\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) = 4\cos\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} = 4\left(-\frac{1}{2}\right)\hat{\mathbf{i}} + 3\left(\frac{\sqrt{3}}{2}\right)\hat{\mathbf{j}} = -2\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos(t)$ and the second component function is $g(t) = 3\sin(t)$. Neither of these functions has a domain restriction, so the domain of $\vec{\mathbf{r}}(t) = 4\cos(t)\,\hat{\mathbf{i}} + 3\sin(t)\,\hat{\mathbf{j}}$ is all real numbers.

(b) To calculate each of the function values, substitute the appropriate value of *t* into the function:

$$\vec{\mathbf{r}}(0) = 3\tan(0)\hat{\mathbf{i}} + 4\sec(0)\hat{\mathbf{j}} + 5(0)\hat{\mathbf{k}} = 0\hat{\mathbf{i}} + 4j + 0\hat{\mathbf{k}} = 4\hat{\mathbf{j}}$$

$$\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) = 3\tan\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} + 5\left(\frac{\pi}{2}\right)\hat{\mathbf{k}} \text{ which does not exist}$$

$$\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) = 3\tan\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} + 5\left(\frac{2\pi}{3}\right)\hat{\mathbf{k}}$$

$$= 3(-\sqrt{3})\hat{\mathbf{i}} + 4(-2)\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}} = (-3\sqrt{3})\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan(t)$, the second component function is $g(t) = 4\sec(t)$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\frac{\pi}{2}$, so the function is not defined for odd multiples of $\frac{\pi}{2}$. Therefore,

$$D_{\vec{r}} = \left\{ t : t \neq \frac{(2n+1)\pi}{2} \right\}$$

where n is any integer.

The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point (the *terminal point)*. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$$

consists of the set of all points $(f(t),\ g(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

consists of the set of all points $(f(t),\ g(t),\ h(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Each plane curve and space curve has an **orientation**, indicated by arrows drawn on the curve, that shows the direction of motion along the curve as the value of the parameter t increases.

Illustration 1.6.2 : Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

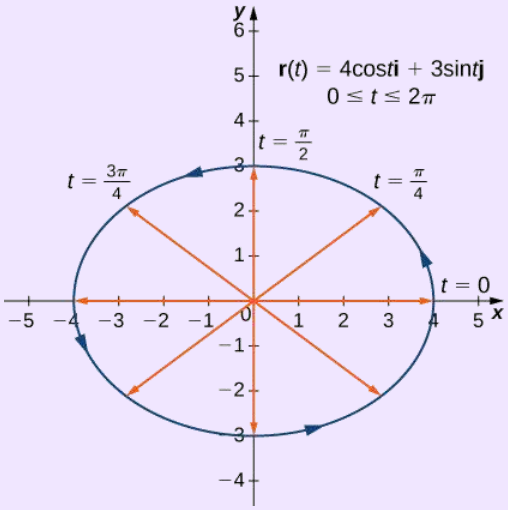
(a) The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t)\,\hat{\mathbf{i}} + 3\sin(t)\,\hat{\mathbf{j}}$, $0 \leq t \leq 2\pi$

(b) The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}}$, $0 \leq t \leq \sqrt[3]{2\pi}$

(c) The space curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t)\,\hat{\mathbf{i}} + 4\sin(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$, $0 \leq t \leq 4\pi$

(a) As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve.

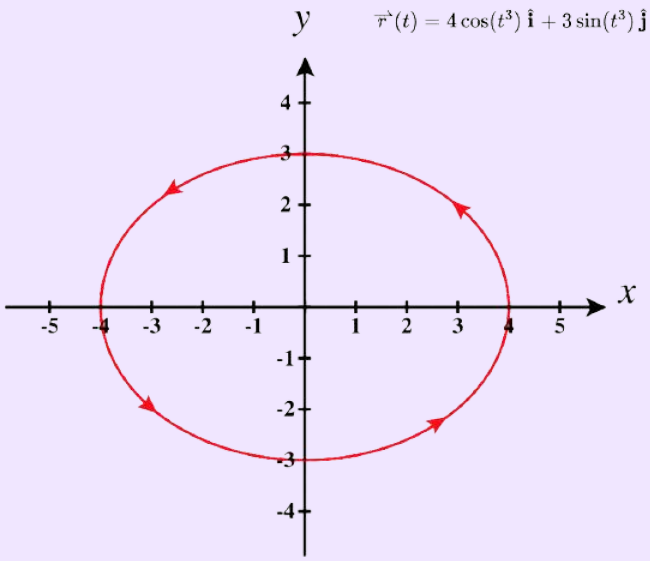
t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\frac{\pi}{2}$	$3\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
π	$-4\hat{\mathbf{i}}$
$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\frac{3\pi}{2}$	$-3\hat{\mathbf{j}}$
$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
2π	$4\hat{\mathbf{i}}$



This curve travels counterclockwise along the ellipse $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$.

(b) The table of values for $\vec{\mathbf{r}}(t)=4\cos(t^3)\,\hat{\mathbf{i}}+3\sin(t^3)\,\hat{\mathbf{j}}$, for $0\le t\le\sqrt[3]{2\pi}$ is as follows:

t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$
$\sqrt[3]{\frac{\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\frac{\pi}{2}}$	$3\hat{\mathbf{j}}$
$\sqrt[3]{\frac{3\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\pi}$	$-4\hat{\mathbf{i}}$
$\sqrt[3]{\frac{5\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\frac{3\pi}{2}}$	$-3\hat{\mathbf{j}}$
$\sqrt[3]{\frac{7\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{2\pi}$	$4\hat{\mathbf{i}}$

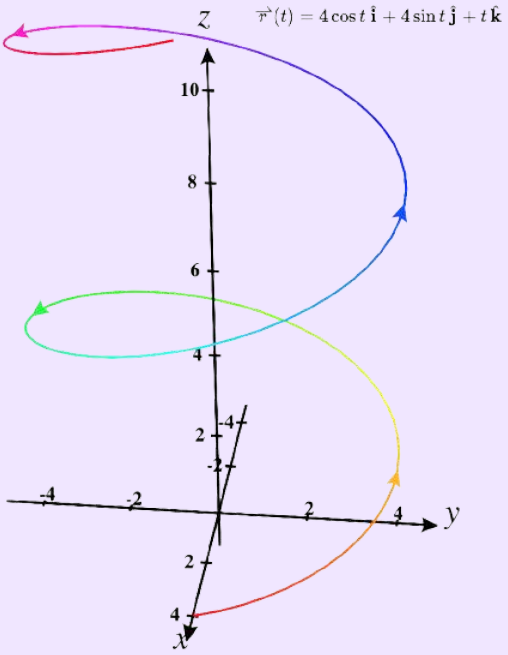


The graph of this curve is the same ellipse, it just does not travel around it at a constant "speed" as it did in part (a)

(c) We go through the same procedure for a three-dimensional vector function.

t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{\pi}{4}\hat{\mathbf{k}}$
$\frac{\pi}{2}$	$4\hat{\mathbf{j}}+\frac{\pi}{2}\hat{\mathbf{k}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{3\pi}{4}\hat{\mathbf{k}}$
π	$-4\hat{\mathbf{i}}+\pi\hat{\mathbf{k}}$
$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{5\pi}{4}\hat{\mathbf{k}}$
$\frac{3\pi}{2}$	$-4\hat{\mathbf{j}}+\frac{3\pi}{2}\hat{\mathbf{k}}$
$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{7\pi}{4}\hat{\mathbf{k}}$
2π	$4\hat{\mathbf{i}}+2\pi\hat{\mathbf{k}}$
$\frac{9\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{9\pi}{4}\hat{\mathbf{k}}$
$\frac{5\pi}{2}$	$4\hat{\mathbf{j}}+\frac{5\pi}{2}\hat{\mathbf{k}}$
$\frac{11\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{11\pi}{4}\hat{\mathbf{k}}$
3π	$-4\hat{\mathbf{i}}+3\pi\hat{\mathbf{k}}$
$\frac{13\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{13\pi}{4}\hat{\mathbf{k}}$
$\frac{7\pi}{2}$	$-4\hat{\mathbf{j}}+\frac{7\pi}{2}\hat{\mathbf{k}}$
$\frac{15\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{15\pi}{4}\hat{\mathbf{k}}$
4π	$4\hat{\mathbf{i}}+4\pi\hat{\mathbf{k}}$

The values then repeat themselves, except for the fact that the coefficient of $\hat{\mathbf{k}}$ is always increasing.



This curve is called a helix. Notice that if the $\hat{\mathbf{k}}$ component is eliminated, then the function becomes $\vec{\mathbf{r}}(t) = 4\cos(t)\hat{\mathbf{i}} + 4\sin(t)\hat{\mathbf{j}}$, which is a circle of radius 4 centered at the origin. That circle is exactly what the view of this space curve would look like if you were looking down from above.

You may notice that the graphs in parts (a) and (b) are identical. This happens because the function describing the curve in part (b) is a so-called **reparameterization** of the function describing the curve in part (a). In fact, any curve has an infinite number of reparameterizations; for example, we can replace t with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which t is defined may change, but that is all. We return to this idea later when we study *arc-length parameterization*.

As mentioned, the name of the shape of the curve of the graph is a *helix*. The curve resembles a spring, with a circular cross-section looking down along the z-axis. The arrows in the graph of this helix indicate the orientation of the curve as t progresses from 0 to 4π. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\vec{\mathbf{r}}(t) = 4\cos(t)\ \hat{\mathbf{i}} + 3\sin(t)\ \hat{\mathbf{j}} + t\ \hat{\mathbf{k}}$ describes an elliptical helix. The projection of this helix into the xy-plane is an ellipse. If you consider each component function parametrically:

$$\begin{cases} x(t) = 4\cos(t) \\ y(t) = 3\sin(t) \\ z(t) = t \end{cases} \longrightarrow \cos(t) = \frac{x}{4} \text{ and } \sin(t) = \frac{y}{3}$$

Since we know $\cos^2(t) + \sin^2(t) = 1$ always holds, we see that this space curve must lie on the cylinder $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$. We see that z moves linearly at the same rate that t increases the angle inside of cosine and sine, which is why our spring is evenly spaced and counterclockwise along this cylinder. We can use our knowledge of previous sections to more easily visualize space curves.

Illustration 1.6.3: Visualizing Space Curves in Different Coordinates

Use different coordinate systems to help sketch each vector valued curve.

(a) $\vec{\mathbf{r}}(t) = (2 + \sin(t))\cos(t)\,\hat{\mathbf{i}} + (2 + \sin(t))\sin(t)\,\hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

(b) $\vec{\mathbf{r}}(t) = t\cos(t)\,\hat{\mathbf{i}} + t\sin(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$, $0 \le t \le 4\pi$

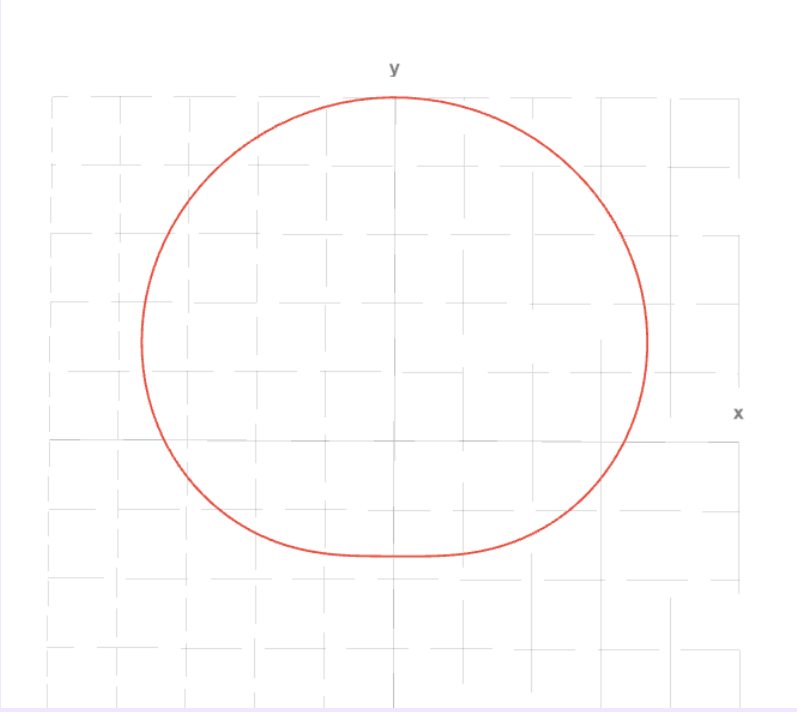
(c) $\vec{\mathbf{r}}(t) = t\cos(t)\,\hat{\mathbf{i}} + t\sin(t)\,\hat{\mathbf{j}} + 2t^2\,\hat{\mathbf{k}}$, $0 \le t \le 2\pi$

(d) $\vec{\mathbf{r}}(t) = \sin(t)\cos(4t)\,\hat{\mathbf{i}} + \sin(t)\sin(4t)\,\hat{\mathbf{j}} + \cos(t)\,\hat{\mathbf{k}}$, $0 \le t \le \pi$

(a) Notice that this curve looks better in polar form:

$$\vec{\mathbf{r}}(t) = R(t)\cos(\theta(t))\hat{\mathbf{i}} + R(t)\sin(\theta(t))\,\hat{\mathbf{j}} \longleftrightarrow R(t) = 2 + \sin(t) \text{ and } \theta(t) = t$$

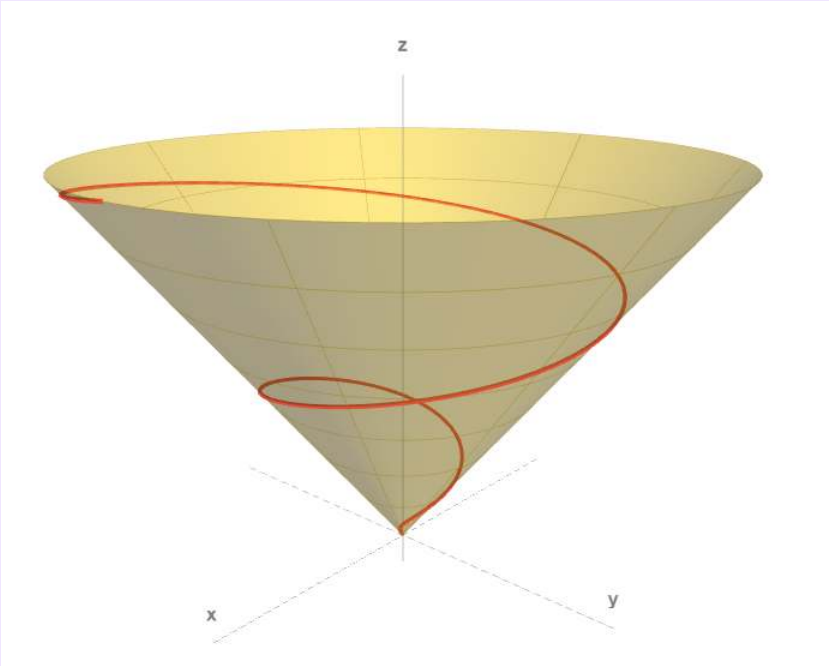
Since $t = \theta$, this is just the polar graph $R = 2 + \sin(\theta)$ traversed in the standard counterclockwise direction:



(b) This space curve is nicer in cylindrical form:

$$\vec{\mathbf{r}}(t) = R(t)\cos(\theta(t))\hat{\mathbf{i}} + R(t)\sin(\theta(t))\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}} \longleftrightarrow R(t) = t \text{ and } \theta(t) = t \text{ and } z(t) = t$$

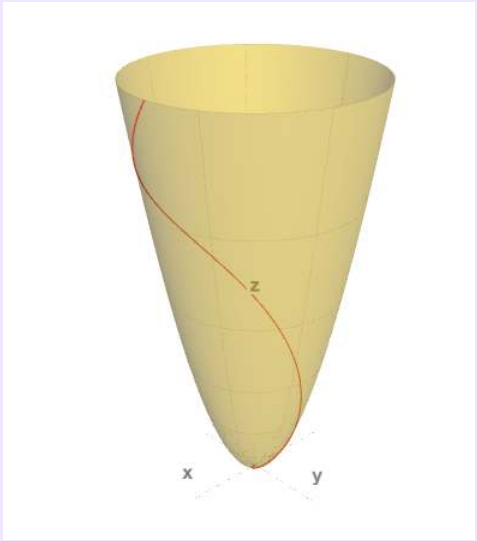
Since $R = t$, this is the cylindrical relationship $R = \theta$ and $z = R$. Note that $z = R$ in cylindrical coordinates is a cone, so we will spiral outward around the surface of this cone in the standard counterclockwise direction while z climbs upward at the same rate we rotate.



(c) This space curve is also nicer in cylindrical form:

$$\vec{\mathbf{r}}(t) = R(t)\cos(\theta(t))\hat{\mathbf{i}} + R(t)\sin(\theta(t))\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \longleftrightarrow R(t) = t \text{ and } \theta(t) = t \text{ and } z(t) = 2t^2$$

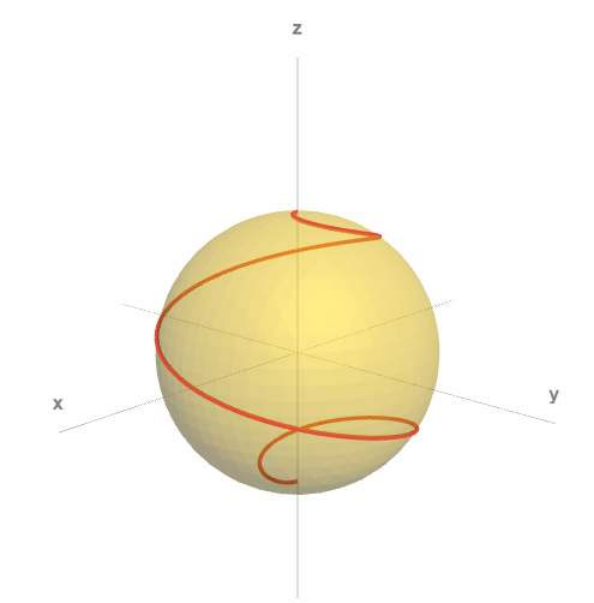
Since $R = t$, this is the cylindrical relationship $R = \theta$ and $z = 2R^2$. Note that $z = 2R^2$ in cylindrical coordinates is a paraboloid, so we will travel around the surface of this paraboloid in the standard counterclockwise direction while z climbs upward at increasing speeds as we rotate.



(d) This space curve is nicer in spherical form:

$$\vec{\mathbf{r}}(t) = \rho(t)\sin(\phi(t))\cos(\theta(t))\hat{\mathbf{i}} + \rho(t)\sin(\phi(t))\sin(\theta(t))\hat{\mathbf{j}} + \rho(t)\cos(\phi(t))\hat{\mathbf{k}}$$
$$\longleftrightarrow \rho(t) = 1 \text{ and } \phi(t) = t \text{ and } \theta(t) = 4t$$

Since $\rho = 1$ is constant, we know this curve travels along the surface of the unit sphere. We have the spherical relationship $\theta = 4\phi$, so we will travel around the surface of this sphere from the top downward and make 2 full counterclockwise rotations in the theta direction.



We can also approach these types of problems in reverse when asked to parameterize the intersections of surfaces.

Illustration 1.6.4

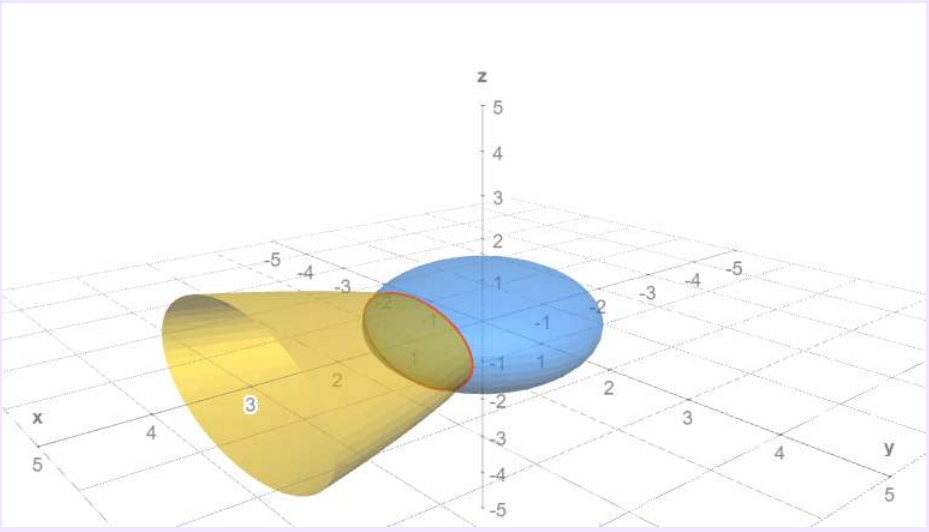
Find a parameterization of the intersection of the surfaces provided.

(a) $x^2 + y^2 + z^2 = 2$ and $x = y^2 + z^2$

(b) $3x^2 + y^2 + z^2 = 1$ and $x + z = 0$

(c) $z = x^2 - y^2$ and $\frac{x^2}{16} + \frac{y^2}{4} = 1$

(a) First we sketch the intersection of the sphere $x^2 + y^2 + z^2 = 2$ and the paraboloid $x = y^2 + z^2$:



We can substitute the paraboloid equation into the sphere to get $x^2 + x = 2$ which has solutions $x = -2, 1$

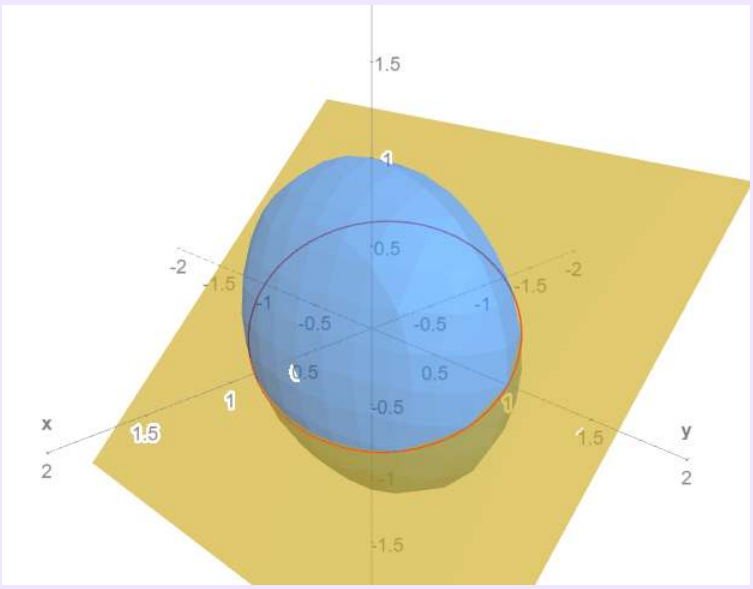
Since $x = -2$ is impossible on the paraboloid, we must have $x(t) = 1$ for all t.

Now we want $1 = (y(t))^2 + (z(t))^2$, which we can parameterize with $y(t) = \sin(t), z(t) = \cos(t)$ for $0 \leq t \leq 2\pi$

Therefore we have $\vec{\mathbf{r}}(t) = 1\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + \cos(t)\,\hat{\mathbf{k}}$ with $0 \leq t \leq 2\pi$

(Note that $\vec{\mathbf{r}}(t) = 1\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}} + \sin(t)\,\hat{\mathbf{k}}$ would also work)

(b) We sketch the intersection of the ellipsoid $3x^2 + y^2 + z^2 = 1$ and the plane $x + z = 0$:



We substitute the plane $z = -x$ into the ellipsoid to get $3x^2 + y^2 + (-x)^2 = 1$

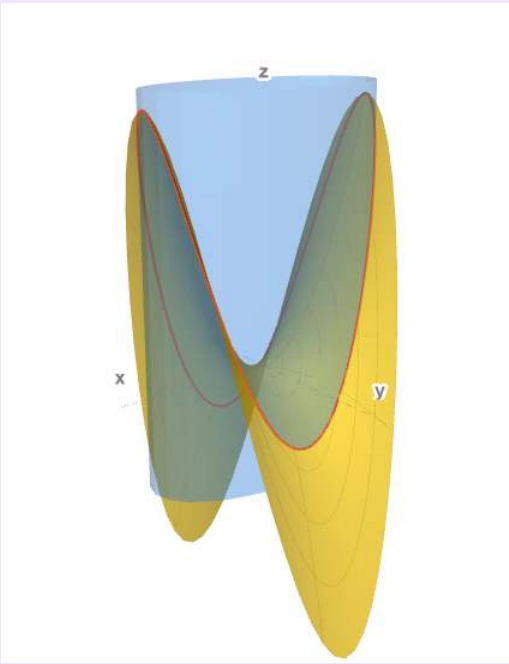
This simplifies to $4x^2 + y^2 = 1$ which is an ellipse. We pick $x(t)$ and $y(t)$ so that $4((x(t))^2 + (y(t))^2 = 1$

One such possibility is $x(t) = \frac{1}{2}\cos(t)$ and $y(t) = \sin(t)$ with $0 \leq t \leq 2\pi$

Since $z = -x$ we must then have $z(t) = -\frac{1}{2}\cos(t)$

Therefore we have $\vec{\mathbf{r}}(t) = \frac{1}{2}\cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} - \frac{1}{2}\cos(t)\,\hat{\mathbf{k}}$ with $0 \leq t \leq 2\pi$

(c) Again we sketch the intersection of the hyperbolic paraboloid $z = x^2 - y^2$ and cylinder $\frac{x^2}{16} + \frac{y^2}{4} = 1$:



Luckily, the cylinder already has only two variables, so we can parameterize $x(t)$ and $y(t)$ along its edge.

One possibility is to pick $x(t) = 4\cos(t)$ and $y(t) = 2\sin(t)$ with $0 \leq t \leq 2\pi$

Since $z = x^2 - y^2$, we have $z(t) = (4\cos(t))^2 - (2\sin(t))^2 = 16\cos^2(t) - 4\sin^2(t)$

Therefore we have $\vec{\mathbf{r}}(t) = 4\cos(t)\,\hat{\mathbf{i}} + 2\sin(t)\,\hat{\mathbf{j}} + \left(16\cos^2(t) - 4\sin^2(t)\right)\,\hat{\mathbf{k}}$ with $0 \le t \le 2\pi$

At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ we can define $x = f(t)$ and $y = g(t)$. If a restriction exists on the values of t (for example, t is restricted to the interval $[a, b]$ for some constants $a < b$), then this restriction is enforced on the parameter. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued graph would represent vectors rather than points. Since we can parameterize a curve defined by a function $y = f(x)$, it is also possible to represent an arbitrary plane curve by a vector-valued function. Just use $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + f(t)\,\hat{\mathbf{j}}$

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1.6: Calculus of Vector-Valued Functions

We now take a look at the limit of a vector-valued function. This is important to understand to study the calculus of vector-valued functions.

Definition: Limit of a Vector-Valued Function

A vector-valued function $\vec{\mathbf{r}}$ approaches the limit $\vec{\mathbf{L}}$ as t approaches a, written

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{L}}$$

provided

$$\lim_{t\to a}\left\|\vec{\mathbf{r}}(t)-\vec{\mathbf{L}}\right\|=0$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following rule:

Rule: Limit of a Vector-Valued Function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ as t approaches a is given by

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=\left[\lim_{t\to a}f(t)\right]\hat{\mathbf{i}}+\left[\lim_{t\to a}g(t)\right]\hat{\mathbf{j}}$$

provided the limits $\lim_{t\to a}f(t)$ and $\lim_{t\to a}g(t)$ exist.

Similarly, the limit of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ as t approaches a is given by

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=\left[\lim_{t\to a}f(t)\right]\hat{\mathbf{i}}+\left[\lim_{t\to a}g(t)\right]\hat{\mathbf{j}}+\left[\lim_{t\to a}h(t)\right]\hat{\mathbf{k}}$$

provided the limits $\lim_{t\to a}f(t)$, $\lim_{t\to a}g(t)$ and $\lim_{t\to a}h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example 1.6.1: Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim_{t\to 3}\vec{\mathbf{r}}(t)$ for

(a) $\vec{\mathbf{r}}(t)=(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}$

(b) $\vec{\mathbf{r}}(t)=\dfrac{2t-4}{t+1}\hat{\mathbf{i}}+\dfrac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}$

Solution

(a) We distribute the limit at $t=3$ into the two component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &= \lim_{t\to 3}\left[(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}\right]\\ &=\left[\lim_{t\to 3}(t^2-3t+4)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}(4t+3)\right]\hat{\mathbf{j}}\\ &=\boxed{4\hat{\mathbf{i}}+15\hat{\mathbf{j}}}\end{aligned}$$

(b) We distribute the limit at $t=3$ into the three component expressions:

$$\lim_{t\to 3}\vec{\mathbf{r}}(t) = \lim_{t\to 3}\left(\frac{2t-4}{t+1}\hat{\mathbf{i}} + \frac{t}{t^2+1}\hat{\mathbf{j}} + (4t-3)\hat{\mathbf{k}}\right)$$
$$= \left[\lim_{t\to 3}\left(\frac{2t-4}{t+1}\right)\right]\hat{\mathbf{i}} + \left[\lim_{t\to 3}\left(\frac{t}{t^2+1}\right)\right]\hat{\mathbf{j}} + \left[\lim_{t\to 3}(4t-3)\right]\hat{\mathbf{k}}$$
$$= \boxed{\frac{1}{2}\hat{\mathbf{i}} + \frac{3}{10}\hat{\mathbf{j}} + 9\hat{\mathbf{k}}}$$

Now that we know how to calculate the limit of a vector-valued function, we can define **continuity at a point** for such a function.

Definition: Continuity of a Vector-Valued Function

A vector-valued function $\vec{\mathbf{r}}(t)$ is **continuous at the point** $t = a$ if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim_{t\to a}\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(a)$

It follows easily that a plane or space curve is continuous at some t if and only if all of its component functions are continuous at that t. While there are many names for the discontinuities of single variable real valued functions in calculus, there are no such names here. Consider that at a single value of t a space curve can have a jump discontinuity in one component, a removable discontinuity in a second component, and an infinite discontinuity in the third. For this reason, we just call such points *discontinuities*.

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, and then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Definition: Derivative of a Vector-Valued Function

The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t\to 0}\frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$$

provided the limit exists. If $\vec{\mathbf{r}}'(t)$ exists, then $\vec{\mathbf{r}}(t)$ is differentiable at t. If $\vec{\mathbf{r}}'(t)$ exists for all t in an open interval (a, b) then $\vec{\mathbf{r}}(t)$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\vec{\mathbf{r}}'(a) = \lim_{\Delta t\to 0^+}\frac{\vec{\mathbf{r}}(a+\Delta t) - \vec{\mathbf{r}}(a)}{\Delta t} \text{ and } \vec{\mathbf{r}}'(b) = \lim_{\Delta t\to 0^-}\frac{\vec{\mathbf{r}}(b+\Delta t) - \vec{\mathbf{r}}(b)}{\Delta t}$$

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

We now demonstrate taking the derivative of a vector-valued function.

Example 1.6.2: Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function

$$\vec{\mathbf{r}}(t) = (3t+4)\,\hat{\mathbf{i}} + (t^2 - 4t + 3)\,\hat{\mathbf{j}}$$

Solution

Let's use the limit definition of derivative above:

$$\begin{aligned}
\vec{\mathbf{r}}\,'(t) &= \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+\Delta t)+4)\,\hat{\mathbf{i}} + ((t+\Delta t)^2 - 4(t+\Delta t) + 3)\,\hat{\mathbf{j}}] - [(3t+4)\,\hat{\mathbf{i}} + (t^2 - 4t + 3)\,\hat{\mathbf{j}}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3t + 3\Delta t + 4)\,\hat{\mathbf{i}} - (3t+4)\,\hat{\mathbf{i}} + (t^2 + 2t\Delta t + (\Delta t)^2 - 4t - 4\Delta t + 3)\,\hat{\mathbf{j}} - (t^2 - 4t + 3)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3\Delta t)\,\hat{\mathbf{i}} + (2t\Delta t + (\Delta t)^2 - 4\Delta t)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} (3\,\hat{\mathbf{i}} + (2t + \Delta t - 4)\,\hat{\mathbf{j}}) \\
&= \boxed{3\,\hat{\mathbf{i}} + (2t-4)\,\hat{\mathbf{j}}}
\end{aligned}$$

Notice that in the calculations above, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following rule. The proof of this rule follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Rule: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ then $\vec{\mathbf{r}}\,'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}}$
2. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ then $\vec{\mathbf{r}}\,'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}$

Example 1.6.3: Calculating the Derivative of a Vector-Valued Function

Use the previous Theorem to calculate the derivative of each of the following functions.

(a) $\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2 + 2t - 3)\,\hat{\mathbf{j}}$

(b) $\vec{\mathbf{r}}(t) = 3\cos(t)\,\hat{\mathbf{i}} + 4\sin(t)\,\hat{\mathbf{j}}$

(c) $\vec{\mathbf{r}}(t) = e^t \sin(t)\,\hat{\mathbf{i}} + e^t \cos(t)\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$

Solution

We use the shortcuts we know about differentiating functions of one variable.

(a) The components of $\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2 + 2t - 3)\,\hat{\mathbf{j}}$ are $f(t) = 6t + 8$ and $g(t) = 4t^2 + 2t - 3$.

These have derivatives $f'(t) = 6$ and $g'(t) = 8t + 2$, so $\boxed{\vec{\mathbf{r}}\,'(t) = 6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}}}$

(b) The components are $f(t) = 3\cos(t)$ and $g(t) = 4\sin(t)$.

Since $f'(t) = -3\sin(t)$ and $g'(t) = 4\cos(t)$, we obtain $\boxed{\vec{\mathbf{r}}\,'(t) = -3\sin(t)\,\hat{\mathbf{i}} + 4\cos(t)\,\hat{\mathbf{j}}}$

(c) The components of $\vec{\mathbf{r}}(t)$ are $f(t) = e^t \sin(t)$, $g(t) = e^t \cos(t)$, and $h(t) = -e^{2t}$.

We find that $f'(t) = e^t(\sin(t) + \cos(t))$, $g'(t) = e^t(\cos(t) - \sin(t))$, and $h'(t) = -2e^{2t}$, so $\boxed{\vec{\mathbf{r}}'(t) = e^t(\sin(t) + \cos(t))\,\hat{\mathbf{i}} + e^t(\cos(t) - \sin(t))\,\hat{\mathbf{j}} - 2e^{2t}\,\hat{\mathbf{k}}}$

We can extend to vector-valued functions the properties of the derivative that we presented previously. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions:

1. for a real-valued function multiplied by a vector-valued function,
2. for the dot product of two vector-valued functions, and
3. for the cross product of two vector-valued functions.

Theorem 1.6.1: Properties of the Derivative of Vector-Valued Functions

Let $\vec{\mathbf{r}}$ and $\vec{\mathbf{u}}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\vec{\mathbf{r}}(t)]$	$=$	$c\vec{\mathbf{r}}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \pm \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \pm \vec{\mathbf{u}}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)]$	$=$	$f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t)$	Scalar product
iv.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}'(t)$	Dot product
v.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}'(t)$	Cross product
vi.	$\frac{d}{dt}[\vec{\mathbf{r}}(f(t))]$	$=$	$\vec{\mathbf{r}}'(f(t)) \cdot f'(t)$	Chain rule
vii.	If $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)$	$=$	c then $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0$	

Proof

The argument is identical for these proofs for both plane and space curves in each part.

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule. Let $\vec{\mathbf{u}}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}
\frac{d}{dt}\left[f(t)\vec{\mathbf{u}}(t)\right] &= \frac{d}{dt}\left[f(t)(g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}})\right] \\
&= \frac{d}{dt}\left[f(t)g(t)\,\hat{\mathbf{i}} + f(t)h(t)\,\hat{\mathbf{j}}\right] \\
&= \frac{d}{dt}\left[f(t)g(t)\right]\hat{\mathbf{i}} + \frac{d}{dt}\left[f(t)h(t)\right]\hat{\mathbf{j}} \\
&= \left(f'(t)g(t) + f(t)g'(t)\right)\hat{\mathbf{i}} + \left(f'(t)h(t) + f(t)h'(t)\right)\hat{\mathbf{j}} \\
&= f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t)
\end{aligned}$$

To prove property iv. let $\vec{\mathbf{r}}(t) = f_1(t)\,\hat{\mathbf{i}} + g_1(t)\,\hat{\mathbf{j}}$ and $\vec{\mathbf{u}}(t) = f_2(t)\,\hat{\mathbf{i}} + g_2(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}
\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)\right] &= \frac{d}{dt}\left[f_1(t)f_2(t) + g_1(t)g_2(t)\right] \\
&= f_1{}'(t)f_2(t) + f_1(t)f_2{}'(t) + g_1{}'(t)g_2(t) + g_1(t)g_2{}'(t) = f_1{}'(t)f_2(t) + g_1{}'(t)g_2(t) + f_1(t)f_2{}'(t) + g_1(t)g_2{}'(t) \\
&= (f_1{}'\,\hat{\mathbf{i}} + g_1{}'\,\hat{\mathbf{j}}) \cdot (f_2\,\hat{\mathbf{i}} + g_2\,\hat{\mathbf{j}}) + (f_1\,\hat{\mathbf{i}} + g_1\,\hat{\mathbf{j}}) \cdot (f_2{}'\,\hat{\mathbf{i}} + g_2{}'\,\hat{\mathbf{j}}) \\
&= \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}'(t)
\end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule.

Lastly, property vii. follows from property iv. since:

$$\frac{d}{dt}\left[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)\right]=\frac{d}{dt}[c]$$
$$\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t)=0$$
$$2\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t)=0$$
$$\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t)=0$$

Now for some examples using these properties.

Example 1.6.4: Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\vec{\mathbf{r}}(t)=(6t+8)\,\hat{\mathbf{i}}+(4t^2+2t-3)\,\hat{\mathbf{j}}+5t\,\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{u}}(t)=(t^2-3)\,\hat{\mathbf{i}}+(2t+4)\,\hat{\mathbf{j}}+(t^3-3t)\,\hat{\mathbf{k}}$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

(a) $\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)]$

(b) $\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)]$

Solution

(a) We have $\vec{\mathbf{r}}'(t)=6\,\hat{\mathbf{i}}+(8t+2)\,\hat{\mathbf{j}}+5\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}'(t)=2t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+(3t^2-3)\,\hat{\mathbf{k}}$. Therefore, according to *property iv.*:

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)]&=\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}'(t)\\&=(6\,\hat{\mathbf{i}}+(8t+2)\,\hat{\mathbf{j}}+5\,\hat{\mathbf{k}})\cdot((t^2-3)\,\hat{\mathbf{i}}+(2t+4)\,\hat{\mathbf{j}}+(t^3-3t)\,\hat{\mathbf{k}})\\&+((6t+8)\,\hat{\mathbf{i}}+(4t^2+2t-3)\,\hat{\mathbf{j}}+5t\,\hat{\mathbf{k}})\cdot(2t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+(3t^2-3)\,\hat{\mathbf{k}})\\&=6(t^2-3)+(8t+2)(2t+4)+5(t^3-3t)\\&+2t(6t+8)+2(4t^2+2t-3)+5t(3t^2-3)\\&=\boxed{20t^3+42t^2+26t-16}\end{aligned}$$

Notice that if we instead calculated directly:

$$\begin{aligned}\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)&=(6t+8)(t^2-3)+(4t^2+2t-3)(2t+4)+(5t)(t^3-3t)\\&=5t^4+14t^3+13t^2-16t-36\end{aligned}$$

This resulting function would clearly have given us the same derivative.

(b) First, we need to adapt *property v.* for this problem:

$$\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)]=\vec{\mathbf{u}}'(t)\times\vec{\mathbf{u}}'(t)+\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore, $\vec{\mathbf{u}}''(t)$ represents the second derivative of $\vec{\mathbf{u}}(t)$:

$$\vec{\mathbf{u}}''(t)=\frac{d}{dt}[\vec{\mathbf{u}}'(t)]=\frac{d}{dt}[2t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+(3t^2-3)\,\hat{\mathbf{k}}]=2\,\hat{\mathbf{i}}+6t\,\hat{\mathbf{k}}$$

Therefore,

$$\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)] = 0 + ((t^2-3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3-3t)\,\hat{\mathbf{k}}) \times (2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t^2-3 & 2t+4 & t^3-3t \\ 2 & 0 & 6t \end{vmatrix}$$

$$= 6t(2t+4)\,\hat{\mathbf{i}} - (6t(t^2-3) - 2(t^3-3t))\,\hat{\mathbf{j}} - 2(2t+4)\,\hat{\mathbf{k}}$$

$$= \boxed{(12t^2+24t)\,\hat{\mathbf{i}} + (12t-4t^3)\,\hat{\mathbf{j}} - (4t+8)\,\hat{\mathbf{k}}}$$

Once again, a direct calculation of $\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)$ gives:

$$\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t^2-3 & 2t+4 & t^3-3t \\ 2t & 2 & 3t^2-3 \end{vmatrix}$$

$$= (4t^3+12t^2-12)\,\hat{\mathbf{i}} + (-t^4+6t^2-9)\,\hat{\mathbf{j}} - (2t^2+8t+6)\,\hat{\mathbf{k}}$$

The derivative of this resulting space curve clearly gives us the same derivative.

Recall that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-valued function

$$\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}}$$

The derivative of this function is

$$\vec{\mathbf{r}}'(t) = -\sin(t)\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}}$$

If we substitute the value $t = \frac{\pi}{6}$ into both functions we get

$$\vec{\mathbf{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\,\hat{\mathbf{i}} + \frac{1}{2}\,\hat{\mathbf{j}} \text{ and } \vec{\mathbf{r}}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}$$

The graph of this function appears below, along with the vectors $\vec{\mathbf{r}}\left(\frac{\pi}{6}\right)$ and $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$:

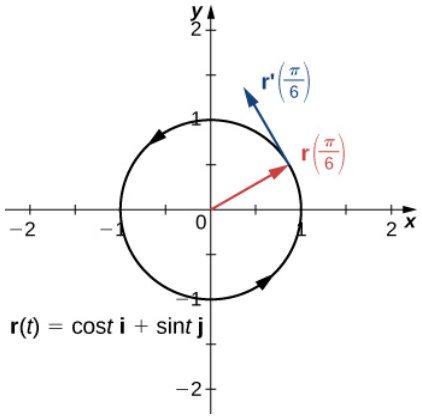


Figure 1.6.1: The tangent line at a point is calculated from the derivative of the vector-valued function $\vec{\mathbf{r}}(t)$.

Notice that the vector $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t = \frac{\pi}{6}$. This is an example of a tangent vector to the plane curve.

Definition: Principal Unit Tangent Vector

Let C be a curve defined by a vector-valued function $\vec{\mathbf{r}}$, and assume that $\vec{\mathbf{r}}'(t)$ exists when $t = t_0$. A tangent vector $\vec{\mathbf{r}}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{r}}$ is tangent to curve C. Vector $\vec{\mathbf{r}}$

$'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\vec{\mathbf{r}}\,'(t) \neq 0$. When $\|\vec{\mathbf{r}}\,'(t)\| \neq 0$, the ***principal unit tangent vector*** at t is defined to be

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}$$

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\vec{\mathbf{r}}\,'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example 1.6.5: Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

(a) $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}}$

(b) $\vec{\mathbf{r}}(t) = (3t^2 + 2t)\,\hat{\mathbf{i}} + (2 - 4t^3)\,\hat{\mathbf{j}} + (6t + 5)\,\hat{\mathbf{k}}$

(c) $\vec{\mathbf{r}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t + 1)\,\hat{\mathbf{j}} + (t - 2)\,\hat{\mathbf{k}}$

Solution

(a) First we find the derivative of $\vec{\mathbf{r}}(t)$: $\vec{\mathbf{r}}\,'(t) = -\sin(t)\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}}$

Next we find its magnitude: $\|\vec{\mathbf{r}}\,'(t)\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$

Lastly we solve for the unit tangent vector: $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} = \boxed{\dfrac{-\sin(t)\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}}}{1} = -\sin(t)\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}}}$

(b) First we find the derivative of $\vec{\mathbf{r}}(t)$: $\vec{\mathbf{r}}\,'(t) = (6t + 2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}$

Next we find its magnitude: $\|\vec{\mathbf{r}}\,'(t)\| = \sqrt{(6t+2)^2 + (-12t^2)^2 + 6^2} = 2\sqrt{36t^4 + 9t^2 + 6t + 10}$

Finally: $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} = \boxed{\dfrac{3t+1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{i}} - \dfrac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{j}} + \dfrac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{k}}}$

(c) First we find the derivative of $\vec{\mathbf{r}}(t)$: $\vec{\mathbf{r}}\,'(t) = (2t)\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 1\,\hat{\mathbf{k}}$

Next we find its magnitude: $\|\vec{\mathbf{r}}\,'(t)\| = \sqrt{(2t)^2 + 2^2 + 1^2} = \sqrt{4t^2 + 5}$

Finally: $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} = \boxed{\dfrac{2t}{\sqrt{4t^2+5}}\,\hat{\mathbf{i}} + \dfrac{2}{\sqrt{4t^2+5}}\,\hat{\mathbf{j}} + \dfrac{1}{\sqrt{4t^2+5}}\,\hat{\mathbf{k}}}$

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time *t*, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Definition: Definite and Indefinite Integrals of Vector-Valued Functions

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}}$$

The definite integral of a vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}}$$

2. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}$$

The definite integral of the vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have

$$\int f(t)\,dt = F(t) + C_1 \text{ and } \int g(t)\,dt = G(t) + C_2$$

where F and G are antiderivatives of f and g, respectively. Then

$$\begin{aligned}\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt &= \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} \\ &= (F(t) + C_1)\,\hat{\mathbf{i}} + (G(t) + C_2)\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}\end{aligned}$$

where $\vec{\mathbf{C}} = C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}}$. Therefore, the *integration constants* become a *constant vector*.

Example 1.6.6: Integrating Vector-Valued Functions

Calculate each of the following integrals:

(a) $\int [(3t^2 + 2t)\,\hat{\mathbf{i}} + (3t - 6)\,\hat{\mathbf{j}} + (6t^3 + 5t^2 - 4)\,\hat{\mathbf{k}}]\,dt$

(b) $\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt$

(c) $\int_0^{\frac{\pi}{3}} [\sin(2t)\,\hat{\mathbf{i}} + \tan(t)\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt$

Solution

(a) We use the first part of the definition of the integral of a space curve:

$$\begin{aligned}\int [(3t^2 + 2t)\,\hat{\mathbf{i}} + (3t - 6)\,\hat{\mathbf{j}} + (6t^3 + 5t^2 - 4)\,\hat{\mathbf{k}}]\,dt &= \left[\int 3t^2 + 2t\,dt\right]\hat{\mathbf{i}} + \left[\int 3t - 6\,dt\right]\hat{\mathbf{j}} \\ &\quad + \left[\int 6t^3 + 5t^2 - 4\,dt\right]\hat{\mathbf{k}} \\ &= \boxed{(t^3 + t^2)\,\hat{\mathbf{i}} + \left(\frac{3}{2}t^2 - 6t\right)\hat{\mathbf{j}} + \left(\frac{3}{2}t^4 + \frac{5}{3}t^3 - 4t\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}}\end{aligned}$$

(b) First calculate $\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle$:

$$\begin{aligned}\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & t^2 & t^3 \\ t^3 & t^2 & t\end{vmatrix} \\ &= (t^2(t) - t^3(t^2))\,\hat{\mathbf{i}} - (t^2 - t^3(t^3))\,\hat{\mathbf{j}} + (t(t^2) - t^2(t^3))\,\hat{\mathbf{k}} \\ &= (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}\end{aligned}$$

Next, substitute this back into the integral and integrate:

$$\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\, dt = \int (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}\, dt$$
$$= \boxed{\left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{i}} + \left(\frac{t^7}{7} - \frac{t^3}{3}\right)\hat{\mathbf{j}} + \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}}$$

(c) Use the second part of the definition of the integral of a space curve:

$$\int_0^{\frac{\pi}{3}} [\sin(2t)\,\hat{\mathbf{i}} + \tan(t)\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\, dt = \left[\int_0^{\frac{\pi}{3}} \sin(2t)\, dt\right]\hat{\mathbf{i}} + \left[\int_0^{\frac{\pi}{3}} \tan(t)\, dt\right]\hat{\mathbf{j}} + \left[\int_0^{\frac{\pi}{3}} e^{-2t}\, dt\right]\hat{\mathbf{k}}$$
$$= \left(-\frac{1}{2}\cos(2t)\right)\Big|_0^{\frac{\pi}{3}}\hat{\mathbf{i}} - (\ln|\cos(t)|)\Big|_0^{\frac{\pi}{3}}\hat{\mathbf{j}} - \left(\frac{1}{2}e^{-2t}\right)\Big|_0^{\frac{\pi}{3}}\hat{\mathbf{k}}$$
$$= \left(-\frac{1}{2}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{2}\cos(0)\right)\hat{\mathbf{i}}$$
$$- \left(\ln\left(\cos\left(\frac{\pi}{3}\right)\right) - \ln(\cos(0))\right)\hat{\mathbf{j}}$$
$$- \left(\frac{1}{2}e^{-\frac{2\pi}{3}} - \frac{1}{2}e^{-2(0)}\right)\hat{\mathbf{k}}$$
$$= \left(\frac{1}{4} + \frac{1}{2}\right)\hat{\mathbf{i}} - (-\ln(2))\hat{\mathbf{j}} - \left(\frac{1}{2}e^{-\frac{2\pi}{3}} - \frac{1}{2}\right)\hat{\mathbf{k}}$$
$$= \boxed{\frac{3}{4}\hat{\mathbf{i}} + (\ln(2))\hat{\mathbf{j}} + \left(\frac{1}{2} - \frac{1}{2}e^{-\frac{2\pi}{3}}\right)\hat{\mathbf{k}}}$$

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1.6: Practice Set

Exercise 1.6.1a

Sketch $\vec{\mathbf{r}}(t) = \langle t, t, t^2 \rangle$ for $t \in \mathbb{R}$

Answer

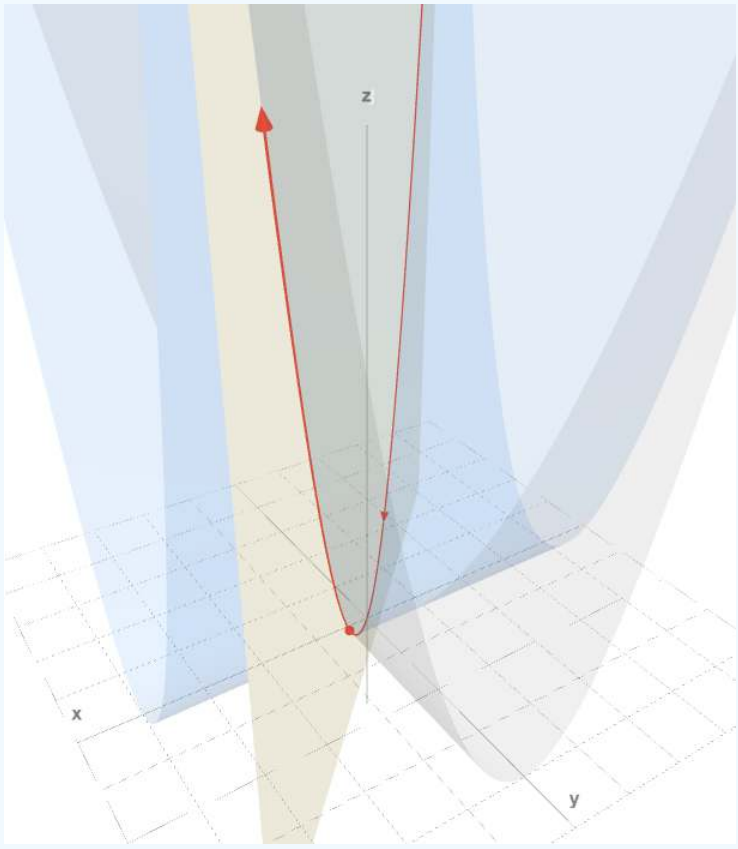
This curve has $x(t) = t$, $y(t) = t$, and $z(t) = t^2$

We can eliminate parameters between y and z to get $z = y^2$ which is a parabolic cylinder.

We can eliminate parameters between x and z to get $z = x^2$ which is a parabolic cylinder.

We can eliminate parameters between x and y to get $x = y$ which is a plane.

This curve travels along the intersections of any two (or all three) of these recognizable shapes:



Exercise 1.6.1b

Sketch $\vec{\mathbf{r}}(t) = \langle 1, \cos(t), -\sin(t) \rangle$ for $0 \leq t \leq 2\pi$

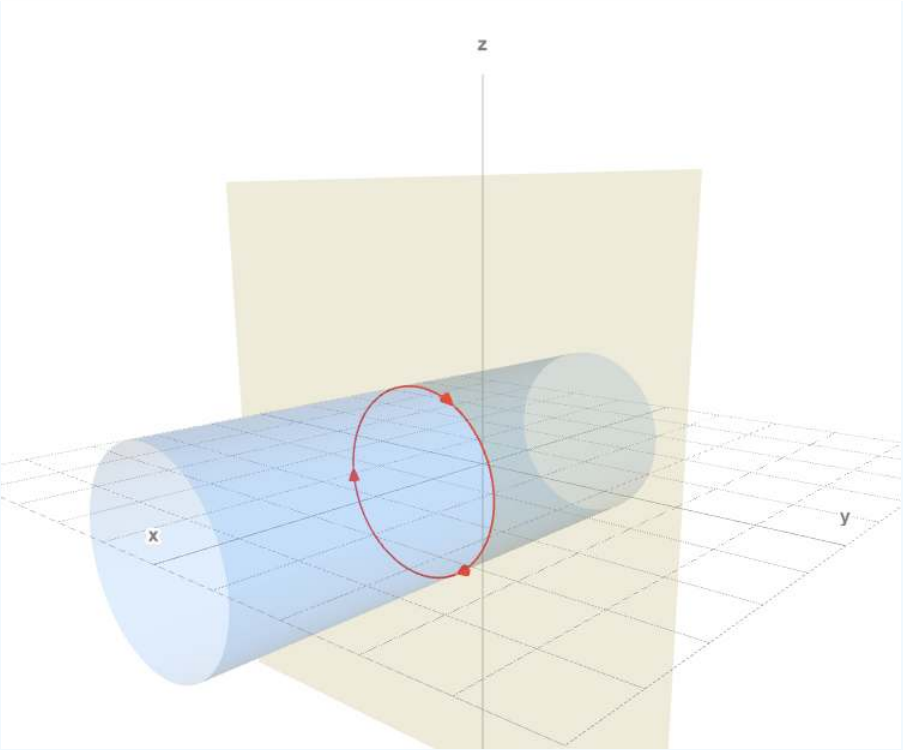
Answer

This curve has $x(t) = 1$, $y(t) = \cos(t)$, and $z(t) = -\sin(t)$

We can eliminate parameters between y and z to get $y^2 + z^2 = 1$ which is a circular cylinder.

The function for x is constant, which defines the plane $x = 1$

This curve travels along the intersections of these two recognizable shapes:



Exercise 1.6.1c

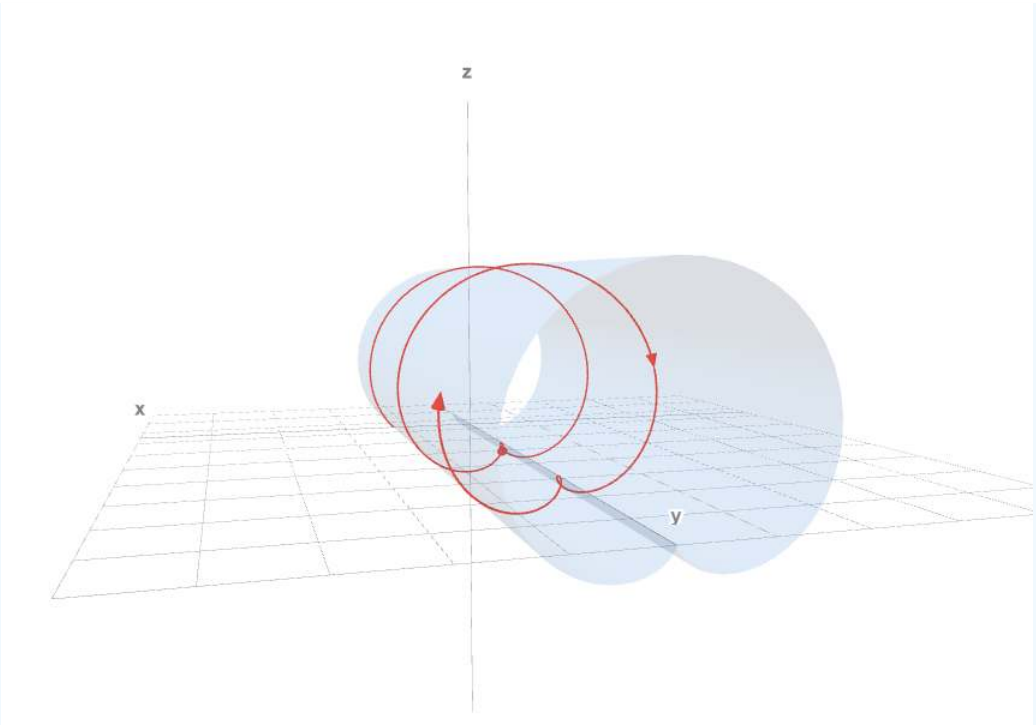
Sketch $\vec{\mathbf{r}}(t) = \langle (1+\sin(\pi t))\cos(\pi t), t, (1+\sin(\pi t))\sin(\pi t) \rangle$ for $t \geq 0$

Answer

This curve has $x(t) = (1+\sin(\pi t))\cos(\pi t)$, $y(t) = t$, and $z(t) = (1+\sin(\pi t))\sin(\pi t)$

If we define $r(\theta) = 1+\sin(\theta)$ we see that the relationship between x and z is polar, where $\theta = \pi t$ and $x = r(\theta)\cos(\theta)$, $z = r(\theta)\sin(\theta)$. This relationship is a cardioid, which means the surface in three-dimensional space is a cylinder with these cardioids as cross sections.

The value of t increases y linearly, so the curve spins counterclockwise (when viewed from the positive y-axis) around the cardioid cylinder while y slowly increases to the right:



Exercise 1.6.1d

Sketch $\vec{\mathbf{r}}(t) = \left\langle 0, t, \frac{|t|}{t} \right\rangle$ for $t \neq 0$

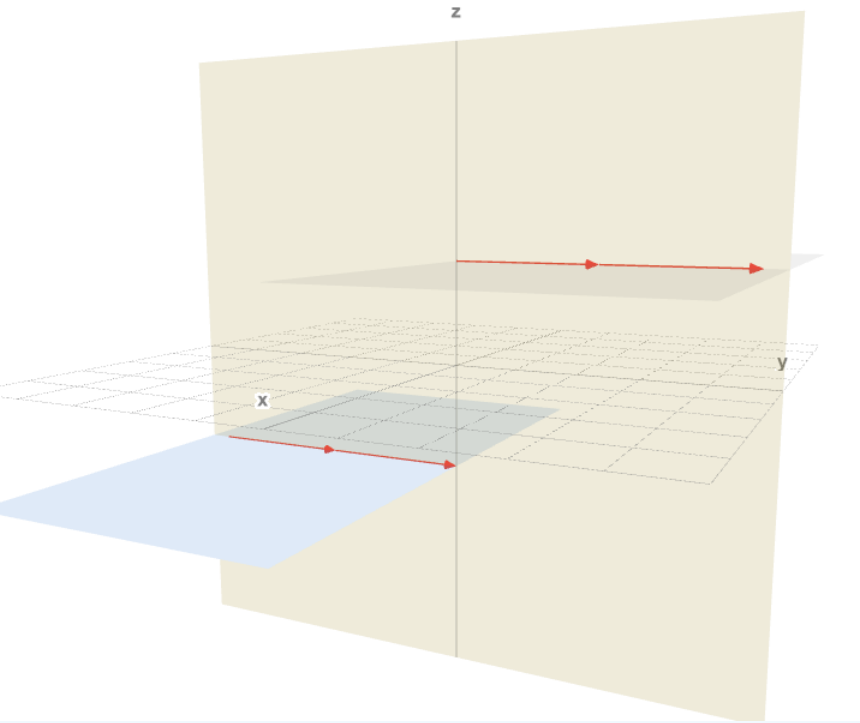
Answer

This curve has $x(t) = 0$, $y(t) = t$, and $z(t) = \frac{|t|}{t} = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$

The function for x is constant, which defines the plane $x = 0$.

The function for z is two different constants, which define the planes $z = 1$ and $z = -1$

Since $y = t$, we know that when $t > 0$ this is a straight line on the intersection of $x = 0$ and $z = 1$, and when $t < 0$ it is a straight line on the intersection of $x = 0$ and $z = -1$:



Exercise 1.6.1e

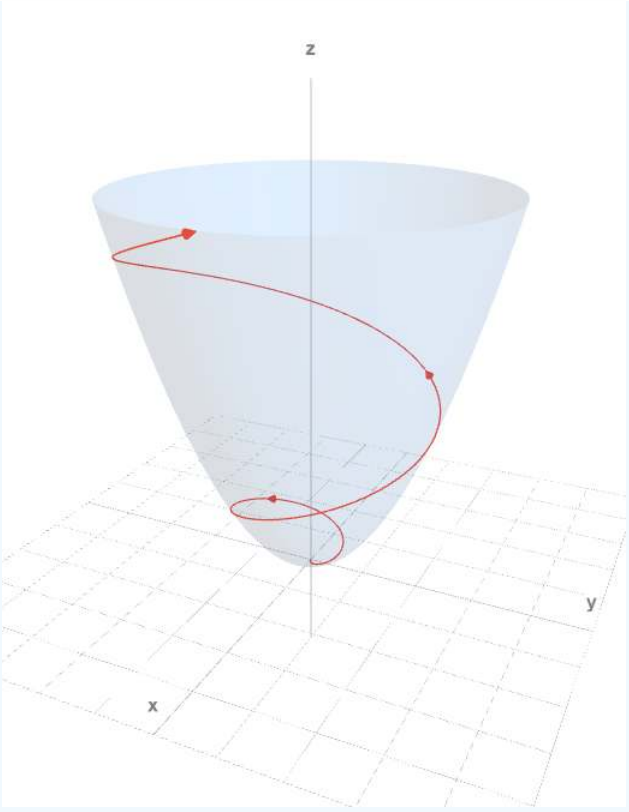
Sketch $\vec{\mathbf{r}}(t) = \langle t\cos(2\pi t), t\sin(2\pi t), t^2\rangle$ for $0 \le t \le 2$

Answer

This curve has $x(t) = t\cos(2\pi t)$, $y(t) = t\sin(2\pi t)$, and $z(t) = t^2$

We can eliminate parameters between x, y and z to get $x^2 + y^2 = z$ which is a paraboloid.

The value of t increases z, so the curve spins counterclockwise (when viewed from the positive z-axis) around the paraboloid while z increases faster and faster, for two rotations:



Exercise 1.6.1f

Sketch $\vec{\mathbf{r}}(t) = \left\langle \sqrt{2}\sin(t), \sqrt{2}\sin(t), \cos(t) \right\rangle$ for $0 \leq t \leq \dfrac{3\pi}{4}$

Answer

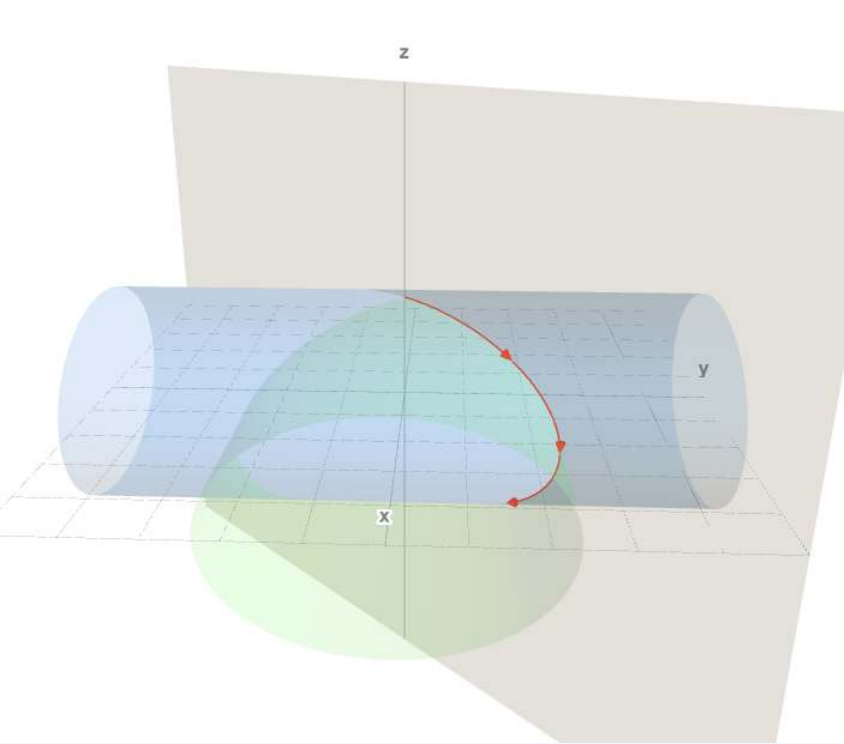
This curve has $x(t) = \sqrt{2}\sin(t)$, $y(t) = \sqrt{2}\sin(t)$, and $z(t) = \cos(t)$

We can eliminate parameters between y and z to get $y^2 + 2z^2 = 2$ which is an elliptical cylinder.

We can eliminate parameters between x and z to get $x^2 + 2z^2 = 2$ which is an elliptical cylinder.

We can eliminate parameters between x and y to get $x = y$ which is a plane.

This curve travels along the intersections of any two (or all three) of these recognizable shapes, starting at $\vec{\mathbf{r}}(0) = (0, 0, 1)$ and making $\dfrac{3}{8}$ of a rotation clockwise (when viewed from the positive x-axis):



Exercise $1.6.1g$

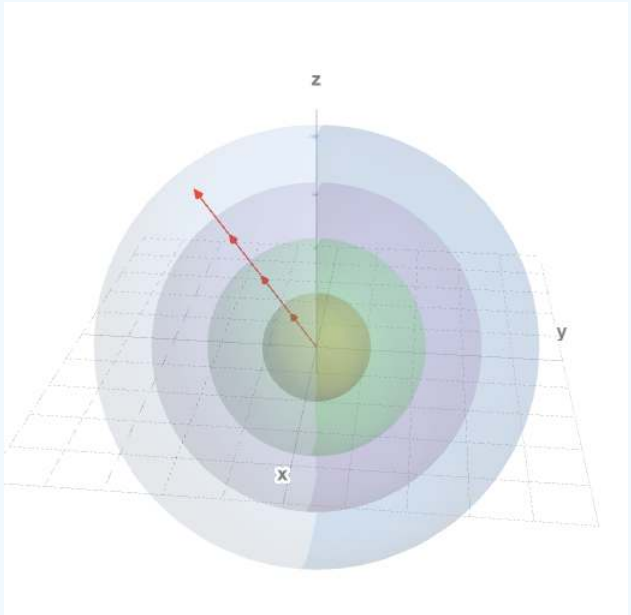
Sketch $\vec{\mathbf{r}}(t) = \left\langle t\sin\left(\frac{\pi}{7}\right)\sqrt{3}, -t\sin\left(\frac{\pi}{7}\right), 2t\cos\left(\frac{\pi}{7}\right)\right\rangle$ for $0 \le t \le \pi$

Answer

This curve has $x(t) = t\sin\left(\frac{\pi}{7}\right)\sqrt{3}$, $y(t) = -t\sin\left(\frac{\pi}{7}\right)$, and $z(t) = 2t\cos\left(\frac{\pi}{7}\right)$

These functions are set up exactly like spherical coordinates, with $\rho = 2t, \phi = \frac{\pi}{7}$, and $\theta = -\frac{\pi}{6}$

Thus, the curve contains points on an expanding sphere with $\phi = \frac{\pi}{7}$ and $\theta = -\frac{\pi}{6}$ such that $0 \le \rho \le 2\pi$:



Exercise 1.6.1h

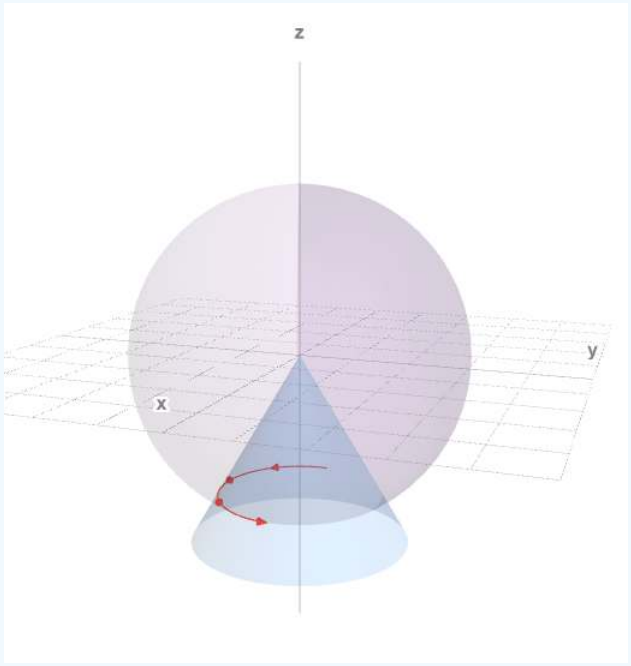
Sketch $\vec{\mathbf{r}}(t) = \langle 3\cos(t), 3\sin(t), -3\sqrt{3} \rangle$ for $\pi \leq t \leq 2\pi$

Answer

This curve has $x(t) = 3\cos(t)$, $y(t) = 3\sin(t)$, and $z(t) = -3\sqrt{3}$

These functions are set up exactly like spherical coordinates, with $\rho = 6, \phi = \frac{5\pi}{6}$, and $\theta = t$

Thus, the curve contains the points on a sphere with $\rho = 6$ at a height determined by $\phi = \frac{5\pi}{6}$ and $\theta = t$ such that $\pi \leq t \leq 2\pi$:



Exercise 1.6.1i

Sketch $\vec{\mathbf{r}}(t) = \langle 1 + \cos(t), 2 + 3\sin(t), 3 - 2\cos(t) \rangle$ for $0 \leq t \leq 2\pi$

Answer

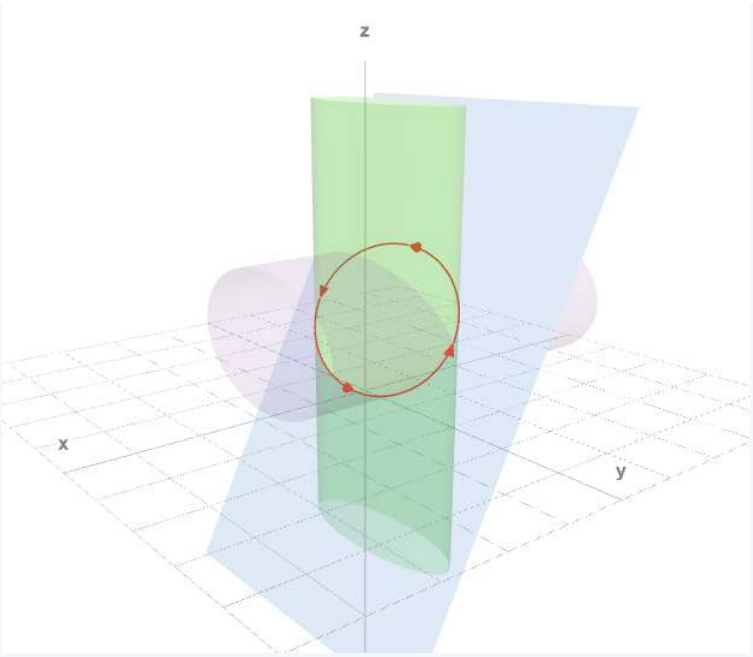
This curve has $x(t) = 1 + \cos(t)$, $y(t) = 2 + 3\sin(t)$, and $z(t) = 3 - 2\cos(t)$

We can eliminate parameters between y and z to get $\frac{(y-2)^2}{9} + \frac{(z-3)^2}{4} = 1$ which is an elliptical cylinder.

We can eliminate parameters between x and z to get $2x + z = 5$ which is a plane.

We can eliminate parameters between x and y to get $(x-1)^2 + \frac{(y-2)^2}{9} = 1$ which is an elliptical cylinder.

This curve travels along the intersections of any two (or all three) of these recognizable shapes:

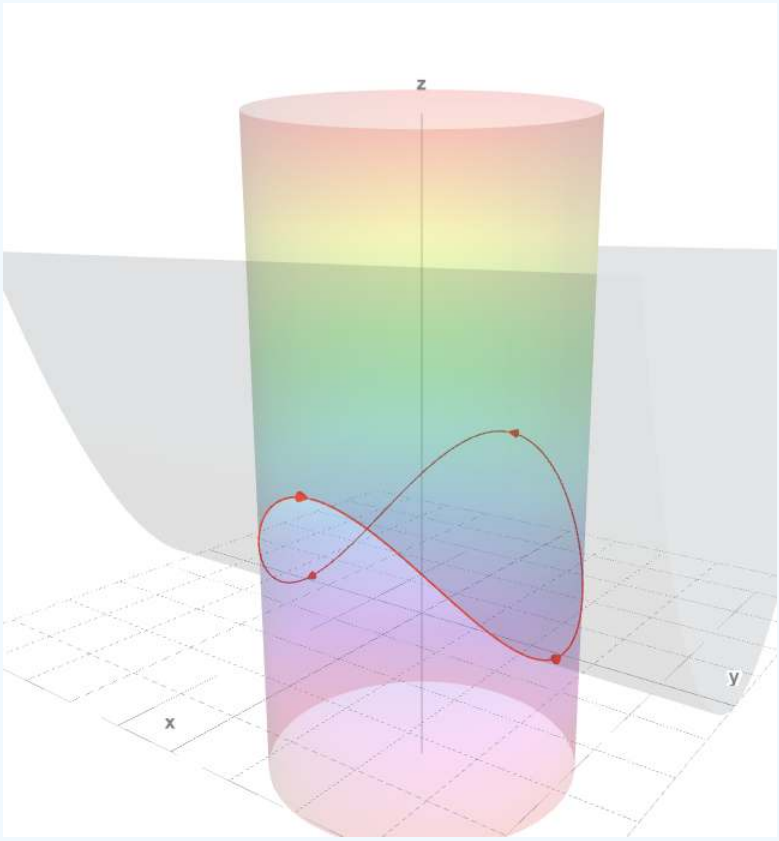


Exercise $1.6.2a$

Parameterize the intersection of $z = x^2$ and $x^2 + y^2 = 1$

Answer

This is the intersection of a parabolic cylinder and a circular cylinder:



The curve defined must travel around the cylinder, so we use $x(t) = \cos(t)$, $y(t) = \sin(t)$

Plugging this into $z = x^2$ we get $z(t) = \cos^2(t)$

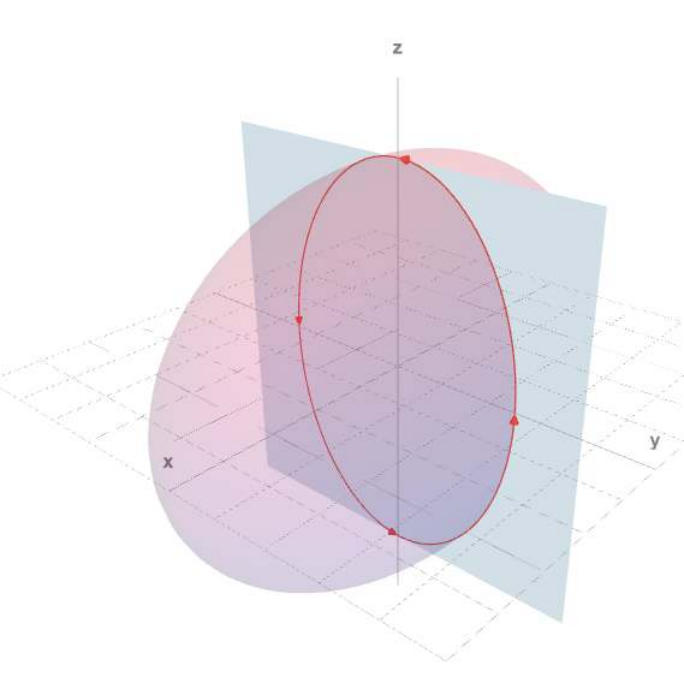
Therefore one such parameterization is $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), \cos^2(t) \rangle$ for $0 \leq t \leq 2\pi$

Exercise 1.6.2b

Parameterize the intersection of $\frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{9} = 1$ and $x = 0$

Answer

This is the intersection of a plane and an ellipsoid:



Plugging in $x = 0$, the ellipsoid becomes $\frac{y^2}{4} + \frac{z^2}{9} = 1$, so we use $y(t) = 2\cos(t),\ z(t) = 3\sin(t)$

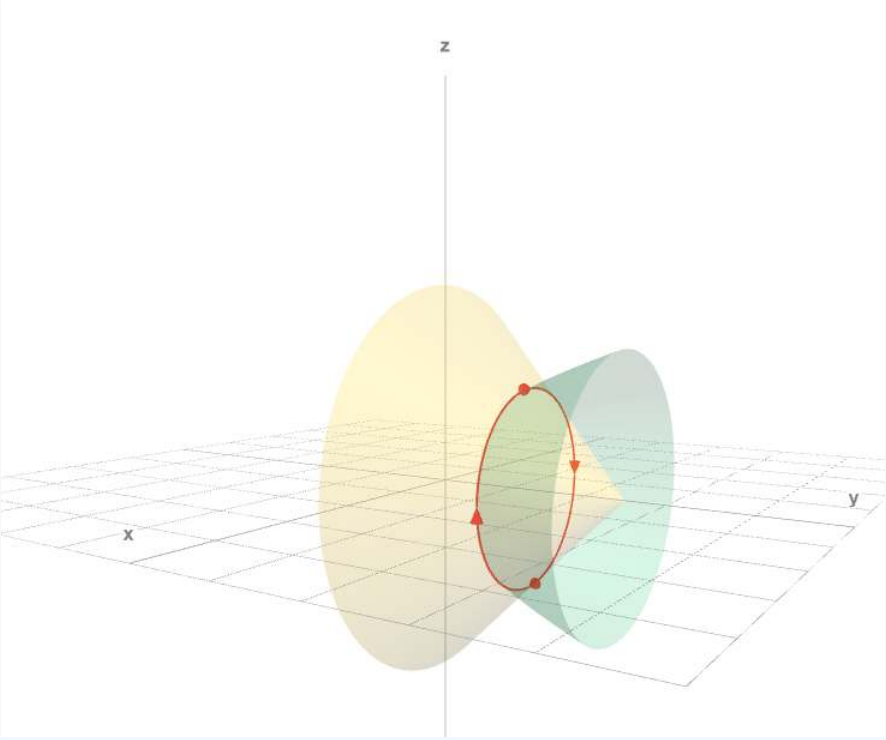
We fix $x(t) = 0$, and one such parameterization is $\vec{\mathbf{r}}(t) = \langle 0, 2\cos(t), 3\sin(t) \rangle$ for $0 \leq t \leq 2\pi$

Exercise 1.6.2c

Parameterize the intersection of $y = 1 - \sqrt{x^2 + z^2}$ and $y = 2x^2 + 2z^2$

Answer

This is the intersection of a cone and a paraboloid:



Substituting one equation into the other gives $1-\sqrt{x^2+z^2}=2x^2+2z^2 \leftrightarrow x^2+z^2=\frac{1}{4}$

We can traverse this circle using $x(t)=\frac{1}{2}\cos(t)$ and $z(t)=\frac{1}{2}\sin(t)$

Plugging these into either $y=1-\sqrt{x^2+z^2}$ or $y=2x^2+2z^2$ gives $y(t)=\frac{1}{2}$

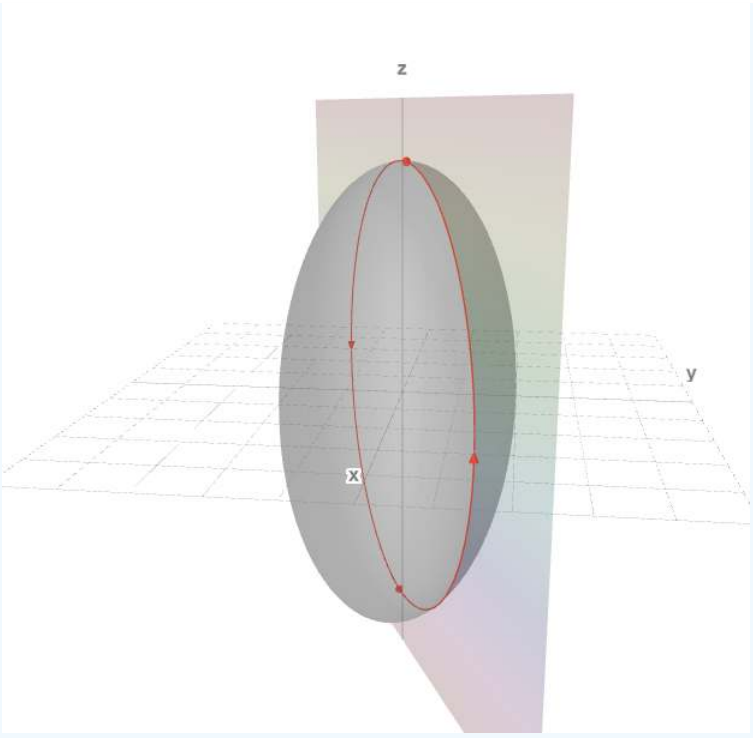
Therefore, one such parameterization is $\vec{\mathbf{r}}(t)=\left\langle \frac{1}{2}\cos(t), \frac{1}{2}, \frac{1}{2}\sin(t) \right\rangle$ for $0 \leq t \leq 2\pi$

Exercise $1.6.2d$

Parameterize the intersection of $x^2+4y^2+z^2=16$ and $x=2y$

Answer

This is the intersection of an ellipsoid and a plane:



Substituting in $x = 2y$, the ellipsoid becomes $8y^2 + z^2 = 16 \longleftrightarrow \frac{y^2}{2} + \frac{z^2}{16} = 1$, so we use $y(t) = \sqrt{2}\cos(t),\ z(t) = 4\sin(t)$

Plugging these into $x = 2y$ gives $x(t) = 2\sqrt{2}\cos(t)$

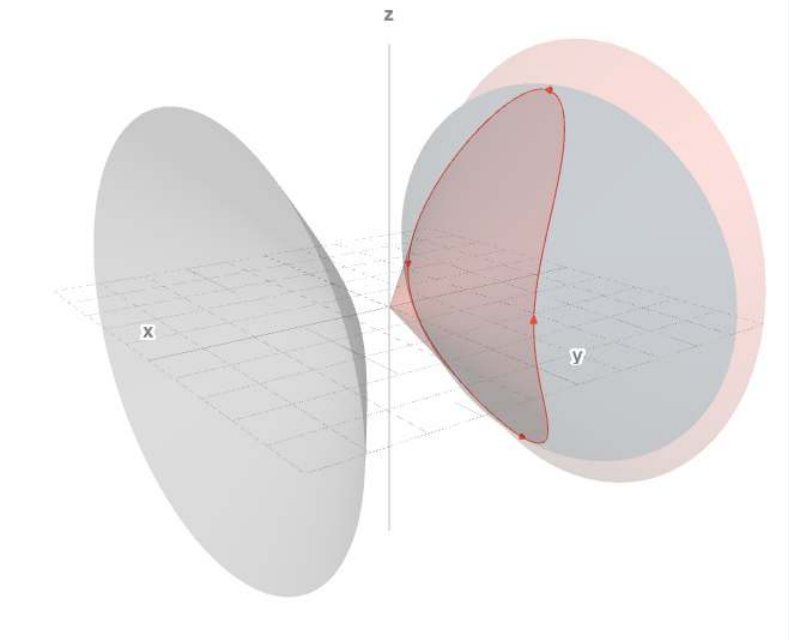
Therefore, one such parameterization is $\vec{\mathbf{r}}(t) = \left\langle 2\sqrt{2}\cos(t), \sqrt{2}\cos(t), 4\sin(t) \right\rangle$ for $0 \leq t \leq 2\pi$

Exercise $1.6.2e$

Parameterize the intersection of $10x^2 - 6y^2 - 9z^2 = 16$ and $x = -\sqrt{y^2 + z^2}$

Answer

This is the intersection of a hyperboloid of two sheets and a cone:



Substituting in $x = -\sqrt{y^2+z^2}$, the hyperboloid becomes $4y^2 + z^2 = 16 \leftrightarrow \frac{y^2}{4} + \frac{z^2}{16} = 1$, so we use $y(t) = 2\cos(t),\ z(t) = 4\sin(t)$

Plugging these into $x = -\sqrt{y^2+z^2}$ gives $x(t) = -\sqrt{4\cos^2(t) + 16\sin^2(t)}$

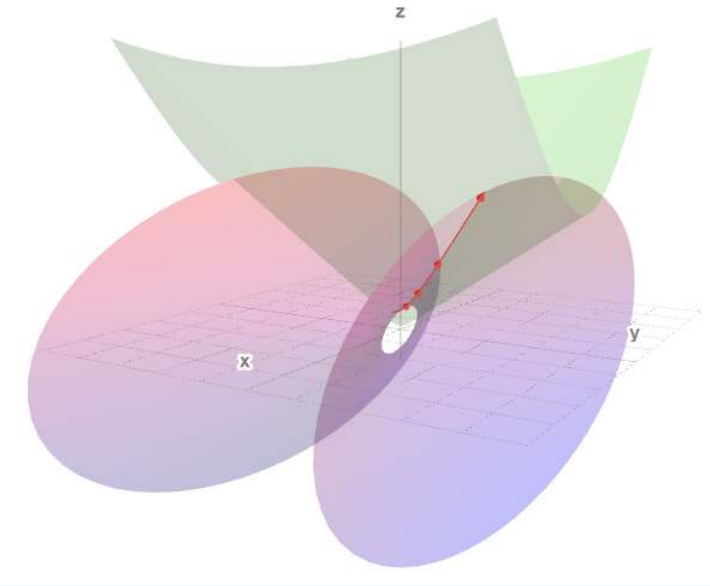
Therefore, one such parameterization is $\vec{\mathbf{r}}(t) = \left\langle -\sqrt{4\cos^2(t) + 16\sin^2(t)}, 2\cos(t), 4\sin(t) \right\rangle$ for $0 \leq t \leq 2\pi$

? Exercise $1.6.2f$

Parameterize the intersection of $z = \sqrt{4x^2+y^2}$ and $x^2 - 6y^2 + 2z^2 = 1$ in the first octant

Answer

This is the intersection of a hyperboloid of one sheet and an elliptical cone:



Substituting in $z = \sqrt{4x^2 + y^2}$, the hyperboloid becomes $9x^2 - 4y^2 = 1 \leftrightarrow \frac{x^2}{\frac{1}{9}} - \frac{y^2}{\frac{1}{4}} = 1$, so we use $x(t) = \frac{1}{3}\sec(t)$, $y(t) = \frac{1}{2}\tan(t)$

Plugging these into $z = \sqrt{4x^2 + y^2}$ gives $z(t) = \sqrt{\frac{4}{9}\sec^2(t) + \frac{1}{4}\tan^2(t)}$

To ensure we are in the first octant (so x, y, and z are all positive), we restrict our domain to $0 \leq t < \frac{\pi}{2}$

Therefore, one such parameterization is $\vec{\mathbf{r}}(t) = \left\langle \frac{1}{3}\sec(t), \frac{1}{2}\tan(t), \sqrt{\frac{4}{9}\sec^2(t) + \frac{1}{4}\tan^2(t)} \right\rangle$ for $0 \leq t < \frac{\pi}{2}$

? Exercise 1.6.3

Evaluate the following limits.

(a) $\lim\limits_{t\to 2^-} \left\langle \frac{t^2-4}{t-2}, \tan(\pi t), \frac{t-2}{\sqrt{t+2}-2} \right\rangle$

(b) $\lim\limits_{t\to 0} \left\langle \frac{\sin(t)}{t}, \frac{e^t-1}{\sin^{-1}(t)}, \frac{\sqrt{t^2+1}-1}{t\sin(t)} \right\rangle$

(c) $\lim\limits_{t\to \infty} \left\langle \frac{3t^2+4t-1}{8t^2-t+5}, \frac{\sqrt{t^2+4}}{2t+1}, \frac{t^3}{e^t-e^{-t}} \right\rangle$

(d) $\lim\limits_{t\to -\infty} \left\langle \frac{1}{t^2+1}, \frac{\sqrt{4t^2+1}}{t+8}, t\cos(t) \right\rangle$

Answer

(a) We calculate each limit separately:

$$\lim_{t\to 2^-} \frac{t^2-4}{t-2} = \lim_{t\to 2^-} \frac{(t-2)(t+2)}{t-2} = \lim_{t\to 2^-} (t+2) = 4$$

$$\lim_{t\to 2^-} \tan(\pi t) = \tan(2\pi) = 0$$

$$\lim_{t\to 2^-} \frac{t-2}{\sqrt{t+2}-2} \overset{L'H}{=} \lim_{t\to 2^-} \frac{1}{\frac{1}{2}(t+2)^{-\frac{1}{2}}} = \lim_{t\to 2^-} 2\sqrt{t+2} = 4$$

Since each limit exists, we know $\lim\limits_{t\to 2^-} \left\langle \frac{t^2-4}{t-2}, \tan(\pi t), \frac{t-2}{\sqrt{t+2}-2} \right\rangle = \langle 4, 0, 4 \rangle$

(b) We calculate each limit separately:

$$\lim_{t\to 0}\frac{\sin(t)}{t}\overset{L'H}{=}\lim_{t\to 2^-}\frac{\cos(t)}{1}=\cos(0)=1$$

$$\lim_{t\to 0}\frac{e^t-1}{\sin^{-1}(t)}\overset{L'H}{=}\lim_{t\to 0}\frac{e^t}{\frac{1}{\sqrt{1-t^2}}}=\frac{1}{1}=1$$

$$\lim_{t\to 0}\frac{\sqrt{t^2+1}-1}{t\sin(t)}\overset{L'H}{=}\lim_{t\to 0}\frac{t(t^2+1)^{-\frac{1}{2}}}{\sin(t)+t\cos(t)}\overset{L'H}{=}\lim_{t\to 0}\frac{(t^2+1)^{-\frac{1}{2}}-t^2(t^2+1)^{-\frac{3}{2}}}{2\cos(t)-t\sin(t)}=\frac{1}{2}$$

Since each limit exists, we know $\lim_{t\to 0}\left\langle\frac{\sin(t)}{t},\frac{e^t-1}{\sin^{-1}(t)},\frac{\sqrt{t^2+1}-1}{t\sin(t)}\right\rangle=\left\langle 1,1,\frac{1}{2}\right\rangle$

(c) We calculate each limit separately:

$$\lim_{t\to\infty}\frac{3t^2+4t-1}{8t^2-t+5}=\lim_{t\to\infty}\frac{3t^2}{8t^2}=\frac{3}{8}$$

$$\lim_{t\to\infty}\frac{\sqrt{t^2+4}}{2t+1}\lim_{t\to\infty}\frac{\sqrt{t^2}}{2t}=\lim_{t\to\infty}\frac{t}{2t}=\frac{1}{2}$$

$$\lim_{t\to\infty}\frac{t^3}{e^t-e^{-t}}\overset{L'H}{=}\lim_{t\to\infty}\frac{3t^2}{e^t+e^{-t}}\overset{L'H}{=}\lim_{t\to\infty}\frac{6t}{e^t-e^{-t}}\overset{L'H}{=}\lim_{t\to\infty}\frac{6}{e^t+e^{-t}}=0$$

Since each limit exists, we know $\lim_{t\to\infty}\left\langle\frac{3t^2+4t-1}{8t^2-t+5},\frac{\sqrt{t^2+4}}{2t+1},\frac{t^3}{e^t-e^{-t}}\right\rangle=\left\langle\frac{3}{8},\frac{1}{2},0\right\rangle$

(d) We calculate each limit separately:

$$\lim_{t\to-\infty}\frac{1}{t^2+1}=\lim_{t\to-\infty}\frac{1}{t^2}=0$$

$$\lim_{t\to-\infty}\frac{\sqrt{4t^2+1}}{t+8},\lim_{t\to-\infty}\frac{\sqrt{4t^2}}{t}=\lim_{t\to-\infty}\frac{-2t}{t}=-2$$

$$\lim_{t\to-\infty}t\cos(t)=\text{DNE}$$

Since one of the limits does not exist, the entire space curve limit **does not exist.**

? Exercise 1.6.4

Let $\vec{\mathbf{r}}(t)$ be a space curve with $\vec{\mathbf{r}}(1)=\langle 2,1,-1\rangle$ and $\vec{\mathbf{r}}\,'(1)=\langle 1,1,-5\rangle$

(a) For $f(t)=\left\langle t^2,3t+1,\sqrt{t}\right\rangle\cdot\vec{\mathbf{r}}(t)$, what is $f'(1)$

(b) For $\vec{\mathbf{g}}(t)=\frac{2}{t^3}\vec{\mathbf{r}}(t^2)$ what is $\vec{\mathbf{g}}\,'(1)$

(c) For $h(t)=\|\vec{\mathbf{r}}(t)\|$ what is $h'(1)$

Answer

(a) Using the product rule for dot products:

$$\begin{aligned}f'(t)&=\frac{d}{dt}\left(\left\langle t^2,3t+1,\sqrt{t}\right\rangle\right)\cdot\vec{\mathbf{r}}(t)+\left\langle t^2,3t+1,\sqrt{t}\right\rangle\cdot\frac{d}{dt}\left(\vec{\mathbf{r}}(t)\right)\\&=\left\langle 2t,3,\frac{1}{2}t^{-\frac{1}{2}}\right\rangle\cdot\vec{\mathbf{r}}(t)+\left\langle t^2,3t+1,\sqrt{t}\right\rangle\cdot\vec{\mathbf{r}}\,'(t)\end{aligned}$$

So plugging in $t=1$ gives:

$$f'(1)=\left\langle 2,3,\frac{1}{2}\right\rangle\cdot\langle 2,1,-1\rangle+\langle 1,4,1\rangle\cdot\langle 1,1,-5\rangle=\frac{13}{2}$$

(b) Using the product rule for scalar products:

$$\begin{aligned}\vec{\mathbf{g}}\,'(t) &= \frac{d}{dt}\left(2t^{-3}\right)\vec{\mathbf{r}}(t^2) + 2t^{-3}\frac{d}{dt}\left(\vec{\mathbf{r}}(t^2)\right)\\ &= -6t^{-4}\cdot\vec{\mathbf{r}}(t^2) + 2t^{-3}\cdot\vec{\mathbf{r}}\,'(t^2)\cdot 2t\end{aligned}$$

So plugging in $t=1$ gives:

$$\vec{\mathbf{g}}\,'(1) = -6\cdot\langle 2,1,-1\rangle + 2\cdot\langle 1,1,-5\rangle\cdot 2 = \langle -12,-6,6\rangle + \langle 4,4,-20\rangle = \langle -8,-2,-14\rangle$$

(c) Since $h(t) = \|\vec{\mathbf{r}}(t)\| = \sqrt{\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)}$, we use the product rule for dot products again:

$$\begin{aligned}h'(t) &= \frac{1}{2}\left(\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)\right)^{-\frac{1}{2}}\cdot\left(\frac{d}{dt}\left(\vec{\mathbf{r}}(t)\right)\cdot\vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t)\cdot\frac{d}{dt}\left(\vec{\mathbf{r}}(t)\right)\right)\\ &= \frac{1}{2\|\vec{\mathbf{r}}(t)\|}\left(\vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)\right)\\ &= \frac{\vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{r}}(t)}{\sqrt{\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)}}\end{aligned}$$

So plugging in $t=1$ gives:

$$h'(1) = \frac{\langle 1,1,-5\rangle\cdot\langle 2,1,-1\rangle}{\sqrt{\langle 2,1,-1\rangle\cdot\langle 2,1,-1\rangle}} = \frac{8}{\sqrt{6}}$$

? Exercise 1.6.5

Evaluate each of the following integrals.

(a) $\displaystyle\int_0^1 \left\langle t^2, e^t, \frac{8t}{1+t^4}\right\rangle dt$

(b) $\displaystyle\int_1^2 \left\langle \frac{10t^4}{(t^5+1)^2}, \frac{20}{t^2-25}, te^{t-1}\right\rangle dt$

(c) $\displaystyle\int_0^{\sqrt{3}} \left\langle \frac{3}{1+t^2}, 4\sqrt{3-t^2}, 6\tan^{-1}(t)\right\rangle dt$

Answer

(a) We calculate each integral separately:

$$\int_0^1 t^2\,dt = \frac{1}{3}t^3\Big|_0^1 = \frac{1}{3}$$

$$\int_0^1 e^t\,dt = e^t\Big|_0^1 = e-1$$

$$\int_0^1 \frac{8t}{1+t^4}\,dt = \int_0^1 \frac{4}{1+w^2}\,dw = 4\tan^{-1}(w)\Big|_0^1 = \pi$$

Since each integral exists, we know $\displaystyle\int_0^1 \left\langle t^2, e^t, \frac{8t}{1+t^4}\right\rangle dt = \left\langle \frac{1}{3}, e-1, \pi\right\rangle$

(b) We calculate each integral separately:

$$\int_1^2 \frac{10t^4}{(t^5+1)^2}\,dt = \int_2^{33} \frac{2}{w^2}\,dw = -2w^{-1}\Big|_2^{33} = \frac{31}{33}$$

$$\int_1^2 \frac{20}{t^2-25}\,dt = \int_1^2 \left(\frac{2}{t-5} - \frac{2}{t+5}\right)dt = \left[2\ln|t-5| - 2\ln|t+5|\right]\Big|_1^2 = 2\ln\left(\frac{9}{14}\right)$$

$$\int_1^2 te^{t-1}\,dt = (t-1)e^{t-1}\Big|_1^2 = e$$

Since each integral exists, we know $\displaystyle\int_1^2 \left\langle \frac{10t^4}{(t^5+1)^2}, \frac{20}{t^2-25}, te^{t-1}\right\rangle dt = \left\langle \frac{31}{33}, 2\ln\left(\frac{9}{14}\right), e\right\rangle$

(c) We calculate each integral separately:

$$\int_0^{\sqrt{3}} \frac{3}{1+t^2}\,dt = 3\tan^{-1}(t)\Big|_0^{\sqrt{3}} = \pi$$

$$\int_0^{\sqrt{3}} 4\sqrt{3-t^2}\,dt = \int_0^{\frac{\pi}{2}} 4\sqrt{3-\left(\sqrt{3}\sin(w)\right)^2}\cdot\left(\sqrt{3}\cos(w)\right)\,dw = \int_0^{\frac{\pi}{2}} 12\sin^2(w)\,dw = \left[6w-3\sin(2w)\right]\Big|_0^{\frac{\pi}{2}} = 3\pi$$

$$\int_0^{\sqrt{3}} 6\tan^{-1}(t)\,dt = \left[6t\tan^{-1}(t)-3\ln\left|1+t^2\right|\right]\Big|_0^{\sqrt{3}} = 2\pi\sqrt{3}-3\ln(4)$$

Since each integral exists, we know $\int_0^{\sqrt{3}} \left\langle \frac{3}{1+t^2}, 4\sqrt{3-t^2}, 6\tan^{-1}(t)\right\rangle\,dt = \left\langle \pi, 3\pi, 2\pi\sqrt{3}-3\ln(4)\right\rangle$

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1.7: Applications of Space Curves

Learning Objectives

- Determine the length of a particle's path in space by using the arc-length function.
- Explain the meaning of the curvature of a curve in space and state its formula.
- Describe the velocity and acceleration vectors of a particle moving in space.
- State Kepler's laws of planetary motion.

In this section, we explore a few applications for space curves. We discuss arc length and re-parametrization of a curve in terms of its length (which is useful for proofs in later chapters). We also define curvature for a space curve and its applications and calculations in different situations. Space curves are seen to be the natural extension of the position/velocity/acceleration relationship in single variable calculus, and we use this to prove some facts about planetary motion discovered by Kepler in the early 15th century.

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1.7: Arc Length and Curvature

In this section, we study formulas related to curves in both two and three dimensions, and see how they are related to various properties of the same curve. For example, suppose a vector-valued function describes the motion of a particle in space. We would like to determine how far the particle has traveled over a given time interval, which can be described by the arc length of the path it follows. Or, suppose that the vector-valued function describes a road we are building and we want to determine how sharply the road curves at a given point. This is described by the curvature of the function at that point. We explore each of these concepts in this section.

We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t), y = y(t), a \leq t \leq b$ is given by

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$$

In a similar fashion, if we define a smooth curve using a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt$$

In three dimensions, if the vector-valued function is described by $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ over the same interval $a \leq t \leq b$, the arc length is given by

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt$$

Rule: Arc-Length Formulas for Plane and Space curves

Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t)$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \|\vec{\mathbf{r}}'(t)\|\,dt$$

Note that the formula is defined for smooth curves: curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

Example 1.7.1: Finding the Arc Length

Calculate the arc length for each of the following vector-valued functions:

(a) $\vec{\mathbf{r}}(t) = (3t - 2)\,\hat{\mathbf{i}} + (4t + 5)\,\hat{\mathbf{j}}, \quad 1 \leq t \leq 5$

(b) $\vec{\mathbf{r}}(t) = \langle \cos(3t), \sin(3t), 4t \rangle, \quad 0 \leq t \leq 2\pi$

Solution

(a) We see that $\vec{\mathbf{r}}\,'(t) = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$, so

$$s = \int_a^b \|\vec{\mathbf{r}}'(t)\|dt = \int_1^5 \sqrt{3^2 + 4^2}\,dt = \int_1^5 5dt = 5t\Big|_1^5 = \boxed{20}$$

(b) We see that $\vec{\mathbf{r}}\,'(t) = \langle -3\sin(3t), 3\cos(3t), 4 \rangle$, so

$$s = \int_a^b \| \vec{\mathbf{r}}'(t) \| \, dt = \int_0^{2\pi} \sqrt{(-3\sin(3t))^2 + (3\cos(3t))^2 + 4^2} dt$$
$$= \int_0^{2\pi} \sqrt{9(\sin^2(3t) + \cos^2(3t)) + 16} dt$$
$$= \int_0^{2\pi} \sqrt{25} dt = \int_0^{2\pi} 5 dt = 5t\Big|_0^{2\pi} = \boxed{10\pi}$$

We now have a formula for the arc length of a curve defined by a vector-valued function. Let's take this one step further and examine what an **arc-length function** is.

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$s = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2} \, du$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable u is to distinguish between time and the variable of integration. Since $s(t)$ measures distance traveled as a function of time, $s'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$, we can differentiate both sides of the equation:

$$s'(t) = \frac{d}{dt}\left[\int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2} du\right] = \frac{d}{dt}\left[\int_a^t \|\vec{\mathbf{r}}'(u)\| du\right] = \|\vec{\mathbf{r}}'(t)\|$$

If we assume that $\vec{\mathbf{r}}(t)$ defines a smooth curve, then the arc length is always increasing, so $s'(t) > 0$ for $t > a$. Last, if $\vec{\mathbf{r}}(t)$ is a curve on which $\|\vec{\mathbf{r}}'(t)\| = 1$ for all t, then

$$s(t) = \int_a^t \|\vec{\mathbf{r}}'(u)\| \, du = \int_a^t 1 \, du = t - a$$

which means that t represents the arc length starting from $t = a$.

Definition: Arc-Length Function

Let $\vec{\mathbf{r}}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$s(t) = \int_a^t \|\vec{\mathbf{r}}'(u)\| \, du$$

Furthermore,

$$\frac{ds}{dt} = \|\vec{\mathbf{r}}'(t)\| > 0$$

If $\|\vec{\mathbf{r}}'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

A useful application of this theorem is to find an alternative parameterization of a given curve, called an **arc-length parameterization**. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function $\vec{\mathbf{r}}(t) = \langle 3\cos(t), 3\sin(t)\rangle, 0 \leq t \leq 2\pi$ that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization $\vec{\mathbf{r}}(t) = \langle 3\cos(4t), 3\sin(4t)\rangle$. The new parameterization still defines a circle of radius 3, but now we only need to use the values $0 \leq t \leq \frac{\pi}{2}$ to traverse the circle once.

Suppose that we find the arc-length function $s(t)$ and are able to solve this function for t as a function of s. We can then reparameterize the original function $\vec{\mathbf{r}}(t)$ by substituting the expression for t back into $\vec{\mathbf{r}}(t)$. The vector-valued function is now written in terms of the parameter s. Since the variable s represents the arc length, we call this an *arc-length parameterization* of the original function $\vec{\mathbf{r}}(t)$. One advantage of finding the arc-length parameterization is that the distance traveled along the curve

starting from $s = 0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Illustration 1.7.1: Finding an Arc-Length Parameterization

Find the arc-length parameterization for each of the following curves:

(a) $\vec{\mathbf{r}}(t) = 4\cos(t)\,\hat{\mathbf{i}} + 4\sin(t)\,\hat{\mathbf{j}}, \quad t \geq 0$

(b) $\vec{\mathbf{r}}(t) = \langle t+3, 2t-4, 2t\rangle, \quad t \geq 3$

(c) $\vec{\mathbf{r}}(t) = \langle 2\cos(2t), 2\sin(2t), 3t\rangle, \quad t \geq 0$

(a) First we find the arc-length function:

$$\begin{aligned} s(t) &= \int_0^t \|\vec{\mathbf{r}}'(u)\|\,du = \int_0^t \sqrt{(-4\sin(u))^2 + (4\cos(u))^2}\,du \\ &= \int_0^t \sqrt{16\sin^2(u) + 16\cos^2(u)}\,du = \int_0^t 4\,du = 4t \end{aligned}$$

which gives the relationship between the arc length s and the parameter t as $s = 4t$; so, $t = \dfrac{s}{4}$.
Next we replace the variable t in the original function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ with the expression $\dfrac{s}{4}$ to obtain

$$\vec{\mathbf{r}}(s) = 4\cos\left(\frac{s}{4}\right)\hat{\mathbf{i}} + 4\sin\left(\frac{s}{4}\right)\hat{\mathbf{j}}$$

This is the arc-length parameterization of $\vec{\mathbf{r}}(t)$. Since the original restriction on t was given by $t \geq 0$, the restriction on s becomes $\dfrac{s}{4} \geq 0$, or $s \geq 0$.

(b) The arc-length function is given by:

$$s(t) = \int_3^t \|\vec{\mathbf{r}}'(u)\|\,du = \int_3^t \sqrt{1^2 + 2^2 + 2^2}\,du = \int_3^t 3\,du = 3t - 9$$

Therefore, the relationship between the arc length s and the parameter t is $s = 3t - 9$, so $t = \dfrac{s}{3} + 3$.
Substituting this into the original function $\vec{\mathbf{r}}(t) = \langle t+3, 2t-4, 2t\rangle$ yields

$$\vec{\mathbf{r}}(s) = \left\langle \left(\frac{s}{3}+3\right)+3,\ 2\left(\frac{s}{3}+3\right)-4,\ 2\left(\frac{s}{3}+3\right)\right\rangle = \left\langle \frac{s}{3}+6, \frac{2s}{3}+2, \frac{2s}{3}+6\right\rangle$$

This is an arc-length parameterization of $\vec{\mathbf{r}}(t)$. The original restriction on the parameter t was $t \geq 3$, so the restriction on s is $\dfrac{s}{3} + 3 \geq 3$, or $s \geq 0$.

(c) The arc-length function is given by:

$$s(t) = \int_0^t \|\vec{\mathbf{r}}'(u)\|\,du = \int_0^t \sqrt{(-4\sin(2t))^2 + (4\cos(2t))^2 + 3^2}\,du = \int_0^t 5\,du = 5t$$

Therefore, the relationship between the arc length s and the parameter t is $s = 5t$, so $t = \dfrac{s}{5}$.
Substituting this into $\vec{\mathbf{r}}(t) = \langle 2\cos(2t), 2\sin(2t), 3t\rangle$ gives

$$\vec{\mathbf{r}}(s) = \left\langle 2\cos\left(\frac{2s}{5}\right), 2\sin\left(\frac{2s}{5}\right), \frac{3s}{5}\right\rangle, \quad s \geq 0$$

An important topic related to arc length is curvature. The concept of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the circle, the greater the curvature.

Think of driving down a road. Suppose the road lies on an arc of a large circle. In this case you would barely have to turn the wheel to stay on the road. Now suppose the radius is smaller. In this case you would need to turn more sharply to stay on the road. In the case of a curve other than a circle, it is often useful first to inscribe a circle to the curve at a given point so that it is tangent to the curve at that point and "hugs" the curve as closely as possible in a neighborhood of the point. The curvature of the graph at that point is then defined to be the same as the curvature of the inscribed circle.

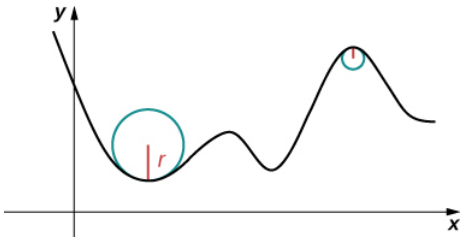


Figure 1.7.1: The graph represents the curvature of a function $y = f(x)$.
The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Definition: Curvature

Let C be a smooth curve in the plane or in space given by $\vec{\mathbf{r}}(s)$, where s is the arc-length parameter. The curvature κ at s is

$$\kappa = \left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \|\vec{\mathbf{T}}'(s)\|$$

The formula in the definition of curvature is not very useful in terms of calculation. In particular, recall that $\vec{\mathbf{T}}(t)$ represents the unit tangent vector to a given vector-valued function $\vec{\mathbf{r}}(t)$, and the formula for $\vec{\mathbf{T}}(t)$ is

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\| \vec{\mathbf{r}}'(t) \|}$$

To use the formula for curvature, it is first necessary to express $\vec{\mathbf{r}}(t)$ in terms of the arc-length parameter s, then find the unit tangent vector $\vec{\mathbf{T}}(s)$ for the function $\vec{\mathbf{r}}(s)$, then take the derivative of $\vec{\mathbf{T}}(s)$ with respect to s. This is a tedious process. Fortunately, there are equivalent formulas for curvature.

Rule: Alternate Formulas of Curvature

If C is a smooth curve given by $\vec{\mathbf{r}}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}$$

If C is a three-dimensional curve, then the curvature can be simplified to the formula

$$\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}$$

If C is the two dimensional graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}}$$

Proof

The first formula follows directly from the chain rule:

$$\frac{d\vec{\mathbf{T}}}{dt} = \frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt}$$

where s is the arc length along the curve C. Dividing both sides by $\frac{ds}{dt}$, and taking the magnitude of both sides gives

$$\left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \left\| \frac{\vec{\mathbf{T}}'(t)}{\frac{ds}{dt}} \right\|$$

Since $\frac{ds}{dt} = \|\vec{\mathbf{r}}'(t)\|$, this gives the formula for the curvature κ of a curve C in terms of any parameterization of C:

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}$$

In the case of a three-dimensional curve, we start with the formulas $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$ and $\dfrac{ds}{dt} = \|\vec{\mathbf{r}}\,'(t)\|$. Therefore, $\vec{\mathbf{r}}\,'(t) = \dfrac{ds}{dt}\vec{\mathbf{T}}(t)$. We can take the derivative of this function using the product rule:

$$\vec{\mathbf{r}}''(t) = \frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}'(t)$$

Using these last two equations we get

$$\begin{aligned}\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) &= \frac{ds}{dt}\vec{\mathbf{T}}(t) \times \left(\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}'(t)\right) \\ &= \frac{ds}{dt}\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) + (\frac{ds}{dt})^2\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}'(t)\end{aligned}$$

Since $\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) = 0$, this reduces to

$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \left(\frac{ds}{dt}\right)^2 \vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}'(t)$$

Since $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}'$ are always orthogonal, this means that $\|\vec{\mathbf{T}} \times \vec{\mathbf{T}}'\| = \|\vec{\mathbf{T}}\|\|\vec{\mathbf{T}}'\|\sin(\pi/2) = 1 \cdot \|\vec{\mathbf{T}}'\| \cdot 1 = \|\vec{\mathbf{T}}'\|$, so

$$\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\vec{\mathbf{T}}'(t)\|$$

Now we solve this equation for $\|\vec{\mathbf{T}}'(t)\|$ and use the fact that $\dfrac{ds}{dt} = \|\vec{\mathbf{r}}'(t)\|$:

$$\|\vec{\mathbf{T}}'(t)\| = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^2}$$

Then, we divide both sides by $\|\vec{\mathbf{r}}'(t)\|$, which finishes our proof:

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}$$

In the case of the planar curve $y = f(x)$, we can define $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + f(t)\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Using the previous formula for curvature:

$$\begin{aligned}\vec{\mathbf{r}}'(t) &= \hat{\mathbf{i}} + f'(t)\,\hat{\mathbf{j}} \\ \vec{\mathbf{r}}''(t) &= f''(t)\,\hat{\mathbf{j}} \\ \vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = f''(t)\,\hat{\mathbf{k}}\end{aligned}$$

Therefore,

$$\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3} = \frac{|f''(t)|}{(1 + [f'(t)]^2)^{\frac{3}{2}}}$$

Example 1.7.2: Finding Curvature

Find the curvature for each of the following curves at the given point:

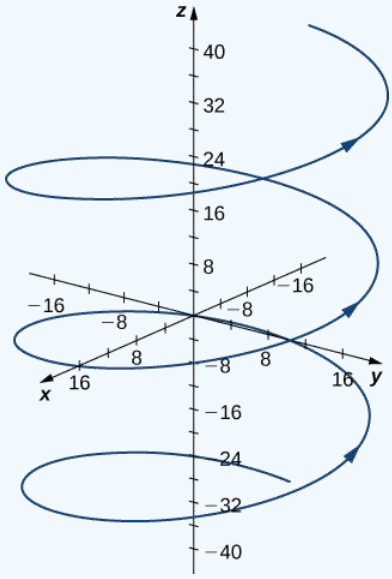
(a) $\vec{\mathbf{r}}(t) = 4\cos(t)\,\hat{\mathbf{i}} + 4\sin(t)\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$ at $t = \dfrac{4\pi}{3}$

(b) $\vec{\mathbf{r}}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$

(c) $f(x) = \sqrt{4x - x^2}$ at $x = 2$

Solution

(a) This function describes a helix.



The curvature of the helix at $t = \dfrac{4\pi}{3}$ can be found by using the equation $\kappa = \dfrac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}$. First, calculate $\vec{\mathbf{T}}(t)$:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\langle -4\sin(t), 4\cos(t), 3\rangle}{\sqrt{(-4\sin(t))^2 + (4\cos(t))^2 + 3^2}} = \left\langle -\frac{4}{5}\sin(t), \frac{4}{5}\cos(t), \frac{3}{5}\right\rangle$$

Next, calculate $\vec{\mathbf{T}}'(t)$:

$$\vec{\mathbf{T}}'(t) = \left\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\right\rangle$$

Therefore our equation becomes:

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\left\| \left\langle -\frac{4}{5}\cos(t), -\frac{4}{5}\sin(t), 0\right\rangle \right\|}{\|\langle -4\sin(t), 4\cos(t), 3\rangle\|}$$
$$= \frac{\sqrt{(-\frac{4}{5}\cos(t))^2 + (-\frac{4}{5}\sin(t))^2 + 0^2}}{\sqrt{(-4\sin(t))^2 + (4\cos(t))^2 + 3^2}} = \frac{\frac{4}{5}}{5} = \boxed{\frac{4}{25}}$$

Since this does not depend on t, we see the curvature of this helix is constant at all points.

(b) For variety, we will use the second equation $\kappa = \dfrac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}$:

$$\vec{\mathbf{r}}'(t) = \langle 1, 2t, 3t^2\rangle \text{ and } \vec{\mathbf{r}}''(t) = \langle 0, 2, 6t\rangle$$

We find the cross product:

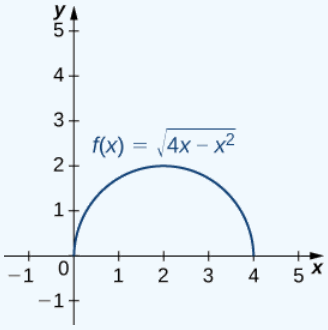
$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2\rangle$$

Therefore we have

$$\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3} = \frac{\sqrt{(6t^2) + (-6t)^2 + 2^2}}{\left(\sqrt{1^2 + (2t)^2 + (3t^2)^2}\right)^3} = \sqrt{\frac{36t^4 + 36t^2 + 4}{(1 + 4t^2 + 9t^4)^3}}$$

Plugging in $t = 1$ makes this simplify to $\kappa = \boxed{\sqrt{\frac{76}{14^3}} \approx .166}$ which is very similar to the curvature of the helix in part (a).

(c) This function describes a semicircle.



To find the curvature of this graph, we will use the third equation: $\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3} = \frac{|f''(t)|}{(1 + [f'(t)]^2)^{\frac{3}{2}}}$.

First, we calculate y' and y'':

$$\begin{aligned} y &= \sqrt{4x - x^2} = (4x - x^2)^{\frac{1}{2}} \\ y' &= \frac{1}{2}(4x - x^2)^{-\frac{1}{2}}(4 - 2x) = (2 - x)(4x - x^2)^{-\frac{1}{2}} \\ y'' &= -(4x - x^2)^{-\frac{1}{2}} + (2 - x)(-\frac{1}{2})(4x - x^2)^{-\frac{3}{2}}(4 - 2x) \\ &= -\frac{4x - x^2}{(4x - x^2)^{\frac{3}{2}}} - \frac{(2 - x)^2}{(4x - x^2)^{\frac{3}{2}}} \\ &= \frac{x^2 - 4x - (4 - 4x + x^2)}{(4x - x^2)^{\frac{3}{2}}} = -\frac{4}{(4x - x^2)^{\frac{3}{2}}} \end{aligned}$$

Plugging these in gives:

$$\begin{aligned} \kappa &= \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} = \frac{\left|-\frac{4}{(4x - x^2)^{\frac{3}{2}}}\right|}{\left[1 + ((2 - x)(4x - x^2)^{-\frac{1}{2}})^2\right]^{\frac{3}{2}}} = \frac{\left|\frac{4}{(4x - x^2)^{\frac{3}{2}}}\right|}{\left[1 + \frac{(2 - x)^2}{4x - x^2}\right]^{\frac{3}{2}}} \\ &= \frac{\left|\frac{4}{(4x - x^2)^{\frac{3}{2}}}\right|}{\left[\frac{4x - x^2 + x^2 - 4x + 4}{4x - x^2}\right]^{\frac{3}{2}}} = \left|\frac{4}{(4x - x^2)^{\frac{3}{2}}}\right| \cdot \frac{(4x - x^2)^{\frac{3}{2}}}{8} = \boxed{\frac{1}{2}} \end{aligned}$$

The curvature of any circle is equal to the reciprocal of its radius.

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1.7: Motion in Space

We have now seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature. All of this leads to the main goal of this section, which is the description of motion along plane curves and space curves. We now have all the tools we need; we put these ideas together and look at how to use them.

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Definition: Speed, Velocity, and Acceleration

Let $\vec{\mathbf{r}}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The velocity vector $\vec{\mathbf{v}}(t)$ of the object is given by

$$\text{Velocity } = \vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t)$$

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined to be

$$\text{Acceleration } = \vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t)$$

The *speed* is defined to be

$$\text{Speed } = v(t) = \|\vec{\mathbf{v}}(t)\| = \|\vec{\mathbf{r}}\,'(t)\| = \frac{ds}{dt}$$

Since $\vec{\mathbf{r}}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ and in three dimensions $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Quantity	Two Dimensions	Three Dimensions
Position	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$
Velocity	$\vec{\mathbf{v}}(t) = x\,'(t)\hat{\mathbf{i}} + y\,'(t)\hat{\mathbf{j}}$	$\vec{\mathbf{v}}(t) = x\,'(t)\hat{\mathbf{i}} + y\,'(t)\hat{\mathbf{j}} + z\,'(t)\hat{\mathbf{k}}$
Acceleration	$\vec{\mathbf{a}}(t) = x\,''(t)\hat{\mathbf{i}} + y\,''(t)\hat{\mathbf{j}}$	$\vec{\mathbf{a}}(t) = x\,''(t)\hat{\mathbf{i}} + y\,''(t)\hat{\mathbf{j}} + z\,''(t)\hat{\mathbf{k}}$
Speed	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y\,'(t))^2}$	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y\,'(t))^2 + (z\,'(t))^2}$

Example 1.7.3: Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5 - t^2}\hat{\mathbf{j}}$, where t measures time in seconds.

(a) Find the velocity, acceleration, and speed as functions of time.

(b) Sketch the curve along with the velocity vector at time $t = 1$.

Solution

(a) These are direct calculations:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = 2t\hat{\mathbf{i}} - \frac{t}{\sqrt{5-t^2}}\hat{\mathbf{j}}$$

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = 2\hat{\mathbf{i}} - 5(5-t^2)^{-\frac{3}{2}}\hat{\mathbf{j}}$$

$$||\vec{\mathbf{v}}(t)|| = ||\vec{\mathbf{r}}\,'(t)|| = \sqrt{(2t)^2 + \left(-\frac{t}{\sqrt{5-t^2}}\right)^2} = \boxed{\sqrt{\frac{21t^2-4t^4}{5-t^2}}}$$

(b) The graph of $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5-t^2}\hat{\mathbf{j}}$ is a portion of a parabola.

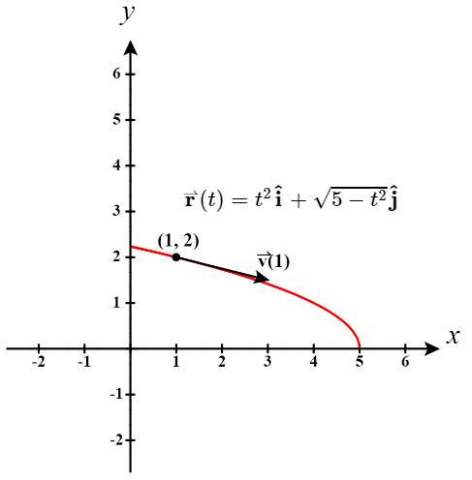
When $t = 1$, $\vec{\mathbf{r}}(1) = (1)^2\hat{\mathbf{i}} + \sqrt{5-(1)^2}\hat{\mathbf{j}} \quad = \quad \hat{\mathbf{i}} + \sqrt{4}\hat{\mathbf{j}} \quad = \quad \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

Thus the particle would be located at the point $(1, 2)$ when $\boxed{t = 1}$

The velocity vector at $t = 1$ is $\vec{\mathbf{v}}(1) = \vec{\mathbf{r}}\,'(1) = 2(1)\hat{\mathbf{i}} - \frac{1}{\sqrt{5-1^2}}\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}$

and the acceleration vector at $t = 1$ is $\vec{\mathbf{a}}(1) = \vec{\mathbf{v}}\,'(1) = 2\hat{\mathbf{i}} - 5(5-1^2)^{-\frac{3}{2}}\hat{\mathbf{j}} = \boxed{2\hat{\mathbf{i}} - \frac{5}{8}\hat{\mathbf{j}}}$

Notice that the velocity vector is tangent to the path, as is always the case.



Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as projectile motion. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive y-axis to be up, the negative y-axis to be down, and the positive x-axis to be forward (i.e., away from the thrower of the object). The effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from gravity, or $\vec{\mathbf{F}}_g = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}_g$ represents the force from gravity and $\vec{\mathbf{a}} = -g\,\hat{\mathbf{j}}$ represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $\vec{\mathbf{F}}_g = -mg\,\hat{\mathbf{j}}$, as shown below:

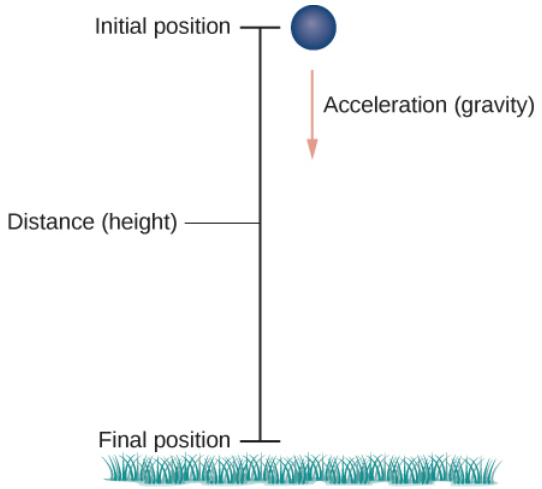


Figure 1.7.2: An object is falling under the influence of gravity.

Newton's second law also tells us that $F = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_g \longleftrightarrow m\vec{\mathbf{a}} = -mg\hat{\mathbf{j}} \longleftrightarrow \vec{\mathbf{a}} = -g\hat{\mathbf{j}}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

$$\vec{\mathbf{v}}\,'(t) = -g\hat{\mathbf{j}}$$

By taking the antiderivative of each side of this equation we obtain

$$\vec{\mathbf{v}}(t) = \int -g\hat{\mathbf{j}}\, dt = -gt\hat{\mathbf{j}} + \vec{\mathbf{C}}_1$$

for some constant vector $\vec{\mathbf{C}}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$. Therefore, $\vec{\mathbf{v}}(0) = -g(0)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$ and $\vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$. This gives the velocity vector as $\vec{\mathbf{v}}(t) = -gt\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0$.

Next we use the fact that velocity $\vec{\mathbf{v}}(t)$ is the derivative of position $\vec{\mathbf{s}}(t)$. This gives the equation

$$\vec{\mathbf{s}}\,'(t) = -gt\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0.$$

Taking the antiderivative of both sides of this equation leads to

$$\vec{\mathbf{s}}(t) = \int -gt\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0\, dt = -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{C}}_2$$

with another unknown constant vector $\vec{\mathbf{C}}_2$. To determine the value of $\vec{\mathbf{C}}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\vec{\mathbf{s}}(0) = \vec{\mathbf{s}}_0$. Therefore, $\vec{\mathbf{s}}(0) = -\frac{1}{2}g(0)^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0(0) + \vec{\mathbf{C}}_2 = \vec{\mathbf{s}}_0$. Clearly $\vec{\mathbf{C}}_2 = \vec{\mathbf{s}}_0$, which gives the position of the object at any time as

$$\vec{\mathbf{s}}(t) = -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{s}}_0$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed $\vec{\mathbf{v}}_0$. How can we modify the previous result to reflect this scenario? First, we can assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\vec{\mathbf{s}}_0 = \vec{\mathbf{0}}$, as shown below.

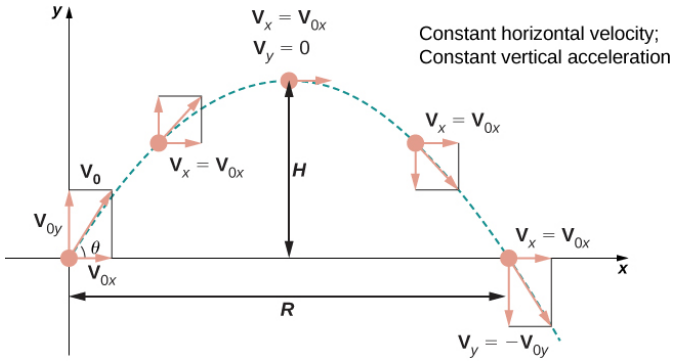


Figure 1.7.3: Projectile motion when the object is thrown upward at an angle θ.
The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

We can rewrite the initial velocity vector in the form $\vec{\mathbf{v}}_0 = v_0\cos(\theta)\,\hat{\mathbf{i}} + v_0\sin(\theta)\,\hat{\mathbf{j}}$. Then the equation for the position function $\vec{\mathbf{s}}(t)$ becomes

$$\vec{\mathbf{s}}(t) = -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + v_0t\cos(\theta)\,\hat{\mathbf{i}} + v_0t\sin(\theta)\,\hat{\mathbf{j}} = v_0t\cos(\theta)\,\hat{\mathbf{i}} + \left(v_0t\sin(\theta) - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}$$

The coefficient of $\hat{\mathbf{i}}$ represents the horizontal component of $\vec{\mathbf{s}}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\hat{\mathbf{j}}$ represents the vertical component of $\vec{\mathbf{s}}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

Illustration 1.7.2: Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of $\sin^{-1}\left(\frac{1}{4}\right)$ above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water.

(a) Find the maximum height of the cannonball.

(b) How long will it take for the cannonball to splash into the sea?

(c) How far out to sea will the cannonball hit the water?

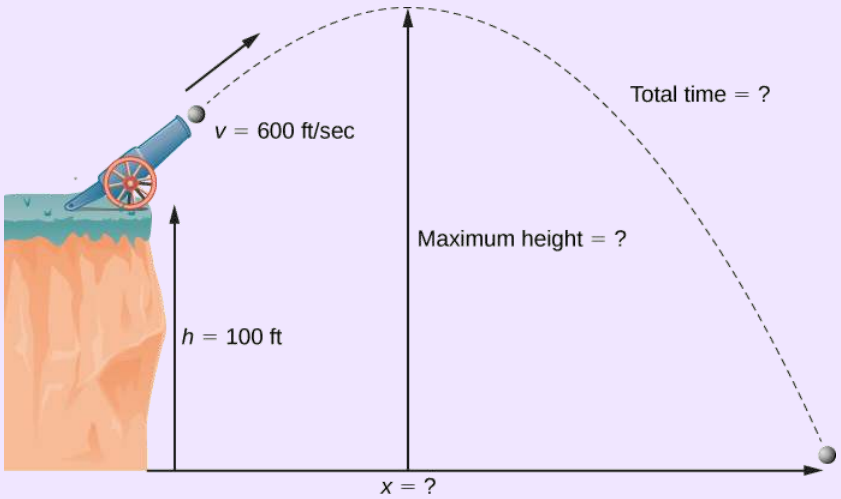


Figure 1.7.4: The flight of a cannonball (ignoring air resistance) is projectile motion.

We use the equation $\vec{\mathbf{s}}(t) = v_0t\cos(\theta)\,\hat{\mathbf{i}} + \left(v_0t\sin(\theta) - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}$ with $\theta = \sin^{-1}\left(\frac{1}{4}\right)$, $g = 32\,\frac{\text{ft}}{\text{sec}^2}$, and $v_0 = 600\,\frac{\text{ft}}{\text{sec}}$.

This position equation becomes

$$\vec{\mathbf{s}}(t) = 600t(\cos\left(\sin^{-1}\left(\frac{1}{4}\right)\right)\,\hat{\mathbf{i}} + \left(600t\sin\left(\sin^{-1}\left(\frac{1}{4}\right)\right) - \frac{1}{2}(32)t^2\right)\,\hat{\mathbf{j}}$$
$$= 150t\sqrt{15}\,\hat{\mathbf{i}} + \left(150t - 16t^2\right)\,\hat{\mathbf{j}}$$

(a) The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{s}}\,'(t) = 150\sqrt{15}\,\hat{\mathbf{i}} + (150 - 32t)\,\hat{\mathbf{j}}$$

Therefore, the vertical component of velocity is given by the expression $150 - 32t$. Setting this expression equal to zero and solving for t gives $t = \frac{75}{16}$ sec. The height of the cannonball at this time is given by the vertical component of the position vector:

$$H = 150\left(\frac{75}{16}\right) - 16\left(\frac{75}{16}\right)^2 = \frac{5625}{16} = 351.5625$$

Therefore, the maximum height of the cannonball is 351.5625 ft above the cannon, or 451.5625 ft above sea level.

(b) When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to -100. Setting the vertical component of $\vec{\mathbf{s}}(t)$ equal to -100 and solving, we obtain

$$\begin{aligned} 150t - 16t^2 &= -100 \\ 16t^2 - 150t - 100 &= 0 \\ 2(t-10)(8t+5) &= 0 \\ t &= 10, -\frac{5}{8} \end{aligned}$$

The positive value of t that solves this equation is 10. Therefore, the cannonball hits the water after 10 sec.

(c) To find the distance out to sea, we simply substitute the answer from part (b) into the horizontal component of $\vec{\mathbf{s}}(t)$:

$$D = 150(10)\sqrt{15} = 1500\sqrt{15} \approx 5809.475$$

Therefore, the ball hits the water about 5809.475 ft away from the base of the cliff.

Note that 5809.475 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

$$\vec{\mathbf{s}} = v_0 t\cos(\theta)\,\hat{\mathbf{i}} + \left(v_0 t\sin(\theta) - \frac{1}{2}gt^2\right)\,\hat{\mathbf{j}}$$

Setting the second component equal to zero and solving for t yields

$$\begin{aligned} v_0 t\sin(\theta) - \frac{1}{2}gt^2 &= 0 \\ t\left(v_0\sin(\theta) - \frac{1}{2}gt\right) &= 0 \end{aligned}$$

Therefore, either $t = 0$ or $t = \frac{2v_0\sin\theta}{g}$. We are interested in the second value of t, so we substitute this into $\vec{\mathbf{s}}(t)$, which gives

$$\vec{\mathbf{s}}\left(\frac{2v_0\sin(\theta)}{g}\right)$$
$$= v_0\left(\frac{2v_0\sin(\theta)}{g}\right)\cos\theta\,\hat{\mathbf{i}} + \left(v_0\left(\frac{2v_0\sin(\theta)}{g}\right)\sin(\theta) - \frac{1}{2}g\left(\frac{2v_0\sin(\theta)}{g}\right)^2\right)\,\hat{\mathbf{j}} = \frac{v_0^2\sin(2\theta)}{g}\,\hat{\mathbf{i}}$$

Thus, the expression for the range of a projectile fired at an angle θ is

$$R = \frac{v_0^2 \sin(2\theta)}{g}\,\hat{\mathbf{i}}$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of $\hat{\mathbf{i}}$ with respect to θ and set it equal to zero:

$$\frac{d}{d\theta}\left(\frac{v_0^2 \sin(2\theta)}{g}\right) = 0 \longleftrightarrow \frac{2v_0^2 \cos(2\theta)}{g} = 0 \longleftrightarrow \theta = 45^\circ$$

This value of θ is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by

$$\vec{\mathbf{s}}\left(\frac{2v_0 \sin(45^\circ)}{g}\right) = \frac{v_0^2 \sin(90^\circ)}{g}\,\hat{\mathbf{i}} = \frac{v_0^2}{g}\,\hat{\mathbf{i}}$$

Therefore, the range for an angle of 45° is $\frac{v_0^2}{g}$ units.

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as Kepler's laws of planetary motion. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Explore: Kepler's Laws of Planetary Motion

1. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
2. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (Figure 1.7.5).
3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the Law of Harmonies).

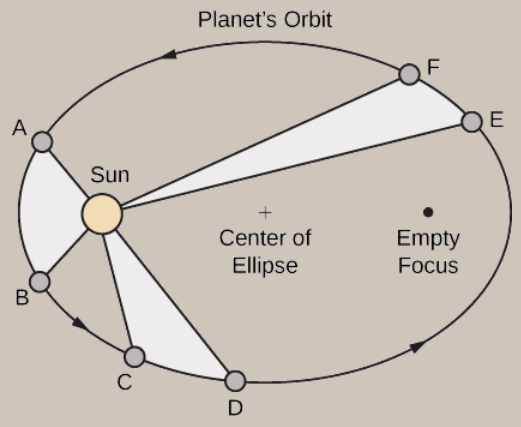


Figure 1.7.5: Kepler's first and second laws are pictured here. The Sun is located at a focus of the elliptical orbit of any planet. Furthermore, the shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

$$T_p^2 = D_p^3$$

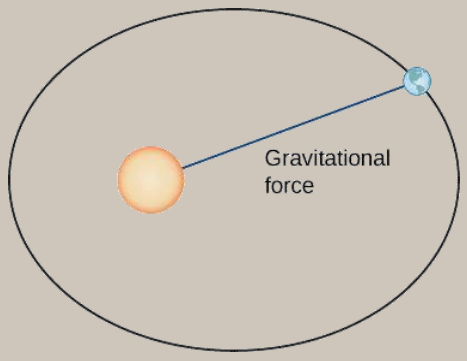
for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in

astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof of Kepler's First Law

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\vec{\mathbf{r}}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}$ represents the net force acting on the planet. His law of universal gravitation can be written in the form $\vec{\mathbf{F}} = -\dfrac{GmM}{||\vec{\mathbf{r}}||^2} \cdot \dfrac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\dfrac{GmM}{||\vec{\mathbf{r}}||^2}$.



Setting these two forces equal to each other, and using the fact that $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t)$, we obtain

$$m\vec{\mathbf{v}}\,'(t) = -\frac{GmM}{\|\vec{\mathbf{r}}\|^2} \cdot \frac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|}$$

which can be rewritten as

$$\frac{d\vec{\mathbf{v}}}{dt} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}}$$

This equation shows that the vectors $\dfrac{d\vec{\mathbf{v}}}{dt}$ and $\vec{\mathbf{r}}$ are parallel to each other, so $\dfrac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$. Next, let's differentiate $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{v}} \times \vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{0}}$$

This proves that $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ is a constant vector, which we call $\vec{\mathbf{C}}$. Since $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are both perpendicular to $\vec{\mathbf{C}}$ for all values of t, they must lie in a plane perpendicular to $\vec{\mathbf{C}}$. Therefore, the motion of the planet lies in a plane.

Next we calculate the expression $\dfrac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}}$:

$$\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \vec{\mathbf{r}})\vec{\mathbf{v}}]$$

The last equality above is from the triple cross product formula. We need an expression for $\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}$. To calculate this, we differentiate $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}} + \vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}$$

Since $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = ||\vec{\mathbf{r}}||^2$, we also have

$$\frac{d}{dt}(\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}) = \frac{d}{dt}||\vec{\mathbf{r}}||^2 = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||$$

Combining these two equations, we get

$$2\vec{\mathbf{r}}\cdot\vec{\mathbf{v}} = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||$$
$$\vec{\mathbf{r}}\cdot\vec{\mathbf{v}} = ||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||$$

Substituting this into the equation for $\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}}$ gives us:

$$\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}} = -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}}\cdot\vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}}\cdot\vec{\mathbf{r}})\vec{\mathbf{v}}]$$

$$= -\frac{GM}{||\vec{\mathbf{r}}||^3}\left[||\vec{\mathbf{r}}||\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - ||\vec{\mathbf{r}}||^2\vec{\mathbf{v}}\right]$$

$$= -GM\left[\frac{1}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - \frac{1}{||\vec{\mathbf{r}}||}\vec{\mathbf{v}}\right]$$

$$= GM\left[\frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\right]$$

However,

$$\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} = \frac{\frac{d}{dt}(\vec{\mathbf{r}})||\vec{\mathbf{r}}|| - \vec{\mathbf{r}}\frac{d}{dt}||\vec{\mathbf{r}}||}{||\vec{\mathbf{r}}||^2}$$
$$= \frac{\frac{d\vec{\mathbf{r}}}{dt}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}||$$
$$= \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}||$$

Which makes the $\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}}$ equation simplify to

$$\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}} = GM\left(\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}\right)$$

Since $\vec{\mathbf{C}}$ is a constant vector, we can integrate both sides and obtain

$$\vec{\mathbf{v}}\times\vec{\mathbf{C}} = GM\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} + \vec{\mathbf{D}}$$

where $\vec{\mathbf{D}}$ is a constant vector. Our goal is to solve for $||\vec{\mathbf{r}}||$. Let's start by calculating $\vec{\mathbf{r}}\cdot\left(\vec{\mathbf{v}}\times\vec{\mathbf{C}}\right)$:

$$\vec{\mathbf{r}}\cdot\left(\vec{\mathbf{v}}\times\vec{\mathbf{C}}\right) = GM\frac{||\vec{\mathbf{r}}||^2}{||\vec{\mathbf{r}}||} + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}}$$

However, $\vec{\mathbf{r}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{C}}) = (\vec{\mathbf{r}}\times\vec{\mathbf{v}})\cdot\vec{\mathbf{C}}$, so

$$(\vec{\mathbf{r}}\times\vec{\mathbf{v}})\cdot\vec{\mathbf{C}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}}$$

Since $\vec{\mathbf{r}}\times\vec{\mathbf{v}} = \vec{\mathbf{C}}$, we have

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}}$$

Note that $\vec{\mathbf{r}}\cdot\vec{\mathbf{D}} = ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}||\cos(\theta)$, where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{D}}$. Therefore,

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}|| \cos(\theta)$$

Solving for $||\vec{\mathbf{r}}||$,

$$||\vec{\mathbf{r}}|| = \frac{||\vec{\mathbf{C}}||^2}{GM + ||\vec{\mathbf{D}}|| \cos(\theta)} = \frac{||\vec{\mathbf{C}}||^2}{GM} \left(\frac{1}{1 + e \cos\theta} \right)$$

where $e = \frac{||\vec{\mathbf{D}}||}{GM}$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e > 1$, a parabola if $e = 1$, or an ellipse if $e < 1$. Since planets have closed orbits, the only possibility is an ellipse. However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

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1.7: Practice Set

Exercise 1.7.1

Consider a particle with position at time t seconds given by $\vec{\mathbf{r}}(t) = \langle 6t^2 - 2, 7, 4t^3 - 4 \rangle$

(a) Find the displacement of the particle between time $t = 0$ and $t = 2$

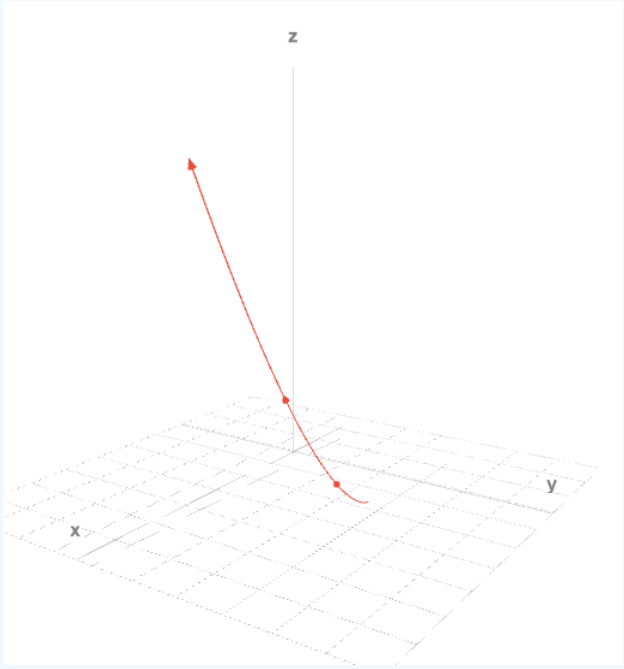
(b) Find the distance traveled for the particle between time $t = 0$ and $t = 2$

(c) Find the acceleration of the particle at time $t = 1$

(d) Find the parameterization by arc length from $\vec{\mathbf{r}}(0)$ of $\vec{\mathbf{r}}(t)$

Answer

The particle travels along the space curve pictured below:



(a) The displacement is given by:

$$\begin{aligned}\int_0^2 \vec{\mathbf{r}}\,'(t)\,dt &= \vec{\mathbf{r}}(2) - \vec{\mathbf{r}}(0) \\ &= \langle 22, 7, 28 \rangle - \langle -2, 7, -4 \rangle \\ &= \langle 24, 0, 32 \rangle\end{aligned}$$

(b) The distance traveled is given by:

$$\begin{aligned}\int_0^2 \|\vec{\mathbf{r}}\,'(t)\|\,dt &= \int_0^2 \| \langle 12t, 0, 12t^2 \rangle \|\,dt \\ &= \int_0^2 \sqrt{(12t)^2 + (12t^2)^2}\,dt \\ &= \int_0^2 12t\sqrt{1+t^2}\,dt \\ &= 4(1+t^2)^{\frac{3}{2}}\Big|_0^2 \\ &= 20\sqrt{5} - 4\end{aligned}$$

(c) The acceleration vector is the second derivative of the position:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}\,''(t) = \langle 12, 0, 24t\rangle \longrightarrow \vec{\mathbf{a}}(1) = \langle 12, 0, 24\rangle$$

(d) To parameterize by arc length, we start by calculating the distance traveled over $0 \le t \le b$:

$$\begin{aligned}\int_0^b \|\vec{\mathbf{r}}\,'(t)\|\,dt &= \int_0^b \|\left\langle 12t, 0, 12t^2\right\rangle\|\,dt \\ &= \int_0^b \sqrt{(12t)^2 + (12t^2)^2}\,dt \\ &= \int_0^b 12t\sqrt{1+t^2}\,dt \\ &= 4(1+t^2)^{\frac{3}{2}}\Big|_0^b \\ &= 4(1+b^2)^{\frac{3}{2}} - 4\end{aligned}$$

We set this equal to s and solve for b:

$$4(1+b^2)^{\frac{3}{2}} - 4 = s \longleftrightarrow b = \sqrt{\left(\frac{s}{4}+1\right)^{\frac{2}{3}} - 1}$$

Now we replace t with this expression for b to get our final answer:

$$\vec{\mathbf{R}}(s) = \vec{\mathbf{r}}\left(\sqrt{\left(\frac{s}{4}+1\right)^{\frac{2}{3}} - 1}\right) = \left\langle 6\left(\frac{s}{4}+1\right)^{\frac{2}{3}} - 8, 7, 4\left(\left(\frac{s}{4}+1\right)^{\frac{2}{3}} - 1\right)^{\frac{3}{2}} - 4\right\rangle$$

? Exercise 1.7.2

A rocket launches from the origin and flies such that its velocity at time t seconds is given by $\vec{\mathbf{v}}(t) = \left\langle 7\sin(2t), \sqrt{15}, 7\cos(2t)\right\rangle$

(a) Find the displacement of the rocket between time $t = 0$ and $t = 2\pi$

(b) Find the distance traveled for the rocket between time $t = 0$ and $t = 2\pi$

(c) Find the maximum curvature of the rocket for $0 \le t \le 2\pi$

(d) Find the parameterization by distance traveled from its initial position for the path of the rocket

Answer

(a) The displacement is given by:

$$\begin{aligned}\int_0^{2\pi} \vec{\mathbf{v}}(t)\,dt &= \int_0^{2\pi} \left\langle 7\sin(2t), \sqrt{15}, 7\cos(2t)\right\rangle\,dt \\ &= \left\langle -\frac{7}{2}\cos(2t), t\sqrt{15}, \frac{7}{2}\sin(2t)\right\rangle\Big|_0^{2\pi} \\ &= \left\langle 0, 2\pi\sqrt{15}, 0\right\rangle\end{aligned}$$

(b) The distance traveled is given by:

$$\begin{aligned}\int_0^{2\pi} \|\vec{\mathbf{v}}(t)\|\,dt &= \int_0^{2\pi} \|\left\langle 7\sin(2t), \sqrt{15}, 7\cos(2t)\right\rangle\|\,dt \\ &= \int_0^{2\pi} \sqrt{(7\sin(2t))^2 + \left(\sqrt{15}\right)^2 + (7\cos(2t))^2}\,dt \\ &= \int_0^{2\pi} \sqrt{7^2+15}\,dt \\ &= \int_0^{2\pi} 8\,dt \\ &= 8t\Big|_0^{2\pi} \\ &= 16\pi\end{aligned}$$

(c) The curvature at time t is given by $\kappa(t) = \dfrac{\|\vec{\mathbf{v}}(t) \times \vec{\mathbf{v}}\,'(t)\|}{\|\vec{\mathbf{v}}(t)\|^3}$, so we find:

It is easy to find the denominator, $\|\vec{\mathbf{v}}(t)\|^3 = \left(\sqrt{(7\sin(2t))^2 + \left(\sqrt{15}\right)^2 + (7\cos(2t))^2}\right)^3 = 64^{\frac{3}{2}} = 512$

Taking the derivative gives $\vec{\mathbf{v}}\,'(t) = \langle 14\cos(2t), 0, -14\sin(2t)\rangle$

Now taking a cross product, we have $\vec{\mathbf{v}}(t) \times \vec{\mathbf{v}}\,'(t) = \left\langle -14\sqrt{15}\sin(2t), 98, -14\sqrt{15}\cos(2t)\right\rangle$

The length of this cross product is $\|\vec{\mathbf{v}}(t) \times \vec{\mathbf{v}}\,'(t)\| = \sqrt{(-14\sqrt{15}\sin(2t))^2 + (98)^2 + (-14\sqrt{15}\cos(2t))^2} = 14\sqrt{15+7^2} = 14 \cdot 8 = 112$

Putting these all together gives our curvature: $\kappa(t) = \dfrac{112}{512} = \dfrac{7}{32}$

Note that this is a constant function, so the maximum curvature is simply $\dfrac{7}{32}$

(d) To parameterize by arc length, we start by calculating the distance traveled over $0 \le t \le b$:

$$\begin{aligned}
\int_0^b \|\vec{\mathbf{v}}(t)\|\,dt &= \int_0^b \left\| \left\langle 7\sin(2t), \sqrt{15}, 7\cos(2t)\right\rangle \right\| dt \\
&= \int_0^b \sqrt{(7\sin(2t))^2 + \left(\sqrt{15}\right)^2 + (7\cos(2t))^2}\,dt \\
&= \int_0^b \sqrt{7^2+15}\,dt \\
&= \int_0^b 8\,dt \\
&= 8t\Big|_0^b \\
&= 8b
\end{aligned}$$

We set this equal to s and solve for b:

$8b = s \longleftrightarrow b = \dfrac{s}{8}$

We integrate to find the position function:

$$\begin{aligned}
\vec{\mathbf{r}}(t) &= \vec{\mathbf{r}}(0) + \int_0^t \vec{\mathbf{v}}(u)\,du \\
&= \langle 0,0,0\rangle + \int_0^t \left\langle 7\sin(2u), \sqrt{15}, 7\cos(2u)\right\rangle\,du \\
&= \left[\left\langle -\frac{7}{2}\cos(2u), u\sqrt{15}, \frac{7}{2}\sin(2u)\right\rangle\right|_0^t \\
&= \left\langle \frac{7}{2}(1-\cos(2t)), t\sqrt{15}, \frac{7}{2}\sin(2t)\right\rangle
\end{aligned}$$

Now we replace t with this expression for b to get our final answer:

$\vec{\mathbf{R}}(s) = \vec{\mathbf{r}}\left(\dfrac{s}{8}\right) = \left\langle \dfrac{7}{2}\left(1-\cos\left(\dfrac{s}{4}\right)\right), \dfrac{s\sqrt{15}}{8}, \dfrac{7}{2}\sin\left(\dfrac{s}{4}\right)\right\rangle$

? Exercise 1.7.3

(a) Let $f(x)$ and $g(x)$ be well defined smooth nonlinear functions. Find the formula for the curvature of the space curve $\vec{\mathbf{r}}(t) = \langle t, f(t), g(t)\rangle$

(b) Let h be a smooth function with local maximum at the point $x = 2$ satisfying $h''(2) = -4$. If C is the curve defined by the intersection of the cylinders $y = h(x)$ and $z = bh(x)$ for some real number $b \neq 0$. If the curvature of C at the point with $x = 2$ is exactly 5, what is b?

Answer

(a) The curvature at time t is given by $\kappa(t) = \dfrac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}$, so we find each component

$\vec{\mathbf{r}}'(t) = \langle 1, f'(t), g'(t)\rangle$ and $\vec{\mathbf{r}}''(t) = \langle 0, f''(t), g''(t)\rangle$

$\|\vec{\mathbf{r}}'(t)\| = \sqrt{1 + (f'(t))^2 + (g'(t))^2}$

$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \langle f'(t)g''(t) - g'(t)f''(t), -g''(t), f''(t)\rangle$

$\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\| = \sqrt{(f'(t)g''(t) - g'(t)f''(t))^2 + (-g''(t))^2 + (f''(t))^2}$

So the curvature is $\kappa = \dfrac{\sqrt{(f'(t)g''(t) - g'(t)f''(t))^2 + (-g''(t))^2 + (f''(t))^2}}{\left(1 + (f'(t))^2 + (g'(t))^2\right)^{\frac{3}{2}}}$

(b) The intersection of these two curves can be parameterized by $\vec{\mathbf{r}}(t) = \langle t, h(t), bh(t)\rangle$

From part (a) above, we plug in $f(x) = h(x),\ g(x) = bh(x)$ to get

$$\kappa = \frac{\sqrt{(h'(t)(bh''(t)) - (bh'(t))h''(t))^2 + (-bh''(t))^2 + (h''(t))^2}}{\left(1 + (h'(t))^2 + (bh'(t))^2\right)^{\frac{3}{2}}} = \frac{|h''(t)|\sqrt{1+b^2}}{(1 + (1+b^2)(h'(t))^2)^{\frac{3}{2}}}$$

At the local maximum $x = t = 2$, we have $h'(2) = 0$, so this curvature becomes $\kappa = |h''(2)|\sqrt{1+b^2} = 4\sqrt{1+b^2}$

Setting this equal to 5 gives $5 = 4\sqrt{1+b^2} \longleftrightarrow b = \pm\dfrac{3}{4}$

Exercise 1.7.4a

Find the length of $\vec{\mathbf{r}}(t) = \langle e^t, e^t\sin(t), e^t\cos(t)\rangle$ over the interval $0 \le t \le \pi$

Answer

First we differentiate to find $\vec{\mathbf{r}}'(t) = \langle e^t, e^t(\sin(t) + \cos(t)), e^t(\cos(t) - \sin(t))\rangle$

Plugging this into the arc length formula we get:

$$\begin{aligned}
\int_0^\pi \|\vec{\mathbf{r}}'(t)\|\,dt &= \int_0^\pi \|\langle e^t, e^t(\sin(t) + \cos(t)), e^t(\cos(t) - \sin(t))\rangle\|\,dt \\
&= \int_0^\pi \sqrt{(e^t)^2 + (e^t(\sin(t) + \cos(t)))^2 + (e^t(\cos(t) - \sin(t)))^2}\,dt \\
&= \int_0^\pi e^t\sqrt{1 + (\sin^2(t) + 2\sin(t)\cos(t) + \cos^2(t)) + (\cos^2(t) - 2\sin(t)\cos(t) + \sin^2(t))}\,dt \\
&= \int_0^\pi e^t\sqrt{3}\,dt \\
&= e^t\sqrt{3}\Big|_0^\pi \\
&= (e^\pi - 1)\sqrt{3}
\end{aligned}$$

Exercise 1.7.4b

Find the length of $\vec{\mathbf{r}}(t) = \left\langle t, \dfrac{6}{5}t^{\frac{5}{3}}, \dfrac{3}{2}t^{\frac{4}{3}}\right\rangle$ over the interval $0 \le t \le 1$

Answer

First we differentiate to find $\vec{\mathbf{r}}'(t) = \left\langle 1, 2t^{\frac{2}{3}}, 2t^{\frac{1}{3}}\right\rangle$

Plugging this into the arc length formula we get:

$$\begin{aligned}\int_0^1 \|\vec{\mathbf{r}}\,'(t)\|\,dt &= \int_0^1 \left\| \left\langle 1, 2t^{\frac{2}{3}}, 2t^{\frac{1}{3}} \right\rangle \right\| dt \\ &= \int_0^1 \sqrt{1^2 + \left(2t^{\frac{2}{3}}\right)^2 + \left(2t^{\frac{1}{3}}\right)^2}\,dt \\ &= \int_0^1 \sqrt{1 + 4t^{\frac{2}{3}} + 4t^{\frac{4}{3}}}\,dt \\ &= \int_0^1 \left(1 + 2t^{\frac{2}{3}}\right)\,dt \\ &= \left[t + \frac{6}{5}t^{\frac{5}{3}}\right]\Big|_0^1 \\ &= \frac{11}{5}\end{aligned}$$

Exercise 1.7.4c

Find the length of $\vec{\mathbf{r}}(t) = \left\langle t\cos(t), t\sin(t), \frac{1}{9}(6t+8)^{\frac{3}{2}} \right\rangle$ over the interval $0 \le t \le 2$

Answer

First we differentiate to find $\vec{\mathbf{r}}\,'(t) = \left\langle \cos(t) - t\sin(t), \sin(t) + t\cos(t), (6t+8)^{\frac{1}{2}} \right\rangle$

Plugging this into the arc length formula we get:

$$\begin{aligned}\int_0^2 \|\vec{\mathbf{r}}\,'(t)\|\,dt &= \int_0^2 \left\| \left\langle \cos(t) - t\sin(t), \sin(t) + t\cos(t), (6t+8)^{\frac{1}{2}} \right\rangle \right\| dt \\ &= \int_0^2 \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2 + \left((6t+8)^{\frac{1}{2}}\right)^2}\,dt \\ &= \int_0^2 \sqrt{\cos^2(t) + \sin^2(t) + t^2\cos^2(t) + t^2\sin^2(t) + 6t + 8}\,dt \\ &= \int_0^2 \sqrt{t^2 + 6t + 9}\,dt \\ &= \int_0^2 (t+3)\,dt \\ &= \left[\frac{1}{2}t^2 + 3t\right]\Big|_0^2 \\ &= 8\end{aligned}$$

Exercise 1.7.4d

Find the length of $\vec{\mathbf{r}}(t) = \left\langle \frac{1}{2}t^2, \frac{2}{3}t^3, \frac{1}{2}t^4 \right\rangle$ over the interval $-4 \le t \le 2$

Answer

First we differentiate to find $\vec{\mathbf{r}}\,'(t) = \langle t, 2t^2, 2t^3 \rangle$

Plugging this into the arc length formula we get:

$$\begin{aligned}
\int_{-4}^{2} \|\vec{\mathbf{r}}\,'(t)\|\,dt &= \int_{-4}^{2} \|\langle t, 2t^2, 2t^3\rangle\|\,dt \\
&= \int_{-4}^{2} \sqrt{t^2 + (2t^2)^2 + (2t^3)^2}\,dt \\
&= \int_{-4}^{2} \sqrt{t^2 + 4t^4 + 4t^6}\,dt \\
&= \int_{-4}^{2} |t|\sqrt{1 + 4t^2 + 4t^4}\,dt \\
&= \int_{-4}^{2} |t|\left(1 + 2t^2\right)\,dt \\
&= \int_{-4}^{0} -t\left(1 + 2t^2\right)\,dt + \int_{0}^{2} t\left(1 + 2t^2\right)\,dt \\
&= \int_{-4}^{0} \left(-t - 2t^3\right)\,dt + \int_{0}^{2} \left(t + 2t^3\right)\,dt \\
&= \left[-\frac{1}{2}t^2 - \frac{1}{2}t^4\right]\Big|_{-4}^{0} + \left[\frac{1}{2}t^2 + \frac{1}{2}t^4\right]\Big|_{0}^{2} \\
&= 136 + 10 \\
&= 146
\end{aligned}$$

Exercise 1.7.4e

Find the length of $\vec{\mathbf{r}}(t) = \langle 2\ln(\sec(t)), \sin(2t), \cos(2t)\rangle$ over the interval $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$

Answer

First we differentiate to find $\vec{\mathbf{r}}\,'(t) = \langle 2\tan(t), 2\cos(2t), -2\sin(2t)\rangle$

Plugging this into the arc length formula we get:

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \|\vec{\mathbf{r}}\,'(t)\|\,dt &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \|\langle 2\tan(t), 2\cos(2t), -2\sin(2t)\rangle\|\,dt \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{(2\tan(t))^2 + (2\cos(2t))^2 + (-2\sin(2t))^2}\,dt \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{4\tan^2(t) + 4}\,dt \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\sqrt{\sec^2(t)}\,dt \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\sec(t)\,dt \\
&= 2\ln|\sec(t) + \tan(t)|\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= 2\ln\left|\sqrt{2} + 1\right| - 2\ln\left|\sqrt{2} - 1\right| \\
&= 2\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)
\end{aligned}$$

Exercise 1.7.5a

An archer on the ground fires an arrow with initial velocity 192 feet per second at an angle θ above horizontal.

(a) If $\theta = \frac{\pi}{6}$, what is the maximum height the arrow will reach?

(b) If $\theta = \frac{\pi}{6}$, how far will it travel?

(c) If the archer wants to hit a target that is on the ground at a distance of 512 feet away, what angle θ could they aim at?

(d) What is the furthest target that the archer could hit at any angle?

You may assume gravity is 32 feet per second squared for your calculations.

Answer

The path of the arrow is defined by a two dimensional curve $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle$ which satisfies $\vec{\mathbf{r}}(0) = \langle 0, 0\rangle$, $\vec{\mathbf{v}}(0) = \langle 192\cos(\theta), 192\sin(\theta)\rangle$ and $\vec{\mathbf{a}}(t) = \langle 0, -32\rangle$

We know that $\vec{\mathbf{v}}\,'(t) = \vec{\mathbf{a}}(t)$, so:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}(0) + \int_0^t \vec{\mathbf{a}}(u)\,du = \langle 192\cos(\theta), 192\sin(\theta)\rangle + \int_0^t \langle 0, -32\rangle\,du = \langle 192\cos(\theta), 192\sin(\theta) - 32t\rangle$$

We also know that $\vec{\mathbf{r}}\,'(t) = \vec{\mathbf{v}}(t)$, so:

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(0) + \int_0^t \vec{\mathbf{v}}(u)\,du = \langle 0, 0\rangle + \int_0^t \langle 192\cos(\theta), 192\sin(\theta) - 32u\rangle\,du = \langle 192t\cos(\theta), 192t\sin(\theta) - 16t^2\rangle$$

(a) If $\theta = \dfrac{\pi}{6}$, we see $\vec{\mathbf{r}}(t) = \langle 96t\sqrt{3}, 96t - 16t^2\rangle$

Therefore the maximum height occurs when $y'(t) = 0 \longleftrightarrow 96 - 32t = 0 \longleftrightarrow t = 3$

This maximum height is the y-coordinate of $\vec{\mathbf{r}}(3) = \langle 288\sqrt{3}, 144\rangle$, so 144 feet.

(b) If $\theta = \dfrac{\pi}{6}$, we see $\vec{\mathbf{r}}(t) = \langle 96t\sqrt{3}, 96t - 16t^2\rangle$

The maximum distance occurs when $y(t) = 0 \longleftrightarrow 96t - 16t^2 = 0 \longleftrightarrow t = 6$

This maximum distance is the x-coordinate of $\vec{\mathbf{r}}\,(6) = \langle 576\sqrt{3}, 0\rangle$, so $576\sqrt{3}$ feet.

(c) To hit a target on the ground at a distance of 512 feet away, the curve $\vec{\mathbf{r}}(t) = \langle 512, 0\rangle$ needs to have a solution.

Setting $y(t) = 0$ gives $192t\sin(\theta) - 16t^2 = 0 \longleftrightarrow t = 0, 12\sin(\theta)$

Since $x(0) = 0$, we need $x\,(12\sin(\theta)) = 512$

$192\cos(\theta)\cdot 12\sin(\theta) = 512 \longleftrightarrow 1152\sin(2\theta) = 512 \longleftrightarrow \sin(2\theta) = \dfrac{4}{9} \longleftrightarrow \theta = \dfrac{1}{2}\sin^{-1}\left(\dfrac{4}{9}\right)$

Therefore the archer should aim at an angle $\theta = \dfrac{1}{2}\sin^{-1}\left(\frac{4}{9}\right)$

(d) The maximum distance occurs when $y(t) = 0 \longleftrightarrow 192t\sin(\theta) - 16t^2 = 0 \longleftrightarrow t = 12\sin(\theta)$

This maximum distance is the x-coordinate of $\vec{\mathbf{r}}\,(6\sin(\theta)) = \langle 192\cos(\theta)\cdot 12\sin(\theta), 0\rangle$

Therefore we want to maximize the function $d(\theta) = 2304\sin(\theta)\cos(\theta) = 1152\sin(2\theta)$

This has derivative $d'(\theta) = 2304\cos(2\theta)$ which equals zero when $2\theta = \dfrac{\pi}{2} \longleftrightarrow \theta = \dfrac{\pi}{4}$

The maximum possible distance is then $d\left(\dfrac{\pi}{4}\right) = 1152\sin\left(\dfrac{\pi}{2}\right) = 1152$ feet.

? Exercise $1.7.5b$

A bullet is shot on a planet where the gravity pulls downward at 48 meters per second squared in the z direction. At the same time, there is a strong wind that blows objects toward the positive y-axis by a factor of 12 meters per second. A bullet is fired from a height of exactly 96 meters at a speed of 72 meters per second in the direction of the unit vector $\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$

(a) If $\phi = 0$, how long does the bullet take to hit the ground?

(b) If $\theta = 0$ and $\phi = \dfrac{\pi}{2}$, where does the bullet hit the ground in relation to the initial position?

(c) If the bullet hits a target at the location $(c, d, 96)$ when aimed in the direction $\theta = \dfrac{\pi}{4}$ and $\phi = \dfrac{\pi}{3}$, what are c and d?

Answer

The path of the bulet is defined by a space curve $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ which satisfies $\vec{\mathbf{r}}(0) = \langle 0, 0, 96\rangle$, $\vec{\mathbf{v}}(0) = \langle 72\sin(\phi)\cos(\theta), 72\sin(\phi)\sin(\theta), 72\cos(\phi)\rangle$ and $\vec{\mathbf{a}}(t) = \langle 0, 0, -48\rangle$

We know that $\vec{\mathbf{v}}\,'(t) = \vec{\mathbf{a}}(t)$, so:

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{v}}(0) + \vec{\mathbf{v}}_{wind} + \int_0^t \vec{\mathbf{a}}(u)\,du = \\ &= \langle 72\sin(\phi)\cos(\theta), 72\sin(\phi)\sin(\theta), 72\cos(\phi)\rangle + \langle 0, 12, 0\rangle + \int_0^t \langle 0, 0, -48\rangle\,dt \\ &= \langle 72\sin(\phi)\cos(\theta), 72\sin(\phi)\sin(\theta) + 12, 72\cos(\phi) - 48t\rangle\end{aligned}$$

We also know that $\vec{\mathbf{r}}\,'(t) = \vec{\mathbf{v}}(t)$, so:

$$\begin{aligned}\vec{\mathbf{r}}(t) &= \vec{\mathbf{r}}(0) + \int_0^t \vec{\mathbf{v}}(u)\,du = \\ &= \langle 0, 0, 96\rangle + \int_0^t \langle 72\sin(\phi)\cos(\theta), 72\sin(\phi)\sin(\theta) + 12, 72\cos(\phi) - 48t\rangle\,dt \\ &= \langle 72t\sin(\phi)\cos(\theta), 72t\sin(\phi)\sin(\theta) + 12t, 72t\cos(\phi) - 24t^2 + 96\rangle\end{aligned}$$

(a) When $\phi = 0$, the position vector is $\vec{\mathbf{r}}(t) = \langle 0, 0, 72t - 24t^2 + 96\rangle$

The bullet hits the ground when $z(t) = 0$, so we solve $72t - 24t^2 + 96 = 0 \longleftrightarrow t = 4$

Therefore it takes four seconds for the bullet to hit the ground when fired straight up into the air.

(b) When $\theta = 0$ and $\phi = \dfrac{\pi}{2}$, the position vector is $\vec{\mathbf{r}}(t) = \langle 72t, 12t, -24t^2 + 96\rangle$

The bullet hits the ground when $z(t) = 0$, so we solve $96 - 24t^2 = 0 \longleftrightarrow t = 2$

So the location will be $\vec{\mathbf{r}}(2) = \langle 144, 24, 0\rangle$

This point represents a displacement of $\langle 144, 24, -96\rangle$ in meters.

(c) When $\theta = \dfrac{\pi}{4}$, and $\phi = \dfrac{\pi}{3}$, the position vector is $\vec{\mathbf{r}}(t) = \langle 18t\sqrt{6}, 18t\sqrt{6} + 12t, 36t - 24t^2 + 96\rangle$

The bullet hits the target when $z(t) = 96$, so we solve $36t - 24t^2 + 96 = 96 \longleftrightarrow t = \dfrac{3}{2}$

Plugging this in gives $\vec{\mathbf{r}}\left(\dfrac{3}{2}\right) = \left\langle 27\sqrt{6}, 27\sqrt{6} + 18, 96\right\rangle$

So we have $c = 27\sqrt{6}$ and $d = 27\sqrt{6} + 18$

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CHAPTER OVERVIEW

2: Functions of Several Variables

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

2.1: Multivariable Functions

- 2.1: Functions of Several Variables
- 2.1: Surfaces
- 2.1: Practice Set

2.2: Multivariable Limits

- 2.2: Limits and Continuity
- 2.2: Practice Set

2.3: Multivariable Differentiation

- 2.3: Partial Derivatives
- 2.3: Tangent Planes and Linear Approximations
- 2.3: Practice Set

2.4: More on Multivariable Derivatives

- 2.4: The Chain Rule for Multivariable Functions
- 2.4: Directional Derivatives and the Gradient
- 2.4: Practice Set

2.5: Vector Fields and Derivatives

- 2.5: Vector Fields
- 2.5: Divergence and Curl
- 2.5: Practice Set

2.6: Multivariable Extrema

- 2.6: Maxima/Minima Problems
- 2.6: Absolute Extrema
- 2.6: Lagrange Multipliers
- 2.6: Practice Set

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2.1: Multivariable Functions

Learning Objectives

- Recognize a function of two variables and identify its domain and range.
- Sketch a graph of a function of two variables.
- Sketch several traces or level curves of a function of two variables.
- Recognize a function of three or more variables and identify its level surfaces.
- Sketch a surface given its parametrization.
- Find a parametrization of a described surface.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

We expand this to examples with more than two variables, and also discuss surfaces as an extension of the graph of functions of two variables.

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2.1: Functions of Several Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: Function of Two Variables

A *function of two variables* $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that have at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown below.

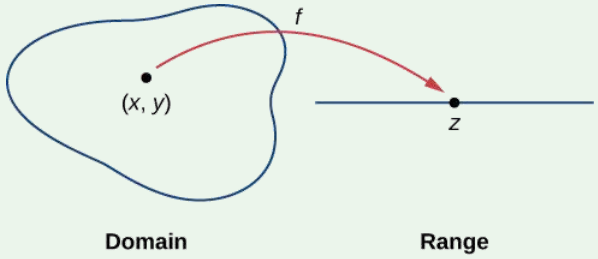


Figure 2.1.1: The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Illustration 2.1.1: Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

(a) $f(x, y) = 3x + 5y + 2$

(b) $g(x, y) = \sqrt{9 - x^2 - y^2}$

(c) $h(x, y) = \ln(36 - 9x^2 - 4y^2)$

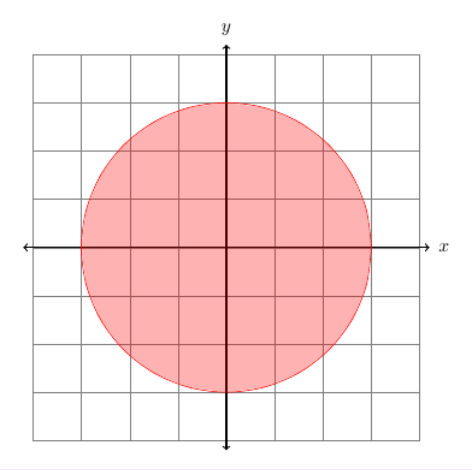
(a) This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is $\mathbb{R}^2$.

To determine the range, first pick a value for z. We hope to find a solution to the equation $f(x, y) = z$, or $3x + 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \frac{z-2}{3}$, which gives the ordered pair $\left(\frac{z-2}{3}, 0\right)$ as a solution to the equation $f(x, y) = z$ for any value of z.
Therefore, the range of the function is all real numbers, or $\mathbb{R}$.

(b) For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0 \longleftrightarrow x^2 + y^2 \leq 9$$

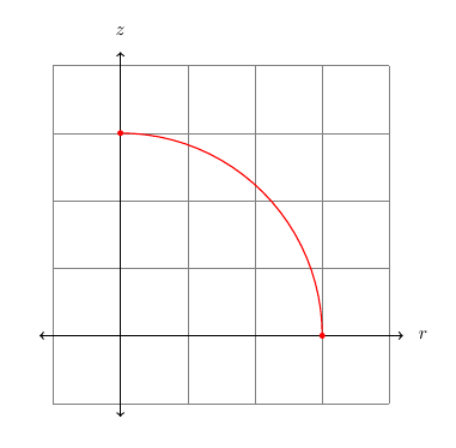
Therefore, the domain of $g(x, y)$ is $\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.



To determine the range of $g(x,y) = \sqrt{9-x^2-y^2}$, once again we pick a real number z. We hope to find a solution to the equation $z = \sqrt{9-x^2-y^2}$. This relationship is easier to think about in cylindrical coordinates:

$$z = \sqrt{9-r^2} \text{ for } 0 \leq r \leq 3$$

Since this is now reduced to a two variable equation, we can use our previous calculus knowledge to sketch the graph:

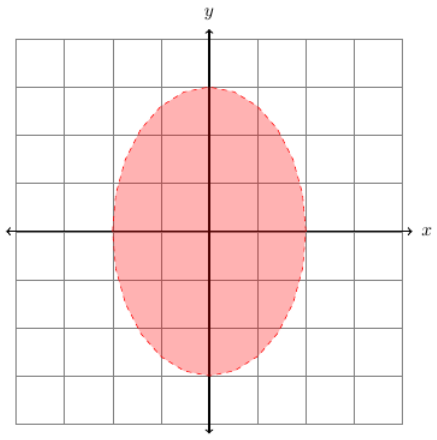


It is clear from this graph that the range of z is the interval $[0,3]$

(c) For the function $h(x,y)$ to have a real value, the quantity inside of the log must be positive:

$$36 - 9x^2 - 4y^2 > 0 \longleftrightarrow \frac{x^2}{4} + \frac{y^2}{9} < 1$$

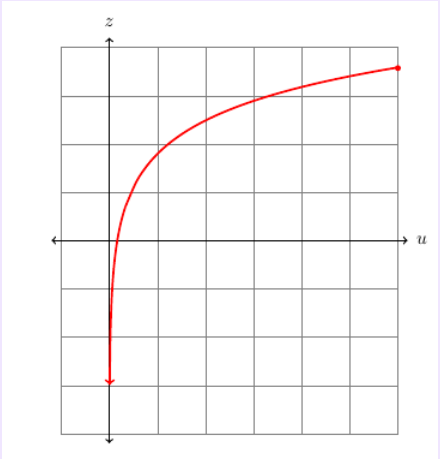
Therefore, the domain of $h(x,y)$ is $\left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{x^2}{4} + \frac{y^2}{9} < 1\right\}$. The graph of this set of points can be described as the interior of an ellipse centered at the origin. The domain does not include the boundary ellipse as shown in the following graph.



To determine the range of $h(x,y) = \ln(36 - 9x^2 - 4y^2)$, once again we pick a real number z. We hope to find a solution to the equation $z = \ln(36 - 9x^2 - 4y^2)$. While cylindrical coordinates do not help here, we can still look at the interior of the log first. Since $9x^2 + 4y^2 \geq 0$ must hold, we know that

$$0 < 36 - 9x^2 - 4y^2 \leq 36$$

So if we define $u = 36 - 9x^2 - 4y^2$, we are looking to solve $z = \ln(u)$ where $0 < u \leq 36$. Once again this is now reduced to a two variable equation, we can use our previous calculus knowledge to sketch the graph:



Clearly the range of this graph, and therefore also $h(x,y)$ is $(-\infty, \ln(36)]$

Suppose we wish to graph the function $z = f(x,y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y = f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x,y) in the plane, and every point in the plane has an ordered pair (x,y) associated with it. With a function of two variables, each ordered pair (x,y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x,y,z). The graph of a function $z = f(x,y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x,y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

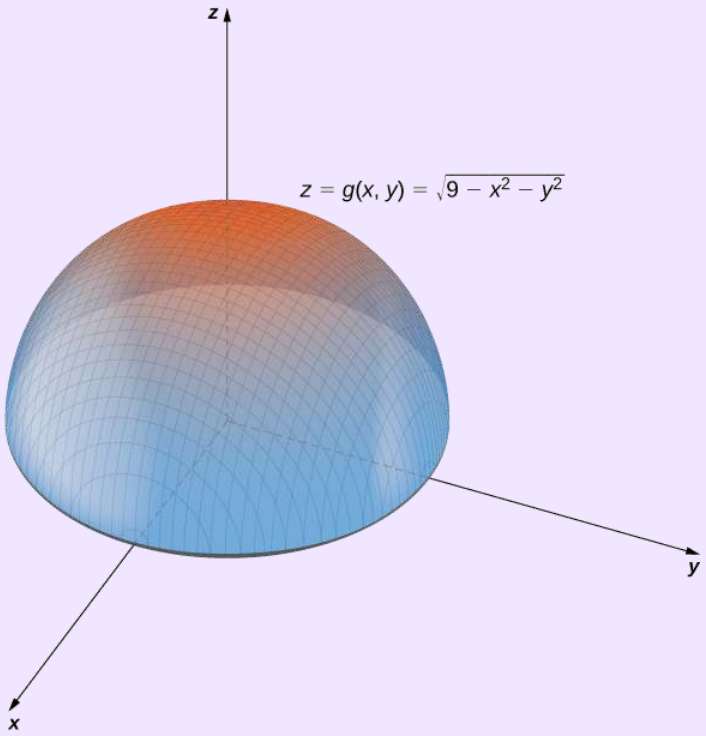
Illustration 2.1.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

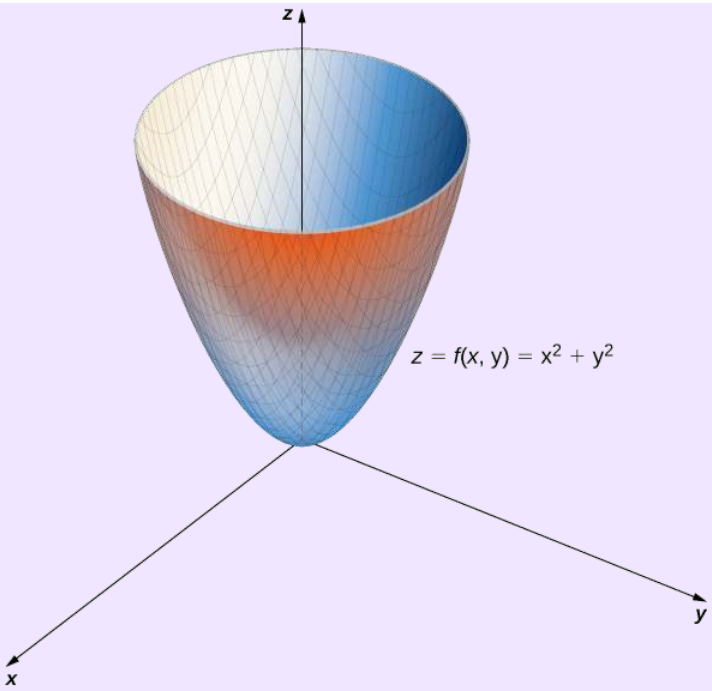
(a) $g(x,y) = \sqrt{9 - x^2 - y^2}$

(b) $f(x,y) = x^2 + y^2$

(a) In Illustration 2.1.2*b*, we determined that the domain of $g(x,y) = \sqrt{9 - x^2 - y^2}$ is $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$ and the range is $\{z \in \mathbb{R} : 0 \leq z \leq 3\}$. When $x^2 + y^2 = 9$ we have $g(x,y) = 0$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z = 0$ in $\mathbb{R}^3$. If $x^2 + y^2 = 8$, then $g(x,y) = 1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy-plane maps to $z = 1$ in $\mathbb{R}^3$. As $x^2 + y^2$ gets closer to zero, the value of z approaches 3. When $x^2 + y^2 = 0$, then $g(x,y) = 3$. This is the origin in the xy-plane If $x^2 + y^2$ is equal to any other value between 0 and 9, then $g(x,y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.



(b) This function also contains the expression $x^2 + y^2$. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x,y) = x^2 + y^2$ is zero (attained when $x = y = 0$.. When $x = 0$, the function becomes $z = y^2$, and when $y = 0$, then the function becomes $z = x^2$. These are cross-sections of the graph, and are parabolas. Recall that the name of the graph of $f(x,y) = x^2 + y^2$ is a **paraboloid**. The graph of f appears in the following graph.



Example 2.1.1: Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,y) = 16 - (x-3)^2 - (y-2)^2$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

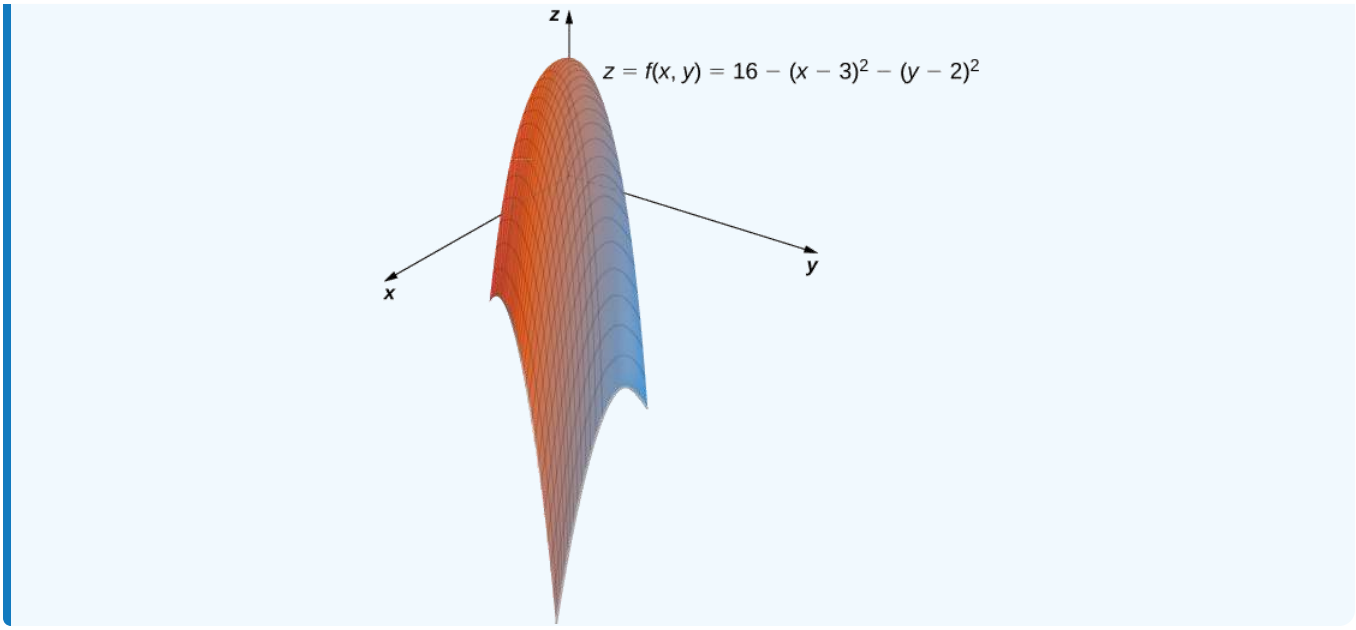
This function is a polynomial function in two variables. The domain of f consists of (x,y) coordinate pairs that yield a nonnegative profit:

$$16 - (x-3)^2 - (y-2)^2 \geq 0$$
$$(x-3)^2 + (y-2)^2 \leq 16$$

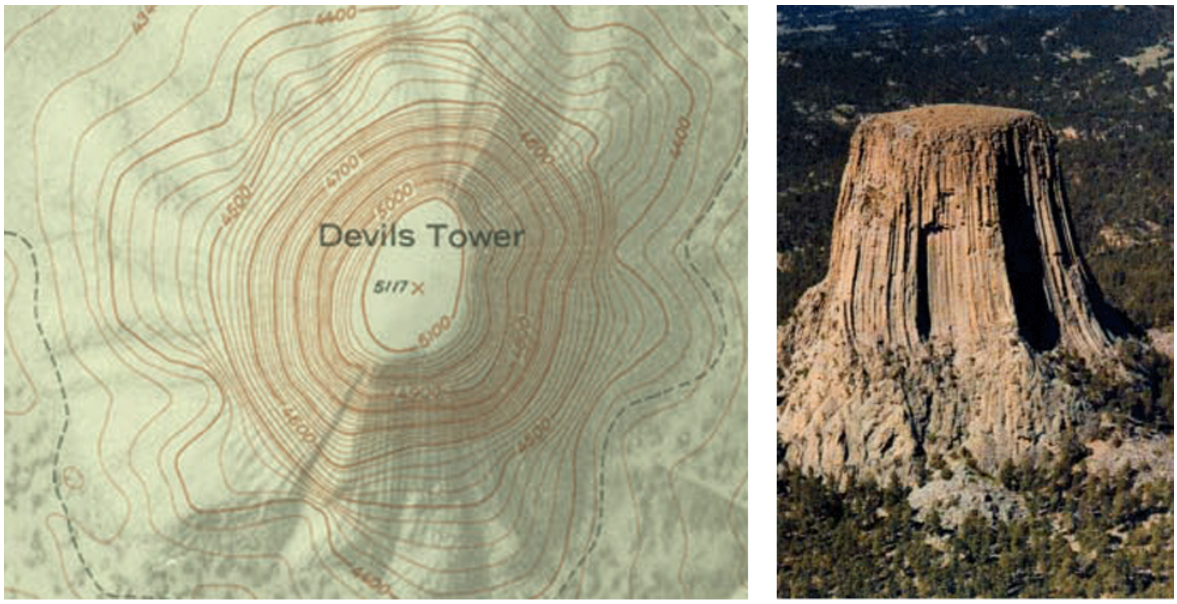
This is a disk of radius 4 centered at $(3,2)$. A further restriction is that both x and y must be nonnegative. When $x = 3$ and $y = 2, f(x,y) = 16$. Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z < 16$, we can solve the equation $f(x,y) = 16$:

$$16 - (x-3)^2 - (y-2)^2 = z$$
$$(x-3)^2 + (y-2)^2 = 16 - z$$

Since $z < 16$, we know that $16 - z > 0$, so the previous equation describes a circle with radius $\sqrt{16-z}$ centered at the point $(3,2)$. Therefore. the range of $f(x,y)$ is $\{z \in \mathbb{R} : z \leq 16\}$. The graph of $f(x,y)$ is also a paraboloid, and this paraboloid points downward as shown.



If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation. A level curve of a function of two variables $f(x, y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 2.1.2: (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Definition: Level Curves

Given a function $f(x, y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

Returning to the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, we can determine the level curves of this function. The range of g is the closed interval $[0, 3]$. First, we choose any number in this closed interval—say, $c = 2$. The level curve corresponding to $c = 2$ is described by the equation

$$\sqrt{9-x^2-y^2}=2$$

To simplify, square both sides of this equation:

$$9-x^2-y^2=4$$

Now, multiply both sides of the equation by -1 and add 9 to each side:

$$x^2+y^2=5$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c=3$, then the circle has radius 0, so it consists solely of the origin. Pictured below is a graph of the level curves of this function corresponding to $c=0,1,2,$ and 3.

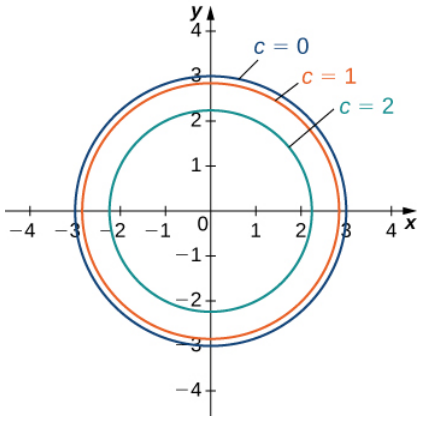


Figure 2.1.3: Level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2,$ and $3 (c=3$ corresponds to the origin).

A graph of the various level curves of a function is called a *contour map*.

Illustration 2.1.3: Making a Contour Map

Given the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$, find the level curve corresponding to $c=0$. Then create a contour map for this function. What are the domain and range of f?

To find the level curve for $c=0$, we set $f(x,y)=0$ and solve. This gives

$$0=\sqrt{8+8x-4y-4x^2-y^2}$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2+y^2-8x+4y-8=0$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2-8x+y^2+4y=8$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2-2x)+(y^2+4y)=8$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2-2x+1)+(y^2+4y+4)=8+4(1)+4$$

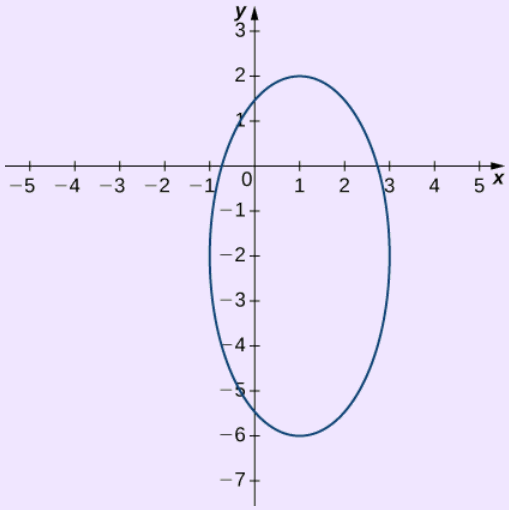
Next, we factor the left-hand side and simplify the right-hand side:

$$4(x-1)^2+(y+2)^2=16$$

Last, we divide both sides by 16 :

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1$$

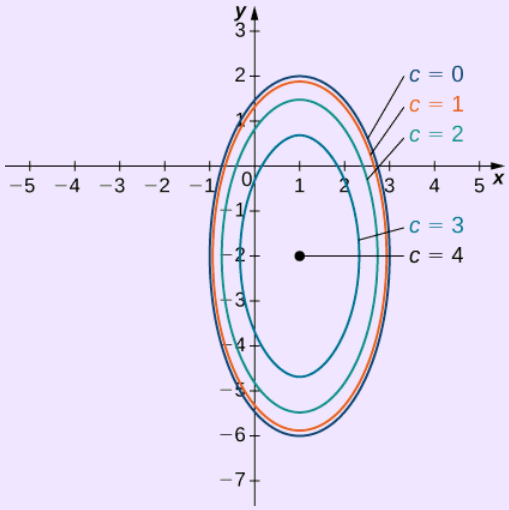
This equation describes an ellipse centered at $(1, -2)$. The graph of this ellipse is provided below.



We can repeat the same derivation for values of c less than 4. Then, each level curve equation becomes

$$\frac{4(x-1)^2}{16-c^2} + \frac{(y+2)^2}{16-c^2} = 1$$

The figure below shows a contour map for $f(x, y)$ using the values $c = 0, 1, 2,$ and 3. When $c = 4$, the level curve is the point $(-1, 2)$.



To find our domain, since this is a square root function, the radicand must not be negative. So we have

$$8 + 8x - 4y - 4x^2 - y^2 \geq 0$$

Recognizing that the boundary of the domain is an ellipse, we repeat the steps we showed above to obtain

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} \leq 1$$

So the domain of f can be written: $\left\{(x, y) \,\middle|\, \frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} \leq 1\right\}$.

To find the range of f, we need to consider the possible outputs of this square root function. We know the output cannot be negative, so we need to next check if its output is ever 0. From the work we completed above to find the level curve for $c = 0$, we know the value of f is 0 for any point on that level curve (on the ellipse, $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$). So we know the lower bound of the range of this function is 0.

To determine the upper bound for the range of the function in this problem, it's easier if we first complete the square under the radical.

$$\begin{aligned} f(x,y) &= \sqrt{8+8x-4y-4x^2-y^2} \\ &= \sqrt{8-4(x^2-2x\quad)-(y^2+4y\quad)} \\ &= \sqrt{8-4(x^2-2x+1-1)-(y^2+4y+4-4)} \\ &= \sqrt{8-4(x^2-2x+1)+4-(y^2+4y+4)+4} \\ &= \sqrt{16-4(x-1)^2-(y+2)^2} \end{aligned}$$

Now that we have f in this form, we can see how large the radicand can be. Since we are subtracting two perfect squares from 16, we know that the value of the radicand cannot be greater than 16. At the point $(1,-2)$, we can see the radicand will be 16 (since we will be subtracting 0 from 16 at this point. This gives us the maximum value of f, that is $f(1,-2)=\sqrt{16}=4$

So the range of this function is the interval $[0,4]$

Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the xy-plane, but as their name implies, vertical traces are graphed in the xz-plane or yz-plane respectively.

Definition: Vertical Traces

Consider a function $z=f(x,y)$ with domain $D\subseteq\mathbb{R}^2$. A *vertical trace* of the function can be either the set of points that solve the equation $f(a,y)=z$ for a given constant $x=a$ or $f(x,b)=z$ for a given constant $y=b$.

Illustration 2.1.4: Finding Vertical Traces

Find vertical traces for the function $f(x,y)=\sin(x)\cos(y)$ corresponding to $x=-\frac{\pi}{4}, 0,$ and $\frac{\pi}{4}$, and $y=-\frac{\pi}{4}, 0,$ and $\frac{\pi}{4}$

First set $x=-\frac{\pi}{4}$ in the equation $z=\sin(x)\cos(y)$:

$$z=\sin\left(-\frac{\pi}{4}\right)\cos(y)=-\frac{\sqrt{2}\cos(y)}{2}$$

This describes a cosine graph in the plane $x=-\frac{\pi}{4}$. The other values of z appear in the following table.

c	Vertical Trace for $x=c$
$-\frac{\pi}{4}$	$z=-\frac{\sqrt{2}\cos(y)}{2}$
0	$z=0$
$\frac{\pi}{4}$	$z=\frac{\sqrt{2}\cos(y)}{2}$

In a similar fashion, we can substitute the y-values in the equation $f(x,y)$ to obtain the traces in the yz-plane, as listed in the following table.

d	Vertical Trace for $y=d$
$\frac{\pi}{4}$	$z=\frac{\sqrt{2}\sin(x)}{2}$
0	$z=\sin(x)$
$-\frac{\pi}{4}$	$z=\frac{\sqrt{2}\sin(x)}{2}$

The three traces in the xz-plane are cosine functions; the three traces in the yz-plane are sine functions. These curves appear in the intersections of the surface with the planes $x = -\frac{\pi}{4}$, $x = 0$, $x = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}$, $y = 0$, $y = \frac{\pi}{4}$ as shown in the following figure.

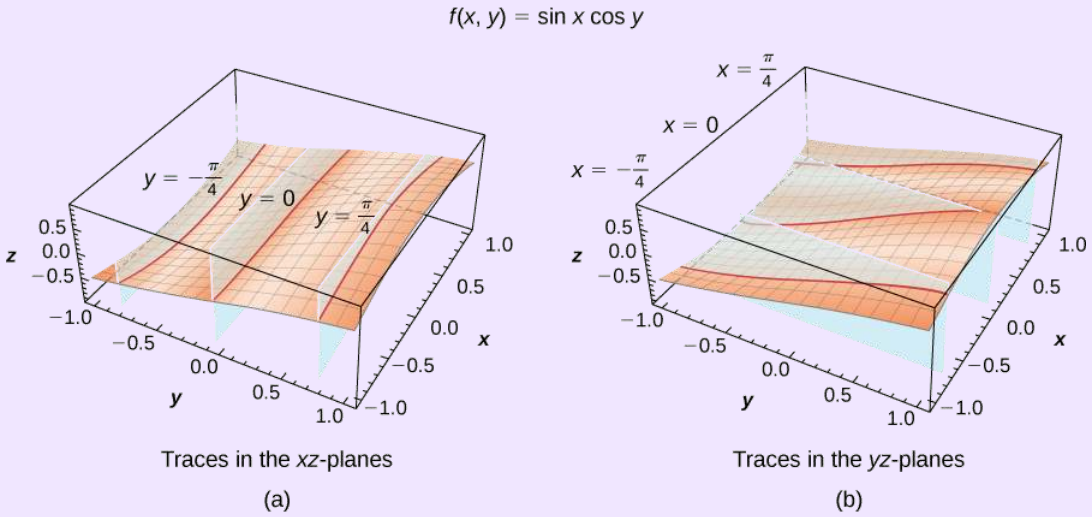


Figure 2.1.4: Vertical traces of the function $f(x, y)$ are (a) cosine curves in the xz-plane and (b) sine curves in the yz-planes

Functions of two variables can produce some striking-looking surfaces. Below we see two such examples.

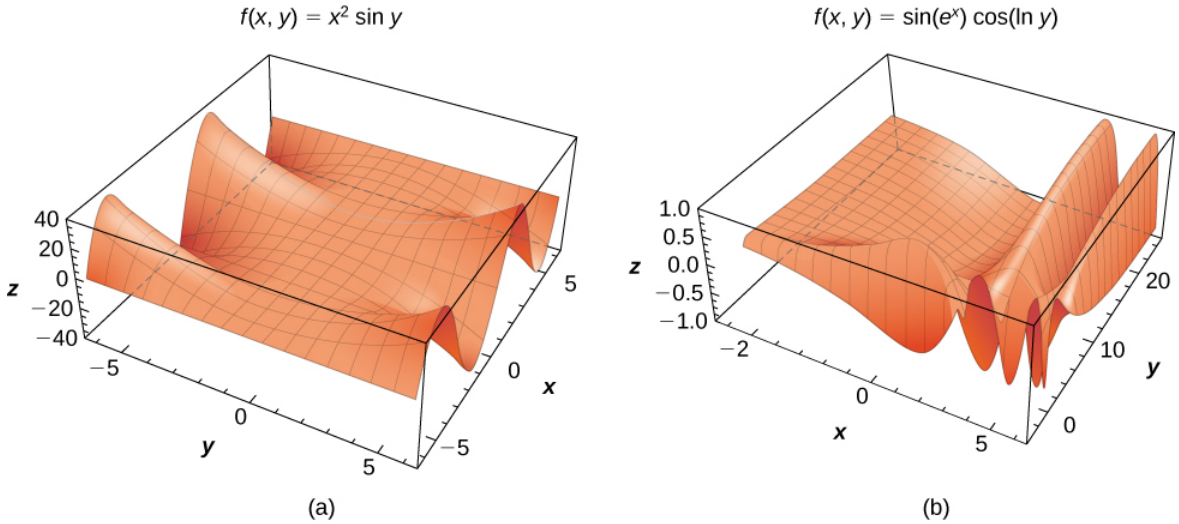


Figure 2.1.5: Examples of surfaces representing functions of two variables:
(a) a combination of a power function and a sine function
(b) a combination of trigonometric, exponential, and logarithmic functions.

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$\underbrace{f(x, y, z) = x^2 - 2xy + y^2 + 3yz - z^2 + 4x - 2y + 3z - 6}_{\text{a polynomial in three variables}}$$

and

$$g(x, y, t) = (x^2 - 4xy + y^2)\sin(t) - (3x + 5y)\cos(t)$$

In the first function, (x, y, z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x, y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 2.1.2: Domains for Functions of Three Variables

Find the domain of each of the following functions:

(a) $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$

(b) $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$

(c) $h(x,y,t)=(3t-6)\sqrt{y-4x^2+4}$

Solution:

(a) For the function $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator cannot be zero.
2. The radicand cannot be negative.

Combining these conditions leads to the inequality

$$9-x^2-y^2-z^2>0$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$\boxed{\text{Domain}(f)=\left\{(x,y,z)\in\mathbb{R}^3\,:\,x^2+y^2+z^2<9\right\}}$$

which describes a ball of radius 3 centered at the origin. (Note: The surface of the ball is not included in this domain.)

(b) For the function $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t-4\geq 0$, and therefore that $t\geq 2$. Since the denominator cannot be zero, $x^2-y^2\neq 0$, or $x^2\neq y^2$, Which can be rewritten as $y=\pm x$, which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$\boxed{\text{Domain}(g)=\{(x,y,t)\,:\,y\neq\pm x,\,t\geq 2\}}$$

(c) For the function $h(x,y,t)=(3t-6)\sqrt{y-4x^2+4}$ we only need to worry about the radicand being non-negative:

$$y-4x^2+4\geq 0\longleftrightarrow y\geq 4x^2-4$$

There is no restriction on the values of t for this function to exist, so we get

$$\boxed{\text{Domain}(h)=\left\{(x,y,t)\,:\,y\geq 4x^2-4,\,t\in\mathbb{R}\right\}}$$

Functions of two variables have level curves, which are shown as curves in the $xy-plane.$ However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: Level Surface of a Function of Three Variables

Given a function $f(x,y,z)$ and a number c in the range of f, a *level surface of a function of three variables* is defined to be the set of points satisfying the equation $f(x,y,z)=c.$

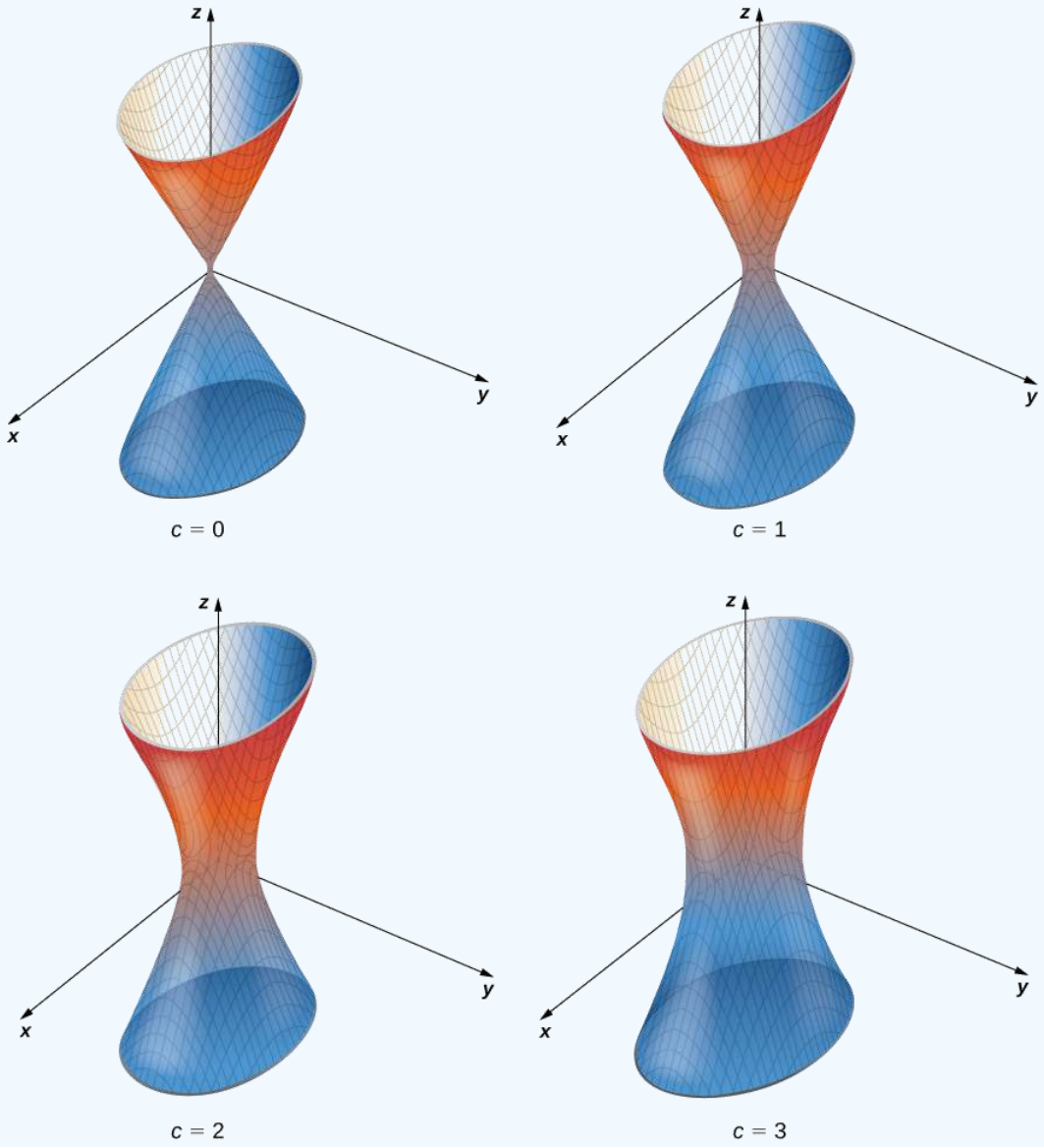
Example 2.1.3: Finding a Level Surface

Find the level surface for the function $f(x,y,z)=4x^2+9y^2-z^2$ corresponding to $c=0,1,2$ and 3.

Solution

The level surface is defined by the equation $4x^2 + 9y^2 - z^2 = c$.

When $c = 0$ this defines a cone, but for $c = 1, 2, 3$ this equation describes a hyperboloid of one sheet as shown below:



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2.1: Surfaces

Now that we have defined a real valued function of two variables, we can immediately discuss one of their most useful applications. The graph of $z = f(x, y)$ in $\mathbb{R}^3$ represents an explicitly defined surface. Many times we prefer to describe these objects parametrically instead.

Definition: Parametric Surface

A **parametric surface** S is a surface in $\mathbb{R}^3$ described by the function

$$\vec{\mathbf{r}}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$$

When this function is defined over a subset D of $\mathbb{R}^2$, the set D is called the **parametric domain** of S.

Note that we can represent the graph of $z = f(x, y)$ by using the parametric surface $\vec{\mathbf{r}}(u, v) = \langle u, v, f(u, v)\rangle$ for any function f. The parametric domain matches exactly with the domain of f described in the previous subsection.

To sketch a parametric surface, we typically consider a few cases of fixing each of u and v as constants and drawing the resulting space curves.

Illustration 2.1.5

Sketch the graph of each of the surfaces defined below.

(a) $\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), u^2\rangle$ over $D = \{(u, v) : 0 \le u \le 3,\ 0 \le v \le \pi\}$

(b) $\vec{\mathbf{r}}(u, v) = \langle u, v, u^2 + v^2\rangle$ over $D = \{(u, v) : u^2 + v^2 \le 9,\ v \ge 0\}$

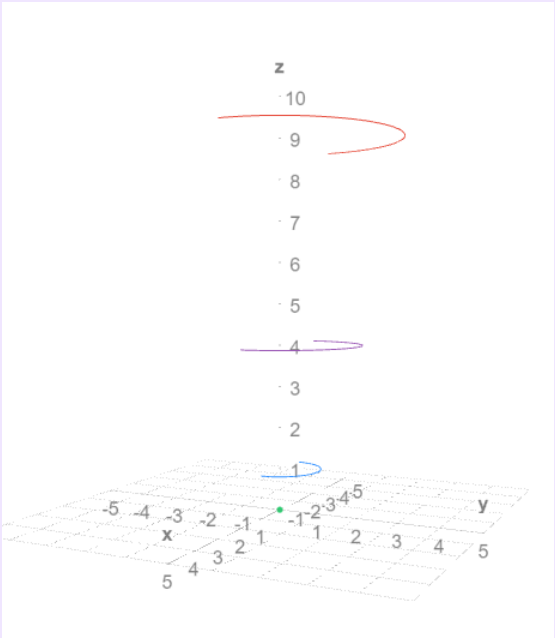
(a) We start by looking at the space curves defined by $\vec{\mathbf{r}}(u_0, v)$ for $u_0 = 0, 1, 2, 3$:

$u = 0$: $\vec{\mathbf{r}}(0, v) = \langle 0, 0, 0\rangle$ for $0 \le v \le 2\pi$ is a point at the origin

$u = 1$: $\vec{\mathbf{r}}(1, v) = \langle \cos(v), \sin(v), 1\rangle$ for $0 \le v \le \pi$ is a semicircle of radius 1 at height $z = 1$

$u = 2$: $\vec{\mathbf{r}}(2, v) = \langle 2\cos(v), 2\sin(v), 4\rangle$ for $0 \le v \le \pi$ is a semicircle of radius 2 at height $z = 4$

$u = 3$: $\vec{\mathbf{r}}(3, v) = \langle 3\cos(v), 3\sin(v), 9\rangle$ for $0 \le v \le \pi$ is a semicircle of radius 3 at height $z = 9$

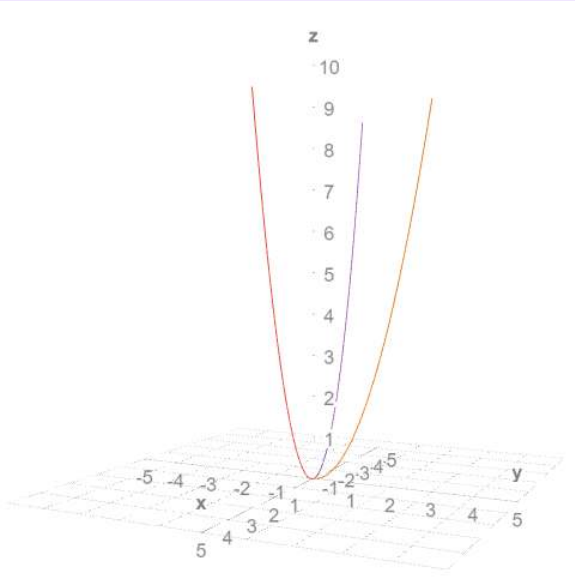


Next we look at the curves defined by $\vec{\mathbf{r}}(u, v_0)$ for $v_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$:

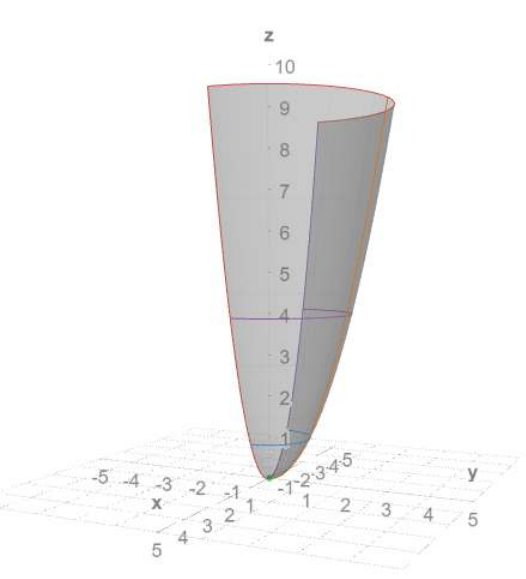
$v = 0$: $\vec{\mathbf{r}}(u, 0) = \langle u, 0, u^2 \rangle$ for $0 \le u \le 3$ is a half parabola on the positive x-axis

$v = \frac{\pi}{2}$: $\vec{\mathbf{r}}\left(u, \frac{\pi}{2}\right) = \langle 0, u, u^2 \rangle$ for $0 \le u \le 3$ is a half parabola on the positive y-axis

$v = \pi$: $\vec{\mathbf{r}}(u, \pi) = \langle -u, 0, u^2 \rangle$ for $0 \le u \le 3$ is a half parabola on the negative x-axis



Putting these curves together, we see they form the skeleton of our surface, which is half of a paraboloid:



(b) We start by looking at the space curves defined by $\vec{\mathbf{r}}(u_0, v)$ for $u_0 = -2, 1, 0, 1, 2$:

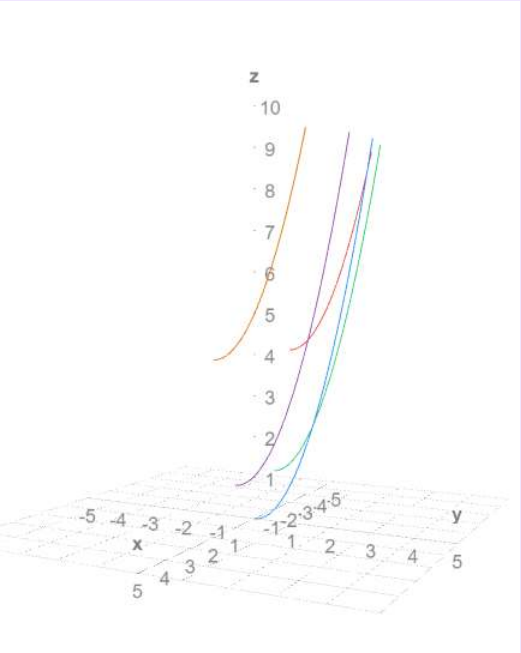
$u = -2$: $\vec{\mathbf{r}}(-2, v) = \langle -2, v, 4 + v^2 \rangle$ for $0 \le v \le \sqrt{5}$ is a half parabola on the $x = -2$ plane.

$u = -1$: $\vec{\mathbf{r}}(-1, v) = \langle -1, v, 1 + v^2 \rangle$ for $0 \le v \le 2\sqrt{2}$ is a half parabola on the $x = -1$ plane.

$u = 0$: $\vec{\mathbf{r}}(0, v) = \langle 0, v, v^2 \rangle$ for $0 \le v \le 3$ is a half parabola on the $x = 0$ plane.

$u = 1$: $\vec{\mathbf{r}}(1, v) = \langle 1, v, 1 + v^2 \rangle$ for $0 \le v \le 2\sqrt{2}$ is a half parabola on the $x = 1$ plane.

$u = -2$: $\vec{\mathbf{r}}(2, v) = \langle 2, v, 4 + v^2 \rangle$ for $0 \le v \le \sqrt{5}$ is a half parabola on the $x = 2$ plane.

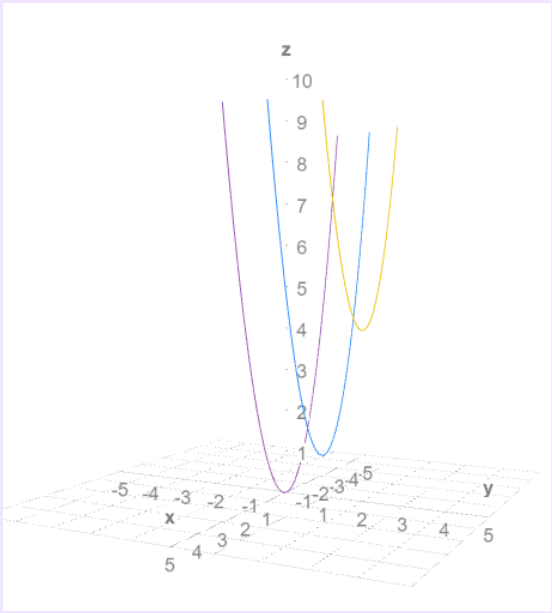


Next we look at the curves defined by $\vec{\mathbf{r}}(u, v_0)$ for $v_0 = 0, 1, 2$:

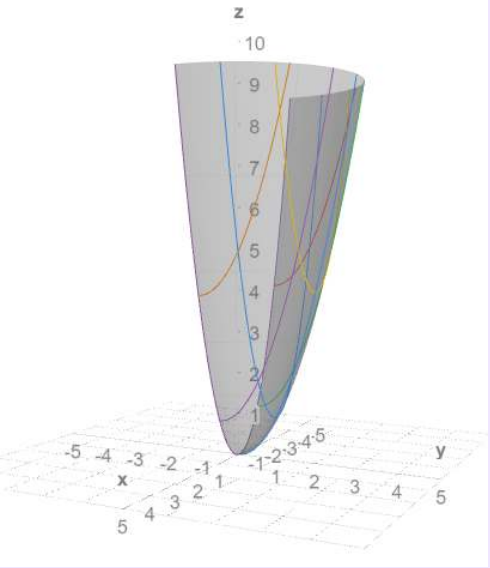
$$v = 0: \quad \vec{\mathbf{r}}(u, 0) = \langle u, 0, u^2 \rangle \text{ for } -3 \le u \le 3 \text{ is a parabola on the } y = 0 \text{ plane.}$$

$$v = 1: \quad \vec{\mathbf{r}}(u, 1) = \langle u, 1, u^2 + 1 \rangle \text{ for } -2\sqrt{2} \le u \le 2\sqrt{2} \text{ is a parabola on the } y = 1 \text{ plane.}$$

$$v = 2: \quad \vec{\mathbf{r}}(u, 2) = \langle u, 2, u^2 + 4 \rangle \text{ for } -\sqrt{5} \le u \le \sqrt{5} \text{ is a parabola on the } y = 2 \text{ plane.}$$



Putting these curves together, we see they form the skeleton of our surface, which is (the same) half of a paraboloid:



Just like with space curves, we see that the same surface can be defined parametrically in different ways. It is sometimes helpful to use elimination of parameters to describe a surface instead of sketching multiple space curves.

Example 2.1.4

Describe the surfaces defined below.

(a) $\vec{\mathbf{r}}(u,v) = \langle 3u\cos(v), u, -2u\sin(v)\rangle$ over $D = \{(u,v) \,:\, u \geq 0,\, 0 \leq v \leq \pi\}$

(b) $\vec{\mathbf{r}}(u,v) = \langle u\sec(v), u\tan(v), u^2\rangle$ over $D = \left\{(u,v) \,:\, 0 \leq u \leq 2,\, -\frac{\pi}{4} < v < \frac{\pi}{4}\right\}$

Solution

(a) We first look to eliminate v by using the identity $cos^2(v) + \sin^2(v) = 1$.

Since $x = 3u\cos(v) \leftrightarrow \cos(v) = \frac{x}{3u}$ and $z = -2u\sin(v) \leftrightarrow \sin(v) = -\frac{z}{2u}$ this gives:

$$\left(\frac{x}{3u}\right)^2 + \left(-\frac{z}{2u}\right)^2 = 1 \leftrightarrow \frac{x^2}{9u^2} + \frac{z^2}{4u^2} = 1$$

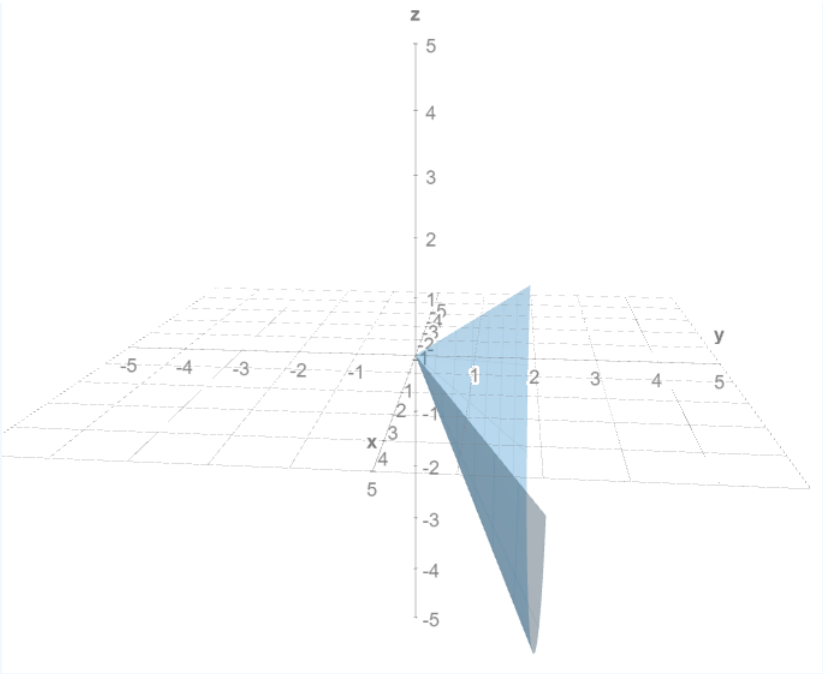
Now we can use $y = u$ to eliminate u from this new equation, giving:

$$\frac{x^2}{9y^2} + \frac{z^2}{4y^2} = 1 \longrightarrow \frac{x^2}{9} + \frac{z^2}{4} = y^2$$

Since $u \geq 0$ in our domain, we know that $y \geq 0$ and can square root both sides to get $y = \sqrt{\frac{x^2}{9} + \frac{z^2}{4}}$

This is recognizable as $\boxed{\text{the cone opening on the positive } y\text{-axis with elliptic cross sections}}$

Since $0 \leq v \leq \pi$ we do not get the full cone! Note that $z = -2u\sin(v) \leq 0$ for all u, so only this is the half of the cone with negative z-values:



(b) We first look to eliminate v by using the identity $\sec^2(v) - \tan^2(v) = 1$.

Since $x = u\sec(v) \leftrightarrow \sec(v) = \dfrac{x}{u}$ and $y = u\tan(v) \leftrightarrow \tan(v) = \dfrac{y}{u}$ this gives:

$$\left(\frac{x}{u}\right)^2 - \left(\frac{y}{u}\right)^2 = 1 \longleftrightarrow \frac{x^2}{u^2} - \frac{y^2}{u^2} = 1$$

Now we can use $z = u^2$ to eliminate u from this new equation, giving:

$$\frac{x^2}{z} - \frac{y^2}{z} = 1 \longrightarrow x^2 - y^2 = z$$

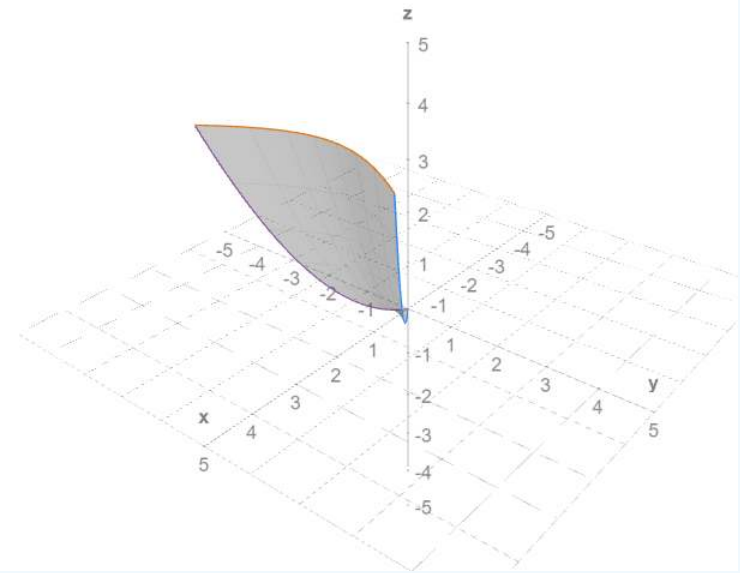
This is recognizable as a $\boxed{\text{hyperbolic paraboloid}}$

Since $-\dfrac{\pi}{4} \leq v \leq \dfrac{\pi}{4}$ we do not get the full hyperbolic paraboloid $z = x^2 - y^2$. On the edges, this is bounded by the curves

$$v = -\frac{\pi}{4}: \quad \vec{\mathbf{r}}\left(u, -\frac{\pi}{4}\right) = \langle u\sqrt{2}, -u, u^2\rangle \text{ which is a half parabola on the plane } x = -y\sqrt{2}$$

$$v = \frac{\pi}{4}: \quad \vec{\mathbf{r}}\left(u, \frac{\pi}{4}\right) = \langle u\sqrt{2}, u, u^2\rangle \text{ which is a half parabola on the plane } x = y\sqrt{2}$$

$$u = 2: \quad \vec{\mathbf{r}}(2, v) = \langle 2\sec(v), 2\tan(v), 4\rangle \text{ which is a partial hyperbola on the plane } z = 4$$



Defining a surface parametrically can be a useful tool when the surface itself is not *well-defined* with respect to any of the variables x, y, or z. Consider the unit sphere

$$x^2 + y^2 + z^2 = 1$$

Solving for $x = \pm\sqrt{1 - y^2 - z^2}$, or $y = \pm\sqrt{1 - x^2 - z^2}$, or $z = \pm\sqrt{1 - x^2 - y^2}$ does not yield a function (because of the plus/minus). This is because the sphere itself does not pass the three dimensional version of the **vertical line test**.

Recall that a relation between x and y can only be written as $y = f(x)$ if every line of the form $x = c$ intersects the curve at most once.

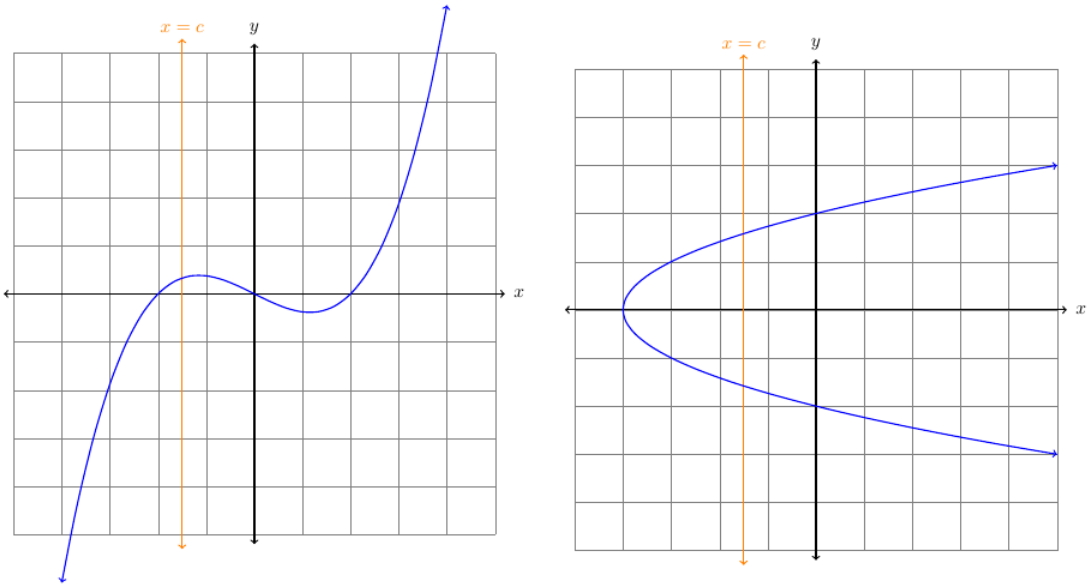


Figure 2.1.6: The graph on the left passes the vertical line test, and can be written as a function $y = f(x)$.
(b) The graph on the right fails the vertical line test, and cannot be written as a function $y = f(x)$ for any well defined function

Similarly, such a relation can be written as a function $x = g(y)$ if and only if every line of the form $y = c$ intersects the curve at most once.

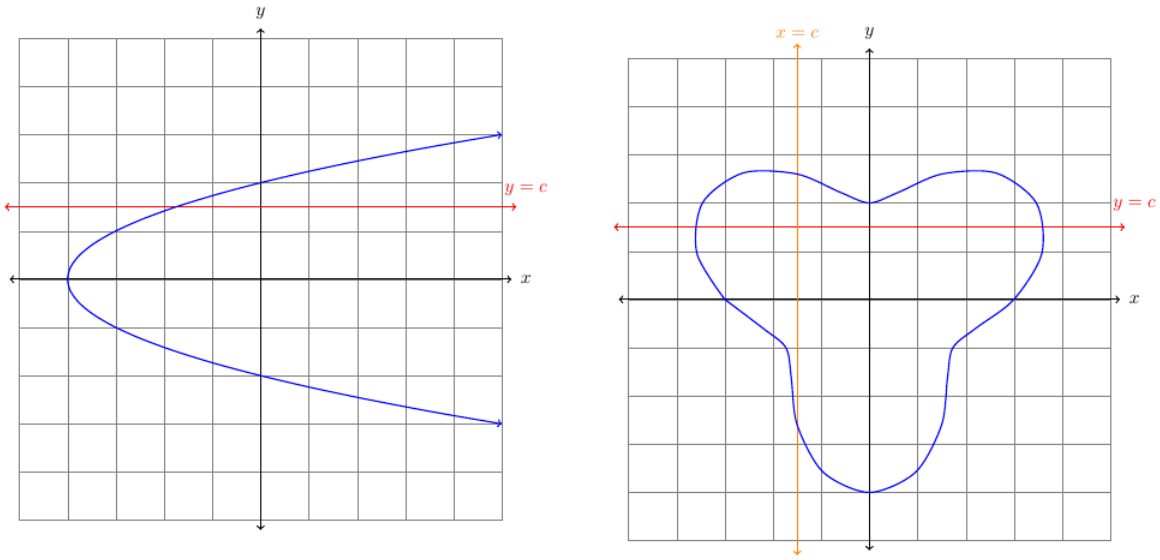
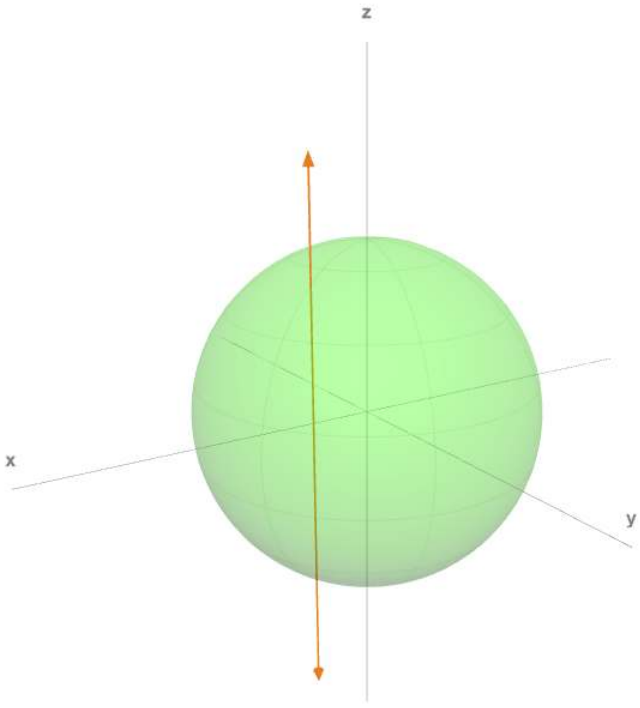


Figure 2.1.7: The graph on the left passes the line test for y, and can be written as a function $x = g(y)$.
(b) The graph on the right fails the line test in both directions, and cannot be written as a function $y = f(x)$ or$x = g(y)$ for any well defined function

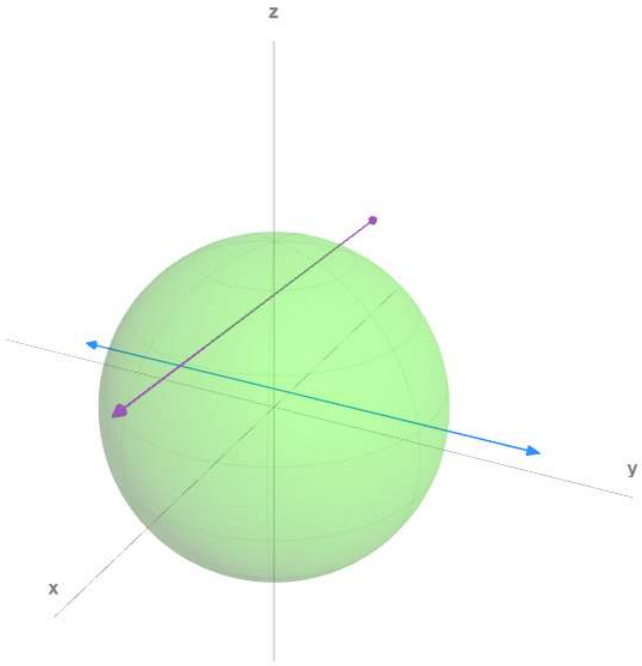
Since the unit circle $x^2 + y^2 = 1$ is not a function in either direction, we had to use trigonometric identities to describe it parametrically:

$$\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t) \rangle \quad \text{for} \quad 0 \leq t \leq 2\pi$$

Returning again to the unit sphere, we see that every line of the form $x = c, y = d$ does not intersect the surface at most once, so it cannot be written $z = f(x, y)$:



Similarly, every line of the form $x = c, z = d$ or $y = c, z = d$ fails this test. This is why we cannot write $y = g(x, z)$ or $x = h(y, z)$ respectively.



Instead we turn again to trigonometry to avoid this issue. Spherical coordinates offer us a fantastic alternative. If we define $u = \theta$ and $v = \phi$ (and set $\rho = 1$), we can write

$$\vec{\mathbf{r}}(u, v) = \langle \sin(v)\cos(u),\ \sin(v)\sin(u),\ \cos(v) \rangle \quad \text{for} \quad 0 \leq u \leq 2\pi,\ 0 \leq v \leq \pi$$

This helps us to define each point on the unit sphere with a one-to-one correspondence to the points in our parametric domain.

Example 2.1.5

Find a parametrization of the surface described by the given relationship in $\mathbb{R}^3$.

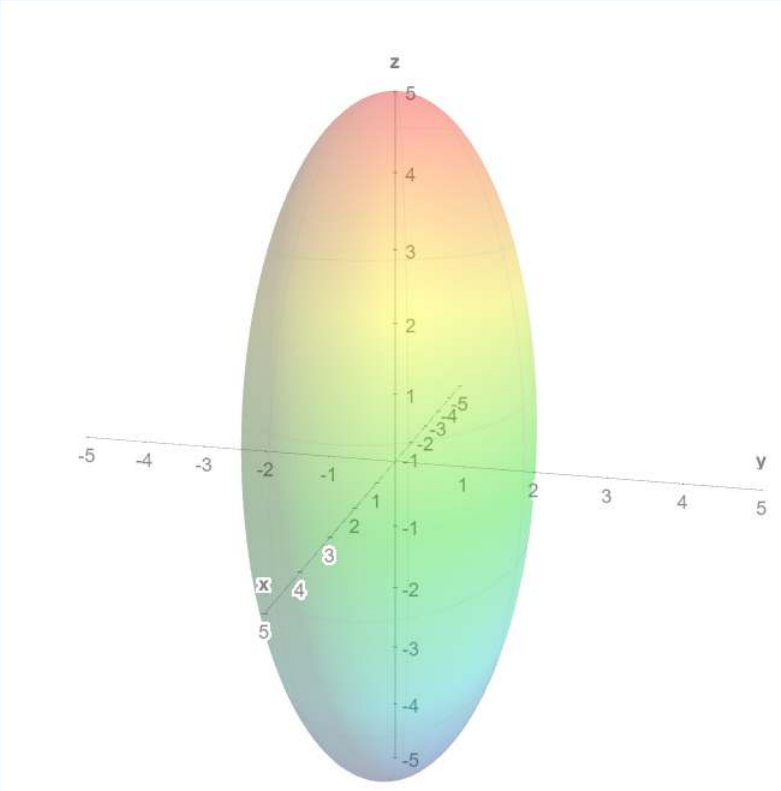
(a) $S = \left\{ (x, y, z) \,:\, \dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{25} = 1 \right\}$

(b) $S = \{ (x, y, z) \,:\, x^2 + z^2 = 4,\ 0 \leq y \leq 2 + z \}$

(c) $S = \{ (x, y, z) \,:\, x = z^2 - y^2,\ 0 \leq y \leq z \leq 2 \}$

Solution

(a) This surface S is an ellipsoid, which is just a sphere with different radii in the x, y and z directions.

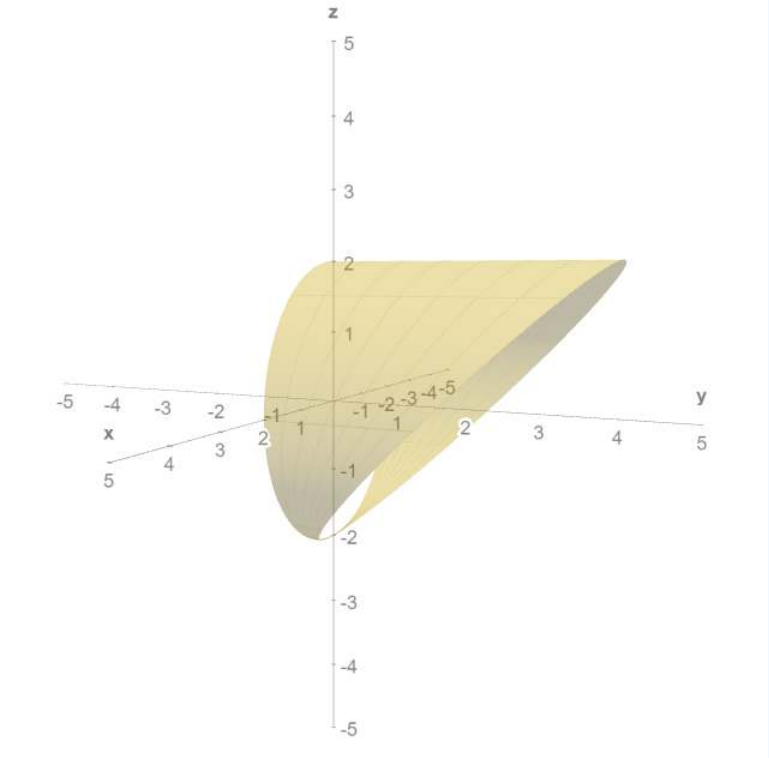


In this case, the x-radius is 3, the y-radis is 2, and the z-radius is 5. Therefore we alter the spherical coordinates to get:

$$\boxed{\vec{\mathbf{r}}(u,v) = \langle 3\sin(v)\cos(u),\ 2\sin(v)\sin(u),\ 5\cos(v)\rangle \quad \text{for} \quad 0 \le u \le 2\pi,\ 0 \le v \le \pi}$$

A quick check shows that this indeed satisfies the equation $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25} = 1$ as desired.

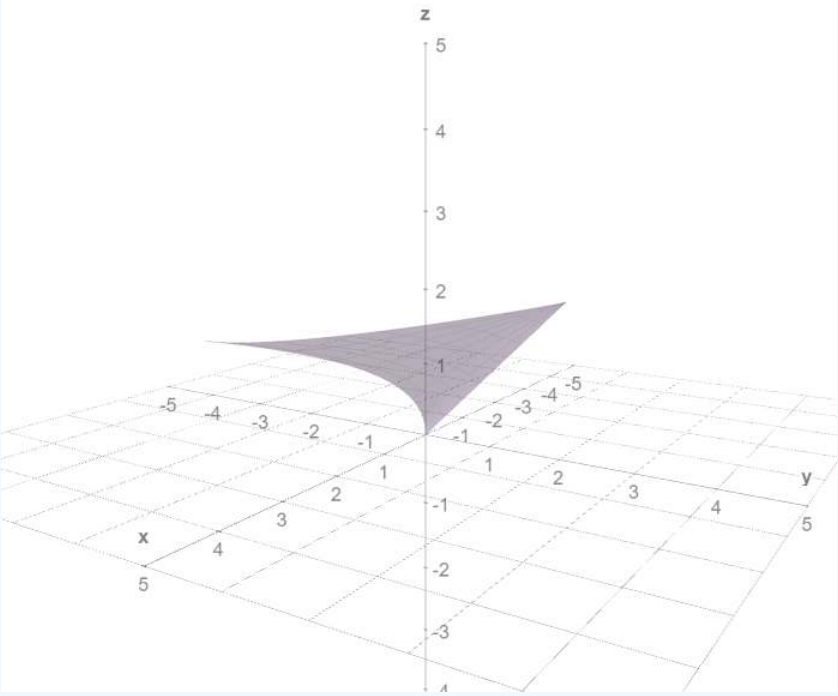
(b) We recognize that S is the part of the cylinder $x^2 + z^2 = 4$ to the right of the plane $y = 0$ and above the plane $y = 2 + z$:



Parameterizing the cylinder can be done by $\boxed{\vec{\mathbf{r}}(u,v) = \langle 2\cos(u), v, 2\sin(u)\rangle}$, and we simply plug this into the restriction $0 \leq y \leq 2 + 2z$ to get our parametric domain: $\boxed{D = \{(u,v) \,:\, 0 \leq u \leq 2\pi,\, 0 \leq v \leq 2 + 2\sin(u)\}}$

(c) While the relationship $x = z^2 - y^2$ (which is a hyperbolic paraboloid) can be parameterized using secants and tangents, we can see that it is already described as a function $x = f(y,z)$. Therefore the easiest parameterization is simply substituting $y = u$ and $z = v$ to get:

$$\boxed{\vec{\mathbf{r}}(u,v) = \langle v^2 - u^2, u, v\rangle \quad \text{over} \quad D = \{(u,v) \,:\, 0 \leq u \leq v \leq 2\}}$$



Describing a surface parametrically can often help while sketching a surface, but we will soon see they can be useful in differentiation and integration as well. If a space curve is considered a way to stretch, shift, or bend an interval $[a, b]$ into three dimensional space, then a surface is analogously a way to stretch, shift, or bend a two dimensional region into $\mathbb{R}^3$.

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2.1: Practice Set

Exercise 2.1.1a

Sketch the level curves for $z = 0, 1, 2, 3$ and draw function $z = \sqrt{(x-y)^2}$ on the domain $D = \{(x, y) : 0 \leq x \leq 4,\ 0 \leq y \leq 4\}$

Answer

Simplifying with each provided value of z we get:

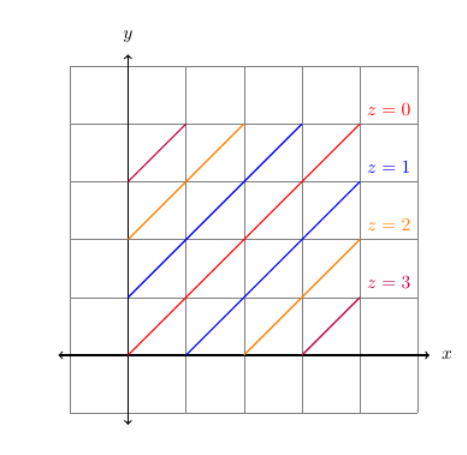
$0 = \sqrt{(x-y)^2} \longleftrightarrow y = x$

$1 = \sqrt{(x-y)^2} \longleftrightarrow y = x \pm 1$

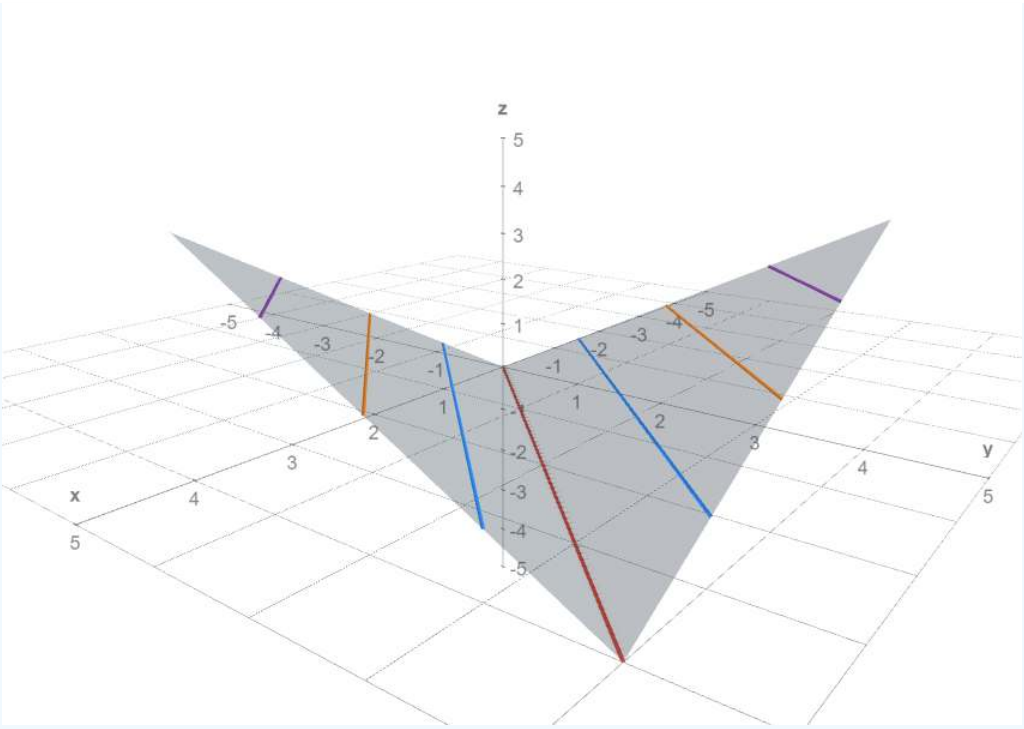
$2 = \sqrt{(x-y)^2} \longleftrightarrow y = x \pm 2$

$3 = \sqrt{(x-y)^2} \longleftrightarrow y = x \pm 3$

The level curves are therefore:



The surface $z = \sqrt{(x-y)^2}$ looks like:



Exercise 2.1.1*b*

Sketch the level curves for $z = 1, 2, 10$ and draw the function $z = x^2 + 9y^2 + 1$

Answer

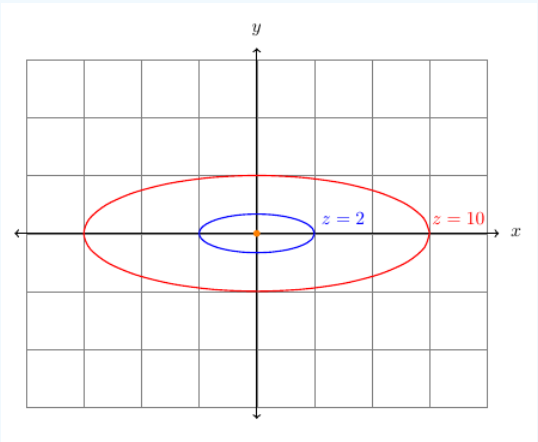
Simplifying with each provided value of z we get:

$1 = x^2 + 9y^2 + 1 \longleftrightarrow x^2 + 9y^2 = 0$

$2 = x^2 + 9y^2 + 1 \longleftrightarrow x^2 + \frac{y^2}{\frac{1}{9}} = 1$

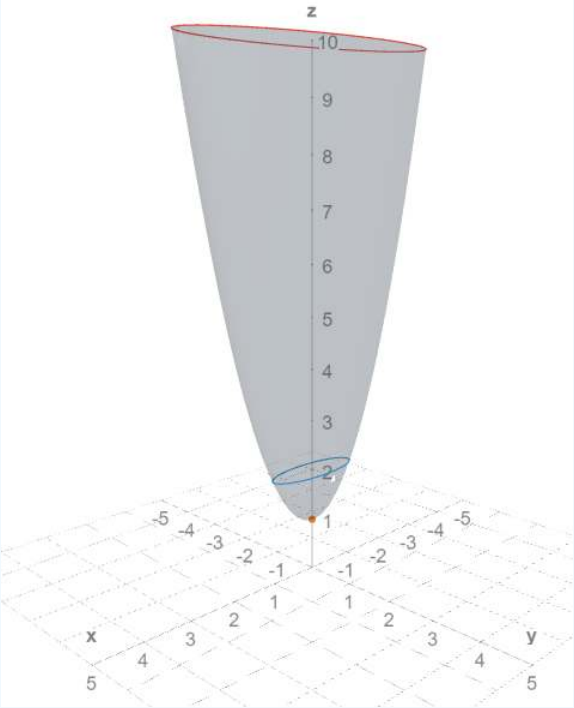
$10 = x^2 + 9y^2 + 1 \longleftrightarrow \frac{x^2}{9} + y^2 = 1$

The level curves are therefore:



The surface $z = x^2 + 9y^2 + 1$ looks like:

https://math.libretexts.org/@go/page/191910



Exercise 2.1.1c

Sketch the level curves for $z = -2, -1, 0, 1, 2$ and draw the function $z = \dfrac{2y}{x^2 + y^2}$

Answer

Simplifying with each provided value of z we get:

$$-2 = \frac{2y}{x^2 + y^2} \longleftrightarrow x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$$

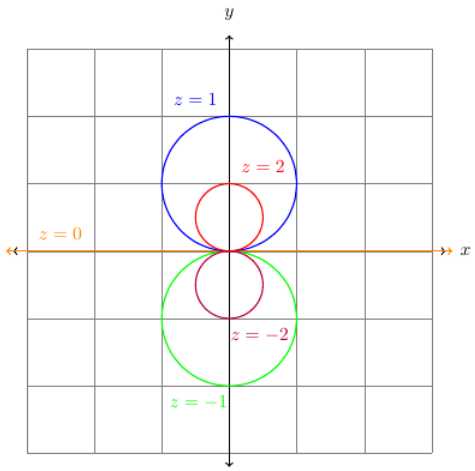
$$-1 = \frac{2y}{x^2 + y^2} \longleftrightarrow x^2 + (y + 1)^2 = 1$$

$$0 = \frac{2y}{x^2 + y^2} \longleftrightarrow y = 0$$

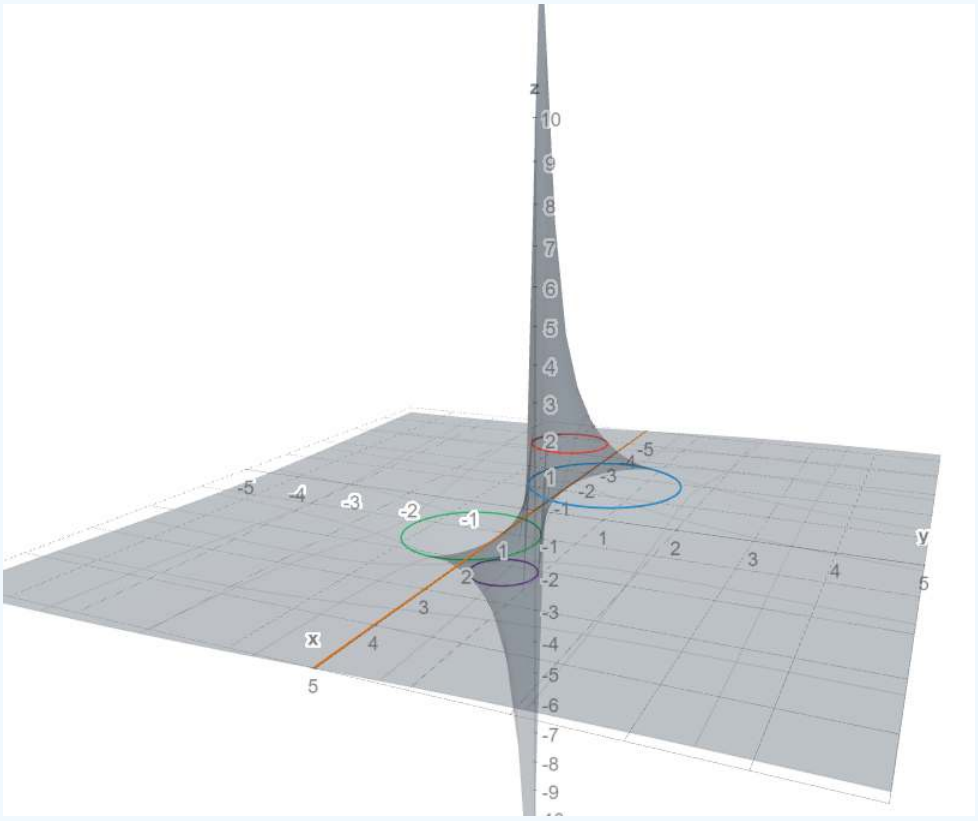
$$1 = \frac{2y}{x^2 + y^2} \longleftrightarrow x^2 + (y - 1)^2 = 1$$

$$2 = \frac{2y}{x^2 + y^2} \longleftrightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

The level curves are therefore:



The surface $z = \dfrac{2y}{x^2 + y^2}$ looks like:



Exercise 2.1.1d

Sketch the level curves for $z = \dfrac{1}{2}, 1, 2$ and draw the function $z = \dfrac{1}{\sqrt[3]{x^2 - y^2}}$

Answer

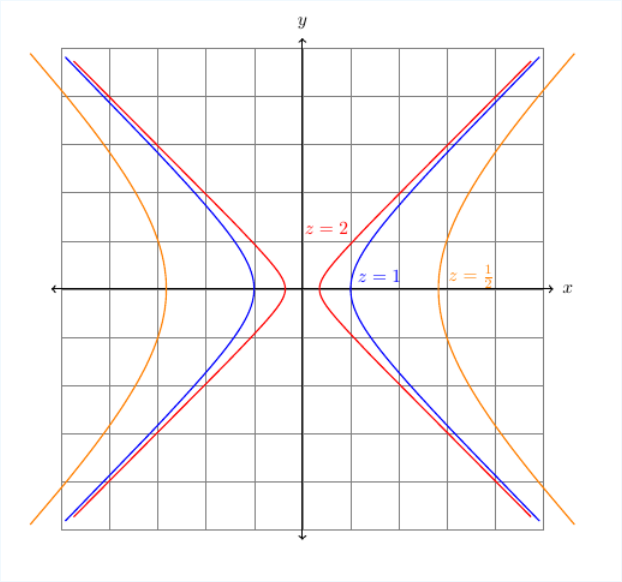
Simplifying with each provided value of z we get:

$$\frac{1}{2} = \frac{1}{\sqrt[3]{x^2 - y^2}} \leftrightarrow \frac{x^2}{8} - \frac{y^2}{8} = 1$$

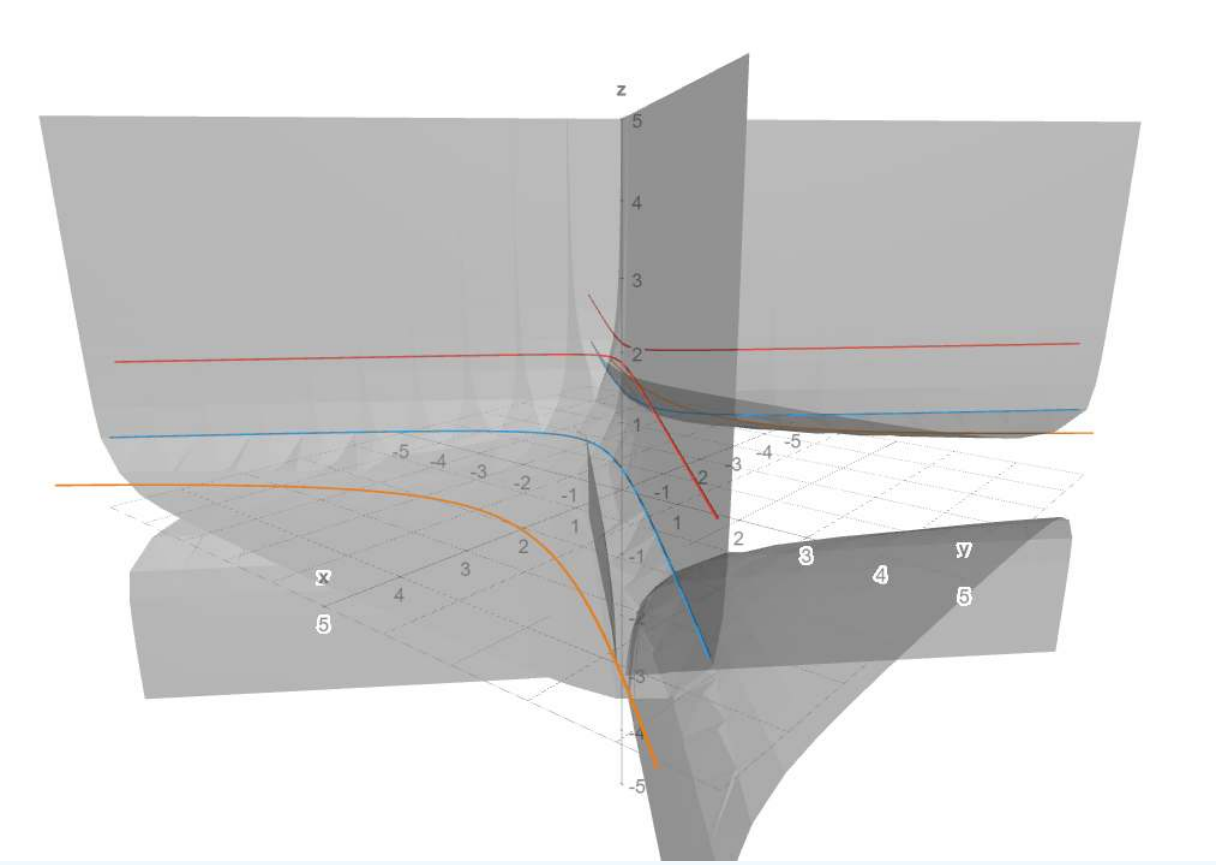
$$1 = \frac{1}{\sqrt[3]{x^2 - y^2}} \longleftrightarrow x^2 - y^2 = 1$$

$$2 = \frac{1}{\sqrt[3]{x^2 - y^2}} \longleftrightarrow \frac{x^2}{\frac{1}{8}} - \frac{y^2}{\frac{1}{8}} = 1$$

The level curves are therefore:



The surface $z = \frac{1}{\sqrt[3]{x^2 - y^2}}$ looks like:



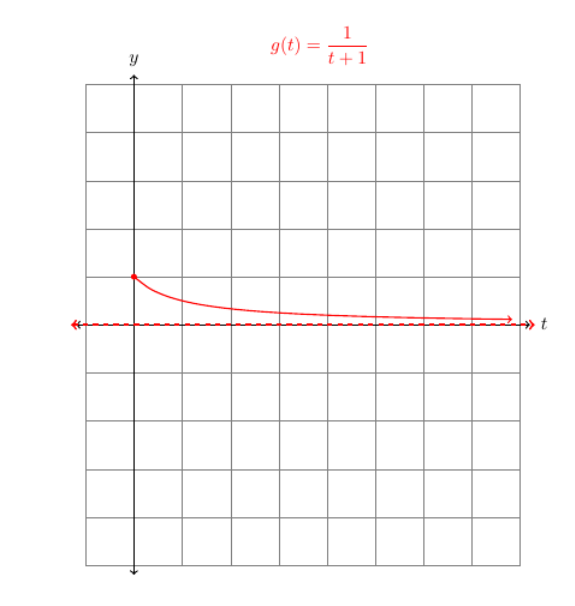
Exercise 2.1.2a

Find the domain and range of the function $f(x,y) = \dfrac{1}{x^2+y^2+1}$

Answer

We can see that the denominator of $f(x,y)$ can never equal zero, so the domain is all of $\mathbb{R}^2$.

Considering that $t = x^2+y^2 \geq 0$, the range of $f(x,y)$ is the same as that of the function $g(t) = \dfrac{1}{t+1}$ for $t \geq 0$:



This function has its maximum at $t = 0$, where $g(0) = 1$ and achieves all values down to (but not including) zero. Therefore the range of $f(x,y)$ is the interval $(0,1]$.

Exercise 2.1.2b

Find the domain and range of the function $f(x,y) = \sin^{-1}\left(3x^2+y^2-2\right)$

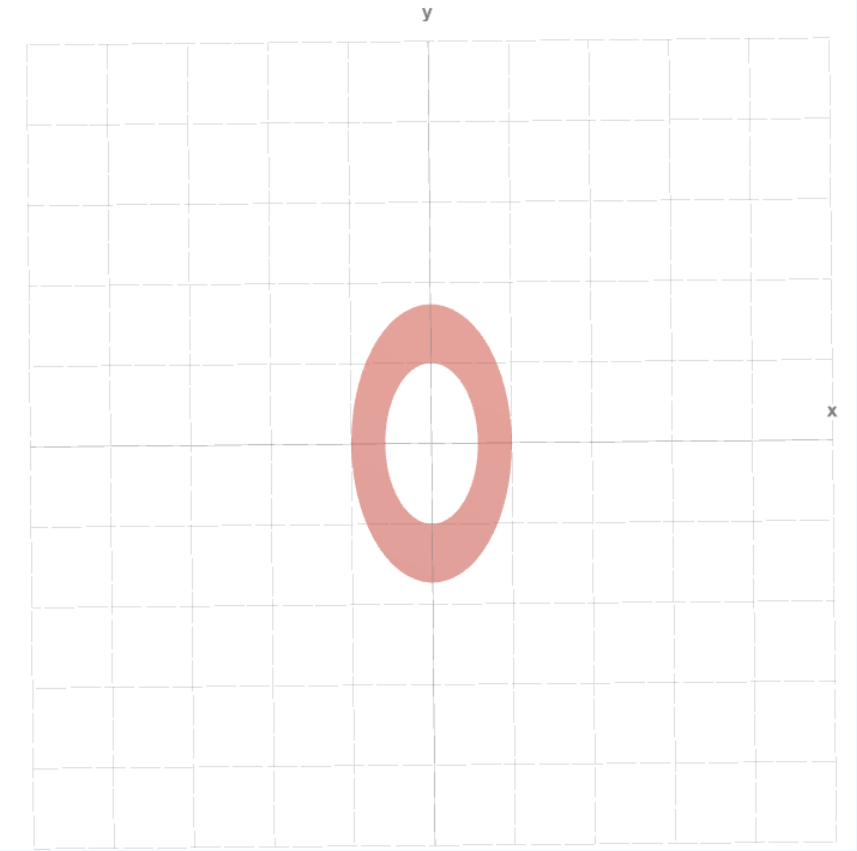
Answer

We know the domain of $\sin^{-1}(t)$ is $[-1,1]$, therefore the domain of $f(x,y)$ is $D = \{(x,y) : -1 \leq 3x^2+y^2-2 \leq 1\}$.

These edge cases can be re-written as:

$$-1 = 3x^2+y^2-2 \longleftrightarrow 3x^2+y^2 = 1 \quad \text{and} \quad 1 = 3x^2+y^2-2 \longleftrightarrow x^2+\frac{y^2}{3} = 1$$

Therefore the domain is the subset of $\mathbb{R}^2$ below:



Considering that $t = 3x^2 + y^2 - 2$ ranges entirely between -1 and 1 in this domain, we see that $sin^{-1}(t)$ reaches its every value in its range, which is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Exercise $2.1.2c$

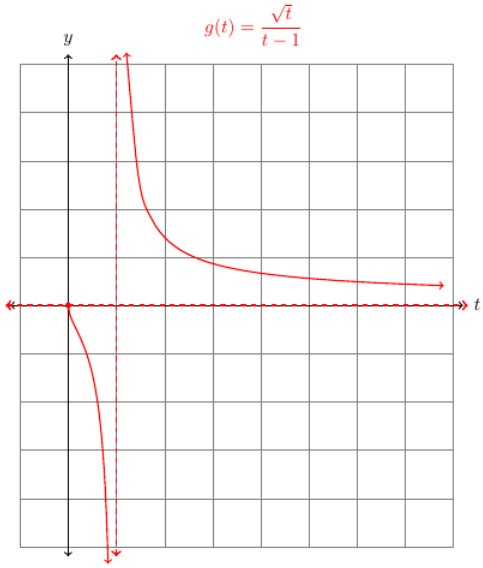
Find the domain and range of the function $f(x, y, z) = \dfrac{\sqrt{x^2 + y^2 + z^2}}{x^2 + y^2 + z^2 - 1}$

Answer

Since the value of $t = x^2 + y^2 + z^2$ is always greater than or equal to zero, there is no domain issue with the square root in $f(x, y, z)$.

To ensure that the denominator is not zero, we simply ensure that $t \neq 1$, therefore the domain of f is $D = \{(x, y, z) : x^2 + y^2 + z^2 \neq 1\}$

The range of $f(x, y, z)$ will match that of the single variable function $g(t) = \dfrac{\sqrt{t}}{t - 1}$:



We can clearly see this range is all of $\mathbb{R}$, so the range of $f(x, y, z)$ is also all of $\mathbb{R}$.

Exercise 2.1.3a

Sketch the surface parameterized by $\vec{\mathbf{r}}(u, v) = \langle u\sin(v), 2, 3u\cos(v)\rangle$ with parametric domain $D = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 2\pi\}$

Answer

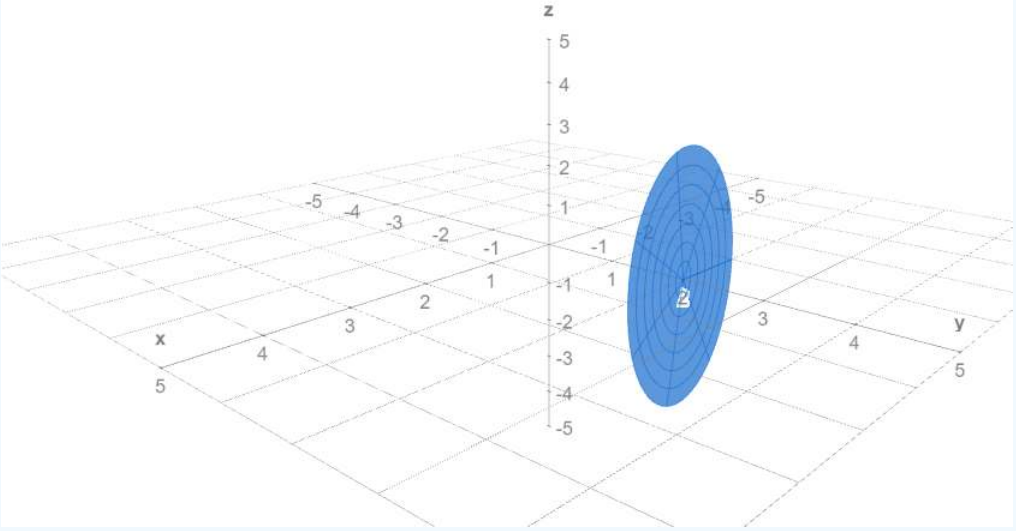
We can rewrite $x = u\sin(v) \longleftrightarrow \frac{x}{u} = \sin(v)$ and $z = 3u\cos(v) \longleftrightarrow \frac{z}{3u} = \cos(v)$

Therefore since $\sin^2(v) + \cos^2(v) = 1$ we know that $\frac{x^2}{u^2} + \frac{z^2}{9u^2} = 1 \longleftrightarrow x^2 + \frac{z^2}{9} = u^2$

Since $y = 2$ is a constant, it does not relate to x and z. Instead, we have $0 \leq u \leq 1$, so:

$0 \leq x^2 + \frac{z^2}{9} \leq 1$ is the interior of an ellipse in the plane $y = 2$. Since $0 \leq v \leq 2\pi$ we have the full ellipse.

Therefore we have a flat elliptical disc in the plane $y = 2$ as pictured below:



Exercise 2.1.3b

Sketch the surface parameterized by $\vec{\mathbf{r}}(u,v)\,\langle \sin(u), \cos(u), v\rangle$ with parametric domain $D = \left\{(u,v) \,:\, 0 \le u \le \frac{\pi}{2},\, 1 \le v \le 2\right\}$

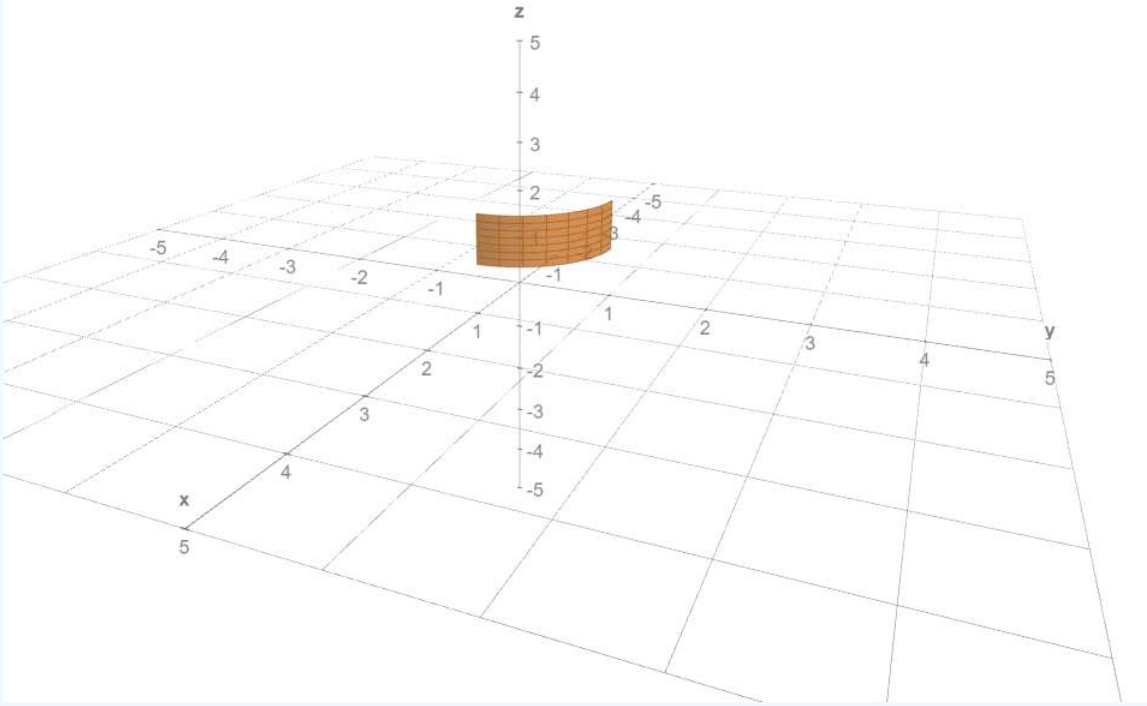
Answer

We can see that $x = \sin(u)$ and $y = \cos(u)$, and thus $\sin^2(u) + \cos^2(u) = 1 \longleftrightarrow x^2 + y^2 = 1$

This is the surface of a cylinder with unit circle cross sections along the z-axis. Since $0 \le u \le \frac{\pi}{2}$, we must have $x \ge 0$ and $y \ge 0$.

Because $z = v$ and $1 \le v \le 2$, we must have $1 \le z \le 2$.

Thus we have a quarter cylinder as pictured below:



Exercise $2.1.3c$

Sketch the surface parameterized by $\vec{\mathbf{r}}(u,v)\,\langle u-v, 2u+v, -u+3v\rangle$ with parametric domain $D=\{(u,v)\,:\,0\le u\le 1,\,0\le v\le 1\}$

Answer

This surface is bounded by four straight line segments:

$u=0:\quad \vec{\mathbf{r}}(0,v)=\langle -v, v, 3v\rangle$ for $0\le v\le 1$

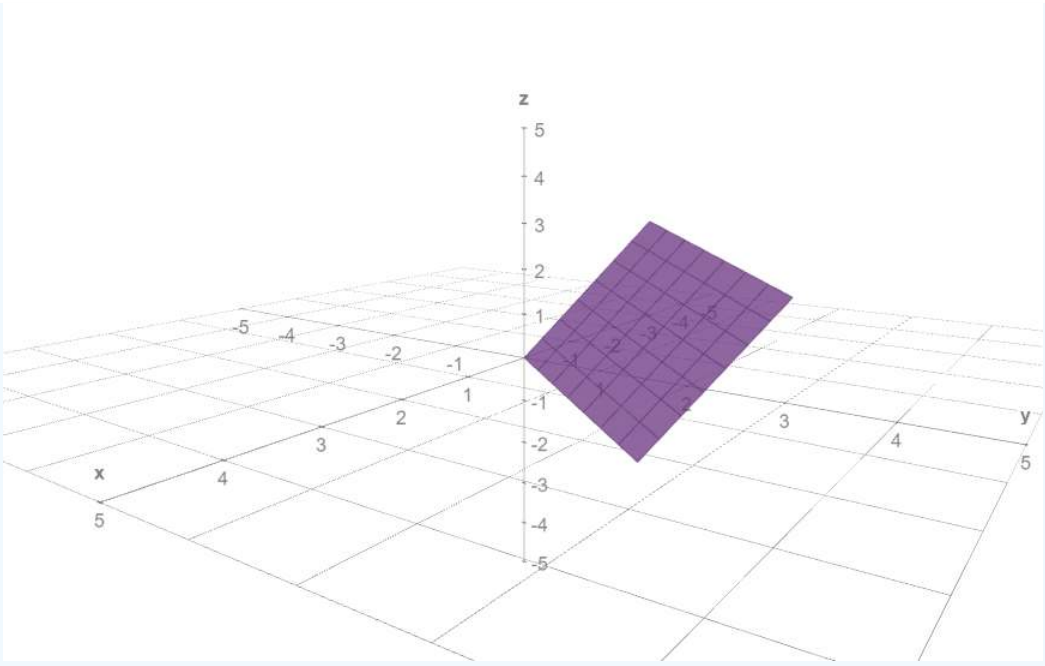
$u=1:\quad \vec{\mathbf{r}}(1,v)=\langle 1-v, 2+v, -1+3v\rangle$ for $0\le v\le 1$

$v=0:\quad \vec{\mathbf{r}}(u,0)=\langle u, 2u, -u\rangle$ for $0\le u\le 1$

$v=1:\quad \vec{\mathbf{r}}(u,1)=\langle u-1, 2u+1, -u+3\rangle$ for $0\le u\le 1$

These must make a quadrilateral with vertices at $\vec{\mathbf{r}}(0,0)=(0,0,0)$, $\vec{\mathbf{r}}(0,1)=(-1,1,3)$, $\vec{\mathbf{r}}(1,0)=(1,2,-1)$, $\vec{\mathbf{r}}(1,1)=(0,3,2)$

Therefore the surface is a parallelogram as pictured below:



Exercise 2.1.3d

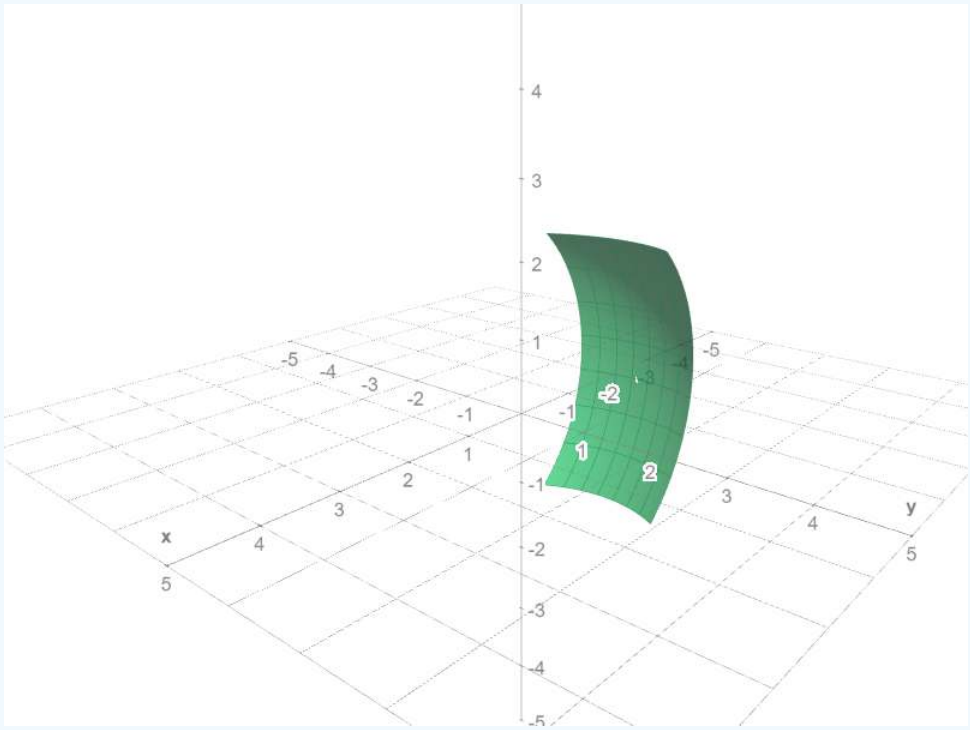
Sketch the surface parameterized by $\vec{\mathbf{r}}(u,v)\left\langle -\sqrt{8-u^2-v^2}, u, v\right\rangle$ with parametric domain $D = \{(u,v) : -1 \le u \le 1, -2 \le v \le 2\}$

Answer

This surface is defined as a function $x = -\sqrt{8-y^2-z^2}$, which is part of the sphere $x^2+y^2+z^2 = 8$ where x is negative.

Since $y = u$ and $z = v$, the parametrization tells us we only want the part where $-1 \le y \le 1$ and $-2 \le z \le 2$.

This gives the picture below:



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2.2: Multivariable Limits

Learning Objectives

- Calculate the limit of a function of two variables.
- Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.
- State the conditions for continuity of a function of two variables.
- Verify the continuity of a function of two variables at a point.
- Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

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2.2: Limits and Continuity

Recall the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| < \varepsilon$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition: δ Disks

Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) — that is,

$$\{(x, y) \in \mathbb{R}^2 \, : \, (x - a)^2 + (y - b)^2 < \delta^2\}$$

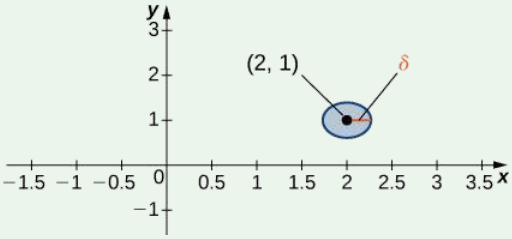


Figure 2.2.1: A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta$$

In more than one dimension, we use a δ disk.

Definition: Limit of a Function of 2 Variables

Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x, y) in a δ disk around (a, b), except possibly for (a, b) itself, the value of $f(x, y)$ is no more than ε away from L.

Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$

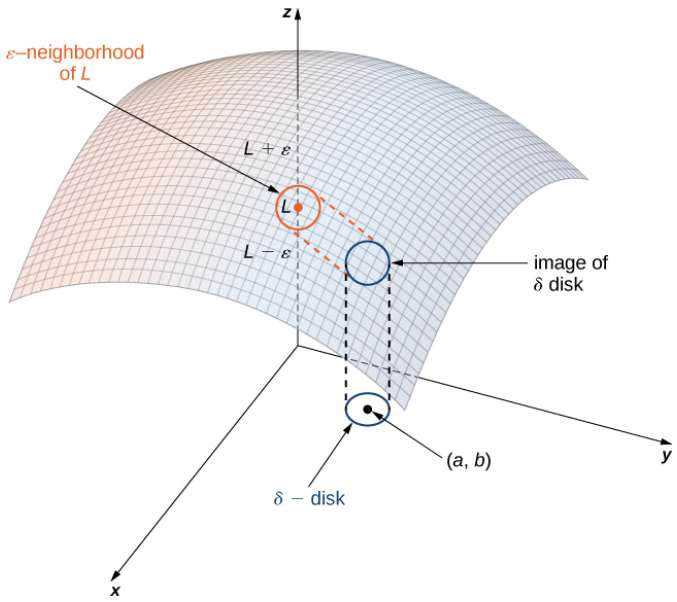


Figure 2.2.2: The limit requires $f(x,y)$ be within ε of L whenever (x,y) is within δ of (a,b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a **limit of a function of two variables** can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the properties of single variable limits that we already know.

Theorem 2.2.1: Limit Laws for Functions of Two Variables

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighborhood around (a,b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = M$$

and let $c \in \mathbb{R}$ be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y)\to(a,b)} c = c$$

Identity Laws:

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

Sum Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$$

Difference Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$$

Constant Multiple Law:

$$\lim_{(x,y)\to(a,b)} (cf(x,y)) = cL$$

Product Law:

$$\lim_{(x,y)\to(a,b)}(f(x,y)g(x,y)) = LM$$

Quotient Law:

$$\lim_{(x,y)\to(a,b)}\frac{f(x,y)}{g(x,y)} = \frac{L}{M} \text{ for } M \neq 0$$

Power Law:

$$\lim_{(x,y)\to(a,b)}(f(x,y))^n = L^n \text{ for any positive integer } n$$

Root Law:

$$\lim_{(x,y)\to(a,b)}\sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 2.2.1: Finding the Limit of a Function of Two Variables

Find each of the following limits:

(a) $\lim_{(x,y)\to(2,-1)}(x^2 - 2xy - 4x + 3y - 6)$

(b) $\lim_{(x,y)\to(2,-1)}\dfrac{2x+3y}{4x-3y}$

(c) $\lim_{(x,y)\to(5,-2)}\sqrt[3]{\dfrac{x^2-y}{y^2-x}}$

Solution

(a) First use the sum and difference laws to separate the terms:

$$\lim_{(x,y)\to(2,-1)}(x^2 - 2xy - 4x + 3y - 6)$$
$$= \left(\lim_{(x,y)\to(2,-1)}x^2\right) - \left(\lim_{(x,y)\to(2,-1)}2xy\right) - \left(\lim_{(x,y)\to(2,-1)}4x\right) + \left(\lim_{(x,y)\to(2,-1)}3y\right) - \left(\lim_{(x,y)\to(2,-1)}6\right)$$

Next, use the constant multiple law on the second, third, and fourth limits:

$$= \left(\lim_{(x,y)\to(2,-1)}x^2\right) - 2\left(\lim_{(x,y)\to(2,-1)}xy\right) - 4\left(\lim_{(x,y)\to(2,-1)}x\right) + 3\left(\lim_{(x,y)\to(2,-1)}y\right) - \left(\lim_{(x,y)\to(2,-1)}6\right)$$

Now, use the power law on the first limit, and the product law on the second limit:

$$\left(\lim_{(x,y)\to(2,-1)}x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)}x\right)\left(\lim_{(x,y)\to(2,-1)}y\right) - 4\left(\lim_{(x,y)\to(2,-1)}x\right) + 3\left(\lim_{(x,y)\to(2,-1)}y\right) - \left(\lim_{(x,y)\to(2,-1)}6\right)$$

Last, use the identity laws on the first five limits and the constant law on the last limit:

$$\lim_{(x,y)\to(2,-1)}(x^2 - 2xy - 4x + 3y - 6) = (2)^2 - 2(2)(-1) - 4(2) + 3(-1) - 6 = \boxed{-9}$$

(b) Before applying the quotient law, we verify that the limit of the denominator is nonzero using the difference law, constant multiple law, and identity law:

$$\lim_{(x,y)\to(2,-1)}(4x-3y)=\lim_{(x,y)\to(2,-1)}4x-\lim_{(x,y)\to(2,-1)}3y$$
$$=4\left(\lim_{(x,y)\to(2,-1)}x\right)-3\left(\lim_{(x,y)\to(2,-1)}y\right)=4(2)-3(-1)=11$$

We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\lim_{(x,y)\to(2,-1)}(2x+3y)=\lim_{(x,y)\to(2,-1)}2x+\lim_{(x,y)\to(2,-1)}3y$$
$$=2\left(\lim_{(x,y)\to(2,-1)}x\right)+3\left(\lim_{(x,y)\to(2,-1)}y\right)=2(2)+3(-1)=1$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)}\frac{2x+3y}{4x-3y}=\frac{\lim\limits_{(x,y)\to(2,-1)}(2x+3y)}{\lim\limits_{(x,y)\to(2,-1)}(4x-3y)}=\boxed{\frac{1}{11}}$$

(c) We start by separately evaluating the limits of the numerator and denominator inside of the radical:

$$\lim_{(x,y)\to(5,-2)}(x^2-y)=\left(\lim_{(x,y)\to(5,-2)}x\right)^2-\left(\lim_{(x,y)\to(5,-2)}y\right)=(5)^2-(-2)=27$$

$$\lim_{(x,y)\to(5,-2)}(y^2-x)=\left(\lim_{(x,y)\to(5,-2)}y\right)^2-\left(\lim_{(x,y)\to(5,-2)}x\right)=(-2)^2-5=-1$$

Since the denominator is nonzero, we can use the quotient law to see that

$$\lim_{(x,y)\to(5,-2)}\frac{x^2-y}{y^2-x}=\frac{\lim\limits_{(x,y)\to(5,-2)}(x^2-y)}{\lim\limits_{(x,y)\to(5,-2)}(y^2-x)}=\frac{27}{-1}=-27$$

Using the root law, we get our final answer:

$$\lim_{(x,y)\to(5,-2)}\sqrt[3]{\frac{x^2-y}{y^2-x}}=\sqrt[3]{-27}=\boxed{-3}$$

Recall that in single variable calculus, limits were just as tedious at the beginning. Then we learn that many functions are continuous, that is:

$$\lim_{x\to a}f(x)=f(a)$$

Which allows us to quickly evaluate a limit by plugging a number directly into $f(x)$. We extend this definition into two variable functions:

Definition: Continuous Functions of 2 variables

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied:

1. $f(a,b)$ exists
2. $\lim\limits_{(x,y)\to(a,b)}f(x,y)$ exists
3. $\lim\limits_{(x,y)\to(a,b)}f(x,y)=f(a,b)$

Illustration 2.2.1: Demonstrating Continuity for a Function of Two Variables

Show that each function is continuous at the given point.

(a) $f(x,y)=x^2-2xy-4x+3y-6$ at $(2,-1)$

(b) $g(x,y)=\dfrac{2x+3y}{4x-3y}$ at $(2,-1)$

(c) $h(x,y) = \sqrt[3]{\dfrac{x^2-y}{y^2-x}}$ at $(5,-2)$

(a) The domain of $f(x,y)$ is clearly all of $\mathbb{R}^2$ so we know $f(2,-1)$ exists, and that

$$f(2,-1) = 2^2 - 2\cdot 2\cdot(-1) - 4\cdot 2 + 3\cdot(-1) - 6 = -9$$

We have shown in Example 1a above that the limit exists, and calculated that $\lim\limits_{(x,y)\to(2,-1)} f(x,y) = -9$

Since these both exist and match, we know that $f(x,y)$ is continuous at $(2,-1)$ (in fact it is continuous at all points in $\mathbb{R}^2$)

(b) The domain of $g(x,y)$ is $D_g = \{(x,y) : 4x \neq 3y\}$ and since $4\cdot 2 \neq 3\cdot(-1)$ we know $g(2,-1)$ exists.

Plugging in this point gives $g(2,-1) = \dfrac{2(2)+3(-1)}{4(2)-3(-1)} = \dfrac{1}{11}$

We have shown in Example 1b above that the limit exists, and calculated that $\lim\limits_{(x,y)\to(2,-1)} g(x,y) = \dfrac{1}{11}$

Since these both exist and match, we know that $g(x,y)$ is continuous at $(2,-1)$ (in fact it is continuous at all points in D_g)

(c) The domain of $h(x,y)$ is $D_h = \{(x,y) : y^2 \neq x\}$ and since $(-2)^2 \neq 5$ we know $h(5,-2)$ exists.

Plugging in this point gives $h(5,-2) = \sqrt[3]{\dfrac{5^2-(-2)}{(-2)^2-5}} = \sqrt[3]{-27} = -3$

We have shown in Example 1c above that the limit exists, and calculated that $\lim\limits_{(x,y)\to(5,-2)} h(x,y) = -3$

Since these both exist and match, we know that $h(x,y)$ is continuous at $(5,-2)$ (in fact it is continuous at all points in D_h)

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0,y_0) in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ it is true, $|f(x,y) - f(a,b)| < \varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

Theorem 2.2.2: The Sum of Continuous Functions is Continuous

If $f(x,y)$ is continuous at (x_0,y_0), and $g(x,y)$ is continuous at (x_0,y_0), then $f(x,y)+g(x,y)$ is continuous at (x_0,y_0).

Theorem 2.2.3: The Product of Continuous Functions is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0,y_0).

Theorem 2.2.4: The Composition of Continuous Functions is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq R$. Suppose g is continuous at some point $(x_0,y_0) \in D$ and define $z_0 = g(x_0,y_0)$. Let f be a function that maps R to R such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0,y_0).

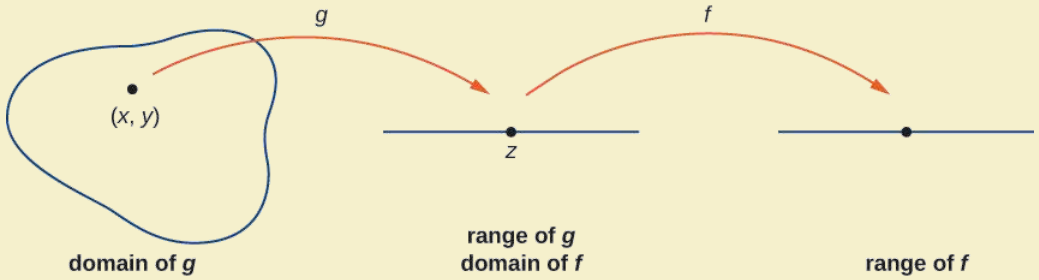


Figure 2.2.3: The composition of two continuous functions is continuous.

Together these theorems tell us that many limits of two variable functions are equal to the value of that function at the limit point. As long as the pieces that make up the function are continuous, *and we don't divide by zero*. Let's now use these theorems to show continuity of functions in the following examples.

Illustration 2.2.2: More Examples of Continuity of a Function of Two Variables

Show that for any $a, b \in \mathbb{R}^2$ we have:

(a) $\lim\limits_{(x,y)\to(a,b)} \cos(4x^3y^2) = \cos(4a^3b^2)$

(b) $\lim\limits_{(x,y)\to(a,b)} (2x^2y^3+3)^4 = (2a^2b^3+3)^4$

(a) The polynomials $p(x) = 4x^3$ and $q(y) = y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane. Since $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane and $g(u) = \cos u$ is continuous at every real number u, the continuity of the composition of functions tells us that $w(x,y) = \cos(4x^3y^2)$ is continuous at every point (x,y) in the xy-plane.

Therefore we must have $\lim\limits_{(x,y)\to(a,b)} \cos(4x^3y^2) = \lim\limits_{(x,y)\to(a,b)} w(x,y) = w(a,b) = \cos(4a^3b^2)$ as desired.

(b) The polynomials $p(x) = 2x^2$ and $q(y) = y^3$ are continuous at every real number; therefore, by the product of continuous functions theorem, $f(x,y) = 2x^2y^3$ is continuous at every point (x,y) in the xy-plane. Furthermore, any constant function is continuous everywhere, so $g(x,y) = 2x^2y^3 + 3$ is continuous at every point (x,y) in the xy-plane. Last, $h(u) = u^4$ is continuous at every real number u, so by the continuity of composite functions theorem $w(x,y) = (2x^2y^3+3)^4$ is continuous at every point (x,y) in the xy-plane.

Therefore we must have $\lim\limits_{(x,y)\to(a,b)} (2x^2y^3+3)^4 = \lim\limits_{(x,y)\to(a,b)} w(x,y) = w(a,b) = (2a^2b^3+3)^4$ as desired.

Of course, not all functions of single variables are continuous everywhere, for instance $f(x) = \dfrac{x^2-1}{x-1}$ at $x = 1$:

$$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{x^2-1}{x-1} = \lim_{x\to1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to1}(x+1) = 2$$

Even though $\lim\limits_{x\to1} f(x)$ exists, it does not match $f(1)$ (which does not exist).

Alternatively, consider $g(x) = \dfrac{|x|}{x}$ at $x = 0$:

$$\lim_{x\to0^+} g(x) = \lim_{x\to0^+} \frac{|x|}{x} = \lim_{x\to0^+} \frac{x}{x} = 1$$

$$\lim_{x\to0^-} g(x) = \lim_{x\to0^-} \frac{|x|}{x} = \lim_{x\to0^-} \frac{-x}{x} = -1$$

Since these two one-sided limits do not agree, we say that $\lim\limits_{x\to0} g(x)$ does not exist.

For functions of two variables, both of these are possible outcomes as well.

We next focus on the difficult task of determining if a limit exists at a discontinuity (typically outside of the domain of the limiting function). In single variable limits like for $g(x)$ above, we say the limit does not exist when the *left-handed* and *right-handed* limits disagree:

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x) \implies \lim_{x\to a} f(x) = \text{DNE}$$

For one variable there are only two "directions" that x can approach a, (the right and left), but for a point (a,b) in $\mathbb{R}^2$ there are an *infinite number* of directions for which $(x,y) \to (a,b)$ can occur. If the answer varies depending on the path taken toward (a,b), then the limit fails to exist. In other words, **the limit must be unique, regardless of path taken in order to exist.**

Illustration 2.2.3: Limits That Fail to Exist

Show that neither of the following limits exist:

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{3x^2+y^2}$

(b) $\lim\limits_{(x,y)\to(2,1)} \dfrac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$

(a) The domain of the function $f(x,y)=\dfrac{2xy}{3x^2+y^2}$ consists of all points in the xy-plane except for the point $(0,0)$. To show that the limit does not exist as (x,y) approaches $(0,0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0,0)$. First, consider the line $y=0$ in the xy-plane. Substituting $y=0$ into $f(x,y)$ gives

$$f(x,0)=\frac{2x(0)}{3x^2+0^2}=0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y=x$. Substituting $y=x$ into $f(x,y)$ gives

$$f(x,x)=\frac{2x(x)}{3x^2+x^2}=\frac{2x^2}{4x^2}=\frac{1}{2}$$

This is true for any point on the line $y=x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\dfrac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $\dfrac{1}{2}$—say, $\dfrac{1}{4}$. Then, no matter how small a δ disk we draw around $(0,0)$, the values of $f(x,y)$ for points inside that δ disk will include both 0 and $\dfrac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

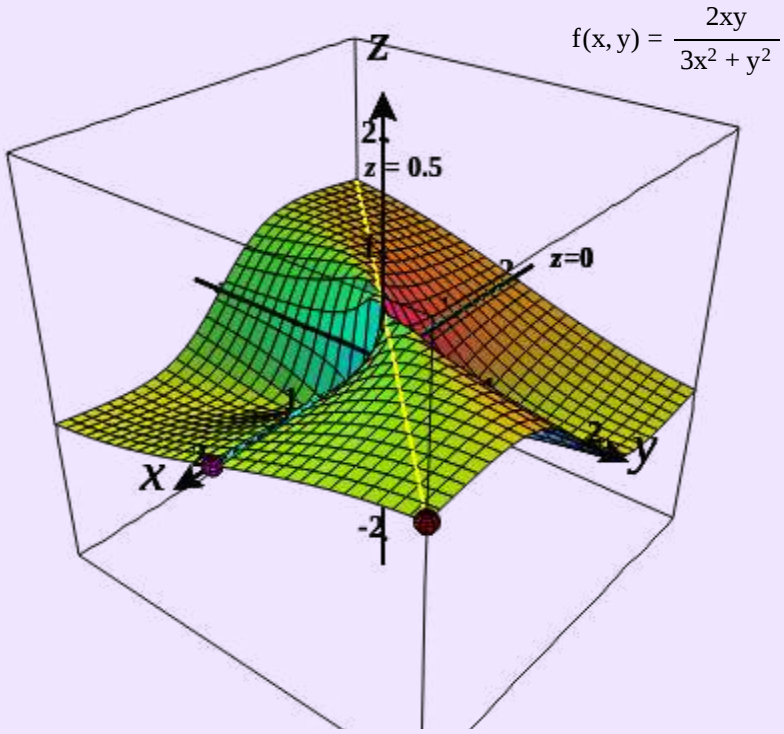


Figure 2.2.5: Graph of the function $f(x,y) = \dfrac{2xy}{3x^2+y^2}$
Along the line $y=0$, the function is equal to zero; along the line $y=x$, the function is equal to $\frac{1}{2}$

(b) Just like in part a, we will approach the point $(2,1)$ along a straight line $y = k(x-2)+1$ for some real number k:

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$
$$= \lim_{x\to 2} \frac{(x-2)([k(x-2)+1]-1)}{(x-2)^2+([k(x-2)+1]-1)^2} = \lim_{x\to 2} \frac{k(x-2)^2}{(x-2)^2+k^2(x-2)^2} = \lim_{x\to 2} \frac{k}{1+k^2} = \frac{k}{1+k^2}$$

Since the result of this limit depends directly on our choice of slope k, this limit cannot exist. For instance when $k=-1$ the function approaches $\dfrac{1}{2}$, but when $k=1$ it approaches $\dfrac{1}{2}$, so any choice of $0<\epsilon<1$ will not be satisfied by any δ.

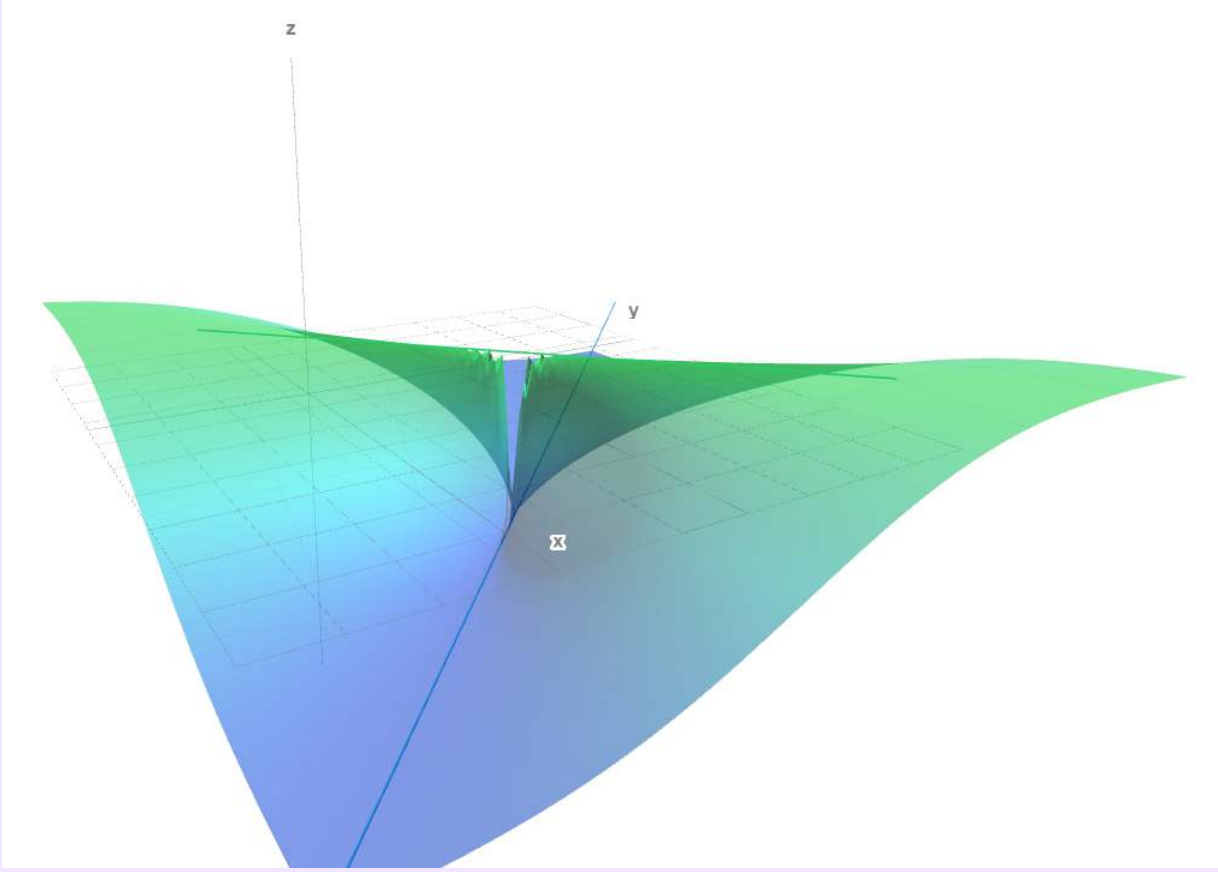


Figure 2.2.6: Graph of the function $f(x,y) = \dfrac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$
Along the blue line $y=-x$, the function is equal to $-\dfrac{1}{2}$; along the green line $y=x$, the function is equal to $\dfrac{1}{2}$

Example 2.2.2

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+3y^4}$

Solution

As in Illustration 2.2.3 above, we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y=0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y=kx$. Then the limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} = \lim_{(x,y)\to(0,0)} \frac{4x(kx)^2}{x^2+3(kx)^4} = \lim_{(x,y)\to(0,0)} \frac{4k^2x^3}{x^2+3k^4x^4}$$
$$= \lim_{(x,y)\to(0,0)} \frac{4k^2x}{1+3k^4x^2} = \frac{\lim\limits_{(x,y)\to(0,0)} (4k^2x)}{\lim\limits_{(x,y)\to(0,0)} (1+3k^4x^2)} = 0$$

Therefore, regardless of the value of k, the limit is equal to zero. But lines are not the only possible path through the origin in $\mathbb{R}^2$!

What if we chose the parabola given by the equation $x = y^2$? Substituting y^2 in place of x in $f(x,y)$ gives:

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} = \lim_{(x,y)\to(0,0)} \frac{4(y^2)y^2}{(y^2)^2+3y^4} = \lim_{(x,y)\to(0,0)} \frac{4y^4}{y^4+3y^4} = \lim_{(x,y)\to(0,0)} 1 = 1$$

By the same logic in part a, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$ Therefore,

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} = \text{DNE}}$$

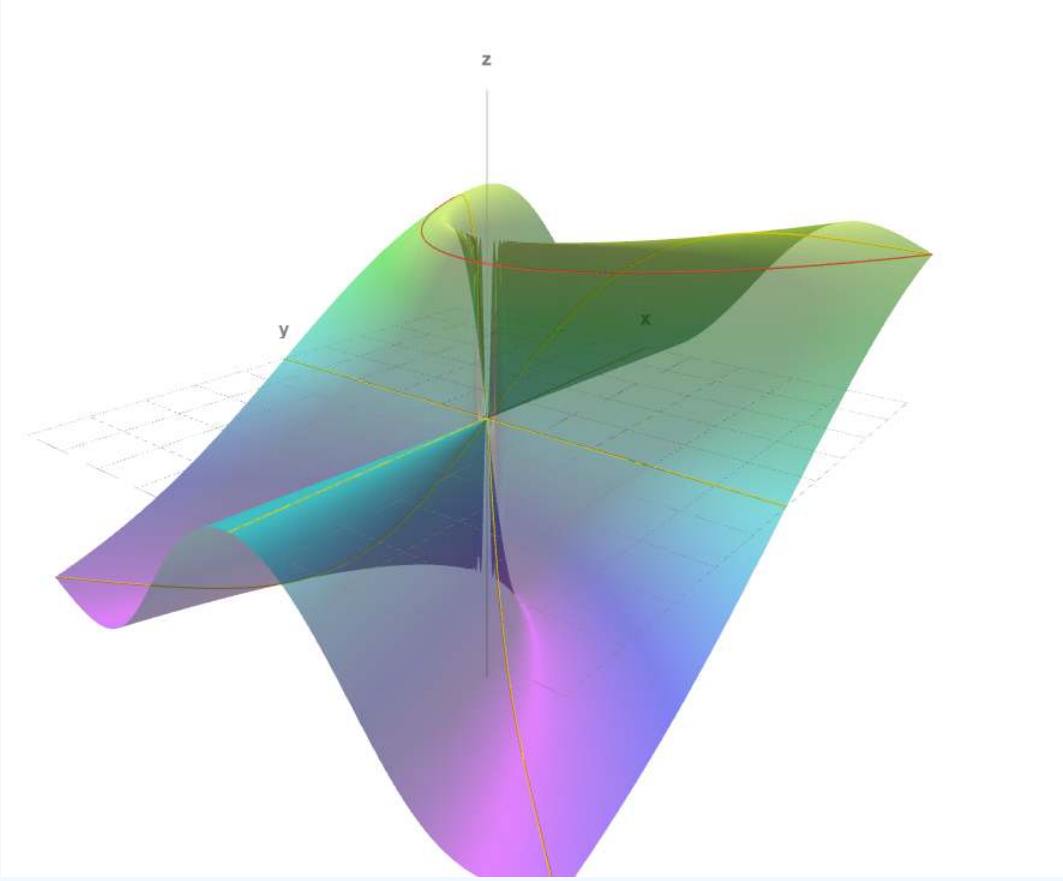


Figure 2.2.6: Graph of the function $f(x,y) = \dfrac{4xy^2}{x^2+3y^4}$
Along each yellow line, the function always approaches 0; along the red parabola $x = y^2$, the function is equal to 1

When evaluating the limit of a multivariable function outside of its domain, it is important to consider **all possible paths through** the limit point.

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x,y,z)$ that gives the temperature at a physical location (x,y,z) in three dimensions. Or perhaps a function $g(x,y,z,t)$ can indicate air

pressure at a location (x, y, z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition: δ-balls

Let (x_0, y_0, z_0) be a point in $\mathbb{R}^3$. Then, a δ-ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0, y_0, z_0) —that is,

$$\left\{(x, y, z) \in \mathbb{R}^3 \, : \, \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\right\}$$

To define a δ-ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P = (w_0, x_0, y_0, z_0)$ in $\mathbb{R}^4$, a δ-ball around P can be described by

$$\left\{(w, x, y, z) \in \mathbb{R}^4 \, : \, \sqrt{(w - w_0)^2 + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\right\}$$

To show that a limit of a function of three variables exists at a point (x_0, y_0, z_0), it suffices to show that for any point in a δ-ball centered at (x_0, y_0, z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 2.2.6: Finding the Limit of a Function of Three Variables

Find each limit below:

(a) $\lim\limits_{(x,y,z)\to(4,1,-3)} \dfrac{x^2y - 3z}{2x + 5y - z}$

(b) $\lim\limits_{(x,y,z)\to(1,0,1)} \dfrac{\sqrt{x^2 + y^2 + z^2 + 2} - 2}{x^2 + y^2 + z^2 - 2}$

Solution

(a) Note that the function $f(x, y, z) = \dfrac{x^2y - 3z}{2x + 5y - z}$ is continuous on its domain $D_f = \{(x, y, z) : z \neq 2x + 5y\}$

Further notice that $-3 \neq 2(4) + 5(1)$ so the point $(4, 1, -3)$ is in the domain, and we must have

$$\lim_{(x,y,z)\to(4,1,-3)} f(x, y, z) = f(4, 1, -3) = \frac{4^2 \cdot 1 - 3(-3)}{2 \cdot 4 + 5 \cdot 1 - (-3)} = \boxed{\frac{25}{16}}$$

(b) The function $g(x, y, z) = \dfrac{\sqrt{x^2 + y^2 + z^2 + 2} - 2}{x^2 + y^2 + z^2 - 2}$ has domain $D_g = \{(x, y, z) : x^2 + y^2 + z^2 \neq 2\}$

Unfortunately, the point $(1, 0, 1)$ is not in this domain, so we cannot use continuity to evaluate this limit.

The numerator and denominator are each separately continuous everywhere, so it is easy to see that:

$$\lim_{(x,y,z)\to(1,0,1)} \sqrt{x^2 + y^2 + z^2 + 2} - 2 = \sqrt{1^2 + 0^2 + 1^2 + 2} - 2 = 0$$

$$\lim_{(x,y,z)\to(1,0,1)} x^2 + y^2 + z^2 - 2 = 1^2 + 0^2 + 1^2 - 2 = 0$$

We do not have a multivariable version of *L'Hopital's Rule* for this $\dfrac{0}{0}$ limit, so we attempt to approach it the old-fashioned way:

$$\frac{\sqrt{x^2+y^2+z^2+2}-2}{x^2+y^2+z^2-2}\cdot\frac{\sqrt{x^2+y^2+z^2+2}+2}{\sqrt{x^2+y^2+z^2+2}+2}=\frac{x^2+y^2+z^2-2}{(x^2+y^2+z^2-2)(\sqrt{x^2+y^2+z^2+2}+2)}$$
$$=\frac{1}{\sqrt{x^2+y^2+z^2+2}+2}$$

Note that $g(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2+z^2+2}+2}$ for all (x,y,z) in D_g, so just like for a single variable limit:

$$\lim_{(x,y,z)\to(1,0,1)} g(x,y,z)=\lim_{(x,y,z)\to(1,0,1)}\frac{1}{\sqrt{x^2+y^2+z^2+2}+2}=\frac{1}{\sqrt{1^2+0^2+1^2+2}+2}=\boxed{\frac{1}{4}}$$

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2.2: Practice Set

Exercise 2.2.1a

Evaluate $\lim\limits_{(x,y)\to(1,1)} \dfrac{y-1}{x-1}$

Answer

Along the path $y=1+m(x-1)$, this limit becomes:

$$\begin{aligned}\lim_{(x,y)\to(1,1)} \frac{y-1}{x-1} &= \lim_{x\to 1} \frac{(1+m(x-1))-1}{x-1} \\ &= \lim_{x\to 1} \frac{m(x-1)}{x-1} = m\end{aligned}$$

Since this result depends on m, the limit value depends on the path taken, and **does not exist**.

Exercise 2.2.1b

Evaluate $\lim\limits_{(x,y)\to(4,2)} \dfrac{x-y^2}{x-2y}$

Answer

Along the path $x=4+m(y-2)$, this limit becomes:

$$\begin{aligned}\lim_{(x,y)\to(4,2)} \frac{x-y^2}{x-2y} &= \lim_{y\to 2} \frac{(4+m(y-2))-y^2}{(4+m(y-2)-2y} \\ &= \lim_{y\to 2} \frac{(4-2m)+my-y^2}{(4-2m)+(m-2)y} \\ &= \lim_{y\to 2} \frac{m-2y}{m-2} \\ &= \frac{m-4}{m-2}\end{aligned}$$

Since this result depends on m, the limit value depends on the path taken, and **does not exist**.

Exercise 2.2.1c

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos(x)}{\cos^{-1}(y)}$

Answer

The function $f(x,y)=\dfrac{\cos(x)}{\cos^{-1}(y)}$ is continuous for all $y\neq 1$, and $f(0,0)=\dfrac{\cos(0)}{\cos^{-1}(0)}=\dfrac{1}{\frac{\pi}{2}}=\dfrac{2}{\pi}$

Therefore we know $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos(x)}{\cos^{-1}(y)}=f(0,0)=\dfrac{2}{\pi}$

Exercise 2.2.1d

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

Answer

For any function $y(x)$ satisfying $\lim\limits_{x\to 0} y(x)=0$, this limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{y(x)+xy'(x)}{\frac{1}{2}(x^2+[y(x)]^2)^{-\frac{1}{2}}\cdot(2x+2y(x)y'(x))}$$
$$= \lim_{x\to 0} \frac{(y(x)+xy'(x))\sqrt{x^2+[y(x)]^2}}{x+y(x)y'(x)}$$
$$= \lim_{x\to 0} \frac{(y(x)+xy'(x))\cdot\frac{1}{2}(x^2+[y(x)]^2)^{-\frac{1}{2}}\cdot(2x+2y(x)y'(x))+(2y'(x)+xy''(x))\cdot\sqrt{x^2+[y(x)]^2}}{1+[y'(x)]^2+y(x)y''(x)}$$
$$= \frac{0}{1+[y'(0)]^2} = 0$$

For any function $x(y)$ satisfying $\lim_{y\to 0} x(y) = 0$, this limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{x'(y)y+x(y)}{\frac{1}{2}([x(y)]^2+y^2)^{-\frac{1}{2}}\cdot(2x(y)x'(y)+2y)}$$
$$= \lim_{x\to 0} \frac{(x'(y)y+x(y))\sqrt{[x(y)]^2+y^2}}{x(y)x'(y)+y}$$
$$= \lim_{x\to 0} \frac{(x'(y)y+x(y))\cdot\frac{1}{2}([x(y)]^2+y^2)^{-\frac{1}{2}}\cdot(2x(y)x'(y)+2y)+(2x'(y)+x''(y)y)\cdot\sqrt{[x(y)]^2+y^2}}{[x'(y)]^2+x(y)x''(y)+1}$$
$$= \frac{0}{1+[x'(0)]^2} = 0$$

Therefore all paths through the origin yields the same limit value, which is 0.

Exercise 2.2.2a

Evaluate $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^2+4y^2}$

Answer

For any function $y(x)$ satisfying $\lim_{x\to 0} y(x) = 0$, this limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^2+4y^2} = \lim_{x\to 0} \frac{2x^2[y(x)]^2}{x^2+4[y(x)]^2}$$
$$= \lim_{x\to 0} \frac{4x[y(x)]^2+4x^2y(x)y'(x)}{2x+8y(x)y'(x)}$$
$$= \lim_{x\to 0} \frac{4[y(x)]^2+16xy(x)y'(x)+4x^2[y'(x)]^2+4x^2y(x)y''(x)}{2+8[y'(x)]^2+8y(x)y''(x)}$$
$$= \frac{0}{2} = 0$$

For any function $x(y)$ satisfying $\lim_{y\to 0} x(y) = 0$, this limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^2+4y^2} = \lim_{y\to 0} \frac{2[x(y)]^2y^2}{[x(y)]^2+4y^2}$$
$$= \lim_{y\to 0} \frac{4x(y)x'(y)y^2+4[x(y)]^2y}{2x(y)x'(y)+8y}$$
$$= \lim_{y\to 0} \frac{4[x'(y)]^2y^2+4x(y)x''(y)y^2+16x(y)x'(y)+4[x(y)]^2}{2[x'(y)]^2+2x(y)x''(y)+8}$$
$$= \frac{0}{8} = 0$$

Therefore all paths through the origin yields the same limit value, which is 0.

Exercise 2.2.2b

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+4y^2}$

Answer

Along the path $y=mx^2$, this limit becomes:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+4y^2} &= \lim_{x\to 0} \frac{2x^2(mx^2)}{x^4+4(mx^2)^2} \\ &= \lim_{x\to 0} \frac{2mx^4}{x^4(1+4m^2)} \\ &= \frac{2m}{1+4m^2}\end{aligned}$$

Since this result depends on m, the limit value depends on the path taken, and **does not exist**.

Exercise 2.2.2c

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-9y^4}{x^2+3y^2}$

Answer

For any function $y(x)$ satisfying $\lim\limits_{x\to 0} y(x)=0$, this limit becomes:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4-9y^4}{x^2+3y^2} &= \lim_{x\to 0} \frac{x^4-9[y(x)]^4}{x^2+3[y(x)]^2} \\ &= \lim_{x\to 0} \frac{4x^3-36[y(x)]^3y'(x)}{2x+6y(x)y'(x)} \\ &= \lim_{x\to 0} \frac{12x^2-108[y(x)]^2[y'(x)]^2-36[y(x)]^3y''(x)}{2+6[y'(x)]^2+6y(x)y''(x)} \\ &= \frac{0}{2+6[y'(0)]^2} = 0\end{aligned}$$

For any function $x(y)$ satisfying $\lim\limits_{y\to 0} x(y)=0$, this limit becomes:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4-9y^4}{x^2+3y^2} &= \lim_{y\to 0} \frac{[x(y)]^4-9y^4}{[x(y)]^2+3y^2} \\ &= \lim_{y\to 0} \frac{4[x(y)]^3x'(y)-36y^3}{2x(y)x'(y)+6y} \\ &= \lim_{y\to 0} \frac{12[x(y)]^2[x'(y)]^2+4[x(y)]^3x''(y)-108y^2}{2[x'(y)]^2+2x(y)x''(y)+6} \\ &= \frac{0}{2[x'(0)]^2+6} = 0\end{aligned}$$

Therefore all paths through the origin yields the same limit value, which is 0.

Exercise 2.2.2d

Evaluate $\lim\limits_{(x,y)\to(\pi,1)} \dfrac{y^2\sin(x)}{x+y-1}$

Answer

Note that $(\pi,1)$ is inside of the domain of $f(x,y)=\dfrac{y^2\sin(x)}{x+y-1}$, and $f(\pi,1)=\dfrac{1^2\cdot 0}{\pi+1-1}=0$.

Since this function is continuous on its domain, we know $\lim_{(x,y)\to(\pi,1)} \dfrac{y^2\sin(x)}{x+y-1} = f(\pi,1) = 0$

Exercise 2.2.2e

Evaluate $\lim_{(x,y)\to(2,-2)} \dfrac{y^2-x^2}{x^2+3xy+2y^2}$

Answer

Notice that the function $f(x,y) = \dfrac{y^2-x^2}{x^2+3xy+2y^2} = \dfrac{(y-x)(y+x)}{(x+y)(x+2y)} = \dfrac{y-x}{x+2y}$ on its domain.

Since $g(x,y) = \dfrac{y-x}{x+2y}$ is continuous at $(2,-2)$, and $g(2,-2) = \dfrac{-4}{-2} = 2$, we know $\lim_{(x,y)\to(2,-2)} g(x,y) = 2$

Therefore, we have $\lim_{(x,y)\to(2,-2)} \dfrac{y^2-x^2}{x^2+3xy+2y^2} = \lim_{(x,y)\to(2,-2)} \dfrac{y-x}{x+2y} = 2$

Exercise 2.2.2f

Evaluate $\lim_{(x,y)\to(3,1)} \dfrac{x^4-5(x+y)^2-1}{x^2+9y^2}$

Answer

Note that the function $f(x,y) = \dfrac{x^4-5(x+y)^2-1}{x^2+9y^2}$ is continuous everywhere, including $(3,1)$.

Since $f(3,1) = \dfrac{0}{18} = 0$, we know that $\lim_{(x,y)\to(3,1)} \dfrac{x^4-5(x+y)^2-1}{x^2+9y^2} = 0$

Exercise 2.2.3a

Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^4+y^4)}{x^4+y^4}$

Answer

Substituting in $a = x^4+y^4$, we know that $a \to 0^+$ along all paths satisfying $(x,y) \to (0,0)$ so our limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^4+y^4)}{x^4+y^4} = \lim_{a\to 0^+} \frac{\sin(a)}{a} = \lim_{a\to 0^+} \frac{\cos(a)}{1} = \cos(0) = 1$$

Therefore this limit must be equal to 1.

Exercise 2.2.3b

Evaluate $\lim_{(x,y,z)\to(\frac{1}{3},\frac{1}{3},\frac{1}{3})} \dfrac{\ln(x+y+z)}{(x+y+z)^2-1}$

Answer

Substituting in $a = x+y+z$, we know that $a \to 1$ along all paths satisfying $(x,y,z) \to \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ so our limit becomes:

$$\lim_{(x,y,z)\to(\frac{1}{3},\frac{1}{3},\frac{1}{3})} \frac{\ln(x+y+z)}{(x+y+z)^2-1} = \lim_{a\to 1} \frac{\ln(a)}{a^2-1} = \lim_{a\to 1} \frac{\frac{1}{a}}{2a} = \lim_{a\to 1} \frac{1}{2a^2} = \frac{1}{2}$$

Therefore this limit must be equal to $\dfrac{1}{2}$.

Exercise 2.2.3c

Evaluate $\lim\limits_{(x,y,z)\to(1,2,-1)} \dfrac{1}{\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}}$

Answer

Substituting in $a=(x-1)^2+(y-2)^2+(z+1)^2$, we know that $a\to 0^+$ along all paths satisfying $(x,y,z)\to(1,2,-1)$ so our limit becomes:

$$\lim_{(x,y,z)\to(1,2,-1)} \frac{1}{\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}} = \lim_{a\to 0^+} \frac{1}{\sqrt{a}} = +\infty$$

Therefore all paths to $(1,2,-1)$ yields a limit of positive infinity.

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2.3: Multivariable Differentiation

Learning Objectives

- Calculate the partial derivatives of a function of two variables.
- Calculate the partial derivatives of a function of more than two variables.
- Determine the higher-order derivatives of a function of two variables.
- Determine the equation of a plane tangent to a given surface at a point.
- Use the tangent plane to approximate a function of two variables at a point.
- Explain when a function of two variables is differentiable.
- Use the tangent plane to approximate a function of two variables.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well. We then consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f'(a)$; what is the "slope" of a tangent plane? We learned about the equation of a plane, now we see how it can be applied to the problem at hand.

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2.3: Partial Derivatives

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is $\frac{dy}{dx}$, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x, y)$ be a function of two variables. Then the ***partial derivative*** of f with respect to x, written as $\partial f/\partial x$ or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

The partial derivative of f with respect to y, written as $\frac{\partial f}{\partial y}$ or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\frac{\partial f}{\partial x}$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 2.3.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$$

Solution

First, calculate $f(x + h, y)$:

$$\begin{aligned} f(x + h, y) &= (x + h)^2 - 3(x + h)y + 2y^2 - 4(x + h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12 \end{aligned}$$

Next, substitute this into the formula for $\frac{\partial f}{\partial x}$ and simplify:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{(x^2 + 2hx + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2 - 3hy - 4h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h - 3y - 4)}{h} \\ &= \lim_{h \to 0} (2x + h - 3y - 4) \\ &= \boxed{2x - 3y - 4} \end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x, y+k)$:

$$\begin{aligned} f(x, y+k) &= x^2 - 3x(y+k) + 2(y+k)^2 - 4x + 5(y+k) - 12 \\ &= x^2 - 3xy - 3xk + 2y^2 + 4yk + 2k^2 - 4x + 5y + 5k - 12 \end{aligned}$$

Next, substitute this into the formula for $\frac{\partial f}{\partial y}$ and simplify:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} \\ &= \lim_{k \to 0} \frac{(x^2 - 3xy - 3xk + 2y^2 + 4yk + 2k^2 - 4x + 5y + 5k - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{k} \\ &= \lim_{k \to 0} \frac{x^2 - 3xy - 3xk + 2y^2 + 4yk + 2k^2 - 4x + 5y + 5k - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{k} \\ &= \lim_{k \to 0} \frac{-3xk + 4yk + 2k^2 + 5k}{k} \\ &= \lim_{k \to 0} \frac{k(-3x + 4y + 2k + 5)}{k} \\ &= \lim_{k \to 0} (-3x + 4y + 2k + 5) \\ &= \boxed{-3x + 4y + 5} \end{aligned}$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x) = f(x, y)$ as a function of x. Then

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y) = f(x, y)$ as a function of y. Then

$$h'(x) = \lim_{k \to 0} \frac{h(x+k) - h(x)}{k} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} = \frac{\partial f}{\partial y}$$

This means that all of the previously learned differentiation rules and shortcuts still apply!

Example 2.3.2: Calculating Partial Derivatives

Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions by holding the opposite variable constant then differentiating:

(a) $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$

(b) $f(x, y) = \sin(x^2 y - 2x + 4)$

(c) $f(x, y) = x^2 e^{xy}$

Solution:

(a) To calculate $\frac{\partial f}{\partial x}$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x using the sum, difference, and power rules:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] = 2x - 3y + 0 - 4 + 0 - 0 = 2x - 3y - 4$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$.

Calculating $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] = 0 - 3x + 4y - 0 + 5 - 0 = -3x + 4y + 5$$

These are the same answers obtained in Example 2.3.1:

$$\boxed{f_x = 2x - 3y - 4 \text{ and } f_y = -3x + 4y + 5}$$

(b) To calculate $\frac{\partial f}{\partial x}$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x^2y - 2x + 4)\right] &= \cos(x^2y - 2x + 4) \cdot \left(\frac{\partial}{\partial x}[x^2y - 2x + 4]\right) \\ &= (2xy - 2)\cos(x^2y - 2x + 4)\end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, treat the variable x as a constant. Then differentiate $f(x, y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y - 2x + 4)\right] \\ = \cos(x^2y - 2x + 4) \cdot \frac{\partial}{\partial y}[x^2y - 2x + 4] &= x^2\cos(x^2y - 2x + 4)\end{aligned}$$

Once again, we see these are two different functions since they represent different rates of change:

$$\boxed{f_x = (2xy - 2)\cos(x^2y - 2x + 4) \text{ and } f_y = x^2\cos(x^2y - 2x + 4)}$$

(c) To calculate $\frac{\partial f}{\partial x}$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x using the product rule and chain rule:

$$\begin{aligned}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2e^{xy}\right] = \frac{\partial}{\partial x}[x^2] \cdot e^{xy} + x^2 \cdot \frac{\partial}{\partial x}[e^{xy}] &= 2xe^{xy} + x^2\left(e^{xy} \cdot \frac{\partial}{\partial x}[xy]\right) \\ = 2xe^{xy} + x^2 \cdot e^{xy} \cdot y &= (2x + x^2y)e^{xy}\end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, treat the variable x as a constant. Then differentiate $f(x, y)$ with respect to y using the product rule and chain rule:

$$\begin{aligned}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2e^{xy}\right] = \frac{\partial}{\partial y}[x^2] \cdot e^{xy} + x^2 \cdot \frac{\partial}{\partial y}[e^{xy}] \\ = 0 \cdot e^{xy} + x^2\left(e^{xy} \cdot \frac{\partial}{\partial y}[xy]\right) = 0 + x^2 \cdot e^{xy} \cdot x = x^3e^{xy}\end{aligned}$$

To summarize, our two partial derivatives are:

$$\boxed{f_x = (2x + x^2y)e^{xy} \text{ and } f_y = x^3e^{xy}}$$

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in $\mathbb{R}^3$. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x + h, y) - f(x, y)}{h}$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 2.3.1 illustrates a surface described by an arbitrary function $z = f(x, y)$.

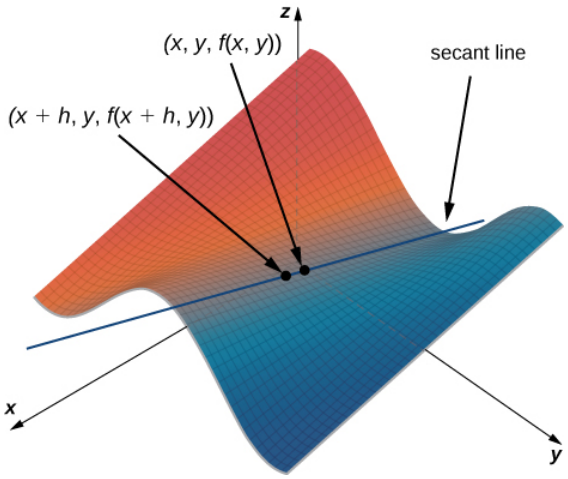


Figure 2.3.1: Secant line passing through the points $(x, y, f(x, y))$ and $(x + h, y, f(x + h, y))$.

In the figure above, the value of h is positive. If we graph $f(x, y)$ and $f(x + h, y)$ for an arbitrary point (x, y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x + h, y) - f(x, y)}{h}$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\frac{\partial f}{\partial x}$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the x-axis and $\frac{\partial f}{\partial y}$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the y-axis. (Later, if we wish to find the slope of a tangent line passing through the same point in any other direction, we will define the concept of a directional derivative.)

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition: Partial Derivatives

Let $f(x, y, z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\frac{\partial f}{\partial x}$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

The *partial derivative* of f with *respect* to y, written as $\frac{\partial f}{\partial y}$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = f_y(x, y, z) = \lim_{k \to 0} \frac{f(x, y + k, z) - f(x, y, z)}{k}$$

The *partial derivative* of f with respect to z, written as $\frac{\partial f}{\partial z}$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = f_z(x, y, z) = \lim_{m \to 0} \frac{f(x, y, z + m) - f(x, y, z)}{m}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\frac{\partial f}{\partial x}$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 2.3.3: Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

(a) $f(x,y,z)=\dfrac{x^2y-4xz+y^2}{x-3yz}$

(b) $g(x,y,z)=\sin(x^2y-z)+\cos(x^2-yz)$

Solution

(a) Treat all variables as constants except the one whose partial derivative you are calculating:

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial}{\partial x}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\&=\frac{\frac{\partial}{\partial x}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial x}(x-3yz)}{(x-3yz)^2}\\&=\frac{(2xy-4z)(x-3yz)-(x^2y-4xz+y^2)(1)}{(x-3yz)^2}\\&=\boxed{\frac{x^2y-6xy^2z+12yz^2-y^2}{(x-3yz)^2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\&=\frac{\frac{\partial}{\partial y}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial y}(x-3yz)}{(x-3yz)^2}\\&=\frac{(x^2+2y)(x-3yz)-(x^2y-4xz+y^2)(-3z)}{(x-3yz)^2}\\&=\boxed{\frac{x^3+2xy-3y^2z-12xz^2}{(x-3yz)^2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z}&=\frac{\partial}{\partial z}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\&=\frac{\frac{\partial}{\partial z}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial z}(x-3yz)}{(x-3yz)^2}\\&=\frac{(-4x)(x-3yz)-(x^2y-4xz+y^2)(-3y)}{(x-3yz)^2}\\&=\boxed{\frac{-4x^2+3x^2y^2+3y^3}{(x-3yz)^2}}\end{aligned}$$

(b) Treat all variables as constants except the one whose partial derivative you are calculating:

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial}{\partial x}\left[\sin(x^2y-z)+\cos(x^2-yz)\right]\\&=(\cos(x^2y-z))\frac{\partial}{\partial x}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial x}(x^2-yz)\\&=\boxed{2xy\cos(x^2y-z)-2x\sin(x^2-yz)}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}[\sin(x^2y-z)+\cos(x^2-yz)] \\ &= (\cos(x^2y-z))\frac{\partial}{\partial y}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial y}(x^2-yz) \\ &= \boxed{x^2\cos(x^2y-z)+z\sin(x^2-yz)} \\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[\sin(x^2y-z)+\cos(x^2-yz)] \\ &= (\cos(x^2y-z))\frac{\partial}{\partial z}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial z}(x^2-yz) \\ &= \boxed{-\cos(x^2y-z)+y\sin(x^2-yz)}\end{aligned}$$

Consider the function

$$f(x,y)=2x^3-4xy^2+5y^3-6xy+5x-4y+12$$

Its partial derivatives are

$$\frac{\partial f}{\partial x}=6x^2-4y^2-6y+5 \text{ and } \frac{\partial f}{\partial y}=-8xy+15y^2-6x-4$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives*, third-order derivatives, and so on. In general, they are referred to as **higher-order partial derivatives.** There are four second-order partial derivatives for any function (provided they all exist):

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=f_{xx} \quad \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=f_{xy}$$
$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]=f_{yx} \quad \frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=f_{yy}$$

Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and f_{yx}, are commonly called **mixed partial derivatives.**

Illustration 2.3.1: Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function $f(x,y)=xe^{-3y}+\sin(2x-5y)$:

To calculate f_{xx} and f_{xy}, we first calculate f_x:

$$\frac{\partial f}{\partial x}=e^{-3y}+2\cos(2x-5y)$$

To calculate f_{xx}, differentiate f_x above with respect to x:

$$f_{xx}=\frac{\partial}{\partial x}[f_x]=\frac{\partial}{\partial x}[e^{-3y}+2\cos(2x-5y)]=-4\sin(2x-5y)$$

To calculate f_{xy}, differentiate f_x with respect to y:

$$f_{xy}=\frac{\partial}{\partial y}[f_x]=\frac{\partial}{\partial y}[e^{-3y}+2\cos(2x-5y)]=-3e^{-3y}+10\sin(2x-5y)$$

To calculate f_{yx} and f_{yy}, first calculate f_y:

$$\frac{\partial f}{\partial y}=-3xe^{-3y}-5\cos(2x-5y)$$

To calculate f_{yx}, differentiate f_y above with respect to x:

$$f_{yx}=\frac{\partial}{\partial x}[f_y]=\frac{\partial}{\partial x}[-3xe^{-3y}-5\cos(2x-5y)]=-3e^{-3y}+10\sin(2x-5y)$$

To calculate f_{yy}, differentiate f_y with respect to y:

$$f_{yy} = \frac{\partial}{\partial y}[f_y] = \frac{\partial}{\partial y}[-3xe^{-3y} - 5\cos(2x-5y)] = 9xe^{-3y} - 25\sin(2x-5y)$$

At this point we should notice that, in the Example above, it was true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Theorem 2.3.1: Equality of Mixed Partial Derivatives (Clairaut's Theorem)

Suppose that $f(x,y)$ is defined on an open disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most Advanced Calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x,y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

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2.3: Tangent Planes and Linear Approximations

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.

Definition: Tangent Planes

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the ***tangent plane*** to S at P_0.

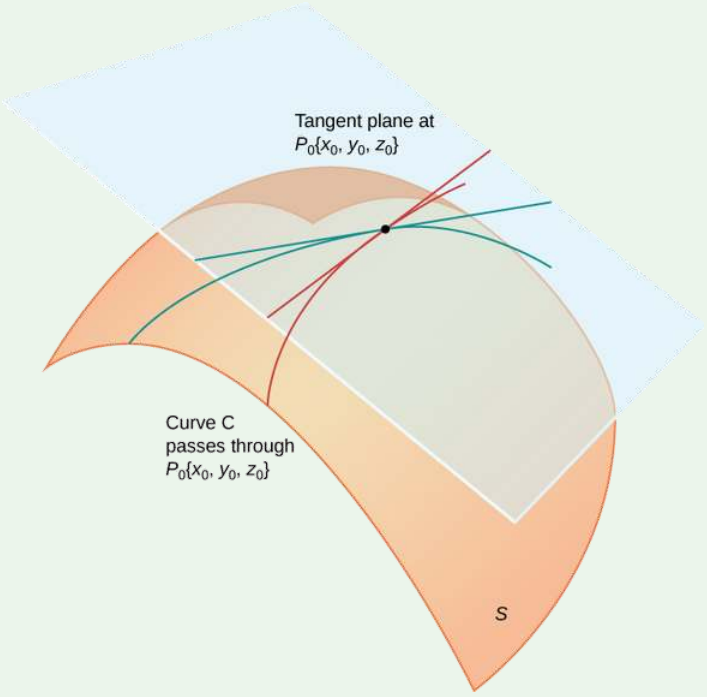


Figure 2.3.2: The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S *that pass through* P_0

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: Tangent Planes to a Surface

Let S be a surface defined by a differentiable function $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, and let $P_0 = (u_0, v_0)$ be a point in the parametric domain of $\vec{\mathbf{r}}$. Then, if one exists, the equation of the tangent plane to S at $\vec{\mathbf{r}}(u_0, v_0)$ must contain the vectors:

$$\vec{\mathbf{r}}_u(u_0,v_0) = \langle x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0)\rangle \text{ and } \vec{\mathbf{r}}_v(u_0,v_0) = \langle x_v(u_0,v_0), y_v(u_0,v_0), z_v(u_0,v_0)\rangle$$

If we define $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$, then for $\vec{\mathbf{n}} \neq \vec{0}$ the equation of the tangent plane can be written as

$$\vec{\mathbf{n}} \cdot \langle x - x(u_0,v_0), y - y(u_0,v_0), z - z(u_0,v_0)\rangle = 0$$

Illustration 2.3.2

Find the equation of the tangent plane to the parametrized surfaces at the given point:

(a) $\vec{\mathbf{r}}(u,v) = \langle u + v^2, 3u - v, u^2 - \sqrt{v}\rangle$ at $\vec{\mathbf{r}}(1,1)$

(b) $\vec{\mathbf{r}}(u,v) = \langle 2u\cos(v), 3u\sin(v), u\rangle$ at $\vec{\mathbf{r}}\left(1, \frac{\pi}{2}\right)$

(a) We start by finding the derivative vectors $\vec{\mathbf{r}}_u$ and $\vec{\mathbf{r}}_v$:

$$\vec{\mathbf{r}}_u(u,v) = \left\langle \frac{\partial}{\partial u}(u+v^2), \frac{\partial}{\partial u}(3u-v), \frac{\partial}{\partial u}(u^2-\sqrt{v})\right\rangle = \langle 1, 3, 2u\rangle$$

$$\vec{\mathbf{r}}_v(u,v) = \left\langle \frac{\partial}{\partial v}(u+v^2), \frac{\partial}{\partial v}(3u-v), \frac{\partial}{\partial v}(u^2-\sqrt{v})\right\rangle = \left\langle 2v, -1, -\frac{1}{2\sqrt{v}}\right\rangle$$

Plugging in the given point $u_0 = 1$ and $v_0 = 1$ this becomes:

$$\vec{\mathbf{r}}_u(1,1) = \langle 1, 3, 2(1)\rangle = \langle 1, 3, 2\rangle$$

$$\vec{\mathbf{r}}_v(1,1) = \left\langle 2v, -1, -\frac{1}{2\sqrt{v}}\right\rangle = \left\langle 2, -1, -\frac{1}{2}\right\rangle$$

Therefore the normal vector to the tangent plane is

$$\vec{\mathbf{n}} = \langle 1, 3, 2\rangle \times \left\langle 2, -1, -\frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, \frac{9}{2}, -7\right\rangle$$

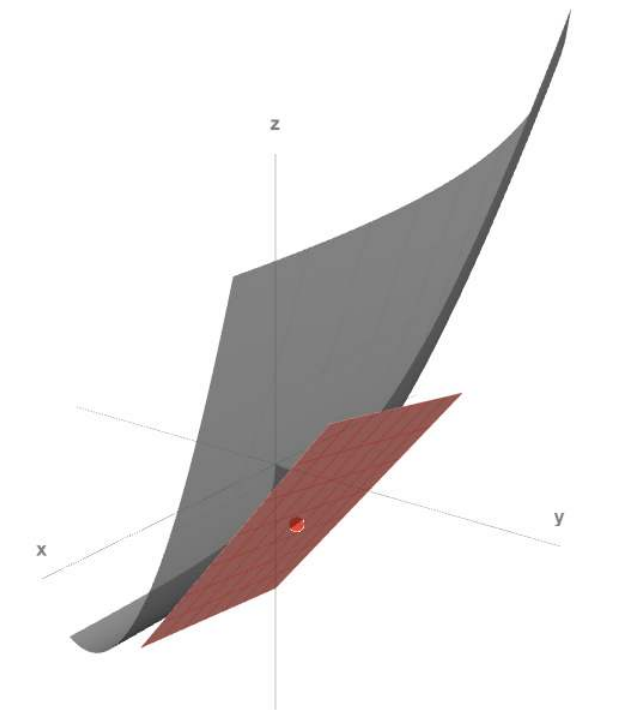
The tangent plane is tangent at the point $\vec{\mathbf{r}}(1,1) = \langle 1+1^2, 3(1)-1, 1^1-\sqrt{1}\rangle = \langle 2, 2, 0\rangle$

Therefore our tangent plane is given by

$$\vec{\mathbf{n}} \cdot \langle x-2, y-2, z-0\rangle = 0$$

$$\frac{1}{2}(x-2) + \frac{9}{2}(y-2) - 7(z-0) = 0$$

This simplifies down to our final answer pictured below: $x + 9y - 14z = 20$



(b) We start by finding the derivative vectors $\vec{\mathbf{r}}_u$ and $\vec{\mathbf{r}}_v$:

$$\vec{\mathbf{r}}_u(u,v) = \left\langle \frac{\partial}{\partial u}(2u\cos(v)), \frac{\partial}{\partial u}(3u\sin(v)), \frac{\partial}{\partial u}(u)\right\rangle = \langle 2\cos(v), 3\sin(v), 1\rangle$$

$$\vec{\mathbf{r}}_v(u,v) = \left\langle \frac{\partial}{\partial v}(2u\cos(v)), \frac{\partial}{\partial v}(3u\sin(v)), \frac{\partial}{\partial v}(u)\right\rangle = \langle -2u\sin(v), 3u\cos(v), 0\rangle$$

Plugging in the given point $u_0 = 1$ and $v_0 = \frac{\pi}{2}$ this becomes:

$$\vec{\mathbf{r}}_u\left(1, \frac{\pi}{2}\right) = \left\langle 2\cos\left(\frac{\pi}{2}\right), 3\sin\left(\frac{\pi}{2}\right), 1\right\rangle = \langle 0, 3, 1\rangle$$

$$\vec{\mathbf{r}}_v\left(1, \frac{\pi}{2}\right) = \left\langle -2(1)\sin\left(\frac{\pi}{2}\right), 3(1)\cos\left(\frac{\pi}{2}\right), 0\right\rangle = \langle -2, 0, 0\rangle$$

Therefore the normal vector to the tangent plane is

$$\vec{\mathbf{n}} = \langle 0, 3, 1\rangle \times \langle -2, 0, 0\rangle = \langle 0, -2, 6\rangle$$

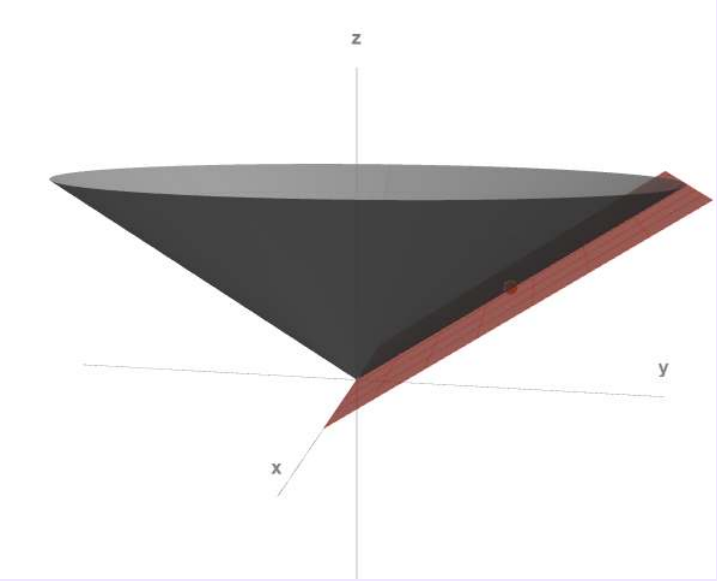
The tangent plane is tangent at the point $\vec{\mathbf{r}}\left(1, \frac{\pi}{2}\right) = \left\langle 2(1)\cos\left(\frac{\pi}{2}\right), 3(1)\sin\left(\frac{\pi}{2}\right), 1\right\rangle = \langle 0, 3, 1\rangle$

Therefore our tangent plane is given by

$$\vec{\mathbf{n}} \cdot \langle x - 0, y - 3, z - 1\rangle = 0$$

$$0(x - 0) - 2(y - 3) + 6(z - 1) = 0$$

This simplifies down to our final answer pictured below: $3z - y = 0$



Note that in Illustration 2.3.2*b*, if we had used the point $(0, 0)$ instead, the vector $\vec{\mathbf{r}}_v(0, 0) = \vec{\mathbf{0}}$. This would have caused $\vec{\mathbf{n}} = \vec{\mathbf{0}}$ and no such tangent plane could have been found. From the picture above it is clear why this is the case, as there cannot be a tangent plane at the sharp point at the base of a cone.

It is entirely possible though that the tangent plane does exist, but our parameterization is insufficient to find it. Instead consider the surface defined by

$$\vec{\mathbf{r}}(u, v) = \langle u^3, v^3, u^3 + v^3\rangle$$

In this case, we can clearly see that the surface defined is the full plane $z = x + y$, so any point on that plane will have tangent plane $x + y - z = 0$. Unfortunately though, since $\vec{\mathbf{r}}_u = \langle 3u^2, 0, 3u^2\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 3v^2, 3v^2\rangle$, we have

$$\vec{\mathbf{r}}_u(0, 0) = \vec{\mathbf{r}}_v(0, 0) = \vec{\mathbf{0}} \text{ and thus } \vec{\mathbf{n}} = \vec{\mathbf{0}}$$

In this case, the tangent plane is hidden from us since the parameterization is singular (aka *complicated*) at the origin.

Many times a surface is defined in the form $z = f(x, y)$. This is just a special case of a parameterized surface where we have $\vec{\mathbf{r}}(u, v) = \langle u, v, f(u, v)\rangle$.

Notice that in this case we have $\vec{\mathbf{r}}_u = \hat{\mathbf{i}} + f_x(u_0, v_0)\,\hat{\mathbf{k}}$ and $\vec{\mathbf{r}}_v = \hat{\mathbf{j}} + f_y(u_0, v_0)\,\hat{\mathbf{k}}$.

We can take the cross product of these two vectors:

$$\vec{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & f_x(u_0, v_0) \\ 0 & 1 & f_y(u_0, v_0) \end{vmatrix} = -f_x(u_0, v_0)\,\hat{\mathbf{i}} - f_y(u_0, v_0)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Since this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{gathered} \vec{\mathbf{n}} \cdot \langle x - x(u_0, v_0), y - y(u_0, v_0), z - z(u_0, v_0) \rangle \\ = -f_x(u_0, v_0)(x - x(u_0, v_0)) - f_y(u_0, v_0)(y - y(u_0, v_0)) + (z - z(u_0, v_0)) = 0 \end{gathered}$$

Plugging in $x(u_0, v_0) = u_0 = x_0$ and $y(u_0, v_0) = v_0 = y_0$ and solving this equation for z gives the tangent plane equation:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Definition: Tangent Planes to a Function

For a surface defined by $z = f(x, y)$, if $f(x, y)$ is differentiable, the tangent plane at the point (x_0, y_0) is given by:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 2.3.4: Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by $z = f(x, y)$ for each function below.

(a) $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$

(b) $f(x, y) = x^3 - x^2y - y^2 - 5x + 6y + 1$ at point $(-1, 2)$

(c) $f(x, y) = \sin(2x)\cos(3y)$ at the point $\left(\frac{\pi}{3}, \frac{\pi}{4}\right)$

Find the equation of the tangent plane to the surface defined by the function

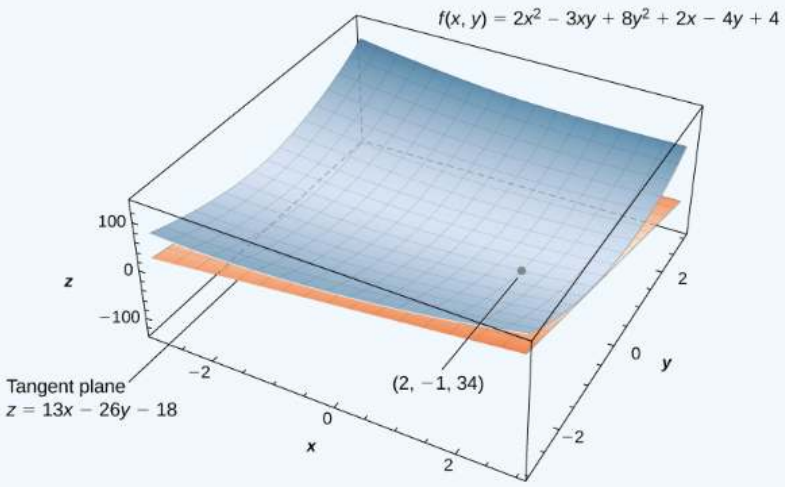
Solution

(a) First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use the tangent plane equation for $z = f(x, y)$ with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned} f_x(x, y) &= 4x - 3y + 2 \\ f_y(x, y) &= -3x + 16y - 4 \\ f(2, -1) &= 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 = 34 \\ f_x(2, -1) &= 4(2) - 3(-1) + 2 = 13 \\ f_y(2, -1) &= -3(2) + 16(-1) - 4 = -26 \end{aligned}$$

Therefore the tangent plane is given by

$$\begin{aligned} z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z &= 34 + 13(x - 2) - 26(y - (-1)) \\ &\boxed{z = 13x - 26y - 18} \end{aligned}$$

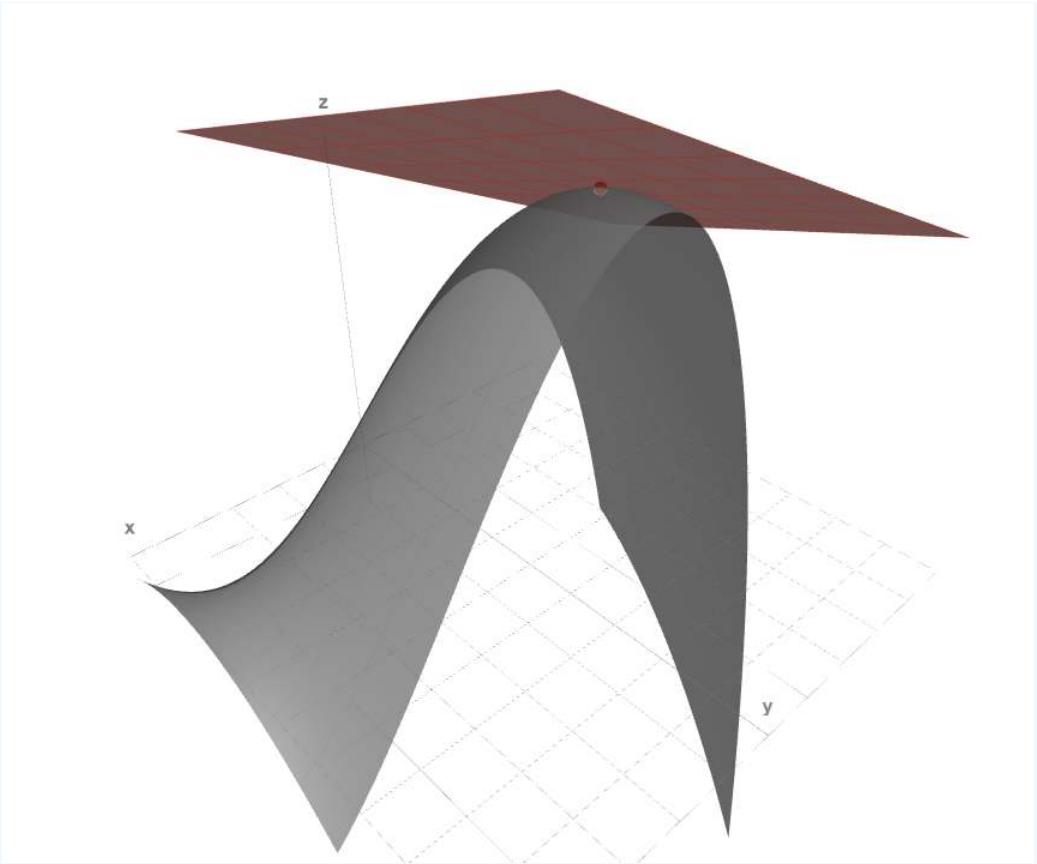


(b) We calculate $f_x(x,y)$ and $f_y(x,y)$, then use the tangent plane equation for $z = f(x,y)$ with $x_0 = -1$ and $y_0 = 2$:

$$f_x(x,y) = 3x^2 - 2xy - 5$$
$$f_y(x,y) = -x^2 - 2y + 6$$
$$f(-1,2) = (-1)^3 - (-1)^2(2) - (2)^2 - 5(-1) + 6(2) + 1 = 11$$
$$f_x(-1,2) = 3(-1)^2 - 2(-1)(2) - 5 = 2$$
$$f_y(-1,2) = -(-1)^2 - 2(2) + 6 = 1$$

Therefore the tangent plane is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$
$$z = 11 + 2(x - (-1)) + 1(y - 2)$$
$$\boxed{z = 2x + y + 11}$$

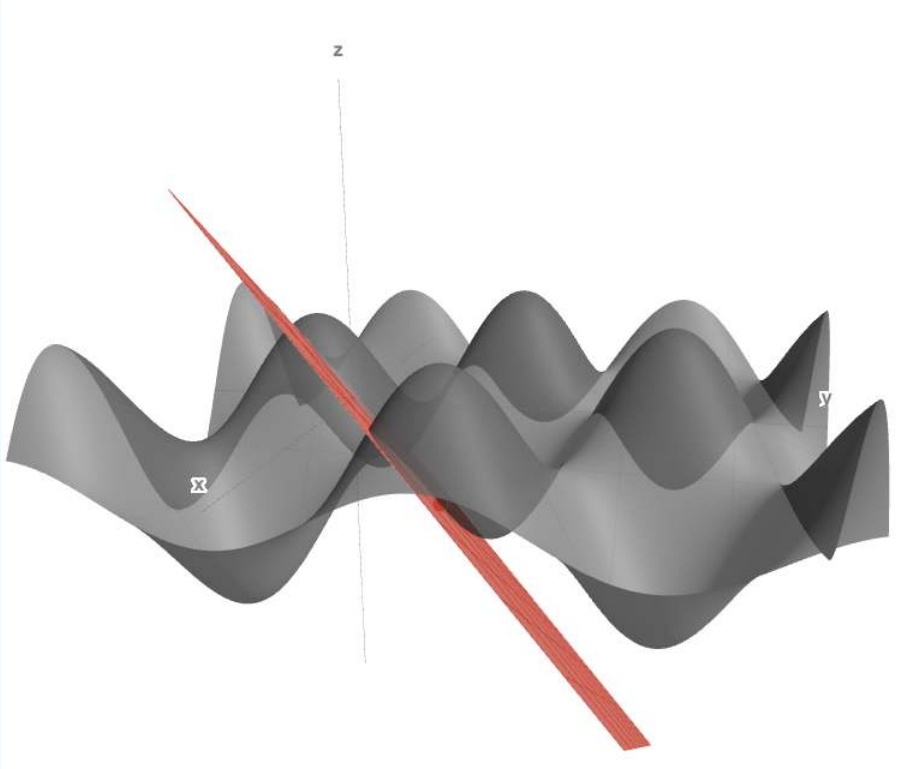


(c) First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use the tangent plane equation for $z = f(x,y)$ with $x_0 = \frac{\pi}{3}$ and $y_0 = \frac{\pi}{4}$:

$$f_x(x,y) = 2\cos(2x)\cos(3y)$$
$$f_y(x,y) = -3\sin(2x)\sin(3y)$$
$$f\left(\frac{\pi}{3},\frac{\pi}{4}\right) = \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4}$$
$$f_x\left(\frac{\pi}{3},\frac{\pi}{4}\right) = 2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$
$$f_y\left(\frac{\pi}{3},\frac{\pi}{4}\right) = -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}$$

Therefore the tangent plane is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{3}\right) - \frac{3\sqrt{6}}{4}\left(y-\frac{\pi}{4}\right)$$
$$\boxed{z = \frac{\sqrt{2}}{2}x - \frac{3\sqrt{6}}{4}y - \frac{\sqrt{6}}{4} - \frac{\pi\sqrt{2}}{6} + \frac{3\pi\sqrt{6}}{16}}$$



Recall in single variables that the formula for the linear approximation of a function $f(x)$ at the point $x = a$ is given by

$$y \approx f(a) + f'(a)(x - a)$$

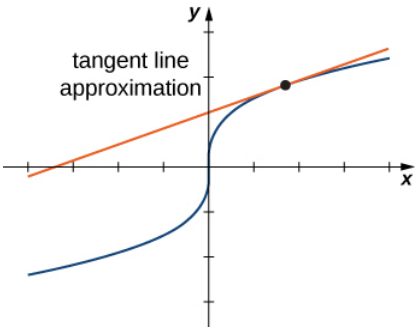


Figure 2.3.3: Linear approximation of a function in one variable.

The tangent plane can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition: Linear Approximation

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the ***linear approximation*** of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ at the point (x_0, y_0). The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to (x_0, y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x, y)$. Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).

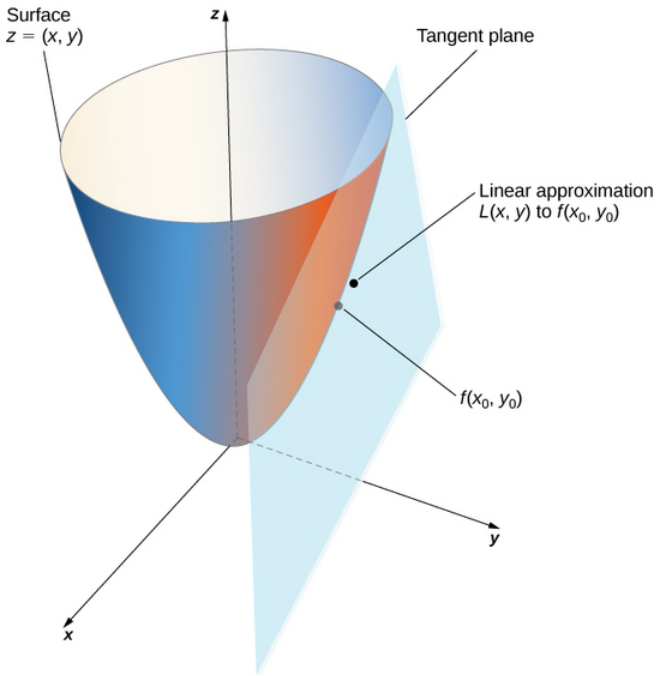


Figure 2.3.4: Using a tangent plane for linear approximation at a point.

Example 2.3.5: Using a Tangent Plane Approximation

Given the function $f(x,y)=\sqrt{41-4x^2-y^2}$, approximate $f(2.05,2.96)$ using point $(2,3)$.

Solution

First must calculate $f(x_0,y_0), f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=2$ and $y_0=3$:

$$f(2,3)=\sqrt{41-4(2)^2-(3)^2}=\sqrt{41-16-9}=\sqrt{16}=4$$

$$f_x(x,y)=-\frac{4x}{\sqrt{41-4x^2-y^2}} \text{ so } f_x(2,3)=-\frac{4(2)}{\sqrt{41-4(2)^2-(3)^2}}=-2$$

$$f_y(x,y)=-\frac{y}{\sqrt{41-4x^2-y^2}} \text{ so } f_y(2,3)=-\frac{3}{\sqrt{41-4(2)^2-(3)^2}}=-\frac{3}{4}$$

Now we substitute these values into our approximation formula:

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=4-2(x-2)-\frac{3}{4}(y-3)$$

Last, we substitute $x=2.05$ and $y=2.96$ into $L(x,y)$:

$$L(2.05,2.96)=4-2(2.05-2)-\frac{3}{4}(2.96-3)=4-0.1+.03=\boxed{3.93}$$

The actual value of $f(2.05,2.96)=\sqrt{41-4(2.05)^2-(2.96)^2}\approx 3.92790$, pretty close!

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function pictured below:

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

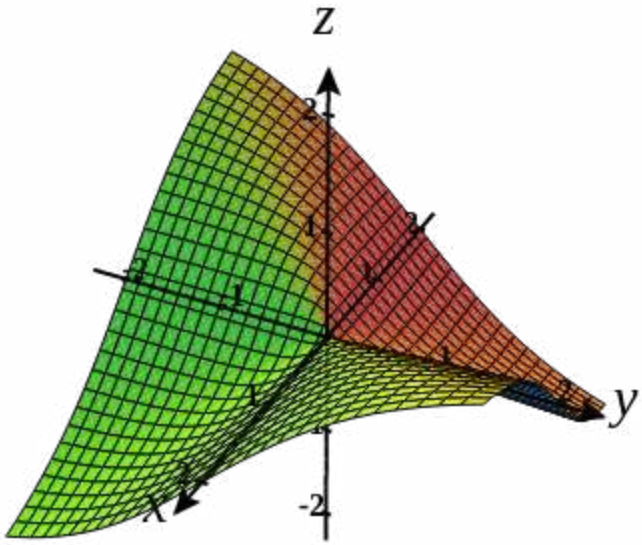


Figure 2.3.5: Graph of a function that does not have a tangent plane at the origin. Dynamic figure powered by CalcPlot3D.

If either $x = 0$ or $y = 0$, then $f(x, y) = 0$, so the value of the function does not change on either the x- or y-axis. Therefore, $f_x(x, 0) = f_y(0, y) = 0$, so as either x or y approach zero, these partial derivatives stay equal to zero. This gives $z = 0$ as the equation of the tangent plane. However, if we approach the origin from a different direction, we get a different situation. For example, suppose we approach the origin along the line $y = x$. If we put $y = x$ into the original function, it becomes

$$f(x, x) = \frac{x(x)}{\sqrt{x^2 + (x)^2}} = \frac{x^2}{\sqrt{2x^2}} = \frac{|x|}{\sqrt{2}}$$

When $x > 0$, the slope of this curve is equal to $\frac{\sqrt{2}}{2}$; when $x < 0$, the slope of this curve is equal to $-\frac{\sqrt{2}}{2}$. This presents a problem. In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here, so there is no tangent plane to this surface. This is why the formula requires $f(x, y)$ to be *differentiable*.

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in the figure above. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition: Differentiable Functions

A function $f(x, y)$ is ***differentiable*** at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y)$$

where the error term $E(x, y)$ satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

The term $E(x, y)$ is referred to as the **error term** and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Illustration 2.3.3: Demonstrating Differentiability

We show that the function $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$

First, we calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$,:

$$\begin{aligned} f(2, -3) &= 2(2)^2 - 4(-3) = 8 + 12 = 20 \\ f_x(x, y) &= 4x \longrightarrow f_x(2, -3) = 4(2) = 8 \\ f_y(x, y) &= -4 \longrightarrow f_y(2, -3) = -4 \end{aligned}$$

Therefore the error term is defined by

$$\begin{aligned} f(x, y) &= f(2, -3) + f_x(2, -3)(x - 2) + f_y(2, -3)(y + 3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8(x - 2) - 4(y + 3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8x - 16 - 4y - 12 + E(x, y) \\ 2x^2 - 4y &= 8x - 4y - 8 + E(x, y) \\ E(x, y) &= 2x^2 - 8x + 8 = 2(x - 2)^2 \end{aligned}$$

Next, we calculate the limit in our differentiable definition:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \lim_{(x,y)\to(2,-3)} \frac{2(x - 2)^2}{\sqrt{(x - 2)^2 + (y + 3)^2}}$$

It is clear that for any path through $(2, -3)$ this limit must go to zero. Therefore, $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

Illustration 2.3.4: Demonstrating Non-Differentiability

Show that the function $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ is not differentiable at point $(0, 0)$

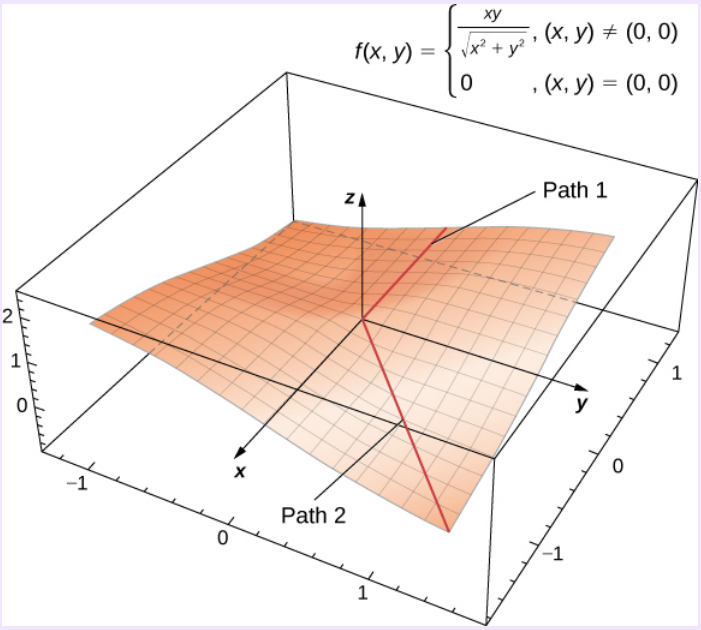
We already calculated $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Therefore the error term is defined by

$$f(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + E(x, y) = E(x, y)$$

Next, we calculate the limit in our differentiable definition:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Approaching the origin along the line $y = kx$ we see that this limit approaches $\frac{k}{1+k^2}$ and therefore does not exist.



Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

Theorem 2.3.2: Differentiability Implies Continuity

Let $z = f(x,y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x,y)$ is differentiable at (x_0, y_0), then $f(x,y)$ is continuous at (x_0, y_0).

Just like in single variable calculus, if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example, the function discussed above

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is *continuous* at the origin, but it is *not differentiable* at the origin.

We can further explore the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem 2.3.3: Continuity of First Partials Implies Differentiability

Let $z = f(x,y)$ be a function of two variables with (x_0, y_0) in the domain of f.

If $f(x,y)$, $f_x(x,y)$, and $f_y(x,y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then $f(x,y)$ is differentiable there.

For our non-differentiable example at the origin, let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} \text{ and } \frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$$

Consider the limit

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}}$$

If we approach the origin along the path $x = ky$ this becomes:

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} = \lim_{y\to 0} \frac{y^3}{((ky)^2+y^2)^{\frac{3}{2}}} = \lim_{y\to 0} \frac{y^3}{(k^2y^2+y^2)^{\frac{3}{2}}} = \frac{1}{(k^2+1)^{\frac{3}{2}}} \lim_{y\to 0} \frac{y^3}{|y|^3}$$

This limit does not exist (it is positive for $y > 0$ and negative for $y < 0$). Therefore f_x is not continuous near the origin, which is why the function $f(x,y)$ was not differentiable there.

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2.3: Practice Set

Exercise 2.3.1a

If $f(x,y) = x^2 \sin(xy)$ find f_x

Answer

We differentiate treating x as the variable and y as a constant:

$$f_x = \frac{\partial}{\partial x}\left(x^2 \sin(xy)\right) = 2x \cdot \sin(xy) + x^2 \cdot (\cos(xy) \cdot y) = 2x\sin(xy) + x^2 y\cos(xy)$$

Exercise 2.3.1b

If $g(s,t) = (2s+3t)^3$ find g_{st}

Answer

We differentiate treating s as the variable and t as a constant to first find g_s:

$$g_s = \frac{\partial}{\partial s}\left((2s+3t)^3\right) = 3(2s+3t)^2 \cdot (2+0) = 6(2s+3t)^2$$

Then we take the derivative again, treating t as a variable and s as a constant:

$$g_{st} = \frac{\partial}{\partial t}\left(6(2s+3t)^2\right) = 12(2s+3t) \cdot (0+3) = 36(2s+3t)$$

Note that doing this is the opposite order would give the same result:

$$g_t = \frac{\partial}{\partial t}\left((2s+3t)^3\right) = 3(2s+3t)^2 \cdot (0+3) = 9(2s+3t)^2$$

$$g_{ts} = \frac{\partial}{\partial s}\left(9(2s+3t)^2\right) = 18(2s+3t) \cdot (2+0) = 36(2s+3t)$$

Exercise 2.3.1c

If $h(a,b) = \frac{a}{b} + \frac{b}{a}$ find h_{bb}

Answer

We differentiate treating b as the variable and a as a constant to first find h_b:

$$h_b = \frac{\partial}{\partial b}\left(ab^{-1} + a^{-1}b\right) = a \cdot (-b^{-2}) + a^{-1} \cdot 1 = -ab^{-2} + a^{-1}$$

Then differentiate again to find the second derivative with respect to b:

$$h_{bb} = \frac{\partial}{\partial b}\left(-ab^{-2} + a^{-1}\right) = -a \cdot (-2b^{-3}) + 0 = 2ab^{-3}$$

Exercise 2.3.1d

If $j(u,v) = e^u\sqrt{u^2 + 5u\sec(u) + 7} - \tan^{-1}(uv)$ find j_{uv}

Answer

Finding j_u is unpleasant, but since we know $j_{uv} = j_{vu}$, we start by finding j_v:

$$j_v = \frac{\partial}{\partial v}\left(e^u\sqrt{u^2 + 5u\sec(u) + 7} - \tan^{-1}(uv)\right) = 0 - \frac{1}{1+(uv)^2} \cdot u = -\frac{u}{1+u^2v^2}$$

Now it is much simpler to differentiate with respect to u to reach our final answer:

$$j_{uv} = j_{vu} = \frac{\partial}{\partial u}\left(\frac{-u}{1+u^2v^2}\right) = \frac{(1+u^2v^2)(-1)-(-u)(0+2u\cdot v^2)}{(1+u^2v^2)^2} = \frac{u^2v^2-1}{(1+u^2v^2)^2}$$

Exercise 2.3.1e

If $A(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ find $A_y(3,4)$

Answer

We differentiate treating y as the variable and x as a constant to find the function $A_y(x,y)$:

$$A_y = \frac{\partial}{\partial y}\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \frac{(x^2+y^2)^{\frac{1}{2}}\cdot x - xy\cdot\left(\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot 2y\right)}{\left(\sqrt{x^2+y^2}\right)^2} = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$$

Then simply plug in the given values of $x=3, y=4$ to get the desired number:

$$A_y(3,4) = \frac{3^3}{(3^2+4^2)^{\frac{3}{2}}} = \frac{27}{125}$$

Exercise 2.3.1f

If $\cos(xz)+\sin(yz) = z$ find z_y

Answer

We differentiate implicitly across the entire equation, remembering that z is a function of our variable y and x is a constant:

$$\frac{\partial}{\partial y}\left(\cos(xz)+\sin(yz) = z\right)$$

$$-\sin(xz)\cdot(xz_y)+\cos(yz)\cdot(z+yz_y) = z_y$$

Moving all z_y terms to the left, and all non z_y terms to the right we get:

$$\left(-x\sin(xz)+y\cos(yz)-1\right)z_y = -z\cos(yz)$$

Then we solve for z_y by dividing away its coefficient to the other side:

$$z_y = \frac{z\cos(yz)}{x\sin(xz)-y\cos(yz)+1}$$

Exercise 2.3.1g

If $f(\rho,\theta,\phi) = \rho^2\cos(\theta)\sin(\phi)$ find $f_{\rho\theta\phi}\left(1,\frac{\pi}{3},\frac{\pi}{3}\right)$

Answer

The order does not matter here, but we will go in the order written. First we treat ρ as the variable:

$$f_\rho = \frac{\partial}{\partial\rho}\left(\rho^2\cos(\theta)\sin(\phi)\right) = 2\rho\cos(\theta)\sin(\phi)$$

Next we differentiate with respect to θ:

$$f_{\rho\theta} = \frac{\partial}{\partial\theta}\left(2\rho\cos(\theta)\sin(\phi)\right) = 2\rho(-\sin(\theta))\sin(\phi) = -2\rho\sin(\theta)\sin(\phi)$$

Third we differentiate with respect to ϕ:

$$f_{\rho\theta\phi} = \frac{\partial}{\partial\phi}\left(-2\rho\sin(\theta)\sin(\phi)\right) = -2\rho\sin(\theta)\cos(\phi)$$

Finally, we plug in $\rho=1, \theta=\dfrac{\pi}{3}$, and $\phi=\dfrac{\pi}{3}$ to get:

$$f_{\rho\theta\phi}\left(1,\frac{\pi}{3},\frac{\pi}{3}\right) = -2\cdot 1\cdot\sin\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Exercise 2.3.1h

If $e^{xyz} = e^x + e^y - e^z$ find $z_x(1,0,1)$

Answer

We differentiate implicitly across the entire equation, remembering that z is a function of our variable x and y is a constant:

$$\frac{\partial}{\partial x}\left(e^{xyz} = e^x + e^y - e^z\right)$$

$$e^{xyz} \cdot (1 \cdot yz + x \cdot 0 \cdot z + x \cdot y \cdot z_x) = e^x + 0 - e^z \cdot z_x$$

Moving all z_x terms to the left, and all non z_x terms to the right we get:

$$\left(xye^{xyz} + e^z\right) z_x = e^x - yze^{xyz}$$

Then we solve for z_y by dividing away its coefficient to the other side:

$$z_x = \frac{e^x - yze^{xyz}}{xye^{xyz} + e^z}$$

Finally we plug in the given points $x = 1, y = 0$, and $z = 1$ to get:

$$z_x(1,0,1) = \frac{e^1 - 0e^0}{0e^0 + e^1} = 1$$

Exercise 2.3.1i

If $B(t,\tau) = \tau\sqrt{t^2+\tau^4} + t$ find $B_{\tau t}(3,-2)$

Answer

We differentiate with respect to τ first, treating t as a constant:

$$B_\tau = \frac{\partial}{\partial \tau}\left(\tau\sqrt{t^2+\tau^4} + t\right) = \sqrt{t^2+\tau^4} + \frac{1}{2}\tau(t^2+\tau^4)^{-\frac{1}{2}} \cdot 4\tau^3 = \frac{t^2+3\tau^4}{\sqrt{t^2+\tau^4}}$$

Next we differentiate with respect to t next, treating τ as a constant:

$$B_{\tau t} = \frac{\partial}{\partial t}\left(\frac{t^2+3\tau^4}{\sqrt{t^2+\tau^4}}\right) = \frac{(t^2+\tau^4)^{\frac{1}{2}} \cdot 2t - (t^2+3\tau^4) \cdot \left(\frac{1}{2}(t^2+\tau^4)^{-\frac{1}{2}} \cdot 2t\right)}{\left(\sqrt{t^2+\tau^4}\right)^2} = \frac{t^3 - t\tau^4}{(t^2+\tau^4)^{\frac{3}{2}}}$$

Lastly we plug in the given values $t = 3, \tau = -2$:

$$B_{\tau t}(3,-2) = \frac{3^3 - 3(-2)^4}{(3^2+(-2)^4)^{\frac{3}{2}}} = -\frac{21}{125}$$

Exercise 2.3.2a

Find the plane tangent to $x + y + z = 1$ at $(1,2,-2)$

Answer

The tangent plane will be given by $z = -2 + z_x(1,2,-2) \cdot (x-1) + z_y(1,2,-2) \cdot (y-2)$

Solving for z is simple here, and we get $z = 1 - x - y$.

Taking each partial derivative gives:

$$z_x = \frac{\partial}{\partial x}(1 - x - y) = -1$$

$$z_y = \frac{\partial}{\partial y}(1 - x - y) = -1$$

Both of these are constant, so plugging in the given point of $x = 1, y = 2$, and $z = -2$ gives us:

$$z_x(1,2,-2) = -1 \quad \text{and} \quad z_y(1,2,-2) = -1$$

Therefore our plane is:

$z = -2 - 1(x-1) - 1(y-2)$

In general form, this answer looks like:

$x + y + z = 1$

Notice: Just like in single variable calculus, where the tangent line to a line it always itself, we see that the tangent plane to a plane is always itself.

? Exercise 2.3.2b

Find the plane tangent to $x^2 + 2y^2 + 3z^2 = 6$ at $(-1, 1, -1)$

Answer

The tangent plane will be given by $z = -1 + z_x(-1,1,-1) \cdot (x+1) + z_y(-1,1,-1) \cdot (y-1)$

It is possible to solve for z here, but we will use implicit derivatives to save time.

$\frac{\partial}{\partial x}\left(x^2 + 2y^2 + 3z^2 = 6\right)$

$2x + 0 + 6z \cdot z_x = 0$

Therefore, we can easily solve for $z_x = -\frac{x}{3z}$.

$\frac{\partial}{\partial y}\left(x^2 + 2y^2 + 3z^2 = 6\right)$

$0 + 4y + 6z \cdot z_y = 0$

Again, we can easily solve for $z_y = -\frac{2y}{3z}$.

Plugging in the given point of $x = -1, y = 1$, and $z = -1$ gives us:

$z_x(-1,1,-1) = -\frac{1}{3} \quad \text{and} \quad z_y(-1,1,-1) = \frac{2}{3}$

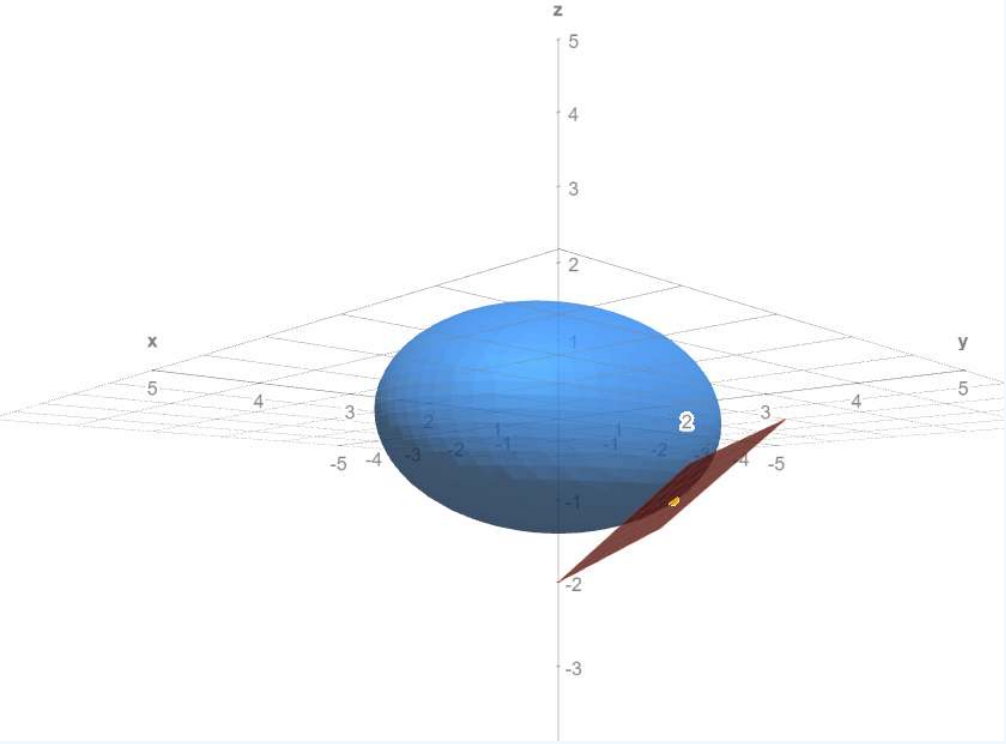
Therefore our plane is:

$z = -1 - \frac{1}{3}(x+1) + \frac{2}{3}(y-1)$

In general form, this answer looks like:

$x - 2y + 3z = -6$

Below is a picture to confirm our solution.



Exercise 2.3.2c

Find the plane tangent to $z = (2x + y)e^{xy}$ at $(0, 5, 5)$

Answer

The tangent plane will be given by $z = 5 + z_x(0, 5, 5) \cdot (x - 0) + z_y(0, 5, 5) \cdot (y - 5)$

The function is given explicitly, so taking each partial derivative gives:

$$z_x = \frac{\partial}{\partial x}\left((2x + y)e^{xy}\right) = 2 \cdot e^{xy} + (2x + y) \cdot (e^{xy} \cdot y) = (2xy + y^2 + 2)e^{xy}$$

$$z_y = \frac{\partial}{\partial y}\left((2x + y)e^{xy}\right) = 1 \cdot e^{xy} + (2x + y) \cdot (e^{xy} \cdot x) = (2x^2 + xy + 1)e^{xy}$$

Plugging in the given point of $x = 0, y = 5$, and $z = 5$ gives us:

$z_x(0, 5, 5) = 27 \quad \text{and} \quad z_y(0, 5, 5) = 1$

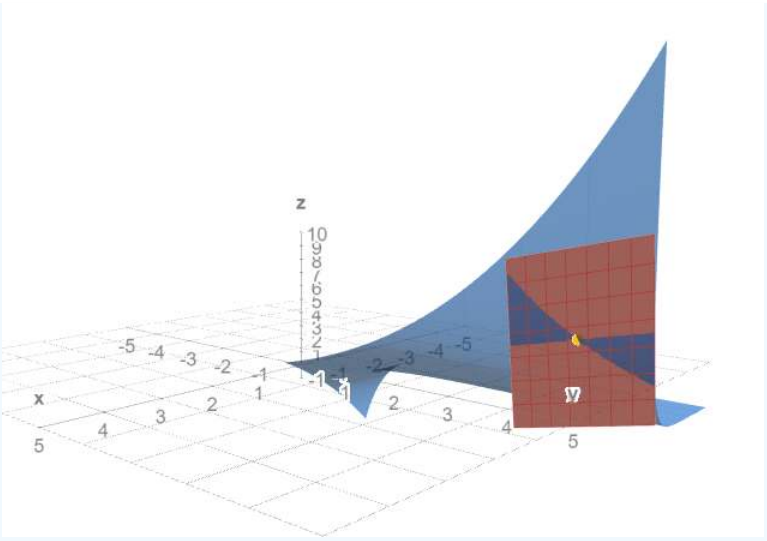
Therefore our plane is:

$z = 5 + 27(x - 0) + 1(y - 5)$

In general form, this answer looks like:

$27x + y - z = 0$

Below is a picture to confirm our solution.



Exercise 2.3.3a

Use a tangent plane to approximate the value of $\dfrac{1+3.02e^{(3.02)^2-9}}{3.02-e^{-.05}}$

Answer

If we define $f(x,y)=\dfrac{1+xe^{x^2-9}}{x-e^y}$, we want to find $f(3.02,-.05)$.

We can use the tangent plane to $z=f(x,y)$ near the point $(3,0)$ to approximate this value.

This tangent plane will be given by $z=f(3,0)+f_x(3,0)\cdot(x-3)+f_y(3,0)\cdot(y-0)$

The function is given explicitly, so taking each partial derivative gives:

$$z_x=\frac{\partial}{\partial x}\left(\frac{1+xe^{x^2-9}}{x-e^y}\right)=\frac{(x-e^y)\cdot(e^{x^2-9}+xe^{x^2-9}\cdot 2x)-(1+xe^{x^2-9})\cdot 1}{(x-e^y)^2}$$

$$z_y=\frac{\partial}{\partial y}\left(\frac{1+xe^{x^2-9}}{x-e^y}\right)=\frac{(x-e^y)\cdot 0-(1+xe^{x^2-9})\cdot(-e^y)}{(x-e^y)^2}$$

Plugging in the given point of $x=3, y=0$, into each of f, f_x, and f_y gives:

$$f(3,0)=2,\quad f_x(3,0)=\frac{17}{2},\quad \text{and}\quad f_y(3,0)=1$$

Therefore our plane is:

$$z=2+\frac{17}{2}(x-3)+(y-0)$$

Plugging in $x=3.02$ and $y=-.05$ into the plane gives us the approximation:

$$f(3.02,-.05)\approx 2+\frac{17}{2}(3.02-3)+(-.05-0)=2.12$$

This is quite close to the exact value of $f(3.02,-.05)=2.12996$

Exercise 2.3.3b

Use a tangent plane to approximate the value of $\pi-4\tan^{-1}\left(\dfrac{\sqrt{4.04}}{1.92}\right)$

Answer

If we define $f(x,y)=\pi-4\tan^{-1}\left(\dfrac{\sqrt{x}}{y}\right)$, we want to find $f(4.04,1.92)$.

We can use the tangent plane to $z = f(x, y)$ near the point $(4, 2)$ to approximate this value.

This tangent plane will be given by $z = f(4, 2) + f_x(4, 2) \cdot (x - 4) + f_y(4, 2) \cdot (y - 2)$

The function is given explicitly, so taking each partial derivative gives:

$$z_x = \frac{\partial}{\partial x}\left(\pi - 4\tan^{-1}\left(\frac{\sqrt{x}}{y}\right)\right) = -\frac{\frac{2}{y\sqrt{x}}}{1 + \frac{x}{y^2}}$$

$$z_y = \frac{\partial}{\partial y}\left(\pi - 4\tan^{-1}\left(\frac{\sqrt{x}}{y}\right)\right) = \frac{\frac{4\sqrt{x}}{y^2}}{1 + \frac{x}{y^2}}$$

Plugging in the given point of $x = 4, y = 2$, into each of f, f_x, and f_y gives:

$$f(4, 2) = 0, \quad f_x(4, 2) = -\frac{1}{4}, \quad \text{and} \quad f_y(4, 2) = 1$$

Therefore our plane is:

$$z = 0 - \frac{1}{4}(x - 4) + (y - 2)$$

Plugging in $x = 4.04$ and $y = 1.92$ into the plane gives us the approximation:

$$f(4.04, 1.92) \approx 0 - \frac{1}{4}(4.04 - 4) + (1.92 - 2) = -0.09$$

This is quite close to the exact value of $f(4.04, 1.92) = -0.0915623$

? Exercise 2.3.3c

Use a tangent plane to approximate the value of $\dfrac{3.02 \cdot 2.01}{(3.02)^2 - (2.01)^3}$

Answer

If we define $f(x, y) = \dfrac{xy}{x^2 - y^3}$, we want to find $f(3.02, 2.01)$.

We can use the tangent plane to $z = f(x, y)$ near the point $(3, 2)$ to approximate this value.

This tangent plane will be given by $z = f(3, 2) + f_x(3, 2) \cdot (x - 3) + f_y(3, 2) \cdot (y - 2)$

The function is given explicitly, so taking each partial derivative gives:

$$z_x = \frac{\partial}{\partial x}\left(\frac{xy}{x^2 - y^3}\right) = \frac{-yx^2 - y^4}{(x^2 - y^3)^2}$$

$$z_y = \frac{\partial}{\partial y}\left(\frac{xy}{x^2 - y^3}\right) = \frac{x^3 + 2xy^3}{(x^2 - y^3)^2}$$

Plugging in the given point of $x = 3, y = 2$, into each of f, f_x, and f_y gives:

$$f(3, 2) = 6, \quad f_x(3, 2) = -34, \quad \text{and} \quad f_y(3, 2) = 75$$

Therefore our plane is:

$$z = 6 - 34(x - 3) + 75(y - 2)$$

Plugging in $x = 3.02$ and $y = 2.01$ into the plane gives us the approximation:

$$f(3.02, 2.01) \approx 6 - 34(3.02 - 3) + 75(2.01 - 2) = 6.07$$

This is quite close to the exact value of $f(3.02, 2.01) = 6.07142$

? Exercise 2.3.3d

Use a tangent plane to approximate the value of $\sqrt{76 + (1.98)^3 + (-2.005)^4}$

Answer

If we define $f(x,y) = \sqrt{76+x^3+y^4}$, we want to find $f(1.98, -2.005)$.

We can use the tangent plane to $z = f(x,y)$ near the point $(2,-2)$ to approximate this value.

This tangent plane will be given by $z = f(2,-2) + f_x(2,-2)\cdot(x-2) + f_y(2,-2)\cdot(y+2)$

The function is given explicitly, so taking each partial derivative gives:

$$z_x = \frac{\partial}{\partial x}\left(\sqrt{76+x^3+y^4}\right) = \frac{3x^2}{2\sqrt{76+x^3+y^4}}$$

$$z_y = \frac{\partial}{\partial y}\left(\sqrt{76+x^3+y^4}\right) = \frac{2y^3}{\sqrt{76+x^3+y^4}}$$

Plugging in the given point of $x=2, y=-2$, into each of f, f_x, and f_y gives:

$$f(2,-2) = 10, \quad f_x(2,-2) = \frac{3}{5}, \quad \text{and} \quad f_y(2,-2) = -\frac{8}{5}$$

Therefore our plane is:

$$z = 10 + \frac{3}{5}(x-2) - \tfrac{8}{5}(y+2)$$

Plugging in $x = 1.98$ and $y = -2.005$ into the plane gives us the approximation:

$$f(1.98,-2.005) \approx 10 + \frac{3}{5}(1.98-2) - \frac{8}{5}(-2.005+2) = 9.996$$

This is quite close to the exact value of $f(1.98,-2.005) = 9.99615$

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2.4: More on Multivariable Derivatives

Learning Objectives

- State the chain rules for one or two independent variables.
- Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
- Use the chain rule to perform implicit differentiation of a function of two or more variables.
- Determine the directional derivative in a given direction for a function of two variables.
- Determine the gradient vector of a given real-valued function.
- Explain the significance of the gradient vector with regard to direction of change along a surface.
- Use the gradient to find the tangent to a level curve of a given function.
- Calculate directional derivatives and gradients in three dimensions.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable. We follow this with a discussion on the directional derivative, which is a derivative taken in a particular direction along a surface (instead of the two default directions which are parallel to the x and y axis, yielding the classic partial derivatives.)

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- **14.6: Directional Derivatives and the Gradient** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

2.4: The Chain Rule for Multivariable Functions

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'\big(g(x)\big)g'(x)$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. For some intuition as to why this formula works, consider the function $h(x) = 2\sin(3x)$. It is easy to see that we have $h(x) = f(g(x))$ where $f(x) = 2\sin(x)$ and $g(x) = 3x$. Since $f'(x) = 2\cos(x)$ and $g'(x) = 3$ the chain rule says that $h'(x) = 2\cos(3x) \cdot 3 = 6\cos(3x)$. At $x = \pi$ notice that we have $g'(\pi) = 3, g(\pi) = 3\pi$, and $f'(3\pi) = f'(g(\pi)) = -2$ which is why we have $h'(\pi) = -6$.

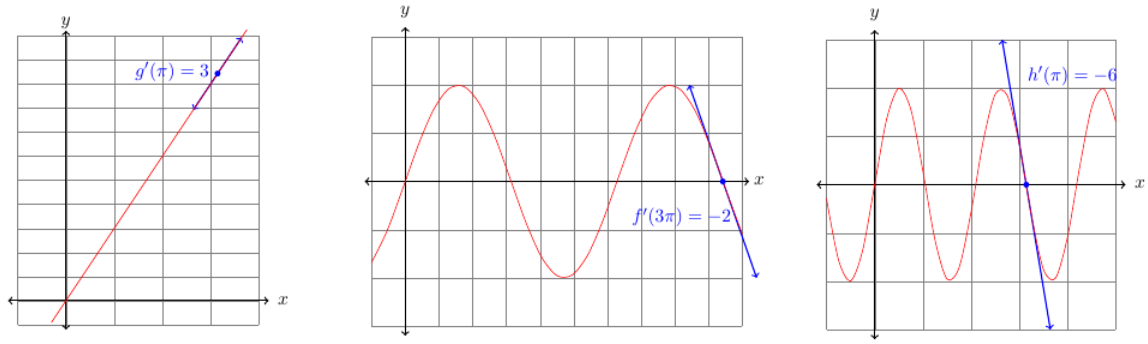


Figure 2.4.1: A visualization of the single variable form of the chain rule.

If x is *changing but* "close" to π, then $g(x)$ is *changing* by a factor of $g'(\pi) = 3$ times as far away from $g(\pi) = 3\pi$. In turn this means $f(x)$ is *changing* by a factor of $f'(3\pi) = -2$ times as far away from $h(\pi) = f(3\pi) = 0$. Thus a small perturbation in x near π yields **six times** as big of a perturbation in $h(x)$ from $h(\pi)$. These factors multiply as a *chain reaction* because of the composition of these functions, which is the origin of the name of the rule!

Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Theorem 2.4.1: Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that f is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f\big(x(t), y(t)\big)$ is differentiable at $t = t_0$ and that the chain rule formula holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y)$$

where $\lim_{(x,y)\to(x_0,y_0)} \dfrac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$

We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$z(t) - z(t_0) = f(x(t), y(t)) - f(x(t_0), y(t_0))$$
$$= f_x(x_0, y_0)(x(t) - x(t_0)) + f_y(x_0, y_0)(y(t) - y(t_0)) + E(x(t), y(t))$$

Next, we divide both sides by $t - t_0$:

$$\frac{z(t) - z(t_0)}{t - t_0} = f_x(x_0, y_0)\frac{x(t) - x(t_0)}{t - t_0} + f_y(x_0, y_0)\frac{y(t) - y(t_0)}{t - t_0} + \frac{E(x(t), y(t))}{t - t_0}$$

Then we take the limit as t approaches t_0:

$$\lim_{t \to t_0} \frac{z(t) - z(t_0)}{t - t_0}$$
$$= f_x(x_0, y_0) \lim_{t \to t_0} \left(\frac{x(t) - x(t_0)}{t - t_0} \right) + f_y(x_0, y_0) \lim_{t \to t_0} \left(\frac{y(t) - y(t_0)}{t - t_0} \right) + \lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0}$$

The left-hand side of this equation is equal to $\dfrac{dz}{dt}$, which leads to

$$\frac{dz}{dt} = f_x(x_0, y_0)\frac{dx}{dt} + f_y(x_0, y_0)\frac{dy}{dt} + \lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0}$$

The last term can be rewritten as

$$\lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0} = \lim_{t \to t_0} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0}$$
$$= \lim_{t \to t_0} \left(\frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right) \lim_{t \to t_0} \left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} \right)$$

As t approaches t_0, $\big(x(t), y(t)\big)$ approaches $\big(x(t_0), y(t_0)\big)$, so we can rewrite the last product as

$$\lim_{(x,y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \lim_{(x,y) \to (x_0, y_0)} \left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} \right)$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\lim_{(x,y) \to (x_0, y_0)} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} = \lim_{(x,y) \to (x_0, y_0)} \sqrt{\frac{(x - x_0)^2 + (y - y_0)^2}{(t - t_0)^2}}$$

$$= \lim_{(x,y) \to (x_0, y_0)} \sqrt{\left(\frac{x - x_0}{t - t_0} \right)^2 + \left(\frac{y - y_0}{t - t_0} \right)^2}$$
$$= \sqrt{\left[\lim_{(x,y) \to (x_0, y_0)} \left(\frac{x - x_0}{t - t_0} \right) \right]^2 + \left[\lim_{(x,y) \to (x_0, y_0)} \left(\frac{y - y_0}{t - t_0} \right) \right]^2}$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t = t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

Closer examination of the chain rule formula above reveals the same pattern we had in the single variable chain rule: a small change in t results in a change in both intermediate variables x and y by the factors $x'(t)$ and $y'(t)$ respectively. Each of these changes propagates to a change in the output of the function $f(x, y)$ by the factors $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. This is why there are two chain rule terms, since changing the variable t results in two different types of change for the eventual function f.

Example 2.4.1: Using the Chain Rule

Calculate $\dfrac{dz}{dt}$ for each of the following functions:

(a) $z = f(x, y) = 4x^2 + 3y^2, \quad x = x(t) = \sin(t), \quad y = y(t) = \cos(t)$

(b) $z = f(x,y) = \sqrt{x^2 - y^2}, \quad x = x(t) = e^{2t}, \quad y = y(t) = e^{-t}$

Solution

(a) To use the chain rule, we need four quantities:

$$\frac{\partial z}{\partial x} = 8x, \quad \frac{dx}{dt} = \cos(t), \quad \frac{\partial z}{\partial y} = 6y, \quad \frac{dy}{dt} = -\sin(t)$$

Now, we substitute each of these into the Chain Rule for One Independent Variable formula:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (8x)(\cos(t)) + (6y)(-\sin(t)) = 8x\cos(t) - 6y\sin(t)$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin(t)$ and $y(t) = \cos(t)$. We obtain

$$\frac{dz}{dt} = 8x\cos(t) - 6y\sin(t) = 8(\sin(t))\cos(t) - 6(\cos(t))\sin(t) = \boxed{2\sin(t)\cos(t)}$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to t:

$$z(t) = f\big(x(t), y(t)\big) = 4(x(t))^2 + 3(y(t))^2 = 4\sin^2(t) + 3\cos^2(t)$$

Then $\frac{dz}{dt} = 2(4\sin(t))(\cos(t)) + 2(3\cos(t))(-\sin(t)) = 2\sin(t)\cos(t)$ which is the same solution.

Note that it may not always be this easy to differentiate in this form.

(b) To use the chain rule, we again need four quantities:

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 - y^2}}, \quad \frac{dx}{dt} = 2e^{2t}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2}}, \quad \frac{dy}{dt} = -e^{-t}$$

We substitute each of these into the Chain Rule for One Independent Variable formula:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \left(\frac{x}{\sqrt{x^2 - y^2}}\right)(2e^{2t}) + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)(-e^{-t}) = \frac{2xe^{2t} + ye^{-t}}{\sqrt{x^2 - y^2}}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\frac{dz}{dt} = \frac{2xe^2 t + ye^{-t}}{\sqrt{x^2 - y^2}} = \frac{2(e^{2t})e^{2t} + (e^{-t})e^{-t}}{\sqrt{e^{4t} - e^{-2t}}} = \boxed{\frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to t:

$$z(t) = f(x(t), y(t)) = \sqrt{(x(t))^2 - (y(t))^2} = \sqrt{e^{4t} - e^{-2t}} = (e^{4t} - e^{-2t})^{\frac{1}{2}}$$

Then $\frac{dz}{dt} = \frac{1}{2}(e^{4t} - e^{-2t})^{-1/2}(4e^{4t} + 2e^{-2t}) = \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}$ which is the same solution.

It is often useful to create a visual representation of the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula. This diagram helps to see how a change in any variable affects another variable or the end function.

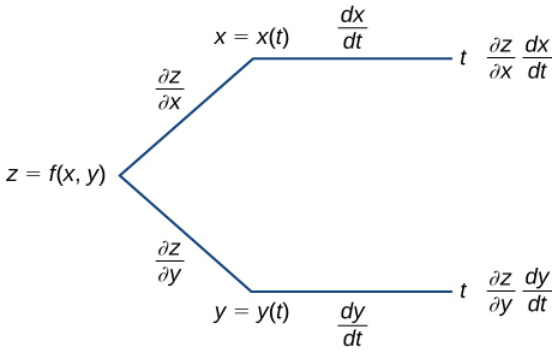


Figure 2.4.2: Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two **independent variables**, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch.

The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $\frac{\partial z}{\partial x}\frac{dx}{dt}$. This product represents the factor which a perturbation of t multiplies by to change the output of z through its effect on x. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $\frac{\partial z}{\partial y}\frac{dy}{dt}$, which is the factor which a change in t multiplies by to change f through its effect on y. To get the formula for $\frac{dz}{dt}$, add all the terms that appear on the rightmost side of the diagram. This gives us chain rule formula we saw above.

The Chain Rule for two independent variables, where $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v stems from a similar tree diagram:

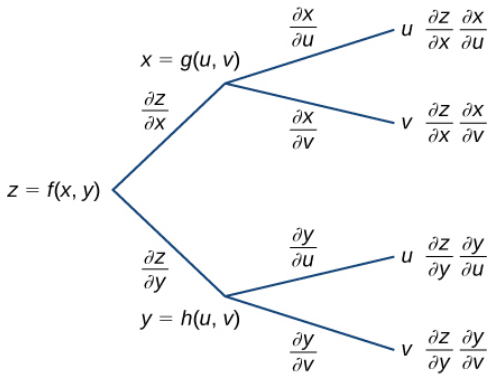


Figure 2.4.3: Tree diagram for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

Theorem 2.4.2: Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

There is an important difference between these two chain rule theorems. In Chain Rule for One Independent Variable, the left-hand side of the formula for the derivative is not a partial derivative, but in Chain Rule for Two Independent Variables it is. The reason is that, in Chain Rule for One Independent Variable, z is ultimately a function of t alone, whereas in Chain Rule for Two Independent Variables, z is a function of both u and v.

Example 2.4.2: Using the Chain Rule for Two Variables

Calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using the following functions:

$$z = f(x,y) = 3x^2 - 2xy + y^2, \quad x = x(u,v) = 3u + 2v, \quad y = y(u,v) = 4u - v^2$$

Solution

To implement the chain rule for two variables, we need six partial derivatives:

$$\frac{\partial z}{\partial x} = 6x - 2y, \quad \frac{\partial z}{\partial y} = -2x + 2y, \quad \frac{\partial x}{\partial u} = 3, \quad \frac{\partial x}{\partial v} = 2, \quad \frac{\partial y}{\partial u} = 4, \quad \frac{\partial y}{\partial v} = -2v$$

To find $\frac{\partial z}{\partial u}$ we use the Chain Rule for Two Independent Variables formula:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 3(6x - 2y) + 4(-2x + 2y) = 10x + 2y$$

Next, we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v^2$:

$$\frac{\partial z}{\partial u} = 10x + 2y = 10(3u + 2v) + 2(4u - v^2) = 38u + 20v - 2v^2$$

To find $\frac{\partial z}{\partial v}$ we again use the Chain Rule for Two Independent Variables formula:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = 2(6x - 2y) + (-2v)(-2x + 2y) = (12 + 4v)x - (4 + 4v)y$$

Then we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v^2$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= (12 + 4v)x - (4 + 4v)y = (12 + 4v)(3u + 2v) - (4 + 4v)(4u - v^2) \\ &= \boxed{-4uv + 20u + 4v^3 + 12v^2 + 24v}\end{aligned}$$

Again, this can be found through direct substitution instead of the chain rule if the reader wishes to check the answer!

Now that we've seen how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

Theorem 2.4.3: Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \ldots, m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in 1, 2, \ldots, n$.

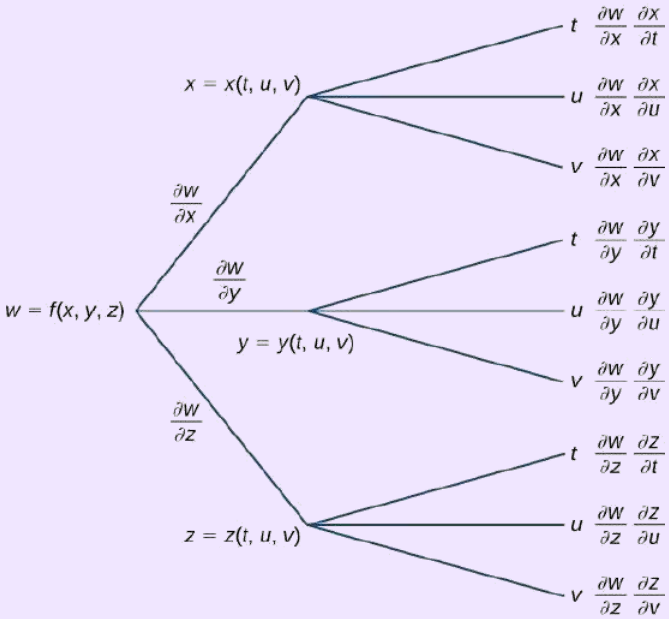
Illustration 2.4.1: Drawing a Tree Diagram

Create a tree diagram for the case when

$$w = f(x, y, z), \quad x = x(t, u, v), \quad y = y(t, u, v), \quad z = z(t, u, v)$$

and write out the formulas for the three partial derivatives of w.

Starting from the left, the function f has three independent variables: x, y, and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables t, u, and v.



The three formulas are:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

Illustration 2.4.2: Using the Generalized Chain Rule

Calculate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ using the following functions:

$$w = f(x, y, z) = 3x^2 - 2xy + 4z^2$$
$$x = x(u, v) = e^u \sin(v)$$
$$y = y(u, v) = e^u \cos(v)$$
$$z = z(u, v) = e^u$$

Solution

The formulas for $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ are

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\frac{\partial w}{\partial x} = 6x - 2y, \quad \frac{\partial w}{\partial y} = -2x, \quad \frac{\partial w}{\partial z} = 8z$$
$$\frac{\partial x}{\partial u} = e^u \sin(v), \quad \frac{\partial y}{\partial u} = e^u \cos(v), \quad \frac{\partial z}{\partial u} = e^u$$
$$\frac{\partial x}{\partial v} = e^u \cos(v), \quad \frac{\partial y}{\partial v} = -e^u \sin(v), \quad \frac{\partial z}{\partial v} = 0$$

Now, we substitute each of them into the first formula to calculate $\frac{\partial w}{\partial u}$:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} = (6x - 2y)e^u \sin(v) - 2xe^u \cos(v) + 8ze^u$$

then substitute $x(u, v) = e^u \sin(v)$, $y(u, v) = e^u \cos(v)$, and $z(u, v) = e^u$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial u} &= (6x - 2y)e^u \sin(v) - 2xe^u \cos(v) + 8ze^u \\ &= (6e^u \sin(v) - 2e^u \cos(v))e^u \sin(v) - 2(e^u \sin(v))e^u \cos(v) + 8e^{2u} \\ &= 6e^{2u} \sin^2(v) - 4e^{2u} \sin(v) \cos(v) + 8e^{2u} \\ &= 2e^{2u}(3 \sin^2(v) - 2 \sin(v) \cos(v) + 4)\end{aligned}$$

Next, we calculate $\frac{\partial w}{\partial v}$:

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v} = (6x - 2y)e^u \cos(v) - 2x(-e^u \sin(v)) + 8z(0)$$

then we substitute $x(u, v) = e^u \sin(v)$, $y(u, v) = e^u \cos(v)$, and $z(u, v) = e^u$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial v} &= (6x - 2y)e^u \cos(v) - 2x(-e^u \sin(v)) \\ &= (6e^u \sin(v) - 2e^u \cos(v))e^u \cos(v) + 2(e^u \sin(v))(e^u \sin(v)) \\ &= 2e^{2u} \sin^2(v) + 6e^{2u} \sin(v) \cos(v) - 2e^{2u} \cos^2(v) \\ &= 2e^{2u}(\sin^2(v) + 3 \sin(v) \cos(v) - \cos^2(v))\end{aligned}$$

Recall implicit differentiation provides a method for finding $\frac{dy}{dx}$ when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for $\frac{dy}{dx}$. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2 + 3y^2 + 4y - 4 = 0$ as follows.

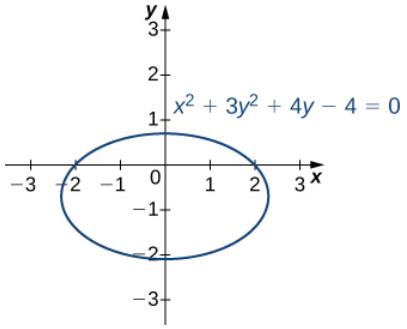


Figure 2.4.4: Graph of the ellipse defined by $x^2 + 3y^2 + 4y - 4 = 0$

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\begin{aligned}\frac{d}{dx}(x^2+3y^2+4y-4) &= \frac{d}{dx}(0)\\ 2x+6y\frac{dy}{dx}+4\frac{dy}{dx} &= 0\\ (6y+4)\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{2x}{6y+4} = -\frac{x}{3y+2}\end{aligned}$$

We can also define a function $z=f(x,y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x,y)=x^2+3y^2+4y-4$. The ellipse $x^2+3y^2+4y-4=0$ can then be described by the equation $f(x,y)=0$. Using this function and the following theorem gives us an alternative approach to calculating $\frac{dy}{dx}$.

Theorem 2.4.4: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z=f(x,y)$ defines y implicitly as a function $y=g(x)$ of x via the equation $f(x,y)=0$. Then when $f_y(x,y)\neq 0$:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

If the equation $f(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then as long as $f_z(x,y,z)\neq 0$:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

These implicit derivative formulas are a direct result of the Chain Rules we have discussed in this section. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x,y)=0$, we can apply the chain rule to find $\frac{dy}{dx}$:

$$\begin{aligned}\frac{d}{dx}f(x,y) &= \frac{d}{dx}(0)\\ \frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} &= 0\\ \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} &= 0.\end{aligned}$$

Solving this equation for $\frac{dy}{dx}$ gives the first of the two implicit formulas above. The second can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Theorem 2.4.4 and the function $f(x,y)=x^2+3y^2+4y-4$, we obtain

$$\frac{\partial f}{\partial x}=2x \quad \text{and} \quad \frac{\partial f}{\partial y}=6y+4$$

Therefore

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{2x}{6y+4} = -\frac{x}{3y+2}$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 2.4.3: Implicit Differentiation by Partial Derivatives

Calculate the desired derivative by using the implicit differentiation formulas:

(a) Find $\dfrac{dy}{dx}$ for $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$

(b) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $x^2e^y - yze^x = 0$.

Solution

(a) Set $f(x, y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$, then calculate f_x and f_y:

$$f_x(x, y) = 6x - 2y + 4 \quad \text{and} \quad f_y(x, y) = -2x + 2y - 6$$

Therefore,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{6x - 2y + 4}{-2x + 2y - 6} = \boxed{\frac{3x - y + 2}{x - y + 3}}$$

(b) We have $f(x, y, z) = x^2e^y - yze^x$. Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x, \quad \frac{\partial f}{\partial y} = x^2e^y - ze^x, \quad \frac{\partial f}{\partial z} = -ye^x$$

Therefore, the partial derivatives are:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{2xe^y - yze^x}{-ye^x} = \boxed{\frac{2xe^y - yze^x}{ye^x}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{x^2e^y - ze^x}{-ye^x} = \boxed{\frac{x^2e^y - ze^x}{ye^x}}$$

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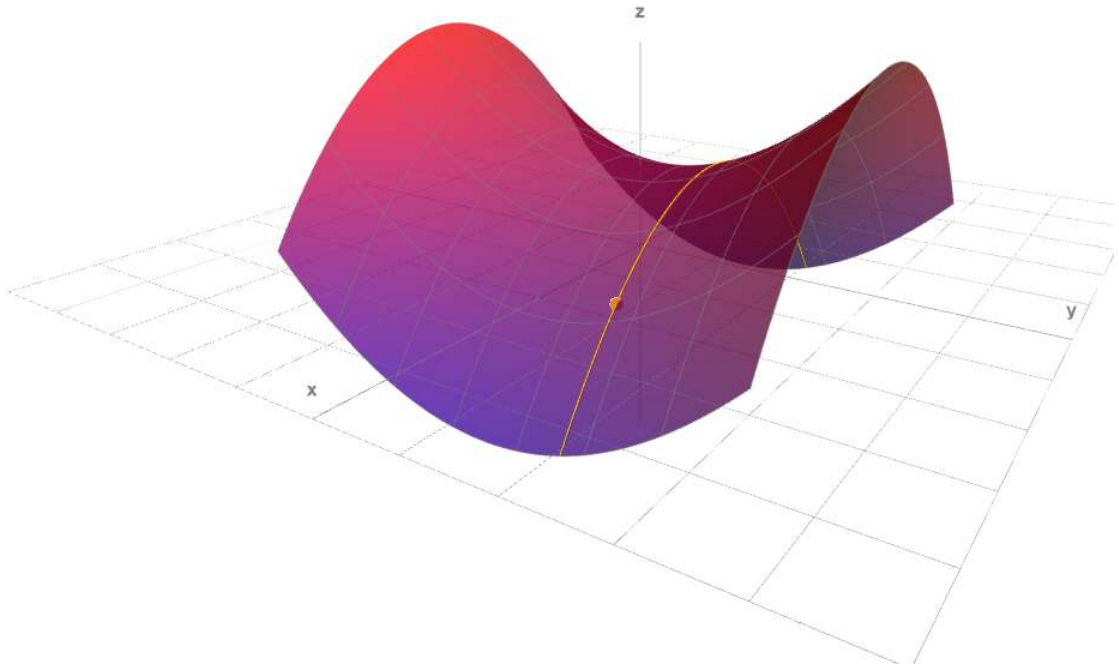
2.4: Directional Derivatives and the Gradient

We start with the graph of a surface defined by the equation $z = f(x, y)$. Given a point (a, b) in the domain of f, we can take two different types of derivatives:

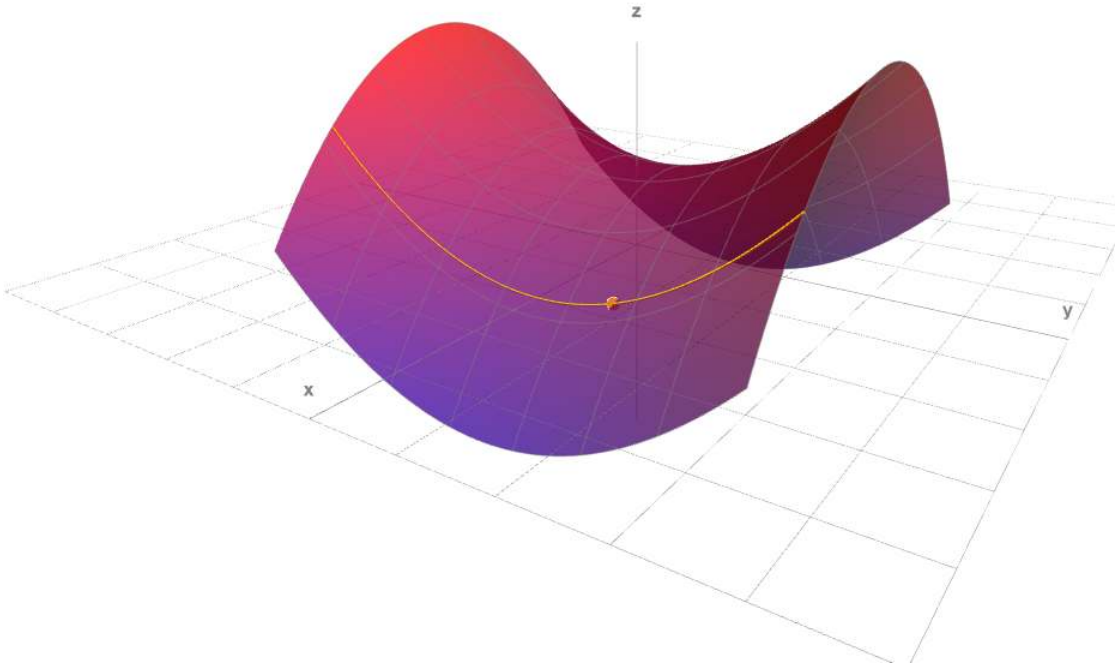
$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} \text{ and } \frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

These are essentially the derivatives along two different paths through (a, b).

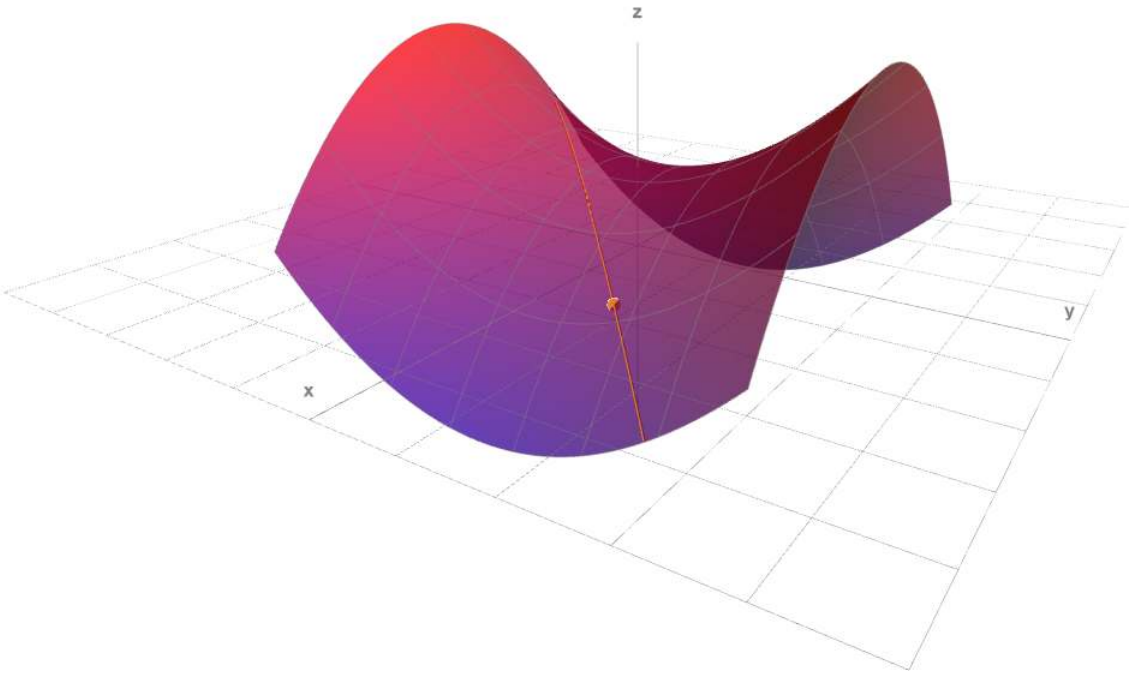
For $\frac{\partial f}{\partial x}(a, b)$, this path is the line $\vec{\mathbf{r}}(t) = \langle t, b, f(t, b) \rangle$ which treats the y-coordinate as a constant. We can consider this as traveling along the surface of $z = f(x, y)$ in the *direction* of the unit vector $\hat{\mathbf{i}}$.



On the other hand, for $\frac{\partial f}{\partial y}(a, b)$, this path is the line $\vec{\mathbf{r}}(t) = \langle a, t, f(a, t) \rangle$ which treats the x-coordinate as a constant. We can consider this as traveling along the surface of $z = f(x, y)$ in the *direction* of the unit vector $\hat{\mathbf{j}}$.



The slope in these two directions is often different, but there are many other linear paths we can take through (a, b). If we travel along $\vec{\mathbf{r}}(t) = \langle a + \alpha t, b + \beta t, f(a + \alpha t, b + \beta t)\rangle$ for some $\alpha^2 + \beta^2 = 1$, then we are traveling through the point $\vec{\mathbf{r}}(0) = (a, b, f(a, b))$ in the direction of the unit vector $\vec{\mathbf{u}} = \langle \alpha, \beta\rangle$. The slope in this direction is referred to as the **directional derivative** in the direction of $\vec{\mathbf{u}}$.



Definition: Directional Derivatives

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define the unit vector $\vec{\mathbf{u}} = \alpha\,\hat{\mathbf{i}} + \beta\,\hat{\mathbf{j}}$. When the limit exists, the ***directional derivative*** of f in the direction of $\vec{\mathbf{u}}$ is defined by

$$D_{\vec{\mathbf{u}}} f(a, b) = \lim_{h \to 0} \frac{f(a + \alpha h, b + \beta h) - f(a, b)}{h}$$

This equation provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Note that since the point (a, b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y. That is,

$$D_{\vec{\mathbf{u}}} f(x, y) = \lim_{h \to 0} \frac{f(x + \alpha h, y + \beta h) - f(x, y)}{h}$$

Illustration 2.4.3: Finding a Directional Derivative from the Definition

Find the directional derivative $D_{\vec{\mathbf{u}}} f(x, y)$ of $f(x, y) = x^2 - x(y + 1) + 2(y - 1)^2$ for $\vec{\mathbf{u}} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$

Then determine $D_{\vec{\mathbf{u}}} f(2, 1)$

Using $f(x, y) = x^2 - x(y + 1) + 2(y - 1)^2$, we first calculate $f\left(x + \frac{4}{5}h, y + \frac{3}{5}h\right)$:

$$\begin{aligned} f\left(x + \tfrac{4}{5}h, y + \tfrac{3}{5}h\right) &= \left(x + \tfrac{4}{5}h\right)^2 - \left(x + \tfrac{4}{5}h\right)\left(y + \tfrac{3}{5}h + 1\right) + 2\left(y + \tfrac{3}{5}h - 1\right)^2 \\ &= \frac{22h^2 + 25hx + 40hy - 80h + 25x^2 - 25xy - 25x + 50y^2 - 100y + 50}{25} \end{aligned}$$

Subtracting $f(x, y)$ from this expression we get:

$$f\left(x + \tfrac{4}{5}h, y + \tfrac{3}{5}h\right) - f(x, y) = \frac{22h^2 + 25hx + 40hy - 80h}{25}$$

We substitute this expression into the limit definition for $D_{\vec{\mathbf{u}}} f(x, y)$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x, y) &= \lim_{h \to 0} \frac{f\left(x + \frac{4}{5}h, y + \frac{3}{5}h\right) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{22h^2 + 25hx + 40hy - 80h}{25h} \\ &= \lim_{h \to 0} \frac{22h + 25x + 40y - 80}{25} \\ &= \frac{25x + 40y - 80}{25} \\ &= \frac{5x + 8y - 16}{5} \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(2, 1)$, we substitute $x = 2$ and $y = 1$ into this answer:

$$D_{\vec{\mathbf{u}}} f(2, 1) = \frac{5(2) + 8(1) - 16}{5} = \frac{2}{5}$$

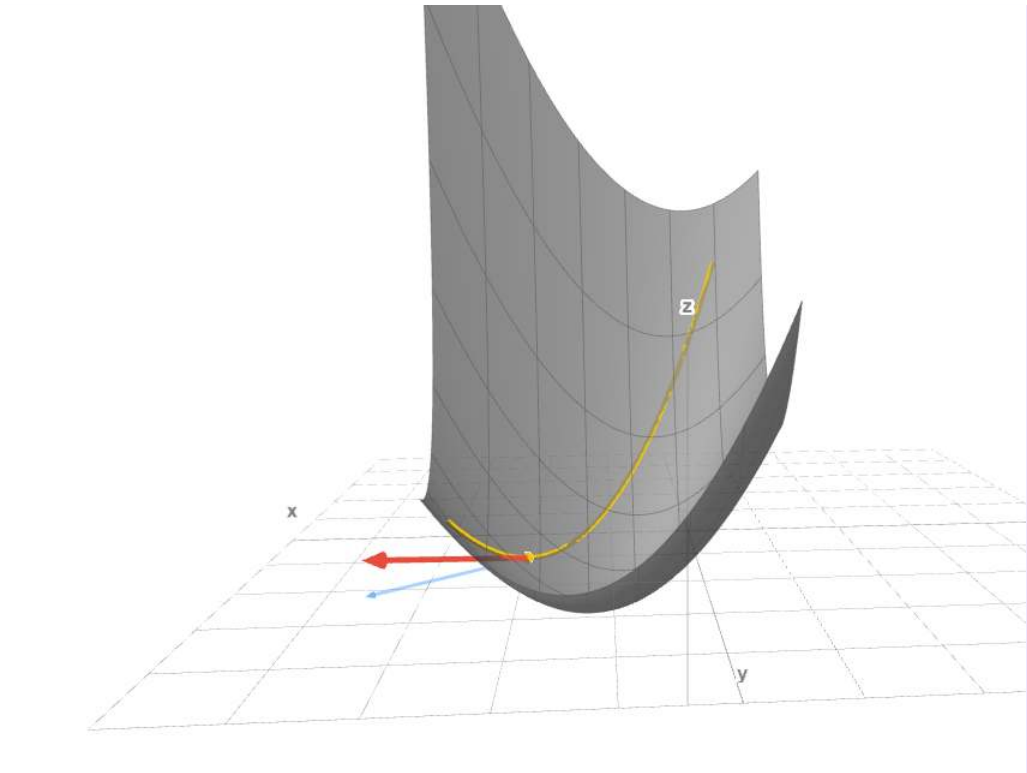


Figure 2.4.5: The surface $z = x^2 - x(y+1) + 2(y-1)^2$ and direction vector $\vec{\mathbf{u}}$ in blue.
The red tangent vector to the yellow path in the direction of $\vec{\mathbf{u}}$ has a z component $D_{\vec{\mathbf{u}}}(2,1) = \frac{2}{5}$.

An easier approach to calculating directional derivatives that involves partial derivatives is outlined in the following theorem.

Theorem 2.4.5: Directional Derivative of a Function of Two Variables

Let $z = f(x,y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of the unit vector $\vec{\mathbf{u}} = \alpha\,\hat{\mathbf{i}} + \beta\,\hat{\mathbf{j}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\alpha + f_y(x,y)\beta$$

Proof

Applying the limit definition of a directional derivative stated above, the directional derivative of f in the direction of $\vec{\mathbf{u}} = \alpha\,\hat{\mathbf{i}} + \beta\,\hat{\mathbf{j}}$ at a point (x_0, y_0) in the domain of f can be written

$$D_{\vec{\mathbf{u}}} f((x_0,y_0)) = \lim_{t\to 0} \frac{f(x_0 + t\alpha, y_0 + t\beta) - f(x_0,y_0)}{t}$$

Let $x(t) = x_0 + t\alpha$ and $y(t) = y_0 + t\beta$, and define $g(t) = f(x(t), y(t))$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x(t),y(t))\alpha + f_y(x(t),y(t))\beta$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0,y_0)\alpha + f_y(x_0,y_0)\beta$$

By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0}\frac{g(t) - g(0)}{t} = \lim_{t\to 0}\frac{f(x_0 + t\alpha, y_0 + t\beta) - f(x_0,y_0)}{t}$$

Therefore, $D_{\vec{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)\alpha + f_y(x_0, y_0)\beta$.

Since the point (x_0, y_0) is an arbitrary point from the domain of f, this result holds for all points in the domain of f for which the partials f_x and f_y exist.

Therefore,

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\alpha + f_y(x, y)\beta$$

□

Illustration 2.4.4: Finding a Directional Derivative: Alternative Method

Find the directional derivative $D_{\vec{\mathbf{u}}} f(x, y)$ of $f(x, y) = x^2 - x(y + 1) + 2(y - 1)^2$ for $\vec{\mathbf{u}} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$, then determine $D_{\vec{\mathbf{u}}} f(2, 1)$.

First, we must calculate the partial derivatives of f:

$$f_x(x, y) = 2x - y - 1 \quad \text{and} \quad f_y(x, y) = -x + 4y - 4$$

Then we use the shortcut formula above to get:

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\alpha + f_y(x, y)\beta = (2x - y - 1)\frac{4}{5} + (-x + 4y - 4)\frac{3}{5} = \frac{5x + 8y - 16}{5}$$

To calculate $D_{\vec{\mathbf{u}}} f(2, 1)$ let $x = 2$ and $y = 1$:

$$D_{\vec{\mathbf{u}}} f(2, 1) = \frac{5(2) + 8(1) - 16}{5} = \frac{2}{5}$$

This is the same answer we obtained using the limit definition.

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in the last example in the direction of the vector $\langle -5, 12 \rangle$, we would first divide by its magnitude to get $\vec{\mathbf{u}}$. This gives us $\vec{\mathbf{u}} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$.

Then

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x, y) &= f_x(x, y)\alpha + f_y(x, y)\beta \\ &= (2x - y - 1)\left(-\frac{5}{13}\right) + (-x + 4y - 4)\frac{12}{13} = \frac{-22x + 53y - 43}{13} \end{aligned}$$

Thanks to our shortcut formula, we can see that $D_{\vec{\mathbf{u}}}(x, y) = f_x(x, y)\alpha + f_y(x, y)\beta$, which can be written as the dot product of two vectors. Define the first vector as $\nabla f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}}$ so that the second vector is just $\vec{\mathbf{u}} = \alpha\,\hat{\mathbf{i}} + \beta\hat{\mathbf{j}}$. Now the directional derivative in the direction of $\vec{\mathbf{u}}$ can be written as:

$$D_{\vec{\mathbf{u}}} f(x, y) = \nabla f(x, y) \cdot \vec{\mathbf{u}}$$

The first vector in this equation has a special name: the gradient of the function f. The symbol ∇ is called **nabla** and the vector ∇f is read "**del** f."

Definition: The Gradient

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. The vector $\nabla f(x, y)$ is called the ***gradient*** of f and is defined as

$$\nabla f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}}$$

The vector $\nabla f(x, y)$ is also written as "**grad** f."

Example 2.4.4: Finding Gradients

Find the gradient $\nabla f(x,y)$ of each of the following functions:

(a) $f(x,y) = x^2 - xy + 3y^2$

(b) $f(x,y) = \sin(3x)\cos(3y)$

(c) $f(x,y) = \dfrac{x^2 - 3y^2}{2x + y}$

Solution

For each part of this example, we first calculate the partial derivatives f_x and f_y, then put them into the vector form as a gradient.

(a) $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$, so

$$\nabla f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} = (2x - y)\,\hat{\mathbf{i}} + (-x + 6y)\,\hat{\mathbf{j}} = \boxed{\langle 2x - y, -x + 6y\rangle}$$

(b) $f_x(x,y) = 3\cos(3x)\cos(3y)$ and $f_y(x,y) = -3\sin(3x)\sin(3y)$, so

$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} = (3\cos(3x)\cos(3y))\,\hat{\mathbf{i}} - (3\sin(3x)\sin(3y))\,\hat{\mathbf{j}} \\ &= \boxed{\langle 3\cos(3x)\cos(3y), -3\sin(3x)\sin(3y)\rangle}\end{aligned}$$

(c) $f_x(x,y) = \dfrac{2x^2 + 2xy + 6y^2}{(2x+y)^2}$ and $f_y(x,y) = -\dfrac{x^2 + 12xy + 3y^2}{(2x+y)^2}$, so

$$\nabla f(x,y) = \frac{2x^2 + 2xy + 6y^2}{(2x+y)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2}{(2x+y)^2}\,\hat{\mathbf{j}} = \boxed{\left\langle \frac{2x^2 + 2xy + 6y^2}{(2x+y)^2}, -\frac{x^2 + 12xy + 3y^2}{(2x+y)^2}\right\rangle}$$

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall that if the angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is φ, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|\cos\varphi$. Therefore, if the angle between $\nabla f(x_0,y_0)$ and $\vec{\mathbf{u}} = \alpha\,\hat{\mathbf{i}} + \beta\,\hat{\mathbf{j}}$ is φ, we have

$$D_{\vec{\mathbf{u}}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{\mathbf{u}} = \|\nabla f(x_0,y_0)\|\|\vec{\mathbf{u}}\|\cos\varphi = \|\vec{\boldsymbol{\nabla}} f(x_0,y_0)\|\cos\varphi$$

The $\|\vec{\mathbf{u}}\|$ disappears because $\vec{\mathbf{u}}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0,y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1.

If $\varphi = 0$, then $\cos\varphi = 1$ and $\vec{\boldsymbol{\nabla}} f(x_0,y_0)$ and $\vec{\mathbf{u}}$ both point in the same direction.

If $\varphi = \pi$, then $\cos\varphi = -1$ and $\vec{\boldsymbol{\nabla}} f(x_0,y_0)$ and $\vec{\mathbf{u}}$ point in opposite directions.

In the first case, the value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is maximized; in the second case, the value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is minimized.

We can also see that if $\nabla f(x_0,y_0) = \vec{\mathbf{0}}$, then

$$D_{\vec{\mathbf{u}}} f(x_0,y_0) = \vec{\boldsymbol{\nabla}} f(x_0,y_0) \cdot \vec{\mathbf{u}} = 0$$

for any vector $\vec{\mathbf{u}}$. These three cases are outlined in the following theorem.

Theorem 2.4.6: Properties of the Gradient

Suppose the function $z = f(x,y)$ is differentiable at (x_0,y_0).

i. If $\nabla f(x_0,y_0) = \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0,y_0) = 0$ for any unit vector $\vec{\mathbf{u}}$.
ii. If $\nabla f(x_0,y_0) \neq \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is **maximized** when $\vec{\mathbf{u}}$ points in the same direction as $\nabla f(x_0,y_0)$.
iii. If $\nabla f(x_0,y_0) \neq \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is **minimized** when $\vec{\mathbf{u}}$ points in the opposite direction from $\nabla f(x_0,y_0)$.

Therefore the maximum value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is $\|\nabla f(x_0,y_0)\|$ and the minimum value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is $-\|\nabla f(x_0,y_0)\|$.

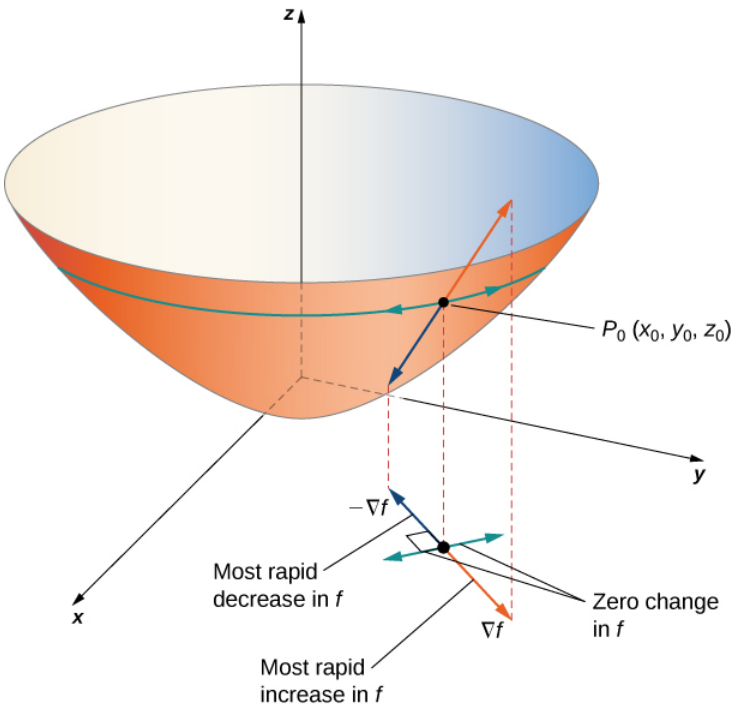


Figure 2.4.6: The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example 2.4.5: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x,y) = -x^2 + xy + y^2 - 4y + 2$ at $(1,1)$ is a maximum. What is the maximum value?

Solution

The maximum value of the directional derivative occurs when ∇f and the unit vector point in the same direction. Therefore, we start by calculating $\nabla f(x,y)$:

$$f_x(x,y) = -2x + y \text{ and } f_y(x,y) = x + 2y - 4$$

so

$$\nabla f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} = (-2x + y)\,\hat{\mathbf{i}} + (x + 2y - 4)\,\hat{\mathbf{j}}$$

Next, we evaluate the gradient at $(1,1)$:

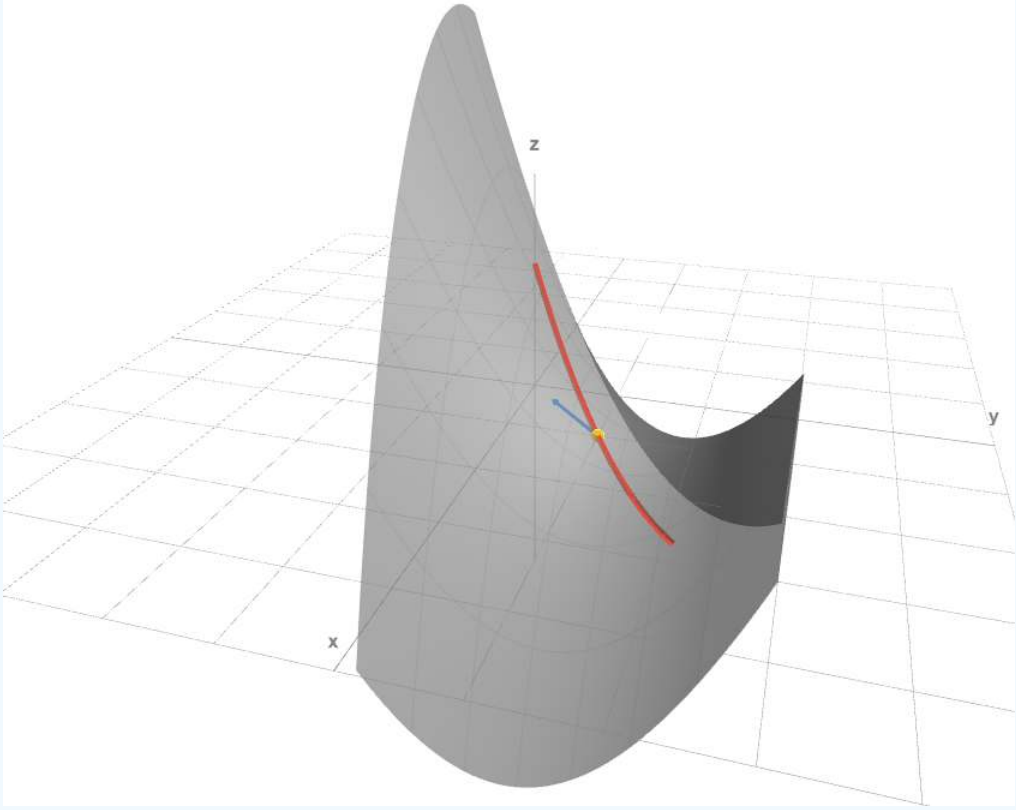
$$\nabla f(1,1) = (-2(1) + 1)\,\hat{\mathbf{i}} + (1 + 2(1) - 4)\,\hat{\mathbf{j}} = -1\,\hat{\mathbf{i}} - 1\,\hat{\mathbf{j}}$$

We need to find a unit vector that points in the same direction as $\nabla f(1,1)$, so the next step is to divide $\nabla f(1,1)$ by its magnitude, which is $\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Therefore,

$$\boxed{\vec{\mathbf{u}} = \frac{\nabla f(1,1)}{\|\nabla f(1,1)\|} = \frac{-1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j}$$

This is the unit vector that points in the same direction as $\nabla f(1,1)$.

The maximum value of the directional derivative at $(1,1)$ is the previous calculated $\|\nabla f(1,1)\| = \boxed{\sqrt{2}}$



Recall that if a curve is defined parametrically by the function pair $(x(t), y(t))$, then the vector $x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}}$ is tangent to the curve for every value of t in the domain. Now let's assume $z = f(x, y)$ is a differentiable function of x and y, and (x_0, y_0) is in its domain. Let's suppose further that $x_0 = x(t_0)$ and $y_0 = y(t_0)$ for some value of t, and consider the level curve $f(x, y) = k$. Define $g(t) = f(x(t), y(t))$ and calculate $g'(t)$ on the level curve. By the chain rule,

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = \nabla f(x, y) \cdot \langle x'(t), y'(t) \rangle$$

But $g'(t) = 0$ because $g(t) = k$ for all t. Therefore,

$$\nabla f(x, y) \cdot \langle x'(t), y'(t) \rangle = 0$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. This will be useful when we later discuss maximums and minimums.

The definition of a gradient and directional derivatives can be extended to functions of more than two variables in the same manner.

Definition: Gradients in 3D

Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\nabla f(x, y, z)$ is called the ***gradient*** of f and is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}}$$

$\nabla f(x, y, z)$ can also be written as grad $f(x, y, z)$

Definition: Directional Derivative of a Function of Three variables

Suppose $w = f(x, y, z)$ is a function of three variables with a domain of D. Let $(x_0, y_0, z_0) \in D$ and let $\vec{\mathbf{u}} = \alpha\,\hat{\mathbf{i}} + \beta\,\hat{\mathbf{j}} + \gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the ***directional derivative*** of f in the direction of u is given by

$$D_{\vec{\mathbf{u}}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + h\alpha, y_0 + h\beta, z_0 + h\gamma) - f(x_0, y_0, z_0)}{h} = \nabla f(x, y, z) \cdot \vec{\mathbf{u}}$$

Example 2.4.6: Finding a Directional Derivative in Three Dimensions

Calculate the indicated directional derivative for the provided function and vector:

(a) $D_{\vec{\mathbf{v}}} f(1, -2, 3)$ for $f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$ and $\vec{\mathbf{v}} = -\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$

(b) $D_{\vec{\mathbf{v}}} g(0, -2, 5)$ for $g(x, y, z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz$ and $\vec{\mathbf{v}} = -3\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$

Solution:

(a) To make $\vec{\mathbf{v}}$ into a unit vector, we start by finding its magnitude:

$$\|\vec{\mathbf{v}}\| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3$$

Therefore, $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \dfrac{-\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}}{3} = -\dfrac{1}{3}\,\hat{\mathbf{i}} + \dfrac{2}{3}\,\hat{\mathbf{j}} + \dfrac{2}{3}\,\hat{\mathbf{k}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$.

Next, we calculate the partial derivatives of f and use them to create our gradient vector:

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \langle 10x - 2y + 3z, -2x + 2y - 4z, -4y + 2z + 3x \rangle$$

Plugging in $x = 1, y = -2, z = 3$ this becomes:

$$\begin{gathered} \nabla f(1, -2, 3) \\ = \langle 10(1) - 2(-2) + 3(3), -2(1) + 2(-2) - 4(3), -4(-2) + 2(3) + 3(1) \rangle = \langle 23, -18, 17 \rangle \end{gathered}$$

Finally, taking the dot product with our unit vector $\vec{\mathbf{u}} = \dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \left\langle -\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3} \right\rangle$ gives us

$$D_{\vec{\mathbf{v}}} f(1, -2, 3) = \nabla f(1, -2, 3) \cdot \vec{\mathbf{u}} = \langle 23, -18, 17 \rangle \cdot \left\langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle = \boxed{-\frac{25}{3}}$$

(b) To make $\vec{\mathbf{v}}$ into a unit vector, we start by finding its magnitude:

$$\|\vec{\mathbf{v}}\| = \sqrt{(-3)^2 + (12)^2 + (-4)^2} = \sqrt{169} = 13$$

Therefore, $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \dfrac{-3\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{13} = -\dfrac{3}{13}\,\hat{\mathbf{i}} + \dfrac{12}{13}\,\hat{\mathbf{j}} - \dfrac{4}{13}\,\hat{\mathbf{k}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$.

Next, we calculate the partial derivatives of g and use them to create our gradient vector:

$$\nabla g(x, y, z) = \langle g_x(x, y, z), g_y(x, y, z), g_z(x, y, z) \rangle = \langle 6x + y + 2z, x - 4y + 4z, 4y - 2z + 2x \rangle$$

Plugging in $x = 0, y = -2, z = 5$ this becomes:

$$\nabla g(0, -2, 5) = \langle 6(0) + (-2) + 2(5), 0 - 4(-2) + 4(5), 4(-2) - 2(5) + 2(0) \rangle = \langle 8, 28, -18 \rangle$$

Finally, taking the dot product with our unit vector $\vec{\mathbf{u}} = \dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \left\langle -\dfrac{3}{13}, \dfrac{12}{13}, -\dfrac{4}{13} \right\rangle$ gives us

$$D_{\vec{\mathbf{v}}} g(0, -2, 5) = \nabla g(0, -2, 5) \cdot \vec{\mathbf{u}} = \langle 8, 28, -18 \rangle \cdot \left\langle -\frac{3}{13}, \frac{12}{13}, -\frac{4}{13} \right\rangle = \boxed{\frac{384}{13}}$$

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2.4: Practice Set

Exercise 2.4.1a

If $f(x,y) = x^2y^2 + 1$ and $x(t) = \sqrt{t^2+1}$ while $y(t) = \dfrac{1}{t^2+1}$, find $\dfrac{df}{dt}$

Answer

By the chain rule: $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dt}$

$\dfrac{\partial f}{\partial x} = 2xy^2 = 2\cdot\sqrt{t^2+1}\cdot\left(\dfrac{1}{t^2+1}\right)^2 = \dfrac{2\sqrt{t^2+1}}{(t^2+1)^2}$

$\dfrac{\partial f}{\partial y} = 2x^2y = 2\cdot(\sqrt{t^2+1})^2\cdot\dfrac{1}{t^2+1} = \dfrac{2(t^2+1)}{t^2+1} = 2$

$\dfrac{dx}{dt} = \dfrac{1}{2}(t^2+1)^{-\frac{1}{2}}\cdot 2t = \dfrac{t}{\sqrt{t^2+1}}$

$\dfrac{dy}{dt} = -1(t^2+1)^{-2}\cdot 2t = -\dfrac{2t}{(t^2+1)^2}$

Therefore $\dfrac{df}{dt} = \dfrac{2\sqrt{t^2+1}}{(t^2+1)^2}\cdot\dfrac{t}{\sqrt{t^2+1}} + 2\cdot\left(-\dfrac{2t}{(t^2+1)^2}\right) = -\dfrac{2t}{(t^2+1)^2}$

Exercise 2.4.1b

If $u = x^3y + y^2z^4$ where $x = rs^2e^t, y = r^2se^{-t}, z = rs\sin(t)$, find u_s at $(r,s,t) = (2,1,0)$

Answer

By the chain rule: $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\cdot\dfrac{\partial z}{\partial s}$

$\dfrac{\partial x}{\partial s} = 2rse^t$ at $r=2, s=1, t=0$ this equals 4

$\dfrac{\partial y}{\partial s} = r^2e^{-t}$ at $r=2, s=1, t=0$ this equals 4

$\dfrac{\partial z}{\partial s} = r\sin(t)$ at $r=2, s=1, t=0$ this equals 0

We can see that $x(2,1,0) = 2, \quad y(2,1,0) = 4$, and $z(2,1,0) = 0$, so

$\dfrac{\partial u}{\partial x} = 3x^2y$ at $x=2, y=4, z=0$ this equals 48

$\dfrac{\partial u}{\partial y} = x^3 + 2yz^4$ at $x=2, y=4, z=0$ this equals 8

$\dfrac{\partial u}{\partial z} = 4y^2z^3$ at $x=2, y=4, z=0$ this equals 0

Therefore $\dfrac{\partial u}{\partial s} = 48\cdot 4 + 8\cdot 4 + 0\cdot 0 = 224$

Exercise 2.4.1c

If $f(t,s) = t^2 + s^2$ and $g(t,s) = \tan^{-1}\left(\dfrac{s}{t}\right)$ and $q(f,g) = \sqrt{f}\cos(g)$, find q_t

Answer

By the chain rule: $\dfrac{\partial q}{\partial t} = \dfrac{\partial q}{\partial f}\cdot\dfrac{\partial f}{\partial t} + \dfrac{\partial q}{\partial g}\cdot\dfrac{\partial g}{\partial t}$

$\dfrac{\partial q}{\partial f} = \dfrac{1}{2}f^{-\frac{1}{2}}\cdot\cos(g) = \dfrac{1}{2}(t^2+s^2)^{-\frac{1}{2}}\cdot\cos\left(\tan^{-1}\left(\dfrac{s}{t}\right)\right) = \dfrac{1}{2\sqrt{t^2+s^2}}\cdot\dfrac{t}{\sqrt{t^2+s^2}} = \dfrac{t}{2(s^2+t^2)}$

$\frac{\partial q}{\partial g} = \sqrt{f} \cdot (-\sin(g)) = -\sqrt{t^2+s^2} \cdot \sin\left(\tan^{-1}\left(\frac{s}{t}\right)\right) = -\sqrt{t^2+s^2} \cdot \frac{s}{\sqrt{t^2+s^2}} = -s$

$\frac{\partial f}{\partial t} = 2t$

$\frac{\partial g}{\partial t} = \frac{1}{\left(\frac{s}{t}\right)^2+1} \cdot \left(-\frac{s}{t^2}\right) = -\frac{s}{s^2+t^2}$

Therefore $\frac{\partial q}{\partial t} = \frac{t}{2(s^2+t^2)} \cdot 2t + (-s) \cdot \left(-\frac{s}{s^2+t^2}\right) = \frac{t^2+s^2}{t^2+s^2} = 1$

Exercise 2.4.1d

If $T = \frac{4v}{2u^2+v}$ and $u = 2pqr, v = 3p^2q^2\sqrt{r}$, find T_q

Answer

By the chain rule: $\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u} \cdot \frac{\partial u}{\partial q} + \frac{\partial T}{\partial v} \cdot \frac{\partial v}{\partial q}$

$\frac{\partial T}{\partial u} = -4v(2u^2+v)^{-2} \cdot 4u = -\frac{16uv}{(2u^2+v)^2} = -\frac{16 \cdot 2pqr \cdot 3p^2q^2\sqrt{r}}{(2 \cdot (2pqr)^2 + 3p^2q^2\sqrt{r})^2} = -\frac{96r^{\frac{1}{2}}}{pq(8r^{\frac{3}{2}}+3)^2}$

$\frac{\partial T}{\partial v} = \frac{4 \cdot (2u^2+v) - 4v \cdot 1}{(2u^2+v)^2} = \frac{8u^2}{(2u^2+v)^2} = \frac{8 \cdot (2pqr)^2}{(2 \cdot (2pqr)^2 + 3p^2q^2\sqrt{r})^2} = \frac{32r}{p^2q^2(8r^{\frac{3}{2}}+3)^2}$

$\frac{\partial u}{\partial q} = 2pr$

$\frac{\partial v}{\partial q} = 6p^2q\sqrt{r}$

Therefore $\frac{\partial T}{\partial q} = \left(-\frac{96r^{\frac{1}{2}}}{pq(8r^{\frac{3}{2}}+3)^2}\right) \cdot 2pr + \frac{32r}{p^2q^2(8r^{\frac{3}{2}}+3)^2} \cdot 6p^2q\sqrt{r} = -\frac{192r^{\frac{3}{2}}}{q(8r^{\frac{3}{2}}+3)^2} + \frac{192r^{\frac{3}{2}}}{q(8r^{\frac{3}{2}}+3)^2} = 0$

Exercise 2.4.1e

If $f(x) = \frac{2(x^2-1)^{2\sqrt{x+2}-1}}{(2\sqrt{x+2}-1)^{x^2-1}-18}$, find $f'(2)$

Answer

Define $u = x^2-1$ and $v = 2\sqrt{x+2}-1$ so that $f = \frac{2u^v}{v^u-18}$

Then we know $\frac{df}{dx} = \frac{\partial f}{\partial u} \cdot \frac{du}{dx} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dx}$

$\frac{du}{dx} = 2x$, so at $x = 2$ this equals 4

$\frac{dv}{dx} = \frac{1}{\sqrt{x+2}}$ so at $x = 2$ this equals $\frac{1}{2}$

We can see that $u(2) = 3$ and $v(2) = 3$

$\frac{\partial f}{\partial u} = \frac{(2vu^{v-1}) \cdot (v^u-18) - (2u^v) \cdot (v^u \ln(v))}{(v^u-18)^2}$, so at $u = 3, v = 3$ this equals $6 - 18\ln(3))$

$\frac{\partial f}{\partial v} = \frac{(2u^v \ln(u)) \cdot (v^u-18) - (2u^v) \cdot (uv^{u-1})}{(v^u-18)^2}$, so at $u = 3, v = 3$ this equals $6\ln(3) - 18$

Therefore we have $f'(2) = (6 - 18\ln(3)) \cdot 4 + (6\ln(3) - 18) \cdot \frac{1}{2} = 15 - 69\ln(3)$

Exercise 2.4.1f

Let $w(x)$ be a function with $w(1)=2, w(2)=1$ and $w'(1)=3, w'(2)=5$ and $\mu(s,t)=s^2t^3$.

If $\zeta(x)=\mu(w(w(x)),w(x))\cdot\mu(w(x),w(w(x)))$, find $\zeta'(2)$

Answer

Define $\ell=\mu(s,t)=s^2t^3$, $\kappa=\mu(t,s)=t^2s^3$, $s=w(w(x))$ and $t=w(x)$ so that $\zeta(x)=\ell\cdot\kappa$

First, at $x=2$ we can easily see that $s=w(w(2))=w(1)=2$ and $t=w(2)=1$

This leads to $\ell=\mu(2,1)=4$ and $\kappa=\mu(1,2)=8$, and by the chain rule:

$\frac{d\zeta}{dx}=\frac{\partial\zeta}{\partial\ell}\cdot\frac{\partial\ell}{\partial s}\cdot\frac{ds}{dx}+\frac{\partial\zeta}{\partial\ell}\cdot\frac{\partial\ell}{\partial t}\cdot\frac{dt}{dx}+\frac{\partial\zeta}{\partial\kappa}\cdot\frac{\partial\kappa}{\partial s}\cdot\frac{ds}{dx}+\frac{\partial\zeta}{\partial\kappa}\cdot\frac{\partial\kappa}{\partial t}\cdot\frac{dt}{dx}$

$\frac{ds}{dx}=w'(w(x))\cdot w'(x)$ so at $x=2$ this is $w'(1)\cdot w'(2)=15$,

$\frac{dt}{dx}=w'(x)$ so at $x=2$ this is $w'(2)=5$

$\frac{\partial\ell}{\partial s}=2st^3$, so at $s=2,t=1$ this is 4

$\frac{\partial\ell}{\partial t}=3s^2t^2$, so at $s=2,t=1$ this is 12

$\frac{\partial\kappa}{\partial s}=3s^2t^2$, so at $s=2,t=1$ this is 12

$\frac{\partial\kappa}{\partial t}=2s^3t$, so at $s=2,t=1$ this is 16

$\frac{\partial\zeta}{\partial\ell}=\kappa$ so at $\ell=4,\kappa=8$ this is 8

$\frac{\partial\zeta}{\partial\kappa}=\ell$ so at $\ell=4,\kappa=8$ this is 4

Putting these all into our chain rule equation above gives:

$\zeta'(2)=8\cdot4\cdot15+8\cdot12\cdot5+4\cdot12\cdot15+4\cdot16\cdot5=2000$

Exercise 2.4.2a

Let $f(x,y)=x^2y+y^3$ and $\vec{\mathbf{u}}=\left\langle\frac{3}{5},\frac{4}{5}\right\rangle$, find $D_{\vec{\mathbf{u}}}f(1,2)$

Answer

First we find the gradient of f: $\nabla f=\langle 2xy, x^2+3y^2\rangle$

Then we plug in the given point $x=1,y=2$ to get: $\nabla f(1,2)=\langle 4,13\rangle$

Lastly we take the dot product with $\vec{\mathbf{u}}$ to get our directional derivative:

$D_{\vec{\mathbf{u}}}f(1,2)=\langle 4,13\rangle\cdot\left\langle\frac{3}{5},\frac{4}{5}\right\rangle=\frac{64}{5}$

Exercise 2.4.2b

Let $g(x,y,z)=(x+(y+z^2)^3)^4$ and $\vec{\mathbf{u}}=\left\langle\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right\rangle$ find $D_{\vec{\mathbf{u}}}g(1,-1,1)$

Answer

First we find the gradient of g: $\nabla g=\langle 4(x+(y+z^2)^3)^3, 4(x+(y+z^2)^3)^3\cdot 3(y+z^2)^2, 4(x+(y+z^2)^3)^3\cdot 3(y+z^2)^2\cdot 2z\rangle$

Then we plug in the given point $x=1,y=-1,z=1$ to get: $\nabla g(1,-1,1)=\langle 4,0,0\rangle$

Lastly we take the dot product with $\vec{\mathbf{u}}$ to get our directional derivative:

$D_{\vec{\mathbf{u}}}g(1,-1,1) = \langle 4,0,0\rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle = \frac{8}{3}$

Exercise 2.4.2c

Let $h(a,b,c) = e^{ab} - e^{bc} + a$ and $\vec{\mathbf{u}} = \left\langle \frac{12}{13}, 0, -\frac{5}{13}\right\rangle$ find $D_{\vec{\mathbf{u}}}h(2,1,-1)$

Answer

First we find the gradient of h : $\nabla h = \langle be^{ab}+1, ae^{ab} - ce^{bc}, -be^{bc}\rangle$

Then we plug in the given point $a=2, b=1, c=-1$ to get: $\nabla h(2,1,-1) = \langle e^2+1, 2e^2+e^{-1}, -e^{-1}\rangle$

Lastly we take the dot product with $\vec{\mathbf{u}}$ to get our directional derivative:

$D_{\vec{\mathbf{u}}}h(2,1,-1) = \langle e^2+1, 2e^2+e^{-1}, -e^{-1}\rangle \cdot \left\langle \frac{12}{13}, 0, -\frac{5}{13}\right\rangle = \frac{12e^2+12+5e^{-1}}{13}$

Exercise 2.4.3a

Find the vector $\vec{\mathbf{w}}$ such that $D_{\vec{\mathbf{w}}}f(2,1)$ is a maximum for the function $f(x,y) = e^{x^2y-4}$

Answer

First we find the gradient of f : $\nabla f = \left\langle 2xye^{x^2y-4}, x^2e^{x^2y-4}\right\rangle$

Then we plug in the given point $x=2, y=1$ to get: $\nabla f(2,1) = \langle 4,4\rangle$

The directional derivative will be a maximum when:

$\vec{\mathbf{w}} = \frac{1}{\|\nabla f(2,1)\|}\nabla f(2,1) = \frac{1}{4\sqrt{2}}\langle 4,4\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

Exercise 2.4.3b

Find the vector $\vec{\mathbf{w}}$ such that $D_{\vec{\mathbf{w}}}g(2,5,-1)$ is a minimum for the function $g(x,y,z) = \frac{x}{\sqrt{2y+z}}$

Answer

First we find the gradient of g : $\nabla g = \left\langle \frac{1}{\sqrt{2y+z}}, -x(2y+z)^{-\frac{3}{2}}, -\frac{1}{2}x(2y+z)^{-\frac{3}{2}}\right\rangle$

Then we plug in the given point $x=2, y=5, z=-1$ to get: $\nabla g(2,5,-1) = \left\langle \frac{1}{3}, -\frac{2}{27}, -\frac{1}{27}\right\rangle$

The directional derivative will be a minimum when:

$\vec{\mathbf{w}} = -\frac{1}{\|\nabla g(2,5,-1)\|}\nabla g(2,5,-1) = -\frac{1}{\frac{\sqrt{86}}{27}}\left\langle \frac{1}{3}, -\frac{2}{27}, -\frac{1}{27}\right\rangle = \left\langle -\frac{9}{\sqrt{86}}, \frac{2}{\sqrt{86}}, \frac{1}{\sqrt{86}}\right\rangle$

Exercise 2.4.3c

Find the vector $\vec{\mathbf{w}}$ such that $D_{\vec{\mathbf{w}}}h(a,b) = 0$ (assume $b \neq 0$) for the function $h(x,y) = \frac{x}{y}$

Answer

First we find the gradient of h : $\nabla h = \left\langle \frac{1}{y}, -\frac{x}{y^2}\right\rangle$

Then we plug in the point $x=a, y=b$ to get: $\nabla h(a,b) = \left\langle \frac{1}{b}, -\frac{a}{b^2}\right\rangle$

Then if $\vec{\mathbf{w}} = \langle u, v\rangle$ the directional derivative will be $D_{\vec{\mathbf{w}}}h(a,b) = \left\langle \frac{1}{b}, -\frac{a}{b^2}\right\rangle \cdot \langle u, v\rangle = \frac{bu-av}{b^2}$

Setting this equal to zero gives $\frac{bu-av}{b^2}=0 \longrightarrow u=\frac{a}{b}v$

Therefore w is the unit vector in the direction of $\left\langle \frac{a}{b}v, v\right\rangle$

Setting this length equal to 1 gives $\sqrt{\left(\frac{a}{b}v\right)^2+v^2}=1 \longrightarrow v=\pm\frac{b}{\sqrt{a^2+b^2}}$

Thus there are two possible vectors: $\vec{\mathbf{w}}_1=\left\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right\rangle$ and $\vec{\mathbf{w}}_2=\left\langle -\frac{a}{\sqrt{a^2+b^2}}, -\frac{b}{\sqrt{a^2+b^2}}\right\rangle$

Exercise 2.4.4a

Find which point(s) (x,y) make $D_{\vec{\mathbf{v}}}k(x,y)=0$ for $k(x,y)=x^2+y^2$ and $\vec{\mathbf{v}}=\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$

Answer

First we find the gradient of k : $\nabla k=\langle 2x, 2y\rangle$

Then we plug in the point $x=a, y=b$ to get: $\nabla k(a,b)=\langle 2a, 2b\rangle$

Then if $\vec{\mathbf{v}}=\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$ the directional derivative will be $D_{\vec{\mathbf{v}}}k(a,b)=\langle 2a, 2b\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle=\frac{6a-8b}{5}$

Setting this equal to zero gives $\frac{6a-8b}{5}=0 \longrightarrow b=\frac{3}{4}a$

Thus all points on the line $y=\frac{3}{4}x$ satisfy $D_{\vec{\mathbf{v}}}k(x,y)=0$

Exercise 2.4.4b

Find which point(s) (x,y) make $D_{\vec{\mathbf{v}}}k(x,y)=0$ for $k(x,y)=\frac{x^2}{y^2}$ and $\vec{\mathbf{v}}=\left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$

Answer

First we find the gradient of k : $\nabla k=\left\langle \frac{2x}{y^2}, -\frac{2x^2}{y^3}\right\rangle$

Then we plug in the point $x=a, y=b$ to get: $\nabla k(a,b)=\left\langle \frac{2a}{b^2}, -\frac{2a^2}{b^3}\right\rangle$

Then if $\vec{\mathbf{v}}=\left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$ the directional derivative will be $D_{\vec{\mathbf{v}}}k(a,b)=\left\langle \frac{2a}{b^2}, -\frac{2a^2}{b^3}\right\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle=\frac{10ab-24a^2}{13b^3}$

Setting this equal to zero gives $\frac{10ab-24a^2}{13b^3}=0 \longrightarrow a=0$ or $b=\frac{12}{5}a$

Thus all points on the lines $x=0$ or $y=\frac{12}{5}x$ satisfy $D_{\vec{\mathbf{v}}}k(x,y)=0$

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2.5: Vector Fields and Derivatives

Learning Objectives

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.
- Explain how to find a potential function for a conservative vector field.
- Explain how to test a vector field to determine whether it is conservative.
- Determine divergence from the formula for a given vector field.
- Determine curl from the formula for a given vector field.
- Use the properties of curl and divergence to determine whether a vector field is conservative.

For a real valued function $f(x,y)$, we have defined the gradient of f to be the vector $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$. This object can be thought of as a function that maps each point (x,y) to a vector in $\mathbb{R}^2$ as opposed to mapping to a real number like the original function f does. Functions that turn points into vectors are called vector fields, and we will study them in this section.

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail.

We also examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

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2.5: Vector Fields

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river?

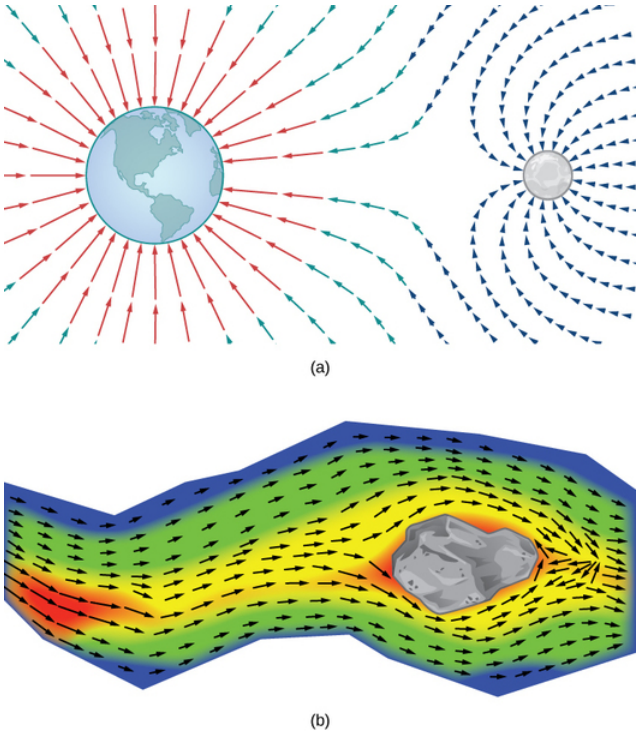


Figure 2.5.1 (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Figure 2.5.1a shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure 2.5.1b shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition: Vector Field

A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) in a domain D (some subset of $\mathbb{R}^2$)

A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) in a domain D (some subset of $\mathbb{R}^3$)

A vector field in $\mathbb{R}^2$ can be represented as a vector with components that are two-variable functions:

$$\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$$

A vector field is said to be *continuous* if its component functions are continuous.

We can now represent a vector field in terms of its components functions, but representing it visually by sketching it is more complex. The domain of a vector field is $\mathbb{R}^2$, and so is the codomain. Therefore the "graph" of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4,-1)$ is $\langle 3,1\rangle$. Then, we would draw vector $\langle 3,1\rangle$ starting from point $(4,-1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and would be useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a radial field, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x,y) is perpendicular to the circle centered at the origin that contains point (x,y), and all other vectors on this circle have the same magnitude.

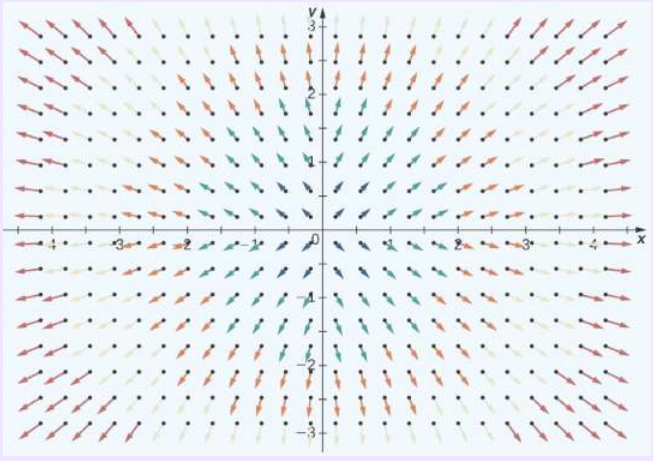
Illustration 2.5.1: Drawing a Radial Vector Field

We sketch the vector field $\vec{\mathbf{F}}(x,y)=\frac{x}{2}\hat{\mathbf{i}}+\frac{y}{2}\hat{\mathbf{j}}$

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\left\langle \frac{1}{2},0\right\rangle$	$(2,0)$	$\langle 1,0\rangle$	$(1,1)$	$\left\langle \frac{1}{2},\frac{1}{2}\right\rangle$
$(0,1)$	$\left\langle 0,\frac{1}{2}\right\rangle$	$(0,2)$	$\langle 0,1\rangle$	$(-1,1)$	$\left\langle -\frac{1}{2},\frac{1}{2}\right\rangle$
$(-1,0)$	$\left\langle -\frac{1}{2},0\right\rangle$	$(-2,0)$	$\langle -1,0\rangle$	$(-1,-1)$	$\left\langle -\frac{1}{2},-\frac{1}{2}\right\rangle$
$(0,-1)$	$\left\langle 0,-\frac{1}{2}\right\rangle$	$(0,-2)$	$\langle 0,-1\rangle$	$(1,-1)$	$\left\langle \frac{1}{2},-\frac{1}{2}\right\rangle$

The graph below shows the vector field:



Note that vector $\vec{\mathbf{F}}(a,b)=\left\langle \frac{a}{2},\frac{b}{2}\right\rangle$ points outward from the origin and is parallel to radial vector $\langle a,b\rangle$. Furthermore, vector $\left\langle \frac{a}{2},\frac{b}{2}\right\rangle$ has length $r=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=\frac{1}{2}\sqrt{a^2+b^2}$. Thus, we have a complete description of this radial vector field: the

vector associated with point (a, b) is the vector with length $\frac{r}{2}$ perpendicular to the circle with radius r, and it points in the outward direction.

In contrast to radial fields, in a *rotational field*, the vector at point (x, y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2 + y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

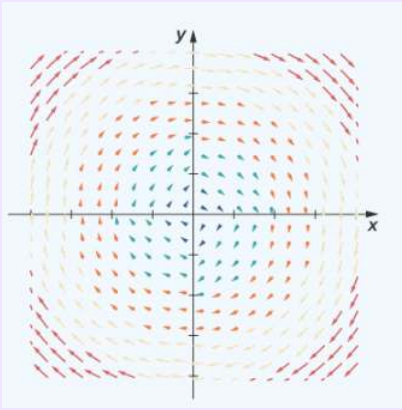
Illustration 2.5.2: Drawing a Rotational Vector Field

We sketch the vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$

Create a table using a representative sample of points in a plane and their corresponding vectors.

(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$

The graph below shows the vector field:



Note that vector $\vec{\mathbf{F}}(a, b) = \langle b, -a \rangle$ points clockwise and is perpendicular to radial vector $\langle a, b \rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0$.) Furthermore, vector $\langle b, -a \rangle$ has length $r = \sqrt{a^2 + b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a, b) is the vector with length r tangent to the circle with radius r, and it points in the clockwise direction.

Sketches of vector fields are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)



Figure 2.5.2: (credit: modification of work by NASA)

Illustration 2.5.3: Sketching a Vector Field

We sketch vector field $\vec{\mathbf{F}}(x,y) = \dfrac{y}{x^2+y^2}\hat{\mathbf{i}}, -\dfrac{x}{x^2+y^2}\hat{\mathbf{j}}$

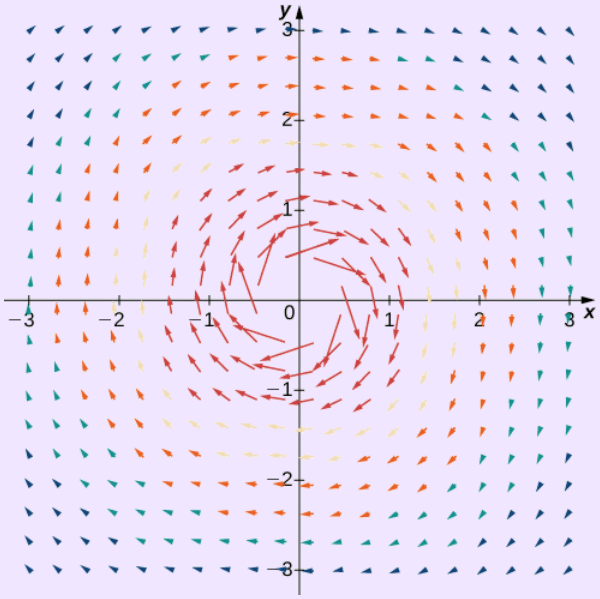
To visualize this vector field, first note that the dot product $\vec{\mathbf{F}}(a,b) \cdot (a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}})$ is zero for any point (a,b). Therefore, each vector is tangent to the circle on which it is located. Also,

$$||\vec{\mathbf{F}}(a,b)|| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \sqrt{\frac{1}{a^2+b^2}}$$

Since $\dfrac{1}{a^2+b^2} \to \infty$ as $(a,b) \to (0,0)$, then $||\vec{\mathbf{F}}(a,b)|| \to \infty$ as $(a,b) \to (0,0)$. This vector field is also rotational, but in this case the magnitudes of the vectors close to the origin are large while those further away are smaller.

Table 2.5.3

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\left\langle 0,-\frac{1}{2}\right\rangle$	$(1,1)$	$\left\langle \frac{1}{2},-\frac{1}{2}\right\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\left\langle \frac{1}{2},0\right\rangle$	$(-1,1)$	$\left\langle \frac{1}{2},\frac{1}{2}\right\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\left\langle 0,\frac{1}{2}\right\rangle$	$(-1,-1)$	$\left\langle -\frac{1}{2},\frac{1}{2}\right\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\left\langle -\frac{1}{2},0\right\rangle$	$(1,-1)$	$\left\langle -\frac{1}{2},-\frac{1}{2}\right\rangle$



Note that this vector field models a whirlpool motion of a river. The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0, 0)$.

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field $\vec{\mathbf{F}}$ is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

Illustration 2.5.4: A Unit Vector Field

Consider the vector field $\vec{\mathbf{F}}(x, y) = \left\langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \right\rangle$

To show that $\vec{\mathbf{F}}$ is a unit vector field, we must show that the magnitude of each vector is 1. Note that

$$\sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{\frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2}} = \sqrt{\frac{x^2+y^2}{x^2+y^2}} = 1$$

Therefore, $\vec{\mathbf{F}}$ is a unit vector field.

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field, then the corresponding unit vector field is $\left\langle \frac{P}{||\vec{\mathbf{F}}||}, \frac{Q}{||\vec{\mathbf{F}}||}, \frac{R}{||\vec{\mathbf{F}}||} \right\rangle$. If $\vec{\mathbf{F}}$ is a vector field, then the process of dividing $\vec{\mathbf{F}}$ by its magnitude to form unit vector field $\frac{\vec{\mathbf{F}}}{||\vec{\mathbf{F}}||}$ is called normalizing the field $\vec{\mathbf{F}}$.

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions:

$$\vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$$

Illusttation 2.5.5: Sketching a Vector Field in Three Dimensions

Describe vector fields

(a) $\vec{\mathbf{F}}(x, y, z) = \langle 2,\ 0,\ 1 \rangle$

(b) $\vec{\mathbf{G}}(x, y, z) = \langle 1,\ 1,\ z \rangle$

(a) For this vector field, all components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x-component 2, y-component 0, and z-component 1. To visualize $\vec{\mathbf{F}}$, we draw the vector $\langle 2, 0, 1 \rangle$ emanating from every point.

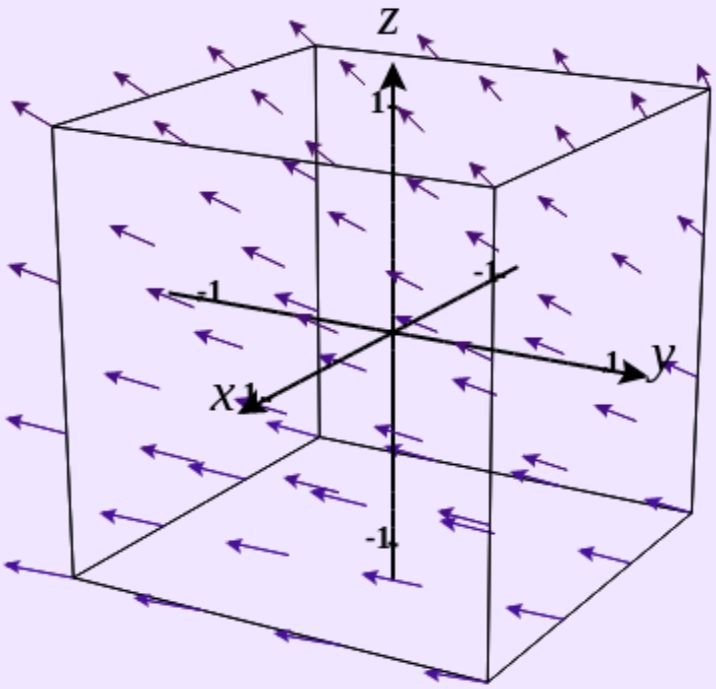


Figure 2.5.4: A visual representation of vector field $\vec{\mathbf{F}}(x, y, z) = \langle 2, 0, 1 \rangle$

(b) For this vector field, the x- and y-components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x- and y-components equal to one. To visualize $\vec{\mathbf{G}}$, we first consider what the field looks like in the xy-plane. In the xy-plane, $z = 0$. Hence, each point of the form $(a, b, 0)$ has vector $\langle 1, 1, 0 \rangle$ associated with it. For points not in the xy-plane but slightly above it, the associated vector has a small but positive z-component, and therefore the associated vector points slightly upward. For points that are far above the xy-plane, the z-component is large, so the vector is almost vertical. When z is negative, these vectors have negative z-component, so they point downward. The figure below shows this vector field.

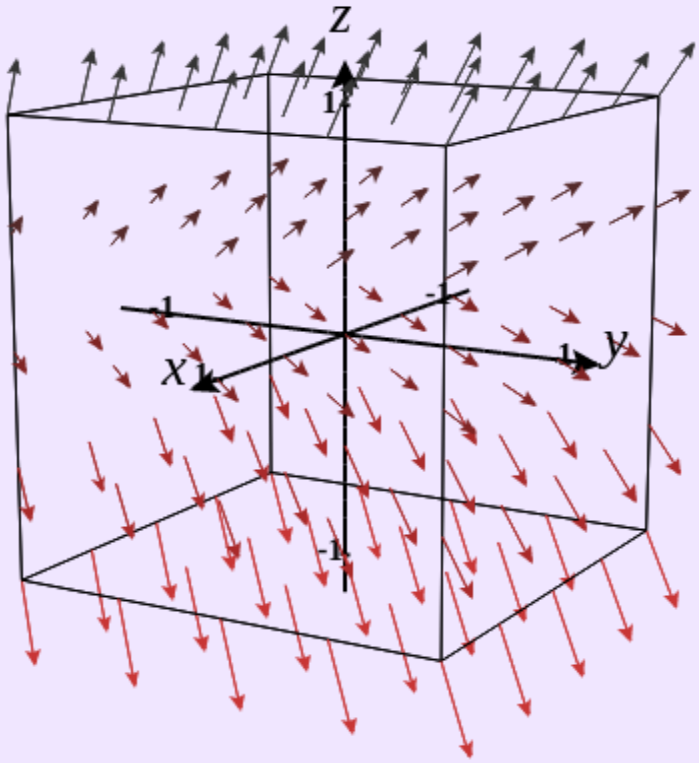


Figure 2.5.5: A visual representation of vector field $\vec{\mathbf{G}}(x, y, z) = \langle 1, 1, z \rangle$

In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Illustration 2.5.6: Describing a Gravitational Vector Field

Newton's law of gravitation states that $\vec{\mathbf{F}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x, y, z). Field $\vec{\mathbf{F}}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2 + y^2 + z^2}$.

The unit vector from object 1 to object 2 is $\hat{\mathbf{r}} = \frac{\langle x, y, z \rangle}{||\langle x, y, z \rangle||}$, and hence $\hat{\mathbf{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$.

Therefore, gravitational vector field $\vec{\mathbf{F}}$ exerted by object 1 on object 2 is

$$\vec{\mathbf{F}} = -Gm_1 m_2 \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle$$

This is an example of a radial vector field in $\mathbb{R}^3$.

The figure below shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

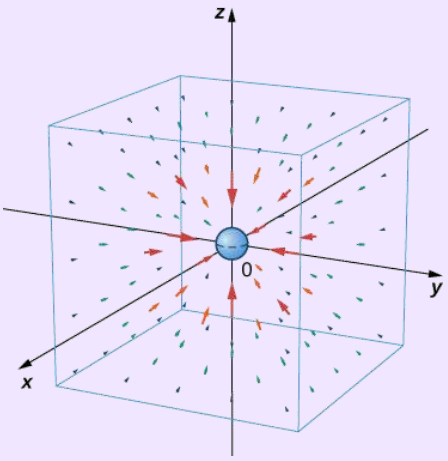


Figure 2.5.6: A visual representation of gravitational vector field for a large mass at the origin.

A special kind of vector field we will learn to appreciate is one that represents the gradient of a real valued function. We call such a vector field a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\text{grad } f = \nabla f(x,y) = f_x(x,y)\hat{\mathbf{i}} + f_y(x,y)\hat{\mathbf{j}}$$

We can see from the form in which the gradient is written that $\vec{\nabla} f$ is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\text{grad } f = \nabla f(x,y,z) = f_x(x,y,z)\hat{\mathbf{i}} + f_y(x,y,z)\hat{\mathbf{j}} + f_z(x,y,z)\hat{\mathbf{k}}$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$. A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

Definition: Gradient Field

A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\nabla f = \vec{\mathbf{F}}$.

Illustration 2.5.7: Sketching a Gradient Vector Field

We can u technology to plot the gradient vector field of $f(x,y) = x^2y^2$

The gradient of f is $\nabla f(x,y) = \langle 2xy^2,\ 2x^2y\rangle$. To sketch the vector field, we plot a large number of points and use color to represent length to make it easier to see (the length of each vector increases as the color changes from dark blue to teal to orange to yellow to red):

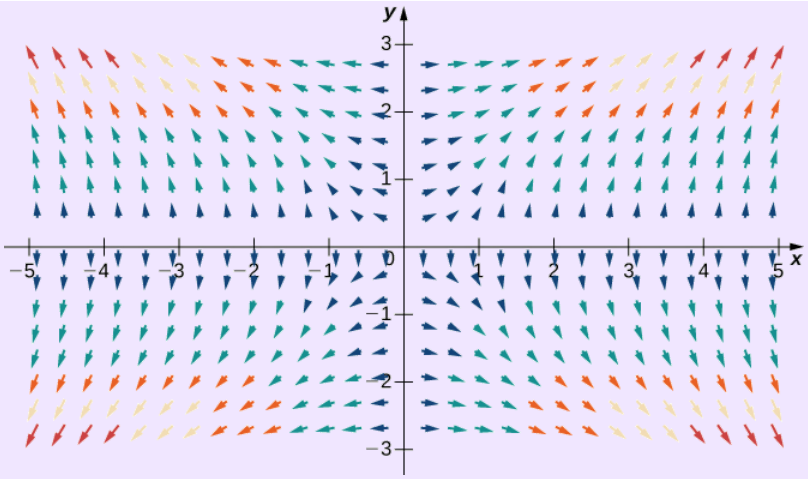


Figure 2.5.7: The gradient vector field is ∇f, where $f(x,y) = x^2y^2$

For the function $f(x,y) = x^2y^2$ in our previous example, if we overlay the level curves of f on the function's gradient vector field, we see the gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

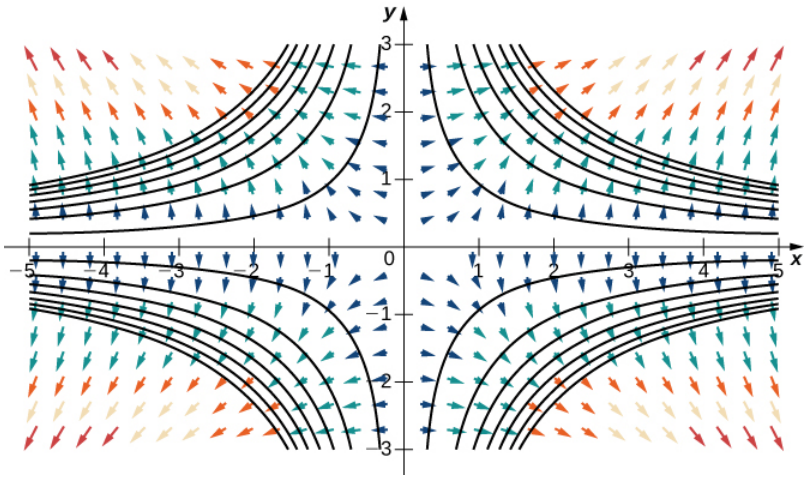


Figure 2.5.8: The gradient field of $f(x,y) = x^2y^2$ and several level curves of f.
Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\vec{\mathbf{F}}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\nabla f = \vec{\mathbf{F}}$. In this situation, f is called a **potential function** for $\vec{\mathbf{F}}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

Example 2.5.1: Verifying a Potential Function

Check if $f(x,y,z) = x^2yz - \sin(xy)$ is a potential function for vector field $\vec{\mathbf{F}}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle$

Solution

We need to confirm whether $\nabla f = \vec{\mathbf{F}}$. We check each partial derivative of f:

$$f_x(x,y,z) = 2xyz - y\cos(xy), \quad f_y(x,y,z) = x^2z - x\cos(xy), \quad f_z(x,y,z) = x^2y$$

Since we have $\vec{\mathbf{F}} = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \nabla f$, we know $\boxed{f \text{ is a potential function for } \vec{\mathbf{F}}}$

Without knowing the potential function, we need to be able to tell if a given vector field is conservative or not.

Theorem 2.5.1: Cross-Partial Property of Conservative Vector Fields

If $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then $\vec{\mathbf{F}}$ is conservative if and only if $P_y = Q_x$.

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a vector field on an open, simply connected region D in $\mathbb{R}^3$, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\vec{\mathbf{F}}$ is conservative.

The proof of this theorem is a result of Clairaut's Theorem, since $f_{xy} = f_{yx}$ must hold, if $\nabla f = \langle P, Q\rangle$, then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} f_x = f_{xy} = f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial Q}{\partial x}$$

The three dimensional formula is justified similarly.

Example 2.5.2: Determining Whether a Vector Field Is Conservative

Determine whether each vector field below is conservative.

(a) $\vec{\mathbf{F}}(x,y) = \left\langle x\ln(y), \dfrac{x^2}{2y}\right\rangle$

(b) $\vec{\mathbf{F}}(x,y,z) = \langle xy^2z, x^2yz, z^2\rangle$

Solution

(a) Note that the domain of $\vec{\mathbf{F}}$ is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of $\vec{\mathbf{F}}$ is part of a plane above the x-axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use *the Cross-Partial Property of Conservative Vector Fields* to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y) = x\ln(y) \quad \text{and} \quad Q(x,y) = \frac{x^2}{2y}$$

Then $P_y(x,y) = \dfrac{x}{y} = Q_x(x,y)$ and thus $\boxed{\vec{\mathbf{F}} \text{ is conservative}}$

(b) Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use *the Cross-Partial Property of Conservative Vector Fields* to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y,z) = xy^2z, \quad Q(x,y,z) = x^2yz, \quad \text{and} \quad R(x,y,z) = z^2$$

Since $Q_z(x,y,z) = x^2y$ and $R_y(x,y,z) = 0$, the vector field $\boxed{\vec{\mathbf{F}} \text{ is not conservative}}$

If $\vec{\mathbf{F}}$ is a conservative vector field, then there is at least one potential function f such that $\nabla f = \vec{\mathbf{F}}$. But could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if F and G are both antiderivatives of $k(x)$, then F and G **differ only by a constant**. That is, there is some number C such that $F(x) = G(x) + C$.

Now let $\vec{\mathbf{F}}$ be a conservative vector field and let f and g be potential functions for $\vec{\mathbf{F}}$. Since the gradient is like a derivative, $\vec{\mathbf{F}}$ being conservative means that $\vec{\mathbf{F}}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x,y) = g(x,y) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

Theorem 2.5.2: Uniqueness of Potential Functions

Let $\vec{\mathbf{F}}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\nabla f = \vec{\mathbf{F}}$ and $\nabla g = \vec{\mathbf{G}}$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\vec{\mathbf{F}}$, then $\nabla(f-g) = \nabla f - \nabla g = \vec{\mathbf{F}} - \vec{\mathbf{F}} = \vec{\mathbf{0}}$. Let $h = f - g$, then we have $\nabla h = \vec{\mathbf{0}}$.We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\nabla h = \vec{\mathbf{0}}$, we have $h_x(x,y) = 0$ and $h_y(x,y) = 0$. The expression $h_x(x,y) = 0$ implies that h is a constant function with respect to x—that is, $h(x,y) = k_1(y)$ for some function k_1. Similarly, $h_y(x,y) = 0$ implies $h(x,y) = k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x,y) = C$ for some constant C on the connected domain of $\vec{\mathbf{F}}$. Note that we really do need connectedness at this point; if the domain of $\vec{\mathbf{F}}$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f = g + C$, as desired.

The question remains, how can we find the potential function of a conservative vector field $\vec{\mathbf{F}}$?

If $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y)\rangle$, then we want $f_x(x,y) = P(x,y)$ and $f_y(x,y) = Q(x,y)$. Using anti-derivatives this means we want a function f such that:

$$f(x,y) = \int P(x,y)\,dx \quad \text{and} \quad f(x,y) = \int Q(x,y)\,dy$$

Above we define the integral dx to treat the variable y as a constant, so $\int P(x,y)\,dx = \tilde{P}(x,y) + h_1(y)$ for some function $h_1(y)$ that is *constant with respect to* x. Similarly the integral dy treats the variable x as a constant, so $\int Q(x,y)\,dy = \tilde{Q}(x,y) + h_2(x)$ for some function $h_2(x)$ that is *constant with respect to* y. If we have $\tilde{P}(x,y) + h_1(y) = \tilde{Q}(x,y) + h_2(x)$ for some functions h_1, h_2, then we have found our potential function $f(x,y)$. A similar process is applied for vector fields of the form $\vec{\mathbf{F}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

Example 2.5.3: Finding a Potential Function

Find a potential function for each vector field below (if possible)

(a) $\vec{\mathbf{F}}(x,y) = \langle e^x y^3 + 2x + y, 3e^x y^2 + x - 1\rangle$

(b) $\vec{\mathbf{F}}(x,y) = \langle 2xy^3 + y, 3x^2y^2 + \cos(y)\rangle$

(c) $\vec{\mathbf{F}}(x,y,z) = \langle yz - 4x^3 e^{\sin(y)}, xz - x^4\cos(y)e^{\sin(y)}, xy + 12\sqrt{z}\rangle$

Solution

(a) We start by doing both integrals separately:

$$\int P(x,y)dx = \int (e^x y^3 + 2x + y)dx = e^x y^3 + x^2 + xy + h_1(y)$$

$$\int Q(x,y)dy = \int (3e^x y^2 + x - 1)dy = e^x y^3 + xy - y + h_2(x)$$

These two are equal when we let $h_1(y) = -y + C$ and $h_2(x) = x^2 + C$ for any constant C.

Therefore $\vec{\mathbf{F}} = \nabla f$ for the potential function $\boxed{f(x,y) = e^x y^3 + x^2 + xy - y + C}$

(b) We start by doing both integrals separately:

$$\int P(x,y)dx = \int (2xy^3 + y)dx = x^2y^3 + xy + h_1(y)$$

$$\int Q(x,y)dy = \int (3x^2y^2 + \cos(y))dy = x^2y^3 + \sin(y) + h_2(x)$$

These two cannot be made equal for any choice of $h_1(y)$ or $h_2(x)$! The term xy appears in the first integral, and cannot be made from $h_2(x)$. This is because the original vector field $\vec{\mathbf{F}}$ is not conservative:

$$P_y = 6xy^2 + 1 \quad \text{and} \quad Q_x = 6xy^2 \implies P_y \neq Q_x$$

There $\boxed{\text{cannot be a potential function}}$ if the vector field is not a gradient field.

(c) We start by doing all three integrals separately:

$$\int P(x,y,z)dx = \int (yz - 4x^3 e^{\sin(y)})dx = xyz - x^4 e^{\sin(y)} + h_1(y,z)$$

$$\int Q(x,y,z)dy = \int (xz - x^4\cos(y)e^{\sin(y)})dy = xyz - x^4 e^{\sin(y)} + h_2(x,z)$$

$$\int R(x,y,z)dz = \int (xy + 12\sqrt{z})dz = xyz + 8z^{\frac{3}{2}} - x^4 e^{\sin(y)} + h_3(x,y)$$

These are equal when we let $h_1(y,z) = 8z^{\frac{3}{2}} + C$, $h_2(x,z) = 8z^{\frac{3}{2}} + C$, and $h_3(x,y) = -x^4 e^{\sin(y)} + C$ for any constant C.

Therefore $\vec{\mathbf{F}} = \nabla f$ for the potential function $\boxed{f(x,y,z) = xyz - x^4 e^{\sin(y)} + 8z^{\frac{3}{2}} + C}$

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2.5: Divergence and Curl

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition: Divergence of a Vector Field

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$ and P_x, and Q_y all exist, then the ***divergence*** of $\vec{\mathbf{F}}$ is defined by

$$\text{div}(\vec{\mathbf{F}}) = P_x + Q_y = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the ***divergence*** of $\vec{\mathbf{F}}$ is defined by

$$\text{div}(\vec{\mathbf{F}}) = P_x + Q_y + R_z = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \text{ or } \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

divergence can be written symbolically as the dot product

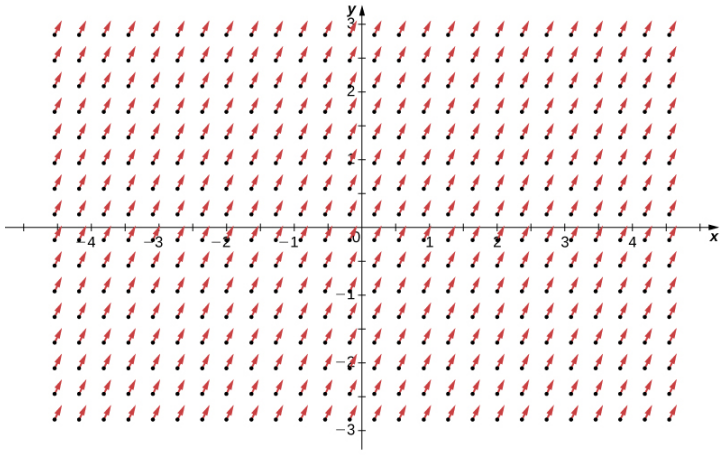
$$\text{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}}$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

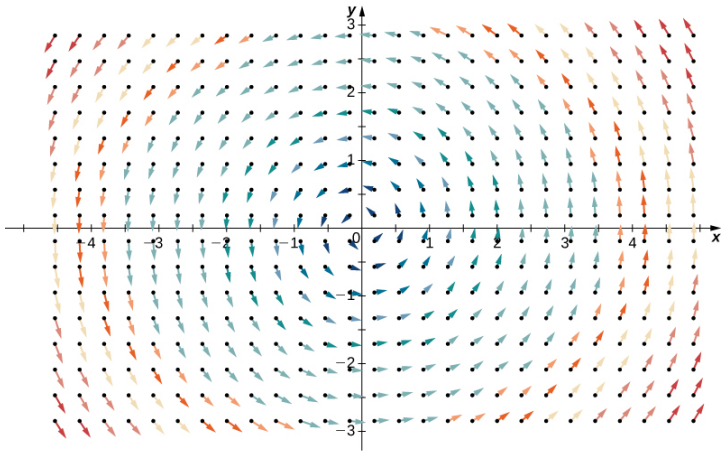
To illustrate this point, consider the two vector fields in $\vec{\mathbf{F}} = \langle 1, 2 \rangle$ and $\vec{\mathbf{G}} = \langle -y, x \rangle$.

$$\text{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0 \text{ and } \text{div}(\vec{\mathbf{G}}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. This is easy to see in the graphs of these vector fields below:



(a)



(b)

Figure 2.5.9: (a) Vector field $\vec{\mathbf{F}} = \langle 1, 2 \rangle$ has zero divergence.
(b) Vector field $\vec{\mathbf{G}} = \langle -y, x \rangle$ also has zero divergence.

By contrast, consider radial vector field $\vec{\mathbf{R}}(x, y) = \langle -x, -y \rangle$.

$$\text{div}(\vec{\mathbf{R}}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2$$

At any given point, more fluid is flowing in than is flowing out, and therefore the “outgoingness” of the field is negative, as we can see below:

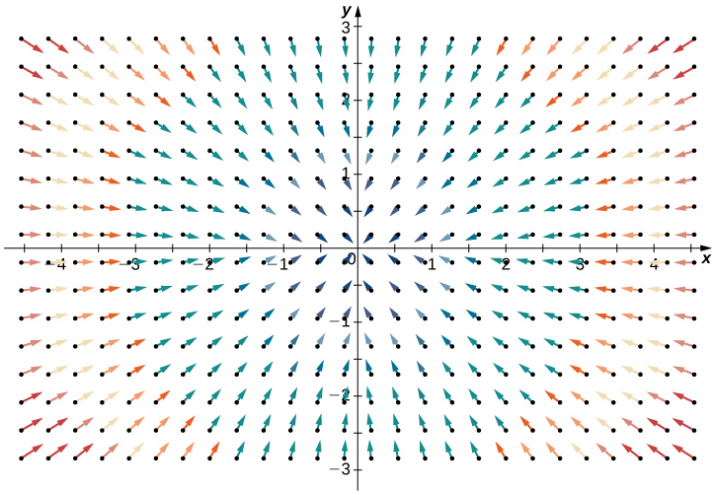


Figure 2.5.10: The vector field $\vec{\mathbf{R}}$ has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in Figure 2.5.9.

On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in Figure 2.5.10 so that the center of the circle lands at point $(3,3)$. The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example 2.5.4: Calculating Divergence at a Point

If $\vec{\mathbf{F}}(x,y,z) = e^x\hat{\mathbf{i}} + yz\hat{\mathbf{j}} - yz^2\hat{\mathbf{k}}$, then find the divergence of $\vec{\mathbf{F}}$ at $(0,2,-1)$.

Solution

The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(yz^2) = e^x + z - 2yz$$

Therefore, the divergence at $(0,2,-1)$ is $e^0 - 1 + 4 = \boxed{4}$

If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0,2,-1)$

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\vec{\mathbf{B}}$ is a magnetic field, then $\nabla \cdot \vec{\mathbf{B}} = 0$; in other words, the divergence of a magnetic field is zero.

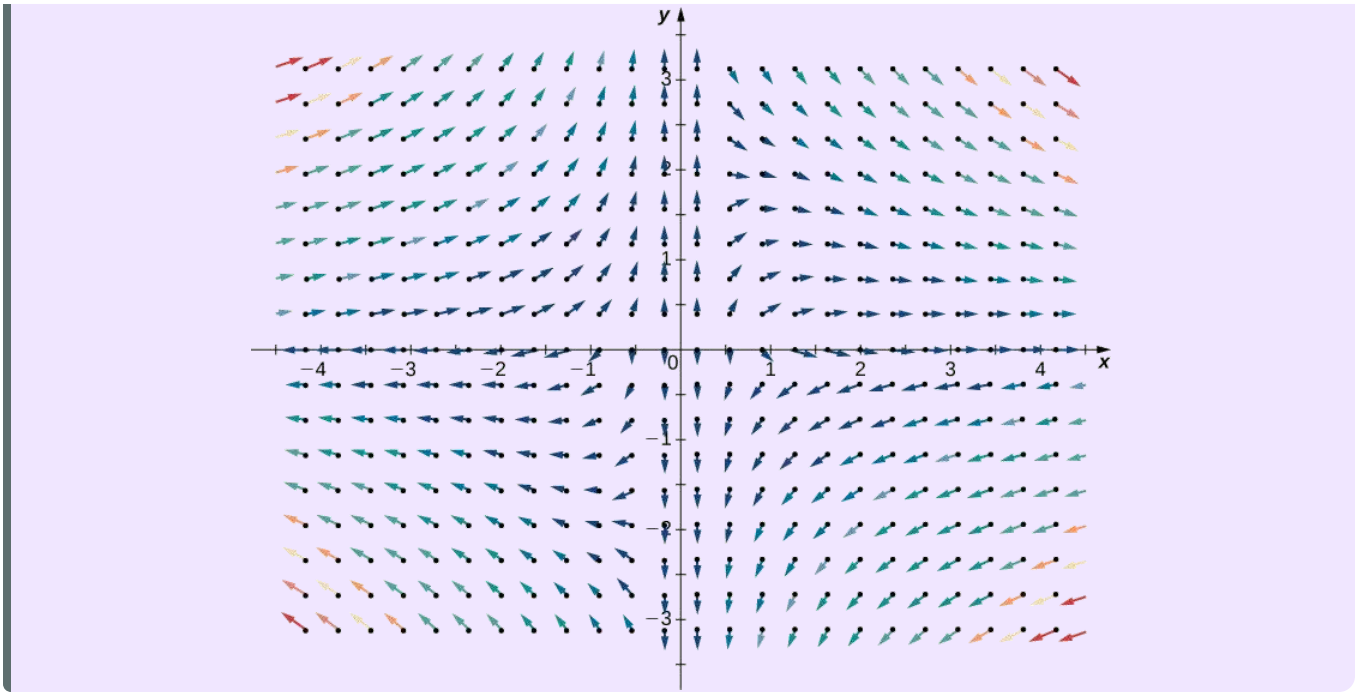
Illustration 2.5.8: Determining Whether a Field Is Magnetic

Is it possible for $\vec{\mathbf{F}}(x,y) = \langle x^2y,\ y - xy^2\rangle$ to be a magnetic field?

If $\vec{\mathbf{F}}$ were magnetic, then its divergence would be zero. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y - xy^2) = 2xy + 1 - 2xy = 1$$

and therefore $\vec{\mathbf{F}}$ cannot model a magnetic field. It is pictured below to confirm this intuitively.



The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\vec{\mathbf{F}}$ represents the velocity field of a fluid. Then, the curl of $\vec{\mathbf{F}}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector as pictured below. The curl measures the tendency of the paddlewheel to rotate.

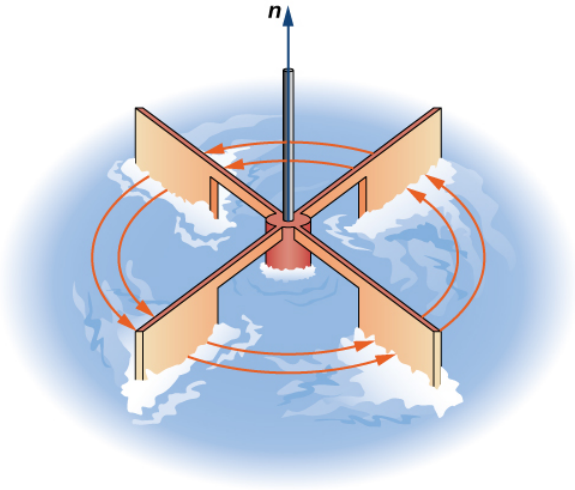


Figure 2.5.11: To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in Figure 2.5.9. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\hat{\mathbf{k}}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition: Curl

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$, and P_x, Q_y, and R_z all exist, then the ***curl*** of $\vec{\mathbf{F}}$ is defined by

$$\begin{aligned} \text{curl}(\vec{\mathbf{F}}) &= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}} \end{aligned}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\nabla \times \vec{\mathbf{F}}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

The determinant of this matrix is

$$(R_y - Q_z)\,\hat{\mathbf{i}} - (R_x - P_z)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = \text{curl}(\vec{\mathbf{F}})$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\vec{\mathbf{F}}$ is not defined.

Example 2.5.5: Finding the Curl of a Three-Dimensional Vector Field

Find the curl of each vector field at the given point:

(a) $\vec{\mathbf{F}} = \langle x^2 z, e^y + xz, xyz \rangle$ at $(1, 2, 3)$

(b) $\vec{\mathbf{F}} = \langle \sin(x)\cos(z), \sin(y)\sin(z), \cos(x)\cos(y) \rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution

(a) The curl is

$$\text{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 z & e^y + xz & xyz \end{vmatrix} = \langle xz - x, x^2 - yz, z - 0 \rangle$$

Plugging in $x = 1, y = 2$, and $z = 3$ this becomes $\boxed{\langle 2, -5, 3 \rangle}$

(b) The curl is

$$\begin{aligned} \text{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin(x)\cos(z) & \sin(y)\sin(z) & \cos(x)\cos(y) \end{vmatrix} \\ &= \langle -\cos(x)\sin(y) - \sin(y)\cos(z)), \sin(x)\cos(y) - \sin(x)\sin(z), 0 - 0 \rangle \end{aligned}$$

Plugging in $x = 0, y = \frac{\pi}{2}$, and $z = \frac{\pi}{2}$ this becomes $\boxed{\langle -1, 0, 0 \rangle}$

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 and an object at (x, y, z) with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by the vector field

$$\vec{\mathbf{F}}(x,y,z) = -Gm_1m_2\left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$$

Illustration 2.5.9: Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

To show that $\vec{\mathbf{F}}$ has no spin, we calculate its curl. Let $\vec{\mathbf{H}} = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$

Then, $\vec{\mathbf{F}} = -Gm_1m_2\vec{\mathbf{H}}$

$$\text{curl}(\vec{\mathbf{H}}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Separately we can see that

$$R_y - Q_z = \frac{-3yz}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{-3yz}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

$$P_z - R_x = \frac{-3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{-3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

$$Q_x - P_y = \frac{-3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{-3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

Therefore $\text{curl}(\vec{\mathbf{H}}) = \vec{\mathbf{0}}$

This means the scalar multiple $\vec{\mathbf{F}}$ also has curl $\vec{\mathbf{0}}$, so a gravitational field has no spin.

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\vec{\mathbf{F}}$ is a vector field in $\mathbb{R}^3$, then the curl of $\vec{\mathbf{F}}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\vec{\mathbf{v}}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $\text{div}(\text{curl}(\vec{\mathbf{v}})) = 0$, the net rate of flow in vector field $\text{curl}(\vec{\mathbf{v}})$ at any point is zero. Taking the curl of vector field $\vec{\mathbf{F}}$ eliminates whatever divergence was present in $\vec{\mathbf{F}}$.

Theorem 2.5.3: Divergence of the Curl

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then,

$$\text{div}(\text{curl}(\vec{\mathbf{F}})) = \nabla \cdot (\nabla \times \vec{\mathbf{F}}) = 0$$

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned} \text{div}(\text{curl}(\vec{\mathbf{F}})) &= \text{div}\left((R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}\right) \\ &= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy} = 0 \end{aligned}$$

Illustration 2.5.10: Showing That a Vector Field Is Not the Curl of Another

Show that $\vec{\mathbf{F}}(x,y,z) = e^x\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$ is not the curl of another vector field. That is, show that there is no other vector $\vec{\mathbf{G}}$ with $\text{curl}(\vec{\mathbf{G}}) = \vec{\mathbf{F}}$.

Solution

Notice that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ and the second-order partials of $\vec{\mathbf{F}}$ are all continuous. Therefore, we can apply the previous theorem to $\vec{\mathbf{F}}$.

The divergence of $\vec{\mathbf{F}}$ is $e^x + z + 2xz$. If $\vec{\mathbf{F}}$ were the curl of vector field $\vec{\mathbf{G}}$, then $\text{div}(\vec{\mathbf{F}}) = \text{div}\left(\text{curl}\left(\vec{\mathbf{G}}\right)\right) = 0$. But, the divergence of $\vec{\mathbf{F}}$ is not zero, and therefore $\vec{\mathbf{F}}$ is not the curl of any other vector field.

With the next two theorems, we show that if $\vec{\mathbf{F}}$ is a conservative vector field then its curl is zero, and if the domain of $\vec{\mathbf{F}}$ is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem 2.5.4: Curl of a Conservative Vector Field

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is conservative, then $\text{curl}(\vec{\mathbf{F}}) = \vec{\mathbf{0}}$

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of $\vec{\mathbf{F}}$ are equal. Therefore,

$$\text{curl}(\vec{\mathbf{F}}) = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $\text{curl}(\nabla f) = \vec{\mathbf{0}}$ for any scalar function f. In terms of our curl notation, $\nabla \times \nabla(f) = \vec{\mathbf{0}}$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\nabla \times \nabla(f) = \vec{\mathbf{0}}$ is simplified as $\nabla \times \nabla = \vec{\mathbf{0}}$.

Theorem 2.5.5: Curl Test for a Conservative Field

Let $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ be a vector field in space on a simply connected domain. If $\text{curl}(\vec{\mathbf{F}}) = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

Proof

Since $\text{curl}(\vec{\mathbf{F}}) = \vec{\mathbf{0}}$, we have that $R_y = Q_z$, $P_z = R_x$, and $Q_x = P_y$. Therefore, $\vec{\mathbf{F}}$ satisfies the cross-partials property on a simply connected domain, and the Cross-Partial Property of Conservative Fields implies that $\vec{\mathbf{F}}$ is conservative.

Illustration 2.5.11: Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy\rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ which is simply connected. Therefore, we can test whether $\vec{\mathbf{F}}$ is conservative by calculating its curl.

The curl of $\vec{\mathbf{F}}$ is

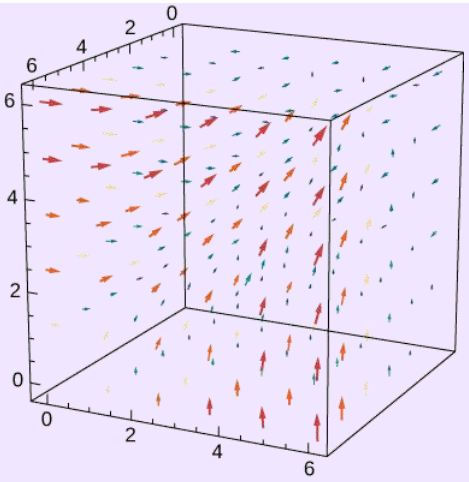
$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\hat{\mathbf{k}} = \langle (x-x), (y-y), (z-z)\rangle = \vec{\mathbf{0}}$$

Thus, $\vec{\mathbf{F}}$ is conservative.

In fact, using the methods of the previous section we can find the potential function of this vector field:

$$\int yzdx = \int xzdy = \int xydz = xyz + C$$

So the gradient of the function $f(x, y, z) = xyz$ is given by $\vec{F}$, pictured below.



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2.5: Practice Set

Exercise 2.5.1a

Determine whether $\vec{\mathbf{F}}(x,y) = \langle 2x, 4y\rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

We check cross partials to determine if $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y)\rangle$ is conservative:

$P_y = \dfrac{\partial}{\partial y}(2x) = 0$

$Q_x = \dfrac{\partial}{\partial x}(4y) = 0$

Since these match, this vector field **is conservative**.

To find the potential function we solve each integral separately:

$\int P dx = \int 2x\, dx = x^2 + h_1(y)$

$\int Q dy = \int 4y\, dy = 2y^2 + h_2(x)$

Letting $h_1(y) = 2y^2$ and $h_2(x) = x^2$ we see that these are equal, giving us the potential function:

$f(x,y) = x^2 + 2y^2$ satisfying $\nabla f = \vec{\mathbf{F}}$

Exercise 2.5.1b

Determine whether $\vec{\mathbf{F}} = \langle 4y, 2x\rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

We check cross partials to determine if $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y)\rangle$ is conservative:

$P_y = \dfrac{\partial}{\partial y}(4y) = 4$

$Q_x = \dfrac{\partial}{\partial x}(2x) = 2$

Since these do not match, this vector field **is not conservative**.

Exercise 2.5.1c

Determine whether $\vec{\mathbf{F}}(x,y) = \langle y\cos(xy), x\cos(xy)\rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

We check cross partials to determine if $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y)\rangle$ is conservative:

$P_y = \dfrac{\partial}{\partial y}(y\cos(xy)) = \cos(xy) - xy\sin(xy)$

$Q_x = \dfrac{\partial}{\partial x}(x\cos(xy)) = \cos(xy) - xy\sin(xy)$

Since these match, this vector field **is conservative**.

To find the potential function we solve each integral separately:

$\int P dx = \int y\cos(xy)\, dx = \sin(xy) + h_1(y)$

$$\int Q dy = \int x\cos(xy)\,dy = \sin(xy) + h_2(x)$$

Letting $h_1(y) = 0$ and $h_2(x) = 0$ we see that these are equal, giving us the potential function:

$f(x,y) = \sin(xy)$ satisfying $\nabla f = \vec{\mathbf{F}}$

Exercise 2.5.1d

Determine whether $\vec{\mathbf{F}}(x,y) = \left\langle \frac{1}{x-y} + \cos(x), \frac{1}{y-x} + \sin(y) \right\rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

We check cross partials to determine if $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y) \rangle$ is conservative:

$$P_y = \frac{\partial}{\partial y}\left(\frac{1}{x-y} + \cos(x)\right) = \frac{1}{(x-y)^2}$$

$$Q_x = \frac{\partial}{\partial x}\left(\frac{1}{y-x} + \sin(y)\right) = \frac{1}{(y-x)^2}$$

Since these match, this vector field **is conservative**.

To find the potential function we solve each integral separately:

$$\int P dx = \int \frac{1}{x-y} + \cos(x)\,dx = \ln|x-y| + \sin(x) + h_1(y)$$

$$\int Q dy = \int \frac{1}{y-x} + \sin(y)\,dy = \ln|y-x| - \cos(y) + h_2(x)$$

Letting $h_1(y) = -\cos(y)$ and $h_2(x) = \sin(x)$ we see that these are equal, giving us the potential function:

$f(x,y) = \ln|x-y| + \sin(x) - \cos(y)$ satisfying $\nabla f = \vec{\mathbf{F}}$

Exercise 2.5.1e

Determine whether $\vec{\mathbf{F}}(x,y,z) = \langle x^2, z^2, y^2 \rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

We check the curl to determine if $\vec{\mathbf{F}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is conservative:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & z^2 & y^2 \end{vmatrix} = \langle 2y - 2z, 0, 0 \rangle$$

Since the curl is not the zero vector, this vector field **is not conservative**.

Exercise 2.5.1f

Determine whether $\vec{\mathbf{F}}(x,y,z) = \left\langle \frac{2x}{(x^2-y^2)^2} - 1, 3 - \frac{2y}{(x^2-y^2)^2}, 4z \right\rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

We check the curl to determine if $\vec{\mathbf{F}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is conservative:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{2x}{(x^2-y^2)^2} - 1 & 3 - \dfrac{2y}{(x^2-y^2)^2} & 4z \end{vmatrix} = \langle 0, 0, 8xy(x^2-y^2)^{-3} - 8xy(x^2-y^2)^{-3} \rangle = \vec{\mathbf{0}}$$

Since the curl is the zero vector, this vector field **is conservative**.

To find the potential function we solve each integral separately:

$$\int P dx = \int \frac{2x}{(x^2-y^2)^2} - 1 \, dx = -(x^2-y^2)^{-1} - x + h_1(y,z)$$

$$\int Q dy = \int 3 - \frac{2y}{(x^2-y^2)^2} \, dy = 3y - (x^2-y^2)^{-1} + h_2(x,z)$$

$$\int R dz = \int 4z \, dz = 2z^2 + h_3(x,y)$$

Letting $h_1(y,z) = 3y + 2z^2$, $h_2(x,z) = -x + 2z^2$, and $h_3(x,y) = -x + 3y$ we see that these are equal, giving us the potential function:

$f(x,y,z) = -(x^2-y^2)^{-1} - x + 3y + 2z^2$ satisfying $\nabla f = \vec{\mathbf{F}}$

Exercise 2.5.1g

Determine whether $\vec{\mathbf{F}}(x,y,z) = \langle 2y + 3z, x + 4z, 3x + 4y + 6z \rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

We check the curl to determine if $\vec{\mathbf{F}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is conservative:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y + 3z & x + 4z & 3x + 4y + 6z \end{vmatrix} = \langle 4 - 4, 3 - 3, 1 - 2 \rangle = \langle 0, 0, -1 \rangle$$

Since the curl is not the zero vector, this vector field **is not conservative**.

Exercise 2.5.1h

Determine whether $\vec{\mathbf{F}}(x,y,z) = \left\langle (1 + xyz^2)e^{xyz^2} + e^x, x^2z^2e^{xyz^2} + 2yz^2, 2x^2yze^{xyz^2} + 2y^2z \right\rangle$ is conservative or not. If it is conservative, find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer

Checking the curl of this vector field is unpleasant, so we attempt the integrals and see if they match:

$$\int P dx = \int (1 + xyz^2)e^{xyz^2} + e^x \, dx = xe^{xyz^2} + e^x + h_1(y,z)$$

$$\int Q dy = \int x^2z^2e^{xyz^2} + 2yz^2 \, dy = xe^{xyz^2} + y^2z^2 + h_2(x,z)$$

$$\int R dz = \int 2x^2yze^{xyz^2} + 2y^2z \, dz = xe^{xyz^2} + y^2z^2 + h_3(x,y)$$

Letting $h_1(y,z) = y^2z^2$, $h_2(x,z) = e^x$, and $h_3(x,y) = e^x$ we see that these are equal, giving us the potential function:

$f(x,y,z) = xe^{xyz^2} + e^x + y^2z^2$ satisfying $\nabla f = \vec{\mathbf{F}}$

Since this potential function exists, we know the vector field **is conservative.**

Exercise 2.5.2a

Find the expression denoted in terms of divergence and curl (if possible) for $\vec{\mathbf{A}} = \langle 2x, 3y, 4z \rangle$

(a) $\text{curl}(\vec{\mathbf{A}})$

(b) $\text{div}(\vec{\mathbf{A}})$

(c) $\text{curl}(\text{curl}(\vec{\mathbf{A}}))$

(d) $\text{curl}(\text{div}(\vec{\mathbf{A}}))$

Answer

(a) $\text{curl}(\vec{\mathbf{A}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 3y & 4z \end{vmatrix} = \langle 0-0, 0-0, 0-0 \rangle = \vec{\mathbf{0}}$

(b) $\text{div}(\vec{\mathbf{A}}) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z) = 2 + 3 + 4 = 9$

(c) $\text{curl}(\text{curl}(\vec{\mathbf{A}})) = \text{curl}(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$

(d) Since $\text{div}(\vec{\mathbf{A}})$ is not a vector, we cannot take its curl. This **does not exist.**

Exercise 2.5.2b

Find the expression denoted in terms of divergence and curl (if possible) for $\vec{\mathbf{B}} = \langle e^{yz+x}, e^{xz+y}, e^{xy+z} \rangle$

(a) $\text{curl}(\vec{\mathbf{B}})$

(b) $\text{div}(\vec{\mathbf{B}})$

(c) $\text{div}(\text{curl}(\vec{\mathbf{B}}))$

(d) $\nabla\text{div}(\vec{\mathbf{B}})$

Answer

(a) $\text{curl}(\vec{\mathbf{B}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{yz+x} & e^{xz+y} & e^{xy+z} \end{vmatrix} = \langle xe^{xy+z} - xe^{xz+y}, ye^{yz+x} - ye^{xy+z}, ze^{xz+y} - ze^{yz+x} \rangle$

(b) $\text{div}(\vec{\mathbf{B}}) = \frac{\partial}{\partial x}(e^{yz+x}) + \frac{\partial}{\partial y}(e^{xz+y}) + \frac{\partial}{\partial z}(e^{xy+z}) = e^{yz+x} + e^{xz+y} + e^{xy+z}$

(c) $\text{div}(\text{curl}(\vec{\mathbf{B}})) = 0$ (since the divergence of a curl is always zero)

(d) To find its gradient, we find each partial derivative of the divergence of $\vec{\mathbf{B}}$:

$\frac{\partial}{\partial x}(e^{yz+x} + e^{xz+y} + e^{xy+z}) = e^{yz+x} + ze^{xz+y} + ye^{xy+z}$

$\frac{\partial}{\partial y}(e^{yz+x} + e^{xz+y} + e^{xy+z}) = ze^{yz+x} + e^{xz+y} + xe^{xy+z}$

$\frac{\partial}{\partial z}(e^{yz+x} + e^{xz+y} + e^{xy+z}) = ye^{yz+x} + xe^{xz+y} + e^{xy+z}$

Therefore, $\nabla\text{div}(\vec{\mathbf{B}}) = \langle e^{yz+x} + ze^{xz+y} + ye^{xy+z}, ze^{yz+x} + e^{xz+y} + xe^{xy+z}, ye^{yz+x} + xe^{xz+y} + e^{xy+z} \rangle$

Exercise 2.5.2c

Find the expression denoted in terms of divergence and curl (if possible) for $\vec{\mathbf{C}} = \left\langle \frac{y}{x}, \frac{z}{y}, \frac{x}{z} \right\rangle$

(a) $\text{curl}(\vec{\mathbf{C}})$

(b) $\text{div}(\vec{\mathbf{C}})$

(c) $\text{div}(\text{div}(\vec{\mathbf{C}}))$

(d) $\text{curl}\left(\left\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\rangle + \vec{\mathbf{C}}\right)$

Answer

(a) $\text{curl}(\vec{\mathbf{C}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x} & \frac{z}{y} & \frac{x}{z} \end{vmatrix} = \left\langle 0 - \frac{1}{y}, 0 - \frac{1}{z}, 0 - \frac{1}{x} \right\rangle = \langle -y^{-1}, -z^{-1}, -x^{-1} \rangle$

(b) $\text{div}(\vec{\mathbf{C}}) = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) + \frac{\partial}{\partial y}\left(\frac{z}{y}\right) + \frac{\partial}{\partial z}\left(\frac{x}{z}\right) = -\frac{y}{x^2} - \frac{z}{y^2} - \frac{x}{z^2}$

(c) Since $\text{div}(\vec{\mathbf{C}})$ is not a vector, we cannot take its divergence. This **does not exist.**

(d) We see that $\text{curl}\left(\left\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\rangle\right) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix} = \langle 0-0, 0-0, 0-0 \rangle = \vec{\mathbf{0}}$

Therefore $\text{curl}\left(\left\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\rangle + \vec{\mathbf{C}}\right) = \vec{\mathbf{0}} + \text{curl}(\vec{\mathbf{C}}) = \langle -y^{-1}, -z^{-1}, -x^{-1} \rangle$

Exercise 2.5.2*d*

Find the expression denoted in terms of divergence and curl (if possible) for $f(x, y, z) = x^2 + 2y^2 - z^3$

(a) $\text{curl}(\nabla f)$

(b) $\text{div}(\nabla f)$

(c) $\text{curl}(\langle x, y, z \rangle \cdot \nabla f)$

(d) $\nabla \text{div}(\nabla f)$

Answer

Each of these involve the gradient of f : $\nabla f = \langle 2x, 4y, -3z^2 \rangle$.

(a) $\text{curl}(\nabla f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 4y & -3z^2 \end{vmatrix} = \langle 0-0, 0-0, 0-0 \rangle = \vec{\mathbf{0}}$

(b) $\text{div}(\nabla f) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z^2) = 2 + 4 - 6z = 6 - 6z$

(c) $\langle x, y, z \rangle \cdot \nabla f = 2x^2 + 4y^2 - 3z^3$ is not a vector, so we cannot take its curl. This **does not exist.**

(d) $\nabla \text{div}(\nabla f) = \left\langle \frac{\partial}{\partial x}(6 - 6z), \frac{\partial}{\partial y}(6 - 6z), \frac{\partial}{\partial z}(6 - 6z) \right\rangle = \langle 0, 0, -6 \rangle$

Exercise 2.5.3

If $\vec{\mathbf{F}}(x, y, z) = \langle |x|, |y|, |z| \rangle$ and $\vec{\mathbf{G}}(x, y, z) = \langle |y|, |z|, |x| \rangle$, at a point (x, y, z) where $x, y, z \neq 0$, find:

(a) $\text{curl}(\vec{\mathbf{F}})$

(b) $\text{div}(\vec{\mathbf{F}})$

(c) $\text{curl}(\vec{\mathbf{G}})$

(d) $\operatorname{div}(\vec{\mathbf{G}})$

Answer

(a) $\operatorname{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ |x| & |y| & |z| \end{vmatrix} = \langle 0-0, 0-0, 0-0 \rangle = \vec{\mathbf{0}}$

(b) $\operatorname{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}(|x|) + \frac{\partial}{\partial y}(|y|) + \frac{\partial}{\partial z}(|z|) = \begin{cases} 1+1+1 & x>0,\ y>0,\ z>0 \\ -1+1+1 & x<0,\ y>0,\ z>0 \\ 1-1+1 & x>0,\ y<0,\ z>0 \\ -1-1+1 & x<0,\ y<0,\ z>0 \\ 1+1-1 & x>0,\ y>0,\ z<0 \\ -1+1-1 & x<0,\ y>0,\ z<0 \\ 1-1-1 & x>0,\ y<0,\ z<0 \\ -1-1-1 & x<0,\ y<0,\ z<0 \end{cases} = \begin{cases} 3 & x>0,\ y>0,\ z>0 \\ 1 & x<0,\ y>0,\ z>0 \\ 1 & x>0,\ y<0,\ z>0 \\ -1 & x<0,\ y<0,\ z>0 \\ 1 & x>0,\ y>0,\ z<0 \\ -1 & x<0,\ y>0,\ z<0 \\ -1 & x>0,\ y<0,\ z<0 \\ -3 & x<0,\ y<0,\ z<0 \end{cases}$

(c) $\operatorname{curl}(\vec{\mathbf{G}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ |y| & |z| & |x| \end{vmatrix} = \begin{cases} \langle 0-1, 0-1, 0-1 \rangle & x>0,\ y>0,\ z>0 \\ \langle 0-1, 0-(-1), 0-1 \rangle & x<0,\ y>0,\ z>0 \\ \langle 0-1, 0-1, 0-(-1) \rangle & x>0,\ y<0,\ z>0 \\ \langle 0-1, 0-(-1), 0-(-1) \rangle & x<0,\ y<0,\ z>0 \\ \langle 0-(-1), 0-1, 0-1 \rangle & x>0,\ y>0,\ z<0 \\ \langle 0-(-1), 0-(-1), 0-1 \rangle & x<0,\ y>0,\ z<0 \\ \langle 0-(-1), 0-1, 0-(-1) \rangle & x>0,\ y<0,\ z<0 \\ \langle 0-(-1), 0-(-1), 0-(-1) \rangle & x<0,\ y<0,\ z<0 \end{cases}$

$= \begin{cases} \langle -1,-1,-1 \rangle & x>0,\ y>0,\ z>0 \\ \langle -1,1,-1 \rangle & x<0,\ y>0,\ z>0 \\ \langle -1,-1,1 \rangle & x>0,\ y<0,\ z>0 \\ \langle -1,1,1 \rangle & x<0,\ y<0,\ z>0 \\ \langle 1,-1,-1 \rangle & x>0,\ y>0,\ z<0 \\ \langle 1,1,-1 \rangle & x<0,\ y>0,\ z<0 \\ \langle 1,-1,1 \rangle & x>0,\ y<0,\ z<0 \\ \langle 1,1,1 \rangle & x<0,\ y<0,\ z<0 \end{cases}$

(d) $\operatorname{div}(\vec{\mathbf{G}}) = \frac{\partial}{\partial x}(|y|) + \frac{\partial}{\partial y}(|z|) + \frac{\partial}{\partial z}(|x|) = 0+0+0 = 0$

Exercise 2.5.4

Find some vector field $\vec{\mathbf{F}}$ with $\operatorname{curl}(\vec{\mathbf{F}}) = \langle 1,2,3 \rangle$

Answer

We need $\vec{\mathbf{F}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ to satisfy: $\begin{cases} R_y - Q_z = 1 \\ P_z - R_x = 2 \\ Q_x - P_y = 3 \end{cases}$

Pick $R_y = 1$ and $Q_z = 0$ so that $R(x,y,z) = \int R_y\, dy = \int 1\, dy = y + r(x,z)$ and $Q(x,y,z) = q(x,y)$

Then our system of equations above becomes: $\begin{cases} P_z - r_x(x,z) = 2 \\ q_x(x,y) - P_y = 3 \end{cases}$

Letting $-r_x(x,z) = 0$ we can see that $r(x,z) = \hat{r}(z)$, and lock in $R(x,y,z) = y + \hat{r}(z)$

This forces $P_z = 2$ so we have $P(x,y,z) = \int P_z\, dz = \int 2\, dz = 2z + p(x,y)$

This turns our final equation $q_x(x,y) - P_y = 3$ into $q_x(x,y) - p_y(x,y) = 3$.

This is simplest when one of them is zero, so $p_y(x,y) = 0 \longrightarrow p(x,y) = \hat{p}(x)$

This allows us to lock in $P(x,y,z) = 2z + \hat{p}(x)$

Which forces $q_x(x,y) = 3$ and gives $q(x,y) = \int q_x\,dx = \int 3\,dx = 3x + \hat{q}(y)$

This allows us to lock in $Q(x,y,z) = 3x + \hat{q}(y)$, so

Therefore one family of solutions is $\vec{\mathbf{F}} = \langle 2z + \hat{p}(x), 3x + \hat{q}(y), y + \hat{r}(z)\rangle$

This gives infinitely many (but not all such) vector fields $\vec{\mathbf{F}}$, but for one example use $\hat{p}(x) = 0, \hat{q}(y) = 0, \hat{r}(z) = 0$:

$$\vec{\mathbf{F}}_1 = \langle 2z, 3x, y\rangle \longrightarrow \text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & y \end{vmatrix} = \langle 1-0, 2-0, 3-0\rangle = \langle 1,2,3\rangle \text{ as desired.}$$

Exercise 2.5.5

Find two different vector fields $\vec{\mathbf{F}}$ with $\text{curl}(\vec{\mathbf{F}}) = \langle x, y, -2z\rangle$

Answer

We need $\vec{\mathbf{F}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ to satisfy: $\begin{cases} R_y - Q_z = x \\ P_z - R_x = y \\ Q_x - P_y = -2z \end{cases}$

Pick $R_y = x$ and $Q_z = 0$ so that $R(x,y,z) = \int R_y\,dy = \int x\,dy = xy + r(x,z)$ and $Q(x,y,z) = q(x,y)$

Then our system of equations above becomes: $\begin{cases} P_z - (y + r_x(x,z)) = y \\ q_x(x,y) - P_y = -2z \end{cases} \longrightarrow \begin{cases} P_z - r_x(x,z) = 2y \\ q_x(x,y) - P_y = -2z \end{cases}$

Letting $-r_x(x,z) = 0$ we can see that $r(x,z) = \hat{r}(z)$, and lock in $R(x,y,z) = xy + \hat{r}(z)$

This forces $P_z = 2y$ so we have $P(x,y,z) = \int P_z\,dz = \int 2y\,dz = 2yz + p(x,y)$

This turns our final equation $q_x(x,y) - P_y = -2z$ into $q_x(x,y) = p_y(x,y)$.

This is simplest when both are zero, so $p(x,y) = \hat{p}(x)$ and $q(x,y) = \hat{q}(y)$

This allows us to lock in $P(x,y,z) = 2yz + \hat{p}(x)$ and $Q(x,y,z) = \hat{q}(y)$, so

Therefore one family of solutions is $\vec{\mathbf{F}} = \langle 2yz + \hat{p}(x), \hat{q}(y), xy + \hat{r}(z)\rangle$

This gives infinitely many vector fields $\vec{\mathbf{F}}$, but for one example use $\hat{p}(x) = 0, \hat{q}(y) = 0, \hat{r}(z) = 0$:

$$\vec{\mathbf{F}}_1 = \langle 2yz, 0, xy\rangle \longrightarrow \text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & 0 & xy \end{vmatrix} = \langle x-0, 2y-y, 0-2z\rangle = \langle x, y, -2z\rangle \text{ as desired.}$$

If we instead started off by picking $R_y = 0$ and $Q_z = -x$, then $Q(x,y,z) = \int Q_z\,dz = \int -x\,dz = -xz + q(x,y)$ and $R(x,y,z) = r(x,z)$

Then our system of equations above becomes: $\begin{cases} P_z - r_x(x,z) = y \\ (q_x(x,y) - z) - P_y = -2z \end{cases} \longrightarrow \begin{cases} P_z - r_x(x,z) = y \\ q_x(x,y) - P_y = -z \end{cases}$

Letting $-r_x(x,z) = 0$ we can see that $r(x,z) = \hat{r}(z)$, and lock in $R(x,y,z) = \hat{r}(z)$

This forces $P_z = y$ so we have $P(x,y,z) = \int P_z\,dz = \int y\,dz = yz + p(x,y)$

This turns our final equation $q_x(x,y) - P_y = -z$ into $q_x(x,y) = p_y(x,y)$.

This is simplest when both are zero, so $p(x,y) = \hat{p}(x)$ and $q(x,y) = \hat{q}(y)$

This allows us to lock in $P(x,y,z) = yz + \hat{p}(x)$ and $Q(x,y,z) = -xz + \hat{q}(y)$, so

Therefore one family of solutions is $\vec{\mathbf{F}} = \langle yz + \hat{p}(x), -xz + \hat{q}(y), \hat{r}(z)\rangle$

This gives infinitely many *more* vector fields $\vec{\mathbf{F}}$, but another example use $\hat{p}(x)=0, \hat{q}(y)=0, \hat{r}(z)=0$:

$$\vec{\mathbf{F}}_2 = \langle yz, -xz, 0\rangle \longrightarrow \operatorname{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & -xz & 0 \end{vmatrix} = \langle 0-(-x), y-0, (-z)-z\rangle = \langle x, y, -2z\rangle \text{ as desired.}$$

Notice that there are infinitely many other such solutions, but they all differ by a vector field with curl zero:

$$\operatorname{curl}\left(\vec{\mathbf{F}}_1 - \vec{\mathbf{F}}_2\right) = \operatorname{curl}\left(\langle yz, xz, xy\rangle\right) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \langle x-x, y-(-y), z-z\rangle = \vec{\mathbf{0}}$$

Exercise 2.5.6

Find three different vector fields $\vec{\mathbf{H}}$ with $\operatorname{div}(\vec{\mathbf{H}}) = xyz$

Answer

There are again infinitely many ways to find such an $\vec{\mathbf{H}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)$. Here are three simple ones:

Setting $P_x = xyz$ gives $P(x,y,z) = \displaystyle\int xyz\,dx = \frac{1}{2}x^2yz + p(y,z)$

Letting $Q_y = 0$ and $R_z = 0$ yield $Q(x,y,z) = q(x,z)$ and $R(x,y,z) = r(x,y)$

Thus $\vec{\mathbf{H}}_1 = \left\langle \frac{1}{2}x^2yz + p(y,z), q(x,z), r(x,y)\right\rangle$ is one such answer.

Setting $Q_y = xyz$ gives $Q(x,y,z) = \displaystyle\int xyz\,dy = \frac{1}{2}xy^2z + q(x,z)$

Letting $P_x = 0$ and $R_z = 0$ yield $P(x,y,z) = p(y,z)$ and $R(x,y,z) = r(x,y)$

Thus $\vec{\mathbf{H}}_2 = \left\langle p(y,z), \frac{1}{2}xy^2z + q(x,z), r(x,y)\right\rangle$ is another such answer.

Setting $R_z = xyz$ gives $R(x,y,z) = \displaystyle\int xyz\,dz = \frac{1}{2}xyz^2 + r(x,y)$

Letting $P_x = 0$ and $Q_y = 0$ yield $P(x,y,z) = p(y,z)$ and $Q(x,y,z) = q(x,z)$

Thus $\vec{\mathbf{H}}_3 = \left\langle p(y,z), q(x,z), \frac{1}{2}xyz^2 + r(x,y)\right\rangle$ is a third such answer.

This is not an exhaustive list!

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2.6: Multivariable Extrema

Learning Objectives

- Use partial derivatives to locate critical points for a function of two variables.
- Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.
- Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.
- Use the method of Lagrange multipliers to solve optimization problems with one constraint.
- Use the method of Lagrange multipliers to solve optimization problems with two constraints.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

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2.6: Maxima/Minima Problems

For functions of a single variable, we defined critical points as the values of the variable at which the function's derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition: Critical Points

Let $z = f(x, y)$ be a function of two variables that is differentiable on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a *critical point (or critical value)* of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 2.6.1: Finding Critical Points

Find the critical points of each of the following functions:

(a) $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$

(b) $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution

(a) First, we define the domain of the function $f(x, y)$:

$$D = \{(x, y) \ : \ 4y^2 - 9x^2 + 24y + 36x + 36 \geq 0\} = \left\{(x, y) \ : \ \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} \leq 1\right\}$$

Now we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$f_x(x, y) = \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-\frac{1}{2}} = \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}$$

$$f_y(x, y) = \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-\frac{1}{2}} = \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}$$

Both derivatives will be zero when the numerators are equal to zero:

$$\begin{cases} -9x + 18 = 0 \\ 4y + 12 = 0 \end{cases}$$

Therefore, $x = 2$ and $y = -3$, and $(2, -3)$ is in D, so it is a critical point of f.

And both derivatives will not exist when their common denominator is equal to zero:

$$4y^2 - 9x^2 + 24y + 36x + 36 = 0$$

This equation represents the hyperbola on the edge of the domain D. Any points on this hyperbola are critical points.

Thus, the critical points of the function f are $\boxed{(2, -3) \text{ and all points on the hyperbola } \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1}$

(b) The domain of $g(x, y)$ is all of $\mathbb{R}^2$. Now, we calculate $g_x(x, y)$ and $g_y(x, y)$:

$$g_x(x, y) = 2x + 2y + 4 \quad \text{and} \quad g_y(x, y) = 2x - 8y - 6$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$\begin{cases} 2x + 2y + 4 &= 0 \\ 2x - 8y - 6 &= 0 \end{cases}$$

Subtracting the second equation from the first gives $10y + 10 = 0$, so $y = -1$. Substituting this into the first equation gives $2x + 2(-1) + 4 = 0$, so $x = -1$.

Therefore $\boxed{(-1, -1)}$ is a critical point of g. There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition: Global and Local Extrema

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0).

The function f has a ***local maximum*** at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a ***global maximum*** (also called an absolute maximum) at (x_0, y_0).

The function f has a ***local minimum*** at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a ***global minimum*** (also called an absolute minimum) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a ***local extremum***.

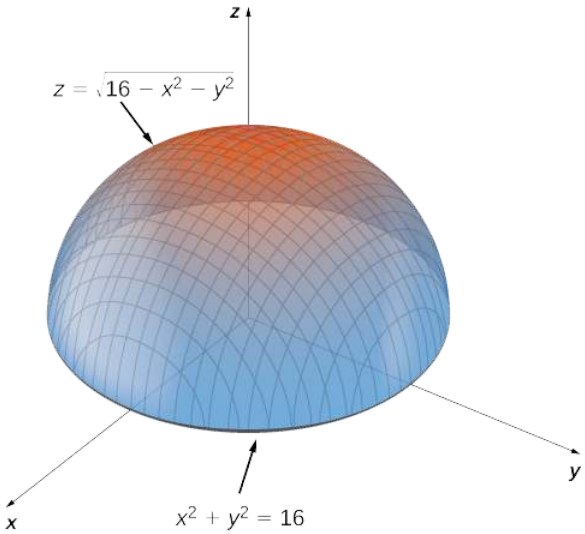


Figure 2.6.1: The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable.

Theorem 2.6.1: Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

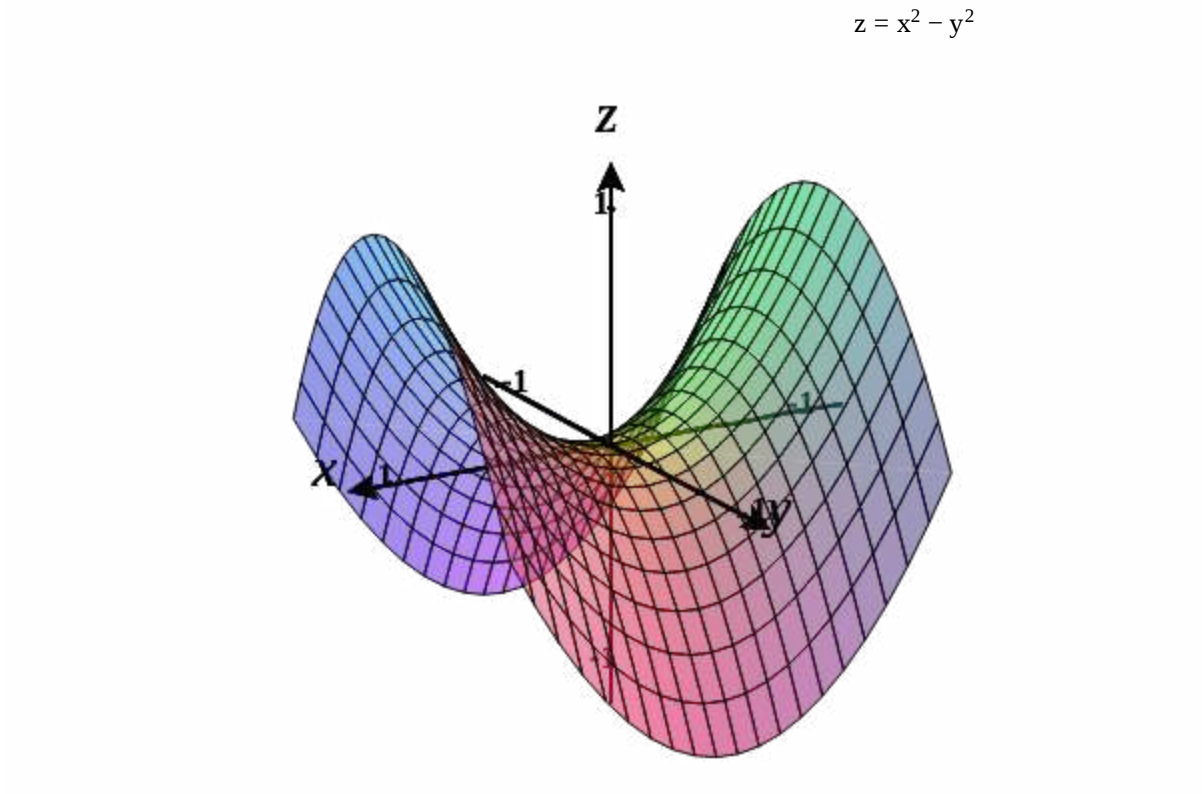


Figure 2.6.2: Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.

In this graph, the origin is a saddle point. This is because the first partial derivatives of $f(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore, the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition: Saddle Point

Given the function $z = f(x, y)$, the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Theorem 2.6.2: Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2$$

Then:

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

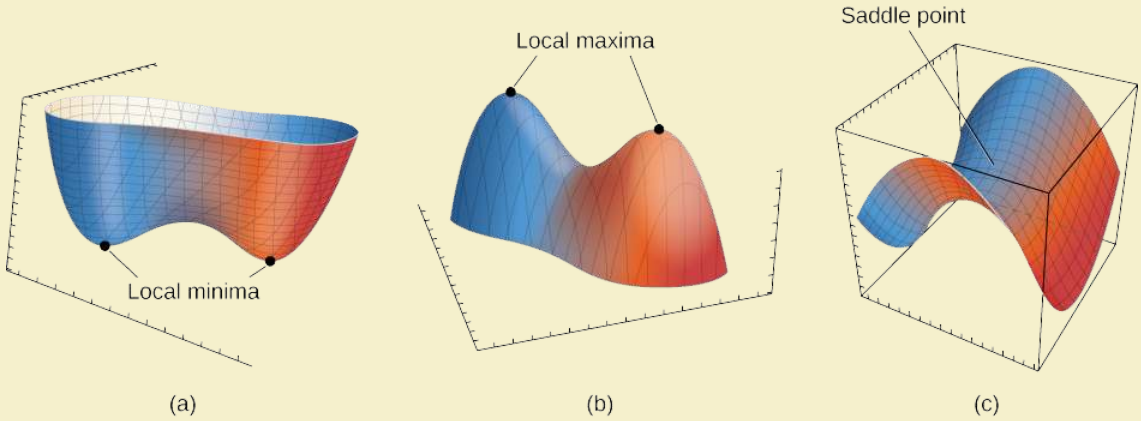


Figure 2.6.3: The second derivative test can often determine whether a function of two variables has local minima (a), local maxima (b), or a saddle point (c).

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$ for each critical point of f.
3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

Example 2.6.2: Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

(a) $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$

(b) $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution

(a) First we find the critical points of f. We calculate $f_x(x, y)$ and $f_y(x, y)$, then set each of them equal to zero:

$$f_x(x, y) = 8x + 8 \quad \text{and} \quad f_y(x, y) = 18y - 36$$

Setting them equal to zero yields the system of equations

$$\begin{cases} 8x + 8 = 0 \\ 18y - 36 = 0 \end{cases}$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is a critical point of f.

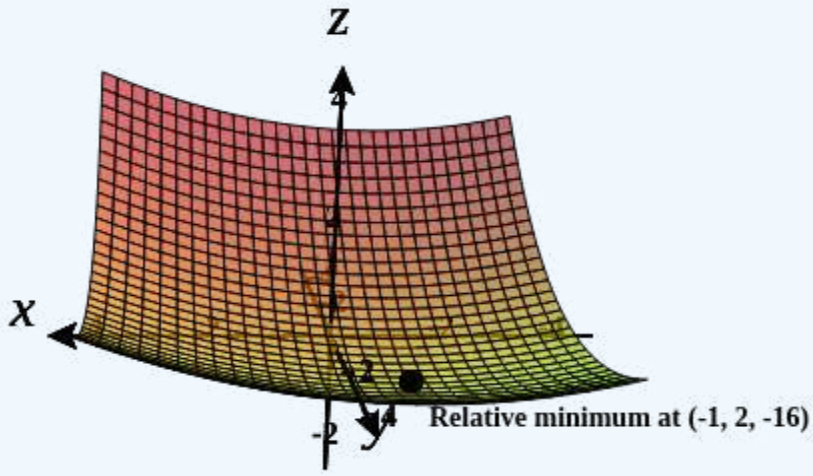
Next we calculate D, for which we will need the second partial derivatives of f:

$$f_{xx}(x,y) = 8, \quad f_{xy}(x,y) = 0, \quad \text{and} \quad f_{yy}(x,y) = 18$$

Therefore, $D = f_{xx}(-1,2)f_{yy}(-1,2) - \left(f_{xy}(-1,2)\right)^2 = (8)(18) - (0)^2 = 144$

Since $D > 0$ and $f_{xx}(-1,2) > 0$, we know \($\boxed{f \text{ has a local minimum at } (-1,2)}$ as shown in the following figure.

$z = 4x^2 + 9y^2 + 8x - 36y + 24$



(b) First we calculate $g_x(x,y)$ and $g_y(x,y)$:

$$g_x(x,y) = x^2 + 2y - 6 \quad \text{and} \quad g_y(x,y) = 2y + 2x - 3$$

Setting them equal to zero yields the system of equations

$$\begin{cases} x^2 + 2y - 6 = 0 \\ 2y + 2x - 3 = 0 \end{cases}$$

To solve this system, first solve the second equation for y. This gives $y = \dfrac{3-2x}{2}$. Substituting this into the first equation gives

$$x^2 + 3 - 2x - 6 = 0 \longrightarrow x^2 - 2x - 3 = 0 \longrightarrow (x-3)(x+1) = 0$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \dfrac{3-2x}{2}$ yields the critical points $\left(-1, \dfrac{5}{2}\right)$ and $\left(3, -\dfrac{3}{2}\right)$.

We next find the value of D, by calculating the second partial derivatives of g:

$$g_{xx}(x,y) = 2x, \quad g_{xy}(x,y) = 2, \quad \text{and} \quad g_{yy}(x,y) = 2$$

Therefore $D(x,y) = g_{xx}(x,y)g_{yy}(x,y) - \left(g_{xy}(x,y)\right)^2 = (2x)(2) - 2^2 = 4x - 4$

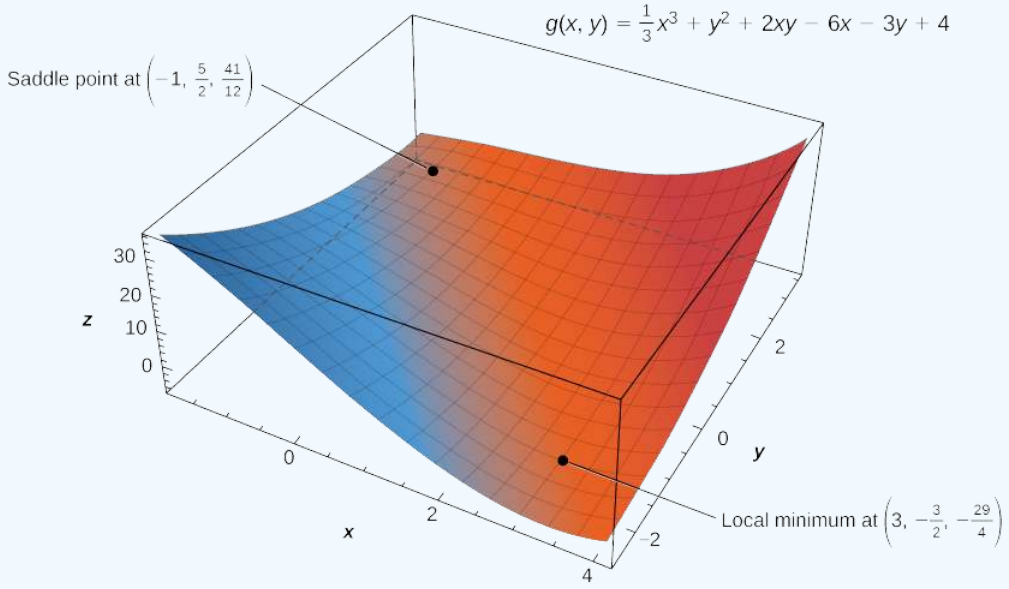
Next, we substitute each critical point into this formula:

$$D\left(-1, \frac{5}{2}\right) = 4(-1) - 4 = -8 \quad \text{and} \quad D\left(3, -\frac{3}{2}\right) = 4(3) - 4 = 8$$

Since $D\left(-1, \frac{5}{2}\right) < 0$ the point $\boxed{\left(-1, \frac{5}{2}, g\left(-1, \frac{5}{2}\right)\right) \text{ is a saddle point}}$

Since $D\left(3, -\frac{3}{2}\right) > 0$ and $g_{yy}\left(3, -\frac{3}{2}\right) > 0$ the point $\boxed{\left(3, -\frac{3}{2}, g\left(3, -\frac{3}{2}\right)\right) \text{ is a local minimum}}$

Both of these are shown in the graph of $z = g(x, y)$ below:



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2.6: Absolute Extrema

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is bounded if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

Theorem 2.6.3: Extreme Value Theorem

A continuous function $f(x,y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

Finding Extreme Values of a Function of Two Variables

Assume $z = f(x,y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in D.
2. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0, y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

Let $z = f(x,y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in the next Example below. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x,y) = c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary (which is the next subsection).

Illustration 2.6.1: Finding Absolute Extrema

We will find the absolute extrema of $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$

We start by finding the critical points of f on its domain. We first calculate $f_x(x,y)$ and $f_y(x,y)$:

$$f_x(x,y) = 2x - 2y - 4 \quad \text{and} \quad f_y(x,y) = -2x + 8y - 2$$

Setting them equal to zero yields the system of equations

$$\begin{cases} 2x - 2y - 4 = 0 \\ -2x + 8y - 2 = 0 \end{cases}$$

Substitution tells us the solution to this system is $x = 3$ and $y = 1$. Therefore $(3,1)$ is a critical point of f. Calculating $f(3,1)$ gives $f(3,1) = 17$

Now we find the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:

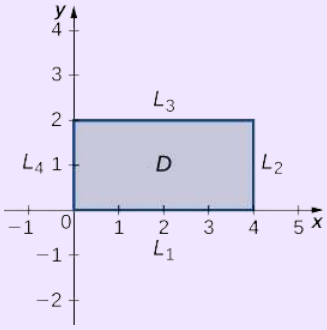


Figure 2.6.4: Graph of the domain of the function $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$

L_1 is the line segment connecting $(0,0)$ and $(4,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 4$. Define $g_1(t) = f\big(x(t), y(t)\big)$. This gives $g_1(t) = f(t,0) = t^2 - 4t + 24$. Differentiating g_1 leads to $g_1{}'(t) = 2t - 4$. Therefore, g_1 has a critical value at $t = 2$, and the endpoints of this single-variable function's domain $[0,4]$ are $t = 0, 4$.

$$g_1(0) = f(0,0) = 24, \quad g_1(2) = f(2,0) = 20, \quad \text{and} \quad g_1(4) = f(4,0) = 24$$

L_2 is the line segment connecting $(4,0)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \le t \le 2$. Define $g_2(t) = f(4,t) = 4t^2 - 10t + 24$ so that $g_2{}'(t) = 8t - 10$. Thus g_2 has a critical value at $t = \dfrac{5}{4}$, and the endpoints are $t = 0, 2$.

$$g_2(0) = f(4,0) = 24, \quad g_2\left(\frac{5}{4}\right) = f\left(4, \frac{5}{4}\right) = \frac{71}{4} = 17.75, \quad \text{and} \quad g_2(2) = f(4,2) = 20$$

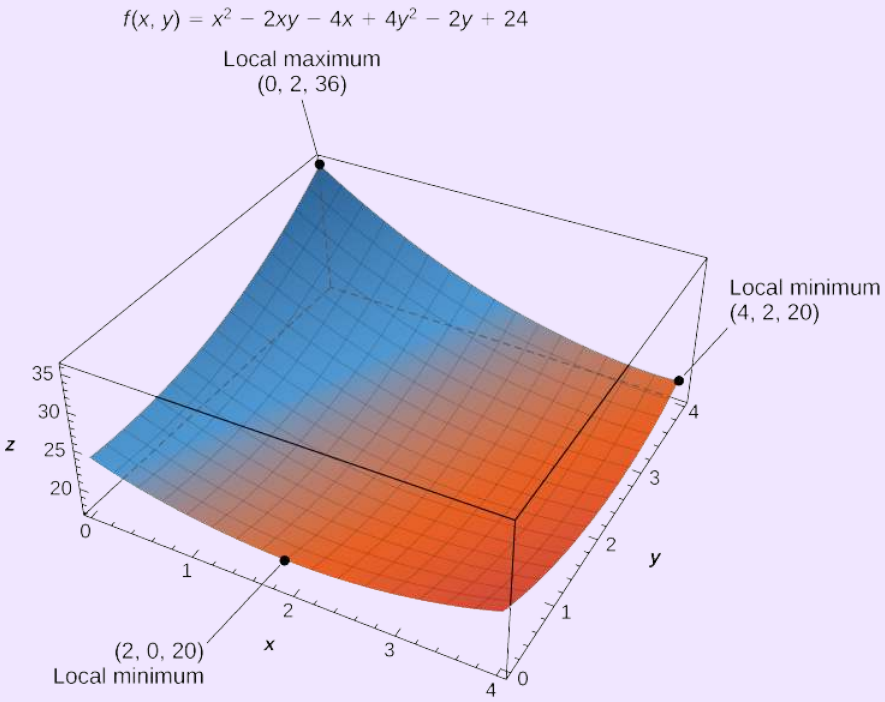
L_3 is the line segment connecting $(0,2)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = t, y(t) = 2$ for $0 \le t \le 4$. Define $g_3(t) = f(t,2) = t^2 - 8t + 36$ so that $g_3'(t) = 2t - 8$. Thus g_3 has a critical value at $t = 4$, and the endpoints are $t = 0, 4$.

$$g_3(0) = f(0,2) = 36 \quad \text{and} \quad g_3(4) = f(4,2) = 20$$

L_4 is the line segment connecting $(0,0)$ and $(0,2)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 2$. Define $g_4(t) = f(0,t) = 4t^2 - 2t + 24$ so that $g_4'(t) = 8t - 2$. Thus g_4 has a critical value at $t = \dfrac{1}{4}$, and the endpoints are $t = 0, 2$.

$$g_4(0) = f(0,0) = 24, \quad g_4\left(\frac{1}{4}\right) = f\left(0, \frac{1}{4}\right) = \frac{95}{4} = 23.75, \quad \text{and} \quad g_4(2) = f(0,2) = 36$$

Comparing all of these extrema *on the boundary* we see that the maximum value is 36, which occurs at $(0,2)$, and the minimum value is 20, which occurs at both $(4,2)$ and $(2,0)$ as shown in the following figure.



Therefore our final answer is that the absolute maximum of $f(x,y)$ on D is at the boundary point $(0,2,36)$ and the absolute minimum of $f(x,y)$ on D is at the interior critical point $(3,1,17)$.

Illustration 2.6.2: Finding Absolute Extrema

We will find the absolute extrema of $f(x,y) = y^3 - 24y - 3x^2y$ on the domain defined by $x^2 + y^2 \leq 4$

We start by finding the critical points of f on its domain. We first calculate $f_x(x,y)$ and $f_y(x,y)$:

$$f_x(x,y) = -6xy \quad \text{and} \quad f_y(x,y) = 3y^2 - 24 - 3x^2$$

Setting them equal to zero yields the system of equations

$$\begin{cases} -6xy = 0 \\ 3y^2 - 24 - 3x^2 = 0 \end{cases}$$

The first equation says that either $x = 0$ or $y = 0$. If $x = 0$, the second equation reduces to $y^2 = 8$, so we have two critical points in $(0, \pm 2\sqrt{2})$, but these are not in the domain so they do not apply in this problem. If $y = 0$, the second equation has no solutions, so there are no additional critical points inside of the domain.

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of a circle of radius 2, which can be parameterized using the functions $x(t) = 2\cos t$, $y(t) = 2\sin t$ for $0 \leq t \leq 2\pi$. Define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) = f\big(2\cos(t), 2\sin(t)\big) &= (2\sin(t))^3 - 24(2\sin(t)) - 3(2\cos(t))^2(2\sin(t)) \\ &= 8\sin^3(t) - 48\sin(t) - 24\cos^2(t)\sin(t) \end{aligned}$$

So the derivative of $g(t)$ is given by:

$$\begin{aligned} g'(t) &= 24\sin^2(t)\cos(t) - 48\cos(t) + 48\cos(t)\sin^2(t) - 24\cos^3(t) \\ &= -48\cos(t) + 72\cos(t)\sin^2(t) - 24\cos^3(t) \\ &= -48\cos(t) + 72\cos(t)(1 - \cos^2(t)) - 24\cos^3(t) \\ &= -96\cos^3(t) + 24\cos(t) \\ &= -24\cos(t)(2\cos(t) - 1)(2\cos(t) + 1) \end{aligned}$$

This equation has six solutions over the interval $0 \leq t \leq 2\pi$: $t = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

Plugging them each and our endpoints into $g(t)$ gives us our possible extrema along the boundary:

$$g(0) = f(2,0) = 0, \quad g\left(\frac{\pi}{3}\right) = f(1,\sqrt{3}) = -24\sqrt{3}, \quad g\left(\frac{\pi}{2}\right) = f(0,2) = -40,$$
$$g\left(\frac{2\pi}{3}\right) = f(-1,\sqrt{3}) = -24\sqrt{3}$$

$$g\left(\frac{4\pi}{3}\right) = f(-1,-\sqrt{3}) = 24\sqrt{3}, \quad g\left(\frac{3\pi}{2}\right) = f(0,-2) = 40,$$
$$g\left(\frac{5\pi}{3}\right) = f(1,-\sqrt{3}) = 24\sqrt{3}, \quad g(2\pi) = f(2,0) = 0$$

Clearly on the boundary we have minimums at $\left(1,\sqrt{3},-24\sqrt{3}\right)$ and $\left(-1,\sqrt{3},-24\sqrt{3}\right)$ and maximums at $\left(-1,-\sqrt{3},24\sqrt{3}\right)$ and $\left(1,-\sqrt{3},24\sqrt{3}\right)$.

Since are no interior critical points, they are also our absolute extrema for $f(x,y)$ on this domain.

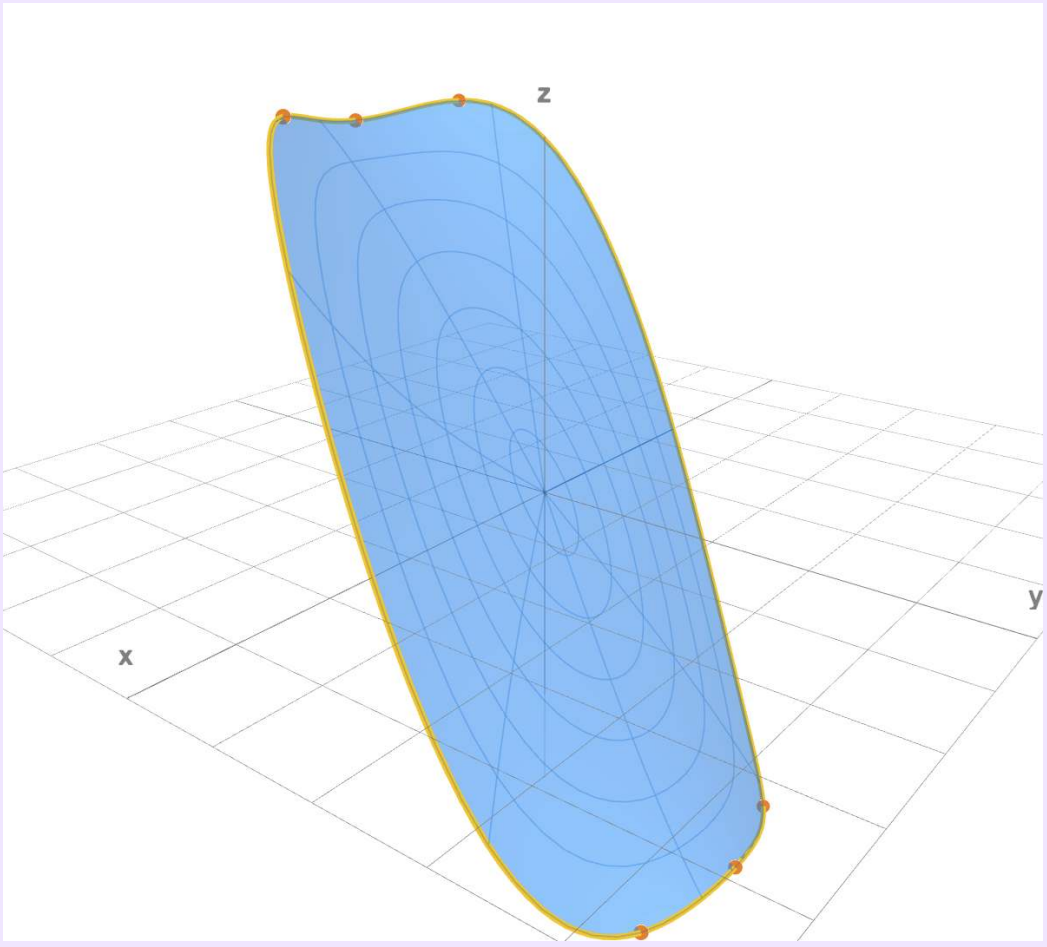


Illustration 2.6.3: Finding Absolute Extrema

We will find the absolute extrema of $f(x,y) = e^{-x^2-4y^2}$ on the domain defined by $\frac{x^2}{16} + \frac{y^2}{4} \leq 1$

We start by finding the critical points of f on its domain. We first calculate $f_x(x,y)$ and $f_y(x,y)$:

$$f_x(x,y) = -2xe^{-x^2-4y^2} \quad \text{and} \quad f_y(x,y) = -8ye^{-x^2-4y^2}$$

Setting them equal to zero yields the system of equations

$$\begin{cases} -2xe^{-x^2-4y^2} = 0 \\ -8ye^{-x^2-4y^2} = 0 \end{cases}$$

The only solution to this system is the point $(0,0)$. Calculating $f(0,0)$, we get

$$f(0,0) = e^{-0^2-4\cdot 0^2} = e^0 = 1$$

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of an ellipse centered at the origin which can be parameterized using the functions $x(t) = 4\cos t,\ y(t) = 2\sin t$ for $0 \le t \le 2\pi$. Define $g(t) = f\big(x(t), y(t)\big)$:

$$g(t) = f\big(4\cos(t), 2\sin(t)\big) = e^{-(4\cos(t))^2-4(2\sin(t))^2} = e^{-16\cos^2(t)-16\sin^2(t)} = e^{-16}$$

So the function $g(t)$ is constant on this boundary, and all points yield the same value: $z = e^{-16}$

This makes the interior critical point the absolute maximum of $f(x,y)$ at $(0,0,1)$ and the absolute minimum of $z = e^{-16}$ occurs at all points on the edge of the domain.

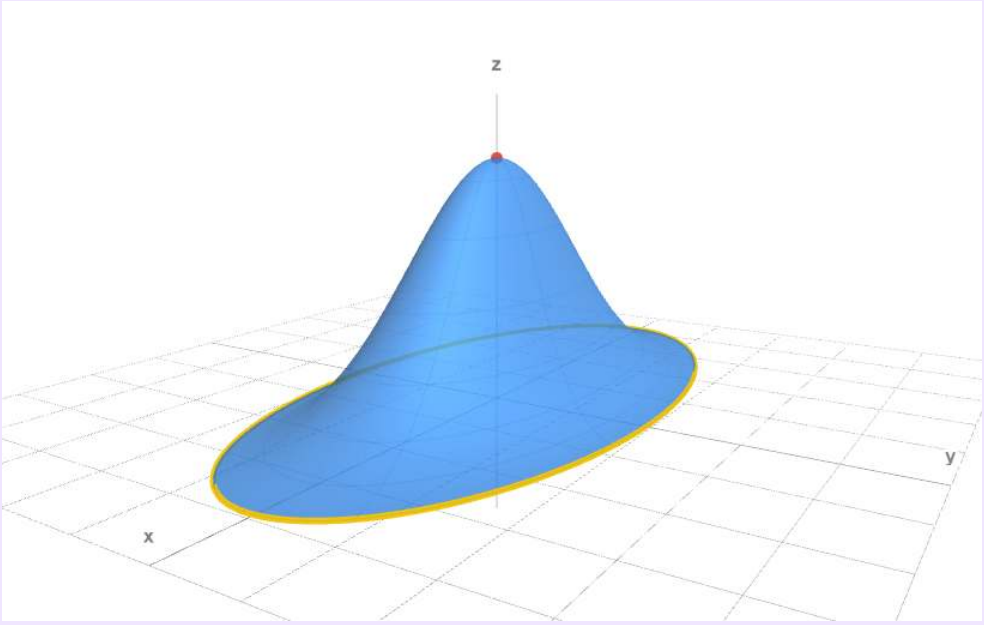


Illustration 2.6.4: Profitable Golf Balls

Pro-T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50,000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.

We start by finding the critical points of f on its domain. We first calculate $f_x(x,y)$ and $f_y(x,y)$:

$$f_x(x,y) = 48 - 2x - 2y \quad \text{and} \quad f_y(x,y) = 96 - 2x - 18y$$

Setting them equal to zero yields the system of equations

$$\begin{cases} 48 - 2x - 2y = 0 \\ 96 - 2x - 18y = 0 \end{cases}$$

Through substitution, we see the solution to this system is $x = 21$ and $y = 3$. Therefore $(21, 3)$ is a critical point of f. Calculating $f(21, 3)$ gives:

$$f(21, 3) = 48(21) + 96(3) - 21^2 - 2(21)(3) - 9(3)^2 = 648$$

The domain of this function is $0 \le x \le 50$ and $0 \le y \le 25$ as shown in the following graph.

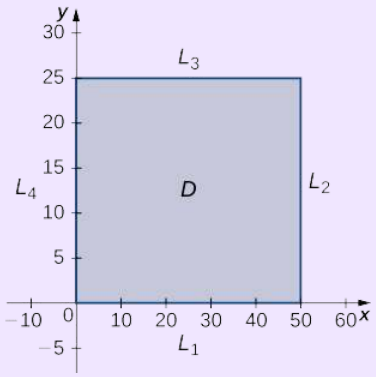


Figure 2.6.7: Graph of the domain of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$

L_1 is the line segment connecting $(0, 0)$ and $(50, 0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 50$.

$$g_1(t) = f(t, 0) = 48t + 96(0) - t^2 - 2(t)(0) - 9(0)^2 = 48t - t^2$$

Finding $g_1'(t) = 48 - 2t$ yields the critical point $t = 24$, and the endpoints of its domain are $t = 0, 50$. Plugging these into $g(t)$ gives us:

$$g_1(0) = f(0, 0) = 0, \quad g_1(24) = f(24, 0) = 576, \quad \text{and} \quad g_1(50) = f(50, 0) = -100$$

L_2 is the line segment connecting $(50, 0)$ and $(50, 25)$, and it can be parameterized by the equations $x(t) = 50, y(t) = t$ for $0 \le t \le 25$.

$$g_2(t) = f(50, t) = 48(50) + 96t - 50^2 - 2(50)t - 9t^2 = -9t^2 - 4t - 100$$

Finding $g_2'(t) = -18t - 4$ yields a critical point at $t = -\frac{2}{9}$, which is not in the domain. We plug in the endpoints $t = 0, 25$:

$$g_2(0) = f(50, 0) = -100 \quad \text{and} \quad g_2(25) = f(50, 25) = -5825$$

L_3 is the line segment connecting $(0, 25)$ and $(50, 25)$, and it can be parameterized by the equations $x(t) = t, y(t) = 25$ for $0 \le t \le 50$.

$$g_3(t) = f(t, 25) = 48t + 96(25) - t^2 - 2t(25) - 9(25^2) = -t^2 - 2t - 3225$$

Finding $g_3'(t) = -2t - 2$ yields a critical point at $t = -1$, which is not in the domain. We plug in the endpoints $t = 0, 50$:

$$g_3(0) = f(0, 25) = -3225 \quad \text{and} \quad g_3(50) = f(50, 25) = -5825$$

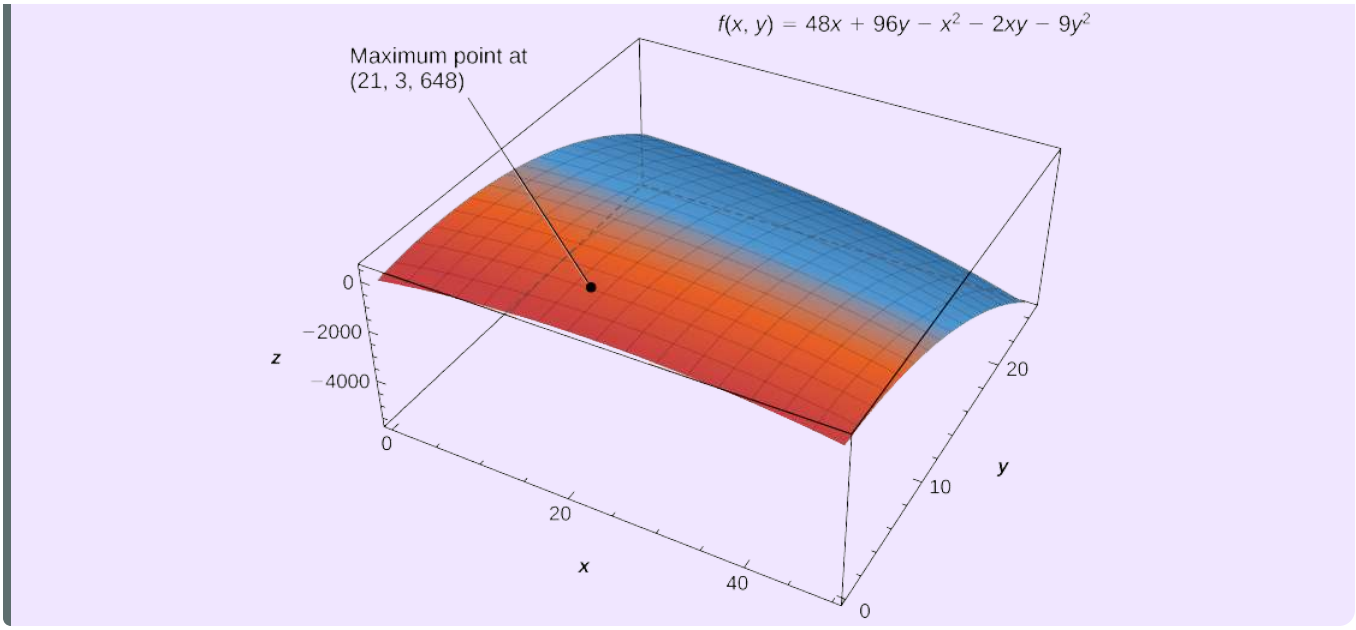
L_4 is the line segment connecting $(0, 0)$ to $(0, 25)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 25$.

$$g_4(t) = f(0, t) = 48(0) + 96t - (0)^2 - 2(0)t - 9t^2 = 96t - 9t^2$$

Finding $g_4'(t) = 96 - 18t$ yields a critical point at $t = \dfrac{16}{3}$, and the endpoints of its domain are $t = 0, 25$. Plugging these into $g(t)$ gives us:

$$g_4(0) = f(0, 0) = 0, \quad g_4\left(\frac{16}{3}\right) = f\left(0, \frac{16}{3}\right) = 256, \quad \text{and} \quad g_4(25) = f(0, 25) = -3225$$

The maximum value occurs at the interior critical point at $f(21, 3) = 648$. Therefore, a maximum profit of $\$648,000$ is realized when $21,000$ golf balls are sold per month and 3 hours of advertising are purchased per month as shown in the following figure.



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2.6: Lagrange Multipliers

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

In the example, an applied situation was explored involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a single budgetary constraint, such as $20x + 4y \leq 216$, that took into account both the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is still to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint and the corresponding profit function

$$f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$$

is an example of an **optimization problem**, and the function $f(x, y)$ is called the **objective function**. A graph of various level curves of the function $f(x, y)$ follows.

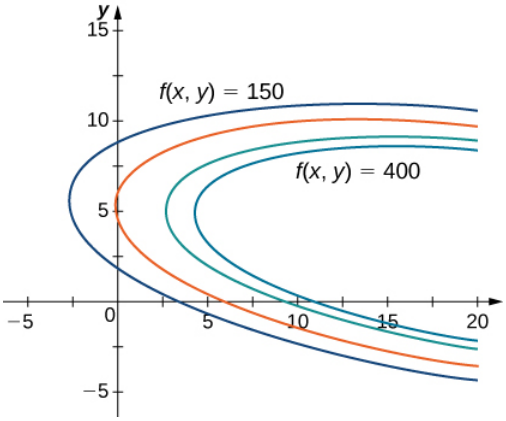


Figure 2.6.5: Graph showing level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 400.

In the figure above, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there were no restrictions on the number of golf balls the company could produce or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x + 4y \leq 216$. To see how this constraint interacts with the profit function, the figure below shows the graph of the line $20x + 4y = 216$ superimposed on the previous graph.

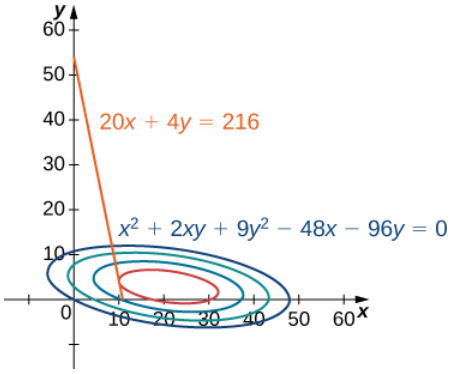


Figure 2.6.6: Graph of level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 395. The line graphed is the constraint function.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the orange line. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around 395, when x and y are both just less than 5. We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers.**

Theorem 2.6.4: Method of Lagrange Multipliers with One Constraint

Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y)=0$. Suppose that f, when restricted to points on the curve $g(x,y)=k$ for some constant k, has a local extremum at the point (x_0,y_0) and that $\nabla g(x_0,y_0)\neq 0$. Then there is a number λ called a **Lagrange multiplier,** for which

$$\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$$

Proof

Assume that a constrained extremum occurs at the point (x_0,y_0). Furthermore, we assume that the equation $g(x,y)=0$ can be smoothly parameterized as

$$\vec{\mathbf{r}}(s)=\langle x(s),y(s)\rangle$$

where s is an arc length parameter with reference point (x_0,y_0) at $s=0$. Therefore, the quantity $z=f(x(s),y(s))$ has a relative maximum or relative minimum at $s=0$, and this implies that $\dfrac{dz}{ds}=0$ at that point. From the chain rule, when $s=0$ we get:

$$0=\frac{d}{ds}(f(x(s),y(s))=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}=\nabla f(x_0,y_0)\cdot\vec{\mathbf{r}}\,'(0)$$

Similarly, the quantity $g(x(s),y(s))$ is the constant value k. This means that

$$0=\frac{d}{ds}(g(x(s),y(s))=\frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial s}=\nabla g(x_0,y_0)\cdot\vec{\mathbf{r}}\,'(0)$$

If $\nabla f(x_0,y_0)=0$ we are already at a critical point, and so as long as $\nabla g(x_0,y_0)\neq 0$, these two gradients are clearly orthogonal to the same nonzero vector. In $\mathbb{R}^2$ this means they must be parallel, so for some λ we have:

$$\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$$

To apply this Theorem to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x,y)$ and the constraint $g(x,y)=k$.
2. Set up a system of equations using the following template: $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$ and $g(x_0,y_0)=k$
3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

Example 2.6.3: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x,y)=x^2+4y^2-2x+8y$ subject to the constraint $x+2y=7$.

Solution

The objective function is $f(x,y)=x^2+4y^2-2x+8y$ and the constraint is $g(x,y)=x+2y=7$.

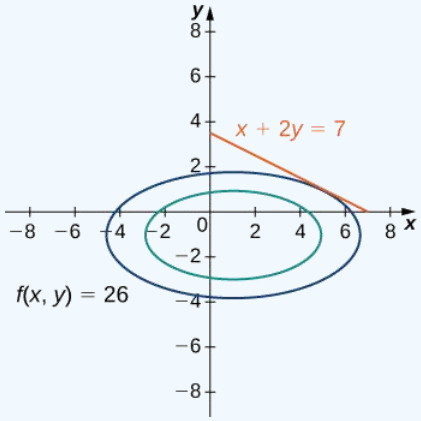


Figure 2.6.3: Graph of level curves of the function $f(x,y) = x^2 + 4y^2 - 2x + 8y$ corresponding to $c = 10$ and 26. The orange graph is the constraint function.

We first must calculate the gradients of both f and g by finding $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$:

$$\nabla f\,(x,y) = \langle 2x - 2, 8y + 8\rangle \quad \text{and} \quad \nabla g\,(x,y) = \langle 1, 2\rangle$$

The equation $\nabla f = \lambda \nabla g$ and $g(x,y) = 7$ yields the system of equations:

$$\begin{cases} 2x - 2 = \lambda(1) \\ 8y + 8 = \lambda(2) \\ x + 2y = 7 \end{cases}$$

This is a linear system of three equations in three variables. We can substitute the first equation into the second to get $8y + 8 = 2(2x - 2)$.
Solving this equation for x gives $x = 2y + 3$, which we can substitute into the third equation and solve:

$$(2y + 3) + 2y = 7 \implies y = 1$$

Since $x = 2y + 3$, this gives $x = 5$.

Now, we evaluate $f(x,y) = x^2 + 4y^2 - 2x + 8y$ at the point $(5,1)$,

$$\boxed{f(5,1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27}$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5,1)$, such as the intercepts of $g(x,y) = 7$, which are $(7,0)$ and $(0,3.5)$.

We get $f(7,0) = 35 > 27$ and $f(0,3.5) = 77 > 27$. So it appears that f has a relative minimum of 27 at $(5,1)$, subject to the given constraint.

Let's now return to the problem posed at the beginning of the section.

Example 2.6.4: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands of golf balls and advertising units is given by $20x + 4y = 216$. Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

The objective function is $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$ and the constraint is $g(x,y) = 5x + y = 54$. This is the reduced version of $20x + 4y = 216$.

We calculate the gradients of both f and g:

$$\nabla f(x,y) = \langle 48 - 2x - 2y, 96 - 2x - 18y \rangle \quad \text{and} \quad \nabla g(x,y) = \langle 5, 1 \rangle$$

The equation $\nabla f = \lambda \nabla g$ and $g(x,y) = 54$ yields the system of equations:

$$\begin{cases} 48 - 2x - 2y = \lambda(5) \\ 96 - 2x - 18y = \lambda(1) \\ 5x + y = 54 \end{cases}$$

We use the left-hand side of the second equation to replace λ in the first equation to get $48 - 2x - 2y = 5(96 - 2x - 18y)$. Solving this equation for x gives $x = 54 - 11y$, which we can substitute into the third equation:

$$5(54 - 11y) + y = 54 \implies y = 4$$

Since $x = 54 - 11y$, this gives $x = 10$.

Now, we evaluate $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$ at the point $(10, 4)$,

$$\boxed{f(10,4) = 48(10) + 96(4) - (10)^2 - 2(10)(4) - 9(4)^2 = 540}$$

To ensure this corresponds to a maximum value on the constraint function, try any other points on the constraint from either side of the point $(10, 4)$, and you will see that $f(x,y) \leq 540$ as expected.

In the case of an objective function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. The method is the same as for the method with a function of two variables.

Example 2.6.5: Lagrange Multipliers with a Three-Variable objective function

Minimize the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$

Solution

The objective function is $f(x,y,z) = x^2 + y^2 + z^2$ and the constraint is $g(x,y,z) = x + y + z = 1$

Next, we calculate $\nabla f(x,y,z)$ and $\nabla g(x,y,z)$:

$$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle \quad \text{and} \quad \nabla g(x,y,z) = \langle 1, 1, 1 \rangle$$

The equation $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 1$ yields the system of equations:

$$\begin{cases} 2x = \lambda(1) \\ 2y = \lambda(1) \\ 2z = \lambda(1) \\ x + y + z = 1 \end{cases}$$

We use the first three equations to see $x = y = z = \dfrac{\lambda}{2}$.

Substituting this into the fourth equation gives: $\dfrac{\lambda}{2} + \dfrac{\lambda}{2} + \dfrac{\lambda}{2} = 1$.

Solving this equation for λ gives $\lambda = \dfrac{2}{3}$, which we can substitute back to get:

$$x = y = z = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

Now, we evaluate $f(x,y,z) = x^2 + y^2 + z^2$ at the point $\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$:

$$\boxed{f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2=\frac{1}{3}}$$

To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. For example,

$$f(1,0,0)=1^2+0^2+0^2=1$$
$$f(0,-2,3)=0^2+(-2)^2+3^2=13$$

Both of these values are greater than $\frac{1}{3}$, so the extremum is a minimum, as desired.

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the objective function, w is a function of three variables:

$$w=f(x,y,z)$$

and it is subject to two constraints:

$$g(x,y,z)=k_1 \text{ and } h(x,y,z)=k_2$$

There are two Lagrange multipliers, λ and γ, and the system of equations stems from

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)+\gamma\nabla h(x_0,y_0,z_0)$$

Illustration 2.6.5: Lagrange Multipliers with Two Constraints

We will find the minimum value of the function $f(x,y,z)=x^2+y^2+z^2$ subject to the constraints $2x+y+2z=9$ and $5x+5y+7z=29$

The objective function is $f(x,y,z)=x^2+y^2+z^2$ and the constraints are $g(x,y,z)=2x+y+2z=9$ and $h(x,y,z)=5x+5y+7z=29$

We then calculate the gradients of f, g, and h:

$$\nabla f(x,y,z)=\langle 2x,2y,2z\rangle,\quad \nabla g(x,y,z)=\langle 2,1,2\rangle,\quad \text{and}\quad \nabla h(x,y,z)=\langle 5,5,7\rangle$$

The equations $\nabla f=\lambda\nabla g+\gamma h$, $g(x,y,z)=9$ and $h(x,y,z)=29$ yield the system of equations:

$$\begin{cases} 2x=\lambda(2)+\gamma(5)\\ 2y=\lambda(1)+\gamma(5)\\ 2z=\lambda(2)+\gamma(7)\\ 2x+y+2z=9\\ 5x+5y+7z=29\end{cases}$$

Dividing the first three equations by 2 each gives us x, y and z in terms of λ and γ. We substitute these into the bottom two equations to get the new system:

$$\begin{cases} 2\left(\frac{2\lambda+5\gamma}{2}\right)+\left(\frac{\lambda+5\gamma}{2}\right)+2\left(\frac{2\lambda+7\gamma}{2}\right)=9\\ 5\left(\frac{2\lambda+5\gamma}{2}\right)+5\left(\frac{\lambda+5\gamma}{2}\right)+7\left(\frac{2\lambda+7\gamma}{2}\right)=29\end{cases}\longrightarrow\begin{cases}9\lambda+29\gamma=18\\29\lambda+99\gamma=58\end{cases}$$

Solving this system can be done with substitution or elimination, and yields $\lambda=2,\gamma=0$, which we can substitute back to get:

$$x=\frac{\lambda+5\gamma}{2}=2,\quad y=\frac{\lambda+5\gamma}{2}=1,\quad z=\frac{2\lambda+7\gamma}{2}=2$$

Now, we evaluate $f(x,y,z)=x^2+y^2+z^2$ at the point $(2,1,2)$,

$$f(2,1,2)=2^2+1^2+2^2=9$$

To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. Like

$$f(5,5,-3) = 5^2 + 5^2 + (-3)^2 = 59$$
$$f(-1,-3,7) = (-1)^2 + (-3)^2 + 7^2 = 59$$

Both of these values are greater than 9, so the extremum is a minimum, as desired.

The method of Lagrange multipliers provides a powerful tool for solving optimization problems subject to constraints, bridging the gap between theoretical calculus and practical applications. By introducing auxiliary variables (Lagrange multipliers), we transform constrained problems into systems of equations that reveal critical points where objective and constraint gradients align. Whether applied to economic models, engineering design, or scientific research, this technique underscores a fundamental principle: optimal solutions often arise precisely where objectives and constraints interact most delicately. As demonstrated in the examples, verifying results by testing nearby points ensures robustness, while extensions to higher dimensions or multiple constraints showcase the method's versatility. Mastery of Lagrange multipliers equips us to tackle real-world problems where resources are limited, but innovation is not.

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2.6: Practice Set

Exercise 2.6.1a

Find and classify all local extrema of the function $f(x,y) = x^2 + 2y^4 + 2xy$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x = 2x + 2y$ and $f_y = 8y^3 + 2x$

Setting these equal to zero gives the system of equations: $\begin{cases} 2x + 2y = 0 \\ 8y^3 + 2x = 0 \end{cases} \longrightarrow \begin{cases} x + y = 0 \\ 4y^3 + x = 0 \end{cases}$

These are satisfied at the critical points $(0,0)$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$, and $\left(-\frac{1}{2}, \frac{1}{2}\right)$

The second partials of f are $\begin{aligned} f_{xx} &= 2 \\ f_{xy} &= 2 \\ f_{yy} &= 24y^2 \end{aligned}$

Therefore we have

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 2 \cdot 24y^2 - (2)^2 = 4(12y^2 - 1)$$

$D(0,0) = 4 \cdot (-1) < 0$

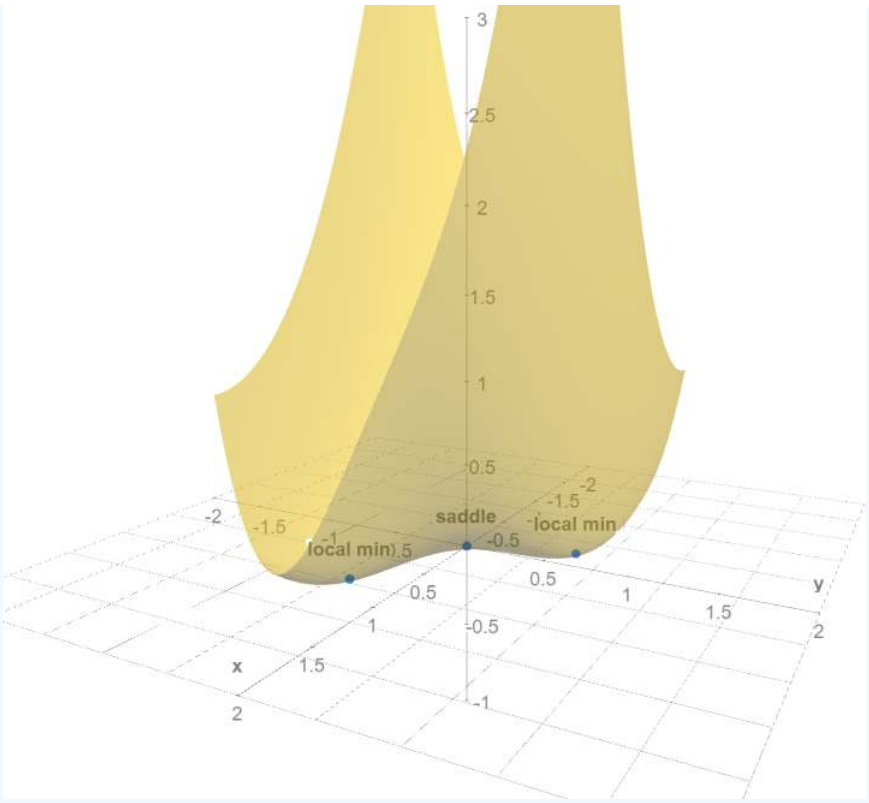
Since $f(0,0) = 0$ we have a **saddle point** at $(0,0,0)$

$D\left(\frac{1}{2}, -\frac{1}{2}\right) = 4 \cdot 2 > 0$ and $f_{xx}\left(\frac{1}{2}, -\frac{1}{2}\right) = 2 > 0$

Since $f\left(\frac{1}{2}, -\frac{1}{2}\right) = -\frac{1}{8}$ we have a **local minimum** at $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{8}\right)$

$D\left(-\frac{1}{2}, \frac{1}{2}\right) = 4 \cdot 2 > 0$ and $f_{xx}\left(-\frac{1}{2}, \frac{1}{2}\right) = 2 > 0$

Since $f\left(-\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{8}$ we have a **local minimum** at $\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{8}\right)$



? Exercise 2.6.1b

Find and classify all local extrema of the function $f(x,y) = 2x^3 - 6x + 6xy^2 + 1$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x = 6x^2 - 6 + 6y^2$ and $f_y = 12xy$

Setting these equal to zero gives the system of equations: $\begin{cases} 6x^2 - 6 + 6y^2 = 0 \\ 12xy = 0 \end{cases} \longrightarrow \begin{cases} x^2 + y^2 = 1 \\ xy = 0 \end{cases}$

These are satisfied at the critical points $(0, \pm 1)$, and $(\pm 1, 0)$

The second partials of f are $\begin{aligned} f_{xx} &= 12x \\ f_{xy} &= 12y \\ f_{yy} &= 12x \end{aligned}$

Therefore we have

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 12x \cdot 12x - (12y)^2 = 144(x^2 - y^2)$$

$D(0,1) = 144 \cdot (-1) < 0$

Since $f(0,1) = 1$ we have a **saddle point** at $(0,1,1)$

$D(0,-1) = 144 \cdot (-1) < 0$

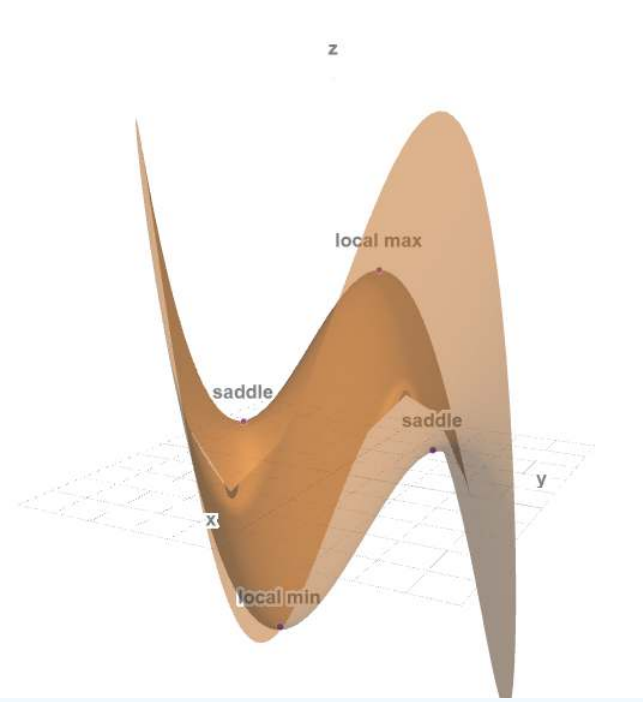
Since $f(0,-1) = 1$ we have a **saddle point** at $(0,-1,1)$

$D(1,0) = 144 \cdot 1 > 0$ and $f_{xx}(1,0) = 12 > 0$

Since $f(1,0) = -3$ we have a **local minimum** at $(1,0,-3)$

$D(-1,0) = 144 \cdot 1 > 0$ and $f_{xx}(-1,0) = -12 < 0$

Since $f(-1,0)=5$ we have a **local maximum** at $(-1,0,5)$



Exercise 2.6.1c

Find and classify all local extrema of the function $f(x,y)=3y-x^4+2x^2-y^3$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x=-4x^3+4x$ and $f_y=3-3y^2$

Setting these equal to zero gives the system of equations: $\begin{cases} -4x^3+4x=0 \\ 3-3y^2=0 \end{cases} \longrightarrow \begin{cases} x(x^2-1)=0 \\ y^2=1 \end{cases}$

These are satisfied at the critical points $(0,\pm1)$, $(1,\pm1)$, and $(-1,\pm1)$

The second partials of f are $\begin{aligned} f_{xx}&=-12x^2+4 \\ f_{xy}&=0 \\ f_{yy}&=-6y \end{aligned}$

Therefore we have

$$D=f_{xx}\cdot f_{yy}-(f_{xy})^2=(-12x^2+4)\cdot(-6y)-(0)^2=24y(3x^2-1)$$

$D(0,1)=24\cdot(-1)<0$

Since $f(0,1)=2$ we have a **saddle point** at $(0,1,2)$

$D(0,-1)=-24\cdot(-1)>0$ and $f_{yy}(0,-1)=6>0$

Since $f(0,-1)=-2$ we have a **local minimum** at $(0,-1,-2)$

$D(1,1)=24\cdot(2)>0$ and $f_{yy}(1,1)=-6<0$

Since $f(1,1)=3$ we have a **local maximum** at $(1,1,3)$

$D(1,-1)=-24\cdot(2)<0$

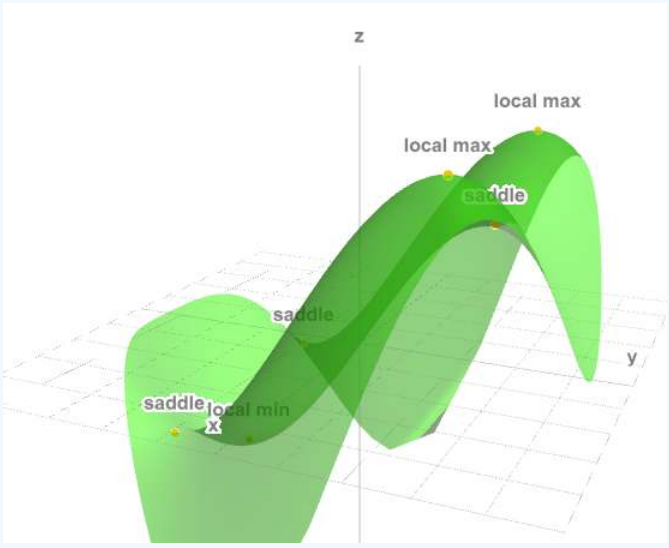
Since $f(1,-1)=-1$ we have a **saddle point** at $(1,-1,-1)$

$D(-1,1) = 24 \cdot (2) > 0$ and $f_{yy}(-1,1) = -6 < 0$

Since $f(-1,1) = 3$ we have a **local maximum** at $(-1,1,3)$

$D(-1,-1) = -24 \cdot (2) < 0$

Since $f(-1,-1) = -1$ we have a **saddle point** at $(-1,-1,-1)$



Exercise 2.6.1d

Find and classify all local extrema of the function $f(x,y) = ye^{-x^2-y^2}$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x = ye^{-x^2-y^2} \cdot (-2x) = -2xye^{-x^2-y^2}$

$f_y = 1 \cdot e^{-x^2-y^2} + y \cdot \left(e^{-x^2-y^2} \cdot (-2y)\right) = (1-2y^2)e^{-x^2-y^2}$

Setting these equal to zero gives the system of equations: $\begin{cases} -2xye^{-x^2-y^2} = 0 \\ (1-2y^2)e^{-x^2-y^2} = 0 \end{cases} \longrightarrow \begin{cases} xy = 0 \\ 1-2y^2 = 0 \end{cases}$

These are satisfied at the critical points $\left(0, \pm\frac{1}{\sqrt{2}}\right)$

The second partials of f are
$f_{xx} = 2y(2x^2-1)e^{-x^2-y^2}$
$f_{xy} = 2x(2y^2-1)e^{-x^2-y^2}$
$f_{yy} = 2y(2y^2-3)e^{-x^2-y^2}$

Therefore we have

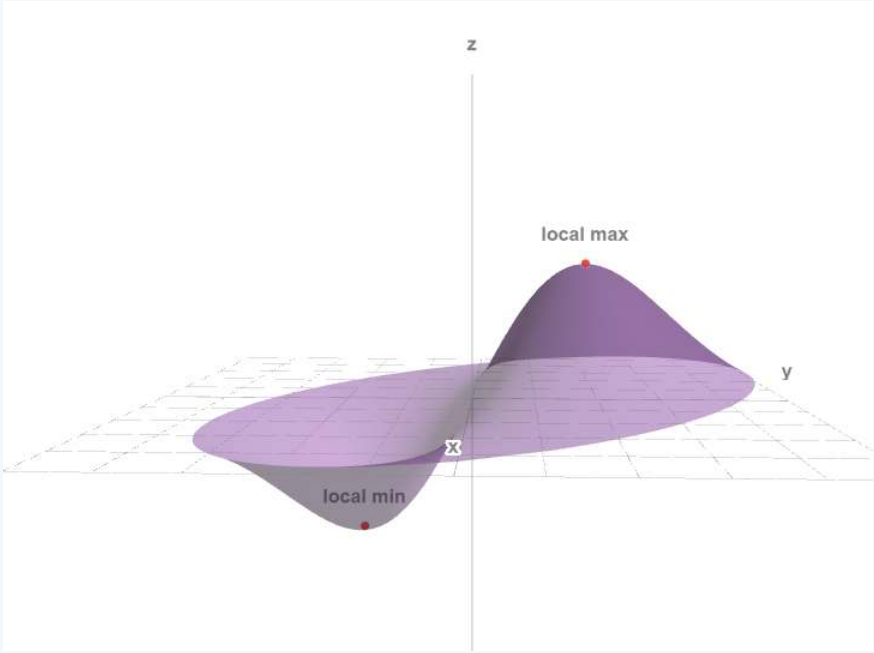
$$\begin{aligned} D &= f_{xx} \cdot f_{yy} - (f_{xy})^2 \\ &= 2y(2x^2-1)e^{-x^2-y^2} \cdot 2y(2y^2-3)e^{-x^2-y^2} - \left(2x(2y^2-1)e^{-x^2-y^2}\right)^2 \\ &= 4\left(y^2(2x^2-1)(2y^2-3) - x^2(2y^2-1)^2\right)e^{-2x^2-2y^2} \end{aligned}$$

$D\left(0, \frac{1}{\sqrt{2}}\right) = 4\left((+)^2(-)(-) - (+)^2(0)^2\right)(+) > 0$ and $f_{xx}\left(0, \frac{1}{\sqrt{2}}\right) < 0$

Since $f\left(0, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2e}}$ we have a **local maximum** at $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$

$D\left(0, -\frac{1}{\sqrt{2}}\right) = 4\left((-)^2(-)(-) - (+)^2(0)^2\right)(+) > 0$ and $f_{xx}\left(0, -\frac{1}{\sqrt{2}}\right) > 0$

Since $f\left(0,-\frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{2e}}$ we have a **local minimum** at $\left(0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2e}}\right)$



Exercise 2.6.2a

Find all extrema for the function $f(x,y)=x^2+y^2-2x$ on the triangle with vertices $(3,0),(0,3)$ and $(0,-3)$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x=2x-2$ and $f_y=2y$

Setting these equal to zero gives the system of equations: $\begin{cases} 2x-2=0 \\ 2y=0 \end{cases} \longrightarrow \begin{cases} x=1 \\ y=0 \end{cases}$

These are satisfied at the critical point $(1,0)$, which is inside of the triangle domain given.

At this critical point we have $f(1,0)=-1$.

Next we determine any maxima or minima on the boundary of the triangle domain.

We can parameterize the edge of the domain between $(0,-3)$ and $(0,3)$ by $\vec{\mathbf{r}}(t)=\langle 0,t\rangle$ for $-3\le t\le 3$.

We consider $g_1(t)=f(0,t)=0^2+t^2-2\cdot 0=t^2$ for $-3\le t\le 3$, which has derivative $g_1'(t)=2t$

Thus we compare the critical point $t=0$ and endpoints $t=-3,t=3$ inside of g_1: $\begin{cases} g_1(-3)=9 \\ g_1(0)=0 \\ g_1(3)=9 \end{cases}$

We can parameterize the edge of the domain between $(3,0)$ and $(0,3)$ by $\vec{\mathbf{r}}(t)=\langle 3-t,t\rangle$ for $0\le t\le 3$.

We consider $g_2(t)=f(3-t,t)=(3-t)^2+t^2-2\cdot(3-t)=2t^2-4t+3$ for $0\le t\le 3$, which has derivative $g_2'(t)=4t-4$

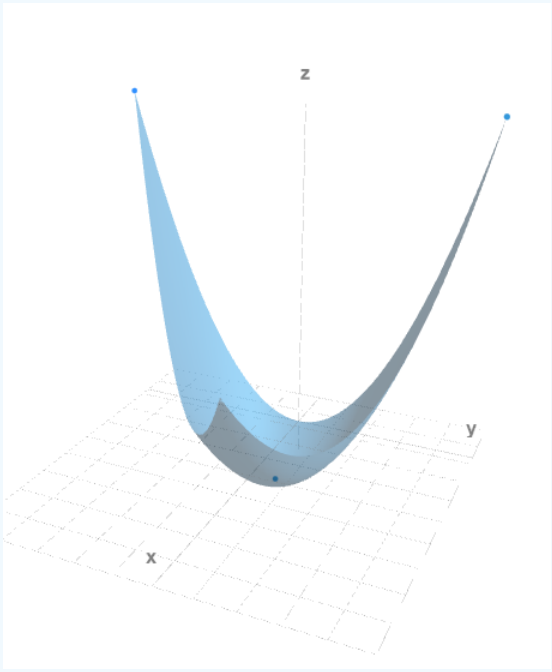
Thus we compare the critical point $t=1$ and endpoints $t=0,t=3$ inside of g_2: $\begin{cases} g_2(0)=3 \\ g_2(1)=1 \\ g_2(3)=9 \end{cases}$

We can parameterize the edge of the domain between $(3,0)$ and $(0,-3)$ by $\vec{\mathbf{r}}(t)=\langle 3-t,-t\rangle$ for $0\le t\le 3$.

We consider $g_3(t)=f(3-t,-t)=(3-t)^2+(-t)^2-2\cdot(3-t)=2t^2-4t+3$ for $0\le t\le 3$, which has derivative $g_3'(t)=4t-4$

Thus we compare the critical point $t=1$ and endpoints $t=0, t=3$ inside of g_3: $\begin{cases} g_3(0)=3 \\ g_3(1)=1 \\ g_3(3)=9 \end{cases}$

Comparing all of these outputs, we see that the maximum occurs at $(0,3,9)$ and $(0,-3,9)$, and the minimum occurs at $(1,0,-1)$



Exercise 2.6.2b

Find all extrema for the function $f(x,y)=y^2-2y\cos(x)$ on the set $\{(x,y)\,:\,0\le x\le\pi,\ -\pi\le y\le\pi\}$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x=2y\sin(x)$ and $f_y=2y-2\cos(x)$

Setting these equal to zero gives the system of equations: $\begin{cases} 2y\sin(x)=0 \\ 2y-2\cos(x)=0 \end{cases} \longrightarrow \begin{cases} y\sin(x)=0 \\ y=\cos(x) \end{cases}$

These are satisfied at the critical points $(0,1)$, $\left(\frac{\pi}{2},0\right)$, and$(\pi,-1)$ inside of the domain given.

At these critical points we have $f(0,1)=-1, f\left(\frac{\pi}{2},0\right)=0$, and $f(\pi,-1)=-1$.

Next we determine any maxima or minima on the boundary of the rectangular domain.

We can parameterize the edge of the domain between $(0,\pi)$ and $(0,-\pi)$ by $\vec{\mathbf{r}}(t)=\langle 0,t\rangle$ for $-\pi\le t\le\pi$.

We consider $g_1(t)=f(0,t)=t^2-2t\cos(0)=t^2-2t$ for $-\pi\le t\le\pi$, which has derivative $g_1'(t)=2t-2$

Thus we compare the critical point $t=1$ and endpoints $t=-\pi, t=\pi$ inside of g_1: $\begin{cases} g_1(-\pi)=\pi^2+2\pi \\ g_1(1)=-1 \\ g_1(\pi)=\pi^2-2\pi \end{cases}$

We can parameterize the edge of the domain between $(0,\pi)$ and (π,π) by $\vec{\mathbf{r}}(t)=\langle t,\pi\rangle$ for $0\le t\le\pi$.

We consider $g_2(t)=f(t,\pi)=\pi^2-2\pi\cos(t)$ for $0\le t\le\pi$, which has derivative $g_2'(t)=2\pi\sin(t)$

The critical points match the endpoints $t=0, t=\pi$, we plug them inside of g_2: $\begin{cases} g_2(0)=\pi^2-2\pi \\ g_2(\pi)=\pi^2+2\pi \end{cases}$

We can parameterize the edge of the domain between $(0,-\pi)$ and $(\pi,-\pi)$ by $\vec{\mathbf{r}}(t)=\langle t,-\pi\rangle$ for $0\le t\le\pi$.

We consider $g_3(t)=f(t,-\pi)=\pi^2+2\pi\cos(t)$ for $0\le t\le\pi$, which has derivative $g_3'(t)=-2\pi\sin(t)$

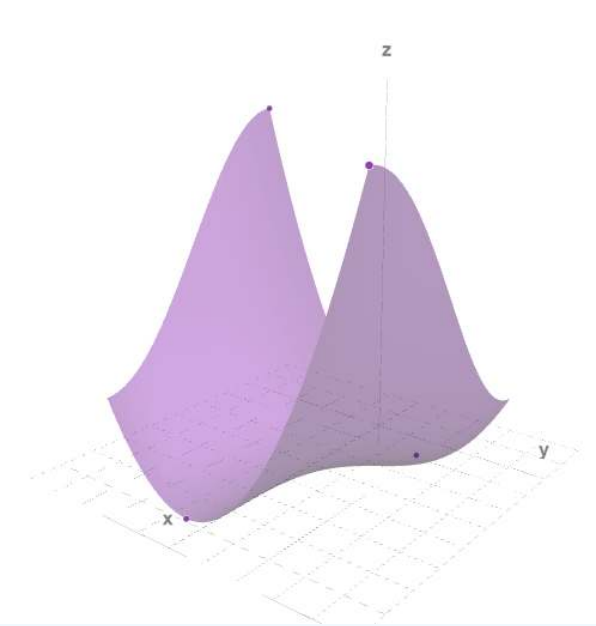
The critical points match the endpoints $t=0, t=\pi$, we plug them inside of g_3: $\begin{cases} g_3(0)=\pi^2+2\pi \\ g_3(\pi)=\pi^2-2\pi \end{cases}$

We can parameterize the edge of the domain between (π,π) and $(\pi,-\pi)$ by $\vec{\mathbf{r}}(t)=\langle \pi,t\rangle$ for $-\pi\le t\le\pi$.

We consider $g_4(t)=f(\pi,t)=t^2-2t\cos(\pi)=t^2+2t$ for $-\pi\le t\le\pi$, which has derivative $g_4'(t)=2t+2$

Thus we compare the critical point $t=-1$ and endpoints $t=-\pi, t=\pi$ inside of g_4: $\begin{cases} g_4(-\pi)=\pi^2-2\pi \\ g_4(-1)=-1 \\ g_4(\pi)=\pi^2+2\pi \end{cases}$

Comparing all of these outputs, we see that the maximum occurs at $(\pi,\pi,\pi^2+2\pi)$ and $(0,-\pi,\pi^2+2\pi)$, and the minimum occurs at $(0,1,-1)$ and $(\pi,-1,-1)$



? Exercise 2.6.2c

Find all extrema for the function $f(x,y)=x^2+y^2+x^2y+4$ on the set $\{(x,y)\ :\ -1\le x\le 1,\ -1\le y\le 1\}$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x=2x+2xy$ and $f_y=2y+x^2$

Setting these equal to zero gives the system of equations: $\begin{cases} 2x+2xy=0 \\ 2y+x^2=0 \end{cases}\longrightarrow\begin{cases} x(1+y)=0 \\ y=-\frac{1}{2}x^2 \end{cases}$

These are satisfied at the critical points $(0,0)$ (inside of domain) and $(\pm\sqrt{2},-1)$ (outside of the domain).

At the interior critical point we have $f(0,0)=4$.

Next we determine any maxima or minima on the boundary of the rectangular domain.

We can parameterize the edge of the domain between $(-1,-1)$ and $(-1,1)$ by $\vec{\mathbf{r}}(t)=\langle -1,t\rangle$ for $-1\le t\le 1$.

We consider $g_1(t)=f(-1,t)=(-1)^2+t^2+(-1)^2\cdot t+4=t^2+t+5$ for $-1\le t\le 1$, which has derivative $g_1'(t)=2t+1$

Thus we compare the critical point $t=-\frac{1}{2}$ and endpoints $t=-1, t=1$ inside of g_1: $\begin{cases} g_1(-1)=5 \\ g_1\left(-\frac{1}{2}\right)=\frac{19}{4} \\ g_1(1)=7 \end{cases}$

We can parameterize the edge of the domain between $(-1,1)$ and $(1,1)$ by $\vec{\mathbf{r}}(t)=\langle t,1\rangle$ for $-1\le t\le 1$.

We consider $g_2(t)=f(t,1)=t^2+1^2+t^2\cdot 1+4=2t^2+5$ for $-1\le t\le 1$, which has derivative $g_2'(t)=4t$

Thus we compare the critical point $t=0$ and endpoints $t=-1, t=1$, we plug them inside of g_2: $\begin{cases} g_2(-1)=7 \\ g_2(0)=5 \\ g_2(1)=7 \end{cases}$

We can parameterize the edge of the domain between $(-1,-1)$ and $(1,-1)$ by $\vec{\mathbf{r}}(t)=\langle t,-1\rangle$ for $-1\le t\le 1$.

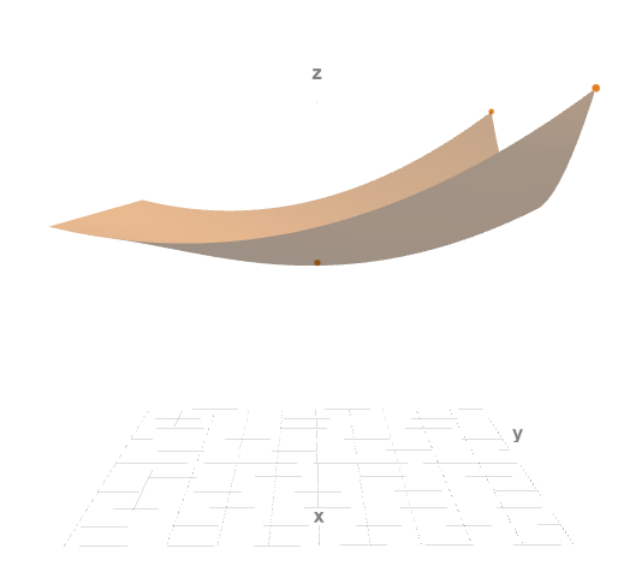
We consider $g_3(t)=f(t,-1)=t^2+(-1)^2+t^2\cdot(-1)+4=5$ for $-1\le t\le 1$, which is constant, so its only output is 5

We can parameterize the edge of the domain between $(1,-1)$ and $(1,1)$ by $\vec{\mathbf{r}}(t)=\langle 1,t\rangle$ for $-1\le t\le 1$.

We consider $g_4(t)=f(1,t)=1^2+t^2+1^2\cdot t+4=t^2+t+5$ for $-1\le t\le 1$, which has derivative $g_4'(t)=2t+1$

Thus we compare the critical point $t=-\frac{1}{2}$ and endpoints $t=-1, t=1$ inside of g_4: $\begin{cases} g_4(-1)=5 \\ g_4\left(-\frac{1}{2}\right)=\frac{19}{4} \\ g_4(1)=7 \end{cases}$

Comparing all of these outputs, we see that the maximum occurs at $(1,1,7)$ and $(-1,1,7)$, and the minimum occurs at $(0,0,4)$



Exercise 2.6.2d

Find all extrema for the function $f(x,y)=x^2+2y^2-2x+3$ on the set $\{(x,y)\,:\,x^2+y^2\le 16\}$

Answer

First we find the partial derivatives of $f(x,y)$:

$f_x=2x-2$ and $f_y=4y$

Setting these equal to zero gives the system of equations: $\begin{cases} 2x-2=0 \\ 4y=0 \end{cases}\longrightarrow\begin{cases} x=1 \\ y=0 \end{cases}$

These are satisfied at the critical point $(1,0)$ which is inside of the circular domain.

At the critical point we have $f(1,0)=2$.

Next we determine any maxima or minima on the boundary of the circle.

We can parameterize the edge of the domain using $\vec{\mathbf{r}}(t) = \langle 4\cos(t), 4\sin(t)\rangle$ for $0 \le t \le 2\pi$ (note: substitution also works!)

We consider

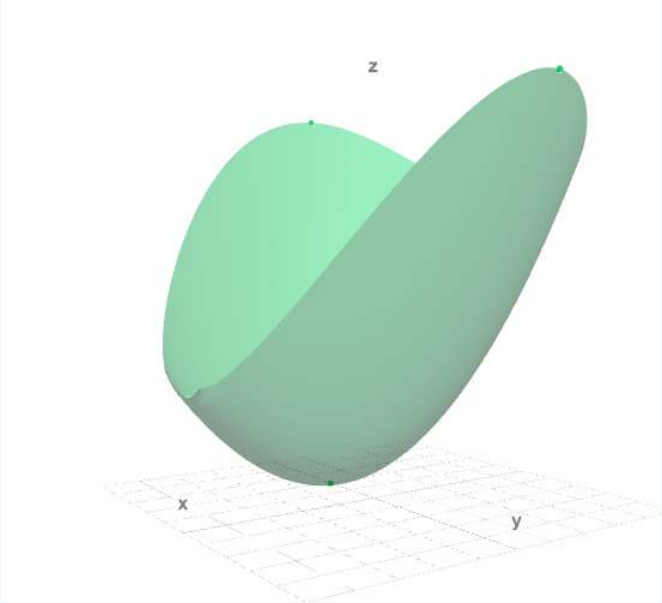
$$\begin{aligned} g(t) &= f\left(4\cos(t), 4\sin(t)\right) \\ &= \left(4\cos(t)\right)^2 + 2\left(4\sin(t)\right)^2 - 2\left(4\cos(t)\right) + 3 \\ &= 16\cos^2(t) + 32\sin^2(t) - 8\cos(t) + 3 \\ &= -16\cos^2(t) - 8\cos(t) + 35 \quad \text{for} \quad 0 \le t \le 2\pi \end{aligned}$$

This has derivative $g'(t) = 32\sin(t)\cos(t) + 8\sin(t)$ which can be written $g'(t) = 8\sin(t)(4\cos(t) + 1)$

This yields critical points at $t = 0, \pi, 2\pi, \cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right)$

Since this list includes our endpoints of the domain, we plug each into $g(t)$ to get: $\begin{cases} g(0) = 11 \\ g\left(\cos^{-1}\left(-\frac{1}{4}\right)\right) = 36 \\ g(\pi) = 27 \\ g\left(2\pi - \cos^{-1}\left(-\frac{1}{4}\right)\right) = 36 \\ g(2\pi) = 11 \end{cases}$

Comparing all of these outputs, we see that the maximum occurs at $(-1, \sqrt{15}, 36)$ and $(-1, -\sqrt{15}, 36)$, and the minimum occurs at $(1, 0, 2)$



? Exercise 2.6.3a

Find the extrema of $f(x, y) = 2x^3 - 6y$ on $x^2 + y^2 = 2$

Answer

We define the constraint function $g(x, y) = x^2 + y^2$ and find the partial derivatives of f and g:

$f_x = 6x^2$ and $f_y = -6$

$g_x = 2x$ and $g_y = 2y$

We set up the system of equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 2$: $\begin{cases} 6x^2 = \lambda \cdot 2x \\ -6 = \lambda \cdot 2y \\ x^2 + y^2 = 2 \end{cases} \longrightarrow \begin{cases} 2x(3x - \lambda) = 0 \\ -3 = \lambda y \\ x^2 + y^2 = 2 \end{cases}$

The first equation is equal to zero in two cases: $x = 0$ or $x = \frac{\lambda}{3}$

Case 1: If we assume $x = 0$, the third equation becomes $0^2 + y^2 = 2$ which has solutions $y = \pm\sqrt{2}$

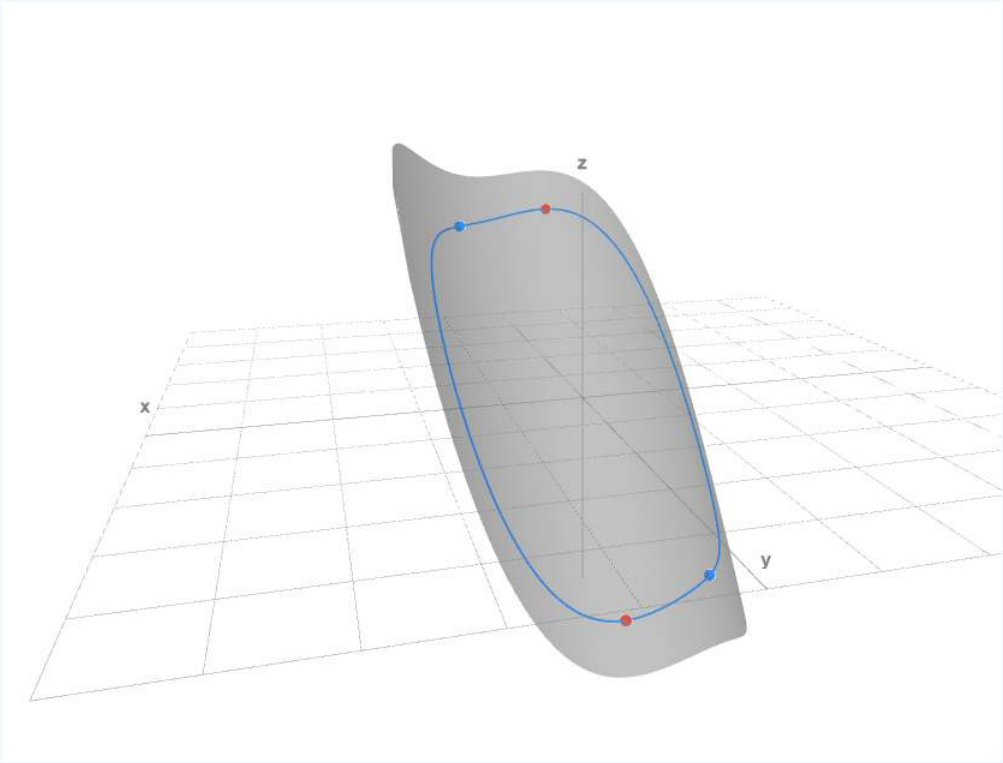
This yields two critical points: $(0, \sqrt{2})$ and $(0, -\sqrt{2})$

Case 2: If we assume $x = \frac{\lambda}{3}$, the second equation becomes $y = -\frac{3}{\lambda}$.

Plugging this into the third equation gives $\left(\frac{\lambda}{3}\right)^2 + \left(-\frac{3}{\lambda}\right)^2 = 2$.

Since this has two solutions, $\lambda = \pm 3$, we get two more critical points: $(1, -1)$ and $(-1, 1)$.

Plugging all four of these critical points into $f(x, y)$ gives us: $\begin{cases} f(0, \sqrt{2}) = -6\sqrt{2} \\ f(0, -\sqrt{2}) = 6\sqrt{2} \\ f(1, -1) = 8 \\ f(-1, 1) = -8 \end{cases}$



Therefore this function under the given constraint has a maximum of $6\sqrt{2}$ at $(0, -\sqrt{2})$, and minimum of $-6\sqrt{2}$ at $(0, \sqrt{2})$

Exercise 2.6.3b

Find the extrema of $f(x, y) = x^2 - xy + y^2$ on $8x^2 + 3y^2 = 84$

Answer

We define the constraint function $g(x, y) = 8x^2 + 3y^2$ and find the partial derivatives of f and g:

$f_x = 2x - y$ and $f_y = 2y - x$

$g_x = 16x$ and $g_y = 6y$

We set up the system of equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 84$: $\begin{cases} 2x - y = \lambda \cdot 16x \\ 2y - x = \lambda \cdot 6y \\ 8x^2 + 3y^2 = 84 \end{cases} \longrightarrow \begin{cases} 2x(1 - 8\lambda) = y \\ 2y(1 - 3\lambda) = x \\ 8x^2 + 3y^2 = 84 \end{cases}$

We substitute the first equation into the second to get: $2\,(2x(1 - 8\lambda))\,(1 - 3\lambda) = x \longrightarrow x\,(4(1 - 8\lambda)(1 - 3\lambda) - 1) = 0$

This is satisfied in two cases: $x = 0$ or $4(1 - 8\lambda)(1 - 3\lambda) = 1$

Case 1: If we assume $x = 0$, the third equation becomes $0^2 + 3y^2 = 84$ which has solutions $y = \pm 2\sqrt{7}$

This yields two critical points: $(0, 2\sqrt{7})$ and $(0, -2\sqrt{7})$

Case 2: If we assume $4(1 - 8\lambda)(1 - 3\lambda) = 1$, this expands to $96\lambda^2 - 44\lambda + 3 = 0$ which has solutions $\lambda = \frac{1}{12}, \frac{3}{8}$.

When $\lambda = \frac{1}{12}$, the first two equations each simplify to $y = \frac{2}{3}x$.

Plugging this into the third equation gives $8x^2 + 3\left(\frac{2}{3}x\right)^2 = 84$, which has two solutions: $x = \pm 3$

This yields the critical points: $(3, 2)$ and $(-3, -2)$.

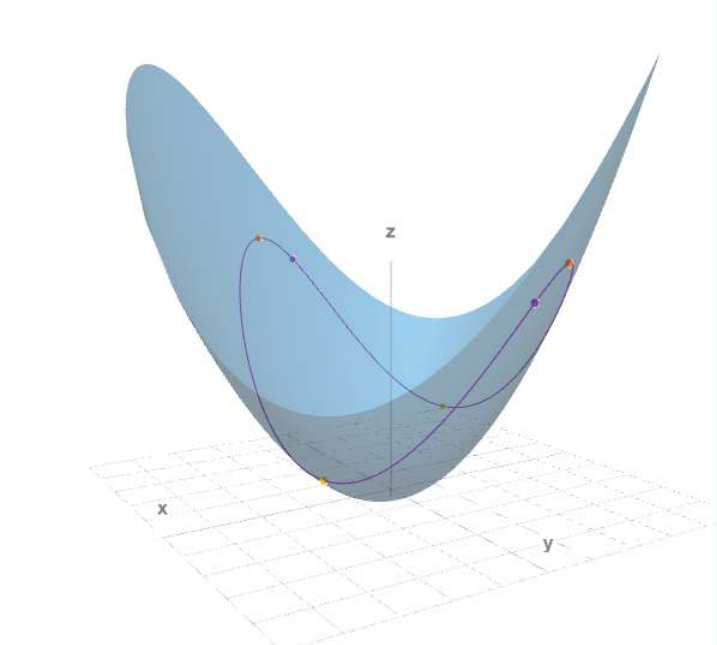
When $\lambda = \frac{3}{8}$, the first two equations each simplify to $y = -4x$.

Plugging this into the third equation gives $8x^2 + 3(-4x)^2 = 84$, which has two solutions: $x = \pm\sqrt{\frac{3}{2}}$

This yields the critical points: $\left(\sqrt{\frac{3}{2}}, -2\sqrt{6}\right)$ and $\left(-\sqrt{\frac{3}{2}}, 2\sqrt{6}\right)$.

Plugging all six of these critical points into $f(x, y)$ gives us:

$$\begin{cases} f(0, 2\sqrt{7}) = 28 \\ f(0, -2\sqrt{7}) = 28 \\ f(3, 2) = 7 \\ f(-3, -2) = 7 \\ f\left(\sqrt{\frac{3}{2}}, -2\sqrt{6}\right) = \frac{63}{2} \\ f\left(-\sqrt{\frac{3}{2}}, 2\sqrt{6}\right) = \frac{63}{2} \end{cases}$$



Therefore this function under the given constraint has a maximum of $\frac{63}{2}$ at $\left(\sqrt{\frac{3}{2}}, 2\sqrt{6}\right)$ and $\left(-\sqrt{\frac{3}{2}}, 2\sqrt{6}\right)$, and minimum of 7 at $(3, 2)$ and $(-3, -2)$

Exercise 2.6.3c

Find the extrema of $f(x,y,z) = 12x + 6y + 6z$ on $x^2 + y^2 + z^2 = 6$

Answer

We define the constraint function $g(x,y,z) = x^2 + y^2 + z^2$ and find the partial derivatives of f and g:

$\nabla f = \langle 12, 6, 6\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$

We set up the system of equations $\nabla f = \lambda \nabla g$ and $g(x,y) = 6$: $\begin{cases} 12 = \lambda \cdot 2x \\ 6 = \lambda \cdot 2y \\ 6 = \lambda \cdot 2z \\ x^2 + y^2 + z^2 = 6 \end{cases} \longrightarrow \begin{cases} x = \dfrac{6}{\lambda} \\ y = \dfrac{3}{\lambda} \\ z = \dfrac{3}{\lambda} \\ x^2 + y^2 + z^2 = 6 \end{cases}$

Note that we were allowed to divide by λ above because it is certain that $\lambda \neq 0$. We substitute everything into the third equation:

$\left(\frac{6}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 6$

This equation simplifies to $6\lambda^2 = 54 \longrightarrow \lambda = \pm 3$

When $\lambda = 3$ we get the critical point $(2,1,1)$, and when $\lambda = -3$ we get $(-2,-1,-1)$

We plug each of these into the function $f(x,y,z)$ and see: $\begin{cases} f(2,1,1) = 36 \\ f(-2,-1-1) = -36 \end{cases}$

Therefore this function under the given constraint has a maximum of 36 at $(2,1,1)$, and minimum of -36 at $(-2,-1,-1)$

Exercise 2.6.3d

Find the extrema of $f(x,y,z) = xyz^2$ on $x^2 + y^2 + z^2 = 4$

Answer

We define the constraint function $g(x,y,z) = x^2 + y^2 + z^2$ and find the partial derivatives of f and g:

$\nabla f = \langle yz^2, xz^2, 2xyz\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$

We set up the system of equations $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 4$: $\begin{cases} yz^2 = \lambda \cdot 2x \\ xz^2 = \lambda \cdot 2y \\ 2xyz = \lambda \cdot 2z \\ x^2 + y^2 + z^2 = 4 \end{cases} \longrightarrow \begin{cases} yz^2 = 2\lambda x \\ xz^2 = 2\lambda y \\ 2z(xy - \lambda) = 0 \\ x^2 + y^2 + z^2 = 4 \end{cases}$

The first three equations can be satisfied by letting any of x, y, z, or λ be zero, but for all such combinations the function $f(x,y,z) = 0$.

We discuss any other possible critical points where $x \neq 0, y \neq 0, z \neq 0, \lambda \neq 0$ holds.

This allows us to divide by any variable, and the third equation gives $\lambda = xy$. Plugging this into the first two equations gives:

$2x^2y = yz^2$ and $2xy^2 = xz^2$

Since we can divide by either of x or y this simplifies to $x^2 = \dfrac{z^2}{2}$ and $y^2 = \dfrac{z^2}{2}$

Plugging these into the fourth (constraint) equation we have:

$\frac{z^2}{2} + \frac{z^2}{2} + z^2 = 4$

This equation simplifies to $z^2 = 2 \longrightarrow z = \pm\sqrt{2}$

When $z=\sqrt{2}$ we have four possible critical points: $(1,1,\sqrt{2}),(1,-1,\sqrt{2}),(-1,1,\sqrt{2})$, and $(-1,-1,\sqrt{2})$

When $z=-\sqrt{2}$ we have four possible critical points: $(1,1,-\sqrt{2}),(1,-1,-\sqrt{2}),(-1,1,-\sqrt{2})$, and $(-1,-1,-\sqrt{2})$

We plug each of these into the function $f(x,y,z)$ and see: $\begin{cases} f(1,1,\pm\sqrt{2})=2 \\ f(1,-1,\pm\sqrt{2})=-2 \\ f(-1,1,\pm\sqrt{2})=-2 \\ f(-1,-1,\pm\sqrt{2})=2 \end{cases}$

Therefore this function under the given constraint has a minimum of -2 at $(-1,1,\sqrt{2}),(1,-1,\sqrt{2}),(-1,1,-\sqrt{2})$, $(1,-1,-\sqrt{2})$, and a maximum of 2 at $(1,1,\sqrt{2}),(-1,-1,\sqrt{2}),(1,1,-\sqrt{2}),(-1,-1,-\sqrt{2})$

Exercise 2.6.3e

Find the extrema of $f(x,y,z)=x^2+y^2+z^2$ when $x^2+y^2-z^2=15$ and $y-2z=6$

Answer

We define the two constraint functions $g(x,y,z)=x^2+y^2-z^2$ and $h(x,y,z)=y-2z$ and find all gradients:

$\nabla f=\langle 2x,2y,2z\rangle, \quad \nabla g=\langle 2x,2y,-2z\rangle, \quad \text{and} \quad \nabla h=\langle 0,1,-2\rangle$

We set up the system of equations $\nabla f=\lambda\nabla g+\gamma\nabla h$ and $g(x,y,z)=15$ and $h(x,y,z)=6$:

$$\begin{cases} 2x=\lambda\cdot 2x+\gamma\cdot 0 \\ 2y=\lambda\cdot 2y+\gamma\cdot 1 \\ 2z=\lambda\cdot(-2z)+\gamma\cdot(-2) \\ x^2+y^2-z^2=15 \\ y-2z=6 \end{cases} \longrightarrow \begin{cases} 2x(1-\lambda)=0 \\ 2y(1-\lambda)=\gamma \\ 2z(1+\lambda)=-2\gamma \\ x^2+y^2-z^2=15 \\ y=2z+6 \end{cases}$$

The top equation would be true in two cases: $\lambda=1$ or $x=0$

Case 1: If $\lambda=1$, the second equation forces $\gamma=0$, which forces the third equation to say $z=0$.

This makes the $h(x,y,z)$ restriction into $y=6$, and thus the $g(x,y,z)$ restriction becomes $x^2+6^2-0^2=15$.

Since this has no solution, there are no critical points for this case.

Case 2: If $x=0$, the second and third equations can be satisfied by any y,z by properly selecting λ and γ.

However, the $g(x,y,z)$ restriction turns into $0^2+y^2-z^2=15$. We can substitute in $y=2z+6$ to make this into:

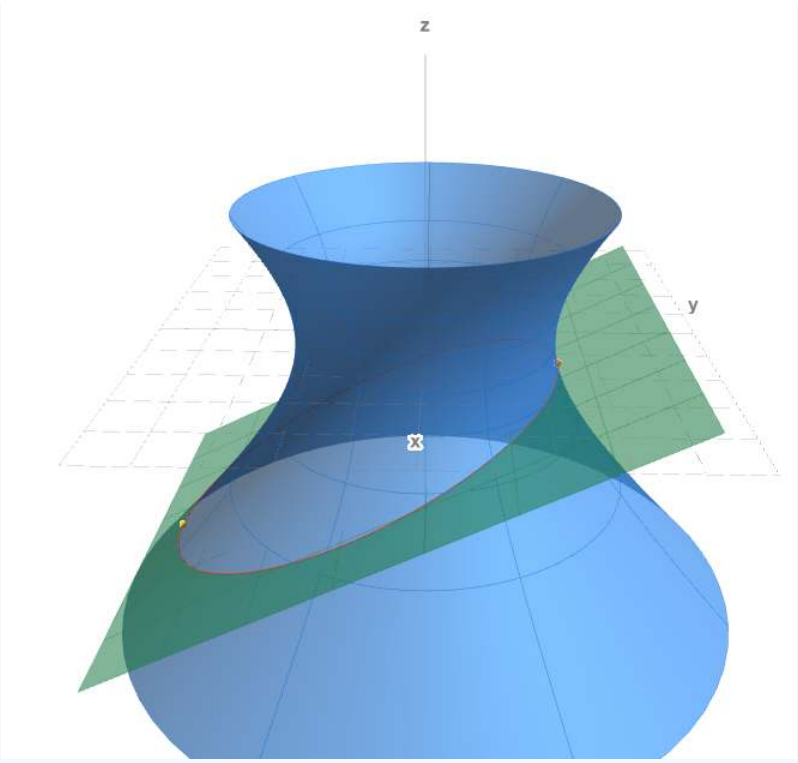
$(2z+6)^2-z^2=15 \longrightarrow 3z^2+24z+21=0 \longrightarrow z=-1,-7$

Plugging in $z=-1$ into $y=2z+6$ gives $y=4$ and plugging in $z=-7$ gives $y=-8$

The outputs at these two points give $f(0,4,-1)=0^2+4^2+(-1)^2=17$ and $f(0,-8,-7)=0^2+(-8)^2+(-7)^2=113$

Thus the maximum of f under these two constraints is 113 at the point $(0,-8,-7)$ and the minimum is 17 at the point $(0,4,-1)$.

Since the function f represents the square of the distance of a point from the origin, these points represent the "furthest" and "closest" points to the origin on the intersection of the surfaces represented by $x^2+y^2-z^2=15$ and $y-2z=6$:



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CHAPTER OVERVIEW

3: Multiple Integration

In this chapter, we extend the concept of a definite integral of a single-variable function to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

3.1: Double Integrals
- 3.1: Iterated Integrals and Area
- 3.1: Double Integration and Volume
- 3.1: Practice Set

3.2: Triple Integrals and Polar Double Integrals
- 3.2: Triple Integrals
- 3.2: Double Integrals in Polar Coordinates
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3.3: Change of Variables in 2 Dimensions
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3.4: Change of Variables in 3 Dimensions
- 3.4: Change of Variables in Triple Integrals
- 3.4: Triple Integrals in Cylindrical and Spherical Coordinates
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3.5: Integrals on a Line
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3.7: Integrals of Vector Fields on a Surface
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3.1: Double Integrals

Learning Objectives

- Evaluate iterated integrals with two variables.
- Find the area of planar regions bounded by curves using double integrals.
- Be able to switch the order of integration in a double integral.
- Use double integrals to determine the signed volume under a surface.

The concept of integration is the natural progression after differentiation. In the same vein as taking partial derivatives by treating all other variables as constants, we review how to take integrals of multivariable functions. We discuss applications of iterated or double integrals for finding areas of two dimensional domains, and extend this to finding volumes of three dimensional regions under surfaces defined by multivariable functions. We define Type I and Type II regions and how to integrate real valued functions over them, as well as how to switch between them when necessary.

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3.1: Iterated Integrals and Area

In the previous chapter we found that we could differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We have seen that one can integrate functions of several variables in a similar way. For instance, if we are told that $f_x(x,y) = 2xy$, we can treat y as staying constant and integrate to obtain $f(x,y)$:

$$f(x,y) = \int f_x(x,y)\,dx = \int 2xy\,dx = x^2y + h(y)$$

Here we see the constant of integration is a function $h(y)$ which is "constant" in the sense that it is something with a derivative of 0 with respect to x. When doing a definite integral of one variable, we typically omit the constant. This is because it would *cancel out* when using the Fundamental Theorem of Calculus:

$$\int_a^b f'(t)dt = [f(t)+C]\Big|_a^b = (f(b)+C) - (f(a)+C) = f(b) - f(a)$$

The same is true when we solve definite integrals of more than one variable.

We can also integrate with respect to y. In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f\big(h_2(y),y\big) - f\big(h_1(y),y\big), \text{ and } \int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f\big(x,g_2(x)\big) - f\big(x,g_1(x)\big)$$

Note that when integrating with respect to x, the bounds are functions of y (of the form $x = h_1(y)$ and $x = h_2(y)$) and the final result is also a function of y. When integrating with respect to y, the bounds are functions of x (of the form $y = g_1(x)$ and $y = g_2(x)$) and the final result is a function of x.

Example 3.1.1: Integrating functions of more than one variable

Evaluate the integral

(a) $\displaystyle\int_1^{2y} 2xy\,dx$

(b) $\displaystyle\int_1^{x} \left(5x^3y^{-3} + 6y^2\right) dy$

Solution

(a) We treat y as a constant and use the Fundamental Theorem of Calculus:

$$\int_1^{2y} 2xy\,dx = \left[x^2y\right]\Big|_1^{2y} = (2y)^2y - (1)^2y = \boxed{4y^3 - y}$$

(b) We treat x as a constant this time:

$$\int_1^{x} \left(5x^3y^{-3} + 6y^2\right) dy = \left[\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right]\Big|_1^{x} = \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3 + 2\right) = \boxed{\frac{9}{2}x^3 - \frac{5}{2}x - 2}$$

Once we have integrated a function with respect to y and ended up with a function of x, we can take the definite integral of this new function with respect to x. This process is known as **iterated integration**, or **multiple integration**.

Example 3.1.2: Integrating an integral

Evaluate $\displaystyle\int_1^2 \left(\int_1^x \left(30x^2y\right) dy\right) dx.$

Solution

We follow a standard "order of operations" and perform the operations inside parentheses first:

$$\begin{aligned}\int_1^2 \left(\int_1^x 30x^2y\,dy\right) dx &= \int_1^2 \left(\left[15x^2y^2\right]\Big|_1^x\right) dx \\ &= \int_1^2 \left(15x^2(x)^2 - 15x^2(1)^2\right) dx \\ &= \int_1^2 \left(15x^4 - 15x^2\right) dx \\ &= \left[3x^5 - 5x^3\right]\Big|_1^2 \\ &= \left(3(2)^5 - 5(2)^3\right) - \left(3(1)^5 - 5(1)^3\right) \\ &= \boxed{58}\end{aligned}$$

The previous example showed how we could perform something called an iterated integral; we do not yet know why we would be interested in doing so nor what the result, such as the number 58, *means*. Before we investigate these questions, we offer some definitions.

Definition: Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let a, b, c and d be numbers and let $g_1(x)$, $g_2(x)$, $h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then:

1. $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right) dy$
2. $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right) dx$

Again make note of the bounds of these iterated integrals.

With $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, x varies from $h_1(y)$ to $h_2(y)$, whereas y varies from c to d. That is, the bounds of x are *curves*, the curves $x = h_1(y)$ and $x = h_2(y)$, whereas the bounds of y are *constants*, $y = c$ and $y = d$. It is useful to remember that when setting up and evaluating such iterated integrals, we integrate "from curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

If you consider the single integral of the constant function 1 on an interval, you easily get:

$$\text{length}([a,b]) = \int_a^b 1\,dx = x\Big|_a^b = b - a$$

Now consider the plane region R bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, shown below.

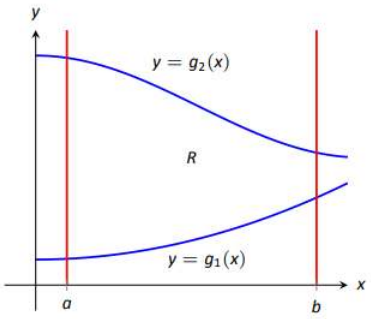


Figure 3.1.1: Calculating the area of a plane region R with an iterated integral.

We learned in single variable calculus that the area of this region can be found via the integral:

$$\text{area}(R) = \int_a^b \big(g_2(x) - g_1(x)\big)\, dx$$

We can view the expression $\big(g_2(x) - g_1(x)\big) = \int_{g_1(x)}^{g_2(x)} 1\, dy$, meaning we can express the area of R as an iterated integral:

$$\text{area}(R) = \int_a^b \big(g_2(x) - g_1(x)\big)\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} 1\, dy\right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dy\, dx$$

In short: the iterated integral with integrand 1 can be viewed as giving the area of a plane region R enclosed by the limits of integration.

A region R could also be defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, as shown below.

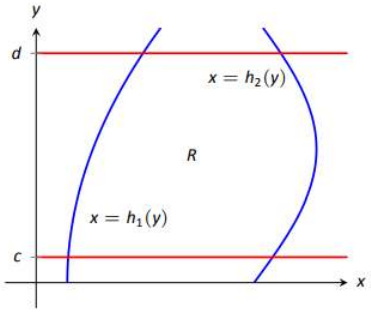


Figure 3.1.2: Calculating the area of a plane region R with an iterated integral.

Using a process similar to that above, we have

$$\text{area}(R) = \int_c^d \int_{h_1(y)}^{h_2(y)} 1\, dx\, dy$$

We state this formally in a theorem.

Theorem 3.1.1: Area of a plane region

1. Let R be a plane region bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$. The area A of R is

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dy\, dx$$

2. Let R be a plane region bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$. The area A of R is

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} 1\, dx\, dy$$

The following example will help us demonstrate this theorem.

Example 3.1.3: Area of a rectangle

Find the area of each region below by setting up and solving two iterated integrals ($dx\, dy$ and $dy\, dx$):

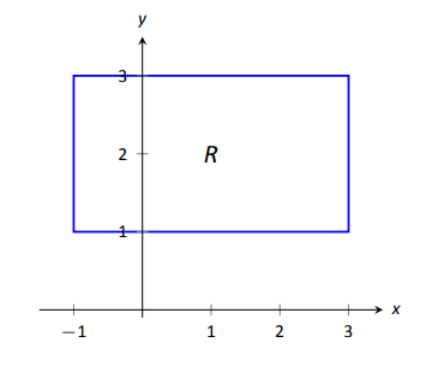
(a) the rectangle with corners $(-1, 1)$ and $(3, 3)$

(b) the triangle with vertices at $(1, 1)$, $(3, 1)$ and $(5, 5)$

(c) the region enclosed by $y = 2x$ and $y = x^2$

Solution

(a) The rectangle we want the area of looks like:



Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region R is bounded by $x = -1$, $x = 3$, $y = 1$ and $y = 3$. Choosing to integrate with respect to y first, we have

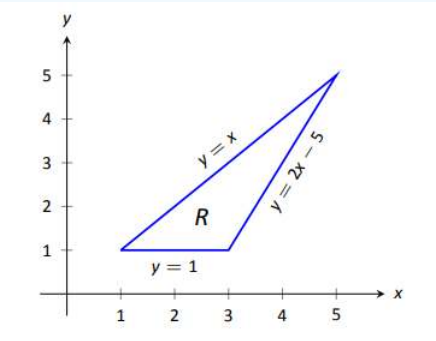
$$A = \int_{-1}^{3}\int_{1}^{3} 1\,dy\,dx = \int_{-1}^{3} y\Big|_{1}^{3}\,dx = \int_{-1}^{3} 2\,dx = 2x\Big|_{-1}^{3} = \boxed{8}$$

We could also integrate with respect to x first, giving:

$$A = \int_{1}^{3}\int_{-1}^{3} 1\,dx\,dy = \int_{1}^{3} x\Big|_{-1}^{3}\,dy = \int_{1}^{3} 4\,dy = 4y\Big|_{1}^{3} = 8$$

Clearly there are simpler ways to find this area, but it is nice to see that this method agrees with them.

(b) We sketch the triangle we want the area of:



The triangle is bounded by the lines as shown in the figure. We will start by doing the iterated integral $dxdy$, meaning we want to integrate x between two functions of y. Since x-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = \dfrac{y+5}{2}$, is the upper bound. This holds for every possible y between the line $y = 1$ on the bottom, and the point of intersection where $y = 5$ at the top. Therefore the area is

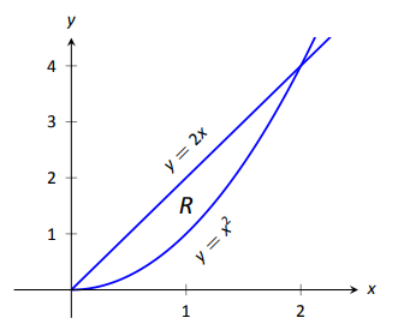
$$A = \int_{1}^{5}\int_{y}^{\frac{y+5}{2}} dx\,dy = \int_{1}^{5} x\Big|_{y}^{\frac{y+5}{2}}\,dy = \int_{1}^{5}\left(-\frac{1}{2}y + \frac{5}{2}\right)\,dy = \left[-\frac{1}{4}y^2 + \frac{5}{2}y\right]\Big|_{1}^{5} = \boxed{4}$$

We can also find the area by integrating with respect to $dydx$. In this situation, we want y to be between two functions of x. The values of y increase from low to high, so the "bottom" curve is piecewise defined: $y_{bot}(x) = \begin{cases} 1 & x \le 3 \\ 2x-5 & x > 3 \end{cases}$, and the "top" curve is simply the line $y = x$. This holds for every possible x between the leftmost point in R (where $x = 1$) and the rightmost point in R (where $x = 5$):

$$
\begin{aligned}
A &= \int_1^3 \int_{y_{bot}(x)}^{x} 1\,dy\,dx \\
&= \int_1^3 \int_1^x 1\,dy\,dx + \int_3^5 \int_{2x-5}^{x} 1\,dy\,dx \\
&= \int_1^3 y\Big|_1^x\,dx + \int_3^5 y\Big|_{2x-5}^{x}\,dx \\
&= \int_1^3 \big(x-1\big)\,dx + \int_3^5 \big(-x+5\big)\,dx \\
&= \frac{1}{2}(x-1)^2\Big|_1^3 - \frac{1}{2}(-x+5)^2\Big|_3^5 \\
&= 2+2=4
\end{aligned}
$$

As expected, we get the same answer both ways.

(c) We sketch the region we want the area of:



Once again we'll find the area of the region using both orders of integration.

Using $dy\,dx$, we see that for each x between the leftmost point $x=0$ and rightmost point $x=2$, the values of y range from the bottom function $y=x^2$ to the top function $y=2x$. Therefore the area is given by:

$$\int_0^2 \int_{x^2}^{2x} 1\,dy\,dx = \int_0^2 (2x-x^2)\,dx = \left[x^2 - \frac{1}{3}x^3\right]\Big|_0^2 = \frac{4}{3}$$

Using $dx\,dy$, we see that for each y between the lowest point $y=0$ and highest point $y=4$, the values of x range from the left function $x=\frac{y}{2}$ to the right function $x=\sqrt{y}$. Therefore the area is given by:

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} 1\,dx\,dy = \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)\,dy = \left[\frac{2}{3}y^{\frac{3}{2}} - \frac{1}{4}y^2\right]\Big|_0^4 = \boxed{\frac{4}{3}}$$

In each of the previous examples, we have been given a region R and found the bounds needed to find the area of R using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle, and so:

$$\int_a^b \int_c^d 1\,dy\,dx = \int_c^d \int_a^b 1\,dx\,dy$$

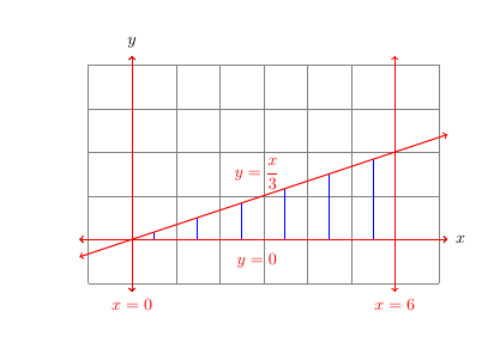
When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Illustration 3.1.1: Changing the order of integration

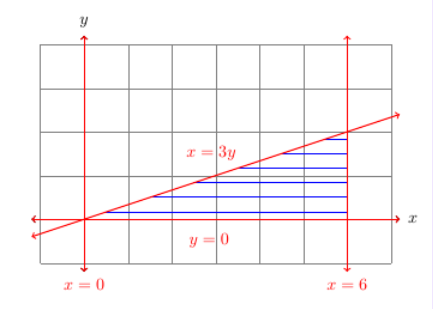
We will rewrite the iterated integral $\int_0^6 \int_0^{\frac{x}{3}} 1 \, dy \, dx$ with the order of integration $dx \, dy$

We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that for each fixed x between 0 and 6, the values of y are bounded by the curves $y = 0$ and $y = \frac{x}{3}$



We are integrating over the triangular domain between the lines $y = 0, x = 0$, and $y = \frac{x}{3}$. Drawing these blue lines from side to side instead:



We can say that for each fixed y between 0 and 2, the values of x are bounded by the curves $x = 3y$ and $x = 6$. Thus, we can rewrite the integral as

$$\int_0^2 \int_{3y}^6 1 \, dx \, dy = \int_0^2 x \Big|_{3y}^6 dy = \int_0^2 (6 - 3y) \, dy = \left[6y - \frac{3}{2}y^2\right] \Big|_0^2 = 6$$

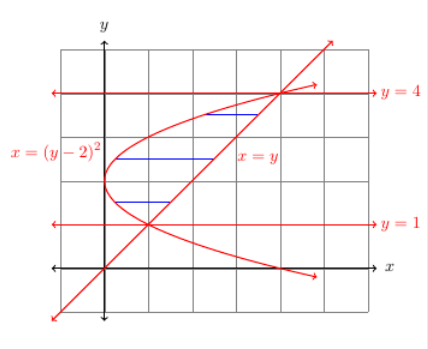
This is exactly the area we would get by solving the original integral (or using the area of a triangle formula).

Illustration 3.1.2: Changing the order of integration

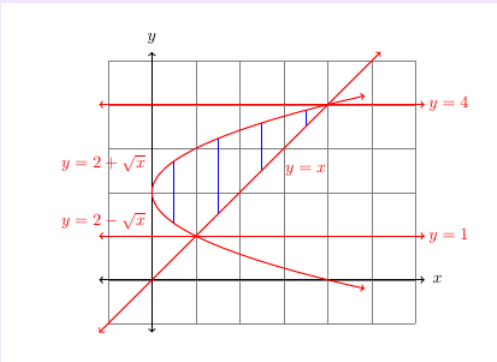
We will change the order of integration of $\int_1^4 \int_{(y-2)^2}^{y} 1 \, dx \, dy$

We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that for each fixed y between 1 and 4, the values of x are bounded by the curves $x = (y-2)^2$ and $x = y$



We are integrating over the domain between the curves $x = (y-2)^2$ and $x = y$. Drawing these blue lines from bottom to top instead:



We can see this region is described in two pieces. For each fixed x between 0 and 1, the values of x are bounded by the curves $y = 2 - \sqrt{x}$ and $y = 2 + \sqrt{x}$, then for each value of y between 1 and 4, the values of x are bounded by the curves $y = x$ and $y = 2 + \sqrt{x}$. Thus, we can rewrite the integral as

$$\int_0^1 \int_{2-\sqrt{x}}^{2+\sqrt{x}} 1 \, dy \, dx + \int_1^4 \int_x^{2+\sqrt{x}} 1 \, dy \, dx$$

The sum of these integrals evaluates to exactly the same number that $\int_1^4 \int_{(y-2)^2}^{y} 1 \, dx \, dy$ does, which is $\frac{9}{2}$

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves. The "real" goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral.

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3.1: Double Integration and Volume

The definite integral of f over $[a, b]$, $\int_a^b f(x)\,dx$, was introduced as "the signed area under the curve." We approximated the value of this area by first subdividing $[a, b]$ into n subintervals, where each subinterval has length $\Delta x = \dfrac{b-a}{n}$, and letting c_i be any value in the ith subinterval. We formed rectangles that approximated part of the region under the curve with width Δx, height $f(c_i)$, and hence with area $f(c_i)\,\Delta x$. Summing all the rectangles' areas gave an approximation of the definite integral, and taking the limit of these approximations gives us its exact value:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(c_i)\,\Delta x$$

We use a similar approach in this section to find volume under a surface of the form $z = f(x, y)$.

Let R be a closed, bounded region in the xy-plane and let $z = f(x, y)$ be a continuous function defined on R. We wish to find the signed volume under the surface of f over a rectangle R. (We use the term "signed volume" to denote that space above the xy-plane, under f, will have a positive volume; space above f and under the xy-plane will have a "negative" volume, similar to the notion of signed area used before.)

We start by partitioning $R = \{(x, y) : a \le x \le b, c \le y \le d\}$ into nm rectangular subregions as shown below. For simplicity's sake, we let all widths be $\Delta x = \dfrac{b-a}{n}$ and all heights be $\Delta y = \dfrac{d-c}{m}$. Note that the sum of the areas of the rectangles is equal to the area of R. Further define the point $(x_i,\ y_j)$ in the ith box on the x axis, and j^{th} box on the y-axis.

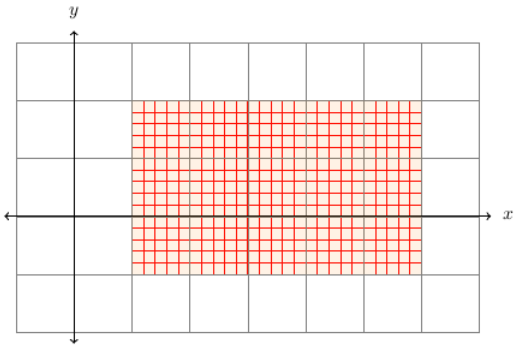


Figure 3.1.3: Approximating the domain R by splitting into nm different Δx by Δy rectangles

The volume of the rectangular solid whose base contains (x_i, y_j) and whose height is $f(x_i,\ y_j)$ is $V_{i,j} = f(x_i,\ y_j)\,\Delta x\,\Delta y$. Such a solid is shown in the figure below. Note how this rectangular solid only approximates the true volume under the surface; part of the solid is above the surface and part is below.

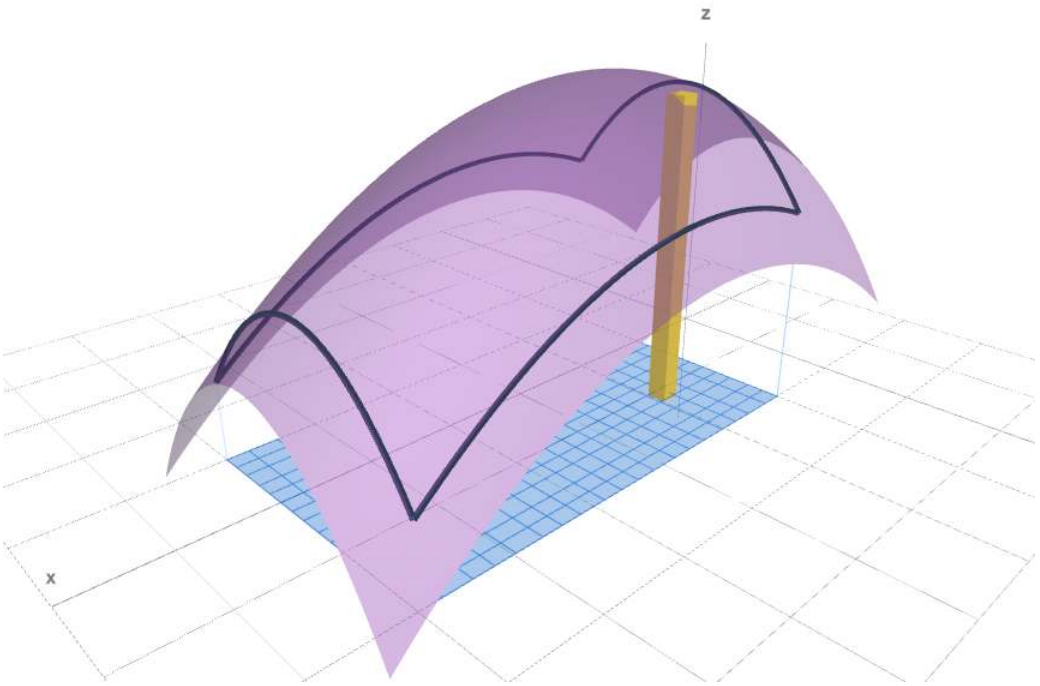


Figure 3.1.4: One of the n rectangular prisms used to approximate the volume underneath $z = f(x, y)$ over R

For each subregion $R_{i,j}$ in R, create the rectangular solid with base area $\Delta x\,\Delta y$ and height $f(x_i,\ y_j)$. The sum of all rectangular solids is

$$\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\,\Delta x\,\Delta y$$

This approximates the signed volume under f over R. As we have done before, to get a better approximation we can use more rectangles to approximate the region R. To get the exact value, we take the limit of these sums as n and m go to infinity (causing both Δx and Δy to tend to zero).

Note

Recall that the integration symbol "$\int$" is an "elongated S," representing the word "sum." We interpreted $\int_a^b f(x)\,dx$ as "take the *sum* of the areas of rectangles over the interval $[a, b]$." The double integral uses two integration symbols to represent a "double sum."

When adding up the volumes of rectangular solids over a partition of a region R, one could first add up the volumes across each row (one type of sum), then add these totals together (another sum), as in

$$\lim_{n\to\infty}\lim_{m\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\,\Delta x\,\Delta y = \lim_{m\to\infty}\sum_{j=1}^{m}\left(\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_j)\,\Delta x\right)\,\Delta y$$

The summation inside the parentheses indicates the sum of heights $\times$ widths, which gives an area; multiplying these areas by the thickness Δy gives a volume. The volume of this "slice" parallel to the y-axis is $\left(\int_a^b f(x, y)dx\right)\,dy$

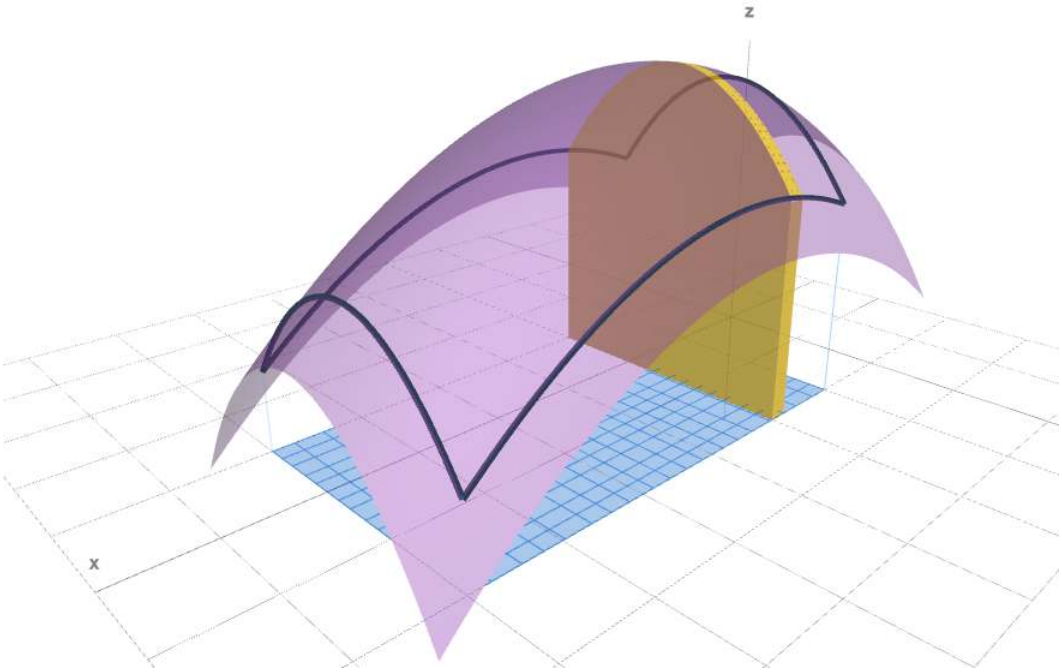


Figure 3.1.5: One of the m slices parallel to the y axis used to find the volume underneath $z = f(x, y)$ over R

Taking the second sum and limit adds up all such slices and gives us the total volume we are looking for:

$$\int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Alternatively, one could first add up the volumes across each column instead of row, then add these totals together, as in

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j)\, \Delta x\, \Delta y = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\lim_{m\to\infty} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta y \right) \Delta x$$

The summation inside the parenthesis indicates the sum of heights $\times$ widths, which gives an area; multiplying these areas by the thickness Δx gives a volume. The volume of this "slice" parallel to the x-axis is $\left(\int_c^d f(x,y)dy \right) dx$

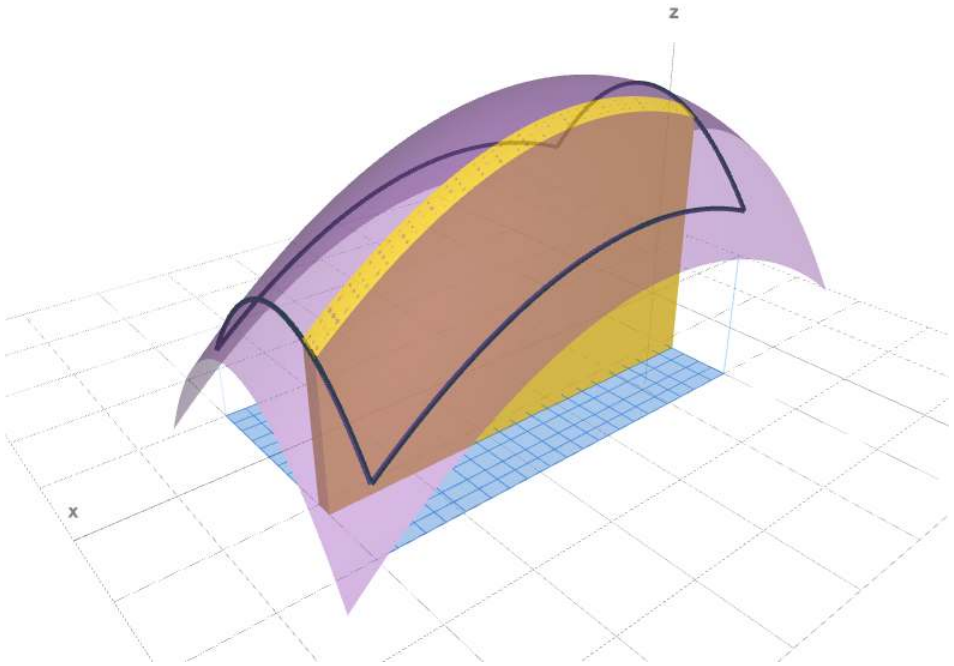


Figure 3.1.6: One of the n slices parallel to the x axis used to find the volume underneath $z = f(x, y)$ over R

Taking the second sum and limit adds up all such slices and gives us the total volume we are looking for in another form:

$$\int_a^b \left(\int_c^d f(x,y) dy \right) dx$$

Definition: Double Integral, Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the xy-plane. The ***signed volume*** V under f over R is denoted by the ***double integral***

$$V = \iint_R f(x,y)\, dA$$

Alternate notations for the double integral are

$$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy = \iint_R f(x,y)\, dy\, dx$$

This gives a concrete method for finding signed volume under a surface. If R is a rectangle, it is easy to see these methods match. But as we have recently seen, not all double integrals need to be over a rectangular domain.

Definition: Type 1 and Type 2 Regions

A region R in $\mathbb{R}^2$ is called a ***Type 1 Region*** if it can be written as $R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ for some constants $a, b \in \mathbb{R}$ and functions $g_1(x), g_2(x)$ which are continuous on $[a, b]$.

A region R in $\mathbb{R}^2$ is called a ***Type 2 Region*** if it can be written as $R = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$ for some constants $c, d \in \mathbb{R}$ and functions $h_1(y), h_2(y)$ which are continuous on $[c, d]$.

Many regions in $\mathbb{R}^2$ are either Type 1 or Type 2, and some qualify as both. One easy example is a rectangle, but more complicated ones also exist (such as a right triangle or circle).

The following theorem states that both $dxdy$ or $dydx$ double integrals give the same result, which is the value of the double integral. It is such an important theorem that it has a name associated with it.

Theorem: Fubini's Theorem

Let R be a closed, bounded region in the xy-plane and let $z = f(x, y)$ be a continuous function on R.

1. If R is a Type 1 region bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, then $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$
2. If R is a Type 2 region bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, then $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Note that once again the bounds of integration follow the "curve to curve, point to point" pattern discussed earlier. In fact, one of the main points of the previous section is developing the skill of describing a region R with the bounds of an iterated integral. Once this skill is developed, we can use double integrals to compute many quantities, not just signed volume under a surface.

Illustration 3.1.3: Evaluating a Double Integral

Let $f(x, y) = xy + e^y$. We will find the signed volume under f on the region R, which is the square with corners $(3, 1), (3, 2), (4, 1)$ and $(4, 2)$ using both orders of integration.

We wish to evaluate $\iint_R \left(xy + e^y\right) dA$. As R is a rectangle, the bounds are easily described as $3 \le x \le 4$ and $1 \le y \le 2$



Using the order $dy\,dx$:

$$\begin{aligned}
\iint_R \left(xy + e^y\right) dA &= \int_3^4 \int_1^2 \left(xy + e^y\right) dy\,dx \\
&= \int_3^4 \left[\frac{1}{2}xy^2 + e^y\right]\Big|_1^2 dx \\
&= \int_3^4 \left(\frac{3}{2}x + e^2 - e\right) dx \\
&= \left[\frac{3}{4}x^2 + \left(e^2 - e\right)x\right]\Big|_3^4 \\
&= \frac{21}{4} + e^2 - e
\end{aligned}$$

Now we check the validity of Fubini's Theorem by using the order $dx\,dy$:

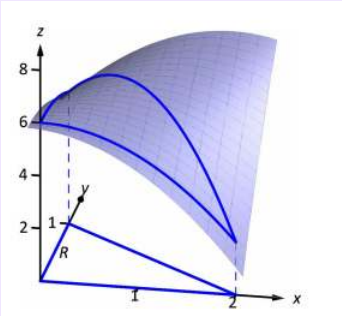
$$\begin{aligned}\iint_R \left(xy+e^y\right)dA &= \int_1^2\int_3^4 \left(xy+e^y\right)dx\,dy \\ &= \int_1^2 \left[\frac{1}{2}x^2y+xe^y\right]\Big|_3^4 dy \\ &= \int_1^2 \left(\frac{7}{2}y+e^y\right)dy \\ &= \left[\frac{7}{4}y^2+e^y\right]\Big|_1^2 \\ &= \frac{21}{4}+e^2-e\end{aligned}$$

Both orders of integration return the same result, as expected.

Illustration 3.1.4: Evaluating a Double Integral

We will evaluate $\iint_R \left(3xy-x^2-y^2+6\right)dA$, where R is the triangle bounded by $x=0$, $y=0$ and $\frac{x}{2}+y=1$ both ways.

A quick sketch of the volume we are looking for:



Using the order $dy\,dx$:

For each fixed x between 0 and 2, we see that y goes from the function $y=0$ to $y=1-\frac{x}{2}$:

$$\begin{aligned}\iint_R (3xy-x^2-y^2+6)\,dA &= \int_0^2\int_0^{1-\frac{x}{2}} (3xy-x^2-y^2+6)\,dy\,dx \\ &= \int_0^2 \left[\frac{3}{2}xy^2-x^2y-\frac{1}{3}y^3+6y\right]\Big|_0^{1-\frac{x}{2}} dx \\ &= \int_0^2 \left(\frac{11}{12}x^3-\frac{11}{4}x^2-x+\frac{17}{3}\right)dx \\ &= \left[\frac{11}{48}x^4-\frac{11}{12}x^3-\frac{1}{2}x^2+\frac{17}{3}x\right]\Big|_0^2 \\ &= \frac{17}{3}\end{aligned}$$

Now let's consider the order $dx\,dy$:

For each fixed y between 0 and 1, we see that x goes from $x=0$ to $x=2-2y$:

$$\begin{aligned}\iint_R (3xy-x^2-y^2+6)\,dA &= \int_0^1\int_0^{2-2y} \left(3xy-x^2-y^2+6\right)dx\,dy \\ &= \int_0^1 \left[\frac{3}{2}x^2y-\frac{1}{3}x^3-xy^2+6x\right]\Big|_0^{2-2y} dy \\ &= \int_0^1 \left(\frac{32}{3}y^3-22y^2+2y+\frac{28}{3}\right)dy \\ &= \left[\frac{8}{3}y^4-\frac{22}{3}y^3+y^2+\frac{28}{3}y\right]\Big|_0^1 \\ &= \frac{17}{3}\end{aligned}$$

Once again, as expected, we obtained the same result using both orders of integration.

Note that in these two examples the bounds of integration depend only on R; *the bounds of integration have nothing to do with* $f(x,y)$.

Before doing another example, we give some properties of double integrals. Each should make sense if we view them in the context of finding signed volume under a surface, over a region.

Theorem: Properties of Double Integrals

Let f and g be continuous functions over a closed, bounded plane region R, and let c be a constant.

1. $\iint_R c\, f(x,y)dA = c\iint_R f(x,y)\,dA$
2. $\iint_R \big(f(x,y) \pm g(x,y)\big)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$
3. If $f(x,y) \geq 0$ on R, then $\iint_R f(x,y)\,dA \geq 0$
4. If $f(x,y) \geq g(x,y)$ on R, then $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$
5. Let R be the union of two nonoverlapping regions, $R = R_1 \bigcup R_2$. Then $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

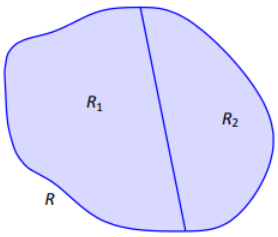


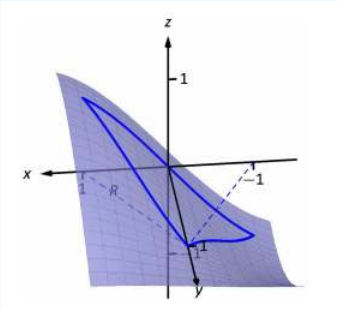
Figure 3.1.7: R is the union of two nonoverlapping regions, R_1 and R_2.

Example 3.1.4: Evaluating a Double Integral

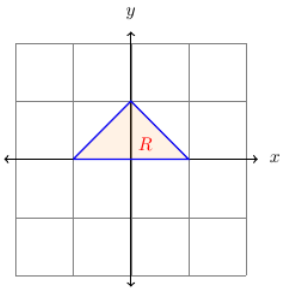
Let $f(x,y) = \sin(x)\cos(y)$ and R be the triangle with vertices $(-1,0)$, $(1,0)$ and $(0,1)$. Evaluate the double integral $\iint_R f(x,y)\,dA$

Solution

We start with a quick sketch of the volume we are looking for:



As well as a sketch of the domain R:



If we attempt to integrate using an iterated integral with the order $dydx$, the values of y range between the lower function $y = 0$ and a piecewise upper function, meaning we'll need to use two iterated integrals. We would need to split the triangle into two regions along the y-axis, then calculate these two pieces separately.

Instead, let's use the order $dxdy$. The curves bounding x are $y - 1 \leq x \leq 1 - y$; the bounds on y are $0 \leq y \leq 1$. This gives us:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 \int_{y-1}^{1-y} \sin(x)\cos(y)\,dx\,dy \\ &= \int_0^1 -\cos(x)\cos(y)\Big|_{y-1}^{1-y} dy \\ &= \int_0^1 \cos(y)\Big(-\cos(1-y) + \cos(y-1)\Big)\,dy\end{aligned}$$

Recall that the cosine function is an even function; that is, $\cos(\theta) = \cos(-\theta)$. Therefore, from the last integral above, we have $\cos(y-1) = \cos(1-y)$. Thus the integrand simplifies to 0, and we have

$$\iint_R f(x,y)\,dA = \int_0^1 0\,dy = 0$$

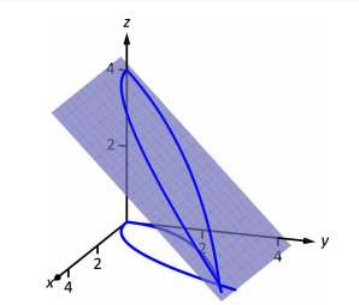
It turns out that over R, there is just as much volume above the xy-plane as below, giving a final signed net volume of $\boxed{0 \text{ cubic units}}$

Example 3.1.5: Evaluating a Double Integral

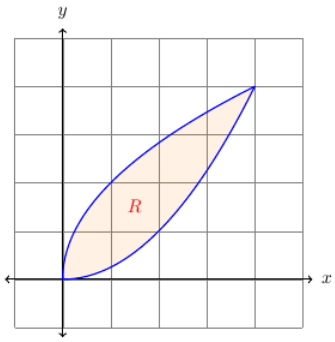
Evaluate $\iint_R (4-y)\,dA$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

Solution

We start with a quick sketch of the volume we are looking for:



As well as a sketch of the domain R:



The regions intersect at $(0,0)$ and $(4,4)$, as shown above.

We now choose an order of integration: $dy\,dx$ or $dx\,dy$? Either order works; since the integrand does not contain x, choosing $dx\,dy$ might be simpler (at least the first integral will be very simple).

Thus we have the following bounds: $\frac{y^2}{4} \le x \le 2\sqrt{y}$, and $0 \le y \le 4$:

$$\begin{aligned}
\iint_R (4-y)\,dA &= \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} (4-y)\,dx\,dy \\
&= \int_0^4 x(4-y)\Big|_{\frac{y^2}{4}}^{2\sqrt{y}}\,dy \\
&= \int_0^4 \left(2\sqrt{y} - \frac{1}{4}y^2\right)(4-y)\,dy \\
&= \int_0^4 \left(\frac{y^3}{4} - y^2 - 2y^{\frac{3}{2}} + 8y^{\frac{1}{2}}\right)\,dy \\
&= \left[\frac{1}{16}y^4 - \frac{1}{3}y^3 - \frac{4}{5}y^{\frac{5}{2}} + \frac{16}{3}y^{\frac{3}{2}}\right]\Big|_0^4 \\
&= \frac{176}{15}
\end{aligned}$$

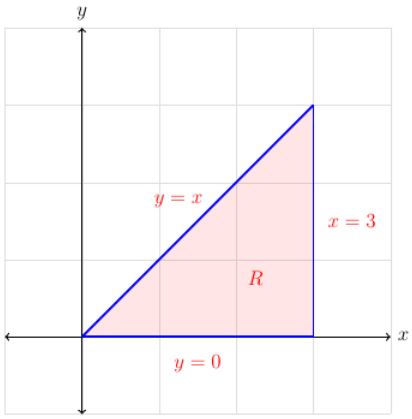
The signed volume under the surface f is $\boxed{\frac{176}{15} \text{ cubic units}}$

We have practiced changing the order of integration of a given iterated integral, where the region R was not explicitly given. Changing the bounds of an integral is more than just a test of understanding. Rather, there are cases where integrating in one order is really hard, if not impossible, whereas integrating with the other order is feasible.

Illustration 3.1.5: Changing the Order of Integration

Rewrite the iterated integral $\int_0^3 \int_y^3 e^{-x^2}\,dx\,dy$ with the order $dy\,dx$.

Once again we make a sketch of the region over which we are integrating to facilitate changing the order. The bounds on x are from $x = y$ to $x = 3$; the bounds on y are from $y = 0$ to $y = 3$. These curves are sketched below, enclosing the region R.



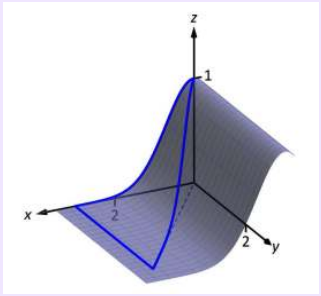
To change the bounds, note that the curves bounding y are $y = 0$ up to $y = x$; the triangle is enclosed between $x = 0$ and $x = 3$. Thus the new bounds of integration are $0 \le y \le x$ and $0 \le x \le 3$, giving the iterated integral $\int_0^3 \int_0^x e^{-x^2}\, dy\, dx$.

How easy is it to evaluate each iterated integral? Consider the order of integrating $dxdy$, as given in the original problem. The first indefinite integral we need to evaluate is $\int e^{-x^2}\, dx$, which cannot be evaluated in terms of elementary functions. We are stuck.

Changing the order of integration makes a big difference here:

$$\begin{aligned}\int_0^3 \int_0^x e^{-x^2}\, dy\, dx &= \int_0^3 ye^{-x^2}\Big|_0^x dx \\ &= \int_0^3 xe^{-x^2}\, dx \\ &= \left[-\frac{1}{2}e^{-x^2}\right]\Big|_0^3 \\ &= -\frac{1}{2}\left(e^{-9} - e^0\right) \\ &= \frac{1 - e^{-9}}{2}\end{aligned}$$

Below is a sketch of the volume we just calculated:



In short, evaluating one iterated integral is impossible; the other iterated integral is relatively simple. This is why being able to switch between Type 1 and Type 2 notation is so important!

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3.1: Practice Set

Exercise 3.1.1

Evaluate $\int_0^1 \int_0^\pi x \sin(xy)\, dy\, dx$

Answer

$$\begin{aligned}\int_0^1 \int_0^\pi x \sin(xy)\, dy\, dx &= \int_0^1 -\cos(xy)\Big|_0^\pi dx \\ &= \int_0^1 (1-\cos(\pi x))\, dx \\ &= \left[x - \frac{1}{\pi}\sin(\pi x)\right]\Big|_0^1 \\ &= 1\end{aligned}$$

Exercise 3.1.2

Evaluate $\int_0^1 \int_0^\pi y \sin(xy)\, dy\, dx$

Answer

$$\begin{aligned}\int_0^1 \int_0^\pi y \sin(xy)\, dy\, dx &= \int_0^1 \left(\left[y \cdot \left(-\frac{1}{x}\cos(xy)\right)\right]\Big|_0^\pi - \int_0^\pi -\frac{1}{x}\cos(xy)\, dy\right) dx \\ &= \int_0^1 \left[y \cdot \left(-\frac{1}{x}\cos(xy)\right) + \frac{1}{x^2}\sin(xy)\right]\Big|_0^\pi dx \\ &= \int_0^1 \left(\frac{1}{x^2}\sin(\pi x) - \frac{\pi}{x}\cos(\pi x)\right) dx\end{aligned}$$

This integral looks difficult! Instead we switch the order of integration (which is easy since we are integrating over a rectangular domain):

$$\begin{aligned}\int_0^1 \int_0^\pi y \sin(xy)\, dy\, dx &= \int_0^\pi \int_0^1 y \sin(xy)\, dx\, dy \\ &= \int_0^\pi -\cos(xy)\Big|_0^1 dy \\ &= \int_0^\pi (1-\cos(y))\, dy \\ &= [y - \sin(y)]\Big|_0^\pi \\ &= \pi\end{aligned}$$

Exercise 3.1.3

Evaluate $\iint_D \frac{12}{y + yx^2}\, dA$ for $D = \{(x,y) : 1 \le x \le \sqrt{3},\ 1 \le y \le e\}$

Answer

$$\begin{aligned}\iint_D \frac{12}{y+yx^2}\,dA &= \int_1^{\sqrt{3}}\int_1^e \frac{12}{y(1+x^2)}\,dy\,dx \\ &= \left(\int_1^{\sqrt{3}} \frac{12}{1+x^2}\,dx\right)\left(\int_1^e \frac{1}{y}\,dy\right) \\ &= \left(12\tan^{-1}(x)\Big|_1^{\sqrt{3}}\right)\cdot\left(\ln|y|\Big|_1^e\right) \\ &= (4\pi - 3\pi)\cdot(\ln(e)-\ln(1)) \\ &= \pi\end{aligned}$$

Exercise 3.1.4

Evaluate $\int_0^1\int_0^x \cos(\pi x^2)\,dy\,dx$

Answer

$$\begin{aligned}\int_0^1\int_0^x \cos(\pi x^2)\,dy\,dx &= \int_0^1 y\cos(\pi x^2)\Big|_0^x\,dx \\ &= \int_0^1 x\cos(\pi x^2)\,dx \\ &= \frac{1}{2\pi}\sin(\pi x^2)\Big|_0^1 \\ &= \frac{1}{2\pi}\sin(\pi) - \frac{1}{2\pi}\sin(0) \\ &= 0\end{aligned}$$

Exercise 3.1.5

Evaluate $\int_1^2\int_1^{x^3} \frac{x^4}{y^3}\,dy\,dx$

Answer

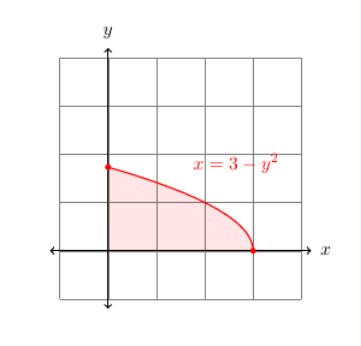
$$\begin{aligned}\int_1^2\int_1^{x^3} \frac{x^4}{y^3}\,dy\,dx &= \int_1^2 \frac{1}{-2}x^4y^{-2}\Big|_1^{x^3}\,dx \\ &= \int_1^2 \left(-\frac{1}{2}x^4\cdot x^{-6} + \frac{1}{2}x^4\cdot 1^{-2}\right)\,dx \\ &= \int_1^2 \left(-\frac{1}{2}x^{-2} + \frac{1}{2}x^4\right)\,dx \\ &= \left[\frac{1}{2}x^{-1} + \frac{1}{10}x^5\right]\Big|_1^2 \\ &= \left(\frac{1}{4}+\frac{16}{5}\right) - \left(\frac{1}{2}+\frac{1}{10}\right) \\ &= \frac{57}{20}\end{aligned}$$

Exercise 3.1.6

Evaluate $\int_0^{\sqrt{3}}\int_0^{3-y^2} 16y\cdot\sqrt[3]{x^2-6x+1}\,dx\,dy$

Answer

This integral dx is not simple to calculate, so we instead change the order of integration. First we sketch the region we are integrating over:



We can write it as $D = \left\{(x,y) \,:\, 0 \le y \le \sqrt{3},\, 0 \le x \le 3-y^2\right\} = \left\{(x,y) \,:\, 0 \le x \le 3,\, 0 \le y \le \sqrt{3-x}\right\}$, so

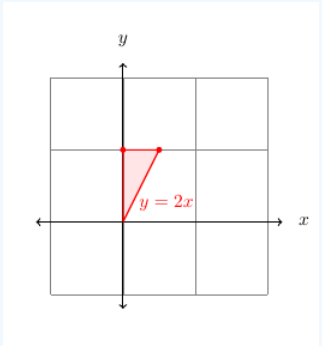
$$\begin{aligned}
\int_0^{\sqrt{3}} \int_0^{3-y^2} 16y \cdot \sqrt[3]{x^2-6x+1}\, dx\, dy &= \int_0^3 \int_0^{\sqrt{3-x}} 16y \cdot \sqrt[3]{x^2-6x+1}\, dy\, dx \\
&= \int_0^3 8y^2 \cdot \sqrt[3]{x^2-6x+1}\Big|_0^{\sqrt{3-x}}\, dx \\
&= \int_0^3 \left(8(3-x) \cdot \sqrt[3]{x^2-6x+1}\right)\, dx \\
&= -3(x^2-6x+1)^{\frac{4}{3}}\Big|_0^3 \\
&= -3\left((-8)^{\frac{4}{3}} - 1\right) \\
&= -45
\end{aligned}$$

Exercise 3.1.7

Evaluate $\int_0^{\frac{1}{2}} \int_{2x}^1 \frac{1}{\sqrt{1-y^4}}\, dy\, dx$

Answer

This integral dy is not simple to calculate, so we instead change the order of integration. First we sketch the region we are integrating over:



We can write it as $D = \left\{(x,y) \,:\, 0 \le x \le \frac{1}{2},\, 2x \le y \le 1\right\} = \left\{(x,y) \,:\, 0 \le y \le 1,\, 0 \le x \le \frac{y}{2}\right\}$, so

$$\int_0^{\frac{1}{2}}\int_{2x}^{1}\frac{1}{\sqrt{1-y^4}}\,dy\,dx = \int_0^{1}\int_0^{\frac{y}{2}}\frac{1}{\sqrt{1-y^4}}\,dx\,dy$$
$$= \int_0^1 \frac{1}{\sqrt{1-y^4}}x\Big|_0^{\frac{y}{2}}\,dy$$
$$= \int_0^1 \frac{y}{2\sqrt{1-y^4}}\,dy$$
$$= \frac{1}{4}\sin^{-1}(y^2)\Big|_0^1$$
$$= \frac{1}{4}\left(\sin^{-1}(1)-\sin^{-1}(0)\right)$$
$$= \frac{\pi}{8}$$

Exercise 3.1.8

Evaluate $\int_\pi^{2\pi}\int_y^{2y}\sin(y)\sin^3(x)\cos(x)\,dx\,dy$

Answer

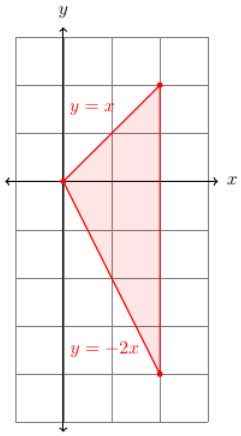
$$\int_\pi^{2\pi}\int_y^{2y}\sin(y)\sin^3(x)\cos(x)\,dx\,dy = \int_\pi^{2\pi}\frac{1}{4}\sin(y)\sin^4(x)\Big|_y^{2y}\,dy$$
$$= \frac{1}{4}\int_\pi^{2\pi}\left(\sin(y)\sin^4(2y)-\sin^5(y)\right)\,dy$$
$$= \frac{1}{4}\int_\pi^{2\pi}\left(16\sin^5(y)\cos^4(y)-\sin^5(y)\right)\,dy$$
$$= \frac{1}{4}\int_\pi^{2\pi}(16\cos^4(y)-1)(1-\cos^2(y))^2\sin(y)\,dy$$
$$= \frac{1}{4}\int_{-1}^{1}(16w^4-1)(1-w^2)^2\,(-dw)$$
$$= -\frac{1}{4}\int_{-1}^{1}(16w^8-32w^6+15w^4+2w^2-1)\,dw$$
$$= -\frac{1}{4}\left[\frac{16}{9}w^9-\frac{32}{7}w^7+3w^5+\frac{2}{3}w^3-w\right]\Big|_{-1}^{1}$$
$$= -\frac{1}{4}\left(\frac{32}{9}-\frac{64}{7}+6+\frac{4}{3}-2\right)$$
$$= \frac{4}{63}$$

Exercise 3.1.9

Evaluate $\iint_T 5xy^2\,dA$ for T the triangle with vertices $(0,0),(2,2)$, and $(2,-4)$

Answer

We see the domain we are integrating over is a triangle:



We can define this triangle most easily as a Type 1 region: $T = \{(x, y) : 0 \le x \le 2, -2x \le y \le x\}$, so

$$\begin{aligned}\iint_T 5xy^2\,dA &= \int_0^2 \int_{-2x}^{x} 5xy^2\,dy\,dx \\ &= \int_0^2 \frac{5}{3}xy^3\Big|_{-2x}^{x}\,dx \\ &= \int_0^2 15x^4\,dx \\ &= 3x^5\Big|_0^2 \\ &= 96\end{aligned}$$

Exercise 3.1.10

Evaluate $\int_1^2 \int_{x+1}^{2x+1} \frac{1}{y^3}\,dy\,dx$

Answer

$$\begin{aligned}\int_1^2 \int_{x+1}^{2x+1} \frac{1}{y^3}\,dy\,dx &= \int_1^2 -\frac{1}{2}y^{-2}\Big|_{x+1}^{2x+1}\,dx \\ &= -\frac{1}{2}\int_1^2 \left((2x+1)^{-2} - (x+1)^{-2}\right)\,dx \\ &= -\frac{1}{2}\left[-\frac{1}{2}(2x+1)^{-1} + (x+1)^{-1}\right]\Big|_1^2 \\ &= -\frac{1}{2}\left(-\frac{1}{10} + \frac{1}{3} + \frac{1}{6} - \frac{1}{2}\right) \\ &= \frac{1}{20}\end{aligned}$$

Exercise 3.1.11

Evaluate $\int_0^1 \int_{x^2}^{x} xe^y\,dy\,dx$

Answer

$$\begin{aligned}\int_0^1 \int_{x^2}^{x} xe^y \, dy \, dx &= \int_0^1 xe^y \Big|_{x^2}^{x} \, dx \\ &= \int_0^1 \left(xe^x - xe^{x^2}\right) \, dx \\ &= \left[(x-1)e^x - \frac{1}{2}e^{x^2}\right]\Big|_0^1 \\ &= \frac{3-e}{2}\end{aligned}$$

Exercise 3.1.12

Evaluate $\int_3^4 \int_0^6 \frac{y}{x^3 - x^2 - 2x} \, dy \, dx$

Answer

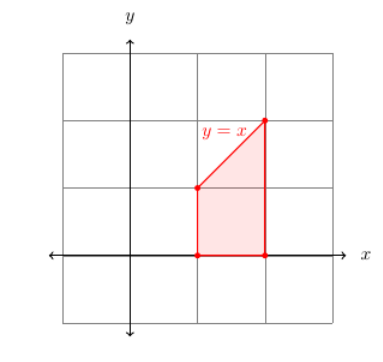
$$\begin{aligned}\int_3^4 \int_0^6 \frac{y}{x^3 - x^2 - 2x} \, dy \, dx &= \int_3^4 \frac{1}{2x(x-2)(x+1)} y^2 \Big|_0^6 \, dx \\ &= \int_3^4 \frac{18}{x(x-2)(x+1)} \, dx \\ &= \int_3^4 \left(-\frac{9}{x} + \frac{3}{x-2} + \frac{6}{x+1}\right) \, dx \\ &= \left[-9\ln|x| + 3\ln|x-2| + 6\ln|x+1|\right]\Big|_3^4 \\ &= (-9\ln(4) + 3\ln(2) + 6\ln(5)) - (-9\ln(3) + 3\ln(1) + 6\ln(4)) \\ &= 6\ln(5) + 9\ln(3) - 27\ln(2)\end{aligned}$$

Exercise 3.1.13

Evaluate $\iint_D \frac{1}{(x+y)^2} \, dA$ for $D = \{(x, y) \, : \, 1 \le x \le 2, \, 0 \le y \le x\}$

Answer

We see the domain we are integrating over is a trapezoid:



This is already given as a Type 1 region, so:

$$\begin{aligned}\iint_D \frac{1}{(x+y)^2}\,dA &= \int_1^2 \int_0^x (x+y)^{-2}\,dy\,dx \\ &= \int_1^2 -(x+y)^{-1}\Big|_0^x\,dx \\ &= \int_1^2 \left(-\frac{1}{2x}+\frac{1}{x}\right)\,dx \\ &= \int_1^2 \frac{1}{2x}\,dx \\ &= \frac{1}{2}\ln|x|\Big|_1^2 \\ &= \frac{1}{2}\ln(2)-\frac{1}{2}\ln(1) \\ &= \frac{1}{2}\ln(2)\end{aligned}$$

Exercise 3.1.14

Evaluate $\int_1^4 \int_0^3 9\sqrt{x+xy}\,dy\,dx$

Answer

$$\begin{aligned}\int_1^4 \int_0^3 9\sqrt{x+xy}\,dy\,dx &= \int_1^4 \int_0^3 9\sqrt{x}\cdot\sqrt{1+y}\,dy\,dx \\ &= \left(\int_1^4 9\sqrt{x}\,dx\right)\left(\int_0^3 \sqrt{1+y}\,dy\right) \\ &= \left(6x^{\frac{3}{2}}\Big|_1^4\right)\cdot\left(\frac{2}{3}(1+y)^{\frac{3}{2}}\Big|_0^3\right) \\ &= 42\cdot\frac{14}{3} \\ &= 196\end{aligned}$$

Exercise 3.1.15

Evaluate $\int_0^\pi \int_{\cos(x)}^1 y^8\sin(x)\,dy\,dx$

Answer

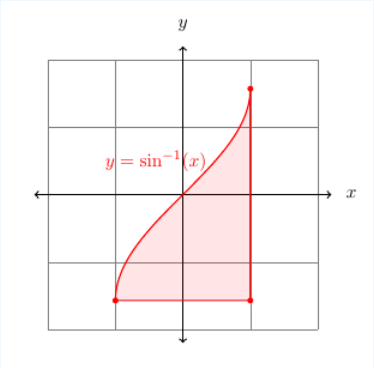
$$\begin{aligned}\int_0^\pi \int_{\cos(x)}^1 y^8\sin(x)\,dy\,dx &= \int_0^\pi \frac{1}{9}y^9\sin(x)\Big|_{\cos(x)}^1\,dx \\ &= \frac{1}{9}\int_0^\pi \left(1-\cos^9(x)\right)\sin(x)\,dx \\ &= \frac{1}{9}\int_1^{-1} \left(1-w^9\right)\,(-dw) \\ &= \frac{1}{9}\int_1^{-1} \left(w^9-1\right)\,dw \\ &= \frac{1}{9}\left[\frac{1}{10}w^{10}-w\right]\Big|_1^{-1} \\ &= \frac{2}{9}\end{aligned}$$

Exercise 3.1.16

Evaluate $\iint_E x^3\,dA$ where E is the region between $y = \sin^{-1}(x), y = -\frac{\pi}{2}$, and $x = 1$

Answer

We sketch the region E with given edges:



This is best described as a Type 2 region $E = \left\{(x, y) : -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \sin(y) \le x \le 1\right\}$, so:

$$\begin{aligned}
\iint_E x^3\,dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\sin(y)}^{1} x^3\,dx\,dy \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}x^4\Big|_{\sin(y)}^{1}\,dy \\
&= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \sin^4(y)\right)\,dy \\
&= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\frac{1}{8}\cos(4y) + \frac{1}{2}\cos(2y) + \frac{5}{8}\right)\,dy \\
&= \frac{1}{4}\left[-\frac{1}{32}\sin(4y) + \frac{1}{4}\sin(2y) + \frac{5}{8}y\right]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{4}\cdot\frac{5\pi}{8} \\
&= \frac{5\pi}{32}
\end{aligned}$$

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3.2: Triple Integrals and Polar Double Integrals

Learning Objectives

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Previously, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. We then extend the definition to more general regions in $\mathbb{R}^3$.

Returning to double integrals, we see that not all regions are Type I or Type II and sometimes they are much easier to evaluate if we change rectangular coordinates to polar coordinates.

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3.2: Triple Integrals

We can define a rectangular box B in $\mathbb{R}^3$ as $B = \{(x, y, z) : a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$.

We follow a similar procedure to what we did previously. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx, divide the interval $[c, d]$ into m subintervals $[y_{j-1}, y_j]$ of equal length Δy, and divide the interval $[e, f]$ into n subintervals $[z_{k-1}, z_k]$ of equal length Δz with

$$\Delta x = \frac{b-a}{l}, \quad \Delta y = \frac{d-c}{m}, \quad \Delta z = \frac{f-e}{n}$$

Then the rectangular box B is subdivided into lmn subboxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ as shown below:

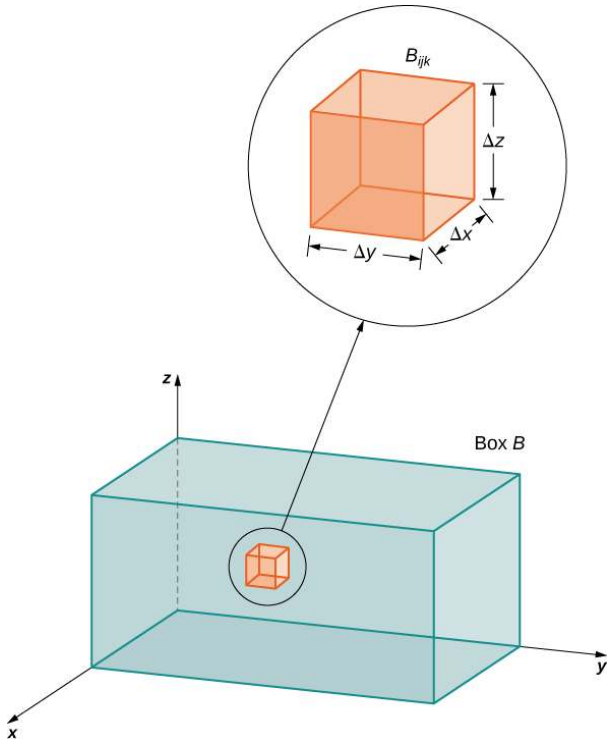


Figure 3.2.1: A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} g(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The Triple Integral

The triple integral of a function $g(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} g(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z = \iiint_B g(x, y, z)\, dV$$

if this limit exists.

When the triple integral exists on B, the function $g(x, y, z)$ is said to be integrable on B. For example, the triple integral always exists if $g(x, y, z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is

sufficient but not necessary; in other words, g is bounded on B and continuous except possibly on the boundary of B. The sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ can be any point in the rectangular sub-box B_{ijk}, and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of **Fubini's theorem** for triple integrals exists.

Theorem 3.2.1: Fubini's Theorem for Triple Integrals

If $g(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\begin{aligned}
\iiint_B g(x,y,z)\,dV &= \int_e^f \left(\int_c^d \left(\int_a^b g(x,y,z)\,dx \right) dy \right) dz \\
&= \int_c^d \left(\int_e^f \left(\int_a^b g(x,y,z)\,dx \right) dz \right) dy \\
&= \int_a^b \left(\int_e^f \left(\int_c^d g(x,y,z)\,dy \right) dz \right) dx \\
&= \int_e^f \left(\int_a^b \left(\int_c^d g(x,y,z)\,dy \right) dx \right) dz \\
&= \int_c^d \left(\int_a^b \left(\int_e^f g(x,y,z)\,dz \right) dx \right) dy \\
&= \int_a^b \left(\int_c^d \left(\int_e^f g(x,y,z)\,dz \right) dy \right) dx
\end{aligned}$$

These six versions all yield the same number.

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 3.2.1: Evaluating a Triple Integral

Evaluate the triple integrals

(a) $\displaystyle\int_0^1 \int_2^4 \int_{-1}^5 (x + yz^2)\,dx\,dy\,dz$

(b) $\displaystyle\iiint_B x^2yz\,dV$ where $B = \{(x, y, z) : -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$

Solution

(a) The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\begin{aligned}
\int_0^1 \int_2^4 \int_{-1}^5 (x + yz^2)\,dx\,dy\,dz &= \int_0^1 \int_2^4 \left[\frac{x^2}{2} + xyz^2 \right] \Big|_{-1}^5 dy\,dz \\
&= \int_0^1 \int_2^4 \left(12 + 6yz^2\right)\,dy\,dz \\
&= \int_0^1 \left[12y + 3y^2z^2\right] \Big|_2^4 dz \\
&= \int_0^1 (24 + 36z^2)\,dz \\
&= \left[24z + 12z^3\right] \Big|_0^1 \\
&= \boxed{36}
\end{aligned}$$

(b) The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 x^2yz\,dy\,dx\,dz\\ &= \int_1^5\int_{-2}^1 \frac{1}{2}x^2y^2z\Big|_0^3\,dx\,dz\\ &= \int_1^5\int_{-2}^1 \frac{9}{2}x^2z\,dx\,dz\\ &= \int_1^5 \frac{3}{2}x^3z\Big|_{-2}^1\,dz\\ &= \int_1^5 \frac{27}{2}z\,dz\\ &= \frac{27}{4}z^2\Big|_1^5\\ &= \boxed{162}\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 x^2yz\,dx\,dz\,dy\\ &= \int_0^3\int_1^5 \frac{1}{3}x^3yz\Big|_{-2}^1\,dz\,dy\\ &= \int_0^3\int_1^5 3yz\,dz\,dy\\ &= \int_0^3 \frac{3}{2}yz^2\Big|_1^5\,dy\\ &= \int_0^3 36y\,dy\\ &= 18y^2\Big|_0^3\\ &= 162\end{aligned}$$

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region** E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \{(x,y,z) : (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

For two functions $z = u_1(x,y)$ and $z = u_2(x,y)$, such that $u_1(x,y) \le u_2(x,y)$ for all (x,y) in D as shown in the following figure.

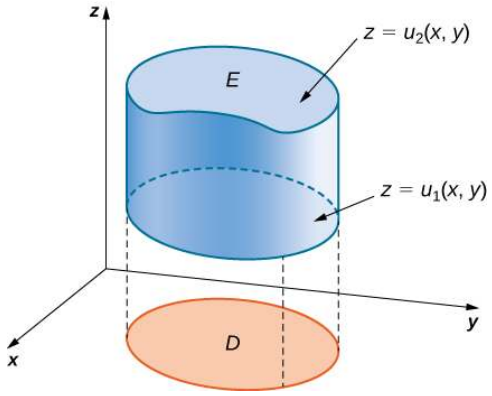


Figure 3.2.2: We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection of E onto the xy-plane, called D.

Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region $E = \{(x, y, z) \ : \ (x, y) \in D_{xy}, \ u_1(x, y) \leq z \leq u_2(x, y)\}$

in $\mathbb{R}^3$, where D_{xy} is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z)\, dV = \iint_{D_{xy}} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region $E = \{(x, y, z) \ : \ (x, z) \in D_{xz}, \ u_1(x, z) \leq y \leq u_2(x, z)\}$

in $\mathbb{R}^3$, where D_{xz} is the projection of E onto the xz-plane, is

$$\iiint_E f(x, y, z)\, dV = \iint_{D_{xz}} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region $E = \{(x, y, z) \ : \ (y, z) \in D_{yz}, \ u_1(y, z) \leq x \leq u_2(y, z)\}$

in $\mathbb{R}^3$, where D_{yz} is the projection of E onto the yz-plane, is

$$\iiint_E f(x, y, z)\, dV = \iint_{D_{yz}} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

Note that the region D in any of the planes may be of Type 1or Type 2 as described previously. If D in the xy-plane is of Type 1 like the figure below, then

$$E = \{(x, y, z) \ : \ a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x), \ u_1(x, y) \leq z \leq u_2(x, y)\}$$

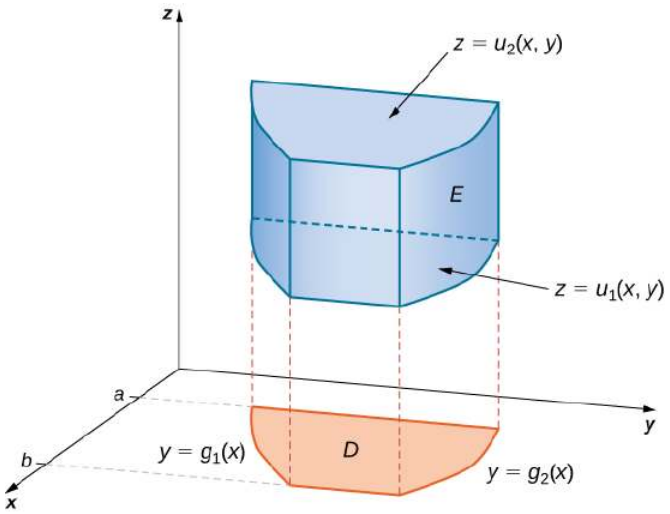


Figure 3.2.3: A box E where the projection D in the xy-plane is of Type 1.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

On the other hand, if D in the xy-plane is of Type 2 like in the following figure, then

$$E = \{(x,y,z) \,:\, c \le y \le d, h_1(y) \le x \le h_2(y),\ u_1(x,y) \le z \le u_2(x,y)\}$$

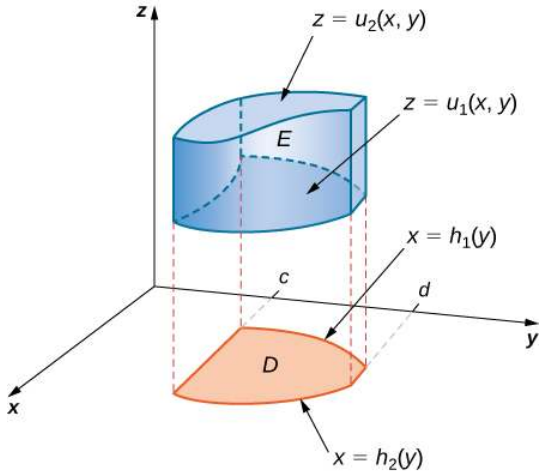


Figure 3.2.4: A box E where the projection D in the xy-plane is of Type 1.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

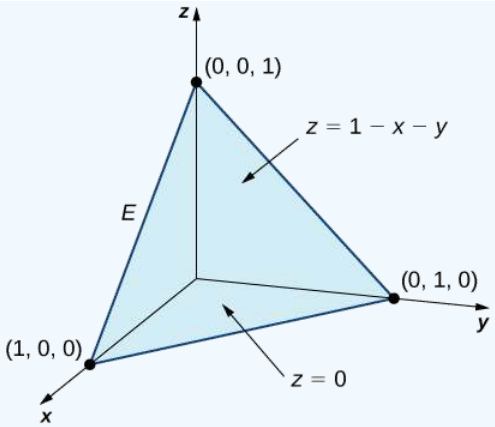
This can be generalized for all six possible orderings of dx, dy, dz.

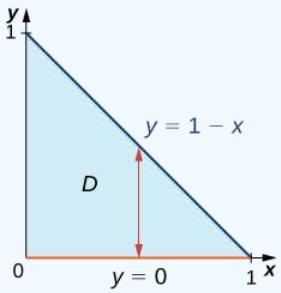
Example 3.2.2: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x,y,z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

We start with a quick sketch of the tetrahedron E and its projection D on the xy-plane.





We can describe D as a Type 1 region, so we write $E = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$

Hence, the triple integral is

$$\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (5x - 3y)\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} [5xz - 3yz]\Big|_0^{1-x-y}\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} (5x(1-x-y) - 3y(1-x-y))\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left(-5x^2 - 2xy + 5x + 3y^2 - 3y\right)\,dy\,dx \\
&= \int_0^1 \left[-5x^2y - xy^2 + 5xy + y^3 - \frac{3}{2}y^2\right]\Big|_0^{1-x}\,dx \\
&= \int_0^1 \left(-5x^2(1-x) - x(1-x)^2 + 5x(1-x) + (1-x)^3 - \frac{3}{2}(1-x)^2\right)\,dx \\
&= \int_0^1 \left(3x^3 - \frac{13}{2}x^2 + 4x - \frac{1}{2}\right)\,dx \\
&= \left[\frac{3}{4}x^4 - \frac{13}{6}x^3 + 2x^2 - \frac{1}{2}x\right]\Big|_0^1\,dx \\
&= \boxed{\frac{1}{12}}
\end{aligned}$$

Just as we used the double integral $\iint_D 1\,dA$ to find the area of a general bounded region D, we can use $\iiint_E 1\,dV$ to find the volume of a general solid bounded region E. The next example illustrates the method.

Example 3.2.3: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1, 1] \times [-1, 1]$ and vertex at the point $(0, 0, 1)$ as shown in the following figure.

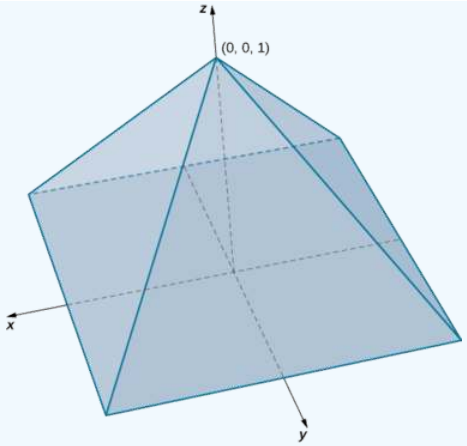


Figure 3.2.5: Find the volume of a pyramid with a square base.

Solution

In this pyramid, the value of z changes from 0 to 1, and at each height z the cross section of the pyramid for any value of z is also a square. Since the sides are defined by straight lines, we know the corners of these squares must be $(\pm s(z), \pm s(z), z)$ for some linear function $s(z)$.

At the bottom when $z = 0$, these corners are $(\pm 1, \pm 1, 0)$, so $s(0) = 1$. At the top when $z = 1$, the corners are all $(\pm 0, \pm 0, 1)$, so $s(1) = 0$. This means the function $s(z) = 1 - z$, and we have the corners at each height z are $(\pm(1-z), \pm(1-z), z)$.

Hence, the volume of the pyramid is

$$\iiint_E 1\,dV \quad \text{where} \quad E = \{(x, y, z) : 0 \le z \le 1,\ -1 + z \le y \le 1 - z,\ -1 + z \le x \le 1 - z\}$$

We evaluate the triple integral such that dz happens last (since the other two limits of integration depend on z):

$$\begin{aligned}
\iiint_E 1\,dV &= \int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} 1\,dx\,dy\,dz \\
&= \int_0^1 \int_{-1+z}^{1-z} x\Big|_{-1+z}^{1-z}\,dy\,dz \\
&= \int_0^1 \int_{-1+z}^{1-z} (2-2z)\,dy\,dz \\
&= \int_0^1 (2-2z)y\Big|_{-1+z}^{1-z}\,dz \\
&= \int_0^1 (2-2z)^2\,dz \\
&= -\frac{1}{6}(2-2z)^3\Big|_0^1 = \frac{4}{3}
\end{aligned}$$

Hence, the volume of the pyramid is $\boxed{\frac{4}{3} \text{ cubic units}}$

As we have already seen in double integrals over general bounded regions, changing the order of integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit.

Illustration 3.2.1: Changing the Order of Integration

Consider the iterated integral

$$\int_0^1 \int_0^{x^2} \int_0^{y^2} f(x,y,z)\,dz\,dy\,dx$$

The order of integration here is first with respect to z, then y, and then x. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. We will verify that the value of the integral is the same if we let $f(x,y,z) = xyz$

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

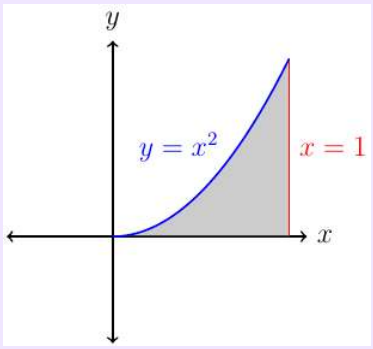
$$\int_0^1 \int_0^{x^2} \int_0^{y^2} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV \text{ where } E$$
$$= \left\{(x,y,z) \,:\, 0 \le x \le 1,\, 0 \le y \le x^2,\, 0 \le z \le y^2\right\}$$

We need to express this region E as

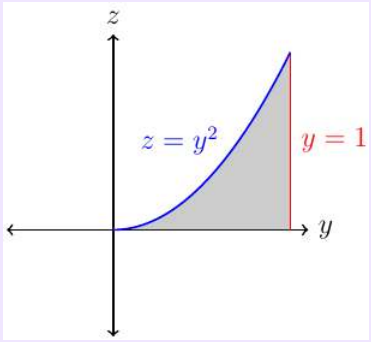
$$E = \left\{(x,y,z) \,:\, c \le y \le d,\, v_1(y) \le z \le v_2(y),\, u_1(y,z) \le x \le u_2(y,z)\right\}$$

Knowing the region E we can calculate and draw the following projections:

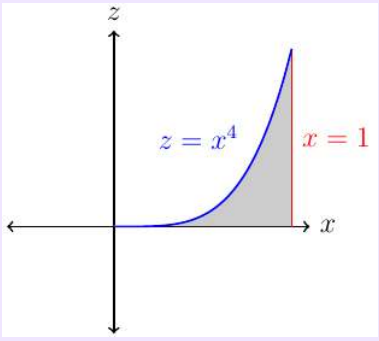
On the xy-plane $D_{xy} = \left\{(x,y) \,:\, 0 \le x \le 1,\, 0 \le y \le x^2\right\} = \left\{(x,y) \,:\, 0 \le y \le 1,\, \sqrt{y} \le x \le 1\right\}$



On the yz-plane $D_{yz} = \left\{(y,z) \,:\, 0 \le y \le 1,\, 0 \le z \le y^2\right\} = \left\{(y,z) \,:\, 0 \le z \le 1,\, \sqrt{z} \le y \le 1\right\}$



On the xz-plane $D_{xz} = \left\{(x,z) \,:\, 0 \le x \le 1,\, 0 \le z \le x^4\right\} = \left\{(x,z) \,:\, 0 \le z \le 1,\, \sqrt[4]{z} \le x \le 1\right\}$



Now we can describe the same region E as $\{(x,y,z) : 0 \le y \le 1,\ 0 \le z \le y^2,\ \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes:

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{x^2} \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

Now assume that $f(x,y,z) = xyz$ in each of the integrals. Then we have

$$\begin{aligned}
\int_0^1 \int_0^{x^2} \int_0^{y^2} (xyz)\,dz\,dy\,dx &= \int_0^1 \int_0^{x^2} \frac{1}{2}xyz^2\Big|_0^{y^2} dy\,dx \\
&= \int_0^1 \int_0^{x^2} \frac{1}{2}xy^5\,dy\,dx \\
&= \int_0^1 \frac{1}{12}x\frac{y^6}{12}\Big|_0^{x^2} dx \\
&= \int_0^1 \frac{1}{12}x^{13}\,dx \\
&= \frac{1}{168}x^{14}\Big|_0^1 \\
&= \frac{1}{168}
\end{aligned}$$

And the original form of the triple integral becomes

$$\begin{aligned}
\int_0^1 \int_0^{y^2} \int_{\sqrt{y}}^1 (xyz)\,dx\,dz\,dy &= \int_0^1 \int_0^{y^2} \frac{1}{2}yzx^2\Big|_{\sqrt{y}}^1 dz\,dy \\
&= \int_0^1 \int_0^{y^2} \left(\frac{1}{2}yz - \frac{1}{2}y^2z\right) dz\,dy \\
&= \int_0^1 \left[\frac{1}{4}yz^2 - \frac{1}{4}y^2z^2\right]\Big|_0^{y^2} dy \\
&= \int_0^1 \left(\frac{1}{4}y^5 - \frac{1}{4}y^6\right) dy \\
&= \left[\frac{1}{24}y^6 - \frac{1}{28}y^7\right]\Big|_0^1 \\
&= \frac{1}{168}
\end{aligned}$$

The answers match, as promised.

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3.2: Double Integrals in Polar Coordinates

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle $R = \{(r, \theta) : a \le r \le b, \alpha \le \theta \le \beta\}$

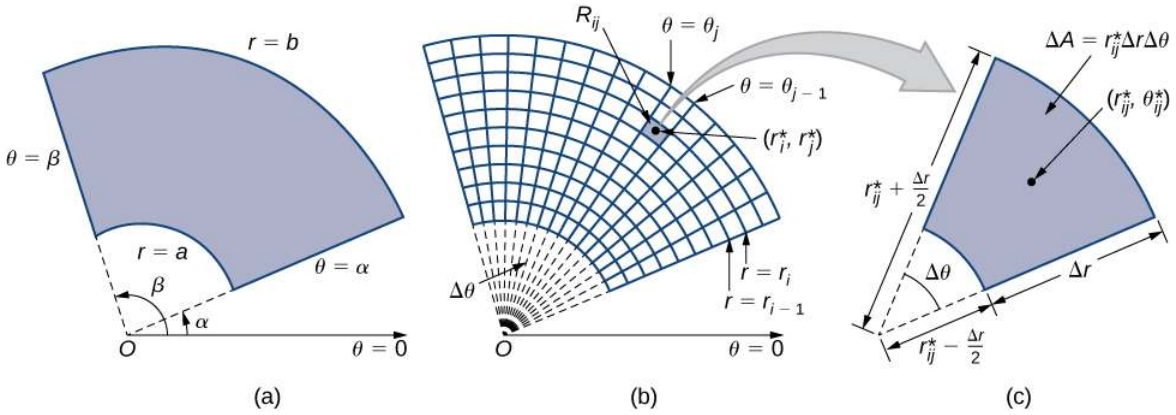


Figure 3.2.6: (a) A polar rectangle R
(b) divided into subrectangles R_{ij}
(c) Close-up of a subrectangle.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = \dfrac{b-a}{m}$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of width $\Delta\theta = \dfrac{\beta-\alpha}{n}$. This means that the circles $r = r_i$ and rays $\theta = \theta_j$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar subrectangles R_{ij}.

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta)$$

Simplifying and letting $r^*_{ij} = \frac{1}{2}(r_{i-1} + r_i)$, we have $\Delta A = r^*_{ij}\Delta r\Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} is $f(r^*_{ij}, \theta^*_{ij})\Delta A = f(r^*_{ij}, \theta^*_{ij})r^*_{ij}\Delta r\Delta\theta$ as pictured below.

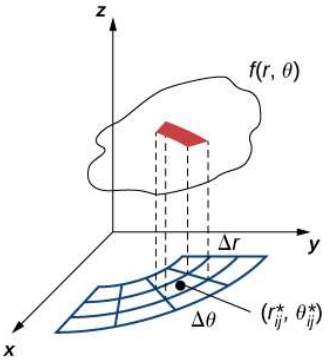


Figure 3.2.7: Finding the volume of the thin box above polar rectangle R_{ij}

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r^*_{ij}, \theta^*_{ij})r^*_{ij}\Delta r\Delta\theta$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta$$

This becomes the expression for the double integral.

Definition: The Double Integral in Polar Coordinates

The double integral of the function $f(r,\theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r,\theta) dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) \Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta$$

Therefore the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\, dr\, d\theta = \int_a^b \int_\alpha^\beta f(r,\theta)\, r\, d\theta\, dr$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $dA = r\, dr\, d\theta$ changes it to

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Note that all the properties listed for double integrals over rectangular regions in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

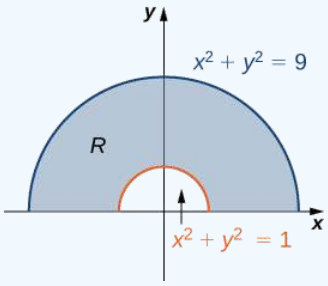
Example 3.2.4: Solving a Polar Double Integral

Evaluate the integral $\iint_R 3r\, dA$ over the region $R = \{(r,\theta) : 1 \le r \le 3,\ 0 \le \theta \le \pi\}$

Solution

First we sketch the polar rectangular region $R = \{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le \pi\}$

We know $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ represents $y \ge 0$. Hence the region R looks like:



Now that we have sketched a polar rectangular region, we can write out and solve the double integral:

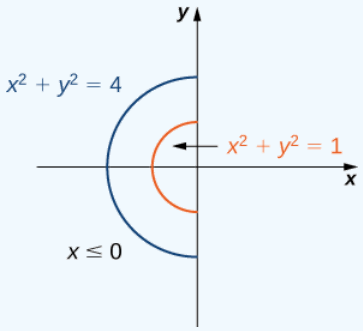
$$\iint_R 3r\, dA = \int_0^\pi \int_1^3 3r\, r\, dr\, d\theta = \int_0^\pi \int_1^3 3r^2\, dr\, d\theta = \int_0^\pi r^3\Big|_1^3 d\theta = \int_0^\pi 26\, d\theta = 26\theta\Big|_0^\pi = \boxed{26\pi}$$

Example 3.2.5: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral $\iint_R (x+y)\,dA$ where $R = \{(x,y) : 1 \le x^2+y^2 \le 4,\ x \le 0\}$ by first converting into polar coordinates.

Solution

The region R is an annular region as pictured below:



In polar form it can be described as $R = \left\{(r,\theta) : 1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$

Using the conversion $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}
\iint_R (x+y)\,dA &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 (r\cos(\theta) + r\sin(\theta))\, r\,dr\,d\theta \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 r^2(\cos(\theta)+\sin(\theta))\,dr\,d\theta \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{3}r^3(\cos(\theta)+\sin(\theta))\Big|_1^2 d\theta \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{7}{3}(\cos(\theta)+\sin(\theta))\,d\theta \\
&= \frac{7}{3}(\sin(\theta)-\cos(\theta))\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
&= \boxed{-\frac{14}{3}}
\end{aligned}$$

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in the section on Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$.

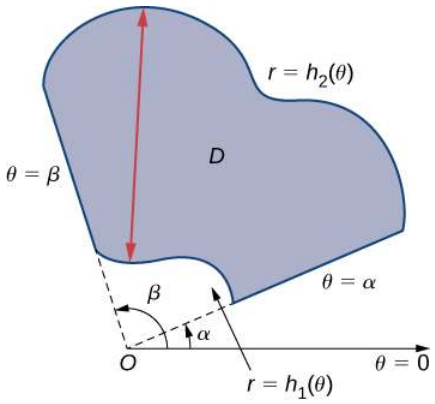


Figure 3.2.8: A general polar region between $\alpha \le \theta \le \beta$ and $h_1(\theta) \le r \le h_2(\theta)$

Theorem 3.2.2: Double Integrals over General Polar Regions

If $f(r,\theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Example 3.2.6: Evaluating a Double Integral over a General Polar Region

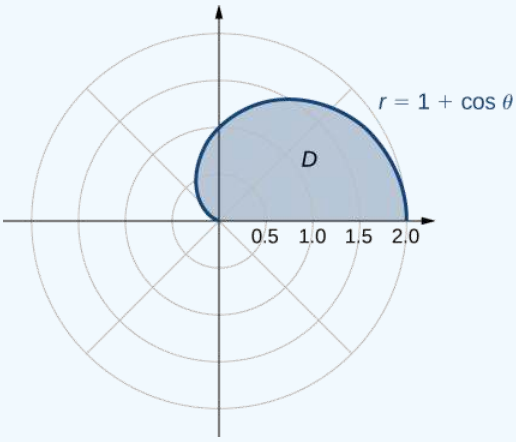
Evaluate the integral

$$\iint_D r^2 \sin(\theta)\, r\, dr\, d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos(\theta)$

Solution

We can describe the region D as $\{(r,\theta) : 0 \le \theta \le \pi,\ 0 \le r \le 1 + \cos(\theta)\}$ as shown below.



Hence, we have

$$\begin{aligned}
\iint_D r^2 \sin(\theta)\, r\, dr\, d\theta &= \int_0^\pi \int_0^{1+\cos(\theta)} r^3 \sin(\theta)\, dr\, d\theta \\
&= \int_0^\pi \frac{1}{4} r^4 \sin(\theta) \Big|_0^{1+\cos(\theta)} d\theta \\
&= \int_0^\pi \frac{1}{4} (1+\cos(\theta))^4 \sin(\theta)\, d\theta \\
&= -\frac{1}{20} (1+\cos(\theta))^5 \Big|_0^\pi \\
&= \boxed{\frac{8}{5}}
\end{aligned}$$

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, with the base of the solid described as $D = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, then the volume is:

$$V = \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Illustration 3.2.2: Finding a Volume Using a Double Integral

We will find the volume of the solid under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane.

Since the base is the unit circle, we know for $R = \{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$, the volume is:

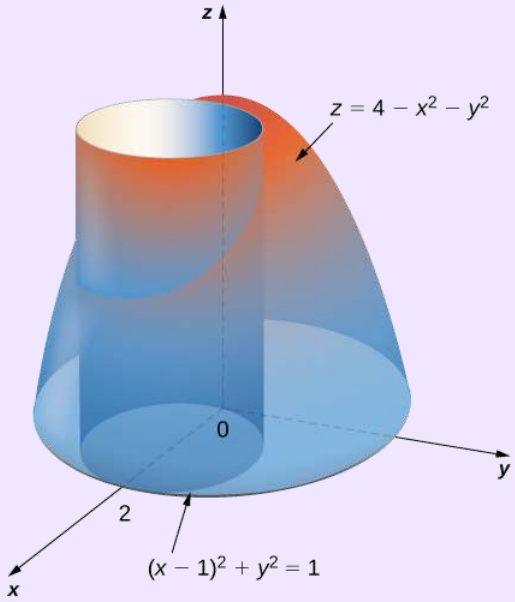
Using the conversion $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $dA = r\,dr\,d\theta$, we are under $z = 1 - r^2$ and the volume is

$$\begin{aligned}
\iint_R (1 - x^2 - y^2)\,dA &= \int_0^{2\pi}\int_0^1 (1 - r^2)\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]\Big|_0^1 d\theta \\
&= \int_0^{2\pi}\frac{1}{4}\,d\theta \\
&= \frac{1}{4}\theta\Big|_0^{2\pi} \\
&= \frac{\pi}{2}\ \text{units}^3
\end{aligned}$$

Illustration 3.2.3: Finding a Volume Using a Double Integral

We will find the volume under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane.

The solid described here is a bit trickier, we start with a sketch:



The base of this solid is the circle $D = \{(x, y) : (x-1)^2 + y^2 \le 1\}$, which we want to change into polar coordinates:

Expanding the square term, we have $x^2 - 2x + 1 + y^2 = 1$. Then simplify to get $x^2 + y^2 = 2x$, which in polar coordinates becomes $r^2 = 2r\cos\theta$ and then either $r = 0$ or $r = 2\cos(\theta)$. Similarly, the equation of the paraboloid changes to $z = 4 - r^2$. Therefore we can describe the disk $(x-1)^2 + y^2 = 1$ on the xy -plane as the region

$$D = \{(r, \theta) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos(\theta)\}$$

Hence the volume of the solid bounded above by the paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ and below by $r = 2\cos(\theta)$ is

$$
\begin{aligned}
V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} (4 - r^2)\, r\, dr\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} (4r - r^3)\, dr\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[2r^2 - \frac{1}{4}r^4\right]\Big|_0^{2\cos(\theta)} d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(8\cos^2(\theta) - 4\cos^4(\theta)\right)\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((4 + 4\cos(2\theta)) - (1 + \cos(2\theta))^2\right)\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2\cos(2\theta) + 3 - \cos^2(2\theta)\right)\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2\cos(2\theta) + \frac{5}{2} - \frac{1}{2}\cos(4\theta)\right)\, d\theta \\
&= \left[\sin(2\theta) + \frac{5}{2}\theta - \frac{1}{8}\sin(4\theta)\right]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{5}{2}\pi \text{ units}^3
\end{aligned}
$$

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

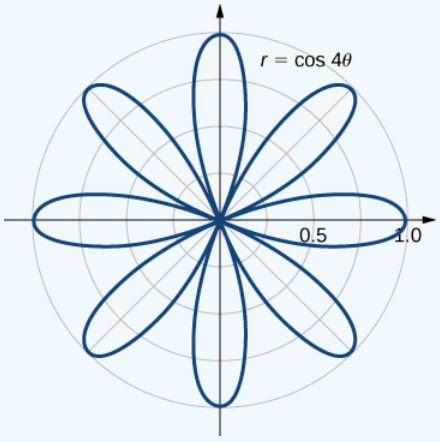
$$\text{Area}(D) = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} 1\, r\, dr\, d\theta$$

Example 3.2.7: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos(4\theta)$

Solution

Sketching the graph of the function $r = \cos(4\theta)$ reveals that it is a polar rose with eight petals (see the following figure).



Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos(4\theta)$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$. Therefore, the area bounded by the curve $r = \cos(4\theta)$ is

$$\begin{aligned}
\text{Area} &= 8\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}\int_{0}^{\cos(4\theta)} 1\, r\, dr\, d\theta \\
&= 8\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2}r^2\Big|_{0}^{\cos(4\theta)}\, d\theta \\
&= 8\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2}\cos^2(4\theta)\, d\theta \\
&= 2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (1+\cos(8\theta))\, d\theta \\
&= 2\left[\theta + \frac{1}{8}\sin(8\theta)\right]\Big|_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \\
&= \boxed{\frac{\pi}{2}\ \text{units}^2}
\end{aligned}$$

We can also use polar coordinates when triple integrals have a circular (or otherwise polar) relationship between two of the three variables in the domain. We will generalize this concept in a future chapter.

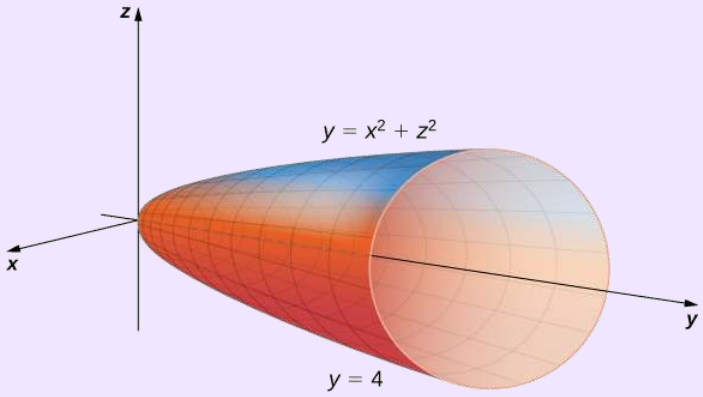
Illustration 3.2.4: Changing to Polar in Triple Integrals

Let E be the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Evaluate the triple integral

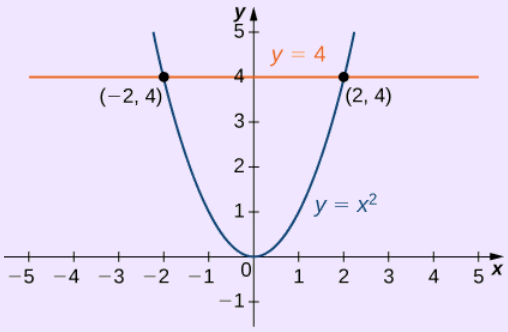
$$\iiint_E \sqrt{x^2+z^2}\, dV$$

Solution

First, a quick sketch of the domain E in $\mathbb{R}^3$:



The projection of the solid region E onto the xy-plane is the region bounded above by $y = 4$ and below by the parabola $y = x^2$:



Thus, we can write $E = \{(x, y, z) : -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y-x^2} \le z \le \sqrt{y-x^2}\}$

In this form, the triple integral becomes:

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2+z^2 \le 4$

Thus, we can write $E = \left\{(x,y,z) \,:\, -2 \le x \le 2,\ -\sqrt{4-x^2} \le z \le \sqrt{4-x^2},\ x^2+z^2 \le y \le 4\right\}$

In this form, the triple integral is instead:

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\,dz\,dx$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. This does not look much easier, but we see that D_{xz} is now a circle, so we can switch it into polar coordinates once y has been integrated away.

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\,dz\,dx &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\sqrt{x^2+z^2}\Big|_{x^2+z^2}^{4}\,dz\,dx \\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx\end{aligned}$$

Now use the polar substitution $x = r\,\cos(\theta)$, $z = r\,\sin(\theta)$, and $dz\,dx = r\,dr\,d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and by using $r^2 = x^2+z^2$, this double integral becomes:

$$\begin{aligned}\int_0^{2\pi}\int_0^2 (4-r^2)\sqrt{r^2}\,r\,dr\,d\theta &= \int_0^{2\pi}\int_0^2 (4r^2-r^4)\,dr\,d\theta \\ &= \int_0^{2\pi}\left[\frac{4}{3}r^3-\frac{1}{5}r^5\right]\Big|_0^2 d\theta \\ &= \int_0^{2\pi}\frac{64}{15}\,d\theta \\ &= \frac{64}{15}\theta\Big|_0^{2\pi} \\ &= \frac{128\pi}{15}\end{aligned}$$

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3.2: Practice Set

Exercise 3.2.1a

Evaluate $\int_1^3 \int_{-1}^2 \int_0^2 (x+y-2z)\,dy\,dx\,dz$

Answer

$$\begin{aligned}\int_1^3 \int_{-1}^2 \int_0^2 (x+y-2z)\,dy\,dx\,dz &= \int_1^3 \int_{-1}^2 \left[xy+\frac{1}{2}y^2-2yz\right]\Big|_0^2 dx\,dz \\ &= \int_1^3 \int_{-1}^2 (2x+2-4z)\,dx\,dz \\ &= \int_1^3 \left[x^2+2x-4xz\right]\Big|_{-1}^2 dz \\ &= \int_1^3 (9-12z)\,dz \\ &= \left[9z-6z^2\right]\Big|_1^3 \\ &= -30\end{aligned}$$

Exercise 3.2.1b

Evaluate $\int_0^3 \int_0^2 \int_0^5 8xyz\,dx\,dz\,dy$

Answer

$$\begin{aligned}\int_0^3 \int_0^2 \int_0^5 8xyz\,dx\,dz\,dy &= \int_0^3 \int_0^2 4x^2yz\Big|_0^5 dz\,dy \\ &= \int_0^3 \int_0^2 100yz\,dz\,dy \\ &= \int_0^3 50yz^2\Big|_0^2 dy \\ &= \int_0^3 200y\,dy \\ &= 100y^2\Big|_0^3 \\ &= 900\end{aligned}$$

Exercise 3.2.1c

Evaluate $\int_0^2 \int_0^2 \int_0^{5-x-2y} 2z\,dz\,dx\,dy$

Answer

$$\begin{aligned}\int_0^2\int_0^2\int_0^{5-x-2y} 2z\,dz\,dx\,dy &= \int_0^2\int_0^2 z^2\Big|_0^{5-x-2y}\,dx\,dy\\ &= \int_0^2\int_0^2 (5-x-2y)^2\,dx\,dy\\ &= \int_0^2 -\frac{1}{3}(5-x-2y)^3\Big|_0^2\,dy\\ &= -\frac{1}{3}\int_0^2 \left((3-2y)^3-(5-2y)^3\right)\,dy\\ &= -\frac{1}{3}\left[-\frac{1}{8}(3-2y)^4+\frac{1}{8}(5-2y)^4\right]\Big|_0^2\\ &= \frac{1}{24}\left((-1)^4-1^4-3^4+5^4\right)\\ &= \frac{68}{3}\end{aligned}$$

Exercise 3.2.1d

Evaluate $\int_0^1\int_{x^2}^x\int_{x^2+y^2}^2 x\,dz\,dy\,dx$

Answer

$$\begin{aligned}\int_0^1\int_{x^2}^x\int_{x^2+y^2}^2 x\,dz\,dy\,dx &= \int_0^1\int_{x^2}^x xz\Big|_{x^2+y^2}^2\,dy\,dx\\ &= \int_0^1\int_{x^2}^x (2x-x^3-xy^2)\,dy\,dx\\ &= \int_0^1\left[2xy-x^3y-\frac{1}{3}xy^3\right]\Big|_{x^2}^x\,dx\\ &= \int_0^1\left(2x^2-2x^3-\frac{4}{3}x^4+x^5+\frac{1}{3}x^7\right)\,dx\\ &= \left[\frac{2}{3}x^3-\frac{1}{2}x^4-\frac{4}{15}x^5+\frac{1}{6}x^6+\frac{1}{24}x^8\right]\Big|_0^1\\ &= \frac{2}{3}-\frac{1}{2}-\frac{4}{15}+\frac{1}{6}+\frac{1}{24}\\ &= \frac{13}{120}\end{aligned}$$

Exercise 3.2.1e

Evaluate $\int_0^1\int_{-z}^z\int_{y+z}^{2z} x\,dx\,dy\,dz$

Answer

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$$\begin{aligned}\int_0^1\int_{-z}^{z}\int_{y+z}^{2z} x\,dx\,dy\,dz &= \int_0^1\int_{-z}^{z}\frac{1}{2}x^2\Big|_{y+z}^{2z}\,dy\,dz\\ &= \int_0^1\int_{-z}^{z}\left(2z^2-\frac{1}{2}(y+z)^2\right)\,dy\,dz\\ &= \int_0^1\left[2yz^2-\frac{1}{6}(y+z)^3\right]\Big|_{-z}^{z}\,dz\\ &= \int_0^1\frac{8}{3}z^3\,dz\\ &= \frac{2}{3}z^4\Big|_0^1\\ &= \frac{2}{3}\end{aligned}$$

Exercise 3.2.1f

Evaluate $\int_0^3\int_0^x\int_0^y xze^{-z}\,dz\,dy\,dx$

Answer

$$\begin{aligned}\int_0^3\int_0^x\int_0^y xze^{-z}\,dz\,dy\,dx &= \int_0^3\int_0^x x(-z-1)e^{-z}\Big|_0^y\,dy\,dx\\ &= \int_0^3\int_0^x x\left(1-(y+1)e^{-y}\right)\,dy\,dx\\ &= \int_0^3 x\left(y+(y+2)e^{-y}\right)\Big|_0^x\,dx\\ &= \int_0^3\left(x^2+(x^2+2x)e^{-x}-2x\right)\,dx\\ &= \left[\frac{1}{3}x^3-x^2-(x^2+4x+4)e^{-x}\right]\Big|_0^3\\ &= 4-25e^{-3}\end{aligned}$$

Exercise 3.2.1g

Evaluate $\iiint_D (y^2-z^2)\cos(x)\,dV$ where $D=\{(x,y,z)\ :\ 0\le x\le\pi,\ \sin(x)\le y\le 1,\ 0\le z\le\sin(x)\}$

Answer

$$\begin{aligned}\iiint_D (y^2-z^2)\cos(x)\,dV &= \int_0^\pi\int_{\sin(x)}^1\int_0^{\sin(x)}(y^2-z^2)\cos(x)\,dz\,dy\,dx\\ &= \int_0^\pi\int_{\sin(x)}^1\left[\left(y^2z-\frac{1}{3}z^3\right)\cos(x)\right]\Big|_0^{\sin(x)}\,dy\,dx\\ &= \int_0^\pi\int_{\sin(x)}^1\left(y^2\sin(x)\cos(x)-\frac{1}{3}\sin^3(x)\cos(x)\right)\,dy\,dx\\ &= \int_0^\pi\left[\frac{1}{3}y^3\sin(x)\cos(x)-\frac{1}{3}\sin^3(x)\cos(x)y\right]\Big|_{\sin(x)}^1\,dx\\ &= \int_0^\pi\frac{1}{3}\left(\sin(x)-\sin^3(x)\right)\cos(x)\,dx\\ &= \frac{1}{3}\left[\frac{1}{2}\sin^2(x)-\frac{1}{4}\sin^4(x)\right]\Big|_0^\pi\\ &= 0\end{aligned}$$

Exercise 3.2.1h

Evaluate $\iiint_E yz^2\,dV$ where $E = \{(x,y,z) : 1 \le z \le 2,\ 0 \le x \le yz,\ z \le y \le 2z\}$

Answer

Since x is bounded by functions of y and z it must be integrated first!

$$\begin{aligned}
\iiint_E yz^2\,dV &= \int_1^2 \int_z^{2z} \int_0^{yz} yz^2\,dx\,dy\,dz \\
&= \int_1^2 \int_z^{2z} xyz^2\Big|_0^{yz}\,dy\,dz \\
&= \int_1^2 \int_z^{2z} y^2z^3\,dy\,dz \\
&= \int_1^2 \frac{1}{3}y^3z^3\Big|_z^{2z}\,dz \\
&= \int_1^2 \frac{7}{3}z^6\,dz \\
&= \frac{1}{3}z^7\Big|_1^2 \\
&= \frac{127}{3}
\end{aligned}$$

Exercise 3.2.1i

Evaluate $\iiint_F ye^x\,dV$ where $F = \{(x,y,z) : 0 \le z \le 25 - x^2 - y^2\}$

Answer

Since z is bounded by functions of x and y it must be integrated first.

We have $25 - x^2 - y^2 \ge 0 \longleftrightarrow x^2 + y^2 \le 25$, which means that F can really be written as:

$F = \left\{(x,y,z) : -5 \le x \le 5,\ -\sqrt{25-x^2} \le y \le \sqrt{25-x^2},\ 0 \le z \le 25 - x^2 - y^2\right\}$

$$\begin{aligned}
\iiint_F ye^x\,dV &= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \int_0^{25-x^2-y^2} ye^x\,dz\,dy\,dx \\
&= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} yze^x\Big|_0^{25-x^2-y^2}\,dy\,dx \\
&= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} ye^x(25-x^2-y^2)\,dy\,dx \\
&= \int_{-5}^{5} -\frac{1}{4}(25-x^2-y^2)^2e^x\Big|_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\,dx \\
&= \int_{-5}^{5} 0\,dx \\
&= 0
\end{aligned}$$

This problem could also have been put into polar form after the z integration is complete:

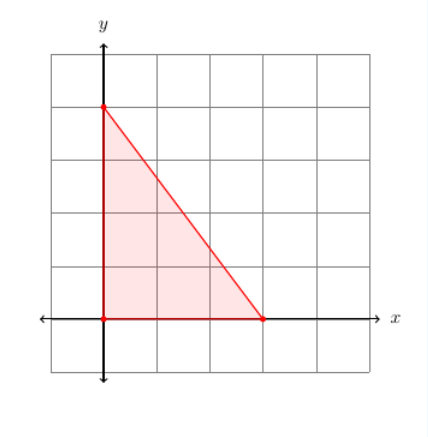
The double integral $\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} ye^x(25-x^2-y^2)\,dy\,dx = \int_0^5 \int_0^{2\pi} r\sin(\theta)e^{r\cos(\theta)}(25-r^2)\cdot r\,d\theta\,dr$ is also easily shown to be zero.

Exercise 3.2.1j

Evaluate $\iiint_G \frac{xz}{3-x-\frac{3}{4}y}\,dV$ where G is the region under $4x+3y+z=12$ with $x,y\geq 0$

Answer

First we sketch the region under the plane $4x+3y+z=12$ in the first quadrant:



Since the intercepts are $(3,0,0)$ and $(0,4,0)$ we see the hypotenuse of this triangle is the line $y=4-\frac{4}{3}x$

Therefore $G=\left\{(x,y,z)\,:\,0\leq x\leq 3\,0\leq y\leq 4-\frac{4}{3}x,\,0\leq z\leq 12-4x-3y\right\}$, and our integral is

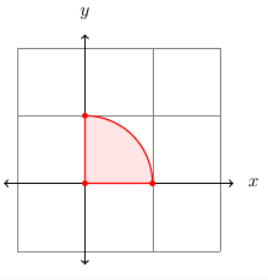
$$\begin{aligned}
\iiint_G \frac{xz}{3-x-\frac{3}{4}y}\,dV &= \int_0^3\int_0^{4-\frac{4}{3}x}\int_0^{12-4x-3y}\frac{xz}{3-x-\frac{3}{4}y}\,dz\,dy\,dx\\
&= \int_0^3\int_0^{4-\frac{4}{3}x}\frac{xz^2}{2(3-x-\frac{3}{4}y)}\Big|_0^{12-4x-3y}\,dy\,dx\\
&= \int_0^3\int_0^{4-\frac{4}{3}x}\frac{x(12-4x-3y)^2}{6-2x-\frac{3}{2}y}\,dy\,dx\\
&= \int_0^3\int_0^{4-\frac{4}{3}x}2x(12-4x-3y)\,dy\,dx\\
&= \int_0^3-\frac{1}{3}x(12-4x-3y)^2\Big|_0^{4-\frac{4}{3}x}\,dx\\
&= \frac{1}{3}\int_0^3 x(12-4x)^2\,dx\\
&= \frac{1}{3}\int_0^3\left(144x-96x^2+16x^3\right)\,dx\\
&= \frac{1}{3}\left[72x^2-32x^3+4x^4\right]\Big|_0^3\\
&= \frac{1}{3}\left(72\cdot 9-32\cdot 27+4\cdot 81\right)\\
&= 36
\end{aligned}$$

Exercise 3.2.2a

Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\frac{4}{1+x^2+y^2}\,dy\,dx$

Answer

The domain $D=\left\{(x,y)\,:\,0\leq x\leq 1,\,0\leq y\leq\sqrt{1-x^2}\right\}$ is a quarter circle.



This can be written in polar form as $D = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le 1,\, 0 \le \theta \le \dfrac{\pi}{2}\right\}$

Therefore the integral is:

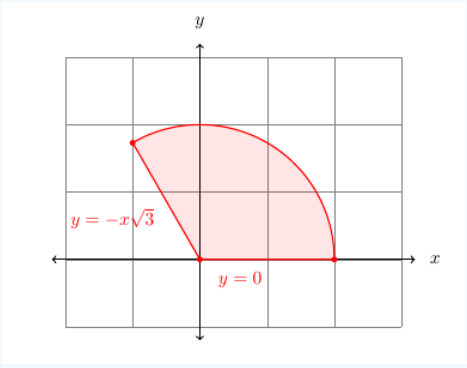
$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{4}{1+x^2+y^2}\,dy\,dx &= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{4}{1+r^2}\cdot r\,d\theta\,dr \\ &= \left(\int_0^1 \frac{4r}{1+r^2}\,dr\right)\left(\int_0^{\frac{\pi}{2}} d\theta\right) \\ &= \left(2\ln\left|1+r^2\right|\Big|_0^1\right)\cdot\left(\theta\Big|_0^{\frac{\pi}{2}}\right) \\ &= (2\ln(2) - 2\ln(1))\cdot\frac{\pi}{2} \\ &= \pi\ln(2)\end{aligned}$$

? Exercise 3.2.2*b*

Evaluate $\displaystyle\int_{-1}^{2}\int_{\max(-x\sqrt{3},0)}^{\sqrt{4-x^2}} y\,dy\,dx$

Answer

The domain $D = \left\{(x,y) \,:\, -1 \le x \le 2,\, \max(-x\sqrt{3},0) \le y \le \sqrt{4-x^2}\right\}$ is a sector.



This can be written in polar form as $D = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le 2,\, 0 \le \theta \le \dfrac{2\pi}{3}\right\}$

Therefore the integral is:

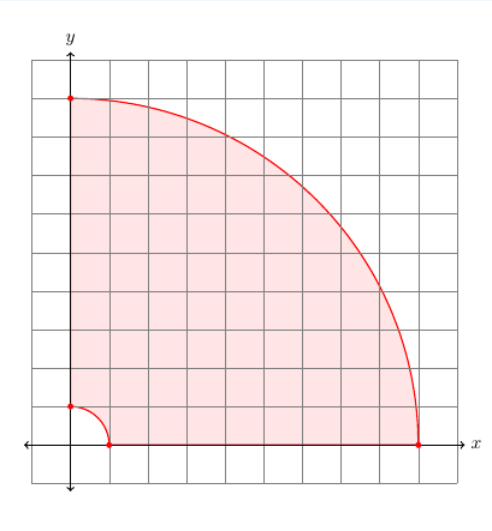
$$\begin{aligned}\int_{-1}^{2}\int_{\max(-x\sqrt{3},0)}^{\sqrt{4-x^2}} y\,dy\,dx &= \int_0^2\int_0^{\frac{2\pi}{3}} r\sin(\theta)\cdot r\,d\theta\,dr\\ &= \left(\int_0^2 r^2\,dr\right)\left(\int_0^{\frac{2\pi}{3}}\sin(\theta)\,d\theta\right)\\ &= \left(\frac{1}{3}r^3\Big|_0^2\right)\cdot\left(-\cos(\theta)\Big|_0^{\frac{2\pi}{3}}\right)\\ &= \frac{8}{3}\cdot\left(\frac{1}{2}+1\right)\\ &= 4\end{aligned}$$

Exercise 3.2.2c

Evaluate $\iint_D \frac{x-y}{\sqrt{x+y}}\,dA$ for $D=\{(x,y)\,:\,1\le x^2+y^2\le 81,\ x\ge 0,\ y\ge 0\}$

Answer

The domain $D=\{(x,y)\,:\,1\le x^2+y^2\le 81,\ x\ge 0,\ y\ge 0\}$ is part of an annulus in the first quadrant.



This can be written in polar form as $D=\left\{(r\cos(\theta),r\sin(\theta))\,:\,1\le r\le 9,\ 0\le\theta\le\frac{\pi}{2}\right\}$

Therefore the integral is:

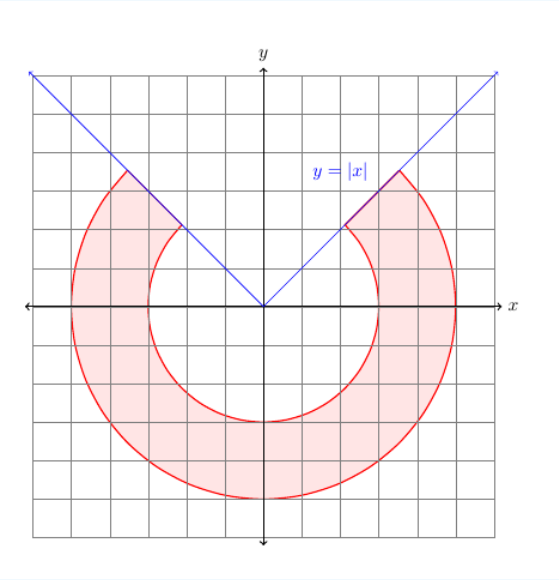
$$\begin{aligned}\iint_D \frac{x-y}{\sqrt{x+y}}\,dA &= \int_1^9\int_0^{\frac{\pi}{2}} \frac{r\cos(\theta)-r\sin(\theta)}{\sqrt{r\cos(\theta)+r\sin(\theta)}}\cdot r\,d\theta\,dr\\ &= \left(\int_1^9 r^{\frac{3}{2}}\,dr\right)\left(\int_0^{\frac{\pi}{2}}\frac{\cos(\theta)-\sin(\theta)}{\sqrt{\cos(\theta)+\sin(\theta)}}\,d\theta\right)\\ &= \left(\frac{2}{5}r^{\frac{5}{2}}\Big|_1^9\right)\cdot\left(2\sqrt{\cos(\theta)+\sin(\theta)}\Big|_0^{\frac{\pi}{2}}\right)\\ &= \frac{2}{5}\left(9^{\frac{5}{2}}-1^{\frac{5}{2}}\right)\cdot\left(2\sqrt{1}-2\sqrt{1}\right)\\ &= 0\end{aligned}$$

Exercise 3.2.2d

Evaluate $\iint_E xy\,dA$ for $E = \{(x,y) : 9 \le x^2+y^2 \le 25,\ y \le |x|\}$

Answer

The domain $E = \{(x,y) : 9 \le x^2+y^2 \le 25,\ y \le |x|\}$ is part of an annulus.



This can be written in polar form as $E = \left\{(r\cos(\theta), r\sin(\theta)) : 3 \le r \le 5,\ \frac{3\pi}{4} \le \theta \le \frac{9\pi}{4}\right\}$

Therefore the integral is:

$$\begin{aligned}
\iint_E xy\,dA &= \int_3^5 \int_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} r\cos(\theta)\cdot r\sin(\theta)\cdot r\,d\theta\,dr \\
&= \left(\int_3^5 r^3\,dr\right)\left(\int_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} \sin(\theta)\cos(\theta)\,d\theta\right) \\
&= \left(\frac{1}{4}r^4\Big|_3^5\right)\cdot\left(\frac{1}{2}\sin^2(\theta)\Big|_{\frac{3\pi}{4}}^{\frac{9\pi}{4}}\right) \\
&= \frac{1}{4}\left(5^4-3^4\right)\cdot\frac{1}{2}\left(\left(\frac{1}{\sqrt{2}}\right)^2-\left(\frac{1}{\sqrt{2}}\right)^2\right) \\
&= 0
\end{aligned}$$

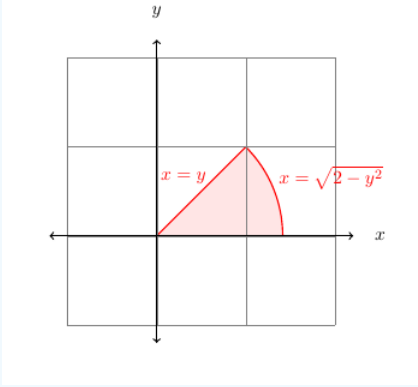
Exercise 3.2.2e

Evaluate $\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \frac{r^2\cos(\theta)}{1-r^2\sin^2(\theta)} e^{r\sin(\theta)}\,dr\,d\theta$

Answer

This integral is already in polar form, but is not separable or easy to evaluate.

We sketch the domain $D = \left\{(r\cos(\theta), r\sin(\theta)) : 0 \le r \le \sqrt{2},\ 0 \le \theta \le \frac{\pi}{4}\right\}$ below:



This can be written as a Type 2 region instead: $D = \left\{(x,y) \,:\, 0 \le y \le 1,\, y \le x \le \sqrt{2-y^2}\right\}$, so the integral becomes:

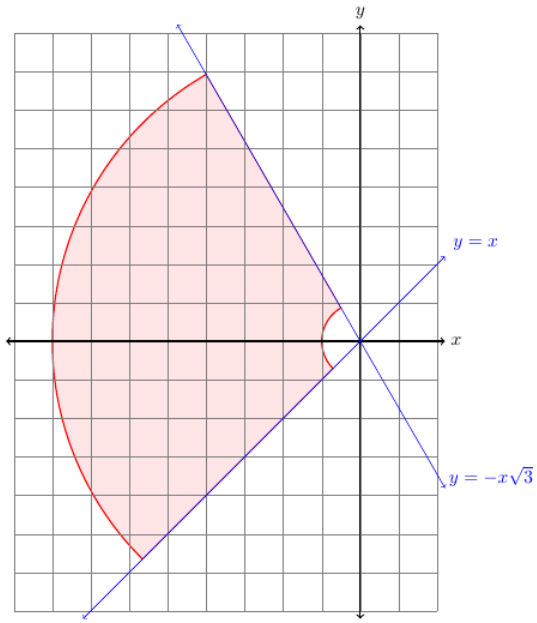
$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \frac{r^2\cos(\theta)}{1-r^2\sin^2(\theta)} e^{r\sin(\theta)}\, dr\, d\theta &= \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \frac{r\cos(\theta)}{1-(r\sin(\theta))^2} e^{r\sin(\theta)} \cdot r\, dr\, d\theta \\
&= \iint_D \frac{x}{1-y^2} e^y\, dA \\
&= \int_0^1 \int_y^{\sqrt{2-y^2}} \frac{x}{1-y^2} e^y\, dx\, dy \\
&= \int_0^1 \frac{x^2}{2(1-y^2)} e^y \Big|_y^{\sqrt{2-y^2}}\, dy \\
&= \int_0^1 \frac{2-2y^2}{2(1-y^2)} e^y\, dy \\
&= \int_0^1 e^y\, dy \\
&= e^y\Big|_0^1 \\
&= e-1
\end{aligned}$$

Exercise 3.2.2f

Evaluate $\iint_E \frac{16}{\sqrt[3]{x^2+y^2}}\, dA$ for $E = \{(x,y) \,:\, 1 \le x^2+y^2 \le 64,\, x \le y,\, y \le -x\sqrt{3}\}$

Answer

The domain $E = \left\{(x,y) \,:\, 1 \le x^2+y^2 \le 64,\, x \le y,\, y \le -x\sqrt{3}\right\}$ is part of an annulus.



This can be written in polar form as $E = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 1 \le r \le 8,\ \frac{2\pi}{3} \le \theta \le \frac{5\pi}{4}\right\}$

Therefore the integral is:

$$\begin{aligned}
\iint_E \frac{16}{\sqrt[3]{x^2+y^2}}\,dA &= \int_1^8 \int_{\frac{2\pi}{3}}^{\frac{5\pi}{4}} \frac{16}{\sqrt[3]{r^2}} \cdot r\,d\theta\,dr \\
&= \left(\int_1^8 16r^{\frac{1}{3}}\,dr\right)\left(\int_{\frac{2\pi}{3}}^{\frac{5\pi}{4}} d\theta\right) \\
&= \left(12r^{\frac{4}{3}}\Big|_1^8\right)\cdot\left(\theta\Big|_{\frac{2\pi}{3}}^{\frac{5\pi}{4}}\right) \\
&= 12\left(8^{\frac{4}{3}} - 1^{\frac{4}{3}}\right)\cdot\frac{7\pi}{12} \\
&= 105\pi
\end{aligned}$$

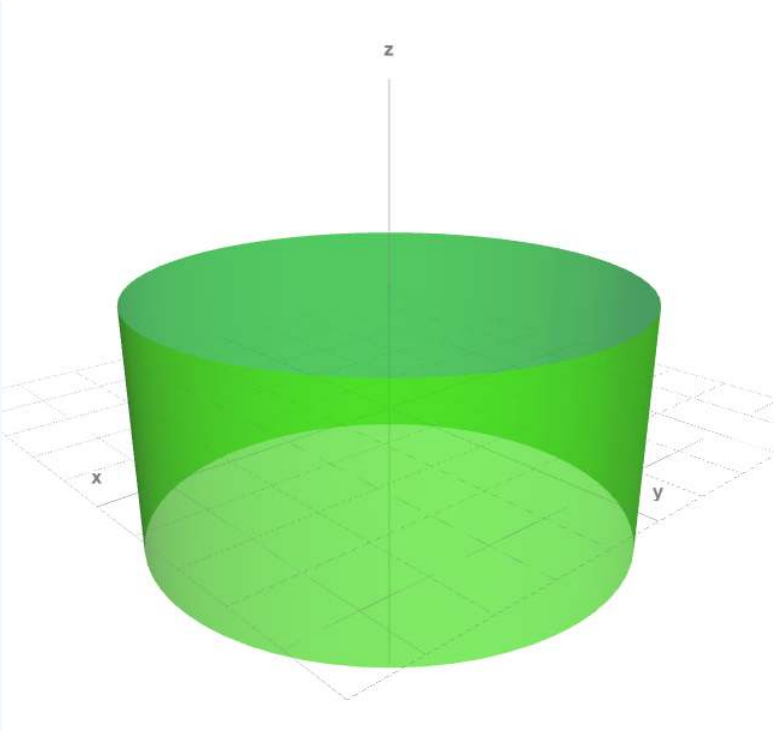
? Exercise $3.2.3a$

Find the volume of $A = \{(x, y, z) \,:\, x^2 + y^2 \le 16,\ -2 \le z \le 2\}$

Answer

The region defined by A is a cylinder:

https://math.libretexts.org/@go/page/194342



We can write it as $A = \{(x, y, z) : -4 \le x \le 4, -\sqrt{16 - x^2} \le y \le \sqrt{16 - x^2}, -2 \le z \le 2\}$

The volume is given by the triple integral:

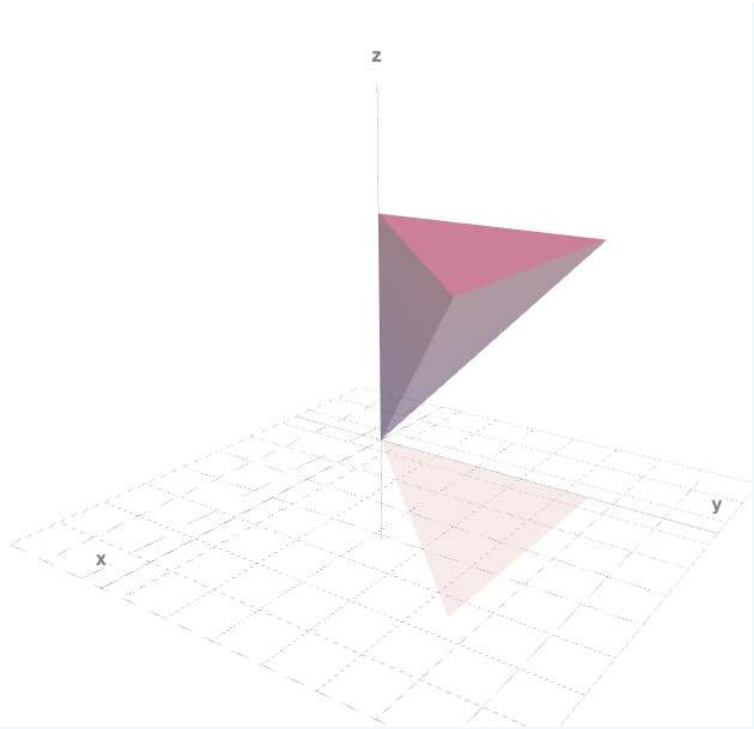
$$\begin{aligned}
\text{Volume}(A) &= \iiint_A 1 \, dV \\
&= \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-2}^{2} 1 \, dz \, dy \, dx \\
&= \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} z \Big|_{-2}^{2} dy \, dx \\
&= \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 4 \, dy \, dx \\
&= 4 \cdot \text{Area}(x^2 + y^2 \le 16) \\
&= 4 \cdot (\pi \cdot 4^2) \\
&= 64\pi
\end{aligned}$$

? Exercise 3.2.3b

Find the volume of $B = \{(x, y, z) : 0 \le x \le y \le z \le 2\}$

Answer

The region B can most easily be written as $B = \{(x, y, z) : 0 \le z \le 2, 0 \le y \le z, 0 \le x \le y\}$



The volume is given by the triple integral:

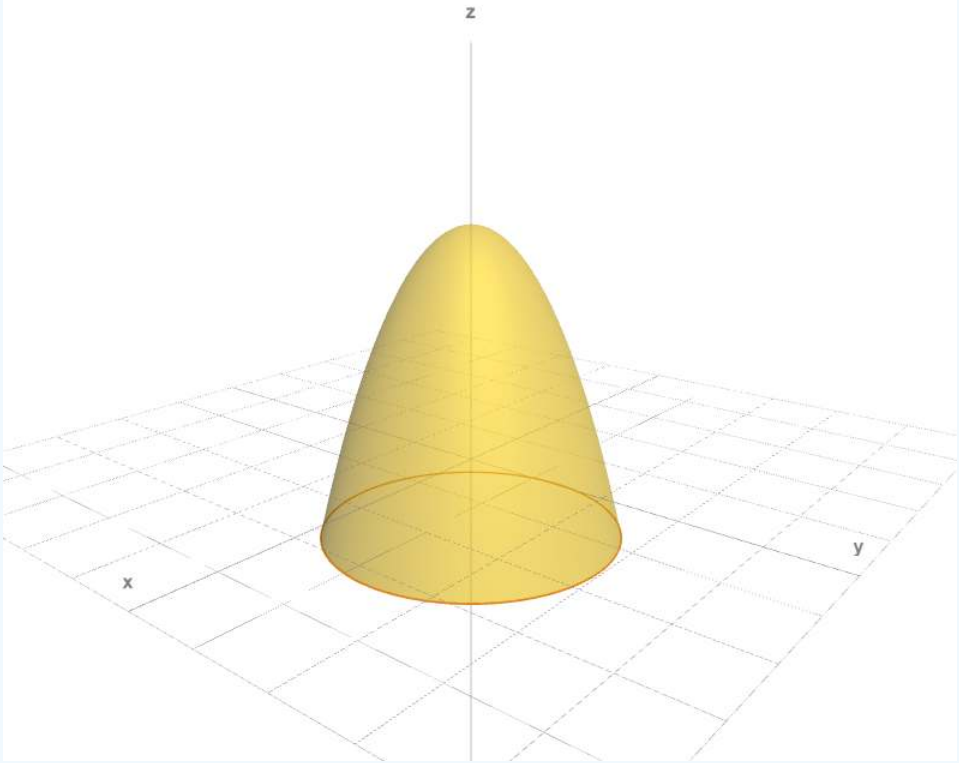
$$\begin{aligned}
\text{Volume}(B) &= \iiint_B 1\,dV \\
&= \int_0^2 \int_0^z \int_0^y 1\,dx\,dy\,dz \\
&= \int_0^2 \int_0^z x\Big|_0^y dy\,dz \\
&= \int_0^2 \int_0^z y\,dy\,dz \\
&= \int_0^2 \frac{1}{2}y^2\Big|_0^z dz \\
&= \int_0^2 \frac{1}{2}z^2\,dz \\
&= \frac{1}{6}z^3\Big|_0^2 \\
&= \frac{4}{3}
\end{aligned}$$

? Exercise 3.2.3c

Find the volume of $C = \{(x, y, z) : -1 \leq z \leq 3 - x^2 - y^2\}$

Answer

The boundary of x and y for C occurs when $-1 = 3 - x^2 - y^2 \longleftrightarrow x^2 + y^2 = 4$



Therefore we write it as $C = \left\{(x, y, z) \,:\, -2 \le x \le 2,\, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\, -1 \le z \le 3 - x^2 - y^2\right\}$

The volume is given by the triple integral:

$$\begin{aligned}
\text{Volume}(C) &= \iiint_C 1\, dV \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{3-x^2-y^2} 1\, dz\, dy\, dx \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z\Big|_{-1}^{3-x^2-y^2} dy\, dx \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx
\end{aligned}$$

This double integral is easier to evaluate in polar form:

$\left\{(x, y) \,:\, -2 \le x \le 2,\, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\right\} = \{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le 2,\, 0 \le \theta \le 2\pi\}$

Therefore our integral becomes:

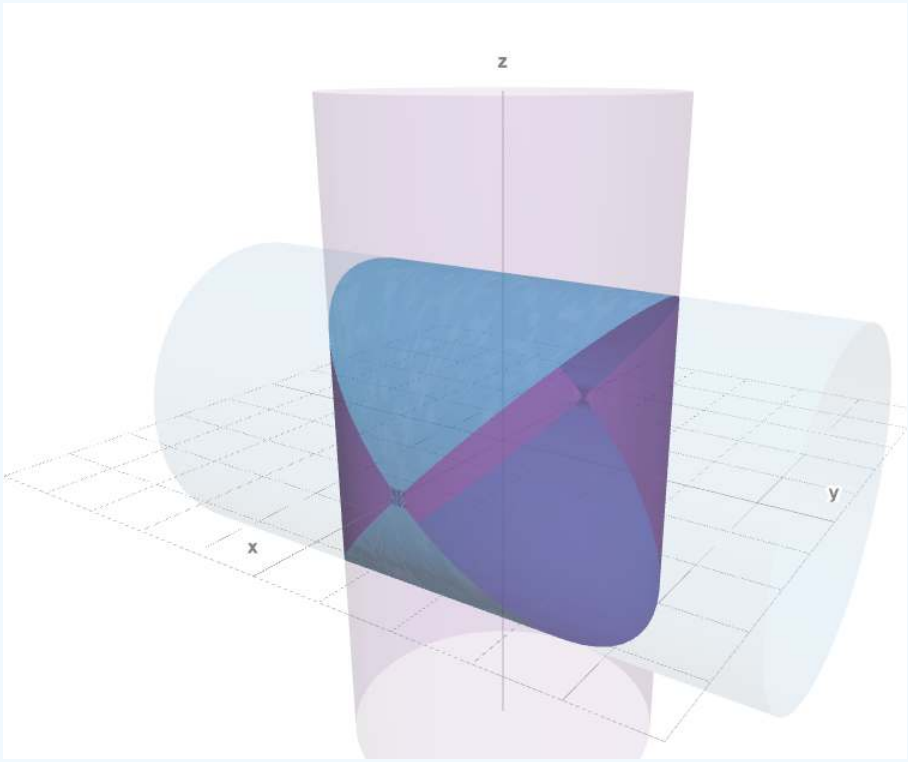
$$\begin{aligned}
\text{Volume}(C) &= \int_0^2 \int_0^{2\pi} (4 - r^2) \cdot r\, d\theta\, dr \\
&= \left(\int_0^2 (4r - r^3)\, dr\right)\left(\int_0^{2\pi} d\theta\right) \\
&= \left[2r^2 - \frac{1}{4}r^4\right]\Big|_0^2 \cdot \left(\theta\Big|_0^{2\pi}\right) \\
&= 4 \cdot 2\pi \\
&= 8\pi
\end{aligned}$$

Exercise $3.2.3d$

Find the volume of the intersection of $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$

Answer

This region D is the intersection of two cylinders:



Since both y and z can be bounded by functions of x, we write:

$D = \left\{(x, y, z) \,:\, -1 \leq x \leq 1,\, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\, -\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}\right\}$

Therefore, the volume is given by the triple integral:

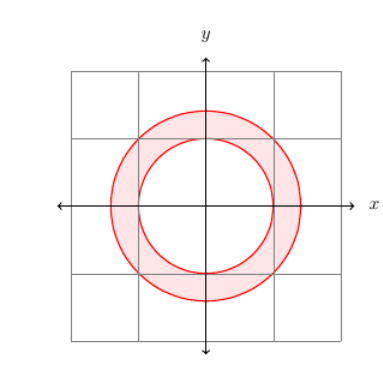
$$\begin{aligned}
\text{Volume}(D) &= \iiint_D 1\, dV \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\, dy\, dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} z\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, dy\, dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\, dy\, dx \\
&= \int_{-1}^{1} 2y\sqrt{1-x^2}\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, dx \\
&= \int_{-1}^{1} 4(1-x^2)\, dx \\
&= 4\left[x - \frac{1}{3}x^3\right]\Big|_{-1}^{1} \\
&= 4 \cdot \frac{4}{3} \\
&= \frac{16}{3}
\end{aligned}$$

Exercise 3.2.3e

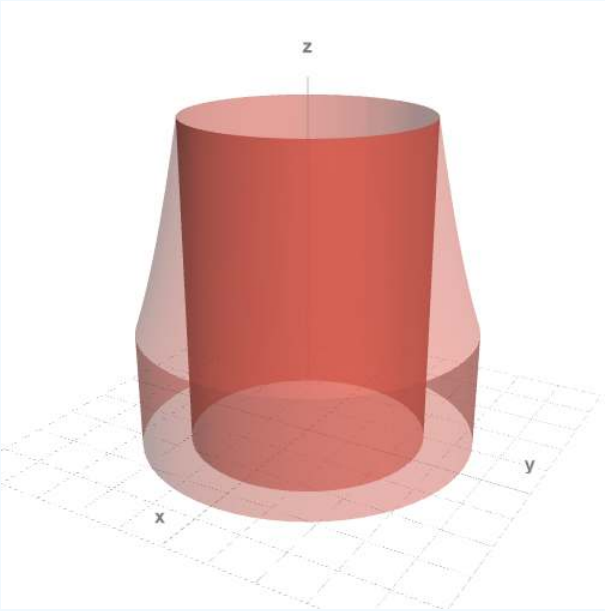
Find the volume of $E = \left\{(x, y, z) \,:\, 1 \le x^2 + y^2 \le 2,\, 0 \le z \le e^{2-x^2-y^2}\right\}$

Answer

The region E is under $z = e^{2-x^2-y^2}$ above the annulus $\hat{E} = \{(x, y) \,:\, 1 \le x^2 + y^2 \le 2\}$:



Therefore we write it as $E = \left\{(x, y, z) \,:\, (x, y) \in \hat{E},\, 0 \le z \le e^{2-x^2-y^2}\right\}$



The volume is given by the triple integral:

$$\begin{aligned}
\text{Volume}(E) &= \iiint_E 1\, dV \\
&= \iint_{\hat{E}} \int_0^{e^{2-x^2-y^2}} 1\, dz\, dA \\
&= \iint_{\hat{E}} z\Big|_0^{e^{2-x^2-y^2}}\, dA \\
&= \iint_{\hat{E}} e^{2-x^2-y^2}\, dA
\end{aligned}$$

This double integral is easier to evaluate in polar form:

$\hat{E} = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 1 \le r \le \sqrt{2},\, 0 \le \theta \le 2\pi\right\}$

Therefore our integral becomes:

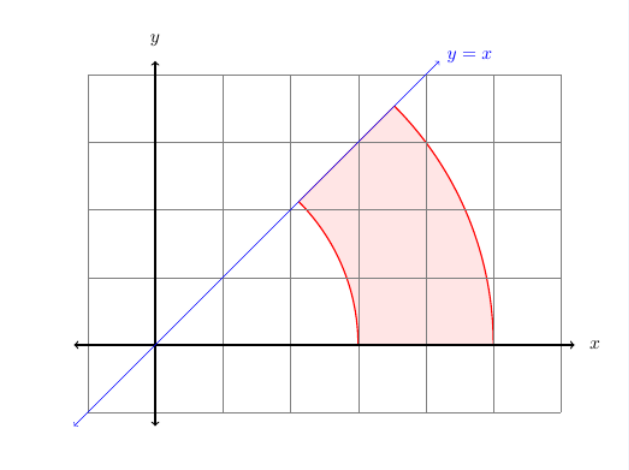
$$\begin{aligned}\text{Volume}(E) &= \int_1^{\sqrt{2}} \int_0^{2\pi} e^{2-r^2} \cdot r \, d\theta \, dr \\ &= \left(\int_1^{\sqrt{2}} r e^{2-r^2} \, dr \right) \left(\int_0^{2\pi} d\theta \right) \\ &= \left(-\frac{1}{2} e^{2-r^2} \Big|_1^{\sqrt{2}} \right) \cdot \left(\theta \Big|_0^{2\pi} \right) \\ &= \left(-\frac{1}{2} + \frac{1}{2} e \right) \cdot 2\pi \\ &= (e-1)\pi\end{aligned}$$

Exercise 3.2.3f

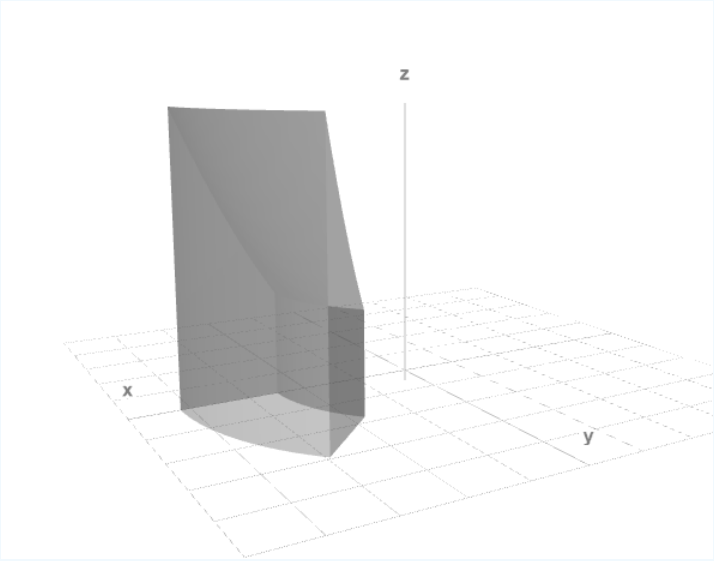
Find the volume of $F = \{(x, y, z) : 9 \leq x^2 + y^2 \leq 25,\ 0 \leq y \leq x,\ 0 \leq z \leq x^2 + y^2\}$

Answer

The region F is under $z = x^2 + y^2$ above the annulus $\hat{F} = \{(x, y) : 9 \leq x^2 + y^2 \leq 25,\ 0 \leq y \leq x\}$:



Therefore we write it as $F = \left\{(x, y, z) : (x, y) \in \hat{F},\ 0 \leq z \leq x^2 + y^2\right\}$



The volume is given by the triple integral:

$$\begin{aligned}\text{Volume}(F) &= \iiint_F 1\,dV \\ &= \iint_{\hat{F}} \int_0^{x^2+y^2} 1\,dz\,dA \\ &= \iint_{\hat{F}} z\Big|_0^{x^2+y^2}\,dA \\ &= \iint_{\hat{F}} (x^2+y^2)\,dA\end{aligned}$$

This double integral is easier to evaluate in polar form:

$\hat{F} = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 3 \le r \le 5,\, 0 \le \theta \le \frac{\pi}{4}\right\}$

Therefore our integral becomes:

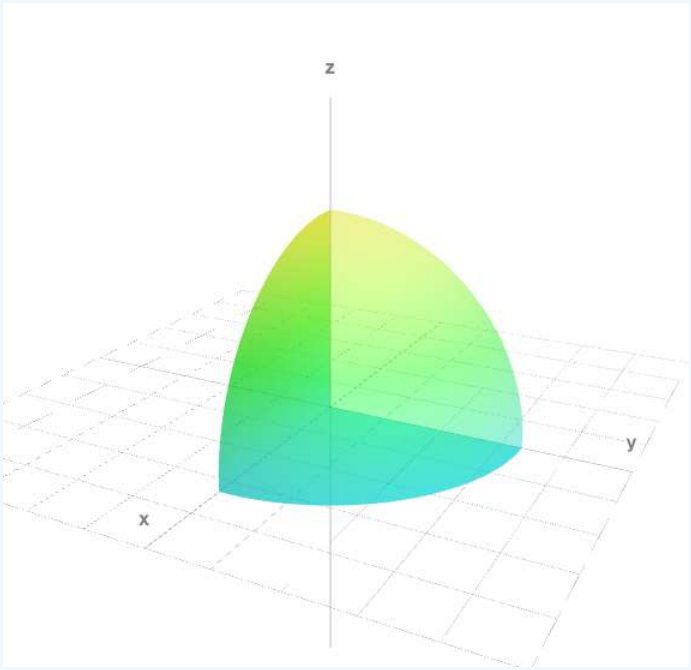
$$\begin{aligned}\text{Volume}(F) &= \int_3^5 \int_0^{\frac{\pi}{4}} r^2 \cdot r\,d\theta\,dr \\ &= \left(\int_3^5 r^3\,dr\right)\left(\int_0^{\frac{\pi}{4}} d\theta\right) \\ &= \left(\frac{1}{4}r^4\Big|_3^5\right)\cdot\left(\theta\Big|_0^{\frac{\pi}{4}}\right) \\ &= \frac{1}{4}\left(5^4 - 3^4\right)\cdot\frac{\pi}{4} \\ &= 34\pi\end{aligned}$$

Exercise 3.2.4a

Find the volume of $E = \{(x, y, z) \,:\, x^2 + y^2 + z^2 \le R^2,\, x, y, z \ge 0\}$

Answer

The region E described is the part of a sphere in the first octant:



This can be written $E = \left\{(x, y, z) \,:\, 0 \le x \le R,\, 0 \le y \le \sqrt{R^2 - x^2},\, 0 \le z \le \sqrt{R^2 - x^2 - y^2}\right\}$

Therefore the volume of E is given by the integral:

$$\begin{aligned}\text{Volume}(E) &= \iiint_E 1\,dV \\ &= \int_0^R \int_0^{\sqrt{R^2-x^2}} \int_0^{\sqrt{R^2-x^2-y^2}} 1\,dz\,dy\,dx \\ &= \int_0^R \int_0^{\sqrt{R^2-x^2}} z\Big|_0^{\sqrt{R^2-x^2-y^2}}\,dy\,dx \\ &= \int_0^R \int_0^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy\,dx\end{aligned}$$

This double integral is easier to evaluate in polar form:

$$\left\{(x,y)\,:\,0\le x\le R,\,0\le y\le\sqrt{R^2-x^2}\right\} = \left\{(r\cos(\theta),r\sin(\theta))\,:\,0\le r\le R,\,0\le\theta\le\frac{\pi}{2}\right\}$$

Therefore our integral becomes:

$$\begin{aligned}\text{Volume}(E) &= \int_0^R \int_0^{\frac{\pi}{2}} \sqrt{R^2-r^2}\cdot r\,d\theta\,dr \\ &= \left(\int_0^R r\sqrt{R^2-r^2}\,dr\right)\left(\int_0^{\frac{\pi}{2}} d\theta\right) \\ &= \left(-\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}\Big|_0^R\right)\cdot\left(\theta\Big|_0^{\frac{\pi}{2}}\right) \\ &= \frac{1}{3}R^3\cdot\frac{\pi}{2} \\ &= \frac{\pi R^3}{6}\end{aligned}$$

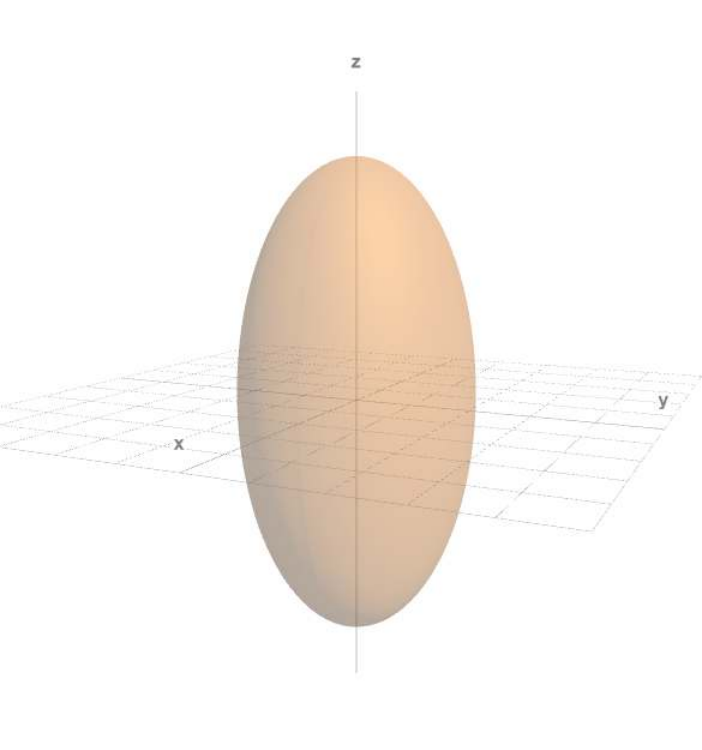
Notice this matches with one eighth of the known volume of a sphere formula: $\frac{1}{8}\cdot\frac{4\pi}{3}R^3 = \frac{\pi R^3}{6}$

? Exercise 3.2.4*b*

Find the volume of $F = \left\{(x,y,z)\,:\,\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}\le 1\right\}$

Answer

The region F in this question is a full ellipsoid with x and y radius $a>0$ and z radius $b>0$:



This can be written $F = \left\{(x,y,z) \,:\, -a \le x \le a,\, -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2},\, -\frac{b}{a}\sqrt{a^2-x^2-y^2} \le z \le \frac{b}{a}\sqrt{a^2-x^2-y^2}\right\}$

Therefore the volume of F is given by the integral:

$$\begin{aligned}
\text{Volume}(F) &= \iiint_F 1\, dV \\
&= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\frac{b}{a}\sqrt{a^2-x^2-y^2}}^{\frac{b}{a}\sqrt{a^2-x^2-y^2}} 1\, dz\, dy\, dx \\
&= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z\Big|_{-\frac{b}{a}\sqrt{a^2-x^2-y^2}}^{\frac{b}{a}\sqrt{a^2-x^2-y^2}} dy\, dx \\
&= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{2b}{a}\sqrt{a^2-x^2-y^2}\, dy\, dx
\end{aligned}$$

This double integral is easier to evaluate in polar form:

$\left\{(x,y) \,:\, -a \le x \le a,\, -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}\right\} = \{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le a,\, 0 \le \theta \le 2\pi\}$

Therefore our integral becomes:

$$\begin{aligned}
\text{Volume}(F) &= \int_0^a\int_0^{2\pi} \frac{2b}{a}\sqrt{a^2-r^2}\cdot r\, d\theta\, dr \\
&= \left(\int_0^a \frac{2b}{a} r\sqrt{a^2-r^2}\, dr\right)\left(\int_0^{2\pi} d\theta\right) \\
&= \left(-\frac{2b}{3a}(a^2-r^2)^{\frac{3}{2}}\Big|_0^a\right)\cdot\left(\theta\Big|_0^{2\pi}\right) \\
&= \frac{2b}{3a}a^3\cdot 2\pi \\
&= \frac{4\pi a^2 b}{3}
\end{aligned}$$

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3.3: Change of Variables in 2 Dimensions

Learning Objectives

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.

In this section, we generalize the concept of substitution in single variable calculus for double integrals. While substitution works fine when $u = f(x)$ or $v = g(y)$, we need a new set of tools when $u = f(x, y)$ and $v = g(x, y)$. These substitutions change the domain we integrate over, as well as the integrand inside.

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3.3: Change of Variables in Double Integrals

Recall the method of integration called substitution. When evaluating an integral such as

$$\int_0^1 \sqrt{3x+1}\,dx$$

we substitute $u = g(x) = 3x + 1$. Then $du = 3\,dx$ or $dx = \frac{1}{3}\,du$ and the limits change to $u = g(0) = 3(0) + 1 = 1$ and $u = g(1) = 3(1) + 1 = 4$.

Thus the integral becomes much easier to evaluate in terms of u:

$$\int_1^4 \sqrt{u}\left(\frac{1}{3}\,du\right)$$

When solving integration problems, making appropriate substitutions can lead to an integral that becomes much simpler than the original integral.

Consider the area of each of these integrals if we did not include the $\frac{1}{3}$ next to the du:

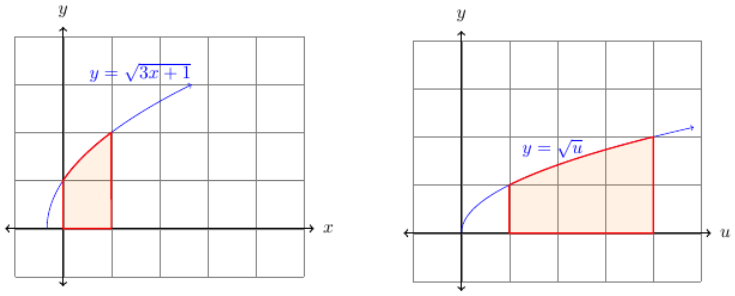


Figure 3.3.1: The relationship between $\int_0^1 \sqrt{3x+1}\,dx$ and $\int_1^4 \sqrt{u}\,du$

Notice that the two areas pictured are not the same. They reach all of the same heights, but the base of the graph in terms of u is three times as wide as the base of the graph in terms of x. This is why the $\frac{1}{3}\,du$ is necessary when we do substitutions, because the graph can be stretched by some factor when switching variables.

This is the main reason why we have a different form of dA when we switch from rectangular coordinates to polar coordinates. We have $dA = dx\,dy$ in rectangular form, but $dA = r\,dr\,d\theta$ in polar form because when we switch to polar there is a "stretching" of the region we are integrating over that we compensate for with the r term.

This section is dedicated to determining how to safely switch between variables in double integrals for effective substitutions. Generally, the function that we use to change the variables to make the integration simpler is called a transformation or mapping.

A planar transformation T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, the figure below shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u, v)$ and $y = h(u, v)$, or sometimes we write $x = x(u, v)$ and $y = y(u, v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, and h_v exist and are also continuous. The need for this requirement will become clear soon.

The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

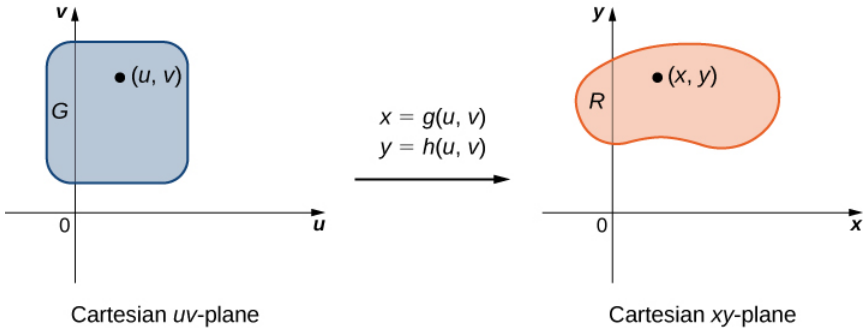


Figure 3.3.2: The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition: One-to-One Transformation

A transformation $T: G \to R$, defined as $T(u, v) = (x, y)$, is said to be a one-to-one transformation if no two points map to the same image point.

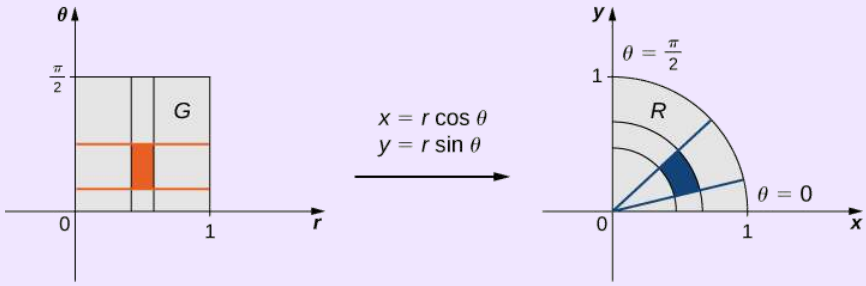
To show that T is a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain $(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Illustration 3.3.1: Determining How the Transformation Works

Suppose a transformation T is defined as $T(r, \theta) = (x, y)$ where $x = r\cos(\theta)$, $y = r\sin(\theta)$.

We will find the image of the polar rectangle $G = \left\{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\right\}$ in the $r\theta$-plane to a region R in the xy-plane, show that T is a one-to-one transformation in G, and find $T^{-1}(x, y)$.

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\frac{\pi}{2}$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane. Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.



In order to show that T is a one-to-one transformation, assume $T(r_1, \theta_1) = T(r_2, \theta_2)$ and show as a consequence that $(r_1, \theta_1) = (r_2, \theta_2)$. In this case, we have

$$T(r_1, \theta_1) = T(r_2, \theta_2) \implies (x_1, y_1) = (x_2, y_2) \implies (r_1\cos(\theta_1), r_1\sin(\theta_1)) = (r_2\cos(\theta_2), r_2\sin(\theta_2))$$

Therefore we have a system of equations:

$$\begin{cases} r_1\cos(\theta_1) = r_2\cos(\theta_2) \\ r_1\sin(\theta_1) = r_2\sin(\theta_2) \end{cases}$$

Dividing these two equations by each other we obtain

$$\frac{r_1\cos(\theta_1)}{r_1\sin(\theta_1)} = \frac{r_2\cos(\theta_2)}{r_2\sin(\theta_2)} \implies \frac{\cos(\theta_1)}{\sin(\theta_1)} = \frac{\cos(\theta_2)}{\sin(\theta_2)} \implies \tan(\theta_1) = \tan(\theta_2)$$

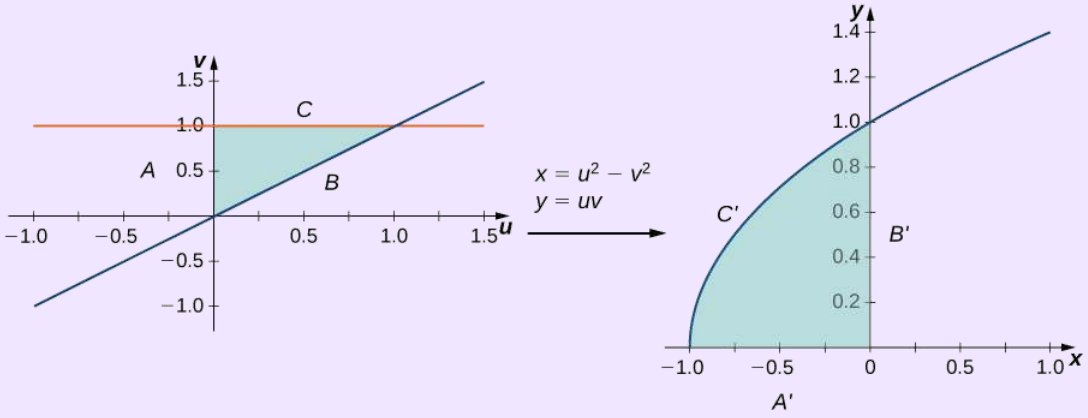
Since the tangent function is a one-to-one function in the interval $0 \le \theta \le \frac{\pi}{2}$, we must have $\theta_1 = \theta_2$. Plugging this into either of the original two equations immediately gives us $r_1 = r_2$ as well. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G to R.

We already know that $r^2 = x^2 + y^2$ and $\tan\ \theta = \frac{y}{x}$. Therefore $T^{-1}(x, y) = (r, \theta)$ is defined by $r(x, y) = \sqrt{x^2 + y^2}$ and $\theta(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

Illustration 3.3.2: Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. We will find the image of the triangle in the uv-plane with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

The triangle and its image are shown below. To understand how the sides of the triangle transform, call the side that joins $(0, 0)$ and $(0, 1)$ side A, the side that joins $(0, 0)$ and $(1, 1)$ side B, and the side that joins $(1, 1)$ and $(0, 1)$ side C.



- For the side A : $u = 0$, $0 \le v \le 1$ transforms to $x = 0^2 - v^2 = -v^2$ and $y = (0)(v) = 0$. This travels from $(x(0, 0), y(0, 0)) = (0, 0)$ to $(x(0, 1), y(0, 1)) = (-1, 0)$ along the line $y = 0$ in the xy-plane, so this is the side A' that joins $(0, 0)$ and $(-1, 0)$.
- For the side B : $v = u$, $0 \le u \le 1$ transforms to $x = u^2 - u^2 = 0$ and $y = (u)(u) = u^2$. This travels from $(x(0, 0), y(0, 0)) = (0, 0)$ to $(x(1, 1), y(1, 1)) = (0, 1)$ along the line $x = 0$ in the xy-plane, so this is the side B' that joins $(0, 0)$ and $(0, 1)$.
- For the side C : $0 \le u \le 1$, $v = 1$ transforms to $x = u^2 - 1^2 = u^2 - 1$ and $y = u(1) = u$. This travels from $(x(0, 1), y(0, 1)) = (-1, 0)$ to $(x(1, 1), y(1, 1)) = (0, 1)$ along the curve $x = y^2 - 1$ in the xy-plane, so this is the side C' that joins $(-1, 0)$ and $(0, 1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u, v) = (g(u, v),\ h(u, v))$, where $x = g(u, v)$ and $y = h(u, v)$ is a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region S that is Δu units by Δv units, in the uv-plane.

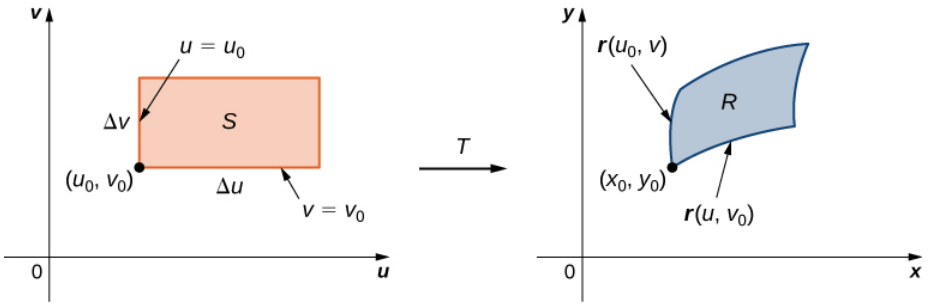


Figure 3.3.3: A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u, v)$ and $y = h(u, v)$ are in the xy-plane, we define the three dimensional position vector $\vec{\mathbf{r}}(u, v) = \langle g(u, v), h(u, v), 0\rangle$ of the image of the point (u, v). Suppose that (u_0, v_0) is the coordinate of the point at the lower left corner that mapped to (x_0, y_0)

$= T(u_0, v_0)$ The line $v = v_0$ maps to the image curve with vector function $\vec{\mathbf{r}}(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$\vec{\mathbf{r}}_u = \langle g_u(u_0, v_0), h_u(u_0, v_0), 0\rangle = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \right\rangle$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $r(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$\vec{\mathbf{r}}_v = \langle g_v(u_0, v_0), h_v(u_0, v_0), 0\rangle = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \right\rangle$$

Now, note that by linearization we know both of the following must hold:

$$\vec{\mathbf{r}}_u = \lim_{\Delta u \to 0} \frac{\vec{\mathbf{r}}(u_0 + \Delta u, v_0) - \vec{\mathbf{r}}(u_0, v_0)}{\Delta u} \longleftrightarrow \vec{\mathbf{r}}(u_0 + \Delta u, v_0) - \vec{\mathbf{r}}(u_0, v_0) \approx \Delta u \vec{\mathbf{r}}_u$$

$$\vec{\mathbf{r}}_v = \lim_{\Delta v \to 0} \frac{\vec{\mathbf{r}}(u_0, v_0 + \Delta v) - \vec{\mathbf{r}}(u_0, v_0)}{\Delta v} \longleftrightarrow \vec{\mathbf{r}}(u_0, v_0 + \Delta v) - \vec{\mathbf{r}}(u_0, v_0) \approx \Delta v \vec{\mathbf{r}}_v$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v\ \vec{\mathbf{r}}_v$ and $\Delta u\ \vec{\mathbf{r}}_u$. By using the cross product of these two vectors, the area ΔA of the image R is approximately $\|\Delta u\ \vec{\mathbf{r}}_u \times \Delta v\ \vec{\mathbf{r}}_v\| = \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\ \Delta u\ \Delta v$. In determinant form, the cross product is

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = (0-0)\hat{\mathbf{i}} + (0-0)\hat{\mathbf{j}} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{\mathbf{k}} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \hat{\mathbf{k}}$$

Therefore we finally get:

$$\Delta A \approx \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| \Delta u \Delta v = \left| \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \right| \|\hat{\mathbf{k}}\| \Delta u \Delta v = \left| \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \right| \Delta u \Delta v$$

Definition: Jacobian

The ***Jacobian*** of the C^1 transformation $T(u, v) = (g(u, v),\ h(u, v))$ is denoted by $J(u, v)$ and is defined by the 2×2 determinant

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right)$$

Using this definition, we have

$$\Delta A \approx |J(u, v)| \Delta u \Delta v = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

This is the "stretching" factor that we need when doing a multivariable substitution in a double integral.

Example 3.3.1: Finding the Jacobian

Find the Jacobian of the transformation $T(r, \theta) = (x, y)$ where $x = r\ \cos(\theta),\ y = r\ \sin(\theta)$

Solution

We have $x(r, \theta) = r\cos(\theta)$ so $\dfrac{\partial x}{\partial r} = \cos(\theta)$ and $\dfrac{\partial x}{\partial \theta} = -r\sin(\theta)$

Similarly, $y(r, \theta) = r\sin(\theta)$ so $\dfrac{\partial y}{\partial r} = \sin(\theta)$ and $\dfrac{\partial y}{\partial \theta} = r\cos(\theta)$

Thus the Jacobian is

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$
$$= r\,\cos^2(\theta) + r\,\sin^2(\theta) = r\left(\cos^2(\theta) + \sin^2(\theta)\right) = \boxed{r}$$

No surprise here, that is exactly what happens when we switch dA into polar form: $dA = dx\,dy$ is r times as big as $dr\,d\theta$

Example 3.3.2: Finding the Jacobian

Find the Jacobian of the transformation $T(u,v) = (u^2 - v^2, uv)$ where $x = u^2 - v^2$ and $y = uv$

Solution

We have $x(u,v) = u^2 - v^2$ so $\frac{\partial x}{\partial u} = 2u$ and $\frac{\partial x}{\partial v} = -2v$

Similarly, $y(u,v) = uv$ so $\frac{\partial y}{\partial u} = v$ and $\frac{\partial y}{\partial v} = u$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = \boxed{2u^2 + 2v^2}$$

We have already seen that, under the change of variables $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx |J(u,v)|\,\Delta u\,\Delta v$$

Now let's go back to the definition of the double integral for a minute:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

Referring to Figure 3.3.4, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v) = (x,y)$.

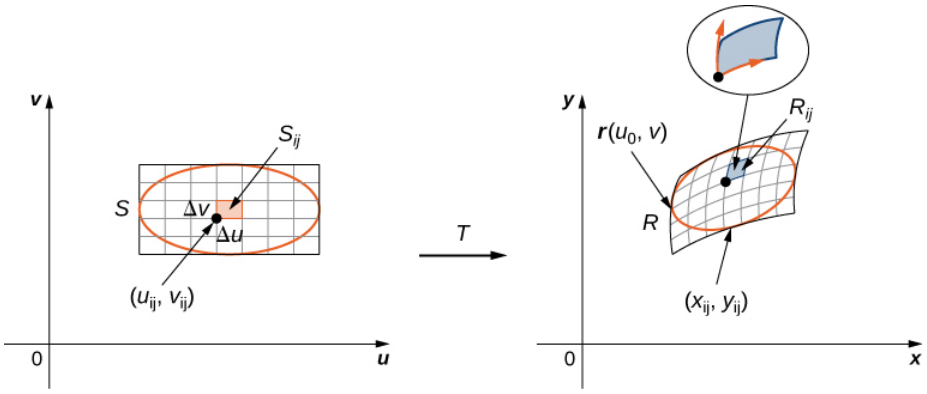


Figure 3.3.4: The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$
$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_{ij}, v_{ij}),\, h(u_{ij}, v_{ij}))|J(u_{ij}, v_{ij})|\Delta u \Delta v$$
$$= \iint_S f(g(u,v),\, h(u,v))\,|J(u,v)|\,du\,dv$$

Theorem 3.3.1: Change of Variables for Double Integrals

Let $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\, h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

With this theorem for double integrals, we can change the variables from (x,y) to (u,v) in a double integral simply by replacing

$$dA = dx\, dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

when we use the substitutions $x = g(u,v)$ and $y = h(u,v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler. In single variable substitution, $\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$ sometimes replacing the numbers makes the "smaller" limit end up on top and the "bigger" limit on the bottom in your new integral, and the du term gains a negative sign. In double integrals, we always set the limits of integration from "small" to "big" instead of tracking which one was originally on top or bottom, and to compensate we remember to include the absolute values on $J(u,v)$.

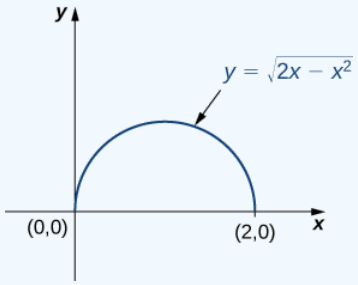
Example 3.3.3: Changing Variables from Rectangular to Polar Coordinates

Use the change of variables $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to evaluate the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$



Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is $y^2 + (x-1)^2 = 1$
In polar coordinates, the circle is $r = 2\,cos(\theta)$ and since it is contained entirely in the first quadrant, we have $0 \le \theta \le \frac{\pi}{2}$.

Therefore, the region of integration in polar coordinates is bounded by $0 \le r \le 2\cos(\theta)$ and $0 \le \theta \le \frac{\pi}{2}$

We have already calculated that in polar the Jacobian is $J(r,\theta) = r$, so our double integral becomes:

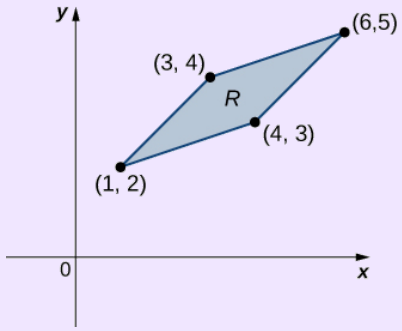
$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} \sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2}\,|J(r,\theta)|\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^2\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{3}r^3\Big|_0^{2\cos(\theta)}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{8}{3}\cos^3(\theta)\,d\theta \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(1-\sin^2(\theta)\right)\cos(\theta)\,d\theta \\
&= \frac{8}{3}\left[\sin(\theta)-\frac{1}{3}\sin^3(\theta)\right]\Big|_0^{\frac{\pi}{2}} \\
&= \frac{16}{9}
\end{aligned}$$

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Illustration 3.3.3: Changing Variables

Let R be the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$. Evaluate

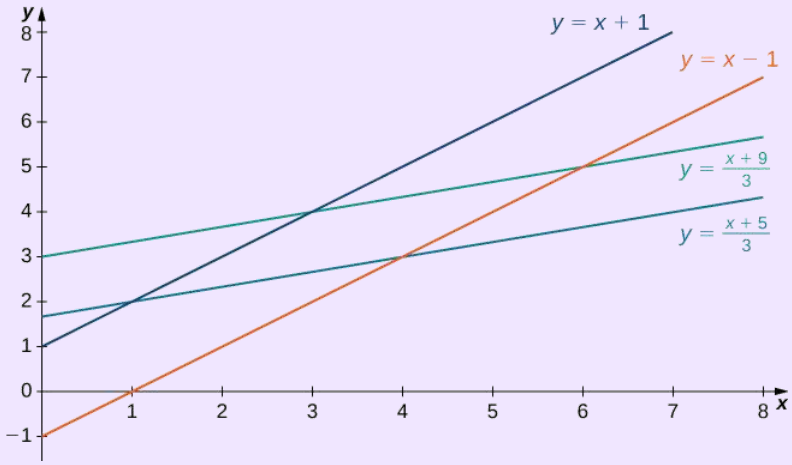
$$\iint_R (x-y)dy\,dx$$



We will do this problem *three different* ways. The first of which is the direct method (writing the region as three distinct Type I regions and adding the resulting integrals together):

Method 1:

If we solve for the lines at the edges of the four sides of this parallelogram, we see they are:



Therefore we can split:

$$\iint_R (x-y)dy\,dx = \int_1^3 \int_{\frac{x+5}{3}}^{x+1} (x-y)\,dy\,dx + \int_3^4 \int_{\frac{x+5}{3}}^{\frac{x+9}{3}} (x-y)\,dy\,dx + \int_4^6 \int_{x-1}^{\frac{x+9}{3}} (x-y)\,dy\,dx$$

Solving these separately we get:

$$\int_1^3 \int_{\frac{x+5}{3}}^{x+1} (x-y)\,dy\,dx = \int_1^3 \left[xy - \frac{1}{2}y^2\right]\Big|_{\frac{x+5}{3}}^{x+1}$$

$$dx = \int_1^3 \left(\frac{2}{9}x^2 - \frac{10}{9}x + \frac{8}{9}\right)\,dx = \left[\frac{2}{27}x^3 - \frac{5}{9}x^2 + \frac{8}{9}x\right]\Big|_1^3 = -\frac{20}{27}$$

$$\int_3^4 \int_{\frac{x+5}{3}}^{\frac{x+9}{3}} (x-y)\,dy\,dx = \int_3^4 \left[xy - \frac{1}{2}y^2\right]\Big|_{\frac{x+5}{3}}^{\frac{x+9}{3}} dx = \int_3^4 \left(\frac{8}{9}x - \frac{28}{9}\right)\,dx = \left[\frac{4}{9}x^2 - \frac{28}{9}x\right]\Big|_3^4 = 0$$

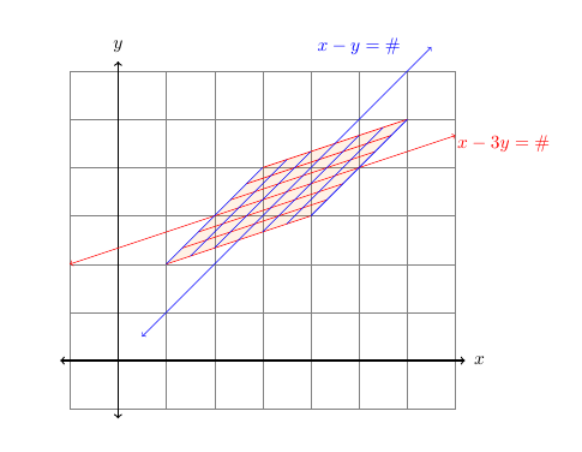
$$\int_4^6 \int_{x-1}^{\frac{x+9}{3}} (x-y)\,dy\,dx = \int_4^6 \left[xy - \frac{1}{2}y^2\right]\Big|_{x-1}^{\frac{x+9}{3}}$$

$$dx = \int_4^6 \left(-\frac{2}{9}x^2 + 2x - 4\right)\,dx = \left[-\frac{2}{27}x^3 + x^2 - 4x\right]\Big|_4^6 = \frac{20}{27}$$

Summing these up, we arrive at our final answer $\iint_R (x-y)\,dy\,dx = 0$

Method 2:

Instead of doing things the hard way, we can do a substitution to make the domain of integration more simple. Clearly the parallelogram is bounded by the lines $y = x+1$, $y = x-1$, $y = \frac{1}{3}(x+5)$, and $y = \frac{1}{3}(x+9)$. Another way to look at them is $x - y = -1$, $x - y = 1$, $x - 3y = -5$, and $x - 3y = -9$. In fact, we can see that lines of the form $x - y = \#$ and $x - 3y = \#$ form a grid inside of this parallelogram:



Therefore if we were to make $u = x - y$ and $v = x - 3y$, then the domain R in terms of u and v would be $-1 \le u \le 1$ and $-9 \le v \le -5$.

Next we need to solve for x and y. We use elimination:

Multiply the first equation by 3 and subtract the second equation, $3u - v = (3x - 3y) - (x - 3y) = 2x$ Then we have $x = \frac{3u - v}{2}$

Moreover, if we simply subtract the second equation from the first, we get $u - v = (x - y) - (x - 3y) = 2y$ and $y = \frac{u - v}{2}$

Thus, we can choose the transformation $T(u, v) = \left(\frac{3u - v}{2}, \frac{u - v}{2}\right)$. The Jacobian is:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u,v)| = \frac{1}{2}$. Plugging this into our original integral we get:

$$\begin{aligned} \iint_R (x-y)\,dy\,dx &= \int_{-9}^{-5}\int_{-1}^{1} \left(\left(\frac{3u-v}{2}\right) - \left(\frac{u-v}{2}\right)\right) |J(u,v)|\;du\,dv \\ &= \int_{-9}^{-5}\int_{-1}^{1} u \cdot \frac{1}{2}\,du\,dv \\ &= \int_{-9}^{-5} \frac{1}{4}u^2\Big|_{-1}^{1}\,dv \\ &= \int_{-9}^{-5} 0 dv = 0 \end{aligned}$$

This was much easier to compute!

Method 3:

Just for fun, our third approach will be a different substitution that works when dealing with parallelograms specifically.

Consider that the interior of the parallelogram is just the bottom left corner $(1,2)$ plus varying multiples of the two adjacent side vectors $\langle 3,1\rangle$ and $\langle 2,2\rangle$. We can think of u and v as the "amount" of these vectors we are adding to this bottom left corner:

$$\langle x,y\rangle = \langle 1,2\rangle + u\langle 3,1\rangle + v\langle 2,2\rangle = \langle 1+3u+2v, 2+u+2v\rangle$$

Thus we can use the parameterization $T(u,v) = (1+3u+2v, 2+u+2v)$ where $0 \le u \le 1$ and $0 \le v \le 1$. The Jacobian is:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 6 - 2 = 4$$

Therefore, $|J(u,v)| = 4$. Plugging this into our original integral we get:

$$\begin{aligned} \iint_R (x-y)\,dy\,dx &= \int_0^1\int_0^1 ((1+3u+2v) - (2+u+2v))\,|J(u,v)|\;du\,dv \\ &= \int_0^1\int_0^1 (-1+2u)\,4\,du\,dv \\ &= \int_0^1 4\left[-u+u^2\right]\Big|_0^1\,dv \\ &= \int_0^1 0\,dv = 0 \end{aligned}$$

As they say, all roads lead to Rome.

We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u,v)$ and $y = h(u,v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u,v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $|J(u,v)|du\,dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 3.3.4: Evaluating an Integral

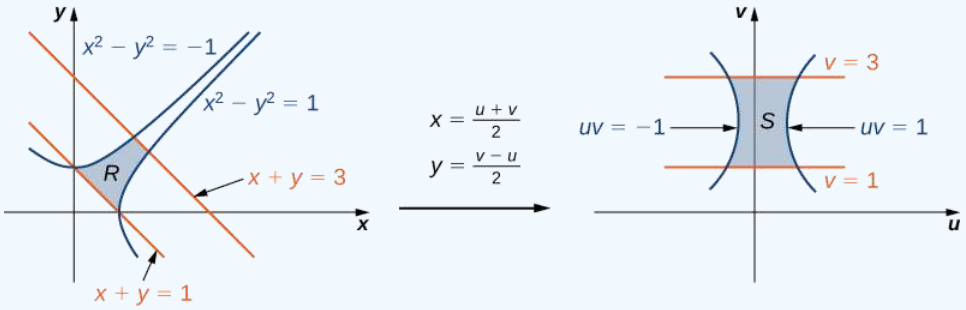
Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_R (x-y)e^{x^2-y^2}\,dA$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made:



Given $u = x - y$ and $v = x + y$, using elimination can we find $x = \dfrac{u+v}{2}$ and $y = \dfrac{v-u}{2}$ and hence the transformation to use is $T(u,v) = \left(\dfrac{u+v}{2}, \dfrac{v-u}{2}\right)$.

The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively.
The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region $S = \left\{(u,v) : 1 \le v \le 3,\ \dfrac{-1}{v} \le u \le \dfrac{1}{v}\right\}$ as pictured above.

The Jacobian for this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Therefore, by using the transformation T, the integral changes to

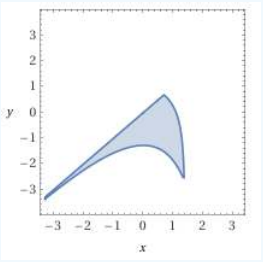
$$\begin{aligned}
\iint_R (x-y)e^{x^2-y^2}\,dA &= \int_1^3 \int_{-\frac{1}{v}}^{\frac{1}{v}} \left(\left(\frac{u+v}{2}\right) - \left(\frac{v-u}{2}\right)\right) e^{\left(\frac{u+v}{2}\right)^2 - \left(\frac{v-u}{2}\right)^2} |J(u,v)|\,du\,dv \\
&= \int_1^3 \int_{-\frac{1}{v}}^{\frac{1}{v}} \frac{1}{2} u e^{uv}\,du\,dv \\
&= \int_1^3 \frac{1}{2}\left(\frac{u}{v} - \frac{1}{v^2}\right) e^{uv}\Big|_{-\frac{1}{v}}^{\frac{1}{v}}\,dv \\
&= \int_1^3 \frac{1}{e} v^{-2}\,dv \\
&= -\frac{1}{e} v^{-1}\Big|_1^3 \\
&= \boxed{\frac{2}{3e}}
\end{aligned}$$

Example 3.3.5: Evaluating an Integral

Find the area of $R = \{(x, y) : 0 \le x - y \le e^x + e^y \le 4\}$

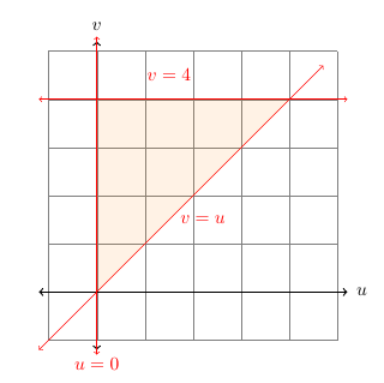
Solution

It is hard to sketch, but in case you are curious the region R is pictured below:



We will not need this picture to solve the question, but at least we believe it exists.

To make things simple, we let $u = x - y$ and $v = e^x + e^y$, which makes the new domain $S = \{(u, v) : 0 \le u \le v \le 4\}$



We clearly know $T^{-1}(x, y) = (x - y, e^x + e^y)$, but solving for $x(u, v)$ and $y(u, v)$ is not straightforward. Since we already have our domain in terms of u and v, all we really need these functions for is $J(u, v)$. Instead of stressing about this, we instead find the **inverse Jacobian** $J^{-1}(x, y) = \dfrac{\partial(u, v)}{\partial(x, y)}$:

$$J^{-1}(x, y) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ e^x & e^y \end{vmatrix} = e^y + e^x = v$$

Since this is the inverse Jacobian, we know that $J(u, v)J^{-1}(x, y) = 1$ must hold, so $J(u, v) = \dfrac{1}{v}$

Therefore, by using the transformation T, the area of R is represented by the double integral:

$$\begin{aligned} \iint_R 1\, dA &= \int_0^4 \int_0^v 1\, |J(u, v)|\, du\, dv \\ &= \int_0^4 \int_0^v \frac{1}{v}\, du\, dv \\ &= \int_0^4 \frac{u}{v}\Big|_0^v\, dv \\ &= \int_0^4 1\, dv \\ &= v\Big|_0^4 \\ &= \boxed{4} \end{aligned}$$

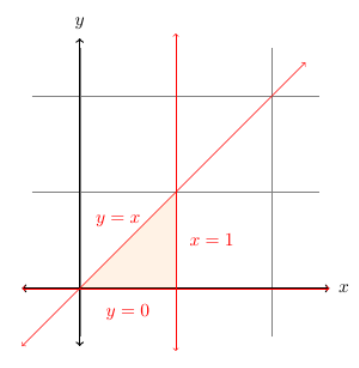
Lastly, we note that sometimes the integrand determines the substitution, instead of the domain.

Example 3.3.6: Evaluating an Integral

Evaluate $\int_0^1 \int_0^x \frac{y-2x}{(y+3x)^2}\,dy\,dx$

Solution

This can be written as $\iint_R \frac{y-2x}{(y+3x)^2}\,dA$ for $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$:



This triangle is easy to draw and clearly already a Type I (and Type II) region, but the function inside is not easy to integrate. Some natural substitutions to clean it up would be: $u = y - 2x, v = y + 3x$. Using elimination to solve for x and y, we can see that this is equivalent to $T(u,v) = \left(\frac{v-u}{5}, \frac{3u+2v}{5}\right)$

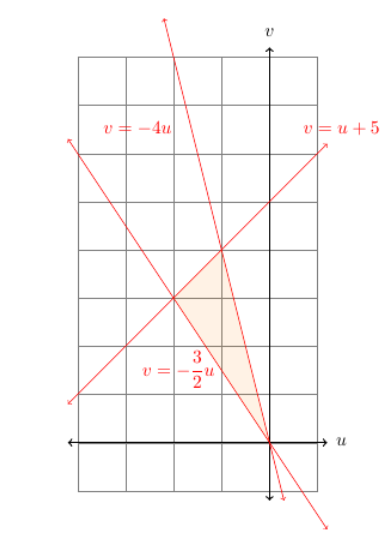
Plugging this into our three lines at the edges of our domain we see:

$y = 0$ becomes $\frac{3u+2v}{5} = 0 \longrightarrow v = -\frac{3}{2}u$

$x = 1$ becomes $\frac{v-u}{5} = 1 \longrightarrow v = 5 + u$

$y = x$ becomes $\frac{3u+2v}{5} = \frac{v-u}{5} \longrightarrow v = -4u$

Thus our new domain S is the triangle with these edges, drawn below:



Written out, this becomes $S = S_1 \cup S_2$ for $S_1 = \left\{(u,v) : -2 \le u \le -1, -\frac{3}{2}u \le v \le u + 5\right\}$ and $S_2 = \left\{(u,v) : -1 \le u \le 0, -\frac{3}{2}u \le v \le -4u\right\}$

The Jacobian for this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{vmatrix} = -\frac{1}{5}$$

Therefore, by using the transformation T, the integral changes to

$$\begin{aligned}
\int_0^1 \int_0^x \frac{y-2x}{(y+3x)^2}\,dy\,dx &= \int_{-2}^{-1} \int_{-\frac{3}{2}u}^{u+5} \frac{u}{v^2} \left|-\frac{1}{5}\right| dv\,du \quad + \quad \int_{-1}^{0} \int_{-\frac{3}{2}u}^{-4u} \frac{u}{v^2} \left|-\frac{1}{5}\right| dv\,du \\
&= \int_{-2}^{-1} -\frac{u}{5v}\Big|_{-\frac{3}{2}u}^{u+5} du \quad + \quad \int_{-1}^{0} -\frac{u}{5v}\Big|_{-\frac{3}{2}u}^{-4u} du \\
&\quad \int_{-2}^{-1} \left(\frac{u}{5(u+5)} - \frac{2}{15}\right) du \quad + \quad \int_{-1}^{0} \left(\frac{1}{20} - \frac{2}{15}\right) du \\
&\quad \int_{-2}^{-1} \left(\frac{1}{u+5} - \frac{1}{3}\right) du \quad + \quad \int_{-1}^{0} \left(-\frac{1}{12}\right) du \\
&\quad \left[\ln(u+5) - \frac{1}{3}u\right]\Big|_{-2}^{-1} \quad + \quad \left[-\frac{1}{12}u\right]\Big|_{-1}^{0} \\
&\quad \left(\ln(4) - \ln(3) - \frac{1}{3}\right) \quad + \quad \left(-\frac{1}{12}\right) \\
&= \boxed{\ln\left(\frac{4}{3}\right) - \frac{5}{12}}
\end{aligned}$$

Although the domain got significantly worse, it allowed us to actually integrate the desired region!

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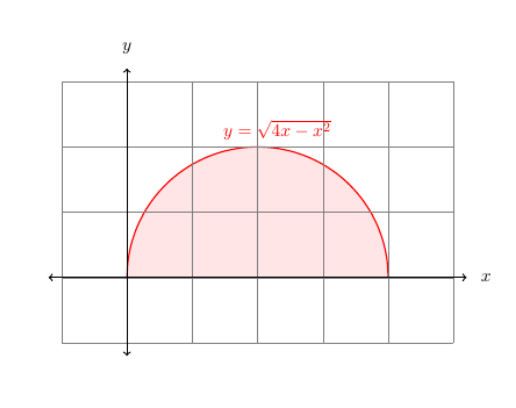
3.3: Practice Set

Exercise 3.3.1

Use a substitution to evaluate $\int_0^4 \int_0^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Answer

The region we are integrating over is $D = \left\{(x,y) \,:\, 0 \le x \le 4,\, 0 \le y \le \sqrt{4x-x^2}\right\}$:



The function inside would benefit from a polar substitution, so we describe the domain in terms of polar.

$y = \sqrt{4x-x^2} \longleftrightarrow x^2 + y^2 = 4x \longleftrightarrow r^2 = 4r\cos(\theta) \longleftrightarrow r = 4\cos(\theta)$

To get the top half of this circle, we see that $0 \le \theta \le \frac{\pi}{2}$

Therefore in polar form $D = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le \theta \le \frac{\pi}{2},\, 0 \le r \le 4\cos(\theta)\right\}$

The parametric domain is therefore $E = \left\{(r,\theta) \,:\, 0 \le \theta \le \frac{\pi}{2},\, 0 \le r \le 4\cos(\theta)\right\}$

Since the Jacobian in polar form is known to be $J(r,\theta) = r$, the double integral becomes:

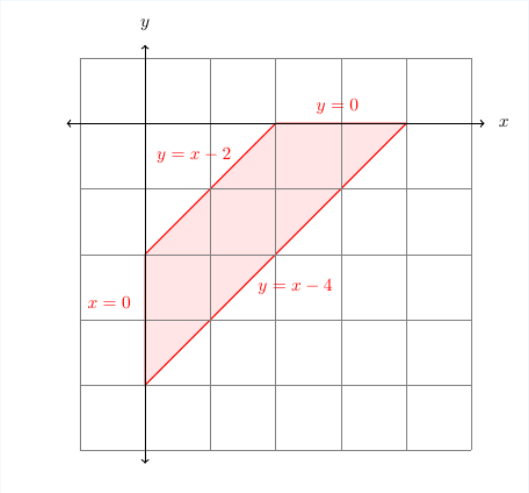
$$\begin{aligned}
\int_0^4 \int_0^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\, dy\, dx &= \iint_D \sqrt{x^2+y^2}\, dA \\
&= \iint_E \sqrt{[x(r,\theta)]^2 + [y(r,\theta)]^2} \cdot |J(r,\theta)|\, dA \\
&= \iint_E \sqrt{(r\cos(\theta))^2 + (r\sin(\theta))^2} \cdot |r|\, dA \\
&= \int_0^{\frac{\pi}{2}} \int_0^{4\cos(\theta)} r^2\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{3} r^3 \Big|_0^{4\cos(\theta)}\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{64}{3}\cos^3(\theta)\, d\theta \\
&= \frac{64}{3} \int_0^{\frac{\pi}{2}} \left(1 - \sin^2(\theta)\right)\cos(\theta)\, d\theta \\
&= \frac{64}{3}\left[\sin(\theta) - \frac{1}{3}\sin^3(\theta)\right]\Big|_0^{\frac{\pi}{2}} \\
&= \frac{64}{3} \cdot \frac{2}{3} \\
&= \frac{128}{9}
\end{aligned}$$

Exercise 3.3.2

Use a substitution to evaluate $\iint_T \frac{4}{(x-y)^3}\,dA$ where T is the trapezoid with vertices $(2,0), (0,-2), (4,0)$, and $(0,-4)$

Answer

The region T is a trapezoid that is not easy to write as Type 1 or 2. We calculate the equations of the lines along its edges:



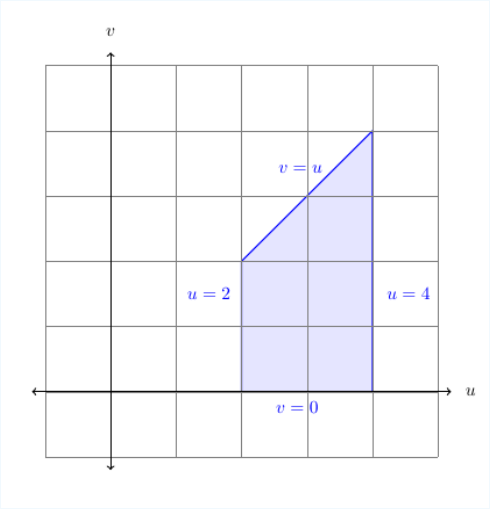
Letting $y = x - u \longleftrightarrow u = x - y$ makes the two diagonal edges into $u = 2$ and $u = 4$.

We could choose v to be many different things, but $v = x$ or $v = y$ are simplest. We will use $v = x$. This makes the left edge $v = 0$.

Now $u = x - y$, $v = x$ and we can solve for x and y to get: $x = v$, $y = v - u$

Therefore the top edge of our trapezoid is $y = 0 \longleftrightarrow v - u = 0 \longleftrightarrow v = u$

The parametric domain is now bounded by the lines $u = 2$, $u = 4$, $v = 0$, and $v = u$:



We write this as a Type 1 region: $G = \{(u,v) : 2 \le u \le 4,\ 0 \le v \le u\}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1$

Therefore, the double integral becomes:

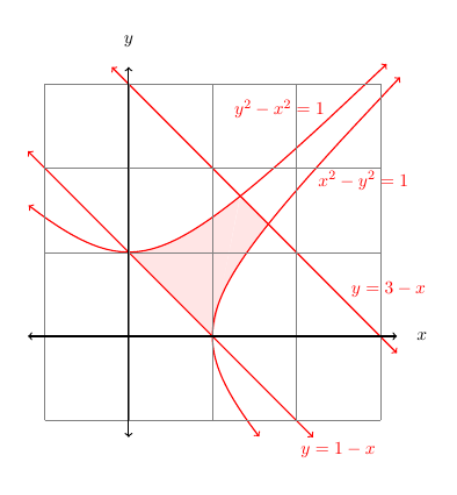
$$\begin{aligned}
\iint_T \frac{4}{(x-y)^3}\,dA &= \iint_G \frac{4}{(x(u,v)-y(u,v))^3}\cdot|J(u,v)|\,dA \\
&= \iint_G \frac{4}{(v-(v-u))^3}\cdot|1|\,dA \\
&= \int_2^4\int_0^u 4u^{-3}\,dv\,du \\
&= \int_2^4 4u^{-3}v\Big|_0^u\,du \\
&= \int_2^4 4u^{-2}\,du \\
&= -4u^{-1}\Big|_2^4 \\
&= 1
\end{aligned}$$

Exercise 3.3.3

Use a substitution to evaluate $\iint_E (x-y)e^{x^2-y^2}\,dA$ where E is the region bounded by $y=1-x,\ y=3-x,\ x^2-y^2=1$, and $y^2-x^2=1$

Answer

The region E described is definitely not Type 1 or Type 2:



Letting $y=u-x \leftrightarrow u=x+y$ makes the parallel lines turn into $u=1$ and $u=3$

Letting $y^2-x^2=v$ makes the hyperbolas turn into $v=1$ and $v=-1$

This makes the parametric domain a rectangle $G=\{(u,v)\,:\,1\le u\le 3,\,-1\le v\le 1\}$

Now $u=x+y$, $v=y^2-x^2$ and we can solve for x and y to get: $x=\dfrac{u^2-v}{2u},\ y=\dfrac{u^2+v}{2u}$

The Jacobian of this transformation is $J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2}+\frac{v}{2u^2} & -\frac{1}{2u}\\ \frac{1}{2}-\frac{v}{2u^2} & \frac{1}{2u}\end{vmatrix}=\frac{1}{2u}$

Therefore, the double integral becomes:

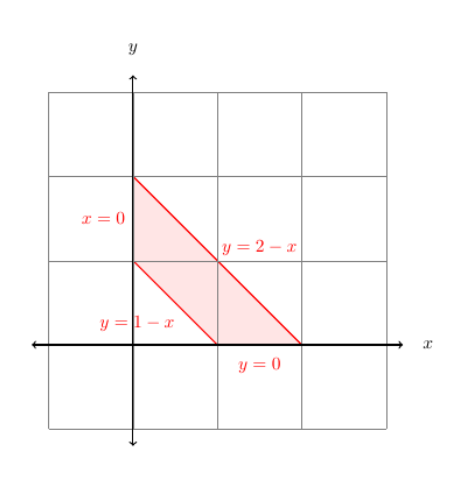
$$\begin{aligned}
\iint_E (x-y)e^{x^2-y^2}\,dA &= \iint_G (x(u,v)-y(u,v))\,e^{[x(u,v)]^2-[y(u,v)]^2}\cdot|J(u,v)|\,dA \\
&= \iint_G -\frac{v}{u}e^{-v}\cdot\left|\frac{1}{2u}\right|\,dA \\
&= \int_1^3\int_{-1}^1 -\frac{v}{2u^2}e^{-v}\,dv\,du \\
&= \left(\int_1^3 -\frac{1}{2}u^{-2}\,du\right)\left(\int_{-1}^1 ve^{-v}\,dv\right) \\
&= \left(\frac{1}{2}u^{-1}\Big|_1^3\right)\cdot\left[-ve^{-v}-e^{-v}\right]\Big|_{-1}^1 \\
&= -\frac{1}{3}\cdot\left(-\frac{2}{e}\right) \\
&= \frac{2}{3e}
\end{aligned}$$

Exercise 3.3.4

Use a substitution to evaluate $\iint_R \sin\left(\frac{\pi(y-x)}{x+y}\right)\,dA$ where R is the trapezoid with vertices $(1,0),(2,0),(0,2)$, and $(0,1)$

Answer

The region R is a trapezoid that is not easy to write as Type 1 or 2. We calculate the equation of the lines along its edges:



The function inside of our integral suggests using $u = y - x$ and $v = x + y$ so that it is easier to simplify.

Luckily, letting $v = x + y$ makes the two diagonal edges into $v = 1$ and $v = 2$.

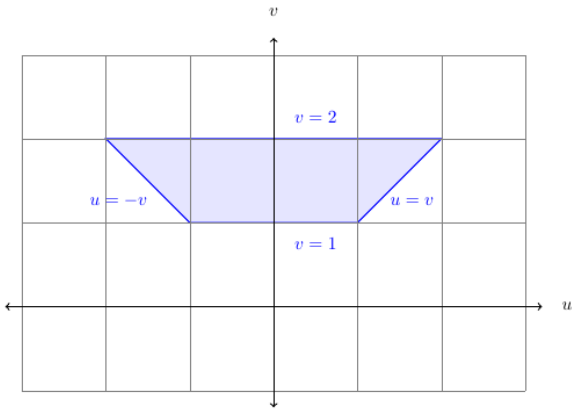
To rewrite the other sides, we need to solve for x and y first.

Since $u = y - x,\ v = x + y$ and solving for x and y gives: $x = \frac{v-u}{2}$, $y = \frac{v+u}{2}$

Therefore the left edge of our trapezoid is $x = 0 \longleftrightarrow \frac{v-u}{2} = 0 \longleftrightarrow u = v$

And the bottom edge of our trapezoid is $y = 0 \longleftrightarrow \frac{v+u}{2} = 0 \longleftrightarrow u = -v$

The parametric domain is now bounded by the lines $v = 1$, $v = 2$, $u = v$, and $u = -v$:



We write this as a Type 2 region: $G = \{(u, v) : 1 \le v \le 2, -v \le u \le v\}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

Therefore, the double integral becomes:

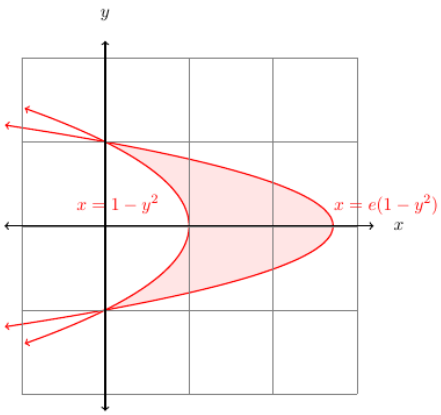
$$\begin{aligned} \iint_R \sin\left(\frac{\pi(y-x)}{x+y}\right) dA &= \iint_G \sin\left(\frac{\pi\left(y(u,v)-x(u,v)\right)}{x(u,v)+y(u,v)}\right) \cdot |J(u,v)|\, dA \\ &= \iint_G \sin\left(\frac{\pi u}{v}\right) \cdot \left|-\frac{1}{2}\right| dA \\ &= \int_1^2 \int_{-v}^{v} \frac{1}{2} \sin\left(\frac{\pi u}{v}\right) du\, dv \\ &= \int_1^2 -\frac{v}{2\pi} \cos\left(\frac{\pi u}{v}\right)\Big|_{-v}^{v} dv \\ &= \int_1^2 \left(-\frac{v}{2\pi}\cos(\pi) + \frac{v}{2\pi}\cos(-\pi)\right) dv \\ &= \int_1^2 0\, dv \\ &= 0 \end{aligned}$$

Exercise 3.3.5

Use a substitution to evaluate $\iint_D \left(\frac{1}{x} + ye^{x^2}\right) dA$ where D is the region between $x = 1 - y^2$ and $x = e - ey^2$

Answer

The region between these two parabolas is easy to write as a Type 2 region:



But this immediately leads to an impossible integral: $\int_{-1}^{1}\int_{1-y^2}^{e(1-y^2)}\left(\frac{1}{x}+ye^{x^2}\right)\,dx\,dy$

Instead we set $x=u(1-y^2)$ so that the boundary functions simplify to $u=1$ and $u=e$.

Since we know $-1\le y\le 1$ in the given region, we simply set $v=y$ so that $-1\le v\le 1$ also holds.

This makes our new domain a rectangle: $G=\{(u,v)\,:\,1\le u\le e,\,-1\le v\le 1\}$

For $u=\dfrac{x}{1-y^2}$ and $v=y$, we can solve for x and y to get: $x=u(1-v^2),\,y=v$

The Jacobian of this transformation is $J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v^2 & -2uv\\ 0 & 1\end{vmatrix}=1-v^2$

Therefore, the double integral becomes:

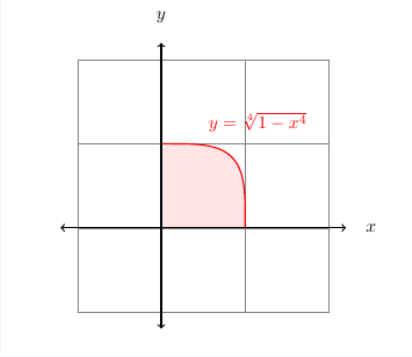
$$\begin{aligned}\iint_D\left(\frac{1}{x}+ye^{x^2}\right)\,dA &= \iint_G\left(\frac{1}{x(u,v)}+y(u,v)e^{[x(u,v)]^2}\right)\cdot|J(u,v)|\,dA\\ &= \iint_G\left(\frac{1}{u(1-v^2)}+ve^{u^2(1-v^2)^2}\right)\cdot|(1-v^2)|\,dA\\ &= \int_1^e\int_{-1}^1\left(\frac{1}{u}+v(1-v^2)e^{u^2(1-v^2)^2}\right)\,dv\,du\\ &= \int_1^e\left[\frac{v}{u}-\frac{1}{4u^2}e^{u^2(1-v^2)^2}\right]\Bigg|_{-1}^{1}\,du\\ &= \int_1^e\frac{2}{u}\,du\\ &= 2\ln|u|\Big|_1^e\\ &= 2\end{aligned}$$

? Exercise 3.3.6

Use a substitution to evaluate $\int_0^1\int_0^{\sqrt[4]{1-x^4}} xy\cos(x^4+y^4)\,dy\,dx$

Answer

The region we are integrating over is $D=\left\{(x,y)\,:\,0\le x\le 1,\,0\le y\le\sqrt[4]{1-x^4}\right\}$:



D is given to us as a Type 1 region, but even if we switch it to Type 2, the integral looks the same (and will be impossible to do directly).

Instead we attempt a similar approach to polar coordinates, and define $x = u\sqrt{\cos(v)}$ and $y = u\sqrt{\sin(v)}$

We do this to ensure that $x^4 + y^4 = u^4(\cos^2(v) + \sin^2(v)) = u^4$, so u acts like $\sqrt{r}$ and v acts like θ in polar coordinates.

Therefore our new domain must be $G = \left\{(u, v) \,:\, 0 \le u \le 1,\, 0 \le v \le \dfrac{\pi}{2}\right\}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sqrt{\cos(v)} & -\dfrac{u\sin(v)}{2\sqrt{cos(v)}} \\ \sqrt{\sin(v)} & \dfrac{u\cos(v)}{2\sqrt{\sin(v)}} \end{vmatrix} = \dfrac{u}{2\sqrt{\cos(v)\sin(v)}}$

Therefore, the double integral becomes:

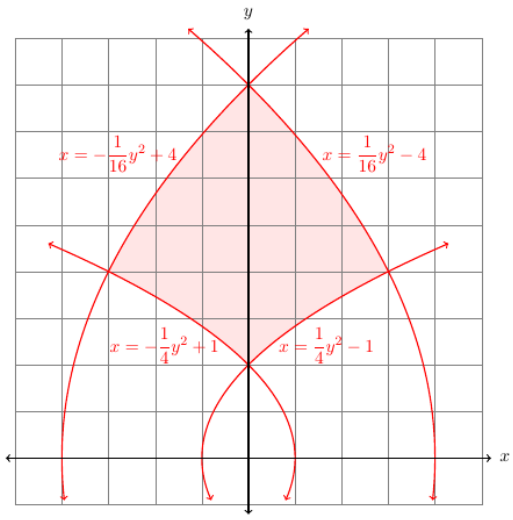
$$\begin{aligned}
\int_0^1 \int_0^{\sqrt[4]{1-x^4}} xy\cos(x^4+y^4)\,dy\,dx &= \iint_D xy\cos(x^4+y^4)\,dA \\
&= \iint_G x(u,v)y(u,v)\cos\left([x(u,v)]^4 + [y(u,v)]^4\right)\cdot|J(u,v)|\,dA \\
&= \iint_G u\sqrt{\cos(v)}\,u\sqrt{\sin(v)}\,\cos\left(u^4\right)\cdot\left|\frac{u}{2\sqrt{\cos(v)\sin(v)}}\right|dA \\
&= \int_0^1\int_0^{\frac{\pi}{2}} \frac{1}{2}u^3\cos(u^4)\,dv\,du \\
&= \left(\int_0^1 \frac{1}{2}u^3\cos(u^4)\,du\right)\left(\int_0^{\frac{\pi}{2}} dv\right) \\
&= \left(\frac{1}{8}\sin(u^4)\Big|_0^1\right)\cdot\left(v\Big|_0^{\frac{\pi}{2}}\right) \\
&= \frac{1}{8}\sin(1)\cdot\frac{\pi}{2} \\
&= \frac{\pi\sin(1)}{16}
\end{aligned}$$

? Exercise 3.3.7

Use a substitution to evaluate $\iint_D y\,dA$ where $D = \{(x,y) \,:\, y \ge 0,\, y^2 - 64 \le 16x \le 4y^2 - 16,\, 16 - 4y^2 \le 16x \le 64 - y^2\}$
(Hint: Use $x = u^2 - v^2, y = 2uv$)

Answer

The domain D is between four parabolas, and is not easy to write as a Type 1 or Type 2 region.



Using the provided hint, we try to rewrite these parabolas to find our new parametric domain.

Since $y = 2uv \longleftrightarrow u = \dfrac{y}{2v}$ we can write $x = u^2 - v^2 \longleftrightarrow x = \left(\dfrac{y}{2v}\right)^2 - v^2 = \dfrac{1}{4v^2}y^2 - v^2$

Notice that $x = \dfrac{1}{4}y^2 - 1$ is in this form for $v = 1$ and $x = \dfrac{1}{16}y^2 - 4$ is in this form for $v = 2$

Since $y = 2uv \longleftrightarrow v = \dfrac{y}{2u}$ we can write $x = u^2 - v^2 \longleftrightarrow x = u^2 - \left(\dfrac{y}{2u}\right)^2 = -\dfrac{1}{4u^2}y^2 + u^2$

Notice that $x = -\dfrac{1}{4}y^2 + 1$ is in this form for $u = 1$ and $x = -\dfrac{1}{16}y^2 + 4$ is in this form for $u = 2$

Therefore our new domain is the rectangle $G = \{(u, v) \,:\, 1 \le u \le 2,\, 1 \le v \le 2\}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$

Therefore, the double integral becomes:

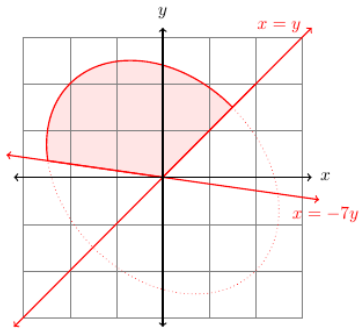
$$\begin{aligned}
\iint_D y\,dA &= \iint_G y(u,v) \cdot |J(u,v)|\; dA \\
&= \iint_G 2uv \cdot \left|4u^2 + 4v^2\right|\; dA \\
&= \int_1^2 \int_1^2 (8u^3v + 8uv^3)\, du\, dv \\
&= \int_1^2 \left[2u^4v + 4u^2v^3\right] \Big|_1^2 dv \\
&= \int_1^2 (30v + 12v^3)\, dv \\
&= \left[15v^2 + 3v^4\right] \Big|_1^2 \\
&= 90
\end{aligned}$$

? Exercise 3.3.8

Find the area of the region $F = \left\{(x, y) \,:\, \dfrac{(x+y)^2}{9} + \dfrac{(x-y)^2}{16} \le 1,\, -7y \le x \le y\right\}$

Answer

The region F described is part of a tilted ellipse:



Clearly this region is not Type 1 or Type 2, so we attempt to do a substitution inspired by polar coordinates.

If we set $\frac{x+y}{3} = u\cos(v)$ and $\frac{x-y}{4} = u\sin(v)$, then $\frac{(x+y)^2}{9} + \frac{(x-y)^2}{16} = u^2(\cos^2(v) + \sin^2(v)) = u^2$

Therefore $\frac{(x+y)^2}{9} + \frac{(x-y)^2}{16} \leq 1$ will give us $0 \leq u \leq 1$

To find the bounds for v we solve for x and y in terms of u and v:

$x + y = 3u\cos(v),\ x - y = 4u\sin(v) \longleftrightarrow x = \frac{3}{2}u\cos(v) + 2u\sin(v),\ y = \frac{3}{2}u\cos(v) - 2u\sin(v)$

Therefore the bound $x \leq y \longleftrightarrow \frac{3}{2}u\cos(v) + 2u\sin(v) \leq \frac{3}{2}u\cos(v) - 2u\sin(v) \longleftrightarrow \sin(v) \leq 0$

And the bound $x \geq -7y \longleftrightarrow \frac{3}{2}u\cos(v) + 2u\sin(v) \geq -\frac{21}{2}u\cos(v) + 14u\sin(v) \longleftrightarrow \cos(v) \geq \sin(v)$

We can see that the values of v satisfying $\cos(v) \geq \sin(v)$ and $\sin(v) \leq 0$ are $\frac{5\pi}{4} \leq v \leq 2\pi$

Therefore the new domain is the rectangle $G = \left\{(u,v) \,:\, 0 \leq u \leq 1,\ \frac{5\pi}{4} \leq v \leq 2\pi\right\}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{2}\cos(v) + 2\sin(v) & -\frac{3}{2}u\sin(v) + 2u\cos(v) \\ \frac{3}{2}\cos(v) - 2\sin(v) & -\frac{3}{2}u\sin(v) - 2u\cos(v) \end{vmatrix} = -6u$

Therefore, the area can be represented by the double integral:

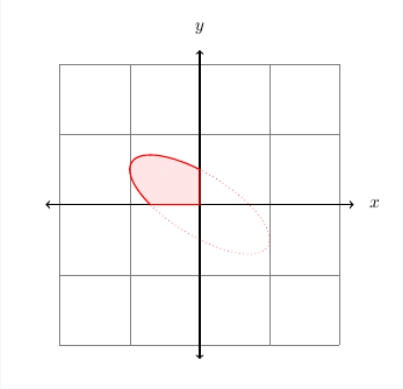
$$\begin{aligned} \text{Area}(F) &= \iint_F 1\,dA \\ &= \iint_G 1 \cdot |J(u,v)|\,dA \\ &= \iint_G |-6u|\,dA \\ &= \int_0^1 \int_{\frac{5\pi}{4}}^{2\pi} 6u\,dv\,du \\ &= \left(\int_0^1 6u\,du\right)\left(\int_{\frac{5\pi}{4}}^{2\pi} dv\right) \\ &= \left(3u^2\Big|_0^1\right) \cdot \left(v\Big|_{\frac{5\pi}{4}}^{2\pi}\right) \\ &= 3 \cdot \frac{3\pi}{4} \\ &= \frac{9\pi}{4} \end{aligned}$$

? Exercise 3.3.9

Use a substitution to evaluate $\iint_D 8(x+y)e^{2y}\,dA$ where $D = \{(x,y) \,:\, 2x^2 + 4xy + 4y^2 \leq 1,\ x \leq 0,\ y \geq 0\}$

Answer

The domain D is the part of a tilted ellipse that is in the second quadrant.



This is not a Type 1 or Type 2 region, so we try a substitution. Notice $2x^2+4xy+4y^2 = x^2+(x+2y)^2$

So, if we let $x = u\cos(v)$ and $x+2y = u\sin(v)$, we have $2x^2+4xy+4y^2 = u^2(\cos^2(v)+\sin^2(v)) = u^2$

This means that $2x^2+4xy+4y^2 \le 1$ implies $0 \le u \le 1$

We also know that $x \le 0$ becomes $u\cos(v) \le 0$ so we need $\cos(v) \le 0$

Solving for y in the substitution equations gives $y = \frac{1}{2}u\sin(v) - \frac{1}{2}u\cos(v)$

Therefore $y \ge 0$ becomes $\frac{1}{2}u\sin(v) - \frac{1}{2}u\cos(v) \ge 0 \longleftrightarrow \sin(v) \ge \cos(v)$

We can see that the values of v satisfying $\cos(v) \le 0$ and $\sin(v) \ge \cos(v)$ are $\frac{\pi}{2} \le v \le \frac{5\pi}{4}$

Therefore the new domain is the rectangle $G = \left\{(u,v) \,:\, 0 \le u \le 1,\ \frac{\pi}{2} \le v \le \frac{5\pi}{4}\right\}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos(v) & -u\sin(v) \\ \frac{1}{2}\sin(v) - \frac{1}{2}\cos(v) & \frac{1}{2}u\cos(v) + \frac{1}{2}u\sin(v) \end{vmatrix} = \frac{1}{2}u$

Therefore, the area can be represented by the double integral:

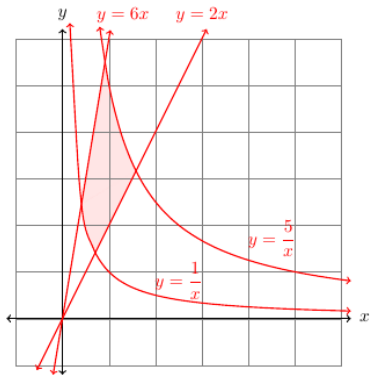
$$\begin{aligned}
\iint_D 8(x+y)e^{2y}\,dA &= \iint_G 8(x(u,v)+y(u,v))e^{2y(u,v)} \cdot |J(u,v)|\,dA \\
&= \iint_G 4\left(u\cos(v)+u\sin(v)\right)e^{u\sin(v)-u\cos(v)} \cdot \left|\frac{1}{2}u\right|\,dA \\
&= \int_0^1 \int_{\frac{\pi}{2}}^{\frac{5\pi}{4}} 2u^2\left(\cos(v)+\sin(v)\right)e^{u(\sin(v)-\cos(v))}\,dv\,du \\
&= \int_0^1 2ue^{u(\sin(v)-\cos(v))}\Big|_{\frac{\pi}{2}}^{\frac{5\pi}{4}}\,du \\
&= \int_0^1 \left(2u-2ue^u\right)\,du \\
&= \left[u^2-2ue^u+2e^u\right]\Big|_0^1 \\
&= -1
\end{aligned}$$

? Exercise 3.3.10

Use a substitution to evaluate $\iint_D xy\,dA$ where $D = \left\{(x,y) \,:\, \frac{1}{x} \le y \le \frac{5}{x},\ 2x \le y \le 6x\right\}$

Answer

The domain D is between four easily identifiable curves:



But it is not a Type 1 or Type 2 region, so evaluating this integral directly is difficult.

Instead we let $y = \dfrac{u}{x}$, so that the domain D uses $1 \le u \le 5$

Next we let $y = vx$, so that $2 \le v \le 6$

This makes the new domain into the rectangle $G = \{(u, v) : 1 \le u \le 5,\ 2 \le v \le 6\}$

Solving for x and y in this substitution gives $x = \sqrt{\dfrac{u}{v}},\ y = \sqrt{uv}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} = \dfrac{1}{2v}$

Therefore, the area can be represented by the double integral:

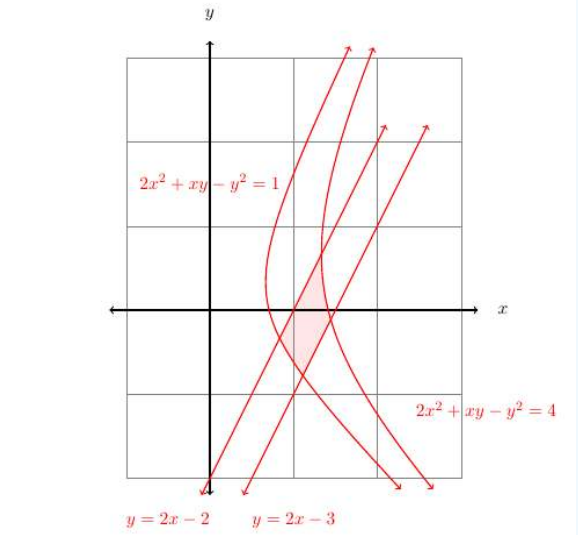
$$\begin{aligned} \iint_D xy\,dA &= \iint_G x(u,v)y(u,v) \cdot |J(u,v)|\ dA \\ &= \iint_G u \cdot \left|\frac{1}{2v}\right| dA \\ &= \int_1^5 \int_2^6 \frac{u}{2v}\,dv\,du \\ &= \left(\int_1^5 \frac{1}{2}u\,du\right)\left(\int_2^6 \frac{1}{v}\,dv\right) \\ &= \left(\frac{1}{4}u^2\Big|_1^5\right) \cdot \left(\ln|v|\Big|_2^6\right) \\ &= 6 \cdot (\ln(6) - \ln(2)) \\ &= 6\ln(3) \end{aligned}$$

? Exercise 3.3.11

Use a substitution to evaluate $\iint_E 3(2x-y)^3\,dA$ where $E = \{(x,y) : 2x - 3 \le y \le 2x - 2,\ 1 \le 2x^2 + xy - y^2 \le 4\}$

Answer

The domain E is between two lines and two hyperbolas:



But it is not a Type 1 or Type 2 region, so evaluating this integral directly is difficult.

Instead we let $y = 2x - u$, so that the domain E uses $2 \le u \le 3$

Next we let $2x^2 + xy - y^2 = v$, so that $1 \le v \le 4$

This makes the new domain into the rectangle $G = \{(u, v) : 2 \le u \le 3,\ 1 \le v \le 4\}$

Solving for x and y in this substitution gives $x = \dfrac{v}{3u} + \dfrac{u}{3}$, $y = \dfrac{2v}{3u} - \dfrac{u}{3}$

The Jacobian of this transformation is $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v}{3u^2} + \frac{1}{3} & \frac{1}{3u} \\ -\frac{2v}{3u^2} - \frac{1}{3} & \frac{2}{3u} \end{vmatrix} = \dfrac{1}{3u}$

Therefore, the area can be represented by the double integral:

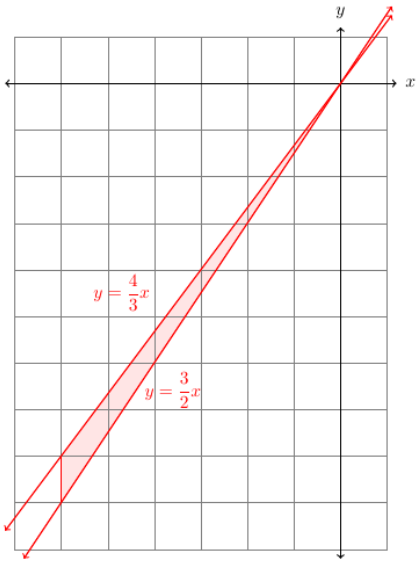
$$\begin{aligned} \iint_E 3(2x-y)^3\,dA &= \iint_G 3(2x(u,v) - y(u,v))^3 \cdot |J(u,v)|\,dA \\ &= \iint_G 3u^3 \cdot \left|\frac{1}{3u}\right| dA \\ &= \int_2^3 \int_1^4 u^2\,dv\,du \\ &= \left(\int_2^3 u^2\,du\right)\left(\int_1^4 dv\right) \\ &= \left(\frac{1}{3}u^3\Big|_2^3\right) \cdot \left(v\Big|_1^4\right) \\ &= \left(9 - \frac{8}{3}\right) \cdot 3 \\ &= 19 \end{aligned}$$

? Exercise 3.3.12

Use a substitution to evaluate $\iint_R \dfrac{1}{(2-2y+3x)\sqrt{4-9x^2-4y^2+12xy}}\,dA$ where R is the triangle with vertices $(0,0), (-6,-8)$, and $(-6,-9)$

Answer

We solve for the lines on the edges of our triangle R:



Although this can easily be written as a Type 1 region, the interior of the radical makes evaluating this integral directly quite difficult.

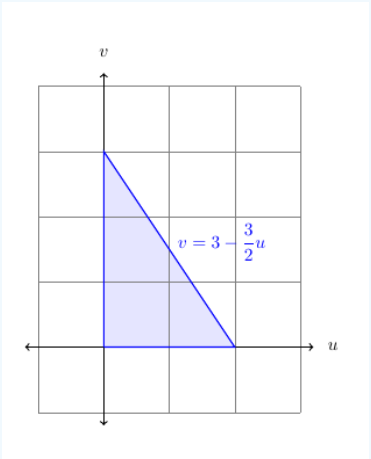
We substitute $u = 2y - 3x$ and $v = 4x - 3y$ to make this simpler.

Solving for x and y in this substitution gives $x = -3u - 2v,\ y = -4u - 3v$

The line $y = \frac{4}{3}x$ becomes $-4u - 3v = \frac{4}{3}(-3u - 2v) \longleftrightarrow v = 0$

The line $y = \frac{3}{2}x$ becomes $-4u - 3v = \frac{3}{2}(-3u - 2v) \longleftrightarrow u = 0$

The line $x = -6$ becomes $-3u - 2v = -6 \longleftrightarrow v = -\frac{3}{2}u + 3$



This makes the new domain into the triangle $G = \left\{(u, v) \ : \ 0 \le u \le 2,\ 0 \le v \le 3 - \frac{3}{2}u\right\}$

The Jacobian of this transformation is $J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ -4 & -3 \end{vmatrix} = 1$

Therefore, the area can be represented by the double integral:

$$\begin{aligned}
\iint_R \frac{1}{(2-2y+3x)\sqrt{4-9x^2-4y^2+12xy}}\,dA &= \iint_G \frac{1}{(2-2y(u,v)+3x(u,v))\sqrt{4-9[x(u,v)]^2-4[y(u,v)]^2+12x(u,v)y(u,v)}}\cdot|J(u,v)|\,dA \\
&= \iint_G \frac{1}{(2-u)\sqrt{4-u^2}}\cdot|1|\,dA \\
&= \int_0^2 \int_0^{3-\frac{3}{2}u} \frac{1}{(2-u)\sqrt{4-u^2}}\,dv\,du \\
&= \int_0^2 \frac{v}{(2-u)\sqrt{4-u^2}}\Bigg|_0^{3-\frac{3}{2}u} du \\
&= \int_0^2 \frac{3}{2\sqrt{4-u^2}}\,du \\
&= \frac{3}{2}\sin^{-1}\left(\frac{u}{2}\right)\Bigg|_0^2 \\
&= \frac{3\pi}{4}
\end{aligned}$$

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3.4: Change of Variables in 3 Dimensions

Learning Objectives

- Evaluate a triple integral using a change of variables.
- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

The previous section showed how to convert a double integral in rectangular coordinates into a double integral in different coordinates. We do the same in this section for triple integrals, with a focus on the two most popular substitutions: cylindrical and spherical coordinates.

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- **15.5: Triple Integrals in Cylindrical and Spherical Coordinates** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

3.4: Change of Variables in Triple Integrals

Changing variables in triple integrals works in exactly the same way as double integrals.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space by a one-to-one C^1 transformation $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$.

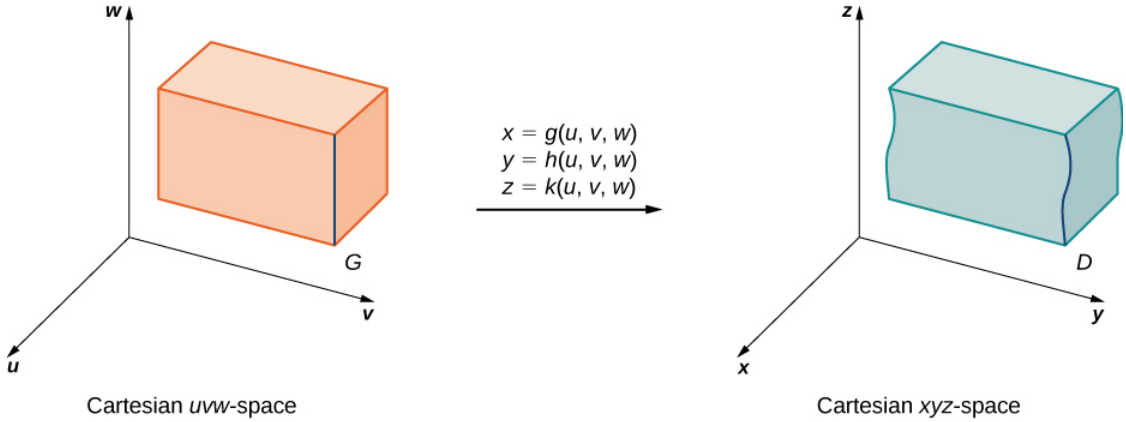


Figure 3.4.1: A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x,y,z)$ defined on D can be thought of as another function $H(u,v,w)$ that is defined on G:

$$F(x,y,z) = F(g(u,v,w),\ h(u,v,w),\ k(u,v,w)) = H(u,v,w)$$

Now we need to define the Jacobian for three variables.

Definition: Jacobian Determinant

The *Jacobian determinant* $J(u,v,w)$ in three variables is defined as follows:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The Jacobian can also be simply denoted as $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Theorem 3.4.1: Change of Variables for Triple Integrals

Let $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-space into the region D in the xyz-space. As in the two-dimensional case, if f is continuous on D, then

$$\iiint_D f(x,y,z)\,dV = \iiint_G f(g(u,v,w),\ h(u,v,w),\ k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

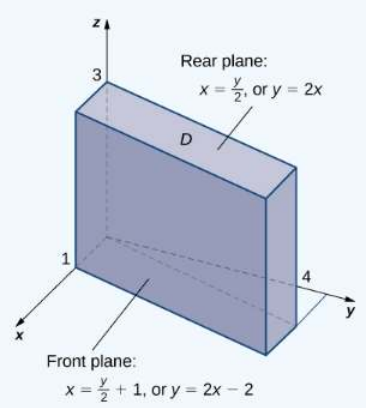
Illustration 3.4.1: Evaluating a Triple Integral with a Change of Variables

We will evaluate the triple integral

$$\int_0^3 \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

In xyz-space by using the transformation $u = \dfrac{2x - y}{2}$, $v = \dfrac{y}{2}$, and $w = \dfrac{z}{3}$

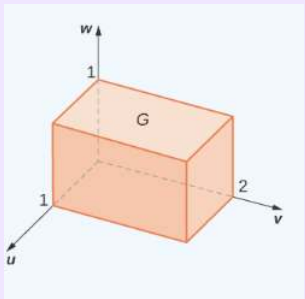
As before, some kind of sketch of the region D in xyz-space over which we have to perform the integration can help identify the region G in uvw-space. Clearly D in xyz-space is bounded by the planes $x = \dfrac{y}{2}$, $x = \dfrac{y}{2} + 1$, $y = 0$, $y = 4$, $z = 0$, and $z = 3$. We also know that we have to use $u = \dfrac{2x - y}{2}$, $v = \dfrac{y}{2}$, and $w = \dfrac{z}{3}$ for the transformations. We need to solve for x, y and z. Here we find that $x = u + v$, $y = 2v$, and $z = 3w$.



Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	**Corresponding equations in uvw for the region G**	**Limits for the integration in uvw**
$x = \dfrac{y}{2}$	$u + v = \dfrac{2v}{2}$	$u = 0$
$x = \dfrac{y}{2} + 1$	$u + v = \dfrac{2v}{2} + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

Therefore the new domain $G = \{(u, v, w) \mid 0 \le u \le 1,\ 0 \le v \le 2,\ 0 \le w \le 1\}$ is a rectangular prism.



Now we can calculate the Jacobian for the transformation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

We are now ready to put everything together and complete the problem.

$$\begin{aligned} \int_0^3 \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz &= \int_0^1 \int_0^2 \int_0^1 \left(u + v + \frac{3w}{3}\right) |J(u,v,w)|\, du\, dv\, dw \\ &= \int_0^1 \int_0^2 \int_0^1 (u + v + w)\, |6|\, du\, dv\, dw \\ &= \int_0^1 \int_0^2 \int_0^1 (6u + 6v + 6w)\, du\, dv\, dw \\ &= \int_0^1 \int_0^2 \left[3u^2 + 6vu + 6wu\right] \Big|_0^1 dv\, dw \\ &= \int_0^1 \int_0^2 (3 + 6v + 6w)\, dv\, dw \\ &= \int_0^1 \left[3v + 3v^2 + 6wv\right] \Big|_0^2 dw \\ &= \int_0^1 (6 + 12 + 12w)\, dw \\ &= \left[18w + 6w^2\right] \Big|_0^1 \\ &= 24 \end{aligned}$$

Example 3.4.1: Evaluating a Triple Integral with a Change of Variables

Let R be the parallelepiped with faces defined by the planes $x + y + z = 1, x + y + z = 6, 2x - y - z = 3, 2x - y - z = 5, x + 3y - z = -1, x + 3y - z = 2$. Evaluate the triple integral

$$\iiint_R \frac{2(x + 3y - z)(2x - y - z)}{x + y + z}\, dV$$

Solution

We are given the planes that bound this surface, and analogous to when we dealt with a two-dimensional parallelogram, a clear choice for substitution should be $u = x + y + z, v = 2x - y - z, w = x + 3y - z$. Clearly these would make our new domain a rectangular prism: $G = \{1 \le u \le 6,\ 3 \le v \le 5,\ -1 \le w \le 2\}$. Since these letters also show up exactly inside of the integrand, we do not need to waste time solving for $x(u,v,w), y(u,v,w), z(u,v,w)$. Instead we can just find the inverse Jacobian:

$$J^{-1}(x,y,z) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 12$$

Thus we know $J(u,v,w) = \frac{1}{12}$ and the triple integral becomes:

$$\begin{aligned}
\iiint_R \frac{2(x+3y-z)(2x-y-z)}{x+y+z}\,dV &= \int_1^6\int_3^5\int_{-1}^2 \frac{2wv}{u}\,|J(u,v,w)|\,dw\,dv\,du \\
&= \int_1^6\int_3^5\int_{-1}^2 \frac{wv}{6u}\,dw\,dv\,du \\
&= \int_1^6\int_3^5 \frac{w^2v}{12u}\Big|_{-1}^{2}\,dv\,du \\
&= \int_1^6\int_3^5 \frac{v}{4u}\,dv\,du \\
&= \int_1^6 \frac{v^2}{8u}\Big|_3^5\,du \\
&= \int_1^6 \frac{2}{u}\,du \\
&= 2\ln(u)\Big|_1^6 \\
&= 2\ln(6)
\end{aligned}$$

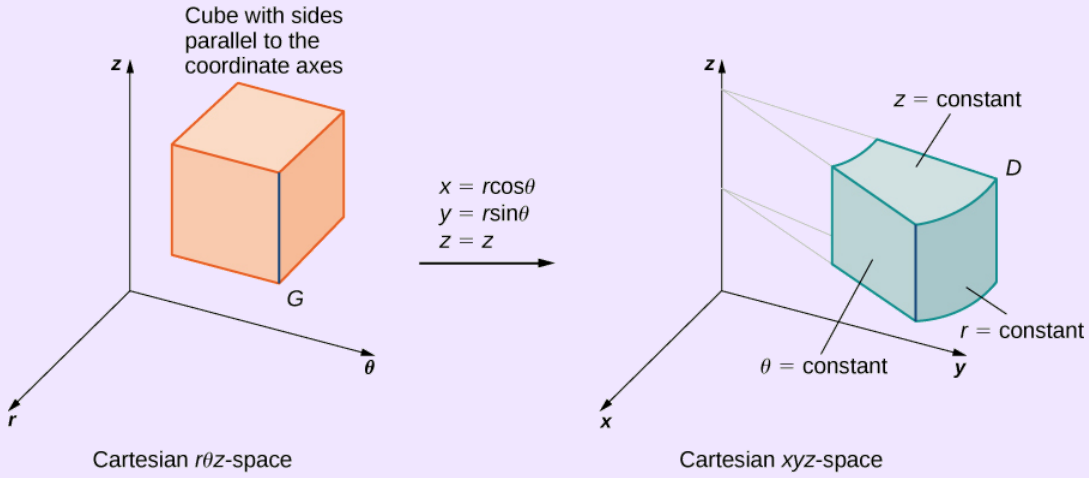
Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem.

Illustration 3.4.2: Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

We will now derive the formula for dV in triple integrals for (a) cylindrical coordinates and (b) spherical coordinates:

(a) For cylindrical coordinates, the transformation is $T(r,\theta,z) = (x,y,z)$ from the Cartesian $r\ \theta z$-space to the Cartesian xyz-space. Here $x = r\ \cos(\theta)$, $y = r\ \sin(\theta)$ and $z = z$. The Jacobian for the transformation is

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\ \cos^2(\theta) + r\ \sin^2(\theta) = r$$



We know that $r \geq 0$, so $|J(r,\theta,z)| = r$. Then the triple integral is

$$\iiint_D f(x,y,z)dV = \iiint_G f(r\ \cos(\theta),\ r\ \sin(\theta),\ z)r\,dr\,d\theta\,dz$$

(b) For spherical coordinates, the transformation is $T(\rho,\theta,\phi)$ from the Cartesian $\rho\ \theta\ \phi$-space to the Cartesian xyz-space. Here $x = \rho\ \sin(\phi)\ \cos(\theta)$, $y = \rho\ \sin(\phi)\ \sin(\theta)$, and $z = \rho\ \cos(\phi)$. The Jacobian for the transformation is

$$J(\rho,\theta,\phi)=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial\rho}&\frac{\partial x}{\partial\theta}&\frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\rho}&\frac{\partial y}{\partial\theta}&\frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial\rho}&\frac{\partial z}{\partial\theta}&\frac{\partial z}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\sin(\phi)\cos(\theta)&-\rho\sin(\phi)\sin(\theta)&\rho\cos(\phi)\cos(\theta)\\ \sin(\phi)\sin(\theta)&\rho\sin(\phi)\cos(\theta)&\rho\cos(\phi)\sin(\theta)\\ \cos(\phi)&0&-\rho\sin(\phi)\end{vmatrix}$$

$$=\cos(\phi)\begin{vmatrix}-\rho\sin(\phi)\sin(\theta)&\rho\cos(\phi)\cos(\theta)\\ \rho\sin(\phi)\cos(\theta)&\rho\cos(\phi)\sin(\theta)\end{vmatrix}-\rho\sin(\phi)\begin{vmatrix}\sin(\phi)\cos(\theta)&-\rho\sin(\phi)\sin(\theta)\\ \sin(\phi)\sin(\theta)&\rho\sin(\phi)\cos(\theta)\end{vmatrix}$$

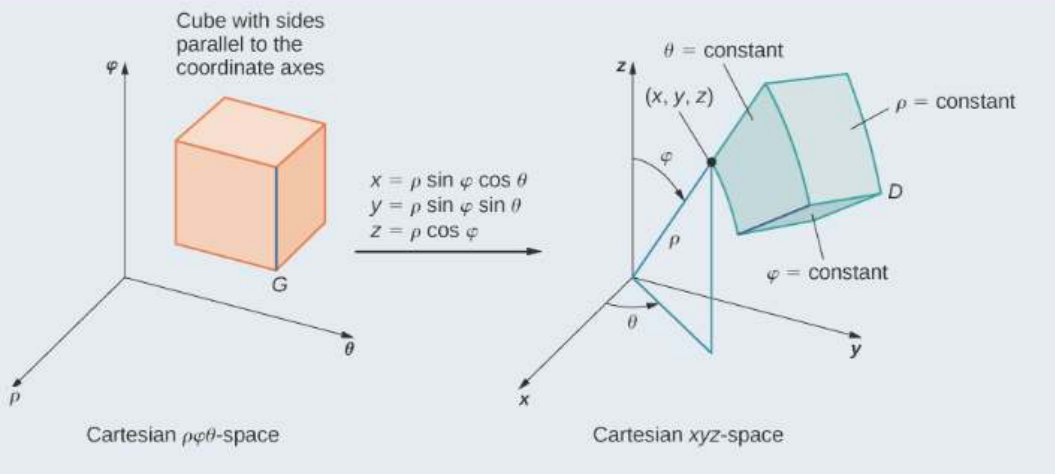
$$=\cos(\phi)\left(-\rho^2\sin(\phi)\cos(\phi)\sin^2(\theta)-\rho^2\sin(\phi)\cos(\phi)\cos^2(\theta)\right)$$
$$-\rho\sin(\phi)\left(\rho\sin^2(\phi)\cos^2(\theta)+\rho\sin^2(\phi)\sin^2(\theta)\right)$$

$$=-\rho^2\sin(\phi)\cos^2(\phi)\left(\sin^2(\theta)+\cos^2(\theta)\right)-\rho^2\sin(\phi)\sin^2(\phi)\left(\sin^2(\theta)+\cos^2(\theta)\right)$$

$$=-\rho^2\sin(\phi)\cos^2(\phi)-\rho^2\sin(\phi)\sin^2(\phi)$$

$$=-\rho^2\sin(\phi)\left(\cos^2(\phi)+\sin^2(\phi)\right)=-\rho^2\sin(\phi)$$

Since $0\le\phi\le\pi$, we must have $\sin\phi\ge 0$. Thus $|J(\rho,\theta,\phi)|=|-\rho^2\sin(\phi)|=\rho^2\sin(\phi)$



Then the triple integral becomes

$$\iiint_D f(x,y,z)dV=\iiint_G f\left(\rho\sin(\phi)\cos(\theta),\rho\sin(\phi)\sin(\theta),\rho\cos(\phi)\right)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

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3.4: Triple Integrals in Cylindrical and Spherical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in the table below. These equations will become handy as we proceed with solving problems using triple integrals.

	Circular cylinder	**Circular cone**	**Sphere**	**Paraboloid**
Rectangular	$x^2+y^2=c^2$	$z^2=c^2(x^2+y^2)$	$x^2+y^2+z^2=c^2$	$z=c(x^2+y^2)$
Cylindrical	$r=c$	$z=cr$	$r^2+z^2=c^2$	$z=cr^2$

Theorem 3.4.2: Fubini's Theorem in Cylindrical Coordinates

Suppose that $f(x,y,z)$ is continuous on a region B which when described in cylindrical coordinates looks like $B=\{(r\cos(\theta),r\sin(\theta),z)\,:\,a\le r\le b,\,\alpha\le\theta\le\beta,\,c\le z\le d\}$. Then

$$\iiint_B f(x,y,z)\,dV=\int_c^d\int_\alpha^\beta\int_a^b f(r\cos(\theta),r\sin(\theta),z)\,r\,dr\,d\theta\,dz$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 3.4.2: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^4 zy\,dz\,dy\,dx$$

Solution

The domain of this triple integral is nicer in cylindrical coordinates: $B=\left\{(r,\theta,z)\,:\,0\le r\le 2,\,0\le\theta\le\frac{\pi}{2},\,0\le z\le 4\right\}$

Therefore we can rewrite the triple integral as the iterated integral

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^4 zy\,dz\,dy\,dx=\int_0^{\frac{\pi}{2}}\int_0^2\int_0^4 zr\,\sin(\theta)\,r\,dz\,dr\,d\theta$$

Since each variable factors out and their limits of integration do not interact, this is equivalent to the product of their separate integrals:

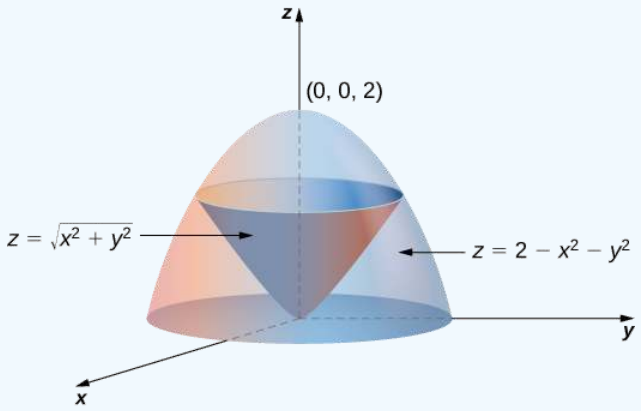
$$\begin{aligned}&=\left(\int_0^{\frac{\pi}{2}}\sin(\theta)\,d\theta\right)\left(\int_0^2 r^2\,dr\right)\left(\int_0^4 z\,dz\right)\\&=\left(-\cos(\theta)\Big|_0^{\frac{\pi}{2}}\right)\cdot\left(\frac{r^3}{3}\Big|_0^2\right)\cdot\left(\frac{z^2}{2}\Big|_0^4\right)\\&=1\cdot\frac{8}{3}\cdot 8=\boxed{\frac{64}{3}}\end{aligned}$$

When we have to integrate over a more general solid, we look at the projections onto the coordinate planes. If that region in cylindrical coordinates looks like $E=\{(r,\theta,z)\,:\,(r,\theta)\in D,u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z)dz\right] r\,dr\,d\theta$$

Example 3.4.3: Finding a Volume with Cylindrical Coordinates

Find the volume of the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$



Solution

In cylindrical form, the cone $z = \sqrt{x^2+y^2}$ is $z = r$ and the paraboloid $z = 2 - x^2 - y^2$ is $z = 2 - r^2$. These intersect when $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r+2)(r-1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E = \{(r\cos(\theta), r\sin(\theta), z) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 2 - r^2\}$.

Hence the volume is given by the triple integral:

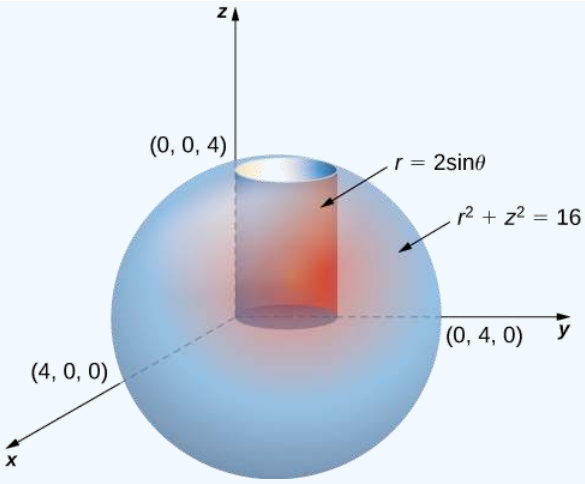
$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 rz\Big|_r^{2-r^2} dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (2r - r^3 - r^2)\,dr\,d\theta \\
&= \int_0^{2\pi}\left[r^2 - \frac{1}{4}r^4 - \frac{1}{3}r^3\right]\Big|_0^1 d\theta \\
&= \int_0^{2\pi}\frac{5}{12}\,d\theta \\
&= \frac{5}{12}\theta\Big|_0^{2\pi} \\
&= \boxed{\frac{5\pi}{6}}
\end{aligned}$$

Example 3.4.4: Triple Integral in Cylindrical Coordinates over a General Region

Consider the region R inside the right circular cylinder with equation $x^2 + (y-1)^2 = 1$, bounded below by the $x\,y$-plane and bounded above by the sphere $x^2 + y^2 + z^2 = 16$. Evaluate the integral $\iiint_R 4z\,dV$

Solution

First, identify that the equation for the sphere is $r^2 + z^2 = 16$ and for the cylinder is $r = 2\sin(\theta)$ for $0 \leq \theta \leq \pi$. We can see that the limits for z are from 0 to $z = \sqrt{16 - r^2}$. Hence we can write R in cylindrical coordinates as the set $E = \{(r,\theta,z) : 0 \leq \theta \leq \pi,\ 0 \leq r \leq 2\sin(\theta),\ 0 \leq z \leq \sqrt{16 - r^2}\}$



Therefore, the triple integral becomes:

$$\begin{aligned}
\iint_R 4z\,dV &= \iiint_E 1\,4rz\,dz\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^{2\sin(\theta)}\int_0^{\sqrt{16-r^2}} 4rz\,dz\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^{2\sin(\theta)} 2rz^2\Big|_0^{\sqrt{16-r^2}}\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^{2\sin(\theta)} \left(32r - 2r^3\right)\,dr\,d\theta \\
&= \int_0^{\pi}\left[16r^2 - \frac{1}{2}r^4\right]\Big|_0^{2\sin(\theta)}\,d\theta \\
&= \int_0^{\pi}\left(64\sin^2(\theta) - 8\sin^4(\theta)\right)\,d\theta \\
&= \int_0^{\pi}\left(29 - 28\cos(2\theta) - \cos(4\theta)\right)\,d\theta \\
&= \left[29 - 14\sin(2\theta) - \frac{1}{4}\sin(4\theta)\right]\Big|_0^{\pi} \\
&= \boxed{29\pi}
\end{aligned}$$

Some circular triple integrals are easier using spherical coordinates instead of rectangular or cylindrical coordinates. Most commonly, this is when the boundary of the domain is made of spheres $\rho = c$, vertical planes $\theta = c$, and cones $\phi = c$.

Theorem 3.4.3: Fubini's Theorem for Spherical Coordinates

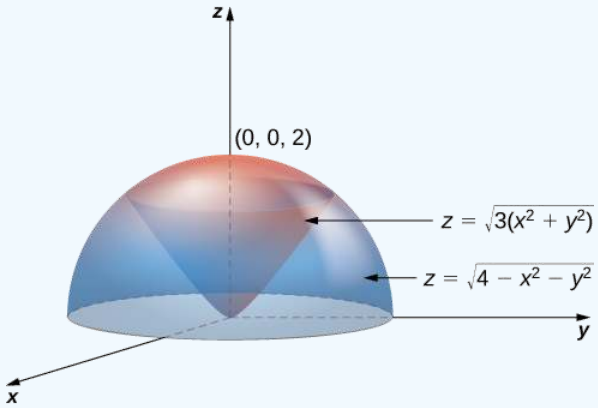
Suppose that $f(x, y, z)$ is continuous on a region B which when described in spherical coordinates looks like $B = \{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) : a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$. Then,

$$\iiint_B f(x, y, z)\,dV = \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 3.4.5: Finding a Volume in Spherical Coordinates

Find the volume of the region bounded by the cone $z = \sqrt{3(x^2 + y^2)}$ and the hemisphere $z = \sqrt{4 - x^2 - y^2}$.



Solution

In spherical form, the hemisphere $z = \sqrt{4 - x^2 - y^2}$ is $\rho = 2$ and since we start from the origin, we know $0 \le \rho \le 2$. The cone $z = \sqrt{3}\sqrt{x^2 + y^2}$ is $\phi = \cot^{-1}(\sqrt{3}) = \dfrac{\pi}{6}$, and since we want the region *above* this cone, we start at $\phi = 0$. We do not need to find the radius of the circular projection of the region onto the xy-plane, but since it is a full circle we know $0 \le \theta \le 2\pi$.

Thus, we can describe the region as $E = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \ : \ 0 \le \theta \le 2\pi,\ 0 \le \rho \le 2,\ 0 \le \phi \le \dfrac{\pi}{6}\right\}$

Hence the volume is given by the triple integral:

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{2}\int_0^{\frac{\pi}{6}} \rho^2\sin(\phi)\, d\phi\, d\rho\, d\theta \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{2}\rho^2 d\rho\right)\left(\int_0^{\frac{\pi}{6}}\sin(\phi)\, d\phi\right) \\ &= \left(\theta\Big|_0^{2\pi}\right)\cdot\left(\frac{1}{3}\rho^3\Big|_0^{2}\right)\cdot\left(-\cos(\phi)\Big|_0^{\frac{\pi}{6}}\right) \\ &= 2\pi\cdot\frac{8}{3}\cdot\left(1 - \frac{\sqrt{3}}{2}\right) \\ &= \boxed{\frac{8\pi(2-\sqrt{3})}{3}} \end{aligned}$$

Example 3.4.6: Converting from Rectangular Coordinates to Spherical Coordinates

Evaluate the triple integral below by transforming to spherical coordinates:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{-\sqrt{2-x^2-y^2}}^{-\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy.$$

Solution

This triple integral is over the region $R = \{(x, y, z) \ : \ -1 \le y \le 1,\ 0 \le x \le \sqrt{1-y^2},\ -\sqrt{1-y^2} \le z \le -\sqrt{2-x^2-y^2}\}$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the range for θ is $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$. The cone $z = -\sqrt{x^2 + y^2}$ is represented by the spherical equation $\phi = \cot^{-1}(-1) = \dfrac{3\pi}{4}$. Since our region is *below* this cone, we know that the other endpoint for ϕ is the negative z-axis, which is $\phi = \pi$. Lastly, we can see the *spherical* radius goes from the origin ($\rho = 0$) to $\rho = \sqrt{2}$. The radius of the projected circle here is r in cylindrical coordinates, but this is not ρ (it is instead the projection of the intersection of the sphere and cone in this question).

Thus we can describe the region as $E = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \ : \ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2},\ 0 \le \rho \le \sqrt{2},\ \dfrac{3\pi}{4} \le \phi \le \pi\right\}$.

Therefore our triple integral becomes:

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{-\sqrt{2-x^2-y^2}}^{-\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\sqrt{2}}\int_{\frac{3\pi}{4}}^{\pi} (\rho\sin(\phi)\cos(\theta))\,(\rho\sin(\phi)\sin(\theta))\\ &\quad (\rho\cos(\phi))\,\rho^2\sin(\phi)d\phi\,d\rho\,d\theta\\ &= \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin(\theta)\cos(\theta)d\theta\right)\left(\int_{0}^{\sqrt{2}}\rho^5 d\rho\right)\\ &\quad \left(\int_{\frac{3\pi}{4}}^{\pi}\sin^3(\phi)\cos(\phi)d\phi\right)\\ &= \left(\frac{1}{2}\sin^2(\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)\cdot\left(\frac{1}{6}\rho^6\Big|_{0}^{\sqrt{2}}\right)\cdot\left(\frac{1}{4}\sin^4(\phi)\Big|_{\frac{3\pi}{4}}^{\pi}\right)\\ &= 0\cdot\frac{4}{3}\cdot\left(-\frac{1}{16}\right) = \boxed{0}\end{aligned}$$

Sometimes things are not perfectly cylindrical or spherical, but are a simple substitution away from being so. The most common example of this are elliptic cylinders (which stretch into circular cylinders and thus cylindrical coordinates), and ellipsoids (which stretch into spheres and thus spherical coordinates).

Illustration 3.4.3: Integrating over an Ellipsoid

Let $R = \{(x,y,z) : x^2+4y^2+9z^2 \le 1,\ x \ge 2y,\ z \ge 0\}$

We will evaluate $\iiint_R \sin\left(\pi(x^2+4y^2+9z^2)^{\frac{3}{2}}\right)\,dV$

The original domain R is clearly a partial ellipsoid, but in terms of $u = x, v = 2y$, and $w = 3z$ it would look like:

$$S = \{(u,v,w) : u^2+v^2+w^2 \le 1,\ u \ge v,\ w \ge 0\}$$

This is easily solved as $x = u, y = \frac{v}{2}, z = \frac{w}{3}$, so the Jacobian of this transformation is:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{6}$$

Now we can turn our original integral into:

$$\iiint_E \sin\left(\pi(x^2+4y^2+9z^2)^{\frac{3}{2}}\right)\,dV = \iiint_S \sin\left(\pi(u^2+v^2+w^2)^{\frac{3}{2}}\right)\cdot\frac{1}{6}\,dV$$

This new domain is easily put into spherical coordinates:
Since $u^2+v^2+w^2 \le 1$, we have $0 \le \rho \le 1$.
Since $u \ge v$ we think of the angles which make $x \ge y$ on the unit circle, which are $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$
Since $w \ge 0$ is the top half of the sphere, we have $0 \le \phi \le \frac{\pi}{2}$.

Therefore our second transformation (into spherical coordinates) turns this into:

$$\begin{aligned}\iiint_S \frac{1}{6}\sin\left(\pi(u^2+v^2+w^2)^{\frac{3}{2}}\right)\,dV &= \int_0^1\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}} \frac{1}{6}\sin\left(\pi\rho^3\right)\rho^2\sin(\phi)\,d\phi\,d\theta\,d\rho \\ &= \left(\int_0^1 \rho^2\sin\left(\pi\rho^3\right)d\rho\right)\left(\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\frac{1}{6}d\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin(\phi)d\phi\right) \\ &= \left(-\frac{1}{3\pi}\cos\left(\pi\rho^3\right)\Big|_0^1\right)\cdot\left(\frac{1}{6}\theta\Big|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\right)\cdot\left(-\cos(\phi)\Big|_0^{\frac{\pi}{2}}\right) \\ &= \frac{2}{3\pi}\cdot\frac{\pi}{6}\cdot 1 = \frac{1}{9}\end{aligned}$$

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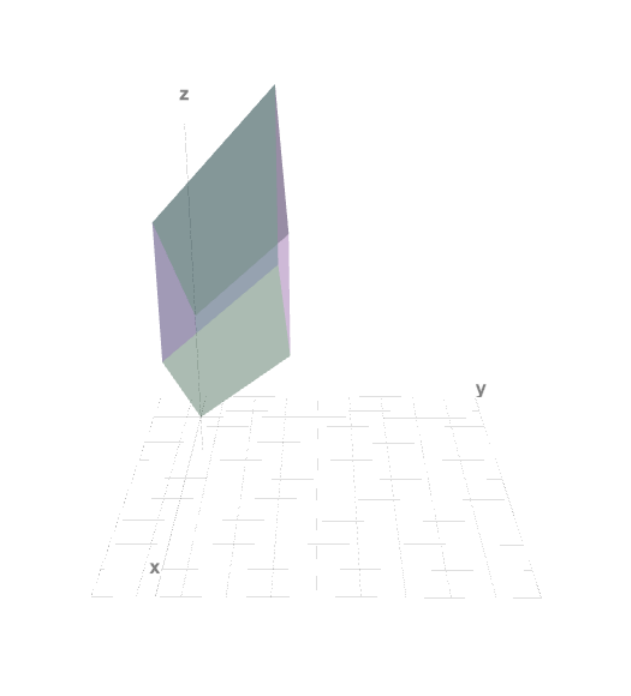
3.4: Practice Set

Exercise 3.4.1a

Use a substitution to evaluate $\int_0^2 \int_y^{y+4} \int_{\frac{y+2x}{2}}^{\frac{y+2x+6}{2}} (2z-2x-y)^2 e^{x-y}\, dz\, dx\, dy$ (Hint: try $u=y,\ v=x-y,\ w=2z-2x-y$)

Answer

The region we are integrating over $D = \left\{(x,y,z) : 0 \le y \le 2,\ y \le x \le y+4,\ \frac{y+2x}{2} \le z \le \frac{y+2x+6}{2}\right\}$ can be written as $D = \{(x,y,z) : 0 \le y \le 2,\ 0 \le x-y \le 4,\ 0 \le 2z-2x-y \le 6\}$, so it is the region between six planes (a parallelepiped):



Using the hinted substitution $u=y,\ v=x-y,\ w=2z-2x-y$, we see the new domain $G = \{(u,v,w) \,|\, 0 \le u \le 2,\ 0 \le v \le 4,\ 0 \le w \le 6\}$ is a rectangular prism.

We can easily solve for x, y, z to get: $x = u+v,\ y = u,\ z = \frac{3}{2}u + v + \frac{1}{2}w$

We see $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & 1 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

Therefore the triple integral becomes:

$$\int_0^2 \int_y^{y+4} \int_{\frac{y+2x}{2}}^{\frac{y+2x+6}{2}} (2z-2x-y)^2 e^{x-y}\,dz\,dx\,dy = \iiint_G \left(2z(u,v,w)-2x(u,v,w)-y(u,v,w)\right)^2 e^{x(u,v,w)-y(u,v,w)} \cdot |J(u,v,w)|\,dV$$

$$= \int_0^2 \int_0^4 \int_0^6 w^2 e^v \cdot \left|-\frac{1}{2}\right| dw\,dv\,du$$

$$= \frac{1}{2} \cdot \left(\int_0^2 du\right)\left(\int_0^4 e^v\,dv\right)\left(\int_0^6 w^2\,dw\right)$$

$$= \frac{1}{2} \cdot \left(u\Big|_0^2\right) \cdot \left(e^v\Big|_0^4\right) \cdot \left(\frac{1}{3}w^3\Big|_0^6\right)$$

$$= \frac{1}{2} \cdot 2 \cdot \left(e^4-1\right) \cdot 72$$

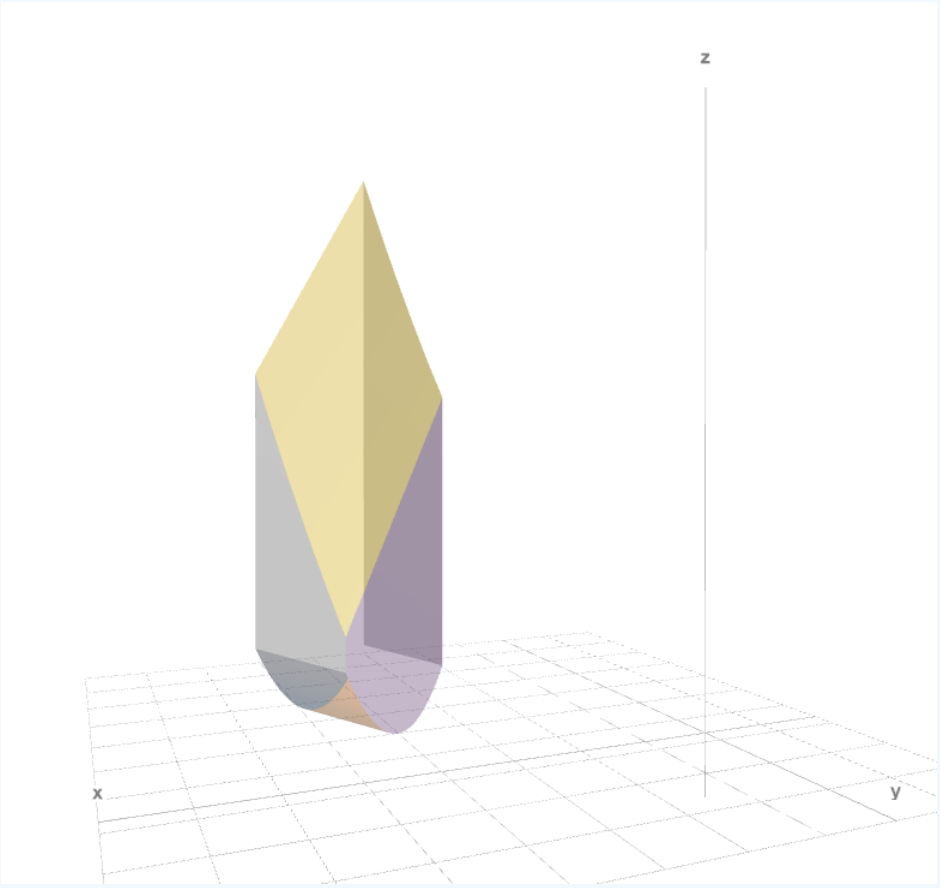
$$= 72\left(e^4-1\right)$$

Exercise 3.4.1*b*

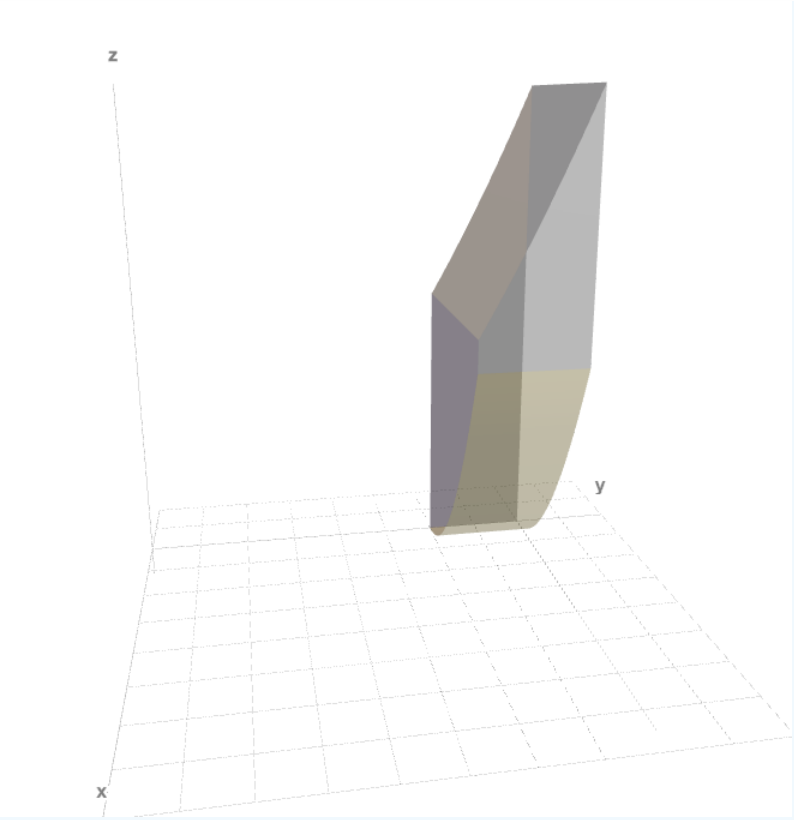
Use a substitution to evaluate $\iiint_D \frac{x+y}{(x-y)^2}\,dV$ over $D = \{(x,y,z) : 0 \le x+y \le 2,\ 3 \le x-y \le 4,\ 2(x+y)^2 \le 3x+4y+2z \le (x-y)^2\}$

Answer

The region we are integrating over $D = \{(x,y,z) : 0 \le x+y \le 2,\ 3 \le x-y \le 4,\ 2(x+y)^2 \le 3x+4y+2z \le (x-y)^2\}$ is not a famous shape.



Using the substitution $u = x + y$, $v = x - y$, $w = 3x + 4y + 2z$, we get the new domain $G = \{(u,v,w) : 0 \le u \le 2,\ 3 \le v \le 4,\ 2u^2 \le w \le v^2\}$ which is much easier to integrate over:



We can easily solve for x, y, z to get: $x = \frac{1}{2}u + \frac{1}{2}v,\ y = \frac{1}{2}u - \frac{1}{2}v,\ z = -\frac{7}{4}u + \frac{1}{4}v + \frac{1}{2}w$

We see $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{7}{4} & \frac{1}{4} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4}$

Therefore the triple integral becomes:

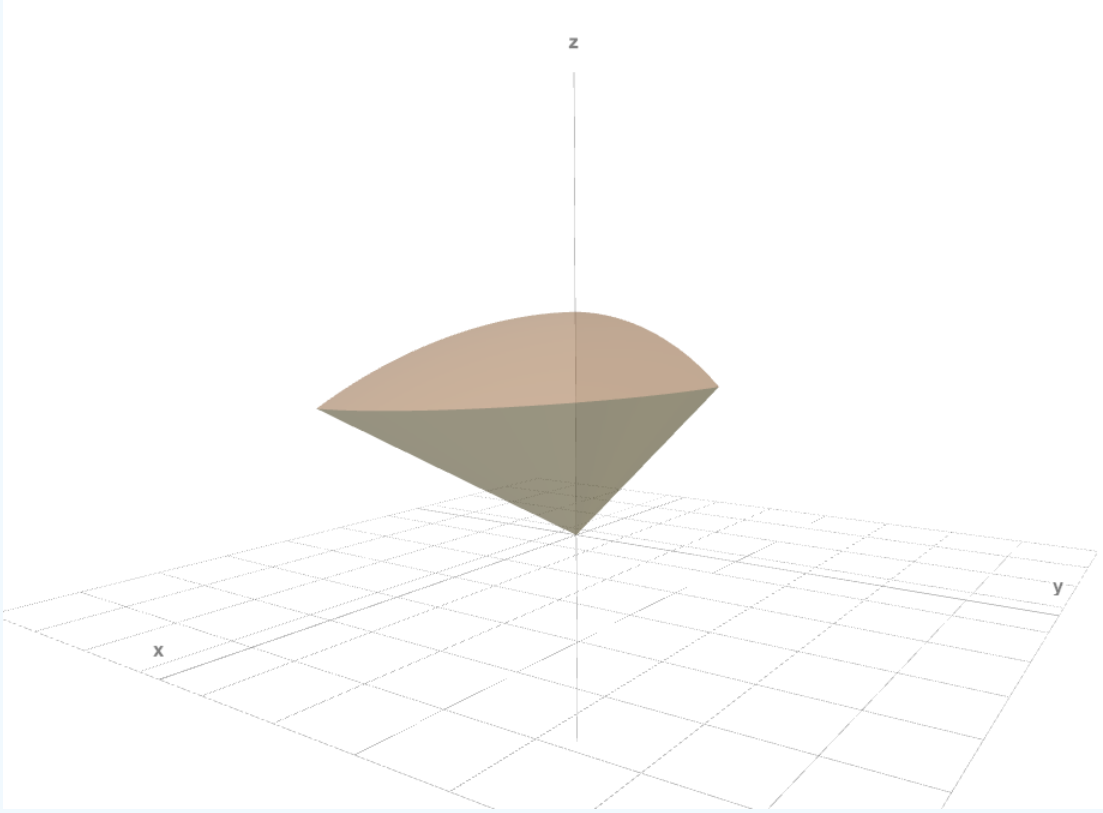
$$\begin{aligned}
\iiint_D \frac{x+y}{(x-y)^2}\,dV &= \iiint_G \frac{x(u,v,w)+y(u,v,w)}{(x(u,v,w)-y(u,v,w))^2}\cdot|J(u,v,w)|\,dV \\
&= \int_0^2\int_3^4\int_{2u^2}^{v^2} \frac{u}{v^2}\cdot\left|-\frac{1}{4}\right|\,dw\,dv\,du \\
&= \frac{1}{4}\int_0^2\int_3^4 \frac{u}{v^2}w\Big|_{2u^2}^{v^2}\,dv\,du \\
&= \frac{1}{4}\int_0^2\int_3^4 \left(u - \frac{2u^3}{v^2}\right)\,dv\,du \\
&= \frac{1}{4}\int_0^2 \left[uv + \frac{2u^3}{v}\right]\Big|_3^4\,du \\
&= \frac{1}{4}\int_0^2 \left(u - \frac{1}{6}u^3\right)\,du \\
&= \frac{1}{4}\left[\frac{1}{2}u^2 - \frac{1}{24}u^4\right]\Big|_0^2 \\
&= \frac{1}{4}\cdot\frac{4}{3} \\
&= \frac{1}{3}
\end{aligned}$$

Exercise 3.4.1c

Use a substitution to evaluate $\int_0^2 \int_0^{\sqrt{1-\frac{x^2}{4}}} \int_{\sqrt{\frac{x^2}{4}+y^2}}^{\sqrt{1-\frac{x^2}{4}-y^2}} \left(\frac{x^2}{4}+y^2+z^2\right)^4 \, dz\, dy\, dx$

Answer

The region we are integrating over $D = \left\{(x,y,z) : 0 \le x \le 2,\, 0 \le y \le \sqrt{1-\frac{x^2}{4}},\, \sqrt{\frac{x^2}{4}+y^2} \le z \le \sqrt{1-\frac{x^2}{4}-y^2}\right\}$ is bounded between the ellipsoid $\frac{x^2}{4}+y^2+z^2=1$ and the cone $z=\sqrt{\frac{x^2}{4}+y^2}$ in the first octant:



Note that an ellipsoid is just a stretched-out sphere, so we try a simple substitution: $x=2u$, $y=v$, $z=w$.

This makes our new domain $G = \left\{(u,v,w) : 0 \le u \le 1,\, 0 \le v \le \sqrt{1-u^2},\, \sqrt{u^2+v^2} \le w \le \sqrt{1-u^2-v^2}\right\}$ a region between a standard sphere and cone!

Notice that G is simple to write in spherical coordinates: $G = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) : 0 \le \rho \le 1,\, 0 \le \phi \le \frac{\pi}{4},\, 0 \le \theta \le \frac{\pi}{2}\right\}$

We see $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2$

Therefore the triple integral becomes:

$$\int_0^2 \int_0^{\sqrt{1-\frac{x^2}{4}}} \int_{\sqrt{\frac{x^2}{4}+y^2}}^{\sqrt{2-\frac{x^2}{4}-y^2}} \left(\frac{x^2}{4}+y^2+z^2\right)^4 dz\,dy\,dx = \iiint_G \left(\frac{[x(u,v,w)]^2}{4}+[y(u,v,w)]^2+[z(u,v,w)]^2\right)^4 \cdot |J(u,v,w)|\,dV$$

$$= \iiint_G \left(u^2+v^2+w^2\right)^4 \cdot 2\,dV$$

$$= \iiint_G 2\left(u^2+v^2+w^2\right)^4 dV$$

$$= \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} 2\left(\rho^2\right)^4 \cdot \rho^2 \sin(\phi)\,d\theta\,d\phi\,d\rho$$

$$= \left(\int_0^1 2\rho^{10}\,d\rho\right)\left(\int_0^{\frac{\pi}{4}} \sin(\phi)\,d\phi\right)\left(\int_0^{\frac{\pi}{2}} d\theta\right)$$

$$= \left(\frac{2}{11}\rho^{11}\Big|_0^1\right)\cdot\left(-\cos(\phi)\Big|_0^{\frac{\pi}{4}}\right)\cdot\left(\theta\Big|_0^{\frac{\pi}{2}}\right)$$

$$= \frac{2}{11}\cdot\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{\pi}{2}$$

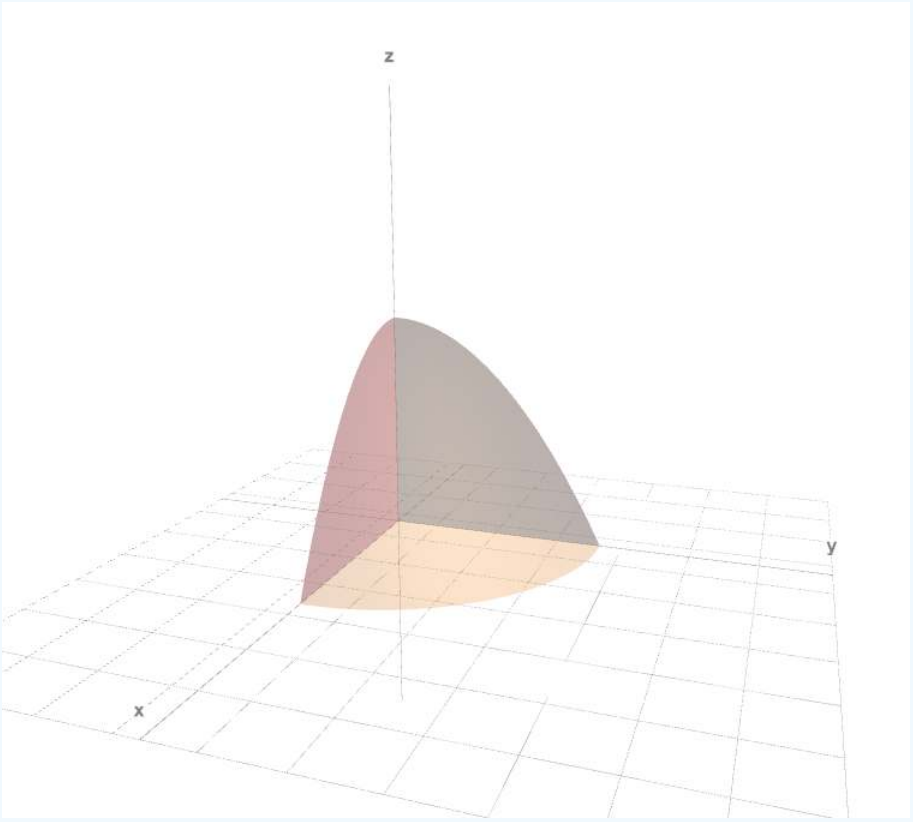
$$= \frac{\pi(2-\sqrt{2})}{22}$$

Exercise 3.4.2a

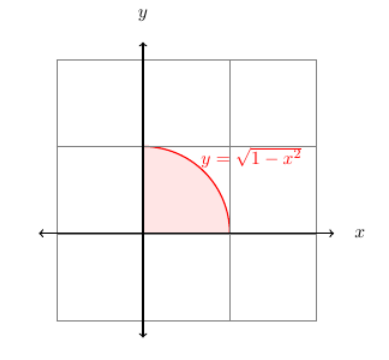
Use cylindrical coordinates to evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} xyz\,dz\,dy\,dx$

Answer

We see the domain $D = \left\{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ 0 \le z \le 1-x^2-y^2\right\}$ is under a paraboloid in the first octant:



The polar base $\hat{D} = \left\{(x,y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\right\} = \left\{(r\cos(\theta), r\sin(\theta)) : 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\right\}$



Therefore in cylindrical form, $D = \left\{(r\cos(\theta), r\sin(\theta), z) \,:\, 0 \le r \le 1,\, 0 \le \theta \le \frac{\pi}{2},\, 0 \le z \le 1 - r^2\right\}$

This makes our triple integral into:

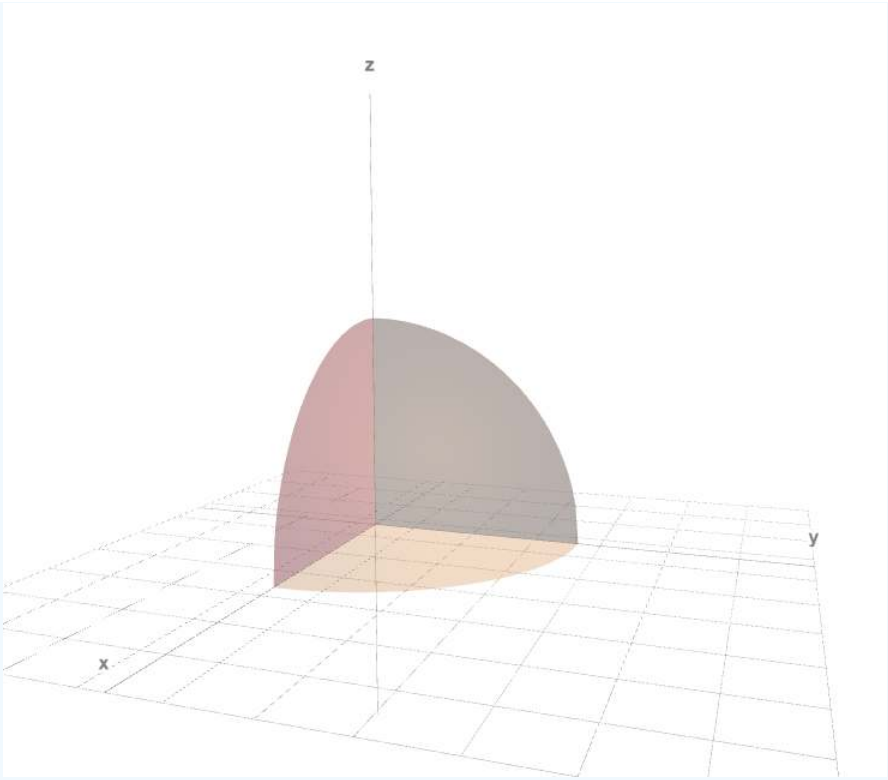
$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} xyz\,dz\,dy\,dx &= \iiint_D xyz\,dV \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{1-r^2} r\cos(\theta)\cdot r\sin(\theta)\cdot z\cdot r\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{1-r^2} r^3\sin(\theta)\cos(\theta)z\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{1}{2}r^3\sin(\theta)\cos(\theta)z^2\Big|_0^{1-r^2}\,d\theta\,dr \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{1}{2}r^3\sin(\theta)\cos(\theta)(1-r^2)^2\,d\theta\,dr \\
&= \frac{1}{2}\left(\int_0^1 (r^3 - 2r^5 + r^7)\,dr\right)\left(\int_0^{\frac{\pi}{2}} \sin(\theta)\cos(\theta)\,d\theta\right) \\
&= \frac{1}{2}\left[\frac{1}{4}r^4 - \frac{1}{3}r^6 + \frac{1}{8}r^8\right]\Big|_0^1 \cdot \left(\frac{1}{2}\sin^2(\theta)\Big|_0^{\frac{\pi}{2}}\right) \\
&= \frac{1}{2}\cdot\left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right)\cdot\frac{1}{2} \\
&= \frac{1}{96}
\end{aligned}$$

? Exercise 3.4.2*b*

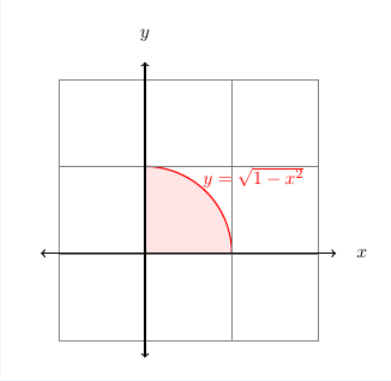
Use cylindrical coordinates to evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

Answer

We see the domain $D = \left\{(x, y, z) \,:\, 0 \le x \le 1,\, 0 \le y \le \sqrt{1-x^2},\, 0 \le z \le \sqrt{1-x^2-y^2}\right\}$ is under the top half of a sphere in the first octant:



The polar base $\hat{D} = \left\{(x, y) \,:\, 0 \le x \le 1,\, 0 \le y \le \sqrt{1 - x^2}\right\} = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le 1,\, 0 \le \theta \le \frac{\pi}{2}\right\}$



Therefore in cylindrical form, $D = \left\{(r\cos(\theta), r\sin(\theta), z) \,:\, 0 \le r \le 1,\, 0 \le \theta \le \frac{\pi}{2},\, 0 \le z \le \sqrt{1 - r^2}\right\}$

This makes our triple integral into:

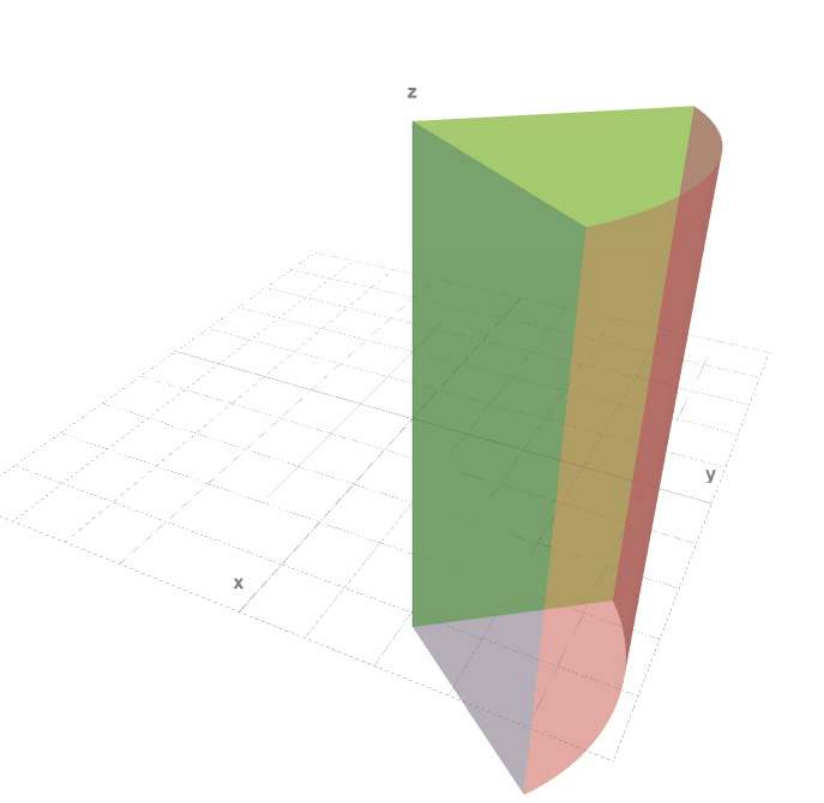
$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx &= \iiint_D xyz\,dV \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{1-r^2}} r\cos(\theta)\cdot r\sin(\theta)\cdot z\cdot r\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{1-r^2}} r^3\sin(\theta)\cos(\theta)z\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{1}{2}r^3\sin(\theta)\cos(\theta)z^2\Big|_0^{\sqrt{1-r^2}}\,d\theta\,dr \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{1}{2}r^3\sin(\theta)\cos(\theta)(1-r^2)\,d\theta\,dr \\
&= \frac{1}{2}\left(\int_0^1 (r^3-r^5)\,dr\right)\left(\int_0^{\frac{\pi}{2}} \sin(\theta)\cos(\theta)\,d\theta\right) \\
&= \frac{1}{2}\left[\frac{1}{4}r^4-\frac{1}{6}r^6\right]\Big|_0^1 \cdot \left(\frac{1}{2}\sin^2(\theta)\Big|_0^{\frac{\pi}{2}}\right) \\
&= \frac{1}{2}\cdot\left(\frac{1}{4}-\frac{1}{6}\right)\cdot\frac{1}{2} \\
&= \frac{1}{48}
\end{aligned}$$

Exercise 3.4.2c

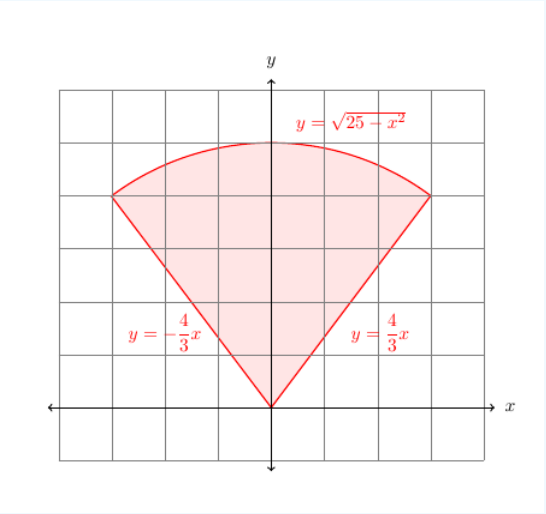
Use cylindrical coordinates to evaluate $\int_{-6}^{6}\int_{-3}^{3}\int_{\frac{4}{3}|x|}^{\sqrt{25-x^2}} \frac{x^2+y^2}{y^2}\,dy\,dx\,dz$

Answer

We see the domain $D=\left\{(x,y,z)\,:\,-6\le z\le 6,\,-3\le x\le 3,\,\frac{4}{3}|x|\le y\le\sqrt{25-x^2}\right\}$ is a partial cylinder:



The polar base is $\hat{D} = \left\{(x,y) : -3 \le x \le 3,\ \frac{4}{3}|x| \le y \le \sqrt{25-x^2}\right\}$
$= \left\{(r\cos(\theta), r\sin(\theta)) : 0 \le r \le 5,\ \tan^{-1}\left(\frac{4}{3}\right) \le \theta \le \pi - \tan^{-1}\left(\frac{4}{3}\right)\right\}$



Therefore in cylindrical form, $D = \left\{(r\cos(\theta), r\sin(\theta), z) : 0 \le r \le 5,\ \tan^{-1}\left(\frac{4}{3}\right) \le \theta \le \pi - \tan^{-1}\left(\frac{4}{3}\right),\ -6 \le z \le 6\right\}$

This makes our triple integral into:

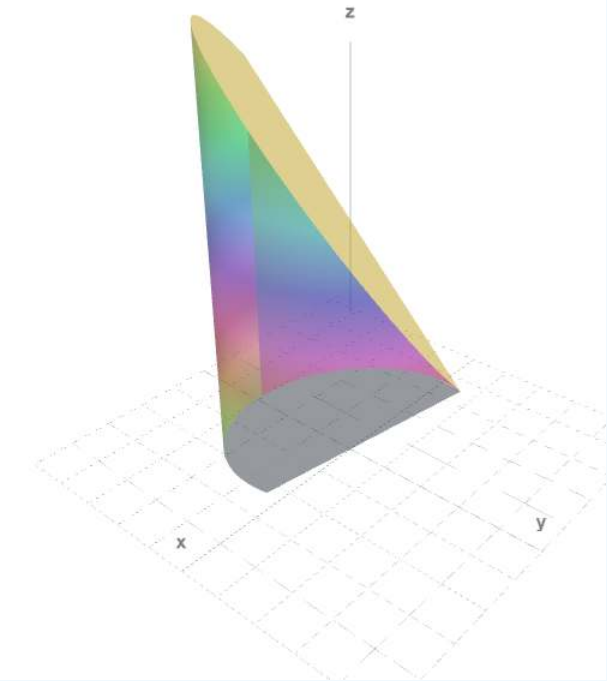
$$\begin{aligned}
\int_{-6}^{6}\int_{-3}^{3}\int_{\frac{4}{3}|x|}^{\sqrt{25-x^2}} \frac{x^2+y^2}{y^2}\,dy\,dx\,dz &= \iiint_D \frac{x^2+y^2}{y^2}\,dV \\
&= \int_0^5 \int_{\tan^{-1}\left(\frac{4}{3}\right)}^{\pi-\tan^{-1}\left(\frac{4}{3}\right)} \int_{-6}^{6} \frac{r^2}{(r\sin(\theta))^2} \cdot r\,dz\,d\theta\,dr \\
&= \left(\int_0^5 r\,dr\right)\left(\int_{\tan^{-1}\left(\frac{4}{3}\right)}^{\pi-\tan^{-1}\left(\frac{4}{3}\right)} \csc^2(\theta)\,d\theta\right)\left(\int_{-6}^{6} dz\right) \\
&= \left(\frac{1}{2}r^2\Big|_0^5\right)\cdot\left(-\cot(\theta)\Big|_{\tan^{-1}\left(\frac{4}{3}\right)}^{\pi-\tan^{-1}\left(\frac{4}{3}\right)}\right)\cdot\left(z\Big|_{-6}^{6}\right) \\
&= \frac{25}{2}\cdot\frac{3}{2}\cdot 12 \\
&= 225
\end{aligned}$$

Exercise 3.4.2*d*

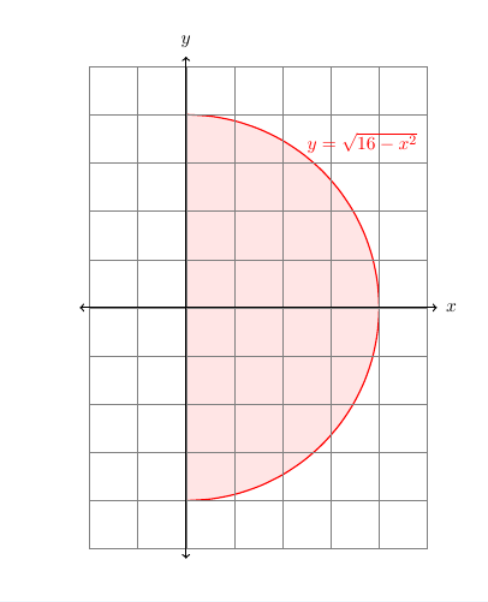
Use cylindrical coordinates to evaluate $\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}}\int_{x+y}^{8+x-y} \frac{1}{\sqrt{x^2+y^2+9}}\,dz\,dx\,dy$

Answer

We see the domain $D = \left\{(x,y,z) : -4 \le y \le 4,\ 0 \le x \le \sqrt{16-y^2},\ x+y \le z \le 8+x-y\right\}$ is a partial cylinder between two planes:



This has a polar base $\hat{D} = \left\{(x,y) \,:\, -4 \le y \le 4,\, 0 \le x \le \sqrt{16-y^2}\right\} = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le 4,\, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$



Therefore in cylindrical form, $D = \left\{(r\cos(\theta), r\sin(\theta), z) \,:\, 0 \le r \le 4,\, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\, r\cos(\theta) + r\sin(\theta) \le z \le 8 + r\cos(\theta) - r\sin(\theta)\right\}$

This makes our triple integral into:

$$\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}}\int_{x+y}^{8+x-y}\frac{1}{\sqrt{x^2+y^2+9}}\,dz\,dx\,dy = \iiint_D \frac{1}{\sqrt{x^2+y^2+9}}\,dV$$

$$= \int_0^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r\cos(\theta)+r\sin(\theta)}^{8+r\cos(\theta)-r\sin(\theta)}\frac{1}{\sqrt{9+r^2}}\cdot r\,dz\,d\theta\,dr$$

$$= \int_0^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r}{\sqrt{9+r^2}}z\Big|_{r\cos(\theta)+r\sin(\theta)}^{8+r\cos(\theta)-r\sin(\theta)}\,d\theta\,dr$$

$$= \int_0^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r}{\sqrt{9+r^2}}(8-2r\sin(\theta))\,d\theta\,dr$$

$$= \int_0^4\frac{r}{\sqrt{9+r^2}}(8\theta+2r\cos(\theta))\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\,dr$$

$$= \int_0^4\frac{8\pi r}{\sqrt{9+r^2}}\,dr$$

$$= 8\pi\sqrt{9+r^2}\Big|_0^4$$

$$= 8\pi\cdot\left(\sqrt{25}-\sqrt{9}\right)$$

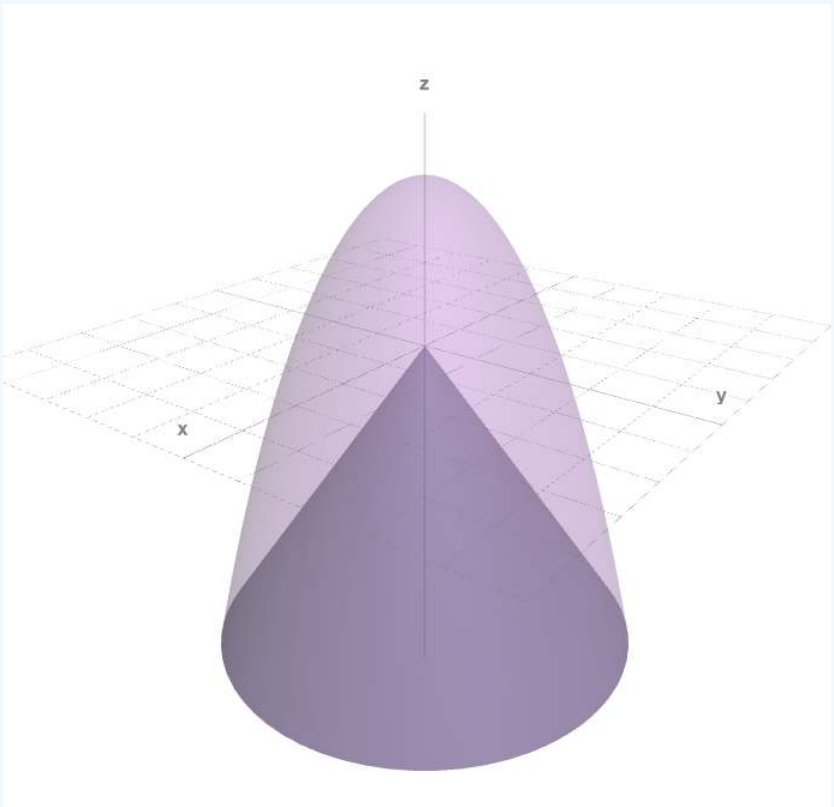
$$= 16\pi$$

Exercise 3.4.2e

Use cylindrical coordinates to evaluate $\iiint_F (3x+2z)\,dV$ for $F = \left\{(x,y,z) : -2\sqrt{x^2+y^2} \le z \le 3-x^2-y^2\right\}$

Answer

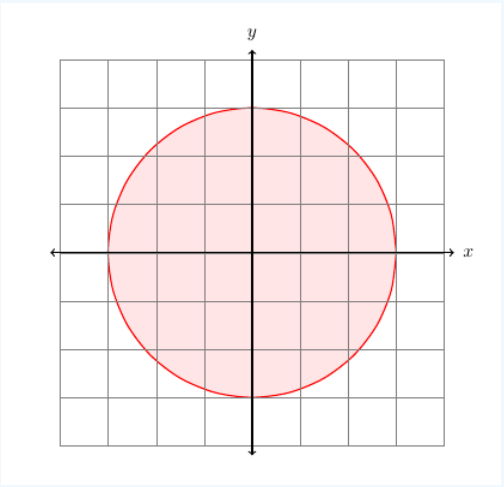
We see the domain $F = \left\{(x,y,z) : -2\sqrt{x^2+y^2} \le z \le 3-x^2-y^2\right\}$ is the region between a cone and paraboloid.



In cylindrical coordinates, the cone is $z = -2r$ and the paraboloid is $z = 3-r^2$. These intersect when:

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$-2r = 3 - r^2 \longleftrightarrow r^2 - 2r - 3 = 0 \longleftrightarrow r = 3, -1$ We choose the positive solution $r = 3$ and see the polar base is $\hat{F} = \{(r\cos(\theta), r\sin(\theta)) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$



Therefore in cylindrical form, $F = \{(r\cos(\theta), r\sin(\theta), z) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ -2r \le z \le 3 - r^2\}$

This makes our triple integral into:

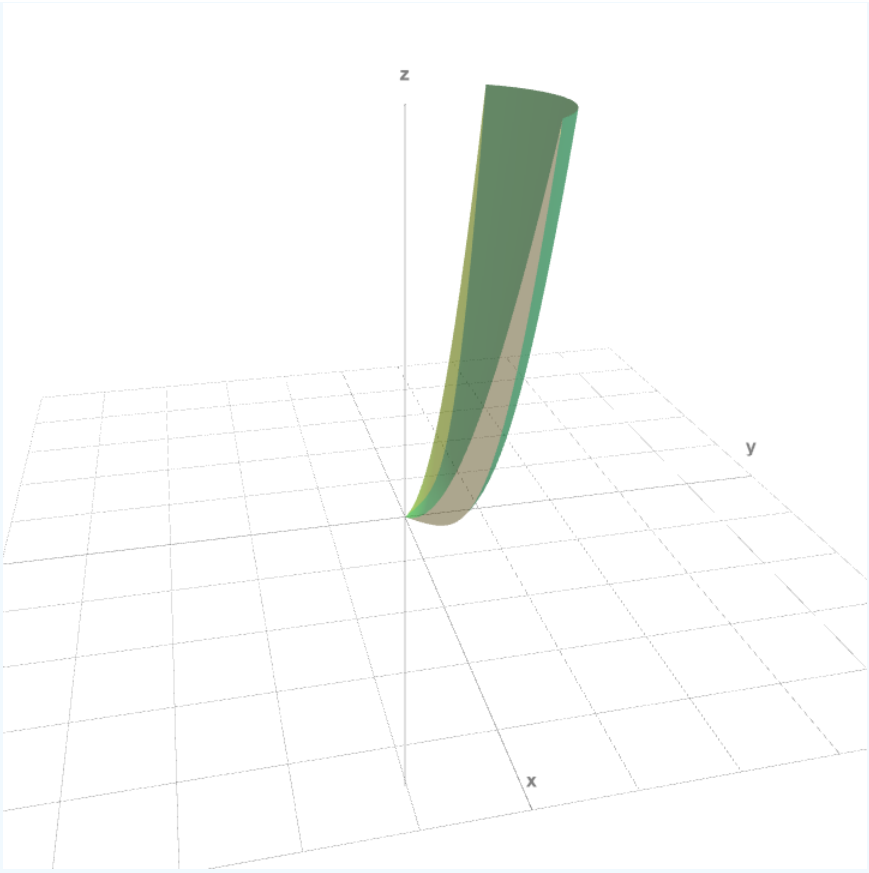
$$\begin{aligned}
\iiint_F (3x + 2z)\,dV &= \int_0^3 \int_0^{2\pi} \int_{-2r}^{3-r^2} (3r\cos(\theta) + 2z) \cdot r\,dz\,d\theta\,dr \\
&= \int_0^3 \int_0^{2\pi} \int_{-2r}^{3-r^2} (3r^2\cos(\theta) + 2rz)\,dz\,d\theta\,dr \\
&= \int_0^3 \int_0^{2\pi} \left[3r^2\cos(\theta)z + rz^2\right]\Big|_{-2r}^{3-r^2} d\theta\,dr \\
&= \int_0^3 \int_0^{2\pi} \left(3r^2(3 - r^2 + 2r)\cos(\theta) + (9r - 10r^3 + r^5)\right)\,d\theta\,dr \\
&= \int_0^3 \left[3r^2(3 - r^2 + 2r)\sin(\theta) + (9r - 10r^3 + r^5)\theta\right]\Big|_0^{2\pi} dr \\
&= \int_0^3 2\pi(9r - 10r^3 + r^5)\,dr \\
&= 2\pi\left[\frac{9}{2}r^2 - \frac{5}{2}r^4 + \frac{1}{6}r^6\right]\Big|_0^3 \\
&= 2\pi \cdot \left(-\frac{81}{2}\right) \\
&= -81\pi
\end{aligned}$$

Exercise 3.4.2f

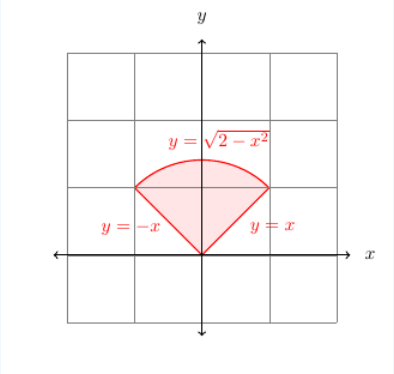
Use cylindrical coordinates to evaluate $\int_{-1}^{1} \int_{|x|}^{\sqrt{2-x^2}} \int_{(x^2+y^2)^2}^{2x^2+2y^2} y^2 z\,dz\,dy\,dx$

Answer

We see the domain $D = \left\{(x, y, z) : -1 \le x \le 1,\ |x| \le y \le \sqrt{2 - y^2},\ (x^2 + y^2)^2 \le z \le 2x^2 + 2y^2\right\}$ is part of the region between the paraboloid $z = 2x^2 + 2y^2$ and the surface $z = (x^2 + y^2)^2$:



This has polar base $\hat{D} = \left\{(x,y) \,:\, -1 \le x \le 1,\ |x| \le y \le \sqrt{2-y^2}\right\} = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le \sqrt{2},\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$



Therefore in cylindrical form, $D = \left\{(r\cos(\theta), r\sin(\theta), z) \,:\, 0 \le r \le \sqrt{2},\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ r^4 \le z \le 2r^2\right\}$

This makes our triple integral into:

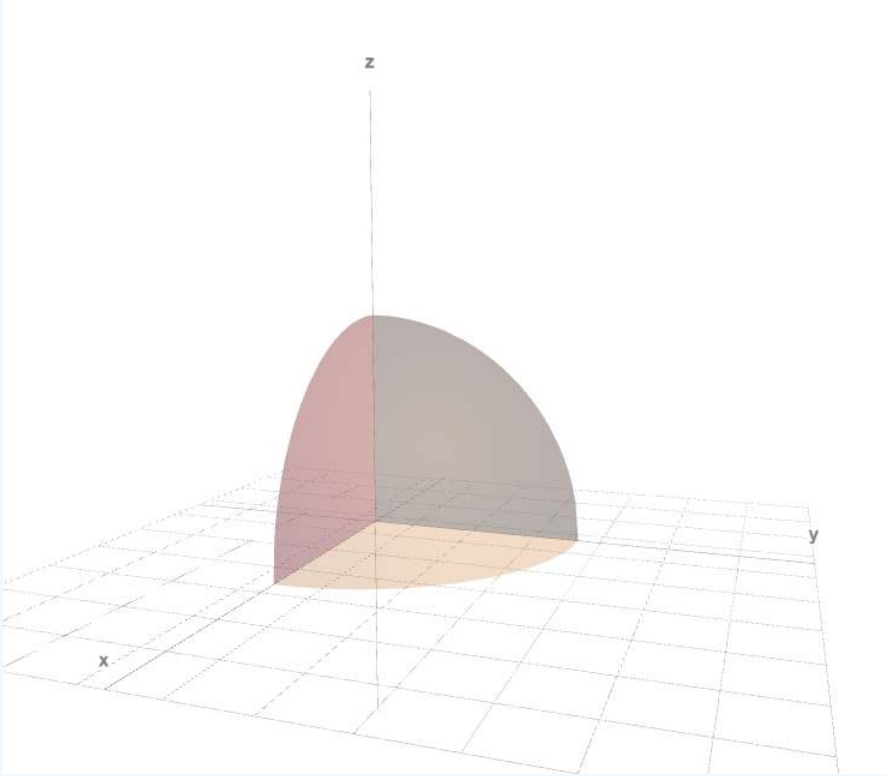
$$\begin{aligned}
\int_{-1}^{1}\int_{|x|}^{\sqrt{2-x^2}}\int_{(x^2+y^2)^2}^{2x^2+2y^2} y^2 z\,dz\,dy\,dx &= \iiint_D y^2 z\,dV \\
&= \int_0^{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{r^4}^{2r^2} (r\sin(\theta))^2\, z\cdot r\,dz\,d\theta\,dr \\
&= \int_0^{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{r^4}^{2r^2} r^3\sin^2(\theta)\, z\,dz\,d\theta\,dr \\
&= \int_0^{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}r^3\sin^2(\theta)\, z^2\Big|_{r^4}^{2r^2}\,d\theta\,dr \\
&= \int_0^{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}r^3\sin^2(\theta)\left(4r^4-r^8\right)\,d\theta\,dr \\
&= \left(\int_0^{\sqrt{2}}(4r^7-r^{11})\,dr\right)\left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{4}(1-\cos(2\theta))\,d\theta\right) \\
&= \left[\frac{1}{2}r^8-\frac{1}{12}r^{12}\right]\Big|_0^{\sqrt{2}}\cdot\left[\frac{1}{4}\theta-\frac{1}{8}\sin(2\theta)\right]\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
&= \frac{8}{3}\cdot\left(\frac{\pi}{8}+\frac{1}{4}\right) \\
&= \frac{\pi+2}{3}
\end{aligned}$$

Exercise 3.4.3a

Use spherical coordinates to evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

Answer

We see the domain $D=\left\{(x,y,z)\,:\,0\le x\le 1,\,0\le y\le\sqrt{1-x^2},\,0\le z\le\sqrt{1-x^2-y^2}\right\}$ is the part of the unit sphere in the first octant:



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Thus, the spherical form is $D = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \,:\, 0 \le \rho \le 1,\, 0 \le \phi \le \frac{\pi}{2},\, 0 \le \theta \le \frac{\pi}{2}\right\}$

This makes our triple integral into:

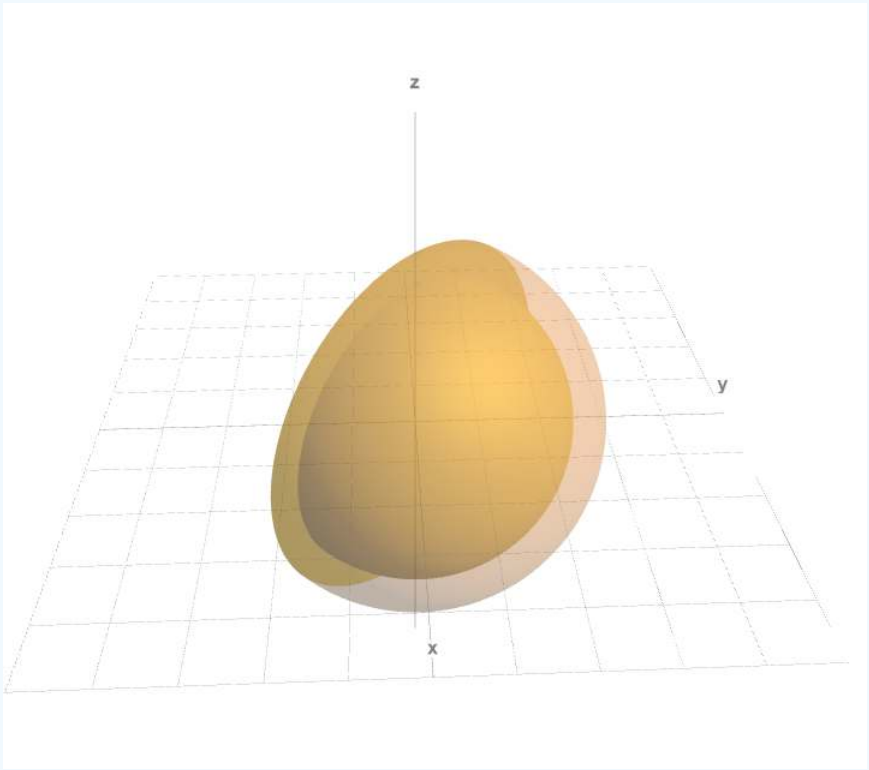
$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx &= \iiint_D xyz\,dV \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\rho\sin(\phi)\cos(\theta))(\rho\sin(\phi)\sin(\theta))(\rho\cos(\phi)) \cdot \rho^2\sin(\phi)\,d\phi\,d\theta\,d\rho \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^5\sin^3(\phi)\cos(\phi)\sin(\theta)\cos(\theta)\,d\phi\,d\theta\,d\rho \\
&= \left(\int_0^1 \rho^5\,d\rho\right)\left(\int_0^{\frac{\pi}{2}} \sin^3(\phi)\cos(\phi)\,d\phi\right)\left(\int_0^{\frac{\pi}{2}} \sin(\theta)\cos(\theta)\,d\theta\right) \\
&= \left(\frac{1}{6}\rho^6\Big|_0^1\right) \cdot \left(\frac{1}{4}\sin^4(\phi)\Big|_0^{\frac{\pi}{2}}\right) \cdot \left(\frac{1}{2}\sin^2(\theta)\Big|_0^{\frac{\pi}{2}}\right) \\
&= \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \\
&= \frac{1}{48}
\end{aligned}$$

? Exercise 3.4.3*b*

Use spherical coordinates to evaluate $\iiint_E \frac{1}{1+(x^2+y^2+z^2)^3}\,dV$ for $E = \left\{(x,y,z) \,:\, 1 \le x^2+y^2+z^2 \le \sqrt[3]{3},\, y \ge -x\right\}$

Answer

We see the domain $E = \left\{(x,y,z) \,:\, 1 \le x^2+y^2+z^2 \le \sqrt[3]{3},\, y \ge -x\right\}$ is part of the region between two spheres:



The restriction $y \ge -x$ tells us that $\sin(\theta) \ge -\cos(\theta)$, so we know $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

Thus, the spherical form is $E = \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \,:\, 1 \le \rho \le \sqrt[6]{3},\, 0 \le \phi \le \pi,\, -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$

This makes our triple integral into:

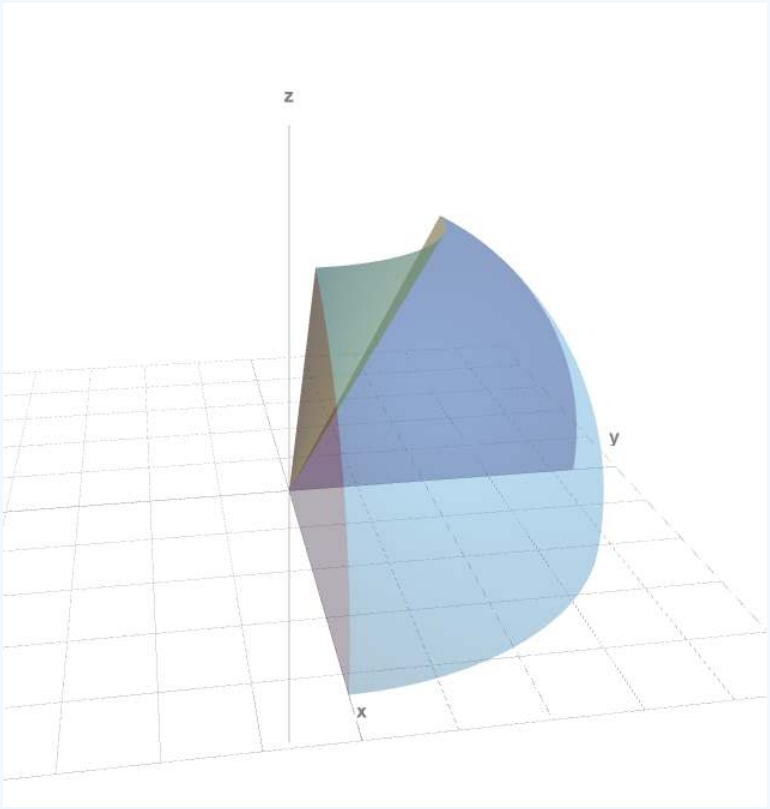
$$\begin{aligned}
\iiint_E \frac{1}{1+(x^2+y^2+z^2)^3}\,dV &= \int_1^{\sqrt[6]{3}}\int_0^{\pi}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1+(\rho^2)^3}\cdot\rho^2\sin(\phi)\,d\theta\,d\phi\,d\rho \\
&= \int_1^{\sqrt[6]{3}}\int_0^{\pi}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\rho^2\sin(\phi)}{1+\rho^6}\,d\theta\,d\phi\,d\rho \\
&= \left(\int_1^{\sqrt[6]{3}} \frac{\rho^2}{1+(\rho^3)^2}\,d\rho\right)\left(\int_0^{\pi}\sin(\phi)\,d\phi\right)\left(\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta\right) \\
&= \left(\frac{1}{3}\tan^{-1}(\rho^3)\Big|_1^{\sqrt[6]{3}}\right)\cdot\left(-\cos(\phi)\Big|_0^{\pi}\right)\cdot\left(\theta\Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\right) \\
&= \frac{1}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\cdot 2\cdot\pi \\
&= \frac{\pi^2}{18}
\end{aligned}$$

? Exercise 3.4.3c

Use spherical coordinates to evaluate $\iiint_E 10xy\,dV$ for $E=\left\{(x,y,z)\,:\,x^2+y^2+z^2\le 3,\,0\le z\le\sqrt{2x^2+2y^2},\,x\ge 0,\,y\ge 0\right\}$

Answer

We see the domain $E=\left\{(x,y,z)\,:\,x^2+y^2+z^2\le 3,\,0\le z\le\sqrt{2x^2+2y^2},\,x\ge 0,\,y\ge 0\right\}$ is part of the region between a sphere and cone in the first octant:



Below the cone $z=\sqrt{2}\cdot\sqrt{x^2+y^2}$ tells us that angle $\phi\ge\cot^{-1}(\sqrt{2})$, and $z\ge 0$ gives $\phi\le\frac{\pi}{2}$

Thus, the spherical form is $E=\left\{(\rho\sin(\phi)\cos(\theta),\rho\sin(\phi)\sin(\theta),\rho\cos(\phi))\,:\,0\le\rho\le\sqrt{3},\,\cot^{-1}(\sqrt{2})\le\phi\le\frac{\pi}{2},\,0\le\theta\le\frac{\pi}{2}\right\}$

This makes our triple integral into:

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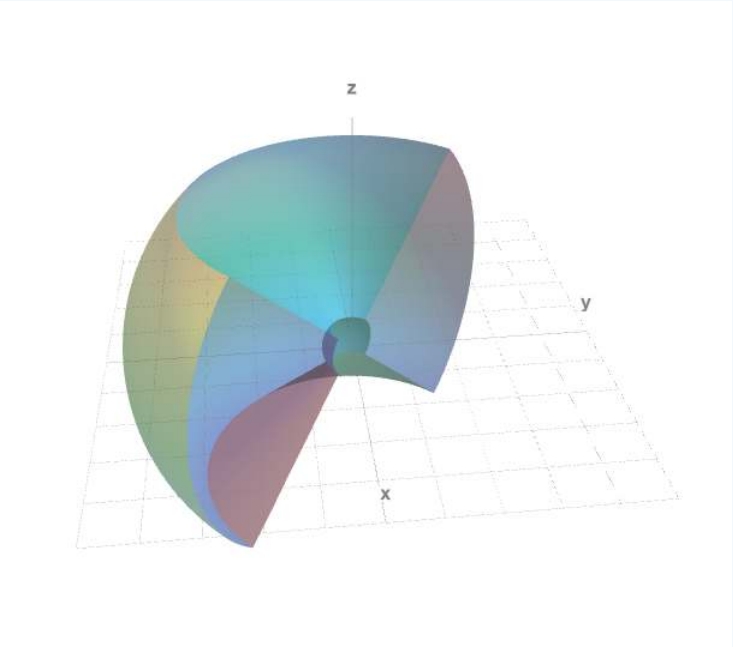
$$\begin{aligned}
\iiint_E 10xy\,dV &= \int_0^{\sqrt{3}} \int_{\cot^{-1}(\sqrt{2})}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 10(\rho\sin(\phi)\cos(\theta))(\rho\sin(\phi)\sin(\theta))\cdot\rho^2\sin(\phi)\,d\theta\,d\phi\,d\rho \\
&= \int_0^{\sqrt{3}} \int_{\cot^{-1}(\sqrt{2})}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 10\rho^4\sin^3(\phi)\sin(\theta)\cos(\theta)\,d\theta\,d\phi\,d\rho \\
&= \left(\int_0^{\sqrt{3}} 10\rho^4\,d\rho\right)\left(\int_{\cot^{-1}(\sqrt{2})}^{\frac{\pi}{2}} (1-\cos^2(\phi))\sin(\phi)\,d\phi\right)\left(\int_0^{\frac{\pi}{2}} \sin(\theta)\cos(\theta)\,d\theta\right) \\
&= \left(2\rho^5\Big|_0^{\sqrt{3}}\right)\cdot\left[-\cos(\phi)+\frac{1}{3}\cos^3(\phi)\right]\Bigg|_{\cot^{-1}(\sqrt{2})}^{\frac{\pi}{2}}\cdot\left(\frac{1}{2}\sin^2(\theta)\Big|_0^{\frac{\pi}{2}}\right) \\
&= 18\sqrt{3}\cdot\left(\frac{\sqrt{2}}{\sqrt{3}}-\frac{1}{3}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^3\right)\cdot\frac{1}{2} \\
&= 7\sqrt{2}
\end{aligned}$$

? Exercise 3.4.3d

Use spherical coordinates to evaluate $\iiint_G \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV$ for $G=\{(x,y,z)\,:\,1\le x^2+y^2+z^2\le e^2,\ z^2\le x^2+y^2,\ y\le -x\}$

Answer

We see the domain $G=\{(x,y,z)\,:\,1\le x^2+y^2+z^2\le e^2,\ y\le -x\}$ is part of the region between two spheres:



The restriction $y\le -x$ tells us that $\sin(\theta)\le -\cos(\theta)$, so we know $\frac{3\pi}{4}\le\theta\le\frac{7\pi}{4}$

The restriction $z^2\le x^2+y^2 \longleftrightarrow -\sqrt{x^2+y^2}\le z\le\sqrt{x^2+y^2}$ tells us that $-1\le\cot(\phi)\le 1$, so we know $\frac{\pi}{4}\le\phi\le\frac{3\pi}{4}$

Thus, the spherical form is $G=\left\{(\rho\sin(\phi)\cos(\theta),\rho\sin(\phi)\sin(\theta),\rho\cos(\phi))\,:\,1\le\rho\le e,\ \frac{\pi}{4}\le\phi\le\frac{3\pi}{4},\ \frac{3\pi}{4}\le\theta\le\frac{7\pi}{4}\right\}$

This makes our triple integral into:

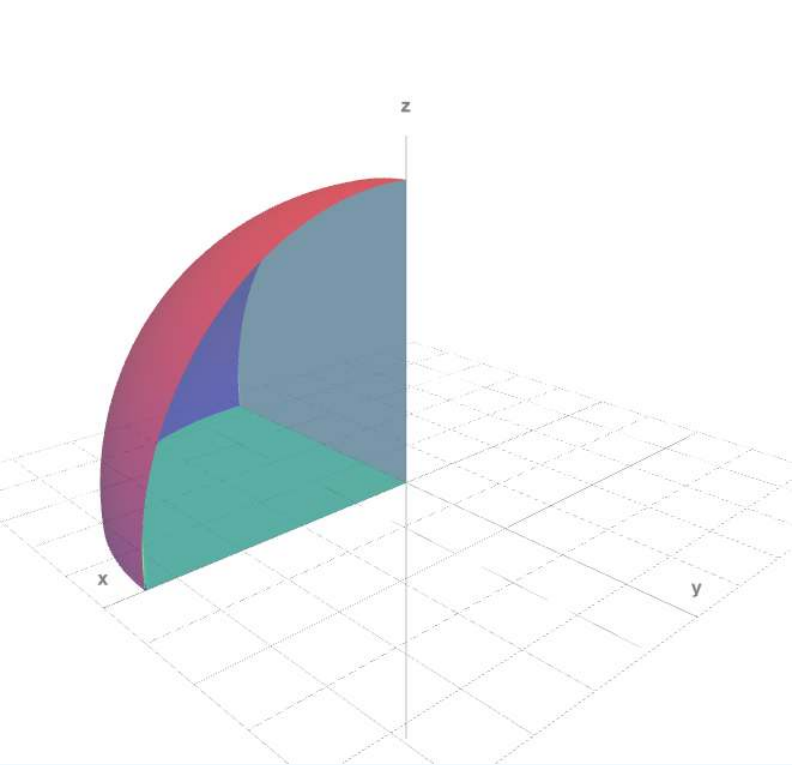
$$\begin{aligned}
\iiint_G \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV &= \int_1^e \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} \frac{1}{(\rho^2)^{\frac{3}{2}}} \cdot \rho^2 \sin(\phi)\,d\theta\,d\phi\,d\rho \\
&= \int_1^e \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} \frac{1}{\rho}\sin(\phi)\,d\theta\,d\phi\,d\rho \\
&= \left(\int_1^e \frac{1}{\rho}\,d\rho\right)\left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(\phi)\,d\phi\right)\left(\int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} d\theta\right) \\
&= \left(\ln|\rho|\Big|_1^e\right)\cdot\left(-\cos(\phi)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\right)\cdot\left(\theta\Big|_{\frac{3\pi}{4}}^{\frac{7\pi}{4}}\right) \\
&= 1\cdot\sqrt{2}\cdot\pi \\
&= \pi\sqrt{2}
\end{aligned}$$

Exercise 3.4.3e

Use spherical coordinates to evaluate $\int_{-5}^{0}\int_{0}^{\sqrt{25-y^2}}\int_{0}^{\sqrt{25-y^2-z^2}} ze^{-z^2}\,dx\,dz\,dy$

Answer

We see the domain $D=\left\{(x,y,z)\,:\,-5\le y\le 0,\,0\le z\le\sqrt{25-y^2},\,0\le x\le\sqrt{25-y^2-z^2}\right\}$ is the part of the sphere with $x\ge 0$, $y\le 0$, $z\ge 0$:



Thus, the spherical form is $D=\left\{(\rho\sin(\phi)\cos(\theta),\rho\sin(\phi)\sin(\theta),\rho\cos(\phi))\,:\,0\le\rho\le 5,\,0\le\phi\le\frac{\pi}{2},\,\frac{3\pi}{2}\le\theta\le 2\pi\right\}$

This makes our triple integral into:

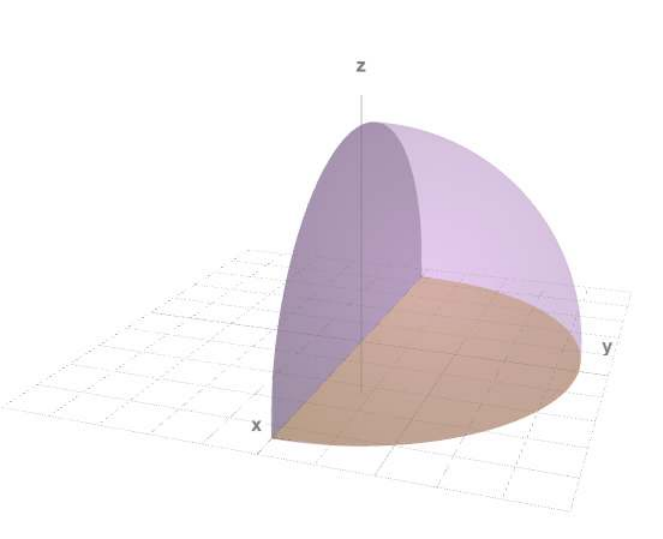
$$\begin{aligned}
\int_{-5}^{0}\int_{0}^{\sqrt{25-y^2}}\int_{0}^{\sqrt{25-y^2-z^2}} ze^{-z^2}\,dx\,dz\,dy &= \iiint_D ze^{-z^2}\,dV \\
&= \int_0^5\int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\frac{\pi}{2}} \rho\cos(\phi)e^{-(\rho\cos(\phi))^2}\cdot\rho^2\sin(\phi)\,d\phi\,d\theta\,d\rho \\
&= \int_0^5\int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\frac{\pi}{2}} \rho^3\sin(\phi)\cos(\phi)e^{-\rho^2\cos^2(\phi)}\,d\phi\,d\theta\,d\rho \\
&= \int_0^5\int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2}\rho e^{-\rho^2\cos^2(\phi)}\Big|_0^{\frac{\pi}{2}}\,d\theta\,d\rho \\
&= \frac{1}{2}\int_0^5\int_{\frac{3\pi}{2}}^{2\pi} \rho\left(1-e^{-\rho^2}\right)\,d\theta\,d\rho \\
&= \frac{1}{2}\left(\int_0^5\left(\rho-\rho e^{-\rho^2}\right)\,d\rho\right)\left(\int_{\frac{3\pi}{2}}^{2\pi}d\theta\right) \\
&= \frac{1}{2}\left[\frac{1}{2}\rho^2+\frac{1}{2}e^{-\rho^2}\right]\Big|_0^5\cdot\left(\theta\Big|_{\frac{3\pi}{2}}^{2\pi}\right) \\
&= \frac{1}{2}\left(\frac{25}{2}+\frac{1}{2}e^{-25}-\frac{1}{2}\right)\cdot\frac{\pi}{2} \\
&= 3\pi+\frac{\pi}{8e^{25}}
\end{aligned}$$

? Exercise 3.4.3f

Use spherical coordinates to evaluate $\int_{-8}^{8}\int_{0}^{\sqrt{64-x^2}}\int_{0}^{\sqrt{64-x^2-y^2}} (3+5\sqrt[3]{z})\,dz\,dy\,dx$

Answer

We see the domain $D=\left\{(x,y,z)\,:\,-8\le x\le 8,\,0\le y\le\sqrt{64-x^2},\,0\le z\le\sqrt{64-x^2-y^2}\right\}$ is part of a sphere:



Since $y\ge 0$ and $-8\le x\le 8$ we know that $0\le\theta\le\pi$

Thus, the spherical form is $D=\left\{(\rho\sin(\phi)\cos(\theta),\rho\sin(\phi)\sin(\theta),\rho\cos(\phi))\,:\,0\le\rho\le 8,\,0\le\phi\le\frac{\pi}{2},\,0\le\theta\le\pi\right\}$

This makes our triple integral into:

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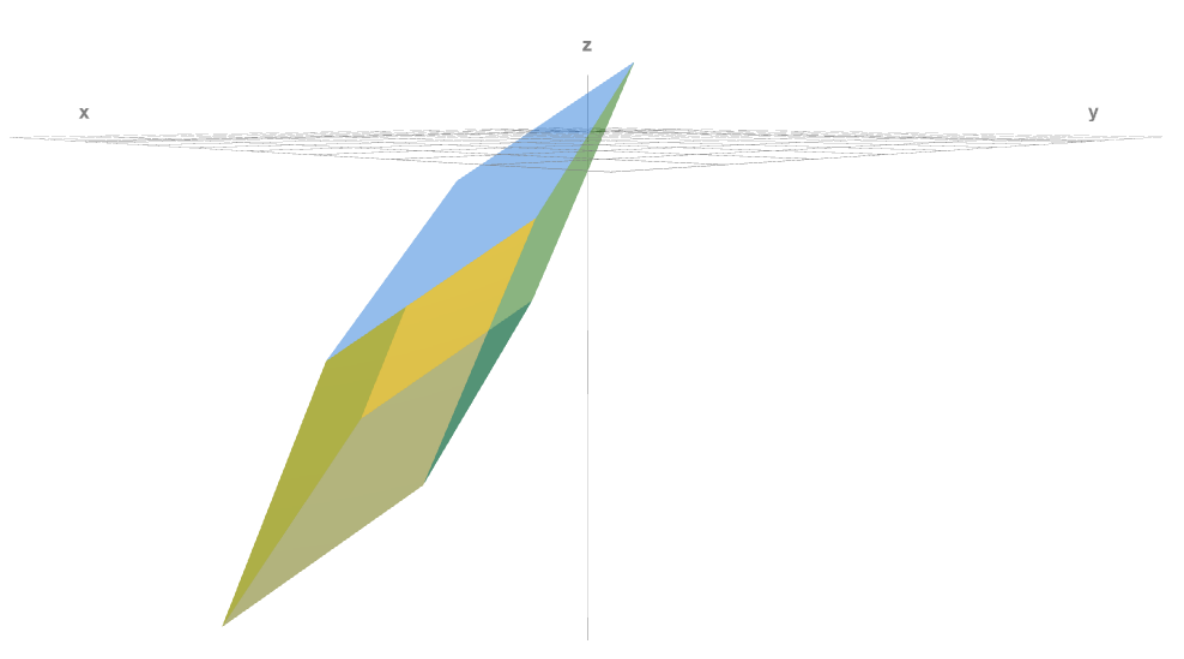
$$\begin{aligned}
\int_{-8}^{8}\int_{0}^{\sqrt{64-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}}(3+5\sqrt[3]{z})\,dz\,dy\,dx &= \iiint_D (3+5\sqrt[3]{z})\,dV \\
&= \int_0^8\int_0^{\frac{\pi}{2}}\int_0^{\pi}\left(3+5\sqrt[3]{\rho\cos(\phi)}\right)\cdot\rho^2\sin(\phi)\,d\theta\,d\phi\,d\rho \\
&= \int_0^8\int_0^{\frac{\pi}{2}}\int_0^{\pi}\left(3\rho^2\sin(\phi)+5\rho^{\frac{7}{3}}2\sin(\phi)\cdot\sqrt[3]{\cos(\phi)}\right)\,d\theta\,d\phi\,d\rho \\
&= \int_0^8\int_0^{\frac{\pi}{2}}\left(3\rho^2\sin(\phi)+5\rho^{\frac{7}{3}}\sin(\phi)\cdot\sqrt[3]{\cos(\phi)}\right)\theta\Big|_0^{\pi}\,d\phi\,d\rho \\
&= \int_0^8\int_0^{\frac{\pi}{2}}\left(3\rho^2\sin(\phi)+5\rho^{\frac{7}{3}}\sin(\phi)\cdot\sqrt[3]{\cos(\phi)}\right)\pi\,d\phi\,d\rho \\
&= \pi\int_0^8\left[-3\rho^2\cos(\phi)-\frac{15}{4}\rho^{\frac{7}{3}}(\cos(\phi))^{\frac{4}{3}}\right]\Big|_0^{\frac{\pi}{2}}\,d\rho \\
&= \pi\int_0^8\left(3\rho^2+\frac{15}{4}\rho^{\frac{7}{3}}\right)\,d\rho \\
&= \pi\left[\rho^3+\frac{9}{8}\rho^{\frac{10}{3}}\right]\Big|_0^8 \\
&= \pi\left(2^9+9\cdot 2^7\right) \\
&= 1664\pi
\end{aligned}$$

Exercise 3.4.4a

Find the volume of $E=\{(x,y,z)\,:\,-1\le 2x-3y+z\le 4,\ 1\le x+2y\le 5,\ -2\le x+z\le 1\}$

Answer

The solid E is bounded by three pairs of parallel planes, so it is a parallelepiped:



While we can visualize this region fairly simply, it is not easy to describe it when setting up a triple integral.

Instead we substitute $u=2x-3y+z$, $v=x+2y$, $w=x+z$ so that our domain becomes $G=\{(u,v,w)\,:\,-1\le u\le 4,\ 1\le v\le 5,\ -2\le w\le 1\}$

We can easily solve for x,y,z to get: $x=\frac{2}{5}u+\frac{3}{5}v-\frac{2}{5}w$, $y=-\frac{1}{5}u+\frac{1}{5}v+\frac{1}{5}w$, $z=-\frac{2}{5}u-\frac{3}{5}v+\frac{7}{5}w$

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We see $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & \frac{7}{5} \end{vmatrix} = \frac{1}{5}$

Therefore the triple integral becomes:

$$\begin{aligned} \text{Volume}(E) &= \iiint_E 1 \, dV \\ &= \iiint_G 1 \cdot |J(u,v,w)| \, dV \\ &= \int_{-1}^{4} \int_{1}^{5} \int_{-2}^{1} 1 \cdot \left|\frac{1}{5}\right| \, dw \, dv \, du \\ &= \frac{1}{5} \left(\int_{-1}^{4} du \right) \left(\int_{1}^{5} dv \right) \left(\int_{-2}^{1} dw \right) \\ &= \frac{1}{5} \left(u\Big|_{-1}^{4} \right) \cdot \left(v\Big|_{1}^{5} \right) \cdot \left(w\Big|_{-2}^{1} \right) \\ &= \frac{1}{5} \cdot 5 \cdot 4 \cdot 3 \\ &= 12 \end{aligned}$$

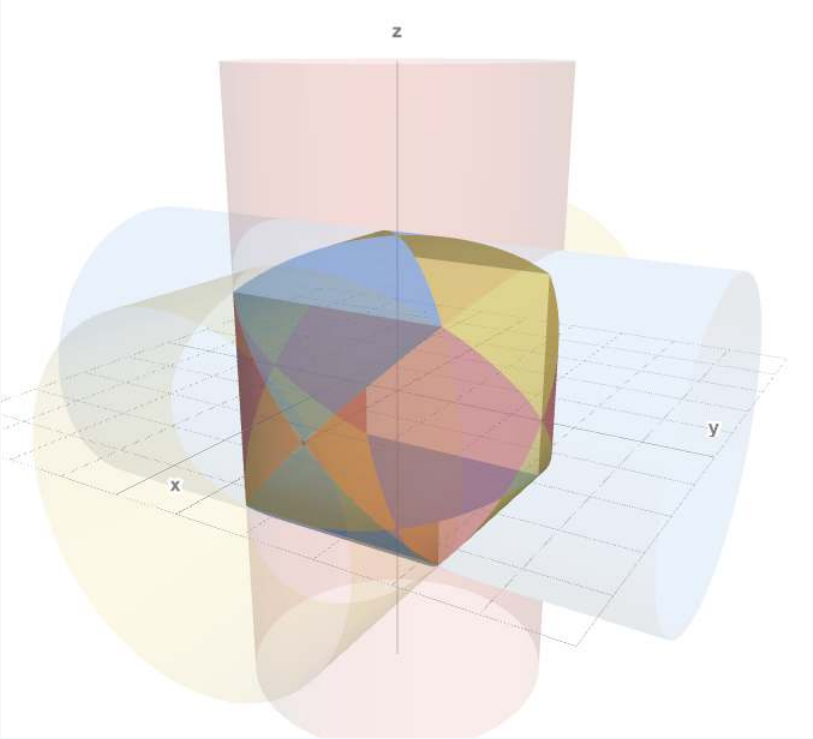
? Exercise 3.4.4*b*

Find the volume of the region $E = \{(x,y,z) \, : \, x^2 + y^2 \leq 1, \, x^2 + z^2 \leq 1, \, y^2 + z^2 \leq 1\}$

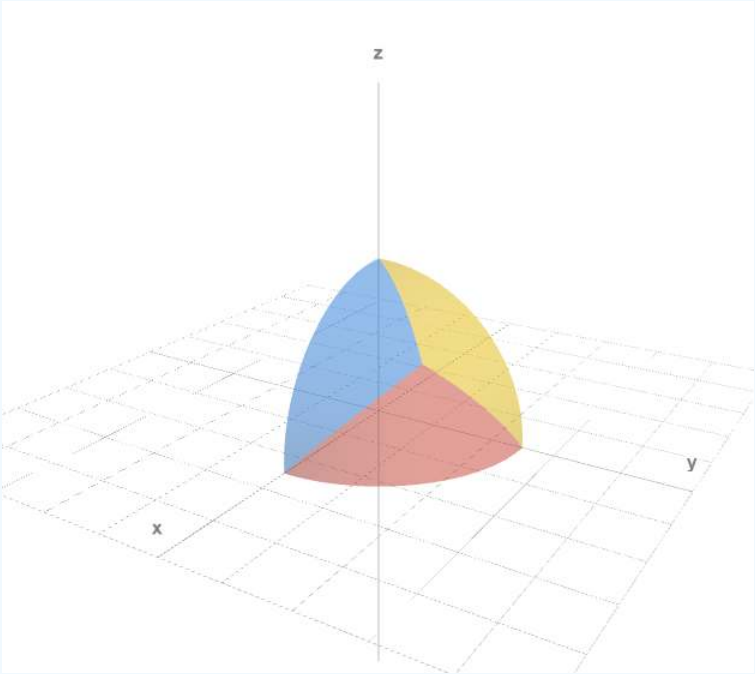
(Hint: Use symmetry and $E^+ = \{(x,y,z) \, : \, x^2 + y^2 \leq 1, \, x^2 + z^2 \leq 1, \, y^2 + z^2 \leq 1, \, x \geq 0, \, y \geq 0, \, z \geq 0\}$)

Answer

This shape is hard to visualize, since it is the intersection of three perpendicular cylinders:



The region E^+ defined in the hint is simply E in the first octant, which looks like:



We can write $E^+ = \left\{(x,y,z) \,:\, 0 \le x \le 1,\, 0 \le y \le \sqrt{1-x^2},\, 0 \le z \le \min\left(\sqrt{1-x^2}, \sqrt{1-y^2}\right)\right\}$

In cylindrical coordinates this becomes $E^+ = \left\{(r\cos(\theta), r\sin(\theta), z) \,:\, 0 \le r \le 1,\, 0 \le \theta \le \frac{\pi}{2},\, 0 \le z \le \min\left(\sqrt{1-(r\cos(\theta))^2}, \sqrt{1-(r\sin(\theta))^2}\right)\right\}$

So the volume of E^+ is given by the triple integral:

$$
\begin{aligned}
\text{Volume}\,(E^+) &= \iiint_{E^+} 1\, dV \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\min\left(\sqrt{1-r^2\cos^2(\theta)}, \sqrt{1-r^2\sin^2(\theta)}\right)} 1 \cdot r\, dz\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 rz \Big|_0^{\min\left(\sqrt{1-r^2\cos^2(\theta)}, \sqrt{1-r^2\sin^2(\theta)}\right)} dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r \cdot \min\left(\sqrt{1-r^2\cos^2(\theta)}, \sqrt{1-r^2\sin^2(\theta)}\right) dr\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^1 r\sqrt{1-r^2\cos^2(\theta)}\, dr\, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r\sqrt{1-r^2\sin^2(\theta)}\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{4}} -\frac{1}{3\cos^2(\theta)}\left(1-r^2\cos^2(\theta)\right)^{\frac{3}{2}} \Big|_0^1 d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{1}{3\sin^2(\theta)}\left(1-r^2\sin^2(\theta)\right)^{\frac{3}{2}} \Big|_0^1 d\theta \\
&= \int_0^{\frac{\pi}{4}} -\frac{1}{3\cos^2(\theta)}\left(\left(1-\cos^2(\theta)\right)^{\frac{3}{2}} - 1\right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{1}{3\sin^2(\theta)}\left(\left(1-\sin^2(\theta)\right)^{\frac{3}{2}} - 1\right) d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{4}} \sec^2(\theta)\left(1-\sin^3(\theta)\right) d\theta + \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2(\theta)\left(1-\cos^3(\theta)\right) d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{4}} \left(\sec^2(\theta) - \sec(\theta)\tan(\theta) + \sin(\theta)\right) d\theta + \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\csc^2(\theta) - \csc(\theta)\cot(\theta) + \cos(\theta)\right) d\theta \\
&= \frac{1}{3}\left[\tan(\theta) - \sec(\theta) - \cos(\theta)\right]\Big|_0^{\frac{\pi}{4}} + \frac{1}{3}\left[-\cot(\theta) + \csc(\theta) + \sin(\theta)\right]\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{1}{3}\left(3 - \frac{3}{2}\sqrt{2}\right) + \frac{1}{3}\left(3 - \frac{3}{2}\sqrt{2}\right) \\
&= 2 - \sqrt{2}
\end{aligned}
$$

Therefore the full volume of E is $8 \cdot \text{Volume}\,(E^+) = 8 \cdot (2 - \sqrt{2}) = 16 - 8\sqrt{2}$

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3.5: Integrals on a Line

Learning Objectives

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)\,dx$, where the domain of integration is an interval $[a, b]$. Such an interval can be thought of as a curve in the xy-plane, since the interval defines a line segment with endpoints $(a, 0)$ and $(b, 0)$—in other words, a line segment located on the x-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the x-axis. Such a task requires a new kind of integral, called a *line integral.*

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3.5: Line Integrals

A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the x-axis.

For a scalar line integral, we let C be a smooth curve in a plane or in space and let f be a function with a domain that includes C. We chop the curve into small pieces. For each piece, we choose point P in that piece and evaluate f at P. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x, y, z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ (Figure 3.5.1). Let t_i^* be a value in the ith interval $[t_{i-1}, t_i]$. Denote the endpoints of $\vec{\mathbf{r}}(t_0)$, $\vec{\mathbf{r}}(t_1)$,…, $\vec{\mathbf{r}}(t_n)$ by P_0,…, P_n. Points P_i divide curve C into n pieces C_1, C_2,…, C_n, with lengths Δs_1, Δs_2,…, Δs_n, respectively. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_i, and therefore P_i^* is in the domain of f. Multiply $f(P_i^*)$ by the length Δs_i of C_i, which gives the area of the "sheet" with base C_i, and height $f(P_i^*)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$.

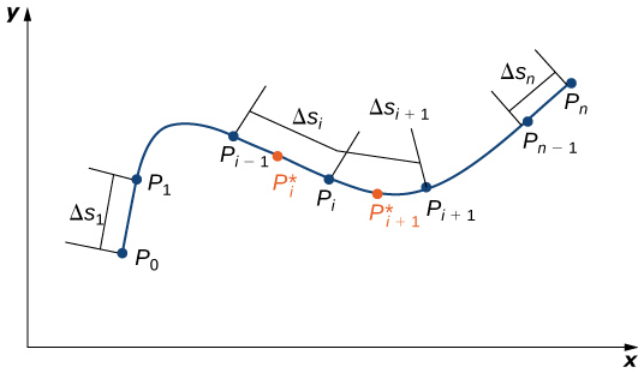


Figure 3.5.1: Curve C has been divided into n pieces, and a point inside each piece has been chosen.

Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)\,dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths Δs_1, Δs_2,…, Δs_n aren't necessarily the same; in the definition of a single-variable integral, the curve on the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

Definition: Scalar Line Integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. The ***scalar line integral*** of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x,y)$ is a function of two variables, then the ***scalar line integral*** of f along C is defined similarly as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\,ds$ always exists. Since $\int_C f\,ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)\,dx$ exists if g is continuous over $[a,b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x,y) \ge 0$ for all points (x,y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x,y)$, thereby creating a new curve C' that lies in the graph of $f(x,y)$. Now we drop a "sheet" from C' down to the xy-plane. The area of this sheet is $\int_C f(x,y)ds$. If $f(x,y) \le 0$ for some points in C, then the value of $\int_C f(x,y)\,ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)\,dx$.)

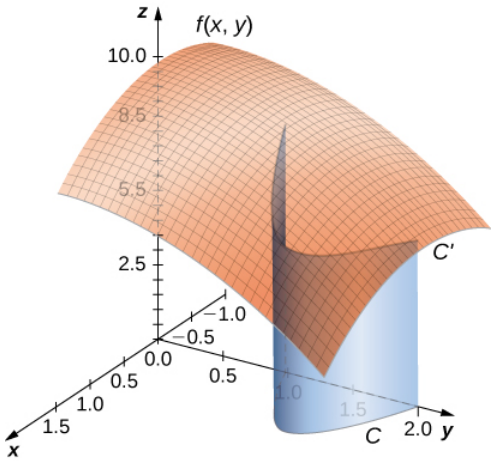


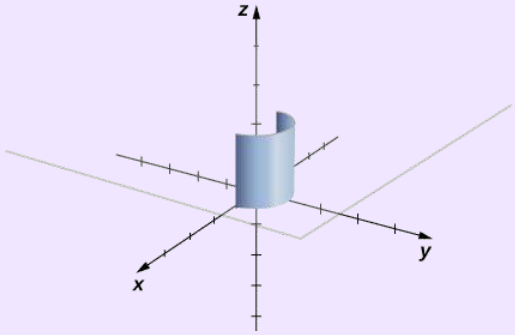
Figure 3.5.2: The area of the blue sheet is $\int_C f(x,y)\,ds$

From this geometry, we can see that line integral $\int_C f(x,y)\,ds$ does not depend on the parameterization $\vec{\mathbf{r}}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

Illustration 3.5.1: Finding the Value of a Line Integral

Consider $\int_C 2\,ds$, where C is the upper half of the unit circle.

The integrand is $f(x,y) = 2$. The graph of $f(x,y) = 2$, curve C, and the sheet formed by them are provided below. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2\,ds = 2\pi$ units^2



To see that $\int_C 2\,ds = 2\pi$ using the definition of line integral, we let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Then, $f(\vec{\mathbf{r}}(t_i)) = 2$ for any number t_i in the domain of $\vec{\mathbf{r}}$. Therefore,

$$\int_C f\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i = \lim_{n\to\infty}\sum_{i=1}^{n} 2\,\Delta s_i = 2\lim_{n\to\infty}\sum_{i=1}^{n}\Delta s_i = 2(\text{length of } C) = 2\pi\,\text{units}^2$$

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f\,ds$ to an integral with a variable of integration that is t.

Let $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$ be a parameterization of C. Since we are assuming that C is smooth, $\vec{\mathbf{r}}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle$ is continuous for all t in $[a, b]$. In particular, $x'(t)$, $y'(t)$, and $z'(t)$ exist for all t in $[a, b]$. According to the arc length formula, we have

$$\text{length}(C_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} \|\vec{\mathbf{r}}\,'(t)\|\,dt$$

If width $\Delta t_i = t_i - t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i} \|\vec{\mathbf{r}}\,'(t)\|\,dt \approx \|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$, $\|\vec{\mathbf{r}}\,'(t)\|$ is almost constant over the interval $[t_{i-1}, t_i]$. Therefore,

$$\int_{t_{i-1}}^{t_i} \|\vec{\mathbf{r}}\,'(t)\|\,dt \approx \|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i, \text{ and therefore } \sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i \approx \sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$$

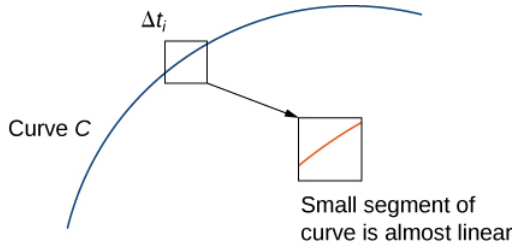


Figure 3.5.3: If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i = \int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt$$

In other words, as the widths of intervals $[t_{i-1}, t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$ converges to the integral $\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt$.

Theorem 3.5.1: Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t)$, for $a \leq t \leq b$. Then

$$\int_C f\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}'(t)\|\,dt$$

Since $\|\vec{\mathbf{r}}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ we obtain the following theorem, which we use to compute scalar line integrals.

Theorem 3.5.2: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

Similarly,

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\vec{\mathbf{r}}'(t)\|\,dt$. In other words, the change in arc length can be viewed as a change in the t-domain, scaled by the magnitude of vector $\vec{\mathbf{r}}'(t)$.

Example 3.5.1: Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\,ds$, where C is part of the helix parameterized by

(a) $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t\rangle$ for $0 \leq t \leq 2\pi$

(b) $\vec{\mathbf{r}}(t) = \langle \cos(2t), \sin(2t), 2t\rangle$ for $0 \leq t \leq \pi$

Solution

(a) To compute a scalar line integral, we start by converting the variable of integration from arc length ds to dt. Note that

$$f(x,y,z) = x^2 + y^2 + z \implies f(\cos(t), \sin(t), t) = \cos^2(t) + \sin^2(t) + t = 1 + t$$

and

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}$$

Therefore,

$$\int_C (x^2 + y^2 + z)\,ds = \int_0^{2\pi} (1+t)\sqrt{2}\,dt = \left[t\sqrt{2} + \frac{t^2\sqrt{2}}{2}\right]\Bigg|_0^{2\pi} = \boxed{2\pi\sqrt{2} + 2\pi^2\sqrt{2}}$$

(b) We repeat with our new parameterization:

$$f(x,y,z) = x^2 + y^2 + z \implies f(\cos(2t), \sin(2t), 2t) = \cos^2(2t) + \sin^2(2t) + 2t = 1 + 2t$$

and

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2 + (2)^2} = \sqrt{8}$$

Therefore,

$$\int_C (x^2 + y^2 + z)\,ds = \int_0^{\pi} (1+2t)\sqrt{8}\,dt = \left[t\sqrt{8} + t^2\sqrt{8}\right]\Big|_0^{\pi} = \pi\sqrt{8} + \pi^2\sqrt{8} = \boxed{2\pi\sqrt{2} + 2\pi^2\sqrt{2}}$$

Notice that these solutions agree. Changing the parameterization did not change the value of the line integral. **Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.**

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x,y,z)=1$, then

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\,\Delta s_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta s_i \\ &= \lim_{n\to\infty}\text{length}(C) \\ &= \text{length}(C)\end{aligned}$$

Therefore, $\int_C 1\,ds$ is the arc length of C.

Example 3.5.2: Calculating Arc Length

Find the length of the wire that can be modeled with the parameterization $\vec{\mathbf{r}}(t)=\left\langle \cos(t),\sin(t),\frac{2}{3}t^{\frac{3}{2}}\right\rangle$, for $0\le t\le 4\pi$

Solution

The length of the wire is given by $\int_C 1\,ds$, where C is the curve with parameterization $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}\text{Length}=\int_C 1\,ds &= \int_0^{4\pi}||\vec{\mathbf{r}}'(t)||\,dt = \int_0^{4\pi}\sqrt{(-\sin(t))^2+(\cos(t))^2+t}\,dt = \int_0^{4\pi}\sqrt{1+t}\,dt \\ &= \frac{2(1+t)^{\frac{3}{2}}}{3}\Bigg|_0^{4\pi} = \boxed{\frac{2}{3}\left((1+4\pi)^{\frac{3}{2}}-1\right)}\end{aligned}$$

A second type of line integrals is *vector line integrals*, in which we integrate along a curve through a vector field. For example, let

$$\vec{\mathbf{F}}(x,y,z)=P(x,y,z)\,\hat{\mathbf{i}}+Q(x,y,z)\,\hat{\mathbf{j}}+R(x,y,z)\,\hat{\mathbf{k}}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\vec{\mathbf{F}}$. How would we compute the work done by $\vec{\mathbf{F}}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an orientation of a curve. The specified direction is called the "positive" direction along C; the opposite direction is the "negative" direction along C. When C has been given an orientation, C is called an oriented curve. The work done on the particle depends on the direction along the curve in which the particle is moving.

A closed curve is one for which there exists a parameterization $\vec{\mathbf{r}}(t)$, $a\le t\le b$, such that $\vec{\mathbf{r}}(a)=\vec{\mathbf{r}}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a,b).

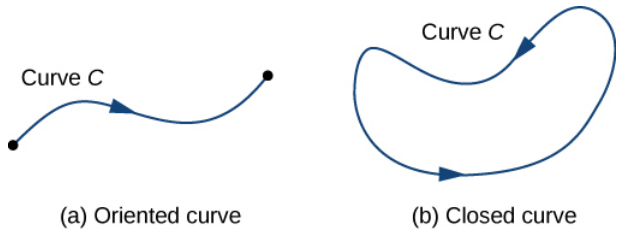


Figure 3.5.4: (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\vec{\mathbf{r}}(t)$ be a parameterization of C for $a\le t\le b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1},t_i]$, $0\le i\le n$, of equal width. Denote the endpoints of $r(t_0)$, $r(t_1)$,…, $r(t_n)$ by P_0,…,P_n. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1},t_i]$. Then, the endpoint of $\vec{\mathbf{r}}(t_i^*)$ is a point in the piece of C between P_{i-1}

and P_i. If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\vec{\mathbf{T}}(P_i^*) = \dfrac{\vec{\mathbf{r}}'(P_i^*)}{\|\vec{\mathbf{r}}'(P_i^*)\|}$, the unit tangent vector at the endpoint of $\vec{\mathbf{r}}(P_i^*)$.

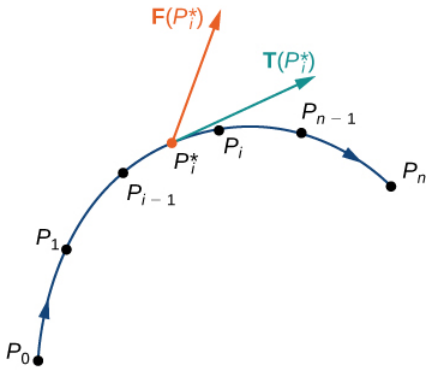


Figure 3.5.5: Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\vec{\mathbf{F}}$ is approximated by $\vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)$.

Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\left\|\text{proj}_{\vec{\mathbf{T}}(P_i^*)}\left(\vec{\mathbf{F}}(P_i^*)\right)\right\| \Delta s_i = \vec{\mathbf{F}}(P_i^*) \cdot (\vec{\mathbf{T}}(P_i^*))\Delta s_i$, so the total work done along C is

$$\sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot (\Delta s_i \vec{\mathbf{T}}(P_i^*)) = \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)\, \Delta s_i$$

Definition: Line Integral of a Vector Field

Let $\vec{\mathbf{F}}$ be a vector field with a domain that includes the smooth curve C that is parameterized by $\vec{\mathbf{r}}(t)$, $a \le t \le b$. The ***line integral*** of $\vec{\mathbf{F}}$ along C is

$$\int_C \vec{\mathbf{F}}(x,y,z) \cdot d\vec{\mathbf{r}} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)\, \Delta s_i$$

if this limit exists.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\vec{\mathbf{F}}$ in moving the particle in the positive direction along C is defined as

$$W = \int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{r}}'}{\|\vec{\mathbf{r}}'\|}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt$$

which gives us the concept of a vector line integral.

Thereom 3.5.3: Evaluating a Line Integral of a Vector Field

The vector line integral of vector field $\vec{\mathbf{F}}$ along oriented smooth curve C parameterized by $\vec{\mathbf{r}}(t)$ on $a \le t \le b$ is

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt$$

Illustration 3.5.2: Evaluating a Vector Line Integral

Let $\vec{\mathbf{F}} = \langle -y, x\rangle$ and find the value of integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is

(a) the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t)\rangle$, for $0 \le t \le \pi$

(b) the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle -\cos(t), \sin(t)\rangle$, for $0 \le t \le \pi$

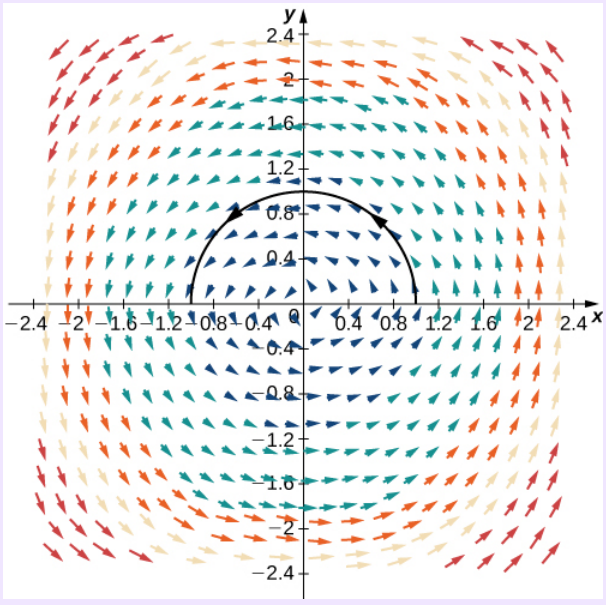
For the curve in (a), we are given $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t)\rangle$ so differentiating each component gives $\vec{\mathbf{r}}'(t) = \langle -\sin(t), \cos(t)\rangle$

Replacing $x = \cos(t)$ and $y = \sin(t)$ into $\vec{\mathbf{F}}(x, y)$ we get $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin(t), \cos(t) \rangle$

Therefore,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^\pi \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt = \int_0^\pi \sin^2(t) + \cos^2(t) \, dt = \int_0^\pi 1 \, dt = \pi$$

If we look at a sketch of this curve over the vector field $\vec{\mathbf{F}}$, we can see that indeed these force arrows push in the same direction as the oriented curve C. Thus, the "current" $\vec{\mathbf{F}}$ "helps" a person swimming along this path by a positive amount:



In (b), we are given $\vec{\mathbf{r}}(t) = \langle -\cos(t), \sin(t) \rangle$ so differentiating each component gives $\vec{\mathbf{r}}\,'(t) = \langle \sin(t), \cos(t) \rangle$

Replacing $x = -\cos(t)$ and $y = \sin(t)$ into $\vec{\mathbf{F}}(x, y)$ we get $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin(t), -\cos(t) \rangle$

Therefore,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin(t), -\cos(t) \rangle \cdot \langle \sin(t), \cos(t) \rangle \, dt \\ &= \int_0^\pi -\sin^2(t) - \cos^2(t) \, dt = \int_0^\pi -1 \, dt = -\pi \end{aligned}$$

Notice this is the same curve as part (a), just traversed in the opposite direction. In this case, the swimmer is fighting *against* the "current" of $\vec{\mathbf{F}}$, so they are "helped" by a negative amount.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

That is, reversing the orientation of a curve changes the sign of a line integral.

Another standard notation for integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is $\int_C P \, dx + Q \, dy + R \, dz$. In this notation, P, Q, and R are functions, and we think of $d\vec{\mathbf{r}}$ as vector $\langle dx, dy, dz \rangle$. To justify this convention, recall that $d\vec{\mathbf{r}} = \vec{\mathbf{r}}\,'(t) \, dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt = \langle dx, dy, dz \rangle$. Therefore,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = \int_C P \, dx + Q \, dy + R \, dz \\ &= \int_a^b \left(P\big(\vec{\mathbf{r}}(t)\big) \frac{dx}{dt} + Q\big(\vec{\mathbf{r}}(t)\big) \frac{dy}{dt} + R\big(\vec{\mathbf{r}}(t)\big) \frac{dz}{dt} \right) dt \end{aligned}$$

Example 3.5.3: Finding the Value of an Integral of the Form $\int_C P\,dx + Q\,dy + R\,dz$

Find the value of integral $\int_C (3z\,dx - 5x\,dy + 3y\,dz)$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t}, t\rangle$, for $1 \le t \le 4$.

Solution

Since $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t}, t\rangle$, we have $\vec{\mathbf{r}}'(t) = \left\langle 2t, \frac{1}{2\sqrt{t}}, 1\right\rangle$

We use the equation derived above to get:

$$\begin{aligned}\int_C 3z\,dx - 5x\,dy + 3y\,dz &= \int_1^4 \left(3t\cdot 2t - 5t^2\cdot\frac{1}{2\sqrt{t}} + 3\sqrt{t}\cdot 1\right)dt \\ &= \int_1^4 \left(6t^2 - \frac{5t^{\frac{3}{2}}}{2} + 3\sqrt{t}\right)dt \\ &= \left[2t^3 - t^{\frac{5}{2}} + 2t^{\frac{3}{2}}\right]\Big|_1^4 \\ &= \boxed{109}\end{aligned}$$

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a piecewise smooth curve. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,…, C_n such that the endpoint of C_i is the starting point of C_{i+1}. When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.

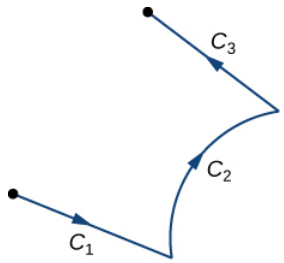


Figure 3.5.6: The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Theorem 3.5.4: Properties of Vector Line Integrals

Let $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\int_C (\vec{\mathbf{F}} + \vec{\mathbf{G}})\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \int_C \vec{\mathbf{G}}\cdot d\vec{\mathbf{r}}$
2. $\int_C k\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = k\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where k is a constant
3. $\int_{-C} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$
4. Suppose instead that C is a piecewise smooth curve in the domains of $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \cdots + \int_{C_n} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$$

Notice the similarities between these items and the properties of single-variable integrals. Properties 1 and 2 say that line integrals are linear, which is true of single-variable integrals as well. Property 3 says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If

the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,…, $[a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \int_{a_2}^{a_3} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx,$$

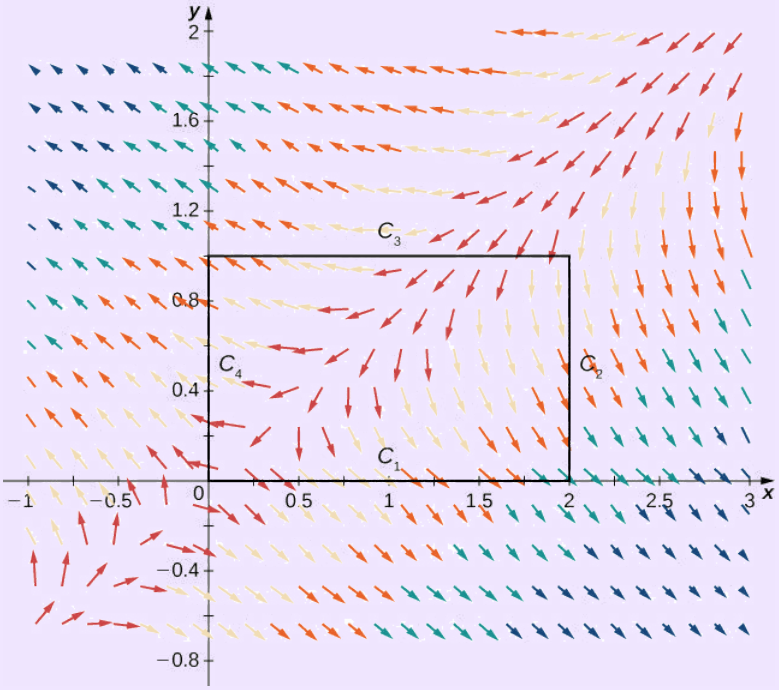
which is analogous to Property 4.

Illustration 3.5.3: Using Properties to Compute a Vector Line Integral

We will find $\int_C (x - 2y)\,dx + (y - x)\,dy$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0, 0)$ to $(2, 0)$, let C_2 represent the side from $(2, 0)$ to $(2, 1)$, let C_3 represent the side from $(2, 1)$ to $(0, 1)$, and let C_4 represent the side from $(0, 1)$ to $(0, 0)$.

We graph $\vec{\mathbf{F}} = \langle x - 2y, y - x \rangle$ and C:



$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

We want to compute each of the four integrals on the right-hand side separately, so we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$\begin{aligned}
C_1 &: \vec{\mathbf{r}}_1(t) = \langle t, 0 \rangle, \quad \text{for} \quad 0 \le t \le 2 \\
C_2 &: \vec{\mathbf{r}}_2(t) = \langle 2, t \rangle, \quad \text{for} \quad 0 \le t \le 1 \\
C_3 &: \vec{\mathbf{r}}_3(t) = \langle 2 - t, 1 \rangle, \quad \text{for} \quad 0 \le t \le 2 \\
C_4 &: \vec{\mathbf{r}}_4(t) = \langle 0, 1 - t \rangle, \quad \text{for} \quad 0 \le t \le 1
\end{aligned}$$

Therefore,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^2 \langle t - 2(0), 0 - t \rangle \cdot \langle 1, 0 \rangle \, dt = \int_0^2 t \, dt = \left[\frac{1}{2} t^2 \right]_0^2 = 2$$

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle 2 - 2t, t - 2 \rangle \cdot \langle 0, 1 \rangle \, dt = \int_0^1 (t - 2) \, dt = \left[\frac{1}{2} t^2 - 2t \right]_0^1 = -\frac{3}{2}$$

$$\int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^2 \langle (2 - t) - 2, 1 - (2 - t) \rangle \cdot \langle -1, 0 \rangle \, dt = \int_0^2 t \, dt = \left[\frac{1}{2} t^2 \right]_0^2 = 2$$

$$\int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle -2(1 - t), 1 - t \rangle \cdot \langle 0, -1 \rangle \, dt = \int_0^1 (t - 1) \, dt = \left[\frac{1}{2} t^2 - t \right]_0^1 = -\frac{1}{2}$$

Thus, we have $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2 + \left(-\frac{3}{2} \right) + 2 + \left(-\frac{1}{2} \right) = 2$

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3.5: Practice Set

Exercise 3.5.1a

Evaluate the line integral $\int_C 4z\,ds$ where C is the straight line from $(1,0,1)$ to $(0,-2,-1)$

Answer

We can parameterize this straight line with $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle = \langle 1-t, -2t, 1-2t\rangle$ for $0 \le t \le 1$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle -1, -2, -2\rangle$

This derivative vector has length $\|\vec{\mathbf{r}}'(t)\| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = 3$

Therefore the line integral becomes:

$$\begin{aligned}\int_C 4z\,ds &= \int_0^1 4z(t)\,\|\vec{\mathbf{r}}'(t)\|\,dt \\ &= \int_0^1 4(1-2t)\cdot 3\,dt \\ &= 12\int_0^1 (1-2t)\,dt \\ &= 12\left[t - t^2\right]\Big|_0^1 \\ &= 12\cdot 0 \\ &= 0\end{aligned}$$

Exercise 3.5.1b

Evaluate the line integral $\int_C e^z\,ds$ where C is given by $\vec{\mathbf{r}}(t) = \langle t\cos(t) - \sin(t), t\sin(t) + \cos(t), t^2\rangle$ over $0 \le t \le 2\pi$

Answer

We are given $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle = \langle t\cos(t) - \sin(t), t\sin(t) + \cos(t), t^2\rangle$ for $0 \le t \le 2\pi$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle -t\sin(t), t\cos(t), 2t\rangle$

This derivative vector has length $\|\vec{\mathbf{r}}'(t)\| = \sqrt{(-t\sin(t))^2 + (t\cos(t))^2 + (2t)^2} = \sqrt{5t^2} = t\sqrt{5}$

Therefore the line integral becomes:

$$\begin{aligned}\int_C e^z\,ds &= \int_0^{2\pi} e^{z(t)}\,\|\vec{\mathbf{r}}'(t)\|\,dt \\ &= \int_0^{2\pi} e^{t^2}\cdot t\sqrt{5}\,dt \\ &= \sqrt{5}\int_0^{2\pi} te^{t^2}\,dt \\ &= \sqrt{5}\cdot\frac{1}{2}e^{t^2}\Big|_0^{2\pi} \\ &= \sqrt{5}\cdot\frac{e^{4\pi^2}-1}{2} \\ &= \frac{(e^{4\pi^2}-1)\sqrt{5}}{2}\end{aligned}$$

Exercise 3.5.1c

Evaluate the line integral $\int_C |x|\,ds$ where C is the curve connecting $(1,2,2)$ to $(-1,-2,-2)$ to $(2,2,-2)$ using straight lines

Answer

Let C_1 be the line connecting $(1,2,2)$ to $(-1,-2,-2)$, and C_2 the line connecting $(-1,-2,-2)$ to $(2,2,-2)$

We can parameterize C_1 with $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle = \langle 1-2t, 2-4t, 2-4t\rangle$ for $0 \le t \le 1$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle -2, -4, -4\rangle$

This derivative vector has length $\|\vec{\mathbf{r}}'(t)\| = \sqrt{(-2)^2+(-4)^2+(-4)^2} = 6$

Therefore the line integral over C_1 becomes:

$$\begin{aligned}\int_{C_1} |x|\,ds &= \int_0^1 |x(t)|\,\|\vec{\mathbf{r}}'(t)\|\,dt \\ &= \int_0^1 |1-2t|\cdot 6\,dt \\ &= 6\left(\int_0^{\frac{1}{2}} (1-2t)\,dt + \int_{\frac{1}{2}}^1 (2t-1)\,dt\right) \\ &= 6\left(\left[t-t^2\right]\Big|_0^{\frac{1}{2}} + \left[t^2-t\right]\Big|_{\frac{1}{2}}^1\right) \\ &= 6\left(\frac{1}{4}+\frac{1}{4}\right) \\ &= 3\end{aligned}$$

We can parameterize C_2 with $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle = \langle -1+3t, -2+4t, -2\rangle$ for $0 \le t \le 1$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle 3, 4, 0\rangle$

Which has length $\|\vec{\mathbf{r}}'(t)\| = \sqrt{3^2+4^2+0^2} = 5$

Therefore the line integral over C_2 becomes:

$$\begin{aligned}\int_{C_2} |x|\,ds &= \int_0^1 |x(t)|\,\|\vec{\mathbf{r}}'(t)\|\,dt \\ &= \int_0^1 |-1+3t|\cdot 5\,dt \\ &= 5\left(\int_0^{\frac{1}{3}} (1-3t)\,dt + \int_{\frac{1}{3}}^1 (3t-1)\,dt\right) \\ &= 5\left(\left[t-\frac{3}{2}t^2\right]\Big|_0^{\frac{1}{3}} + \left[\frac{3}{2}t^2-t\right]\Big|_{\frac{1}{3}}^1\right) \\ &= 5\left(\frac{1}{6}+\frac{2}{3}\right) \\ &= \frac{25}{6}\end{aligned}$$

Adding these two together we get our final answer

$$\int_C |x|\,ds = \int_{C_1} |x|\,ds + \int_{C_2} |x|\,ds = 3 + \frac{25}{6} = \frac{43}{6}$$

Exercise 3.5.2a

Evaluate the line integral $\int_C x^2 y\,dy$ where C is the curve starting at $(0,1)$ and ending at $(1,0)$ clockwise around the unit circle

Answer

This curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle = \langle \sin(t), \cos(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t)\rangle = \langle \cos(t), -\sin(t)\rangle$

Therefore the line integral over C becomes:

$$\begin{aligned}\int_C x^2 y\,dy &= \int_0^{\frac{\pi}{2}} [x(t)]^2 \cdot y(t) \cdot y'(t)\,dt \\ &= \int_0^{\frac{\pi}{2}} \sin^2(t)\cos(t) \cdot (-\sin(t))\,dt \\ &= \int_0^{\frac{\pi}{2}} -\sin^3(t)\cos(t)\,dt \\ &= -\frac{1}{4}\sin^4(t)\Big|_0^{\frac{\pi}{2}} \\ &= -\frac{1}{4}\end{aligned}$$

Exercise 3.5.2b

Evaluate the line integral $\int_C x^2 y\,dy$ where C is the curve starting at $(0,1)$ and ending at $(1,0)$ clockwise around $x^4 + y^4 = 1$

Answer

This curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle = \left\langle \sqrt{\sin(t)}, \sqrt{\cos(t)}\right\rangle$ for $0 \le t \le \frac{\pi}{2}$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t)\rangle = \left\langle \frac{\cos(t)}{2\sqrt{\sin(t)}}, -\frac{\sin(t)}{2\sqrt{\cos(t)}}\right\rangle$

Therefore the line integral over C becomes:

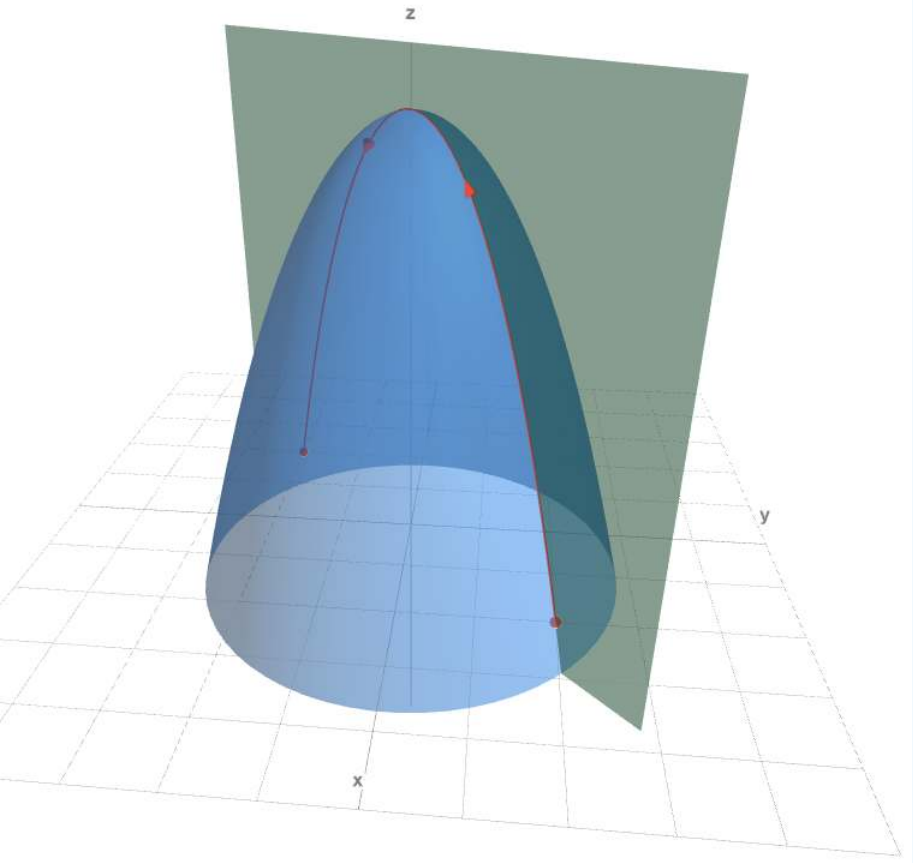
$$\begin{aligned}\int_C x^2 y\,dy &= \int_0^{\frac{\pi}{2}} [x(t)]^2 \cdot y(t) \cdot y'(t)\,dt \\ &= \int_0^{\frac{\pi}{2}} \sin(t)\sqrt{\cos(t)} \cdot \left(-\frac{\sin(t)}{2\sqrt{\cos(t)}}\right) dt \\ &= \int_0^{\frac{\pi}{2}} -\frac{1}{2}\sin^2(t)\,dt \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{4}\cos(2t) - \frac{1}{4}\right) dt \\ &= \left[\frac{1}{8}\sin(2t) - \frac{1}{4}t\right]\Big|_0^{\frac{\pi}{2}} \\ &= -\frac{\pi}{8}\end{aligned}$$

Exercise 3.5.2c

Evaluate the line integral $\int_C (x+y)\,dz$ where C connects $(2,2,0)$ to $(-2,-2,0)$ along the intersection of $z = 8 - x^2 - y^2$ and $x = y$

Answer

The curve C described in the intersection of a plane and paraboloid:



We can parameterize it using $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle = \langle -t, -t, 8 - 2t^2 \rangle$ for $-2 \leq t \leq 2$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle -1, -1, -4t \rangle$

Therefore the line integral over C becomes:

$$\begin{aligned} \int_C (x + y)\, dz &= \int_{-2}^{2} (x(t) + y(t)) \cdot z'(t)\, dt \\ &= \int_{-2}^{2} (-2t) \cdot (-4t)\, dt \\ &= \int_{-2}^{2} 8t^2\, dt \\ &= \frac{8}{3} t^3 \Big|_{-2}^{2} \\ &= \frac{128}{3} \end{aligned}$$

Exercise 3.5.3a

Evaluate the line integral $\int_C xy\, dx + y^2\, dy$ where C is the straight line from $(1, 2)$ to $(2, 5)$

Answer

We can parameterize C with $\vec{\mathbf{r}}(t) = \langle x(t), y(t) \rangle = \langle 1 + t, 2 + 3t \rangle$ for $0 \leq t \leq 1$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t) \rangle = \langle 1, 3 \rangle$

Therefore the line integral over C becomes:

$$\begin{aligned}\int_C xy\,dx + y^2\,dy &= \int_0^1 \left(x(t)\cdot y(t)\cdot x'(t) + [y(t)]^2\cdot y'(t)\right)\,dt \\ &= \int_0^1 \left((1+t)(2+3t)\cdot 1 + (2+3t)^2\cdot 3\right)\,dt \\ &= \int_0^1 (30t^2+41t+14)\,dt \\ &= \left[10t^3+\frac{41}{2}t^2+14t\right]\Big|_0^1 \\ &= \frac{89}{2}\end{aligned}$$

Exercise 3.5.3b

Evaluate the line integral $\int_C (x-y)\,dx + (2x+y)\,dy$ where C is given by $\vec{\mathbf{r}}(t) = \langle t^2-2t+3, t^2+4t-7\rangle$ over $1\le t\le 3$

Answer

We are given $\vec{\mathbf{r}}(t) = \langle t^2-2t+3, t^2+4t-7\rangle$ over $1\le t\le 3$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t)\rangle = \langle 2t-2, 2t+4\rangle$

Therefore the line integral over C becomes:

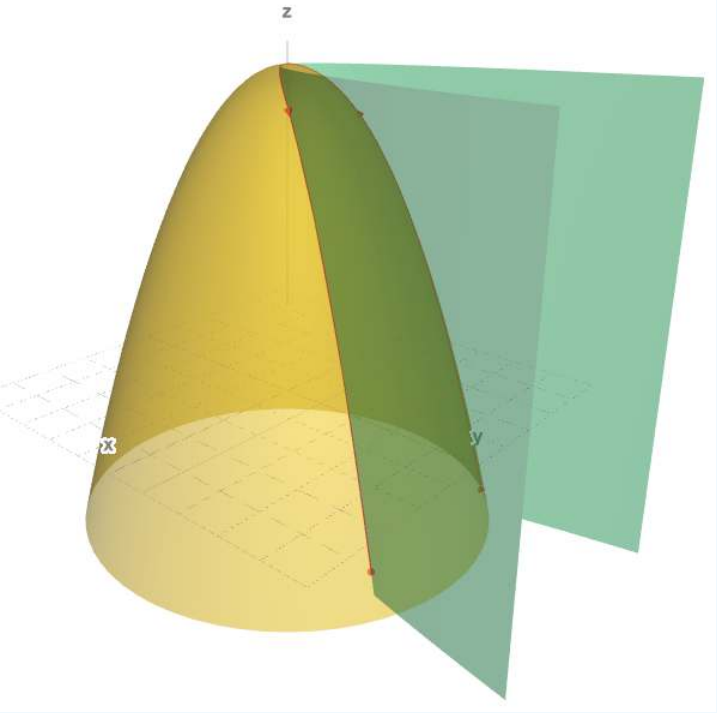
$$\begin{aligned}\int_C (x-y)\,dx + (2x+y)\,dy &= \int_1^3 \left((x(t)-y(t))\cdot x'(t) + (2x(t)+y(t))\cdot y'(t)\right)\,dt \\ &= \int_1^3 \left(((t^2-2t+3)-(t^2+4t-7))\cdot(2t-2) + (2(t^2-2t+3)+(t^2+4t-7))\cdot(2t+4)\right)\,dt \\ &= \int_1^3 (6t^3+30t-24)\,dt \\ &= \left[\frac{3}{2}t^4+15t^2-24t\right]\Big|_1^3 \\ &= 192\end{aligned}$$

Exercise 3.5.3c

Evaluate the line integral $\int_C z\,dx + x\,dz$ where C connects $(-2,4,-4)$ to $(2,4,-4)$ along the intersection of $z = 16-x^2-y^2$ and $y = x^2$

Answer

The curve C described in the intersection of a parabolic cylinder and a paraboloid:



We can parameterize it using $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle = \langle t, t^2, 16 - t^2 - t^4 \rangle$ for $-2 \leq t \leq 2$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle 1, 2t, -2t - 4t^3 \rangle$

Therefore the line integral over C becomes:

$$\begin{aligned}
\int_C z\,dx + x\,dz &= \int_{-2}^{2} \left(z(t) \cdot x'(t) + x(t) \cdot z'(t) \right)\,dt \\
&= \int_{-2}^{2} \left((16 - t^2 - t^4) \cdot 1 + t \cdot (-2t - 4t^3) \right)\,dt \\
&= \int_{-2}^{2} (-5t^4 - 3t^2 + 16)\,dt \\
&= \left[-t^5 - t^3 + 16t \right] \Big|_{-2}^{2} \\
&= -16
\end{aligned}$$

Exercise $3.5.3d$

Evaluate the line integral $\int_C |x|\,dx + |y|\,dy$ where C is the unit circle traversed clockwise once

Answer

This curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t) \rangle = \langle \sin(t), \cos(t) \rangle$ for $0 \leq t \leq 2\pi$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t) \rangle = \langle \cos(t), -\sin(t) \rangle$

Therefore the line integral over C becomes:

$$\begin{aligned}\int_C |x|\,dx + |y|\,dy &= \int_0^{2\pi} \left(|x(t)|\cdot x'(t) + |y(t)|\cdot y'(t)\right)\,dt \\ &= \int_0^{2\pi} \left(|\sin(t)|\cdot\cos(t) + |\cos(t)|\cdot(-\sin(t))\right)\,dt \\ &= \left(\int_0^{\frac{\pi}{2}} 0\,dt + \int_{\frac{\pi}{2}}^{\pi} 2\sin(t)\cos(t)\,dt + \int_{\pi}^{\frac{3\pi}{2}} 0\,dt + \int_{\frac{3\pi}{2}}^{2\pi} -2\sin(t)\cos(t)\,dt\right) \\ &= \left(\sin^2(t)\Big|_{\frac{\pi}{2}}^{\pi} + \left[-\sin^2(t)\right]\Big|_{\frac{3\pi}{2}}^{2\pi}\right) \\ &= (-1+1) \\ &= 0\end{aligned}$$

Exercise 3.5.3e

Evaluate the line integral $\int_C |y|dx + |x|dy$ where C is the unit circle traversed clockwise once

Answer

This curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle = \langle \sin(t), \cos(t)\rangle$ for $0 \le t \le 2\pi$

This gives derivative $\vec{\mathbf{r}}'(t) = \langle x'(t), y'(t)\rangle = \langle \cos(t), -\sin(t)\rangle$

Therefore the line integral over C becomes:

$$\begin{aligned}\int_C |y|\,dx + |x|\,dy &= \int_0^{2\pi} \left(|y(t)|\cdot x'(t) + |x(t)|\cdot y'(t)\right)\,dt \\ &= \int_0^{2\pi} \left(|\cos(t)|\cdot\cos(t) + |\sin(t)|\cdot(-\sin(t))\right)\,dt \\ &= \left(\int_0^{\frac{\pi}{2}} \cos(2t)\,dt + \int_{\frac{\pi}{2}}^{\pi} -1\,dt + \int_{\pi}^{\frac{3\pi}{2}} -\cos(2t)\,dt + \int_{\frac{3\pi}{2}}^{2\pi} 1\,dt\right) \\ &= \left(\frac{1}{2}\sin(2t)\Big|_0^{\frac{\pi}{2}} + [-t]\Big|_{\frac{\pi}{2}}^{\pi} + \left[-\frac{1}{2}\sin(2t)\right]\Big|_{\pi}^{\frac{3\pi}{2}} + t\Big|_{\frac{3\pi}{2}}^{2\pi}\right) \\ &= \left(0 + \left(-\frac{\pi}{2}\right) + 0 + \frac{\pi}{2}\right) \\ &= 0\end{aligned}$$

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3.6: Integrals of Real Functions on a Surface

Learning Objectives

- Find the parametric representations of a cylinder, a cone, and a sphere.
- Describe the surface integral of a scalar-valued function over a parametric surface.
- Use a surface integral to calculate the area of a given surface.

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

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3.6: Surface Integrals

Before we can define surface integrals, we need to generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\vec{\mathbf{r}}(t),\ a \le t \le b$ is regular (or **smooth**) if $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t in $[a,b]$. For a curve, this condition ensures that the image of $\vec{\mathbf{r}}$ really is a curve, and not just a point. For example, consider curve parameterization $\vec{\mathbf{r}}(t) = \langle 1, 2\rangle,\ 0 \le t \le 5$. The image of this parameterization is simply point $(1,2)$, which is not a curve. Notice also that $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$. The fact that the derivative is the zero vector indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity (or smoothness) for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider the parameterization

$$\vec{\mathbf{r}}(u,v) = \langle 0,\ \cos v,\ 1\rangle \quad \text{for} \quad 0 \le u \le 1,\ 0 \le v \le \pi$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line:

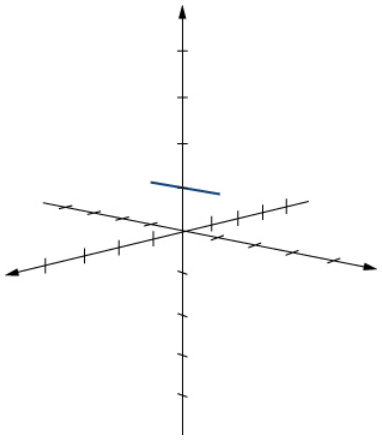


Figure 3.6.1: The image of parameterization $\vec{\mathbf{r}}(u,v) = \langle 0, \cos v, 1\rangle$ for $0 \le u \le 1,\ 0 \le v \le \pi$ is a line.

How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\vec{\mathbf{r}}_u = \langle 0,0,0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, -\sin(v), 0\rangle$, and the corresponding cross product is zero. The analog of the condition $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$ is that $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any point (u,v) in the parameter domain.

Definition: Smooth Surfaces

A surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is ***smooth*** if vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain.

A surface may also be *piecewise smooth* if it has smooth faces but also has locations where the directional derivatives do not exist.

Illustration 3.6.1: Identifying Smooth and Nonsmooth Surfaces

Which of the following surfaces are smooth?

(a) $\vec{\mathbf{r}}(u,v) = \langle (2+\cos(v))\cos(u),\ (2+\cos(v))\sin(u),\ \sin(v)\rangle$ for $0 \le u < 2\pi,\ 0 \le v < 2\pi$

(b) $\vec{\mathbf{r}}(u,v) = \langle (1+\cos(v))\cos(u),\ (1-\cos(v))\sin(u),\ v\rangle$ for $-\pi \le u < \pi,\ -2 \le v \le 2$

For the surface in (a), notice that vectors

$$\vec{\mathbf{r}}_u = \langle -(2+\cos(v))\sin(u),\ (2+\cos(v))\cos(u), 0\rangle \quad \text{and}$$
$$\vec{\mathbf{r}}_v = \langle -\sin(v)\ \cos(u),\ -\sin(v)\ \sin(u),\ \cos(v)\rangle$$

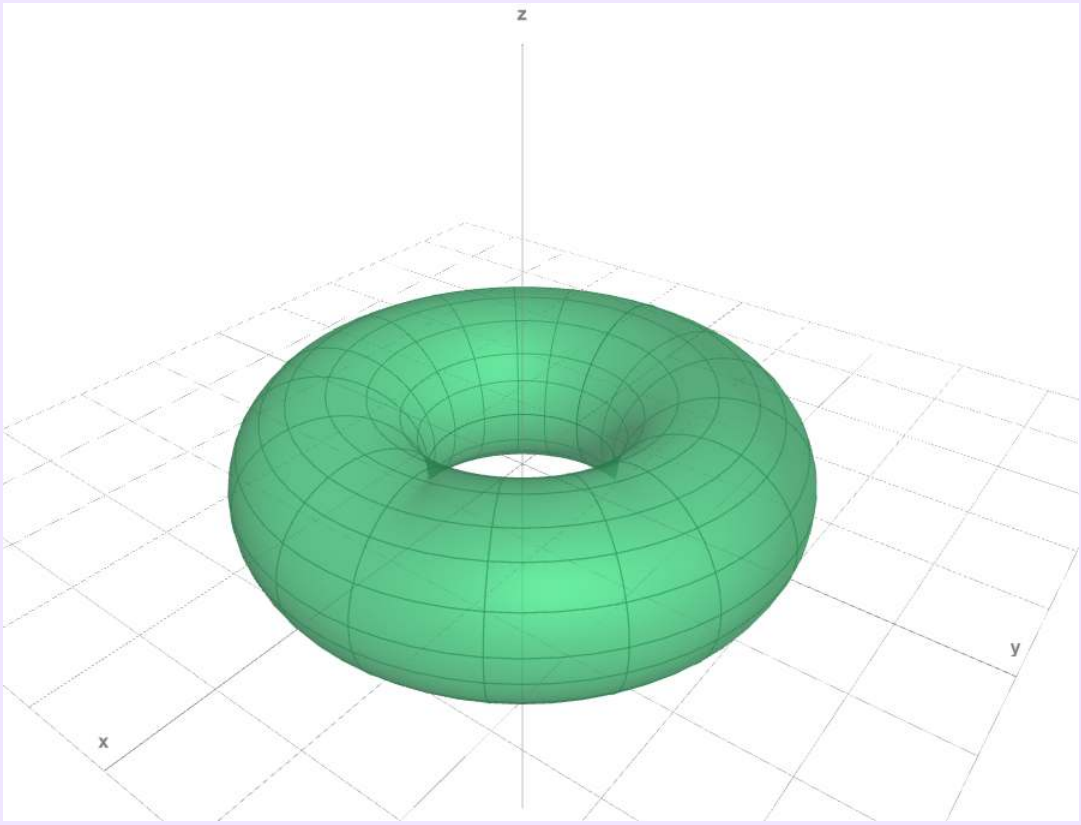
exist for any choice of u and v in the parameter domain, and have cross product

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(2+\cos(v))\sin(u) & (2+\cos(v))\cos(u) & 0 \\ -\sin(v)\ \cos(u) & -\sin(v)\ \sin(u) & \cos(v) \end{vmatrix}$$

$$= [(2+\cos(v))\cos(u)\ \cos(v)]\hat{\mathbf{i}} + [(2+\cos(v))\sin(u)\ \cos(v)]\hat{\mathbf{j}} + [(2+\cos(v))\sin(v)\ \sin^2(u) + (2+\cos(v))\sin(v)\ \cos^2(u)]\hat{\mathbf{k}}$$

$$= [(2+\cos(v))\cos(u)\ \cos(v)]\hat{\mathbf{i}} + [(2+\cos(v))\sin(u)\ \cos(v)]\hat{\mathbf{j}} + [(2+\cos(v))\sin(v)]\hat{\mathbf{k}}$$

The $\hat{\mathbf{k}}$ component of this vector is zero only if $v = 0$ or $v = \pi$. If $v = 0$ or $v = \pi$, then the only choices for u that make the $\hat{\mathbf{j}}$ component zero are $u = 0$ or $u = \pi$. But these choices of u do not make the $\hat{\mathbf{i}}$ component zero. Therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain, and the parameterization is smooth. Here is a graph:



We see that the corresponding surface has no sharp corners.

In (b), notice that vectors $\langle (1+\cos(v))\cos(u),\ (1-\cos(v))\sin(u),\ v\rangle$ for $-\pi \le u < \pi,\ -2 \le v \le 2$

$$\vec{\mathbf{r}}_u = \langle -(1+\cos(v))\sin(u),\ (1-\cos(v))\cos(u), 0\rangle \quad \text{and}$$
$$\vec{\mathbf{r}}_v = \langle -\sin(v)\ \cos(u),\ \sin(v)\ \sin(u),\ 1\rangle$$

exist for any choice of u and v in the parameter domain, and have cross product

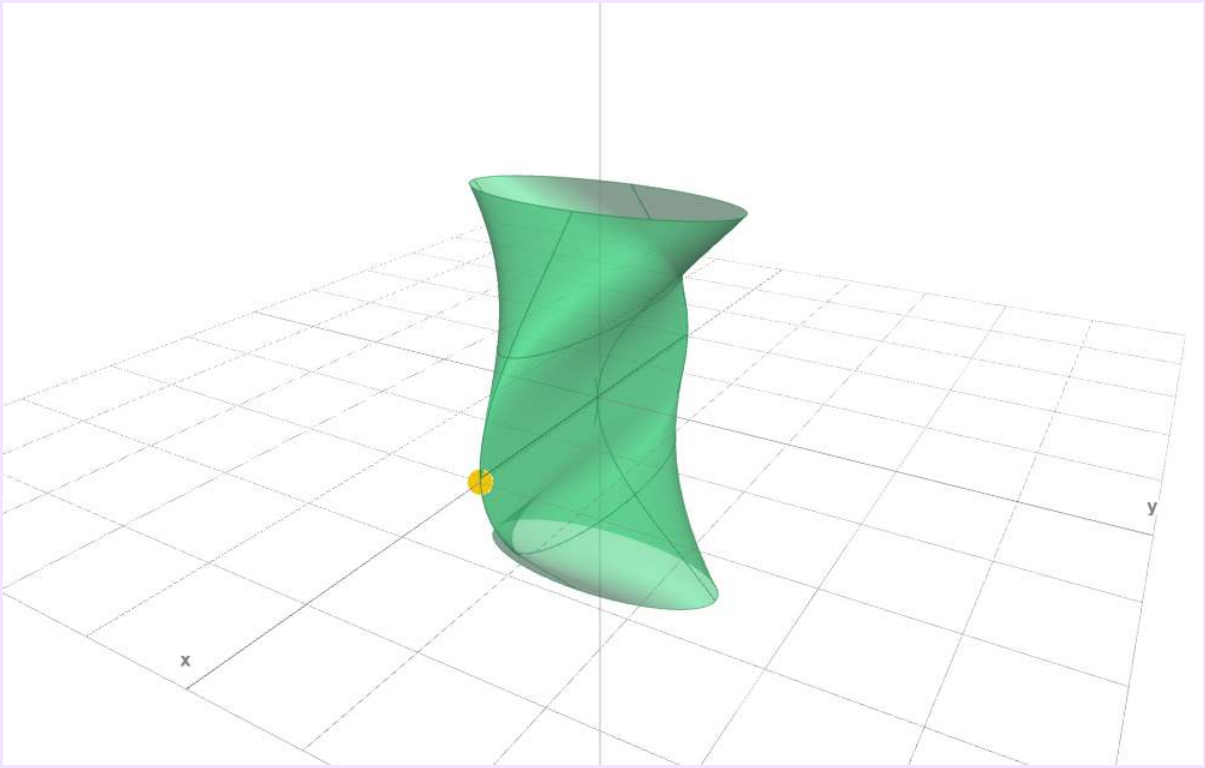
$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(1+\cos(v))\sin(u) & (1-\cos(v))\cos(u) & 0 \\ -\sin(v)\ \cos(u) & \sin(v)\ \sin(u) & 1 \end{vmatrix}$$

$$= [(1-\cos(v))\cos(u)]\hat{\mathbf{i}} + [(1+\cos(v))\sin(u)]\hat{\mathbf{j}} + [-(1+\cos(v))\sin(v)\ \sin^2(u) + (1-\cos(v))\sin(v)\ \cos^2(u)]\hat{\mathbf{k}}$$

$$= [((1-\cos(v))\cos(u)]\hat{\mathbf{i}} + [(1+\cos(v))\sin(u)]\hat{\mathbf{j}} + [(\cos(2u) - \cos(v))\sin(v)]\hat{\mathbf{k}}$$

The $\hat{\mathbf{j}}$ component of this vector is zero only if $u = 0, \pm\pi$, while the $\hat{\mathbf{i}}$ component of this vector is zero only if $v = 0$ or $u = \pm\frac{\pi}{2}$. These can only both be zero at the points $(u, v) \in \{(0,0), (\pm\pi, 0)\}$. Checking these points in the $\hat{\mathbf{k}}$ component we see that they

all cause it to equal zero. Therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is zero for at least one choice of u and v in the parameter domain, and the parameterization is not smooth. Here is a graph:



Clearly at $u = 0, v = 0$, which maps to $\vec{\mathbf{r}}\,(0,0) = (2,0,0)$ this surface is not smooth. The same is true at $u = \pm\pi, v = 0$ which maps to $\vec{\mathbf{r}}\,(\pm\pi,0) = (-2,0,0)$

Our goal is to define a surface integral, and as a first step, we have examined how to parameterize a smooth surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that *i* ranges from 1 to *m* and *j* ranges from 1 to *n* so that D is subdivided into *mn* rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of grid curves on surface S that is parameterized by $\vec{\mathbf{r}}(u,v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in Figure 3.6.9. If we think of $\vec{\mathbf{r}}$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\vec{\mathbf{r}}$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i,\ y = v$; the other is given by $x = u,\ y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through (u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\vec{\mathbf{r}}(u_i, v)$ and (u, v_j) and these curves intersect at point P_{ij}.

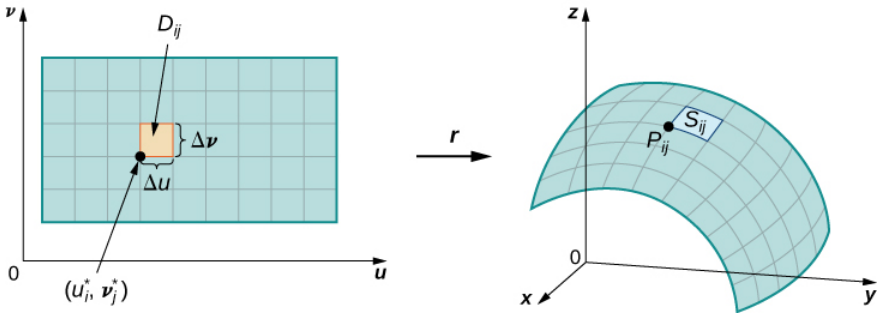


Figure 3.6.2: Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves at the point P_{ij}:

$$\vec{\mathbf{t}}_u(P_{ij}) = \vec{\mathbf{r}}_u(u_i, v_j) = \langle x_u(u_i, v_j),\ y_u(u_i, v_j),\ z_u(u_i, v_j)\rangle \text{ and } \vec{\mathbf{t}}_v(P_{ij}) = \vec{\mathbf{r}}_v(u_i, v_j) = \langle x_v(u_i, v_j),\ y_v(u_i, v_j),\ z_v(u_i, v_j)\rangle$$

If vector $\vec{\mathbf{N}} = \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (Figure 3.6.10). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

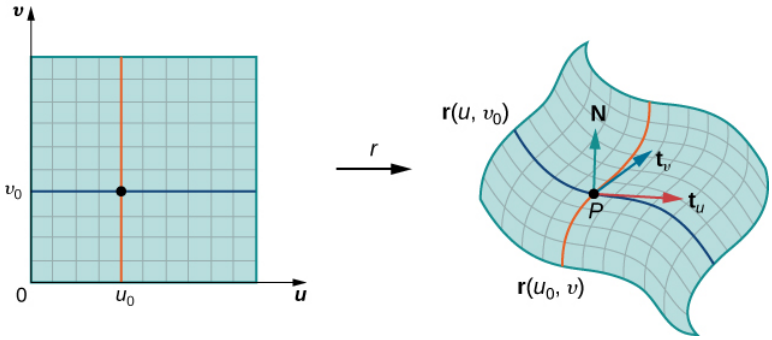


Figure 3.6.3: If the cross product of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ exists and is non-zero, then there is a tangent plane.

The tangent plane at P_{ij} contains vectors $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ and therefore the parallelogram spanned by $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u\vec{\mathbf{t}}_u(P_{ij})$ and $\Delta v\vec{\mathbf{t}}_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx ||(\Delta u\vec{\mathbf{t}}_u(P_{ij})) \times (\Delta v\vec{\mathbf{t}}_v(P_{ij}))|| = ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\Delta u\,\Delta v$$

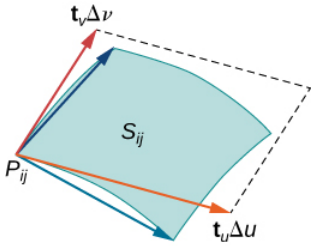


Figure 3.6.4: The parallelogram spanned by $\vec{\mathbf{t}}_u$ *and* $\vec{\mathbf{t}}_v$ *approximates the piece of surface* S_{ij}.

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface.

Theorem 3.6.1: Smooth Parameterization of Surface

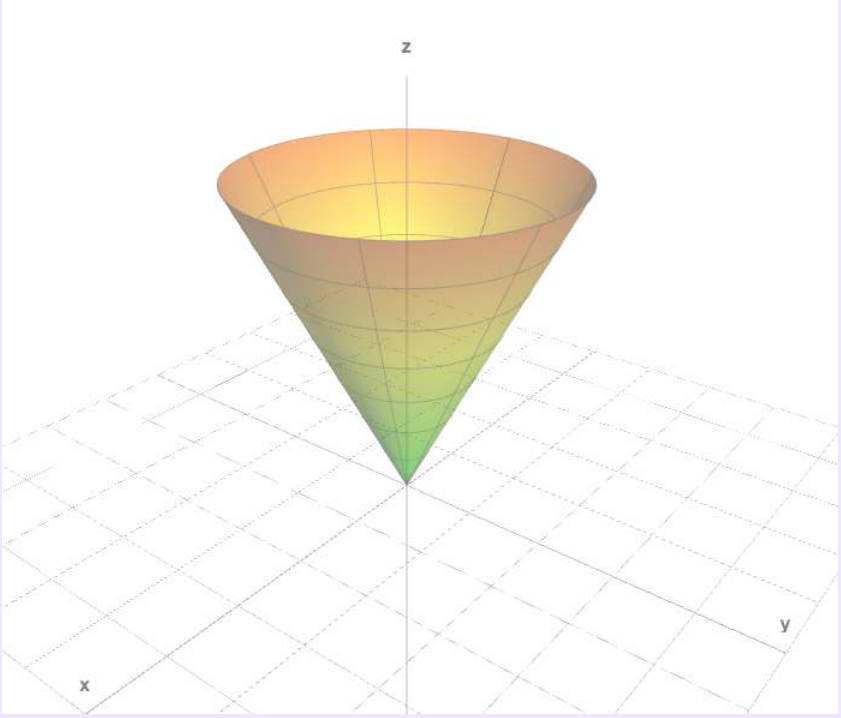
Let $\vec{\mathbf{r}}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$ with parameter domain D be a smooth parameterization of surface S. Furthermore, assume that S is traced out only once as (u, v) varies over D. The surface area of S is

$$\iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| \, dA$$

Illustration 3.6.2: Calculating Surface Area

We can use this to calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height H and radius R.

Before calculating the surface area of this cone, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin as below.



To obtain a parameterization, we know the rectangular form of such a cone is $z = k\sqrt{x^2 + y^2}$ for some k. We want the height $z = H$ to occur when the radius $\sqrt{x^2 + y^2} = R$ so clearly $k = \frac{H}{R}$. This looks a bit nicer in terms of cylindrical coordinates: $z = \frac{H}{R} r$.

Therefore one such parameterization is given by setting $v = r$ and $u = \theta$:

$$\vec{\mathbf{r}}(u, v) = \left\langle v\cos(u),\, v\sin(u),\, \frac{H}{R}v \right\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le R$$

With a parameterization in hand, we can calculate the tangent vectors:

$$\vec{\mathbf{r}}_u = \langle -v\sin(u), v\cos(u), 0\rangle \text{ and } \vec{\mathbf{r}}_v = \left\langle \cos(u), \sin(u), \frac{H}{R} \right\rangle$$

Next we find their cross product:

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -v\sin(u) & v\cos(u) & 0 \\ \cos(u) & \sin(u) & \frac{H}{R} \end{vmatrix} = \left\langle \frac{H}{R}v\cos(u), \frac{H}{R}v\sin(u), -v\sin^2(u) - v\cos^2(u) \right\rangle$$
$$= \left\langle \frac{H}{R}v\cos(u), \frac{H}{R}v\sin(u), -v \right\rangle$$

The magnitude of this vector is

$$\left\| \left\langle \frac{H}{R} v\cos(u), \frac{H}{R} v\sin(u), -v \right\rangle \right\| = \sqrt{\frac{H^2}{R^2} v^2\cos^2(u) + \frac{H^2}{R^2} v^2 \sin^2(u) + v^2} = v\sqrt{\frac{H^2}{R^2} + 1}$$

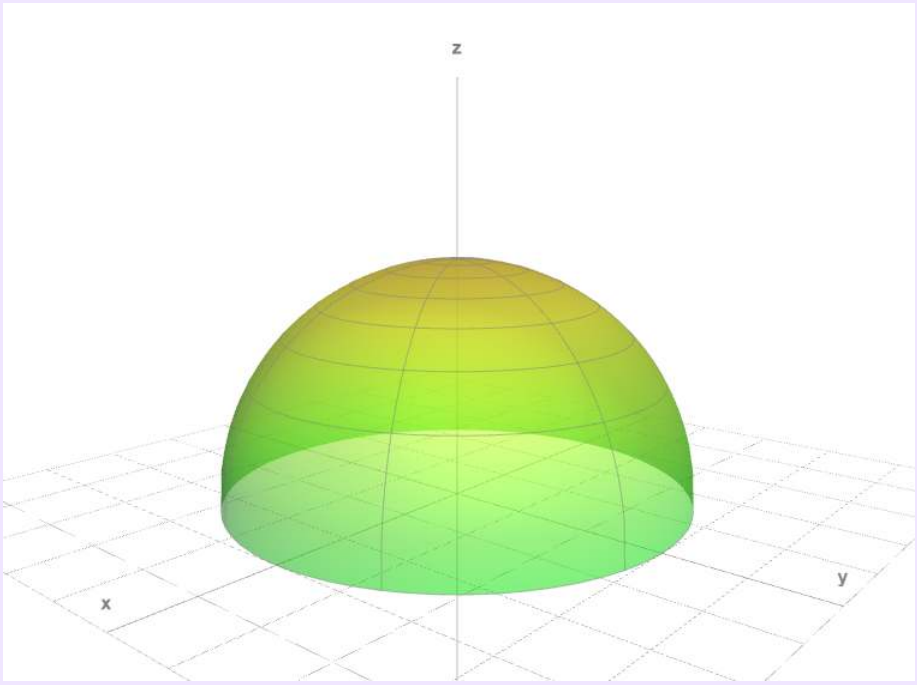
So the surface area is represented by the integral

$$\iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| \, dA = \int_0^R \int_0^{2\pi} v\sqrt{\frac{H^2}{R^2} + 1} \, du \, dv = \sqrt{\frac{H^2}{R^2} + 1} \left(\int_0^R v dv \right) \left(\int_0^{2\pi} du \right)$$
$$= \sqrt{\frac{H^2}{R^2} + 1} \cdot \frac{R^2}{2} \cdot 2\pi = \pi R \sqrt{H^2 + R^2}$$

Note that in geometry classes the surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2 + r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2 + r^2}$ that we found.

Illustration 3.6.3: Calculating Surface Area

We will derive the surface area of the hemisphere $x^2 + y^2 + z^2 = R^2$ with $z \geq 0$



This problem can be solved by remaining in rectangular or switching to cylindrical or spherical coordinates.

For rectangular, define the parameterization with $x = u, y = v$:

$$\vec{\mathbf{r}}(u, v) = \left\langle u, v, \sqrt{R^2 - u^2 - v^2} \right\rangle \quad \text{for } u^2 + v^2 \leq R^2$$

The tangent vectors are

$$\vec{\mathbf{r}}_u = \left\langle 1, 0, \frac{-u}{\sqrt{R^2 - u^2 - v^2}} \right\rangle \text{ and } \vec{\mathbf{r}}_v = \left\langle 0, 1, \frac{-v}{\sqrt{R^2 - u^2 - v^2}} \right\rangle$$

So their cross product is

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle \frac{u}{\sqrt{R^2 - u^2 - v^2}}, \frac{v}{\sqrt{R^2 - u^2 - v^2}}, 1 \right\rangle$$

Which has length

$$\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{\frac{u^2}{R^2-u^2-v^2} + \frac{v^2}{R^2-u^2-v^2} + 1} = \sqrt{\frac{R^2}{R^2-u^2-v^2}} = \frac{R}{\sqrt{R^2-u^2-v^2}}$$

Therefore, the surface area of the hemisphere is given by integrating over the parametric domain $D = \{(u,v) : u^2+v^2 \le R^2\}$:

$$\iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\, dA = \iint_D \frac{R}{\sqrt{R^2-u^2-v^2}}\, dA$$

This domain lends itself to polar coordinates, so our integral becomes:

$$\int_0^{2\pi}\int_0^R \frac{R}{\sqrt{R^2-r^2}}\, r\, dr\, d\theta = R\left(\int_0^{2\pi} d\theta\right)\left(\int_0^R \frac{r}{\sqrt{R^2-r^2}} dr\right) = R\cdot 2\pi \cdot R = 2\pi R^2$$

We have derived the familiar formula for the surface area of a sphere (divided by two) using surface integrals.

We will do this calculation again, but using cylindrical coordinates instead. We let $r = u, \theta = v$ and use the parameterization

$$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), \sqrt{R^2-u^2}\rangle, \quad \text{for } 0 \le u \le R,\ 0 \le v < 2\pi$$

The tangent vectors are

$$\vec{\mathbf{r}}_u = \left\langle \cos(v), \sin(v), -\frac{u}{\sqrt{R^2-u^2}} \right\rangle \text{ and } \vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$$

So, their cross product is

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle \frac{u^2\cos(v)}{\sqrt{R^2-u^2}}, \frac{u^2\sin(v)}{\sqrt{R^2-u^2}}, u\cos^2(v) + u\sin^2(v) \right\rangle = \left\langle \frac{u^2\cos(v)}{\sqrt{R^2-u^2}}, \frac{u^2\sin(v)}{\sqrt{R^2-u^2}}, u \right\rangle$$

Which has length

$$\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{\frac{u^4\cos^2(v)}{R^2-u^2} + \frac{u^4\sin^2(v)}{R^2-u^2} + u^2}$$
$$= \sqrt{\frac{u^4 + u^2(R^2-u^2)}{R^2-u^2}} = \sqrt{\frac{u^2R^2}{R^2-u^2}} = \frac{Ru}{\sqrt{R^2-u^2}}$$

The surface area of the hemisphere is

$$\int_0^R \int_0^{2\pi} \frac{Ru}{\sqrt{R^2-u^2}} dv\, du = R\left(\int_0^{2\pi} v\right)\left(\int_0^R \frac{u}{\sqrt{R^2-u^2}} du\right) = R\cdot 2\pi\cdot R = 2\pi R^2$$

We get the same answer. In fact, the same result occurs if we parameterize in terms of spherical coordinates!

Notice in the illustration above that when we parameterized in cylindrical coordinates, the Jacobian for cylindrical coordinates shows up naturally in $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|$. This will always occur, so substitutions can occur simultaneously with parameterization when it is convenient. (But it is not necessary to do so if the reader is willing to calculate a Jacobian *after* doing the parameterization).

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object). Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges

from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij} evaluate P_{ij} at f, and multiply by area S_{ij} to form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\Delta S_{ij}$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

Definition: Surface Integral of a Scalar-Valued Function

The **surface integral of a scalar-valued function** of f over a piecewise smooth surface S is

$$\iint_S f(x,y,z)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$:

$$\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \quad \text{and} \quad \vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\,\Delta u\,\Delta v$$

Therefore,

$$\iint_S f(x,y,z)\,dS \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\,\Delta u\,\Delta v$$

This approximation becomes arbitrarily close to $\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting m and n go to infinity. Therefore, we have the following equation to calculate scalar surface integrals:

Theorem 3.6.2: Evaluating a Scalar Surface Integral

Let f be a continuous function with a domain that includes the surface S with parameterization $\vec{\mathbf{r}}(u,v)$ with parametric domain D. Then,

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))||\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v||\,dA$$

This equation allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to the equation for line integrals.

$$\iint_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))||\vec{\mathbf{r}}'(t)||\,dt$$

In this case, vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is perpendicular to the surface, whereas vector $\vec{\mathbf{r}}'(t)$ is tangent to the curve.

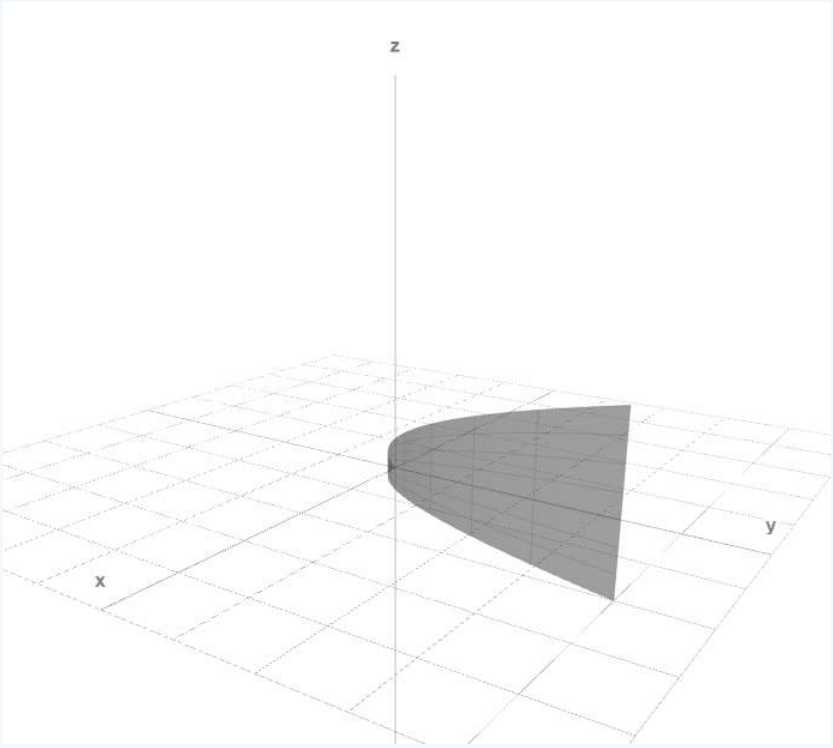
Example 3.6.1: Calculating a Surface Integral

If S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u,\, u^2,\, v\rangle$ for $0 \le u \le 2$ and $0 \le v \le u$, find

$$\iint_S \sqrt{1+4y}\,dS$$

Solution

We see the surface sketched below.



Notice that this parameter domain D is a triangle, and therefore the parameter domain is not rectangular. We can see that $D = \{(u,v) : 0 \le u \le 2,\, 0 \le v \le u\}$ is a Type 1 region in the uv-plane.

To calculate the surface integral, we first find vectors $\vec{\mathbf{r}}_u$ and $\vec{\mathbf{r}}_v$. Note that $\vec{\mathbf{r}}_u = \langle 1, 2u, 0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 0, 1\rangle$. Therefore,

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u,\, -1,\, 0\rangle$$

Taking its length gives:

$$\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{(2u)^2 + (-1)^2 + 0^2} = \sqrt{1+4u^2}$$

Therefore our desired surface integral becomes:

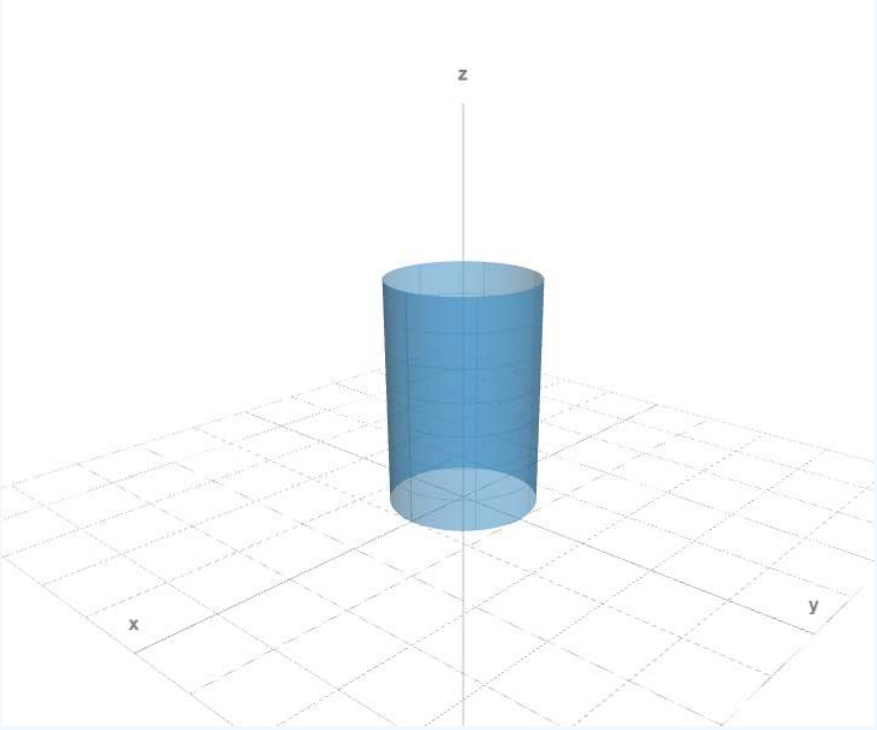
$$\begin{aligned}\iint_S \sqrt{1+4y}\,dS &= \iint_D \sqrt{1+4y(u,v)}\,\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|, dA \\ &= \iint_D \sqrt{1+4u^2}\cdot\sqrt{1+4u^2}\,dA \\ &= \int_0^2 \int_0^u (1+4u^2)\,dv\,du \\ &= \int_0^2 (1+4u^2)v\Big|_0^u\,du \\ &= \int_0^2 (u+4u^3)\,du \\ &= \left[\frac{1}{2}u^2+u^4\right]\Big|_0^2 \\ &= 2+16 = \boxed{18}\end{aligned}$$

Example 3.6.2: Calculating the Surface Integral of a Cylinder

Calculate surface integral $\iint_S (x+2y^2)\,dS$, where S is cylinder $x^2+y^2=1,\ 0\le z\le 3$

Solution

To calculate the surface integral, we first need a parameterization of the cylinder pictured below.



One such parameterization is to use cylindrical coordinates, with $r=1, \theta=u, z=v$:

$$\vec{\mathbf{r}}(u,v) = \langle \cos(u),\ \sin(u),\ v\rangle, \qquad \text{for } D=\{(u,v)\ :\ 0\le u\le 2\pi,\ 0\le v\le 3\}$$

The tangent vectors are $\vec{\mathbf{r}}_u = \langle -\sin(u),\ \cos(u),\ 0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0,0,1\rangle$. The cross product is,

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos(u),\ \sin(u),\ 0\rangle$$

The length of which is easily $||\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|| = \sqrt{\cos^2(u)+\sin^2(u)} = 1$

Therefore the surface integral becomes:

$$\begin{aligned}\iint_S f(x,y,z)dS &= \iint_D \left(x(u,v)+2[y(u,v)]^2\right)\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\ &= \int_0^3\int_0^{2\pi}(\cos(u)+2\sin^2(u))\,du\,dv \\ &= \left(\int_0^3 dv\right)\left(\int_0^{2\pi}(\cos(u)+1-\cos(2u))\,du\right) \\ &= \left(v\Big|_0^3\right)\cdot\left[\sin(u)+u-\frac{1}{2}\sin(2u)\right]\Big|_0^{2\pi}dv \\ &= 3\cdot 2\pi = \boxed{6\pi}\end{aligned}$$

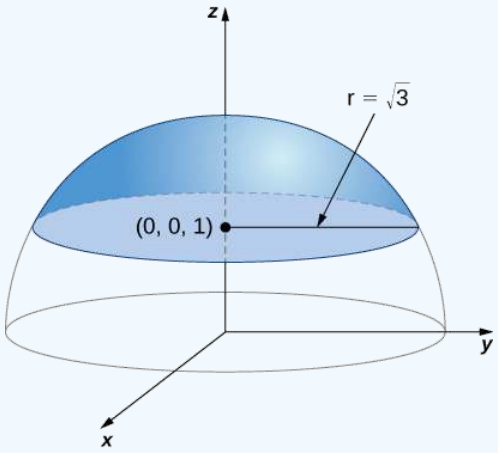
Example 3.6.3: Calculating the Surface Integral of a Piecewise Smooth Surface

Let S be the surface along the boundary of the region $E=\{(x,y,z)\,:\,x^2+y^2+z^2\le 4, z\ge 1\}$ and calculate

$$\iint_S z^2\,dS$$

Solution

We sketch E and notice that its boundary surface S is not smooth but is piecewise smooth.



S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore,

$$\iint_S z^2 dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS$$

First, we calculate $\iint_{S_1} z^2\,dS$:

To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z=1$ centered at $(0,0,1)$. To parameterize this disk, we have to find its radius. Since the disk is formed where plane $z=1$ intersects sphere $x^2+y^2+z^2=4$, we substitute $z=1$ into equation $x^2+y^2+z^2=4$:

$$x^2+y^2+1=4 \Rightarrow x^2+y^2=3$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\vec{\mathbf{r}}(u,v)=\langle u\,\cos(v),\,u\,\sin(v),\,1\rangle$, $0\le u\le\sqrt{3}$, $0\le v\le 2\pi$. The tangent vectors are $\vec{\mathbf{r}}_u=\langle\cos(v),\,\sin(v),\,0\rangle$ and $\vec{\mathbf{r}}_v=\langle -u\,\sin(v),\,u\,\cos(v),\,0\rangle$, and thus

$$\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \cos(v) & \sin(v) & 0\\ -u\sin(v) & u\cos(v) & 0\end{vmatrix}=\langle 0,\,0,\,u\,\cos^2(v)+u\,\sin^2(v)\rangle=\langle 0,0,u\rangle$$

The magnitude of this vector is $\|\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v\|=u$. Therefore, our first surface integral is:

$$\begin{aligned}\iint_{S_1} z^2\,dS &= \int_0^{\sqrt{3}}\int_0^{2\pi} [z(u,v)]^2\,\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dv\,du \\ &= \int_0^{\sqrt{3}}\int_0^{2\pi} 1^2\cdot u\,dv\,du \\ &= \left(\int_0^{2\pi} dv\right)\left(\int_0^{\sqrt{3}} u\,du\right) \\ &= \left(v\Big|_0^{2\pi}\right)\cdot\left(\frac{1}{2}u^2\Big|_0^{\sqrt{3}}\right) \\ &= 2\pi\cdot\frac{3}{2} = 3\pi\end{aligned}$$

Now we calculate $\iint_{S_2} z^2\,dS$:

To calculate this integral, we need a parameterization of S_2. We could use any of rectangular, cylindrical, or spherical coordinates to parameterize here, since the surface is a well defined function. Since we have already shown both rectangular and cylindrical parameterizations of a sphere in previous examples, we will use spherical this time for instructive purposes. Since $z = 2\cos(\phi) \geq 1$, we know that $\phi \leq \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$. Letting $\rho = 2, \theta = u, \phi = v$, our parameterization will be

$$\vec{\mathbf{r}}(u,v) = \langle 2\sin(v)\cos(u), 2\sin(v)\sin(u), 2\cos(v)\rangle \quad \text{for} \quad 0 \leq u \leq 2\pi,\ 0 \leq v \leq \frac{\pi}{3}$$

The tangent vectors are $\vec{\mathbf{r}}_u = \langle -2\sin(v)\sin(u), 2\sin(v)\cos(u), 0\rangle$ and $\vec{\mathbf{r}}_v = \langle 2\cos(v)\cos(u), 2\cos(v)\sin(u), -2\sin(v)\rangle$, and have cross product

$$\begin{aligned}\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2\sin(v)\sin(u) & 2\sin(v)\cos(u) & 0 \\ 2\cos(v)\cos(u) & 2\cos(v)\sin(u) & -2\sin(v) \end{vmatrix} \\ &= \langle -4\sin^2(v)\cos(u), -4\sin^2(v)\sin(u), -4\sin(v)\cos(v)\rangle\end{aligned}$$

The magnitude of this vector is

$$\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{16\sin^4(v)\cos^2(u) + 16\sin^4(v)\sin^2(u) + 16\sin^2(v)\cos^2(v)} = 4\sin(v)$$

Therefore,

$$\begin{aligned}\iint_{S_2} z^2\,dS &= \int_0^{\frac{\pi}{3}}\int_0^{2\pi} [z(u,v)]^2\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,du\,dv \\ &= \int_0^{\frac{\pi}{3}}\int_0^{2\pi} (2\cos(v))^2\cdot 4\sin(v)\,du\,dv \\ &= \left(\int_0^{\frac{\pi}{3}} \cos^2(v)\sin(v)dv\right)\left(\int_0^{2\pi} 16du\right) \\ &= \left(-\frac{1}{3}\cos^3(v)\Big|_0^{\frac{\pi}{3}}\right)\cdot\left(16u\Big|_0^{2\pi}\right) \\ &= \frac{7}{24}\cdot 32\pi = \frac{28\pi}{3}\end{aligned}$$

So our final answer is

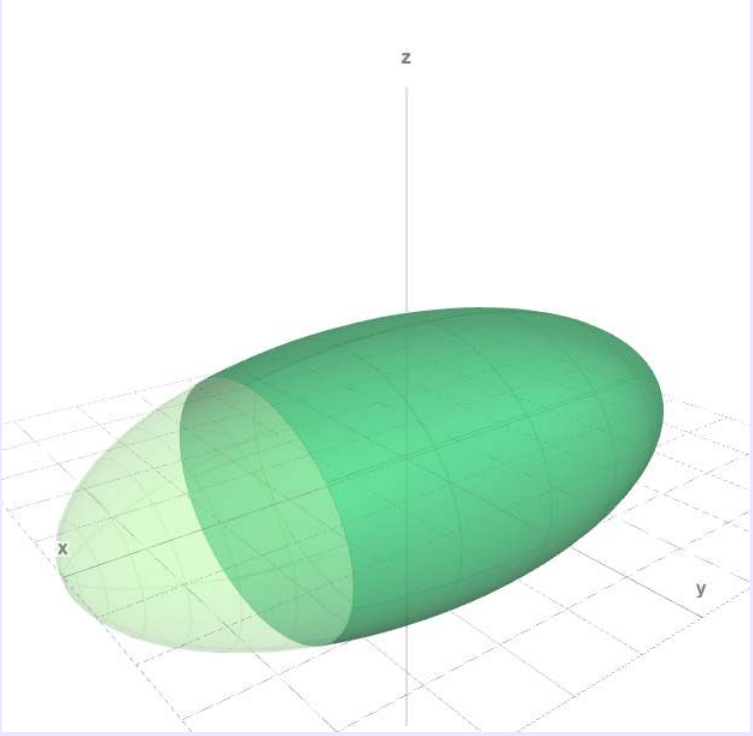
$$\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS = 3\pi + \frac{28\pi}{3} = \boxed{\frac{37\pi}{3}}$$

One application of surface integrals is finding the total mass of a surface when given a density function. If the density of a given surface is some function $\delta(x,y,z) \geq 0$ at the location (x,y,z), then its total mass is simply $\iint_S \delta\,dS$.

Illustration 3.6.4: Calculating the Surface Integral of a Piece of an Ellipsoid

We will calculate the total weight of the surface where $S = \{(x, y, z) : x^2 + 4y^2 + 9z^2 \leq 36,\ x \leq 3\}$ if it has density $\delta(x, y, z) = \dfrac{3 - x}{\sqrt{3 + y^2 + 6z^2}}$.

The surface S described is part of the ellipsoid $\dfrac{x^2}{36} + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$, as pictured below.



We can use a parameterization similar to spherical coordinates with two minor tweaks. First, we will switch the variables x and z in spherical, so that we can more easily "cut off" the front piece of the ellipsoid by changing ϕ. Second, we will replace ρ with the individual radius in each direction. This means we will use $x = 6\cos(\phi)$, and thus $x \leq 3 \implies \phi \geq \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$. Letting $u = \theta$ and $v = \phi$ this gives us:

$$\vec{\mathbf{r}}(u, v) = \langle 6\cos(v), 3\sin(v)\sin(u), 2\sin(v)\cos(u)\rangle \quad \text{for} \quad 0 \leq u \leq 2\pi,\ \frac{\pi}{3} \leq v \leq \pi$$

Now we can find $\vec{\mathbf{r}}_u = \langle 0, 3\sin(v)\cos(u), -2\sin(v)\sin(u)\rangle$ and $\vec{\mathbf{r}}_v = \langle -6\sin(v), 3\cos(v)\sin(u), 2\cos(v)\cos(u)\rangle$, which gives:

$$\begin{aligned}\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 3\sin(v)\cos(u) & -2\sin(v)\sin(u) \\ -6\sin(v) & 3\cos(v)\sin(u) & 2\cos(v)\cos(u) \end{vmatrix} \\ &= \langle 6\sin(v)\cos(v),\ 12\sin^2(v)\sin(u),\ 18\sin^2(v)\cos(u)\rangle\end{aligned}$$

Next, we find the length:

$$\begin{aligned}\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| &= \sqrt{(6\sin(v)\cos(v))^2 + (-12\sin^2(v)\sin(u))^2 + (18\sin^2(v)\cos(u))^2} \\ &= 6\sqrt{\sin^2(v)\cos^2(v) + 4\sin^4(v)\sin^2(u) + 9\sin^4(v)\cos^2(u)} \\ &= 6\sin(v)\sqrt{\cos^2(v) + \sin^2(v)(\sin^2(u) + \cos^2(u)) + 3\sin^2(v)\sin^2(u) + 8\sin^2(v)\cos^2(u)} \\ &= 6\sin(v)\sqrt{\cos^2(v) + \sin^2(v) + 3\sin^2(v)\sin^2(u) + 8\sin^2(v)\cos^2(u)} \\ &= 6\sin(v)\sqrt{1 + 3\sin^2(v)\sin^2(u) + 8\sin^2(v)\cos^2(u)}\end{aligned}$$

The above expression has many different ways to be written, but one that looks like a multiple of the denominator of our density function:

$$\begin{aligned}\sqrt{3+[y(u,v)]^2+6[z(u,v)]^2} &= \sqrt{3+(3\sin(v)\sin(u))^2+6(2\sin(v)\cos(u))^2} \\ &= \sqrt{3\left(1+3\sin^2(v)\sin^2(u)+8\sin^2(v)\cos^2(u)\right)}\end{aligned}$$

Now we can calculate the total weight via the surface integral:

$$\begin{aligned}\iint_S \frac{3-x}{\sqrt{3+y^2+6z^2}}\,dS &= \int_{\frac{\pi}{3}}^{\pi}\int_0^{2\pi} \frac{3-x(u,v)}{\sqrt{3+[y(u,v)]^2+6[z(u,v)]^2}}\|\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v\|\,du\,dv \\ &= \int_{\frac{\pi}{3}}^{\pi} \int_0^{2\pi} \frac{(3-6\cos(v))\cdot 6\sin(v)\sqrt{1+3\sin^2(v)\sin^2(u)+8\sin^2(v)\cos^2(u)}}{\sqrt{3\left(1+3\sin^2(v)\sin^2(u)+8\sin^2(v)\cos^2(u)\right)}} \\ &\quad du\,dv \\ &= \int_{\frac{\pi}{3}}^{\pi}\int_0^{2\pi} 6\sqrt{3}\,\sin(v)-12\sqrt{3}\,\sin(v)\cos(v)\,du\,dv \\ &= \left(\int_{\frac{\pi}{3}}^{\pi} 6\sqrt{3}\,\sin(v)-12\sqrt{3}\,\sin(v)\cos(v)\,dv\right)\left(\int_0^{2\pi} du\right) \\ &= \left[-6\sqrt{3}\,\cos(v)-6\sqrt{3}\,\sin^2(v)\right]\Big|_{\frac{\pi}{3}}^{\pi}\cdot\left(u\Big|_0^{2\pi}\right) \\ &= \frac{27\sqrt{3}}{2}\cdot 2\pi \\ &= 27\pi\sqrt{3}\end{aligned}$$

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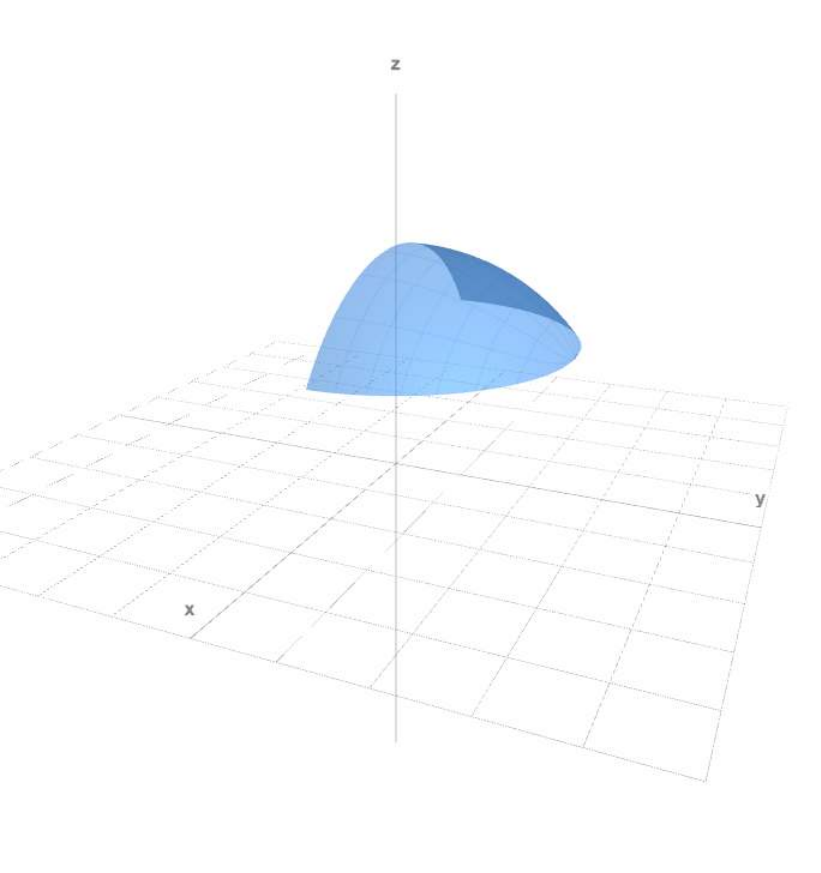
3.6: Practice Set

Exercise 3.6.1a

Find the surface area of $S = \{(x, y, z) : x^2 + y^2 + z^2 \leq 25,\ z \geq 3,\ y \geq 0\}$

Answer

S is part of the top half of a sphere of radius 5



This can be written $z = \sqrt{25 - x^2 - y^2}$ over a semi-circular domain, $z \geq 3 \longrightarrow x^2 + y^2 \leq 16$

We will parameterize it using cylindrical coordinates, with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u, v) = \left\langle u\cos(v), u\sin(v), \sqrt{25 - u^2} \right\rangle$ over $D = \{(u, v) : 0 \leq u \leq 4,\ 0 \leq \theta \leq \pi\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \left\langle \cos(v), \sin(v), -\dfrac{u}{\sqrt{25 - u^2}} \right\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle \dfrac{u^2\cos(v)}{\sqrt{25 - u^2}}, \dfrac{u^2\sin(v)}{\sqrt{25 - u^2}}, u \right\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{\left(\dfrac{u^2\cos(v)}{\sqrt{25 - u^2}}\right)^2 + \left(\dfrac{u^2\sin(v)}{\sqrt{25 - u^2}}\right)^2 + (u)^2} = \sqrt{\dfrac{25u^2}{25 - u^2}} = \dfrac{5u}{\sqrt{25 - u^2}}$

Therefore the surface area of S is given by:

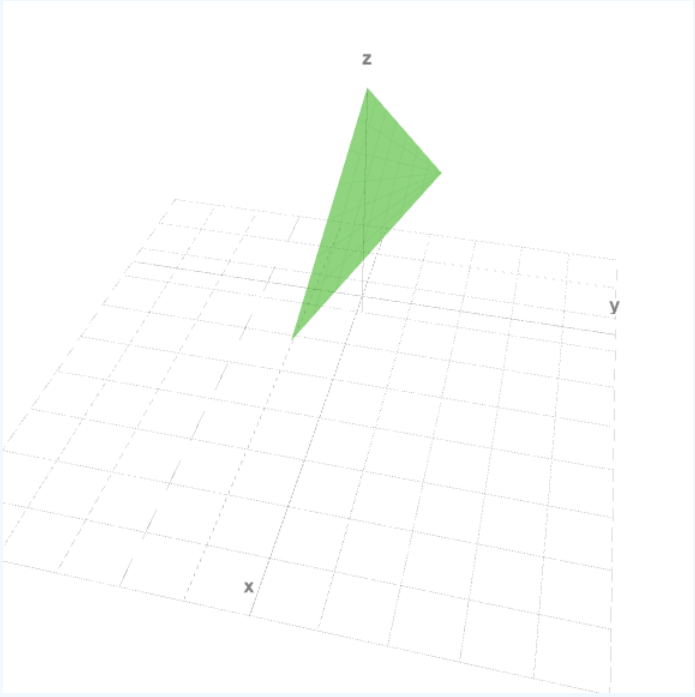
$$\begin{aligned}\iint_S 1\,dS &= \iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\ &= \int_0^{\pi}\int_0^4 \frac{5u}{\sqrt{25-u^2}}\,du\,dv \\ &= \left(\int_0^{\pi} 5\,dv\right)\left(\int_0^4 \frac{u}{\sqrt{25-u^2}}\,du\right) \\ &= \left(5v\Big|_0^{\pi}\right)\cdot\left(-\sqrt{25-u^2}\Big|_0^4\right) \\ &= 5\pi\cdot 2 \\ &= 10\pi\end{aligned}$$

Exercise 3.6.1b

Find the surface area of the part of the plane $2x-2y+z=12$ above the triangle with vertices $(0,0),(1,1)$, and $(2,0)$

Answer

Let S be the part of the plane that is described above.



The triangular domain is easiest to describe as Type 2 as $\{(x,y)\,:\,0\le y\le 1,\,y\le x\le 2-y\}$

Since we can solve for $z=12-2x+2y$, we can easily parameterize using $u=x$ and $v=y$:

$\vec{\mathbf{r}}(u,v)=\langle u,v,12-2u+2v\rangle$ over $D=\{(u,v)\,:\,0\le v\le 1,\,v\le u\le 2-v\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u=\langle 1,0,-2\rangle$ and $\vec{\mathbf{r}}_v=\langle 0,1,2\rangle$

These have cross product $\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v=\langle 2,-2,1\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v\|=\sqrt{2^2+(-2)^2+1^2}=3$

Therefore the surface area of S is given by:

$$\begin{aligned}\iint_S 1\,dS &= \iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\ &= \int_0^1 \int_v^{2-v} 3\,du\,dv \\ &= \int_0^1 [3u\,]\Big|_v^{2-v}\,dv \\ &= \int_0^1 (6-6v)\,dv \\ &= \left[6v-3v^2\right]\Big|_0^1 \\ &= 3\end{aligned}$$

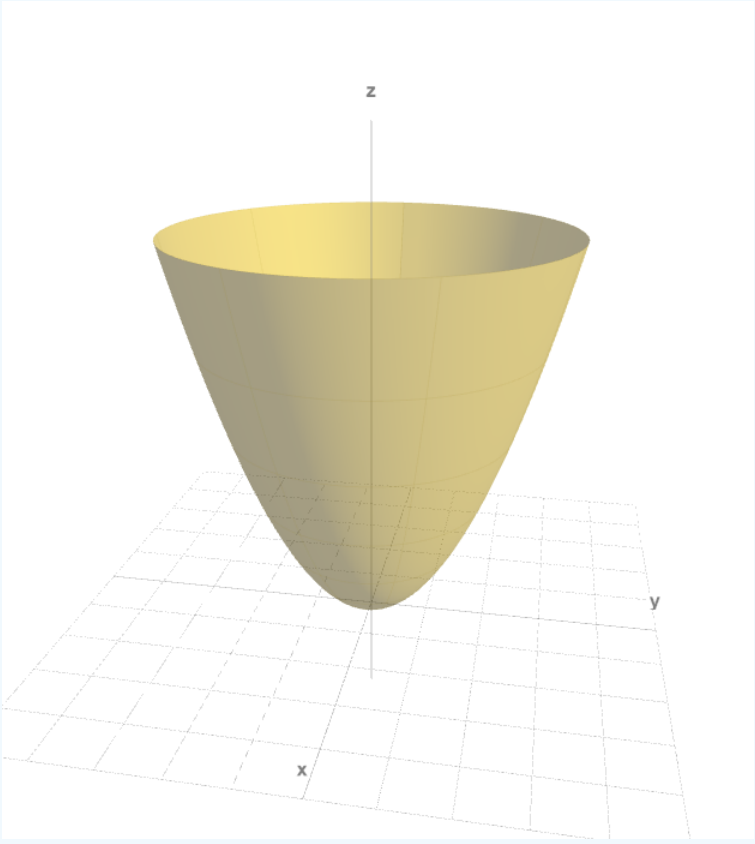
(Note: this could have been done using the shortcut $\iint_D 3\,dA = 3\cdot \text{area}(D) = 3\cdot 1 = 3$)

? Exercise 3.6.1c

Find the surface area of the part of $z = x^2+y^2$ above the circle $D = \{(x,y) : x^2+y^2 \le 2\}$

Answer

This surface S is a paraboloid with circular cross sections and $0 \le z \le 2$:



We can parameterize it using cylindrical coordinates, with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), u^2\rangle$ over $D = \left\{(u,v) : 0 \le u \le \sqrt{2},\ 0 \le \theta \le 2\pi\right\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 2u\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -2u^2\cos(v), -2u^2\sin(v), u\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{(-2u^2\cos(v))^2 + (-2u^2\sin(v))^2 + (u)^2} = \sqrt{4u^4+u^2} = u\sqrt{4u^2+1}$

Therefore the surface area of S is given by:

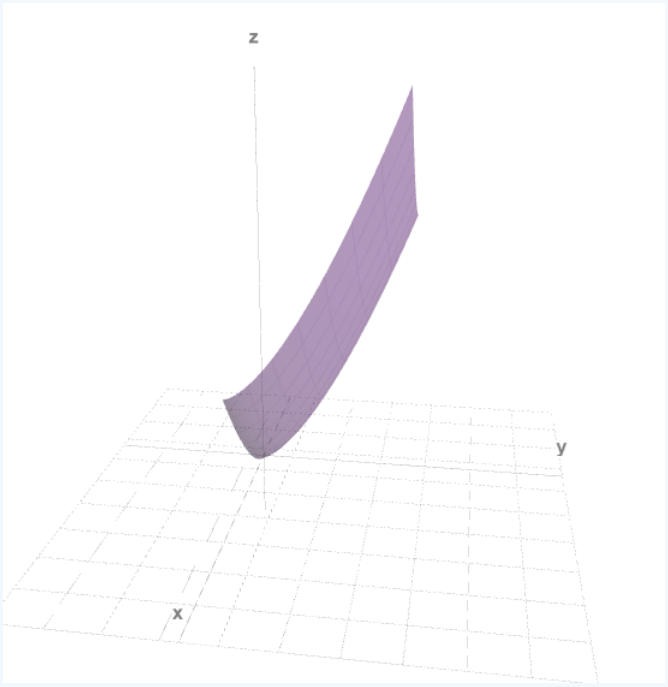
$$\begin{aligned}\iint_S 1\,dS &= \iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\ &= \int_0^{2\pi}\int_0^{\sqrt{2}} u\sqrt{4u^2+1}\,du\,dv \\ &= \left(\int_0^{2\pi} 1\,dv\right)\left(\int_0^{\sqrt{2}} u\sqrt{4u^2+1}\,du\right) \\ &= \left(v\Big|_0^{2\pi}\right)\cdot\left(\frac{1}{12}(4u^2+1)^{\frac{3}{2}}\Big|_0^{\sqrt{2}}\right) \\ &= 2\pi\cdot\frac{9^{\frac{3}{2}}-1^{\frac{3}{2}}}{12} \\ &= \frac{13\pi}{3}\end{aligned}$$

Exercise 3.6.1d

Find the surface area of the part of $z = \frac{2\sqrt{15}}{3}x^{\frac{3}{2}} + \frac{8\sqrt{3}}{3}y^{\frac{3}{2}}$ over the square $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$

Answer

This surface S is not a famous shape, but it is given as an explicitly defined function $z = f(x,y)$:



Therefore we can parameterize it using $u = x$ and $v = y$:

$\vec{\mathbf{r}}(u,v) = \left\langle u, v, \frac{2\sqrt{15}}{3}u^{\frac{3}{2}} + \frac{8\sqrt{3}}{3}v^{\frac{3}{2}}\right\rangle$ over $D = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \left\langle 1, 0, \sqrt{15}\cdot u^{\frac{1}{2}}\right\rangle$ and $\vec{\mathbf{r}}_v = \left\langle 0, 1, 4\sqrt{3}\cdot v^{\frac{1}{2}}\right\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -\sqrt{15u}, -4\sqrt{3v}, 1\right\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{(-\sqrt{15u})2^2 + \left(-4\sqrt{3v}\right)^2 + 1^2} = \sqrt{15u + 48v + 1}$

Therefore the surface area of S is given by:

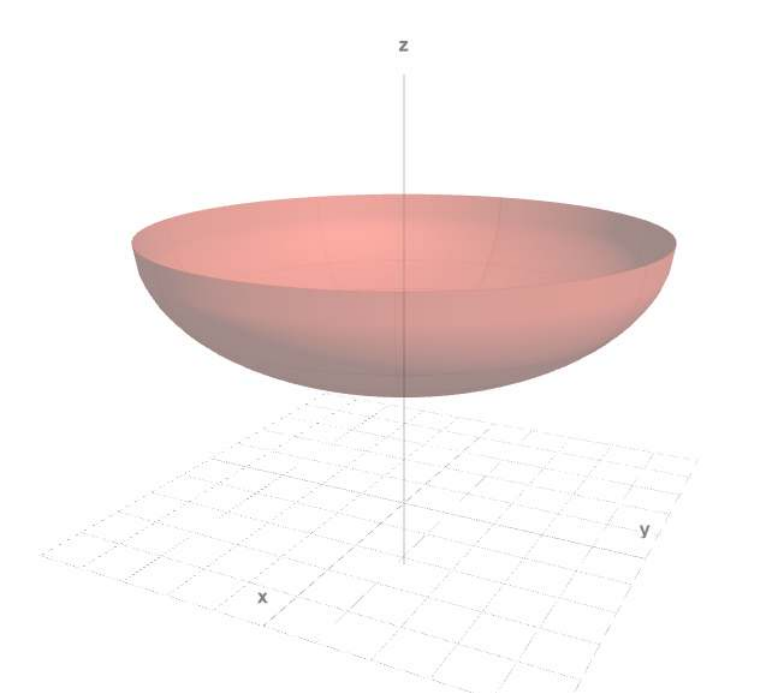
$$
\begin{aligned}
\iint_S 1\,dS &= \iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\
&= \int_0^1 \int_0^1 \sqrt{15u+48v+1}\,dv\,du \\
&= \int_0^1 \frac{1}{72}(15u+48v+1)^{\frac{3}{2}}\Big|_0^1\,du \\
&= \int_0^1 \frac{1}{72}\left((15u+49)^{\frac{3}{2}} - (15u+1)^{\frac{3}{2}}\right)du \\
&= \frac{1}{72}\cdot\frac{2}{75}\left[(15u+49)^{\frac{5}{2}} - (15u+1)^{\frac{5}{2}}\right]\Big|_0^1 \\
&= \frac{1}{75\cdot 36}\left(64^{\frac{5}{2}} - 16^{\frac{5}{2}} - 49^{\frac{5}{2}} + 1^{\frac{5}{2}}\right) \\
&= \frac{8^5 - 4^5 - 7^5 + 1}{75\cdot 36} = \frac{7469}{1350}
\end{aligned}
$$

Exercise 3.6.1e

Find the surface area of $S = \left\{(x,y,z) \,:\, z = 4 - \sqrt{4-x^2-y^2}\right\}$

Answer

This surface S is a half sphere shifted up on the z-axis.



We can parameterize it using cylindrical coordinates, with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u,v) = \left\langle u\cos(v), u\sin(v), 4 - \sqrt{4-u^2}\right\rangle$ over $D = \{(u,v) \,:\, 0 \le u \le 2,\, 0 \le \theta \le 2\pi\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \left\langle \cos(v), \sin(v), \frac{u}{\sqrt{4-u^2}}\right\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -\frac{u^2\cos(v)}{\sqrt{4-u^2}}, -\frac{u^2\sin(v)}{\sqrt{4-u^2}}, u\right\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{\left(-\frac{u^2\cos(v)}{\sqrt{4-u^2}}\right)^2 + \left(-\frac{u^2\sin(v)}{\sqrt{4-u^2}}\right)^2 + (u)^2} = \sqrt{\frac{4u^2}{4-u^2}} = \frac{2u}{\sqrt{4-u^2}}$

Therefore the surface area of S is given by:

$$\begin{aligned}\iint_S 1\,dS &= \iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\ &= \int_0^{2\pi}\int_0^2 \frac{2u}{\sqrt{4-u^2}}\,du\,dv \\ &= \left(\int_0^{2\pi} 1\,dv\right)\left(\int_0^2 \frac{2u}{\sqrt{4-u^2}}\,du\right) \\ &= \left(v\Big|_0^{2\pi}\right)\cdot\left(-2(4-u^2)^{\frac{1}{2}}\Big|_0^2\right) \\ &= 2\pi\cdot 4 \\ &= 8\pi\end{aligned}$$

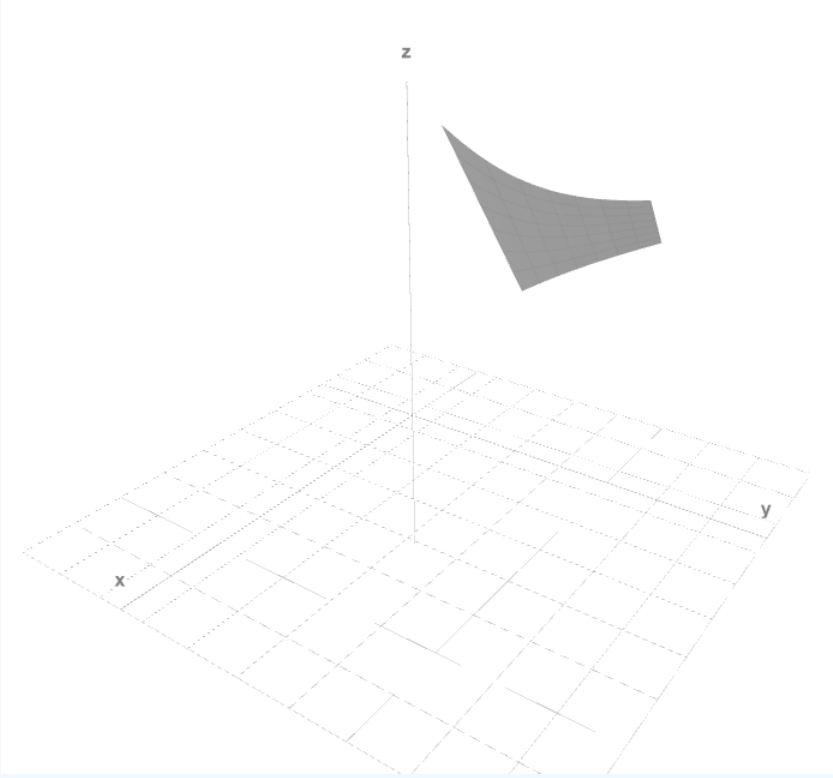
This matches with the surface area formula of a sphere (multiplied in half): $SA = \frac{1}{2}\cdot 4\pi\cdot 2^2 = 8\pi$

Exercise 3.6.1f

Find the surface area of the part of $z = x + \sqrt{y}$ over $D = \left\{(x,y)\ :\ 0 \le x \le \frac{1}{y^2},\ 1 \le y \le 2\right\}$

Answer

This surface S is defined explicitly as a function $z = f(x,y)$:



Therefore we can parameterize it using $u = x$ and $v = y$:

$\vec{\mathbf{r}}(u,v) = \langle u, v, u + \sqrt{v}\rangle$ over $D = \left\{(u,v)\ :\ 1 \le v \le 2,\ 0 \le u \le \frac{1}{v^2}\right\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \langle 1, 0, 1\rangle$ and $\vec{\mathbf{r}}_v = \left\langle 0, 1, \frac{1}{2}v^{-\frac{1}{2}}\right\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -1, -\frac{1}{2\sqrt{v}}, 1\right\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{(-1)^2 + \left(-\frac{1}{2\sqrt{v}}\right)^2 + 1^2} = \sqrt{2 + \frac{1}{4v}}$

Therefore the surface area of S is given by:

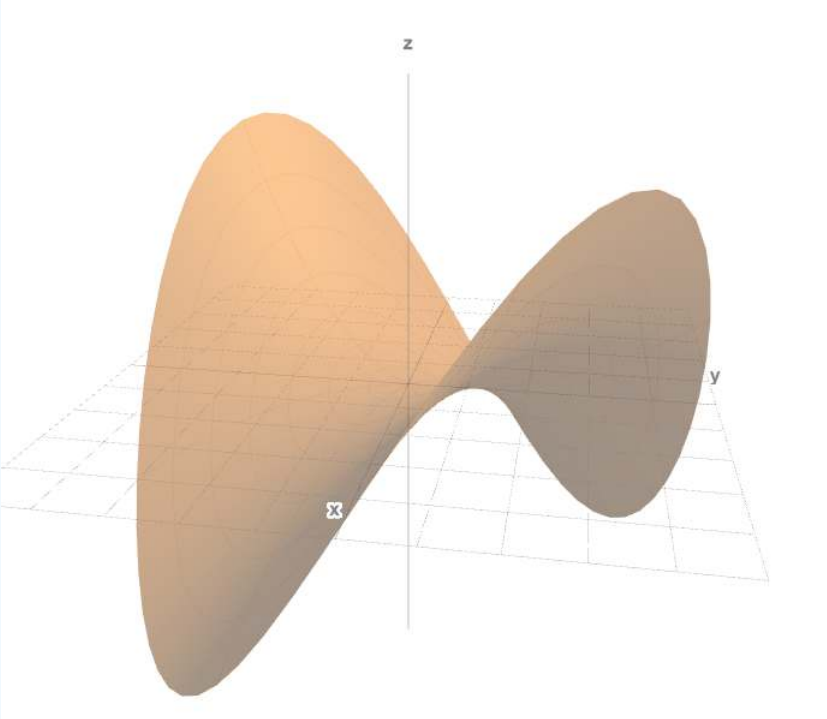
$$\begin{aligned}
\iint_S 1\, dS &= \iint_D \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\, dA \\
&= \int_1^2 \int_0^{\frac{1}{v^2}} \sqrt{2 + \frac{1}{4v}}\, du\, dv \\
&= \int_1^2 u\sqrt{2 + \frac{1}{4v}}\Big|_0^{\frac{1}{v^2}}\, dv \\
&= \int_1^2 \frac{1}{v^2}\sqrt{2 + \frac{1}{4v}}\, dv \\
&= -\frac{8}{3}\left(2 + \frac{1}{4v}\right)^{\frac{3}{2}}\Big|_1^2 \\
&= -\frac{8}{3}\left(\left(\frac{17}{8}\right)^{\frac{3}{2}} - \left(\frac{9}{4}\right)^{\frac{3}{2}}\right) \\
&= 9 - \frac{17\sqrt{17}}{6\sqrt{2}}
\end{aligned}$$

? Exercise 3.6.2a

Evaluate $\iint_S (5 + 5x^2 + 5y^2)\, dS$ where $S = \{(x, y, z) : z = xy,\ x^2 + y^2 \leq 8\}$

Answer

This surface S is a circular chunk of a function that resembles a rotated hyperbolic paraboloid.



Because of its domain, we use cylindrical coordinates, with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), u^2\cos(v)\sin(v)\rangle$ over $D = \left\{(u, v) : 0 \leq u \leq 2\sqrt{2},\ 0 \leq \theta \leq 2\pi\right\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 2u\cos(v)\sin(v) \rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), u^2(\cos^2(v) - \sin^2(v)) \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -u^2\sin(v), -u^2\cos(v), u \rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{(-u^2\cos(v))^2 + (-u^2\sin(v))^2 + (u)^2} = \sqrt{u^4 + u^2} = u\sqrt{u^2+1}$

Therefore the surface integral over S is given by:

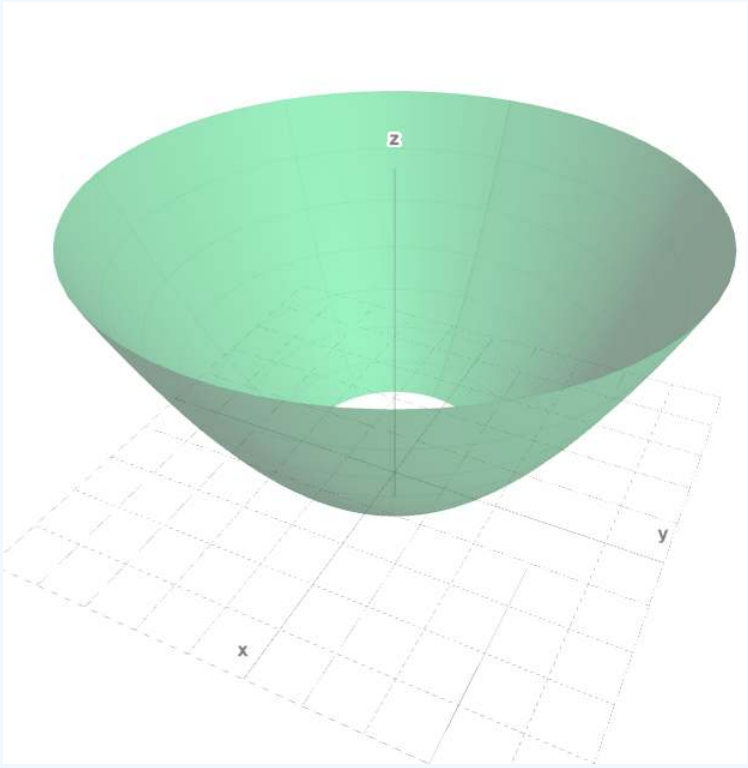
$$\begin{aligned}
\iint_S (5 + 5x^2 + 5y^2)\, dS &= \iint_D (5 + 5u^2) \cdot \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\, dA \\
&= \int_0^{2\pi} \int_0^{2\sqrt{2}} (5 + 5u^2) \cdot u\sqrt{u^2+1}\, du\, dv \\
&= \left(\int_0^{2\pi} 5\, dv \right) \left(\int_0^{2\sqrt{2}} u(u^2+1)^{\frac{3}{2}}\, du \right) \\
&= \left(5v \Big|_0^{2\pi} \right) \cdot \left(\frac{1}{5}(u^2+1)^{\frac{5}{2}} \Big|_0^{2\sqrt{2}} \right) \\
&= 10\pi \cdot \frac{9^{\frac{5}{2}} - 1^{\frac{5}{2}}}{5} \\
&= 484\pi
\end{aligned}$$

? Exercise 3.6.2*b*

Evaluate $\iint_S \frac{1}{\sqrt{4z+1}}\, dS$ where $S = \{(x, y, z) : z = x^2 + y^2,\ 1 \le z \le 9\}$

Answer

This surface S is a standard paraboloid with domain between two concentric circles.



Because of its domain, we use cylindrical coordinates, with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), u^2 \rangle$ over $D = \{(u, v) : 1 \le u \le 3,\ 0 \le v \le 2\pi\}$

https://math.libretexts.org/@go/page/194357

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 2u \rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -2u^2\cos(v), -2u^2\sin(v), u \rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{(-2u^2\sin(v))^2 + (-2u^2\cos(v))^2 + (u)^2} = \sqrt{4u^4 + u^2} = u\sqrt{4u^2+1}$

Therefore the surface integral over S is given by:

$$\begin{aligned}
\iint_S \frac{1}{\sqrt{4z+1}}\, dS &= \iint_D \frac{1}{\sqrt{4u^2+1}} \cdot \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\, dA \\
&= \int_0^{2\pi}\int_1^3 \frac{1}{\sqrt{4u^2+1}} \cdot u\sqrt{4u^2+1}\, du\, dv \\
&= \left(\int_0^{2\pi} 1\, dv\right)\left(\int_1^3 u\, du\right) \\
&= \left(v\Big|_0^{2\pi}\right) \cdot \left(\frac{1}{2}u^2\Big|_1^3\right) \\
&= 2\pi \cdot 4 \\
&= 8\pi
\end{aligned}$$

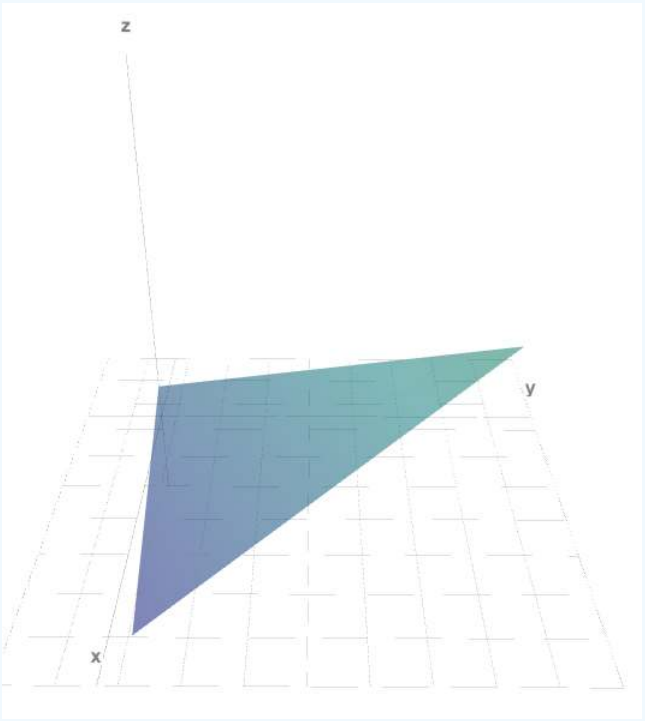
(Note: this could have been done in rectangular coordinates and would reduce to $\iint_D 1\, dA = \text{area}(D) = \pi \cdot 3^2 - \pi \cdot 1^2 = 8\pi$)

? Exercise 3.6.2c

Evaluate $\iint_S (2x + 3y + 5z)\, dS$ where S is the part of $3x - 4y + 5z = 2$ above the triangle with vertices $(0,0), (2,0)$, and $(2,4)$

Answer

This surface is a triangular cut out of the given plane $z = \frac{2}{5} - \frac{3}{5}x + \frac{4}{5}y$, as seen below:



The domain is easy to write as Type 1, so we can parameterize it using $u = x$ and $v = y$:

$\vec{\mathbf{r}}(u,v) = \left\langle u, v, \frac{2}{5} - \frac{3}{5}u + \frac{4}{5}v \right\rangle$ over $D = \{(u,v) : 0 \le u \le 2,\ 0 \le v \le 2u\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \left\langle 1, 0, -\frac{3}{5} \right\rangle$ and $\vec{\mathbf{r}}_v = \left\langle 0, 1, \frac{4}{5} \right\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle \frac{3}{5}, -\frac{4}{5}, 1 \right\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 + 1^2} = \sqrt{2}$

Therefore the surface integral of S is given by:

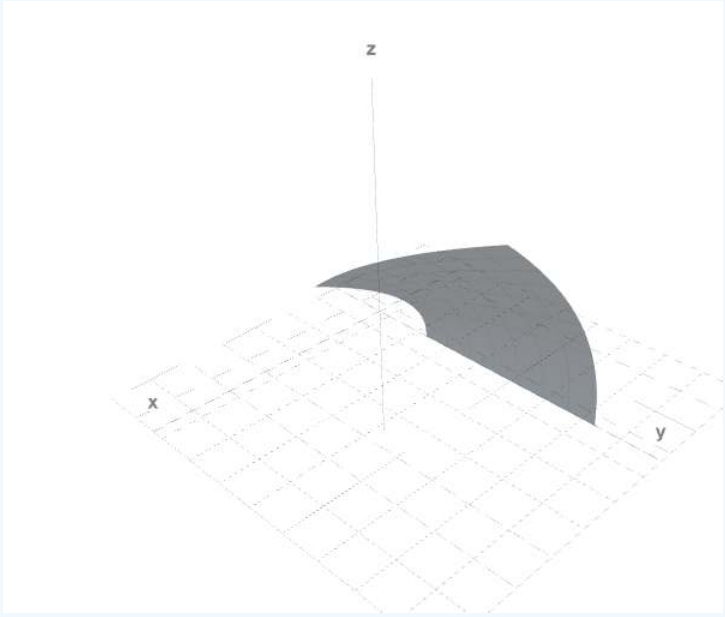
$$\begin{aligned}
\iint_S (2x + 3y + 5z)\, dS &= \iint_D \left(2u + 3v + 5\left(\frac{2}{5} - \frac{3}{5}u + \frac{4}{5}v\right)\right) \cdot \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\, dA \\
&= \int_0^2 \int_0^{2u} (-u + 7v + 2) \cdot \sqrt{2}\, dv\, du \\
&= \sqrt{2} \cdot \int_0^2 \left[-uv + \frac{7}{2}v^2 + 2v\right]\Big|_0^{2u} du \\
&= \sqrt{2} \cdot \int_0^2 4u + 12u^2\, du \\
&= \sqrt{2} \cdot \left[2u^2 + 4u^3\right]\Big|_0^2 \\
&= 40\sqrt{2}
\end{aligned}$$

Exercise 3.6.2d

Evaluate $\iint_S z\, dS$ where $S = \left\{(x, y, z) \,:\, z = \sqrt{2xy},\ 0 \le x \le 1,\ 1 \le y \le 4\right\}$

Answer

This surface is given as an explicit function $z = f(x, y)$:



The domain is a rectangle, so we can parameterize it using $u = x$ and $v = y$:

$\vec{\mathbf{r}}(u, v) = \left\langle u, v, \sqrt{2uv} \right\rangle$ over $D = \{(u, v) \,:\, 0 \le u \le 1,\ 1 \le v \le 4\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \left\langle 1, 0, \sqrt{\frac{v}{2u}} \right\rangle$ and $\vec{\mathbf{r}}_v = \left\langle 0, 1, \sqrt{\frac{u}{2v}} \right\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -\sqrt{\frac{v}{2u}}, -\sqrt{\frac{u}{2v}}, 1 \right\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{\left(-\sqrt{\frac{v}{2u}}\right)^2 + \left(-\sqrt{\frac{u}{2v}}\right)^2 + 1^2} = \sqrt{\frac{v^2+2uv+u^2}{2uv}} = \frac{u+v}{\sqrt{2uv}}$

Therefore the surface integral of S is given by:

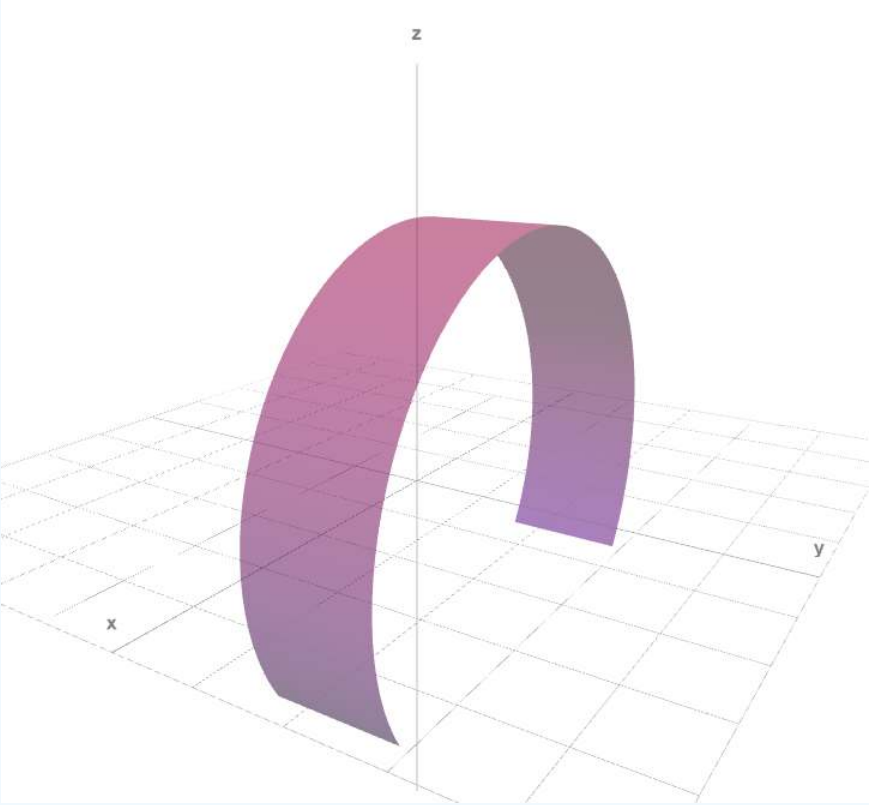
$$\begin{aligned}
\iint_S z\,dS &= \iint_D \sqrt{2uv} \cdot \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\
&= \int_0^1 \int_1^4 \sqrt{2uv} \cdot \frac{u+v}{\sqrt{2uv}}\,dv\,du \\
&= \int_0^1 \int_1^4 (u+v)\,dv\,du \\
&= \int_0^1 \left[uv + \frac{1}{2}v^2\right]\Big|_1^4 du \\
&= \int_0^1 \left(3u + \frac{15}{2}\right) du \\
&= \left[\frac{3}{2}u^2 + \frac{15}{2}u\right]\Big|_0^1 \\
&= 9
\end{aligned}$$

? Exercise 3.6.2e

Evaluate $\iint_S (x^2 - y^2)\,dS$ where $S = \{(x, y, z) : x^2 + z^2 \leq 4,\ z \geq -1,\ 0 \leq y \leq 1\}$

Answer

This surface is part of a cylinder along the y-axis. We cannot solve for z as a function of x and y since it does not pass the line test.



Therefore we parameterize it similarly to cylindrical coordinates with y and z switched. The radius $r = 2$ is fixed, and y equals itself.

Note $z \geq -1 \longleftrightarrow 2 \cdot \sin(\theta) \geq -1 \longleftrightarrow -\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$. Let $u = y$ and $v = \theta$:

$\vec{\mathbf{r}}(u, v) = \langle 2\cos(v), u, 2\sin(v) \rangle$ over $D = \left\{ (u, v) \, : \, 0 \leq u \leq 1, \, -\frac{\pi}{6} \leq v \leq \frac{7\pi}{6} \right\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \langle 0, 1, 0 \rangle$ and $\vec{\mathbf{r}}_v = \langle -2\sin(v), 0, 2\cos(v) \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2\cos(v), 0, 2\sin(v) \rangle$

The length of this cross product is $\| \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v \| = \sqrt{(2\cos(v))^2 + 0^2 + (2\sin(v))^2} = \sqrt{4} = 2$

Therefore the surface integral of S is given by:

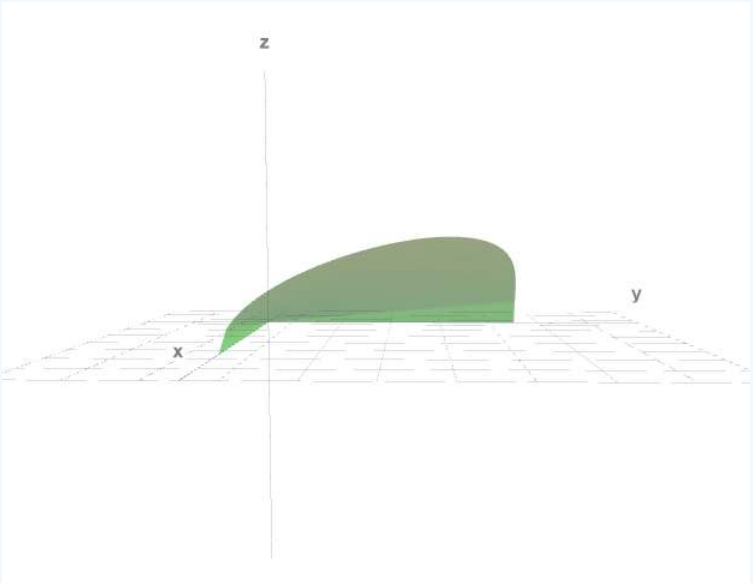
$$\begin{aligned}
\iint_S (x^2 - y^2) \, dS &= \iint_D \left((2\cos(v))^2 - u^2 \right) \cdot \| \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v \| \, dA \\
&= \int_0^1 \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left(4\cos^2(v) - u^2 \right) \cdot 2 \, dv \, du \\
&= \int_0^1 \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left(4 + 4\cos(2v) - 2u^2 \right) \, dv \, du \\
&= \int_0^1 \left[4v + 2\sin(2v) - 2u^2 v \right] \Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \, du \\
&= \int_0^1 \left(\frac{16\pi}{3} + 2\sqrt{3} - \frac{8\pi}{3} u^2 \right) \, du \\
&= \left[\frac{16\pi}{3} u + 2\sqrt{3} \cdot u - \frac{8\pi}{9} u^3 \right] \Big|_0^1 \\
&= \frac{40\pi}{9} + 2\sqrt{3}
\end{aligned}$$

? Exercise 3.6.2f

Evaluate $\iint_S (x^2 - y^2) z \, dS$ where $S = \left\{ (x, y, z) \, : \, z = \sqrt{\frac{xy}{2}}, \, 0 \leq x \leq \sqrt{1 - y^2} \right\}$

Answer

This surface is given as an explicit function $z = f(x, y)$ over a quarter circle domain:



Because of its domain, we use cylindrical coordinates, with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u,v) = \left\langle u\cos(v), u\sin(v), \sqrt{\frac{u^2\cos(v)\sin(v)}{2}} \right\rangle$ over $D = \left\{(u,v) \,:\, 0 \le u \le 1,\, 0 \le v \le \frac{\pi}{2}\right\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \left\langle \cos(v), \sin(v), \sqrt{\frac{\cos(v)\sin(v)}{2}} \right\rangle$ and $\vec{\mathbf{r}}_v = \left\langle -u\sin(v), u\cos(v), \frac{u\left(\cos^2(v) - \sin^2(v)\right)}{2\sqrt{2\cos(v)\sin(v)}} \right\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -\frac{u\sin(v)}{2\sqrt{2\cos(v)\sin(v)}}, \frac{-u\cos(v)}{2\sqrt{2\cos(v)\sin(v)}}, u \right\rangle$

The length of this cross product is

$$\begin{aligned}\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| &= \sqrt{\left(-\frac{u\sin(v)}{2\sqrt{2\cos(v)\sin(v)}}\right)^2 + \left(-\frac{u\cos(v)}{2\sqrt{2\cos(v)\sin(v)}}\right)^2 + (u)^2} \\ &= \sqrt{\frac{u^2}{8\cos(v)\sin(v)} + u^2} = u\sqrt{\frac{1}{8\cos(v)\sin(v)} + 1}\end{aligned}$$

Therefore the surface integral over S is given by:

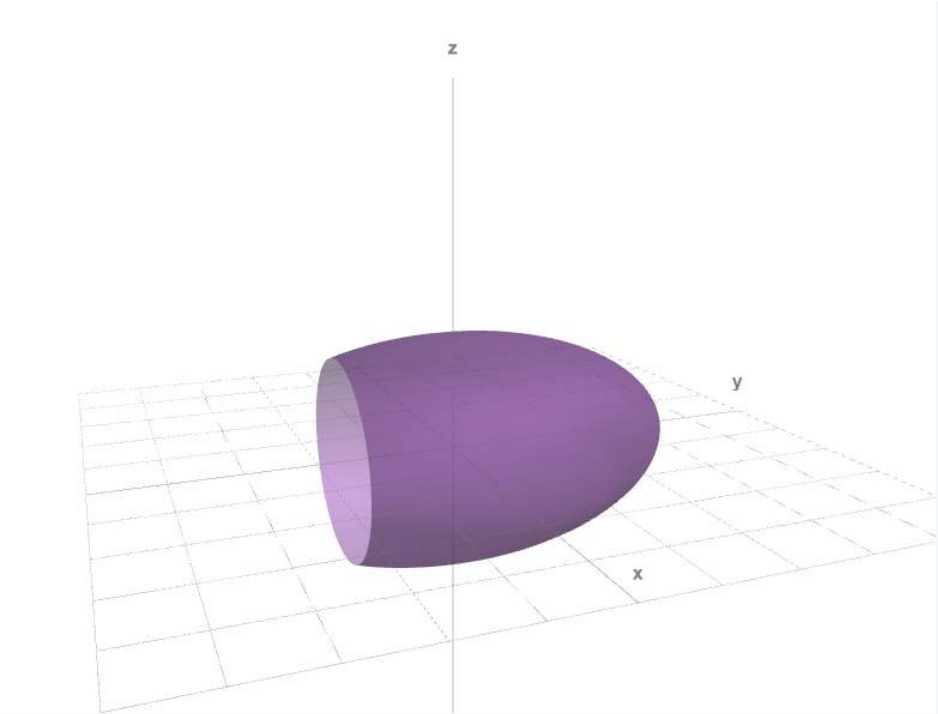
$$\begin{aligned}\iint_S (x^2 - y^2)z\, dS &= \iint_D \left((u\cos(v))^2 - (u\sin(v))^2\right) \cdot u\sqrt{\frac{\cos(v)\sin(v)}{2}} \cdot \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\, dA \\ &= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{u^3(\cos^2(v) - \sin^2(v))\sqrt{\cos(v)\sin(v)}}{\sqrt{2}} \cdot u\sqrt{\frac{1}{8\cos(v)\sin(v)} + 1}\, du\, dv \\ &= \left(\int_0^{\frac{\pi}{2}} (\cos^2(v) - \sin^2(v)) \cdot \sqrt{\frac{1}{8} + \cos(v)\sin(v)}\, dv\right)\left(\int_0^1 \frac{1}{\sqrt{2}} u^4\, du\right) \\ &= \left(\frac{2}{3}\left(\frac{1}{8} + \cos(v)\sin(v)\right)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}}\right) \cdot \left(\frac{1}{5\sqrt{2}} u^5\Big|_0^1\right) \\ &= \frac{2}{3}\left(\left(\frac{1}{8}\right)^{\frac{3}{2}} - \left(\frac{1}{8}\right)^{\frac{3}{2}}\right) \cdot \frac{1}{5\sqrt{2}} \\ &= 0\end{aligned}$$

? Exercise 3.6.2g

Evaluate $\iint_S x\, dS$ where $S = \left\{(x,y,z) \,:\, x^2 + \frac{y^2}{4} + z^2 = 1,\, y \ge -1\right\}$

Answer

This surface is part of an ellipsoid as pictured below. It does not pass the line test in any direction, so we cannot solve for an explicit function.



Instead we will use something similar to spherical coordinates, with the role of y and z switched (this allows us to use ϕ restrictions in the y direction), and the y will use $\rho = 2$ while the other two coordinates will use $\rho = 1$ to yield our desired shape.

Thus, $y \geq -1 \longleftrightarrow 2\cos(\phi) \geq -1 \longleftrightarrow \phi \leq \dfrac{2\pi}{3}$

Define $u = \theta$ and $v = \phi$, then our parameterization is:

$\vec{\mathbf{r}}(u,v) = \langle \sin(v)\cos(u), 2\cos(v), \sin(v)\sin(u) \rangle$ over $D = \left\{ (u,v) \,:\, 0 \leq u \leq 2\pi,\, 0 \leq v \leq \dfrac{2\pi}{3} \right\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \langle -\sin(v)\sin(u), 0, \sin(v)\cos(u) \rangle$ and $\vec{\mathbf{r}}_v = \langle \cos(v)\cos(u), -2\sin(v), \cos(v)\sin(u) \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2\cos(u)\sin^2(v), \cos(v)\sin(v), 2\sin(u)\sin^2(v) \rangle$

The length of this cross product is

$$\begin{aligned} \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| &= \sqrt{\left(2\cos(u)\sin^2(v)\right)^2 + \left(\cos(v)\sin(v)\right)^2 + \left(2\sin(u)\sin^2(v)\right)^2} \\ &= \sqrt{4\sin^4(v) + \cos^2(v)\sin^2(v)} = \sin(v)\sqrt{4\sin^2(v) + \cos^2(v)} \end{aligned}$$

Therefore the surface integral over S is given by:

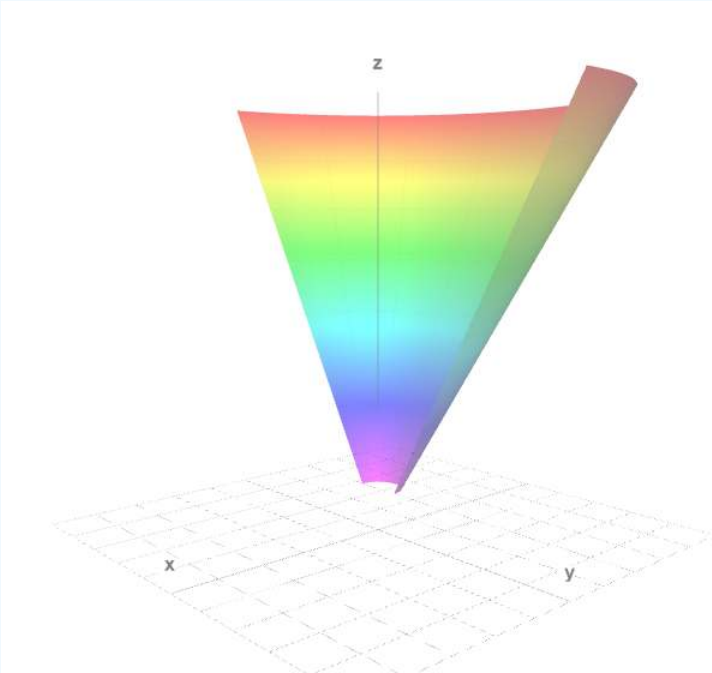
$$\begin{aligned} \iint_S x \, dS &= \iint_D \sin(v)\cos(u) \cdot \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| \, dA \\ &= \int_0^{\frac{2\pi}{3}} \int_0^{2\pi} \sin(v)\cos(u) \cdot \sin(v)\sqrt{4\sin^2(v) + \cos^2(v)} \, du \, dv \\ &= \int_0^{\frac{2\pi}{3}} \sin^2(v)\sin(u)\sqrt{4\sin^2(v) + \cos^2(v)}\Big|_0^{2\pi} \, dv \\ &= \int_0^{\frac{2\pi}{3}} 0 \, dv \\ &= 0 \end{aligned}$$

Exercise 3.6.2h

Evaluate $\iint_S (3x+4y+2z)dS$ where $S = \left\{(x,y,z) \,:\, z = \sqrt{x^2+y^2},\, 1 \le z \le 10,\, x \le 0\right\}$

Answer

This surface S is the back half of a partial cone. It is defined as an explicit function $z = f(x,y)$:



We use cylindrical coordinates, and $x \le 0 \longleftrightarrow \cos(\theta) \le 0 \longleftrightarrow \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$. Setting $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), u\rangle$ over $D = \left\{(u,v) \,:\, 1 \le u \le 10,\, \frac{\pi}{2} \le v \le \frac{3\pi}{2}\right\}$

Next we calculate the tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 1\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -u\cos(v), -u\sin(v), u\rangle$

The length of this cross product is $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{(-u\cos(v))^2 + (-u\sin(v))^2 + (u)^2} = \sqrt{2u^2} = u\sqrt{2}$

Therefore the surface integral over S is given by:

$$\begin{aligned}
\iint_S (3x+4y+2z)\,dS &= \iint_D (3u\cos(v) + 4u\sin(v) + 2u)\cdot \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\,dA \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^{10} u(3\cos(v) + 4\sin(v) + 2)\cdot u\sqrt{2}\,du\,dv \\
&= \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (3\cos(v) + 4\sin(v) + 2)\,dv\right)\left(\int_1^{10} u^2\sqrt{2}\,du\right) \\
&= [3\sin(v) - 4\cos(v) + 2v]\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cdot \left(\frac{\sqrt{2}}{3}u^3\Big|_1^{10}\right) \\
&= (2\pi - 6)\cdot 333\sqrt{2} \\
&= 666(\pi - 3)\sqrt{2}
\end{aligned}$$

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3.7: Integrals of Vector Fields on a Surface

Learning Objectives

- Explain the meaning of an oriented surface, giving an example.
- Describe the surface integral of a vector field.
- Use surface integrals to solve applied problems and find flux.

In this section, we define the orientation of a parametrized surface S and discuss its meaning. We also expand the definition of a surface integral to apply to vector fields instead of scalar functions, and define the concept of flux and its notation. It is important that the reader is able to properly distinguish the flux surface integral $\iint_S \vec{\mathbf{F}}(x,y,z)\cdot d\vec{\mathbf{S}}$ from the previous scalar surface integrals $\iint_S f(x,y,z)\,dS$.

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3.7: Orientation and Flux

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation or, in the case of surfaces without a constant sense of "up and down" (such as a sphere or cylinder), it is given an "outward" or "inward" orientation.

Let S be a smooth surface. For any point (x, y, z) on S, we can identify two unit normal vectors $\vec{\mathbf{N}}$ and $-\vec{\mathbf{N}}$. If it is possible to choose a unit normal vector $\vec{\mathbf{N}}$ at every point (x, y, z) on S so that $\vec{\mathbf{N}}$ varies continuously over S, then S is "*orientable*." Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives S an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface*. We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

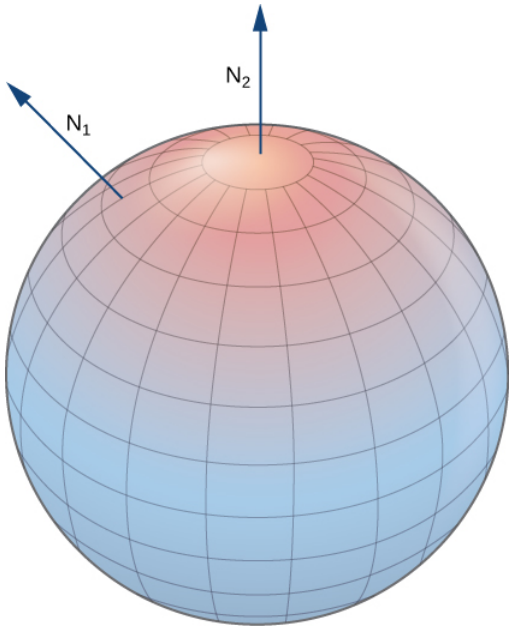


Figure 3.7.1: An oriented sphere with positive orientation.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points "above" the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points "below" the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\vec{\mathbf{r}}(u, v) = \langle u,\ v,\ f(u, v)\rangle$, where u and v vary over the domain of f. Then, $\vec{\mathbf{r}}_u = \langle 1, 0, f_x(u, v)\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 1, f_y(u, v)\rangle$, and therefore the cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ (which is normal to the surface at any point on the surface) is $\langle -f_x(u, v),\ -f_y(u, v),\ 1\rangle$. Since the z-component of this vector is one, the corresponding unit normal vector points "upward," and the upward side of the surface is chosen to be the "positive" side.

Let S be a smooth orientable surface with parameterization $\vec{\mathbf{r}}(u, v)$. For each point $\vec{\mathbf{r}}(a, b)$ on the surface, vectors $\vec{\mathbf{r}}_u$ and $\vec{\mathbf{r}}_v$ lie in the tangent plane at that point. Vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is normal to the tangent plane at $\vec{\mathbf{r}}(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v}{\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|}$$

gives an orientation of surface S.

Illustration 3.7.1: Choosing an Orientation

We will find both an inward and outward orientation of the cylinder $x^2 + y^2 = R^2,\ 0 \le z \le H$:

Using cylindrical coordinates with $r = R, \theta = u, z = v$, this surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle R\cos(u), R\sin(u), v\rangle$ for $0 \le u < 2\pi,\ 0 \le v \le H$

The tangent vectors are $\vec{\mathbf{r}}_u = \langle -R\sin(u), R\cos(u), 0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 0, 1\rangle$.

To get the orientation of the surface with this parameterization, we compute the unit normal vector $\vec{\mathbf{N}} = \frac{\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v}{\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|}$

In this case, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle R\cos(u), R\sin(u), 0\rangle$ which has length $\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\| = \sqrt{R^2\cos^2(u) + R^2\sin^2(u)} = R$

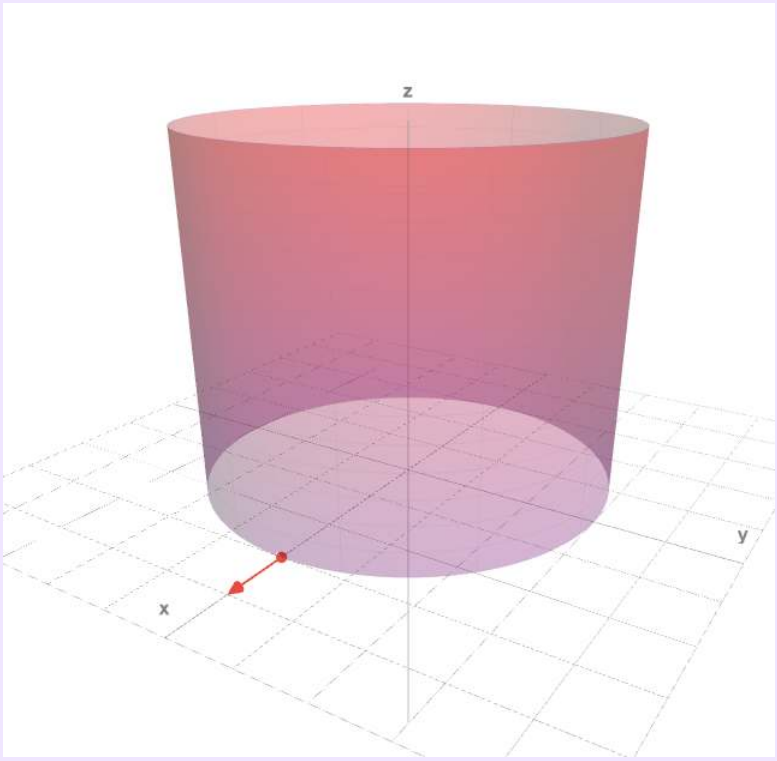
Therefore our orientation is

$$\vec{\mathbf{N}}(u, v) = \frac{\langle R\cos(u), R\sin(u), 0\rangle}{R} = \langle \cos(u),\ \sin(u),\ 0\rangle$$

Notice that all vectors are parallel to the xy-plane, which should be the case with vectors that are normal to the cylinder. If we plug in any point (u, v) in our domain to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{N}}$ we can determine if it is inward or outward. We use $u = 0, v = 0$ to get:

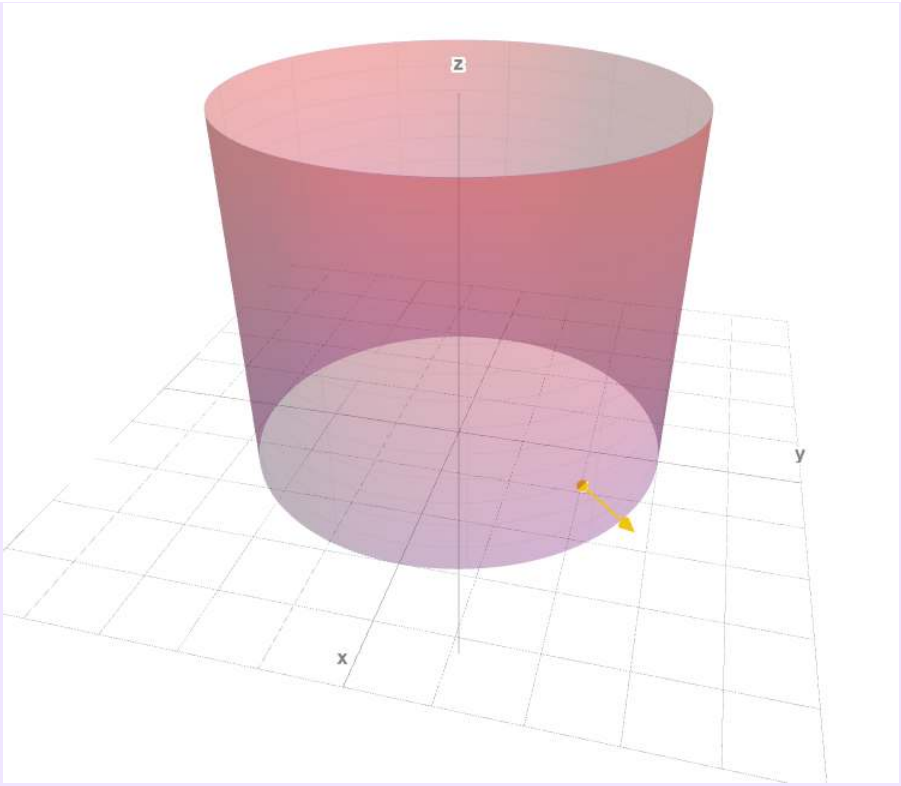
$$\vec{\mathbf{r}}\,(0, 0) = (R, 0, 0) \quad \text{and} \quad \vec{\mathbf{N}}\,(0, 0) = \langle 1, 0, 0\rangle$$

We place the start of the vector $\vec{\mathbf{N}}$ at the location $\vec{\mathbf{r}}$ on our surface to see that it points **outward**.



If we had chosen *any* point in the domain of our parameterization, we would see it is outward. For instance if we had used $u = \frac{\pi}{4}, v = 1$:

$$\vec{\mathbf{r}}\left(\frac{\pi}{4}, 1\right) = \left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, 1\right) \quad \text{and} \quad \vec{\mathbf{N}}\left(\frac{\pi}{4}, 1\right) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle$$



Now that we have found that our parameterization is outward, we will try to find an alternate parameterization that is inward. Thanks to the cross product property $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$, this can be done by simply switching the roles of u and v.

Consider $\vec{\mathbf{s}}(u, v) = \langle R\cos(v), R\sin(v), u\rangle$ for $0 \leq v < 2\pi,\ 0 \leq u \leq H$

The tangent vectors are $\vec{\mathbf{s}}_u = \langle 0, 0, 1\rangle$ and $\vec{\mathbf{s}}_v = \langle -R\sin(v), R\cos(v), 0\rangle$.

It is easy to verify that now, $\vec{\mathbf{s}}_u \times \vec{\mathbf{s}}_v = \langle -R\cos(v), -R\sin(v), 0\rangle$ and $\|\vec{\mathbf{s}}_u \times \vec{\mathbf{s}}_v\| = R$

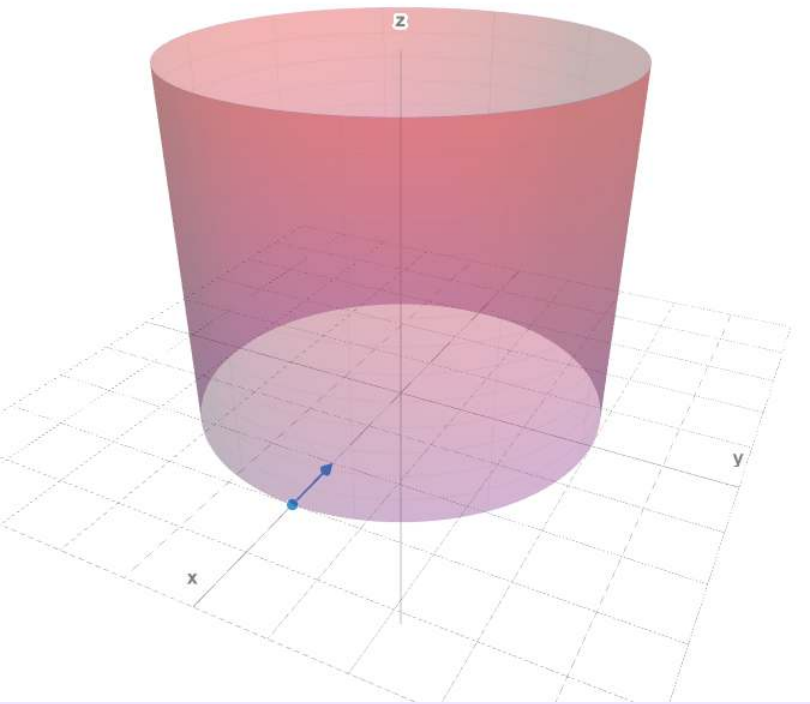
Therefore our orientation is

$$\vec{\mathbf{N}}(u, v) = \frac{\langle -R\cos(v), -R\sin(v), 0\rangle}{R} = \langle -\cos(v),\ -\sin(v),\ 0\rangle$$

And plugging any point in our domain, such as $u = 0, v = 0$, into $\vec{\mathbf{s}}$ and $\vec{\mathbf{N}}$ gives:

$$\vec{\mathbf{s}}\,(0, 0) = (R, 0, 0) \quad \text{and} \quad \vec{\mathbf{N}}\,(0, 0) = \langle -1, 0, 0\rangle$$

We place the start of the vector $\vec{\mathbf{N}}$ at the location $\vec{\mathbf{s}}$ on our surface to see that it points **inward**.



Since every curve has a “forward” and “backward” direction (or, in the case of a closed curve, a clockwise and counterclockwise direction), it is possible to give an orientation to any curve. Hence, it is possible to think of every curve as an oriented curve. This is not the case with surfaces, however. Some surfaces cannot be oriented; such surfaces are called *nonorientable*. Essentially, a surface can be oriented if the surface has an “inner” side and an “outer” side, or an “upward” side and a “downward” side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an “inner” or “outer” side.

The classic example of a nonorientable surface is the Möbius strip (pictured below). To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and then glue the ends together. Because of the half-twist in the strip, the surface has no “outer” side or “inner” side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.

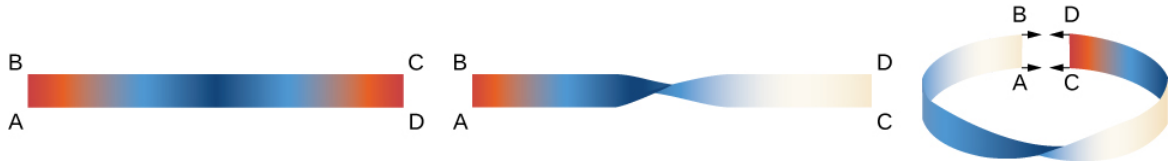


Figure 3.7.2: The construction of a Möbius strip.

Since some surfaces are *nonorientable*, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve. Thus, these types of surfaces will not be used when finding surface integrals over a vector field.

With the idea of orientable surfaces in place, we are now ready to define a surface integral of a vector field. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional vector field and C is a plane curve, then the definition of the flux of $\vec{\mathbf{F}}$ along C involved chopping C into small pieces, choosing a point inside each piece, and calculating $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point (where $\vec{\mathbf{N}}$ is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface S into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point.

To place this definition in a real-world setting, let S be an oriented surface with unit normal vector $\vec{\mathbf{N}}$. Let $\vec{\mathbf{F}}$ be a vector field of wind hitting a parachute in the shape of S, at each point on the surface. How could we calculate the total "push" the wind does across S?

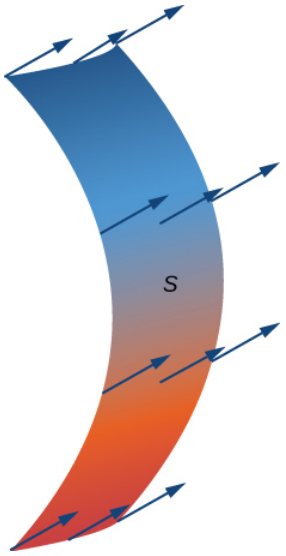


Figure 3.7.3: Vector flows across a surface S

To calculate the total wind push (known as *flux*) across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector at P can be used to approximate $\vec{\mathbf{N}}(x,y,z)$ across the entire piece S_{ij} because the normal vector to a plane does not change as we move across the plane. The amount of the vector $\vec{\mathbf{F}}$ at P in the direction of $\vec{\mathbf{N}}$ is $\text{proj}_{\vec{\mathbf{N}}}\left(\vec{\mathbf{F}}\right) = \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\right)\vec{\mathbf{N}}$ at P. Since S_{ij} is small, the dot product $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ changes very little as we vary across S_{ij} and therefore $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ can be taken as approximately constant across S_{ij}.

To approximate the total push of wind across S_{ij} (and not just locally at point P), we need to multiply $\left(\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\right)(P)$ by the area of S_{ij}. Therefore, the total amount of push across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $\left(\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\right)\Delta S_{ij}$ where $\vec{\mathbf{N}}$ and $\vec{\mathbf{F}}$ all evaluated at P.

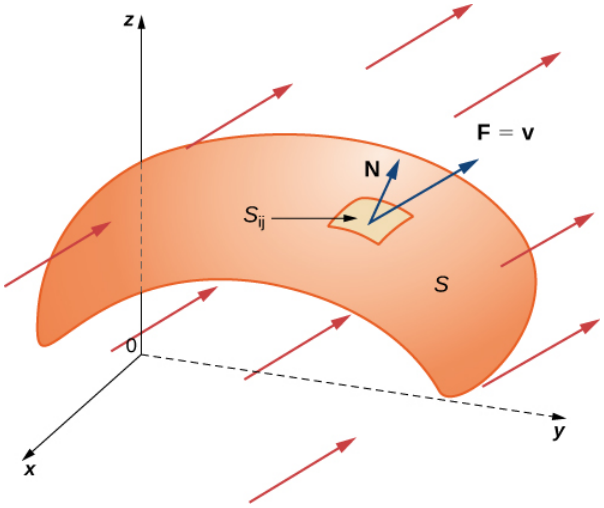


Figure 3.7.4: The push of wind across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})\Delta S_{ij}$

To approximate the total flux across S, form the sum

$$\sum_{i=1} m \sum_{j=1}^{n} (\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})\Delta S_{ij}$$

As pieces S_{ij} get smaller, the sum above gets arbitrarily close to the total flux. Therefore, the flux is

$$\iint_s \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}) \Delta S_{ij}$$

Definition: Flux Surface Integrals

Let $\vec{\mathbf{F}}$ be a continuous vector field defined on an oriented surface S with unit normal vector $\vec{\mathbf{N}}$. The **flux** of $\vec{\mathbf{F}}$ over S is

$$\iint_s \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}) \Delta S_{ij}$$

Theorem 3.7.1: Evaluating Flux Integrals

Let $\vec{\mathbf{F}}$ be a continuous vector field defined on an oriented surface S that is positively oriented by the parameterization $\vec{\mathbf{r}}(u,v)$ with parametric domain D, the flux of $\vec{\mathbf{F}}$ over S is:

$$\begin{aligned}\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS \\ &= \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot \frac{\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v}{\|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|} \|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\|\, dA \\ &= \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA\end{aligned}$$

Example 3.7.1: Calculating a Flux Integral

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS,$$

where $\vec{\mathbf{F}} = \langle -y, x, 0\rangle$ and S is the surface with parameterization

$$\vec{\mathbf{r}}(u,v) = \langle u, v^2 - u,\ u+v\rangle, \quad \text{for} \quad 0 \le u \le 3,\ 0 \le v \le 4$$

Solution

The tangent vectors are $\vec{\mathbf{r}}_u = \langle 1, -1, 1\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 2v, 1\rangle$. Therefore,

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -1-2v, -1, 2v\rangle$$

Plugging this into the formula for flux integrals, we get:

$$\begin{aligned}\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^4 \int_0^3 F(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, du\, dv \\ &= \int_0^4 \int_0^3 \langle u - v^2,\ u,\ 0\rangle \cdot \langle -1-2v,\ -1,\ 2v\rangle\, du\, dv \\ &= \int_0^4 \int_0^3 [(u-v^2)(-1-2v) - u]\, du\, dv \\ &= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\, du\, dv \\ &= \int_0^4 \left[2v^3 u + v^2 u - v u^2 - u^2\right]\Big|_0^3 dv \\ &= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, dv \\ &= \left[\frac{3}{2}v^4 + v^3 - \frac{9}{2}v^2 - 9v\right]\Big|_0^4 \\ &= 340\end{aligned}$$

Therefore, $\boxed{\text{the flux is } 340}$

Example 3.7.2: Flux on a Sphere

Find the flux of $\vec{\mathbf{F}}(x,y,z) = \langle 2x,\, 2y,\, -z\rangle$ across $S = \{(x,y,z) \mid x^2+y^2+z^2 = 9,\, z \le 2\}$ with outward orientation.

Solution

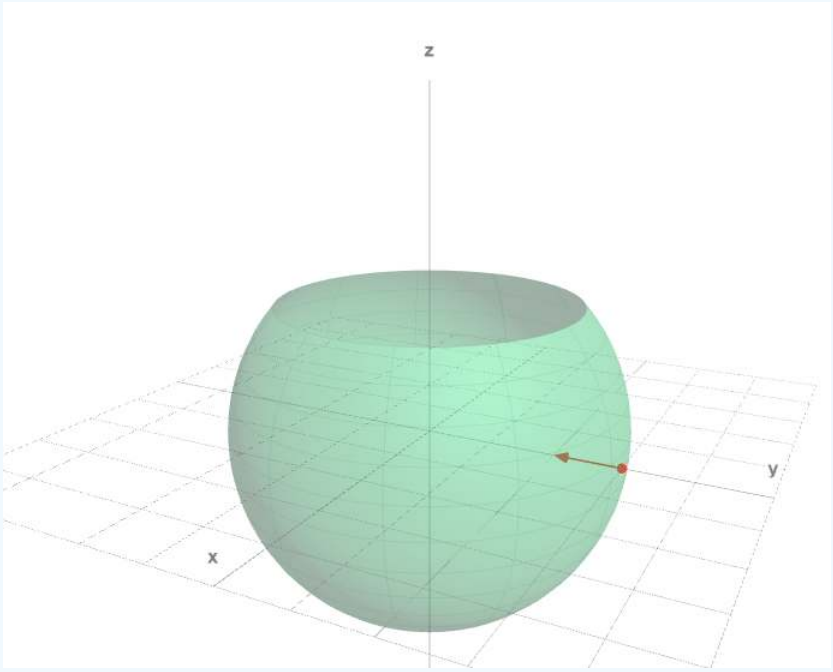
The easiest parameterization of the surface is to use spherical coordinates with $\rho = 3, \theta = u, \phi = v$. Since $z = 3\cos(\phi) \le 2 \implies \phi \ge \cos^{-1}\left(\frac{2}{3}\right)$, we have:

$$\vec{\mathbf{r}}(u,v) = \langle 3\sin(v)\cos(u), 3\sin(v)\sin(u), 3\cos(v)\rangle \text{ with}$$
$$D = \left\{(u,v) \,:\, 0 \le u \le 2\pi,\, \cos^{-1}\left(\frac{2}{3}\right) \le v \le \pi\right\}$$

We calculate $\vec{\mathbf{r}}_u = \langle -3\sin(v)\sin(u), 3\sin(v)\cos(u), 0\rangle$ and $\vec{\mathbf{r}}_v = \langle 3\cos(v)\cos(u), 3\cos(v)\sin(u), -3\sin(v)\rangle$.

The cross product of these vectors is $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -9\sin^2(v)\cos(u), -9\sin^2(v)\sin(u), -9\sin(v)\cos(v)\rangle$

Notice that for $u = \frac{\pi}{2}, v = \frac{\pi}{2}$ we have $\vec{\mathbf{r}}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = (0,3,0)$ and $\vec{\mathbf{N}}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{\langle 0,-9,0\rangle}{9} = \langle 0,-1,0\rangle$ so this parameterization is actually pointing *inward*.



Since we want outward orientation, we simply use $-(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)$ in the flux integral formula to compensate.

Thus, we evaluate

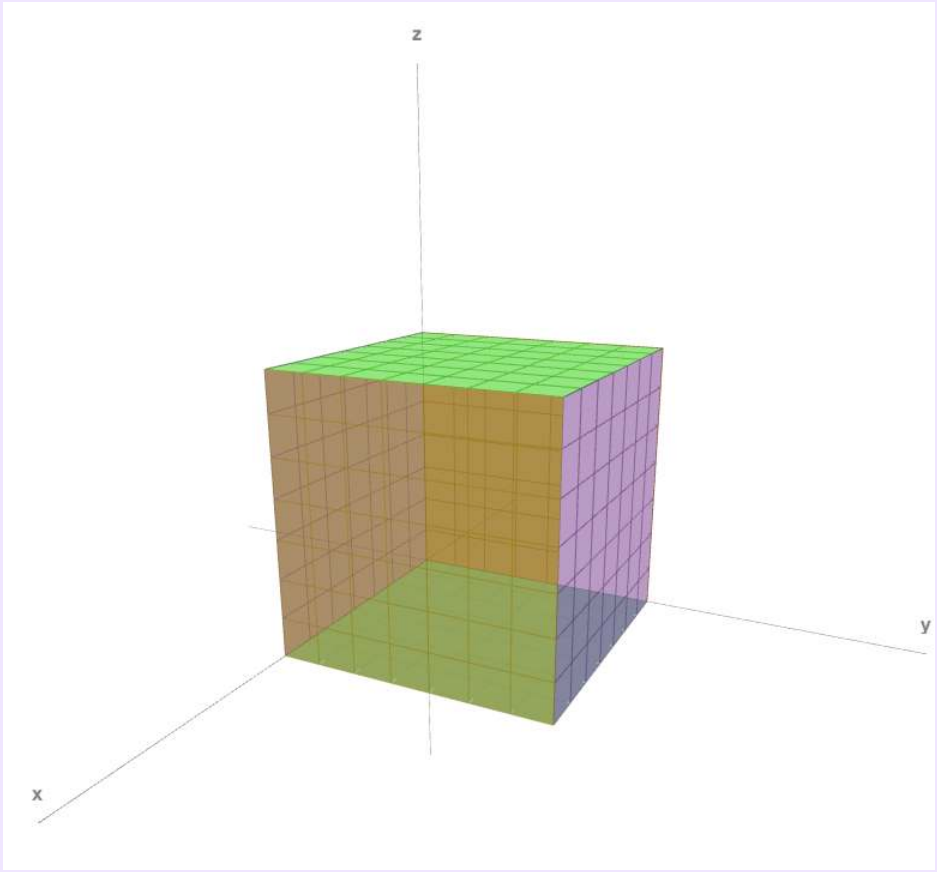
$$
\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_{\cos^{-1}\left(\frac{2}{3}\right)}^{\pi} \int_0^{2\pi} \langle 6\sin(v)\cos(u), 6\sin(v)\sin(u), -3\cos(v)\rangle \\
&\qquad \cdot \left\langle 9\sin^2(v)\cos(u), 9\sin^2(v)\sin(u), 9\sin(v)\cos(v)\right\rangle \, du\, dv \\
&= \int_{\cos^{-1}\left(\frac{2}{3}\right)}^{\pi} \int_0^{2\pi} \left(54\sin^3(v)\cos^2(u) + 54\sin^3(v)\sin^2(u) - 27\sin(v)\cos^2(v)\right) du\, dv \\
&= \int_{\cos^{-1}\left(\frac{2}{3}\right)}^{\pi} \int_0^{2\pi} \left(54\sin^3(v) - 27\sin(v)\cos^2(v)\right) du\, dv \\
&= \int_{\cos^{-1}\left(\frac{2}{3}\right)}^{\pi} 27\left(2\sin^3(v) - \sin(v)\cos^2(v)\right) u \Big|_0^{2\pi} dv \\
&= \int_{\cos^{-1}\left(\frac{2}{3}\right)}^{\pi} 54\pi\left(2\sin^3(v) - \sin(v)\cos^2(v)\right) dv \\
&= \int_{\cos^{-1}\left(\frac{2}{3}\right)}^{\pi} 54\pi\left(2 - 3\cos^2(v)\right)\sin(v)\, dv \\
&= 54\pi\left[-2\cos(v) + \cos^3(v)\right] \Big|_{\cos^{-1}\left(\frac{2}{3}\right)}^{\pi} \\
&= 54\pi \cdot \frac{55}{27} \\
&= 110\pi
\end{aligned}
$$

Therefore, $\boxed{\text{the flux is } 110\pi}$

Illustration 3.7.2: Flux on a Piecewise Smooth Surface

We will find the flux of $\vec{\mathbf{F}} = \langle x^2, y^2, z^2\rangle$ on the surface of the cube $E = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ oriented outward.

Let S denote the boundary of E.



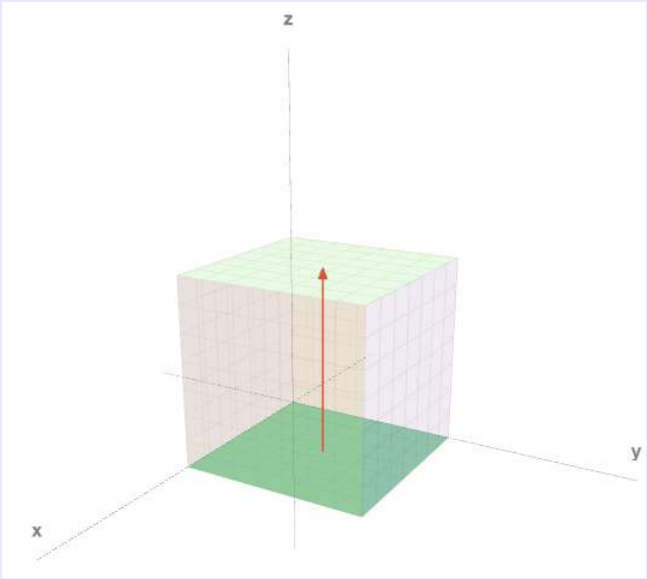
This cube has six flat surfaces. We will name them as follows:

$$S_1 = \{(x, y, 0) \,:\, 0 \le x \le 1,\, 0 \le y \le 1\}$$
$$S_2 = \{(x, y, 1) \,:\, 0 \le x \le 1,\, 0 \le y \le 1\}$$
$$S_3 = \{(x, 0, z) \,:\, 0 \le x \le 1,\, 0 \le z \le 1\}$$
$$S_4 = \{(x, 1, z) \,:\, 0 \le x \le 1,\, 0 \le z \le 1\}$$
$$S_5 = \{(0, y, z) \,:\, 0 \le y \le 1,\, 0 \le z \le 1\}$$
$$S_6 = \{(1, y, z) \,:\, 0 \le y \le 1,\, 0 \le z \le 1\}$$

We will parametrize each of them separately, find the individual flux on each surface, and then add them up to get our total flux.

S_1 can be parameterized by $\vec{\mathbf{r}}(u, v) = \langle u, v, 0\rangle$ with domain $D = \{(u, v) \,:\, 0 \le u \le 1, 0 \le v \le 1\}$

We see that $\vec{\mathbf{r}}_u = \langle 1, 0, 0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 1, 0\rangle$ so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, 0, 1\rangle$



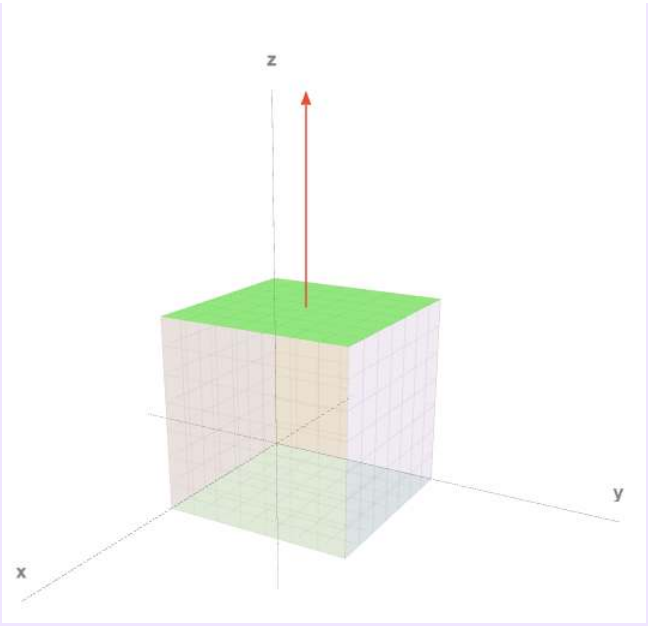
This vector points upward, which is *inward* on this surface (since it is the bottom face of the cube), so we will replace it with $\langle 0, 0, -1\rangle$

The flux across the bottom face S_1 is therefore:

$$\iint_{S_1} \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{S}} = \iint_D \langle u^2, v^2, 0^2\rangle \cdot \langle 0, 0, -1\rangle \, dA = \iint_D 0 \, dA = 0$$

S_2 can be parameterized by $\vec{\mathbf{r}}(u, v) = \langle u, v, 1\rangle$ with domain $D = \{(u, v) \,:\, 0 \le u \le 1, 0 \le v \le 1\}$

We see that $\vec{\mathbf{r}}_u = \langle 1, 0, 0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 1, 0\rangle$ so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, 0, 1\rangle$



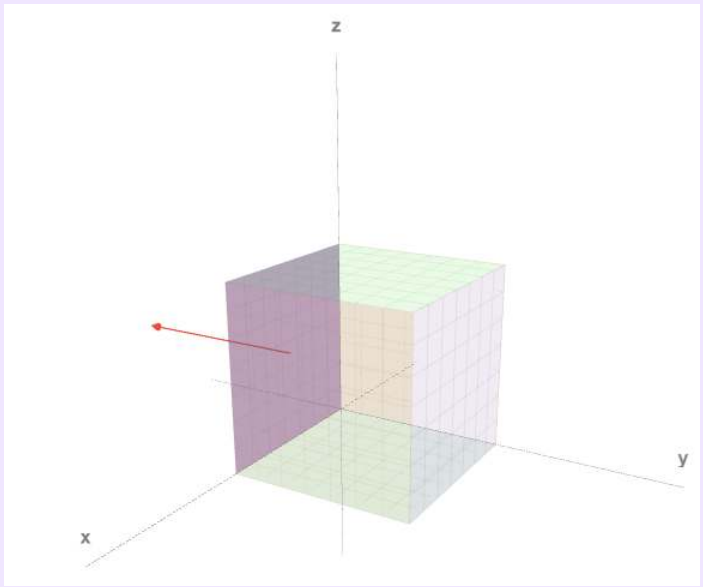
This vector points upward, which is outward on this surface (since it is the top face of the cube).

The flux across the top face S_2 is therefore:

$$\iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \langle u^2, v^2, 1^2 \rangle \cdot \langle 0, 0, 1 \rangle \, dA = \iint_D 1 \, dA = \text{area}(D) = 1$$

S_3 can be parameterized by $\vec{\mathbf{r}}(u, v) = \langle u, 0, v \rangle$ with domain $D = \{(u, v) \, : \, 0 \leq u \leq 1, 0 \leq v \leq 1\}$

We see that $\vec{\mathbf{r}}_u = \langle 1, 0, 0 \rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 0, 1 \rangle$ so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, -1, 0 \rangle$



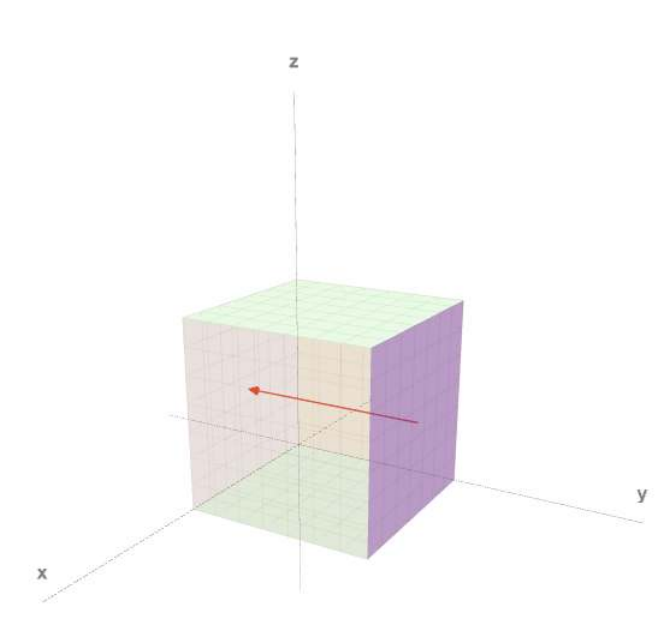
This vector points leftward, which is outward on this surface (since it is the left face of the cube).

The flux across the left face S_3 is therefore:

$$\iint_{S_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \langle u^2, 0^2, v^2 \rangle \cdot \langle 0, -1, 0 \rangle \, dA = \iint_D 0 \, dA = 0$$

S_4 can be parameterized by $\vec{\mathbf{r}}(u, v) = \langle u, 1, v \rangle$ with domain $D = \{(u, v) \, : \, 0 \leq u \leq 1, 0 \leq v \leq 1\}$

We see that $\vec{\mathbf{r}}_u = \langle 1, 0, 0 \rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 0, 1 \rangle$ so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, -1, 0 \rangle$



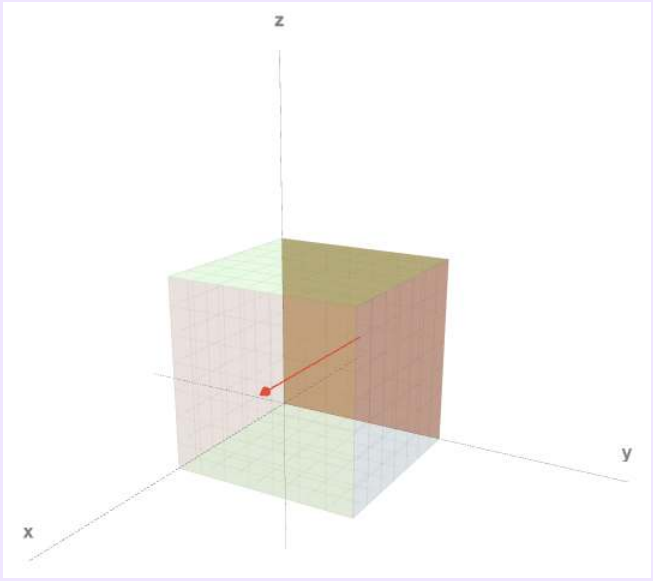
This vector points leftward, which is *inward* on this surface (since it is the right face of the cube), so we will replace it with $\langle 0,1,0\rangle$

The flux across the right face S_4 is therefore:

$$\iint_{S_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \langle u^2, 1^2, v^2\rangle \cdot \langle 0,1,0\rangle \, dA = \iint_D 1 \, dA = \text{area}(D) = 1$$

S_5 can be parameterized by $\vec{\mathbf{r}}(u,v) = \langle 0,u,v\rangle$ with domain $D = \{(u,v) \,:\, 0 \le u \le 1, 0 \le v \le 1\}$

We see that $\vec{\mathbf{r}}_u = \langle 0,1,0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0,0,1\rangle$ so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 1,0,0\rangle$



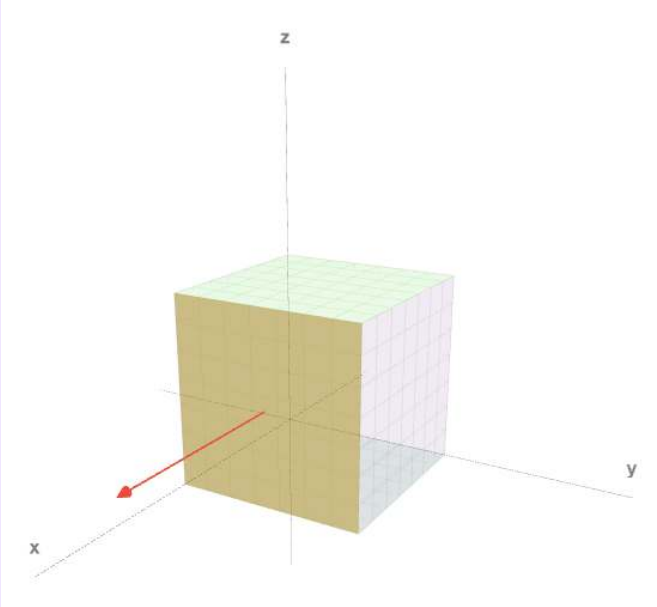
This vector points frontward, which is *inward* on this surface (since it is the back face of the cube), so we will replace it with $\langle -1,0,0\rangle$

The flux across the back face S_5 is therefore:

$$\iint_{S_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \langle 0^2, u^2, v^2\rangle \cdot \langle -1,0,0\rangle \, dA = \iint_D 0 \, dA = 0$$

S_6 can be parameterized by $\vec{\mathbf{r}}(u,v) = \langle 1, u, v\rangle$ with domain $D = \{(u,v) \,:\, 0 \le u \le 1, 0 \le v \le 1\}$

We see that $\vec{\mathbf{r}}_u = \langle 0,1,0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0,0,1\rangle$ so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 1,0,0\rangle$



This vector points frontward, which is outward on this surface (since it is the front face of the cube).

The flux across the front face S_6 is therefore:

$$\iint_{S_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \langle 1^2, u^2, v^2\rangle \cdot \langle 1,0,0\rangle \, dA = \iint_D 1 \, dA = \text{area}(D) = 1$$

Adding all of these six surfaces together, we see the total flux is

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0 + 1 + 0 + 1 + 0 + 1 = 3$$

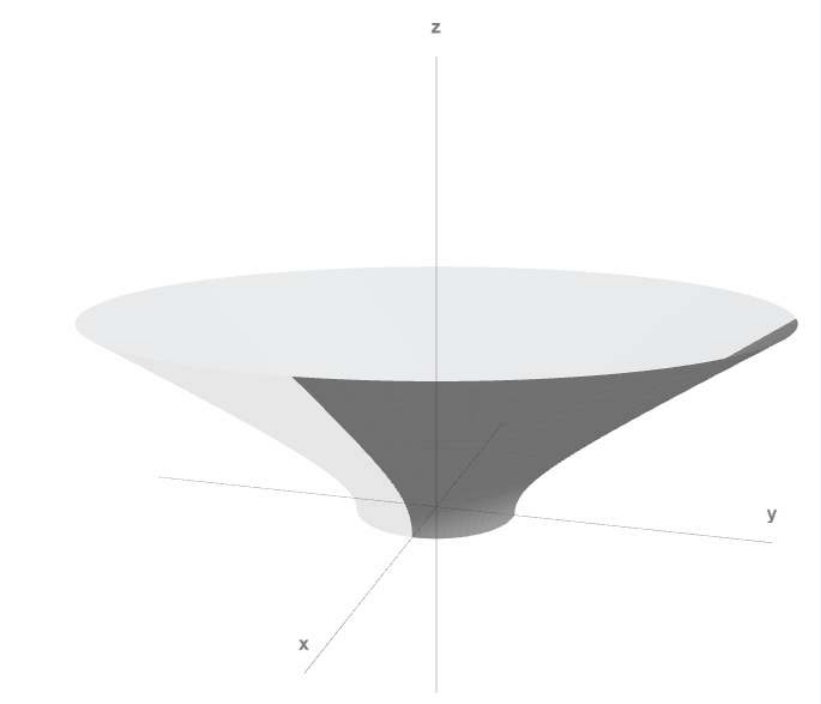
Example 3.7.3: Flux on a Variable Surface

If $S_h = \{(x,y,z) \,:\, x^2 + y^2 - 4z^2 = 1,\ x \ge 0,\ y \ge 0,\ 0 \le z \le h\}$ and $\vec{\mathbf{F}} = \langle y, -x, 2z\rangle$

What value(s) of h make the flux of $\vec{F}$ across S with upward orientation exactly equal to 36π?

Solution

The surface S is part of a hyperboloid of one sheet. In cylindrical coordinates, this is $z = \frac{1}{2}\sqrt{r^2 - 1}$



Clearly the radius at the bottom when $z = 0$ is $r = 1$, and at the top when $z = h$ is $r = \sqrt{1 + 4h^2}$
Since we only want the part of this hyperboloid in the first quadrant, we know $0 \le \theta \le \frac{\pi}{2}$

Therefore we parameterize S in terms of cylindrical coordinates, using $r = u$ and $\theta = v$:

$$\vec{\mathbf{r}}(u, v) = \left\langle u\cos(v), u\sin(v), \frac{1}{2}\sqrt{u^2 - 1} \right\rangle \text{ for } D = \left\{ (u, v) \,:\, 1 \le u \le \sqrt{1 + 4h^2},\ 0 \le v \le \frac{\pi}{2} \right\}$$

We find the tangent vectors $\vec{\mathbf{r}}_u = \left\langle \cos(v), \sin(v), \frac{u}{2\sqrt{u^2 - 1}} \right\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$.

The cross product is therefore

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -\frac{u^2\cos(v)}{2\sqrt{u^2 - 1}}, -\frac{u^2\sin(v)}{2\sqrt{u^2 - 1}}, u\cos^2(v) + u\sin^2(v) \right\rangle = \left\langle -\frac{u^2\cos(v)}{2\sqrt{u^2 - 1}}, -\frac{u^2\sin(v)}{2\sqrt{u^2 - 1}}, u \right\rangle$$

Since u is positive, this vector has positive z-coordinate, so this clearly points upward.

We calculate the flux integral on S_h:

$$\begin{aligned}
\iint_{S_h} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle u\sin(v), -u\cos(v), \sqrt{u^2 - 1} \right\rangle \cdot \left\langle -\frac{u^2\cos(v)}{2\sqrt{u^2 - 1}}, -\frac{u^2\sin(v)}{2\sqrt{u^2 - 1}}, u \right\rangle dA \\
&= \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{1+4h^2}} u\sqrt{u^2 - 1}\, du\, dv \\
&= \left(\int_0^{\frac{\pi}{2}} dv \right) \left(\int_1^{\sqrt{1+4h^2}} u\sqrt{u^2 - 1}\, du \right) \\
&= \left(v\Big|_0^{\frac{\pi}{2}} \right) \cdot \left(\frac{1}{3}(u^2 - 1)^{\frac{3}{2}} \Big|_1^{\sqrt{1+4h^2}} \right) \\
&= \frac{\pi}{2} \cdot \frac{8h^3}{3} \\
&= \frac{4h^3\pi}{3}
\end{aligned}$$

Setting this equal to 36π gives $\frac{4h^3\pi}{3} = 36\pi \longrightarrow h^3 = 27 \longrightarrow h = 3$

Thus, $\boxed{h = 3}$ yields the surface which gives our desired flux for this vector field.

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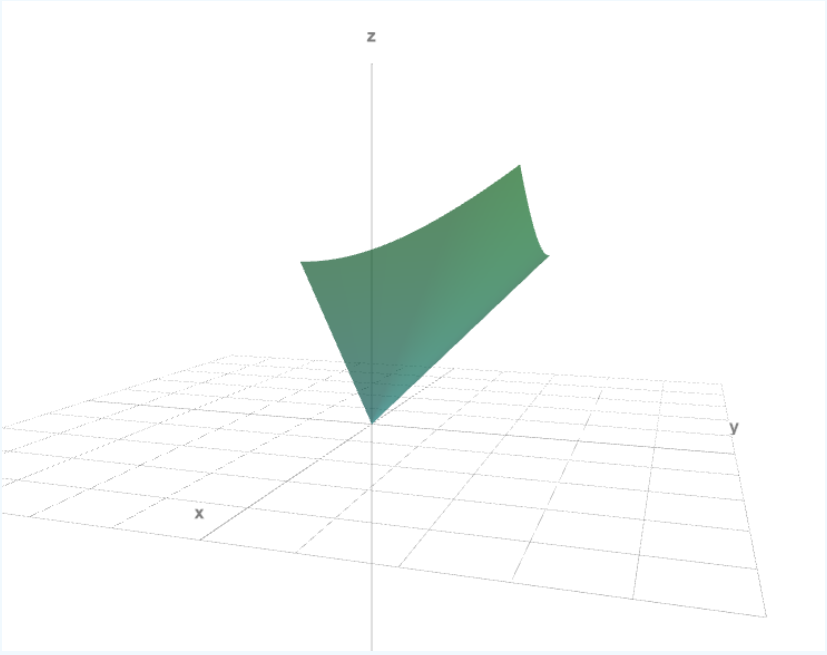
3.7: Practice Set

Exercise 3.7.1

Find the flux of $\vec{\mathbf{A}} = \langle xz, yz, xy \rangle$ over the surface $S = \left\{(x, y, z) \,:\, z = \sqrt{x^2 + y^2},\, 0 \le x \le 1,\, 0 \le y \le 1\right\}$ oriented upwards

Answer

While this surface is a cone, its domain is over a square region:



Because of this domain, we stick to parametrizing using $u = x$ and $v = y$:

$\vec{\mathbf{r}}(u, v) = \left\langle u, v, \sqrt{u^2 + v^2} \right\rangle$ over $D = \{(u, v) \,:\, 0 \le u \le 1,\, 0 \le v \le 1\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \left\langle 1, 0, \dfrac{u}{\sqrt{u^2 + v^2}} \right\rangle$ and $\vec{\mathbf{r}}_v = \left\langle 0, 1, \dfrac{v}{\sqrt{u^2 + v^2}} \right\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -\dfrac{u}{\sqrt{u^2 + v^2}}, -\dfrac{v}{\sqrt{u^2 + v^2}}, 1 \right\rangle$

Notice that the z-component faces upward, so this has upward orientation as desired.

Therefore, the flux integral over S is given by:

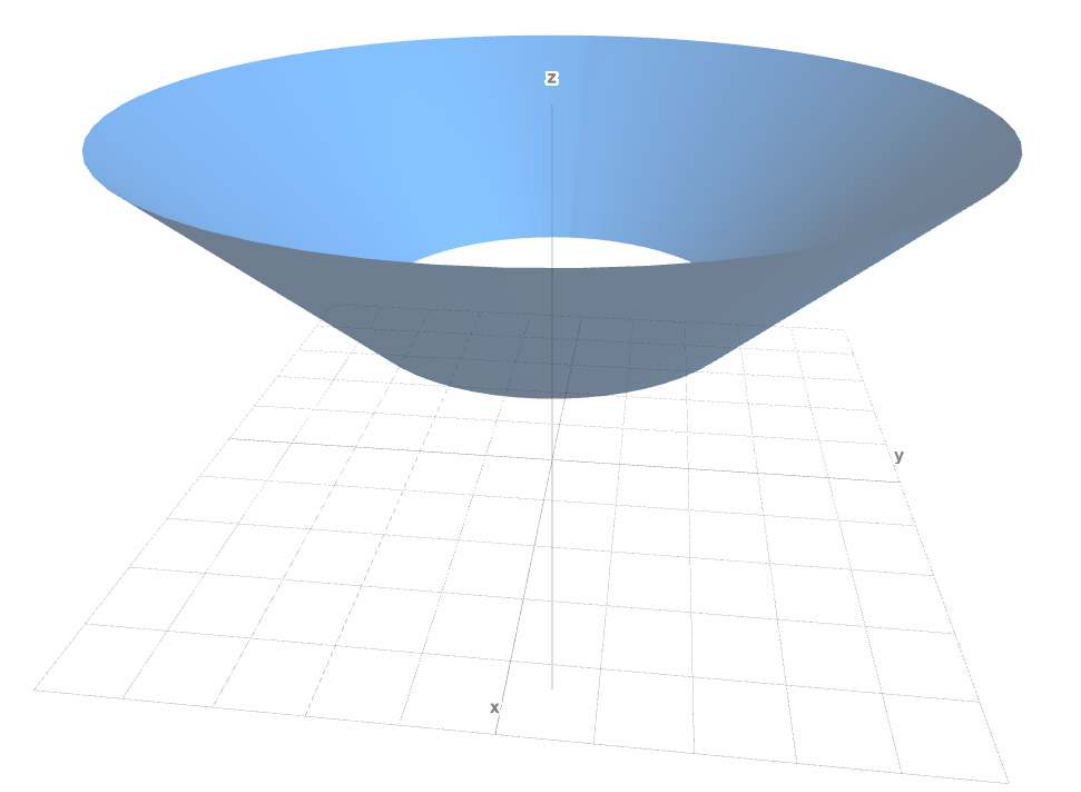
$$\begin{aligned}
\iint_S \vec{\mathbf{A}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle u\sqrt{u^2 + v^2}, v\sqrt{u^2 + v^2}, uv \right\rangle \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA \\
&= \int_0^1 \int_0^1 \left\langle u\sqrt{u^2 + v^2}, v\sqrt{u^2 + v^2}, uv \right\rangle \cdot \left\langle -\frac{u}{\sqrt{u^2 + v^2}}, -\frac{v}{\sqrt{u^2 + v^2}}, 1 \right\rangle du\, dv \\
&= \int_0^1 \int_0^1 \left(-u^2 - v^2 + uv\right) du\, dv \\
&= \int_0^1 \left[-\frac{1}{3}u^3 - v^2 u + \frac{1}{2}u^2 v\right]\Big|_0^1 dv \\
&= \int_0^1 \left(-\frac{1}{3} - v^2 + \frac{1}{2}v\right) dv \\
&= \left[-\frac{1}{3}v - \frac{1}{3}v^3 + \frac{1}{4}v^2\right]\Big|_0^1 \\
&= -\frac{5}{12}
\end{aligned}$$

Exercise 3.7.2

Find the flux of $\vec{\mathbf{B}} = \langle x, y, z^2 \rangle$ over the surface $S = \left\{(x, y, z) \,:\, z = \sqrt{x^2 + y^2},\ 1 \le z \le 2\right\}$ oriented upwards

Answer

This surface S is part of the same cone, but with a domain between concentric circles:



This time we parametrize using cylindrical coordinates with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), u \rangle$ over $D = \{(u, v) \,:\, 1 \le u \le 2,\ 0 \le v \le 2\pi\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 1 \rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -u\cos(v), -u\sin(v), u \rangle$

Notice that the z-component faces upward (since $u \ge 0$), so this has upward orientation as desired.

Therefore, the flux integral over S is given by:

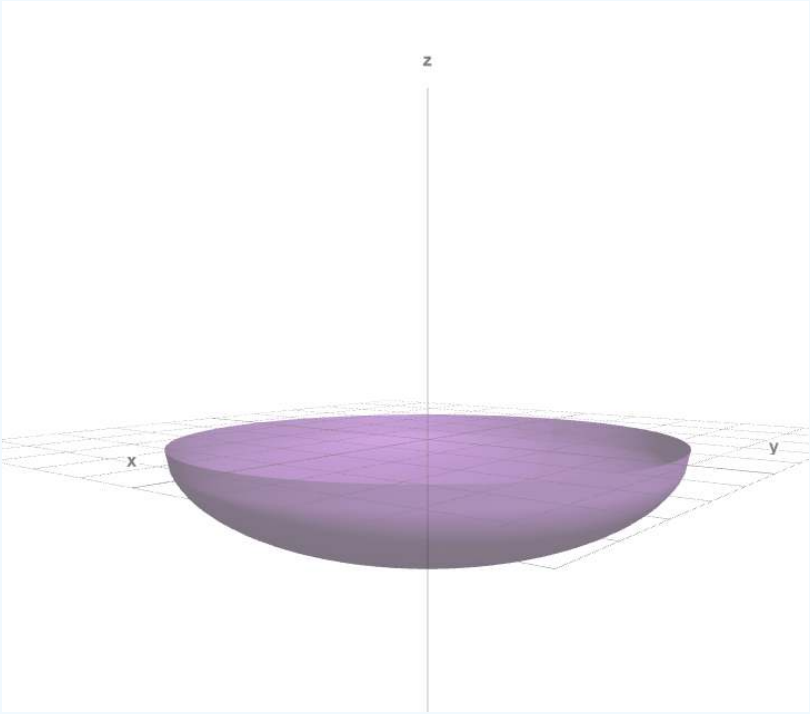
$$\begin{aligned}
\iint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle u\cos(v), u\sin(v), u^2 \right\rangle \cdot \left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right)\, dA \\
&= \int_1^2 \int_0^{2\pi} \left\langle u\cos(v), u\sin(v), u^2 \right\rangle \cdot \langle -u\cos(v), -u\sin(v), u \rangle \, dv\, du \\
&= \int_1^2 \int_0^{2\pi} \left(-u^2 + u^3\right)\, dv\, du \\
&= \left(\int_1^2 \left(u^3 - u^2\right)\, du\right)\left(\int_0^{2\pi} 1\, dv\right) \\
&= \left[\frac{1}{4}u^4 - \frac{1}{3}u^3\right]\Big|_1^2 \cdot \left(v\Big|_0^{2\pi}\right) \\
&= \frac{17}{12} \cdot 2\pi \\
&= \frac{17\pi}{6}
\end{aligned}$$

Exercise 3.7.3

Find the flux of $\vec{\mathbf{C}} = \left\langle y, -x, e^{-x^2-y^2} \right\rangle$ over the surface $S = \{(x, y, z) : x^2 + y^2 + 9z^2 = 9,\ z \leq 0\}$ oriented upwards

Answer

This surface S is the bottom half of an ellipsoid, and it can be written explicitly as $z = -\sqrt{1 - \dfrac{x^2+y^2}{9}}$:



Because the domain that the function z is defined over is a circle, we will use cylindrical coordinates with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u, v) = \left\langle u\cos(v), u\sin(v), -\sqrt{1 - \dfrac{u^2}{9}} \right\rangle$ over $D = \{(u, v) : 0 \leq u \leq 3,\ 0 \leq v \leq 2\pi\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \left\langle \cos(v), \sin(v), \dfrac{u}{9\sqrt{1 - \frac{u^2}{9}}} \right\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle -\dfrac{u^2\cos(v)}{9\sqrt{1 - \frac{u^2}{9}}}, -\dfrac{u^2\sin(v)}{9\sqrt{1 - \frac{u^2}{9}}}, u \right\rangle$

Notice that the z-component faces upward (since $u \geq 0$), so this has upward orientation as desired.

Therefore, the flux integral over S is given by:

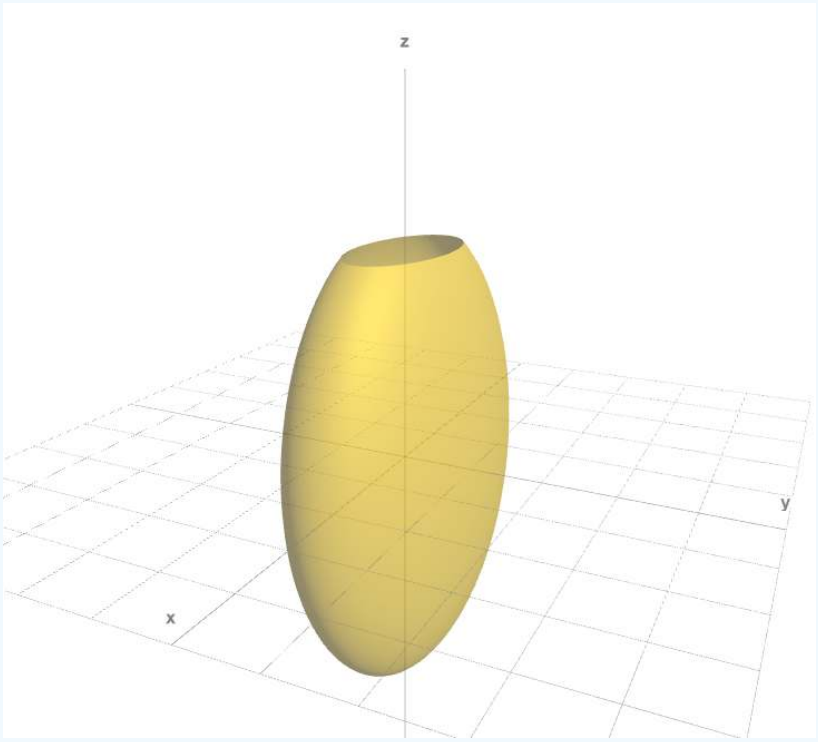
$$\begin{aligned}
\iint_S \vec{\mathbf{C}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle u\sin(v), -u\cos(v), e^{-u^2} \right\rangle \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA \\
&= \int_0^3 \int_0^{2\pi} \left\langle u\sin(v), -u\cos(v), e^{-u^2} \right\rangle \cdot \left\langle -\frac{u^2\cos(v)}{9\sqrt{1-\frac{u^2}{9}}}, -\frac{u^2\sin(v)}{9\sqrt{1-\frac{u^2}{9}}}, u \right\rangle dv\, du \\
&= \int_0^3 \int_0^{2\pi} ue^{-u^2}\, dv\, du \\
&= \left(\int_0^3 ue^{-u^2}\, du\right)\left(\int_0^{2\pi} 1\, dv\right) \\
&= \left(-\frac{1}{2}e^{-u^2}\Big|_0^3\right) \cdot \left(v\Big|_0^{2\pi}\right) \\
&= \frac{1-e^{-9}}{2} \cdot 2\pi \\
&= (1-e^{-9})\pi
\end{aligned}$$

Exercise 3.7.4

Find the flux of $\vec{\mathbf{D}} = \langle 3x, 3y, 12\rangle$ over the surface $S = \left\{(x,y,z) : x^2+4y^2+z^2 = 4,\ z \le \sqrt{3}\right\}$ oriented outwards

Answer

This surface is also a partial ellipsoid, but it cannot be solved explicitly as $z = f(x,y)$:



We instead parametrize using modified spherical coordinates, using $\rho = 2$ for the x and z coordinates, and $\rho = 1$ for y. Since $z \le \sqrt{3}$, we have $2\cos(\phi) \le \sqrt{3} \longleftrightarrow \phi \ge \frac{\pi}{6}$. Therefore we set $u = \phi$ and $v = \theta$ to get:

$\vec{\mathbf{r}}(u,v) = \langle 2\sin(u)\cos(v), \sin(u)\sin(v), 2\cos(u)\rangle$ over $D = \left\{(u,v) : \frac{\pi}{6} \le u \le \pi,\ 0 \le v \le 2\pi\right\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \langle 2\cos(u)\cos(v), \cos(u)\sin(v), -2\sin(u)\rangle$ and $\vec{\mathbf{r}}_v = \langle -2\sin(u)\sin(v), \sin(u)\cos(v), 0\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2\sin^2(u)\cos(v), 4\sin^2(u)\sin(v), 2\sin(u)\cos(u)\rangle$

Notice that at $u = \frac{\pi}{2}, v = 0$ we have $\vec{\mathbf{r}}\left(\frac{\pi}{2}, 0\right) = (2,0,0)$ and $(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\left(\frac{\pi}{2}, 0\right) = \langle 2,0,0\rangle$. This vector clearly points outward along the positive x-axis from the front of the ellipsoid, so this has outward orientation as desired.

Therefore, the flux integral over S is given by:

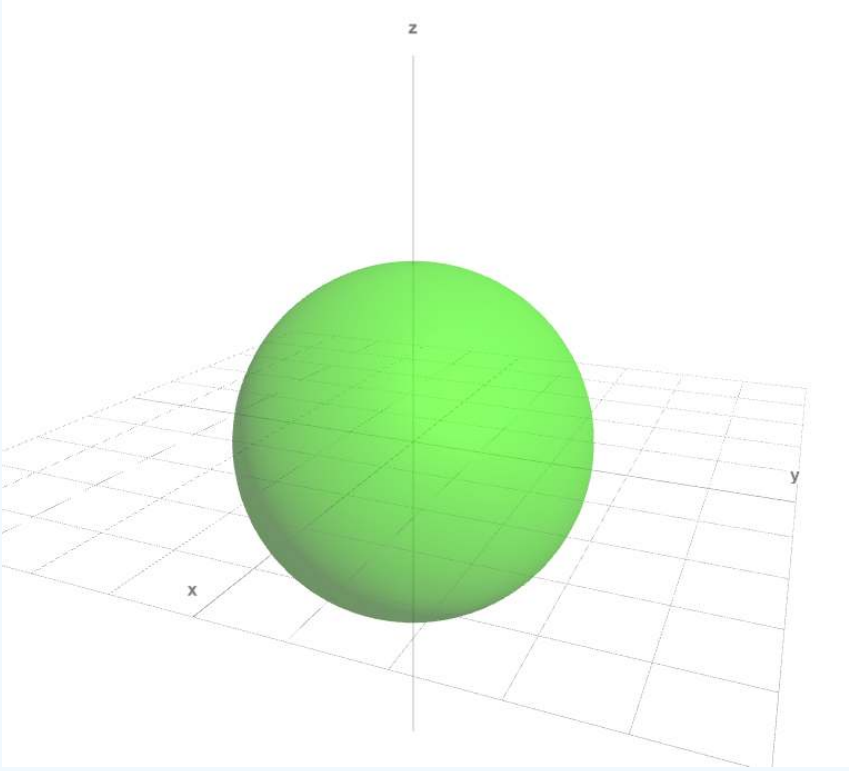
$$\begin{aligned}
\iint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{S}} &= \iint_D \langle 6\sin(u)\cos(v), 3\sin(u)\sin(v), 12\rangle \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA \\
&= \int_{\frac{\pi}{6}}^{\pi}\int_0^{2\pi} \langle 6\sin(u)\cos(v), 3\sin(u)\sin(v), 12\rangle \cdot \left\langle 2\sin^2(u)\cos(v), 4\sin^2(u)\sin(v), 2\sin(u)\cos(u)\right\rangle \, dv\, du \\
&= \int_{\frac{\pi}{6}}^{\pi}\int_0^{2\pi} \left(12\sin^3(u) + 24\sin(u)\cos(u)\right)\, dv\, du \\
&= \left(\int_{\frac{\pi}{6}}^{\pi} \left(12 - 12\cos^2(u) + 24\cos(u)\right)\sin(u)\, du\right)\left(\int_0^{2\pi} 1\, dv\right) \\
&= \left[-12\cos(u) + 4\cos^3(u) - 12\cos^2(u)\right]\Big|_{\frac{\pi}{6}}^{\pi} \cdot \left(v\Big|_0^{2\pi}\right) \\
&= \left(5 + \frac{9\sqrt{3}}{2}\right) \cdot 2\pi \\
&= (10 + 9\sqrt{3})\pi
\end{aligned}$$

Exercise 3.7.5

Find the flux of $\vec{\mathbf{E}} = \langle xz^2, yz^2, z^2\rangle$ over the surface $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ oriented outwards

Answer

This surface is the classic unit sphere, which we will parametrize using spherical coordinates



Setting $\rho = 1,\ u = \phi$, and $v = \theta$ we have:

$\vec{\mathbf{r}}(u, v) = \langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\rangle$ over $D = \{(u, v) : 0 \le u \le \pi\, 0 \le v \le 2\pi\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)\rangle$ and $\vec{\mathbf{r}}_v = \langle -\sin(u)\sin(v), \sin(u)\cos(v), 0\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u)\rangle$

Notice that at $u = \frac{\pi}{2}, v = 0$ we have $\vec{\mathbf{r}}\left(\frac{\pi}{2}, 0\right) = (1, 0, 0)$ and $(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\left(\frac{\pi}{2}, 0\right) = \langle 1, 0, 0\rangle$. This vector clearly points outward along the positive x-axis from the front of the sphere, so this has outward orientation as desired.

Therefore, the flux integral over S is given by:

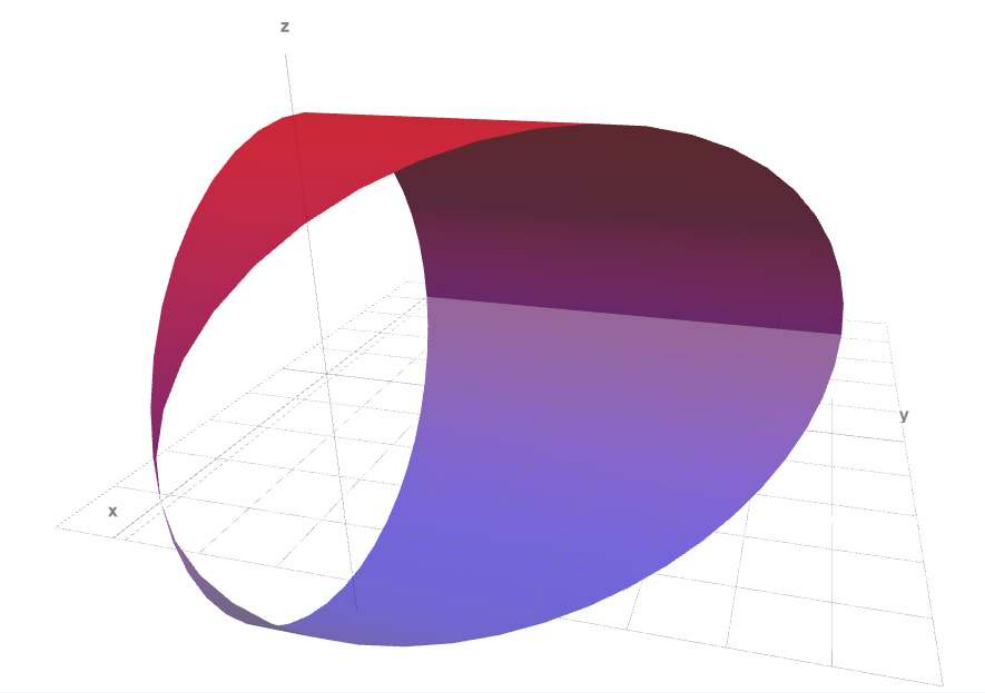
$$\begin{aligned}
\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle \sin(u)\cos(v)\cdot(\cos(u))^2, \sin(u)\sin(v)\cdot(\cos(u))^2, (\cos(u))^2 \right\rangle \cdot \left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right)\, dA \\
&= \int_0^{\pi}\int_0^{2\pi} \left\langle \sin(u)\cos^2(u)\cos(v), \sin(u)\cos^2(u)\sin(v), \cos^2(u) \right\rangle \cdot \left\langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u) \right\rangle \, dv\, du \\
&= \int_0^{\pi}\int_0^{2\pi} \left(\sin^3(u)\cos^2(u) + \cos^3(u)\sin(u)\right)\, dv\, du \\
&= \left(\int_0^{\pi} \left(\cos^2(u) - \cos^4(u) + \cos^3(u)\right)\sin(u)\, du\right)\left(\int_0^{2\pi} 1\, dv\right) \\
&= \left[-\frac{1}{3}\cos^3(u) + \frac{1}{5}\cos^5(u) - \frac{1}{4}\cos^4(u)\right]\Big|_0^{\pi} \cdot \left(v\Big|_0^{2\pi}\right) \\
&= \frac{4}{15}\cdot 2\pi \\
&= \frac{8\pi}{15}
\end{aligned}$$

Exercise 3.7.6

Find the flux of $\vec{\mathbf{F}} = \langle x, xyz, y\rangle$ over the surface $S = \{(x,y,z) : x^2 + z^2 = 25,\ 0 \le y \le 5 - x\}$ oriented outwards

Answer

This surface S is part of a cylinder, and it cannot be written explicitly in any direction:



We parametrize the circle $x^2 + z^2 = 25$ by setting $x = 5\cos(u), z = 5\sin(u)$, then use the remaining variable $v = y$:

$\vec{\mathbf{r}}(u,v) = \langle 5\cos(u), v, 5\sin(u)\rangle$ over $D = \{(u,v) : 0 \le u \le 2\pi,\ 0 \le v \le 5 - 5\cos(u)\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \langle -5\sin(u), 0, 5\cos(u)\rangle$ and $\vec{\mathbf{r}}_v = \langle 0,1,0\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -5\cos(u), 0, -5\sin(u)\rangle$

Notice that at $u = 0, v = 0$ we have $\vec{\mathbf{r}}(0,0) = (5,0,0)$ and $\left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right)(0,0) = \langle -5,0,0\rangle$. This vector clearly points inward along the positive x-axis from the front of the cylinder, so this has inward orientation. To force the outward orientation, we use $-\left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right)$ in our flux calculation.

Therefore, the flux integral over S is given by:

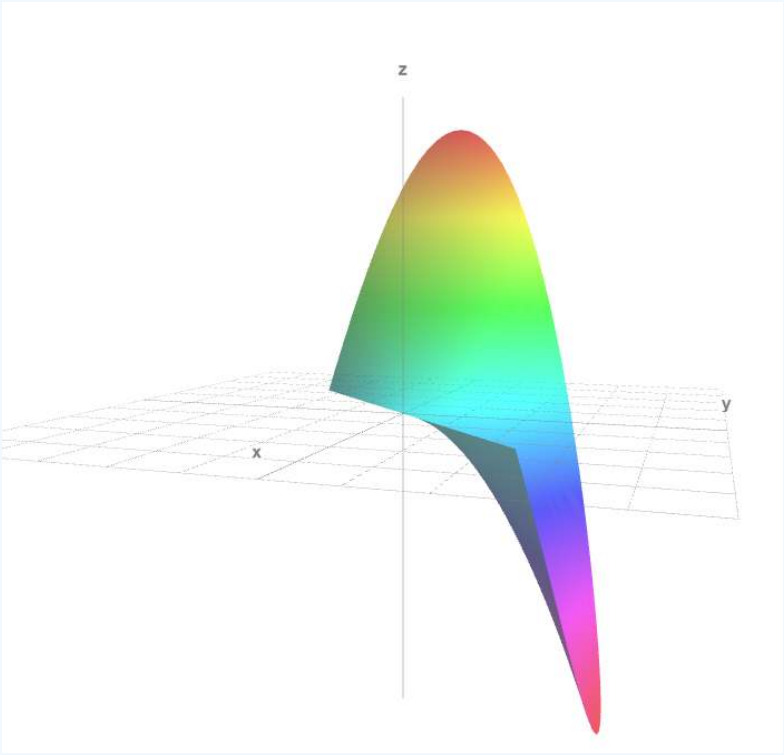
$$
\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D \langle 5\cos(u), 25v\cos(u)\sin(u), v\rangle \cdot -\left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right)\, dA \\
&= \int_0^{2\pi} \int_0^{5-5\cos(u)} \langle 5\cos(u), 25v\cos(u)\sin(u), v\rangle \cdot \langle 5\cos(u), 0, 5\sin(u)\rangle \, dv\, du \\
&= \int_0^{2\pi} \int_0^{5-5\cos(u)} \left(25\cos^2(u) + 5v\sin(u)\right) \, dv\, du \\
&= \int_0^{2\pi} \left[25v\cos^2(u) + \frac{5}{2}v^2\sin(u)\right]\Big|_0^{5-5\cos(u)} du \\
&= \int_0^{2\pi} \left(125\cos^2(u) - 125\cos^3(u) + \frac{125}{2}(1-\cos(u))^2\sin(u)\right) du \\
&= \int_0^{2\pi} \left(\frac{125}{2}(1+\cos(2u)) - 125(1-\sin^2(u))\cos(u) + \frac{125}{2}(1-\cos(u))^2\sin(u)\right) du \\
&= \left[\frac{125}{2}u + \frac{125}{4}\sin(2u) - 125\sin(u) + \frac{125}{3}\sin^3(u) + \frac{125}{6}(1-\cos(u))^3\right]\Big|_0^{2\pi} \\
&= 125\pi
\end{aligned}
$$

Exercise 3.7.7

Find the flux of $\vec{\mathbf{G}} = \left\langle \frac{x}{\sqrt{z+2y^2}}, -\frac{y}{\sqrt{z+2y^2}}, 3+4x-5y \right\rangle$ over the surface $S = \{(x,y,z) : z = x^2 - y^2,\ x^2+y^2 \le 4,\ y \ge x\}$ oriented upwards

Answer

This surface is explicitly given as a hyperbolic paraboloid over a semicircular domain:



Because of the domain we will parametrize it in cylindrical coordinates. Note that $y \ge x \longleftrightarrow \sin(\theta) \ge \cos(\theta) \longleftrightarrow \frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$. Setting $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), u^2(\cos^2(v) - \sin^2(v))\rangle$ over $D = \left\{(u,v) \,\middle|\, 0 \le u \le 2,\ \frac{\pi}{4} \le v \le \frac{5\pi}{4}\right\}$

To shorten the writing, we will make use of the double angle identities $\cos(2v) = \cos^2(v) - \sin^2(v)$ and $\sin(2v) = 2\sin(v)\cos(v)$:

$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), u^2\cos(2v)\rangle$ over $D = \left\{(u,v) \,:\, 0 \le u \le 2,\ \frac{\pi}{4} \le v \le \frac{5\pi}{4}\right\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 2u\cos(2v)\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), -2u^2\sin(2v)\rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -2u^2\cos(v)\cos(2v) - 2u^2\sin(v)\sin(2v), -2u^2\cos(2v)\sin(v) + 2u^2\cos(v)\sin(2v), u\rangle$

Notice that the z-component faces upward (since $u \ge 0$), so this has upward orientation as desired.

We can see that $\cos(v) = \cos(2v - v) = \cos(2v)\cos(v) + \sin(2v)\sin(v)$ and $sin(v) = \sin(2v - v) = \sin(2v)\cos(v) - \sin(v)\cos(2v)$, so the cross product simplifies to $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -2u^2\cos(v), 2u^2\sin(v), u\rangle$

Therefore, the flux integral over S is given by:

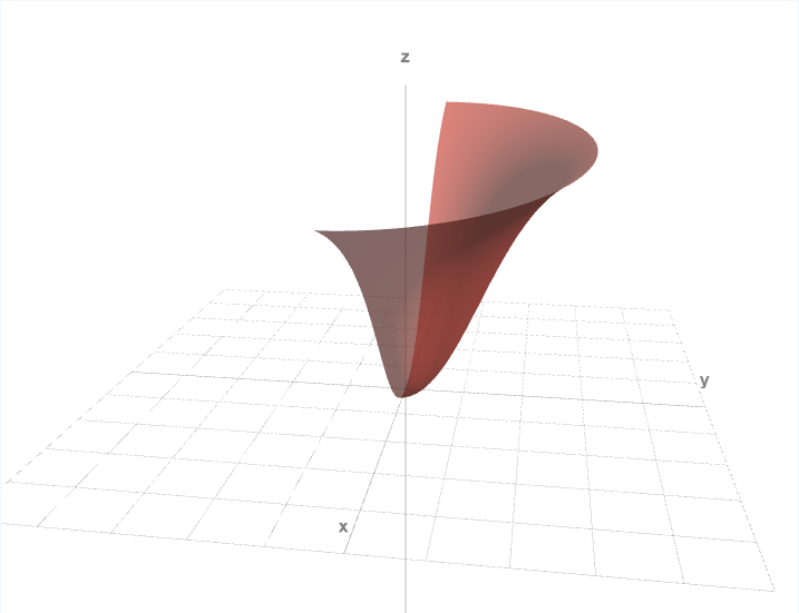
$$\begin{aligned}
\iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle \frac{u\cos(v)}{\sqrt{u^2(\cos^2(v)+\sin^2(v))}}, -\frac{u\sin(v)}{\sqrt{u^2(\cos^2(v)+\sin^2(v))}}, 3 + 4u\cos(v) - 5u\sin(v)\right\rangle \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA \\
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^2 \langle \cos(v), -\sin(v), 3 + 4u\cos(v) - 5u\sin(v)\rangle \cdot \langle -2u^2\cos(v), 2u^2\sin(v), u\rangle\ du\,dv \\
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^2 \left(u^2\left(-2\sin^2(v) - 2\cos^2(v) - 5\sin(v) + 4\cos(v)\right) + 3u\right)\ du\,dv \\
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left[\frac{1}{3}u^3\left(-2 - 5\sin(v) + 4\cos(v)\right) + \frac{3}{2}u^2\right]\Bigg|_0^2 dv \\
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\frac{8}{3}\left(-2 - 5\sin(v) + 4\cos(v)\right) + 6\right)\ dv \\
&= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\frac{2}{3} - \frac{40}{3}\sin(v) + \frac{32}{3}\cos(v)\right)\ dv \\
&= \left[\frac{2}{3}v + \frac{40}{3}\cos(v) + \frac{32}{3}\sin(v)\right]\Bigg|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= \frac{2}{3}\pi - 24\sqrt{2}
\end{aligned}$$

? Exercise 3.7.8

Find the flux of $\vec{\mathbf{H}} = \langle 4yz^2, -xz^2, 6y\rangle$ over the surface $S = \{(x,y,z) \,:\, x^2 + 4y^2 \le 3,\ z = \tan^{-1}(x^2 + 4y^2),\ y \ge 0\}$ oriented upwards

Answer

This is not a famous surface, but it is provided as an explicit function $z = f(x,y)$:



Since the domain is half of the ellipse $\frac{x^2}{3}+\frac{y^2}{\frac{3}{4}}\le 1$, we will parametrize it using $x=\sqrt{3}\cdot u\cos(v)$ and $y=\frac{\sqrt{3}}{2}\cdot u\sin(v)$

This allows us to have $\frac{x^2}{3}+\frac{y^2}{\frac{3}{4}}=u^2$ so $0\le u\le 1$ and to ensure $y\ge 0$ we restrict to $0\le v\le \pi$:

$\vec{\mathbf{r}}(u,v)=\left\langle \sqrt{3}\cdot u\cos(v),\frac{\sqrt{3}}{2}\cdot u\sin(v),\tan^{-1}(3u^2)\right\rangle$ over $D=\{(u,v)\,:\,0\le u\le 1,\,0\le v\le \pi\}$

This has tangent vectors $\vec{\mathbf{r}}_u=\left\langle \sqrt{3}\cos(v),\frac{\sqrt{3}}{2}\sin(v),\frac{6u}{1+9u^4}\right\rangle$ and $\vec{\mathbf{r}}_v=\left\langle -\sqrt{3}\cdot u\sin(v),\frac{\sqrt{3}}{2}\cdot u\cos(v),0\right\rangle$

These have cross product $\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v=\left\langle -\frac{6u^2\sqrt{3}}{2(1+9u^4)}\cos(v),-\frac{6u^2\sqrt{3}}{1+9u^4}\sin(v),\frac{3}{2}u\right\rangle$

Notice that the z-component faces upward (since $u\ge 0$), so this has upward orientation as desired.

Therefore, the flux integral over S is given by:

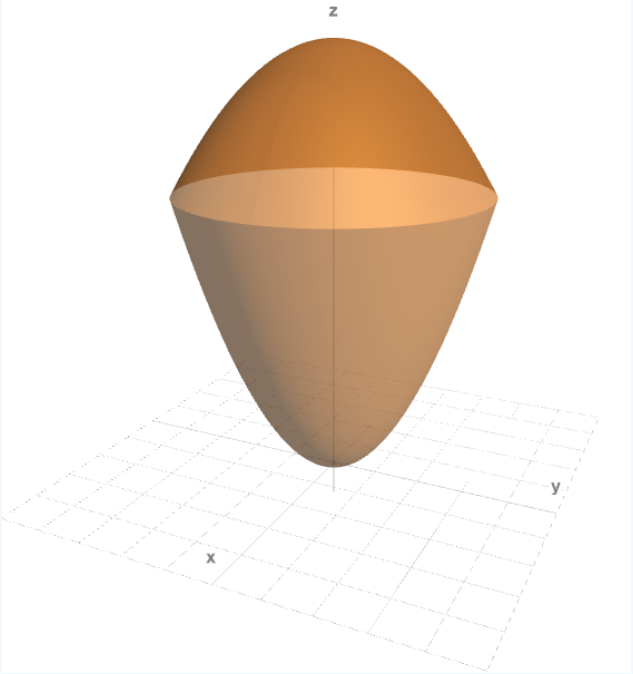
$$\begin{aligned}\iint_S \vec{\mathbf{H}}\cdot d\vec{\mathbf{S}} &= \iint_D \left\langle 2\sqrt{3}u\sin(v)\left(\tan^{-1}(3u^2)\right)^2,-\sqrt{3}u\cos(v)\left(\tan^{-1}(3u^2)\right)^2,3\sqrt{3}u\sin(v)\right\rangle\cdot\left(\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v\right)\,dA\\ &=\int_0^{\pi}\int_0^1 \left\langle 2\sqrt{3}u\sin(v)\left(\tan^{-1}(3u^2)\right)^2,-\sqrt{3}u\cos(v)\left(\tan^{-1}(3u^2)\right)^2,3\sqrt{3}u\sin(v)\right\rangle\cdot\left\langle -\frac{6u^2\sqrt{3}}{2(1+9u^4)}\cos(v),-\frac{6u^2\sqrt{3}}{1+9u^4}\sin(v),\frac{3}{2}u\right\rangle\,du\,dv\\ &=\int_0^{\pi}\int_0^1 \frac{9\sqrt{3}}{2}u^2\sin(v)\,du\,dv\\ &=\left(\int_0^{\pi}\sin(v)\,dv\right)\left(\int_0^1\frac{9\sqrt{3}}{2}u^2\,du\right)\\ &=\left(-\cos(v)\Big|_0^{\pi}\right)\cdot\left(\frac{3\sqrt{3}}{2}u^3\Big|_0^1\right)\\ &=2\cdot\frac{3\sqrt{3}}{2}\\ &=3\sqrt{3}\end{aligned}$$

Exercise 3.7.9

Find the flux of $\vec{\mathbf{I}}=\left\langle\frac{x}{z},\frac{y}{z},z\right\rangle$ over the boundary of $E=\{(x,y,z)\,:\,2x^2+2y^2\le z\le 12-x^2-y^2\}$ oriented outwards

Answer

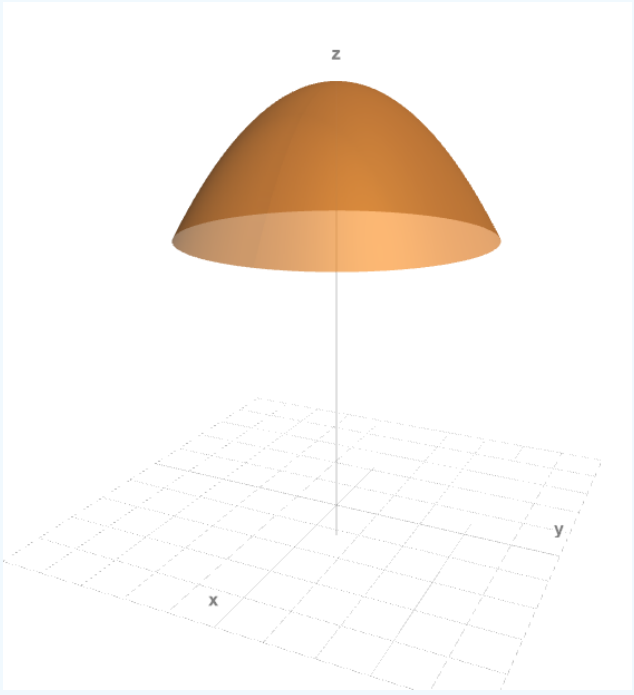
The region E described is between the two paraboloids $z=2x^2+2y^2$ and $z=12-x^2-y^2$ as pictured below:



The surface of its boundary is made of two smooth pieces: the top piece, denoted S_1, and the bottom piece, denoted S_2.

It is easy to calculate their intersection as the circle $x^2 + y^2 = 4$, so each of them is over the circle of radius 2.

We start by finding the flux across $S_1 = \{(x, y, z) : z = 12 - x^2 - y^2,\ x^2 + y^2 \leq 4\}$



We parametrize it using cylindrical coordinates with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), 12 - u^2\rangle$ over $D = \{(u, v) : 0 \leq u \leq 2,\ 0 \leq v \leq 2\pi\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), -2u\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$

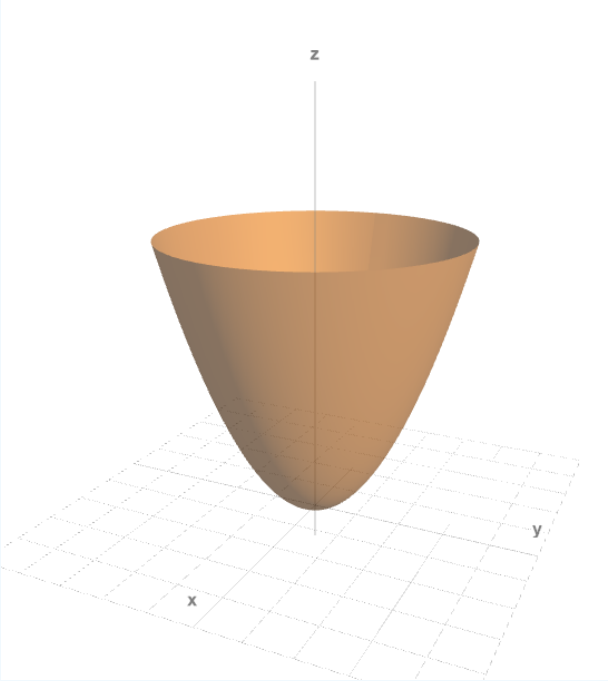
These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2u^2\cos(v), 2u^2\sin(v), u\rangle$

Notice that the z-component faces upward (since $u \geq 0$), so this has outward orientation as desired.

Therefore, the flux integral over S_1 is given by:

$$\begin{aligned}
\iint_{S_1} \vec{\mathbf{I}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle \frac{u\cos(v)}{12-u^2}, \frac{u\sin(v)}{12-u^2}, 12-u^2 \right\rangle \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA \\
&= \int_0^2 \int_0^{2\pi} \left\langle \frac{u\cos(v)}{12-u^2}, \frac{u\sin(v)}{12-u^2}, 12-u^2 \right\rangle \cdot \left\langle 2u^2\cos(v), 2u^2\sin(v), u \right\rangle \, dv\, du \\
&= \int_0^2 \int_0^{2\pi} \left(\frac{2u^3}{12-u^2} + 12u - u^3 \right) dv\, du \\
&= \left(\int_0^{2\pi} 1\, dv \right) \left(\int_0^2 \left(-u^3 - \frac{24u}{u^2-12} + 10u \right) du \right) \\
&= \left(v\Big|_0^{2\pi} \right) \cdot \left[-\frac{1}{4}u^4 - 12\ln|u^2-12| + 5u^2 \right] \Big|_0^2 \\
&= 2\pi \cdot \left(16 + 12\ln\left(\frac{3}{2}\right) \right) \\
&= 32\pi + 24\pi\ln\left(\frac{3}{2}\right)
\end{aligned}$$

Now we repeat this process to find the flux on the bottom half, $S_2 = \{(x,y,z) : z = 2x^2+2y^2,\ x^2+y^2 \leq 4\}$:



We parametrize it using cylindrical coordinates with $u = r$ and $v = \theta$:

$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), 2u^2 \rangle$ over $D = \{(u,v) : 0 \leq u \leq 2,\ 0 \leq v \leq 2\pi\}$

This has tangent vectors $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 4u \rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

These have cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -4u^2\cos(v), -4u^2\sin(v), u \rangle$

Notice that the z-component faces upward (since $u \geq 0$), but this is *inward* for the full surface. We will use $-(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)$ instead when calculating the flux.

Therefore, the flux integral over S_2 is given by:

$$\begin{aligned}
\iint_{S_2} \vec{\mathbf{I}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle \frac{u\cos(v)}{2u^2}, \frac{u\sin(v)}{2u^2}, 2u^2 \right\rangle \cdot -\left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right)\, dA \\
&= \int_0^2 \int_0^{2\pi} \left\langle \frac{\cos(v)}{2u}, \frac{\sin(v)}{2u}, 2u^2 \right\rangle \cdot \left\langle 4u^2\cos(v), 4u^2\sin(v), -u \right\rangle \, dv\, du \\
&= \int_0^2 \int_0^{2\pi} \left(2u - 2u^3\right)\, dv\, du \\
&= \left(\int_0^{2\pi} 1\, dv\right)\left(\int_0^2 \left(2u - 2u^3\right)\, du\right) \\
&= \left(v\Big|_0^{2\pi}\right) \cdot \left[u^2 - \frac{1}{2}u^4\right]\Big|_0^2 \\
&= 2\pi \cdot (-4) \\
&= -8\pi
\end{aligned}$$

Therefore our final answer is the sum of these two flux integrals:

$$\iint_S \vec{\mathbf{I}} \cdot d\vec{\mathbf{S}} = \iint_{S_1} \vec{\mathbf{I}} \cdot d\vec{\mathbf{S}} + \iint_{S_2} \vec{\mathbf{I}} \cdot d\vec{\mathbf{S}} = 24\pi\left(1 + \ln\left(\frac{3}{2}\right)\right)$$

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CHAPTER OVERVIEW

4: Vector Calculus Theorems

This chapter gives us four analogous theorems to the classic Fundamental Theorem of Calculus in single variable calculus. Each of them gives us a way to switch between the types of integrals in Chapter 3 by using different forms of differentiation that we have discussed in Chapter 2.

- 4.1: The Fundamental Theorem of Line Integrals
  - 4.1: FTCLI
  - 4.1: Practice Set
- 4.2: Green's Theorem
  - 4.2: Green’s Theorem
  - 4.2: Practice Set
- 4.3: Stokes' Theorem
  - 4.3: Stokes’ Theorem
  - 4.3: Practice Set
- 4.4: The Divergence Theorem
  - 4.4: The Divergence Theorem
  - 4.4: Practice Set

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4.1: The Fundamental Theorem of Line Integrals

Learning Objectives

- Describe simple and closed curves; define connected and simply connected regions.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. This will be the first of four major theorems generalizing the standard Fundamental Theorem of Calculus which allows us to relate integration and differentiation.

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4.1: FTCLI

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here. We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve.

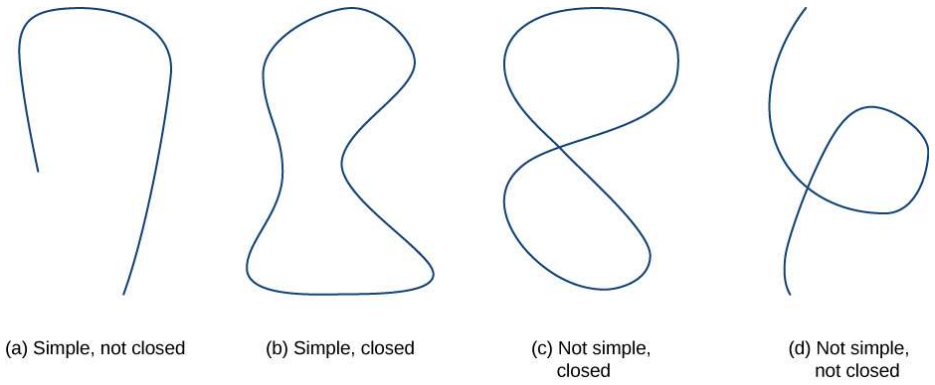


Figure 4.1.1. Types of curves that are simple or not simple and closed or not closed.

Definition: Closed Curves

Curve C is a *closed curve* if there is a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that the parameterization traverses the curve exactly once and $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Curve C is a simple curve if C does not cross itself. That is, C is simple if there exists a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that $\vec{\mathbf{r}}$ is one-to-one over (a, b). It is possible for $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, meaning that the simple curve is also closed.

Illustration 4.1.1: Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\vec{\mathbf{r}}(t) = \left\langle \cos(t), \dfrac{\sin(2t)}{2} \right\rangle$, $0 \leq t \leq 2\pi$ a simple closed curve?

Note that $\vec{\mathbf{r}}(0) = \langle 1, 0 \rangle = \vec{\mathbf{r}}(2\pi)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\vec{\mathbf{r}}\left(\dfrac{\pi}{2}\right) = \langle 0, 0 \rangle = \vec{\mathbf{r}}\left(\dfrac{3\pi}{2}\right)$ and therefore the curve crosses itself at the origin.

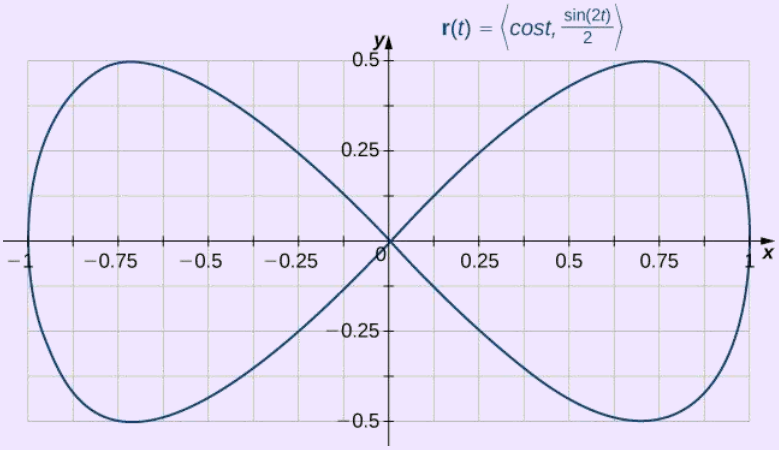


Figure 4.1.2. *A curve that is closed but not simple.*

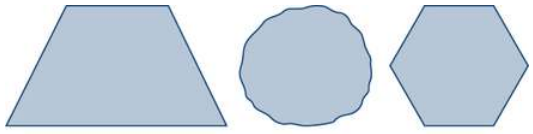
Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there

is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

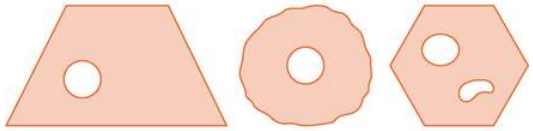
Definition: Connected Region

A region D is a **connected region** if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a simply connected region if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

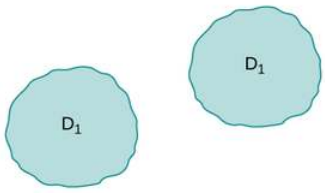
All simply connected regions are connected, but not all connected regions are simply connected, as we can see below.



(a) Simply connected regions



(b) Connected regions that are not simply connected



(c) A region that is not connected

Figure 4.1.3: Not all connected regions are simply connected. (a) Simply connected regions have no holes.
(b) Connected regions that are not simply connected may have holes, but you can still find a path in the region between any two points.
(c) A region that is not connected has some points that cannot be connected by a path in the region.

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

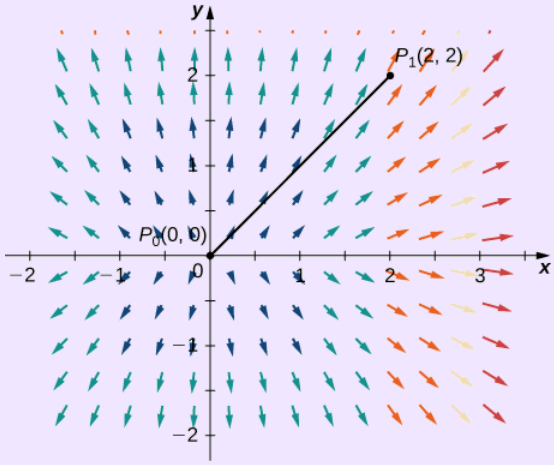
Illustration 4.1.2: Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\vec{\mathbf{F}}(x,y) = \langle 2x, 4y\rangle$. Calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the line segment from $(0,0)$ to $(2,2)$.

Curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle 2t, 2t\rangle$, for $0 \le t \le 1$

Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle 4t, 8t\rangle$ and $\vec{\mathbf{r}}\,'(t) = \langle 2, 2\rangle$, which implies that

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle 4t, 8t\rangle \cdot \langle 2, 2\rangle\, dt = \int_0^1 (8t + 16t)\, dt = \int_0^1 24t\, dt = 12t^2\Big|_0^1 = 12$$



Notice that $\vec{\mathbf{F}} = \nabla f$, where $f(x, y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of $\vec{\mathbf{F}}$. In the case of single-variable integrals, the integral of derivative $g'(x)$ is $g(b) - g(a)$, where a is the start point of the interval of integration and b is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral $\vec{\mathbf{F}}$ to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point.

Notice that this is the case for this example, $f(2, 2) - f(0, 0) = 4 + 8 - 0 = 12$. This matches the line integral we solved above!

In other words, the integral of a "derivative" can be calculated by evaluating an "antiderivative" at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals**.

Theorem 4.1.1: The Fundamental Theorem for Line Integrals

Let C be a piecewise smooth curve with parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$$

Proof

First, by the chain rule:

$$\frac{d}{dt}(f(\vec{\mathbf{r}}(t)) = \nabla f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = \int_a^b \nabla f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt = \int_a^b \frac{d}{dt}(f(\vec{\mathbf{r}}(t))\, dt = f(\vec{\mathbf{r}}(t))\Big|_a^b = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$$

We know that if $\vec{\mathbf{F}}$ is a conservative vector field, there is a potential function f such that $\nabla f = \vec{\mathbf{F}}$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is conservative, is a two-step process:

1. Find a potential function ("antiderivative") f for $\vec{\mathbf{F}}$

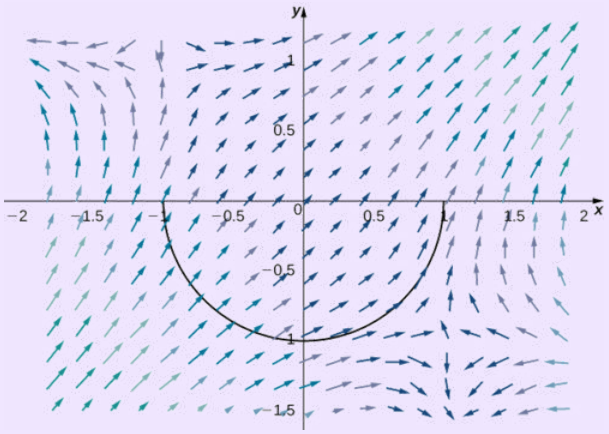
2. Compute the value of f at the endpoints of C and calculate their difference $f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$

Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals:

A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.

Illustration 4.1.3: Applying the Fundamental Theorem

We will calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x \rangle$ and C is the lower half of the unit circle oriented counterclockwise (a) without using the Fundamental Theorem for Line Integrals, and (b) by using the Fundamental Theorem for Line Integrals.



(a) First, we calculate the integral without the Fundamental Theorem for Line Integrals.
We parameterize the curve C using $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t) \rangle$ for $\pi \le t \le 2\pi$

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_\pi^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\
&= \int_\pi^{2\pi} \big\langle 2\cos(t)\sin(t) - 2\sin(t) + (\sin(t)+1)^2, \\
&\qquad (\cos(t)-1)^2 + 2\sin(t)\cos(t) + 2\cos(t) \big\rangle \cdot \langle -\sin(t), \cos(t) \rangle\, dt \\
&= \int_\pi^{2\pi} \left(3\cos^2(t)\sin(t) - 2\sin(t) + 2\cos(t) - 3\sin^2(t)\cos(t)\right)\, dt \\
&= \left[-\cos^3(t) + 2\cos(t) + 2\sin(t) - \sin^3(t) \right] \Big|_\pi^{2\pi} \\
&= 2
\end{aligned}$$

(b) We need to find the potential function $f(x,y)$ which satisfies $\nabla f = \vec{\mathbf{F}}$:

$$\int \left(2xy - 2y + (y+1)^2\right)\, dx = x^2y - 2xy + x(y+1)^2 + h_1(y)$$

$$\int \left((x-1)^2 + 2yx + 2x\right)\, dy = (x-1)^2 y + xy^2 + 2xy + h_2(x)$$

We see that if $h_1(y) = y$ and $h_2(x) = x$, these are both equal to $f(x,y) = (x-1)^2 y + x(y+1)^2$

This f is a potential function for $\vec{\mathbf{F}}$, and now the Fundamental Theorem for Line Integrals gives:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(2\pi)) - f(\vec{\mathbf{r}}(\pi)) = f(1,0) - f(-1,0) = 2$$

Example 4.1.1: Applying the Fundamental Theorem

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z) = \left\langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz \right\rangle$ and C is a curve with parameterization $\vec{\mathbf{r}}(t) = \langle t^2, t, t \rangle$, for $1 \le t \le e$

(a) without using the Fundamental Theorem of Line Integrals

(b) by using the Fundamental Theorem of Line Integrals

Solution

(a) First, we calculate the integral without the Fundamental Theorem for Line Integrals and instead use the method we learned in the previous chapter:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot dr &= \int_1^e \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\ &= \int_1^e \left\langle 2t^2\ln(t), \frac{t^4}{t} + t^2, 2t^2 \right\rangle \cdot \langle 2t, 1, 1 \rangle\, dt \\ &= \int_1^e (4t^3\ln(t) + t^3 + 3t^2)\, dt \\ &= \left[t^4\ln(t) + t^3\right]\Big|_1^e \\ &= \boxed{e^4 + e^3 - 1}\end{aligned}$$

(b) We need to find the potential function $f(x,y,z)$ which satisfies $\nabla f = \vec{\mathbf{F}}$:

$$\int 2x\ln(y)\, dx = x^2\ln(y) + h_1(y,z)$$

$$\int \left(\frac{x^2}{y} + z^2\right) dy = x^2\ln(y) + yz^2 + h_2(x,z)$$

$$\int 2yz\, dz = yz^2 + h_3(x,y)$$

We see that $f(x,y,z) = x^2\ln(y) + yz^2$ is a potential function for $\vec{\mathbf{F}}$, and now the Fundamental Theorem for Line Integrals gives:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(e)) - f(\vec{\mathbf{r}}(1)) = f(e^2, e, e) - f(1,1,1) = \boxed{e^4 + e^3 - 1}$$

The previous example illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate many vector line integrals more easily. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\vec{\mathbf{F}}$ is conservative and C is a closed curve, then the line integral of $\vec{\mathbf{F}}$ along C is zero—that is, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. To see why this is true, let f be a potential function for $\vec{\mathbf{F}}$. Since C is a closed curve, the terminal point $\vec{\mathbf{r}}(b)$ of C is the same as the initial $\vec{\mathbf{r}}(a)$ of C—that is, $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)) = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(b)) = 0$$

Recall that the reason a conservative vector field $\vec{\mathbf{F}}$ is called "conservative" is because such vector fields model forces in which energy is conserved. We know that gravity is an example of such a force. If we think of vector field $\vec{\mathbf{F}}$ in integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ as a gravitational field, and if a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero: $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$

The second important consequence of the Fundamental Theorem for Line Integrals is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path

between the endpoints.

Definition: Path Independence

Let $\vec{\mathbf{F}}$ be a vector field with domain D; it is independent of path (or ***path independent***) if

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

for any paths C_1 and C_2 in D with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem 4.1.2: Conservative Vector Fields

If $\vec{\mathbf{F}}$ is a conservative vector field, then $\vec{\mathbf{F}}$ is independent of path.

Proof

Let D denote the domain of $\vec{\mathbf{F}}$, and let C_1 and C_2 be two paths in D with the same initial and terminal points. Call the initial point P_1 and the terminal point P_2. Since $\vec{\mathbf{F}}$ is conservative, there is a potential function f for $\vec{\mathbf{F}}$. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(P_2) - f(P_1) = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Therefore, $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ is independent of path.

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore, he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers. While this may seem incorrect in the case of Hiker 3, consider that gravity actually "helps" them on their way down (after they changed their mind) the exact amount that it worked against them on their way up!

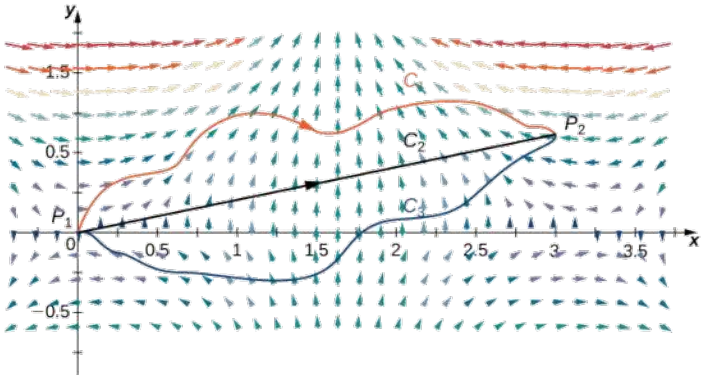


Figure 4.1.4: The vector field is conservative, and therefore independent of path.

We have shown that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ is independent of path. It turns out that if the domain of $\vec{\mathbf{F}}$ is open and connected, then the converse is also true. That is, if $\vec{\mathbf{F}}$ is independent of path and the domain of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is

conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Theorem 4.1.3: Path Independence Test for Conservative Vector Fields

If $\vec{\mathbf{F}}$ is a continuous vector field that is independent of path and the domain D of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y)\rangle$ is conservative, we must find a potential function f for $\vec{\mathbf{F}}$. To that end, let X be a fixed point in D. For any point (x,y) in D, let C be a path from X to (x,y). Define $f(x,y)$ by $f(x,y) = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. (Note that this definition of f makes sense only because $\vec{\mathbf{F}}$ is independent of path. If $\vec{\mathbf{F}}$ was not independent of path, then it might be possible to find another path C' from X to (x,y) such that $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, and in such a case $f(x,y)$ would not be a function.) We want to show that f has the property $\nabla f = \vec{\mathbf{F}}$. Since domain D is open, it is possible to find a disk centered at (x,y) such that the disk is contained entirely inside D. Let (a,y) with $a < x$ be a point in that disk. Let C be a path from X to (x,y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from X to (a,y) that stays inside D; C_2 is the horizontal line segment from (a,y) to (x,y) as pictured below.

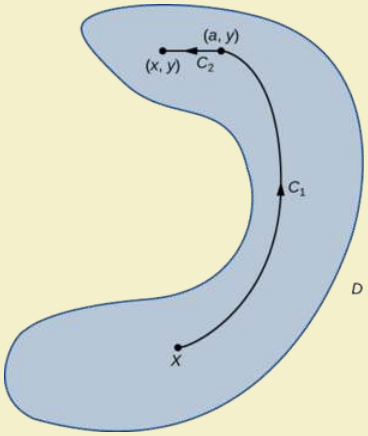


Figure 4.1.5. Here, C_1 is any path from X to (a,y) that stays inside D, and C_2 is the horizontal line segment from (a,y) to (x,y).

Then

$$f(x,y) = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

The first integral does not depend on x, so

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}\right) + \frac{\partial}{\partial x}\left(\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}\right) = \frac{\partial}{\partial x}\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

If we parameterize C_2 by $\vec{\mathbf{r}}(t) = \langle t, y\rangle$, with $a \le t \le x$, then

$$\frac{\partial}{\partial x}\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{\partial}{\partial x}\int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}$$
$$'(t)\, d = \frac{\partial}{\partial x}\int_a^x \langle P(t,y), Q(t,y)\rangle \cdot \langle 1, 0\rangle\, dt = \frac{\partial}{\partial x}\int_a^x P(t,y)\, dt$$

Therefore, by the Fundamental Theorem of Calculus,

$$f_x(x,y) = \frac{\partial}{\partial x}\int_a^x P(t,y)\, dt = P(x,y)$$

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y(x,y) = Q(x,y)$

Therefore, $\nabla f = \vec{\mathbf{F}}$ and $\vec{\mathbf{F}}$ is conservative.

We have spent a lot of time discussing and proving the theorems above, but we can summarize them simply: a vector field $\vec{\mathbf{F}}$ on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

Illustration 4.1.4: Showing That a Vector Field Is Not Conservative

We can show the vector field $\vec{\mathbf{F}}(x, y) = \langle x^2 y, y + 5\rangle$ is not conservative three different ways:

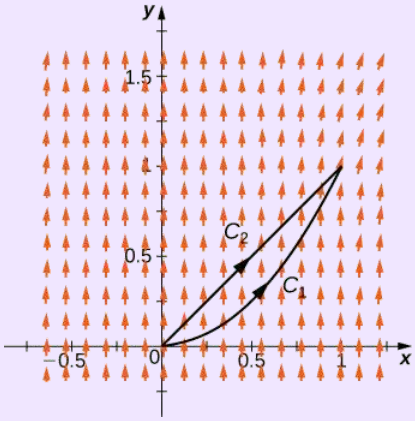
(a) Using path independence

(b) Using the derivative property of conservative vector fields

(c) By showing no potential function exists using the integral property

(a) We can indicate that $\vec{\mathbf{F}}$ is not conservative by showing that $\vec{\mathbf{F}}$ is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0, 0)$ and end at $(1, 1)$, and yet $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Let C_1 be the curve with parameterization $\vec{\mathbf{r}}_1(t) = \langle t, t\rangle$ for $0 \leq t \leq 1$, and let C_2 be the curve with parameterization $\vec{\mathbf{r}}_2(t) = \langle t, t^2\rangle$ for $0 \leq t \leq 1$



The value of the line integral along C_1 is given by:

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1{}'(t)\, dt = \int_0^1 \langle t^3, t + 5\rangle \cdot \langle 1, 1\rangle\, dt$$
$$= \int_0^1 (t^3 + t + 5)\, dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2 + 5t\right]\Big|_0^1 = \frac{23}{4}$$

Meanwhile, the value of the line integral along C_2 is:

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2{}'(t)\, dt = \int_0^1 \langle t^4, t^2 + 5\rangle \cdot \langle 1, 2t\rangle\, dt$$
$$= \int_0^1 (t^4 + 2t^3 + 10t)\, dt = \left[\frac{1}{5}t^5 + \frac{1}{2}t^4 + 5t^2\right]\Big|_0^1 = \frac{57}{10}$$

Since $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, the value of a line integral of $\vec{\mathbf{F}}$ depends on the path between two given points. Therefore, $\vec{\mathbf{F}}$ is not independent of path, and $\vec{\mathbf{F}}$ is not conservative.

(b) Recall that we can check to see if $\vec{\mathbf{F}} = \langle P(x, y), Q(x, y)\rangle$ is conservative by seeing if $P_y = Q_x$:

$$P_y = \frac{\partial}{\partial y}\left(x^2 y\right) = x^2 \quad \text{and} \quad Q_x = \frac{\partial}{\partial x}(y + 5) = 0$$

Since these are not equal, we know $\vec{\mathbf{F}}$ is not a conservative vector field.

(c) Lastly, we check to see if $\vec{\mathbf{F}}$ is conservative by attempting to find the potential function via integration:

$$\int P(x,y)dx = \int (x^2y)\,dx = \frac{1}{3}x^3y + h_1(y) \quad \text{and}$$
$$\int Q(x,y)dy = \int (y+5)\,dy = \frac{1}{2}y^2 + 5y + h_2(x)$$

Notice that the term $\frac{1}{3}x^3y$ from the left integral does not appear in the right integral (and cannot be part of $h_2(x)$ since it has y terms in it). There is no potential function $f(x,y)$ which could satisfy $\nabla f = \vec{\mathbf{F}}$

We now have three methods to determine if a given vector field is conservative. In practice, the checking for path independence is the least efficient and most time-consuming, while checking using partial derivatives is the quickest, but does not yield the potential function. Checking by directly finding the potential function can be the most efficient, but obviously that depends on the student's personal comfort with the integrals that appear.

Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, then our first question should be: Is $\vec{\mathbf{F}}$ conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals cannot help us and we have to use other methods, such as using the method from the previous chapter (using $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$ and $\vec{\mathbf{r}}\,'(t)$).

Example 4.1.2: Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z) = \langle 2xe^yz + e^xz,\ x^2e^yz,\ x^2e^y + e^x\rangle$ and C is any smooth curve that goes from the origin to $(1,1,1)$

Solution

Before trying to compute the integral, we need to determine whether $\vec{\mathbf{F}}$ is conservative and whether the domain of $\vec{\mathbf{F}}$ is simply connected. The domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of $\vec{\mathbf{F}}$ is simply connected. Let

$$P(x,y,z) = 2xe^yz + e^xz, \quad Q(x,y,z) = x^2e^yz, \quad \text{and} \quad R(x,y,z) = x^2e^y + e^x$$

so that $\vec{\mathbf{F}}(x,y,z) = \langle P,Q,R\rangle$. Since the domain of $\vec{\mathbf{F}}$ is simply connected, we can check the cross partials to determine whether $\vec{\mathbf{F}}$ is conservative. Note that

$$\begin{aligned} P_y(x,y,z) &= 2xe^yz && \text{and} && Q_x(x,y,z) = 2xe^yz \\ P_z(x,y,z) &= 2xe^y + e^x && \text{and} && R_x(x,y,z) = 2xe^y + e^x \\ Q_z(x,y,z) &= x^2e^y && \text{and} && R_y(x,y,z) = x^2e^y \end{aligned}$$

Therefore, $\vec{\mathbf{F}}$ is conservative. Now we are confident the following integrals will all match:

$$\int P(x,y,z)\,dx = \int (2xe^yz + e^xz)\,dx = x^2e^yz + e^xz + h_1(y,z)$$
$$\int Q(x,y,z)\,dy = \int (x^2e^yz)\,dy = x^2e^yz + h_2(x,z)$$
$$\int R(x,y,z)\,dz = \int (x^2e^y + e^x)\,dz = x^2e^yz + e^xz + h_3(x,y)$$

We have found a potential function for $\vec{\mathbf{F}}$ is $f(x,y,z) = x^2e^yz + e^xz$. Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(1,1,1) - f(0,0,0) = 2e - 0 = \boxed{2e}$$

Notice that if we hadn't recognized that $\vec{\mathbf{F}}$ is conservative, we would have had to parameterize C and use the method from the previous section. Since curve C is unknown, this would have been impossible to determine (and each choice of C may have led

to different final answers).

Illustration 4.1.5: Work Done on a Particle

Let $\vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2y\rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point (a,b) in $\mathbb{R}^2$. Furthermore, the particle's motion can be modeled with a smooth parameterization. For which value(s) of $a, b \in \mathbb{R}$ does $\vec{\mathbf{F}}$ do positive work on the particle?

We find the work done by $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y)\rangle$ on the particle by showing that it is conservative and then by using the Fundamental Theorem for Line Integrals. It is easy to check the mixed partials match:

$$P_y = \frac{\partial}{\partial y}\left(2xy^2\right) = 4xy \quad \text{and} \quad Q_x = \frac{\partial}{\partial x}\left(2x^2y\right) = 4xy$$

Now we integrate to find a potential function:

$$\int P(x,y)\,dx = \int (2xy^2)\,dx = x^2y^2 + h_1(y) \quad \text{and} \quad \int Q(x,y)\,dy = \int 2x^2y\,dy = x^2y^2 + h_2(x)$$

Letting $h_1(y) = h_2(x) = 0$, we see that we can use the potential function $f(x,y) = x^2y^2$

Let (a,b) be the point at which the particle stops its motion, and let C denote the curve that models the particle's motion. The work done by $\vec{\mathbf{F}}$ on the particle is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. By the Fundamental Theorem for Line Integrals,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(a,b) - f(0,0) = a^2b^2$$

Thus, as long as $a \neq 0$ and $b \neq 0$, we have $a^2b^2 > 0$, which means $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} > 0$. Therefore, $\vec{\mathbf{F}}$ does positive work on the particle as long as the path does not end on the x-axis or y-axis.

Illustration 4.1.6: Almost Conservative Line Integral

Let C be the curve connecting $\left(-\frac{\pi}{2}, 0\right)$ to $\left(\frac{\pi}{2}, 0\right)$ along the path $y = \cos(x)$. We will evaluate the line integral

$$\int_C \left(\sin(y)e^x - ye^{\sin(x)}\right)\,dx + \cos(y)e^x\,dy$$

First, we consider the vector field $\vec{\mathbf{F}} = \langle \sin(y)e^x - ye^{\sin(x)}, \cos(y)e^x\rangle$ and determine if it is conservative:

$$\frac{\partial}{\partial y}\left(\sin(y)e^x - ye^{\sin(x)}\right) = \cos(y)e^x - e^{\sin(x)} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\cos(y)e^x\right) = \cos(y)e^x$$

Clearly this is not a conservative vector field, so we attempt to solve the integral directly.

We use the parameterization $\vec{\mathbf{r}}(t) = \langle t, \cos(t)\rangle$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, which gives us:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle \sin(\cos(t))e^t - \cos(t)e^{\sin(t)}, -\cos(\cos(t))e^t\rangle \cdot \langle 1, -\sin(t)\rangle \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -e^{\sin(t)}\cos(t) + e^t\sin(t)\cos(\cos(t)) + e^t\sin(\cos(t))dt\end{aligned}$$

This integral is not simple to solve, so we look to solve this problem by breaking it into smaller parts. Define $\vec{\mathbf{F}} = \vec{\mathbf{G}} + \vec{\mathbf{H}}$ for:

$$\vec{\mathbf{G}} = \langle \sin(y)e^x, \cos(y)e^x\rangle \quad \text{and} \quad \vec{\mathbf{H}} = \langle -ye^{\sin(x)}, 0\rangle$$

This way, we can solve for $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \left(\vec{\mathbf{G}} + \vec{\mathbf{H}}\right) \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}} + \int_C \vec{\mathbf{H}} \cdot d\vec{\mathbf{r}}$

It is easy to see that $\vec{\mathbf{G}}$ is conservative, with $\vec{\mathbf{G}} = \nabla g$ for $g(x,y) = e^x\sin(y)$. Therefore,

$$\int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}} = \int_C \nabla g \cdot d\vec{\mathbf{r}} = g\left(\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)\right) - g\left(\vec{\mathbf{r}}\left(-\frac{\pi}{2}\right)\right) = 0$$

While $\vec{\mathbf{H}}$ is not conservative, it is easier to integrate on its own, with $\vec{\mathbf{r}}(t) = \langle t, \cos(t) \rangle$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$:

$$\int_C \vec{\mathbf{H}} \cdot d\vec{\mathbf{r}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -e^{\sin(t)} \cos(t)\, dt = -e^{\sin(t)} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -e^1 + e^{-1}$$

Putting these together, we get our final answer:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0 - e^1 + e^{-1} = \frac{1 - e^2}{e}$$

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4.1: Practice Set

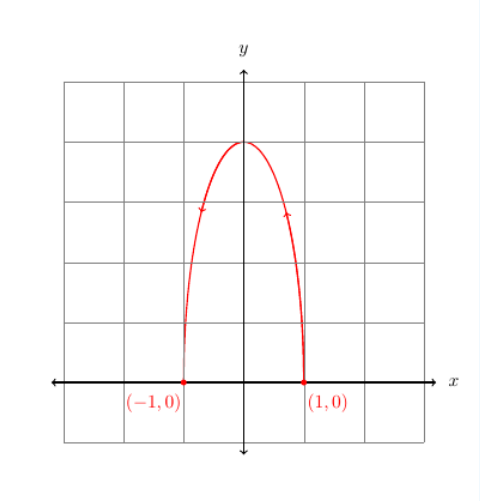
Exercise 4.1.1

Let C be the curve connecting $(1,0)$ to $(-1,0)$ counterclockwise along the ellipse $x^2 + \frac{y^2}{16} = 1$

(a) Evaluate $\int_C 2y\,dx + 4x\,dy$

(b) Evaluate $\int_C 2x\,dx + 4y\,dy$

(c) Evaluate $\int_C \left(x^8 - 2xy\sin(x^2)\right)\,dx + \left(\sqrt{y^3+1} + \cos(x^2)\right)\,dy$



Answer

(a) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle 2y, 4x\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}(2y) = 2 \quad \text{and} \quad \frac{\partial}{\partial x}(4x) = 4$$

Therefore, $\vec{\mathbf{F}}$ is not conservative, and we must solve the integral by parameterization.

$\vec{\mathbf{r}}(t) = \langle \cos(t), 4\sin(t)\rangle$ for $0 \le t \le \pi$ defines this curve, so we have:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{\pi} \left(2(4\sin(t)) \cdot (-\sin(t)) + 4(\cos(t)) \cdot (4\cos(t))\right)\,dt \\ &= \int_0^{\pi} 16\cos^2(t) - 8\sin^2(t)\,dt \\ &= \left[4t + 6\sin(2t)\right]\Big|_0^{\pi} \\ &= 4\pi\end{aligned}$$

(b) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle 2x, 4y\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}(2x) = 0 \quad \text{and} \quad \frac{\partial}{\partial x}(4y) = 0$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int 2x\,dx = x^2 + h_1(y)$$

$$\int 4y\,dy = 2y^2 + h_2(x)$$

These are equivalent when we pick $h_1(y) = 2y^2$ and $h_2(x) = x^2$

Thus $\vec{\mathbf{F}} = \nabla f$ for $f(x, y) = x^2 + 2y^2$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(-1, 0) - f(1, 0) = 1 - 1 = 0$$

(c) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \left\langle x^8 - 2xy\sin(x^2), \sqrt{y^3 + 1} + \cos(x^2) \right\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}\left(x^8 - 2xy\sin(x^2)\right) = -2x\sin(x^2) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\sqrt{y^3 + 1} + \cos(x^2)\right) = -2x\sin(x^2)$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int \left(x^8 - 2xy\sin(x^2)\right)\,dx = \frac{1}{9}x^9 + y\cos(x^2) + h_1(y)$$

$$\int \left(\sqrt{y^3 + 1} + \cos(x^2)\right)\,dy = \left(\int_0^y \sqrt{w^3 + 1}\,dw\right) + y\cos(x^2) + h_2(x)$$

These are equivalent when we pick $h_1(y) = \int_0^y \sqrt{w^3 + 1}\,dw$ and $h_2(x) = \frac{1}{9}x^9$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x, y) = y\cos(x^2) + \frac{1}{9}x^9 + \left(\int_0^y \sqrt{w^3 + 1}\,dw\right)$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

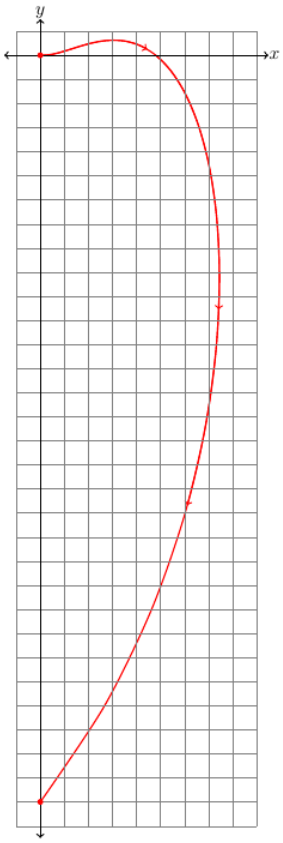
$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(-1, 0) - f(1, 0) = -\frac{1}{9} - \frac{1}{9} = -\frac{2}{9}$$

? Exercise 4.1.2

Let C be the curve defined by $\vec{\mathbf{r}}(t) = \langle 8e^t \sin(t), t^3\cos(t)\rangle$ over $0 \le t \le \pi$

(a) Evaluate $\int_C \left(2x + (x + y^2)^3\right)\,dx + 2y(x + y^2)^3\,dy$

(b) Evaluate $\int_C (1 + y)\,dx + (x - \pi)\,dy$



Answer

(a) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle 2x + (x+y^2)^3, 2y(x+y^2)^3 \rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}\left(2x + (x+y^2)^3\right) = 6y(x+y^2)^2 \quad \text{and} \quad \frac{\partial}{\partial x}\left(2y(x+y^2)^3\right) = 6y(x+y^2)^2$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int \left(2x + (x+y^2)^3\right)\, dx = x^2 + \frac{1}{4}(x+y^2)^4 + h_1(y)$$

$$\int 2y(x+y^2)^3\, dy = \frac{1}{4}(x+y^2)^4 + h_2(x)$$

These are equivalent when we pick $h_1(y) = 0$ and $h_2(x) = x^2$

Thus $\vec{\mathbf{F}} = \nabla f$ for $f(x,y) = \frac{1}{4}(x+y^2)^4 + x^2$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}(\pi)\right) - f\left(\vec{\mathbf{r}}(0)\right) = f\left(0, -\pi^3\right) - f(0,0) = \frac{\pi^{24}}{4} - 0 = \frac{\pi^{24}}{4}$$

(b) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle 1 + y, x - \pi \rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}(1+y)=1 \quad \text{and} \quad \frac{\partial}{\partial x}(x-\pi)=1$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int (1+y)\,dx = x+xy+h_1(y)$$

$$\int (x-\pi)\,dy = xy-\pi y+h_2(x)$$

These are equivalent when we pick $h_1(y)=-\pi y$ and $h_2(x)=x$

Thus $\vec{\mathbf{F}}=\nabla f$ for $f(x,y)=xy+x-\pi y$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C \nabla f\cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}(\pi)\right)-f\left(\vec{\mathbf{r}}(0)\right)=f\left(0,-\pi^3\right)-f(0,0)=\pi^4-0=\pi^4$$

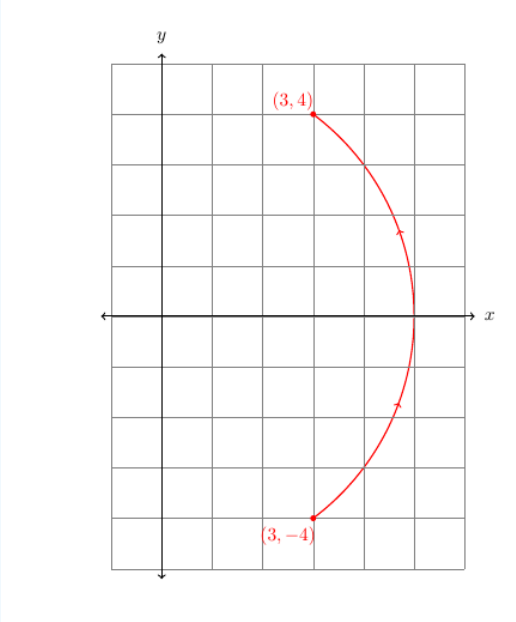
Exercise 4.1.3

Let C be the curve starting at $(3,-4)$ and ending at $(3,4)$ counter-clockwise around the circle $x^2+y^2=25$

(a) Evaluate $\int_C 3y^2\,dx-3x^2\,dy$

(b) Evaluate $\int_C \frac{xy\,dx-x^2\,dy}{(x^2+y^2)^{\frac{3}{2}}}$

(c) Evaluate $\int_C \frac{dx+dy}{(x^2+y^2)^{\frac{1}{2}}}$



Answer

(a) We want $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}}=\langle 3y^2,-3x^2\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}\left(3y^2\right)=6y \quad \text{and} \quad \frac{\partial}{\partial x}\left(-3x^2\right)=-6x$$

Therefore, $\vec{\mathbf{F}}$ is not conservative, and we must solve the integral by parameterization.

$\vec{\mathbf{r}}(t) = \langle 5\cos(t), 5\sin(t)\rangle$ for $\tan^{-1}\left(-\frac{4}{3}\right) \le t \le \tan^{-1}\left(\frac{4}{3}\right)$ defines this curve, so we have:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{\tan^{-1}\left(-\frac{4}{3}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} \left(3(5\sin(t))^2 \cdot (-5\sin(t)) - 3(5\cos(t))^2 \cdot (5\cos(t))\right)\, dt \\ &= \int_{\tan^{-1}\left(-\frac{4}{3}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} -125\left(3\sin^3(t) + 3\cos^3(t)\right)\, dt \\ &= -125\left[-3\cos(t) + \cos^3(t) + 3\sin(t) - \sin^3(t)\right]\Big|_{\tan^{-1}\left(-\frac{4}{3}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} \\ &= -472\end{aligned}$$

(b) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \left\langle \frac{xy}{(x^2+y^2)^{\frac{3}{2}}}, -\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}\left(\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\right) = \frac{x(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}} \quad \text{and} \quad \frac{\partial}{\partial x}\left(-\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}\right) = \frac{x(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}}$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\begin{aligned}\int \frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\, dx &= -\frac{y}{\sqrt{x^2+y^2}} + h_1(y) \\ \int -\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}\, dy &= -\frac{y}{\sqrt{x^2+y^2}} + h_2(x)\end{aligned}$$

These are equivalent when we pick $h_1(y) = 0$ and $h_2(x) = 0$

Thus $\vec{\mathbf{F}} = \nabla f$ for $f(x,y) = -\frac{y}{\sqrt{x^2+y^2}}$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(3,4) - f(3,-4) = -\frac{4}{5} - \frac{4}{5} = -\frac{8}{5}$$

(c) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \left\langle \frac{1}{(x^2+y^2)^{\frac{1}{2}}}, \frac{1}{(x^2+y^2)^{\frac{1}{2}}} \right\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with differentiation:

$$\frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2)^{\frac{1}{2}}}\right) = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2)^{\frac{1}{2}}}\right) = -\frac{x}{(x^2+y^2)^{\frac{3}{2}}}$$

Therefore, $\vec{\mathbf{F}}$ is not conservative, and we must solve the integral by parameterization.

$\vec{\mathbf{r}}(t) = \langle 5\cos(t), 5\sin(t)\rangle$ for $\tan^{-1}\left(-\frac{4}{3}\right) \le t \le \tan^{-1}\left(\frac{4}{3}\right)$ defines this curve, so we have:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{\tan^{-1}\left(-\frac{4}{3}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} \frac{1 \cdot (-5\sin(t)) + 1 \cdot (5\cos(t))}{((5\cos(t))^2 + (5\sin(t))^2)^{\frac{1}{2}}}\, dt \\ &= \int_{\tan^{-1}\left(-\frac{4}{3}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} (-\sin(t) + \cos(t))\, dt \\ &= [\cos(t) + \sin(t)]\Big|_{\tan^{-1}\left(-\frac{4}{3}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} \\ &= \frac{8}{5}\end{aligned}$$

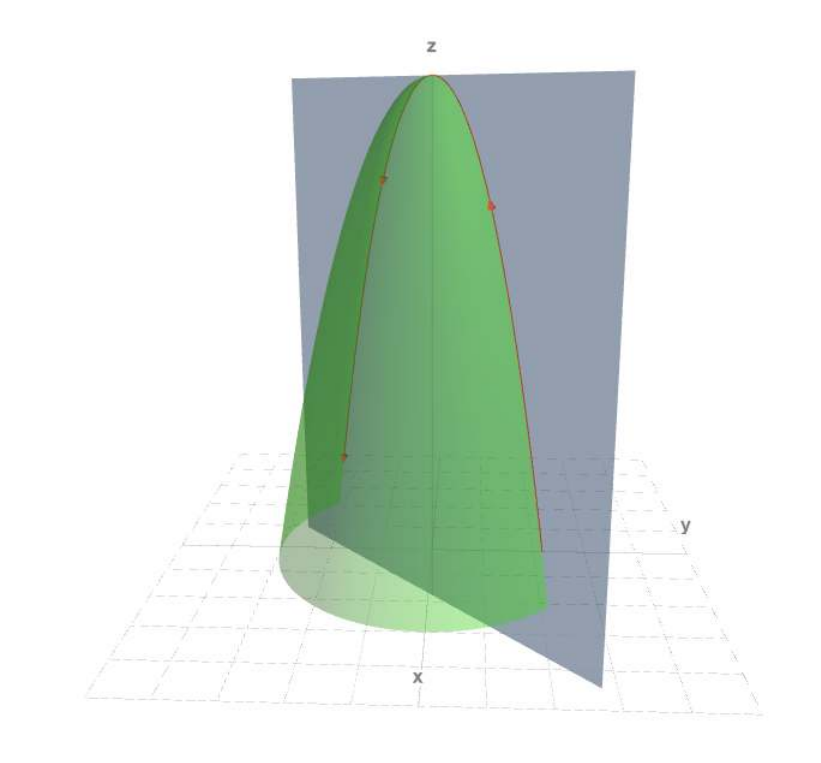
Exercise 4.1.4

Let C be the curve connecting $(2,2,1)$ to $(-2,-2,1)$ along the intersection of $9-x^2-y^2$ and $x=y$

(a) Evaluate $\int_C z\,dx + x\,dz$

(b) Evaluate $\int_C (x+y)dz$

(c) Evaluate $\int_C 2x\,dx - 4y^3\,dy + 9z^8\,dz$



Answer

(a) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle z, 0, x\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with curl:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & x \end{vmatrix} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int z\,dx = xz + h_1(y,z)$$
$$\int 0\,dy = 0 + h_2(x,z)$$
$$\int x\,dz = xz + h_3(x,y)$$

These are equivalent when we pick $h_1(y,z) = 0$, $h_2(x,z) = xz$, and $h_3(x,y) = 0$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x,y,z) = xz$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(-2,-2,1) - f(2,2,1) = -2 - 2 = -4$$

(b) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle 0, 0, x+y \rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with curl:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x+y \end{vmatrix} = 1\,\hat{\mathbf{i}} - 1\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} \neq \vec{\mathbf{0}}$$

Therefore, $\vec{\mathbf{F}}$ is not conservative, and we must solve the integral by parameterization.

$\vec{\mathbf{r}}(t) = \langle -t, -t, 9 - 2t^2 \rangle$ for $-2 \leq t \leq 2$ defines this curve, so we have:

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{-2}^{2} ((-t) + (-t)) \cdot (-4t)\, dt \\ &= \int_{-2}^{2} 8t^2\, dt \\ &= \frac{8}{3} t^3 \Big|_{-2}^{2} \\ &= \frac{128}{3} \end{aligned}$$

(c) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle 2x, -4y^3, 9z^8 \rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with curl:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -4y^3 & 9z^8 \end{vmatrix} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int 2x\, dx = x^2 + h_1(y,z)$$

$$\int -4y^3\, dy = -y^4 + h_2(x,z)$$

$$\int 9z^8\, dz = z^9 + h_3(x,y)$$

These are equivalent when we pick $h_1(y,z) = -y^4 + z^9$, $h_2(x,z) = x^2 + z^9$, and $h_3(x,y) = x^2 - y^4$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x,y,z) = x^2 - y^4 + z^9$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

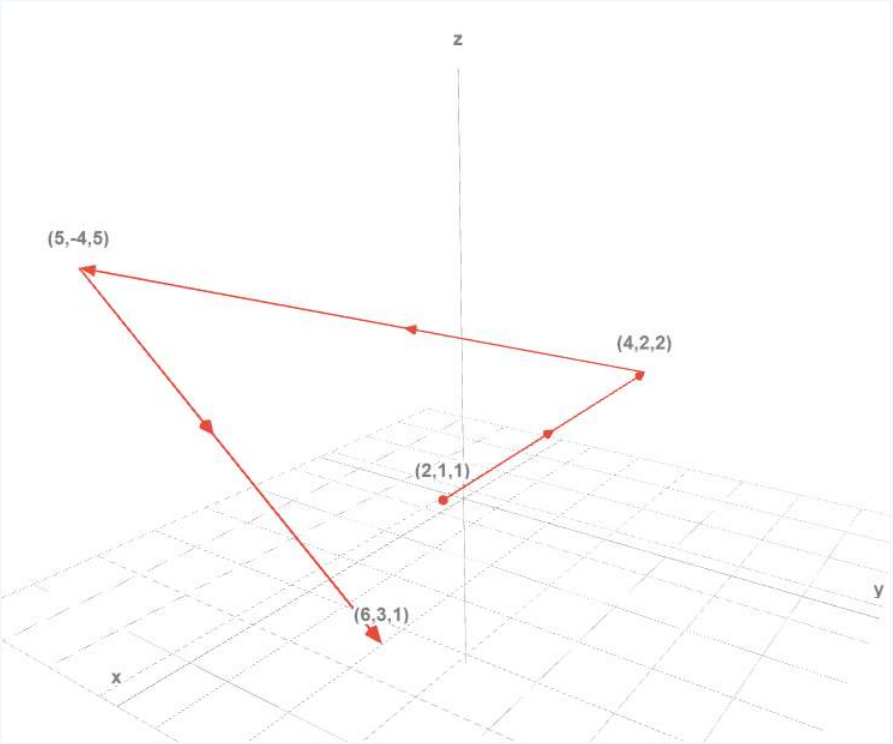
$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(-2,-2,1) - f(2,2,1) = -11 - (-11) = 0$$

? Exercise 4.1.5

Let C be the curve defined as straight lines from $(2,1,1)$ to $(-4,2,2)$ to $(5,-4,5)$ to $(6,3,1)$

(a) Evaluate $\int_C (2x - yz)\, dx + (4 - xz)\, dy - xy\, dz$

(b) Evaluate $\int_C (1+y)\,dx + (1+x)\,dy + z^{\tan^{-1}(z)}\,dz$



Answer

(a) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle 2x - yz, 4 - xz, -xy \rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with curl:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - yz & 4 - xz & -xy \end{vmatrix} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int (2x - yz)\,dx = x^2 - xyz + h_1(y,z)$$

$$\int (4 - xz)\,dy = 4y - xyz + h_2(x,z)$$

$$\int -xy\,dz = -xyz + h_3(x,y)$$

These are equivalent when we pick $h_1(y,z) = 4y$, $h_2(x,z) = x^2$, and $h_3(x,y) = x^2 + 4y$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x,y,z) = x^2 + 4y - xyz$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(6,3,1) - f(2,1,1) = 30 - 6 = 24$$

(b) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \left\langle 1 + y, 1 + x, z^{\tan^{-1}(z)} \right\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with curl:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1+y & 1+x & z^{\tan^{-1}(z)} \end{vmatrix} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int (1+y)\,dx = x + xy + h_1(y,z)$$

$$\int (1+x)\,dy = y + xy + h_2(x,z)$$

$$\int z^{\tan^{-1}(z)}\,dz = \left(\int_1^z w^{\tan^{-1}(w)}\,dw\right) + h_3(x,y)$$

These are equivalent when we pick $h_1(y,z) = y + \int_1^z w^{\tan^{-1}(w)}\,dw$, $h_2(x,z) = x + \int_1^z w^{\tan^{-1}(w)}\,dw$, and $h_3(x,y) = x + y$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x,y,z) = xy + x + y + \int_1^z w^{\tan^{-1}(w)}\,dw$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

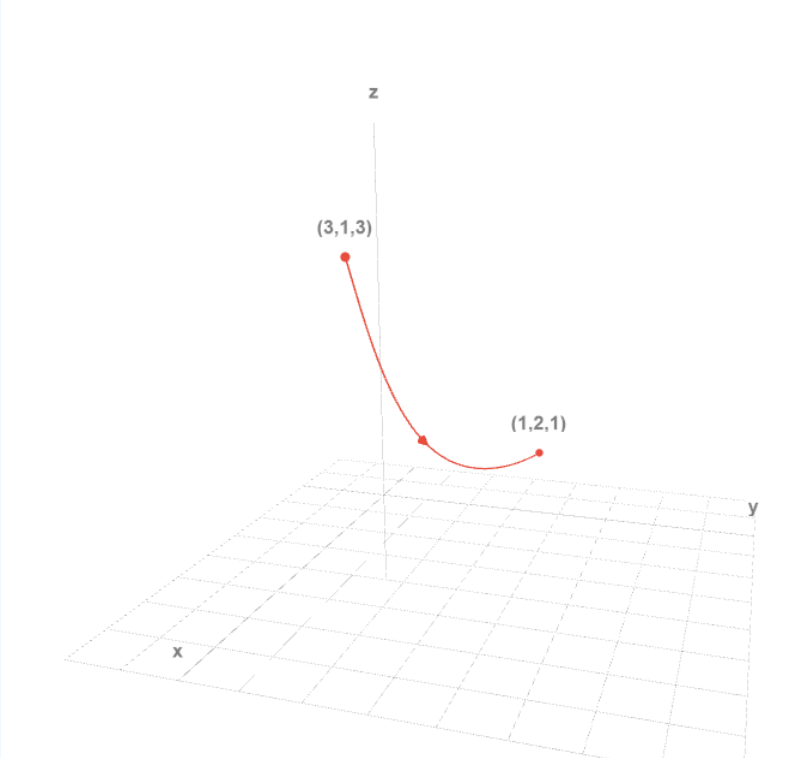
$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(6,3,1) - f(2,1,1) = 27 - 5 = 22$$

? Exercise 4.1.6

Let C be the curve defined by $\vec{\mathbf{r}}(t) = \left\langle \sqrt{1 + 8\cos\left(\frac{\pi t}{4}\right)}, \log_3(t^2+t+3), \frac{t^2-t+3}{t^2+1} \right\rangle$ over $0 \le t \le 2$

(a) Evaluate $\int_C (xy+1)e^{xy+2z}\,dx + x^2 e^{xy+2z}\,dy + 2xe^{xy+2z}\,dz$

(b) Evaluate $\int_C \sqrt[3]{64xyz^2}dx + \sqrt[3]{\frac{x^4z^2}{y^2}}dy + \left(\sqrt[3]{\frac{8x^4y}{z}} - 4z\right)dz$



https://math.libretexts.org/@go/page/195729

Answer

(a) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \langle (xy+1)e^{xy+2z}, x^2e^{xy+2z}, 2xe^{xy+2z} \rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with curl:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (xy+1)e^{xy+2z} & x^2e^{xy+2z} & 2xe^{xy+2z} \end{vmatrix} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int (xy+1)e^{xy+2z}\,dx = xe^{xy+2z} + h_1(y,z)$$

$$\int x^2e^{xy+2z}\,dy = xe^{xy+2z} + h_2(x,z)$$

$$\int 2xe^{xy+2z}\,dz = xe^{xy+2z} + h_3(x,y)$$

These are equivalent when we pick $h_1(y,z) = 0$, $h_2(x,z) = 0$, and $h_3(x,y) = 0$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x,y,z) = xe^{xy+2z}$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}(2)\right) - f\left(\vec{\mathbf{r}}(0)\right) = f(1,2,1) - f(3,1,3) = e^4 - 3e^9$$

(b) We want $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the vector field $\vec{\mathbf{F}} = \left\langle 4x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{2}{3}}, x^{\frac{4}{3}}y^{-\frac{2}{3}}z^{\frac{2}{3}}, 2x^{\frac{4}{3}}y^{\frac{1}{3}}z^{-\frac{1}{3}} - 4z \right\rangle$

First, we check if $\vec{\mathbf{F}}$ is conservative with curl:

$$\text{curl}(\vec{\mathbf{F}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{2}{3}} & x^{\frac{4}{3}}y^{-\frac{2}{3}}z^{\frac{2}{3}} & 2x^{\frac{4}{3}}y^{\frac{1}{3}}z^{-\frac{1}{3}} - 4z \end{vmatrix} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Therefore, $\vec{\mathbf{F}}$ is conservative, and we look to find its potential function:

$$\int 4x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{2}{3}}\,dx = 3x^{\frac{4}{3}}y^{\frac{1}{3}}z^{\frac{2}{3}} + h_1(y,z)$$

$$\int x^{\frac{4}{3}}y^{-\frac{2}{3}}z^{\frac{2}{3}}\,dy = 3x^{\frac{4}{3}}y^{\frac{1}{3}}z^{\frac{2}{3}} + h_2(x,z)$$

$$\int \left(2x^{\frac{4}{3}}y^{\frac{1}{3}}z^{-\frac{1}{3}} - 4z\right)\,dz = 3x^{\frac{4}{3}}y^{\frac{1}{3}}z^{\frac{2}{3}} - 2z^2 + h_3(x,y)$$

These are equivalent when we pick $h_1(y,z) = -2z^2$, $h_2(x,z) = -2z^2$, and $h_3(x,y) = 0$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x,y,z) = 3x^{\frac{4}{3}}y^{\frac{1}{3}}z^{\frac{2}{3}} - 2z^2$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}(2)\right) - f\left(\vec{\mathbf{r}}(0)\right) = f(1,2,1) - f(3,1,3) = (3\sqrt[3]{2} - 2) - (27 - 18) = 3\sqrt[3]{2} - 11$$

? Exercise $4.1.7a$

Use the Fundamental Theorem of Line Integrals to evaluate $\int_{-1}^{1} \left(\dfrac{2t^5(3t^3+1)}{\left(t^{10} + (3t^3+1)^2\right)^2} \cdot 5t^4 + \dfrac{(3t^3+1)^2 - t^{10}}{\left(t^{10} + (3t^3+1)^2\right)^2} \cdot 9t^2 \right) dt$

Answer

This definite integral looks intimidating, but we see it has a very distinct shape!

If we set $x(t) = t^5$, and $y(t) = (3t^3 + 1)$ we see that it is exactly:

$$\int_{-1}^{1} \left(\frac{2x(t)y(t)}{([x(t)]^2 + [y(t)]^2)^2} \cdot x'(t) + \frac{[y(t)]^2 - [x(t)]^2}{([x(t)]^2 + [y(t)]^2)^2} \cdot y'(t) \right) dt$$

Therefore, it is the parameterized version of the line integral

$$\int_C \frac{2xy}{(x^2+y^2)^2}\, dx + \frac{y^2 - x^2}{(x^2+y^2)^2}\, dy$$

for the curve C defined by $\vec{\mathbf{r}}(t) = \langle t^5, 3t^3 + 1 \rangle$ over $-1 \le t \le 1$

The vector field $\vec{\mathbf{F}} = \left\langle \frac{2xy}{(x^2+y^2)^2}, \frac{y^2 - x^2}{(x^2+y^2)^2} \right\rangle$ is conservative, so we can find its potential function:

$$\int \frac{2xy}{(x^2+y^2)^2}\, dx = -\frac{y}{x^2+y^2} + h_1(y)$$

$$\int \frac{y^2 - x^2}{(x^2+y^2)^2}\, dy = -\frac{y}{x^2+y^2} + h_2(x)$$

These are equivalent when we pick $h_1(y, z) = 0$, $h_2(x, z) = 0$, and $h_3(x, y) = 0$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x, y, z) = -\frac{y}{x^2+y^2}$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}\,(1)\right) - f\left(\vec{\mathbf{r}}(-1)\right) = f(1, 4) - f(-1, -2) = -\frac{4}{17} - \frac{2}{5} = -\frac{54}{85}$$

? Exercise 4.1.7b

Use the Fundamental Theorem of Line Integrals to evaluate $\int_0^{\pi} \left((2t\cos(t) - 4\cos(4t) - 2)\sin(t) + \left(-16\cos(t) + 12\cos^2(4t) - 16\cos(4t)t^2\right)\sin(4t) + \left(4t\cos^2(4t) - \cos^2(t)\right)\right) dt$

Answer

This definite integral looks intimidating, but we see it has a very distinct shape!

If we set $x(t) = \cos(t)$, $y(t) = \cos(4t)$, and $z(t) = t$, we see that it is exactly:

$$\int_0^{\pi} \left((-2x(t)z(t) + 4y(t) + 2) \cdot x'(t) + \left(4x(t) - 3[y(t)]^2 + 4y(t)[z(t)]^2\right) \cdot y'(t) + \left(4[y(t)]^2 z(t) - [x(t)]^2\right) \cdot z'(t)\right) dt$$

Therefore, it is the parameterized version of the line integral

$$\int_C (-2xz + 4y + 2)\, dx + (4x - 3y^2 + 4yz^2)\, dy + (4y^2 z - x^2)\, dz$$

for the curve C defined by $\vec{\mathbf{r}}(t) = \langle \cos(t), \cos(4t), t \rangle$ over $0 \le t \le \pi$

The vector field $\vec{\mathbf{F}} = \langle -2xz + 4y + 2, 4x - 3y^2 + 4yz^2, 4y^2 z - x^2 \rangle$ is conservative, so we can find its potential function:

$$\int (-2xz + 4y + 2)\, dx = 4xy - x^2 z + 2x + h_1(y, z)$$

$$\int (4x - 3y^2 + 4yz^2)\, dy = 4xy + 2y^2 z^2 - y^3 + h_2(x, z)$$

$$\int (4y^2 z - x^2)\, dz = 2y^2 z^2 - x^2 z + h_3(x, y)$$

These are equivalent when we pick $h_1(y, z) = 2y^2 z^2 - y^3$, $h_2(x, z) = 2x - x^2 z$, and $h_3(x, y) = 4xy + 2x - y^3$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x, y, z) = 4xy + 2y^2 z^2 - x^2 z + 2x - y^3$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}\left(\pi\right)\right) - f\left(\vec{\mathbf{r}}(0)\right) = f\left(-1,1,\pi\right) - f(1,1,0) = \left(2\pi^2 - \pi - 7\right) - 5 = 2\pi^2 - \pi - 12$$

Exercise 4.1.7c

Use the Fundamental Theorem of Line Integrals to evaluate $\int_0^{\frac{\pi}{2}} \left(\frac{t^3 \sin(t)(e^t \sin(t) + 2)}{(e^t \sin(t) + 1)^2} \cdot \cos(t) + \frac{\sin^2(t)}{e^t \sin(t) + 1} \cdot 3t^2 - \frac{t^3 \sin^3(t)}{(e^t \sin(t) + 1)^2} \cdot e^t \right) dt$

Answer

This definite integral looks intimidating, but we see it has a very distinct shape!

If we set $x(t) = \sin(t)$, $y(t) = t^3$, and $z(t) = e^t$, we see that it is exactly:

$$\int_0^{\frac{\pi}{2}} \left(\frac{x(t)y(t)\left(x(t)z(t) + 2\right)}{\left(x(t)z(t) + 1\right)^2} \cdot x'(t) + \frac{[x(t)]^2}{x(t)z(t) + 1} \cdot y'(t) - \frac{[x(t)]^3 y(t)}{\left(x(t)z(t) + 1\right)^2} \cdot z'(t) \right) dt$$

Therefore, it is the parameterized version of the line integral

$$\int_C \frac{xy(xz+2)}{(xz+1)^2}\,dx + \frac{x^2}{xz+1}\,dy - \frac{x^3 y}{(xz+1)^2}\,dz$$

for the curve C defined by $\vec{\mathbf{r}}(t) = \langle \sin(t), t^3, e^t \rangle$ over $0 \le t \le \frac{\pi}{2}$

The vector field $\vec{\mathbf{F}} = \left\langle \frac{xy(xz+2)}{(xz+1)^2}, \frac{x^2}{xz+1}, -\frac{x^3 y}{(xz+1)^2} \right\rangle$ is conservative, so we can find its potential function:

$$\int \frac{xy(xz+2)}{(xz+1)^2}\,dx = \frac{x^2 y}{xz+1} + h_1(y,z)$$

$$\int \frac{x^2}{xz+1}\,dy = \frac{x^2 y}{xz+1} + h_2(x,z)$$

$$\int -\frac{x^3 y}{(xz+1)^2}\,dz = \frac{x^2 y}{xz+1} + h_3(x,y)$$

These are equivalent when we pick $h_1(y,z) = 0$, $h_2(x,z) = 0$, and $h_3(x,y) = 0$

Thus, $\vec{\mathbf{F}} = \nabla f$ for $f(x,y,z) = \frac{x^2 y}{xz+1}$

This allows us to use the Fundamental Theorem of Calculus for Line Integrals to evaluate the desired line integral:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)\right) - f\left(\vec{\mathbf{r}}(0)\right) = f\left(1, \frac{\pi^3}{8}, e^{\frac{\pi}{2}}\right) - f(0,0,1) = \frac{\frac{\pi^3}{8}}{e^{\frac{\pi}{2}} + 1} - 0 = \frac{\pi^3}{8e^{\frac{\pi}{2}} + 8}$$

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4.2: Green's Theorem

Learning Objectives

- Apply Green's Theorem to solve closed line integrals as double integrals.
- Apply Green's Theorem to solve double integrals as line integrals.
- Apply Green's Theorem to line integrals of non-closed curves.

In this section, we examine Green's Theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's Theorem requires the region D in the double integral to be simply connected. However, we will extend Green's Theorem to regions that are not simply connected.

Put simply, Green's Theorem relates a line integral around a simple closed plane curve C and a double integral over the region enclosed by C. The theorem is useful because it allows us to translate difficult line integrals into simpler double integrals, or difficult double integrals into simpler line integrals.

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4.2: Green's Theorem

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$, depends only on the value of F at the endpoints a and b of that segment. Since the numbers a and b are the boundaries of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F'(x)\,dx$ based on information about the boundary of line segment $[a, b]$.

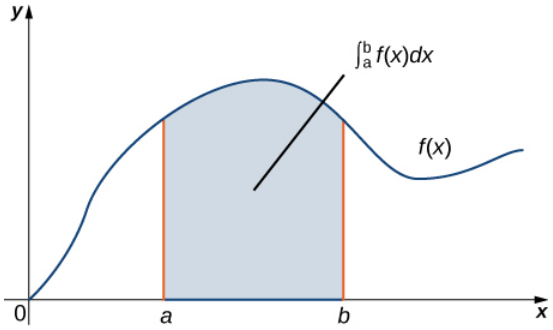


Figure 4.2.1: The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve C.

Green's Theorem takes this idea and extends it to calculating double integrals. Green's Theorem says that we can calculate a double integral over region D based solely on information about the boundary of D. Green's Theorem also says we can calculate a line integral over a simple closed curve C based solely on information about the region that C encloses. In particular, Green's Theorem connects a double integral over region D to a line integral around the boundary of D.

The first form of Green's Theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the line integral of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

Theorem 4.2.1: Green's Theorem

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise.

Let $\vec{\mathbf{F}} = \langle P(x, y), Q(x, y)\rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

Notice that Green's Theorem can be used only for a two-dimensional vector field $\vec{\mathbf{F}}$.

The proof of Green's Theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\vec{\mathbf{F}} = \langle P, Q\rangle$ is conservative. In this case,

$$\int_C P\,dx + Q\,dy = 0$$

because any closed line integral is zero in conservative vector fields: $\vec{\mathbf{F}}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y)\,dA = \int_D 0\,dA = 0 = \int_C P\,dx + Q\,dy$$

which confirms Green's Theorem in the case of **conservative vector fields**.

Proof

Let's now prove that Green's Theorem is true when the region D is a rectangle. Let D be the rectangle $[a,b] \times [c,d]$, and let its boundary C be oriented counterclockwise. Then, C consists of four smooth pieces C_1, C_2, C_3, and C_4. We parameterize each side of D as follows:

$$C_1 : \vec{\mathbf{r}}_1(t) = \langle t, c\rangle, \text{ for } a \le t \le b$$
$$C_2 : \vec{\mathbf{r}}_2(t) = \langle b, t\rangle, \text{ for } c \le t \le d$$
$$-C_3 : \vec{\mathbf{r}}_3(t) = \langle t, d\rangle, \text{ for } a \le t \le b$$
$$-C_4 : \vec{\mathbf{r}}_4(t) = \langle a, t\rangle, \text{ for } c \le t \le d$$

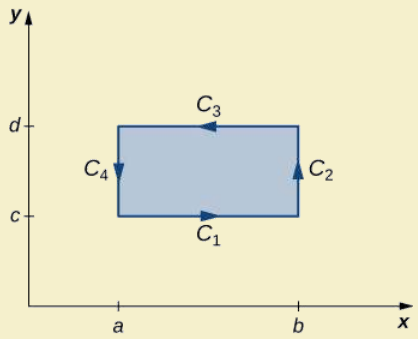


Figure 4.2.3: Rectangle D is oriented counterclockwise.

Then,

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1'(t)\,dt + \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2'(t)\,dt \\
&\quad - \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_3(t)) \cdot \vec{\mathbf{r}}_3'(t)\,dt - \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_4(t)) \cdot \vec{\mathbf{r}}_4'(t)\,dt \\
&= \int_a^b P(t,c)\,dt + \int_c^d Q(b,t)\,dt - \int_a^b P(t,d)\,dt - \int_c^d Q(a,t)\,dt \\
&= \int_a^b (P(t,c) - P(t,d))\,dt + \int_c^d (Q(b,t) - Q(a,t))\,dt \\
&= -\int_a^b (P(t,d) - P(t,c))\,dt + \int_c^d (Q(b,t) - Q(a,t))\,dt
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t,d) - P(t,c) = \int_c^d \frac{\partial}{\partial y} P(t,y)dy \quad \text{and} \quad Q(b,t) - Q(a,t) = \int_a^b \frac{\partial}{\partial x} Q(x,t)\,dx$$

Therefore,

$$
\begin{aligned}
-\int_a^b (P(t,d)-P(t,c))\,dt + \int_c^d (Q(b,t)-Q(a,t))\,dt &= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(t,y)\,dy\,dt \\
&\quad + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,t)\,dx\,dt \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(x,y)\,dy\,dx \\
&\quad + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,y)\,dx\,dy \\
&= \int_a^b \int_c^d (Q_x - P_y)\,dy\,dx \\
&= \iint_D (Q_x - P_y)\,dA
\end{aligned}
$$

Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D (Q_x - P_y)\,dA$, and we have proved Green's Theorem in the case of a rectangle.

To prove Green's Theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

Illustration 4.2.1: Applying Green's Theorem over a Rectangle

If C is a rectangle with vertices $(1,1)$, $(1,5)$, $(4,5)$, and $(4,1)$ oriented counterclockwise, evaluate

$$\int_C x^2 y\,dx + (y-4)\,dy$$

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 y, y-4\rangle$. Then, $Q_x(x,y) = 0$ and $P_y(x,y) = x^2$. Therefore, $Q_x - P_y = -x^2$

Let D be the rectangular region enclosed by C. By Green's Theorem,

$$
\begin{aligned}
\int_C x^2 y\,dx + (y-4)\,dy &= \iint_D (Q_x - P_y)\,dA \\
&= \iint_D -x^2\,dA \\
&= \int_1^5 \int_1^4 -x^2\,dx\,dy \\
&= \int_1^5 -\frac{x^3}{3}\Big|_1^4\,dy \\
&= \int_1^5 (-21)\,dy \\
&= -21y\Big|_1^5 \\
&= -84
\end{aligned}
$$

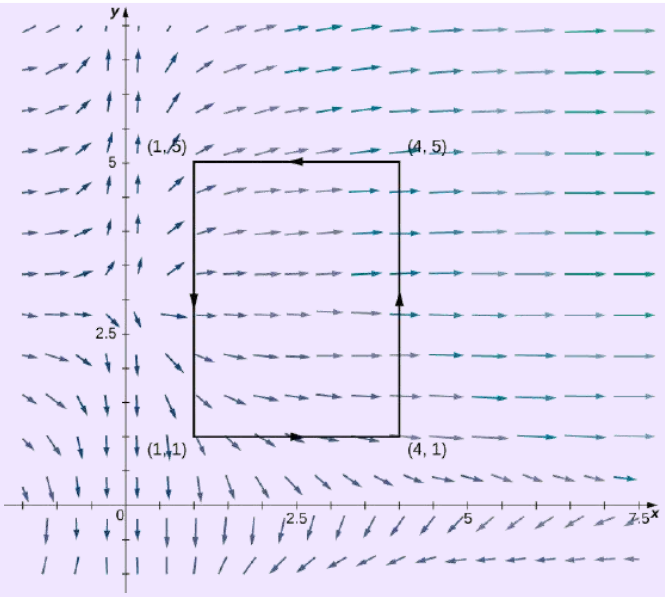


Figure 4.2.4: The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Let's compare this to the direct calculation of each of the four straight lines that C is made of:

Going right along the bottom edge $C_1 : \vec{\mathbf{r}}_1(t) = \langle t, 1\rangle$ for $1 \le t \le 4$, so we have

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_1^4 \langle t^2(1), (1-4)\rangle \cdot \langle 1, 0\rangle \, dt = \int_1^4 t^2 \, dt = \frac{1}{3}t^3\Big|_1^4 = 21$$

Going up the right edge $C_2 : \vec{\mathbf{r}}_2(t) = \langle 4, t\rangle$ for $1 \le t \le 5$, so we have

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_1^5 \langle 4^2(t), (t-4)\rangle \cdot \langle 0, 1\rangle \, dt = \int_1^5 (t-4) \, dt = \frac{1}{2}(t-4)^2\Big|_1^5 = -4$$

Going left along the top edge $C_3 : \vec{\mathbf{r}}_3(t) = \langle 4-t, 5\rangle$ for $0 \le t \le 3$, so we have

$$\int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^3 \langle (4-t)^2(5), (5-4)\rangle \cdot \langle -1, 0\rangle \, dt = \int_0^3 -5(4-t)^2 \, dt = \frac{5}{3}(4-t)^3\Big|_0^3 = -105$$

Finally, going downward along the left edge $C_4 : \vec{\mathbf{r}}_4(t) = \langle 1, 5-t\rangle$ for $0 \le t \le 4$, so we have

$$\int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^4 \langle 1^2(5-t), ((5-t)-4)\rangle \cdot \langle 0, -1\rangle \, dt = \int_0^4 (t-1) \, dt = \frac{1}{2}(t-1)^2\Big|_0^4 = 4$$

Adding these up, we get the same total of -84 that we got by using Green's Theorem.

Example 4.2.1: Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by the force field

$$\vec{\mathbf{F}}(x, y) = \langle y + \sin x, e^y - x\rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

Solution

Let C denote the circle, and let D be the disk enclosed by C. The work done on the particle is

$$W = \int_C (y + \sin x) \, dx + (e^y - x) \, dy$$

As with the previous example, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's Theorem.

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle y + \sin x, e^y - x\rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$

By Green's Theorem,

$$\begin{aligned} W &= \int_C (y + \sin(x))\,dx + (e^y - x)\,dy \\ &= \iint_D (Q_x - P_y)\,dA \\ &= \iint_D -2\,dA \\ &= -2 \cdot \text{area}(D) \\ &= -2 \cdot 4\pi \\ &= \boxed{-8\pi} \end{aligned}$$

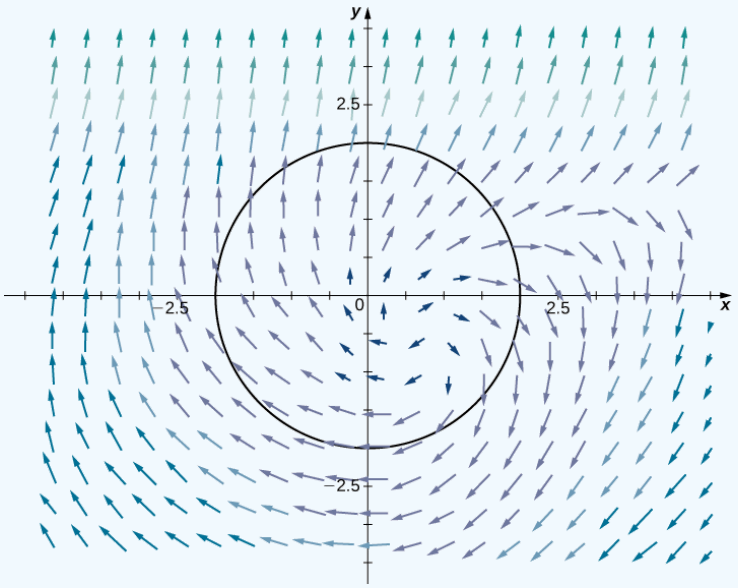


Figure 4.2.5: The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.

In the preceding two problems, the double integral in Green's Theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's Theorem to translate a double integral into a line integral.

Illustration 4.2.2: Applying Green's Theorem over an Ellipse

We can use Green's Theorem to calculate the area enclosed by ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

$$\vec{\mathbf{r}}(t) = \langle a\cos(t), b\sin(t)\rangle \quad \text{for} \quad 0 \le t \le 2\pi$$

Calculating the area of D is equivalent to computing double integral $\iint_D dA$. To calculate this integral without Green's Theorem, we would have to solve:

$$\int_{-a}^{a} \int_{-\sqrt{b^2 - \frac{b^2x^2}{a^2}}}^{\sqrt{b^2 - \frac{b^2x^2}{a^2}}} dy\,dx$$

This integral is not straightforward to calculate. Instead of trying to calculate it directly, we use Green's Theorem to transform $\iint_D dA$ into a line integral around the boundary C.

Consider vector field

$$\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \left\langle -\frac{y}{2}, \frac{x}{2}\right\rangle$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\vec{\mathbf{F}}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's Theorem transforms the line integral of $\vec{\mathbf{F}}$ over C into the double integral of 1 over D.

By Green's Theorem,

$$\begin{aligned}
\iint_D dA &= \iint_D (Q_x - P_y)\, dA \\
&= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{1}{2}\int_C -y\, dx + x\, dy \\
&= \frac{1}{2}\int_0^{2\pi} -b\sin(t)(-a\sin(t)) + a(\cos(t))b\cos(t)\, dt \\
&= \frac{1}{2}\int_0^{2\pi} (ab\cos^2(t) + ab\sin^2(t))\, dt \\
&= \frac{1}{2}\int_0^{2\pi} ab\, dt \\
&= \frac{1}{2}abt\Big|_0^{2\pi} \\
&= \pi ab
\end{aligned}$$

Therefore, the area of the ellipse is πab units^2

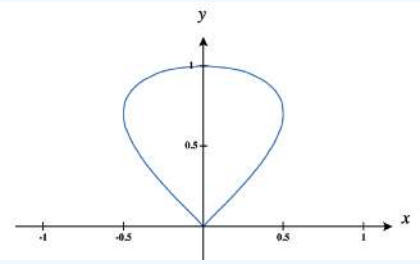
Notice that we could have instead chosen the vector field $\vec{\mathbf{G}} = \langle 0, x\rangle$, which also satisfies $Q_x - P_y = 1$

$$\begin{aligned}
\iint_D dA &= \iint_D (Q_x - P_y)\, dA \\
&= \int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}} = \int_C 0\, dx + x\, dy \\
&= \int_0^{2\pi} a(\cos(t))b\cos(t)\, dt \\
&= \int_0^{2\pi} ab\cos^2(t)\, dt \\
&= \frac{ab}{2}\int_0^{2\pi} (1 + \cos(2t))\, dt \\
&= \frac{ab}{2}\left[t + \frac{1}{2}\sin(2t)\right]\Big|_0^{2\pi} \\
&= \pi ab
\end{aligned}$$

It's worth noting that if $\vec{\mathbf{F}} = \langle P, Q\rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous illustration works.

Example 4.2.2: Applying Green's Theorem to find Area

Find the area of the region enclosed by the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \sin(t)\cos(t), \sin(t)\rangle$, $0 \le t \le \pi$



Solution

Let D be the region enclosed by C. Consider the vector field we used in Illustration 4.2.2:

$$\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \left\langle -\frac{y}{2}, \frac{x}{2}\right\rangle$$

For the curve $r(t) = \langle \sin(t)\cos(t), \sin(t)\rangle$ with $0 \le t \le \pi$, we have

$$\vec{\mathbf{r}}\,'(t) = \langle \cos^2(t) - \sin^2(t), \cos(t)\rangle \quad \text{and} \quad \vec{\mathbf{F}}(t) = \left\langle -\frac{1}{2}\sin(t), \frac{1}{2}\sin(t)\cos(t)\right\rangle$$

By Green's Theorem,

$$\begin{aligned}
\text{Area}(D) = \iint_D dA = \iint_D (Q_x - P_y)\, dA &= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \frac{1}{2}\int_C -y\, dx + x\, dy \\
&= \frac{1}{2}\int_0^{\pi} \Big(-\sin(t)(\cos^2(t) - \sin^2(t)) + \sin(t)\cos^2(t)\Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(-\sin(t)\cos^2(t) + \sin(t)\sin^2(t) + \sin(t)\cos^2(t)\Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(\sin(t) - \cos^2(t)\sin(t)\Big)\, dt \\
&= \frac{1}{2}\left[-\cos(t) + \frac{1}{3}\cos^3(t)\right]\Big|_0^{\pi} \\
&= \frac{1}{2}\cdot\frac{4}{3} \\
&= \frac{2}{3}\ \text{units}^2
\end{aligned}$$

Therefore, the area of this region is $\boxed{\frac{2}{3}\ \text{units}^2}$

Green's Theorem, as stated so far, applies only to regions that are simply connected, excluding regions with holes. Here, we extend Green's Theorem so that it does work on regions with finitely many holes.

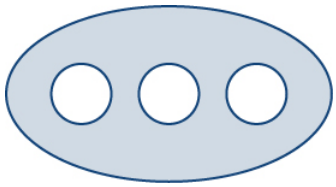


Figure 4.2.6: Green's Theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's Theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D. To extend Green's Theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes.

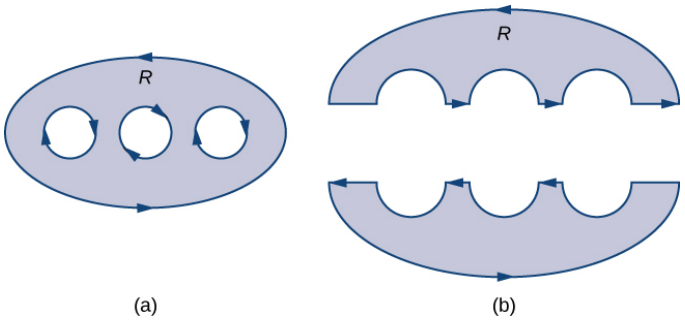


Figure 4.2.7: (a) Region D with an oriented boundary has three holes. (b) Region D split into two simply connected regions has no holes.

Assume the boundary of D is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If $\vec{\mathbf{F}}$ is a vector field defined on D, then Green's Theorem says that

$$\int_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$
$$= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA = \iint_{D} (Q_x - P_y) \, dA$$

Therefore, Green's Theorem still works on a region with holes.

To see how this works in practice, consider annulus D pictured below.

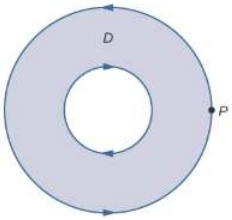


Figure 4.2.8: The annulus D with a hole.

Suppose that $F = \langle P, Q \rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

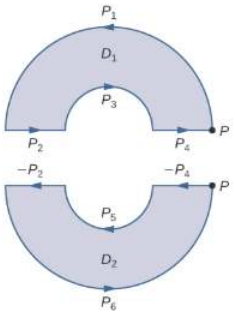


Figure 4.2.9: The annulus D split in half.

We label each piece of these new boundaries as P_i for some i, as above. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region D. If we had only traversed one portion of the boundary of D, then we cannot apply Green's Theorem to D.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's Theorem implies

$$\begin{aligned}
\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&\quad + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&\quad + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA \\
&= \iint_{D} (Q_x - P_y)\, dA
\end{aligned}$$

Therefore, we arrive at the equation found in Green's Theorem:

$$\int_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D (Q_x - P_y)\, dA$$

Example 4.2.3: Using Green's Theorem on a Region with Holes

Let D be the annulus given by the polar inequalities $1 \le r \le 2$, $0 \le \theta \le 2\pi$. Calculate the integral

$$\int_{\partial D} \left(\sin(x) - \frac{1}{3}y^3\right) dx + \left(\frac{1}{3}x^3 + \sin(y)\right) dy$$

Solution

Although D is not simply connected, we can use the extended form of Green's Theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned}
\int_{\partial D} \left(\sin(x) - \frac{1}{3}y^3\right) dx + \left(\frac{1}{3}x^3 + \sin(y)\right) dy &= \iint_D (Q_x - P_y)\, dA \\
&= \iint_D (x^2 + y^2)\, dA \\
&= \int_0^{2\pi} \int_1^2 (r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_1^2 r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{1}{4} r^4 \Big|_1^2 d\theta \\
&= \int_0^{2\pi} \frac{15}{4}\, d\theta \\
&= \frac{15}{4}\theta \Big|_0^{2\pi} \\
&= \boxed{\frac{15\pi}{2}}
\end{aligned}$$

Notice that this does indeed match the direct calculation we would get without using Green's Theorem:

Let C_1 be the inner circle, oriented *clockwise*. We can parameterize it using $\vec{\mathbf{r}}(t) = \langle \sin(t), \cos(t) \rangle$ for $0 \le t \le 2\pi$

$$\begin{aligned}\int_{C_1}\left(\sin x-\frac{1}{3}y^3\right)\,dx+\left(\frac{1}{3}x^3+\sin y\right)\,dy &= \int_0^{2\pi}\Big\langle \sin(\sin(t))-\frac{1}{3}\cos^3(t),\\ &\qquad \frac{1}{3}\sin^3(t)+\sin(\cos(t))\Big\rangle\cdot\langle\cos(t),-\sin(t)\rangle\,dt\\ &= \int_0^{2\pi}\left(-\frac{1}{3}\cos^4(t)-\frac{1}{3}\sin^4(t)+\sin(\sin(t))\cos(t)\right.\\ &\qquad \left.-\sin(\cos(t))\sin(t)\right)\,dt\\ &= \int_0^{2\pi}\left(-\frac{1}{4}-\frac{1}{12}\cos(4t)+\sin(\sin(t))\cos(t)\right.\\ &\qquad \left.-\sin(\cos(t))\sin(t)\right)\,dt\\ &= \left[-\frac{1}{4}t-\frac{1}{48}\sin(4t)-\cos(\sin(t))-\cos(\cos(t))\right]\Big|_0^{2\pi}\\ &= -\frac{\pi}{2}\end{aligned}$$

Let C_2 be the outer circle, oriented *counter-clockwise*. We parameterize it using $\vec{\mathbf{r}}(t)=\langle 2\cos(t),2\sin(t)\rangle$ for $0\le t\le 2\pi$

$$\begin{aligned}\int_{C_2}\left(\sin(x)-\frac{1}{3}y^3\right)\,dx &\\ +\left(\frac{1}{3}x^3+\sin(y)\right)\,dy &= \int_0^{2\pi}\left\langle \sin(2\cos(t))-\frac{8}{3}\sin^3(t),\frac{8}{3}\cos^3(t)+\sin(2\sin(t))\right\rangle\\ &\qquad\cdot\langle -2\sin(t),2\cos(t)\rangle\,dt\\ &= \int_0^{2\pi}\left(\frac{16}{3}\cos^4(t)+\frac{16}{3}\sin^4(t)+2\sin(2\sin(t))\cos(t)\right.\\ &\qquad \left.-2\sin(2\cos(t))\sin(t)\right)\,dt\\ &= \int_0^{2\pi}\left(4+\frac{4}{3}\cos(4t)+2\sin(2\sin(t))\cos(t)\right.\\ &\qquad \left.-2\sin(2\cos(t))\sin(t)\right)\,dt\\ &= \left[4t+\frac{1}{16}\sin(4t)-\cos(2\sin(t))\cos(t)-\cos(2\cos(t))\right]\Big|_0^{2\pi}\\ &= 8\pi\end{aligned}$$

Adding these up gives us the same result that Green's Theorem did. This also demonstrates the important condition that *interior boundaries* are traversed in the clockwise direction while *exterior boundaries* are traversed counterclockwise.

Illustration 4.2.3: Using the Extended Form of Green's Theorem

Let $\vec{\mathbf{F}}=\langle P,Q\rangle=\left\langle\frac{y}{x^2+y^2},-\frac{x}{x^2+y^2}\right\rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$?

Notice that this vector field $\vec{\mathbf{F}}$ is conservative:

$$P_y=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{x^2-y^2}{(x^2+y^2)^2}\quad\text{and } Q_x=\frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right)=\frac{x^2-y^2}{(x^2+y^2)^2}$$

Since $\vec{\mathbf{F}}$ has domain $E=\{(x,y)\mid x^2+y^2\neq 0\}$, we consider two cases: Curves that enclose the origin, and curves that do not. Define D to be the interior of the curve C.

In the first case, if $(0,0)$ is not inside of D, we know that we can immediately use Green's Theorem on a simply connected domain:

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\iint_D (Q_x-P_y)\,dA=\iint_D 0\,dA=0$$

But if $(0,0)$ *is* inside of D, we need to exclude it in order to use Green's Theorem. Define a second curve C_1 going around the origin clockwise.

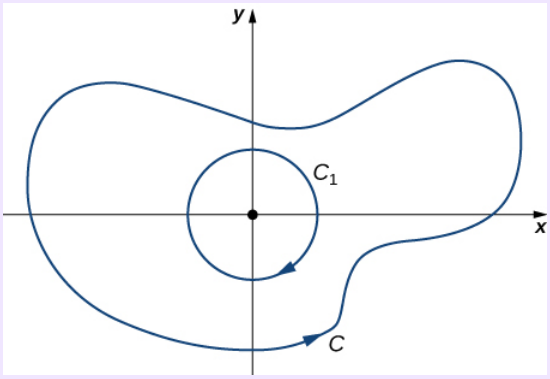


Figure 4.2.10: Choose circle C_1 centered at the origin that is contained entirely inside C.

Now we can use Green's Theorem on the region between these two curves, D':

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D'} (Q_x - P_y)\, dA = \iint_{D'} 0\, dA = 0$$

Clearly we must then have $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

Since C_1 is a specific curve (let's say of radius $a > 0$), we can parameterize it: $\vec{\mathbf{r}}(t) = \langle a\sin(t), a\cos(t)\rangle$ for $0 \le t \le 2\pi$.

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{2\pi} \left\langle \frac{a\cos(t)}{a^2}, -\frac{a\sin(t)}{a^2} \right\rangle \cdot \langle a\cos(t), -a\sin(t)\rangle\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

This shows that whenever C encompasses the origin, we must have $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -2\pi$

Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is either 0 or -2π, no matter how crazy the curve C is.

An important ingredient in using Green's Theorem is a closed curve C. When solving a line integral on a curve is that not closed, sometimes you can *add* another curve to it to make it closed. This can sometimes be useful when the integral along your new piece of curve is easier to evaluate than the original one.

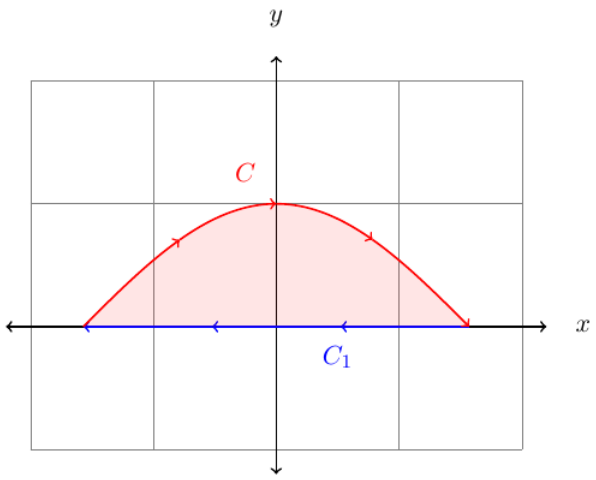
Illustration 4.2.4: Closing a Curve to Use Green's Theorem

Let C be the curve connecting $\left(-\frac{\pi}{2}, 0\right)$ to $\left(\frac{\pi}{2}, 0\right)$ along the path $y = \cos(x)$. Evaluate the line integral

$$\int_C \left(\sin(y)e^x - ye^{\sin(x)}\right)\, dx + \left(\cos(y)e^x\right)\, dy$$

We solved this exact question using the Fundamental Theorem of Calculus for Line Integrals in the previous section. We will show how to approach this question by using Green's Theorem instead. Let $\vec{\mathbf{F}} = \langle P, Q\rangle$ for $P(x,y) = \sin(y)e^x - ye^{\sin(x)}$ and $Q(x,y) = \cos(y)e^x$

Notice first that the curve C is *not closed*, it starts at $\left(-\frac{\pi}{2}, 0\right)$ and ends at $\left(\frac{\pi}{2}, 0\right)$. Green's Theorem can only be used on a closed curve. To help, we will close the curve by adding an additional curve C_1 that connects $\left(\frac{\pi}{2}, 0\right)$ to $\left(-\frac{\pi}{2}, 0\right)$ with a straight line along the x-axis.



Together, C and C_1 form the boundary of the region $D = \left\{(x,y) : -\frac{\pi}{2} \le x \le \frac{\pi}{2},\ 0 \le y \le \cos(x)\right\}$, traversed in the *clockwise* direction. Because the direction is clockwise, we add a minus sign to Green's Theorem and write:

$$\int_C P\,dx + Q\,dy + \int_{C_1} P\,dx + Q\,dy = -\iint_D (Q_x - P_y)\,dA$$

We want to know the left-most integral above, so we will solve for the other two.

We parameterize C_1 using $\vec{\mathbf{r}}(t) = \left\langle \frac{\pi}{2} - t, 0 \right\rangle$ for $0 \le t \le \pi$, therefore:

$$\int_{C_1} P\,dx + Q\,dy = \int_0^{\pi} \left(P\left(\frac{\pi}{2} - t, 0\right) \cdot (-1) + Q\left(\frac{\pi}{2} - t, 0\right) \cdot 0\right)\,dt = \int_0^{\pi} 0\,dt = 0$$

Now we can easily solve the double integral:

$$\begin{aligned}
-\iint_D (Q_x - P_y)\,dA &= -\iint_D \left(\cos(y)e^x - \cos(y)e^x + e^{\sin(x)}\right)\,dA \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos(x)} e^{\sin(x)}\,dy\,dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -ye^{\sin(x)}\Big|_0^{\cos(x)}\,dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\cos(x)e^{\sin(x)}\,dx \\
&= -e^{\sin(x)}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= -e^1 + e^{-1}
\end{aligned}$$

Since the line integral along C_1 was zero, we directly get

$$\int_C P\,dx + Q\,dy = -\iint_D (Q_x - P_y)\,dA = -e^1 + e^{-1} = \frac{1 - e^2}{e}$$

Notice that this matches the answer we got previously!

Example 4.2.4: Closing a Curve to Use Green's Theorem

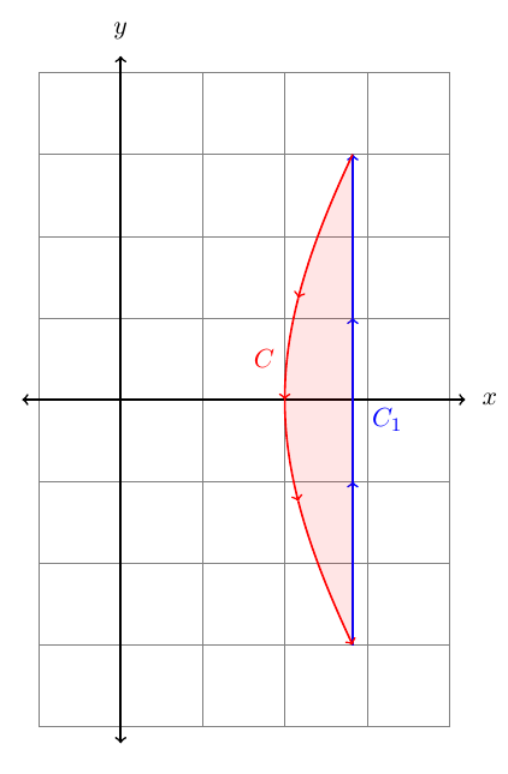
Let C be the curve parameterized by $\vec{\mathbf{r}}(t) = \left\langle 2\sec\left(\frac{t}{4}\right), -3\tan\left(\frac{t}{4}\right)\right\rangle$ for $-\pi \le t \le \pi$ and evaluate:

$$\int_C \left(2xy\sin(x^2 - 8) - y^2\right)\,dx + \left(x^2 - \cos(x^2 - 8)\right)\,dy$$

Solution

Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ for $P(x,y) = 2xy\sin(x^2-8) - y^2$ and $Q(x,y) = x^2 - \cos(x^2-8)$

Notice again that the curve C is *not closed*, it starts at $\left(2\sqrt{2}, 3\right)$ and ends at $\left(2\sqrt{2}, -3\right)$. Green's Theorem can only be used on a closed curve. We will close the curve by adding an additional curve C_1 that connects $\left(2\sqrt{2}, -3\right)$ to $\left(2\sqrt{2}, 3\right)$ with a straight line along the line $x = 2\sqrt{2}$



Together, C and C_1 form the boundary of the region $D = \left\{(x,y) \,:\, 2 \le x \le 2\sqrt{2},\, -3\sqrt{\frac{x^2}{4} - 1} \le y \le 3\sqrt{\frac{x^2}{4} - 1}\right\}$, traversed in the *counter-clockwise* direction. Therefore, using Green's Theorem:

$$\int_C P\,dx + Q\,dy + \int_{C_1} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

We want to know the left-most integral above, so we will solve for the other two.

We parameterize C_1 using $\vec{\mathbf{r}}(t) = \left\langle 2\sqrt{2}, t - 3 \right\rangle$ for $0 \le t \le 6$, therefore:

$$\int_{C_1} P\,dx + Q\,dy = \int_0^6 \left(P\left(2\sqrt{2}, t-3\right) \cdot 0 + Q\left(2\sqrt{2}, t-3\right) \cdot 1\right)\,dt = \int_0^6 7\,dt = 42$$

Now we can easily solve the double integral:

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \iint_D \left((2x + 2x\sin(x^2-8)) - (2x\sin(x^2-8) - 2y)\right)\,dA \\ &= \int_2^{2\sqrt{2}} \int_{-3\sqrt{\frac{x^2}{4}-1}}^{3\sqrt{\frac{x^2}{4}-1}} (2x + 2y)\,dy\,dx \\ &= \int_2^{2\sqrt{2}} \left[2xy + y^2\right] \Big|_{-3\sqrt{\frac{x^2}{4}-1}}^{3\sqrt{\frac{x^2}{4}-1}} dx \\ &= \int_2^{2\sqrt{2}} 12x\sqrt{\frac{x^2}{4} - 1}\,dx \\ &= \int_2^{2\sqrt{2}} 6x\sqrt{x^2 - 4}\,dx \\ &= 2(x^2-4)^{\frac{3}{2}} \Big|_2^{2\sqrt{2}} \\ &= 2(4)^{\frac{3}{2}} - 2(0)^{\frac{3}{2}} \\ &= 16\end{aligned}$$

Putting these together, we directly get

$$\left(\int_C P\,dx + Q\,dy\right) + 42 = 16 \longrightarrow \int_C P\,dx + Q\,dy = \boxed{-26}$$

Green's Theorem (Using Flux Notation)

Let D be an open, simply connected region in $\mathbb{R}^2$ with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise. Although this is a two-dimensional region, we can visualize inside of $\mathbb{R}^3$.

Consider the surface S defined by $\vec{\mathbf{r}}(u,v) = \langle u, v, 0\rangle$ for (u,v) in D. Then it is easy to calculate $\vec{\mathbf{r}}_u = \hat{\mathbf{i}}$ and $\vec{\mathbf{r}}_v = \hat{\mathbf{j}}$

Therefore, the normal vector to this surface is $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$.

Green's Theorem says that the line integral

$$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

Define the vector field $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y), 0\rangle$, and notice that $\text{curl}(\vec{\mathbf{F}}) = \langle 0, 0, Q_x - P_y\rangle$

Thus, one particularly messy way of writing Green's Theorem is:

$$\int_C P\,dx + Q\,dy + 0\,dz = \iint_D \text{curl}(\vec{\mathbf{F}}) \cdot \hat{\mathbf{k}}\,dA = \iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

The very next section will show this formula applies in general for all surfaces, not just those in the xy-plane.

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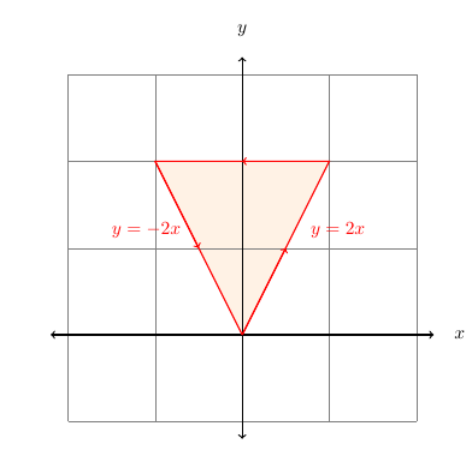
4.2: Practice Set

Exercise 4.2.1a

Use Green's Theorem to evaluate $\int_C e^x\,dy$ for C connecting $(0,0)$ to $(1,2)$ to $(-1,2)$ to $(0,0)$ using straight lines.

Answer

The lines connecting these points enclose a triangle:



This triangle can be described as a Type 2 region: $D = \left\{(x,y) \,:\, 0 \le y \le 2,\, -\frac{y}{2} \le x \le \frac{y}{2}\right\}$

Since the curve travels around the boundary of D in the counterclockwise direction we have:

$$\begin{aligned}
\int_C e^x\,dy &= \iint_D \left(\frac{\partial}{\partial x}\left(e^x\right) - \frac{\partial}{\partial y}\left(0\right)\right) dA \\
&= \iint_D e^x\,dA \\
&= \int_0^2 \int_{-\frac{y}{2}}^{\frac{y}{2}} e^x\,dx\,dy \\
&= \int_0^2 e^x\Big|_{-\frac{y}{2}}^{\frac{y}{2}}\,dy \\
&= \int_0^2 \left(e^{\frac{y}{2}} - e^{-\frac{y}{2}}\right) dy \\
&= \left[2e^{\frac{y}{2}} + 2e^{-\frac{y}{2}}\right]\Big|_0^2 \\
&= 2e + 2e^{-1} - 4
\end{aligned}$$

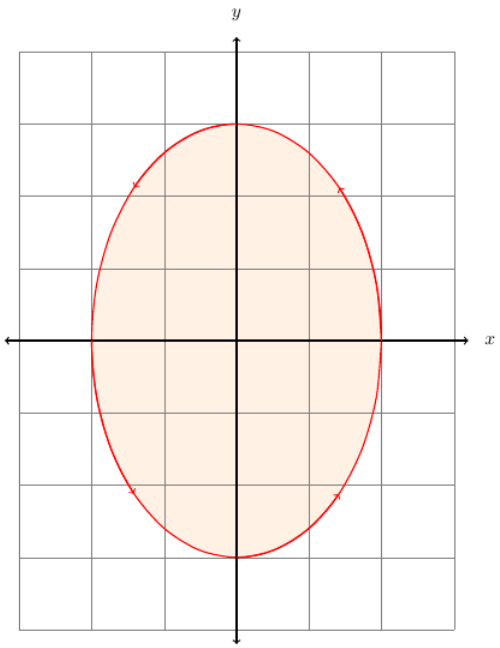
Exercise 4.2.1b

Use Green's Theorem to evaluate $\int_C (4x^2 - y)\,dx + 2x\,dy$ for C parameterized by $\vec{\mathbf{r}}(t) = \langle 2\cos(t), 3\sin(t)\rangle$ with $0 \le t \le 2\pi$

Answer

The region enclosed by this curve is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} \le 1$:

https://math.libretexts.org/@go/page/195733



This ellipse can be described as a Type 1 or 2 region: $D = \left\{(x,y) \,:\, -2 \le x \le 2,\, -3\sqrt{1-\frac{x^2}{4}} \le y \le 3\sqrt{1-\frac{x^2}{4}}\right\}$

Since the curve travels around the boundary of D in the counterclockwise direction we have:

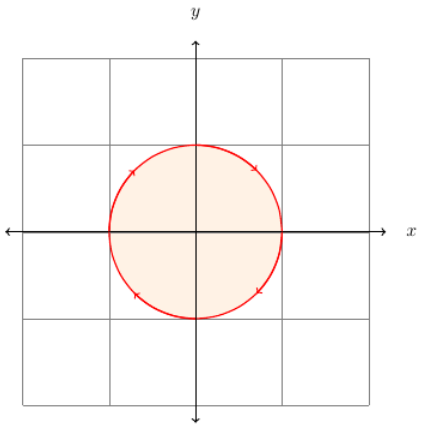
$$\begin{aligned}\int_C (4x^2 - y)\,dx + 2x\,dy &= \iint_D \left(\frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(4x^2 - y)\right) dA \\ &= \iint_D 3\,dA \\ &= 3\iint_D 1\,dA \\ &= 3 \cdot \text{Area}(D) \\ &= 3 \cdot (\pi \cdot 2 \cdot 3) \\ &= 18\pi\end{aligned}$$

? Exercise 4.2.1c

Use Green's Theorem to evaluate $\int_C e^{-x^3}\,dx + \left(3x^5 + 10x^3y^2 + 15xy^4\right)\,dy$ for C parameterized by $\vec{\mathbf{r}}(t) = \langle \sin(t), \cos(t)\rangle$ with $0 \le t \le 2\pi$

Answer

The region enclosed by this curve is the unit circle:



The unit circle can be described in polar form: $D = \{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le 1,\, 0 \le \theta \le 2\pi\}$

Since the curve travels around the boundary of D in the *clockwise* direction we have:

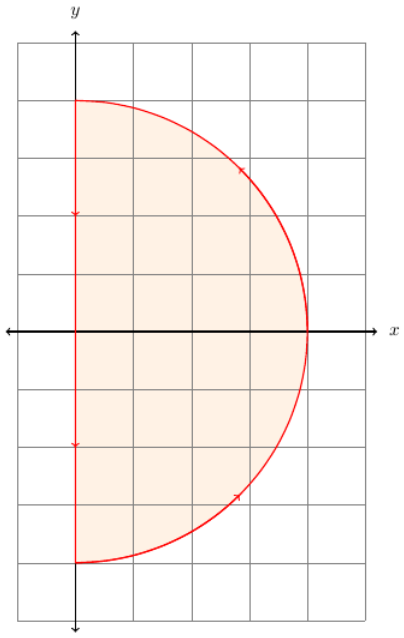
$$\begin{aligned}
\int_C e^{-x^3}\,dx + \left(3x^5 + 10x^3y^2 + 15xy^4\right)\,dy &= -\iint_D \left(\frac{\partial}{\partial x}\left(3x^5 + 10x^3y^2 + 15xy^4\right) - \frac{\partial}{\partial y}\left(e^{-x^3}\right)\right)\,dA \\
&= \iint_D -\left(15x^4 + 30x^2y^2 + 15y^4\right)\,dA \\
&= \iint_D -15(x^2+y^2)^2\,dA \\
&= \int_0^1 \int_0^{2\pi} -15(r^2)^2 \cdot r\,d\theta\,dr \\
&= \left(\int_0^1 -15r^5\,dr\right)\left(\int_0^{2\pi} d\theta\right) \\
&= \left(-\frac{5}{2}r^6\Big|_0^1\right) \cdot \left(\theta\Big|_0^{2\pi}\right) \\
&= -\frac{5}{2} \cdot 2\pi \\
&= -5\pi
\end{aligned}$$

? Exercise 4.2.1d

Use Green's Theorem to evaluate $\int_C 3xy\,dx$ for C connecting $(0,4)$ to $(0,-4)$ with a straight line, then back to $(0,4)$ counter-clockwise along the curve $x^2 + y^2 = 16$

Answer

The region enclosed by this curve is the right half of the circle $x^2 + y^2 \le 16$:



This semicircle can be described in polar form: $D = \left\{(r\cos(\theta), r\sin(\theta)) \,:\, 0 \le r \le 4,\, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$

Since the curve travels around the boundary of D in the counterclockwise direction we have:

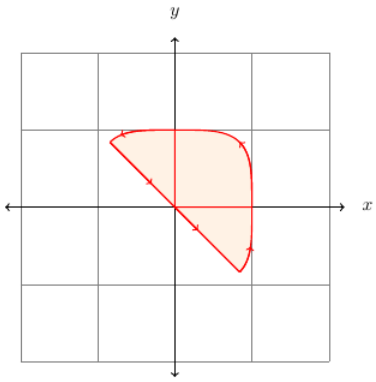
$$\begin{aligned}
\int_C 3xy\,dx &= \iint_D \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(3xy)\right) dA \\
&= \iint_D -3x\,dA \\
&= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -3r\cos(\theta)\cdot r\,d\theta\,dr \\
&= \left(\int_0^4 -3r^2\,dr\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\,d\theta\right) \\
&= \left(-r^3\Big|_0^4\right)\cdot\left(\sin(\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) \\
&= -64\cdot 2 \\
&= -128
\end{aligned}$$

Exercise 4.2.1e

Use Green's Theorem to evaluate $\int_C 5y^2\,dx + 10xy\,dy$ for C counterclockwise path along the boundary of $D = \{(x,y) \,:\, x^4 + y^4 \le 2,\, y \ge -x\}$

Answer

The region D described is half of a squircle (sometimes called a superellipse):



This is not a Type 1 or Type 2 region, so we would have to define it using substitution.
Before deciding exactly how, we look at the double integral we will attempt:

$$\begin{aligned}\int_C 5y^2\,dx + 10xy\,dy &= \iint_D \left(\frac{\partial}{\partial x}(10xy) - \frac{\partial}{\partial y}(5y^2)\right) dA \\ &= \iint_D 0\,dA \\ &= 0\end{aligned}$$

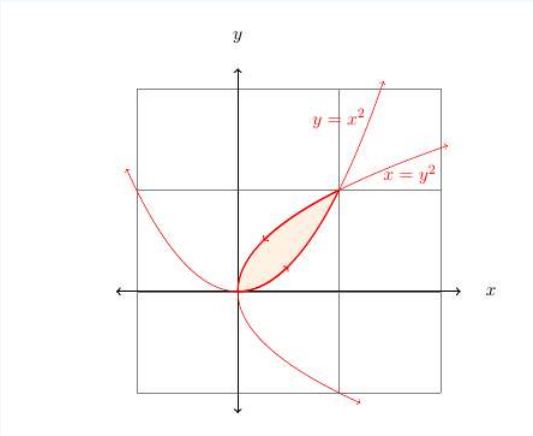
Note that the substitution we would have used would have required treating each quadrant separately:
In the first quadrant $x = u\sqrt{\cos(v)},\ y = u\sqrt{\sin(v)}$ for $0 \le u \le 1,\ 0 \le v \le \frac{\pi}{2}$
In the second quadrant $x = u\sqrt{-\cos(v)},\ y = u\sqrt{\sin(v)}$ for $0 \le u \le 1,\ \frac{\pi}{2} \le v \le \frac{3\pi}{4}$
In the fourth quadrant $x = u\sqrt{\cos(v)},\ y = u\sqrt{-\sin(v)}$ for $0 \le u \le 1,\ \frac{7\pi}{4} \le v \le 2\pi$
Thankfully, the zero integrand saved us from this large amount of work!

? Exercise 4.2.1f

Use Green's Theorem to evaluate $\int_C e^{\sqrt{x}}\,dx + x\,dy$ for C the counterclockwise path along the boundary the region between $y = x^2$ and $x = y^2$

Answer

The region D between these two parabolas is contained entirely in the first quadrant:



D can be described as either Type 1 or Type 2: $D = \{(x, y) : 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$

Since the curve travels around the boundary of D in the counterclockwise direction we have:

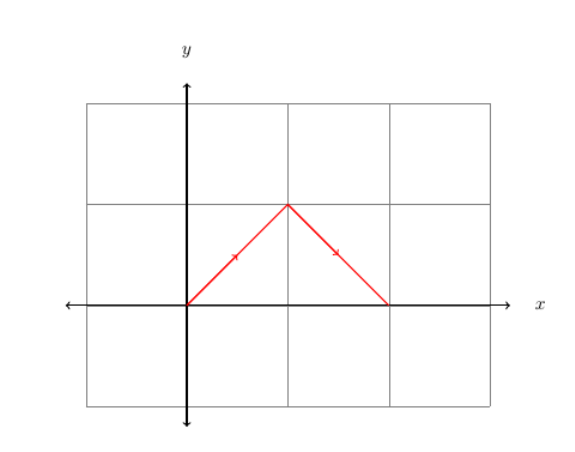
$$\begin{aligned}\int_C e^{\sqrt{x}}\,dx + x\,dy &= \iint_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}\left(e^{\sqrt{x}}\right)\right) dA \\ &= \iint_D 1\,dA \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} 1\,dy\,dx \\ &= \int_0^1 y\Big|_{x^2}^{\sqrt{x}}\,dx \\ &= \int_0^1 \left(\sqrt{x} - x^2\right)\,dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]\Big|_0^1 \\ &= \frac{1}{3}\end{aligned}$$

Exercise 4.2.1g

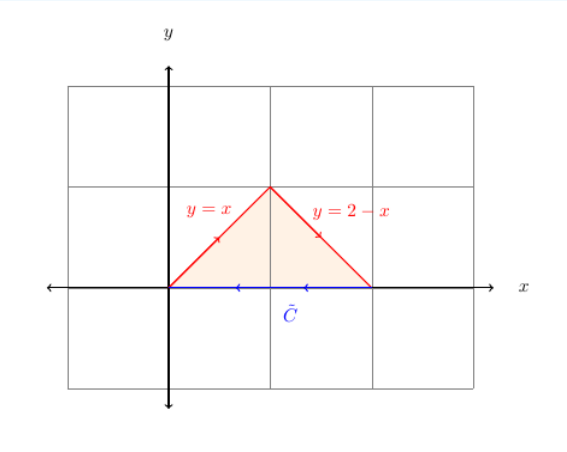
Use Green's Theorem to evaluate $\int_C y^2\,dx + 5xy\,dy$ where C connects $(0,0)$ to $(1,1)$ to $(2,0)$

Answer

The given curve C is not closed:



We add on an additional curve $\tilde{C}$ along the bottom to enclose a triangle $D = \{(x,y) : 0 \le y \le 1,\ y \le x \le 2 - y\}$:



When these curves are combined, they travel *clockwise* around the boundary of D, so we can use Green's Theorem to say

$$\begin{aligned}\int_{C\cup\tilde{C}} y^2\,dx+5xy\,dy &= -\iint_D \left(\frac{\partial}{\partial x}\left(5xy\right)-\frac{\partial}{\partial y}\left(y^2\right)\right)\,dA\\ &= \iint_D -3y\,dA\\ &= \int_0^1\int_y^{2-y} -3y\,dx\,dy\\ &= \int_0^1 -3xy\Big|_y^{2-y}\,dy\\ &= \int_0^1 \left(-6y+6y^2\right)\,dy\\ &= \left[-3y^2+2y^3\right]\Big|_0^1\\ &= -1\end{aligned}$$

Next, we manually calculate the line integral along $\tilde{C}$, which can be parameterized by $\vec{\mathbf{r}}(t)=\langle 2-t,0\rangle$ for $0\le t\le 2$:

$$\int_{\tilde{C}} y^2\,dx+5xy\,dy=\int_0^2 \left(0^2\cdot(-1)+(5(2-t)\cdot 0)\cdot 0\right)\,dt=\int_0^2 0\,dt=0$$

We know that $\int_C y^2\,dx+5xy\,dy+\int_{\tilde{C}} y^2\,dx+5xy\,dy=\int_{C\cup\tilde{C}} y^2\,dx+5xy\,dy$

Therefore, $\left(\int_C y^2\,dx+5xy\,dy\right)+0=\int_{C\cup\tilde{C}} y^2\,dx+5xy\,dy=-1$

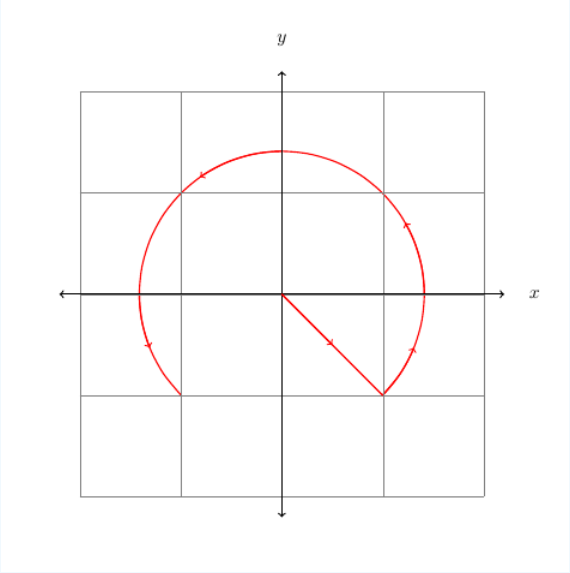
So our final answer is $\int_C y^2\,dx+5xy\,dy=-1$

? Exercise 4.2.1h

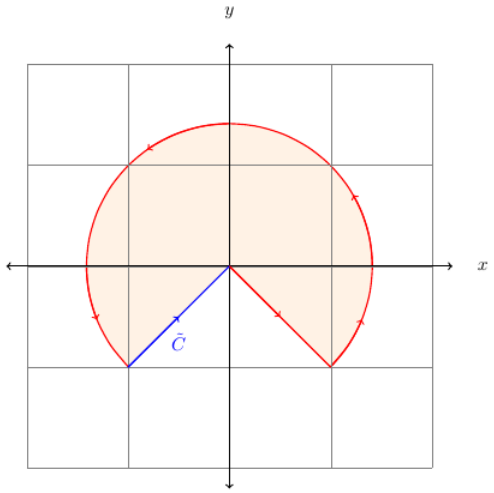
Use Green's Theorem to evaluate $\int_C 12x^3\,dx+12x^3\,dy$ where C connects $(0,0)$ to $(1,-1)$ in a straight line, then counterclockwise along $x^2+y^2=2$ to $(-1,-1)$

Answer

The given curve C is not closed:



We add on the line $\tilde{C}$ to the origin to enclose $D=\left\{(r\cos(\theta),r\sin(\theta))\,:\,0\le r\le\sqrt{2},\,-\frac{\pi}{4}\le\theta\le\frac{5\pi}{4}\right\}$:



When these curves are combined, they travel counterclockwise around the boundary of D, so we can use Green's Theorem to say

$$\begin{aligned}\int_{C\cup\tilde{C}} 12x^3\,dx + 12x^3\,dy &= \iint_D \left(\frac{\partial}{\partial x}\left(12x^3\right) - \frac{\partial}{\partial y}\left(12x^3\right)\right) dA \\ &= \iint_D 36x^2\,dA \\ &= \int_0^{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{5\pi}{4}} 36(r\cos(\theta))^2 \cdot r\,d\theta\,dr \\ &= \left(\int_0^{\sqrt{2}} 36r^3\,dr\right)\left(\int_{-\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos^2(\theta)\,d\theta\right) \\ &= \left(9r^4\Big|_0^{\sqrt{2}}\right)\cdot\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]\Big|_{-\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= 36\cdot\left(\frac{3\pi}{4}+\frac{1}{2}\right) \\ &= 27\pi + 18\end{aligned}$$

Next, we manually calculate the line integral along $\tilde{C}$, which can be parameterized by $\vec{\mathbf{r}}(t) = \langle t-1, t-1\rangle$ for $0 \le t \le 1$:

$$\int_{\tilde{C}} 12x^3\,dx + 12x^3\,dy = \int_0^1 \left(12(t-1)^3\cdot 1 + 12(t-1)^3\cdot 1\right)\,dt = \int_0^1 24(t-1)^3\,dt = 6(t-1)^4\Big|_0^1 = -6$$

We know that $\int_C 12x^3\,dx + 12x^3\,dy + \int_{\tilde{C}} 12x^3\,dx + 12x^3\,dy = \int_{C\cup\tilde{C}} 12x^3\,dx + 12x^3\,dy$

Therefore, $\left(\int_C 12x^3\,dx + 12x^3\,dy\right) + (-6) = 27\pi + 18$

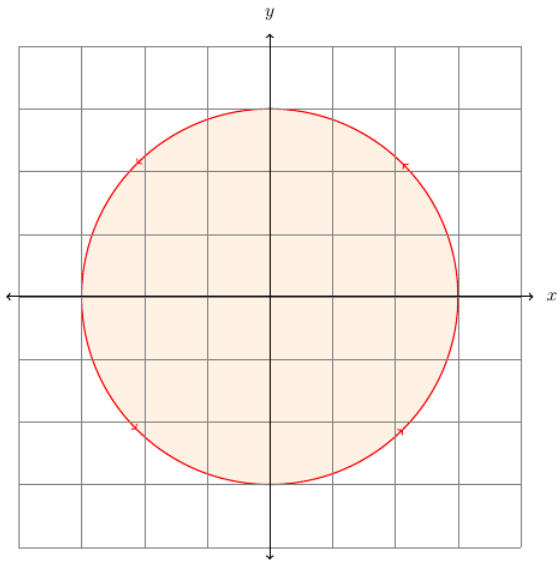
So our final answer is $\int_C 12x^3\,dx + 12x^3\,dy = 27\pi + 24$

? Exercise 4.2.2a

Use Green's Theorem to find the area of the regions enclosed by $\vec{\mathbf{r}}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le 2\pi$

Answer

The region D enclosed by this curve is a circle of radius 3:



We can find its area by evaluating $\iint_D 1\,dA = \int_C P\,dx + Q\,dy$ for any $P(x,y)$ and $Q(x,y)$ which satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

We choose $P(x,y) = 0, Q(x,y) = x$ and get:

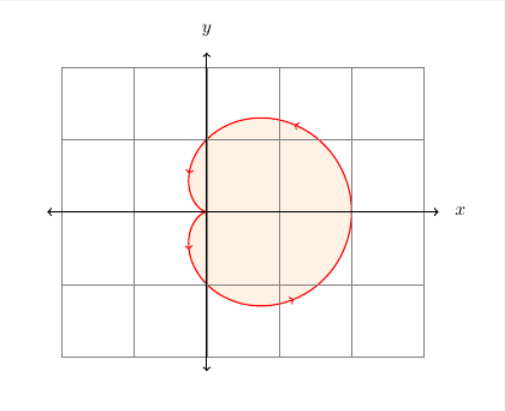
$$\begin{aligned}
\text{Area}(D) &= \int_C 0\,dx + x\,dy \\
&= \int_0^{2\pi} (3\cos(t)) \cdot (3\cos(t))\,dt \\
&= \int_0^{2\pi} 9\cos^2(t)\,dt \\
&= \left[\frac{9}{2}t + \frac{9}{4}\sin(2t)\right]\Big|_0^{2\pi} \\
&= 9\pi
\end{aligned}$$

Exercise 4.2.2b

Use Green's Theorem to find the area of the regions enclosed by $\vec{\mathbf{r}}(t) = \langle \cos(t) + \cos^2(t), \sin(t) + \cos(t)\sin(t) \rangle$ for $0 \le t \le 2\pi$

Answer

The region D enclosed by this curve is a cardioid:



We can find its area by evaluating $\iint_D 1\,dA = \int_C P\,dx + Q\,dy$ for any $P(x,y)$ and $Q(x,y)$ which satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

We choose $P(x,y) = -y, Q(x,y) = 0$ and get:

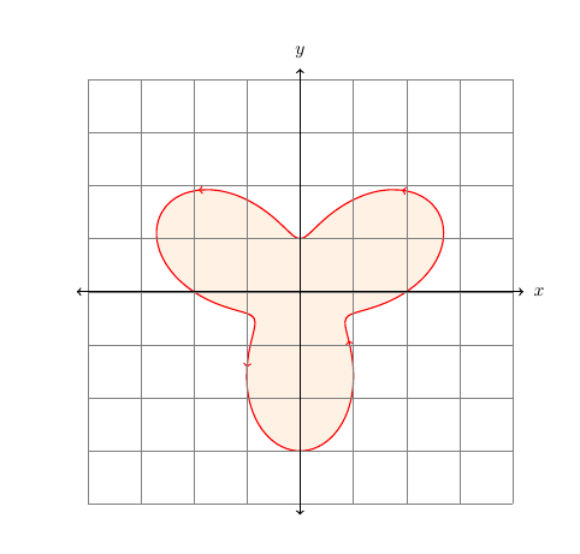
$$\begin{aligned}
\text{Area}(D) &= \int_C -y\,dx + 0\,dy \\
&= \int_0^{2\pi} -(\sin(t) + \cos(t)\sin(t)) \cdot (-\sin(t) - 2\cos(t)\sin(t))\ dt \\
&= \int_0^{2\pi} \left(\sin^2(t) + 2\cos^2(t)\sin^2(t) + 3\sin^2(t)\cos(t)\right)\ dt \\
&= \int_0^{2\pi} \left(\frac{3}{4} - \frac{1}{2}\cos(2t) - \frac{1}{4}\cos(4t) + 3\sin^2(t)\cos(t)\right)\ dt \\
&= \left[\frac{3}{4}t - \frac{1}{4}\sin(2t) - \frac{1}{16}\sin(4t) + \sin^3(t)\right]\Bigg|_0^{2\pi} \\
&= \frac{3\pi}{2}
\end{aligned}$$

Exercise 4.2.2c

Use Green's Theorem to find the area of the regions enclosed by $\vec{\mathbf{r}}(t) = \langle \cos(t)(2+\sin(3t)), \sin(t)(2+\sin(3t)) \rangle$ for $0 \le t \le 2\pi$

Answer

The region D enclosed by this curve is strange:



We can find its area by evaluating $\iint_D 1\,dA = \int_C P\,dx + Q\,dy$ for any $P(x,y)$ and $Q(x,y)$ which satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

We choose $P(x,y) = 0, Q(x,y) = x$ and get:

$$
\begin{aligned}
\text{Area}(D) &= \int_C 0\,dx + x\,dy \\
&= \int_0^{2\pi} \cos(t)\,(2+\sin(3t)) \cdot (\cos(t)(2+\sin(3t)) + \sin(t)(3\cos(3t)))\;dt \\
&= \int_0^{2\pi} \left(\frac{9}{4} + \frac{9}{4}\cos(2t) + \frac{1}{4}\cos(4t) - \frac{1}{4}\cos(6t) - \frac{1}{2}\cos(8t) - \frac{1}{2}\sin(t) + 2\sin(3t) + \frac{5}{2}\sin(5t)\right)\;dt \\
&= \left[\frac{9}{4}t + \frac{9}{8}\sin(2t) + \frac{1}{16}\sin(4t) - \frac{1}{24}\sin(6t) - \frac{1}{16}\sin(8t) + \frac{1}{2}\cos(t) - \frac{2}{3}\cos(3t) - \frac{1}{2}\cos(5t)\right]\Bigg|_0^{2\pi} \\
&= \frac{9\pi}{2}
\end{aligned}
$$

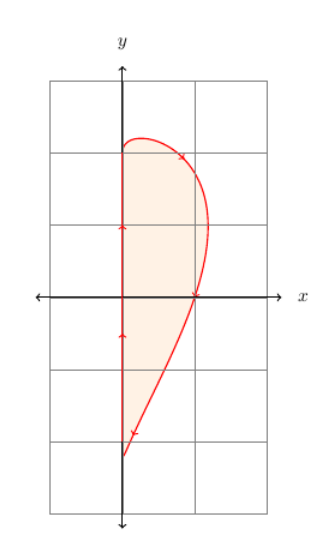
(Note that this approach is very heavy in trigonometric identities, and the area would be significantly easier to find as a polar double integral!)

Exercise 4.2.2*d*

Use Green's Theorem to find the area of the regions enclosed by $\vec{\mathbf{r}}(t) = \langle (\cos(t)+1)\sin^2(t), (\sin(t)+2)\cos(t)\rangle$ for $0 \le t \le \pi$, plus the line returning to the point $(0,2)$

Answer

The region D enclosed by this strange curve is pictured below:



We can find its area by evaluating $\iint_D 1\,dA = \int_C P\,dx + Q\,dy$ for any $P(x,y)$ and $Q(x,y)$ which satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1$ (since C is clockwise!)

We choose $P(x,y) = y, Q(x,y) = 0$ so that the line integral along the straight line will be zero.

Thus, the area matches the line integral along the curved part of C, and we get:

$$
\begin{aligned}
\text{Area}(D) &= \int_C y\,dx + 0\,dy \\
&= \int_0^{\pi} (\sin(t)+2)\cos(t)\cdot\left(-\sin^3(t)+2(\cos(t)+1)\sin(t)\cos(t)\right)\,dt \\
&= \int_0^{\pi} \left(4\cos^2(t)\sin(t)+4\cos^3(t)\sin(t)-2\cos(t)\sin^3(t)+2\cos^2(t)\sin^2(t)+2\cos^3(t)\sin^2(t)-\sin^4(t)\cos(t)\right)\,dt \\
&= \left[-\frac{4}{3}\cos^3(t)-\cos^4(t)-\frac{1}{2}\sin^4(t)+\frac{1}{4}t-\frac{1}{16}\sin(4t)+\frac{2}{3}\sin^3(t)-\frac{3}{5}\sin^5(t)\right]\Bigg|_0^{\pi} \\
&= \frac{8}{3}+\frac{\pi}{4}
\end{aligned}
$$

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4.3: Stokes' Theorem

Learning Objectives

- Explain the meaning of Stokes' theorem.
- Use Stokes' theorem to evaluate a line integral.
- Use Stokes' theorem to calculate a surface integral.

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a *vector* surface integral over a surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S.

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4.3: Stokes' Theorem

Stokes' Theorem says we can calculate the flux of $\text{curl}(\vec{\mathbf{F}})$ across surface S by knowing information only about the values of $\vec{\mathbf{F}}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\vec{\mathbf{F}}$ along the boundary of surface S by translating to a double integral of the curl of $\vec{\mathbf{F}}$ over S.

Let S be a smooth, oriented surface with unit normal vector $\vec{\mathbf{N}}$. Suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, when traversing C with your head pointing in the direction of $\vec{\mathbf{N}}$, the surface is always on your left. With this definition in place, we can state *Stokes' Theorem*.

Theorem 4.3.1: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation. If $\vec{\mathbf{F}}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

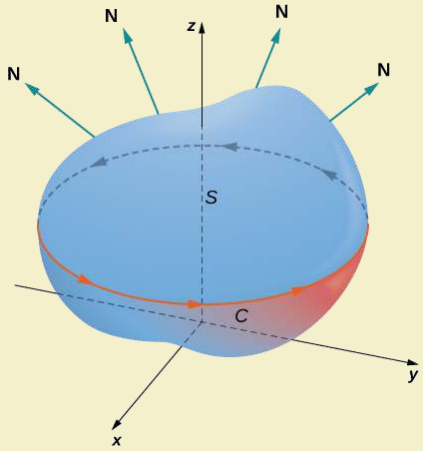


Figure 4.3.1: Stokes' Theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

We saw in the previous section that Green's Theorem is a special case of Stokes' Theorem. Green's Theorem can only handle surfaces in a plane, but Stokes' Theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' Theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\vec{\mathbf{F}}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_E \text{curl}(\vec{\mathbf{F}}) \cdot \vec{\mathbf{N}}\, dS = \iint_E \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface.

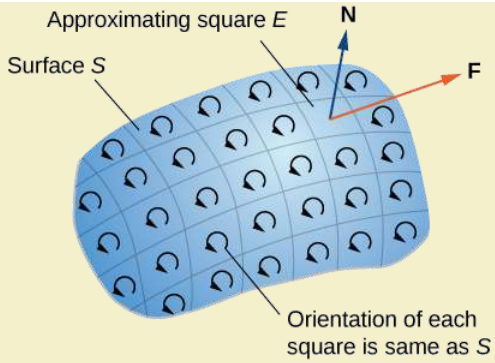


Figure 4.3.2: Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

As we add up all the fluxes over all the squares approximating surface S, line integrals

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \quad \text{and} \quad \int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E. After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

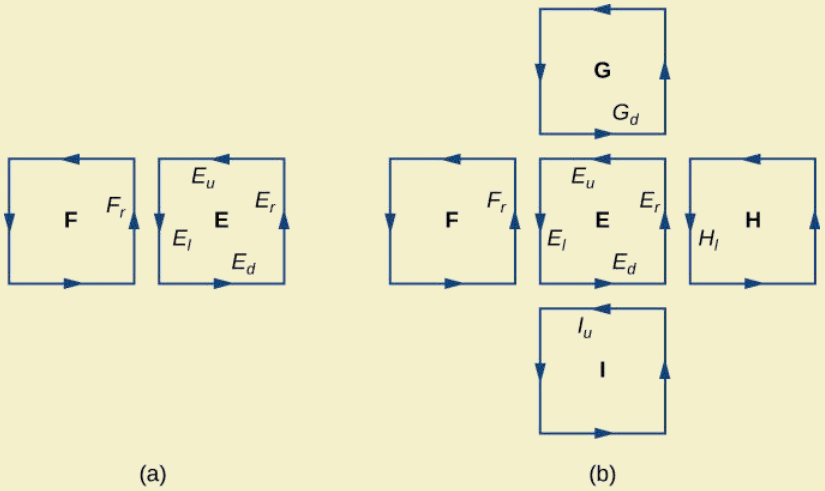


Figure 4.3.3: (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area. Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C as shown below. Suppose that S is oriented upward. The counterclockwise orientation of C is positive,

as is the counterclockwise orientation of C'. Let $\vec{\mathbf{F}}(x,y,z) = \langle P, Q, R\rangle$ be a vector field with component functions that have continuous partial derivatives.

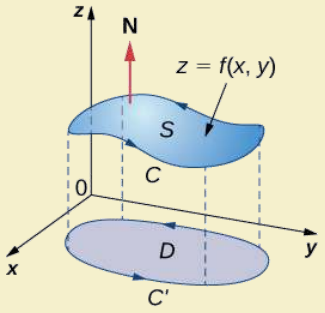


Figure 4.3.4: D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of S: $\vec{\mathbf{r}}(u,v) = \langle u, v, f(u,v)\rangle$.
The tangent vectors are $\vec{\mathbf{r}}_u = \langle 1, 0, f_x(u,v)\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, 1, f_y(u,v)\rangle$, and therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -f_x(u,v), -f_y(u,v), 1\rangle$.

$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA$$

where the partial derivatives are all evaluated at $(u, v, f(u,v))$, making the integrand depend on x and y only. Suppose $\langle x(t),\ y(t)\rangle,\ a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t),\ y(t),\ f(x(t),\ y(t))\rangle,\ a \le t \le b$. Armed with these parameterizations, the Chain rule, and Green's Theorem, and keeping in mind that P, Q and R are all functions of x and y, we can evaluate line integral

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b (Px'(t) + Qy'(t) + Rz'(t))\, dt \\
&= \int_a^b \left[Px'(t) + Qy'(t) + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)x'(t) + \left(Q + R\frac{\partial z}{\partial y}\right)y'(t)\right] dt \\
&= \int_{C'} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA \\
&= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y}\right) \\
&\quad - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right)
\end{aligned}$$

By Clairaut's Theorem, $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$.

Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

which equals

$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

We have shown that Stokes' Theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\vec{\mathbf{F}}$ is a conservative field. If $\vec{\mathbf{F}} = \nabla f$ is conservative, the curl of $\vec{\mathbf{F}}$ is zero, so

$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = 0$$

Since the boundary of S is a closed curve, we know any parameterization of it $\vec{\mathbf{r}}(t)$ for $a \le t \le b$ will have $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Therefore, the integral

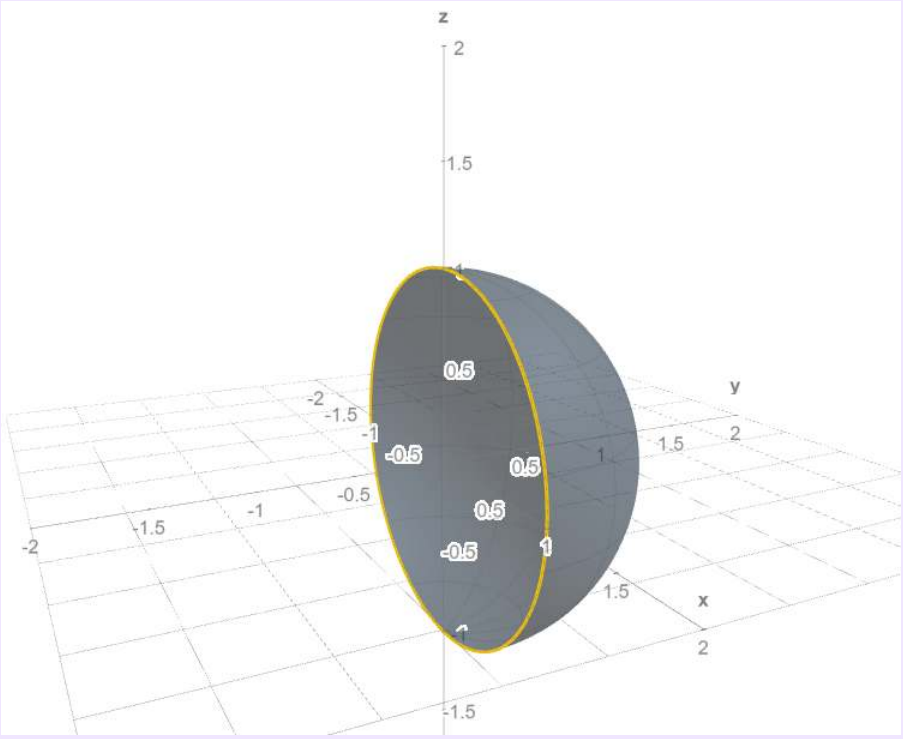
$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f\left(\vec{\mathbf{r}}(b)\right) - f\left(\vec{\mathbf{r}}(a)\right) = 0$$

Once again, these match, and Stokes' Theorem holds.

Illustration 4.3.1: Verifying Stokes' Theorem for a Specific Case

We will verify that Stokes' Theorem is true for vector field $\vec{\mathbf{F}}(x, y) = \langle -z, x, 0\rangle$ and surface S, where S is the hemisphere, oriented outward, with parameterization $\vec{\mathbf{r}}(u, v) = \langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\rangle$ over $D = \{(u, v) : 0 \le u \le \pi,\ 0 \le v \le \pi\}$

Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane $y = 0$.



We want it to look counterclockwise when viewed from the positive y-axis (where the surface orientation is pointing), so we use the parameterization $\vec{\mathbf{r}}(t) = \langle \sin(t), 0, \cos(t)\rangle$ for $0 \le t \le 2\pi$. Thus the line integral on C is:

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \langle -\cos(t), \sin(t), 0\rangle \cdot \langle \cos(t), 0, -\sin(t)\rangle \, dt \\
&= \int_0^{2\pi} -\cos^2(t)\, dt \\
&= \int_0^{2\pi} -\frac{1}{2} - \frac{1}{2}\cos(2t)\, dt \\
&= \left[-\frac{1}{2}t - \frac{1}{4}\sin(2t)\right]\Big|_0^{2\pi} \\
&= -\pi
\end{aligned}$$

We know that $\vec{\mathbf{r}}_u = \langle \cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)\rangle$ and $\vec{\mathbf{r}}_v = \langle -\sin(u)\sin(v), \sin(u)\cos(v), 0\rangle$.

Therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u)\rangle$
This points *outward*, so we have positive orientation in our surface integral.

We easily calculate that $\text{curl}(\vec{\mathbf{F}}) = \langle 0, -1, 1\rangle$. The flux of this over S is:

$$\begin{aligned}
\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \iint_D \langle 0,-1,1\rangle \cdot \langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u)\rangle \, dA \\
&= \int_0^{\pi}\int_0^{\pi} (\sin(u)\cos(u) - \sin^2(u)\sin(v))\, du\, dv \\
&= \int_0^{\pi} \left[v\sin(u)\cos(u) + \sin^2(u)\cos(v)\right]\Big|_0^{\pi} du \\
&= \int_0^{\pi} (\pi\sin(u)\cos(u) - 2\sin^2(u))\, du \\
&= \int_0^{\pi} (\pi\sin(u)\cos(u) - 1 + \cos(2u))\, du \\
&= \left[\frac{\pi}{2}\sin^2(u) - u + \frac{1}{2}\sin(2u)\right]\Big|_0^{\pi} \\
&= -\pi
\end{aligned}$$

We have verified Stokes' Theorem for this example.

Stokes' Theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example 4.3.1: Calculating a Surface Integral

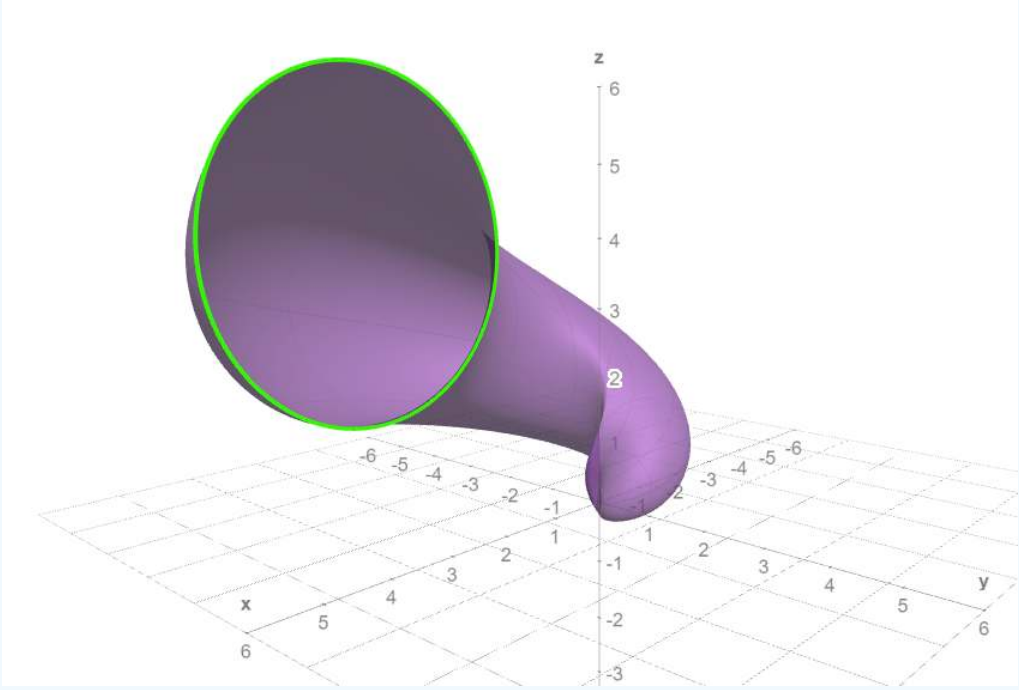
Let S be the surface defined by $\vec{\mathbf{r}}(u,v) = \langle (v^2 + v\cos(u))\cos(\pi v), (v^2 + v\cos(u))\sin(\pi v), v^2 + v\sin(u)\rangle$ for $0 \le u \le 2\pi$ and $0 \le v \le 2$, oriented outward.

If $\vec{\mathbf{F}} = \langle x - z, e^{xz}, y^3 + x\rangle$, calculate surface integral

$$\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

Solution

Note that to calculate $\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ without using Stokes' Theorem, we would need to find $\vec{\mathbf{r}}_u$, $\vec{\mathbf{r}}_v$ and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$, as well as the curl of $\vec{\mathbf{F}}$.



Instead of doing all of that work, we use Stokes' Theorem, noting that the boundary C of the surface is merely a single circle at $v = 2$. Replacing u with t gives us its parameterization:

$$\vec{\mathbf{r}}(t) = \langle 4 + 2\cos(t), 0, 4 + 2\sin(t)\rangle \quad \text{for} \quad 0 \le t \le 2\pi$$

Note that as t increases, we walk *clockwise* around this circle when viewed from the positive y-axis, but that ensures that the surface is on our left side if our head points out the *outward* direction of the surface. This means our orientations match.

Thus, we know that the desired surface integral is equal to:

$$\begin{aligned}
\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_0^{2\pi} \langle (4 + 2\cos(t)) - (4 + 2\sin(t)), e^{(4+2\cos(t))(4+2\sin(t))}, 0^3 + (4 + 2\cos(t))\rangle \\
&\quad \cdot \langle -2\sin(t), 0, 2\cos(t)\rangle \, dt \\
&= \int_0^{2\pi} \left(-4\cos(t)\sin(t) + 4\sin^2(t) + 8\cos(t) + 4\cos^2(t)\right) \, dt \\
&= \int_0^{2\pi} \left(-4\cos(t)\sin(t) + 8\cos(t) + 4\right) \, dt \\
&= \left[-2\sin^2(t) + 8\sin(t) + 4t\right] \Big|_0^{2\pi} \\
&= \boxed{8\pi}
\end{aligned}$$

This is significantly faster than calculating the surface integral directly!

An amazing consequence of Stokes' Theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then for any conservative vector field $\vec{\mathbf{F}}$,

$$\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

because Stokes' Theorem says the surface integral depends on the line integral around the boundary only.

In the example above, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' Theorem,

$$\iint_{S_1} \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S_2} \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$

Therefore, if $\iint_{S_1} \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ is difficult to calculate but $\iint_{S_2} \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ is easy to calculate, Stokes' Theorem allows us to calculate the easier surface integral.

This relationship shows that flux integrals of curl vector fields are surface independent in the same way that line integrals of gradient fields are path independent. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a *potential function* for $\vec{\mathbf{F}}$, and C is a curve in the domain of $\vec{\mathbf{F}}$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

depends only on the endpoints of C. Therefore, if C' is any other curve with the same starting point and endpoint as C (and thus same orientation as C), then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

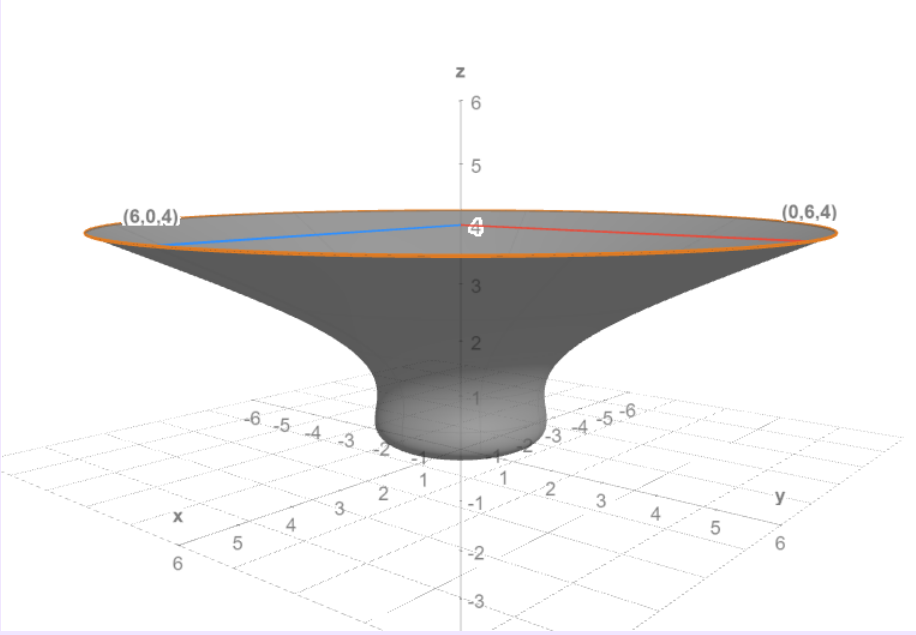
Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\vec{\mathbf{G}}$ is a three-dimensional vector field that can be written as the curl of another vector field $\vec{\mathbf{F}}$ (so that $\vec{\mathbf{F}}$ is like a "potential field" of $\vec{\mathbf{G}}$). We can see that

$$\iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} = \iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S'} \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \iint_{S'} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}$$

Therefore, the flux integral of $\vec{\mathbf{G}}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

Illustration 4.3.2: Using an Equivalent Surface

We can use Stokes' Theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{G}}(x, y, z) = \langle x^2 z^2 \cos(y), -2xz^2 \sin(y), 1 \rangle$ and S is the surface as shown in the following figure.

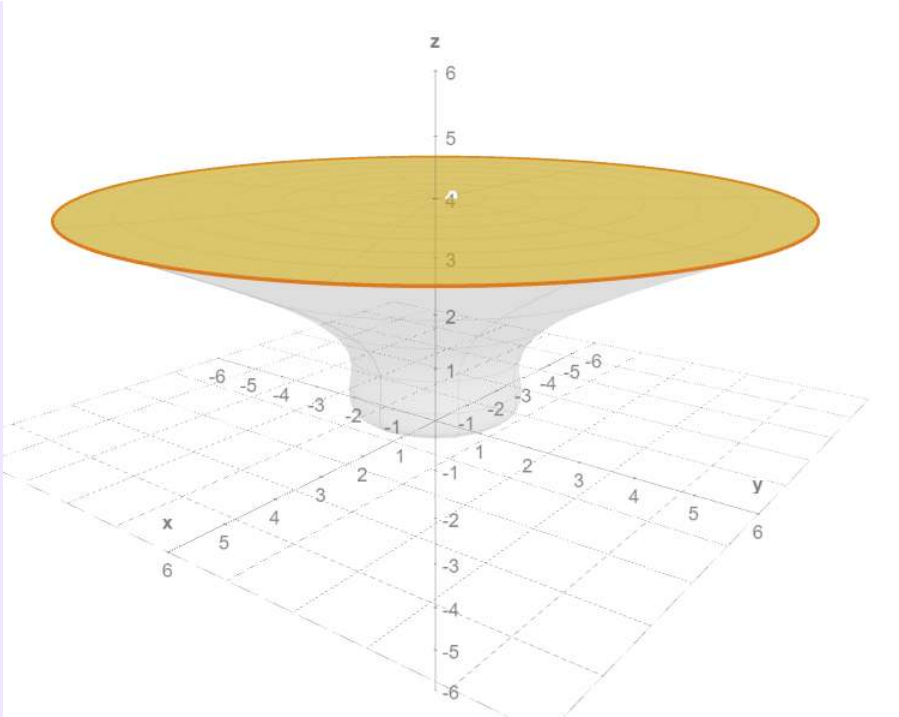


In the picture provided, we can see that the boundary of the surface is simply a circle of radius 6, centered at $(0, 0, 4)$ in the plane $z = 4$

Notice that $\text{div}(\vec{\mathbf{G}}) = 2xz^2 \cos(y) - 2xz^2 \cos(y) = 0$, so $\vec{\mathbf{G}} = \text{curl}\left(\vec{\mathbf{F}}\right)$ for some vector field $\vec{\mathbf{F}}$.

This means that we can evaluate the flux of $\vec{\mathbf{G}}$ over S by using any surface with the same boundary curve.

We choose the flat disc $S' = \{(x, y, 4) : x^2 + y^2 \leq 36\}$



This can be parameterized by $\vec{\mathbf{r}}(u,v) = \langle u, v, 4\rangle$ over $D = \{(u,v) \,:\, u^2 + v^2 \le 36\}$. We can easily calculate

$$\vec{\mathbf{r}}_u = \hat{\mathbf{i}}, \quad \vec{\mathbf{r}}_v = \hat{\mathbf{j}}, \quad \text{and} \quad \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \hat{\mathbf{k}}$$

Therefore, we know

$$\begin{aligned}
\iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} &= \iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\
&= \iint_{S'} \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\
&= \iint_{S'} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} \\
&= \iint_D \langle u^2(4)^2\cos(v), -2u(4)^2\sin(v), 1\rangle \cdot \hat{\mathbf{k}}\, dA \\
&= \iint_D 1\, dA \\
&= \operatorname{Area}(D) = 36\pi
\end{aligned}$$

Stokes' Theorem works in both directions. We have seen it used to turn surface integrals into line integrals, but it can also turn line integrals into surface integrals.

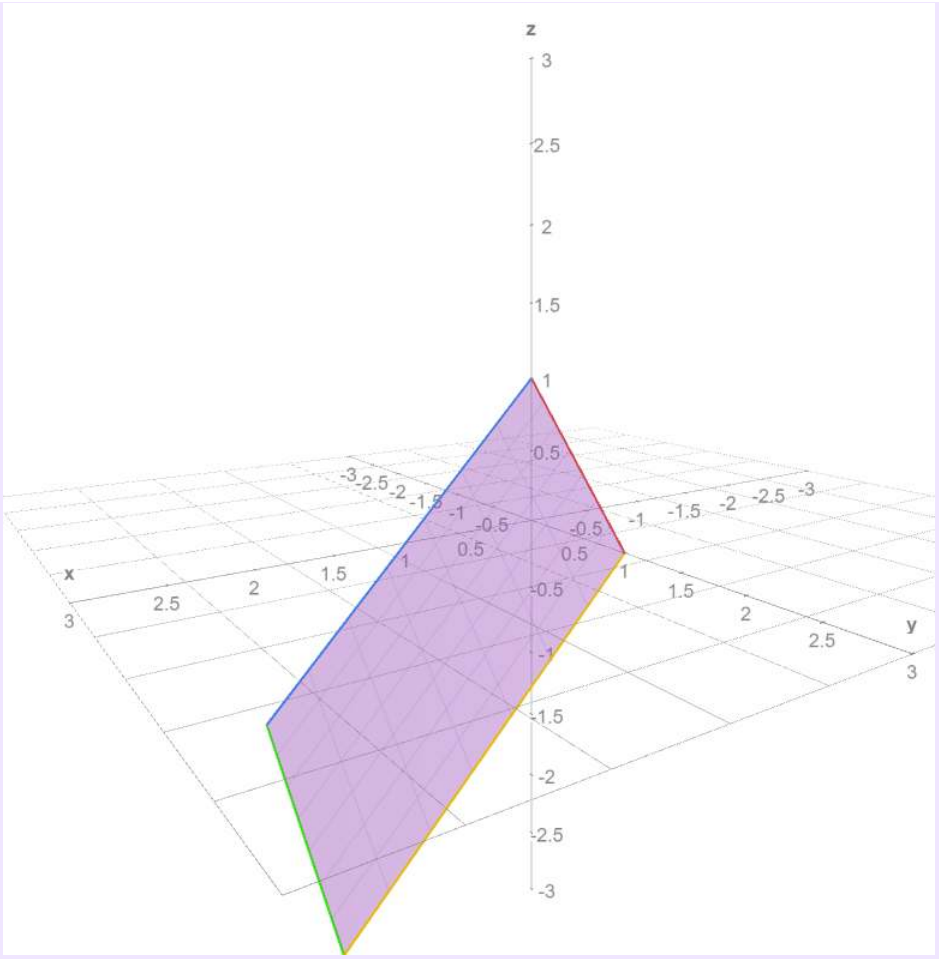
Illustration 4.3.3: Calculating a Line Integral

We will calculate the line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle xy,\ x^2 + y^2 + z^2,\ yz\rangle$ and C is the boundary of the parallelogram with vertices $(0,0,1)$, $(0,1,0)$, $(2,0,-1)$, and $(2,1,-2)$

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the results. This is not overly complicated, but it is time-consuming.



By contrast, let's calculate the line integral using Stokes' Theorem. Let S denote the surface of the parallelogram.

We can see the parallelogram has adjacent sides determined by the vectors $\vec{\mathbf{a}} = \langle 0, 1, -1\rangle$ and $\vec{\mathbf{b}} = \langle 2, 0, -2\rangle$. The normal vector to the plane containing this parallelogram must therefore be $\vec{\mathbf{n}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \langle -2, -2, -2\rangle$. Plugging in any of our four vertices and simplifying gives the plane as $x + y + z = 1$.

Therefore, S is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 0)$, and $(2, 1)$ in the xy-plane. Therefore, a parameterization of S is $\vec{\mathbf{r}}(u, v) = \langle u, v, 1 - u - v\rangle$ with $0 \le u \le 2$, $0 \le v \le 1$.

$$\vec{\mathbf{r}}_u = \langle 1, 0, -1\rangle, \quad \vec{\mathbf{r}}_v = \langle 0, 1, -1\rangle, \quad \text{and} \quad \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 1, 1, 1\rangle$$

Finally, the curl of $\vec{\mathbf{F}}$ is easily calculated to be $\langle -z,\ 0,\ x\rangle$, so by Stokes' Theorem:

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\
&= \int_0^2 \int_0^1 \langle -(1-u-v), 0, u \rangle \cdot \langle 1,1,1 \rangle \, dv \, du \\
&= \int_0^2 \int_0^1 (2u + v - 1) \, dv \, du \\
&= \int_0^2 \left[2uv + \frac{1}{2}v^2 - v \right] \Big|_0^1 du \\
&= \int_0^2 \left(2u - \frac{1}{2} \right) du \\
&= \left[u^2 - \frac{1}{2}u \right] \Big|_0^2 \\
&= 3
\end{aligned}$$

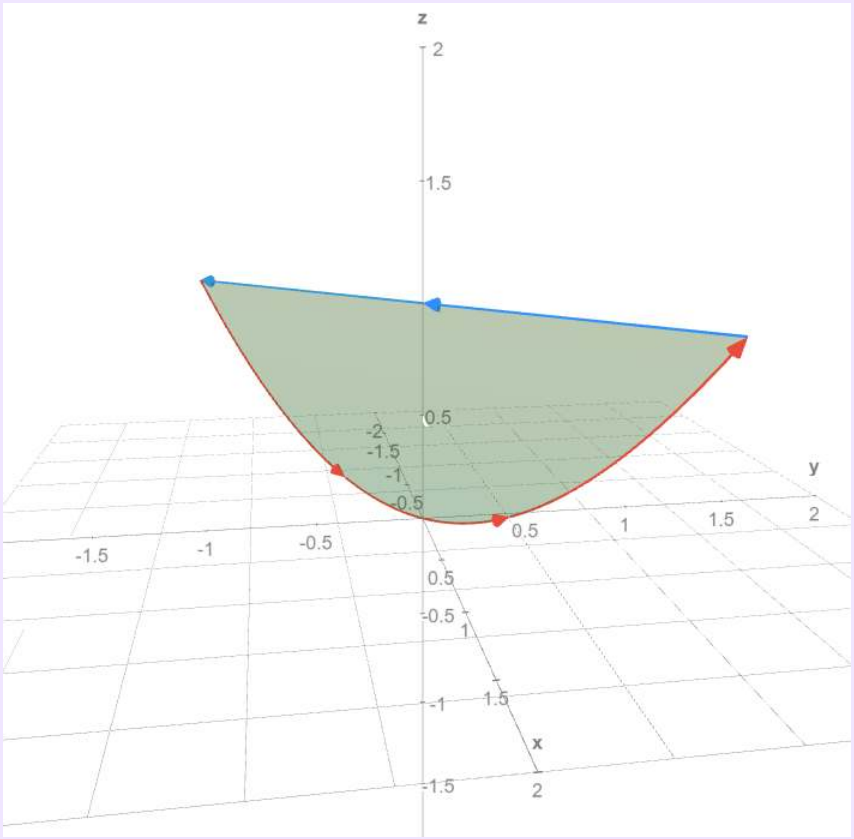
Just like in Green's Theorem, if the curve of a line integral in $\mathbb{R}^3$ is not closed, we can close it in order to use Stokes' Theorem.

Illustration 4.3.4: Closing a Curve for Stokes' Theorem

If C is the curve defined by the straight lines $\vec{\mathbf{r}}(t) = \langle t, t, t^2 \rangle$ for $-1 \leq t \leq 1$, we evaluate the line integral

$$\int_C y \, dx - xy \, dy + z^z \, dz$$

This line integral is not pleasant to solve directly, and also is not closed so we cannot immediately use Stokes' Theorem. Instead we look to add another curve to it (that we can evaluate the line integral of) to close it, and then use Stokes' Theorem on them together.



We define C_1 to be the curve connecting these endpoints. It is best to use the straight line with a constant z value to simplify the new line integral. So this can be parameterized by $\vec{\mathbf{s}}(t) = \langle -t, -t, 1\rangle$ for $-1 \le t \le 1$

For the vector field $\vec{\mathbf{F}} = \langle y, -xy, z^z\rangle$ we can evaluate:

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{-1}^{1} \langle -t, -t^2, 1^1\rangle \cdot \langle -1, -1, 0\rangle \, dt = \int_{-1}^{1} t^2 + t \, dt = \left[\frac{1}{3}t^3 + \frac{1}{2}t^2\right]\Big|_{-1}^{1} = \frac{2}{3}$$

Now since $C \cup C_1$ is closed, we can consider any surface S with boundary $C \cup C_1$ to use Stokes' Theorem on.

One such surface is the plane $x = y$ on which these curves all lie. This is parameterized by $\vec{\mathbf{r}}(u, v) = \langle u, u, v\rangle$ over $D = \{(u, v) : -1 \le u \le 1,\ u^2 \le v \le 1\}$

$$\vec{\mathbf{r}}_u = \langle 1, 1, 0\rangle, \quad \vec{\mathbf{r}}_v = \langle 0, 0, 1\rangle, \quad \text{and} \quad \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 1, -1, 0\rangle$$

This points *forward* in the picture above, which matches the orientation of the boundary curve $C \cup C_1$

We calculate $\text{curl}(\vec{\mathbf{F}}) = \langle 0, 0, -1 - y\rangle$ and now can find the flux on S:

$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \iint_D \langle -0, 0, -1 - u\rangle \cdot \langle 1, -1, 0\rangle \, dA = \iint_D 0 \, dA = 0$$

Now we put these together using Stokes' Theorem:

$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Therefore:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} - \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0 - \frac{2}{3} = -\frac{2}{3}$$

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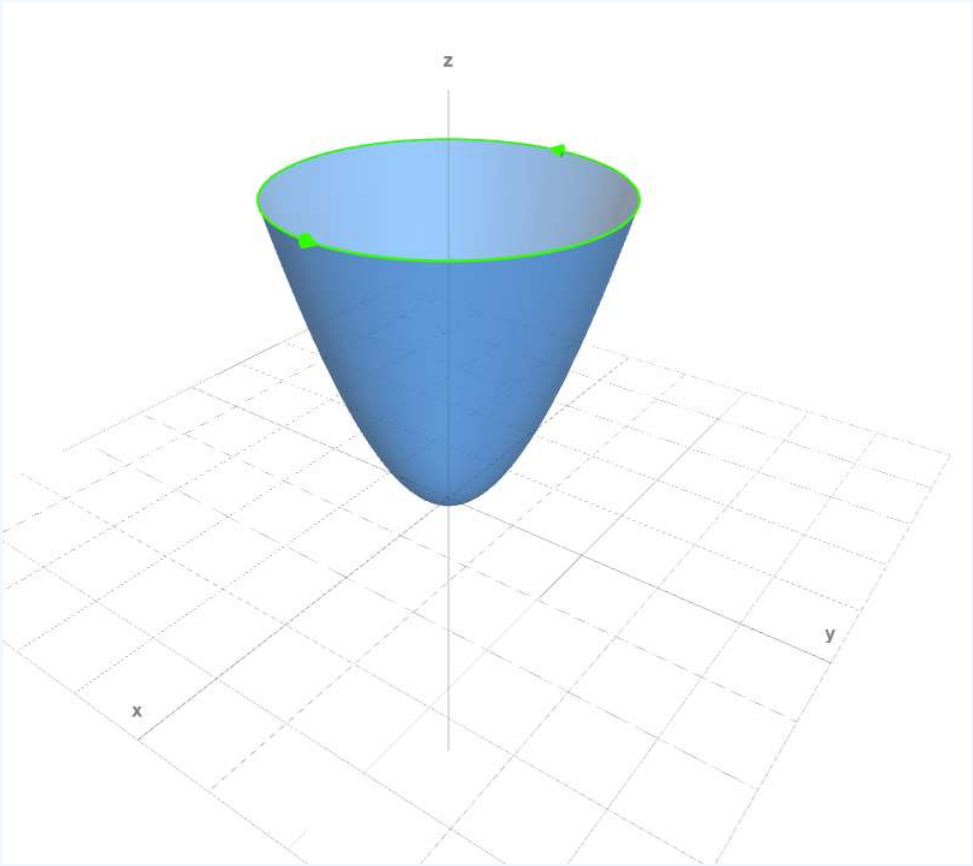
4.3: Practice Set

Exercise 4.3.1a

Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = \langle 2x, 2xy, e^z \rangle$ and $S = \{(x, y, z) : z = x^2 + y^2, 0 \leq z \leq 4\}$ oriented upward

Answer

The surface S is a paraboloid, and its boundary the circle $x^2 + y^2 = 4$ at height $z = 4$:



We can parameterize this boundary curve $C = \partial S$ by $\vec{\mathbf{r}}(t) = \langle 2\cos(t), 2\sin(t), 4 \rangle$ over $0 \leq t \leq 2\pi$

Therefore by Stokes' Theorem we have:

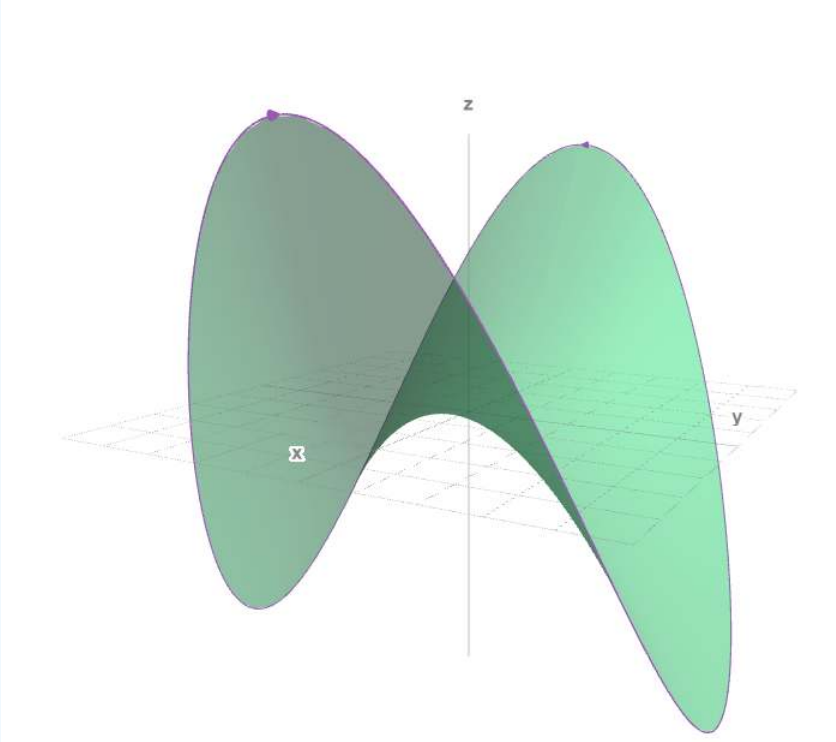
$$\begin{aligned}
\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) \\
&= \int_0^{2\pi} \left\langle 2(2\cos(t)), 2(2\cos(t))(2\sin(t)), e^4 \right\rangle \cdot \langle -2\sin(t), 2\cos(t), 0 \rangle \, dt \\
&= \int_0^{2\pi} \left(-8\cos(t)\sin(t) + 16\cos^2(t)\sin(t) \right) dt \\
&= \left[4\cos^2(t) - \frac{16}{3}\cos^3(t) \right] \Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

Exercise 4.3.1b

Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = \langle y, y^2, z^2 \rangle$ and $S = \{(x, y, z) : z = x^2 - y^2,\ x^2 + y^2 \leq 100\}$ oriented upward

Answer

The surface S is a hyperbolic paraboloid, and its boundary is the circle $x^2 + y^2 = 100$ at height $z = x^2 - y^2$:



We can parameterize this boundary curve $C = \partial S$ by $\vec{\mathbf{r}}(t) = \langle 10\cos(t), 10\sin(t), 100\cos^2(t) - 100\sin^2(t) \rangle$ over $0 \leq t \leq 2\pi$

Therefore by Stokes' Theorem we have:

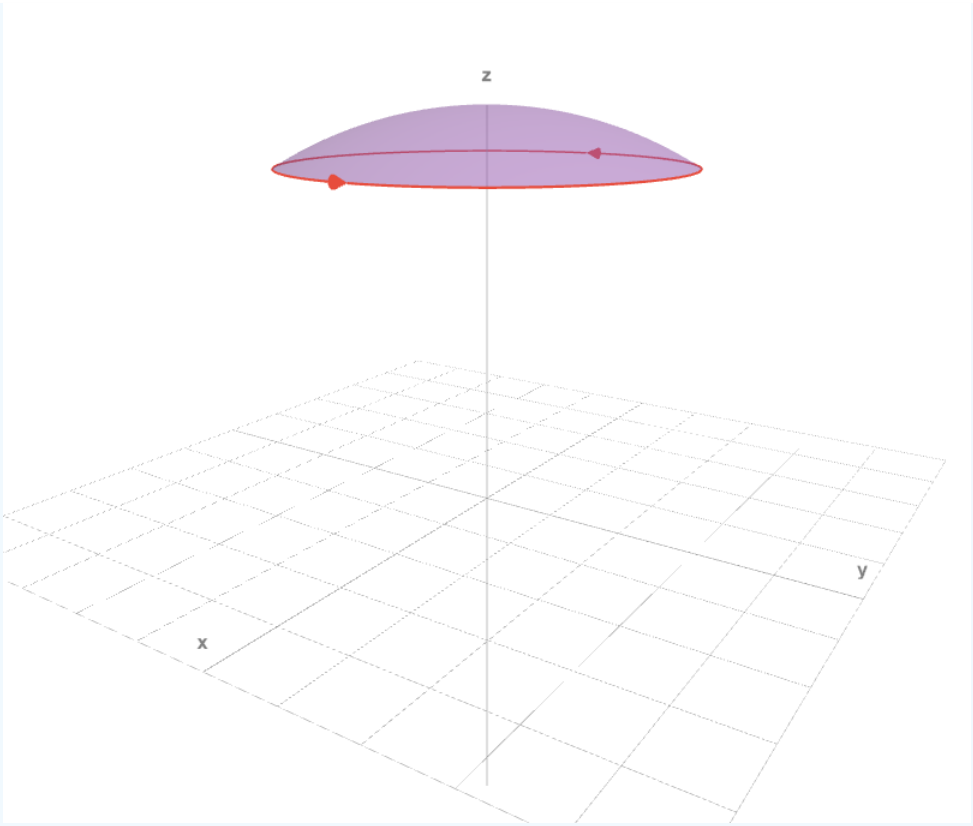
$$\begin{aligned}
\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\
&= \int_0^{2\pi} \langle 10\sin(t), (10\sin(t))^2, (100\cos^2(t) - 100\sin^2(t))^2 \rangle \cdot \langle -10\sin(t), 10\cos(t), -400\cos(t)\sin(t) \rangle\, dt \\
&= \int_0^{2\pi} \left(-100\sin^2(t) + 1000\sin^2(t)\cos(t) - 4000000(\cos^5(t)\sin(t) - 2\cos^3(t)\sin^3(t) + \sin^5(t)\cos(t)) \right) dt \\
&= \left[-50t + 25\sin(2t) + \frac{1000}{3}\sin^3(t) + \frac{1000000}{3}\cos^3(2t) \right] \Bigg|_0^{2\pi} \\
&= -100\pi
\end{aligned}$$

Exercise 4.3.1c

Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = \langle yz, -xz, x^3y^3z^2 \rangle$ and $S = \{(x, y, z) : x^2 + y^2 + z^2 = 4,\ x^2 + y^2 \leq 1,\ z \geq 0\}$ oriented upward

Answer

The surface S is a small part of a sphere, and its boundary is the circle $x^2 + y^2 = 1$ at height $z = \sqrt{3}$:



We can parameterize this boundary curve $C = \partial S$ by $\vec{\mathbf{r}}(t) = \left\langle \cos(t), \sin(t), \sqrt{3} \right\rangle$ over $0 \le t \le 2\pi$

Therefore by Stokes' Theorem we have:

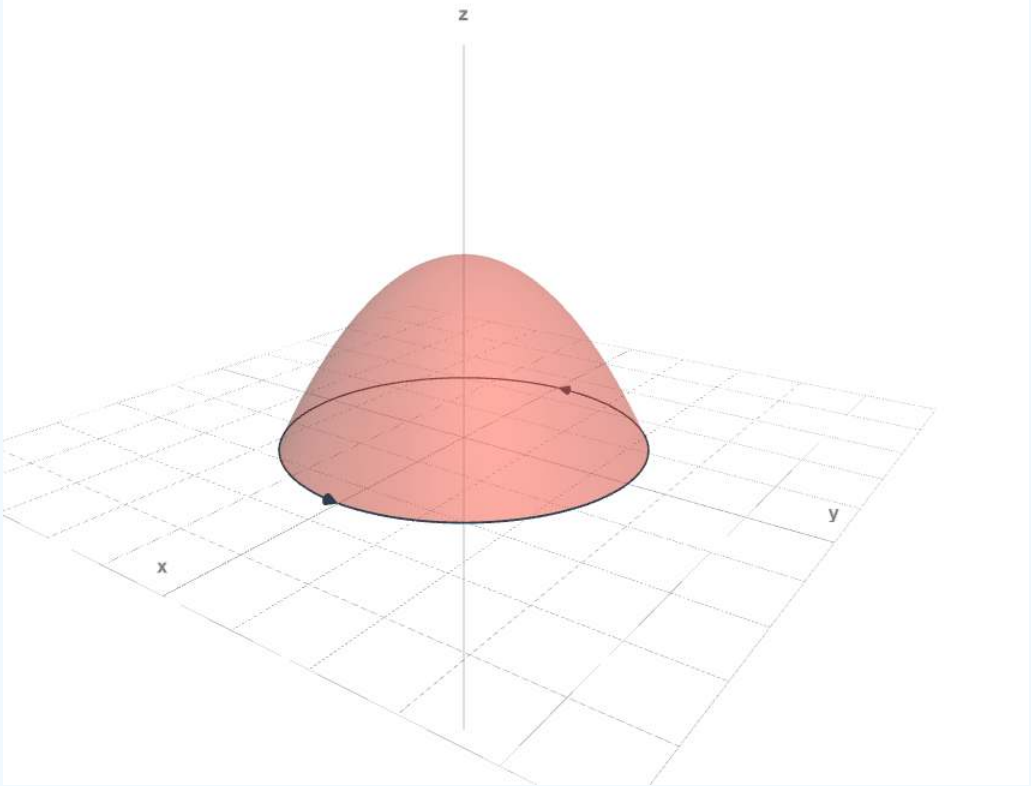
$$\begin{aligned}
\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t) \\
&= \int_0^{2\pi} \left\langle (\sin(t))(\sqrt{3}), -(\cos(t))(\sqrt{3}), (\cos(t))^3 (\sin(t))^3 (\sqrt{3})^2 \right\rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \, dt \\
&= \int_0^{2\pi} \left(-\sin^2(t)\sqrt{3} - \cos^2(t)\sqrt{3} \right) dt \\
&= \int_0^{2\pi} -\sqrt{3}\, dt \\
&= -\sqrt{3}\, t \Big|_0^{2\pi} \\
&= -2\pi\sqrt{3}
\end{aligned}$$

? Exercise $4.3.1d$

Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = \langle -x^2 y, xy^2, \sqrt{z} \rangle$ and $S = \{(x, y, z) : z = 1 - x^2 - y^2,\ z \ge 0\}$ oriented upward

Answer

The surface S is a paraboloid, and its boundary is the circle $x^2 + y^2 = 1$ at height $z = 0$:



We can parameterize this boundary curve $C = \partial S$ by $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), 0 \rangle$ over $0 \leq t \leq 2\pi$

Therefore by Stokes' Theorem we have:

$$\begin{aligned}
\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t) \\
&= \int_0^{2\pi} \left\langle -(\cos(t))^2(\sin(t)), (\cos(t))(\sin(t))^2, \sqrt{0} \right\rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \; dt \\
&= \int_0^{2\pi} \left(2\cos^2(t)\sin^2(t)\right) \; dt \\
&= \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{4}\cos(4t)\right) \; dt \\
&= \left[\frac{1}{4}t - \frac{1}{16}\sin(4t)\right]\Big|_0^{2\pi} \\
&= \frac{\pi}{2}
\end{aligned}$$

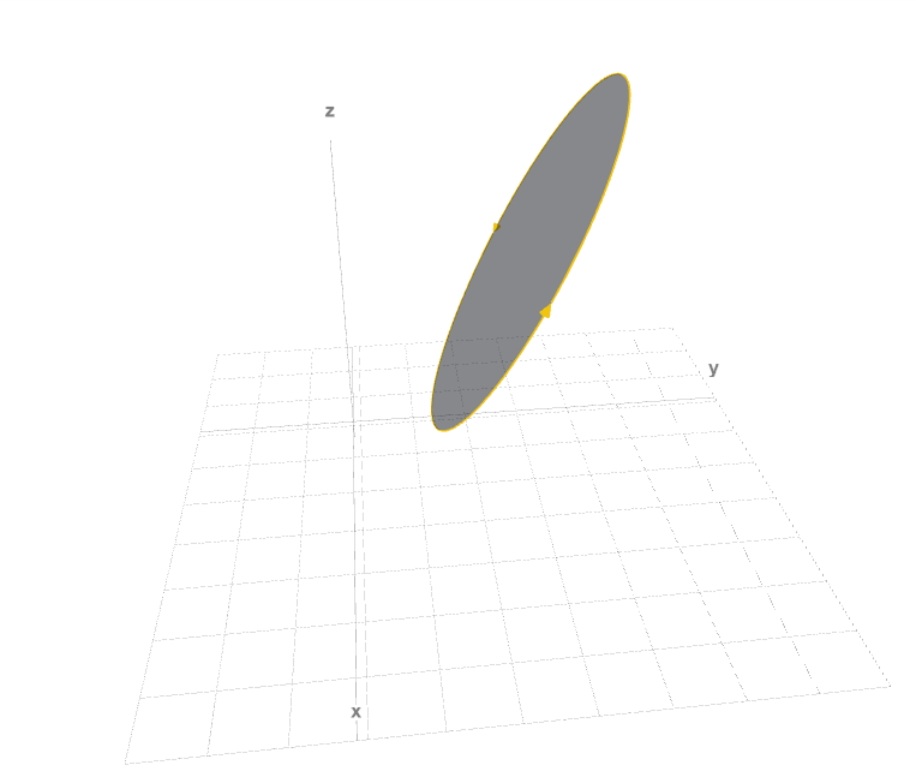
? Exercise 4.3.1e

Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = \langle z, x, y \rangle$ and $S = \{(x, y, z) \,:\, z = 2x + 16y - 13,\, x^2 + 4y^2 \leq z\}$ oriented upward

Answer

This surface is an elliptical cut-out of a plane. The boundary can be found by substituting $z = 2x + 16y - 13$ into $z = x^2 + 4y^2$:

$$x^2 + 4y^2 = 2x + 16y - 13 \longleftrightarrow (x-1)^2 + 4(y-2)^2 = 4 \longleftrightarrow \frac{(x-1)^2}{4} + (y-2)^2 = 1$$



We can parameterize this boundary curve $C = \partial S$ by $\vec{\mathbf{r}}(t) = \langle 1 + 2\cos(t), 2 + \sin(t), 2(1 + 2\cos(t)) + 16(2 + \sin(t)) - 13\rangle$ $= \langle 1 + 2\cos(t), 2 + \sin(t), 21 + 4\cos(t) + 16\sin(t)\rangle$ over $0 \le t \le 2\pi$

Therefore by Stokes' Theorem we have:

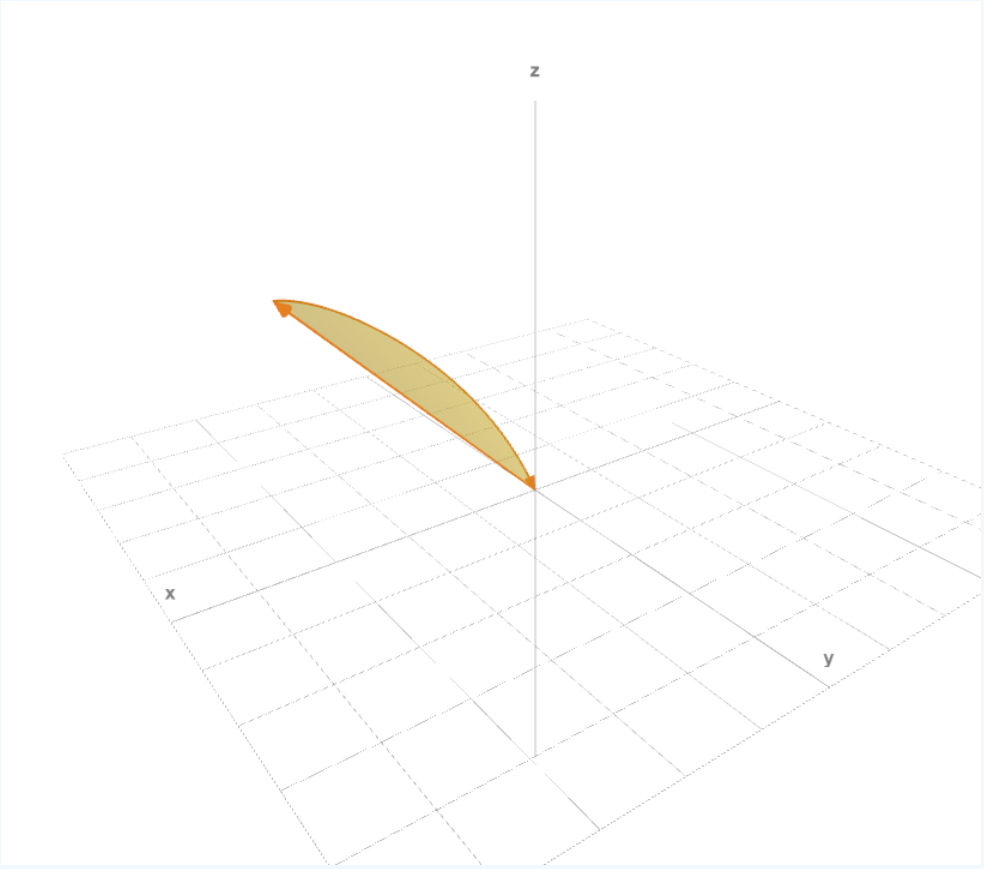
$$\begin{aligned}
\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) \\
&= \int_0^{2\pi} \langle (21 + 4\cos(t)) + 16\sin(t)), (1 + 2\cos(t)), (2 + \sin(t))\rangle \cdot \langle -2\sin(t), \cos(t), -4\sin(t) + 16\cos(t)\rangle \; dt \\
&= \int_0^{2\pi} \left(-50\sin(t) - 32\sin^2(t) + 2\cos^2(t) + 33\cos(t) - 4\sin^2(t) + 8\cos(t)\sin(t)\right) \; dt \\
&= \int_0^{2\pi} \left(-50\sin(t) - 17 + 19\cos(2t) + 33\cos(t) + 8\cos(t)\sin(t)\right) \; dt \\
&= \left[50\cos(t) - 17t + \frac{19}{2}\sin(2t) + 33\sin(t) + 4\sin^2(t)\right] \Big|_0^{2\pi} \\
&= -34\pi
\end{aligned}$$

Exercise 4.3.1f

Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = \langle x, \sin(x - z^2), 2\rangle$ and $S = \{(x, y, z) : y = e^x - e^z,\ z^2 \le x \le z\}$ oriented toward the negative y-axis

Answer

This surface is a function $y = f(x, z)$ over a domain between a line and parabola:



The boundary of this domain connects the points $(0,0,0)$ and $(1,0,1)$ along two separate curves.

For this to *look counterclockwise from the negative y-axis*, we need the arrows to point in the directions they do in the picture above.

We let C_1 be the path from $(0,0,0)$ to $(1,0,1)$ along the straight line $z=x$

We parameterize this using $\vec{\mathbf{r}}(t)=\langle t,0,t\rangle$ over $0\le t\le 1$

The line integral of $\vec{\mathbf{F}}$ on this path is given by:

$$\begin{aligned}\int_{C_1}\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}'(t) &= \int_0^1 \langle t,\sin(t-t^2),2\rangle\cdot\langle 1,0,1\rangle\ dt\\ &= \int_0^1 (t+2)\ dt\\ &= \left[\frac{1}{2}t^2+2t\right]\Big|_0^1\\ &= \frac{5}{2}\end{aligned}$$

We let C_2 be the path back from $(1,0,1)$ to $(0,0,0)$ along $x=z^2$

We parameterize this using $\vec{\mathbf{r}}(t)=\left\langle (1-t)^2, e^{(1-t)^2}-e^{1-t}, 1-t\right\rangle$ over $0\le t\le 1$

The line integral of $\vec{\mathbf{F}}$ on this path is given by:

$$\int_{C_2} \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) = \int_0^1 \left\langle (1-t)^2, \sin(0), 2 \right\rangle \cdot \left\langle -2(1-t), -2(1-t)e^{(1-t)^2} + e^{1-t}, -1 \right\rangle \, dt$$
$$= \int_0^1 \left(-2(1-t)^3 - 2 \right) \, dt$$
$$= \frac{1}{2}(1-t)^4 - 2t \Big|_0^1$$
$$= -\frac{5}{2}$$

Finally by Stokes' Theorem we know that

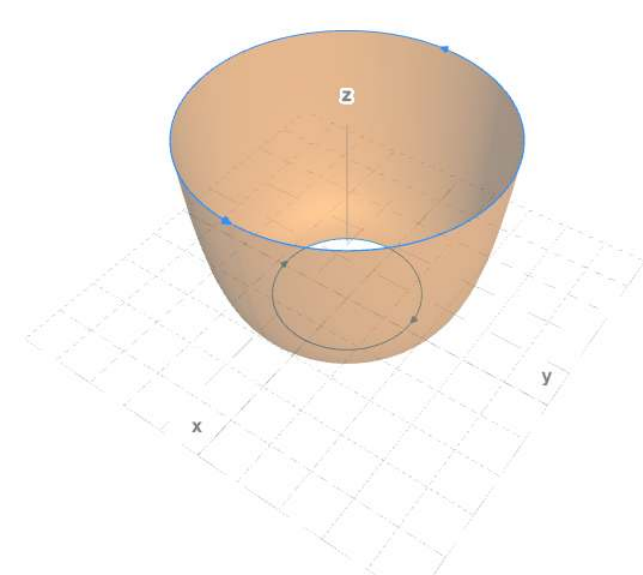
$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_{C_1 \cup C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{5}{2} - \frac{5}{2} = 0$$

Exercise 4.3.1g

Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = \langle -y, x, \ln(z) \rangle$ and $S = \left\{ (x, y, z) \, : \, z = e^{x^2+y^2}, \, 1 \leq x^2 + y^2 \leq 4 \right\}$ oriented upward

Answer

The surface S is defined by a function $z = f(x, y)$ over an annulus domain:



The boundary of this surface is also in two pieces, the outer circle $x^2 + y^2 = 4$ (which we must traverse counterclockwise) and the inner circle $x^2 + y^2 = 1$ (which we want to traverse clockwise).

Let C_1 be the outer boundary, which we can parameterize using $\vec{\mathbf{r}}(t) = \langle 2\cos(t), 2\sin(t), e^4 \rangle$ over $0 \leq t \leq 2\pi$

The line integral of $\vec{\mathbf{F}}$ on this path is given by:

$$\int_{C_1} \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) = \int_0^{2\pi} \langle -2\sin(t), 2\cos(t), 4 \rangle \cdot \langle -2\sin(t), 2\cos(t), 0 \rangle \, dt$$
$$= \int_0^{2\pi} 4 \, dt$$
$$= 4t \Big|_0^{2\pi}$$
$$= 8\pi$$

Let C_2 be the inner boundary, which we can parameterize using $\vec{\mathbf{r}}(t) = \langle \sin(t), \cos(t), e^1 \rangle$ over $0 \leq t \leq 2\pi$

The line integral of $\vec{\mathbf{F}}$ on this path is given by:

$$\begin{aligned}\int_{C_2} \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) &= \int_0^{2\pi} \langle -\cos(t), \sin(t), 1\rangle \cdot \langle \cos(t), -\sin(t), 0\rangle \, dt \\ &= \int_0^{2\pi} -1 \, dt \\ &= -t\Big|_0^{2\pi} \\ &= -2\pi\end{aligned}$$

Finally by Stokes' Theorem we know that

$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_{C_1 \cup C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \iint_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 8\pi + (-2\pi) = 6\pi$$

Exercise $4.3.2a$

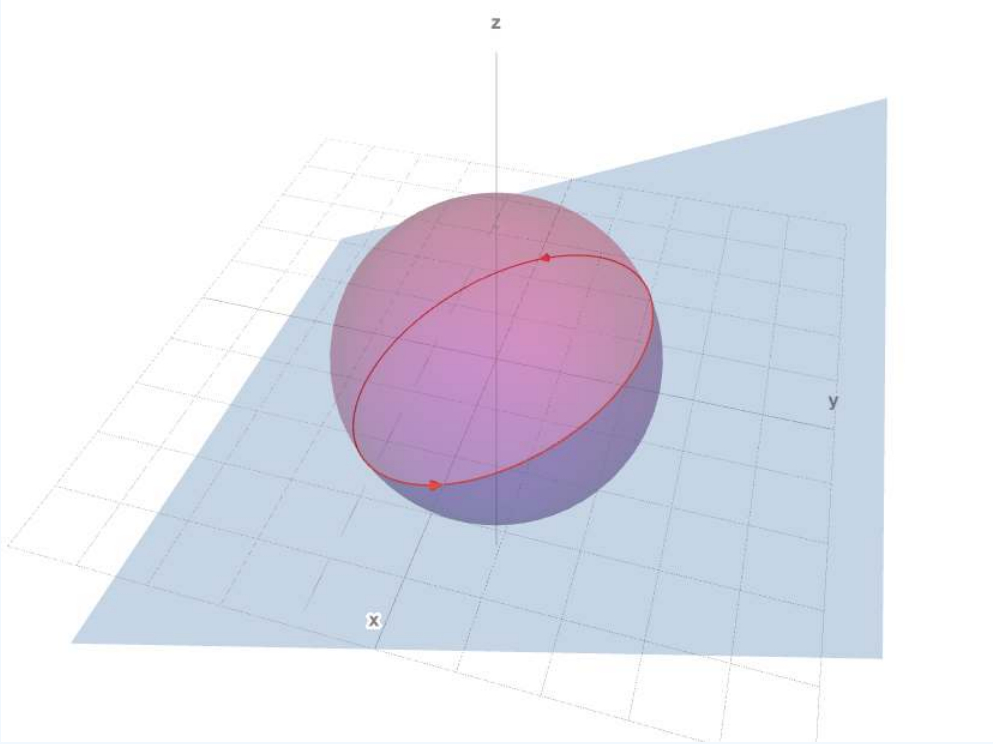
Use Stokes' Theorem to evaluate $\int_C y\,dx + (x+z)\,dy + z^2\,dz$ for C the intersection of the unit sphere and the plane $z = \frac{1}{2}y$ in the counterclockwise direction when viewed from above.

Answer

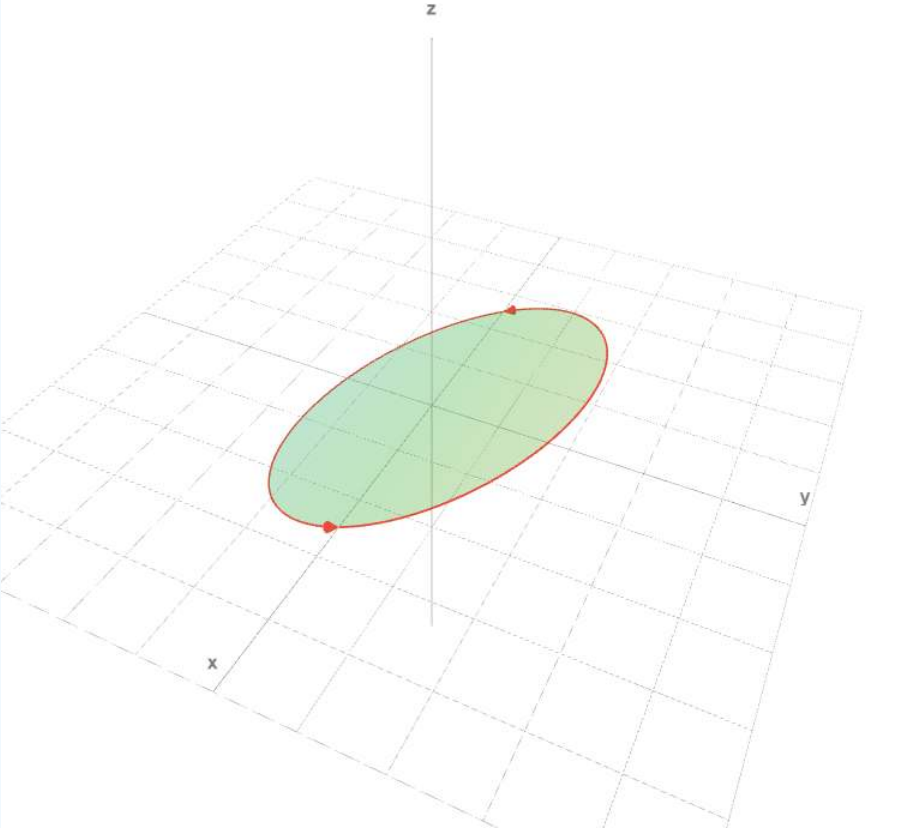
The line integral is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle y, x+z, z^2\rangle$

This vector field has curl $\text{curl}(\vec{\mathbf{F}}) = \langle -1, 0, 0\rangle$

The line C is the intersection of the unit sphere and the plane $z = \frac{1}{2}y$ in the counterclockwise direction when viewed from above:



One surface with this boundary is $S = \{(x, y, z) : x^2 + 5z^2 \leq 1,\ y = 2z\}$



We can parameterize this surface using $\vec{\mathbf{r}}(u,v) = \left\langle u\cos(v), \frac{2}{\sqrt{5}}u\sin(v), \frac{1}{\sqrt{5}}u\sin(v)\right\rangle$ over $D = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 2\pi\}$

We calculate $\vec{\mathbf{r}}_u = \left\langle \cos(v), \frac{2}{\sqrt{5}}\sin(v), \frac{1}{\sqrt{5}}\sin(v)\right\rangle$, $\vec{\mathbf{r}}_v = \left\langle -u\sin(v), \frac{2}{\sqrt{5}}u\cos(v), \frac{1}{\sqrt{5}}u\cos(v)\right\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle 0, -\frac{1}{\sqrt{5}}u, \frac{2}{\sqrt{5}}u\right\rangle$

This is clearly oriented upward, since the z component is positive. This matches the orientation of C in regards to Stokes' Theorem.

Therefore, we can calculate the line integral as:

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \iint_S \operatorname{curl}(\vec{\mathbf{F}})\cdot d\vec{\mathbf{S}} \\ &= \iint_D \langle -1,0,0\rangle \cdot \left\langle 0, -\frac{1}{\sqrt{5}}u, \frac{2}{\sqrt{5}}u\right\rangle dA \\ &= \iint_D 0\, dA \\ &= 0\end{aligned}$$

Exercise 4.3.2*b*

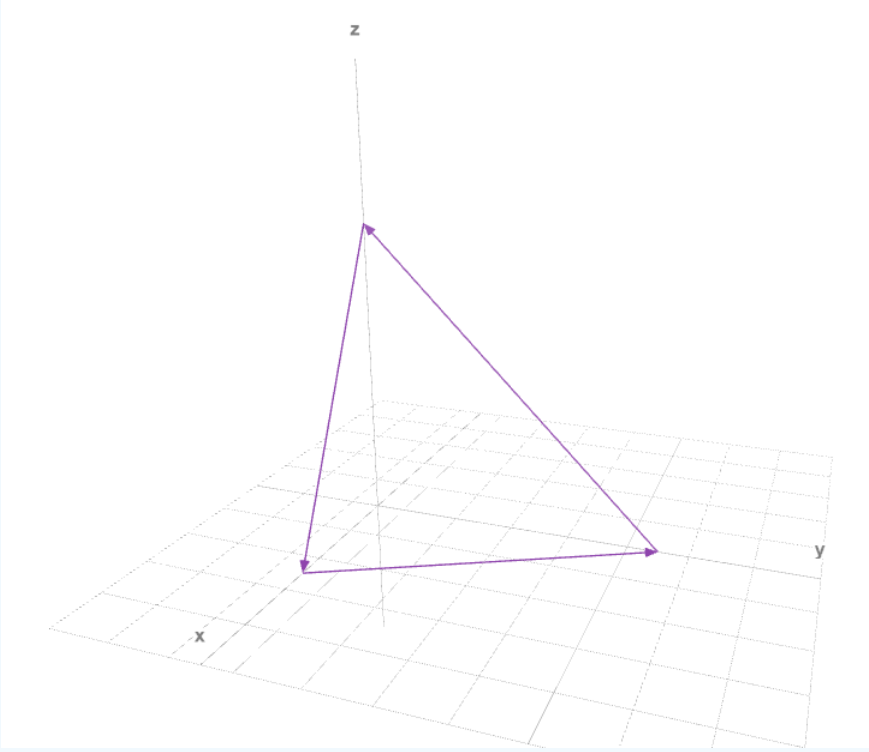
Use Stokes' Theorem to evaluate $\int_C z^2\,dx + y^2\,dy + xy\,dz$ for C the curve connecting $(1,0,0)$ to $(0,2,0)$ to $(0,0,2)$ to $(1,0,0)$ with straight lines.

Answer

The line integral is $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle z^2, y^2, xy\rangle$

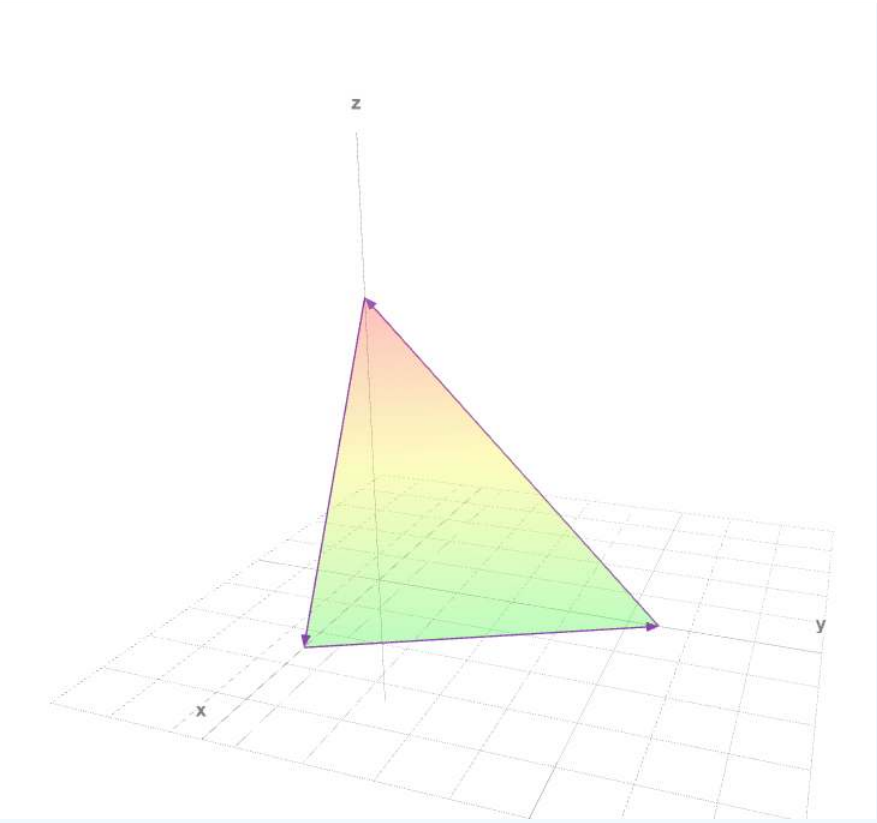
This vector field has curl $\text{curl}(\vec{\mathbf{F}}) = \langle x, 2z - y, 0 \rangle$

The curve C traces a triangle in some plane which moves along its edge in the counterclockwise direction when viewed from above:



The equation of the plane containing this curve can easily be calculated to be $2x + y + z = 2$, so one surface with this as its boundary is $S = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 2 - 2x,\ z = 2 - 2x - y\}$

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We can parameterize this surface using $\vec{\mathbf{r}}(u,v) = \langle u, v, 2-2u-v\rangle$ over $D = \{(u,v) \,:\, 0 \le u \le 1,\, 0 \le v \le 2-2u\}$

We calculate $\vec{\mathbf{r}}_u = \langle 1, 0, -2\rangle$, $\vec{\mathbf{r}}_v = \langle 0, 1, -1\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2, 1, 1\rangle$

This is clearly oriented upward, since the z component is positive. This matches the orientation of C in regards to Stokes' Theorem.

Therefore, we can calculate the line integral as:

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\
&= \iint_D \langle u, 2(2-2u-v)-v, 0\rangle \cdot \langle 2,1,1\rangle \; dA \\
&= \int_0^1 \int_0^{2-2u} (4-2u-3v)\, dv\, du \\
&= \int_0^1 \left[4v - 2uv - \frac{3}{2}v^2\right]\Big|_0^{2-2u} du \\
&= \int_0^1 \left(2-2u^2\right)\, du \\
&= \left[2u - \frac{2}{3}u^3\right]\Big|_0^1 \\
&= \frac{4}{3}
\end{aligned}$$

Exercise $4.3.2c$

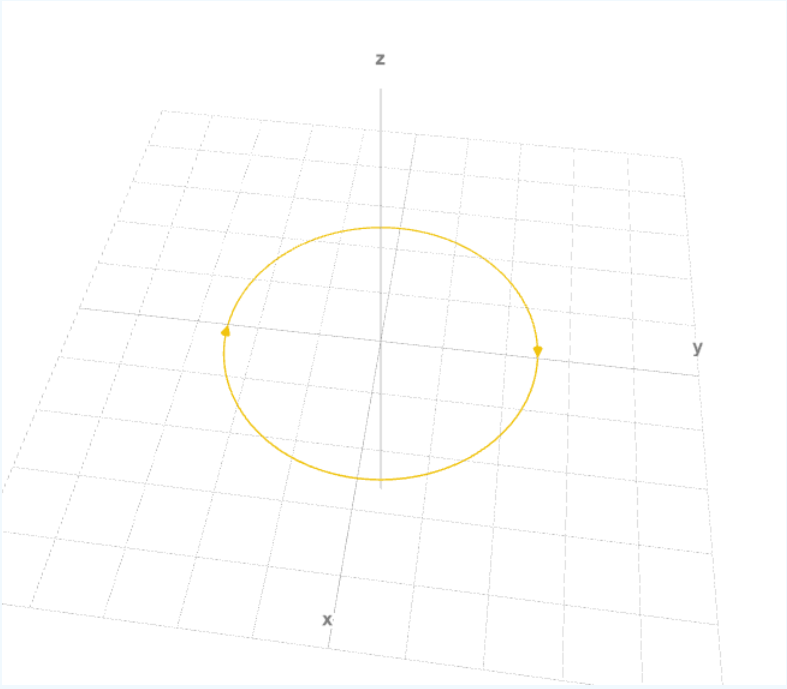
Use Stokes' Theorem to evaluate $\int_C x^2\,dx + (2xy + x)\,dy + z\,dz$ for C parameterized by $\vec{\mathbf{r}}(t) = \langle \sin(\pi t), \cos(\pi t), 0\rangle$ over $0 \le t \le 2$

Answer

The line integral is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle x^2, 2xy + x, z\rangle$

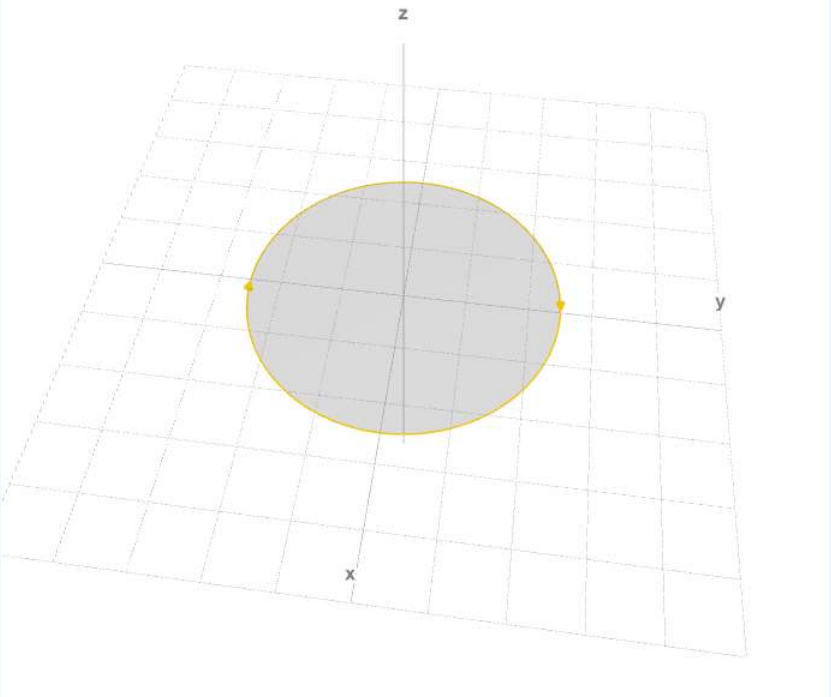
This vector field has curl $\text{curl}(\vec{\mathbf{F}}) = \langle 0, 0, 2y + 1\rangle$

The curve C traces a circle in the $z = 0$ plane which moves along its edge in the clockwise direction when viewed from above:



One surface with this as its boundary is $S = \{(x, y, z) : x^2 + y^2 \le 1\}$

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We can parameterize this surface using $\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), 0\rangle$ over $D = \{(u,v) \,:\, 0 \le u \le 1,\ 0 \le v \le 2\pi\}$

We calculate $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 0\rangle$, $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, 0, u\rangle$

This is clearly oriented upward, since the z component is positive. This *does not match* the orientation of C in regards to Stokes' Theorem, so we multiply the flux integral by a negative sign to compensate.

We can calculate the line integral as:

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= -\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\
&= -\iint_D \langle 0, 0, 2u\sin(v) + 1\rangle \cdot \langle 0, 0, u\rangle \; dA \\
&= \int_0^1 \int_0^{2\pi} (-2u^2\sin(v) - u)\, dv\, du \\
&= \int_0^1 \left[2u^2\cos(v) - uv\right]\Big|_0^{2\pi} du \\
&= \int_0^1 -2\pi u \, du \\
&= -\pi u^2\Big|_0^1 \\
&= -\pi
\end{aligned}$$

? Exercise $4.3.2d$

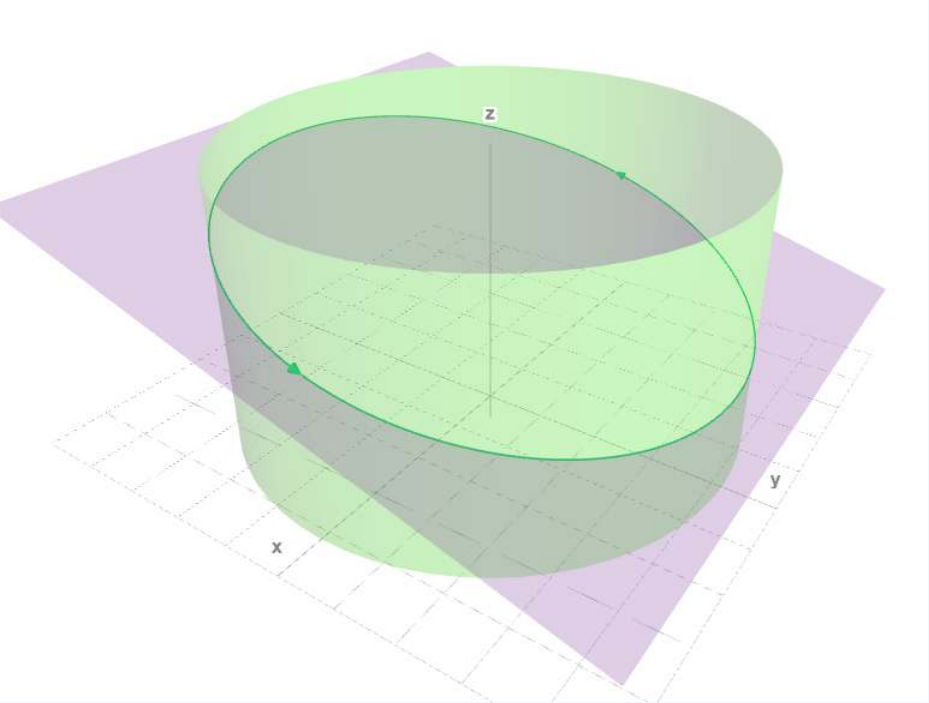
Use Stokes' Theorem to evaluate $\int_C y^2\,dx + 2x\,dy - 4z^2\,dz$ for C the intersection of $y + z = 12$ and $x^2 + y^2 = 25$ going counterclockwise when viewed from above.

Answer

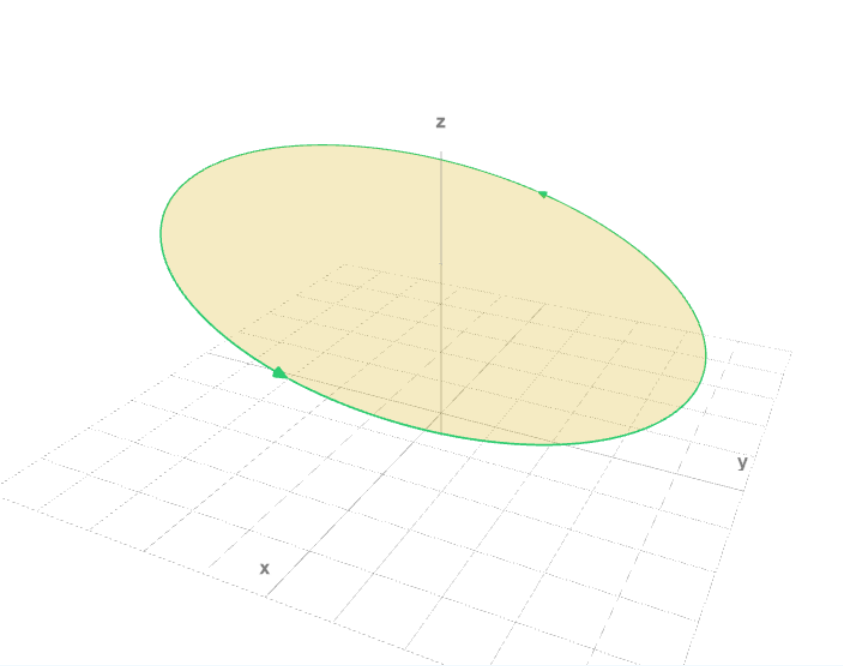
The line integral is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle y^2, 2x, -4z^2 \rangle$

This vector field has curl $\text{curl}(\vec{\mathbf{F}}) = \langle 0, 0, 2 - 2y \rangle$

The curve C is the intersection of the plane $y + z = 12$ and cylinder $x^2 + y^2 = 25$ going counterclockwise when viewed from above.



One surface with this as its boundary is $S = \{(x, y, z) : x^2 + y^2 \leq 25,\ z = 12 - y\}$



We can parameterize this surface using $\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), 12 - u\sin(v) \rangle$ over $D = \{(u, v) : 0 \leq u \leq 5,\ 0 \leq v \leq 2\pi\}$

We calculate $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), -\sin(v) \rangle$, $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), -u\cos(v) \rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, u, u \rangle$

This is clearly oriented upward, since the z component is positive. This matches the orientation of C in regards to Stokes' Theorem.

Therefore, we can calculate the line integral as:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\ &= \iint_D \langle 0, 0, 2 - 2u\sin(v)\rangle \cdot \langle 0, u, u\rangle \; dA \\ &= \int_0^5 \int_0^{2\pi} (2u - 2u^2\sin(v))\, dv\, du \\ &= \int_0^5 \left[2uv + 2u^2\cos(v)\right]\Big|_0^{2\pi} du \\ &= \int_0^5 4\pi u \, du \\ &= 2\pi u^2\Big|_0^5 \\ &= 50\pi\end{aligned}$$

Exercise $4.3.2e$

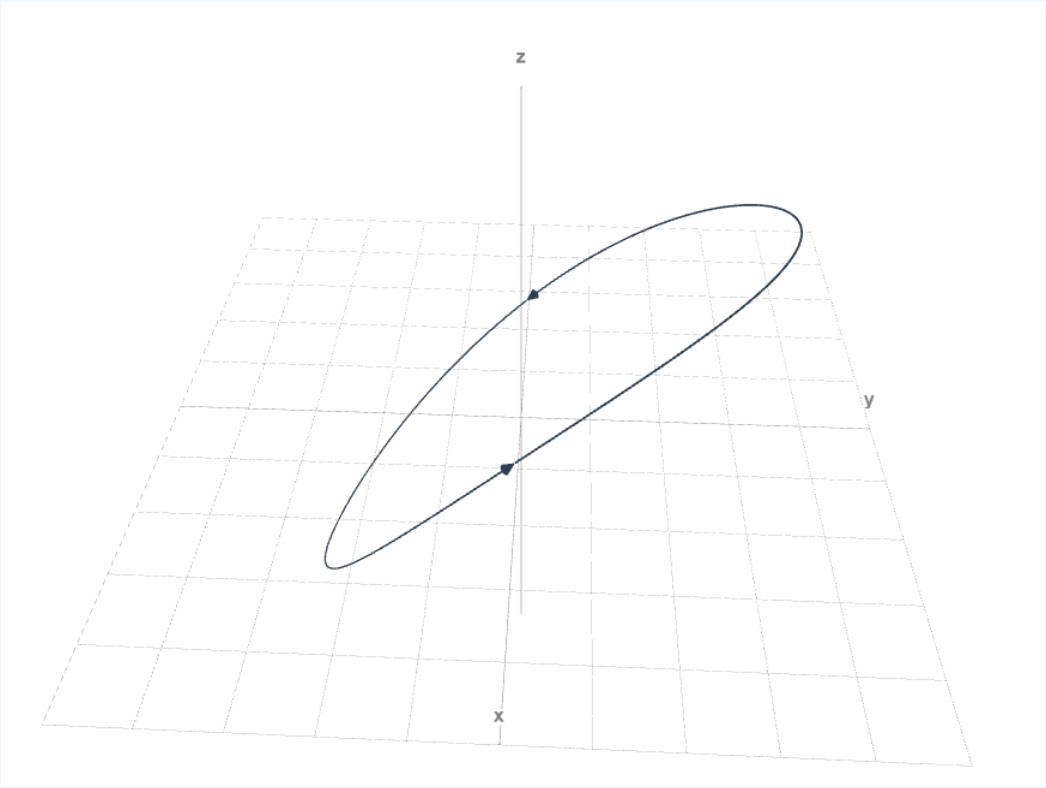
Use Stokes' Theorem to evaluate $\int_C \cos(x^2)\,dx + e^{|y|}\,dy + (x - \ln(z^4+1))\,dz$ for C parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(t), 2\sin(t), \cos^2(t) + 4\sin(t)\rangle$ over $0 \le t \le 2\pi$

Answer

This line integral is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle \cos(x^2), e^{|y|}, x - \ln(z^4+1)\rangle$

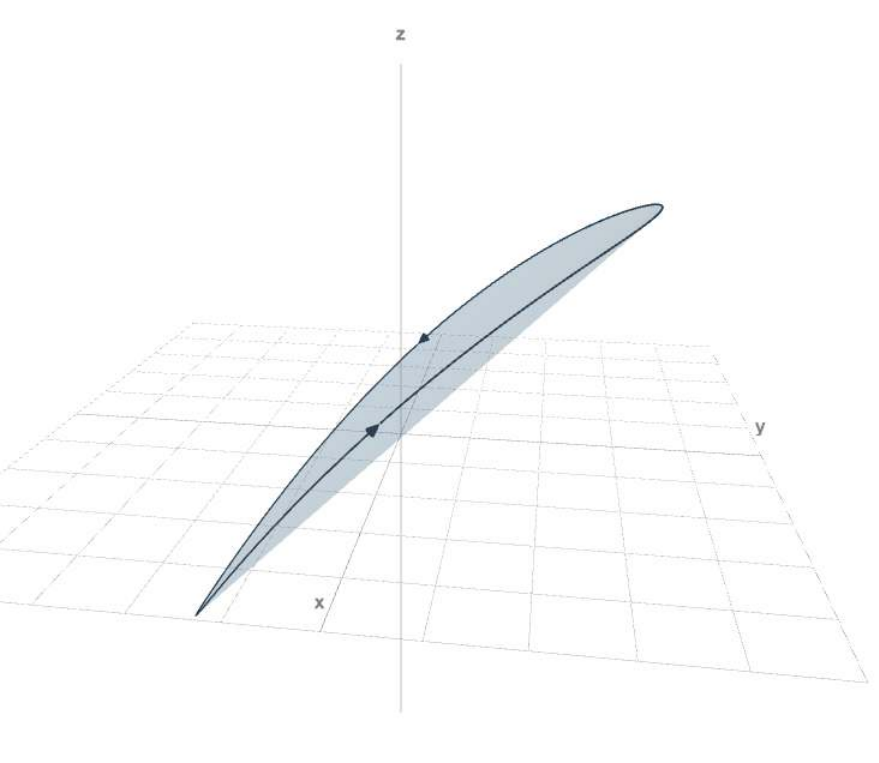
This vector field has curl $\operatorname{curl}(\vec{\mathbf{F}}) = \langle 0, -1, 0\rangle$

The curve C traces an ellipse $x^2 + \frac{y^2}{4} = 1$ while $z = x^2 + 2y$



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One surface with this as its boundary is $S = \left\{(x, y, z) \, : \, x^2 + \frac{y^2}{4} \leq 1, \, z = x^2 + 2y\right\}$



We can parameterize this surface using $\vec{\mathbf{r}}(u, v) = \langle u\cos(v), 2u\sin(v), u^2\cos^2(v) + 4u\sin(v)\rangle$ over $D = \{(u, v) \, : \, 0 \leq u \leq 1, \, 0 \leq v \leq 2\pi\}$

We calculate $\vec{\mathbf{r}}_u = \langle \cos(v), 2\sin(v), 2u\cos^2(v) + 4\sin(v)\rangle$, $\vec{\mathbf{r}}_v = \langle -u\sin(v), 2u\cos(v), -2u^2\cos(v)\sin(v) + 4u\cos(v)\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -4u^2\cos^2(v), -4u, 2u\rangle$

This is clearly oriented upward, since the z component is positive. This matches the orientation of C in regards to Stokes' Theorem.

Therefore, we can calculate the line integral as:

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} \\
&= \iint_D \langle 0, -1, 0\rangle \cdot \left\langle -4u^2\cos^2(v) - 2u^2\sin^2(v), -4u - u^2\cos(v)\sin(v), 2u\right\rangle \, dA \\
&= \int_0^1 \int_0^{2\pi} (4u + u^2\cos(v)\sin(v)) \, dv \, du \\
&= \int_0^1 \left[4uv + \frac{1}{2}u^2\sin^2(v)\right]\Big|_0^{2\pi} du \\
&= \int_0^1 8\pi u \, du \\
&= 4\pi u^2\Big|_0^1 \\
&= 4\pi
\end{aligned}$$

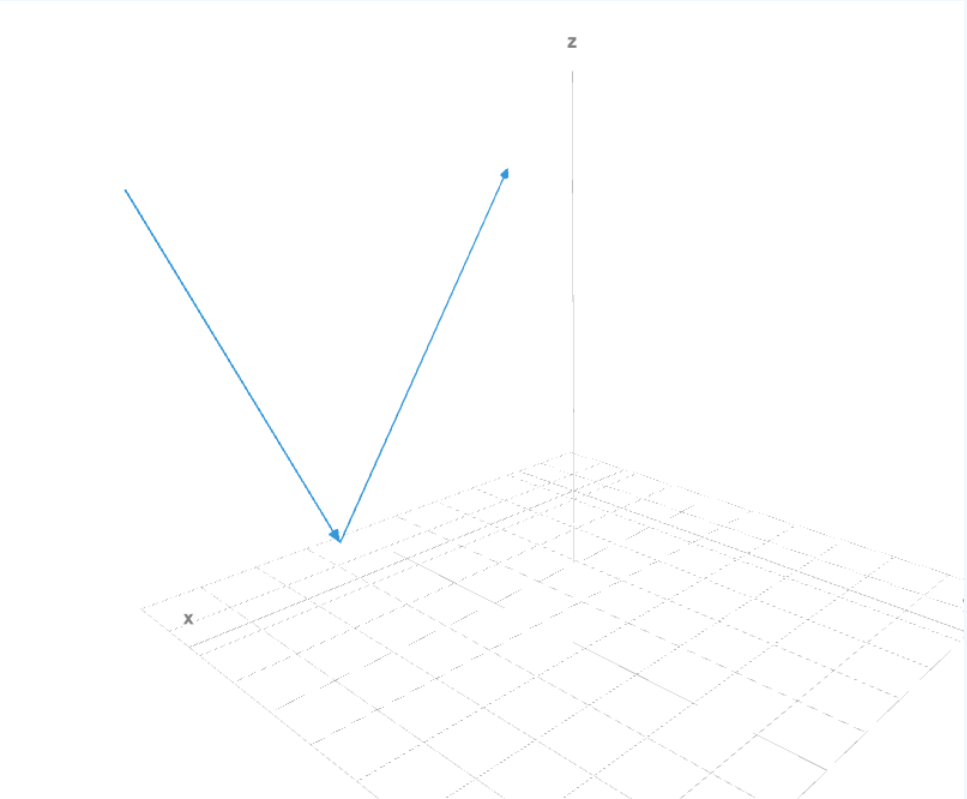
? Exercise 4.3.3a

Use Stokes' Theorem to evaluate $\int_C e^y \, dx + e^{\sin(\pi z)} \, dy + \sqrt[4]{z} \, dz$ where C is the curve connecting $(5, 0, 4)$ to $(4, 1, 1)$ to $(1, 0, 4)$ using straight lines.

Answer

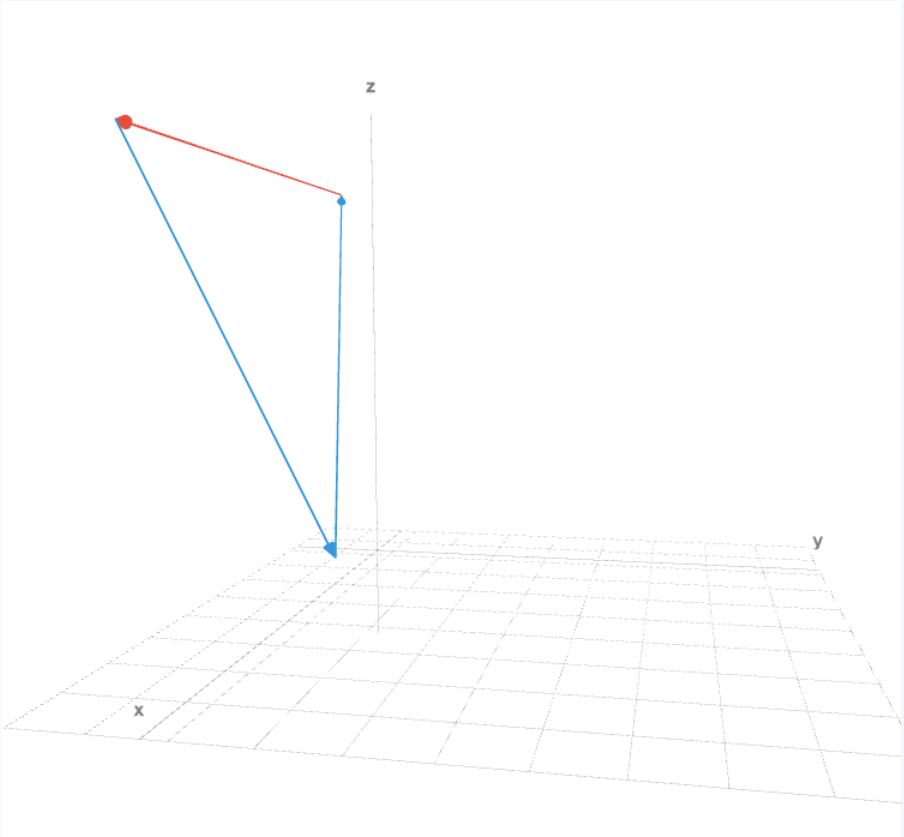
The desired line integral is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle e^y, e^{\sin(\pi z)}, \sqrt[4]{z} \rangle$

Notice that this path C is not closed, so we need to close it to use Stokes' Theorem.



We will add on a return path $\tilde{C}$, which is easiest as a straight line back to $(5, 0, 4)$

We can parameterize this path using $\vec{\mathbf{s}}(t) = \langle 1 + 4t, 0, 4 \rangle$ for $0 \leq t \leq 1$

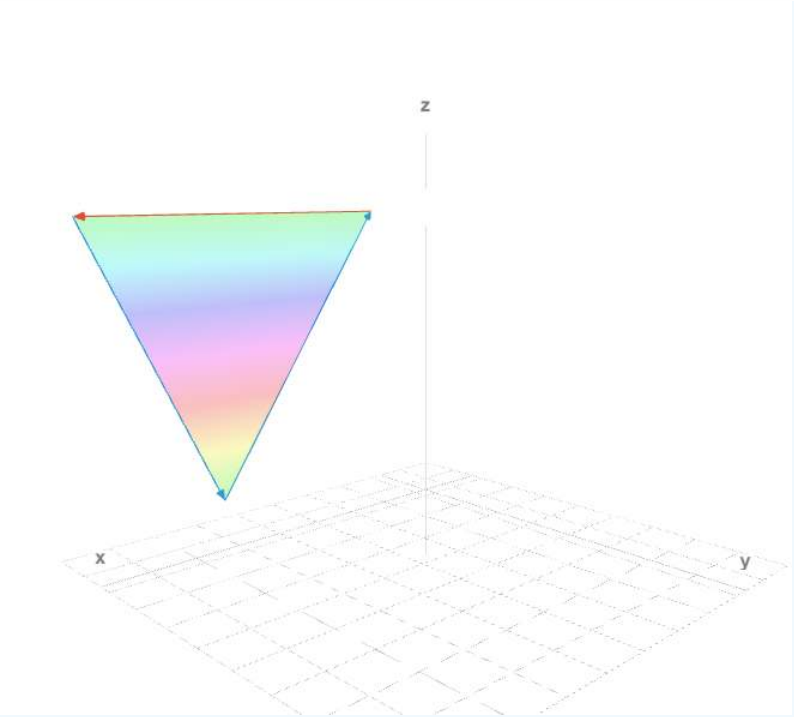


We calculate the return line integral:

$$\begin{aligned}\int_{\tilde{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \left(e^0 \cdot 4 + e^0 \cdot 0 + \sqrt[4]{4} \cdot 0\right)\, dt \\ &= \int_0^1 4\, dt \\ &= 4t\Big|_0^1 \\ &= 4\end{aligned}$$

It is easy to find these points (and lines) all lie within the same plane, which is $3y + z = 4$

One surface with this as its boundary is thus $S = \{(x, y, z) \,:\, 0 \le y \le 1,\, 3y + 1 \le x \le 5 - y,\, z = 4 - 3y\}$



We can parameterize this surface using $\vec{\mathbf{r}}(u,v) = \langle u, v, 4-3v\rangle$ over $D = \{(u,v) \,:\, 0 \le v \le 1,\ 3v+1 \le u \le 5-v\}$

We calculate $\vec{\mathbf{r}}_u = \langle 1,0,0\rangle$, $\vec{\mathbf{r}}_v = \langle 0,1,-3\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0,3,1\rangle$

This is clearly oriented upward, since the z component is positive. This matches the orientation of C in regards to Stokes' Theorem, since the curve travels around the boundary of S counterclockwise when viewed from above.

This vector field has $\text{curl}(\vec{\mathbf{F}}) = \left\langle -\pi\cos(\pi z)e^{\sin(\pi z)}, 0, -e^y\right\rangle$

Therefore, we can calculate the surface integral as:

$$\begin{aligned}
\iint_S \text{curl}(\vec{\mathbf{F}})\cdot d\vec{\mathbf{S}} &= \iint_D \left\langle -\pi\cos(\pi(4-3v))e^{\sin(\pi(4-3v))}, 0, -e^v\right\rangle \cdot \langle 0,3,1\rangle \, dA \\
&= \int_0^1 \int_{3v+1}^{5-v} -e^v \, du\, dv \\
&= \int_0^1 -ue^v\Big|_{3v+1}^{5-v} \, dv \\
&= \int_0^1 (4v-4)e^v \, dv \\
&= 4(v-2)e^v\Big|_0^1 \\
&= 8-4e
\end{aligned}$$

Stokes' Theorem says that $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \int_{\tilde{C}} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \iint_S \text{curl}(\vec{\mathbf{F}})\cdot d\vec{\mathbf{S}}$

Therefore, our final answer is $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = 4-4e$

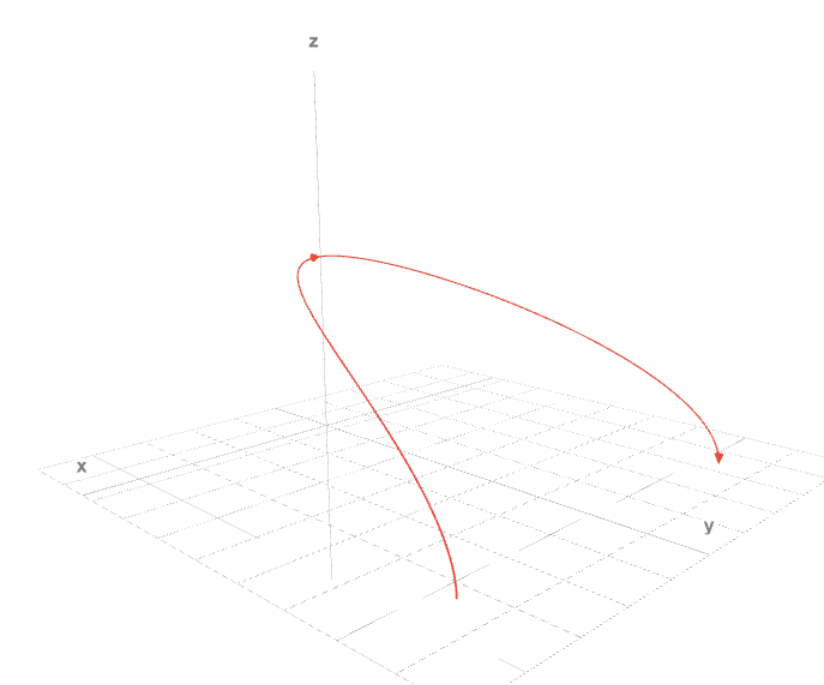
Exercise 4.3.3b

Use Stokes' Theorem to evaluate $\int_C \left(2x^{17}+3x^2+yz\right)\,dx - y^8\,dy + \left(xy+(1+z)^z\right)\,dz$ where C is the path parameterized by $\vec{\mathbf{r}}(t) = \langle 2\cos(\pi t), 4\cos^2(\pi t), \sin(\pi t)\rangle$ for $0 \le t \le 1$

Answer

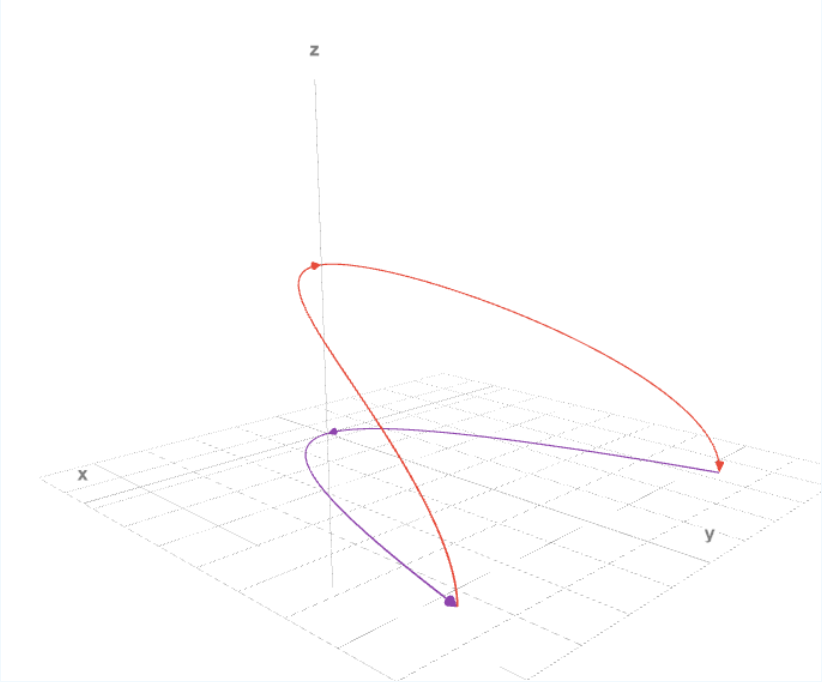
The desired line integral is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle 2x^{17} + 3x^2 + yz, -y^8, xy + (1+z)^z \rangle$

Notice that this path C traces a half-ellipse $\frac{x^2}{4} + z^2 = 1$ while $y = x^2$ and is not closed, so we need to close it to use Stokes' Theorem.



We will add on a return path $\tilde{C}$, which is easiest along the cylinder $y = x^2$ with $z = 0$

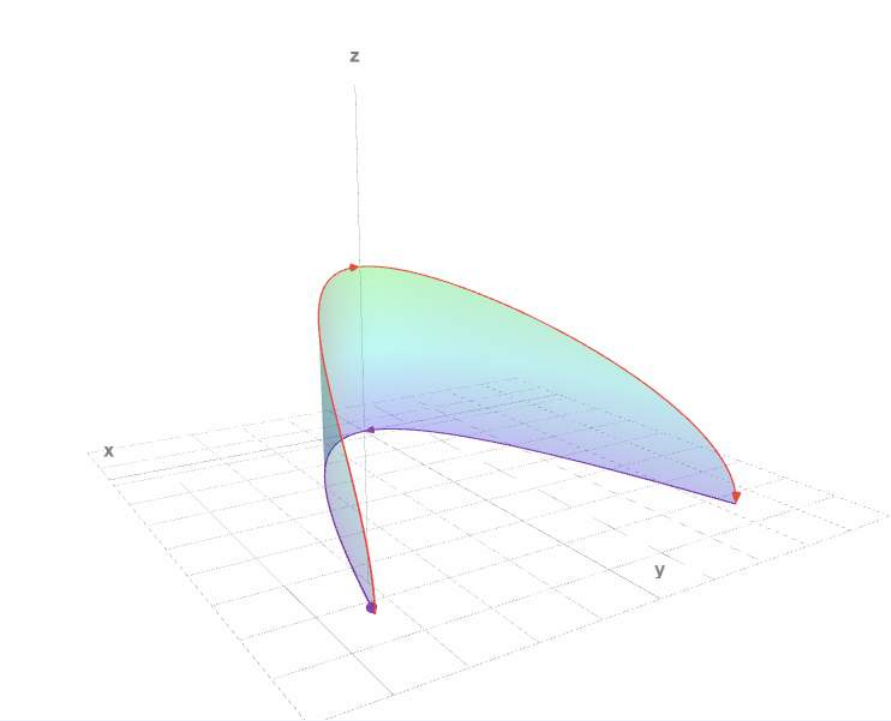
We can parameterize this path using $\vec{\mathbf{s}}(t) = \langle t, t^2, 0 \rangle$ for $-2 \leq t \leq 2$



We calculate the return line integral:

$$\begin{aligned}\int_{\vec{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} &= \int_{-2}^{2} \left(\left(2t^{17} + 3t^2 + t^2 \cdot 0\right) \cdot 1 + \left(-(t^2)^8\right) \cdot 2t + \left(t \cdot t^2 + (1+0)^0\right) \cdot 0 \right) \, dt \\ &= \int_{-2}^{2} 3t^2 \, dt \\ &= t^3 \Big|_{-2}^{2} \\ &= 16\end{aligned}$$

One surface with this as its boundary is $S = \left\{ (x, y, z) \, : \, \frac{x^2}{4} + z^2 \leq 1, \, z \geq 0, \, y = x^2 \right\}$



We can parameterize this surface using $\vec{\mathbf{r}}(u, v) = \langle 2u\cos(v), 4u^2\cos^2(v), u\sin(v) \rangle$ over $D = \{(u, v) \, : \, 0 \leq u \leq 1, \, 0 \leq v \leq \pi\}$

We calculate $\vec{\mathbf{r}}_u = \langle 2\cos(v), 8u\cos^2(v), \sin(v) \rangle$, $\vec{\mathbf{r}}_v = \langle -2u\sin(v), -8u\cos(v)\sin(v), u\cos(v) \rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 8u^2\cos(v), -2u, 0 \rangle$

This is clearly oriented toward the negative y-axis, since the y component is negative. This matches the orientation of C in regards to Stokes' Theorem, since the curve travels around the boundary of S counterclockwise when viewed from the negative y-axis.

This vector field has $\text{curl}(\vec{\mathbf{F}}) = \langle x, 0, -z \rangle$

Therefore, we can calculate the surface integral as:

$$\begin{aligned}\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} &= \iint_D \langle 2u\cos(v), 0, -u\sin(v) \rangle \cdot \langle 8u^2\cos(v), -2u, 0 \rangle \, dA \\ &= \int_0^1 \int_0^\pi 16u^3\cos^2(v) \, dv \, du \\ &= \left(\int_0^1 16u^3 \, du \right) \left(\int_0^\pi \cos^2(v) \, dv \right) \\ &= \left(4u^4 \Big|_0^1 \right) \cdot \left[\frac{1}{2}v + \frac{1}{4}\sin(2v) \right] \Big|_0^\pi \\ &= 4 \cdot \frac{\pi}{2} \\ &= 2\pi\end{aligned}$$

Stokes' Theorem says that $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{\tilde{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$

Therefore, our final answer is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2\pi - 16$

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4.4: The Divergence Theorem

Learning Objectives

- Explain the meaning of the divergence theorem.
- Use the divergence theorem to calculate the flux of a vector field.
- Apply the divergence theorem to find the volume of a region contained inside of a closed surface.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a "derivative" of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals or find volumes within surfaces.

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4.4: The Divergence Theorem

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

   This theorem relates the integral of derivative f' over line segment $[a, b]$ along the x-axis to a difference in f evaluated at the boundary points.

2. The Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(P_1) - f(P_0)$$

   where P_0 is the initial point of C and P_1 is the terminal point of C. The Fundamental Theorem for Line Integrals allows a curve C to be any path in a plane or in space, not just a line segment on the x-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative ∇f over path C to a difference of f evaluated on the boundary of C.

3. Green's theorem:

$$\iint_D (Q_x - P_y)\,dA = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

   Since $Q_x - P_y = \text{curl}\ (\vec{\mathbf{F}}) \cdot \hat{\mathbf{k}}$ and curl is a derivative-like operator, Green's theorem relates the integral of the derivative $\text{curl}(\vec{\mathbf{F}})$ over planar region D to an integral of $\vec{\mathbf{F}}$ over the boundary of D.

4. Stokes' theorem:

$$\iint_S \text{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

   If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative $\text{curl}(\vec{\mathbf{F}})$ over surface S (not necessarily planar) to an integral of $\vec{\mathbf{F}}$ over the boundary of S.

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative $\text{div}(\vec{\mathbf{F}})$ over a solid to a flux integral of $\vec{\mathbf{F}}$ over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field $\vec{\mathbf{F}}$ over a closed surface S to a triple integral of the divergence of $\vec{\mathbf{F}}$ over the solid enclosed by S.

Theorem 4.4.1: The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\vec{\mathbf{F}}$ be a vector field with continuous partial derivatives on an open region containing E. Then

$$\iiint_E \text{div}(\vec{\mathbf{F}})\,dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

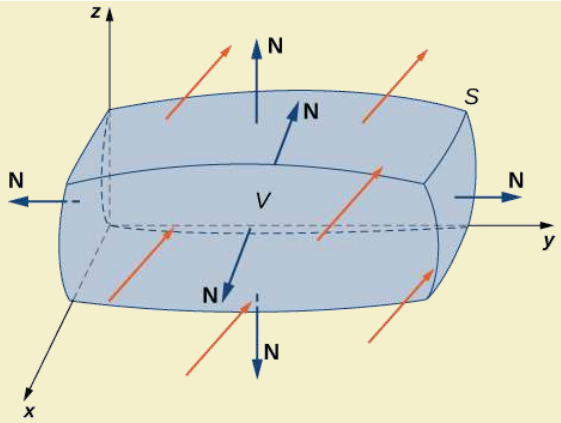


Figure 4.4.1: The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E. Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz. The normal vector out of the top of the box is $\hat{\mathbf{k}}$ and the normal vector out of the bottom of the box is $-\hat{\mathbf{k}}$. The dot product of $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ with $\hat{\mathbf{k}}$ is R and the dot product with $-\hat{\mathbf{k}}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x \Delta y$.

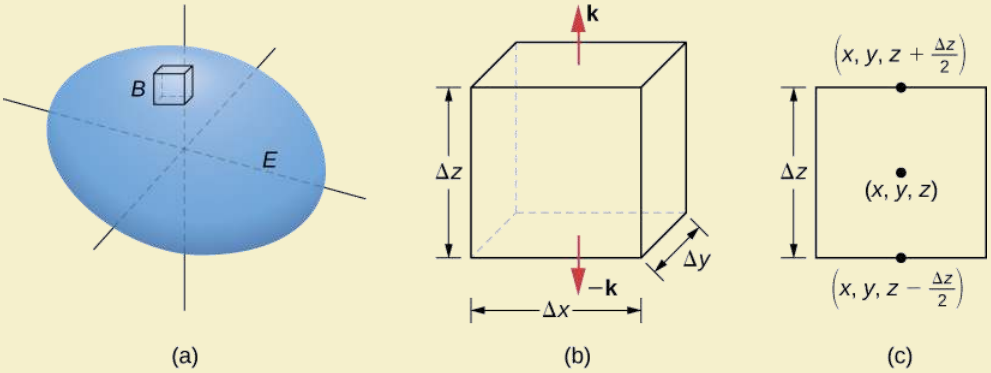


Figure 4.4.2: (a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengths Δx, Δy, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of $\frac{\Delta z}{2}$ up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance $\frac{\Delta z}{2}$ down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x,\, y,\, z + \dfrac{\Delta z}{2}\right) \Delta x\, \Delta y$ and the flux out of the bottom of the box is $-R\left(x,\, y,\, z - \dfrac{\Delta z}{2}\right) \Delta x\, \Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\, \Delta x\, \Delta y$. However,

$$\Delta R\, \Delta x\, \Delta y = \left(\frac{\Delta R}{\Delta z}\right) \Delta x\, \Delta y \Delta z \approx \left(\frac{\partial R}{\partial z}\right) \Delta V$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\dfrac{\partial R}{\partial z}\right) \Delta V$. Similarly, the net flux in the x-direction can be approximated by $\left(\dfrac{\partial P}{\partial x}\right) \Delta V$ and the net flux in the y-direction can be approximated by $\left(\dfrac{\partial Q}{\partial y}\right) \Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

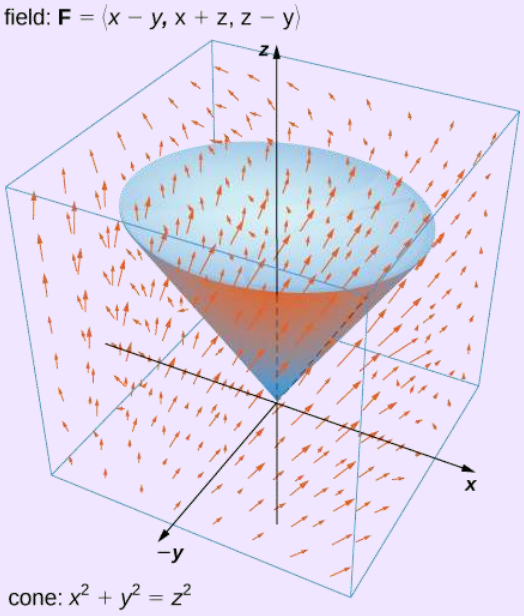
$$\text{Total flux} \approx \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Delta V = \operatorname{div}(\vec{\mathbf{F}})\,\Delta V$$

This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

The sum of $\operatorname{div}(\vec{\mathbf{F}})\,\Delta V$ over all the small boxes approximating E is approximately $\iiint_E \operatorname{div}(\vec{\mathbf{F}})\,dV$. On the other hand, the sum of $\operatorname{div}(\vec{\mathbf{F}})\,\Delta V$ over all the small boxes approximating E is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancellation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of E. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over S.

Illustration 4.4.1: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\vec{\mathbf{F}} = \langle x-y,\ x+z,\ z-y\rangle$ and surface S that consists of cone $x^2+y^2=z^2,\ 0\le z\le 1$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that

$$\iiint_E \operatorname{div}(\vec{\mathbf{F}})\,dV = \iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$$

by calculating each integral separately.

To compute the triple integral, note that $\operatorname{div}(\vec{\mathbf{F}}) = P_x + Q_y + R_z = 2$, and therefore the triple integral is

$$\iiint_E \operatorname{div}(\vec{\mathbf{F}})\,dV = \iiint_E 2dV = 2\cdot\text{Volume}(E) = 2\cdot\frac{\pi}{3}(1)^2(1) = \frac{2\pi}{3}$$

(Recall the volume of a right circular cone is given by $\frac{\pi}{3}r^2h$ and this cone has $h=r=1$)

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), 1\rangle \quad \text{for} \quad 0\le u\le 1,\ 0\le v\le 2\pi$$

Then, the tangent vectors are $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 0\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$. Thus $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, 0, u\rangle$

This has upward orientation, which is *outward* for our full surface S and therefore is positively oriented. Now we find the flux across S_1 is:

$$\begin{aligned}
\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^1 \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA \\
&= \int_0^1 \int_0^{2\pi} \langle u\cos(v) - u\sin(v), u\cos(v) + 1, 1 - u\sin(v)\rangle \cdot \langle 0, 0, u\rangle \, dv\, du \\
&= \int_0^1 \int_0^{2\pi} \left(u - u^2\sin(v)\right)\, dv\, du \\
&= \int_0^1 \left[uv + u^2\cos(v)\right]\Big|_0^{2\pi} du \\
&= \int_0^1 2\pi u\, du \\
&= \pi u^2\Big|_0^1 \\
&= \pi
\end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$\vec{\mathbf{r}}(u,v) = \langle u\cos(v), u\sin(v), u\rangle \quad \text{for} \quad 0 \le u \le 1,\ 0 \le v \le 2\pi$$

The tangent vectors are $\vec{\mathbf{r}}_u = \langle \cos(v),\ \sin(v),\ 1\rangle$ and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$, so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -u\cos(v), -u\sin(v), u\rangle$

This points upward, which is *inward* for our full surface S and therefore is negatively oriented. We multiply it by -1 to make it outward, and now we find the flux across S_2 is:

$$\begin{aligned}
\iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^1 \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot \left(-(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\right)\, dA \\
&= \int_0^1 \int_0^{2\pi} \langle u\cos(v) - u\sin(v), u\cos(v) + u, u - u\sin(v)\rangle \\
&\qquad \cdot \langle u\cos(v), u\sin(v), -u\rangle \, dv\, du \\
&= \int_0^1 \int_0^{2\pi} \left(u^2\cos^2(v) + 2u^2\sin(v) - u^2\right)\, dv\, du \\
&= \int_0^1 \int_0^{2\pi} \left(\frac{1}{2}u^2\cos(2v) + 2u^2\sin(v) - \frac{1}{2}u^2\right)\, dv\, du \\
&= \int_0^1 \left[\frac{1}{4}u^2\sin(2v) - 2u^2\cos(v) - \frac{1}{2}u^2 v\right]\Big|_0^{2\pi} du \\
&= \int_0^1 -\pi u^2\, du \\
&= -\frac{\pi}{3}u^3\Big|_0^1 \\
&= -\frac{\pi}{3}
\end{aligned}$$

We have found total flux across S is

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} + \iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \pi + \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3} = \iiint_E \operatorname{div}\left(\vec{\mathbf{F}}\right)\, dV$$

and we have verified the divergence theorem for this example.

Illustration 4.4.2: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\vec{\mathbf{F}} = \langle 2y^2, 3x^2, 5z^3 \rangle$ and surface S that consists of the unit sphere.

Let E be the solid unit sphere enclosed by S. To verify the theorem for this example, we show that

$$\iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

by calculating each integral separately.

To compute the triple integral, note that $\operatorname{div}(\vec{\mathbf{F}}) = P_x + Q_y + R_z = 15z^2$, and therefore the triple integral is

$$\begin{aligned}
\iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV &= \iiint_E 15z^2 dV \\
&= \int_0^1 \int_0^{\pi} \int_0^{2\pi} 15(\rho\cos(\phi))^2 \rho^2 \sin(\phi)\, d\theta\, d\phi\, d\rho \\
&= \int_0^1 \int_0^{\pi} \int_0^{2\pi} 15\rho^4 \cos^2(\phi)\sin(\phi)\, d\theta\, d\phi\, d\rho \\
&= \left(\int_0^1 15\rho^4 d\rho\right)\left(\int_0^{\pi} \cos^2(\phi)\sin(\phi)\, d\phi\right)\left(\int_0^{2\pi} d\theta\right) \\
&= \left(3\rho^5\Big|_0^1\right)\cdot\left(-\frac{1}{3}\cos^3(\phi)\Big|_0^{\pi}\right)\cdot\left(\theta\Big|_0^{2\pi}\right) \\
&= 3\cdot\left(\frac{2}{3}\right)\cdot 2\pi \\
&= 4\pi
\end{aligned}$$

To compute the flux integral, first note that S is smooth and can be parameterized using spherical coordinates (with $\rho = 1$) as:

$$\vec{\mathbf{r}}(u,v) = \langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u) \rangle \quad \text{for} \quad 0 \le u \le \pi,\ 0 \le v \le 2\pi$$

We find $\vec{\mathbf{r}}_u = \langle \cos(u)\cos(v), \cos(u)\sin(v), -\sin(u) \rangle$ and $\vec{\mathbf{r}}_v = \langle -\sin(u)\sin(v), \sin(u)\cos(v), 0 \rangle$

Next we calculate $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u) \rangle$

This has *outward orientation* for our surface S and therefore is positively oriented. Now we find the flux across S is:

$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^{\pi} \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA \\
&= \int_0^{\pi} \int_0^{2\pi} \langle 2\sin^2(u)\sin^2(v), 3\sin^2(u)\cos^2(v), 5\cos^3(u) \rangle \\
&\quad \cdot \langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u) \rangle\, dv\, du \\
&= \int_0^{\pi} \int_0^{2\pi} \left(2\sin^4(u)\sin^2(v)\cos(v) + 2\sin^4(u)\cos^2(v)\sin(v) + 5\cos^4(u)\sin(u)\right)\, dv\, du \\
&= \int_0^{\pi} \left[\frac{2}{3}\sin^4(u)\sin^3(v) - \frac{2}{3}\sin^4(u)\cos^3(v) + 5v\cos^4(u)\sin(u)\right]\Big|_0^{2\pi} du \\
&= \int_0^{\pi} 10\pi\cos^4(u)\sin(u)\, du \\
&= -2\pi\cos^5(u)\Big|_0^{\pi} \\
&= 4\pi
\end{aligned}$$

Once more, we have verified the divergence theorem holds in this example.

The divergence theorem relates the flux integral of closed surface S to a triple integral over the solid E which is enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 4.4.1: Applying the Divergence Theorem

If S is cylinder $x^2+y^2=1,\ 0\le z\le 2$, including the circular top and bottom, and $\vec{\mathbf{F}}=\left\langle \frac{x^3}{3}+yz,\ \frac{y^3}{3}-\sin(xz),\ z-x-y\right\rangle$, find

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$$

Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product.

By contrast, the divergence theorem allows us to calculate the single triple integral over the solid enclosed by the cylinder, which we denote as E.

In cylindrical coordinates, we have $E=\{(r\cos(\theta),r\sin(\theta),z)\ :\ 0\le r\le 1,\ 0\le\theta\le 2\pi,\ 0\le z\le 2\}$. Therefore

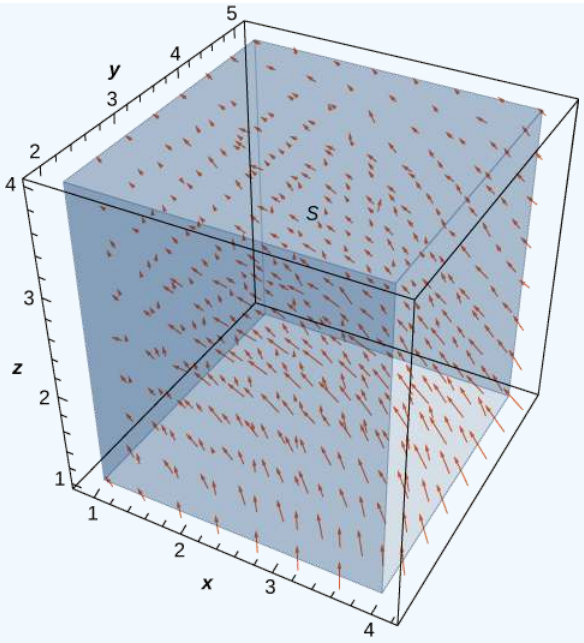
$$\begin{aligned}
\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\,dV \\
&= \iiint_E (x^2+y^2+1)\,dV \\
&= \int_0^{2\pi}\int_0^1\int_0^2 (r^2+1)\,r\,dz\,dr\,d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (r^2+1)\,r\,dr\right)\left(\int_0^2 dz\right) \\
&= \left(\theta\Big|_0^{2\pi}\right)\cdot\left(\frac{1}{4}(r^2+1)^2\Big|_0^1\right)\cdot\left(z\Big|_0^2\right) \\
&= 2\pi\cdot\frac{3}{4}\cdot 2 \\
&= 3\pi
\end{aligned}$$

Example 4.4.2: Applying the Divergence Theorem

Let $\vec{\mathbf{F}}=\left\langle -\frac{y}{z},\ \frac{x}{z},\ 0\right\rangle$ and $E=\{(x,y,z)\ :\ 1\le x\le 4,\ 2\le y\le 5,\ 1\le z\le 4\}$. If S is the boundary of this cube oriented outward, find the flux of $\vec{\mathbf{F}}$ on S.

Solution

The flux across the boundary of this cube requires six separate parameterizations and calculations.



Instead, we use the Divergence Theorem:

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV = \iiint_E 0\, dV = 0$$

Therefore the flux is zero.

The example above illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\vec{\mathbf{F}}$ be a vector field defined on an open region containing the surface enclosed by S. If $\vec{\mathbf{F}}$ has the form $F = \langle f(y,z),\ g(x,z),\ h(x,y)\rangle$, then the divergence of $\vec{\mathbf{F}}$ is zero. By the divergence theorem, the flux of $\vec{\mathbf{F}}$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where S is any closed surface and

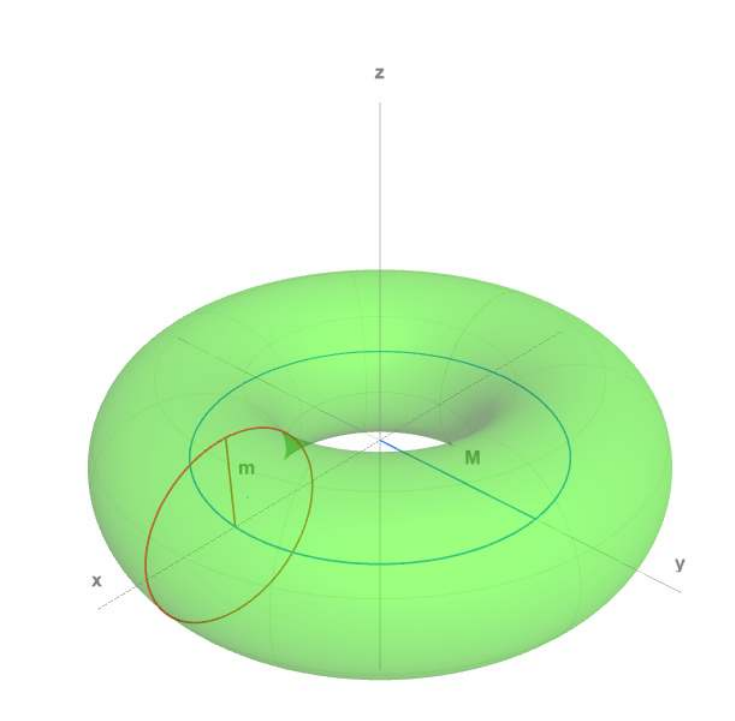
$$\vec{\mathbf{F}} = \langle \sin(y)\, e^{yz},\ x^2 z^2,\ \cos(xy)\, e^{\sin(x)} \rangle$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to parameterize a surface (or many surfaces, if we are unlucky). But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

Another way to consider this is in terms of Stokes' Theorem. Any vector field $\vec{\mathbf{F}}$ with a divergence of zero can be written as the curl of some $\vec{\mathbf{G}}$. If S is a closed surface, then its "boundary" can be considered to be empty (or a single point), which means the line integral along this boundary should be zero.

Illustration 4.4.3: Finding Volume through the Divergence Theorem

Find the volume of the region contained inside of the torus with major radius M and minor radius m (where $M > m$)



The volume of any region E can be represented by a triple integral $\iiint_E 1\,dV$

Unfortunately, this region is incredibly difficult to describe in rectangular, cylindrical, and spherical coordinates. Instead, we look to parameterize the surface of E and use the Divergence Theorem!

We parameterize using $\vec{\mathbf{r}}(u,v) = \langle (m\cos(v)+M)\cos(u), (m\cos(v)+M)\sin(u), m\sin(v)\rangle$ for $0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

$$\vec{\mathbf{r}}_u = \langle -(m\cos(v)+M)\sin(u), (m\cos(v)+M)\cos(u), 0\rangle \quad \text{and}$$
$$\vec{\mathbf{r}}_v = \langle -m\sin(v)\cos(u), -m\sin(v)\sin(u), m\cos(v)\rangle$$

The cross product of these vectors is therefore:

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$$
$$= \langle m(m\cos(v)+M)\cos(v)\cos(u), m(m\cos(v)+M)\cos(v)\sin(u), m(m\cos(v)+M)\sin(v)\rangle$$

Plugging in $u = 0, v = 0$ we see that at $\vec{\mathbf{r}}(0,0) = (M+m, 0, 0)$ we have $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle m(m+M), 0, 0\rangle$, so this surface is parameterized *outward*, as needed for the Divergence Theorem. Next we need to find a vector field $\vec{\mathbf{F}}$ such that $\operatorname{div}(\vec{\mathbf{F}}) = 1$, and $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ is reasonable.

Looking at the normal vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$, the easiest choice should be $\vec{\mathbf{F}} = \langle 0, 0, z\rangle$:

$$\begin{aligned}
\iiint_E 1\,dV &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\,dV \\
&= \iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} \\
&= \int_0^{2\pi}\int_0^{2\pi} \langle 0,0,m\sin(v)\rangle \cdot \langle m(m\cos(v)+M)\cos(v)\cos(u), \\
&\quad\ m(m\cos(v)+M)\cos(v)\sin(u), m(m\cos(v)+M)\sin(v)\rangle\,dA \\
&= \int_0^{2\pi}\int_0^{2\pi} m^2(m\cos(v)+M)\sin^2(v)\,du\,dv \\
&= \left(\int_0^{2\pi} m^2\,du\right)\left(\int_0^{2\pi}\left(m\sin^2(v)\cos(v)+M\left(\frac{1}{2}-\frac{1}{2}\cos(2v)\right)\right)\,dv\right) \\
&= \left(m^2u\Big|_0^{2\pi}\right)\cdot\left[\frac{m}{3}\sin^3(v)+\frac{M}{2}\left(v-\frac{1}{2}\sin(2v)\right)\right]\Bigg|_0^{2\pi} \\
&= 2\pi m^2\cdot M\pi \\
&= 2Mm^2\pi^2
\end{aligned}$$

Thus the volume of a torus is $2Mm^2\pi^2$. Notice this can be written as the product of the area of the smaller circles (πm^2) and the circumference of the larger circle $2\pi M$. Intuitively, the torus is simply a circular cylinder of radius m and height $2\pi M$ bent around so that its top and bottom are touching.

Illustration 4.4.4: A Stubborn Torus

Let $M>0$ be a constant, and $g(t), f(t)$ be functions bounded by $0\le g(t)\le M$ and $f(t)\ge 0$, which satisfy $g(0)=g(2\pi)$ and $f(0)=f(2\pi)$. Find the volume of the region contained inside of the surface parameterized by

$$\vec{\mathbf{r}}(u,v)=\langle (g(u)\cos(v)+M)\cos(u),(g(u)\cos(v)+M)\sin(u),f(u)\sin(v)\rangle \quad \text{for} \quad 0\le u\le 2\pi,\ 0\le v\le 2\pi$$

This region is similar to a torus, but the smaller circular cross sections are replaced by ellipses with varying height and width.

Again, this region is incredibly difficult to describe in rectangular, cylindrical, and spherical coordinates. We find the tangent vectors:

$$\begin{aligned}
\vec{\mathbf{r}}_u &= \langle g'(u)\cos(v)\cos(u)-(g(u)\cos(v)+M)\sin(u), \\
&\quad\ g'(u)\cos(v)\sin(u)+(g(u)\cos(v)+M)\cos(u),f'(u)\sin(v)\rangle \\
\vec{\mathbf{r}}_v &= \langle -g(u)\sin(v)\cos(u),-g(u)\sin(v)\sin(u),f(u)\cos(v)\rangle
\end{aligned}$$

The cross product of these vectors is therefore:

$$\begin{aligned}
\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v &= \left[g'(u)f(u)\cos^2(v)\sin(u)+g(u)f(u)\cos^2(v)\cos(u)+Mf(u)\cos(v)\cos(u)\right. \\
&\quad \left.+g(u)f'(u)\sin^2(v)\sin(u)\right]\hat{\mathbf{i}} \\
&\quad -\left[g'(u)f(u)\cos^2(v)\cos(u)-g(u)f(u)\cos^2(v)\sin(u)-Mf(u)\cos(v)\sin(u)\right. \\
&\quad \left.+g(u)f'(u)\sin^2(v)\cos(u)\right]\hat{\mathbf{j}} \\
&\quad +\left[g(u)(g(u)\cos(v)+M)\sin(v)\right]\hat{\mathbf{k}}
\end{aligned}$$

Once again, this surface is parameterized *outward*, as needed for the Divergence Theorem.

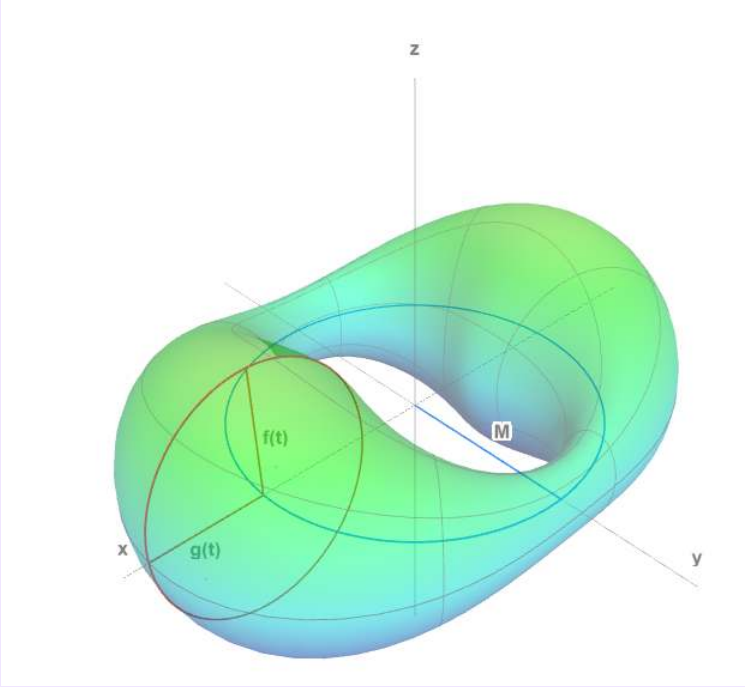
Next, we need to find a vector field $\vec{\mathbf{F}}$ such that $\operatorname{div}(\vec{\mathbf{F}})=1$, and $\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$ is reasonable.

Looking at the normal vector $\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v$, the obvious choice should again be $\vec{\mathbf{F}}=\langle 0,0,z\rangle$:

$$
\begin{aligned}
\text{Volume} &= \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \\
&= \int_0^{2\pi} \int_0^{2\pi} \langle 0, 0, f(u)\sin(v)\rangle \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA \\
&= \int_0^{2\pi} \int_0^{2\pi} f(u)g(u)(g(u)\cos(v) + M)\sin^2(v)\, dv\, du \\
&= \int_0^{2\pi} \int_0^{2\pi} \left(f(u)\left[g(u)\right]^2 \sin^2(v)\cos(v) + Mf(u)g(u)\left(\frac{1}{2} - \frac{1}{2}\cos(2v)\right)\right) dv\, du \\
&= \int_0^{2\pi} \left[\frac{1}{3}f(u)\left[g(u)\right]^2 \sin^3(v) + Mf(u)g(u)\left(\frac{1}{2}v - \frac{1}{4}\sin(2v)\right)\right]\Bigg|_0^{2\pi} du \\
&= \int_0^{2\pi} M\pi f(u)g(u)\, du
\end{aligned}
$$

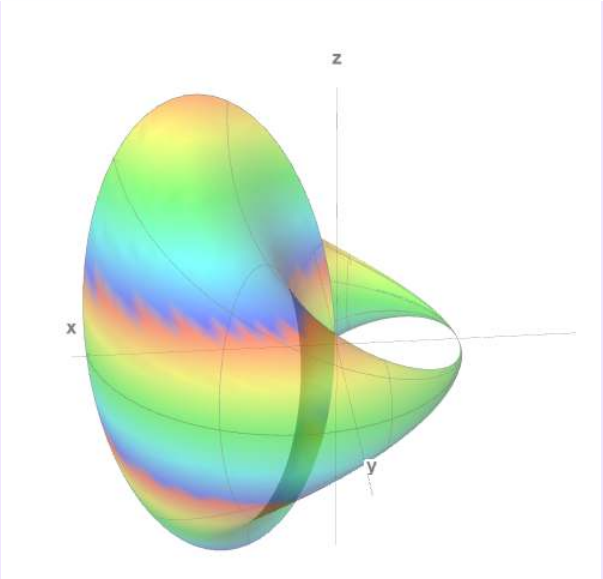
Notice that in our previous example, the torus has $h(t) = g(t) = m$, which matches the result we would get from this more general formula.

In the case where $M = 5$, $f(t) = \cos(2t) + 3$ and $g(t) = \cos(2t) + 2$, we get a shape like the one below:



And since $\int_0^{2\pi} f(u)g(u)\, du = \int_0^{2\pi} \left(\cos^2(2u) + 5\cos(2u) + 6\right)\, du = 13\pi$, we know the volume of this solid is $V = 65\pi^2$

If instead we use $M = 10$, $f(t) = 2(t-\pi)^2$ and $g(t) = (t-\pi)^2$ the shape looks like:



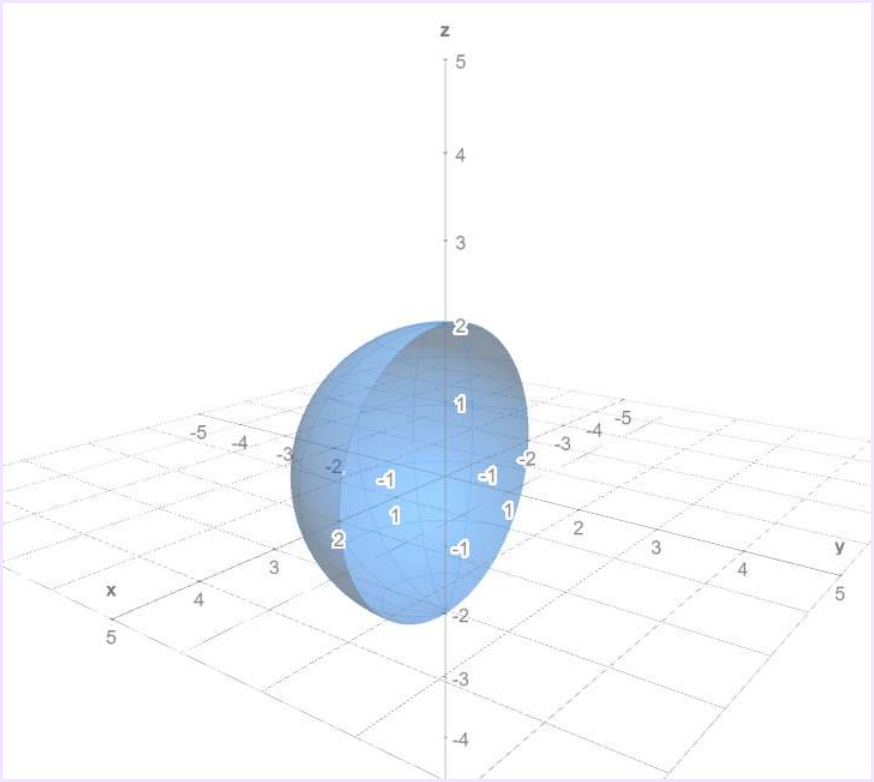
And since $\int_0^{2\pi} f(u)g(u)\,du = \int_0^{2\pi} 2(u-\pi)^4\,du = \frac{4\pi^5}{5}$, we know the volume of this solid is $V = 8\pi^6$

It is important to recognize that the Divergence Theorem only applies to closed surfaces. If a surface S is not closed, however, we can always close it by adding on additional surfaces.

Illustration 4.4.5: Closing a Surface to Use Divergence Theorem

Find the flux of $\vec{\mathbf{F}} = \langle e^{y^2}, 1, 3yz\rangle$ over the hemisphere $S = \{(x,y,z) : x^2+y^2+z^2 = 4,\ y \le 0\}$ oriented outward.

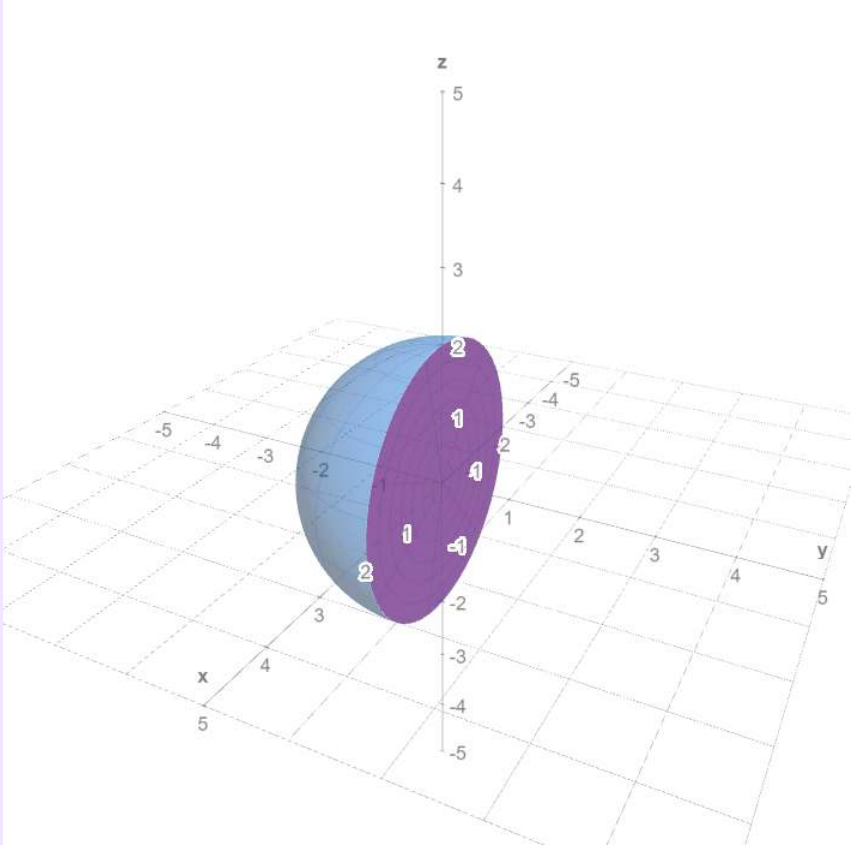
While we can parameterize S a few different ways, none of them yield a surface integral that is pleasant to solve directly.



Since S is not closed, we cannot immediately use the Divergence Theorem to avoid this flux integral. We remedy this by adding on a surface S_1 to close the object. As long as it is oriented outwards, if we denote E as the region between them, we will have:

$$\iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} + \iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

The simplest such surface is a flat disc on the $y = 0$ plane.



This surface is easy to parameterize, we can use $\vec{\mathbf{r}}(u, v) = \langle u, 0, v\rangle$ over $D = \{(u, v) : u^2 + v^2 \leq 4\}$

To find the flux over S_1 we first find its tangent vectors: $\vec{\mathbf{r}}_u = \hat{\mathbf{i}}$ and $\vec{\mathbf{r}}_v = \hat{\mathbf{k}}$

Therefore we have $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = -\hat{\mathbf{j}}$. Note that this points *inward*, so we will replace it with $\hat{\mathbf{j}}$:

$$\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \langle 1, 1, 0\rangle \cdot \hat{\mathbf{j}}\, dA = \iint_D 1\, dA = \operatorname{area}(D) = 4\pi$$

The region between these surfaces is $E = \{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) : 0 \leq \rho \leq 2,\ 0 \leq \phi \leq \pi,\ \pi \leq \theta \leq 2\pi\}$

It is easy to calculate $\operatorname{div}(\vec{\mathbf{F}})) = 3y$, so our triple integral becomes:

$$\begin{aligned}\iiint_E \operatorname{div}(\vec{\mathbf{F}})\,dV &= \iiint_E 3y\,dV \\ &= \int_0^2 \int_0^{\pi} \int_{\pi}^{2\pi} 3\rho\sin(\phi)\sin(\theta)\,\rho^2\sin(\phi)\,d\theta\,d\phi\,d\rho \\ &= \left(\int_0^2 3\rho^3\,d\rho\right)\left(\int_0^{\pi}\sin^2(\phi)\,d\phi\right)\left(\int_{\pi}^{2\pi}\sin(\theta)\,d\theta\right) \\ &= \left(\frac{3}{4}\rho^4\Big|_0^2\right)\cdot\left[\frac{1}{2}\phi-\frac{1}{4}\sin(2\phi)\right]\Big|_0^{\pi}\cdot\left(-\cos(\theta)\Big|_{\pi}^{2\pi}\right) \\ &= 12\cdot\frac{\pi}{2}\cdot(-2) = -12\pi\end{aligned}$$

Putting these all together, we see that

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \iiint_E \operatorname{div}(\vec{\mathbf{F}})\,dV - \iint_{S_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = -12\pi - 4\pi = -16\pi$$

The Divergence Theorem is a powerful tool that helps us to avoid many undesirable flux integral calculations. It acts as the capstone theorem of this text and demonstrates that the Fundamental Theorem of Calculus generalizes to many forms (as we summarized in the Overview of Theorems above). While integrals may come in many forms in multivariable calculus, we see that the "right kind" of derivative can cancel out with one layer of the "right kind" of integral in almost every one of them.

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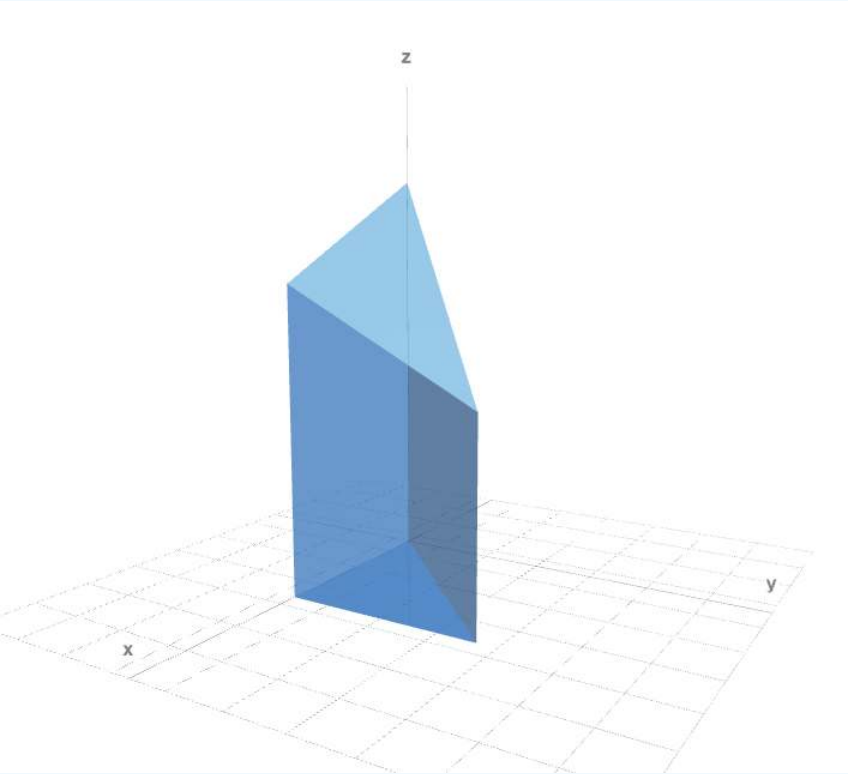
4.4: Practice Set

Exercise 4.4.1a

Find the flux of $\vec{\mathbf{F}} = \langle x+y+z, y-x, x-2y \rangle$ over $S = \partial E$ for $E = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 4-x-y\}$ oriented outward.

Answer

This S is defined as the five planes that make up the boundary of $E = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 4-x-y\}$



Instead of parameterizing each face separately, we find the divergence of $\vec{\mathbf{F}}$:

$$\operatorname{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}(x+y+z) + \frac{\partial}{\partial y}(y-x) + \frac{\partial}{\partial z}(x-2y) = 2$$

Therefore, by the Divergence Theorem we know:

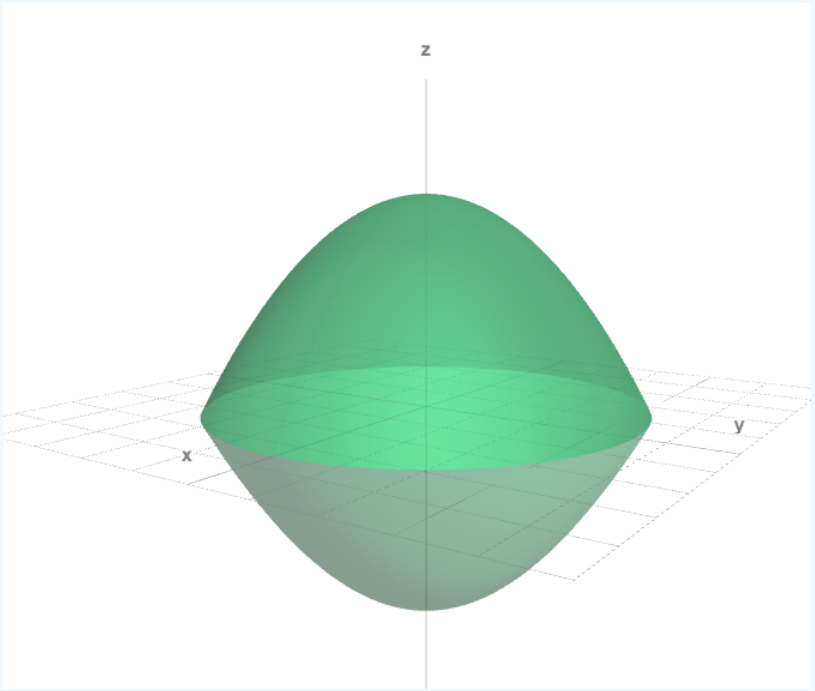
$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV \\
&= \int_0^1 \int_0^x \int_0^{4-x-y} 2\, dz\, dy\, dx \\
&= \int_0^1 \int_0^x 2z \Big|_0^{4-x-y} dy\, dx \\
&= \int_0^1 \int_0^x (8 - 2x - 2y)\, dy\, dx \\
&= \int_0^1 \left[8y - 2xy - y^2\right] \Big|_0^x dx \\
&= \int_0^1 (8x - 3x^2)\, dx \\
&= \left[4x^2 - x^3\right] \Big|_0^1 \\
&= 3
\end{aligned}$$

Exercise 4.4.1*b*

Find the flux of $\vec{\mathbf{F}} = \langle x^3, y^3, xyz \rangle$ over $S = \{(x, y, z) : z = 2 - x^2 - y^2,\ 0 \le z \le 2\} \cup \{(x, y, z) : z = x^2 + y^2 - 2,\ -2 \le z \le 0\}$ oriented outward.

Answer

This S is defined as two paraboloids facing toward each other. Together they form the boundary of the solid region E
$= \{(x, y, z) : x^2 + y^2 \le 2,\ x^2 + y^2 - 2 \le z \le 2 - x^2 - y^2\}$
$= \left\{(r\cos(\theta), r\sin(\theta), z) : 0 \le r \le \sqrt{2},\ 0 \le \theta \le 2\pi,\ r^2 - 2 \le z \le 2 - r^2\right\}$



Instead of parameterizing each paraboloid separately, we find the divergence of $\vec{\mathbf{F}}$:

$$\operatorname{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(xyz\right) = 3x^2 + 3y^2 + xy$$

Therefore, by the Divergence Theorem we know:

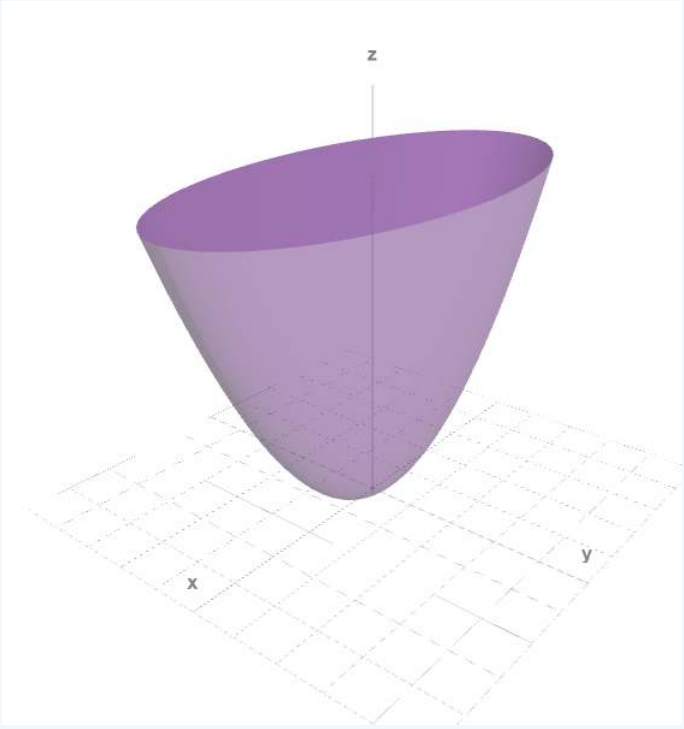
$$
\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV \\
&= \iiint_E (3x^2 + 3y^2 + xy)\, dV \\
&= \int_0^{\sqrt{2}} \int_0^{2\pi} \int_{r^2-2}^{2-r^2} \left(3r^2 + r^2\cos(\theta)\sin(\theta)\right) \cdot r\, dz\, d\theta\, dr \\
&= \int_0^{\sqrt{2}} \int_0^{2\pi} r^3(3 + \cos(\theta)\sin(\theta)) z \Big|_{r^2-2}^{2-r^2} d\theta\, dr \\
&= \int_0^{\sqrt{2}} \int_0^{2\pi} r^3(3 + \cos(\theta)\sin(\theta))(4 - 2r^2)\, d\theta\, dr \\
&= \left(\int_0^{\sqrt{2}} (4r^3 - 2r^5)\, dr\right)\left(\int_0^{2\pi} (3 + \cos(\theta)\sin(\theta))\, d\theta\right) \\
&= \left[r^4 - \frac{1}{3}r^6\right]\Big|_0^{\sqrt{2}} \cdot \left[3\theta + \frac{1}{2}\sin^2(\theta)\right]\Big|_0^{2\pi} \\
&= \frac{4}{3} \cdot 6\pi \\
&= 8\pi
\end{aligned}
$$

? Exercise 4.4.1c

Find the flux of $\vec{\mathbf{F}} = \langle yz^2, xz^3, x^2 + 2y^2 + 3z^2 \rangle$ over $S = \{(x, y, z) : z = x^2 + 4y^2,\ z \le 4\} \cup \{(x, y, z) \mid x^2 + 4y^2 \le 4,\ z = 4\}$ oriented outward.

Answer

This S is defined as an elliptical paraboloid closed by a flat disc. Together they form the boundary of $E = \{(x, y, z) : x^2 + 4y^2 \le 4,\ x^2 + 4y^2 \le z \le 4\}$



Instead of parameterizing each face separately, we find the divergence of $\vec{\mathbf{F}}$:

$$\text{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}\left(yz^2\right) + \frac{\partial}{\partial y}\left(xz^3\right) + \frac{\partial}{\partial z}\left(x^2 + 2y^2 + 3z^2\right) = 6z$$

A change of variables using $x = 2u\cos(v), y = u\sin(v), z = w$ changes E into $\tilde{E} = \{(u, v, w) : 0 \le u \le 1,\ 0 \le v \le 2\pi,\ 4u^2 \le w \le 4\}$

The Jacobian of this transformation is $J(u, v, w) = \begin{vmatrix} 2\cos(v) & -2u\sin(v) & 0 \\ \sin(v) & u\cos(v) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2u$

Therefore, by the Divergence Theorem we know:

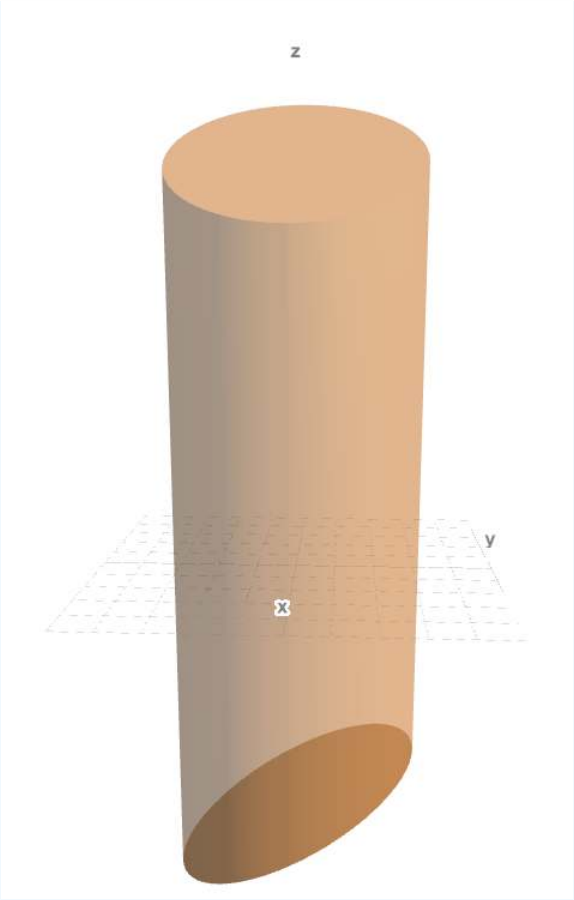
$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \text{div}(\vec{\mathbf{F}})\, dV \\
&= \iiint_E 6z\, dV \\
&= \iiint_{\tilde{E}} 6w \cdot |2u|\, dV \\
&= \int_0^1 \int_0^{2\pi} \int_{4u^2}^4 12uw\, dw\, dv\, du \\
&= \int_0^1 \int_0^{2\pi} 6uw^2 \Big|_{4u^2}^4 dv\, du \\
&= \int_0^1 \int_0^{2\pi} (96u - 96u^5)\, dv\, du \\
&= \left(\int_0^1 (96u - 96u^5)\, du\right)\left(\int_0^{2\pi} dv\right) \\
&= \left[48u^2 - 16u^6\right]\Big|_0^1 \cdot \left(v\Big|_0^{2\pi}\right) \\
&= 32 \cdot 2\pi \\
&= 64\pi
\end{aligned}$$

? Exercise 4.4.1d

Find the flux of $\vec{\mathbf{F}} = \langle 2x + y, 2y + z, 2z + x\rangle$ over $S = \partial E$ for $E = \{(x, y, z) : x^2 + y^2 \le 25,\ y - 20 \le z \le 30 - x\}$ oriented outward.

Answer

This S is defined as the two planes and cylinder that make up the boundary of the cylindrical solid $E = \{(x, y, z) : x^2 + y^2 \le 25,\ y - 20 \le z \le 30 - x\}$
$= \{(r\cos(\theta), r\sin(\theta), z) : 0 \le r \le 5,\ 0 \le \theta \le 2\pi,\ r\sin(\theta) - 20 \le z \le 30 - r\cos(\theta)\}$



Instead of parameterizing each face and the sides separately, we find the divergence of $\vec{\mathbf{F}}$:

$$\operatorname{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}(2x+y) + \frac{\partial}{\partial y}(2y+z) + \frac{\partial}{\partial z}(2z+x) = 6$$

Therefore, by the Divergence Theorem we know:

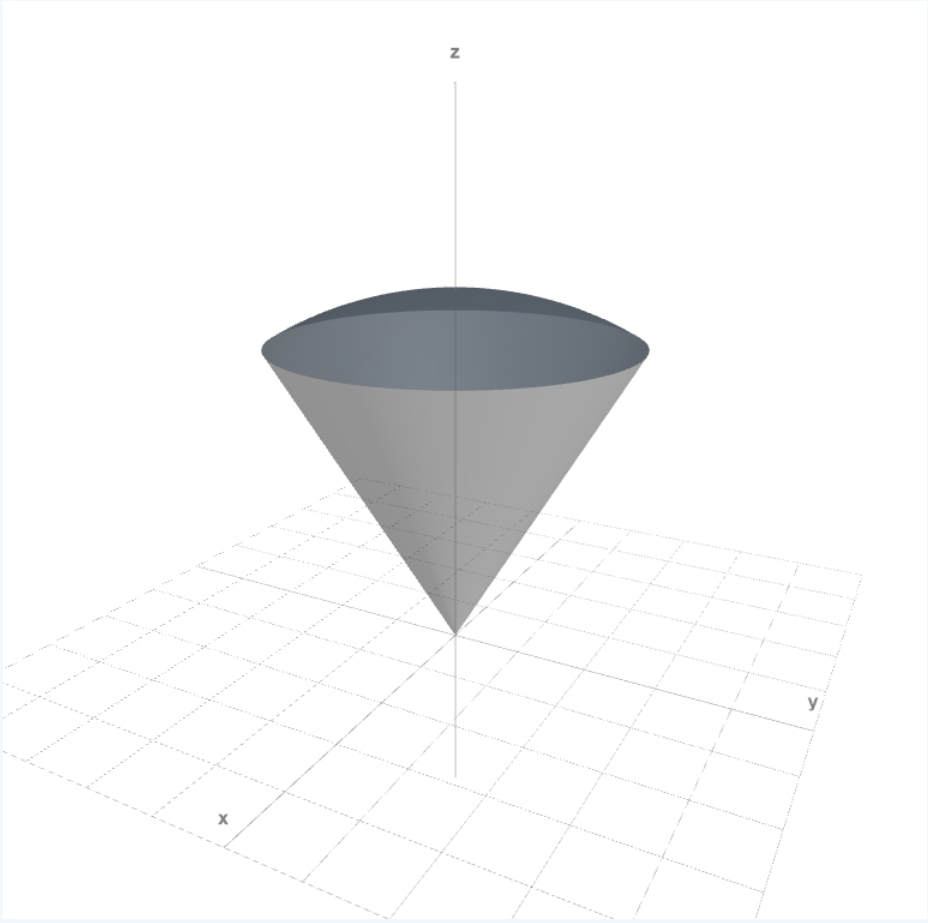
$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV \\
&= \iiint_E 6\, dV \\
&= \int_0^5 \int_0^{2\pi} \int_{r\sin(\theta)-20}^{30-r\cos(\theta)} 6 \cdot r\, dz\, d\theta\, dr \\
&= \int_0^5 \int_0^{2\pi} 6rz \Big|_{r\sin(\theta)-20}^{30-r\cos(\theta)} d\theta\, dr \\
&= \int_0^5 \int_0^{2\pi} \left(300r - 6r^2\cos(\theta) - 6r^2\sin(\theta)\right) d\theta\, dr \\
&= \int_0^5 \left[300r\theta - 6r^2\sin(\theta) + 6r^2\cos(\theta)\right] \Big|_0^{2\pi} dr \\
&= \int_0^5 600\pi r\, dr \\
&= 300\pi r^2 \Big|_0^5 \\
&= 7500\pi
\end{aligned}$$

Exercise 4.4.1e

Find the flux of $\vec{\mathbf{F}} = \langle x+y, x-y, 5z^3 \rangle$ over $S = \left\{(x,y,z) : z = \sqrt{3x^2+3y^2},\, z \le \sqrt{3}\right\} \cup \left\{(x,y,z) : z = \sqrt{4-x^2-y^2},\, z \ge \sqrt{3}\right\}$ oriented outward.

Answer

This S is defined as the cone and sphere that make up the boundary of the solid E
$= \left\{(x,y,z) : x^2+y^2+z^2 \le 4,\, \sqrt{3x^2+3y^2} \le z\right\}$
$= \left\{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) : 0 \le \rho \le 2,\, 0 \le \theta \le 2\pi,\, 0 \le \phi \le \frac{\pi}{6}\right\}$



Instead of parameterizing each face and the sides separately, we find the divergence of $\vec{\mathbf{F}}$:

$$\text{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(x-y) + \frac{\partial}{\partial z}(5z^3) = 15z^2$$

Therefore, by the Divergence Theorem we know:

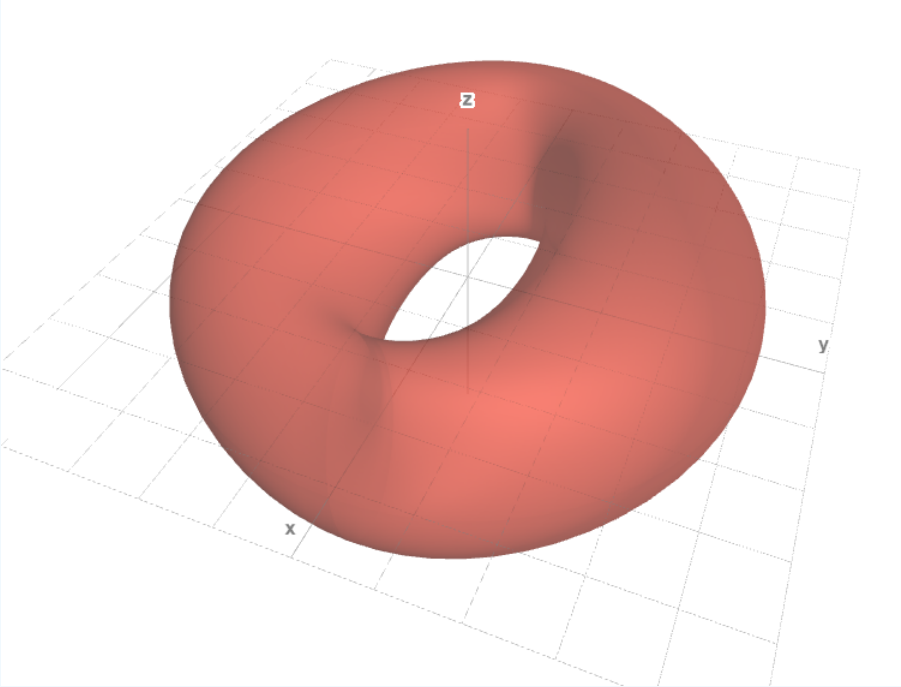
$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV \\
&= \iiint_E 15z^2\, dV \\
&= \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{6}} 15(\rho\cos(\phi))^2 \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho \\
&= \left(\int_0^2 15\rho^4\, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{6}} \cos^2(\phi)\sin(\phi)\, d\phi\right) \\
&= \left(3\rho^5\Big|_0^2\right)\cdot\left(\theta\Big|_0^{2\pi}\right)\cdot\left(-\frac{1}{3}\cos^3(\phi)\Big|_0^{\frac{\pi}{6}}\right) \\
&= 96\cdot 2\pi\cdot\left(\frac{1}{3}-\frac{\sqrt{3}}{8}\right) \\
&= 8\pi\left(8-3\sqrt{3}\right)
\end{aligned}$$

Exercise 4.4.1f

Find the flux of $\vec{\mathbf{F}} = \langle y^7, z^8, x^9\rangle$ over S parameterized by $\vec{\mathbf{r}}(u,v) = \langle(4+\cos(v))\cos(u), (3+2\cos(2v))\sin(u), 2\sin(2v)\rangle$ with $0 \le u \le 2\pi$, $0 \le v \le \pi$ oriented outward.

Answer

This surface S forms something like a torus. The region E that it encloses is not easy to describe in any coordinate system.



We are given the parameterization of S, but the direct calculation of this flux seems intimidating. Thankfully, the divergence of $\vec{\mathbf{F}}$ is:

$$\operatorname{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}\left(y^7\right) + \frac{\partial}{\partial y}\left(z^8\right) + \frac{\partial}{\partial z}\left(x^9\right) = 0$$

Therefore, by the Divergence Theorem we know:

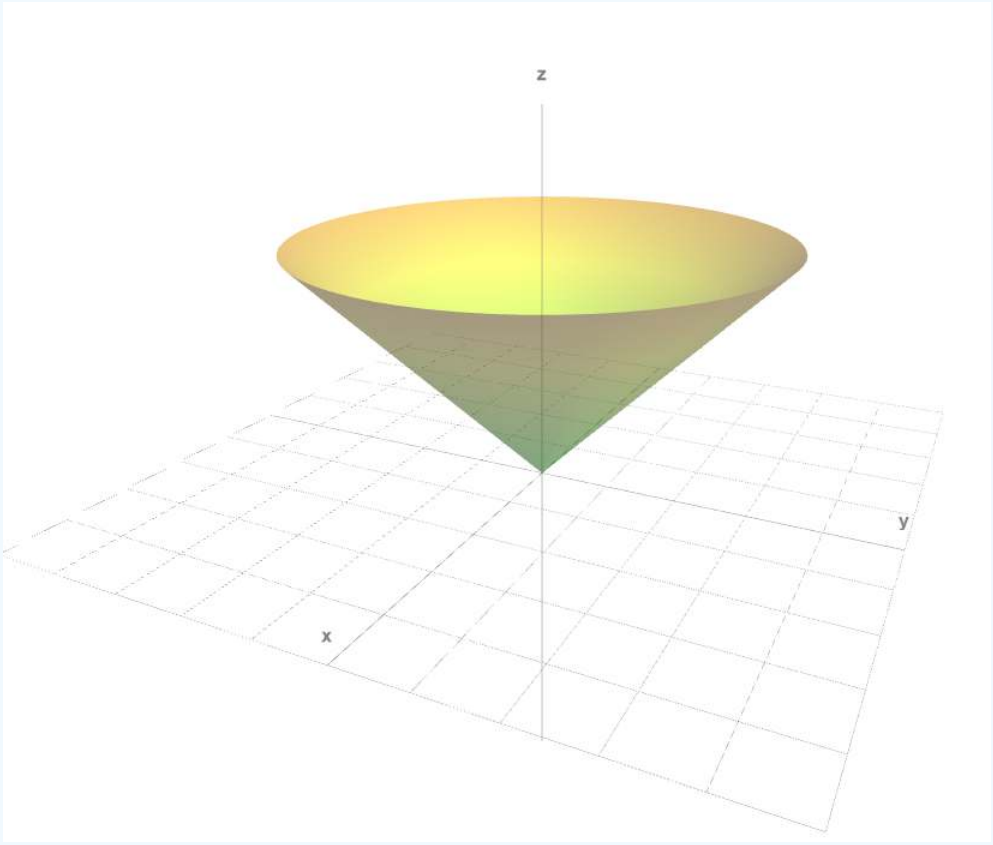
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV = \iiint_E 0\, dV = 0$$

Exercise $4.4.2a$

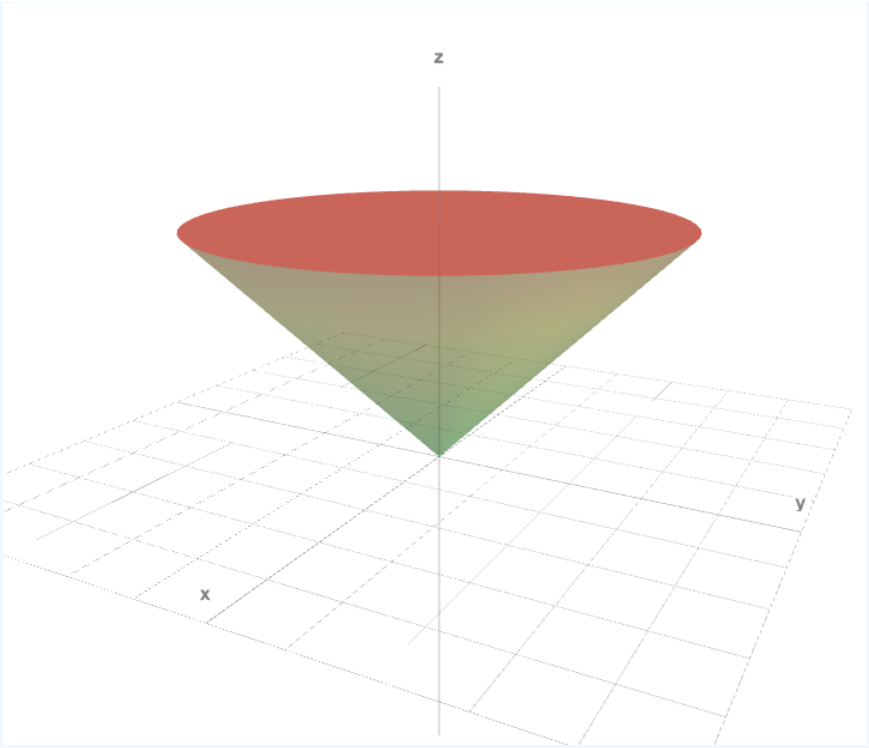
Find the flux of $\vec{\mathbf{F}} = \left\langle 12x^3y^3 + e^{z^2}, \cos(x^3) - 9x^2y^4, 3z \right\rangle$ over $S = \left\{(x, y, z) \,:\, z = \sqrt{x^2 + y^2}, 0 \leq z \leq 2\right\}$ oriented downward.

Answer

The given surface S is a cone, but we notice that it is **not closed**.



In order to apply the Divergence Theorem, we must add on a surface which closes it. For simplicity, we use the flat disc $\tilde{S} = \{(x, y, z) \,:\, x^2 + y^2 \leq 4,\ z = 2\}$



We find the flux of $\vec{\mathbf{F}}$ across this disc oriented *upward* (so that together $S \cup \tilde{S}$ is oriented *outward*).

We can parameterize this disc using $\vec{\mathbf{r}}(u,v) = \langle u, v, 2\rangle$ over $D = \{(u,v) \,:\, u^2 + v^2 \leq 4\}$

This gives $\vec{\mathbf{r}}_u = \langle 1,0,0\rangle$, $\vec{\mathbf{r}}_v = \langle 0,1,0\rangle$ and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0,0,1\rangle$

This is indeed oriented upward as needed, so the flux across $\tilde{S}$ is given by:

$$\begin{aligned}
\iint_{\tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle 12u^3v^3 + e^4, \cos(u^3) - 9u^2v^4, 3\cdot 2\right\rangle \cdot \langle 0,0,1\rangle \, dA \\
&= \iint_D 6 \, dA \\
&= 6 \cdot \text{Area}(D) \\
&= 6 \cdot \left(\pi \cdot 2^2\right) \\
&= 24\pi
\end{aligned}$$

Now that the surface $S \cup \tilde{S}$ is closed and oriented outward, we find the divergence of $\vec{\mathbf{F}}$:

$$\text{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}\left(12x^3y^3 + e^{z^2}\right) + \frac{\partial}{\partial y}\left(\cos(x^3) - 9x^2y^4\right) + \frac{\partial}{\partial z}(3z) = 3$$

Letting $E = \left\{(x,y,z) \,:\, x^2 + y^2 \leq 4,\ \sqrt{x^2+y^2} \leq z \leq 2\right\}$ be the enclosed region, we use the Divergence Theorem to calculate:

$$\begin{aligned}
\iint_{S \cup \tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \text{div}(\vec{\mathbf{F}}) \, dV \\
&= \iiint_E 3 \, dV \\
&= 3 \cdot \text{Volume}(E) \\
&= 3 \cdot \left(\frac{1}{3}\pi \cdot 2^2 \cdot 2\right) \\
&= 8\pi
\end{aligned}$$

(Note that we used the volume of a cone formula, $V_{cone} = \frac{1}{3}\pi r^2 h$ above)

Putting this together we get our final answer:

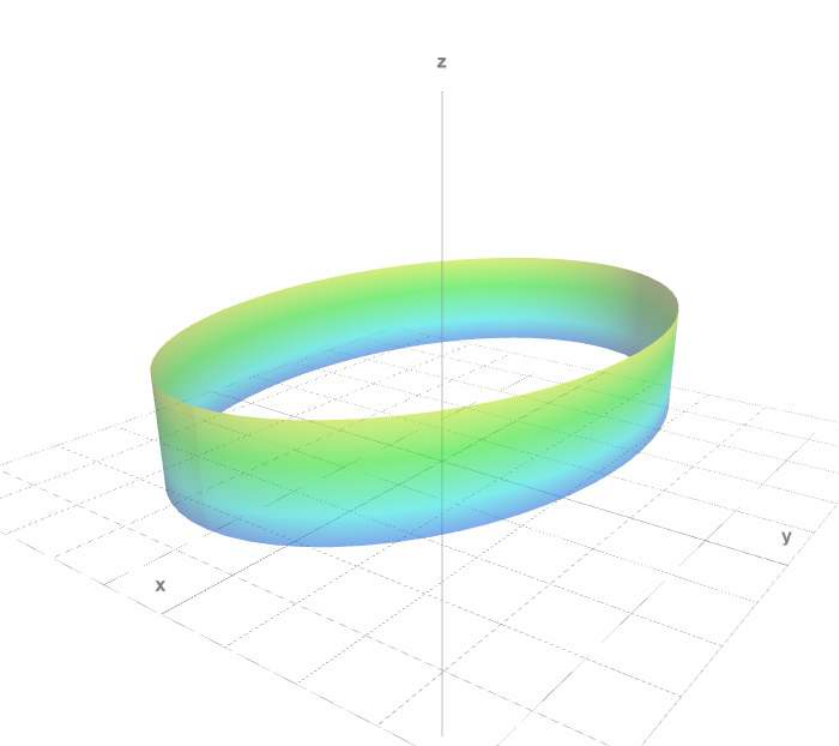
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S \cup \tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} - \iint_{\tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 8\pi - 24\pi = -16\pi$$

Exercise 4.4.2*b*

Find the flux of $\vec{\mathbf{F}} = \left\langle \frac{1}{4}x^4 + \frac{1}{5}y^3 - z, 3x^5 - 2y^2 + z, z^2 \right\rangle$ over $S = \left\{ (x, y, z) \, : \, \frac{x^2}{36} + \frac{y^2}{16} = 1, \, 1 \leq z \leq 3 \right\}$ oriented outward.

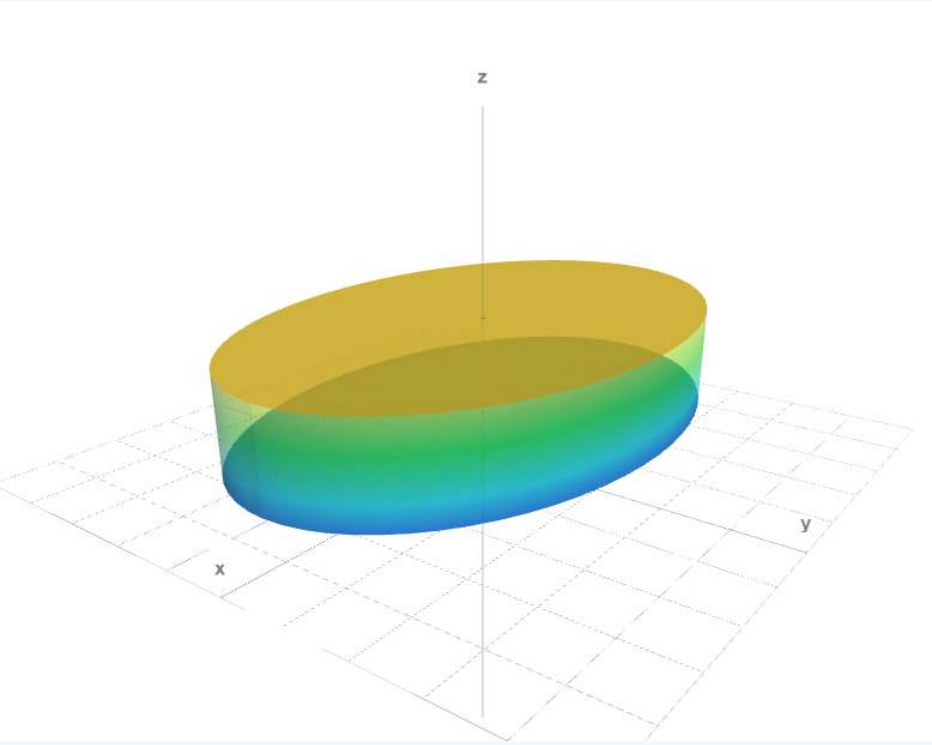
Answer

The given surface S is an elliptical cylinder, but we notice that it is **not closed**.



In order to apply the Divergence Theorem, we must add on two surfaces which close it.

For simplicity, we use the flat elliptical discs $S_{top} = \left\{ (x, y, z) \, : \, \frac{x^2}{36} + \frac{y^2}{16} \leq 1, \, z = 3 \right\}$ and $S_{bot} = \left\{ (x, y, z) \, : \, \frac{x^2}{36} + \frac{y^2}{16} \leq 1, \, z = 1 \right\}$



We find the flux of $\vec{\mathbf{F}}$ across S_{top} oriented *upward* and S_{bot} oriented *downward* (so that together $S \cup S_{top} \cup S_{bot}$ is oriented *outward*).

We can parameterize S_{top} using $\vec{\mathbf{r}}(u,v) = \langle u, v, 3\rangle$ over $D = \left\{(u,v) \,:\, \frac{u^2}{36} + \frac{v^2}{16} \leq 1\right\}$

This gives $\vec{\mathbf{r}}_u = \langle 1,0,0\rangle$, $\vec{\mathbf{r}}_v = \langle 0,1,0\rangle$ and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0,0,1\rangle$

This is indeed oriented upward as needed, so the flux across S_{top} is given by:

$$\begin{aligned}\iint_{S_{top}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle \frac{1}{4}u^4 + \frac{1}{5}v^3 - 3, 3u^5 - 2v^2 + 3, 3^2 \right\rangle \cdot \langle 0,0,1\rangle \, dA \\ &= \iint_D 9 \, dA \\ &= 9 \cdot \text{Area}(D) \\ &= 9 \cdot (\pi \cdot 6 \cdot 4) \\ &= 216\pi\end{aligned}$$

We can parameterize S_{bot} using $\vec{\mathbf{s}}(u,v) = \langle u, v, 1\rangle$ over $D = \left\{(u,v) \,:\, \frac{u^2}{36} + \frac{v^2}{16} \leq 1\right\}$

This gives $\vec{\mathbf{s}}_u = \langle 1,0,0\rangle$, $\vec{\mathbf{s}}_v = \langle 0,1,0\rangle$ and $\vec{\mathbf{s}}_u \times \vec{\mathbf{s}}_v = \langle 0,0,1\rangle$

This is oriented upward, but we want it downward. We compensate by using $-(\vec{\mathbf{s}}_u \times \vec{\mathbf{s}}_v) = \langle 0,0,-1\rangle$, so the flux across S_{bot} is given by:

$$\begin{aligned}\iint_{S_{bot}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle \frac{1}{4}u^4 + \frac{1}{5}v^3 - 1, 3u^5 - 2v^2 + 1, 1^2 \right\rangle \cdot \langle 0,0,-1\rangle \, dA \\ &= \iint_D -1 \, dA \\ &= -1 \cdot \text{Area}(D) \\ &= -1 \cdot (\pi \cdot 6 \cdot 4) \\ &= -24\pi\end{aligned}$$

Now that the surface $S \cup S_{top} \cup S_{bot}$ is closed and oriented outward, we find the divergence of $\vec{\mathbf{F}}$:

$$\operatorname{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}\left(\frac{1}{4}x^4 + \frac{1}{5}y^3 - z\right) + \frac{\partial}{\partial y}\left(3x^5 - 2y^2 + z\right) + \frac{\partial}{\partial z}\left(z^2\right) = x^3 - 4y + 2z$$

Letting E be the enclosed cylinder, a change of variables using $x = 6u\cos(v), y = 4u\sin(v), z = w$ changes E into $\tilde{E} = \{(u, v, w) : 0 \le u \le 1, 0 \le v \le 2\pi, 1 \le w \le 3\}$

The Jacobian of this transformation is $J(u, v, w) = \begin{vmatrix} 6\cos(v) & -6u\sin(v) & 0 \\ 4\sin(v) & 4u\cos(v) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 24u$

$$\begin{aligned}
\iint_{S \cup S_{top} \cup S_{bot}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV \\
&= \iiint_E (x^3 - 4y + 2z)\, dV \\
&= \iiint_{\tilde{E}} \left((6u\cos(v))^3 - 4(4u\sin(v)) + 2w\right) \cdot |24u|\, dV \\
&= 24 \cdot \int_0^1 \int_0^{2\pi} \int_1^3 \left(216u^4\cos^3(v) - 16u^2\sin(v) + 2uw\right)\, dw\, dv\, du \\
&= 24 \cdot \int_0^1 \int_0^{2\pi} \left[216u^4 w\cos^3(v) - 16u^2 w\sin(v) + uw^2\right]\Big|_1^3 dv\, du \\
&= 24 \cdot \int_0^1 \int_0^{2\pi} \left(432u^4\cos^3(v) - 32u^2\sin(v) + 8u\right) dv\, du \\
&= 24 \cdot \int_0^1 \left[432u^4\left(\sin(v) - \frac{1}{3}\sin^3(v)\right) + 32u^2\cos(v) + 8uv\right]\Big|_0^{2\pi} du \\
&= 24 \cdot \int_0^1 16\pi u\, du \\
&= 24 \cdot 8\pi u^2\Big|_0^1 \\
&= 192\pi
\end{aligned}$$

Putting this together we get our final answer:

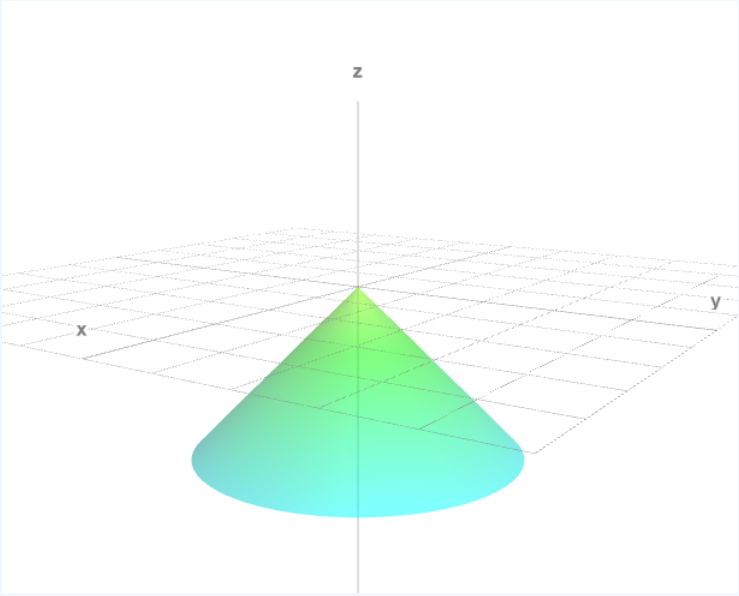
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S \cup S_{top} \cup S_{bot}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} - \iint_{S_{top}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} - \iint_{S_{bot}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 192\pi - 216\pi - (-24\pi) = 0$$

Exercise 4.4.2c

Find the flux of $\vec{\mathbf{F}} = \langle 2x^3 z, 2y^3 z, -3(x^2 + y^2 + 1)z^2 \rangle$ over $S = \left\{(x, y, z) : z = -\sqrt{x^2 + y^2}, z \ge -1\right\}$ oriented upward.

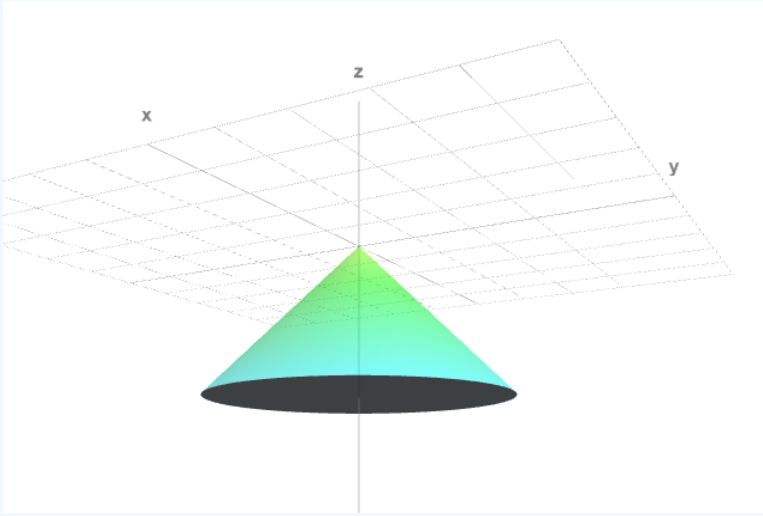
Answer

The given surface S is a cone, but we notice that it is **not closed**.



In order to apply the Divergence Theorem, we must add on a surface which closes it.

The simplest such surface is the flat disc $\tilde{S} = \{(x, y, z) : x^2 + y^2 \leq 1,\ z = -1\}$



We find the flux of $\vec{\mathbf{F}}$ across this disc oriented *downward* (so that together $S \cup \tilde{S}$ is oriented *outward*).

We can parameterize this disc using $\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), -1\rangle$ over $D = \{(u, v) : 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi\}$

This gives $\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 0\rangle$, $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$ and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, 0, u\rangle$

This is oriented upward, but we want downward, so we will use $-(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) = \langle 0, 0, -u\rangle$ and calculate the flux across $\tilde{S}$ to be:

$$\begin{aligned}\iint_{\tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D \left\langle 2(u\cos(v))^3(-1), 2(u\sin(v))^3(-1), -3((u\cos(v))^2 + (u\sin(v))^2 + 1)(-1)^2 \right\rangle \cdot \langle 0, 0, -u \rangle \, dA \\ &= \int_0^1 \int_0^{2\pi} 3u(u^2+1)\, dv\, du \\ &= \left(\int_0^{2\pi} dv \right) \left(\int_0^1 (3u^3 + 3u)\, du \right) \\ &= \left(v\Big|_0^{2\pi} \right) \cdot \left[\frac{3}{4}u^4 + \frac{3}{2}u^2 \right] \Big|_0^1 \\ &= 2\pi \cdot \frac{9}{4} \\ &= \frac{9\pi}{2}\end{aligned}$$

Now that the surface $S \cup \tilde{S}$ is closed and oriented outward, we find the divergence of $\vec{\mathbf{F}}$:

$$\operatorname{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}\left(2x^3 z\right) + \frac{\partial}{\partial y}\left(2y^3 z\right) + \frac{\partial}{\partial z}\left(-3(x^2+y^2+1)z^2\right) = -6z$$

Letting $E = \left\{(x,y,z) \mid -\sqrt{x^2+y^2} \geq z \geq -1\right\} = \{(r\cos(\theta), r\sin(\theta), z) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ -1 \leq z \leq -r\}$ be the enclosed region, we use the Divergence Theorem to calculate:

$$\begin{aligned}\iint_{S \cup \tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV \\ &= \iiint_E -6z\, dV \\ &= \int_0^1 \int_0^{2\pi} \int_{-1}^{-r} -6z \cdot r\, dz\, d\theta\, dr \\ &= \int_0^1 \int_0^{2\pi} -3rz^2 \Big|_{-1}^{-r} d\theta\, dr \\ &= \int_0^1 \int_0^{2\pi} \left(-3r^3 + 3r\right)\, d\theta\, dr \\ &= \left(\int_0^1 \left(-3r^3 + 3r\right)\, dr \right) \left(\int_0^{2\pi} d\theta \right) \\ &= \left[-\frac{3}{4}r^4 + \frac{3}{2}r^2 \right] \Big|_0^1 \cdot \left(\theta \Big|_0^{2\pi} \right) \\ &= \frac{3}{4} \cdot 2\pi \\ &= \frac{3\pi}{2}\end{aligned}$$

Putting this together we get our final answer:

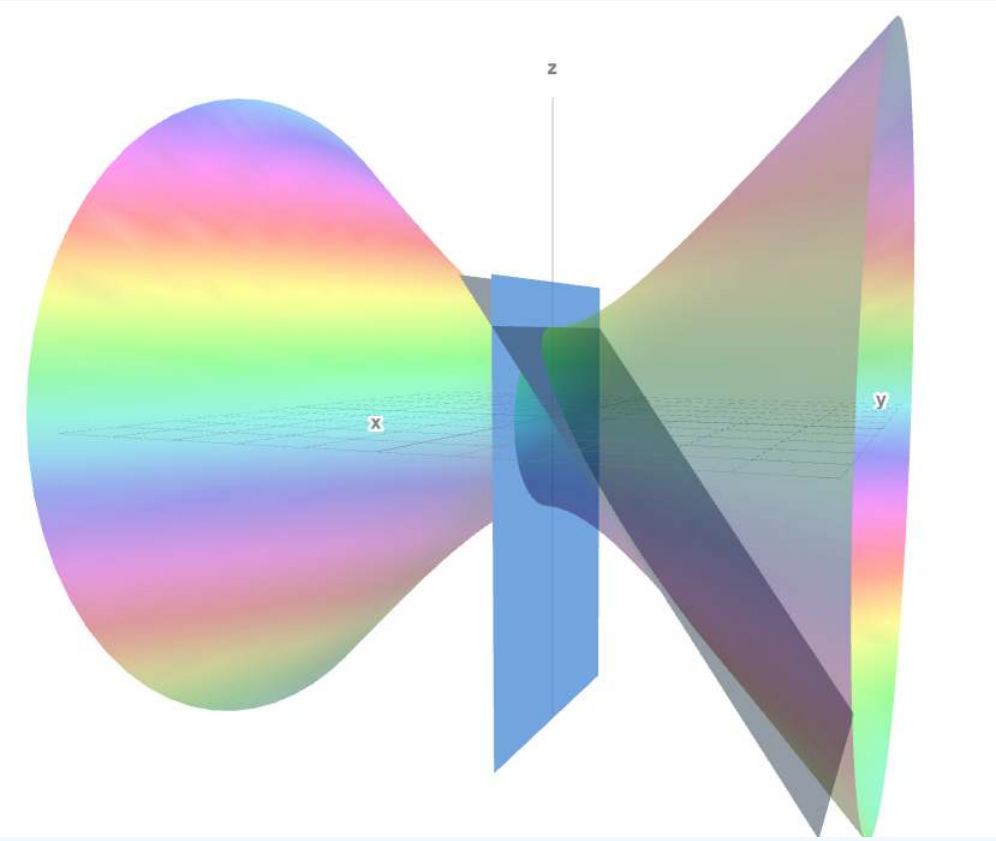
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S \cup \tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} - \iint_{\tilde{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \frac{3\pi}{2} - \frac{9\pi}{2} = -3\pi$$

? Exercise 4.4.2d

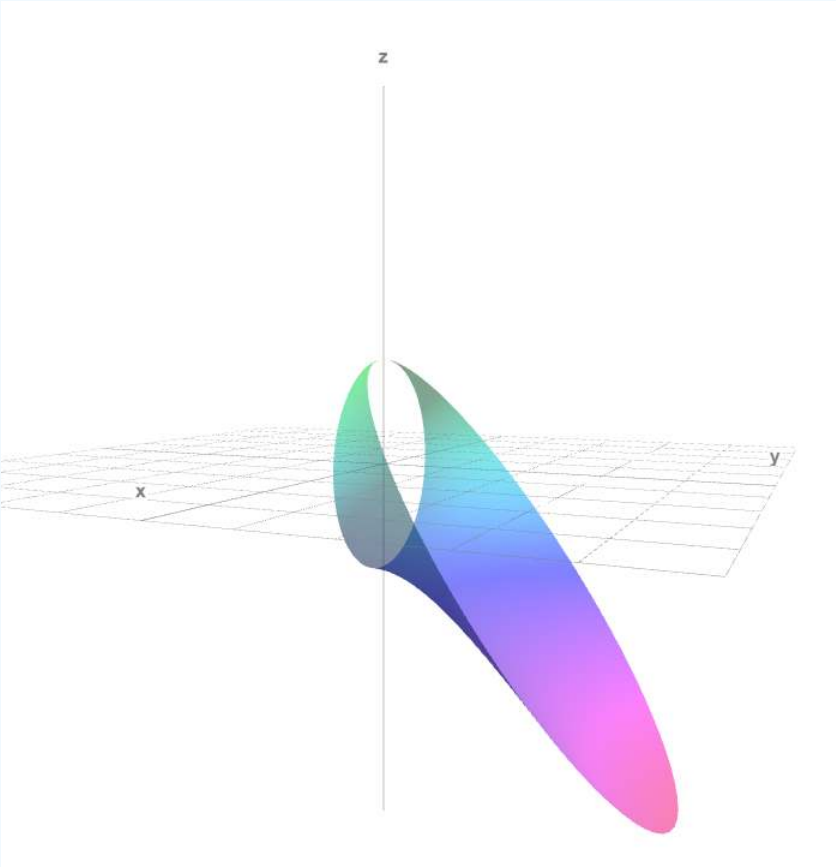
Find the flux of $\vec{\mathbf{F}} = \left\langle y^3 z^7 \sqrt{3}, -x^2, x^2\sqrt{2} \right\rangle$ over $S = \left\{ (x,y,z) : x^2 - y^2 + z^2 = 4,\ 0 \leq y \leq \frac{2-z}{\sqrt{2}} \right\}$ oriented outward.

Answer

The surface described is part of a hyperboloid of one sheet that is trapped between two planes, as shown below:

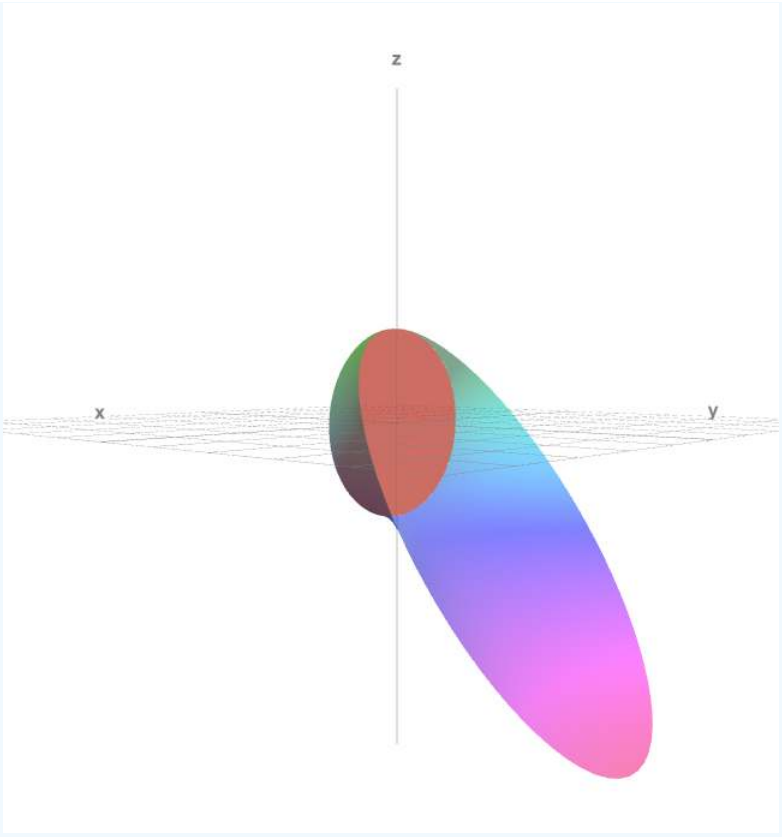


This surface is very challenging to parameterize. Note that the planes themselves are not part of the definition of S, so it is **not closed.**



In order to apply the Divergence Theorem, we must add the surfaces which close it. The simplest such choices are the surfaces that lie on the planes above.

The left one is the flat disc $S_{left} = \{(x, y, z) : x^2 + z^2 \leq 4,\ y = 0\}$



We can parameterize this surface using $\vec{\mathbf{r}}(u,v) = \langle u\cos(v), 0, u\sin(v)\rangle$ over $D_{left} = \{(u,v) : 0 \le u \le 2,\ 0 \le v \le 2\pi\}$

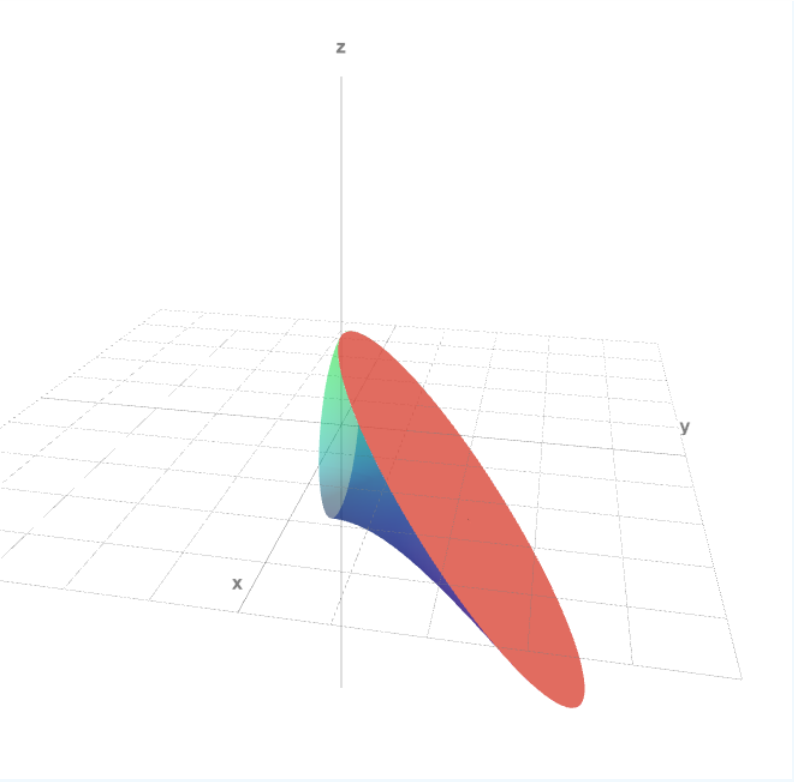
This has $\vec{\mathbf{r}}_u = \langle \cos(v), 0, \sin(v)\rangle$, $\vec{\mathbf{r}}_v = \langle -u\sin(v), 0, u\cos(v)\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, -u, 0\rangle$

Note that this is oriented toward the negative y-axis, which is outward when we close the surface S as desired.

The flux across S_{left} is given by:

$$\begin{aligned}
\iint_{S_{left}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_{D_{left}} \left\langle 0, -(u\cos(v))^2, (u\cos(v))^2\sqrt{2}\right\rangle \cdot \langle 0, -u, 0\rangle \, dA \\
&= \int_0^2 \int_0^{2\pi} u^3 \cos^2(v)\, dv\, du \\
&= \left(\int_0^{2\pi} \cos^2(v)\, dv\right)\left(\int_0^2 u^3\, du\right) \\
&= \left[\frac{1}{2}v + \frac{1}{4}\sin(2v)\right]\Big|_0^{2\pi} \cdot \left(\frac{1}{4}u^4\Big|_0^2\right) \\
&= \pi \cdot 4 \\
&= 4\pi
\end{aligned}$$

The right one is the slanted elliptical disc where $x^2 - y^2 + z^2 = 4$ intersects $y = \dfrac{2-z}{\sqrt{2}} \longleftrightarrow z = 2 - y\sqrt{2}$.



Substituting this in, we see that $x^2 - y^2 + (2 - y\sqrt{2})^2 = 4 \longleftrightarrow x^2 + (y - 2\sqrt{2})^2 = 8$

We can parameterize this surface using $\vec{\mathbf{r}}(u, v) = \langle u\cos(v), 2\sqrt{2} + u\sin(v), -2 - u\sqrt{2}\sin(v)\rangle$ over $D_{right} = \left\{(u, v) : 0 \le u \le 2\sqrt{2},\ 0 \le v \le 2\pi\right\}$

This has $\vec{\mathbf{r}}_u = \langle\cos(v), \sin(v), -\sqrt{2}\sin(v)\rangle$, $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), -u\sqrt{2}\cos(v)\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 0, u\sqrt{2}, u\rangle$

Note that this is oriented upward, which is outward when we close the surface S as desired.

The flux across S_{right} is given by:

$$\begin{aligned}\iint_{S_{right}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_{D_{right}} \left\langle \left(2\sqrt{2} + u\sin(v)\right)^3 \left(-2 - u\sqrt{2}\sin(v)\right)^7 \sqrt{3}, -(u\cos(v))^2, (u\cos(v))^2\sqrt{2} \right\rangle \cdot \left\langle 0, u\sqrt{2}, u \right\rangle \, dA \\ &= \iint_{D_{right}} 0 \, dA \\ &= 0\end{aligned}$$

Now that the surface $S \cup S_{left} \cup S_{right}$ is closed and oriented outward, we can find the divergence of $\vec{\mathbf{F}}$:

$$\text{div}(\vec{\mathbf{F}}) = \frac{\partial}{\partial x}\left(y^3 z^7 \sqrt{3}\right) + \frac{\partial}{\partial y}\left(-x^2\right) + \frac{\partial}{\partial z}\left(x^2\sqrt{2}\right) = 0$$

Therefore for the enclosed region $E = \left\{(x, y, z) : x^2 - y^2 + z^2 \le 4,\ 0 \le y \le \frac{2 - z}{\sqrt{2}}\right\}$, the Divergence Theorem says:

$$\begin{aligned}\iint_{S \cup S_{left} \cup S_{right}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \text{div}(\vec{\mathbf{F}}) \, dV \\ &= \iiint_E 0 \, dV \\ &= 0\end{aligned}$$

Putting this all together we have our final answer:

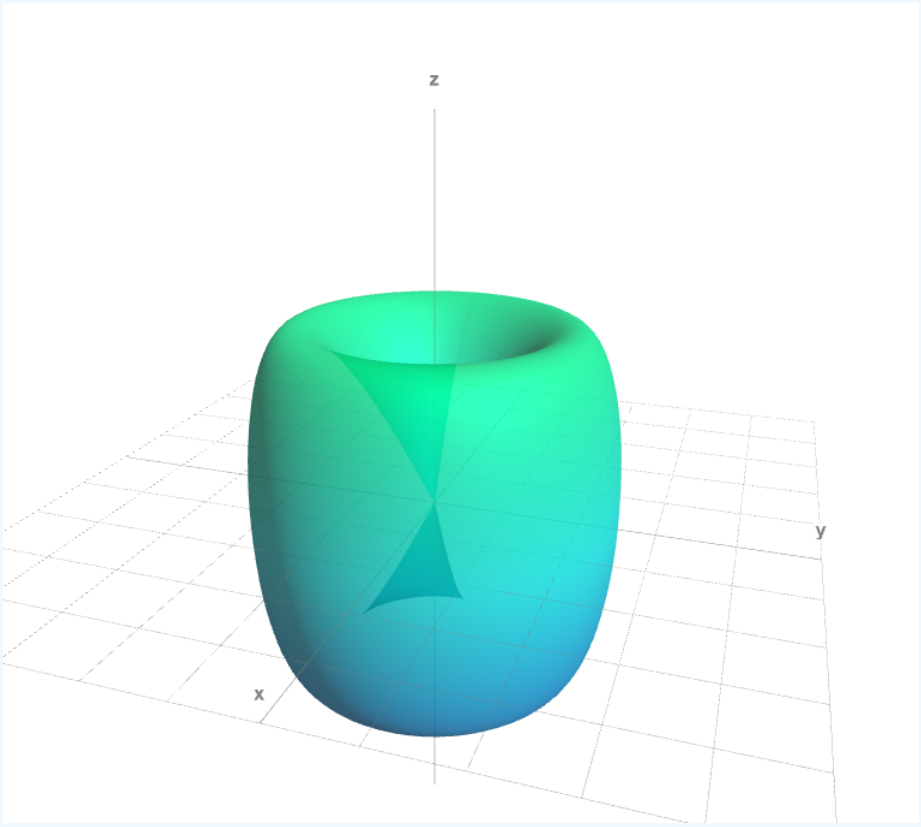
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S \cup S_{left} \cup S_{right}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} - \iint_{S_{left}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} - \iint_{S_{right}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0 - 4\pi - 0 = -4\pi$$

Exercise $4.4.3a$

Find the volume inside of the surface S parameterized by $\vec{\mathbf{r}}(u,v) = \langle \cos(u)\cos(v), \cos(u)\sin(v), 2\sin(u)\cos(u)\rangle$ over $D = \left\{(u,v) : -\frac{\pi}{2} \le u \le \frac{\pi}{2}, 0 \le v \le 2\pi\right\}$

Answer

We want the volume of the region E enclosed by S pictured below:



This region is possible to write E in cylindrical coordinates, it is not necessary to do so to find its volume.

Instead we find a vector field $\vec{\mathbf{F}}$ such that $\operatorname{div}(\vec{\mathbf{F}}) = 1$ so that

$$\text{Volume}(E) = \iiint_E 1\, dV = \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Given our parameterization $\vec{\mathbf{r}}(u,v) = \langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(2u)\rangle$ we can find $\vec{\mathbf{r}}_u = \langle -\sin(u)\cos(v), -\sin(u)\sin(v), 2\cos(2u)\rangle$, $\vec{\mathbf{r}}_v = \langle -\cos(u)\sin(v), \cos(u)\cos(v), 0\rangle$, and $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -2\cos(u)\cos(2u)\cos(v), -2\cos(u)\cos(2u)\sin(v), -\cos(u)\sin(u)\rangle$

Note that this is oriented inward, so we will use $-(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) = \langle 2\cos(u)\cos(2u)\cos(v), 2\cos(u)\cos(2u)\sin(v), \cos(u)\sin(u)\rangle$ when calculating the flux.

Since we can choose any vector field with divergence 1, we avoid the messy x and y coordinates and choose $\vec{\mathbf{F}} = \langle 0, 0, z\rangle$:

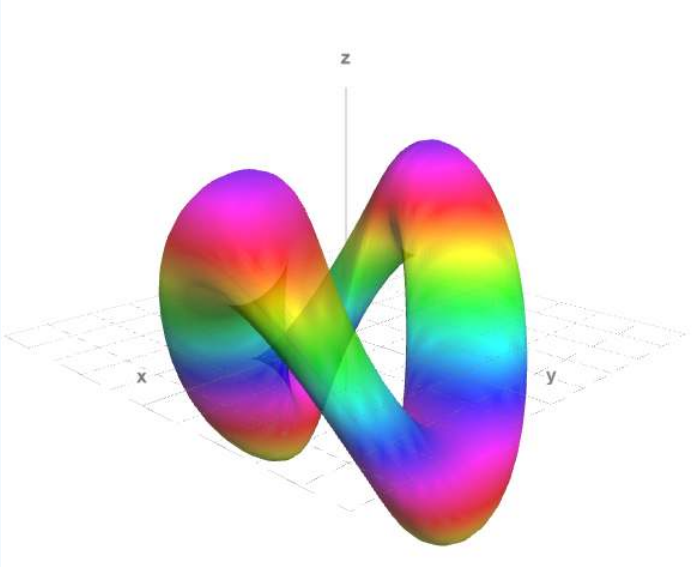
$$
\begin{aligned}
\text{Volume}(E) &= \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \\
&= \iint_D \langle 0, 0, 2\sin(u)\cos(u)\rangle \cdot \langle 2\cos(u)\cos(2u)\cos(v), 2\cos(u)\cos(2u)\sin(v), \cos(u)\sin(u)\rangle \; dA \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} 2\sin^2(u)\cos^2(u)\, dv\, du \\
&= \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin^2(u)\cos^2(u)\, du \right) \left(\int_0^{2\pi} dv \right) \\
&= \left[\frac{1}{4}u - \frac{1}{16}\sin(4u) \right] \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \left(v \Big|_0^{2\pi} \right) \\
&= \frac{\pi}{4} \cdot 2\pi \\
&= \frac{\pi^2}{2}
\end{aligned}
$$

? Exercise 4.4.3*b*

Let $a > 1$ and b be any real number. Find the volume inside of the surface S parameterized by $\vec{\mathbf{r}}(u,v) = \langle (a+\cos(2v))\cos(u), (a+\cos(2v))\sin(u), \sin(2v) + b\cos(2u)\rangle$ over $D = \{(u,v) \,:\, 0 \le u \le 2\pi,\ 0 \le v \le \pi\}$

Answer

We want the volume of the region E enclosed by S pictured below:



This region E is possible to write in cylindrical coordinates, it is not necessary to do so to find its volume.

Instead we find a vector field $\vec{\mathbf{F}}$ such that $\text{div}(\vec{\mathbf{F}}) = 1$ so that

$$\text{Volume}(E) = \iiint_E 1\, dV = \iiint_E \text{div}(\vec{\mathbf{F}})\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Given our parameterization $\vec{\mathbf{r}}(u,v) = \langle (a+\cos(2v))\cos(u), (a+\cos(2v))\sin(u), \sin(2v) + b\cos(2u)\rangle$, we know $\vec{\mathbf{r}}_u = \langle -(a+\cos(2v))\sin(u), (a+\cos(2v))\cos(u), -2b\sin(2u)\rangle$, and $\vec{\mathbf{r}}_v = \langle -2\sin(2v)\cos(u), -2\sin(2v)\sin(u), 2\cos(2v)\rangle$, so

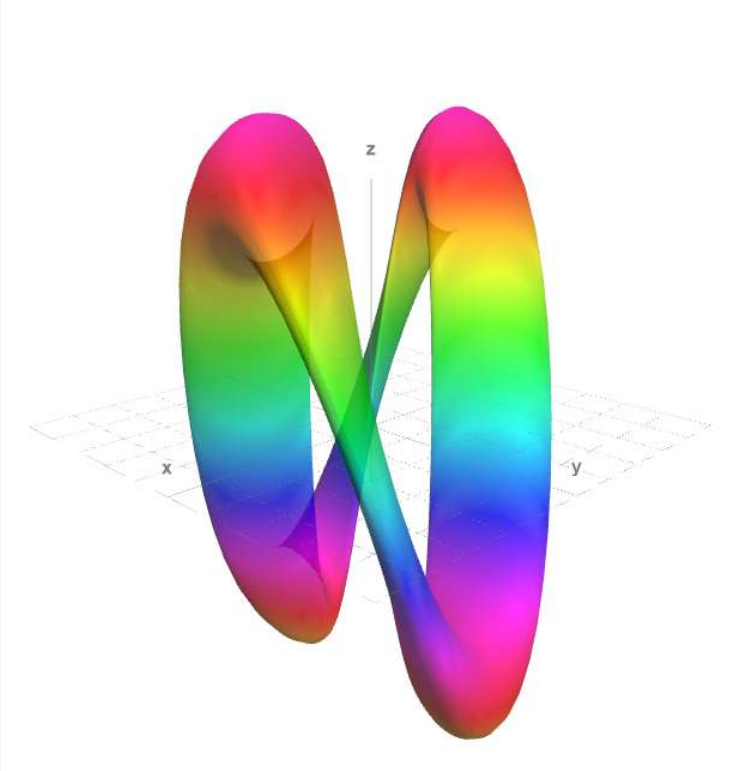
$$
\begin{aligned}
\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = &\left(2\cos(2v)(a+\cos(2v))\cos(u) - 4b\sin(2v)\sin(u)\sin(2u)\right)\hat{\mathbf{i}} \\
&+ \left(4b\sin(2v)\cos(u)\sin(2u) + 2\cos(2v)(a+\cos(2v))\sin(u)\right)\hat{\mathbf{j}} \\
&+ 2\sin(2v)(a+\cos(2v))\hat{\mathbf{k}}
\end{aligned}
$$

Note that this is oriented outward, as needed for the Divergence Theorem.

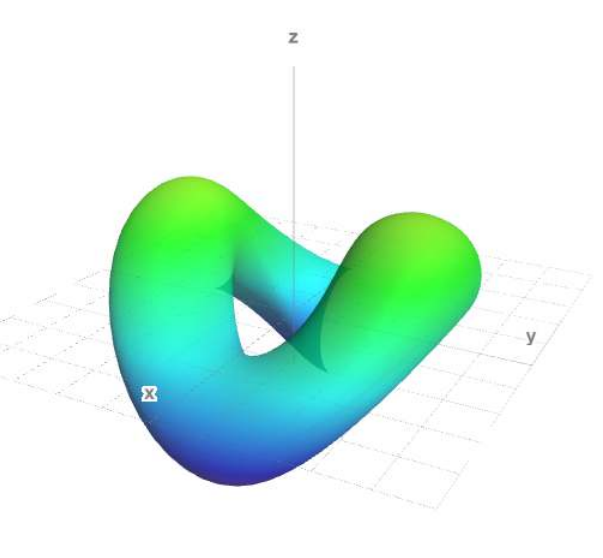
Since we can choose any vector field with divergence 1, we avoid the messy x and y coordinates and choose $\vec{\mathbf{F}} = \langle 0, 0, z \rangle$:

$$\begin{aligned}
\text{Volume}(E) &= \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \\
&= \iint_D \langle 0, 0, \sin(2v) + b\cos(2u) \rangle \cdot \langle \star, \star, 2\sin(2v)(a + \cos(2v)) \rangle \; dA \\
&= \int_0^{\pi} \int_0^{2\pi} (2ab\cos(2u)\sin(2v) + 2a\sin^2(2v) + 2b\cos(2u)\cos(2v)\sin(2v) + 2\sin^2(2v)\cos(2v)) \, du \, dv \\
&= \int_0^{\pi} \left[ab\sin(2u)\sin(2v) + 2au\sin^2(2v) + b\sin(2u)\cos(2v)\sin(2v) + 2u\sin^2(2v)\cos(2v) \right] \Big|_0^{2\pi} dv \\
&= \int_0^{\pi} \left(4\pi a \sin^2(2v) + 4\pi \sin^2(2v)\cos(2v) \right) \, dv \\
&= \left[2\pi a v - \frac{\pi a}{2} \pi \sin(4v) + \frac{2\pi}{3} \sin^3(2v) \right] \Big|_0^{\pi} \\
&= 2\pi^2 a
\end{aligned}$$

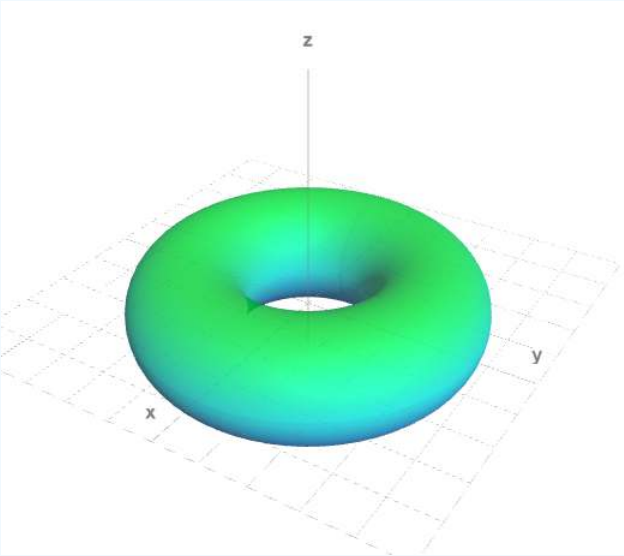
Notice that the value of b does not affect the volume! Below is a picture with a larger value of b:



If b is a negative value, instead you get:



And of course when $b = 0$, we get the standard torus:

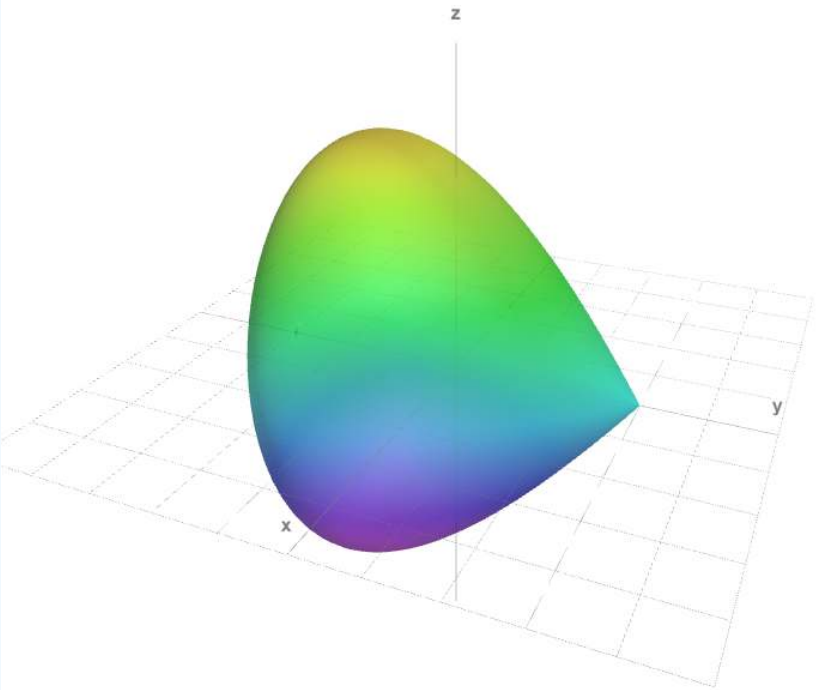


? Exercise $4.4.3c$

Let $a, b, c > 0$ be constants and find the volume inside of the surface S parameterized by $\vec{\mathbf{r}}(u, v) = \langle au\sin(u)\cos(v), b\cos(u), cu\sin(u)\sin(v)\rangle$ over $D = \{(u, v) : 0 \le u \le \pi, 0 \le v \le 2\pi\}$

Answer

We want the volume of the region E enclosed by S pictured below:



This region E is difficult to describe, it is not necessary to do so to find its volume.

Instead we find a vector field $\vec{\mathbf{F}}$ such that $\operatorname{div}(\vec{\mathbf{F}}) = 1$ so that

$$\text{Volume}(E) = \iiint_E 1\, dV = \iiint_E \operatorname{div}(\vec{\mathbf{F}})\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Given our parameterization $\vec{\mathbf{r}}(u,v) = \langle au\sin(u)\cos(v), b\cos(u), cu\sin(u)\sin(v)\rangle$, we know $\vec{\mathbf{r}}_u = \langle a(\sin(u) + u\cos(u))\cos(v), -b\sin(u), c(\sin(u) + u\cos(u))\sin(v)\rangle$, and $\vec{\mathbf{r}}_v = \langle -au\sin(u)\sin(v), 0, cu\sin(u)\cos(v)\rangle$, so $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -bcu\sin^2(u)\cos(v), -acu\sin(u)(\sin(u) + u\cos(u)), -abu\sin^2(u)\sin(v)\rangle$

Note that this is oriented inward, so we will use $-(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)$ when using the Divergence Theorem.

Since we can choose any vector field with divergence 1, we avoid the messy x and z coordinates and choose $\vec{\mathbf{F}} = \langle 0, y, 0\rangle$:

$$\begin{aligned}
\text{Volume}(E) &= \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \\
&= \iint_D \langle 0, b\cos(u), 0\rangle \cdot \left\langle bcu\sin^2(u)\cos(v), acu\sin(u)(\sin(u) + u\cos(u)), abu\sin^2(u)\sin(v)\right\rangle\, dA \\
&= \int_0^{\pi}\int_0^{2\pi} abcu\sin(u)\cos(u)(\sin(u) + u\cos(u))\, dv\, du \\
&= abc\left(\int_0^{\pi} \left(u\sin^2(u)\cos(u) + u^2\sin(u)\cos^2(u)\right)\, du\right)\left(\int_0^{2\pi} dv\right) \\
&= abc\cdot\left[-\frac{1}{4}(u^2-3)\cos(u) + \frac{1}{108}(9u^2+1)\cos(3u) - \frac{1}{36}u\sin(3u) - u\sin(u)\right]\Big|_0^{\pi} \cdot \left(v\Big|_0^{2\pi}\right) \\
&= \frac{2\pi abc\,(9\pi^2 - 40)}{27}
\end{aligned}$$

(The integral of u above is calculated using lots of integration by parts and trig identities!)

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