MATH 375: PRE-CALCULUS

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Math 375

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This text was compiled on 04/22/2026

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Detailed Licensing

Licensing

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CHAPTER OVERVIEW

Chapter 1: Triangles and Circles

Circles and triangles are the most straightforward geometric figures. A circle is the simplest curve, and a triangle is the simplest polygon - the one with the fewest sides. Each has properties that make it useful in many fields of endeavor. A triangle is the most stable polygon because once its sides are fixed in length, its angles cannot change. A triangular truss bridge is a stable structure. A three-legged stool will not wobble.

Figure 1

A circle encloses more area than any figure of the same length or perimeter, and a sphere encloses more space. A geodesic dome is a portion of a sphere constructed with triangles. It has been called "the strongest, lightest, and most efficient means of enclosing space known to man."

Geodesic domes may also help us learn how to live on another planet. In 2017, the United Arab Emirates began construction on Mars Science City, a series of interconnecting geodesic domes designed to be a realistic model for living on Mars. The city will cover 1.9 million (1.9×10^6) square feet, and its walls will be 3D-printed using sand from the desert.

Figure 2

The city will contain laboratories to address food, sustainability, and energy issues worldwide. Finally, the project will implement an experiment where a team will spend a year living on the simulated planet.

Figure 3

Activity: Properties of Triangles

Materials

You will need paper and pencil, scissors, a ruler and compass, and two plastic straws.

Prepare

Cut the straw into pieces of length 2 inches, 3 inches, and 6 inches. Use the ruler to draw a large triangle and cut it out.

A. What do we know about the sides of a triangle?

1. a. Can you make a triangle with sides of length 2 inches, 3 inches, and 6 inches?
   b. Use the pieces of length 2 inches and 3 inches to form two sides of a triangle. What happens to the length of the third side as you increase the angle between the first two sides?
   c. What is the longest that the third side could be? What is the smallest?
2. Two sides of a triangle are 6 centimeters and 8 centimeters long. What are the possible lengths of the third side?
3. Two sides of a triangle are p units and q units long. What are the possible lengths of the third side?

B. What do we know about the angles of a triangle?

1. Use a protractor to measure the three angles of the paper triangle in degrees. Now, add up the three angles. What is their sum?
2. Tear off the three corners of the triangle. Place them side-by-side with their vertices (tips) at the same point. What do you find?
3. How are your answers to parts (1) and (2) related?

C. How are the sides and angles of a triangle related?

1. A standard way to label a triangle is to call the angles A, B, and C. The side opposite angle A is called a, the side opposite angle B is called b, and the side opposite angle C is called c. Sketch a triangle and label it with standard notation.
2. Using a ruler, carefully draw a triangle and label it with standard notation so that $a > b > c$. Now, use a protractor to measure the angles and list them from largest to smallest. What do you observe?
3. Using a ruler, carefully draw another triangle and label it with standard notation so that $A > B > C$. Now, use a ruler to measure the sides and list them in order from largest to smallest. What do you observe?

D. What do we know about right triangles?

1. The **hypotenuse** is the side opposite the $90°$ angle in a right triangle. Why is the hypotenuse always the longest side of a right triangle?
2. The **Pythagorean Theorem** states the following:
   IF: a, b, and c are the sides of a right triangle, and c is the hypotenuse $a^2 + b^2 = c^2$
   THEN: $a^2 + b^2 = c^2$
   The "if" part of the theorem is called the **hypothesis**, and the "then" part is called the **conclusion**. The **converse** of a theorem is the new statement you obtain when you interchange the hypothesis and the conclusion. Write the converse of the Pythagorean Theorem.
3. The converse of the Pythagorean Theorem is also true.[1] Use the converse to decide whether each of the following triangles is a right triangle. Support your conclusions with calculations.
   a. $a = 9, b = 16, c = 25$
   b. $a = 12, b = 16, c = 20$
   c. $a = \sqrt{8}, b = \sqrt{5}, c = \sqrt{13}$
   d. $a = \frac{\sqrt{3}}{2}, b = \frac{1}{2}, c = 1$

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Footnotes

[1] In general, the converse of a theorem is not necessarily true. For example, someone might say, "If it is raining, it must be cloudy." While this is true, the converse, "If it is cloudy, then it must be raining," is not necessarily true. A theorem in which the converse is also true is called a **bijection** (but most students know this as an "if and only if").

1.1: Angles and Basic Geometry

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Note to the Instructor (click to expand)

The material in this section focuses on the basic Geometry needed for the rest of this course. Angles (in degrees), circles, and areas of circles are the main focus (anything to do with triangles is reserved for the next section).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Sets and Numbers**
  - The Language of Algebra: Needed for understanding and using variables (e.g., x, y, r, d, Greek letters like α, β, θ) and algebraic expressions throughout the section and in homework problems.
  - Sets of Numbers: Irrational numbers are referenced and occur frequently throughout Trigonometry.
- **Arithmetic**
  - Mathematical Operations - Addition, Subtraction, Multiplication, and Division: Essential for calculations such as finding complements and supplements, determining angle rotations (Example 2), and applying formulas for circumference and area (Examples 4, 5, and several homework problems).
  - The Order of Operations: Required for correctly evaluating expressions, particularly in formulas like the area of a circle $A = \pi r^2$ (Example 5 and several homework problems).
- **Simplifying Expressions**
  - Algebraic Expressions - An Introduction and Using in Applications: Helpful for translating geometric relationships into algebraic expressions, as required in homework problems involving finding unknown angles.
- **Solving Equations**
  - Solving Linear Equations: Directly required for solving for unknown variables in many homework problems where geometric properties (e.g., vertical angles, angles on a straight line, properties of parallel lines) are used to set up equations.
- **Systems**
  - Solving 2x2 Systems of Linear Equations: This skill is needed for the "Skills Refresher" exercises and for several homework problems where two variables must be found

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using relationships between angles formed by parallel lines and a transversal.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Identify acute, obtuse, right, and straight angles.
- Compute the complement and supplement of an angle.
- Determine the number of degrees in a partial rotation.
- Use vertical, alternate interior, or corresponding angles to solve for an angle.
- Find the circumference and area of a circle.

Angles

Before we can jump into Trigonometry, we must define angles; however, before defining angles, we need to define what makes up the components of an angle.

Definition: Ray

A **ray** is a line segment that has one endpoint and extends infinitely in one direction. The point from which the ray originates is called the **initial point** of the ray. A ray can also be described as a **half-line**.

Figure 1.1.1: *A ray with initial point P*

Note that we label the initial point P. The choice of letter is arbitrary, so we could just have easily chosen Q, R, or S. The choice of using capital letters for points on a line or ray, however, is tradition and we stick with that convention in this text. We now move on to defining an angle.

Definition: Angle

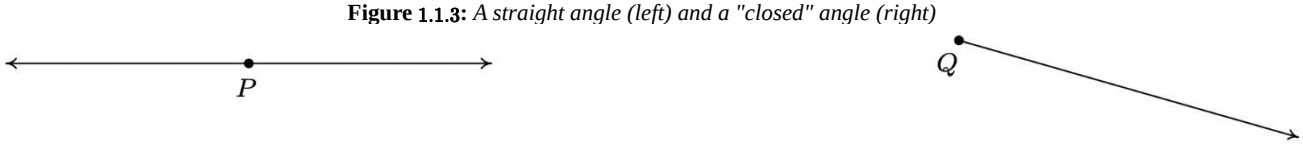
When two rays share a common initial point, they form an **angle**, and the shared initial point is called the angle's **vertex**.

Figure 1.1.2 shows two examples of what are commonly considered as angles.

Figure 1.1.2: *An angle with vertex P (left) and an angle with vertex Q (right)*

Figure 1.1.3 also depicts angles - albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other, forming what is known as a **straight angle**; in the second, the rays are identical, so the "angle" is indistinguishable from the ray itself.

Figure 1.1.3: *A straight angle (left) and a "closed" angle (right)*



Definition: Straight Angle

A **straight angle** is defined as an angle that is formed when the two rays forming the angle extend in opposite directions.

Now that we have an idea of what makes an angle, and a "beginner's notation," it's time we define the **measure of an angle**.

Definition: Angle Measure

The **measure of an angle** is a number that indicates the amount of rotation that separates the rays of the angle.

There is one immediate problem with this definition, as should be evident from Figure **1.1.4** below.

Figure 1.1.4: *The ambiguity of angles*

Which amount of rotation are we attempting to quantify?

We have just discovered that this diagram describes at least two angles. Clearly, these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently.

In Trigonometry, we often use lowercase Greek letters to represent the measure of the angle. For example, we use the Greek letters α (alpha) and β (beta) as representations of the unknown angle measures in Figure **1.1.5**.

Figure 1.1.5

Identifying the lowercase Greek letters will be beneficial in this course. Table **1.1.1** summarizes the (lowercase) Greek alphabet. The letters in **bold** are the ones most commonly used within Trigonometry.

Table 1.1.1: *Lowercase Letters in the Greek Alphabet*

Greek Alphabet

α	**alpha**	β	**beta**	γ	**gamma**
δ	delta	ϵ	epsilon	ζ	zeta
η	eta	θ	**theta**	ι	iota
κ	kappa	λ	lambda	μ	mu
ν	nu	ξ	xi	o	omicron
π	pi	ρ	rho	σ	sigma
τ	**tau**	υ	upsilon	ϕ	**phi**
χ	chi	ψ	psi	ω	**omega**

(1.1.1)

Caution

The lowercase Greek letter π is *never* used as a symbolic representation of an *unknown* angle. π is reserved for the universal constant $\pi \approx 3.141592654$ (which is an **irrational number**).

Another ambiguity that sneaks into angular measure is which direction the angle is "opening." We need to extend our notion of "angle" from merely measuring an extent of rotation to quantities associated with real numbers. To that end, we introduce the concept of an **oriented angle**.

Definition: Oriented Angle

An **oriented angle** refers to the measure of rotation between two rays with a common vertex, taking into account the direction of rotation (clockwise or counterclockwise). The angle is swept out starting from an **initial side** and ending at a **terminal side**. When the rotation is counterclockwise from the initial side to the terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

As its name suggests, in an oriented angle, the direction of the rotation is essential.

Figure 1.1.6: *A positive angle (left) and a negative angle (right)*

Alternative Angle Notations

We have already mentioned that we will frequently use lowercase Greek letters when referencing angles; however, this is not the only convention that exists when referencing an angle. Angles may be labeled with a single letter at the vertex as long as it is clear that there is only one such angle. For example, in Figure 1.1.7 (below), it is clear that angle A is the vertex. There is no ambiguity about this fact. Therefore, an acceptable notation to reference this angle is $\angle A$ (some texts might use the notation $m\angle A$, meaning "the measure of angle A").

Figure 1.1.7

However, if there is the possibility for confusion, angles are labeled by specifying 3 points, with the center point being the angle's vertex. Therefore, we could reference the angle in Figure 1.1.7 as $\angle BAC$. The first letter indicates the initial side of the angle, the middle is the vertex, and the last letter indicates the terminal side. It is crucial that the center letter in this notation is the angle's vertex.

Common Angle Notations

The common notations used for angles are as follows:

- Lowercase Greek letters (most commonly α, β, and θ)
- $\angle A$ or $m\angle A$, where A is the vertex of the rays where the angle is formed
- $\angle BAC$, where A is the vertex where the rays emanating from A and going through B and C meet.

Degree Measure

One commonly used system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the familiar "$^\circ$" symbol.

Definition: Degree Measure

We define **degree measure** by dividing the full rotation around the circle into 360 segments. The angular measure of one of these segments is defined to be one degree, denoted $1°$. That is, $1°$ represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution.

Figure 1.1.8 shows the degree measures for a full, half, and quarter revolution. Since we define a full revolution as being $360°$, half of a revolution measures $\frac{1}{2}(360°) = 180°$, a quarter of a revolution measures $\frac{1}{4}(360°) = 90°$, and so on.

Figure 1.1.8: *A full revolution (left), one-half a revolution (middle), and one-quarter of a revolution (right)*

The following theorem is a direct result of Figure 8.

Theorem: Degree Measure of a Straight Angle

The degree measure of a straight angle is $180°$.

In Figure 1.1.8, we have used the small square "□" to denote a very special angle - the **right angle**.

Definition: Right Angle

A **right angle** is defined to be the angle measure of a quarter-circle, which is $90°$.

The right angle is a very special angle in Trigonometry. As such, we use it as a sort of "measuring stick" against which we judge other angles. One such judgement is whether the measure of an angle is smaller or larger than the measure of a right angle.

Definition: Acute Angle

An angle that measures strictly between $0°$ and $90°$ is called an **acute angle**.

Definition: Obtuse Angle

An angle that measures strictly between $90°$ and $180°$ is called an **obtuse angle**.

It is important to note that, theoretically, we can know the measure of any angle as long as we know the proportion it represents of the entire revolution. For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}(360°) = 240°$, the measure of an angle which constitutes only $\frac{1}{12}$ of a revolution measures $\frac{1}{12}(360°) = 30°$, and an angle which indicates no rotation at all is measured as $0°$ (see Figure 1.1.9).

Figure 1.1.9: *An obtuse angle (left), an acute angle (middle), and a "closed" angle (right)*

We now investigate the relationship between two angles whose measures add to $90°$ or $180°$.

Definition: Complementary Angles

Two acute angles are called **complementary angles** if their measures add to $90°$.

Definition: Supplementary Angles

Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to $180°$.

In the diagram below, the angles α and β are supplementary while the pair γ and θ are complementary angles.

Figure 1.1.10: *Supplementary angles, α and β (left) and complementary angles, γ and θ (right)*

Caution

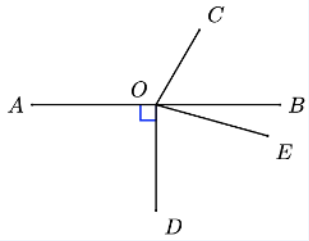
In this text, we will restrict the definition of complementary and supplementary angles to positive angles only. That is, despite their sum being $90°$, we will *not* say that $\alpha = -20°$ and $\beta = 110°$ are complementary. Other textbooks allow for this silliness, but in truth there is no need to expand these definitions to include negative angles.

Finally, in practice, the distinction between the angle itself and its measure is blurred so that the sentence "α is an angle measuring $42°$" is often abbreviated as "$\alpha = 42°$."

Example 1.1.1

Consider Figure 1.1.11 below.

Figure 1.1.11



a. Which angle pairs are supplementary?
b. Which angle pairs are complementary?
c. Which angles are obtuse?
d. Which angles are acute?

Solutions

a. $\angle AOC$ and $\angle BOC$ are supplementary, $\angle AOD$ and $\angle BOD$ are supplementary, and $\angle AOE$ and $\angle BOE$ are supplementary.
b. $\angle DOE$ and $\angle BOE$ are complementary.
c. $\angle AOC$, $\angle AOE$, and $\angle DOC$ are each obtuse angles.

d. $\angle BOC$, $\angle BOE$, and $\angle DOE$ are each acute angles.

It could be argued in Example 1.1.1*d* that $\angle COE$ is acute; however, there is no valid reference for the measure of $\angle COB$ and $\angle BOE$ - only a visual cue. Therefore, we avoid assuming that $\angle COE$ is acute.

? Checkpoint 1.1.1

Find the complement of the angle $31°$.

$\square$ °

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Find the supplement of the angle $29°$.

$\square$ °

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The measuring "system" we use to measure angles is formally called the **Decimal Degree** (**DD**) system.

Definition: Decimal Degree

An angle written as a decimal is said to belong to the **decimal degree (DD)** system of measuring angles.

Therefore, angles like $38°$, $23.1°$, and $123.456°$ belong to the decimal degree system. There are other angle measure systems:

- **Degree-Minute-Second (DMS):** While we will eventually use this angular measurement system (specifically, in some applications), it is not necessary to introduce it at this time.
- **Radians:** We will introduce this "better" angular measurement system later. Without this system, we could not perform Calculus.

Angles in Rotations

Now that we have some basic information about angles and have reintroduced ourselves to a measurement system for angles, it's time to introduce an application of angular measure. Specifically, we will learn to use angles to describe rotation. For example, think of the minute hand on a clock. The minute hand moves through one complete rotation every hour, or $360°$. In two hours, the minute hand rotates through $720°$.

✓ Example 1.1.2

Through how many degrees does the minute hand rotate in an hour and a half? In forty minutes?

Solution

Figure 1.1.12

Look at Figure **1.1.12**. An hour and a half represents 1.5 complete rotations, or

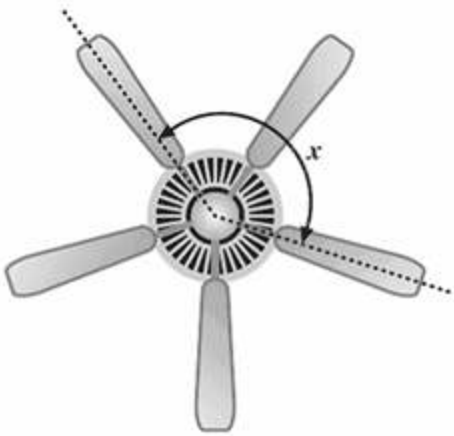
$$1.5(360^\circ) = 540^\circ.$$

Forty minutes is two-thirds of an hour, so the minute hand rotates through

$$\frac{2}{3}(360^\circ) = 240^\circ$$

? Checkpoint 1.1.2

A ceiling fan with five equally-spaced blades.



What is the degree measure of angle x? ____°

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Geometry Necessary for Trigonometry

Trigonometry requires a decent grasp of basic Geometry. As such, let's review some pertinent topics.

Definition: Vertical Angles

Non-adjacent angles formed by the intersection of two straight lines are called **vertical angles**.

For example, $\angle 1$ and $\angle 3$ in Figure **1.1.14** are vertical angles, as are the angles labeled $\angle 2$ and $\angle 4$.

Figure 1.1.14

Before introducing a theorem relating vertical angles, we need to define what it means for angles to be **equal**.

Definition: Equality of Angles

Two angles are defined to be **equal** if their measures are equal.

Having clarified this, we introduce a theorem that is very useful for deriving other concepts from Geometry.

Theorem: Congruency of Vertical Angles

Vertical angles are equal (also known in Geometry as **congruent**).

Proof

Suppose two lines intersect, as shown in the figure below.

Since $\angle 1 + \angle 2$ forms a straight angle,

$$\angle 1 + \angle 2 = 180^\circ.$$

By a similar argument,

$$\angle 1 + \angle 4 = 180^\circ.$$

Therefore,

$$\angle 1 + \angle 2 = 180^\circ = \angle 1 + \angle 4.$$

That is,

$$\angle 2 = \angle 4.$$

By a similar set of arguments, we can show that

$$\angle 1 = \angle 3.$$

Hence, it must always be the case that vertical angles are congruent.

A line intersecting two parallel lines forms eight angles, as shown in Figure 1.1.15.

Figure 1.1.15: *Two parallel lines "cut" by a line*

The line cutting through the parallel lines in Figure 1.1.15 gets a special name.

Definition: Transversal

A line intersecting two parallel lines is called a **transversal**.

Figure 1.1.15 shows that there are four pairs of vertical angles formed by a transversal - $\angle 1$ and $\angle 4$, $\angle 2$ and $\angle 3$, $\angle 5$ and $\angle 8$, and finally $\angle 6$ and $\angle 7$; however, there are other relationships that can be seen in Figure 1.1.15.

Definition: Corresponding Angles

Angles in the same position relative to the transversal on each parallel line are called **corresponding angles**.

For example, $\angle 1$ and $\angle 5$ are corresponding angles, as are $\angle 4$ and $\angle 8$. This naturally leads to the following theorem.

Theorem: Congruency of Corresponding Angles

Corresponding angles are congruent.

Our final definition concerning the transversals is one of the most popular from Geometry.

Definition: Alternate Interior Angles

Angles that lie inside two parallel lines and on opposite sides of a transversal line that intersects those parallel lines are called **alternate interior angles**.

In Figure 1.1.15, $\angle 3$ and $\angle 6$ are alternate interior angles, as are $\angle 4$ and $\angle 5$.

Theorem: Alternate Interior Angles

If a transversal intersects parallel lines, the alternate interior angles are congruent.

Proof

Suppose a transversal cuts two parallel lines as shown below.

We know that the corresponding and vertical angles are congruent. Therefore,

$$\angle 1 = \angle 5$$

and

$$\angle 1 = \angle 4.$$

Hence,

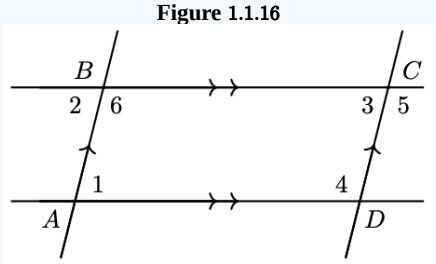
$$\angle 5 = \angle 4.$$

Since our choice of angles was arbitrary, this shows that alternate interior angles are always congruent.

Example 1.1.3

The parallelogram $ABCD$ shown below is formed by the intersection of two sets of parallel lines. Show that the opposite angles of the parallelogram are equal.

Figure 1.1.16

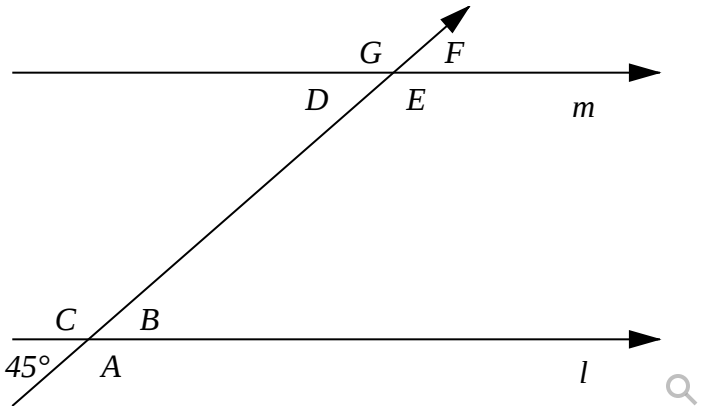


Solution

Angles 1 and 2 are equal because they are alternate interior angles, and angles 2 and 3 are equal because they are corresponding angles. Therefore, angles 1 and 3, the opposite angles of the parallelogram, are equal. A similar argument shows that angles 4, 5, and 6 are equal.

Checkpoint 1.1.3

Given the angle in the image below and that $l \parallel m$, find the value of each of the other angles.



A. ☐
B. ☐
C. ☐
D. ☐
E. ☐
F. ☐
G. ☐

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Circles

We end this section with one of the defining geometric structures in all of Mathematics - the **circle**.

Definition: Circle

The shape created from all points in a plane that are equidistant from a given point is called a **circle**. The given point is called the **center** of the circle, and the distance from the center to any point along the circle is called the **radius** of the circle.

Figure 1.1.17: *A circle with center O and radius r*

Figure 1.1.17 shows a circle of radius r centered at a point O. In fact, the word "radius" plays two roles in Geometry. By definition, the radius is the distance between the center of a circle and its edge; however, *any* line segment joining the center of a circle with any single point on the circle itself is called a radius. Therefore, the word radius simultaneously refers to a distance and a line segment.

Definition: Radius (as a line segment)

A line segment with one endpoint at the center of a circle and the other on the circle is called a **radius** of the circle.

Just like the radius, the **diameter** of a circle, denoted as d in Figure 1.1.17, has two interpretations. Our first is in terms of distance.

Definition: Diameter (as a distance)

The **diameter** of a circle is the length of any line segment going through the center of a circle and whose endpoints are on the circle.

This definition naturally leads to the following theorem.

Theorem: Diameter in Terms of Radius

Given a circle of radius r, the diameter, d, of the circle satisfies

$$d = 2r.$$

As with the radius of a circle, the second interpretation of the diameter is in terms of a line segment.

Definition: Diameter (as a line segment)

A line segment going through the center of a circle and whose endpoints are on the circle is called a **diameter** of the circle.

The terminology from Geometry related to circles extends beyond center, radius, and diameter (e.g., arc, chord, sector, and tangent); however, we will introduce these only as needed.

Circumference

Now that we have the basic definition of a circle, let's dive into some measurements.

Definition: Circumference

The **circumference** is the distance around a circle.

In Figure **1.1.17**, the circumference is denoted using the letter C.

Theorem: Circumference

A circle's circumference, C, is proportional to its radius, r. That is,

$$C = kr.$$

The constant of proportionality is 2π. Hence,

$$C = 2\pi r.$$

A proof of this fundamental theorem in Mathematics is beyond the scope of this course.

In the theorem presented for the circumference, it would be more impactful to state the circumference is proportional to the diameter. That is,

$$C = \text{ some number } \times d.$$

It turns out that the value of "some number" (which is commonly called the **constant of proportionality**) is the universal constant π. That is,

$$C = \pi d \implies \pi = \frac{C}{d}.$$

This fascinating fact tells us that no matter the size of the circle, the ratio of its circumference to its diameter is always this magical number that we call π.

Language Note: "Circumference of a Circle" is Redundant

It is common to hear someone say, "the circumference of a circle." However, the word circumference *only applies to circles*. Therefore, saying "the circumference of a circle" is redundant. For non-circular two-dimensional shapes, we use the word **perimeter** instead of circumference.

Example 1.1.4

Loi buys a ring light for her YouTube videos. The light is circular and has a radius of 7 inches. What is the circumference of her ring light?

Solution

We are given the fact that the radius is 7 inches. Therefore, $r = 7$ (inches). Using the formula for circumference, we find that Loi's ring light has a circumference of

$$C = 2\pi r = 2\pi(7 \text{ inches}) = 14\pi \text{ inches} \approx 44 \text{ inches}.$$

? Checkpoint 1.1.4

The circumference of the circular swimming pool shown is 63.6 feet. What are the diameter and radius of the pool?
Round to the nearest hundredth.

Diameter: [ ] feet

Radius: [ ] feet

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Area of a Circle

The area of a circle is proportional to the square of its radius.

Theorem: Area of a Circle

The area of a circle of radius r is given by

$$A = \pi r^2.$$

Again, the proof of this theorem is beyond the scope of this course.

✓ Example 1.1.5

Phuong is making a circular tablecloth for her grandfather's antique oak table. If she wants the tablecloth to have a diameter of at least 7 feet (the bare minimum to cover the table), but no more than 9 feet (allowing a little bit of tablecloth to hang over the edge of the table), between what two values must the area of the tablecloth be? Round your answers to the nearest tenth of a square foot.

Solution

To barely cover the table, Phuong needs a tablecloth with radius 3.5 feet. Therefore, the minimum area of the tablecloth should be

$$A_{\text{minimum}} = \pi r^2_{\text{minimum}} = \pi(3.5)^2 \approx 38.5 \text{ square feet.}$$

At most, if she wants a little bit of the tablecloth to hang over the edge, Phuong will want the radius to be 4.5 feet. This leads to an area of

$$A_{\text{maximum}} = \pi r^2_{\text{maximum}} = \pi(4.5)^2 \approx 63.6 \text{ square feet.}$$

Any amount of circular tablecloth between 38.5 and 63.6 square feet should work nicely for Phuong's project.

Checkpoint 1.1.5

The length of a side of the square is 6 inches.

What is the length of the radius of the circle?

inches

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1.1.1: Resources and Key Concepts

Resources

Videos

- Lines, Rays, Vertices, Angles, Oriented Angles, Initial Side, Terminal Side
  - Angles - Creating Angles
- Degree Measure
  - Angles - Angle Measure: Degrees
- Circles and Their Properties
  - Intermediate Algebra - Circles: The Concept

Key Concepts

Definitions

- **Ray:** A line segment with one starting point (the initial point) that extends infinitely in one direction. Also called a half-line.
- **Angle:** A figure formed when two rays share a common initial point.
- **Vertex:** The shared initial point of the two rays that form an angle.
- **Straight Angle:** An angle formed when its two rays extend in opposite directions, forming a straight line.
- **Angle Measure:** A numerical value indicating the amount of rotation that separates the two rays of an angle.
- **Oriented Angle:** An angle whose measure considers the direction of rotation from an initial side to a terminal side.
- **Initial Side:** The starting ray of an oriented angle.
- **Terminal Side:** The ending ray of an oriented angle after rotation.
- **Positive Angle:** An angle formed by a counterclockwise rotation from the initial side to the terminal side.
- **Negative Angle:** An angle formed by a clockwise rotation from the initial side to the terminal side.
- **Degree Measure:** A system for measuring angles where a full rotation is divided into 360 equal parts, with each part being one degree (1°).
- **Right Angle:** An angle that measures exactly 90°, representing a quarter-circle rotation.
- **Acute Angle:** An angle whose measure is strictly between 0° and 90°.
- **Obtuse Angle:** An angle whose measure is strictly between 90° and 180°.
- **Complementary Angles:** Two positive acute angles whose measures sum to 90°.
- **Supplementary Angles:** Two positive angles whose measures sum to 180°.
- **Decimal Degree (DD) system:** A system of measuring angles where the angle is written as a decimal number.
- **Vertical Angles:** The non-adjacent angles that are formed by the intersection of two straight lines.
- **Equality of Angles:** A condition where two angles are considered equal if their measures are equal.
- **Transversal:** A line that intersects two parallel lines.
- **Corresponding Angles:** Angles that are in the same relative position at each intersection where a transversal crosses two lines.
- **Alternate Interior Angles:** A pair of angles on opposite sides of the transversal and inside the two parallel lines.
- **Circle:** The set of all points in a plane that are the same distance (equidistant) from a central point.
- **Center:** The given point from which all points on a circle are equidistant.
- **Radius:** The distance from the center to any point on the circle. The term also refers to any line segment connecting the center to a point on the circle.
- **Diameter:** The length of a line segment that passes through the center of a circle and has both endpoints on the circle. The term also refers to the line segment itself.
- **Circumference:** The distance around a circle.

Theorems

- **Degree Measure of a Straight Angle:** The degree measure of a straight angle is 180°.
- **Congruency of Vertical Angles:** Vertical angles are equal (congruent).
- **Congruency of Corresponding Angles:** Corresponding angles are congruent.

- **Alternate Interior Angles:** If a transversal intersects parallel lines, the alternate interior angles are congruent.
- **Diameter in Terms of Radius:** For a circle with radius r and diameter d, the relationship is $d = 2r$.
- **Circumference:** The circumference C of a circle with radius r is given by the formula $C = 2\pi r$.
- **Area of a Circle:** The area A of a circle with radius r is given by the formula $A = \pi r^2$.

Common Mistakes

- **Using π for an angle:** The lowercase Greek letter π is reserved for the constant approximately equal to 3.14159 and should never be used to represent an unknown angle.
- **Using negative angles for complements/supplements:** In this textbook, complementary and supplementary angles are restricted to be positive angles only.
- **Relying on visual appearance:** Do not assume angles are acute, obtuse, or have a specific measure based on a diagram alone. Conclusions must be supported by definitions or theorems. For example, in Example 1.1.1, one cannot conclude $\angle COE$ is acute without more information.
- **Ambiguous angle notation:** Using a single letter to name an angle can be ambiguous if multiple angles share the same vertex. Use the three-point notation (e.g., $\angle BAC$) to be clear, ensuring the middle letter is the vertex.
- **Redundant language:** The word "circumference" applies only to circles, so saying "the circumference of a circle" is redundant. For other shapes, the term "perimeter" is used.

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1.1.2: Homework

Skills Refresher

The following is a set of review exercises you will need for this section.

? Skills Refresher

For the following exercises, solve the equation.

1. $x-8=19-2x$
2. $2x-9=12-x$
3. $13x+5=2x-28$
4. $4+9x=-7+x$

For the following exercises, solve the system.

5. $\begin{cases} 5x-2y &= -13 \\ 2x+3y &= -9 \end{cases}$
6. $\begin{cases} 4x+3y &= 9 \\ 3x+2y &= 8 \end{cases}$

Answers

1. 9
2. 2
3. -3
4. -2
5. $x=-3, y=-1$
6. $x=6, y=-5$

Homework

Reading Questions

1. What is a "ray"?
2. How is the "vertex" of an angle defined?
3. What distinguishes a "positive angle" from a "negative angle"?
4. How many degrees constitute a full revolution?
5. What is an "obtuse angle"?
6. What are "supplementary angles"?
7. According to the theorem presented, what is true about "vertical angles"?
8. What does a "transversal" line do?
9. What is the formula for the circumference of a circle in terms of its radius, r?
10. If a circle has a diameter d, how is its radius r related to d?

Vocabulary Check

11. (MyOpenMath) When two rays share a common initial point they form a(n) ___.

12. (MyOpenMath) Trigonometry focuses on oriented angles. An oriented angle starts from an ___ side and ends at a ___ side.
13. (MyOpenMath) ___ angles start at an initial side and rotate clockwise to the terminal side.
14. (MyOpenMath) A positive angle opens with a ___ rotation.
15. We define ___ measure by dividing the full rotation around the circle into 360 segments.
16. One-quarter of a complete revolution around a circle forms a ___ angle, which measures ___ degrees.
17. ___ angles measure between $0°$ and $90°$, while ___ angles measure between $90°$ and $180°$.
18. The degree system for measuring the size of an angle is also known as the ___ system.
19. When two lines intersect, the non-adjacent angles formed from this intersection are called ___ angles.
20. A line intersecting two parallel lines is called a ___.
21. The set of all points equidistant to a single point on a plane is called a ___.
22. The ___ of a circle is simultaneously the largest distance between any two points on a circle and is the name of the line segment that passes through the center of the circle whose endpoints lie on the circle.
23. π is a(n) ___ number (unlike 3, which is a natural number).

Concept Check

24. Greek letters like α, β, and θ are often used to represent angles in Trigonometry. Name one Greek letter that is *never* used as a label for an angle.
25. If a positive angle rotates from its initial to its terminal side, what direction is the rotation?
26. If a negative angle rotates from its initial to its terminal side, what direction is the rotation?
27. Suppose you have a straight angle and you subtract a right angle. What type of angle is the result?
28. Can a right angle be acute? How about obtuse?
29. Suppose the vertex of an angle is at the point P, and the points R and S lie on the initial and terminal sides of the angle, respectively. Which of the following are correct notations for the angle?
    i. $\angle P$
    ii. P
    iii. $m\angle P$
    iv. $\triangle P$
    v. $\angle PRS$
    vi. $\angle RPS$
    vii. $\angle SPR$
30. Quantities measured in degrees use the ___ symbol.
31. According to this text, can $107°$ have a complement? If so, what is the value of this complement?
32. According to this text, can $107°$ have a supplement? If so, what is the value of this supplement?
33. Can two acute angles be supplementary?
34. Choose two of the eight angles formed by a pair of parallel lines cut by a transversal. Those two angles are either equal or ___.
35. Write two formulas for the circumference of a circle - one involving the radius of the circle and the other involving the diameter.
36. The formula for the circumference of a circle is ___.
37. The formula for the area of a circle is ___.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

38. If α and β are vertical angles, then $\alpha + \beta = 90°$.

39. (MyOpenMath) The complement to $47°$ is $43°$.

40. The supplement to $190°$ is $-10°$.

41. Alternate interior angles sum to $180°$.

42. The distance from the center to the edge of a circle is called the radius of the circle.

43. The line connecting the center of a circle to its edge is called a radius.

44. The circumference of a square is the sum of the lengths of its sides.

45. $\pi = 3.14$

46. $\pi = \frac{22}{7}$

Basic Skills

47. (MyOpenMath) State (if possible) if the angle is acute or obtuse.

48. (MyOpenMath) Which is the closest measure to the angle shown below: $40°$, $65°$, $95°$, or 120^{ \circ }?

For the following exercises, state (if possible) which angles are acute and which are obtuse. Give the complement and the supplement of each angle, if applicable.

49. (MyOpenMath) $30°$

50. $45°$

51. $60°$

52. $43°$

53. $90°$

54. $120°$

55. $135°$

56. $143°$

57. $150°$

For the following exercises, give the complement of each angle.

58. $60°$

59. $80°$

60. $25°$

61. $18°$

62. $64°$

63. $47°$

For the following exercises, give the supplement of each angle.

64. $30°$

65. $45°$

66. $120°$

67. $25°$

68. $165°$

69. $110°$

70. (MyOpenMath) If the measure of $\angle CAB = 62°$, then find the measure of $\angle CAD$.

71. (MyOpenMath)

    a. Which angle is supplementary to $\angle BOC$?

    b. Which angle is complementary to $\angle BOC$?

    c. What is the measure of $\angle EOF$?

    d. What is the measure of $\angle AOE$?

    e. What is the measure of $\angle BOF$?

For the following exercises, use the following graph to answer the question. Assume lines r and s are parallel.

72. (MyOpenMath) $\angle 3$ is alternate interior to which other angle(s)?

73. (MyOpenMath) $\angle 5$ is a corresponding angle to which other angle(s)?

74. (MyOpenMath) $\angle 7$ and what other angle(s) are vertical angles?

75. (MyOpenMath) $\angle 8$ is alternate exterior to which other angle(s)?

76. (MyOpenMath) Find $m\angle 5$ given that $m\angle 1 = 46°$.

77. (MyOpenMath) Find $m\angle 8$ given that $m\angle 6 = 154°$.

78. (MyOpenMath) Find $m\angle 5$ given that $m\angle 3 = 82°$.

79. (MyOpenMath) Find $m\angle 3$ given that $m\angle 6 = 105°$.

80. (MyOpenMath) Find $m\angle 8$ given that $m\angle 3 = 76°$.

81. (MyOpenMath) Find the remaining angles given that $m\angle 1 = 15°$.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

82.

83.

84. (MyOpenMath) The lines x and y are parallel, and are cut by transversals z and w. Given $\angle E = 57°$, find the measure of all remaining angles.

85. (MyOpenMath) The lines x and y are parallel, and are cut by transversals z and w. Given $\angle H = 36°$ and $\angle E = 40°$, find the measure of all remaining angles.

86.

    a. Among the angles labeled 1 through 5 in the figure below, find two pairs of equal angles.

    b. $\angle 4 + \angle 2 + \angle 5 =$_____.

    c. Use parts (a) and (b) to explain why the sum of the angles of a triangle is $180°$.

87. $ABCD$ is a rectangle. The diagonals of a rectangle bisect each other. In the figure, $\angle AQD = 130°$. Find the angles labeled 1 through 5 in order, and give a reason for each answer.

For the following exercises, give an exact answer, and then round your answer to hundredths.

88. What is the area of a circle whose radius is 5 inches?

89. What is the circumference of a circle whose radius is 5 meters?
90. (MyOpenMath) What is the circumference of the following circle?
91. (MyOpenMath) What is the circumference of the following circle?
92. (MyOpenMath) What is the area of the following circle?
93. (MyOpenMath) What is the area of the following circle?

For the following exercises, find the circumference and area for each circle having the given radius r or diameter d.

94. (MyOpenMath) $r = 7$ feet
95. $r = 8.4$ centimeters
96. $d = 16$ yards
97. $d = 12$ meters
98. Find the diameter of a circle having a circumference of 4π kilometers.
99. (MyOpenMath) Find the radius of a circle having a circumference of 10 inches.
100. (MyOpenMath) Find the radius of a circle having an area of 24π square meters.
101. (MyOpenMath) Find the radius of a circle having an area of 272 square meters.
102. (MyOpenMath) Find the diameter of a circle having an area of 64 square feet.
103. Find the circumference of a circle having an area of 81π square miles.

Synthesis Questions

104. (MyOpenMath) Find the area of the shaded region. Round your answer to the nearest tenth.
105. (MyOpenMath) Find the area of the shaded region. Round your answer to the nearest tenth.
106. (MyOpenMath) Suppose the sides of the square are 8 miles in length.
    a. What is the area of the square?
    b. What is the area of the circle? Round your answer to the nearest tenth.
    c. What is the area of the blue shaded region? Round your answer to the nearest tenth.
107. State the condition(s) for which α is an acute angle, and for which α is an obtuse angle. In each case, give the complement and the supplement of α.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

108.
109.
110.
111.
112. (MyOpenMath) The lines x and y are parallel and cut by a transversal z. Given that $\angle A = k^2 + 22k$ and $\angle B = 14k + 48$, find k.
113. **Climbing Stairs.** In each of the following problems, refer to the following image of a staircase and rails. Assume the rails are perpendicular to the ground, they are parallel to each other, and that the top rail (handrail) is parallel to the bottom rail (the black line along the steps).

a. What is the relationship between α and β?

b. What is the relationship between β and θ?

c. What is the relationship between β and ϕ?

d. What is the relationship between θ and ϕ?

e. Find α if $\beta = 48°$.

f. Find θ if $\alpha = 30°$.

Applications

114. **Emergency Light.** An ambulance has a rotating light on its roof. The light rotates through one complete revolution every 2 seconds. How long does it take the light to rotate through $90°$?

115. **Rotation of the Earth (MyOpenMath).** The Earth goes through a complete circuit around the sun in approximately 365.25 days. Through how many degrees does the Earth move in one week?

116. **Earth's Rotation About the Sun.** The Earth takes 24 hours to make a complete rotation on its axis. If your math class is 2 hours and 20 minutes, through how many degrees does the Earth turn during your math class?

117. **Circumference of the Earth.** The radius of the Earth is approximately 3,960 miles. Using this approximation, find the circumference of the Earth.

118. **Earth's Rotation About the Sun.** The Earth has a nearly circular orbit about the sun. The radius of this near-circular orbit is approximately 93 million miles. Find the distance the Earth travels when it completes one full orbit about the sun.

Challenge Problems

119. (MyOpenMath) A straight piece of wire, x centimeters long, is bent into a perfect circle. Express the area of the circle, A, in terms of x.

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1.2: Triangles

https://math.libretexts.org/@go/page/197465

Note to the Instructor (click to expand)

Triangles, being the focus of Trigonometry, deserve their own full section. Focus here is on types of triangles, the Pythagorean Theorem, the special triangles, and the area of a triangle (not requiring trigonometric functions).
Many exercises involving the **30°-60°-90°** and **45°-45°-90°** that would normally be presented in Right Triangle Trigonometry are found in this section and its homework (but without reference to the trigonometric functions). Similar triangles are held off until next section.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Sets and Numbers**
  - Inequalities and Inequality Notation: This is required to understand the notations used with the Triangle Inequality.
- **Geometry**
  - Common Volumes: This skill is needed for homework problems that require finding the side length of a cube given its volume.
- **Simplifying Expressions**
  - Properties of Radicals and Simplifying Expressions Involving Roots: This skill is essential for working with the Pythagorean Theorem and special triangles, as seen in Example 8 and the theorems for special triangles.
- **Solving Equations**
  - Solving Quadratic Equations (by Extraction of Roots): This is needed for many problems within this section.
  - Solving Radical Equations: This skill is required when using the Pythagorean Theorem to find the length of a missing side of a right triangle, as the process often involves isolating a variable that is squared (e.g., Example 8).
- **Solving Inequalities**
  - Solving Linear Inequalities: This is needed for the Triangle Inequality.
  - Solving Compound Inequalities: This skill is necessary for applying the Triangle Inequality Theorem to find the possible range of lengths for a triangle's third side, as demonstrated in Example 3.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

- Find the missing angle within a triangle using the Triangle Sum.
- Use the Triangle Inequality to determine constraints on an unknown side length of a triangle.
- Use the Pythagorean Theorem to find the third side of a right triangle.
- Find the missing sides of a $30°$-$60°$-$90°$ or $45°$-$45°$-$90°$ triangle given one side.
- Solve real-world applications requiring the Pythagorean Theorem or the special triangles.

Triangle Basics

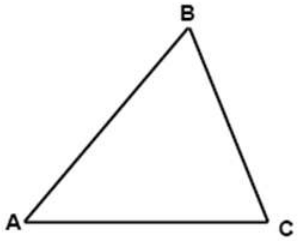
The study of Trigonometry inevitably involves understanding everything we possibly can about **triangles**.

Definition: Triangle

When three line segments bound a portion of the plane, the resulting shape is called a **triangle** (also known as a **three-sided polygon**). The line segments are called the **sides** of the triangle. A point where two sides meet is called a **vertex** of the triangle. Finally, the angle formed in the triangle's interior where two sides meet is called an **angle** of the triangle.

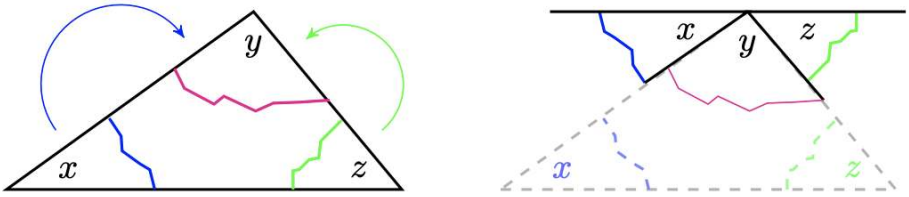
In Figure 1.2.1, the sides are denoted $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. The vertices (plural of vertex) are A, B, and C. The angles are denoted $\angle A$, $\angle B$, and $\angle C$. Using the notation $\triangle ABC$, we refer to the triangle by its vertices.

Figure 1.2.1: $\triangle ABC$



If one tears off the corners of any triangle and lines them up, as shown in Figure 1.2.2, they will always form a straight angle.

Figure 1.2.2: *Tearing corners from a triangle to form a straight angle*



Theorem: Triangle Sum

The sum of the interior angles in a triangle is $180°$.

Proof

Given the triangle, $\triangle ABC$, draw a line parallel to side $\overline{BC}$ going through vertex A (see the figure below).

Since $\angle q + \angle a + \angle p$ forms a straight angle,

$$\angle q + \angle a + \angle p = 180^\circ.$$

Moreover, the line segment $\overline{AC}$ is a transversal for the parallel lines $\overline{BC}$ and $\overline{DE}$. Therefore, $\angle c = \angle p$. By a similar argument, $\angle b = \angle q$. Hence,

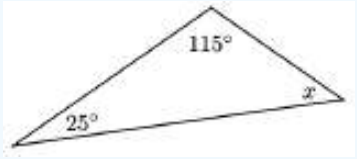
$$\angle b + \angle a + \angle c = 180^\circ.$$

Thus, the sum of the angles of a triangle is 180°.

Example 1.2.1

Two of the angles in the triangle shown below are 25° and 115°. Find the third angle.

Figure 1.2.3



Solution

To find the third angle, we write an equation.

$$\begin{array}{rrrcrl} & x + 25 + 115 & = & 180 & \text{(Triangle Sum)} \\ \text{Arithmetic} \implies & x + 140 & = & 180 & \text{(add)} \\ \cancel{\text{Arithmetic}} \rightarrow \text{Algebra} \implies & x & = & 40 & \text{(subtract 140 from both sides)} \end{array}$$

The third angle is 40°.

What does Arithmetic and ~~Arithmetic~~ mean?

To get you used to the Mathematical Mantra, every once in a while I will include the "thought processes" during solutions. Just to review, we perform Mathematics in the order we learned it - Arithmetic, Algebra, Trigonometry, ... At each step during a "mechanical" solution process, we should pause and ask ourselves if there is some simple Arithmetic to be done. If not, we move on to any Algebra that can be done. If there is no Algebra to be done, we then move on to Trigonometry.

Thus,

- "Arithmetic $\implies$" in a solution means that there is some Arithmetic we could do to clean up the previous expression
- "~~Arithmetic~~ $\rightarrow$ Algebra $\implies$" means there isn't any Arithmetic but there *is* some Algebra we could use to clean up the previous expression or equation

- "$\cancel{\text{Arithmetic}} \to \cancel{\text{Algebra}} \to \text{Trigonometry} \implies$" means there isn't any Arithmetic nor is there any Algebra that can be done to clean up the previous step, so we need to start looking at doing something from Trigonometry.

Checkpoint 1.2.1

Find all the unknown angles in each figure.

a.

Figure 1.2.4

b.

Figure 1.2.5

Answer

a. $x = 39°, 2x = 78°, 2x - 15 = 63°$

b. $\alpha = 40°, \beta = 140°, \gamma = 75°, \delta = 65°$

In any triangle, the longest side is always opposite the largest angle, and the shortest side is opposite the smallest, as illustrated in Figure **1.2.6**. Although obvious to most people, this fact will become invaluable as we move through Trigonometry.

Figure 1.2.6

Notation: Angles and Sides of a Triangle

It is common (though not necessary) to label the angles of a triangle with capital letters and the side opposite each angle with the corresponding lowercase letter, as shown in the Figure **1.2.7**. We will follow this practice unless indicated otherwise.

Figure 1.2.7

Example 1.2.2

In $\triangle FGH, \angle F = 48°$, and $\angle G$ is obtuse. Side f is 6 feet long. What can be concluded about the other sides?

Solution

Because $\angle G$ is greater than $90°$, we know that $\angle F + \angle G$ is greater than $90° + 48° = 138°$, so $\angle H$ is less than $180° - 138° = 42°$. Thus, $\angle H < \angle F < \angle G$, and consequently $h < f < g$. We can conclude that $h < 6$ feet long, and $g > 6$ feet long.

Checkpoint 1.2.2

In triangle $\triangle RST$, $\angle S$ is $72°$. The remaining two angles have the same size. Which side is longer, s or t?

Answer

s is longer

The Triangle Inequality

The sum of the lengths of any two sides of a triangle must be greater than the third side; otherwise, the two sides will not meet to form a triangle. This fact is known as the **triangle inequality**. The theorem is presented without proof (however, you can always find a suitable proof online).

Theorem: Triangle Inequality

In any triangle, we must have that

$$p + q > r,$$

where p, q, and r are the lengths of the sides of the triangle (see the figure below).

That is, the sum of the lengths of any two sides of a triangle is greater than the length of the other side.

We cannot use the triangle inequality to find the *exact* lengths of a triangle's sides, but we can find the largest and smallest possible values for the length.

Example 1.2.3

Two sides of a triangle have lengths 7 inches and 10 inches, as shown below. What can be said about the length of the third side?

Figure 1.2.8

Solution

We let x represent the length of the third side of the triangle. By looking at each side in turn, we can apply the triangle inequality in three different ways to get

$$7 < x + 10, \quad 10 < x + 7, \quad \text{and} \quad x < 10 + 7$$

We solve each of these inequalities to find

$$-3 < x, \quad 3 < x, \quad \text{and} \quad x < 17$$

We already know that $x > -3$ because x must be positive, but the other two inequalities do give us new information. The third side must be greater than 3 inches but less than 17 inches long.

Checkpoint 1.2.3

Can a triangle be made with three wooden sticks of lengths 14 feet, 26 feet, and 10 feet? Sketch a picture, and explain why or why not.

Answer

No, $10 + 14$ is not greater than 26.

Types of Triangles

There are three types of triangles classified by whether they have three equal side lengths, two equal side lengths, or no side lengths that are equal. These are called the equilateral, isosceles, and scalene triangles, respectively.

Equilateral Triangles

Definition: Equilateral Triangle

A triangle in which all three sides have the same length is called an **equilateral triangle**.

Since all three sides of an equilateral triangle have the same length, there is no smallest side for which the smallest angle will be opposite, and there is no largest side for which the largest angle will be opposite. This gives us the following theorem.

Theorem: Angles of an Equilateral Triangle

All the angles of an equilateral triangle are equal.

Example 1.2.4

All three sides of a triangle are 4 feet long. Find the angles.

Figure 1.2.9

Solution

The triangle is equilateral, so all of its angles are equal. Thus

$$\begin{aligned} 3x &= 180 && \text{(Triangle Sum)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} \quad x &= 60 && \text{(divide both sides by 3)} \end{aligned}$$

Each of the angles is $60°$.

Checkpoint 1.2.4

Find x, y, and z in the triangle.

Figure 1.2.10

Answer

$x = 60°, y = 8, z = 8$

Isosceles Triangles

Definition: Isosceles Triangle

A triangle in which at least two sides are of equal length is called an **isosceles triangle**. The angle between the equal sides is called the **vertex angle**, and the other two angles are called the **base angles**.

Theorem: Angles of an Isosceles Triangle

The base angles of an isosceles triangle are equal.

In Example 1.2.5, we introduce another common notation used throughout Geometry - the **tick mark**. We use tick marks on line segments to denote they have equal length. Similarly, tick marks on angles denote that the angles have the same measure.

Example 1.2.5

Explain why $\alpha = \beta$ in the triangle below.

Figure 1.2.11

Solution

Because they are the base angles of an isosceles triangle, θ (theta) and ϕ (phi) are equal. Also, $\alpha = \theta$ because they are vertical angles, and similarly $\beta = \phi$. Therefore, $\alpha = \beta$ because they are equal to equal quantities.

Example 1.2.6

Find x and y in the triangle.

Figure 1.2.12

Solution

The triangle is isosceles, so the base angles are equal. Therefore, $y = 38°$. To find the vertex angle, we solve

$$\begin{aligned} & x + 38 + 38 = 180 && \text{(Triangle Sum)} \\ \text{Arithmetic} \implies & x + 76 = 180 && \text{(add)} \\ \text{Arithmetic} \to \text{Algebra} \implies & x = 104 && \text{(subtract 76 from both sides)} \end{aligned}$$

The vertex angle is $104°$.

Checkpoint 1.2.6

Find x and y.

Figure 1.2.13

Answer

$x = 140°, y = 9$

Scalene Triangles

Finally, we have the **scalene** triangles.

Definition: Scalene Triangle

A triangle in which none of the sides have equal length (and, therefore, none of the angles are equal) is called a **scalene triangle**.

Figure 1.2.14 shows a general scalene triangle. The different "hash marks" on the sides indicate the side lengths are different. The same is true for the angles.

Figure 1.2.14: *A general scalene triangle*

Scalene triangles become important in this course when we learn the Law of Sines and the Law of Cosines.

Classification of Triangles by Largest Angle

Most people are not likely to refer to a triangle by its type (equilateral, isosceles, or scalene); however, knowing the features of each type of triangle is important in Trigonometry. Another classification system for triangles, and one that will be referenced

constantly in Trigonometry, is based on the size of the triangle's largest angle. Three classifications exist: right, obtuse, and acute triangles.

Obtuse Triangles

We start with the class of triangles having their largest angle greater than $90°$.

Definition: Obtuse Triangle

A triangle which contains an obtuse angle is called an **obtuse triangle**.

Recall that an obtuse angle has measure strictly between $90°$ and $180°$ (meaning that we exclude $90°$ and $180°$). Since the sum of the interior angles of a triangle must be $180°$, and one of the angles in an obtuse triangle is greater than $90°$, the two smaller angles in an obtuse triangle must sum to strictly less than $90°$.

Theorem: Angles of an Obtuse Triangle

In an obtuse triangle, one angle must be greater than $90°$, and the remaining two angles must each be less than $90°$.

Right Triangles

Reducing the largest possible angle to *exactly* $90°$, we arrive at arguably the most essential triangle classification in Trigonometry - the **right triangles**.

Definition: Right Triangle

A triangle containing a right angle is called a **right triangle**.

Example 1.2.7

One of the smaller angles of a right triangle is $34°$. What is the third angle?

Figure 1.2.15

Solution

The sum of the two smaller angles in a right triangle is $90°$. So

$$\begin{aligned} x + 34 &= 90 \\ \text{Arithmetic} \to \text{Algebra} \quad \Longrightarrow \quad x &= 56 \quad \text{(subtract 34 from both sides)} \end{aligned}$$

The unknown angle must be $56°$.

? Checkpoint 1.2.7

Two angles of a triangle are $35°$ and $45°$. Can it be a right triangle?

Answer

No.

Right Triangles: The Pythagorean Theorem

Arguably, the most important theorem in Trigonometry is the **Pythagorean Theorem**, mainly because without it, Trigonometry would not exist.

Theorem: Pythagorean Theorem

In a right triangle, if c is the length of the hypotenuse, and the lengths of the two legs are denoted by a and b (as shown in the figure below), then

$$a^2 + b^2 = c^2.$$

The proof of the Pythagorean Theorem is left as a guided exercise in the homework.

✓ Example 1.2.8

A 25-foot ladder is placed against a wall so that its foot is 7 feet from the base of the wall. How far up the wall does the ladder reach?

Solution

We sketch the situation, as shown below, and label any known dimensions. We will call the unknown height h.

Figure 1.2.16

The ladder forms the hypotenuse of a right triangle, so we can apply the Pythagorean Theorem. Substituting 25 for c, 7 for b, and h for a, we get the following:

$$
\begin{array}{lrcll}
 & a^2+b^2 & = & c^2 & \\
\text{Arithmetic}\to\text{Algebra} \implies & h^2+7^2 & = & 25^2 & \text{(substitution)} \\
\text{Arithmetic} \implies & h^2+49 & = & 625 & \text{(apply powers)} \\
\text{Arithmetic}\to\text{Algebra} \implies & h^2 & = & 576 & \text{(subtract 49 from both sides)} \\
\text{Arithmetic}\to\text{Algebra} \implies & h & = & \pm\sqrt{576} & \text{(Extraction of Roots)} \\
\text{Arithmetic}\to\text{Algebra} \implies & h & = & \pm 24 & \text{(simplify the radical)}
\end{array}
$$

The height must be a positive number, so the solution -24 does not make sense for this problem. The ladder reaches 24 feet up the wall.

? Checkpoint 1.2.8

A baseball diamond is a square with sides that are 90 feet long. When the catcher sees a runner on first trying to steal second, he throws the ball to the second baseman. Find the straight-line distance from home plate to second base.

Answer

$90\sqrt{2} \approx 127.3$ feet

Caution

The Pythagorean Theorem is valid only for right triangles!

Most people do not realize that the Pythagorean Theorem is a bijection. That is, the converse of the theorem is also true. If the sides of a triangle satisfy the relationship $a^2+b^2=c^2$, then the triangle must be a right triangle with hypotenuse c. We can use this fact to test whether a given triangle has a right angle.

✓ Example 1.2.9

Mike is paving a patio in his backyard and would like to know if the corner at C is a right angle. He measures 20 cm along one side from the corner and 48 cm along the other, placing pegs P and Q at each position, as shown below. The line joining those two pegs is 52 cm long. Is the corner a right angle?

Figure 1.2.17

Solution

If it is a right triangle, its sides must satisfy $p^2+q^2=c^2$. We find

$$
\begin{aligned}
p^2+q^2 &= 20^2+48^2 \\
&= 400+2304 \\
&= 2704
\end{aligned}
$$

and

$$
\begin{aligned}
c^2 &= 52^2 \\
&= 2704
\end{aligned}
$$

Therefore, yes, because $p^2+q^2=c^2$, the corner at C is a right angle.

Checkpoint 1.2.9

The sides of a triangle measure 15, 25, and 30 inches long. Is the triangle a right triangle?

Answer

No

The Pythagorean Theorem relates the sides of *right* triangles. However, for information about the sides of other triangles, the best we can do (without Trigonometry) is the triangle inequality. Moreover, the Pythagorean Theorem doesn't help us find the *angles* in a triangle - that comes later.

Acute Triangles

Finally, we reduce the largest possible angle within a triangle to strictly less than $90°$ and obtain the **acute triangles**.

Definition: Acute Triangle

A triangle in which all angles are less than $90°$ is called an **acute triangle**.

Special Triangles

We finally arrive at what could be considered the most essential triangles in elementary Trigonometry - the **special triangles**. These triangles are revered in Trigonometry because they allow the learner (read as "you") to perform computations without needing a calculator. This, in turn, allows you to focus on concepts rather than computations.

In order to derive one of the theorems we are about to introduce, we need a quick definition.

Definition: Altitude of a Triangle

The **altitude** of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side (or the line containing the opposite side).

The altitude of a triangle introduces a right angle, as the following theorem states.

Theorem: Altitudes and Right Angles

The altitude of a triangle represents the shortest distance from the vertex to the opposite side and forms a $90°$ angle with that side.

The $30°$-$60°$-$90°$ Triangle

Our first of our two special triangles is the **$30°$-$60°$-$90°$ triangle**.

Definition: $30°$-$60°$-$90°$ Triangle

A right triangle in which one angle is $30°$ and another angle is $60°$ is called a **$30°$-$60°$-$90°$ triangle**.

Working with this triangle allows us to know the length of all three sides if only given the length of one side.

Theorem: Side Relationships for a 30°-60°-90° Triangle

In any right triangle in which the two acute angles are $30°$ and $60°$, the hypotenuse is always twice the length of the shortest side (the side opposite the $30°$ angle), and the remaining side (opposite the $60°$ angle) is always $\sqrt{3}$ times the shortest side.

Proof

Consider an equilateral triangle with side lengths $2a$, as shown in the figure below.

Since this triangle is equilateral, each angle is $60°$. Drawing an altitude divides the triangle into two smaller triangles. Since the altitude is perpendicular to the base, each of these smaller triangles is a right triangle. Moreover, since one angle in each of these smaller right triangles is $60°$, the remaining angle must be $30°$. Furthermore, the side opposite the $30°$ angle must have length a (in the original equilateral triangle, the side opposite $60°$ had length $2a$, so it should make sense that the side opposite $30°$ is half of this). We now consider one of these smaller right triangles (see the figure below).

The missing side length, which we will call x for now, can be found using the Pythagorean Theorem.

$$\begin{aligned} & x^2 + a^2 &&= (2a)^2 \\ \Longrightarrow\ & x^2 + a^2 &&= 4a^2 \\ \Longrightarrow\ & x^2 &&= 3a^2 \\ \Longrightarrow\ & x &&= a\sqrt{3} \end{aligned}$$

Before we go any further, notice in the proof of this theorem the convention of writing $x = a\sqrt{3}$ instead of $x = \sqrt{3}a$. This is purposeful. The latter often leads the reader to think that the a is "under" the square root. It's good form always to write the radical

at the end of a term to avoid this visual ambiguity.

Example 1.2.10

Matterhorn Chocolate bars are sold in boxes shaped like triangular prisms. The two triangular ends of a box are equilateral triangles with 8-centimeter sides. When stacking the boxes on a shelf, the second level will be a triangle's altitude above the bottom level. Find the exact length of the triangle's altitude, h.

Solution

The altitude divides the triangle into two $30° - 60° - 90°$ right triangles, as shown in Figure 1.2.18 below.

Figure 1.2.18

We see that the hypotenuse is $2a = 8$, which implies the shortest side (opposite the $30°$ angle) is $a = 4$. Hence, the side opposite the $60°$ angle is $a\sqrt{3} = 4\sqrt{3} \approx 6.9282$ centimeters long.

Checkpoint 1.2.10

A ladder is leaning against a house so that the top is 12 feet above the ground, and the bottom makes a $60°$ angle with the ground. Assuming the wall of the house is perpendicular to the ground (and the ground is level), how long is the ladder? State your answer in exact form and round to two decimal places.

Answer

$\frac{24}{\sqrt{3}} \approx 13.86$ feet

Note that the exact form of the answer in Checkpoint 1.2.10 was left as $\frac{24}{\sqrt{3}}$. While it is true that you could rationalize the denominator to get

$$\frac{24}{\sqrt{3}} = \frac{24}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{24\sqrt{3}}{3} = 8\sqrt{3},$$

we will commonly leave radicals in the denominators of purely numeric expressions.

The 45°-45°-90° Triangle

The second of our our two special triangles is the **45°-45°-90° triangle**. This is also known as an **isosceles right triangle**.

Definition: Isosceles Right Triangle

An isosceles triangle in which the vertex angle is $90°$ is called an **isosceles right triangle**.

Since the right angle takes $90°$, the remaining two angles must share the remaining $90°$ evenly. That is, the remaining two angles must be $45°$ each.

Definition: 45°-45°-90° Triangle

Since an isosceles right triangle has two angles that are **45°**, it is commonly called a **45°-45°-90° triangle**.

Just as with our first special triangle, knowing one side of a **45°-45°-90°** triangle allows us to easily compute the lengths of the remaining sides.

Theorem: Side Relationship for a 45°-45°-90° Triangle

In any right triangle in which the two acute angles are **45°**, the hypotenuse is always $\sqrt{2}$ times the side length.

Proof

The sides opposite the **45°** angles in a **45°-45°-90°** triangle must have the same length. Let a be the lengths of these sides and x be the length of the hypotenuse, as shown in the figure below.

Since this is a right triangle, we can use the Pythagorean Theorem to find the length of the hypotenuse in terms of a.

$$\begin{aligned} a^2 + a^2 &= x^2 \\ \Longrightarrow \quad 2a^2 &= x^2 \\ \Longrightarrow \quad a\sqrt{2} &= x \end{aligned}$$

Example 1.2.11

You are hired to build a truss to support a balcony. Figure **1.2.19** shows the design plans you have submitted for the project. What is the length of the beam needed from point A to point D?

Figure 1.2.19

Solution

$\triangle ABD$ is a $45°$-$45°$-$90°$ triangle. Therefore, if the leg length (opposite the $45°$ angle) is $a = 4$ meters, then beam between points A and D must measure $a\sqrt{2} = 4\sqrt{2} \approx 5.66$ meters.

Checkpoint 1.2.11

Suppose, instead of knowing the 4-meter lengths in Example 1.2.11, we knew that the length of the beam from A to D was 12 meters. How long is the beam from B to D?

Answer

$\frac{12}{\sqrt{2}} \approx 8.49$ meters

Area of a Triangle

Our final review topic for triangles in this section is something necessary at multiple points throughout Trigonometry and Calculus - the area of a triangle.

Theorem: Area of a Triangle

Let b be the length of one side of a triangle and h be the altitude height from that side to its opposite vertex. Then the area of the triangle is given by

$$A = \frac{1}{2}bh.$$

Proof

To be done in homework.

Example 1.2.12

Compute the area of the triangle in Figure 1.2.20.

Figure 1.2.20

Solution

We are given the length of one leg of this triangle as 11 inches. Let's call this leg the base. Thus, $b = 11$. The altitude from the base to its opposite vertex has length 8 inches. Hence, $h = 8$. Therefore, the area of this triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(11)(8) = 44 \text{ square inches.}$$

Checkpoint 1.2.12

Compute the area of the triangle in Figure 1.2.21 if we know that $c = 5.24$ and $h = 1.25$.

Figure 1.2.21

Answer

3.28125

1.2.1: Resources and Key Concepts

Resources

Videos

- Triangles, their types, and the Pythagorean Theorem
  - Right Triangle Trigonometry - Triangles
- Special Triangles
  - Right Triangle Trigonometry - Special Triangles in Trigonometry

Key Concepts

Definitions

- **Triangle:** A three-sided polygon formed by three line segments bounding a portion of a plane. The line segments are its sides, the points where sides meet are its vertices, and the interior angles are its angles.
- **Equilateral Triangle:** A triangle where all three sides have the same length.
- **Isosceles Triangle:** A triangle where at least two sides have equal length. The angle between the two equal sides is the vertex angle, and the other two angles are the base angles.
- **Scalene Triangle:** A triangle where no sides have equal length.
- **Obtuse Triangle:** A triangle that contains one obtuse angle (an angle greater than 90°).
- **Right Triangle:** A triangle that contains one right angle (an angle of exactly 90°).
- **Acute Triangle:** A triangle in which all three angles are acute (less than 90°).
- **30°-60°-90° Triangle:** A right triangle whose acute angles measure 30° and 60°.
- **Isosceles Right Triangle:** An isosceles triangle where the vertex angle is 90°.
- **45°-45°-90° Triangle:** A common name for an isosceles right triangle, since its two other angles must be 45°.

Theorems

- **Triangle Sum:** The sum of the interior angles in a triangle is 180°.
- **Triangle Inequality:** In any triangle with side lengths p, q, and r, the sum of the lengths of any two sides is greater than the length of the third side ($p + q > r$).
- **Angles of an Equilateral Triangle:** All the angles of an equilateral triangle are equal.
- **Angles of an Isosceles Triangle:** The base angles of an isosceles triangle are equal.
- **Angles of an Obtuse Triangle:** In an obtuse triangle, one angle must be greater than 90°, and the remaining two angles must each be less than 90°.
- **Pythagorean Theorem:** In a right triangle, if c is the length of the hypotenuse and the lengths of the two legs are a and b, then $a^2 + b^2 = c^2$.
- **Converse of the Pythagorean Theorem:** If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle must be a right triangle with hypotenuse c.
- **Side Relationships for a 30°-60°-90° Triangle:** In any 30°-60°-90° triangle, the hypotenuse is twice the length of the shortest side (the side opposite the 30° angle), and the side opposite the 60° angle is $\sqrt{3}$ times the length of the shortest side.
- **Side Relationship for a 45°-45°-90° Triangle:** In any 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times the length of a leg.

Common Mistakes

- **Misapplying the Pythagorean Theorem:** The Pythagorean Theorem is only valid for right triangles. It cannot be used to find side lengths in acute or obtuse triangles.
- **Confusing the Triangle Inequality with the Pythagorean Theorem:** The Triangle Inequality theorem determines if a triangle can be formed from three side lengths. The Converse of the Pythagorean Theorem determines if a triangle is a right triangle. A triangle can exist without being a right triangle.
- **Radical Notation Ambiguity:** When a term includes a variable and a radical, it is best to write the radical last to avoid confusion. For example, write $a\sqrt{3}$ instead of $\sqrt{3}a$, as the latter might be misinterpreted as $\sqrt{3a}$.

- **Confusing Isosceles Triangle Angles:** The vertex angle of an isosceles triangle is the angle between the two equal sides. The base angles are opposite the equal sides. The base angles are equal to each other, but not necessarily equal to the vertex angle.

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1.2.2: HOMEWORK

SKILLS REFRESHER

The following is a set of review exercises you will need for this section.

? SKILLS REFRESHER

For the following exercises, solve the inequality.

1. $6 - x > 3$
2. $\frac{-3x}{4} \geq -6$
3. $3x - 7 \leq -10$
4. $4 - 3x < 2x + 9$

For the following exercises, assume that $x < 0$. If this is the case, is the expression positive or negative?

5. $-x$
6. $-(-x)$
7. $|x|$
8. $-|x|$
9. $-|-x|$
10. x^{-1}

Answers

1. $x < 3$
2. $x \leq 8$
3. $x \leq -1$
4. $x > -1$
5. Positive
6. Negative
7. Positive
8. Negative
9. Negative
10. Negative

HOMEWORK

READING QUESTIONS

1. What is the sum of the interior angles in any triangle?
2. What does the Triangle Inequality Theorem state about the side lengths of a triangle?
3. What is the definition of an equilateral triangle?
4. In an isosceles triangle, which angles are referred to as the "base angles," and what is true about their measures?
5. What is the name for the side opposite the right angle in a right triangle?
6. State the Pythagorean Theorem using the terms hypotenuse and legs.
7. If you are given the three side lengths of a triangle, how can you test if it is a right triangle?
8. Is it possible for a triangle to contain two right angles? Why or why not?
9. In a 30°-60°-90° triangle, which side is the longest?
10. What is the relationship between the hypotenuse and the shortest side in a 30°-60°-90° triangle?
11. What is another name for a 45°-45°-90° triangle?
12. If the two legs of a 45°-45°-90° triangle each have length a, what is the length of the hypotenuse?

VOCABULARY CHECK

13. A three-sided polygon is another way of saying ___.
14. (MyOpenMath) If given the triangle $\triangle ABC$, it is common notation to label the side opposite $\angle A$ with the letter ___.
15. (MyOpenMath) A triangle where all three sides have the same length is called a ___ triangle.
16. An isosceles triangle has two ___ angles.
17. The ___ angle of an isosceles triangle lies between the two sides of equal length.
18. (MyOpenMath) A(n) ___ triangle must have one angle ___ **90°**.
19. A(n) ___ triangle must have all angles ___ **90°**.
20. The two special triangles are the ___-**90°** and ___-**90°** triangles.
21. Another name for the **45°-45°-90°** triangle is a(n) ___ triangle.

CONCEPT CHECK

22. Is it always true that the hypotenuse is the longest side in a right triangle? Why or why not?
23. In $\triangle DEF$, is it possible that $d + e > f$ and $e + f > d$ are both true? Explain your answer.
24. In a right triangle with hypotenuse c, we know that $a^2 + b^2 = c^2$. Is it also true that $a + b = c$? Why or why not?
25. Is having more than one obtuse angle in a triangle possible? Why or why not?
26. Draw any quadrilateral (a four-sided polygon) and divide it into two triangles by connecting two opposite vertices by a diagonal. What is the sum of the angles in your quadrilateral?
27. What is the difference between a vertex and vertical angles?
28. In any triangle, the sum of the three angles is ___.
29. The longest side in a triangle is opposite the ___ angle, and the shortest side is opposite the ___ angle.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem

stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

30. A right triangle always has one angle of $90°$.
31. Given a triangle with side lengths a, b, and c, where c is the longest side,
$$a^2 + b^2 = c^2.$$
32. All of the angles of an equilateral triangle are equal.
33. The vertex angles of an isosceles triangle are equal.
34. If the sum of the lengths of any two sides is greater than the length of the third side, then the triangle must be a right triangle.
35. In $\triangle ABC$, $a^2 + b^2 = c^2$.
36. If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle is a right triangle.

BASIC SKILLS

For the following exercises, sketch and label a triangle with the given properties.

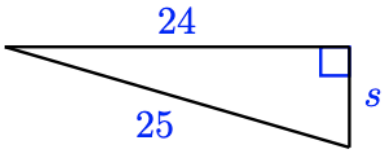
37. An isosceles triangle with vertex angle $30°$
38. A scalene triangle with one obtuse angle
39. (MyOpenMath) A right triangle with legs 4 and 7
40. (MyOpenMath) A right triangle with hypotenuse 12, where one leg is of length 8.
41. An isosceles right triangle
42. An isosceles triangle with one obtuse angle
43. A right triangle with one angle $20°$
44. (MyOpenMath) The triangle $\triangle ABC$ has $m\angle A = 88°$ and $m\angle B = 87°$, find $m\angle C$.

For the following exercises, find the unknown angle.

45. (MyOpenMath)
46.
47. (MyOpenMath)
48.
49.
50. (MyOpenMath)
51. (MyOpenMath)
52. If two sides of a triangle are 6 feet and 10 feet long, what are the largest and smallest possible values for the length of the third side?
53. Two adjacent sides of a parallelogram are 3 cm and 4 cm long. What are the largest and smallest possible values for the length of the diagonal?
54. If one of the equal sides of an isosceles triangle is 8 millimeters long, what are the largest and smallest possible values for the length of the base?

For the following exercises, find the unknown side of the triangle.

55.
56. (MyOpenMath)
57.
58. (MyOpenMath)
59.



60. Find α and β.
61. (MyOpenMath) Find x and y.
62. (MyOpenMath) Find x and y.

For the following exercises, decide whether a triangle with the given sides is a right triangle.

63. (MyOpenMath) 9 in, 16 in, 25 in
64. 12 m, 16 m, 20 m
65. 5 m, 12 m, 13 m
66. 5 ft, 8 ft, 13 ft
67. 5^2 ft, 8^2 ft, 13^2 ft
68. $\sqrt{5}$ ft, $\sqrt{8}$ ft, $\sqrt{13}$ ft
69. (MyOpenMath) Compute the exact value of the altitude h in an equilateral triangle, where all sides are 14 inches long.

SYNTHESIS QUESTIONS

70. (MyOpenMath) In a certain triangle, angle A is three times as big as angle C and angle B is $55°$ more than angle C. What is the degree measure of angle B?
71. The two shorter sides of an obtuse triangle are 3 in and 4 in. What are the possible lengths for the third side?

For the following exercises, explain why the measurements shown cannot be accurate.

72.
73.
74.
75.
76.

77.

78.

79.

80.

For the following exercises, find the unknown angle. Arrows on a pair of lines indicate that they are parallel.

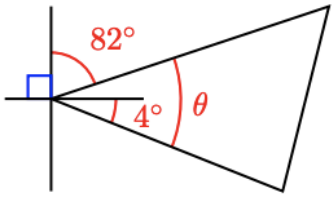
81.

82.

83.

84.

85.



86.

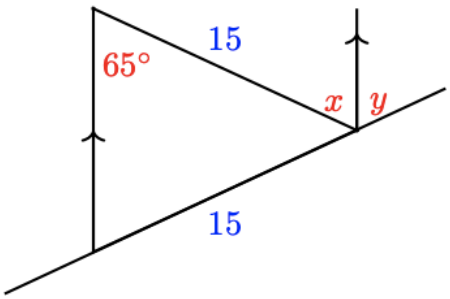
87.

88.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

89.

90.



91.

92.

93. How long is the diagonal of a square whose side is 6 centimeters?

94. How long is the side of a cube whose volume is 80 cubic feet?

For the following exercises, find the value of x.

95. (MyOpenMath)

96. (MyOpenMath)

97. (MyOpenMath)

For the following exercises, find the length of the indicated side(s).

98. (MyOpenMath) x and y

99. (MyOpenMath) $\overline{TY}$ and $\overline{TX}$

100. (MyOpenMath) $\overline{AY}$ and $\overline{YB}$

101. (MyOpenMath) x and y

For the following exercises, the angle labeled ϕ is called an exterior angle of the triangle, formed by one side and the extension of an adjacent side. Find ϕ.

102.

103.

104.

105.

For the following exercises, make a sketch of the situation described and label a right triangle. Then use the Pythagorean Theorem to solve the problem.

106. The diagonal of a square is 12 inches long. How long is the side of the square?

107. The length of a rectangle is twice its width, and its diagonal is meters long. Find the dimensions of the rectangle.

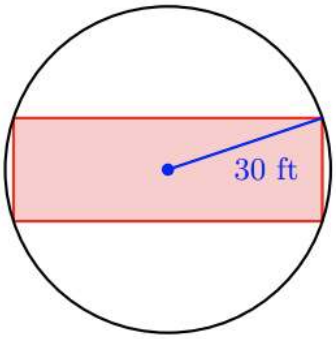
108. (MyOpenMath) The hypotenuse of right triangle is 146 centimeters long. The difference between the other two sides is 14 centimeters. Find the exact values of the missing sides.

109. (MyOpenMath) The shortest side of a right triangle is 9 centimeters long. The difference between the lengths of the other two sides is 1 centimeter. Find the exact values of the missing side and hypotenuse.

110. (MyOpenMath) The shaded, half circular disk is drawn on the side of a right triangle, whose legs are labeled 5.3 meters and 10.1 meters. Find the area of this half disk. Round your answer to one decimal place.

111. (MyOpenMath) Imagine that for the right triangle shown below, each of its three sides is bent into a perfect circle. What would be the *combined total area* of all three circles? Round your answer to one decimal place.

112. What size rectangle can be inscribed in a circle of radius 30 feet if the length of the rectangle must be three times its width?



113. What size square can be inscribed inside a circle of radius 8 inches, so that its vertices just touch the circle?

114. Find α, β and h.

115. Find α, β and d.

116. (MyOpenMath) Find the slant height (labeled x) of the right circular cone.

117. (MyOpenMath) Compute the radius of the base of the right circular cone if given the slant height of 20 cm and the cone height of 16 cm.

118. (MyOpenMath) A right square pyramid is shown. The height of the pyramid is 8 units. The distance from the center of the base of the pyramid to vertex B is 15 units, as shown. Find the length of segment $\overline{AB}$.

119. (MyOpenMath) Compute the exact value of the height h of the square-based straight pyramid, given that the base is a square with sides $a = 36$ inches long, and all other edges are $b = 54$ inches long.

120. Find the diagonal of a cube of side 8 inches. Hint: Find the diagonal of the base first.

121. Find the diagonal of a rectangular box whose sides are 6 cm by 8 cm by 10 cm. Hint: Find the diagonal of the base first.

For the following exercises, $\triangle ABC$ is equilateral. Find the unknown angles.

122.

123.

124. Consider the following figure.

    a. Compute $2\theta + 2\phi$.

    b. Compute $\theta + \phi$.

    c. What must be true about $\triangle ABC$?

125. Consider the following figure.

    a. Explain why $\angle OAB$ and $\angle ABO$ are equal in measure.

    b. Explain why $\angle OBC$ and $\angle BCO$ are equal in measure.

    c. Explain why $\angle ABC$ is a right angle.

126. a. In the figure below, find θ, and justify your answer.

    b. Write an algebraic expression for θ in the figure below.

APPLICATIONS

127. **Bounding Distances.** The town of Madison is 15 miles from Newton, and 20 miles from Lewis. What are the possible values for the distance from Lewis to Newton?

For the following exercises, make a sketch of the situation described and label a right triangle. Then use the Pythagorean Theorem to solve the problem.

128. **TV Dimensions.** The size of a TV screen is the length of its diagonal. If the width of a 35-inch TV screen is 28 inches, what is its height?

129. **TV Dimensions (MyOpenMath).** Piyali goes to the store to buy a new TV. She tells the salesperson that she needs a TV that has a maximum width of 37 inches and a maximum height of 28 inches. Assuming that she can get a TV with these maximum dimensions, and knowing that TVs are measured by the diagonal of the screen, what size TV can Piyali buy?

130. **Tree Shadows.** If a 30-meter pine tree casts a shadow of 30 meters, how far is the tip of the shadow from the top of the tree?

131. **Utility Pole (MyOpenMath).** A utility pole is 9 m high. A cable is stretched from the top of the pole to a point in the ground that is 8 m from the bottom of the pole. How long is the cable?

132. **Distance Between Towns (MyOpenMath).** If Nhat is 22 miles due west of Sacramento and Loi is 30 miles due north of Sacramento, what is the shortest distance from Loi to Nhat?

133. **Altitude (MyOpenMath).** What is the altitude of the plane?

134. **Ladder Length (MyOpenMath).** What is the minimum length the ladder must be to reach the roof?

135. A 24-foot flagpole is being raised by a rope and pulley, as shown in the figure. The loose end of the rope can be secured to a ring on the ground 7 feet from the base of the pole. From the ring to the top of the pulley, how long should the rope be when the flagpole is vertical?

136. To check whether the corners of a frame are square, carpenters sometimes measure the sides of a triangle, with two sides meeting at the join of the boards. Is the corner shown in the figure square?

For the following exercises, make a sketch and solve.

137. **Pesky Pipe.** The back of Ron's pickup truck is five feet wide and seven feet long. He wants to bring home a 9-foot length of copper pipe. Will it lie flat on the floor of the truck?

138. **Pesky Pipe.** Mary has a pickup truck with a camper shell. The camper shell floor is five feet wide and seven feet long, but the shell is only three feet tall. Will a 9-foot copper pipe fit diagonally across the back of the truck?

139. What is the longest curtain rod that will fit inside a box 60 inches long by 10 inches wide by 4 inches tall?

CHALLENGE PROBLEMS

140. (MyOpenMath) A sidewalk goes all the way around a rectangular field, as shown below (not drawn to scale). The field is 65 feet long and 45 feet wide. You need to get from the bottom-left corner to the top-right corner. Your walking speed is 7 feet per second along the sidewalk and 4 feet per second through the field.

    a. How many seconds will it take you if you walk from the bottom-left corner to the top-right corner directly across the field?

    b. How many seconds will it take you if you walk all 65 feet along the sidewalk (from bottom-left to bottom-right) and then 45 feet along the sidewalk (from bottom-right to top-right)?

    c. Suppose you walk x feet along the sidewalk and then turn to walk the remaining distance through the field directly to the top-right corner. Write an expression in x representing the number of seconds it would take you to walk this path.

141. (MyOpenMath) Find the values of x, y, and z to two decimal places.

142. (MyOpenMath) Find the values of x, y, and z to two decimal places.

143. (MyOpenMath) Find the values of x, y, and z to two decimal places.

144. (MyOpenMath) Find the values of x, y, and z to two decimal places.

145. (MyOpenMath) Find the values of x, y, and z to two decimal places.

146. (MyOpenMath) An arch is in the shape of a semicircle. At a point along the base 4 feet from an end of the arch, the height of the arch is 8 feet. Find the exact value of the maximum height of the arch.

147. Consider the following figure.

    a. Compare θ with $\alpha + \beta$. (Hint: What do you know about supplementary angles and the sum of angles in a triangle?

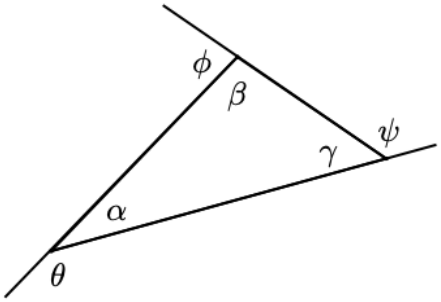
    b. Compare α and β.

    c. Explain why the inscribed angle $\angle BAO$ is half the size of the central angle $\angle BOC$.

148. Find an algebraic expression for ϕ and use your answer to write a rule for finding an exterior angle of a triangle.

149. a. Find the three exterior angles of the triangle. What is the sum of the exterior angles?

    b. Write an algebraic expression for each exterior angle in terms of one of the angles of the triangle. What is the sum of the exterior angles?



150. (MyOpenMath) The area of square $ABCD$ is one square unit. Triangle $\triangle PCQ$ is equilateral. What are the exact lengths of the sides of the triangle?

151. **Proving the Pythagorean Theorem.** There are many proofs of the Pythagorean Theorem. Here is a simple visual argument.

    a. What is the length of the side of the large square in the figure? Write an expression for its area.

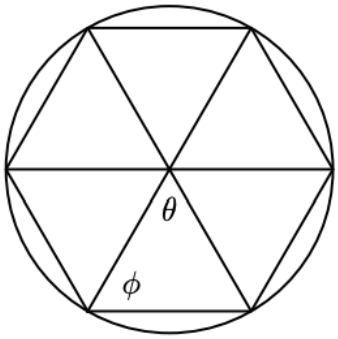
    b. Write another expression for the area of the large square by adding the areas of the four right triangles and the smaller central square.

    c. Equate your two expressions for the area of the large square, and deduce the Pythagorean Theorem.

For the following exercises, the figures inscribed in each problem are regular polygons, which means that all their sides are the same length, and all the angles have the same measure. Find the angles θ and ϕ.

152.

153.



154. A tangent meets the radius of a circle at a right angle. In the figure, $\angle AOB = 140°$. Find the angles labeled 1 through 5 in order, and give a reason for each answer.

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1.3: Similar Triangles

Note to the Instructor (click to expand)

This entire section focuses solely on similar triangles and their applications.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Arithmetic**
  - Fractions: Working with similar triangles often requires a strong grasp of fractions and fractional arithmetic.
- **Solving Equations**
  - Solving Rational Equations: This skill is fundamental to this section. It is required to solve the proportions that arise from similar triangles, as seen in the "Skills Refresher" and throughout the examples and homework (e.g., Example 4, where the proportion $\frac{x}{4} = \frac{15}{6}$ is solved).

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Determine when two triangles are congruent.
- Use proportion to find a side given two similar triangles.
- Solve application problems involving similar right triangles.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Congruent Triangles

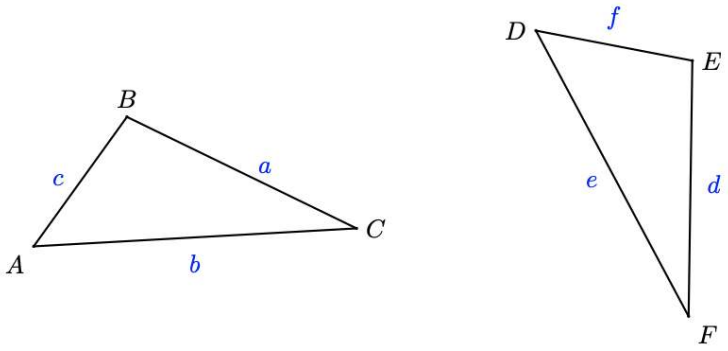
We begin this section by identifying what it means for two triangles to be **congruent**.

Definition: Congruent Triangles

Two triangles are said to be **congruent** if their corresponding angles are equal, and their corresponding sides have the same lengths. Symbolically, we use the congruency symbol $\cong$ to state that two triangles are congruent.

In simple terms, two triangles are considered congruent if they have the same size and shape. For example, it looks as though $\triangle ABC$ and $\triangle DEF$ in Figure **1.3.1** are identical in both shape and size ($\triangle DEF$ has been rotated from $\triangle ABC$), so they are congruent. We write $\triangle ABC \cong \triangle DEF$.

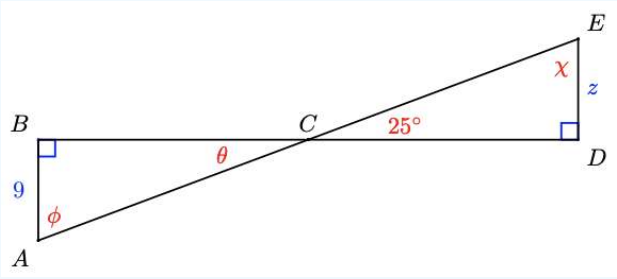
Figure 1.3.1: *Congruent triangles* $\triangle ABC$ *and* $\triangle DEF$



Example 1.3.1

The two triangles in Figure 1.3.2 below are congruent. List the corresponding parts, and find the angles θ, ϕ, and χ, and side z.

Figure 1.3.2



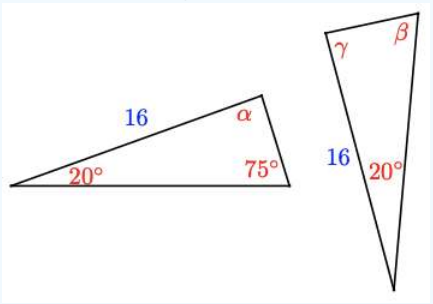
Solution

In these triangles, $\angle B = \angle D$ because they are both right angles and $\angle BCA = \angle DCE$ because they are vertical angles, so $\theta = 25°$. The third angles, $\angle A$ and $\angle E$, must also be equal, so $\phi = \chi = 65°$. (Do you see why?) The sides opposite each pair of corresponding angles are equal, so $AB = DE, BC = CD$ and $AC = CE$. In particular, we find that $z = 9$.

Checkpoint 1.3.1

The two triangles in Figure 1.3.3 below are congruent. Find the values of α, β, and γ.

Figure 1.3.3



Answer

$\alpha = 85°, \beta = 75°, \gamma = 85°$

Recall that the altitude of a triangle is the segment from one vertex of the triangle perpendicular to the opposite side.

Example 1.3.2

Show that the altitude of an equilateral triangle divides it into two congruent right triangles.

Solution

Figure 1.3.4

Consider, for example, an equilateral triangle of side 8 inches, as shown above. The altitude is perpendicular to the base, so each half of the original triangle is a right triangle. Because each right triangle contains a $60°$ angle, the remaining angle in each triangle must be $90° - 60° = 30°$. Both triangles have a side of length 8 between the angles of $30°$ and $60°$, so they are congruent. (Consequently, the short sides of the congruent triangles are equal, so each is half the original base.)

For the following Checkpoint, you should recall the following definition from your Geometry class.

Definition: Parallelogram

A **parallelogram** is a quadrilateral with two pairs of parallel sides. The opposite or facing sides of a parallelogram are of equal length and the opposite angles of a parallelogram are of equal measure.

Checkpoint 1.3.2

The diagonal of a parallelogram divides it into two congruent triangles, as shown in Figure 1.3.5 below. List the corresponding parts of the two triangles and explain why each pair is equal.

Figure 1.3.5

Answer

$\angle BCA = \angle CAD$ and $\angle BAC = \angle ACD$ because they are alternate interior angles. If two angles in a triangle are equal, so is the third pair, so $\angle B = \angle D$. $BC = AD$ and $AB = CD$ because they are opposite sides of a parallelogram, and $AC = AC$.

Similar Triangles

The most common comparison of triangles in both Trigonometry and Calculus will be with **similar triangles**.

Definition: Similar Triangles

Two triangles are **similar** if their corresponding angles are equal.

The definition of similar triangles looks almost like the definition of congruent triangles; however, similarity between triangles is a weaker form of congruence - the triangles have the same shape but might be sized differently. This flexibility grants us much power in mathematics.

Fun Fact!

Congruent triangles are similar, but similar triangles need not be congruent.

The corresponding angles between similar triangles are equal, and the corresponding sides are proportional. We can think of one similar triangle as an enlargement or a reduction of the other (see Figure **1.3.6** below).

Figure 1.3.6: *Proportional relationships between similar triangles*

To decide whether two triangles are similar, we must verify *only one* of the two similarity conditions (the other condition will be true automatically).

Theorem: Similarity Conditions

Two triangles are similar if either their corresponding angles are equal or their corresponding sides are proportional.

Example 1.3.3

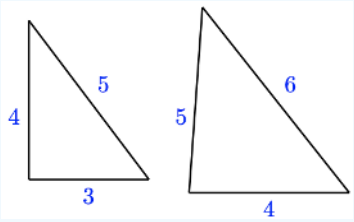
Which of the pairs of triangles shown below are similar?

a.

Figure 1.3.7

b.

Figure 1.3.8



c.

Figure 1.3.9

Solutions

a. We will check whether the corresponding sides are proportional. We compute the ratios of the corresponding sides, making sure to write each ratio in the same order, $\frac{\text{larger triangle}}{\text{smaller triangle}}$. (The other order, $\frac{\text{smaller triangle}}{\text{larger triangle}}$, would also work, as long as we use the same order for all the ratios.)

$$\text{shorter legs: } \frac{9}{6} \quad \text{longer legs: } \frac{12}{8} \quad \text{hypotenuses: } \frac{15}{10}$$

Because all of these ratios are equal to 1.5, the triangles are similar.

b. The ratios of corresponding sides are not equal: the ratio of the longest sides is $\frac{6}{5}$, but the ratio of the smallest sides is $\frac{4}{3}$. The triangles are not similar.
c. The missing angle of the first right triangle is $48°$, and the missing angle in the second right triangle is $42°$, so three pairs of angles match. The triangles are similar.

Checkpoint 1.3.3

Are the triangles below similar? Explain why or why not in each case.

a.

Figure 1.3.10

b.

Figure 1.3.11

Answers

a. The triangles are similar because $\frac{4}{6} = \frac{6}{9} = \frac{8}{12}$, so the sides are proportional.
b. The third angle in both triangles is $80°$, so the triangles are similar because their corresponding angles are equal.[a]

In part (b) of Checkpoint 1.3.3, the third angle in each triangle must be $80°$ because the sum of the angles is $180°$. Thus, we only need to show that *two* pairs of angles are equal to show that two triangles are similar.

Using Proportions with Similar Triangles

Figure 1.3.12 shows a parallelogram $ABCD$ and two triangles, $\triangle ABE$ and $\triangle FCE$. Can we find the unknown lengths x and y in the larger triangle?

Figure 1.3.12

First, note that two pairs of corresponding angles in the triangles are equal: $\angle BEA$ and $\angle FEC$ are vertical angles, and $\angle EFC$ and $\angle BAE$ are alternate interior angles. However, if two pairs of corresponding angles are equal, the third pair must also be equal. This means that the two triangles are similar, and we can use the fact that their corresponding sides are proportional to find x and y.

Example 1.3.4

Find the value of x in Figure 1.3.12.

Solution

We see that x is the length of the shortest side in $\triangle ABE$. We know the short side in $\triangle FCE$ and the lengths of the medium sides in each triangle. If we form the ratios of the short and medium sides, we obtain the following proportion.

$$\frac{\text{larger triangle}}{\text{smaller triangle}} : \frac{x}{4} = \frac{15}{6}$$

To solve the proportion, we multiply both sides by the LCD to get

$$\begin{aligned} & & 6x &= 4(15) & \\ \text{Arithmetic} \quad & \Longrightarrow & 6x &= 60 & \text{(multiply)} \\ \cancel{\text{Arithmetic}} \rightarrow \text{Algebra} \quad & \Longrightarrow & x &= 10 & \text{(divide both sides by 6)} \end{aligned}$$

What does the Arithmetic and ~~Arithmetic~~ mean?

To get you used to the Mathematical Mantra, every once in a while I will include the "thought processes" during solutions. Just to review, we perform Mathematics in the order we learned it - Arithmetic, Algebra, Trigonometry, ... At each step during a "mechanical" solution process, we should pause and ask ourselves if there is some simple Arithmetic to be done. If not, we move on to any Algebra that can be done. If there is no Algebra to be done, we then move on to Trigonometry.

Thus,

- "Arithmetic $\Longrightarrow$ " in a solution means that there is some Arithmetic we could do to clean up the previous expression
- "~~Arithmetic~~ $\rightarrow$ Algebra $\Longrightarrow$ " means there isn't any Arithmetic but there *is* some Algebra we could use to clean up the previous expression or equation
- "~~Arithmetic~~ $\rightarrow$ ~~Algebra~~ $\rightarrow$ Trigonometry $\Longrightarrow$ " means there isn't any Arithmetic nor is there any Algebra that can be done to clean up the previous step, so we need to start looking at doing something from Trigonometry.

Checkpoint 1.3.4

Find the value of y in the previous example.

Answer

$y = 20$

Similar Right Triangles

If two *right* triangles have one pair of corresponding acute angles with the same measure, they are similar. We can use this fact about right triangles to make indirect measurements.

Example 1.3.5

Ivan wants to know the height of a particular building. He asks Kim to hold up a 5-foot pole near the building and measure the length of its shadow. The shadow of the pole is 3 feet long, and the shadow of the building is 12 feet long.

Figure 1.3.13

a. Use similar triangles to write a proportion involving the height of the building.
b. Solve the proportion to find the height of the building.

Solutions

a. In Figure **1.3.13**, we see two right triangles: One triangle is formed by the building and its shadow, and the pole and its shadow form the other. Because the light rays from the sun are parallel, the two angles at the tips of the shadows are equal. Thus, the two right triangles are similar, and their corresponding sides are proportional. The ratios of heights and bases in the two triangles yield the proportion

$$\frac{\text{larger triangle}}{\text{smaller triangle}} : \frac{h}{5} = \frac{12}{3}$$

b. To solve the proportion, we multiply both sides by the LCD to get

$$\begin{aligned} &\xRightarrow{\text{Arithmetic}\to\text{Algebra}} && 3h &&= 5(12) && \text{(multiply both sides by the LCD of all terms in the equation)} \\ &\xRightarrow{\text{Arithmetic}} && 3h &&= 60 && \text{(multiply)} \\ &\xRightarrow{\text{Arithmetic}\to\text{Algebra}} && h &&= 20 && \text{(divide both sides by 3)} \end{aligned}$$

The building is 20 feet tall.

? Checkpoint 1.3.5

In Example **1.3.2**, we created a $30° - 60° - 90°$ triangle in which the shorter leg was 4 inches and the hypotenuse was 8 inches. The hypotenuse of another $30° - 60° - 90°$ triangle is 5 feet. What is the length of the side opposite the $30°$ angle?

Answer

2.5 feet

Overlapping Triangles

In some applications, similar triangles may share a side or an angle.

✓ Example 1.3.6

Identify two similar triangles in Figure **1.3.14** below, and write a proportion to find H.

Figure 1.3.14

Solution

The two triangles overlap, sharing the marked angle, as shown below. Because each triangle also has a right angle, they are similar.

Figure 1.3.15

Note that the base of the larger triangle is $24+12=36$. The ratio of the heights and the ratio of the bases must be equal, so we write the following proportion.

$$\frac{H}{10} = \frac{36}{24} \quad \text{(start with the proportion)}$$

$$\xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad 24H = 360 \quad \text{(multiply both sides by the LCD)}$$

$$\xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad H = \frac{360}{24} \quad \text{(divide both sides by 24)}$$

$$\xRightarrow{\text{Arithmetic}} \quad H = 15 \quad \text{(division)}$$

? Checkpoint 1.3.6

Min wants to know the height of a street lamp. She discovers that when she is 12 feet from the lamp, her shadow is 6 feet long. Find the height of the street lamp.

Figure 1.3.16

Answer

15 feet

1.3.1: RESOURCES AND KEY CONCEPTS

KEY CONCEPTS

DEFINITIONS

- **Congruent Triangles:** Two triangles that have equal corresponding angles and identical corresponding side lengths. They are the same shape and size.
- **Parallelogram:** A four-sided figure (quadrilateral) in which both pairs of opposite sides are parallel and have equal length.
- **Similar Triangles:** Two triangles whose corresponding angles are equal. This implies they have the same shape but may be different sizes.

THEOREMS

- **Similarity Conditions:** Two triangles are determined to be similar if one of two conditions is met: either their corresponding angles are equal, or their corresponding sides are proportional. If one condition is true, the other is automatically true.

COMMON MISTAKES

- **Confusing Congruent and Similar:** While congruent triangles are always similar, similar triangles are not necessarily congruent. Similarity requires only the same shape (equal angles), not necessarily the same size.
- **Incorrectly Setting Up Proportions:** When creating ratios of corresponding sides, the order must be consistent. For example, if the first ratio is $\frac{\text{larger triangle side}}{\text{smaller triangle side}}$, all subsequent ratios in the proportion must follow the same order.
- **Using Incorrect Side Lengths in Overlapping Triangles:** When triangles overlap, be careful to identify the full side length of the larger triangle. For instance, in Example 6, the base of the larger triangle is the sum of two segments ($24 + 12$), not just one of the individual segments.

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1.3.2: HOMEWORK

READING QUESTIONS

1. What two conditions must be met for two triangles to be congruent?
2. What is the symbol used to indicate that two triangles are congruent?
3. What is the key difference between congruent triangles and similar triangles?
4. Are all congruent triangles similar? Are all similar triangles congruent?
5. What does it mean for the corresponding sides of two triangles to be "proportional"?
6. According to the Similarity Conditions theorem, what are the two ways to verify that two triangles are similar?
7. If two right triangles share one pair of equal acute angles, are the triangles similar?
8. In Example 2, how is the altitude of an equilateral triangle used to create two congruent right triangles?
9. In Example 5, what assumption is made about the sun's light rays that allows us to conclude the triangles are similar?
10. When triangles overlap, as in Example 6, what must you be careful about when determining the side lengths for your proportion?

SKILLS REFRESHER

The following is a set of review exercises you will need for this section.

? SKILLS REFRESHER

For each of the following exercises, solve the equation.

1. $\frac{x}{12} = \frac{3}{x}$
2. $1 + \frac{x}{2} = \frac{2x}{5}$
3. $\frac{x}{6} = \frac{1}{6} + \frac{x}{6}$
4. $x + \frac{6}{x} = -5$
5. $\frac{5}{2x} = \frac{17}{18} - \frac{1}{3x}$

Answers

1. $x = \pm 6$
2. $x = -10$
3. $x = 4$
4. $x = -3$ and $x = -2$
5. $x = 3$

HOMEWORK

VOCABULARY CHECK

1. If given two triangles that are the same size and shape, we say they are ___; however, if their size is different but their shape remains the same, we call them ___ triangles.
2. The ___ of a triangle is the segment from one vertex of the triangle perpendicular to the ___ side.

CONCEPT CHECK

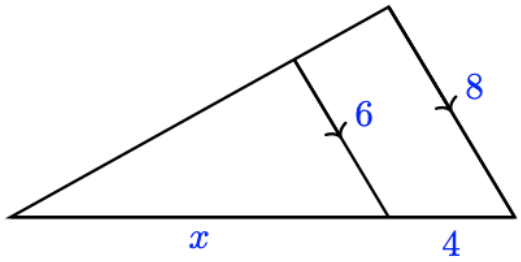
3. What is the difference between congruent triangles and similar triangles?
4. What is the name of the short-cut method for solving proportions? Why does the method work?
5. In two triangles, are the triangles similar if two corresponding pairs of angles are equal? How do you know?

True or False? For each of the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. The diagonal of a parallelogram splits the shape into two congruent triangles.
7. In a **30°-60°-90°** triangle, the leg opposite the **60°** angle is half the length of the hypotenuse.
8. If two right triangles have one pair of angles with the same measure, then the triangles are similar.

BASIC SKILLS

9. For the triangles shown, which of the following equations is true? Explain why.

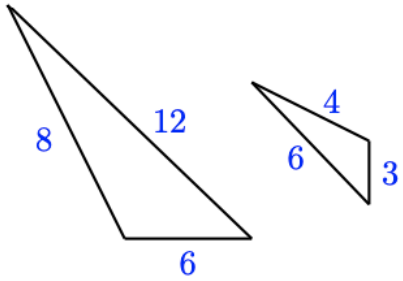


a. $\frac{4}{x} = \frac{6}{8}$
b. $\frac{x}{4} = \frac{6}{8}$
c. $\frac{x}{x+4} = \frac{6}{8}$
d. $\frac{x}{x+4} = \frac{6}{14}$

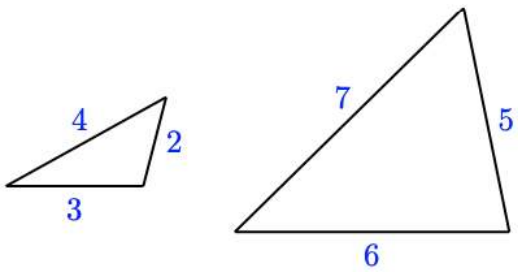
For each of the following exercises, decide whether the triangles are similar and explain why or why not.

10.

11.

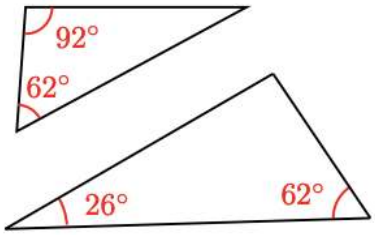


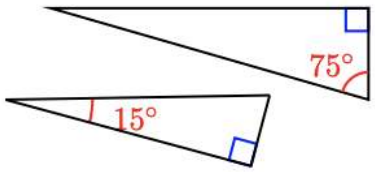
12.



13.

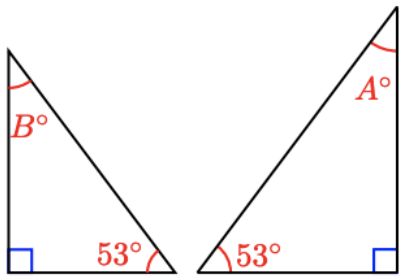
MyOpenMath





For each of the following exercises, assume the triangles are similar. Solve for the variables. (Figures are not drawn to scale.)

14.



15.

16.

17.

For each of the following exercises, use properties of similar triangles to solve for the variable.

18.

19.

20.

21.

22.

23.

For each of the following exercises, use properties of similar triangles to solve for the variable.

24.

25.

26.

27.

28.

29.

SYNTHESIS QUESTIONS

For each of the following exercises, name two congruent triangles and find the unknown quantities.

30. $PQRS$ is an isosceles trapezoid.

31.

32. ΔPRU is isosceles.

33. ΔARN is isosceles and $OR = NG$. Find $\angle RNG$ and $\angle RNO$.

For each of the following exercises, explain why the measurements shown are inaccurate.

34.

35.

36.

37. The follow pairs of triangles are similar. Solve for y in terms of x.

For each of the following exercises, solve for y in terms of x.

38.

39.

40.

41.

42. Triangle ABC is a right triangle, and AD meets the hypotenuse BC at a right angle.

    a. If $\angle ACB = 20°$, find $\angle B$, $\angle CAD$, and $\angle DAB$.

    b. Find two triangles similar to ΔABC. List the corresponding sides in each of the triangles.

APPLICATIONS

43. **Measuring Distances.** Ivan and Kim want to measure the distance across a stream. They mark point A directly across the stream from a tree at point T on the opposite bank. Ivan walks from point A down the bank a short distance to point B and sights the tree. He measures the angle between his line of sight and the stream bank.

    a. Draw a figure showing the stream, the tree, and the right triangle $\triangle ABT$.

    b. Meanwhile, Kim, still standing at point A, walks away from the stream at right angles to Ivan's path. Ivan watches her progress and tells her to stop at point C when the angle between the stream bank and his line of sight to Kim is the same as the angle from the stream bank to the tree. Add triangle $\triangle ABC$ to your figure.

    c. Ivan now measures the distance from point A to Kim at point C. Explain why this distance is the same as the distance across the stream.

44. **Measuring Distances.** If you have a baseball cap, here is another way to measure the distance across a river. Stand at point A directly across the river from a convenient landmark, say a large rock, on the other side. Tilt your head down so that the brim of the cap points directly at the base of the rock, R.

    a. Draw a figure showing the river, the rock, and the right triangle $\triangle ABR$, where B is the location of your baseball cap on your head.

    b. Now, without changing the angle of your head, rotate $90°$ and sight along the bank on your side of the river. Have a friend mark the spot C on the ground where the brim of your cap points. Add triangle $\triangle ABC$ to your figure.

    c. Finally, you can measure the distance from point A to point C. Explain why this distance is the same as the distance across the river.

For each of the following exercises, use properties of similar triangles to solve.

45. **Heights.** A rock climber estimates the height of a cliff she plans to scale. She places a mirror on the ground to see the top of the cliff in the mirror while she stands straight. The angles 1 and 2 formed by the light rays are equal, as shown in the figure. She then measures the distance to the mirror (2 feet) and the distance from the mirror to the base of the cliff (56 feet). How high is the cliff if she is 5 feet 6 inches tall?

46. **Heights.** Lap wants to estimate the height of the Washington Monument. He notices that he can see the reflection of the top of the monument in the reflecting pool. He is 35 feet from the tip of the reflection, and that point is 1080 yards from the base of the monument, as shown below. From his physics class, Lap knows that the angles marked (in the image below) are equal. If Lap is 6 feet tall, what is his estimate for the height of the Washington Monument?

47. **Distances.** In the sixth century BC, the Greek philosopher and mathematician Thales used similar triangles to measure the

distance to a ship at sea. Two observers on the shore at points A and B would sight the ship and measure the angles formed, as shown in Figure (a). They would then construct a similar triangle, as shown in Figure (b), with the same angles at A and B, and measure its sides. (This method is called *triangulation*.) Use the lengths given in the figures to find the distance from the observer at location B to the ship.

48. **Distances.** The Capilano Suspension Bridge is a footbridge that spans a 230-foot gorge north of Vancouver, British Columbia. Before crossing the bridge, you decide to estimate its length. You walk 100 feet downstream from the bridge and sight its far end, noting the angle formed by your line of sight, as shown in Figure (a). You then construct a similar right triangle with a two-centimeter base, as shown in Figure (b). You find that the height of your triangle is 8.98 centimeters. How long is the Capilano Suspension Bridge?

49. **Surface Area.** A conical tank is 12 feet deep, and the top's diameter is 8 feet. If the tank is filled with water to a depth of 7 feet, as shown in the figure at right, what is the area of the exposed surface of the water?

50. **Distances.** To measure the distance EC across the lake shown in the figure at right, stand at A and sight point C across the lake, then mark point B. Then sight to point E and mark point D so that DB is parallel to EC. If $AD = 25$ yards, $AE = 60$ yards, and $BD = 30$ yards, how wide is the lake?

CHALLENGE PROBLEMS

51. **Distances.** Here is a way to find the distance across a gorge using a carpenter's square and a five-foot pole. Plant the pole vertically on one side of the gorge at point A and place the angle of the carpenter's square on top of the pole at point B, as shown in the figure. Sight along one side of the square so that it points to the opposite side of the gorge at point P. Without moving the square, sight along the other side and mark point Q. If the distance from Q to A is six inches, calculate the width of the gorge. Explain your method.

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1.4: Circles and Angles in the Rectangular Coordinate System

Note to the Instructor (click to expand)

Up to this point, all of your instruction should have omitted the Cartesian coordinate system and equations depicting solutions on that plane. We now glue our beloved coordinate system to the Geometry reviewed up to point. Much of the content here cannot be done without mastering the previous sections.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Sets and Numbers**
  - Sets of Numbers: The set of integers is needed from this point forward in Trigonometry.
- **Solving Equations**
  - Solving Quadratic Equations: This skill is required when finding coordinates of points on a circle, such as in Example 6 where solving for y in $(-1)^2 + y^2 = 4$ is necessary.
- **Graphs**
  - The Cartesian Coordinate System and the Quadrants: This section is built upon the coordinate plane, where concepts like the Distance Formula, circles, and angles in standard position are defined.
  - Ordered Pairs and Graphing Relations by Point-Plotting: This skill is used to locate points, graph circles, and understand the position of angles in the plane.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Compute the distance between two points on the Cartesian plane.
- Verify a point lies on the unit circle.
- Graph a circle given its equation.
- Determine in which quadrant an angle in standard position terminates.
- Sketch an angle in standard position.
- Find an angle coterminal with a given angle.

Hawk A.I. Section-Specific Tutor

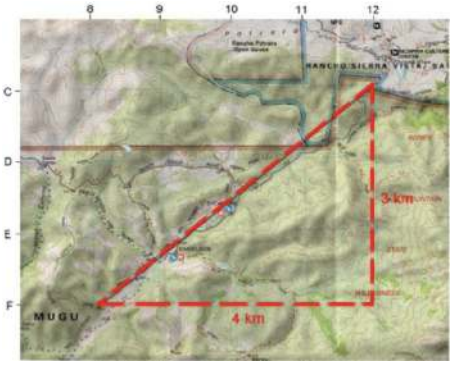
COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

The Distance Formula

Matt is hiking in the Santa Monica mountains. He would like to know the distance from the Sycamore Canyon trailhead, located at $12-\mathrm{C}$ on his map, to the Coyote Trail junction, located at $8-\mathrm{F}$, as shown below.

Figure 1.4.1



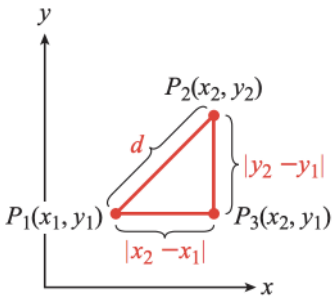
Each interval on the map represents one kilometer. Matt remembers the Pythagorean Theorem and uses the map coordinates to label the sides of a right triangle. The distance he wants is the hypotenuse of the triangle, so

$$\begin{aligned} & d^2 &=& 4^2+3^2 \\ \Longrightarrow\ & d^2 &=& 16+9 \\ \Longrightarrow\ & d^2 &=& 25 \\ \Longrightarrow\ & d &=& \sqrt{25} \\ \Longrightarrow\ & d &=& 5 \end{aligned}$$

The straight-line distance to Coyote junction is about 5 kilometers. The formula for the distance between two points is obtained similarly.

We first label a right triangle with points P_1 and P_2 on opposite ends of the hypotenuse (see Figure **1.4.2**). The sides of the triangle have lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. We can use the Pythagorean Theorem to calculate the distance between P_1 and P_2: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Figure 1.4.2



Taking the (positive) square root of each side of this equation gives us the **Distance Formula**.

Theorem: Distance Formula

The distance d between two points $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

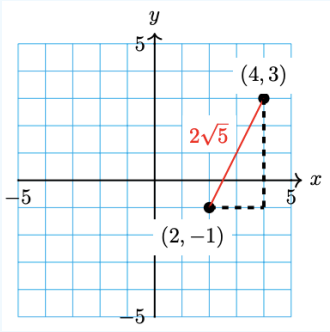
Example 1.4.1

Find the distance between (2, −1) and (4, 3).

Solution

We substitute $(2, -1)$ for (x_1, y_1) and $(4, 3)$ for (x_2, y_2) in the Distance Formula to obtain

Figure 1.4.3



$$
\begin{aligned}
d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} && \text{(Distance Formula)}\\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad &= \sqrt{(4-2)^2+[3-(-1)]^2} && \text{(substitution)}\\
\xRightarrow{\text{Arithmetic}} \quad &= \sqrt{2^2+4^2} && \text{(addition and subtraction)}\\
\xRightarrow{\text{Arithmetic}} \quad &= \sqrt{4+16} && \text{(applying powers)}\\
\xRightarrow{\text{Arithmetic}} \quad &= \sqrt{20} && \text{(addition)}\\
\xRightarrow{\text{Arithmetic}} \quad &= 2\sqrt{5} && \text{(simplify the radical)}
\end{aligned}
$$

The exact value of the distance, shown at right, is $2\sqrt{5}$ units. We obtain the same answer if we use $(4,3)$ for P_1 and $(2,-1)$ for P_2:

$$
\begin{aligned}
d &= \sqrt{(2-4)^2+[(-1)-3]^2}\\
&= \sqrt{4+16}\\
&= \sqrt{20}\\
&= 2\sqrt{5}
\end{aligned}
$$

We can use a calculator to obtain an approximation for this value and find $2\sqrt{5} \approx 2(2.236) = 4.472$.

Critical Mistake

In Example 1.4.1, the radical $\sqrt{4+16}$ cannot be simplified to $\sqrt{4}+\sqrt{16}$. (Do you remember why not?)

Checkpoint 1.4.1

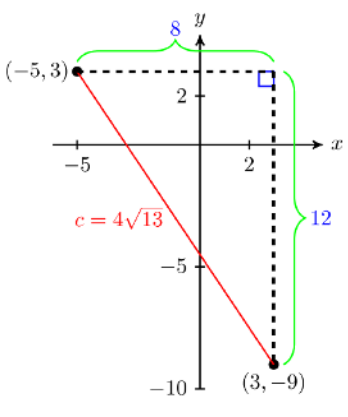
a. Find the distance between the points (−5, 3) and (3, −9).

b. Plot the points on a Cartesian coordinate system and show how the Pythagorean Theorem is used to calculate the distance.

Answers

a. $4\sqrt{13}$

b. The points (−5, 3) and (3, −9) are two vertices of a right triangle with one horizontal leg of length 8 and one vertical leg of length 12. The distance between (−5, 3) and (3, −9) is the hypotenuse c.



Equation for a Circle

The Distance Formula is required for us to find the equation of a circle.

Theorem: Equation of a Circle Centered at the Origin

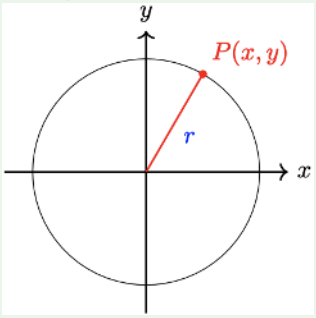
The equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Proof

Consider a circle of radius r centered at the origin, as shown in the following figure.

The circle of radius r centered at the origin



The distance from the origin to any point $P(x, y)$ on the circle is r. Therefore, by the Distance Formula,

$$\sqrt{(x-0)^2 + (y-0)^2} = r$$

or, squaring both sides,

$$(x-0)^2 + (y-0)^2 = r^2.$$

Because every point on the circle must satisfy this equation, we have found an equation for the circle.

Example 1.4.2

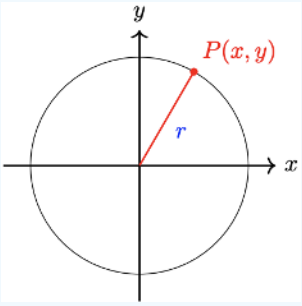
Find two points on the circle $x^2 + y^2 = 4$ with an x-coordinate of -1.

Solution

We substitute $x = -1$ into the equation for the circle and solve for y.

$$
\begin{array}{lrcll}
\text{Arithmetic} \to \text{Algebra} & \Longrightarrow & (-1)^2 + y^2 & = & 4 & \text{(substitution)} \\
\text{Arithmetic} & \Longrightarrow & 1 + y^2 & = & 4 & \text{(applying the power)} \\
\text{Arithmetic} \to \text{Algebra} & \Longrightarrow & y^2 & = & 3 & \text{(subtract 1 from both sides)} \\
\text{Arithmetic} \to \text{Algebra} & \Longrightarrow & y & = & \pm\sqrt{3} & \text{(Extraction of Roots)}
\end{array}
$$

Figure 1.4.4



The points are $(-1, \sqrt{3})$ and $(-1, \sqrt{3})$, as shown at right. Note that $\sqrt{3} \approx 1.732$.

? Checkpoint 1.4.2

Find the coordinates of two points on the circle $x^2 + y^2 = 1$ with y-coordinate $\frac{1}{2}$.

Answer

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

The circle in Checkpoint **1.4.2** holds a special place in Trigonometry.

Definition: Unit Circle

The circle centered at the origin and having radius 1, is called the **unit circle**. The unit circle has equation

$$x^2 + y^2 = 1.$$

While the definition of the unit circle is only mentioned in passing here, it will become one of the foundational pieces of Trigonometry. We will revisit this later.

Later in this course, we will need to work with circles that do not have centers at the origin.

Theorem: Equation of a Circle Centered Off-Origin

The equation for a circle of radius r centered at the point (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is also know as the **standard form** of the equation of a circle.

Proof

Consider a circle of radius r centered at the point (h, k), as shown in the following figure.

A circle of radius r centered at the point (h,k)

Since this is a circle of radius r, the point (x,y) is on the circle if and only if its distance to (h,k) is r. By the Distance Formula, this means

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

By squaring both sides of this equation, we get

$$r^2 = (x-h)^2 + (y-k)^2.$$

Example 1.4.3

Write the standard equation of the circle with center $(-2,3)$ and radius 5.

Solution

Here, $(h,k) = (-2,3)$ and $r = 5$, so we get

$$\begin{aligned} & (x-(-2))^2 + (y-3)^2 &&= (5)^2 \\ \Longrightarrow \quad & (x+2)^2 + (y-3)^2 &&= 25 \end{aligned}$$

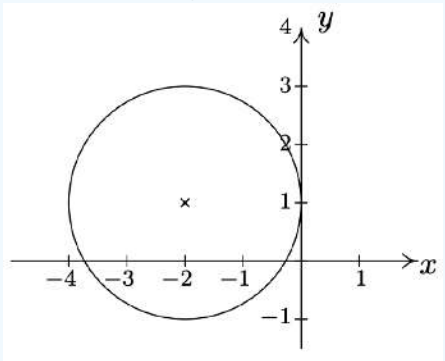
Checkpoint 1.4.3

Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Answer

From the standard form of a circle we have that $x+2$ is $x-h$, so $h=-2$ and $y-1$ is $y-k$ so $k=1$. This tells us that our center is $(-2,1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2,1)$ with a radius of 2. Graphing gives us

Figure 1.4.5



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Angles in the Cartesian Coordinate System

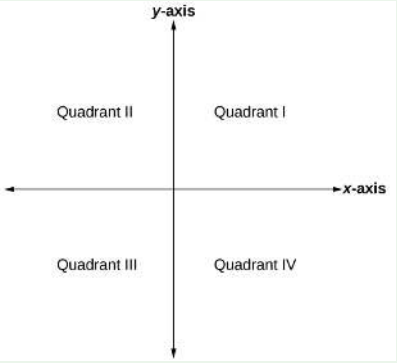
Now that we have reacquainted ourselves with angles, circles, and the Cartesian coordinate system, the next natural evolution toward Trigonometry is introducing angles to the Cartesian coordinate plane. Before we begin, however, it will be instrumental in this course to divide the Cartesian coordinate system into four sections, called **quadrants**.

Definition: Quadrants

In the Cartesian coordinate system, the section of the plane where

- x and y are both positive is called **Quadrant I** (**QI**),
- $x < 0$ and $y > 0$ is called **Quadrant II** (**QII**),
- x and y are both negative is called **Quadrant III** (**QIII**), and
- $x > 0$ and $y < 0$ is called **Quadrant IV** (**QIV**).

The four quadrants of the Cartesian coordinate system



Knowing this definition and understanding how the signs of x and y relate to these quadrants will be critical as we move forward.

Angles in Standard Position

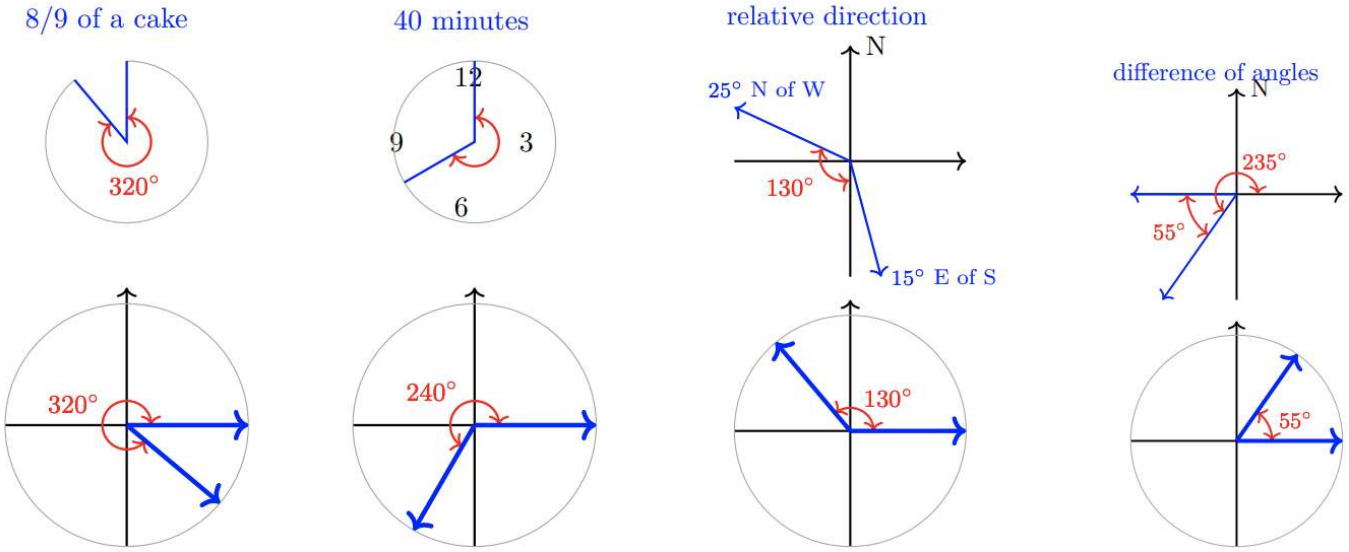
In general, and without worrying about a coordinate system, the degree measure of an angle depends only on the fraction of a whole rotation between its sides, not on the angle's location or position. To compare and analyze angles, we place them in **standard position**.

Definition: Standard Position of an Angle

An angle is said to be in **standard position** if the vertex of the angle is at the origin and the initial side of the angle is on the positive x-axis.

Figure **1.4.6** below shows several angles moved into standard position.

Figure 1.4.6: *Angles in nonstandard position (top row) and their equivalent angles in standard position (bottom row)*

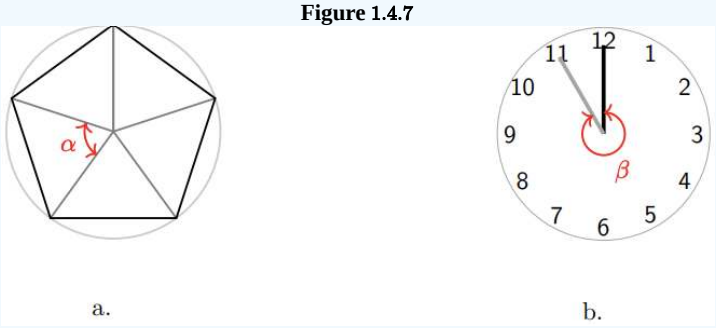


The $320°$ in Figure 1.4.6 has its terminal side in quadrant IV. We denote this as $320° \in \text{QIV}$. Likewise, the remaining angles in Figure 1.4.6 are listed as $240° \in \text{QIII}$, $130° \in \text{QII}$, and $55° \in \text{QI}$.

Example 1.4.4

Find the degree measure of each angle shown below, state the quadrant where the angle terminates, and sketch the angle in standard position.

Figure 1.4.7



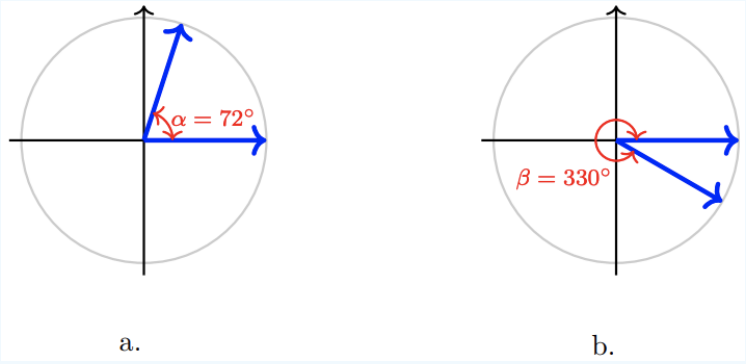
Solutions

a. The angle α is one-fifth of a complete revolution, or

$$\frac{1}{5}(360°) = 72°.$$

In standard position, $\alpha = 72° \in \text{QI}$, as shown in Figure 1.4.8(a) below.

Figure 1.4.8



b. The angle β is $\frac{11}{12}$ of a complete revolution, or

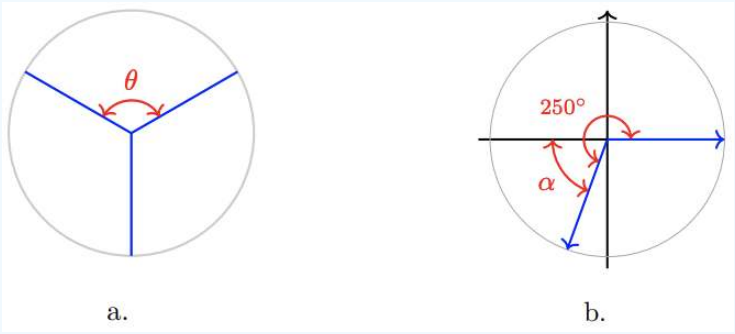
$$\frac{11}{12}(360°) = 330°.$$

In standard position, $\beta = 330° \in$ QIV. (See Figure 1.4.8(b).)

Checkpoint 1.4.4

Find the degree measure of each angle below, sketch the angle in standard position, and state the quadrant in which the angle terminates.

Figure 1.4.9



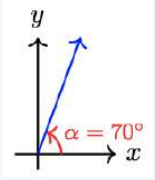
Answers

a. $\theta = 120° \in$ QII

Figure 1.4.10

b. $\alpha = 250° - 180° = 70° \in$ QI

Figure 1.4.11



Quadrantal Angles

Now that we are acquainted with angles in standard position, it's time we start classifying these angles by type. First up are the **quadrantal angles**.

Definition: Quadrantal Angle

An angle in standard position having terminal side on an axis is called a **quadrantal angle**.

For example, 270° has a terminal side on the negative y-axis. Therefore, we would call 270° a quadrantal angle. Figure 1.4.12 illustrates a few quadrantal angles; however, as we will learn shortly, there are infinite quadrantal angles.

Figure 1.4.12: *Some quadrantal angles*

Negative Angles

Aligning with what we said earlier in this textbook, an angle is positive if it opens counterclockwise. If an angle opens clockwise, the angle is considered negative. For example, Figure 1.4.13 shows the angle $\alpha = -135^\circ$, which terminates in QIII.

Figure 1.4.13: *A negative angle*

Graphing positive and negative angles in standard position will be a critical skill in Trigonometry.

Coterminal Angles

Because 360° represents one complete revolution, we can add or subtract a multiple of 360° to any angle and the terminal side of the resulting angle will arrive at the same position. For example, the angles 70° and 430° have the same terminal side because $430^\circ = 70^\circ + 360^\circ$. Such angles are called **coterminal**.

Definition: Coterminal Angles

Angles in standard position that share a terminal side are called **coterminal angles**.

The angle 790° is also coterminal with 70°, because $790^\circ = 70^\circ + 2\,(360^\circ)$ (see Figure 1.4.14 below).

Figure 1.4.14: *Graphical representations of coterminal angles*

Coterminal angles need not be positive. For example, we could have been given $\alpha = 140^\circ$ and chosen to *subtract* 360° to get the coterminal angle -220° as shown in Figure 1.4.15.

Figure 1.4.15

In fact, we could denote all angles coterminal to $\alpha = 140°$ using the notation

$$140° + 360°k, \text{ where } k \in \mathbb{Z}.$$

The symbol, $\mathbb{Z}$, denotes the set of all integers (the set of numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$). This notation gets used frequently in Mathematics from this point forward, so it is best to get used to it quickly.

Theorem: Cotermial Angle Formula

All angles coterminal to the angle α (in standard position) have the form

$$\alpha + 360°k, \text{ where } k \in \mathbb{Z}.$$

Previously, we mentioned that the angles illustrated in Figure 1.4.12 were only a "few quadrantal angles," implying there were more than what we listed. This was because, for example, given the quadrantal angle $90°$, we could generate an infinite number of other quadrantal angles with the same terminal side by considering

$$90° + 360°k, \text{ where } k \in \mathbb{Z}.$$

Moreover, since the quadrantal angles only differ from each other by $90°$, we could generate *all* quadrantal angles with the single statement

$$0° + 90°k, \text{ where } k \in \mathbb{Z}.$$

There is no need for the $0°$ angle in that formula, so we could shorten it up to just

$$90°k, \text{ where } k \in \mathbb{Z}.$$

Example 1.4.5

Graph each of the angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

a. $\alpha = 60°$
b. $\beta = -225°$
c. $\gamma = 540°$
d. $\phi = -750°$

Solutions

a. To graph $\alpha = 60°$, we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{60°}{360°} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find coterminal angles, we look for angles θ of the form $\theta = \alpha + 360° \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60° + 360° = 420°$. Substituting $k = -1$ gives $\theta = 60° - 360° = -300°$. Finally, if we let $k = 2$, we get $\theta = 60° + 720° = 780°$.

b. Since $\beta = -225°$ is negative, we start at the positive x-axis and rotate *clockwise* $\frac{225°}{360°} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225° + 360° \cdot k$ for integer values of k. We find $135°$, $-585°$ and $495°$ are all coterminal with $-225°$.

Figure 1.4.16

c. Since $\gamma = 540°$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $360°$, with $180°$, or $\frac{1}{2}$ of a revolution remaining. Since the terminal side of γ lies on the negative x-axis, γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540° + 360° \cdot k$, where k is an integer. Working through the arithmetic, we find three such angles: $180°$, $-180°$ and $900°$.

d. The Greek letter ϕ is pronounced "fee" or "fie," and since ϕ is negative, we begin our rotation clockwise from the positive x-axis. Two full revolutions account for $720°$, with just $30°$ or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750° + 360° \cdot k$ for a few integers k and obtain $-390°$, $-30°$ and $330°$.

Figure 1.4.17

? Checkpoint 1.4.5

Find two angles coterminal with $102°$, one positive and one negative.

Answer

$462°, -258°$

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1.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Distance Formula
  - Intermediate Algebra - Distance Formula: The Concept
  - Intermediate Algebra - Distance Formula: General and Advanced Uses
- Equation of a Circle
  - Intermediate Algebra - Circles: The Concept
  - Intermediate Algebra - Circles: Equations and Graphs

KEY CONCEPTS

DEFINITIONS

- **Unit Circle:** The circle with a radius of 1 that is centered at the origin of the Cartesian plane. Its equation is $x^2 + y^2 = 1$.
- **Quadrants:** The four regions into which the Cartesian coordinate system is divided by the x- and y-axes. They are numbered I, II, III, and IV, starting from the top right and moving counterclockwise.
- **Standard Position of an Angle:** An angle is in standard position if its vertex is located at the origin and its initial side lies along the positive x-axis.
- **Quadrantal Angle:** An angle in standard position whose terminal side lies on either the x-axis or the y-axis.
- **Coterminal Angles:** Angles that are in standard position and share the same terminal side.

THEOREMS

- **Distance Formula:** The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- **Equation of a Circle Centered at the Origin:** The equation for a circle with radius r centered at the origin is $x^2 + y^2 = r^2$.
- **Equation of a Circle Centered Off-Origin:** The standard form for the equation of a circle with radius r centered at the point (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.
- **Coterminal Angle Formula:** All angles that are coterminal with an angle α can be expressed in the form $\alpha + 360°k$, where k is any integer.

COMMON MISTAKES

- **Distributing a Radical over a Sum:** A square root cannot be distributed across terms that are added or subtracted. For example, $\sqrt{a^2 + b^2}$ is not equal to $a + b$, and $\sqrt{4 + 16}$ is not equal to $\sqrt{4} + \sqrt{16}$. This is a critical mistake when using the Distance Formula.
- **Direction of Rotation for Angles:** Positive angles are generated by a counterclockwise rotation from the initial side. Negative angles are generated by a clockwise rotation.

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1.4.2: HOMEWORK

READING QUESTIONS

1. What fundamental theorem is used as the basis for deriving the Distance Formula?
2. Write the equation for a circle that has its center at the origin and a radius of r.
3. What is a "unit circle"?
4. What is the "standard form" of the equation for a circle with center (h, k) and radius r?
5. In which quadrant are the x-coordinates negative and the y-coordinates positive?
6. What two conditions define an angle in "standard position"?
7. Where is the terminal side of a 90° angle in standard position?
8. What is a "quadrantal angle"?
9. What does it mean for two angles to be "coterminal"?
10. Describe the direction of rotation (clockwise or counterclockwise) for a negative angle.
11. If you are given an angle of 50°, how would you find a negative angle that is coterminal with it?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, find the slope and y-intercept of the graph for the given equation (do not graph).

1. $y = 5.2x - 1.9$
2. $y = x$
3. $y = -8$
4. $x = 7$
5. $12y - 10x = 5$

For the following exercises, write the equation of the line that satisfies the given conditions.

6. The line goes through the points $(-2, 6)$ and $(3, -4)$.
7. The line goes through the points $(5, -7)$ and $(5, -5)$.

For the following exercises, determine if the statement is true or false.

8. $\sqrt{a^2 + b^2} = a + b$
9. $\sqrt{36 + 64} = 6 + 8$
10. $\sqrt{16x^4} = 4x^2$
11. $\sqrt{2x}\sqrt{3y} = \sqrt{6xy}$
12. $\sqrt{5x} + \sqrt{3x} = \sqrt{8x}$
13. $\sqrt{4 + N} = 2 + \sqrt{N}$
14. $\sqrt{\frac{x}{4}} = \frac{\sqrt{x}}{2}$
15. $\sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

For the following exercises, solve the equation. If no solution exists, state why.

16. $x^2 - 81 = 0$
17. $2y^2 - 98 = 0$
18. $x^2 + 12 = 8$
19. $-3x^2 + 19 = 10$

Answers

1. $m = 5.2 \quad (0, -1.9)$
2. $m = 1 \quad (0, 0)$
3. $m = 0 \quad (0, -8)$
4. The slope is undefined and there is not a y-intercept.
5. $m = \frac{5}{6} \quad (0, \frac{5}{12})$
6. $y = -2x + 2$
7. $x = 5$
8. False
9. False
10. True
11. True
12. False
13. False
14. True
15. True
16. $x = \pm 9$
17. $y = \pm 7$
18. No solution because x^2 should never be negative, but when solving, we get $x^2 = -4$.
19. $x = \pm\sqrt{3}$

HOMEWORK

CONCEPT CHECK

1. Explain why the Distance Formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

cannot be simplified to

$$(x_2 - x_1) + (y_2 - y_1).$$

2. What is a unit circle, and what is its equation?
3. In which quadrant are the x-values for points negative but the y-values are positive?
4. Explain what it means for angles to be coterminal. Demonstrate by creating three coterminal angles, one of which is a negative angle.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

5. Positive angles open clockwise.

BASIC SKILLS

For the following exercises, find the distance between the points. Give your answer as an exact value, then as a decimal rounded to hundredths.

6.

7.

8. (MyOpenMath)

For the following exercises, find the distance between the points.

9. (1, 1), (4, 5)

10. (−1, 1), (5, 9)

11. (MyOpenMath) (2, −3), (−2, −1)

12. (5, −4), (−1, 1)

13. (3, 5), (−2, 5)

14. (−2, −5), (−2, 3)

15.

a. Write an expression for the distance between the points (x, y) and $(-3, 4)$.

b. Write an equation that says "The distance between the points (x, y) and $(-3, 4)$ is 5 units."

16.

a. Write an expression for the distance between the points $(-6, -1)$ and (h, k).

b. Write an equation that says, "The point $(-6, -1)$ is 3 units from the point (h, k)."

17.

a. Complete the table of values for the equation $x^2 + y^2 = 25$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

b. Plot the points and graph the equation.

18.

a. Complete the table of values for the equation $x^2 + y^2 = 1$.

x	-1	-0.8	-0.6	0	0.6	0.8	2
y							

b. Plot the points and graph the equation.

19.

a. Sketch a graph of all points in the plane that lie 6 units from the origin.

b. Write an equation for your graph.

20.

a. Sketch a graph of all points in the plane that lie $\sqrt{12}$ units from the origin.

b. Write an equation for your graph.

21.

a. Sketch a graph of all points in the plane that lie less than 3 units from the origin.

b. Write an inequality for your graph.

22.

a. Sketch a graph of all points in the plane that lie more than 5 units from the origin.

b. Write an inequality for your graph.

23.

a. Explain why the solutions of the equation $x^2 + y^2 = 16$ must have $-4 \le x \le 4$.

b. What does part (a) tell you about the graph of the equation?

24.

a. Explain why the solutions of the equation $x^2 + y^2 = 100$ must have $-10 \le x \le 10$

b. What does part (a) tell you about the graph of the equation?

25. The point $(1, -3)$ lies on a circle centered at the origin. What is the radius of the circle?

26. The point $(-2, \sqrt{6})$ lies on a circle centered at the origin. What is the radius of the circle?

27. Find the coordinates of the points on the unit circle.

28.

a. Use the Distance Formula to write an equation for the circle of radius 6 centered at the point $(3, -2)$.

b. Use the Distance Formula to derive an equation for the circle of radius r centered at the point (h, k).

For the following exercises, find the coordinates of the indicated points on each circle.

29. $(x-4)^2+(y-5)^2=9$

30. $(x+3)^2+(y-1)^2=25$

31. How many degrees are in each angle?
    a. $\frac{3}{5}$ of one rotation
    b. $\frac{3}{10}$ of one rotation
    c. $\frac{4}{3}$ of one rotation
    d. $\frac{8}{3}$ of one rotation

32. How many degrees are in each angle?
    a. $\frac{5}{6}$ of one rotation
    b. $\frac{3}{8}$ of one rotation
    c. $\frac{7}{4}$ of one rotation
    d. $\frac{7}{12}$ of one rotation

33. What fraction of a complete rotation is represented by each angle?
    a. $45°$
    b. $300°$
    c. $540°$
    d. $420°$

34. What fraction of a complete rotation is represented by each angle?
    a. $60°$
    b. $240°$
    c. $450°$
    d. $150°$

For the following exercises, find two angles, one positive and one negative, that are coterminal with the given angle.

35. $40°$

36. $160°$

37. $215°$

38. $250°$

39. $305°$

40. $340°$

For the following exercises, find a positive angle between $0°$ and $360°$ that is coterminal with the given angle.

41. $-65°$

42. $-140°$

43. $-290°$

44. $-325°$

45. (MyOpenMath) $655°$

46. (MyOpenMath) $-750°$

47. (MyOpenMath) Write an expression describing all the angles that are coterminal with $252°$.

SYNTHESIS QUESTIONS

47. Show that the triangle with vertices $(0,0)$, $(6,0)$, and $(3,3)$ is an isosceles right triangle, that is, a right triangle with two sides of the same length.

48. Two opposite vertices of a square are $A(-9,-5)$ and $C(3,3)$.
    a. Find the length of a diagonal of the square.
    b. Find the length of the side of the square.

49. Find all of the points on the line $y=2x+1$ which are 4 units from the point $(-1,3)$.

50. Sketch a triangle with vertices $(10,1)$, $(3,1)$, and $(5,9)$, and find its perimeter. Round your answer to tenths.

51. Sketch a triangle with vertices $(-1,5)$, $(8,-7)$, and $(4,1)$, and find its perimeter. Round your answer to tenths.

For the following exercises, interpret the equations as statements about distance.

52. $\sqrt{(x-4)^2+(y+1)^2}=3$

53. $\sqrt{(-2-h)^2+(5-k)^2}=l$

For the following exercises, use the graph to answer the question.

54. Find the circumference of the circle.

MyOpenMath

55. Find the area of the circle.

MyOpenMath

For the following exercises, graph the equation and find the circumference of the circle.

56. $x^2+y^2=36$

57. $x^2+y^2=16$

58. $4x^2+4y^2=16$

59. Give the coordinates of two points on the circle in the previous exercise that have $y=-4$. Plot those points on your graph.

60. $2x^2+2y^2=18$

61. Give the coordinates of two points on the circle in the previous exercise that have $x = -2$. Plot those points on your graph.

APPLICATIONS

62. **Distance.** Paige is sailing and is currently 3 miles west and 5 miles south of the harbor. She heads directly towards an island 8 miles west and 7 miles north of the harbor. How far is Paige from the island?

63. **Distance.** Nam is 100 meters east and 250 meters north of Kristen. He walks directly towards a tree 220 meters east and 90 meters north of Kristen. How far is Nam from the tree?

64. **Computing Angles (MyOpenMath).**
    a. Through what angle does the hour hand of a clock rotate between 2 pm and 10 pm?
    b. Through what angle does the hour hand of a clock rotate between 2 am and 10 pm?

65. **Computing Angles.**
    a. Through what angle does the hour hand of a clock rotate between 3:25 am and 3:50 am?
    b. Through what angle does the hour hand of a clock rotate between 4:10 pm and 6:25 pm?

66. **Astronomy.** The Distance Formula can be modified to compute distances between objects in three-dimensional space. To do so, we must expand the Cartesian coordinate system to include one more spatial dimension. In three-space, coordinates are listed in **ordered triples**, such as (x, y, z). If we have two points, $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, in three-dimensional space, the distance between these points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Suppose our Sun is given the very special coordinate of $(0, 0, 0)$ (thereby affirming that we are the center of the universe). Compute the distance between each star (these computed distances are in light-years). Round answers to the nearest hundredth of a light-year.

a. Our Sun, $(0, 0, 0)$, and Alpha Centauri, $(-1.8, 0, 3.9)$.
b. Our Sun, $(0, 0, 0)$, and Sirius, $(-3.4, -3.1, 7.3)$.
c. Our Sun, $(0, 0, 0)$, and Polaris, $(99.6, 28.2, 376.0)$.
d. Alpha Centauri, $(-1.8, 0, 3.9)$, and Sirius, $(-3.4, -3.1, 7.3)$.
e. Zubenelgenubi, $(64.6, -22.0, 23.0)$, and Polaris, $(99.6, 28.2, 376.0)$.

Computing Angles. For the following exercises, calculate the degree measure of the unknown angle and sketch the angle in standard position.

67.

68.

69.

70.

71.

72.

CHALLENGE PROBLEMS

73. In this problem, we'll show that any angle inscribed in a semi-circle must be a right angle. The figure shows a triangle inscribed in a unit circle, one side lying on the diameter of the circle and the opposite vertex at point (p, q) on the circle.

    a. What are the coordinates of the other two vertices of the triangle? What is the length of the side joining those vertices?
    b. Use the Distance Formula to compute the lengths of the other two sides of the triangle.
    c. Show that the sides of the triangle satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$.

74. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0, 0)$, $P(1, m_1)$ and $Q(1, m_2)$ gives us the following set up.

The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

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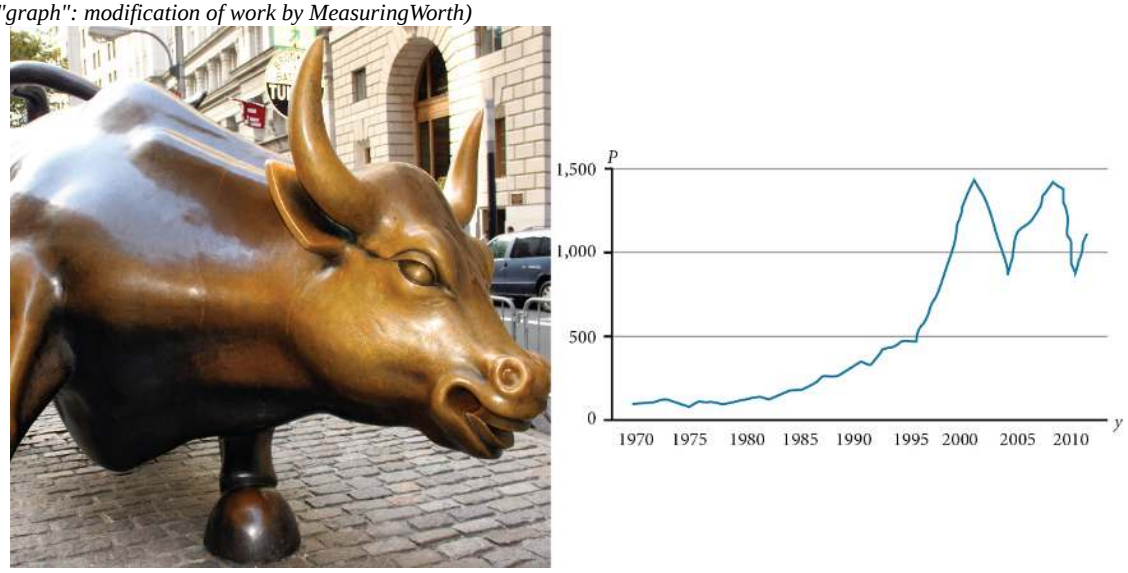
CHAPTER OVERVIEW

Chapter 2: Functions

Figure 1: *Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)*



Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The chart on the right tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1,100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

2.1: Functions and Function Notation

2.1.1: Resources and Key Concepts

2.1.2: Homework

2.2: Domain and Range

2.2.1: Resources and Key Concepts

2.2.2: Homework

2.3: The Arithmetic and Composition of Functions

2.3.1: Resources and Key Concepts

2.3.2: Homework

2.4: Behavior of Graphs of Functions

2.4.1: Resources and Key Concepts

2.4.2: Homework

2.5: Transformations of Functions

2.5.1: Resources and Key Concepts

2.5.2: Homework

2.6: Absolute Value Functions
2.6.1: Resources and Key Concepts
2.6.2: Homework
2.7: Inverse Functions
2.7.1: Resources and Key Concepts
2.7.2: Homework

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2.1: Functions and Function Notation

Note to the Instructor (click to expand)

Treat this section as a review of functions (represented as tables, algebraic functions, word problems, and graphs). As far as computation is concerned, the algebraic functions used within this section are restricted to linear, quadratic, radical, and very simple rational functions. Finally, the eight base graphs are introduced, but do not cover transformations (as this is held off for later).

We're bringing back some skills from earlier courses that are essential for the rest of your math journey. If you're rusty, see **CRC's Corequisite Codex** for a quick review. (Don't rely solely on support classes, as they may not cover everything!)

Prerequisite Skills and Support Topics (click to expand)

- **Sets and Numbers**
  - Sets of Numbers (natural numbers are mentioned)
- **Arithmetic**
  - The Order of Operations (Example 6c)
- **Geometry**
  - Circles - Properties, Area, and Circumference (Example 13)
- **Factoring Techniques**
  - Greatest Common Factor (Example 6)
- **Simplifying Expressions**
  - Algebraic Vocabulary and Evaluating Expressions (most examples)
  - Adding and Subtracting Polynomials (Example 6)
  - Multiplying Polynomials and the Distributive Property (Example 6)
  - Simplifying Ratios of Polynomials (Example 6)
- **Solving Equations**
  - Solving Linear Equations - One or Two-Steps (Example 9)
  - Solving Quadratic Equations by Factoring (Example 8)
  - Solving Quadratic Equations Using Extraction of Roots (Example 10)
- **Graphs**
  - The Cartesian Coordinate System and the Quadrants (for VLT and HLT)
  - Ordered Pairs and Graphing Equations by Point-Plotting (ordered pairs are mentioned)
  - Reading Values from Graphs (Examples 8 and 12)

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the Vertical Line Test to identify functions.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

We begin our journey with an exploration followed by a definition (and from this, progress to more definitions, then theorems, then more theorems, and so on until we will have derived all of College Algebra).

Consider the following set of ordered pairs:

$$D = \{(1,2),(2,4),(3,6),(4,8),(5,10)\}. \tag{2.1.1}$$

The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first. It is logical to want to describe this link between the first and second values of these ordered pairs. We do so by defining this link as a **relation**.

Definition: Relation

A **relation** is any set of ordered pairs, (x, y).

Relations *might* be able to be described in terms of formulas (like, $y = 2x$ in the set 2.1.1), but this property need not exist. That is, we might be given a set of ordered pairs where there is no formula to the relationship between x and y.

Note

Before we go to far, it is important to mention that relations can involve more than two variables, but until you reach Calulus III, we will try our best to restrict our exploration to relations between two unknowns.

Now that we have a name for the "machine" linking the first and second values of these ordered pairs, let's introduce more language so that we can stop saying, "the first and second values of the ordered pairs."

Definition: Domain and Range (of a relation)

In a relation defined between two variables, x and y, of the form (x, y), all the x-values in the ordered pairs make up the **domain** of the relation. All the y-values in the ordered pairs make up the **range**.

Therefore, the domain of the relation 2.1.1 is $\{1,2,3,4,5\}$ and the range is $\{2,4,6,8,10\}$.

Definition: Input and Output Values

Each value in the domain is also known as an **input value**, and each value in the range is also known as an **output value**.

Input and output values are traditionally considered constants. That is, in 2.1.1 the input value of 3 mapped to an output value of 6. However, when referring to an *arbitrary* element from the set of input or output values, we use a slightly modified language (because, in Mathematics, the word "arbitrary" means "variable" - or non-constant).

Definition: Independent and Dependent Variables

If we are treating the input value as a variable, x, then we call x the **independent variable**. Likewise, the variable y is often used when referencing an arbitrary element from the set of output values, and y is called the **dependent variable**.

A Variable By Any Other Name...

While x and y are traditionally used in lower-level math courses as the independent and dependent variables, respectively, this need not always be the case. We can use any letter (or picture, for that matter) to represent these variables. The choice is literally limitless.

Relations are incredibly flexible - as long as there is some link between two variables, we can form a relation between them. For example, there is a relation between social security number and name. That is, the social security number 123-45-6789 (which, in truth, is not a real social security number) is related to a person's name. A natural question that arises when we do some thinking is whether the relation is unique. That is, if we grab a random social security number, will it be related to one name? This type of special relation is called a **function**.

Definition: Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say the output is a function of the input. The set of input values makes up the **domain** of the function, and the corresponding set of output values makes up the **range** of the function.

In a very real sense, functions are a special subclass of relations. While a relation can relate the input of 5 to the output 7 and then the input of 5 to the output of 10, functions are not allowed such silliness.

The relation from 2.1.1 is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$, is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$. For the social security numbers, the relation in one direction is a function - each social security number leads to a single name. However, if we reversed the direction of the relation so that the domain is the set of all names and the range is the set of all social security numbers, the relation would *not* be a function. This is because the name, Roy Simpson, leads to more than one social security number (there are more Roy Simpsons in the United States than just me!).

Let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd}, 1), (\text{even}, 2), (\text{odd}, 3), (\text{even}, 4), (\text{odd}, 5)\}.$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$, is not paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the range, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function. Figure $2.1.1$ compares relations that are functions and not functions.

Figure 2.1.1

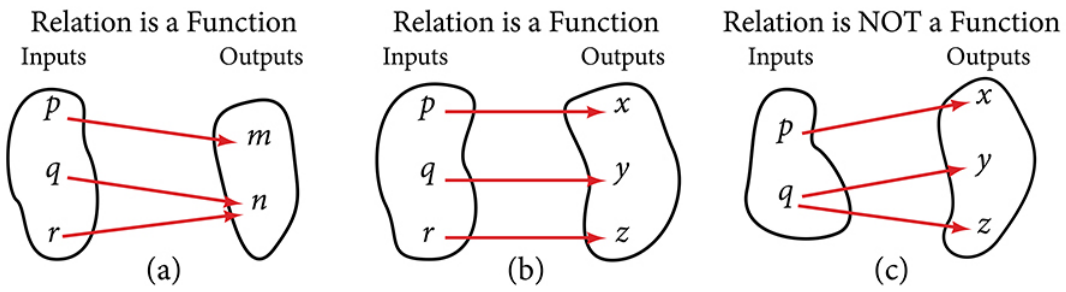


Figure $2.1.1(a)$ is a function because each input is associated with a single output. Note that inputs q and r both give output n. This is perfectly fine. The definition of a function only restricts a single input from leading to two distinctly different outputs. Figure $2.1.1(b)$ is also a function. In this case, each input is associated with a single output. Finally, Figure $2.1.1(c)$ is *not* a function because input q is associated with two different outputs (y and z).

Example 2.1.1: Determining If Menu Price Lists Are Functions

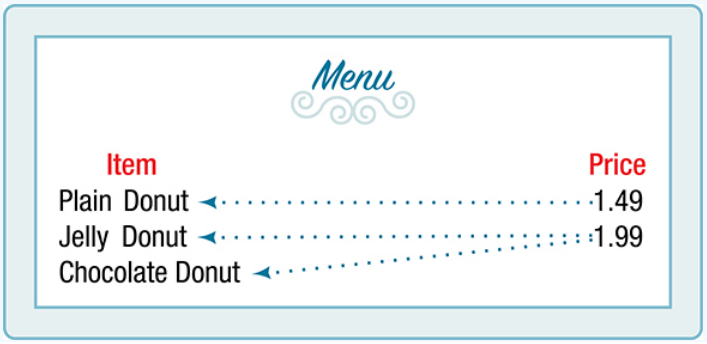
The coffee shop menu, shown in Figure **2.1.2**, consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

Figure 2.1.2

Solutions

a. Let's begin by considering the input as the items on the menu. The output values are then the prices (see Figure **2.1.2**). Each item on the menu has only one price, so the price is a function of the item.
b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it (see the figure below).



Example 2.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table **2.1.1** shows a possible rule for assigning grade points.

Table 2.1.1

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Checkpoint 2.1.2

Table 2.1.2 lists the five greatest baseball players of all time in order of rank.

Table 2.1.2

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables: h for height and a for age. The letters f, g, and h are often used to represent functions just as we use x, y, and z to represent numbers and A, B, and C to represent sets.

We then build a structure for the function notation. Since height is a function of age, we would like our notation to match.

$$\begin{array}{rccc} & \textbf{height} & \textbf{is} & \textbf{a function of age} \\ \Longrightarrow & h & = & f(a) \end{array}$$

Remember, we can use any letter to name the function. The notation $f(a)$ shows us that f depends on a. The value a must be put into the function f to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example, $f(a+b)$ means, "first add a and b, and then input the result into the function f." The operations must be performed in this order to obtain the correct result.

Definition: Function Notation

For the notation $y = f(x)$, f is the **name** of the function, x is the **domain value**, also known as the **independent variable**, and $f(x)$ is the **range value** of the function f at x. Since $y = f(x)$, we can also say that the range value of the function f at x is y. In this case, we say that y is the **dependent variable** (as its value *depends* on x). We read $f(x)$ as "f of x" or the value of f at x.

Example 2.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution

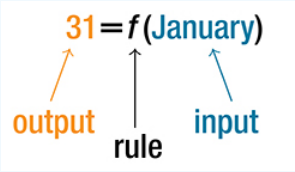
The number of days in a month is a function of the name of the month, so if we name the function f, we write

$$\text{days} = f(\text{month})$$

or

$$d = f(m).$$

The name of the month is the input to a "rule" that associates a specific number (the output) with each input.



For example, $f(\text{March}) = 31$ because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 2.1.4: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Checkpoint 2.1.4

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as Physics and Engineering. However, in exploring Mathematics itself, we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 2.1.3 lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table 2.1.3

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 2.1.4 defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table 2.1.4

n	1	2	3	4	5
Q	8	6	7	6	8

Table 2.1.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 2.1.5

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Example 2.1.5: Identifying Tables that Represent Functions

Which table, Table 2.1.6A, Table 2.1.6B, or Table 2.1.6C, represents a function (if any)?

Table 2.1.6A

Input	Output
2	1
5	3
8	6

Table 2.1.6B

Input	Output
–3	5
0	1
4	5

Table 2.1.6C

Input	Output
1	0
5	2
5	4

Solution

Table 2.1.6A and Table 2.1.6B define functions. In both, each input value corresponds to exactly one output value. Table 2.1.6C does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 2.1.6A can be represented by writing

$$f(2) = 1, f(5) = 3, \text{ and } f(8) = 6.$$

Similarly, the statements

$$g(-3) = 5, g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table 2.1.6B.

Table 2.1.6C cannot be expressed in a similar way because it does not represent a function.

? Checkpoint 2.1.5

Select all of the following tables which represent y as a function of x.

x	y
-2	-4
3	2
6	5
9	8
3	10

☐

x	y
-2	-5
1	2
1	4
7	8
13	11

☐

x	y
-1	0
1	3
4	3
7	9
11	15

☐

x	y
-3	-1
2	3
4	6
7	8
14	11

☐

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Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Preview of Future Material

Each of the *types* of functions in the following examples will be studied in detail as we move through the textbook. The functions in Examples **2.1.6**, **2.1.7**, and **2.1.8** are called **quadratic functions**, and are the focus of an entire section later on. The functions in Checkpoints **2.1.7** and **2.1.8** are called **square root functions** (a subset of the **radical functions**). These also get their own sections later in the text.

Example 2.1.6: Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at:

a. 2
b. a
c. $a + h$
d. Now evaluate $\frac{f(a+h)-f(a)}{h}$

Solutions

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple Algebra to simplify.

$$\begin{aligned} f(2) &= (2)^2 + 3(2) - 4 \qquad \text{(substituting)} \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \qquad \text{(substituting)} \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \qquad \text{(distributing)} \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4.$$

Now we combine the results and simplify.

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} && \text{(substituting)}\\
&= \frac{a^2+2ah+h^2+3a+3h-4-a^2-3a+4}{h} && \text{(distributing)}\\
&= \frac{2ah+h^2+3h}{h} && \text{(combining like terms)}\\
&= \frac{h(2a+h+3)}{h} && \text{(factoring out the GCF)}\\
&= \frac{\cancel{h}^{1}(2a+h+3)}{\cancel{h}^{1}} && \text{(canceling like factors)}\\
&= 2a+h+3
\end{aligned}$$

Example 2.1.7: Evaluating Functions

Given the function $h(p)=p^2+2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned}
h(4) &= (4)^2+2(4) && \text{(substituting)}\\
&= 16+8 && \text{(simplifying)}\\
&= 24 && \text{(simplifying)}
\end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Checkpoint 2.1.7

If $f(x)=\sqrt{3x-15}$, then $f(15)=$ [ ] (Leave your answer in exact form. Enter DNE if the function is undefined or not a real number)

Question Help: Video Written Example

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Example 2.1.8: Solving Functions

Given the function $h(p)=p^2+2p$, solve $h(p)=3$.

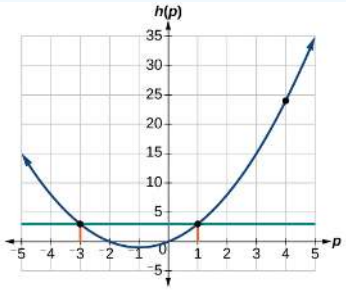
Solution

$$\begin{aligned}
& h(p) = 3 && \\
\Longrightarrow\ & p^2+2p = 3 && \text{(substituting)}\\
\Longrightarrow\ & p^2+2p-3 = 0 && \text{(subtracting 3 from both sides)}\\
\Longrightarrow\ & (p+3)(p-1) = 0 && \text{(factoring)}
\end{aligned}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case (recall from Elementary Algebra, this is called the **Zero Factor Property**).

$$(p+3)=0 \implies p=-3$$
$$(p-1)=0 \implies p=1$$

This gives us two solutions, so we know that $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing (see the figure below). The graph verifies that $h(1)=h(-3)=3$ and (for Example 2.1.7) $h(4)=24$.



p	-3	-2	0	1	4
$h(p)$	3	0	0	3	24

? Checkpoint 2.1.8

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

✔ Example 2.1.9: Finding the Algebraic Form of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as

$$p = [\text{expression involving } n].$$

$$\begin{aligned} & 2n+6p = 12 & \\ \implies \quad & 6p = 12-2n & \text{(subtracting } 2n \text{ from both sides)} \\ \implies \quad & p = \frac{12-2n}{6} & \text{(dividing both sides by 6)} \\ \implies \quad & p = \frac{12}{6}-\frac{2n}{6} & \\ \implies \quad & p = 2-\frac{n}{3} & \text{(canceling like factors)} \end{aligned}$$

Therefore, p as a function of n is written as

$$p=f(n)=2-\frac{1}{3}n.$$

Caution: Not all relations are functions

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 2.1.10: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution

First we subtract x^2 from both sides, to get

$$y^2 = 1 - x^2.$$

We now try to solve for y in this equation.

$$\begin{aligned} y^2 &= 1 - x^2 \\ \Rightarrow \quad y &= \pm\sqrt{1 - x^2} \qquad \text{(Extraction of Roots)} \end{aligned}$$

Thus, either $y = \sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$. That is, we get two outputs corresponding to the same input. Therefore, this relationship cannot be represented as a single function $y = f(x)$.

Checkpoint 2.1.10

Consider the relation defined by the equation $x = y^3$.

(a) Does this equation represent y as a function of x? If so, express y as a function of x.

- (●) Yes, $y = f(x) =$ [ ]
- ( ) No, y cannot be represented as a function of x.

(b) Does this equation represent x as a function of y? If so, express x as a function of y.

- (●) Yes, $x = g(y) =$ [ ]
- ( ) No, x cannot be represented as a function of y.

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Q&A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table.

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. For the pet-memory span information, let's call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Example 2.1.11: Evaluating and Solving a Tabular Function

Use the provided table to answer the following questions.

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

n	1	2	3	4	5
g	8	6	7	6	8

Solutions

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. The table shows two solutions: 2 and 4. That is, when we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Checkpoint 2.1.11

Answer each question based on the given table.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	33	69	16	74	2	65	77	1	47	54

(a) Evaluate $f(1)$.

$f(1) =$ _____

(b) Solve $f(x) = 2$.

$x =$ _____

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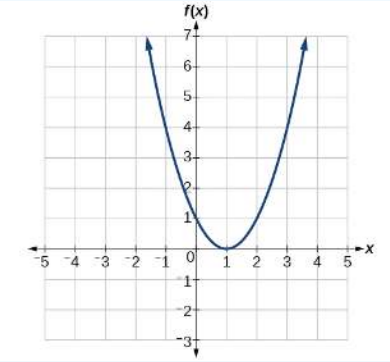
Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 2.1.12: Reading Function Values from a Graph

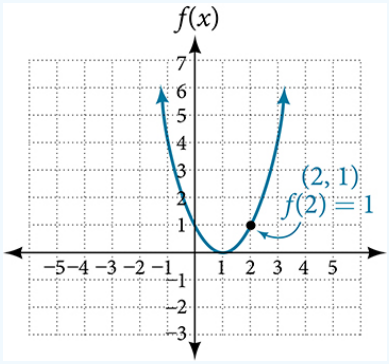
Use the graph below to answer the following questions.

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

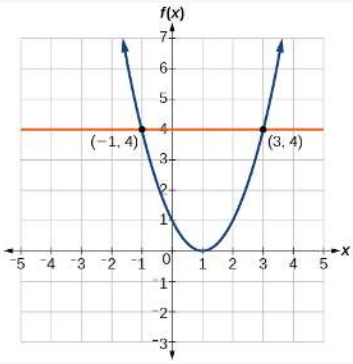


Solutions

a. To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$ (see the following figure).

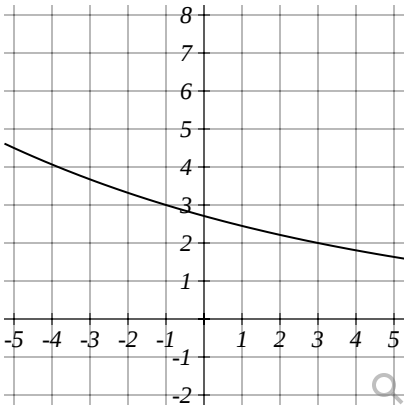


b. To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4 (see the figure below).



? Checkpoint 2.1.12

The graph below represents the function $f(x)$.



(a) Evaluate $f(3)$.

$f(3) =$ _____

(b) Solve $f(x) = 3$.

$x =$ _____

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Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input.

Definition: One-to-One Function

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

As an example, consider a school that uses only letter grades and decimal equivalents, as listed in the following table.

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figure $2.1.1(a)$ and Figure $2.1.1(b)$. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

Example 2.1.13: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value (the area) must correspond to a unique input value (the radius). Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Checkpoint 2.1.13A

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Checkpoint 2.1.13B

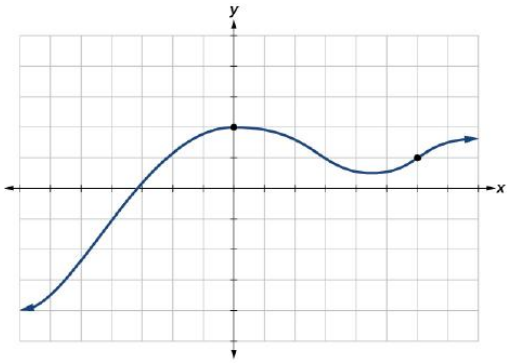
Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

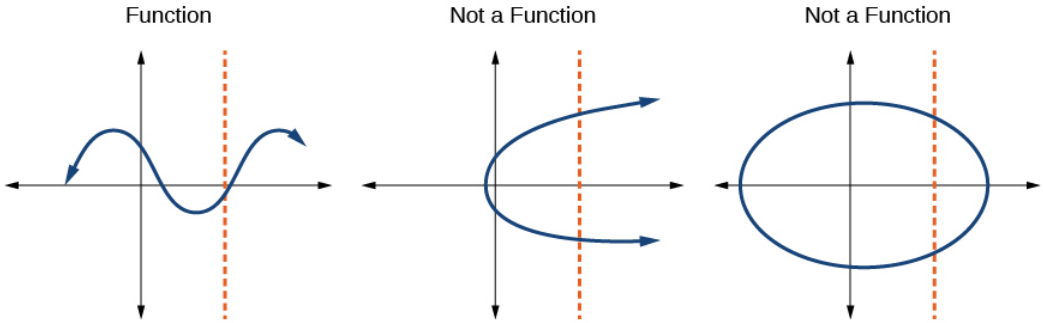
Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in the following figure tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points.



If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value (see the following figure).



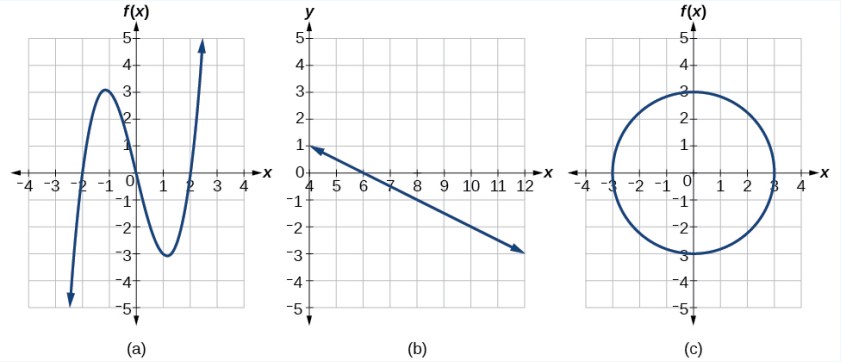
This leads us to the **vertical line test**.

Theorem: Vertical Line Test

A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

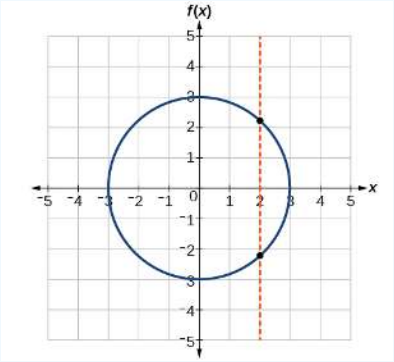
Example 2.1.14: Applying the Vertical Line Test

Which of the graphs in the following figure represent(s) a function $y = f(x)$?



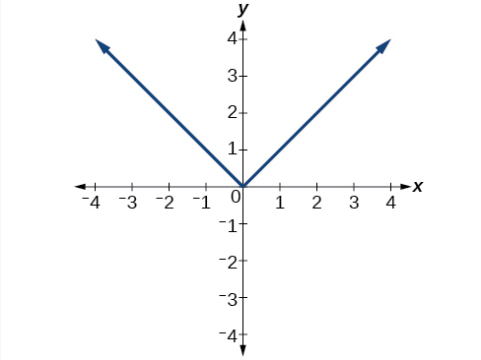
Solutions

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of the figure. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in the figure below.



? Checkpoint 2.1.14

Does the graph in the following figure represent a function?



Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **Horizontal Line Test**.

Theorem: Horizontal Line Test

A set of points representing a function in a rectangular coordinate system is the graph of a one-to-one function if every horizontal line intersects the graph in at most one point. If any horizontal line intersects the graph in more than one point, the graph does not represent a one-to-one function.

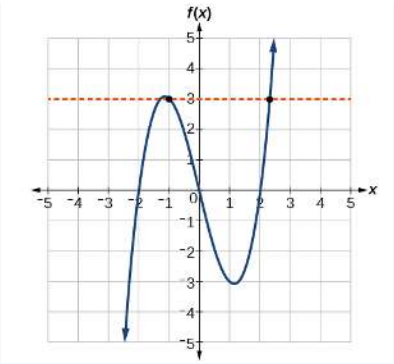
To use the Horizontal Line Test, draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Example 2.1.15: Applying the Horizontal Line Test

Consider the functions shown in parts (a) and (b) of Example **2.1.14**. Are either of the functions one-to-one?

Solution

The function in part (a) is not one-to-one. The horizontal line shown in the following figure intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)



The function in part (b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.

? Checkpoint 2.1.15

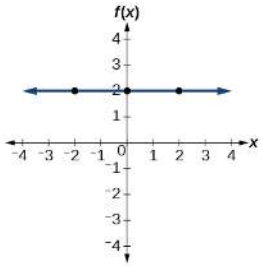
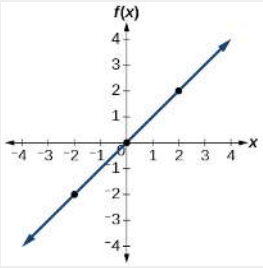
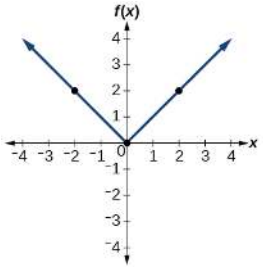
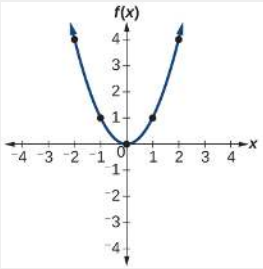
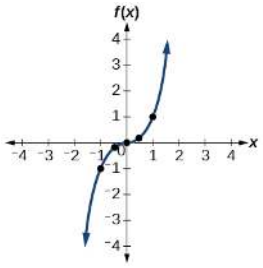
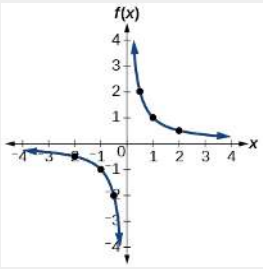
Is the graph shown in part (c) of Example **2.1.14** one-to-one?

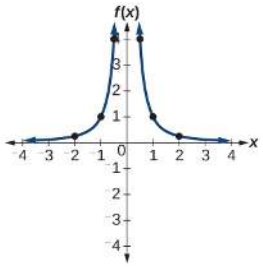
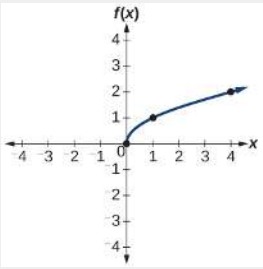
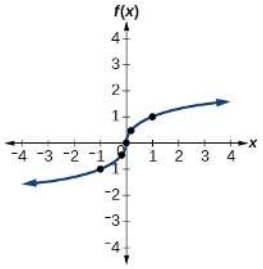
Identifying Basic Toolkit Functions

In this text, we will be exploring functions - the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do Arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "**toolkit functions**," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in the following table.

Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$, where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>
Identity	$f(x) = x$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Absolute value	$f(x) = \lvert x \rvert$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Quadratic	$f(x) = x^2$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table>
Cubic	$f(x) = x^3$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table>
Reciprocal	$f(x) = \frac{1}{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−0.5</td></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table>

Name	Function	Graph
Reciprocal squared	$f(x) = \frac{1}{x^2}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>0.25</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>−0.5</td><td>4</td></tr><tr><td>0.5</td><td>4</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.25</td></tr></table>
Square root	$f(x) = \sqrt{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table>
Cube root	$f(x) = \sqrt[3]{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table>

2.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Relations
  - Functions - Overview - A Review for College Algebra: Relations and Functions
- Functions
  - Algebra in Review: Functions - Overview
  - Intermediate Algebra - Functions: An Introduction
- Function notation
  - Algebra in Review: Functions - Notation
- Evaluating functions
  - Functions - Overview - A Review for College Algebra: Evaluating Functions
  - Functions - Overview - A Review for Precalculus: Evaluating Functions
  - Algebra in Review: Functions - Evaluation
- Representation of functions
  - Intermediate Algebra - Functions: The Four Representations
- One-to-one function
  - Functions - Overview - A Review for College Algebra: One-to-One Functions
- Vertical Line Test
  - Functions - Overview - A Review for College Algebra: The Vertical Line Test
- Horizontal Line Test
  - Functions - Overview - A Review for College Algebra: The Horizontal Line Test
- Toolkit functions
  - Intermediate Algebra - Functions: Base Graphs

KEY CONCEPTS

DEFINITIONS

- **Relation:** Any set of ordered pairs, denoted as (x, y).
- **Domain (of a relation/function):** The set of all first components (x-values) from the ordered pairs in a relation; it represents all possible input values.
- **Range (of a relation/function):** The set of all second components (y-values) from the ordered pairs in a relation; it represents all possible output values.
- **Input Value:** A value from the domain of a relation or function.
- **Output Value:** A value from the range of a relation or function, resulting from a given input value.
- **Independent Variable:** The variable (commonly x) that represents the input values of a function.
- **Dependent Variable:** The variable (commonly y) that represents the output values of a function, as its value depends on the input.
- **Function:** A specific type of relation where each possible input value is associated with exactly one output value.
- **Function Notation:** A convention, typically written as $y = f(x)$, used to denote a function, where f is the name of the function, x is the input (independent variable), and $f(x)$ is the output (dependent variable) corresponding to the input x.
- **One-to-One Function:** A function for which every distinct input value produces a distinct output value, meaning each output value corresponds to exactly one input value.

THEOREMS

- **Vertical Line Test:** A graph in a rectangular coordinate system represents a function if and only if no vertical line intersects the graph at more than one point. If any vertical line intersects the graph at more than one point, the graph does not represent a function.
- **Horizontal Line Test:** The graph of a function represents a one-to-one function if and only if no horizontal line intersects the graph at more than one point. If any horizontal line intersects the graph at more than one point, the function is not one-to-one.

COMMON MISTAKES

- **Misinterpreting Parentheses in Function Notation:** The parentheses in function notation, such as in $f(a)$, signify that a is the input to the function f. They do not indicate multiplication of f by a.
- **Assuming All Equations Define Functions:** Not every relationship expressed by an equation can be written as a single function where one variable is a function of the other. For example, the equation of a circle $x^2 + y^2 = 1$ does not define y as a single function of x because for a given x-value (other than 1 or -1), there are two corresponding y-values.

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2.1.2: HOMEWORK

READING QUESTIONS

1. What is a relation?
2. What are the domain and range of a relation?
3. How does a function differ from a general relation?
4. In the function notation $y = f(x)$, what do x, f, and $f(x)$ represent?
5. What does it mean to "evaluate" a function at a specific value, for example, $f(2)$?
6. Describe the Vertical Line Test. What is it used for?
7. What is a one-to-one function?
8. How can the Horizontal Line Test be used?
9. Does the equation $x^2 + y^2 = 1$ represent y as a function of x? Why or why not?
10. Name three of the "toolkit functions" mentioned in the section.

HOMEWORK

1. What is the difference between a relation and a function?
2. What is the difference between the input and the output of a function?
3. Why does the Vertical Line Test tell us whether the graph of a relation represents a function?
4. How can you determine if a relation is a one-to-one function?
5. Why does the Horizontal Line Test tell us whether the graph of a function is one-to-one?

For the following exercises, determine whether the relation represents a function.

6. $\{(a,b),(c,d),(a,c)\}$
7. $\{(a,b),(b,c),(c,c)\}$

For the following exercises, determine whether the relation represents y as a function of x.

8. $5x + 2y = 10$
9. $y = x^2$
10. $x = y^2$
11. $3x^2 + y = 14$
12. $2x + y^2 = 6$
13. $y = -2x^2 + 40x$
14. $y = \frac{1}{x}$
15. $x = \frac{3y+5}{7y-1}$
16. $x = \sqrt{1-y^2}$
17. $y = \frac{3x+5}{7x-1}$
18. $x^2 + y^2 = 9$
19. $2xy = 1$
20. $x = y^3$
21. $y = x^3$
22. $y = \sqrt{1-x^2}$
23. $x = \pm\sqrt{1-y}$
24. $y = \pm\sqrt{1-x}$
25. $y^2 = x^2$
26. $y^3 = x^2$

For the following exercises, evaluate $f(-3)$, $f(2)$, $f(-a)$, $-f(a)$, and $f(a+h)$.

27. $f(x) = 2x - 5$
28. $f(x) = -5x^2 + 2x - 1$
29. $f(x) = \sqrt{2-x} + 5$
30. $f(x) = \frac{6x-1}{5x+2}$
31. $f(x) = |x-1| - |x+1|$
32. Given the function $g(x) = 5 - x^2$, evaluate

$$\frac{g(x+h) - g(x)}{h}, \text{ where } h \neq 0.$$

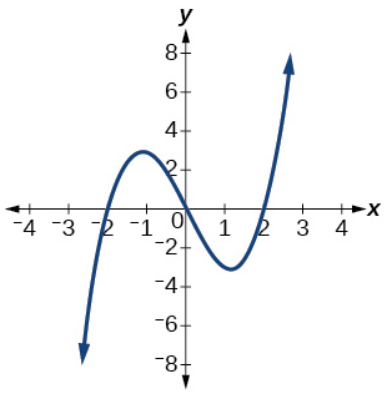
33. Given the function $g(x) = x^2 + 2x$, evaluate

$$\frac{g(x) - g(a)}{x - a}, \text{ where } x \neq a.$$

34. Given the function $k(t) = 2t - 1$:
    a. Evaluate $k(2)$.
    b. Solve $k(t) = 7$.
35. Given the function $f(x) = 8 - 3x$:
    a. Evaluate $f(-2)$.
    b. Solve $f(x) = -1$.
36. Given the function $p(c) = c^2 + c$:
    a. Evaluate $p(-3)$.
    b. Solve $p(c) = 2$.
37. Given the function $f(x) = x^2 - 3x$:
    a. Evaluate $f(5)$.
    b. Solve $f(x) = 4$.
38. Given the function $f(x) = \sqrt{x+2}$:
    a. Evaluate $f(7)$.
    b. Solve $f(x) = 4$.
39. Consider the relationship $3r + 2t = 18$.
    a. Write the relationship as a function $r = f(t)$.
    b. Evaluate $f(-3)$.
    c. Solve $f(t) = 2$.

For the following exercises, use the Vertical Line Test to determine which graphs show relations that are functions.

40.



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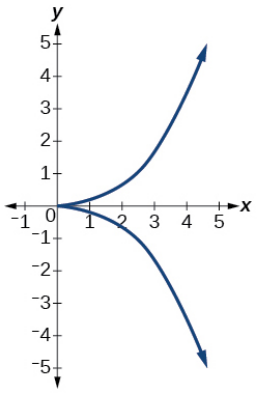

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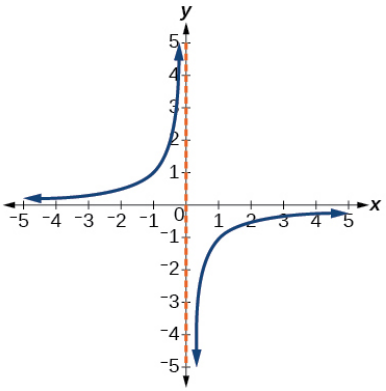

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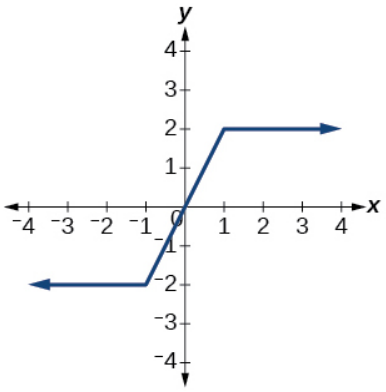

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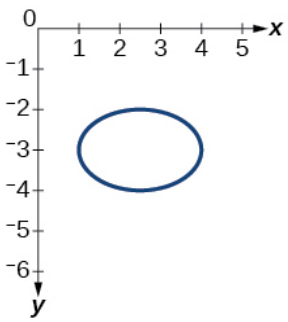

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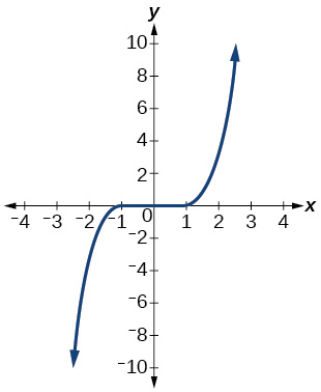

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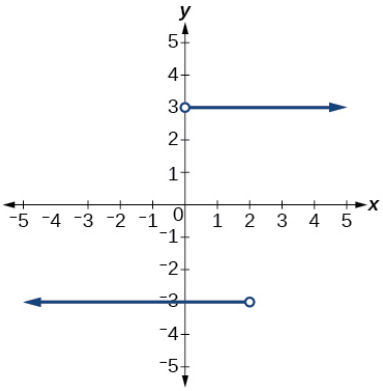

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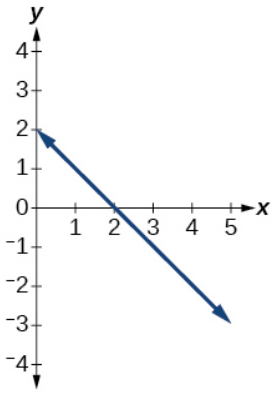

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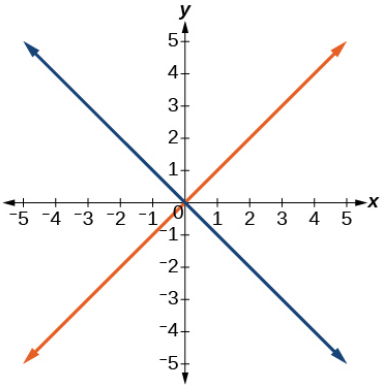

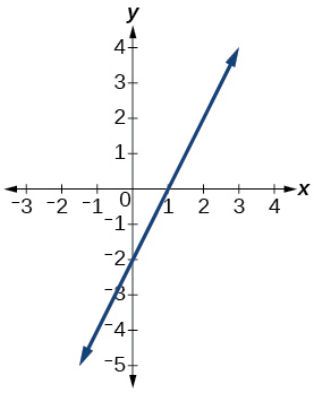

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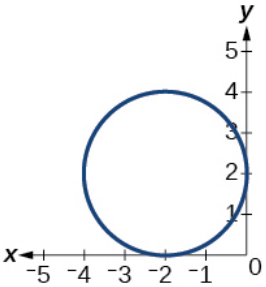

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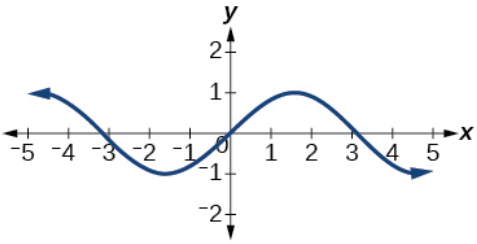
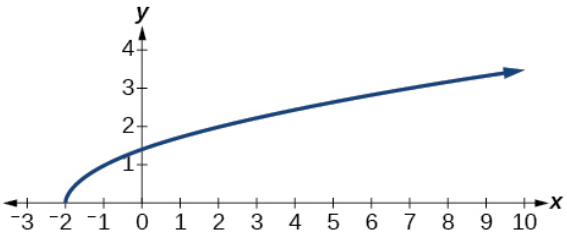


52. Given the following graph,

a. Evaluate $f(-1)$.

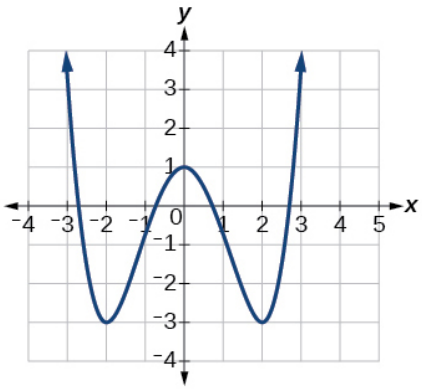
b. Solve for $f(x) = 3$.



53. Given the following graph,

a. Evaluate $f(0)$.

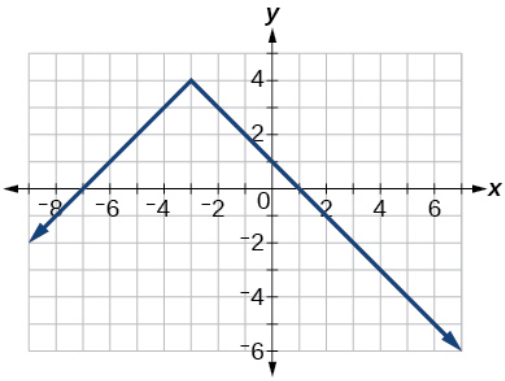
b. Solve for $f(x) = -3$.



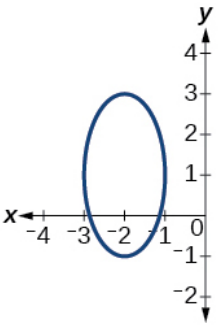
54. Given the following graph,

a. Evaluate $f(4)$.

b. Solve for $f(x) = 1$.



For the following exercises, determine if the given graph is a one-to-one function.



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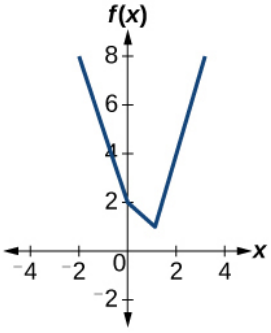

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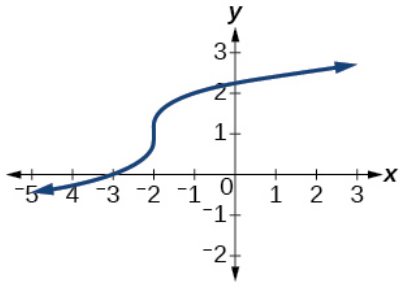
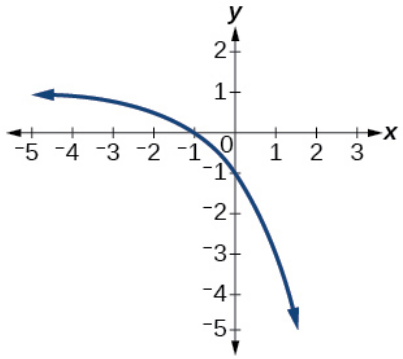
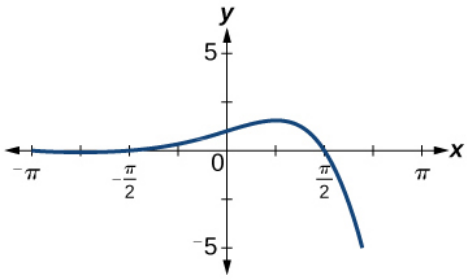
57. 

58. 

59. 

For the following exercises, determine whether the relation represents a function.

60. $\{(-1,-1),(-2,-2),(-3,-3)\}$
61. $\{(3,4),(4,5),(5,6)\}$
62. $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

63.

x	5	10	15
y	3	8	14

64.

x	5	10	15
y	3	8	8

65.

x	5	10	10
y	3	8	14

For the following exercises, use the function f represented in the following table.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

66. Evaluate $f(3)$.

67. Solve $f(x)=1$.

For the following exercises, evaluate the function f at the values $-2, -1, 0, 1,$ and 2.

68. $f(x)=4-2x$
69. $f(x)=8-3x$
70. $f(x)=8x^2-7x+3$
71. $f(x)=3+\sqrt{x+3}$
72. $f(x)=\frac{x-2}{x+3}$
73. $f(x)=3^x$

For the following exercises, evaluate the expressions, given functions f, g, and h:

- $f(x)=3x-2$
- $g(x)=5-x^2$
- $h(x)=-2x^2+3x-1$

74. $3f(1)-4g(-2)$
75. $f\left(\frac{7}{3}\right)-h(-2)$

For the following exercises, graph $y=x^2$ on the given domain. Determine the corresponding range. Show each graph.

76. $[-0.1, 0.1]$
77. $[-10, 10]$
78. $[-100, 100]$

For the following exercises, graph $y=x^3$ on the given domain. Determine the corresponding range. Show each graph.

79. $[-0.1, 0.1]$
80. $[-10, 10]$
81. $[-100, 100]$

For the following exercises, graph $y=\sqrt{x}$ on the given domain. Determine the corresponding range. Show each graph.

82. $[0, 0.01]$
83. $[0, 100]$
84. $[0, 10000]$

For the following exercises, graph $y=\sqrt[3]{x}$ on the given domain. Determine the corresponding range. Show each graph.

85. $[-0.001, 0.001]$
86. $[-1000, 1000]$
87. $[-1000000, 1000000]$
88. The amount of garbage, G, produced by a city with population p is given by $G=f(p)$. G is measured in tons per week, and p is measured in thousands of people.
    a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
    b. Explain the meaning of the statement $f(5)=2$.
89. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D=g(a)$.
    a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.
    b. Explain the meaning of the statement $g(100)=1$.

90. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
    a. $f(5) = 30$
    b. $f(10) = 40$
91. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
    a. $h(1) = 200$
    b. $h(2) = 350$
92. Show that the function $f(x) = 3(x-5)^2 + 7$ is not one-to-one.

2.2: Domain and Range

Note to the Instructor (click to expand)

The algebraic functions encountered in this section should *not* involve compositions of functions (e.g., the inverse square function composed with a square root composed with a quadratic). Focus on the domains of basic functions (involving shifts). Piecewise functions and the absolute value represented as a piecewise function are critical topics.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Sets and Numbers**
  - Inequalities and Inequality Notation: Understanding and using inequality notation is fundamental for defining domains, especially when working with radicands of even roots (which must be non-negative, e.g., $7 - x \geq 0$) and for interpreting interval notation. This section extensively uses inequalities to define valid inputs for functions.
  - Intervals and Interval Notation: Although reviewed in Figure 3 and the "Set-Builder Notation and Interval Notation" box, immediate application in homework for expressing domains and ranges makes prior familiarity beneficial for fluency.
- **Factoring Techniques**
  - Factoring Difference of Squares: Useful for expressions like $36 - x^2$ found in radicands.
- **Solving Inequalities**
  - Solving Linear Inequalities: Necessary for determining the domain of functions involving even roots where the radicand is a linear expression, such as in Example 4 ($7 - x \geq 0$) and within several homework questions.
  - Solving Compound Inequalities: Useful for understanding and expressing domains that consist of multiple, disjoint intervals, often connected by "or," such as the union of sets (e.g., $\{x \mid |x| \geq 3\}$ which translates to $x \leq -3$ or $x \geq 3$) and in Example 5.
  - Solving Quadratic Inequalities (sign chart method): Required for finding the domain of functions where an even root contains a quadratic expression or where a rational expression's denominator under a root must be strictly positive, as seen in the homework ($f(x) = \frac{1}{\sqrt{49 - x^2}}$, requiring $49 - x^2 > 0$).
- **Graphs**
  - Closed- and Open-Circle Notations: Essential for correctly interpreting and graphing piecewise functions, particularly at the boundary points of their domains, as

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

discussed in the solution to Example 13 and required for some homework questions.

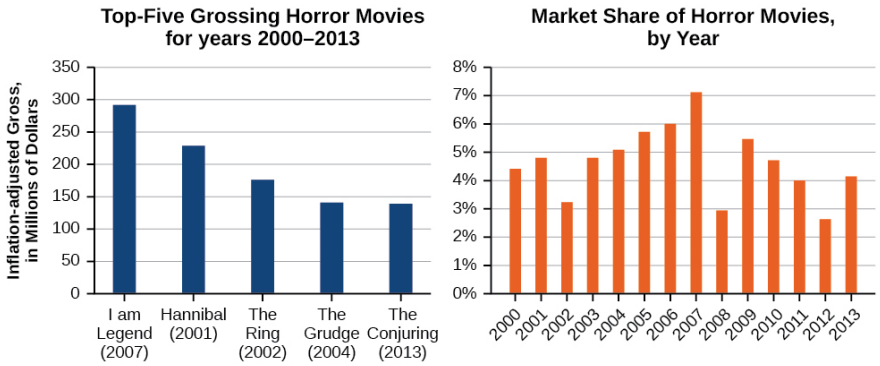
The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time - *I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. Figure **2.2.1** shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

Figure 2.2.1

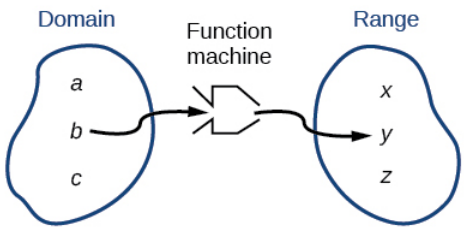


Finding the Domain of a Function Defined by an Equation

In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (see Figure **2.2.2**).

Figure 2.2.2



We can write the domain and range in interval notation, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket, [, when the set includes the endpoint and a parenthesis, (, to indicate that the endpoint is either

not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will review interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an odd root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called **exclusive**.
- Brackets, [or], are used to indicate that an endpoint is included, called **inclusive**.

Figure 2.2.3 provides a summary of interval notation.

Figure 2.2.3

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Example 2.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$.

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

? Checkpoint 2.2.1

Find the domain of the function: $\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}$

✓ Example 2.2.2: Finding the Domain of a Function

Find the domain of the function $f(x)=x^2-1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers. In interval form, the domain of f is $(-\infty,\infty)$.

? Checkpoint 2.2.2

Find the domain of the function: $f(x)=5-x+x^3$.

✓ Example 2.2.3: Finding the Domain of a Function Involving a Denominator

Find the domain of the function

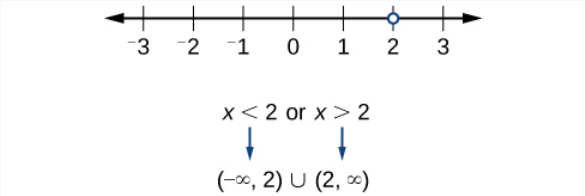
$$f(x)=\frac{x+1}{2-x}$$

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2-x &= 0 \\ \Rightarrow \quad 2 &= x \qquad \text{(adding } x \text{ to both sides)} \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$. We can use a symbol known as the union, $\cup$, to combine the two sets. In interval notation, we write the solution: $(-\infty,2)\cup(2,\infty)$.



Thus, in interval form, the domain of f is $(-\infty,2)\cup(2,\infty)$.

? Checkpoint 2.2.3

Find the domain of the function:

$$f(x)=\frac{1+4x}{2x-1}.$$

✓ Example 2.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function

$$f(x)=\sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand. To do this, we set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7-x &\geq 0 && \text{(even-indexed radicals must have nonnegative arguments)} \\ \Longrightarrow \quad 7 &\geq x && \text{(adding } x \text{ to both sides)} \end{aligned}$$

Therefore, x must be less than or equal to 7. Hence, we need to exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

? Checkpoint 2.2.4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Preview of Future Material

Functions of the types found in Examples **2.2.2** through **2.2.4** will be explored in more detail later in the textbook. For now, any domain questions involving these functions will be fairly straightforward; however, later we will see them "composed" with other functions.

Q&A

Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function

$$f(x) = -\frac{1}{\sqrt{x}}$$

has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**.

Definition: Set-Builder Notation

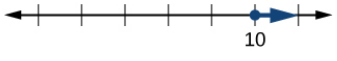
The notation

$$\{x \mid x \text{ satisfies some condition}\}$$

is called **set-builder notation**. The braces, { }, are read literally as "the set of." The vertical bar is read as, "such that." Hence, set-builder notation is read as "the set of x such that x satisfies some given condition."

For example, $\{x \mid 10 \leq x < 30\}$ describes the behavior of x in set-builder notation. So we would read $\{x \mid 10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

The following figure compares inequality notation, set-builder notation, and interval notation.

	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$, to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x \mid |x| \geq 3\} = (-\infty, -3] \cup [3, \infty).$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x \mid \text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

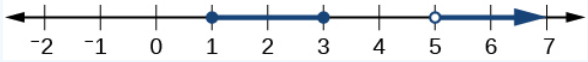
$$\{x \mid 4 < x \leq 12\}.$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12].$$

Example 2.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in the following figure using inequality notation, set-builder notation, and interval notation.



Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and

less than or equal to 3, or a real number greater than 5."

Inequality	$1 \leq x \leq 3$ or $x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

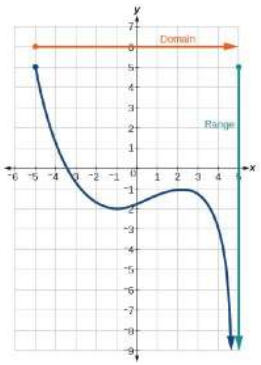
? Checkpoint 2.2.5

Use the following figure to specify the graphed set in

a. words
b. set-builder notation
c. interval notation

Finding Domain and Range from Graphs

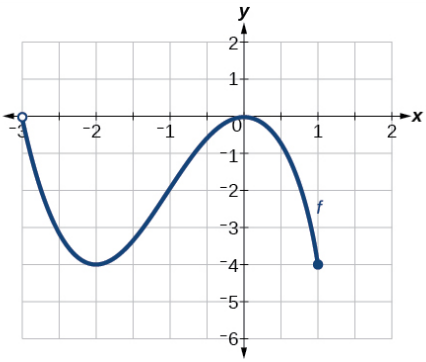
Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values.



We can observe that the graph in the figure above extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

✓ Example 2.2.6: Finding Domain and Range from a Graph

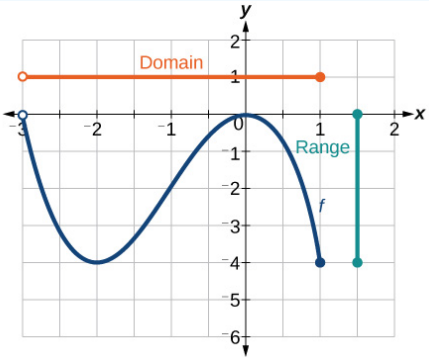
Find the domain and range of the function f whose graph is shown below.



Solution

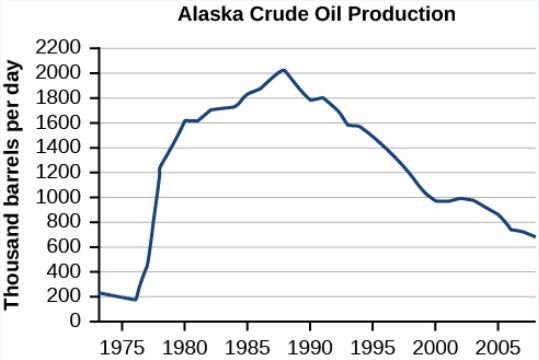
We can observe that the horizontal extent of the graph is –3 (exclusive) to 1 (inclusive), so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 (inclusive) to –4 (inclusive), so the range is $[-4, 0]$. See the figure below.



Example 2.2.7: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown below.



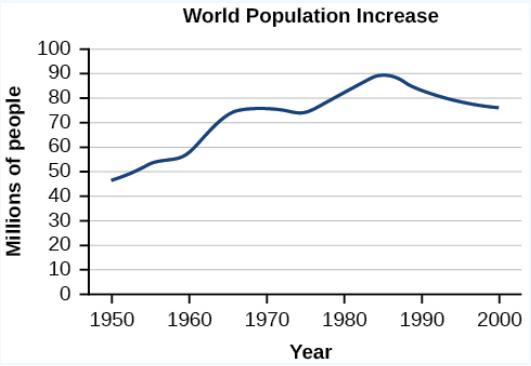
Solution

The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Checkpoint 2.2.7

Identify the domain and range of the graph in the figure below using interval notation.



Q&A

Can a function's domain and range be the same?

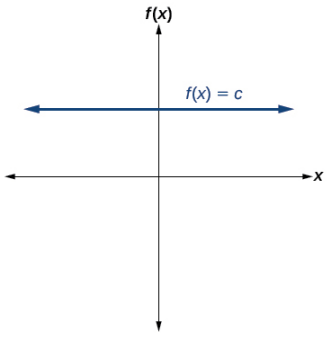
Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

Figure 2.2.4

For the constant function $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



Domain: $(-\infty, \infty)$
Range: $[c,c]$

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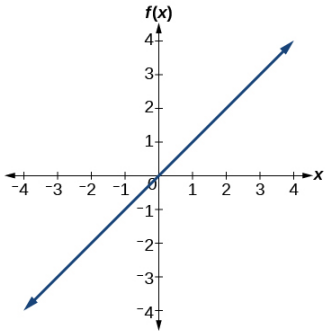
Figure 2.2.5

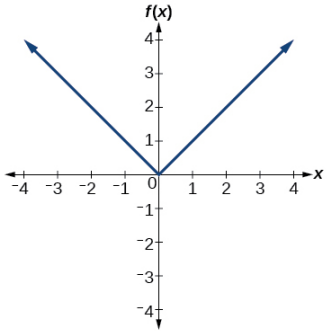
For the identity function $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.6

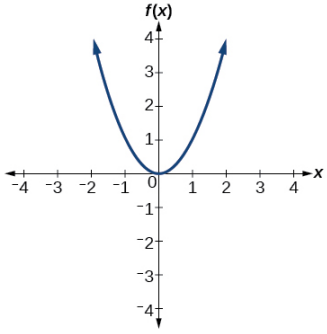
For the absolute value function $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.7

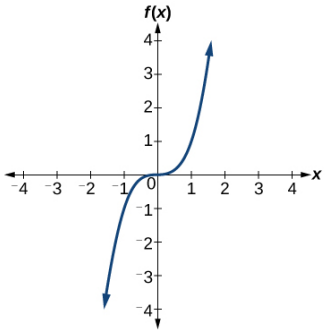
For the quadratic function $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.8

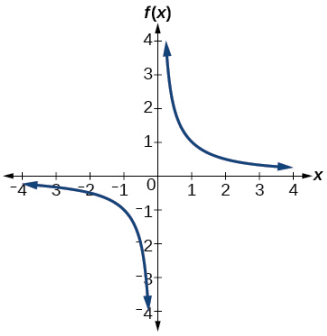
For the cubic function $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.9

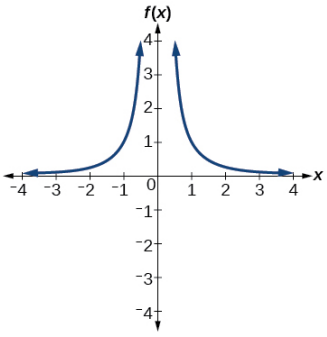
For the reciprocal function $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x \mid x \neq 0\}$, the set of all real numbers that are not zero.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 2.2.10

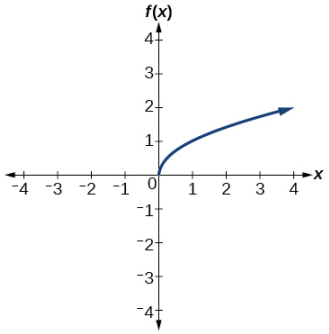
For the reciprocal squared function $f(x) = \frac{1}{x^2}$, we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 2.2.11

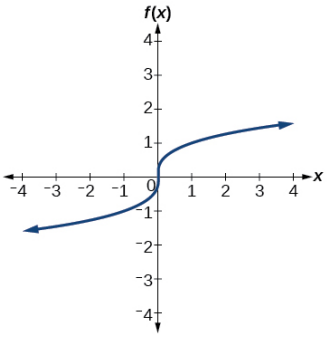
For the square root function $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: $[0, \infty)$
Range: $[0, \infty)$

Figure 2.2.12

For the cube root function $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Example 2.2.8: Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result. Therefore, the domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 2.2.9: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 2.2.10: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

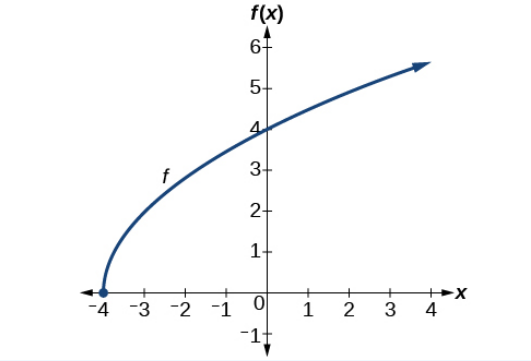
$$x + 4 \geq 0 \text{ when } x \geq -4.$$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

The following figure represents the function f.



Checkpoint 2.2.10

Find the domain and range of $f(x) = -\sqrt{2-x}$.

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0.$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0.$$

Because this requires two different processes, or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

Definition: Piecewise Function

A **piecewise function** is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea as follows:

$$f(x) = \begin{cases} \text{formula 1}, & \text{if } x \text{ is in domain 1} \\ \text{formula 2}, & \text{if } x \text{ is in domain 2} \\ \text{formula 3}, & \text{if } x \text{ is in domain 3} \\ \vdots & \vdots \end{cases}$$

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to $10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \le \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Example 2.2.11: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

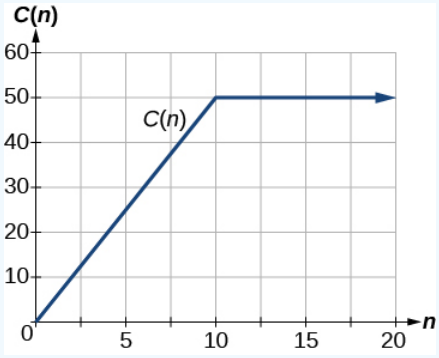
Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n, & \text{if } 0 < n < 10 \\ 50, & \text{if } n \ge 10 \end{cases}$$

Analysis

The function is represented in the figure below. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.



Example 2.2.12: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25, & \text{if } 0 < g < 2 \\ 25 + 10(g - 2), & \text{if } g \ge 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

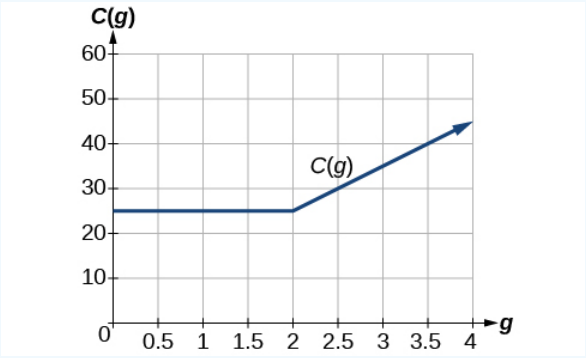
$$C(1.5) = \$25.$$

To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4 - 2) = \$45.$$

Analysis

The function is represented in the figure below. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.



While I am not a huge fan of "algorithmic" instructions on how to do each problem (such habits lead to an overwhelming amount of memorization and a subpar level of understanding), such instruction is incredibly helpful when it comes to graphing piecewise functions.

How To

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 2.2.13: Graphing a Piecewise Function

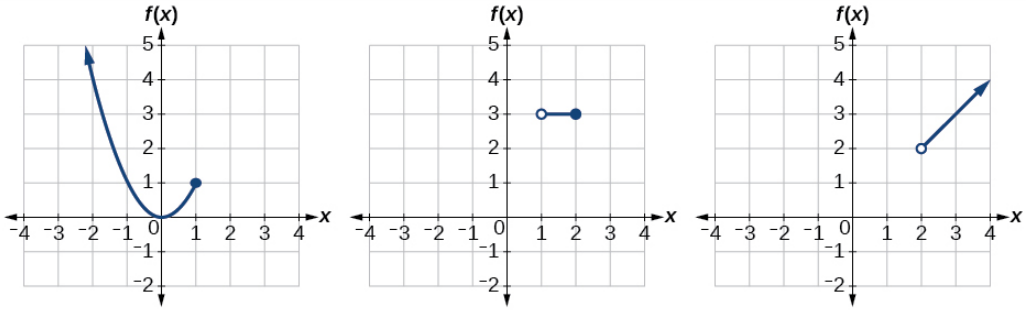
Sketch a graph of the function.

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ 3, & \text{if } 1 < x \leq 2 \\ x, & \text{if } x > 2 \end{cases}$$

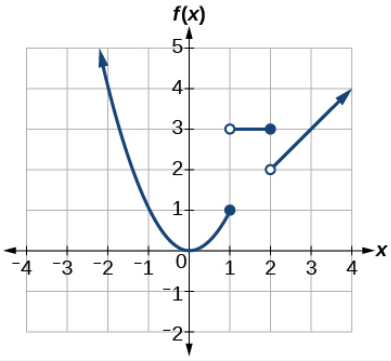
Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

The figure below shows the three components of the piecewise function graphed on separate coordinate systems.



Now that we have sketched each piece individually, we combine them in the same coordinate plane.



Analysis

Note that the graph does pass the Vertical Line Test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

Checkpoint 2.2.13

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3, & \text{if } x \leq -1 \\ -2, & \text{if } -1 < x \leq 4 \\ \sqrt{x}, & \text{if } x > 4 \end{cases}$$

Before we leave this section, it's best to preview the *real* definition of the absolute value as a piecewise function.

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

Q&A

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

2.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Domain and Range of a function
  - Algebra in Review: Functions - Determining Domain
  - Domain and Range - Finding Domain and Range from Graphs of Functions
  - Domain and Range - Finding the Domain of a Function Defined by an Equation
- Interval notation
  - Logic & Set Theory in Review: Sets - Notation (Interval Notation)
- Graphing piecewise functions
  - Domain and Range - Graphing Piecewise-Defined Functions
  - Functions - Overview - A Review for Precalculus: Piecewise-Defined Functions - Graphing

KEY CONCEPTS

DEFINITIONS

- **Interval Notation:** A method of writing sets of numbers using parentheses ( ) to indicate endpoints that are not included (exclusive) and square brackets [ ]) to indicate endpoints that are included (inclusive). Unbounded intervals use ∞ or $-\infty$.
- **Exclusive Endpoint:** An endpoint of an interval that is not included in the set, denoted by a parenthesis in interval notation.
- **Inclusive Endpoint:** An endpoint of an interval that is included in the set, denoted by a square bracket in interval notation.
- **Union ($\cup$):** A symbol used to combine two or more sets, representing all elements that belong to any of the original sets. For intervals, it connects two or more disjoint segments of the number line.
- **Set-Builder Notation:** A notation used to describe a set by stating the properties that its members must satisfy, typically in the form $\{x \mid \text{condition(s) involving } x\}$, read as "the set of all x such that x satisfies some given condition."
- **Piecewise Function:** A function defined by multiple sub-functions, where each sub-function applies to a different interval in the domain.
- **Magnitude (or Modulus):** For a real number, its absolute value, representing its distance from 0 on the number line, regardless of sign.
- **Absolute Value Function (as a piecewise function):** The function $f(x) = |x|$ defined as $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$.

COMMON MISTAKES

- **Graphing Multiple Function Pieces Over the Same Interval:** When sketching a piecewise function, graphing more than one of its defining equations over the same segment of the domain violates the definition of a function (as each input would have multiple outputs).
- **Misinterpreting Open and Closed Circles in Piecewise Graphs:** Incorrectly determining whether boundary points are included in a piece of a piecewise function can lead to errors. For example, in Example 13, points like $(1, 3)$ and $(2, 2)$ are excluded (open circles) while $(1, 1)$ and $(2, 3)$ are included (closed circle or part of a continuous segment).
- **Applying Multiple Formulas to a Single Input in a Piecewise Function:** For any given input value in the domain of a piecewise function, only one of its defining formulas/equations applies.

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2.2.2: HOMEWORK

READING QUESTIONS

1. What are the two main algebraic restrictions you must consider when determining the domain of a function consisting of real numbers given by an equation?
2. Explain the difference between a parenthesis and a square bracket in interval notation.
3. If a graph of a function has an open circle at $x = 3$, is $x = 3$ included in the domain? What if it has a closed circle?
4. What is a piecewise function?
5. How do you find the domain of a function like $f(x) = \frac{5}{x-7}$?
6. How do you find the domain of a function like $g(x) = \sqrt{x+4}$?
7. Can a single input value in the domain of a piecewise function correspond to more than one output value or be defined by more than one of its formulas?
8. What is the domain and range of the identity function, $f(x) = x$?
9. What is the domain and range of the absolute value function, $f(x) = |x|$?
10. When graphing a piecewise function, what do you draw at an endpoint of an interval if the inequality is strict (e.g., $<$ or $>$)? What if it is inclusive (e.g., $\leq$ or $\geq$)?

HOMEWORK

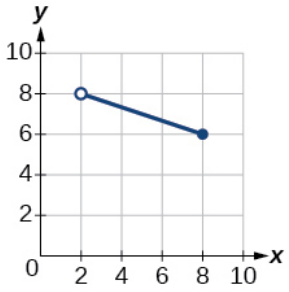
1. Why does the domain differ for different functions?
2. How do we determine the domain of a function defined by an equation?
3. Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.
4. When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?
5. How do you graph a piecewise function?

For the following exercises, find the domain of each function using interval notation.

6. $f(x) = -2x(x-1)(x-2)$
7. $f(x) = 5 - 2x^2$
8. $f(x) = 3\sqrt{x-2}$
9. $f(x) = 3 - \sqrt{6-2x}$
10. $f(x) = \sqrt{4-3x}$
11. $f(x) = \sqrt{x^2+4}$
12. $f(x) = \sqrt[3]{1-2x}$
13. $f(x) = \sqrt[3]{x-1}$
14. $f(x) = \frac{9}{x-6}$
15. $f(x) = \frac{3x+1}{4x+2}$
16. $f(x) = \frac{\sqrt{x+4}}{x-4}$
17. $f(x) = \frac{x-3}{x^2+9x-22}$
18. $f(x) = \frac{1}{x^2-x-6}$
19. $f(x) = \frac{2x^3-250}{x^2-2x-15}$
20. $f(x) = \frac{5}{\sqrt{x-3}}$
21. $f(x) = \frac{2x+1}{\sqrt{5-x}}$
22. $f(x) = \frac{\sqrt{x-4}}{\sqrt{x-6}}$
23. $f(x) = \frac{\sqrt{x-6}}{\sqrt{x-4}}$
24. $f(x) = \frac{x}{x}$
25. $f(x) = \frac{x^2-9x}{x^2-81}$

For the following exercises, write the domain and range of each function using interval notation.

26.



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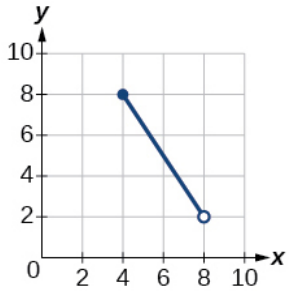

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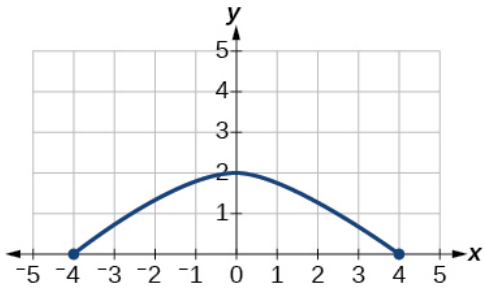

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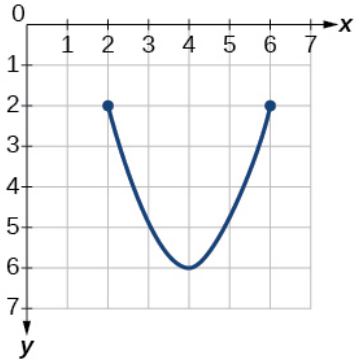

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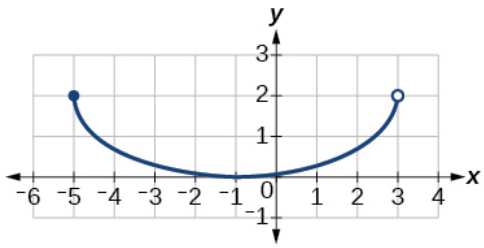

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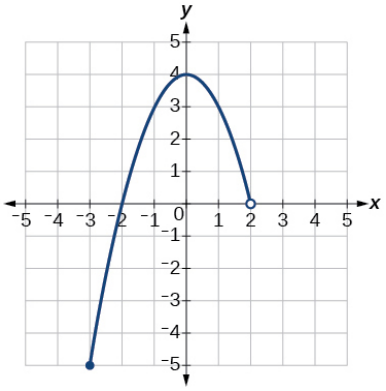

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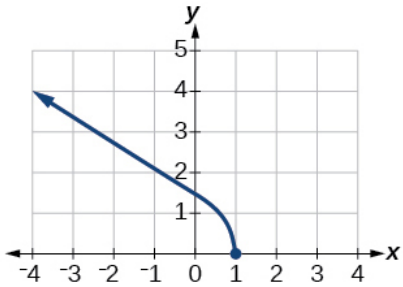

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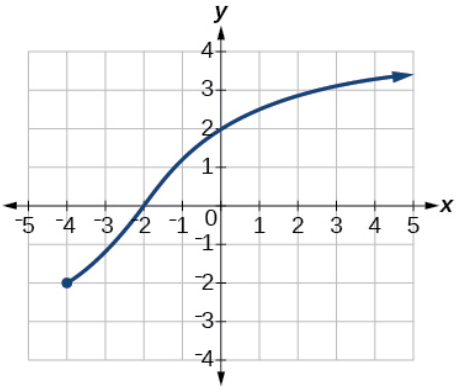

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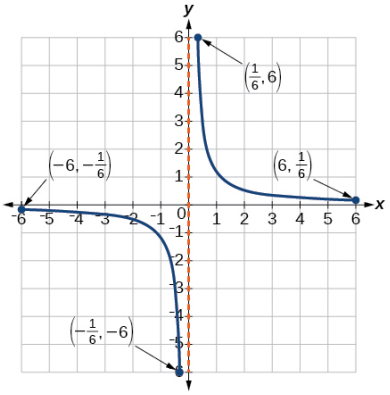

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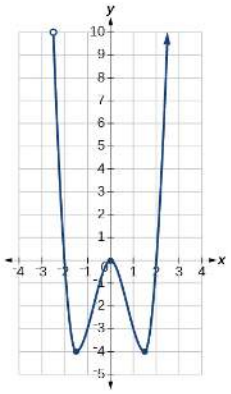

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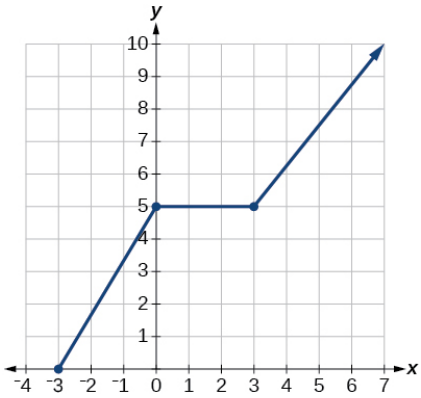

For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

37. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$

38. $f(x) = \begin{cases} 2x-1, & \text{if } x < 1 \\ 1+x, & \text{if } x \geq 1 \end{cases}$

39. $f(x) = \begin{cases} x+1, & \text{if } x < 0 \\ x-1, & \text{if } x > 0 \end{cases}$

40. $f(x) = \begin{cases} 3, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases}$

41. $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 1-x, & \text{if } x > 0 \end{cases}$

42. $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x+2, & \text{if } x \geq 0 \end{cases}$

43. $f(x) = \begin{cases} x+1, & \text{if } x < 1 \\ x^3, & \text{if } x \geq 1 \end{cases}$

44. $f(x) = \begin{cases} |x|, & \text{if } x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$

For the following exercises, given each function f, evaluate $f(-3), f(-2), f(-1)$, and $f(0)$.

45. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$
46. $f(x) = \begin{cases} 1, & \text{if } x \leq -3 \\ 0, & \text{if } x > -3 \end{cases}$
47. $f(x) = \begin{cases} -2x^2+3, & \text{if } x \leq -1 \\ 5x-7, & \text{if } x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1), f(0), f(2)$, and $f(4)$.

48. $f(x) = \begin{cases} 7x+3, & \text{if } x < 0 \\ 7x+6, & \text{if } x \geq 0 \end{cases}$
49. $f(x) = \begin{cases} x^2-2, & \text{if } x < 2 \\ 4+|x-5|, & \text{if } x \geq 2 \end{cases}$
50. $f(x) = \begin{cases} 5x, & \text{if } x < 0 \\ 3, & \text{if } 0 \leq x \leq 3 \\ x^2, & \text{if } x > 3 \end{cases}$

For the following exercises, write the domain for the piecewise function in interval notation.

51. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$
52. $f(x) = \begin{cases} x^2-2, & \text{if } x < 1 \\ -x^2+2, & \text{if } x > 1 \end{cases}$
53. $f(x) = \begin{cases} 2x-3, & \text{if } x < 0 \\ -3x^2, & \text{if } x \geq 2 \end{cases}$
54. Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.
55. Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.
56. Suppose the range of a function f is $[-5, 8]$. What is the range of $|f(x)|$?
57. Create a function in which the range is all nonnegative real numbers.
58. Create a function in which the domain is $x > 2$.
59. The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?
60. The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.
    a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.
    b. What is the cost of making 25 items?
    c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

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2.3: The Arithmetic and Composition of Functions

Note to the Instructor (click to expand)

In addition to introducing compositions (and arithmetic combinations) of functions, you should focus on finding the domains of these new functions. Furthermore, the difference quotient is introduced (and will be revisited throughout the remainder of the text whenever a new function is introduced).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Simplifying Ratios of Polynomials: This is directly applied in Example 1 when simplifying $(\frac{g}{f})(x) = \frac{x^2-1}{x-1}$ by canceling common factors.
  - Adding and Subtracting Rational Expressions: Required when combining functions that might result in rational expressions, or when simplifying parts of a composite function. For example, if $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x-1}$, finding $f(x) + g(x)$ would require this skill. (While not explicitly in an example, this is a natural extension of function arithmetic with rational functions like those in the homework).
  - Multiplying and Dividing Rational Expressions: Necessary for performing multiplication or division of functions that are rational expressions. These types of problems occur in the homework.
  - Simplifying Compound Rational Expressions: This skill is necessary when simplifying the result of a composition of rational functions, such as in $f(x) = \frac{x}{x-2}$ and $g(x) = \frac{6}{x}$, and one computes $f(g(x))$ or $g(f(x))$.
- **Solving Equations**
  - Solving Rational Equations: Needed to find values of x for which $g(x)$ would be an excluded value from the domain of f when determining the domain of $f(g(x))$, as seen in Example 9 where we solve $g(x) = 1$, which is $\frac{4}{3x-2} = 1$. Also used in the homework.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Perform arithmetic on functions
- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Hawk A.I. Section-Specific Tutor

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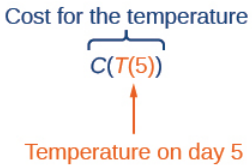
- Compute the Difference Quotient for a given function.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. This cost will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, \(\text{Cost\} = C (T (d)) \) means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C(T(5))$.



By combining these two relationships into one function, we have performed **function composition**, which is the focus of this section.

Combining Functions Using Algebraic Operations

Before we get to function composition, we need to backtrack a little to learn how to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y).$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w.$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of Algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

Definition: Function Arithmetic

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \text{ if } g(x)\neq 0\end{aligned}$$

Example 2.3.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions. For $(g-f)(x)$, we have

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= x^2-1-(x-1) && \text{(substituting)}\\ &= x^2-1-x+1 && \text{(distributing)}\\ &= x^2-x && \text{(combining like terms)}\\ &= x(x-1) && \text{(factoring out the GCF)}\end{aligned}$$

For $\left(\frac{g}{f}\right)(x)$, we have

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ &= \frac{x^2-1}{x-1} && \text{(substituting)}\\ &= \frac{(x-1)(x+1)}{x-1} && \text{(factoring)}\\ &= \frac{\cancel{(x-1)}^{1}(x+1)}{\cancel{(x-1)}^{1}} && \text{(canceling like terms and assuming } x\neq 1)\\ &= x+1 && \text{(where } x\neq 1)\end{aligned}$$

Therefore, the functions are not the same.

In Example 2.3.1, for $\left(\frac{g}{f}\right)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Checkpoint 2.3.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$ given

$$f(x)=x-1 \text{ and } g(x)=x^2-1.$$

Are they the same function?

Focus on Calculus - Difference Quotients

Now that we know how to perform arithmetic with functions, it's time to turn our attention to the primary building block of a huge concept in Calculus. This primary building block is called the **difference quotient**.

Definition: Difference Quotient

Given a function f, the **difference quotient** of f is the expression

$$\frac{f(x+h)-f(x)}{h}.$$

You might see this written online and in other texts as

$$\frac{f(a+h)-f(a)}{h},$$

but fear not - this is the same expression (just with a instead of x).

We will revisit this concept in more detail in a future section, but for now we use it as a way to practice function notation and function arithmetic. For reasons that will become clear in Calculus, "simplifying" a difference quotient means rewriting it in a form where the h cancels from the denominator. Once that happens, we consider our work to be done.

Example 2.3.2

Find and simplify the difference quotient for the following function.

$$f(x) = x^2 - x - 2$$

Solution

To find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 && \text{(substituting)} \\ &= x^2 + 2xh + h^2 - x - h - 2 && \text{(distributing)} \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x^2 + 2xh + h^2 - x - h - 2) - (x^2 - x - 2)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} && \text{(distributing)} \\ &= \frac{2xh + h^2 - h}{h} && \text{(combining like terms)} \\ &= \frac{h(2x + h - 1)}{h} && \text{(factoring out the GCF)} \\ &= \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} && \text{(canceling like factors)} \\ &= 2x + h - 1. \end{aligned}$$

We know we are done once the original h in the denominator is canceled out.

A word of caution before we continue: watch your negatives! Notice that at some point in the solution of Example **2.3.2** we had the expression $(x^2 + 2xh + h^2 - x - h - 2) - (x^2 - x - 2)$ in the numerator. Those parentheses on the second group are critically important and necessary. We are subtracting *the entire function* $f(x)$. As such, we needed $f(x+h) - f(x) = f(x+h) - (x^2 - x - 2)$. If we didn't have those parentheses, we would (incorrectly) have written $f(x+h) - f(x) = f(x+h) - x^2 - x - 2$, which I hope you would agree is not the same thing.

Let's state that one more time for clarity.

Caution

The most common mistake made on a difference quotient problem is not using grouping symbols properly!

Checkpoint 2.3.2

Find and simplify the difference quotient for $g(x) = 1 - 3x$.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input

and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**.

Definition: Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)).$$

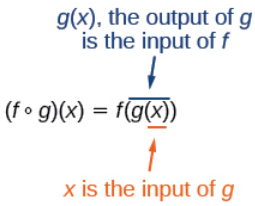
We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the **composition operator**. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value.

Caution

The product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication take two numbers and give a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the Order of Operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.



In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) && \text{(substituting)} \\ &= (x+2)^2 && \text{(substituting)} \\ &= x^2 + 4x + 4 && \text{(distributing)} \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) && \text{(substituting)} \\ &= x^2 + 2 && \text{(substituting)} \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Example 2.3.3: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.

$$f(x) = 2x + 1, \quad g(x) = 3 - x.$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= f(3-x)+1 && \text{(substituting)} \\ &= 2(3-x)+1 && \text{(substituting)} \\ &= 6-2x+1 && \text{(distributing)} \\ &= 7-2x && \text{(combining like terms)} \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= g(2x+1) && \text{(substituting)} \\ &= 3-(2x+1) && \text{(substituting)} \\ &= 3-2x-1 && \text{(distributing)} \\ &= 2-2x-1 && \text{(combining like terms)} \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example 2.3.4: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in T minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 2.3.5: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles} = f(\text{number of hours}).$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons} = g(\text{number of miles}).$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Q&A

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

Checkpoint 2.3.5

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 2.3.6: Using a Table to Evaluate a Composite Function

Using the following table, evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$\begin{aligned} g(3) &= 2 \\ \implies f(g(3)) &= f(2) = 8 \end{aligned}$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2.$$

The following table shows the composite functions $f \circ g$ and $g \circ f$ as tables.

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

? Checkpoint 2.3.6

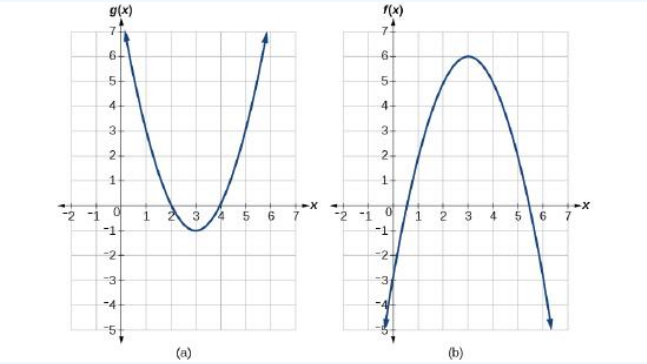
Use the table from Example 2.3.6 to evaluate $f(g(1))$ and $g(f(4))$.

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

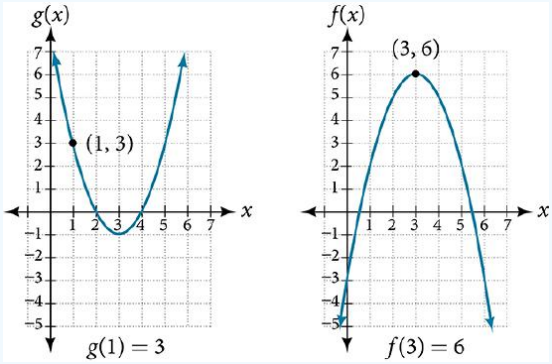
✔ Example 2.3.7: Using a Graph to Evaluate a Composite Function

Use the following figure to evaluate $f(g(1))$.



Solution

To evaluate $f(g(1))$, we start with the inside evaluation.



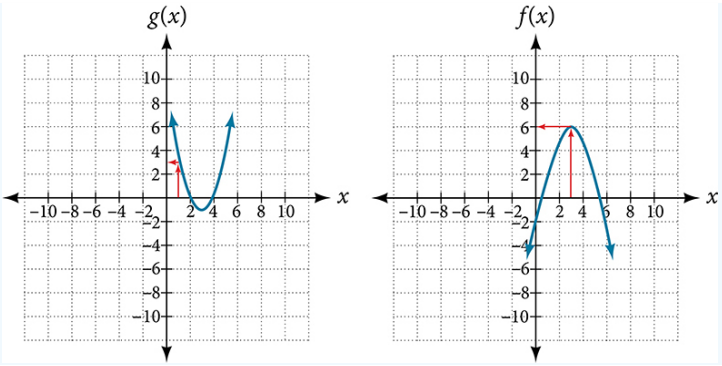
We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

The following figure shows how we can mark the graphs with arrows to trace the path from the input value to the output value.



Checkpoint 2.3.7

Use the graph from Example **2.3.7** to evaluate $g(f(2))$.

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

Example 2.3.8: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$\begin{aligned} h(1) &= 3(1) + 2 && \text{(substituting)} \\ &= 5 && \text{(simplifying)} \end{aligned}$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$\begin{aligned} f(h(1)) &= f(5) && \text{(substituting)} \\ &= 5^2 - 5 && \text{(substituting)} \\ &= 25 - 5 && \text{(squaring)} \\ &= 20 && \text{(simplifying)} \end{aligned}$$

Analysis

It makes no difference what the input variables T and x were called in this problem because we evaluated for specific numerical values.

Checkpoint 2.3.8

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot. That is, it is important to know the domain of a function such as $f \circ g$.

Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful. Otherwise, we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed and the expression is still undefined. Thus, the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Theorem: Domain of a Composite Function

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

To find the domain of the composite function $f(g(x))$, we start by finding the domain of g. Those values of x for which g is defined will be the largest *possible* domain of the composition; however, this set might get restricted once we take the function f into consideration. That said, note the domain of f. Our goal is to find the elements, x, in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 2.3.9: Finding the Domain of a Composite Function

Find the domain of $(f \circ g)(x)$ where $f(x) = \frac{5}{x-1}$ and $g(x) = \frac{4}{3x-2}$.

Solution

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} & g(x) = 1 & & \\ \Rightarrow \quad & \frac{4}{3x-2} = 1 & & \text{(substituting)} \\ \Rightarrow \quad & 4 = 1(3x-2) & & \text{(multiplying both sides by the LCD)} \\ \Rightarrow \quad & 4 = 3x-2 & & \text{(distributing)} \\ \Rightarrow \quad & 6 = 3x & & \text{(adding 2 to both sides)} \\ \Rightarrow \quad & 2 = x & & \text{(dividing both sides by 3)} \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that $x \neq \frac{2}{3}$ o $x \neq 2$.We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty).$$

Example 2.3.10: Finding the Domain of a Composite Function Involving Radicals

Find the domain of $(f \circ g)(x)$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.

Solution

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}.$$

Since the radicand of a square root must be positive, $\sqrt{3-x}+2 \geq 0$. Hence,

$$\begin{aligned} \sqrt{3-x}+2 &\geq 0 \\ \implies \quad \sqrt{3-x} &\geq -2 \qquad \text{(subtracting 2 from both sides)} \end{aligned}$$

Since square roots never return negative numbers, the inequality $\sqrt{3-x} \geq -2$ is always true. Thus, our only restriction on this composition is the original restriction from the domain of g - namely that the values of x must be in the interval $(-\infty, 3]$. This gives a domain of $(-\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

If we change the function f very slightly in Example 2.3.10 to $f(x) = \sqrt{x-2}$, then the composition, $(f \circ g)(x)$ becomes

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}-2}.$$

Again, since the radicand is never negative, we know

$$\begin{aligned} \sqrt{3-x}-2 &\geq 0 \\ \implies \quad \sqrt{3-x} &\geq 2 \qquad \text{(adding 2 to both sides)} \end{aligned}$$

This makes the problem of finding the domain a more complex. If $\sqrt{3-x} \geq 2$, then

$$\left(\sqrt{3-x}\right)^2 \geq 2^2 \implies 3-x \geq 4 \implies 3 \geq 4+x \implies -1 \geq x.$$

Hence, we would require $x \leq 3$ (from the domain of g) and, simultaneously, $-1 \geq x$. The common intersection of these two domain restrictions is $x \leq -1$. Thus, the domain (in this adjusted case) would be $(-\infty, -1]$.

? Checkpoint 2.3.10

Find the domain of $(f \circ g)(x)$ where $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

✔ Example 2.3.11: Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5-x^2 \text{ and } g(x) = \sqrt{x}.$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5-x^2) = \sqrt{5-x^2}.$$

Checkpoint 2.3.11

Write $f(x) = \frac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

https://math.libretexts.org/@go/page/197481

2.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Difference quotient
  - Algebra Mini-Lecture: The Difference Quotient
  - Functions - Overview - A Review for Precalculus: The Difference Quotient
- Composition of functions
  - Combining Functions - Domains of Compositions of Functions
  - Combining Functions - Applications of Compositions

KEY CONCEPTS

DEFINITIONS

- **Function Arithmetic (Sum, Difference, Product, Ratio):**
  - $(f+g)(x) = f(x) + g(x)$ (Sum of functions)
  - $(f-g)(x) = f(x) - g(x)$ (Difference of functions)
  - $(fg)(x) = f(x)g(x)$ (Product of functions)
  - $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$ (Quotient or Ratio of functions
- **Difference Quotient:** For a given function f, the expression $\frac{f(x+h)-f(x)}{h}$ (or $\frac{f(a+h)-f(a)}{h}$).
- **Composition of Functions:** The process of combining functions such that the output of one function becomes the input of another. The resulting function is known as a composite function.
- **Composite Function (Notation):** For functions f and g, the composite function, denoted $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.
- **Composition Operator ($\circ$):** The open circle symbol used to denote function composition, as in $f \circ g$.

THEOREMS

- **Domain of a Composite Function:** The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

COMMON MISTAKES

- **Incorrectly Using Grouping Symbols in Difference Quotients:** When calculating $f(x+h) - f(x)$, failing to use parentheses around $f(x)$ if $f(x)$ consists of multiple terms will lead to errors in subtraction. For example, for $f(x) = x^2 - x - 2$, writing $f(x+h) - x^2 - x - 2$ instead of $f(x+h) - (x^2 - x - 2)$.
- **Confusing Function Composition with Function Multiplication:** The product of functions $(fg)(x) = f(x)g(x)$ is not the same as the function composition $(f \circ g)(x) = f(g(x))$. In general, $f(x)g(x) \neq f(g(x))$.

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2.3.2: HOMEWORK

READING QUESTIONS

1. If $f(x) = 2x + 3$ and $g(x) = x - 5$, what is $(f + g)(x)$?
2. If $f(x) = x^2$ and $g(x) = x + 1$, what is $(fg)(x)$?
3. What is the formula for the difference quotient of a function $f(x)$?
4. When simplifying a difference quotient, what is the main goal regarding the variable h in the denominator?
5. What does the notation $(f \circ g)(x)$ mean, and how is it evaluated?
6. Is function composition generally commutative? Explain with an example if $f(x) = x^2$ and $g(x) = x + 2$.
7. If $C(T)$ is the cost of heating a house at temperature T, and $T(d)$ is the temperature on day d, what does $C(T(d))$ represent?
8. What two conditions must be met for an input x to be in the domain of the composite function $(f \circ g)(x)$?
9. How would you decompose the function $H(x) = \sqrt{x^2 + 5}$ into two simpler functions $f(x)$ and $g(x)$ such that $H(x) = f(g(x))$?
10. If you are given functions as tables, how do you evaluate $(f \circ g)(3)$?
11. How does one find the domain of the quotient of two functions, $\frac{f}{g}$?
12. What is the composition of two functions, $f \circ g$?
13. If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.
14. How do you find the domain for the composition of two functions, $f \circ g$?

HOMEWORK

ALGEBRAIC

For the following exercises, use the pair of functions f and g to find the following values if they exist. Simplify your answers.

- $(f + g)(2)$
- $(f - g)(-1)$
- $(g - f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

15. $f(x) = 3x + 1$ and $g(x) = 4 - x$
16. $f(x) = x^2$ and $g(x) = -2x + 1$
17. $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
18. $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
19. $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$
20. $f(x) = \sqrt{4 - x}$ and $g(x) = \sqrt{x + 2}$
21. $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$
22. $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$
23. $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$
24. $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$

For the following exercises, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f + g)(x)$
- $(f - g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

25. $f(x) = 2x + 1$ and $g(x) = x - 2$
26. $f(x) = 1 - 4x$ and $g(x) = 2x - 1$
27. $f(x) = x^2$ and $g(x) = 3x - 1$
28. $f(x) = x^2 - x$ and $g(x) = 7x$
29. $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
30. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
31. $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$
32. $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$
33. $f(x) = x$ and $g(x) = \sqrt{x + 1}$
34. $f(x) = \sqrt{x - 5}$ and $g(x) = f(x) = \sqrt{x - 5}$
35. Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
36. Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
37. Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
38. Given $f(x) = \frac{1}{x-4}$ and $g(x) = \frac{1}{6-x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
39. Given $f(x) = 3x^2$ and $g(x) = \sqrt{x - 5}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
40. Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$. Determine the domain of the function in interval notation.
41. Given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$, find the following:
    a. $f(g(2))$
    b. $f(g(x))$
    c. $g(f(x))$
    d. $(g \circ g)(x)$
    e. $(f \circ f)(-2)$

For the following exercises, find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the given function.

42. $f(x) = 2x - 5$
43. $f(x) = -3x + 5$
44. $f(x) = 6$
45. $f(x) = 3x^2 - x$
46. $f(x) = -x^2 + 2x - 1$
47. $f(x) = 4x^2$
48. $f(x) = x - x^2$
49. $f(x) = x^3 + 1$
50. $f(x) = mx + b$ where $m \neq 0$
51. $f(x) = ax^2 + bx + c$ where $a \neq 0$
52. $f(x) = \frac{2}{x}$
53. $f(x) = \frac{3}{1-x}$
54. $f(x) = \frac{1}{x^2}$
55. $f(x) = \frac{2}{x+5}$

56. $f(x)=\frac{1}{4x-3}$
57. $f(x)=\frac{3x}{x+1}$
58. $f(x)=\frac{x}{x-9}$
59. $f(x)=\frac{x^2}{2x+1}$
60. $f(x)=\sqrt{x-9}$
61. $f(x)=\sqrt{2x+1}$
62. $f(x)=\sqrt{-4x+5}$
63. $f(x)=\sqrt{4-x}$
64. $f(x)=\sqrt{ax+b}$, where $a\neq 0$.
65. $f(x)=x\sqrt{x}$
66. $f(x)=\sqrt[3]{x}$.
    HINT: $(a-b)\left(a^2+ab+b^2\right)=a^3-b^3$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

67. $f(x)=x^2+1$, $g(x)=\sqrt{x+2}$
68. $f(x)=\sqrt{x}+2$, $g(x)=x^2+3$
69. $f(x)=|x|$, $g(x)=5x+1$
70. $f(x)=\sqrt[3]{x}$, $g(x)=\frac{x+1}{x^3}$
71. $f(x)=\frac{1}{x-6}$, $g(x)=\frac{7}{x}+6$
72. $f(x)=\frac{1}{x-4}$, $g(x)=\frac{2}{x}+4$

For the following exercises, use each set of functions to find $f(g(h(x)))$. Simplify your answers.

73. $f(x)=x^4+6$, $g(x)=x-6$, and $h(x)=\sqrt{x}$
74. $f(x)=x^2+1$, $g(x)=\frac{1}{x}$, and $h(x)=x+3$
75. Given $f(x)=\frac{1}{x}$ and $g(x)=x-3$, find the following:
    a. $(f\circ g)(x)$
    b. the domain of $(f\circ g)(x)$ in interval notation
    c. $(g\circ f)(x)$
    d. the domain of $(g\circ f)(x)$
    e. $\left(\frac{f}{g}\right)x$
76. Given $f(x)=\sqrt{2-4x}$ and $g(x)=-\frac{3}{x}$, find the following:
    a. $(g\circ f)(x)$
    b. the domain of $(g\circ f)(x)$ in interval notation
77. Given the functions $f(x)=\frac{1-x}{x}$ and $g(x)=\frac{1}{1+x^2}$, find the following:
    a. $(g\circ f)(x)$
    b. $(g\circ f)(2)$
78. Given functions $p(x)=\frac{1}{\sqrt{x}}$ and $m(x)=x^2-4$, state the domain of each of the following functions using interval notation:
    a. $p(x)m(x)$
    b. $p(m(x))$
    c. $m(p(x))$
79. Given functions $q(x)=\frac{1}{\sqrt{x}}$ and $h(x)=x^2-9$, state the domain of each of the following functions using interval notation.
    a. $\frac{q(x)}{h(x)}$
    b. $q(h(x))$
    c. $h(q(x))$
80. For $f(x)=\frac{1}{x}$ and $g(x)=\sqrt{x-1}$, write the domain of $(f\circ g)(x)$ in interval notation.

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f(g(x))$.

81. $h(x)=(x+2)^2$
82. $h(x)=(x-5)^3$
83. $h(x)=\frac{3}{x-5}$
84. $h(x)=\frac{4}{(x+2)^2}$
85. $h(x)=4+\sqrt[3]{x}$
86. $h(x)=\sqrt[3]{\frac{1}{2x-3}}$
87. $h(x)=\frac{1}{(3x^2-4)^{-3}}$
88. $h(x)=\sqrt[4]{\frac{3x-2}{x+5}}$
89. $h(x)=\left(\frac{8+x^3}{8-x^3}\right)^4$
90. $h(x)=\sqrt{2x+6}$
91. $h(x)=(5x-1)^3$
92. $h(x)=\sqrt[3]{x-1}$
93. $h(x)=|x^2+7|$
94. $h(x)=\frac{1}{(x-2)^3}$
95. $h(x)=\left(\frac{1}{2x-3}\right)^2$
96. $h(x)=\sqrt{\frac{2x-1}{3x+4}}$

GRAPHICAL

For the following exercises, use the graphs of f, shown in Figure 2.3.2.1, and g, shown in Figure 2.3.2.2, to evaluate the expressions.

Figure 2.3.2.1

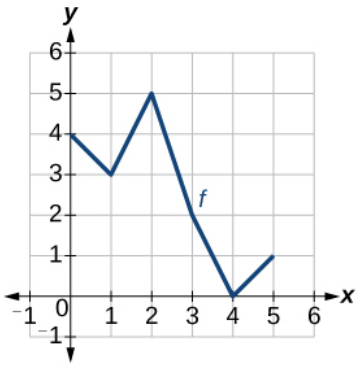
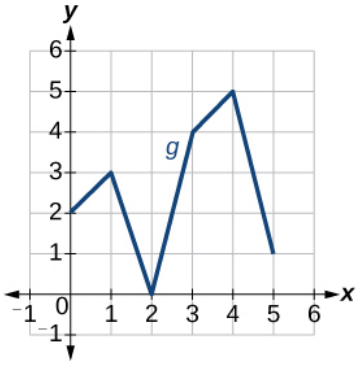


Figure 2.3.2.2



97. $f(g(3))$
98. $f(g(1))$
99. $g(f(1))$

100. $g(f(0))$
101. $f(f(5))$
102. $f(f(4))$
103. $g(g(2))$
104. $g(g(0))$

For the following exercises, use graphs of $f(x)$, shown in Figure 2.3.2.3, $g(x)$, shown in Figure 2.3.2.4, and $h(x)$, shown in Figure 2.3.2.5, to evaluate the expressions.

Figure 2.3.2.3

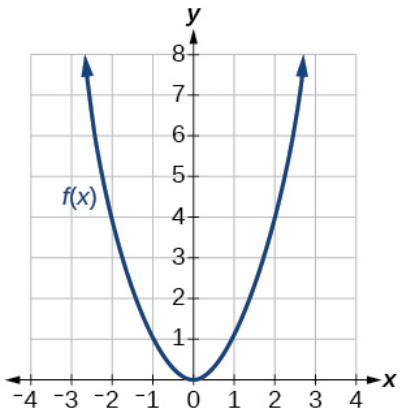


Figure 2.3.2.4

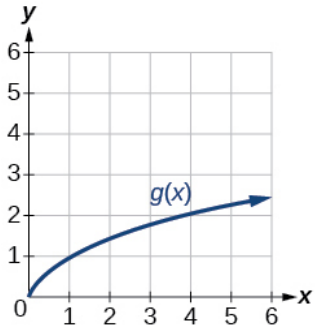
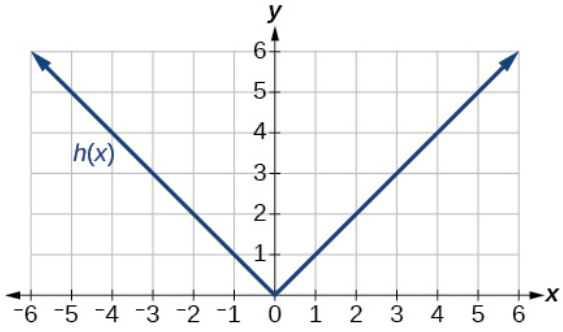


Figure 2.3.2.5



105. $g(f(1))$
106. $g(f(2))$
107. $f(g(4))$
108. $f(g(1))$
109. $f(h(2))$
110. $h(f(2))$
111. $f(g(h(4)))$
112. $f(g(f(-2)))$

NUMERIC

For the following exercises, use the function values for f and g shown in Table 2.3.2.1 to evaluate each expression.

Table 2.3.2.1

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

113. $f(g(8))$
114. $f(g(5))$
115. $g(f(5))$
116. $g(f(3))$
117. $f(f(4))$
118. $f(f(1))$
119. $g(g(2))$
120. $g(g(6))$

For the following exercises, use the function values for f and g shown in Table 2.3.2.2 to evaluate the expressions.

Table 2.3.2.2

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3
3	-1	-8

121. $(f \circ g)(1)$
122. $(f \circ g)(2)$
123. $(g \circ f)(2)$
124. $(g \circ f)(3)$
125. $(g \circ g)(1)$
126. $(f \circ f)(3)$

For the following exercises, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

127. $f(x) = 4x + 8$, $g(x) = 7 - x^2$

128. $f(x) = 5x + 7$, $g(x) = 4 - 2x^2$
129. $f(x) = \sqrt{x+4}$, $g(x) = 12 - x^3$
130. $f(x) = \frac{1}{x+2}$, $g(x) = 4x + 3$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

131. $f(g(2))$
132. $f(g(x))$
133. $g(f(-3))$
134. $(g \circ g)(x)$

EXTENSIONS

For the following exercises, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

Compute the indicated value if it exists.

135. $(f+g)(-3)$
136. $(f-g)(2)$
137. $(fg)(-1)$
138. $(g+f)(1)$
139. $(g-f)(3)$
140. $(gf)(-3)$
141. $\left(\frac{f}{g}\right)(-2)$
142. $\left(\frac{f}{g}\right)(-1)$
143. $\left(\frac{f}{g}\right)(2)$
144. $\left(\frac{g}{f}\right)(-1)$
145. $\left(\frac{g}{f}\right)(3)$
146. $\left(\frac{g}{f}\right)(-3)$

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$.

147. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.
148. Find $(f \circ g)(2)$ and $(g \circ f)(2)$.
149. What is the domain of $(g \circ f)(x)$?
150. What is the domain of $(f \circ g)(x)$?
151. Let $f(x) = \frac{1}{x}$.
    a. Find $(f \circ f)(x)$.
    b. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x + 1$.

152. True or False: $(g \circ f)(x) = F(x)$.
153. True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

154. $(f \circ g)(6)$; $(g \circ f)(6)$
155. $(g \circ f)(a)$; $(f \circ g)(a)$
156. $(f \circ g)(11)$; $(g \circ f)(11)$

REAL-WORLD APPLICATIONS

157. The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is \$6, you would do which of the following?
    a. Evaluate $D(C(6))$.
    b. Evaluate $C(D(6))$.
    c. Solve $D(C(x)) = 6$.
    d. Solve $C(D(p)) = 6$.
158. The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?
    a. Evaluate $A(m(4))$.
    b. Evaluate $m(A(4))$.
    c. Solve $A(m(t)) = 4$.
    d. Solve $m(A(d)) = 4$.
159. A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)
160. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.
161. A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).
162. Use the function you found in the previous exercise to find the total area burned after 5 minutes.
163. The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.
    a. Find the composite function $r(V(t))$.
    b. Find the *exact* time when the radius reaches 10 inches.
164. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.
    a. Find the composite function $N(T(t))$.
    b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

For the following exercises, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- Find and interpret $C(0)$.

- Find and interpret $\overline{C}(10)$. ($\overline{C}$ is called the **average cost**)
- Find and interpret $p(5)$.
- Find and simplify $R(x)$. ($R(x)$ is the **revenue function**)
- Find and simplify $P(x)$. ($P(x)$ is the **profit function**)
- Solve $P(x) = 0$ and interpret.

165. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26,\ x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x,\ 0 \leq x \leq 15$.
166. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100,\ x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x, 0 \leq x \leq 35$.
167. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240,\ x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x, 0 \leq x \leq 30$.
168. The daily cost, in dollars, to produce x Sasquatch Berry Pies $C(x) = 3x + 36,\ x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x,\ 0 \leq x \leq 24$.
169. The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000,\ x \geq 0$ and the price-demand function, in hundreds of dollars per scooter, is $p(x) = 140 - 2x,\ 0 \leq x \leq 70$.

2.4: Behavior of Graphs of Functions

Note to the Instructor (click to expand)

The first half of this section is meant to get the student used to (and to compute) the average rate of change of a function. Use this as an opportunity to impress upon the student the fact that this concept is a fundamental building block for Calculus. The remainder of the section is dedicated to interpreting the behavior of the graph of a function - specifically, increasing, decreasing, and local/global extrema. It is important to note that we adopt the convention that intervals of increase and decrease are always open. Finally, we finish the section with the important conversation about symmetry.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Graphing Functions**
  - Base Graphs: Familiarity with the shapes of toolkit functions is helpful for understanding their increasing/decreasing behavior and symmetry.
- **Linear Functions**
  - Slope and Average Rate of Change: The concept of average rate of change introduced in this section is a generalization of the slope of a linear function. Prior understanding of slope as a rate of change is beneficial.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.
- Determine whether a function is even, odd, or neither from its graph.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table **2.4.1** lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

Table 2.4.1

y	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

Average Rate of Change

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table **2.4.1** did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\text{Average rate of change} = \frac{\text{Change in output}}{\text{Change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Definition: Average Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The Greek letter Δ (Delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.}$$

On average, the price of gas increased by about 19.6 cents each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Example 2.4.1: Computing an Average Rate of Change

Using the data in Table **2.4.1**, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution

In 2007, the price of gasoline was $2.84. In 2009, the cost was $2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} && \text{(substituting)} \\ &= \frac{-\$0.43}{2\,\text{years}} && \text{(simplifying)} \\ &\approx -\$0.22 \text{ per year}\end{aligned}$$

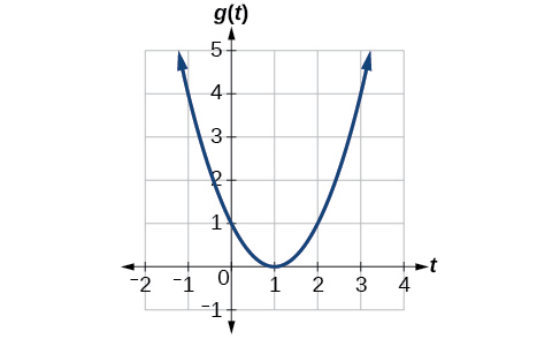
Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Checkpoint 2.4.1

Using the data in Table **2.4.1**, find the average rate of change between 2005 and 2010.

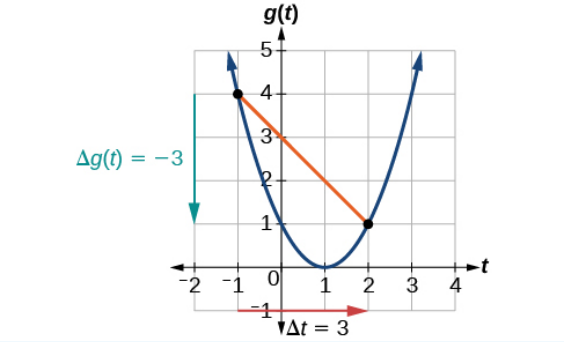
Example 2.4.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in the following figure, find the average rate of change on the interval $[-1, 2]$.



Solution

The following figure shows $g(-1) = 4$. At $t = 2$, the graph shows $g(2) = 1$.



The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1.$$

Example **2.4.2** illustrates the fact that the order we choose is very important. If, for example, we used $\frac{y_2 - y_1}{x_1 - x_2}$, we would not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Example 2.4.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in the following table. Find her average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0}=\frac{282}{6}=47.$$

The average speed is 47 miles per hour.

In Example 2.4.3, because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2,3]$, the average speed is 63 miles per hour.

Example 2.4.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x)=x^2-\frac{1}{x}$ on the interval $[2,4]$.

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2-\frac{1}{2} & f(4) &= 4^2-\frac{1}{4} \\ &= 4-\frac{1}{2} & &= 16-\frac{1}{4} \\ &= \frac{7}{2} & &= \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4)-f(2)}{4-2} \\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2} && \text{(substituting)} \\ &= \frac{4}{4}\cdot\left(\frac{\frac{63}{4}-\frac{7}{2}}{2}\right) && \text{(simplifying the compound fraction)} \\ &= \frac{63-14}{8} && \text{(distributing)} \\ &= \frac{49}{8} && \text{(simplifying)} \end{aligned}$$

Checkpoint 2.4.4

Find the average rate of change of $f(x)=x-2\sqrt{x}$ on the interval $[1,9]$.

Example 2.4.5: Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $\frac{2}{d^2}$ on the interval $[2, 6]$.

$$\begin{aligned}
\text{Average rate of change} &= \frac{F(6) - F(2)}{6 - 2} \\
&= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{4} && \text{(substituting and simplifying)} \\
&= \frac{\frac{2}{36} - \frac{2}{4}}{4} && \text{(simplifying)} \\
&= \frac{\frac{1}{18} - \frac{1}{2}}{4} && \text{(simplifying)} \\
&= \frac{18}{18} \cdot \left(\frac{\frac{1}{18} - \frac{1}{2}}{4} \right) && \text{(simplifying the compound fraction)} \\
&= \frac{1 - 9}{72} && \text{(distributing)} \\
&= -\frac{8}{72} && \text{(simplifying)} \\
&= -\frac{1}{9} && \text{(simplifying)}
\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 2.4.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Solution

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} \\
&= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a} && \text{(substituting)} \\
&= \frac{a^2 + 3a + 1 - 1}{a} && \text{(distributing)} \\
&= \frac{a^2 + 3a}{a} && \text{(combining like terms)} \\
&= \frac{a(a + 3)}{a} && \text{(factoring } a \text{ out of the numerator)} \\
&= \frac{\cancel{a}^{1}(a + 3)}{\cancel{a}^{1}} && \text{(canceling like factors)} \\
&= a + 3 && \text{(simplifying)}
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $5 + 3 = 8$.

The answer in Example 6 is a repeated concept from the previous section - it is an alternative form of the difference quotient!

Checkpoint 2.4.6

Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

Focus on Calculus - Local Extrema and Intervals of Increase and Decrease

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is **increasing** on an interval if the function values increase as the input values increase within that interval. Similarly, a function is **decreasing** on an interval if the function values decrease as the input values increase over that interval. The following definitions codifies these statements.

Definition: Increasing

A function f is called an **increasing function** on an open interval I if $f(b) > f(a)$ for every a and b in the interval where $b > a$.

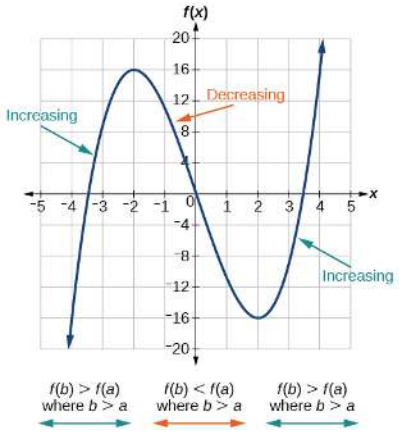
Definition: Decreasing

A function f is called a **decreasing function** on an open interval I if $f(b) < f(a)$ for every a and b in the interval where $b > a$.

The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 2.4.1 shows examples of increasing and decreasing intervals on a function.

Figure 2.4.1

The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.



While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is the location of a **local maximum**. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is the location of a **local minimum**.

Definition: Local Extrema

A function f has a **local maximum** at $x = b$ in an open interval (a, c) if $f(b) \geq f(x)$ for all $x \neq b$ in the interval. Likewise, f has a **local minimum** at $x = b$ in (a, c) if $f(b) \leq f(x)$ for all $x \neq b$ in the interval.

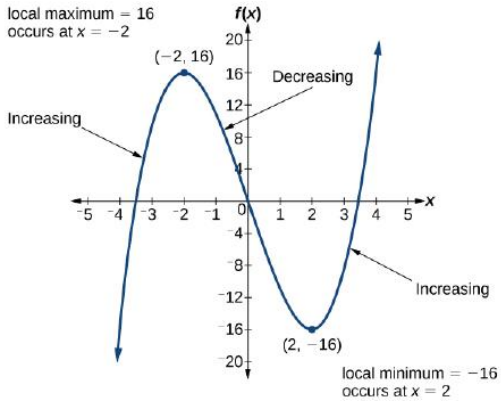
If f has more than one local maximum on a given interval, we call the collection of these maximums the **local maxima**. Likewise, if f has more than one local minimum on a given interval, we call the collection of these minimums the **local minima**.

Together, local maxima and minima are called **local extrema** (or **local extreme values**).

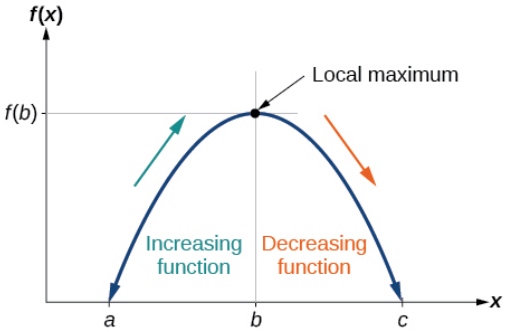
Some texts use the word "relative" instead of "local." We will use the latter throughout this text.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown below, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

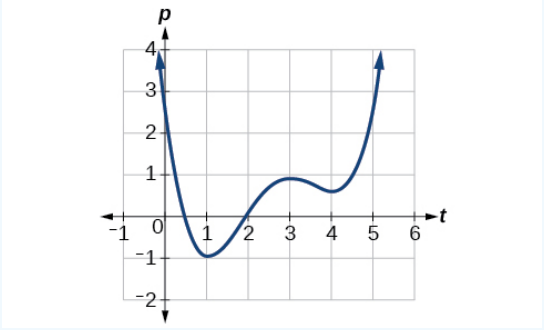


To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an *open interval*. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. The following figure illustrates these ideas for a local maximum.



Example 2.4.7: Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in the following figure, identify the intervals on which the function appears to be increasing.



Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to

$t = 3$ and from $t = 4$ on. In interval notation, we would say the function appears to be increasing on the interval $(1, 3) \cup (4, \infty)$.

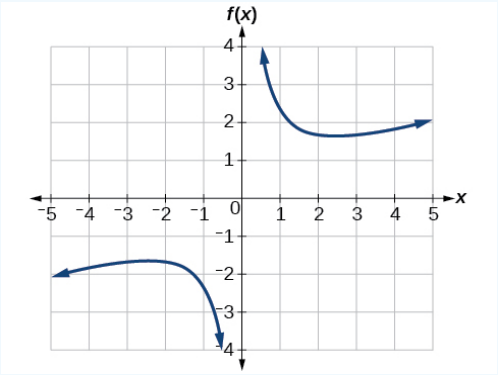
Notice in Example 2.4.7 that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example 2.4.8: Finding Local Extrema from a Graph

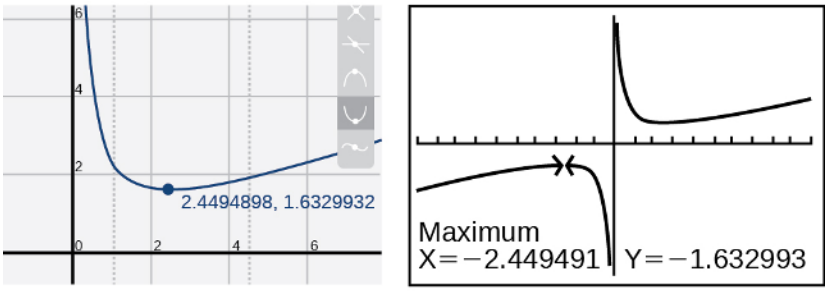
Use graphing technology to graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in the figure below. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$.



Most graphing calculators and graphing utilities can *estimate* the location of maxima and minima. The following figure provides screen images from two different technologies, showing the estimate for the local maximum and minimum.



(a) (b)

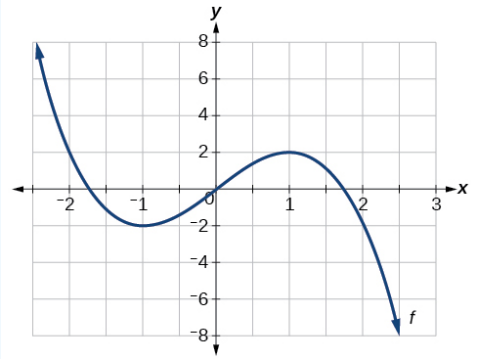
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires Calculus.)

Checkpoint 2.4.8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Example 2.4.9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown below, find all local maxima and minima.



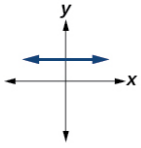
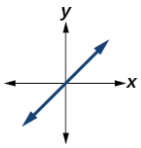
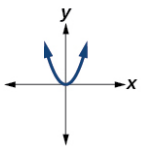
Solution

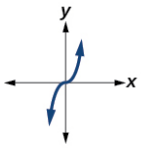
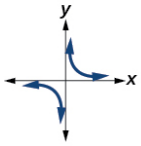
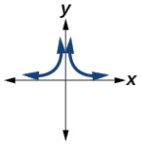
The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$. The local maximum is the y-coordinate at $x = 1$, which is 2.

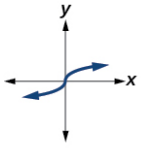
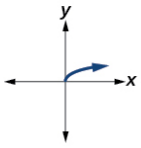
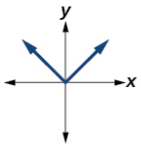
The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is -2.

Revisiting the Toolkit Functions

We will now return to our toolkit functions and discuss their graphical behavior.

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

Function	Increasing/Decreasing	Example
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

Focus on Calculus - Absolute Extrema

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively.

Definition: Absolute Extrema

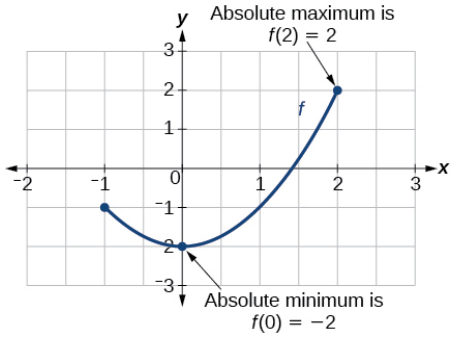
A function f has an **absolute maximum** at $x = b$ if $f(b) \geq f(x)$ for all x in the domain of f. Likewise, a function f has an **absolute minimum** at $x = b$ if $f(b) \leq f(x)$ for all x in the domain of f.

If f has more than one local absolute maximum, we call the collection of these maximums the **absolute maxima**. Likewise, if f has more than one absolute minimum, we call the collection of these minimums the **absolute minima**.

Together, absolute maxima and minima are called **absolute extrema**.

Some texts use the word "global" instead of "absolute." We will use the latter throughout this text.

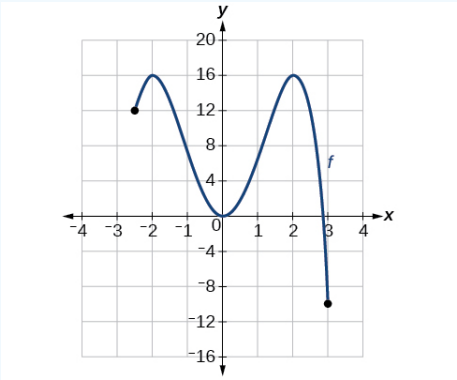
To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function.



Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Example 2.4.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown below, find all absolute maxima and minima.



Solution

The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at $x = 3$, which is -10.

Symmetry - Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are **symmetric about the y-axis**. Functions whose graphs are symmetric about the y-axis are called **even functions**.

Definition: Even and Odd Functions

The function f is called an **even function** if for every input x in the domain of f

$$f(-x) = f(x).$$

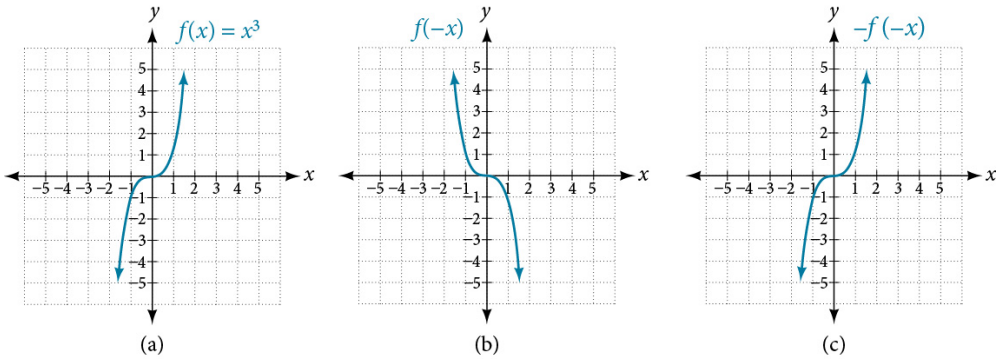
The function f is called an **odd function** if for every input x in the domain of f

$$f(-x) = -f(x).$$

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in Figre 2.4.2.

Figure 2.4.2

(a) The cubic toolkit function. (b): Horizontal reflection of the cubic toolkit function. (c): Horizontal and vertical reflections reproduce the original cubic function.



We say that these graphs are **symmetric about the origin**. A function with a graph that is symmetric about the origin is called an **odd function**.

Theorem: Symmetry of Even and Odd Functions

The graph of an even function is symmetric about the y- axis.

The graph of an odd function is symmetric about the origin.

A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Example 2.4.11: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

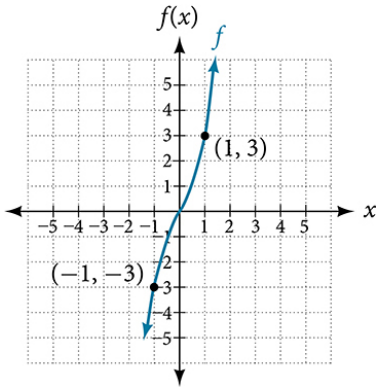
Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$\begin{aligned} f(-x) &= (-x)^3 + 2(-x) && \text{(substituting)} \\ &= -x^3 - 2x && \text{(simplifying)} \\ &= -(x^3 + 2x) && \text{(factoring out } -1) \\ &= -f(x) \end{aligned}$$

Because $f(-x) = -f(x)$, this is an odd function.

The graph of the function from Example 2.4.11 is shown below. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.



Checkpoint 2.4.11

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

https://math.libretexts.org/@go/page/197484

2.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Average rate of change
  - CRC Mathematics in Review: Linears - Slope and Rate of Change
- Intervals of Increase and Decrease
  - Rates of Change and Behavior of Graphs - Intervals of Increase/Decrease and Local Extrema
  - Intermediate Algebra - Functions: Intro to Graphs
- Even and Odd functions
  - Transformations of Functions - Determining Even and Odd Functions
- Using Technology to Graph
  - Functions - Overview - A Review for Precalculus: Graphs - Using Desmos to Graph

KEY CONCEPTS

DEFINITIONS

- **Rate of Change:** A description of how an output quantity changes relative to a change in the input quantity. The units are "output units per input units."
- **Average Rate of Change:** The total change in function values (output values) divided by the change in the input values over a specified interval. Formula: $\frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$.
- **Increasing Function (on an interval):** A function f is increasing on an open interval I if $f(b) > f(a)$ for every a and b in I where $b > a$.
- **Decreasing Function (on an interval):** A function f is decreasing on an open interval I if $f(b) < f(a)$ for every a and b in I where $b > a$.
- **Local Maximum (Relative Maximum):** A function f has a local maximum at $x = b$ in an open interval (a, c) if $f(b) \geq f(x)$ for all $x \neq b$ in the interval. It is a point where the function changes from increasing to decreasing.
- **Local Minimum (Relative Minimum):** A function f has a local minimum at $x = b$ in an open interval (a, c) if $f(b) \leq f(x)$ for all $x \neq b$ in the interval. It is a point where the function changes from decreasing to increasing.
- **Local Maxima:** The collection of all local maximum values on a given interval.
- **Local Minima:** The collection of all local minimum values on a given interval.
- **Local Extrema (Local Extreme Values):** The collective term for all local maxima and local minima.
- **Absolute Maximum (Global Maximum):** A function f has an absolute maximum at $x = b$ if $f(b) \geq f(x)$ for all x in the entire domain of f.
- **Absolute Minimum (Global Minimum):** A function f has an absolute minimum at $x = b$ if $f(b) \leq f(x)$ for all x in the entire domain of f.
- **Absolute Maxima:** The collection of all absolute maximum values of a function.
- **Absolute Minima:** The collection of all absolute minimum values of a function.
- **Absolute Extrema (Global Extrema):** The collective term for all absolute maxima and absolute minima.
- **Even Function:** A function f for which $f(-x) = f(x)$ for every input x in its domain. Its graph is symmetric about the y-axis.
- **Odd Function:** A function f for which $f(-x) = -f(x)$ for every input x in its domain. Its graph is symmetric about the origin.
- **Symmetric about the y-axis:** A characteristic of the graph of an even function, where reflecting the graph horizontally across the y-axis results in the original graph.
- **Symmetric about the origin:** A characteristic of the graph of an odd function, where reflecting the graph over both the x-axis and y-axis (or rotating 180 degrees about the origin) results in the original graph.

THEOREMS

- **Symmetry of Even and Odd Functions:**
  - The graph of an even function is symmetric about the y-axis.
  - The graph of an odd function is symmetric about the origin.

COMMON MISTAKES

- **Incorrect Order of Subtraction in Average Rate of Change:** Using an inconsistent order for coordinates when calculating average rate of change, for example, $\frac{y_2-y_1}{x_1-x_2}$ instead of $\frac{y_2-y_1}{x_2-x_1}$.
- **Confusing Local Extrema with Absolute Extrema:** A local maximum/minimum is the highest/lowest point in an open interval around it, but not necessarily the highest/lowest point over the entire domain of the function.
- **Assuming Endpoints of Intervals are Extrema for Increasing/Decreasing Intervals:** When defining intervals of increase or decrease, open intervals are used because the function is neither increasing nor decreasing *at* the local extrema points themselves.

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2.4.2: HOMEWORK

READING QUESTIONS

1. What does the average rate of change of a function represent, and how is it calculated?
2. If a function is increasing on an interval, what can you say about its average rate of change on that interval?
3. Describe the visual characteristic of a local maximum on a graph.
4. What is the difference between a local minimum and an absolute minimum?
5. How can you determine algebraically if a function $f(x)$ is an even function?
6. What type of symmetry does the graph of an odd function exhibit?
7. Can a function be both even and odd? If so, give an example.
8. Give an example of a toolkit function that is decreasing on the interval $(-\infty, 0)$ and increasing on $(0, \infty)$.
9. What does it mean if the average rate of change of a function over an interval is zero?
10. If a graph has a local maximum at $x = a$, what is the value of the local maximum?
11. Can the average rate of change of a function be constant?
12. If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?
13. How are the absolute maximum and minimum similar to and different from the local extrema?
14. How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?
15. How can you determine whether a function is odd or even from the formula of the function?

HOMEWORK

ALGEBRAIC

For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h.

16. $f(x) = 4x^2 - 7$ on $[1, b]$
17. $g(x) = 2x^2 - 9$ on $[4, b]$
18. $p(x) = 3x + 4$ on $[2, 2 + h]$
19. $k(x) = 4x - 2$ on $[3, 3 + h]$
20. $f(x) = 2x^2 + 1$ on $[x, x + h]$
21. $g(x) = 3x^2 - 2$ on $[x, x + h]$
22. $a(t) = \frac{1}{t+4}$ on $[9, 9 + h]$
23. $b(x) = \frac{1}{x+3}$ on $[1, 1 + h]$
24. $j(x) = 3x^3$ on $[1, 1 + h]$
25. $r(t) = 4t^3$ on $[2, 2 + h]$
26. $f(x) = 2x^2 - 3x$ on $[x, x + h]$

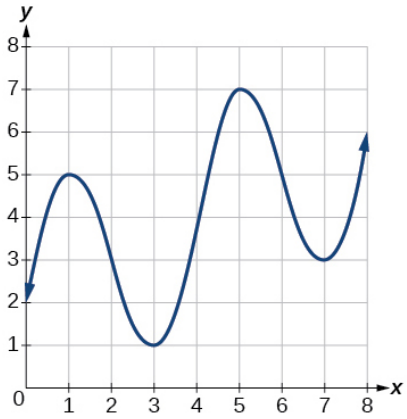
For the following exercises, determine whether the function is odd, even, or neither.

27. $f(x) = 3x^4$
28. $g(x) = \sqrt{x}$
29. $h(x) = \frac{1}{x} + 3x$
30. $f(x) = (x - 2)^2$
31. $g(x) = 2x^4$
32. $h(x) = 2x - x^3$

GRAPHICAL

For the following exercises, consider the graph of f shown in Figure 2.4.2.1.

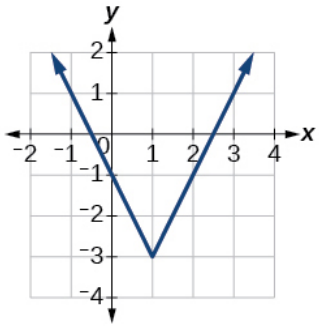
Figure 2.4.2.1



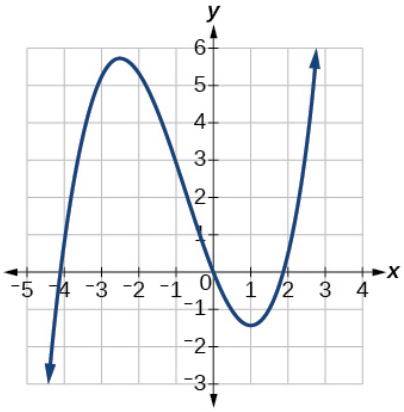
33. Estimate the average rate of change from $x = 1$ to $x = 4$.
34. Estimate the average rate of change from $x = 2$ to $x = 5$.

For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.

35.



36.



37.

38.

For the following exercises, consider the graph shown in Figure 2.4.2.2.

Figure 2.4.2.2

40. Estimate the intervals where the function is increasing or decreasing.
41. Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

For the following exercises, consider the graph in Figure 2.4.2.3.

Figure 2.4.2.3

42. If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.
43. If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

NUMERIC

44. Table **2.4.2.1** gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Table 2.4.2.1

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

45. Table **2.4.2.2** gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Table 2.4.2.2

Year	Population (thousands)
2000	87
2001	84
2002	83
2003	80
2004	77
2005	76
2006	78
2007	81
2008	85

For the following exercises, find the average rate of change of each function on the interval specified.

46. $f(x) = x^2$ on $[1, 5]$
47. $h(x) = 5 - 2x^2$ on $[-2, 4]$
48. $q(x) = x^3$ on $[-4, 2]$
49. $g(x) = 3x^3 - 1$ on $[-3, 3]$
50. $y = \frac{1}{x}$ on $[1, 3]$
51. $p(t) = \frac{(t^2-4)(t+1)}{t^2+3}$ on $[-3, 1]$
52. $k(t) = 6t^2 + \frac{4}{t^3}$ on $[-1, 3]$

TECHNOLOGY

For the following exercises, use graphing technology to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

53. $f(x) = x^4 - 4x^3 + 5$
54. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$
55. $g(t) = t\sqrt{t+3}$
56. $k(t) = 3t^{2/3} - t$
57. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$
58. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

EXTENSION

59. The graph of the function f is shown in Figure 2.4.2.4. Based on the calculator screen shot, the point $(1.333, 5.185)$ is which of the following?

    a. a relative (local) maximum of the function
    b. the vertex of the function
    c. the absolute maximum of the function
    d. a zero of the function

Figure 2.4.2.4

60. Let $f(x) = \frac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.
61. Let $f(x) = \frac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

REAL-WORLD APPLICATIONS

62. At the start of a trip, the odometer on a car read 21,395. At the end of the trip, 13.5 hours later, the odometer read 22,125. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?
63. A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?
64. Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.
65. The graph in Figure 2.4.2.5 illustrates the decay of a radioactive substance over t days.

Figure 2.4.2.5

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

2.5: Transformations of Functions

Note to the Instructor (click to expand)

This is the typical section on transformations of functions; however, it is important that students get a strong exposure to understanding horizontal stretches and compressions that are combined with horizontal shifts (everything is assumed to have been seen before).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Arithmetic**
  - The Order of Operations: Essential for correctly applying sequences of transformations, especially when combining shifts, stretches/compressions, and reflections. For example, understanding whether to apply a vertical stretch before or after a vertical shift.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Graph functions using compressions and stretches.
- Combine transformations.

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Figure 2.5.1 (credit: "Misko"/Flickr)

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Graphing Functions Using Vertical and Horizontal Shifts

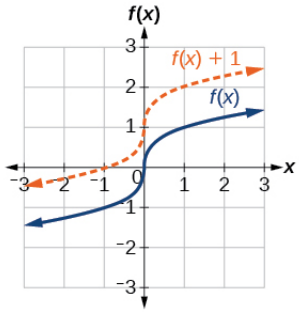
Often when given a problem, we try to model the scenario using Mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest of these is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure 2.5.2 for an example.

Figure 2.5.2

Vertical shift by $k = 1$ of the cube root function $f(x) = x^3$



To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

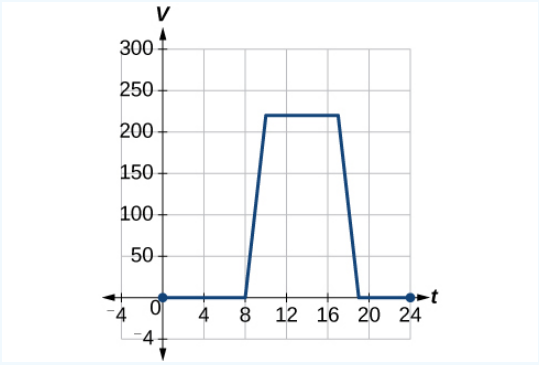
Theorem: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of $f(x)$. All the output values change by k units. If k is positive, the graph will shift up by k units. If k is negative, the graph will shift down by $|k|$ units.

Example 2.5.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 2.5.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

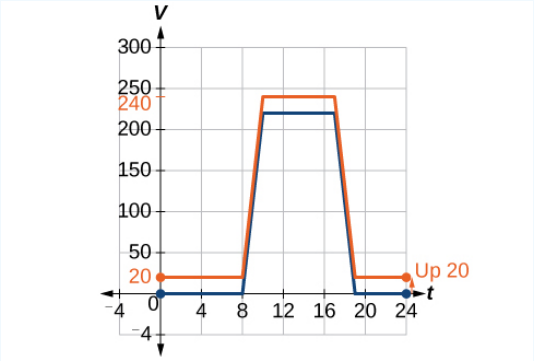
Figure 2.5.3



Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 2.5.4.

Figure 2.5.4



Notice that in Figure 2.5.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20.$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 2.5.1.

Table 2.5.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Example 2.5.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 2.5.2. Create a table for the function $g(x) = f(x) - 3$.

Table 2.5.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example, we know (from Table 2.5.2) that $f(2) = 1$, and we were given $g(x) = f(x) - 3$. Therefore,

$$\begin{aligned} g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 2.5.3.

Table 2.5.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	−2	0	4	8

As with Example **2.5.1**, the input values in Example **2.5.2** stay the same and only the output values change.

? Checkpoint 2.5.2

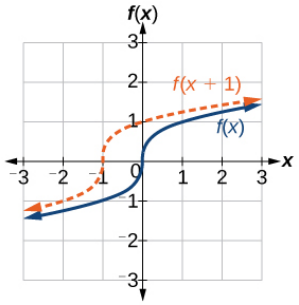
The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure **2.5.5**.

Figure 2.5.5

Horizontal shift of the function $f(x) = x^3$. Note that $(x + 1)$ means $h = -1$ which shifts the graph to the left, that is, towards negative values of x.



For example, if $f(x) = x^2$, then $g(x) = (x - 2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Theorem: Horizontal Shift

Given a function $f(x)$, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of $f(x)$. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

✓ Example 2.5.3: Adding a Constant to an Input

Returning to our building airflow example from Figure **2.5.3**, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

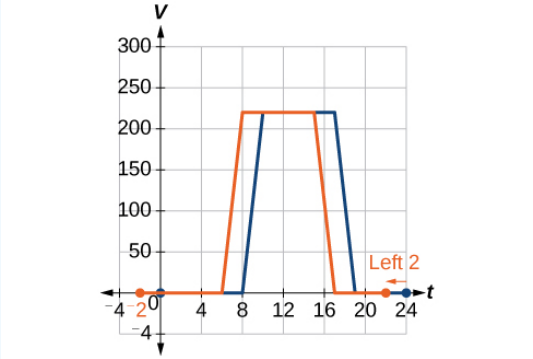
We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{the original venting plan}$$
$$F(t) = \text{starting 2 hours sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$ (see Figure 2.5.6). Notice also that the vents first opened to $220\ \text{ft}^2$ at 10 a.m. under the original plan, while under the new plan the vents reach $220\ \text{ft}^2$ at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

Figure 2.5.6



It is important to note that, in Example 2.5.3, $V(t+2)$ has the effect of shifting the graph to the *left*.

Horizontal changes, or "inside changes," affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F(t) = V(t+2)$.

Example 2.5.4: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 2.5.4. Create a table for the function $g(x) = f(x-3)$.

Table 2.5.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$g(5) = f(5-3)$$
$$= f(2)$$

We continue with the other values to create Table 2.5.5.

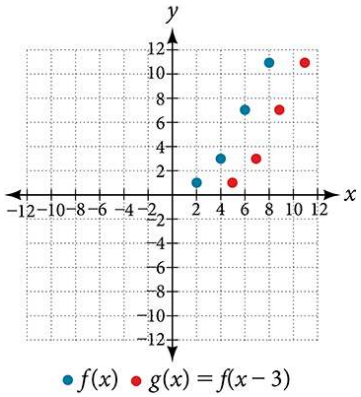
Table 2.5.5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x-3)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Figure 2.5.7 represents both of the functions. We can see the horizontal shift in each point.

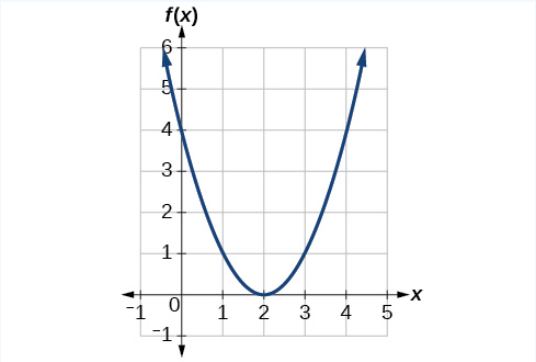
Figure 2.5.7



Example 2.5.5: Identifying a Horizontal Shift of a Toolkit Function

Figure 2.5.8 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

Figure 2.5.8



Solution

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x-2) g(x) = f(x-2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$. That is, since $f(x) = x^2$, $g(x) = f(x - 2)$. Therefore,

$$g(x) = f(x - 2) = (x - 2)^2$$

In Example 2.5.5, to determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the **vertex** point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to *subtract* 2 units from our input values.

✔ Example 2.5.6: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$.

Solution

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

? Checkpoint 2.5.6

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output y-axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input x-axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left. Note that these shifts can be performed in either order.

✔ Example 2.5.7: Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a "V" shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 2.5.9.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0, 0)$ is transformed first by shifting left 1 unit: $(0, 0) \to (-1, 0)$
- The point $(-1, 0)$ is transformed next by shifting down 3 units: $(-1, 0) \to (-1, -3)$

Figure 2.5.9

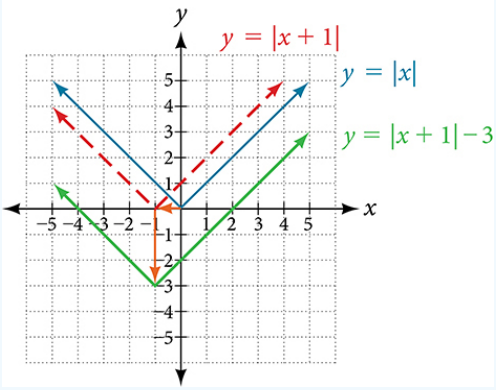
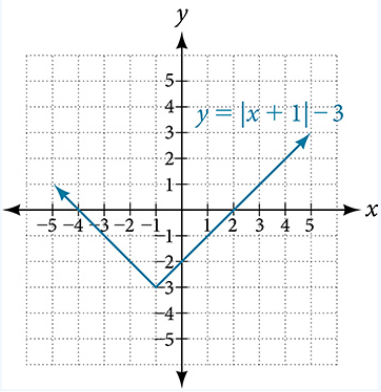


Figure **2.5.10** shows the graph of h.

Figure 2.5.10



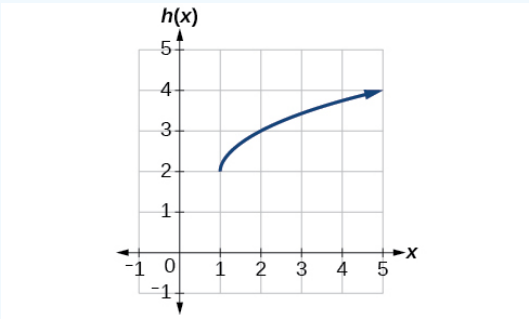
Checkpoint 2.5.7

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Example 2.5.8: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure **2.5.11**, which is a transformation of the toolkit square root function.

Figure 2.5.11



Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1) + 2.$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1} + 2.$$

Note that the transformation in Example 2.5.8 has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

? Checkpoint 2.5.8

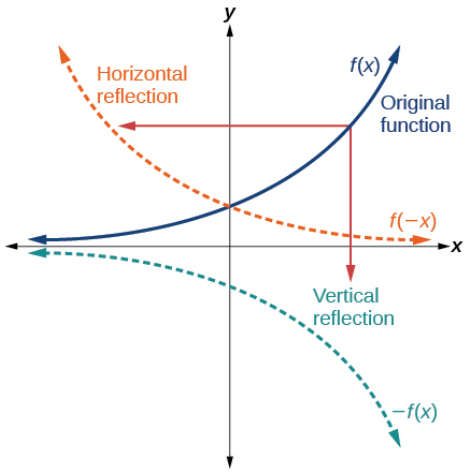
Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 2.5.12.

Figure 2.5.12

Vertical and horizontal reflections of a function



Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Theorem: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a vertical reflection of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a horizontal reflection of the function $f(x)$, sometimes called a reflection about the y-axis.

Thus, we multiply all outputs by –1 for a vertical reflection, and we multiply all inputs by –1 for a horizontal reflection.

✓ Example 2.5.9: Reflecting a Graph Horizontally and Vertically

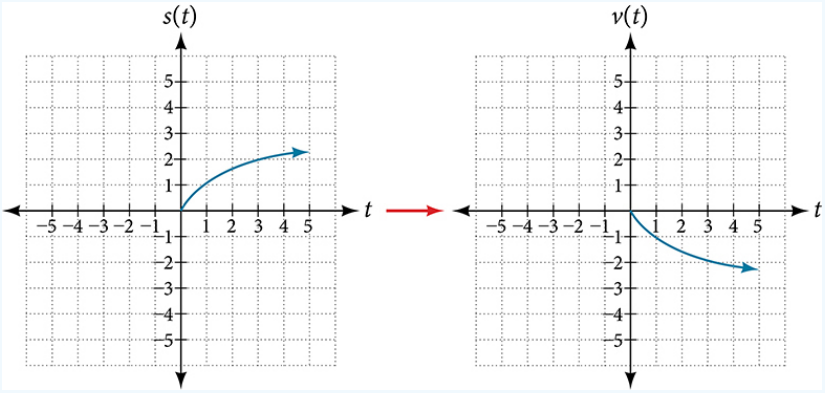
Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure **2.5.13**.

Figure 2.5.13

Vertical reflection of the square root function



Because each output value is the opposite of the original output value, we can write

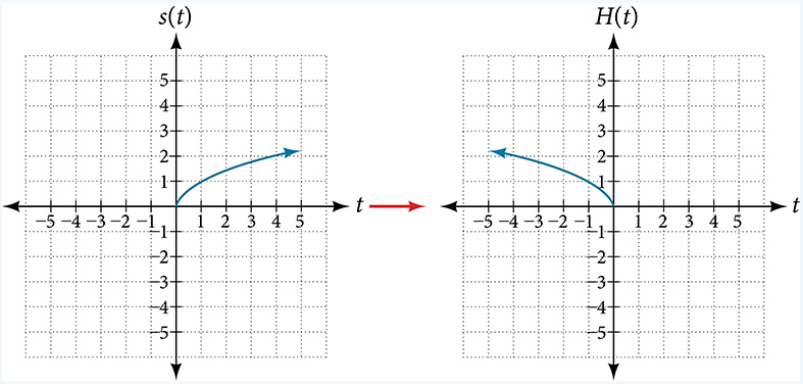
$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}.$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure **2.5.14**.

Figure 2.5.14

Horizontal reflection of the square root function



Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}.$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that transformations like those in Example **2.5.9** can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)H(t)$ function the domain $(-\infty, 0]$.

Checkpoint 2.5.9

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Example 2.5.10: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 2.5.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.5.6

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 2.5.7.

Table 2.5.7

x	2	4	6	8
$g(x)$	–1	–3	–7	–11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 2.5.8.

Table 2.5.8

x	−2	−4	−6	−8
$h(x)$	1	3	7	11

Checkpoint 2.5.10

A function $f(x)$ is given as Table 2.5.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

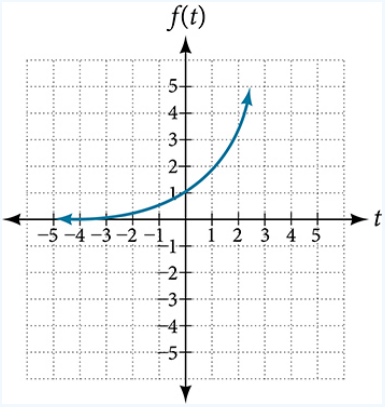
Table 2.5.9

x	−2	0	2	4
$f(x)$	5	10	15	20

Example 2.5.11: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 2.5.15. Sketch a graph of $k(t)$.

Figure 2.5.15



Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

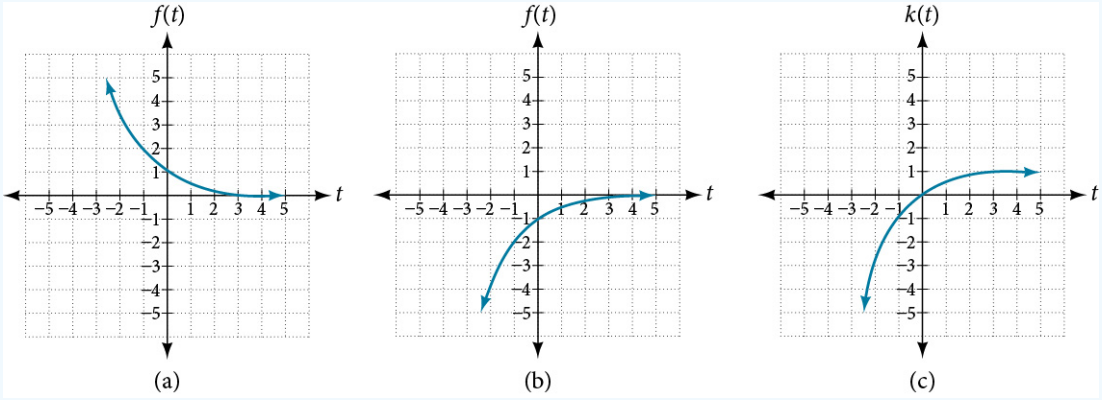
We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

1. First, we apply a horizontal reflection: $(0, 1) \to (0, 1)$ and $(1, 2) \to (-1, 2)$.
2. Then, we apply a vertical reflection: $(0, 1) \to (0, -1)$ and $(-1, 2) \to (-1, -2)$.
3. Finally, we apply a vertical shift: $(0, -1) \to (0, 0)$ and $(-1, -2) \to (-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 2.5.16, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

Figure 2.5.16



As a model for learning, the function in Example 2.5.11 would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

? Checkpoint 2.5.11

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

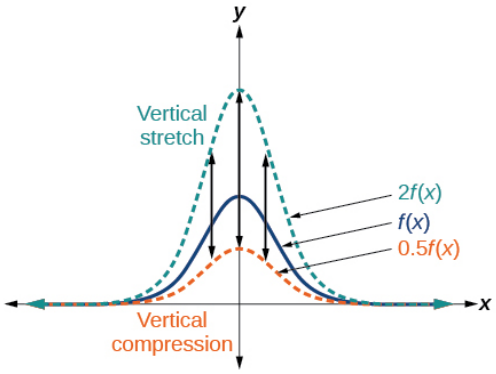
We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 2.5.17 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

Figure 2.5.17

Vertical stretch and compression



Theorem: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = a\,f(x)$, where a is a constant, is a vertical stretch or vertical compression of the function $f(x)$.

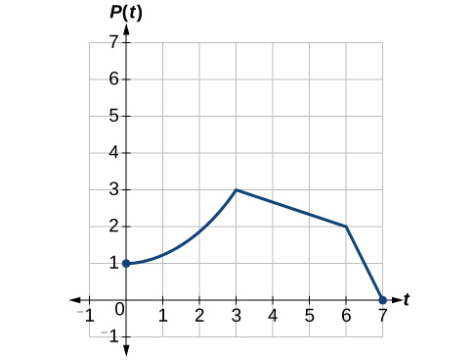
- If $a > 1$, then the graph will be stretched vertically by a factor of a.
- If $0 < a < 1$, then the graph will be compressed vertically by a factor of a.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

A good summary is that, if you have a function multiplied by a, you multiply all range values by a.

Example 2.5.12: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 2.5.18.

Figure 2.5.18



A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

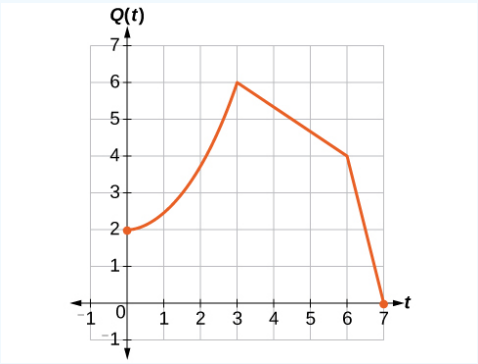
Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure **2.5.19**.

If we choose four reference points, $(0,1)$, $(3,3)$, $(6,2)$, and $(7,0)$, we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0,1) \rightarrow (0,2)$$
$$(3,3) \rightarrow (3,6)$$
$$(6,2) \rightarrow (6,4)$$
$$(7,0) \rightarrow (7,0)$$

Figure 2.5.19



Symbolically, the relationship is written as

$$Q(t) = 2P(t).$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Example 2.5.13: Finding a Vertical Compression of a Tabular Function

A function f is given as Table **2.5.10**. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 2.5.10

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}.$$

We do the same for the other values to produce Table 2.5.11.

Table 2.5.11

x	22	44	66	88
$g(x)$	1212	3232	7272	112112

The result in Example 2.5.13 is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Checkpoint 2.5.13

A function f is given as Table 2.5.12. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

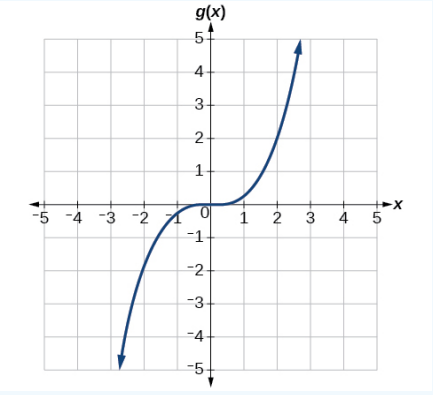
Table 2.5.12

x	2	4	6	8
$f(x)$	12	16	20	0

Example 2.5.14: Recognizing a Vertical Stretch

The graph in Figure 2.5.20 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

Figure 2.5.20



Solution

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can (somewhat) safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3.$$

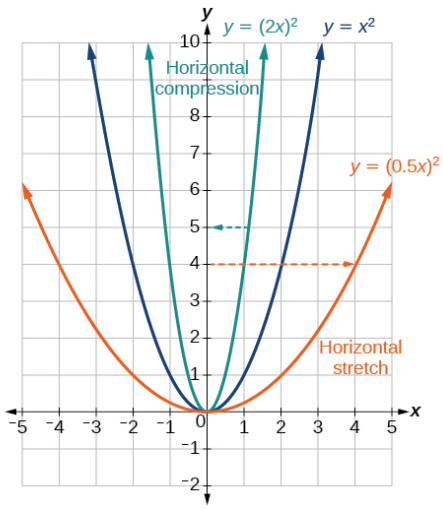
? Checkpoint 2.5.14

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

Figure 2.5.21



Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.5.21. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of $\frac{1}{2}$.

Theorem: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f(x)$.

- If $b > 1$, then the graph will be compressed horizontally by a factor of $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched horizontally by a factor of $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

In effect, we multiply all x-values by $\frac{1}{b}$.

Example 2.5.15: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2),$$

$$R(2) = P(4),$$

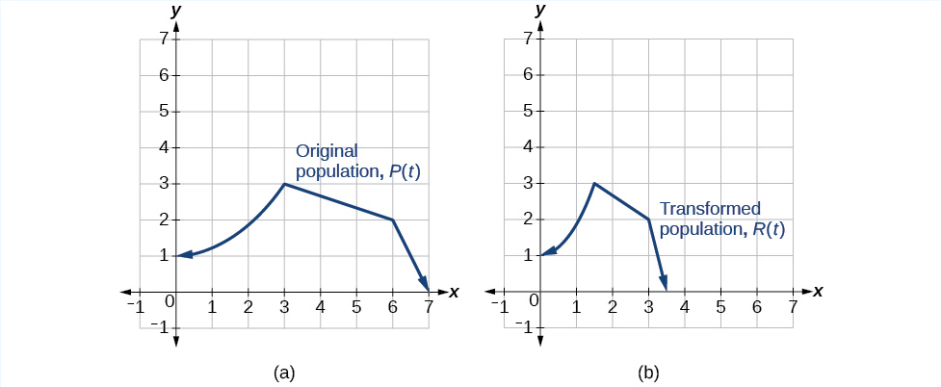
and in general,

$$R(t) = P(2t).$$

See Figure 2.5.22 for a graphical comparison of the original population and the compressed population.

Figure 2.5.22

(a) Original population graph (b) Compressed population graph



Note that, in Example 2.5.15, the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.5.16: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.5.13. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

Table 2.5.13

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1.$$

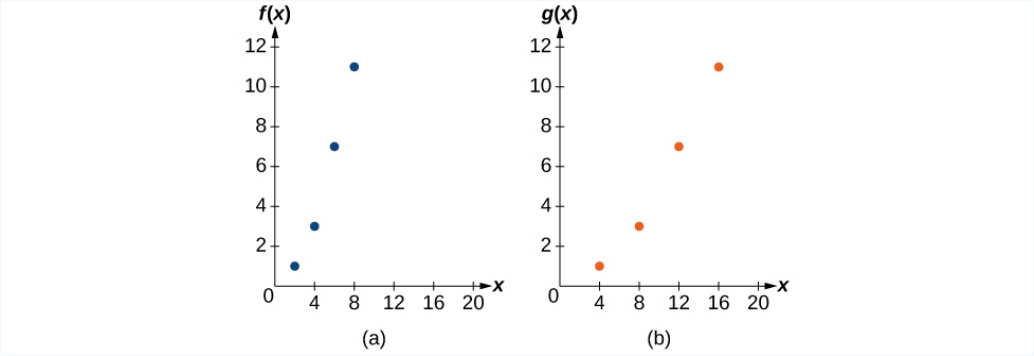
We do the same for the other values to produce Table 2.5.14.

Table 2.5.14

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 2.5.23 shows the graphs of both of these sets of points.

Figure 2.5.23

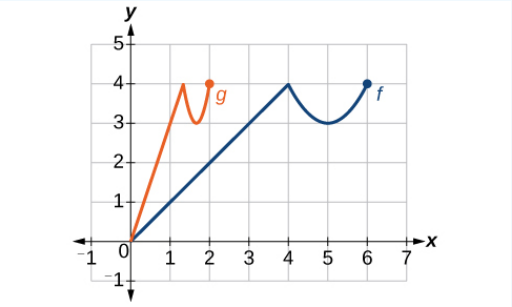


Because each input value has been doubled, the result in Example 2.5.16 is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 2.5.17: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.5.24.

Figure 2.5.24



Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6, 4)$ and $g(x)$ ends at $(2, 4)$, we can see that the x-values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

? Checkpoint 2.5.17

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Performing a Sequence of Transformations

When combining transformations, it is *very* important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the Order of Operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f\left(b\left(x+\frac{p}{b}\right)\right).$$

Let's work through an example. If

$$f(x)=(2x+4)^2,$$

then we can factor out a 2 to get

$$f(x)=(2(x+2))^2.$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

Given a function $g(x)$ of the form

$$g(x)=af(bx+p)+k,$$

where the base graph of $y=f(x)$ is known, we factor b from the argument $bx+p$ to get $b\left(x+\frac{p}{b}\right)$. This gives

$$g(x)=af(bx+p)+k=af\left(b\left(x+\frac{p}{b}\right)\right)+k.$$

With the function rewritten in this specific order, we perform transformations on the graph of f in the order written - from left to right - to arrive at the graph of $g(x)$.

If your original function is not in the form $af(bx+p)+k$, then you will have to "precondition" the function using arithmetic. For example,

$$g(x)=-2-\frac{1}{3}f(7x-8)=-\frac{1}{3}f(7x-8)-2=-\frac{1}{3}f\left(7\left(x-\frac{8}{7}\right)\right)-2.$$

✓ Example 2.5.18: Finding a Triple Transformation of a Tabular Function

Given Table 2.5.15 for the function $f(x)$, create a table of values for the function $g(x)=2f(3x)+1$.

Table 2.5.15

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

Using the advice given previously, we make sure the function is written in the form

$$af(bx+p)+k.$$

In this case, we were given $g(x)=2f(3x)+1$, which is already in this form. We then start applying the transformations by reading from left to right.

We start with the vertical stretch, which will multiply the output values by 2. Table **2.5.16**.

Table 2.5.16

x	6	12	18	24
$2f(x)$	20	28	30	34

We then move to the right in the function $g(x)=2f(3x)+1$ and deal with the 3. This causes a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. We apply these transformations to the x-values from Table **2.5.16** to arrive at Table **2.5.17**.

Table 2.5.17

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table **2.5.18**.

Table 2.5.18

x	2	4	6	8
$g(x)=2f(3x)+1$	21	29	31	35

Example 2.5.19: Graphing a Function with Many Transformations

Sketch a graph of the function

$$f(x)=-4-\frac{1}{2}\sqrt{12-3x}.$$

Solution

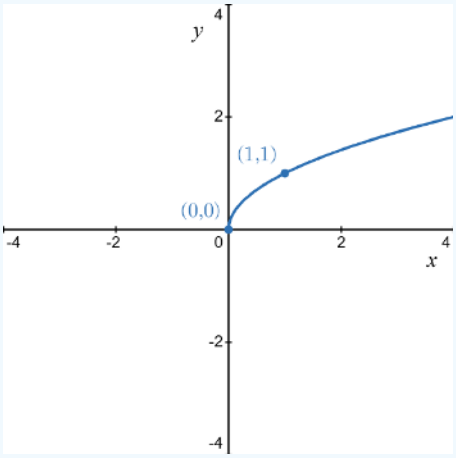
The function given to us needs to be rewritten in the form $af(bx+p)+k=af\left(b\left(x+\frac{p}{b}\right)\right)+k$. Let's start by simply rearranging terms.

$$f(x)=-4-\frac{1}{2}\sqrt{12-3x}=-\frac{1}{2}\sqrt{-3x+12}-4$$

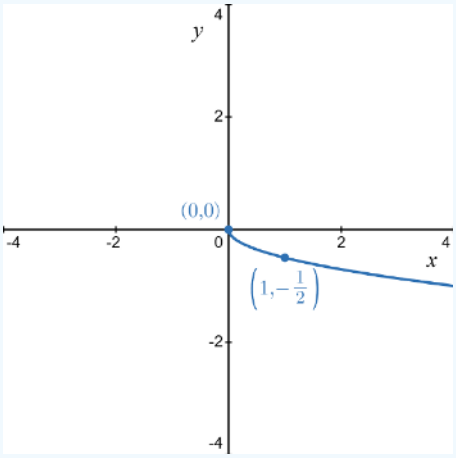
We now take the time to factor the lead coefficient from both terms inside the function.

$$f(x)=-\frac{1}{2}\sqrt{-3\left(x-4\right)}-4.$$

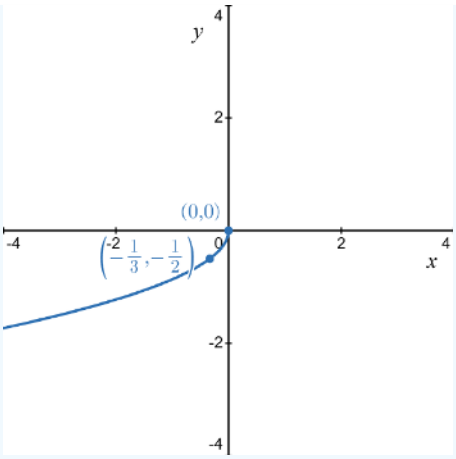
The parent function is $y = \sqrt{x}$. It's best to label two "anchor" points (which will almost always be $(0, 0)$ and $(1, 1)$ for the Toolkit functions).



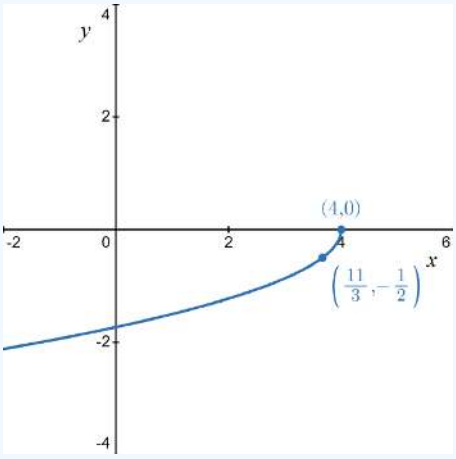
Reading our "cleaned" function from left to right, the coefficient $-\frac{1}{2}$ reflects the graph of $y = \sqrt{x}$ vertically about the x-axis and compresses it vertically (squishing towards the x-axis) by a factor of $\frac{1}{2}$. Another way to think of this is that all of the y-values on the graph of $y = \sqrt{x}$ are multiplied by $-\frac{1}{2}$.



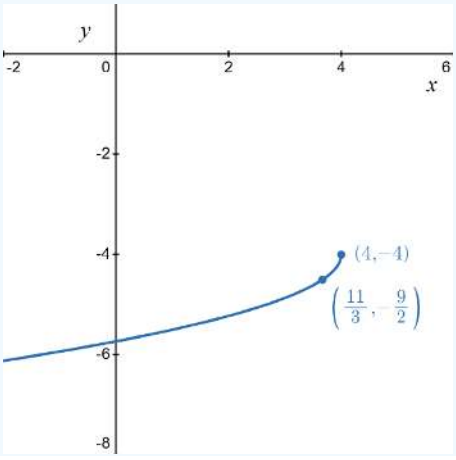
Continuing reading from left to right, we have the factor of -3 on the *inside* of the function. Remember, anything on the inside of the function affects the x-values (and in an opposite way from how they affect the outside of the function). Specifically, the factor of -3 reflects the graph of $y = -\frac{1}{2}\sqrt{x}$ about the y-axis and *compresses* the graph horizontally (squishing it) towards the y-axis by a factor of $\frac{1}{3}$. Another way to think of this is that the factor -3 *divides* all of the x-values on the graph of $y = -\frac{1}{2}\sqrt{x}$ by -3.



Continuing reading our original function from left to right, we have $(x-4)$ on the *inside* of the function. This causes a shift of 4 units horizontally to the right for the graph of $y=-\frac{1}{2}\sqrt{3x}$. Another way to think of this is that we *add* 4 units to each of the x-values.



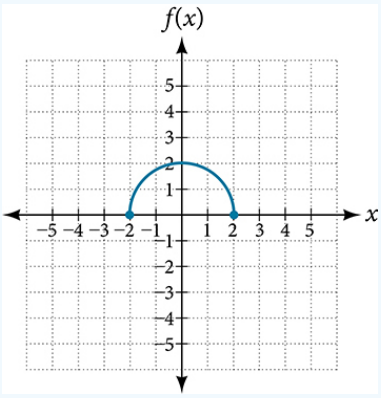
Finally, we have the subtraction of 4 with which to contend. This subtract occurs *outside* the function (the square root), so it shifts the graph of $y=-\frac{1}{2}\sqrt{3(x-4)}$ vertically *down* 4 units. Our final graph is given below.



Example 2.5.20: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure $2.5.25$ to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

Figure 2.5.25



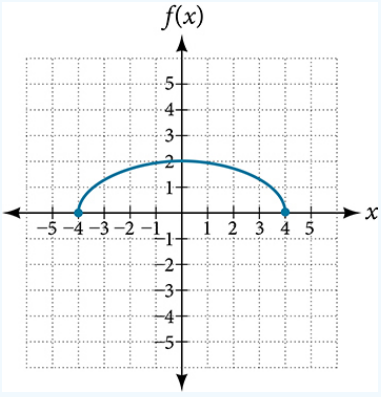
Solution

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3.$$

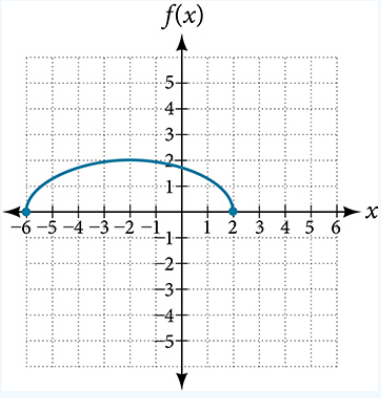
Reading from left to right, we first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 2.5.26.

Figure 2.5.26



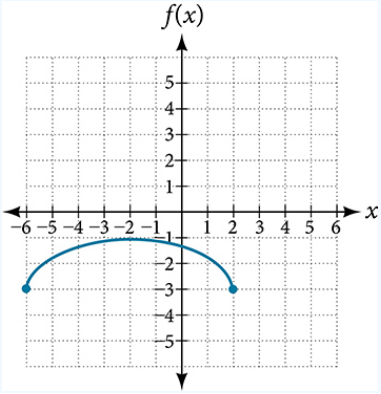
Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 2.5.27.

Figure 2.5.27



Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 2.5.28.

Figure 28



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2.5.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Transformations of Functions
  - Functions - Graphs - Rigid Transformations - Horizontal and Vertical Shifts
  - Functions - Graphs - Rigid Transformations - Horizontal and Vertical Reflections
  - Functions - Graphs - Non-Rigid Transformations - Stretches and Compressions
  - Functions - Graphs - Performing a Sequence of Rigid and Non-Rigid Transformations (Standard)
  - Functions - Graphs - Sequences of Rigid and Non-Rigid Transformations on Non-Standard Functions
  - Intermediate Algebra - Functions: Transformations (older video)

KEY CONCEPTS

DEFINITIONS

- **Vertical Shift:** A transformation that moves the graph of a function up or down by adding a constant k to the output: $g(x) = f(x) + k$. If $k > 0$, the shift is upward; if $k < 0$, the shift is downward.
- **Horizontal Shift:** A transformation that moves the graph of a function left or right by adding or subtracting a constant h from the input: $g(x) = f(x - h)$. If $h > 0$, the shift is to the right; if $h < 0$ (e.g., $f(x + c)$ where $c > 0$), the shift is to the left.
- **Vertical Reflection:** A transformation that reflects the graph of a function vertically across the x-axis, given by $g(x) = -f(x)$.
- **Horizontal Reflection:** A transformation that reflects the graph of a function horizontally across the y-axis, given by $g(x) = f(-x)$.
- **Vertical Stretch:** A transformation where the graph of a function is stretched away from the x-axis, caused by multiplying the output by a constant a where $|a| > 1$. If $a > 1$, it's a pure stretch.
- **Vertical Compression:** A transformation where the graph of a function is compressed toward the x-axis, caused by multiplying the output by a constant a where $0 < |a| < 1$. If $0 < a < 1$, it's a pure compression.
- **Horizontal Stretch:** A transformation where the graph of a function is stretched away from the y-axis, caused by multiplying the input by a constant b where $0 < |b| < 1$. If $0 < b < 1$, it's a pure stretch (e.g., $f(\frac{1}{2}x)$).
- **Horizontal Compression:** A transformation where the graph of a function is compressed toward the y-axis, caused by multiplying the input by a constant b where $|b| > 1$. If $b > 1$, it's a pure compression (e.g., $f(2x)$).

THEOREMS

- **Vertical Shift:** Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.
- **Horizontal Shift:** Given a function $f(x)$, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of $f(x)$. If h is positive, the graph will shift right. If h is negative, the graph will shift left.
- **Reflections:**
  - Given a function $f(x)$, a new function $g(x) = -f(x)$ is a vertical reflection of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.
  - Given a function $f(x)$, a new function $g(x) = f(-x)$ is a horizontal reflection of the function $f(x)$, sometimes called a reflection about the y-axis
- **Vertical Stretches and Compressions:** Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a vertical stretch or vertical compression of the function $f(x)$.
  - If $a > 1$, then the graph will be stretched vertically by a factor of a.
  - If $0 < a < 1$, then the graph will be compressed vertically by a factor of a.
  - If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection
- **Horizontal Stretches and Compressions:** Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f(x)$.
  - If $b > 1$, then the graph will be compressed horizontally by a factor of $\frac{1}{b}$.
  - If $0 < b < 1$, then the graph will be stretched horizontally by a factor of $\frac{1}{b}$.
  - If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

COMMON MISTAKES

- **Counterintuitive Nature of Horizontal Shifts:** Horizontal changes often seem counterintuitive. For example, $f(x - h)$ shifts right if $h > 0$, and $f(x + h)$ (which is $f(x - (-h))$) shifts left if $h > 0$.
- **Order of Transformations:** When combining multiple transformations, the order in which they are applied is very important, especially when mixing shifts with stretches/compressions. For horizontal transformations, factoring inside the function (e.g., writing $f(b(x + p))$) can make the order easier to manage.
- **Misinterpreting Coefficients for Horizontal Stretch/Compression:** The coefficient b in $f(bx)$ leads to a stretch/compression by a factor of $\frac{1}{b}$, not b. For example, $f(2x)$ is

a horizontal compression by a factor of $\frac{1}{2}$, and $f(\frac{1}{2}x)$ is a horizontal stretch by a factor of 2.

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2.5.2: HOMEWORK

READING QUESTIONS

1. How does the transformation $g(x) = f(x) + 5$ affect the graph of $f(x)$?
2. If the graph of $f(x)$ is shifted 3 units to the left, what is the equation of the new function in terms of $f(x)$?
3. What is the difference in the graphical effect of $g(x) = -f(x)$ versus $h(x) = f(-x)$?
4. If $f(x) = x^2$, describe the transformation that results in $g(x) = 3x^2$. Is this a stretch or a compression? In which direction?
5. If $f(x) = \sqrt{x}$, describe the transformation that results in $g(x) = \sqrt{2x}$. Is this a stretch or a compression? In which direction? By what factor?
6. When transforming a function using the form $y = af(bx + p) + k$, why is it recommended to first factor out b from the term $(bx + p)$?
7. Explain why $f(x - 2)$ shifts the graph of $f(x)$ to the right, even though a subtraction is involved.
8. If you are given the graph of $y = f(x)$ and want to graph $y = 2f(x) - 3$, describe the sequence of transformations you would apply.
9. What is the domain and range of $g(t) = \sqrt{-t}$ if the original function is $s(t) = \sqrt{t}$ with domain $[0, \infty)$ and range $[0, \infty)$?
10. Consider the function $k(t) = -2^{-t} + 1$. Identify the base function and list the transformations applied to it in the correct order to obtain $k(t)$.
11. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?
12. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?
13. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?
14. When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

HOMEWORK

ALGEBRAIC

15. Write a formula for the function obtained when the graph of $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.
16. Write a formula for the function obtained when the graph of $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.
17. Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x}$ is shifted down 4 units and to the right 3 units.
18. Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

19. $y = f(x - 49)$
20. $y = f(x + 43)$
21. $y = f(x + 3)$
22. $y = f(x - 4)$
23. $y = f(x) + 5$
24. $y = f(x) + 8$
25. $y = f(x) - 2$
26. $y = f(x) - 7$
27. $y = f(x - 2) + 3$
28. $y = f(x + 4) - 1$

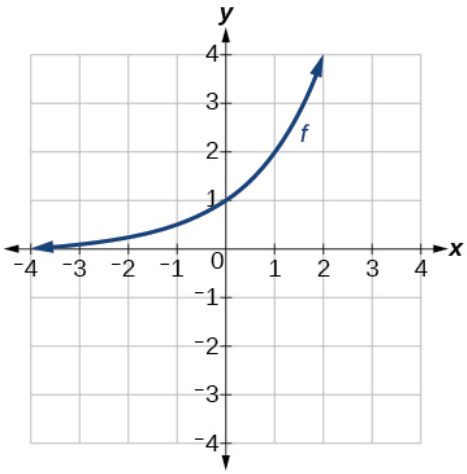
For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

29. $f(x) = 4(x + 1)^2 - 5$
30. $g(x) = 5(x + 3)^2 - 2$
31. $a(x) = \sqrt{-x + 4}$
32. $k(x) = -3\sqrt{x} - 1$

GRAPHICAL

For the following exercises, use the graph of $f(x) = 2^x$ shown in Figure 2.5.2.1 to sketch a graph of each transformation of $f(x)$.

Figure 2.5.2.1



33. $g(x) = 2^x + 1$
34. $h(x) = 2^x - 3$
35. $w(x) = 2^{x-1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

36. $f(t) = (t + 1)^2 - 3$
37. $h(x) = |x - 1| + 4$
38. $k(x) = (x - 2)^3 - 1$
39. $m(t) = 3 + \sqrt{t + 2}$

NUMERIC

40. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	−2	−1	0	1	2
$f(x)$	−2	−1	−3	1	2

x	−1	0	1	2	3
$g(x)$	−2	−1	−3	1	2

x	−2	−1	0	1	2
$h(x)$	−1	0	−2	2	3

41. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	−2	−1	0	1	2
$f(x)$	−1	−3	4	2	1

x	−3	−2	−1	0	1
$g(x)$	−1	−3	4	2	1

x	−2	−1	0	1	2
$h(x)$	−2	−4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

42.

43.

44.

45.

46.

47.

48.

49.

For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

50.

51.

For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

52.

53.

54.

55.

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

56. $g(x) = -f(x)$
57. $g(x) = f(-x)$
58. $g(x) = 4f(x)$
59. $g(x) = 6f(x)$
60. $g(x) = f(5x)$
61. $g(x) = f(2x)$
62. $g(x) = f\left(\frac{1}{3}x\right)$
63. $g(x) = f\left(\frac{1}{5}x\right)$
64. $g(x) = 3f(-x)$
65. $g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

66. The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.
67. The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.
68. The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.
69. The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.
70. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.
71. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

72. $g(x) = 4(x+1)^2 - 5$
73. $g(x) = 5(x+3)^2 - 2$
74. $h(x) = -2|x-4| + 3$
75. $k(x) = -3\sqrt{x} - 1$
76. $m(x) = \frac{1}{2}x^3$
77. $n(x) = \frac{1}{3}|x-2|$
78. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$
79. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$
80. $a(x) = \sqrt{-x+4}$

For the following exercises, use the graph in Figure 2.5.2.2 to sketch the given transformations.

Figure 2.5.2.2

81. $g(x) = f(x) - 2$
82. $g(x) = -f(x)$
83. $g(x) = f(x + 1)$
84. $g(x) = f(x - 2)$

2.6: Absolute Value Functions

Note to the Instructor (click to expand)

Although it was introduced when we introduced piecewise functions, the definition of the absolute value function as a piecewise function is reintroduced (it is that important). The middle part of this section focuses on graphing transformations of the absolute value function, which is a nice bridge from the previous section. The remainder of the section is focused on solving absolute value equations and inequalities (needed for Calculus); however, they only should involve linear expressions (other types are treated in a future section).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Arithmetic**
  - The Absolute Value of a Number: A fundamental understanding of what absolute value means numerically is essential before discussing absolute value functions.
- **Absolute Value Functions**
  - The Concept of Absolute Value: A deep understanding of absolute value as distance is used in contextual problems like Example 1 and 2.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Graph an absolute value function.
- Solve an absolute value equation.
- Solve an absolute value inequality.

Hawk A.I. Section-Specific Tutor

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To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Figure 2.6.1

Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate absolute value functions.

Focus on Calculus - Piecewise Definition of the Absolute Value

Recall that in its basic form $f(x) = |x|$, the absolute value function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

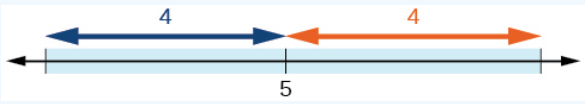
Example 2.6.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in Figure 2.6.2, to represent the condition to be satisfied.

Figure 2.6.2



The distance from x to 5 can be represented using the absolute value as $|x - 5|$. We want the values of x that satisfy the condition $|x - 5| \leq 4$.

Note that

$$\begin{aligned} -4 &\le x-5 & \qquad x-5 &\le 4 \\ 1 &\le x & \qquad x &\le 9. \end{aligned}$$

So $|x-5| \le 4$ is equivalent to $1 \le x \le 9$.

However, mathematicians generally prefer absolute value notation.

? Checkpoint 2.6.1

Describe all values x within a distance of 3 from the number 2.

✓ Example 2.6.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often $\pm 1\%$, $\pm 5\%$, or $\pm 10\%$.

Suppose we have a resistor rated at 680 ohms, $\pm 5\%$. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \le 34.$$

? Checkpoint 2.6.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

A theme of this textbook is to loop back to previous concepts that constantly reappear in Calculus. The difference quotient is one such concept.

✓ Example 2.6.3: Revisiting the Difference Quotient

Find, and simplify, the difference quotient for $f(x) = -\frac{1}{2}|4x-5|+3$ assuming that $x < \frac{5}{4}$ and $h < 0$.

Solution

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\left(-\frac{1}{2}|4(x+h)-5|+3\right)-\left(-\frac{1}{2}|4x-5|+3\right)}{h} && \text{(substituting)} \\ &= \frac{-\frac{1}{2}|4(x+h)-5|+3+\frac{1}{2}|4x-5|-3}{h} && \text{(distributing)} \\ &= \frac{-\frac{1}{2}|4(x+h)-5|+\frac{1}{2}|4x-5|}{h} && \text{(simplifying)} \\ &= \frac{2}{2}\cdot\frac{-\frac{1}{2}|4(x+h)-5|+\frac{1}{2}|4x-5|}{h} && \text{(simplifying the compound fraction)} \\ &= \frac{-|4(x+h)-5|+|4x-5|}{2h} && \text{(distributing)} \\ &= \frac{-|4x+4h-5|+|4x-5|}{2h} && \text{(distributing)} \end{aligned}$$

At this point, we are stuck and need to refer to the piecewise definition of the absolute value function.

$$|4x-5| = \begin{cases} 4x-5, & \text{if } 4x-5 \geq 0 \implies x \geq \frac{5}{4} \\ -(4x-5), & \text{if } 4x-5 < 0 \implies x < \frac{5}{4} \end{cases}$$

Since we were told $x < \frac{5}{4}$, $|4x-5| = -(4x-5) = -4x+5$. Turning our attention to the other absolute value, we get the following:

$$|4x+4h-5| = \begin{cases} 4x+4h-5, & \text{if } 4x+4h-5 \geq 0 \implies 4x \geq 5-4h \implies x \geq \frac{5}{4}-h \\ -(4x+4h-5), & \text{if } 4x+4h-5 < 0 \implies 4x < 5-4h \implies x < \frac{5}{4}-h \end{cases}$$

Since we are told that $x < \frac{5}{4}$ and $h < 0$, it must be true that $x + h < \frac{5}{4}$. Hence, $x < \frac{5}{4} - h$ and so $|4x+4h-5| = -(4x+4h-5) = -4x-4h+5$. Combining these two results, we get the following:

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{-|4x+4h-5|+|4x-5|}{2h} \\ &= \frac{-(-4x-4h+5)+(-4x+5)}{2h} && \text{(substituting)} \\ &= \frac{4x+4h-5-4x+5}{2h} && \text{(distributing)} \\ &= \frac{4h}{2h} && \text{(combining like terms)} \\ &= \frac{4\cancel{h}^{1}}{2\cancel{h}^{1}} && \text{(canceling like factors)} \\ &= 2 \end{aligned}$$

? Checkpoint 2.6.3

Find, and simplify, the difference quotient for $f(x) = -\frac{1}{2}|4x-5|+3$ assuming that $x > \frac{5}{4}$ and $h > 0$.

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 2.6.3.

Figure 2.6.3

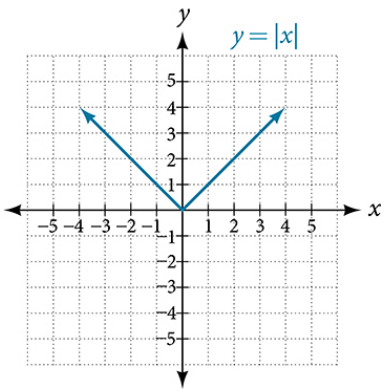
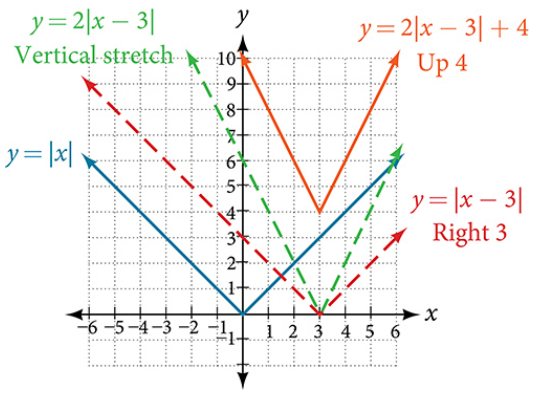


Figure 2.6.4 shows the graph of $y = 2|x-3|+4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

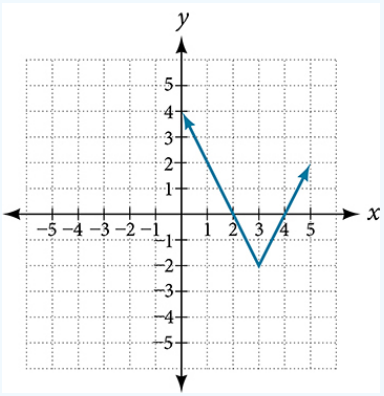
Figure 2.6.4



Example 2.6.4: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 2.6.5.

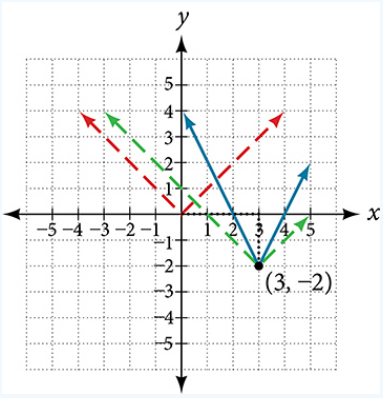
Figure 2.6.5



Solution

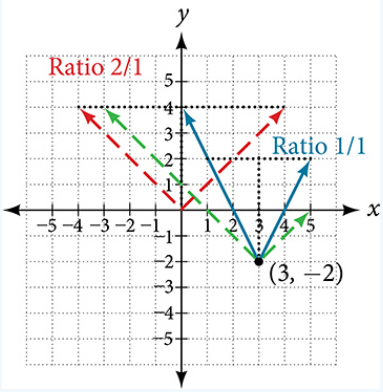
The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 2.6.6.

Figure 2.6.6



We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 2.6.7.

Figure 2.6.7



From this information we can write the equation as $f(x) = 2|x - 3| - 2$, treating the stretch as a vertical stretch.

In Example 2.6.4, we could have opted to treat the stretch as a horizontal compression. In that case, the function would have been $f(x) = |2(x - 3)| - 2$. Note that this equation is algebraically equivalent to what we arrived at in our solution. This result, that the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression, is not a rule for all functions - it just happens to work for absolute value (and linear) functions. Note also that if the vertical stretch factor is negative, there is also a reflection about the x-axis.

Q&A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x - 3| - 2.$$

Now substituting in the point $(1, 2)$, we get

$$\begin{aligned} & 2 = a|1 - 3| - 2 && \text{(substituting)} \\ \Longrightarrow\ & 4 = 2a && \text{(simplifying and adding 2 to both sides)} \\ \Longrightarrow\ & 2 = a && \text{(dividing both sides by 2)} \end{aligned}$$

Checkpoint 2.6.4A

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Q&A

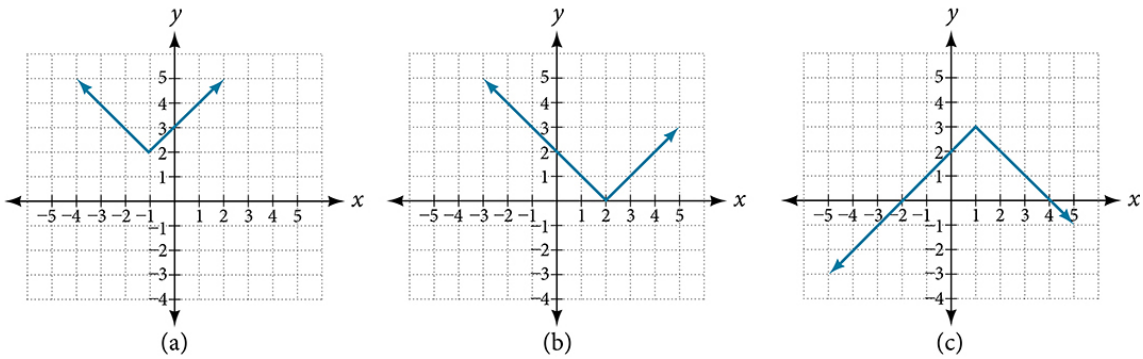
Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see Figure 2.6.8).

Figure 2.6.8

(a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.



? Checkpoint 2.6.4*B*

Is the absolute value function one-to-one?

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{aligned} 2x-6 &= 8 \quad \text{or} \quad 2x-6 = -8 \\ \Longrightarrow \quad 2x &= 14 \qquad\quad 2x = -2 \\ \Longrightarrow \quad x &= 7 \qquad\quad x = -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

Definition: Absolute Value Equation

An **absolute value equation** is an equation in which the unknown variable appears within the absolute value bars.

For example,

$$\begin{aligned} |x| &= 4, \\ |2x-1| &= 3, \text{ and} \\ |5x+2|-4 &= 9. \end{aligned}$$

How To: Solve Absolute Value Equations

To solve an absolute value equation, $|\blacksquare| = k$, where $\blacksquare$ is an algebraic expression involving a variable, we rely on the piecewise function definition of the absolute value. Namely,

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

This means that to solve $|\blacksquare| = k$, we have two cases to consider:

If $\blacksquare \geq 0$,	If $\blacksquare < 0$,
then we solve	then we solve
$\blacksquare = k$	$-\blacksquare = k$

However, if $k < 0$, we can safely say (without any effort) that the original absolute value equation has no solution. Remember, an absolute value cannot be a negative number.

I specifically chose to use $\blacksquare$ in this version of the definition instead of x because most students new to the concept of the piecewise definition of the absolute value tend to think that the conditions must always be $x \geq 0$ and $x < 0$ - **this is not true**.

Let's illustrate the process through an example.

Example 2.6.5: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x + 1| - 7$, find the values of x such that $f(x) = 0$.

Solution

We set $f(x) = 0$ and solve.

$$\begin{aligned} 0 &= |4x+1| - 7 && \text{(setting } f(x) = 0\text{)} \\ \Longrightarrow \quad 7 &= |4x+1| && \text{(adding 7 to both sides)} \end{aligned}$$

Now that we have the absolute value isolated, let's recall the piecewise function definition of the absolute value.

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

In our case, $\blacksquare = 4x + 1$. Hence,

$$|4x+1| = \begin{cases} 4x+1, & \text{if } 4x+1 \geq 0 \\ -(4x+1), & \text{if } 4x+1 < 0 \end{cases}$$

That is,

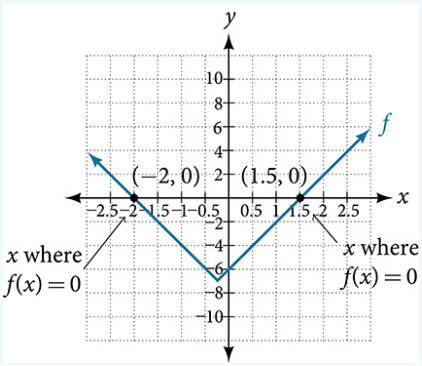
$$|4x+1| = \begin{cases} 4x+1, & \text{if } x \geq -\frac{1}{4} \\ -(4x+1), & \text{if } x < -\frac{1}{4} \end{cases}$$

This tells us that our absolute value equation splits into two possibilies:

For $x \geq -\frac{1}{4}$			For $x < -\frac{1}{4}$		
7	$=$	$\lvert 4x+1 \rvert$	7	$=$	$\lvert 4x+1 \rvert$
$\Longrightarrow \quad 7$	$=$	$4x+1$	$\Longrightarrow \quad 7$	$=$	$-(4x+1)$
$\Longrightarrow \quad 6$	$=$	$4x$	$\Longrightarrow \quad -7$	$=$	$4x+1$
$\Longrightarrow \quad \frac{3}{2}$	$=$	x	$\Longrightarrow \quad -8$	$=$	$4x$
			$\Longrightarrow \quad -2$	$=$	x

The function outputs 0 when $x = 1.5$ or $x = -2$. See Figure 2.6.9.

Figure 2.6.9



Checkpoint 2.6.5

For the function $f(x) = |2x - 1| - 3$, find the values of x such that $f(x) = 0$.

Q&A

Should we always expect two answers when solving $|A| = B$?

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

Example 2.6.6: Solving an Absolute Value Equation

Solve $1 = 4|x - 2| + 2$.

Solution

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x-2|+2 \\ \Longrightarrow \quad -1 &= 4|x-2| \\ \Longrightarrow \quad -\frac{1}{4} &= |x-2| \end{aligned}$$

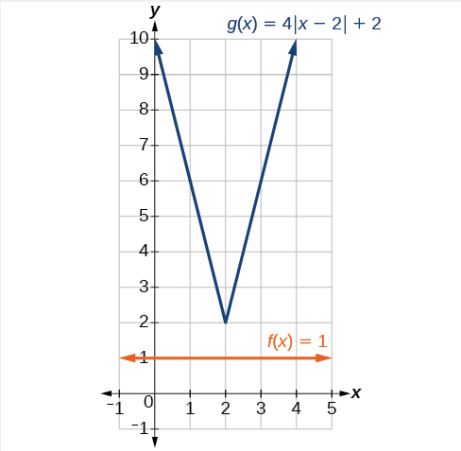
The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Q&A

In Example 2.6.6, if $f(x) = 1$ and $g(x) = 4|x - 2| + 2$ were graphed on the same set of axes, would the graphs intersect?

No. The graphs of f and g would not intersect, as shown in Figure 2.6.10. This confirms, graphically, that the equation $1 = 4|x - 2| + 2$ has no solution.

Figure 2.6.10



Checkpoint 2.6.6

Find where the graph of the function $f(x) = -|x + 2| + 3$ intersects the horizontal and vertical axes.

Focus on Calculus - Solving Absolute Value Inequalities

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an absolute value inequality to solve such an equation. An **absolute value inequality** is an equation of the form

$$|A| < B, \quad |A| \leq B, \quad |A| > B, \quad \text{or } |A| \geq B,$$

where an expression A (and possibly, but not usually, B) depends on a variable x.

Definition: Absolute Value Inequality

An **absolute value inequality** is an inequality in which the unknown variable appears within the absolute value bars.

Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The disadvantage of this approach is that, when graphing by hand, the solutions are likely not going to be accurate. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200 \text{ or } -200 < x < 200.$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of values x such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x - 600|$. Thus, we want $|x - 600| < 200$, or, using inequality notation, $-200 < x - 600 < 200$. Solving this inequality, we get

$$\begin{aligned} & -200 < x - 600 < 200 \\ \Rightarrow \quad & 400 < x < 800 \qquad \text{(adding 600 to all three 'sides')} \end{aligned}$$

This means our returns would be between \$400 and \$800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

Example 2.6.7: Solving an Absolute Value Inequality

Solve $|x-5|<4$.

Solution

Our approach will duplicate our work in Example 2.6.6. Since the inequality is already isolated, we immediately dive into the piecewise definition of the absolute value:

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

In our case, $\blacksquare = x-5$. Hence,

$$|x-5| = \begin{cases} x-5, & \text{if } x-5 \geq 0 \implies x \geq 5 \\ -(x-5), & \text{if } x-5 < 0 \implies x < 5 \end{cases}$$

This tells us that our absolute value inequality splits into two possibilies:

For $x \geq 5$				For $x < 5$			
	$\lvert x-5\rvert$	$<$	4		$\lvert x-5\rvert$	$<$	4
$\implies$	$x-5$	$<$	4	$\implies$	$-(x-5)$	$<$	4
$\implies$	x	$<$	9	$\implies$	$x-5$	$>$	-4
				$\implies$	x	$>$	1

To summarize, we are okay with $x \geq 5$ as long as $x < 9$. Additionally, we are okay with $x < 5$ as long as $x > 1$. Combining this information, our solution interval is $(1, 9)$.

The approach we used in our solution to Example 2.6.7 is what I consider to be a "preferred method." This is because it reinforces the piecewise definition of the absolute value (which is something you must know to be successful in Calculus); however, it is best to show you another approach used by many other instructors and students.

How To: Solve Absolute Value Inequalities

To solve an absolute value inequality, $|\blacksquare| \lessgtr k$, where $\blacksquare$ is an algebraic expression involving a variable and $\lessgtr$ is either $>$, $\geq$, $<$, or $\leq$, we rely on the piecewise function definition of the absolute value. Namely,

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

This means that to solve $|\blacksquare| \lessgtr k$, we have two cases to consider:

If $\blacksquare \geq 0$,	If $\blacksquare < 0$,
then we solve	then we solve
$\blacksquare \lessgtr k$	$-\blacksquare \lessgtr k$

Time can be saved by paying attention to the inequality and asking yourself if it is always true or never true. Remember, an absolute value cannot be a negative number. Therefore, $|\blacksquare| < k$, where k is negative, never has a solution. Moreover, $|\blacksquare| > k$, where k is negative, will have a solution for any value of the variable. This is because an absolute value is always greater than a negative number.

First, we will need to know where the corresponding equality is true. That is, we need to solve $|x-5| = 4$. The process is almost identical to what we did in Example 2.6.7.

$$|x-5| = \begin{cases} x-5, & \text{if } x-5 \geq 0 \implies x \geq 5 \\ -(x-5), & \text{if } x-5 < 0 \implies x < 5 \end{cases}$$

This tells us that our absolute value equation splits into two possibilities:

For $x \geq 5$				For $x < 5$			
	$\lvert x-5 \rvert$	$=$	4		$\lvert x-5 \rvert$	$=$	4
$\implies$	$x-5$	$=$	4	$\implies$	$-(x-5)$	$=$	4
$\implies$	x	$=$	9	$\implies$	$x-5$	$=$	-4
				$\implies$	x	$=$	1

After determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x < 1, \quad 1 < x < 9, \quad \text{and } x > 9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table **2.6.1**.

Table 2.6.1

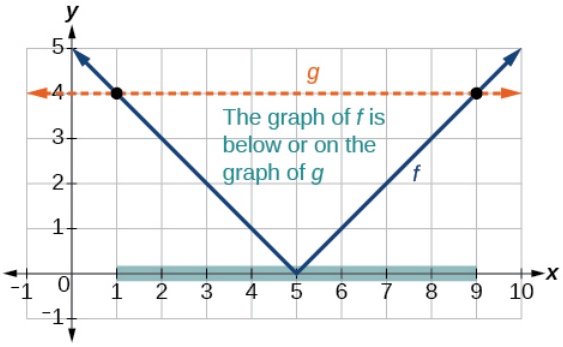
Interval test x	Test value for x	< 4 or > 4?	
$x < 1$	0	$\lvert 0-5 \rvert = 5$	Greater than
$1 < x < 9$	6	$\lvert 6-5 \rvert = 1$	Less than
$x > 9$	11	$\lvert 11-5 \rvert = 6$	Greater than

Because $1 < x < 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5| < 4$ is $1 < x < 9$, or $(1, 9)$.

To use a graph, we can sketch the function $f(x) = |x-5|$. To help us see where the outputs are 4, the line $g(x) = 4$ could also be sketched as in Figure **2.6.11**.

Figure 2.6.11

Graph to find the points satisfying an absolute value inequality.



We can see the following:

- The output values of the absolute value are equal to 4 at $x = 1$ and $x = 9$.
- The graph of f is below the graph of g on $1 < x < 9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1 < x < 9$. In interval notation, this would be the interval $(1, 9)$.

For absolute value inequalities,

$$\begin{array}{rrcl} & |x-A| & < & C \\ \Longrightarrow & -C < & x-A & < C \end{array} \qquad \begin{array}{rrcl} & |x-A| & > & C \\ & x-A & < & -C \quad \text{or} \quad x-A>C \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for Example **2.6.7**, we could use this alternative approach.

$$\begin{array}{rrcl} & |x-5| & < & 4 \\ \Longrightarrow & -4 < & x-5 & < 4 \qquad \text{(rewriting by removing the absolute value bars)} \\ \Longrightarrow & -4+5 < & x-5+5 & < 4+5 \qquad \text{(adding 5 to all three 'sides')} \\ \Longrightarrow & 1 < & x & < 9 \end{array}$$

? Checkpoint 2.6.7

Solve $|x+2| \leq 6$.

✔ Example 2.6.8: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine the x-values for which the function values are negative.

Solution

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$\begin{array}{rrcll} & -\frac{1}{2}|4x-5| & < & -3 & \text{(subtracting 3 from both sides)} \\ \Longrightarrow & |4x-5| & > & 6 & \text{(multiplying both sides by } -2 \text{ and switching the inequality sign)} \end{array}$$

Using the definition of the absolute value, we get

$$|4x-5| = \begin{cases} 4x-5, & \text{if } 4x-5 \geq 0 \implies x \geq \frac{5}{4} \\ -(4x-5), & \text{if } 4x-5 < 0 \implies x < \frac{5}{4} \end{cases}$$

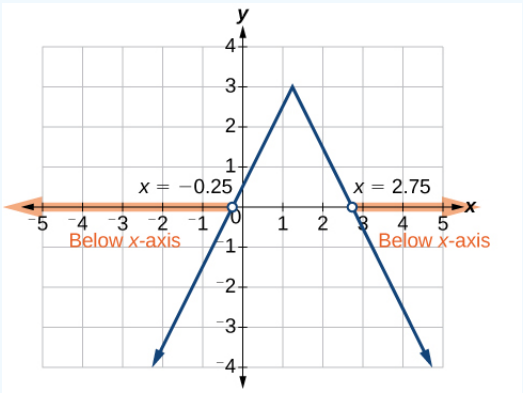
Thus,

$$\begin{array}{rrcl} & \text{For } x \geq \frac{5}{4} & & \\ & |4x-5| & > & 6 \\ \Longrightarrow & 4x-5 & > & 6 \\ \Longrightarrow & 4x & > & 11 \\ \Longrightarrow & x & > & \frac{11}{4} \end{array} \qquad \begin{array}{rrcl} & \text{For } x < \frac{5}{4} & & \\ & |4x-5| & > & 6 \\ \Longrightarrow & -(4x-5) & > & 6 \\ \Longrightarrow & 4x-5 & < & -6 \\ \Longrightarrow & 4x & < & -1 \\ \Longrightarrow & x & < & -\frac{1}{4} \end{array}$$

The leftmost solution tells us that as long as $x \geq \frac{5}{4}$, the original inequality is satisfied for $x > \frac{11}{4}$. The rightmost solution tells us that as long as $x < \frac{5}{4}$, the original inequality is satisfied for $x < -\frac{1}{4}$. Thus, our solution is $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$.

Had we decided to using the graphing method, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure **2.6.12**.

Figure 2.6.12



We observe that the graph of the function is below the x-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > \frac{11}{4}.$$

In interval notation, this would be $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$.

? Checkpoint 2.6.8

Solve $-2|k-4| \leq -6$.

https://math.libretexts.org/@go/page/197490

2.6.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Absolute value as a concept
  - Intermediate Algebra - Absolute Value: The Concepts
- Creating absolute value equations
  - Intermediate Algebra - Absolute Value: Forming Absolute Value Equations
- Solving absolute value equations
  - Equations - Overview - A Review for College Algebra - Solving Absolute Value Equations
- Solving absolute value inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Absolute Value Inequalities
- Interval notation for solutions
  - Logic & Set Theory in Review: Sets - Notation (Interval Notation)

KEY CONCEPTS

DEFINITIONS

- **Absolute Value Function:** The function $f(x) = |x|$, which gives the distance of x from zero on a number line. Algebraically, $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$.
- **Absolute Value Equation:** An equation in which the unknown variable appears within absolute value bars.
- **Absolute Value Inequality:** An inequality in which the unknown variable appears within the absolute value bars, typically of the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$.

COMMON MISTAKES

- **Assuming Absolute Value Equations Always Have Two Solutions:** While many absolute value equations yield two solutions (e.g., $|2x - 6| = 8$ in the introduction to solving equations), they can also have one solution (e.g., $|x - 3| = 0$) or no solutions (e.g., $|x - 2| = -\frac{1}{4}$ in Example 5).
- **Incorrectly Handling Negative Numbers in Absolute Value Equations/Inequalities:**
  - An equation of the form $|A| = k$ where $k < 0$ has no solution, because an absolute value cannot be negative.
  - An inequality of the form $|A| < k$ where $k < 0$ has no solution.
  - An inequality of the form $|A| > k$ where $k < 0$ is true for all values of A for which A is defined, because an absolute value is always non-negative and thus always greater than a negative number.
- **Forgetting to Switch Inequality Sign When Multiplying/Dividing by a Negative Number:** When solving absolute value inequalities algebraically, if a step involves multiplying or dividing both sides by a negative number, the direction of the inequality sign must be reversed.
- **Misinterpreting Conditions for Piecewise Definition:** When using the piecewise definition of absolute value, $|■| = ■$ if $■ \geq 0$ and $|■| = -■$ if $■ < 0$, students might incorrectly assume the condition always refers to $x \geq 0$ or $x < 0$ instead of the expression inside the absolute value bars ($■$).

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2.6.2: HOMEWORK

READING QUESTIONS

1. How is the absolute value function $f(x) = |x|$ defined as a piecewise function? What does each piece represent?
2. What does the expression $|x - a|$ represent geometrically on a number line?
3. Explain how the parameters a, h, and k in the function $g(x) = a|x - h| + k$ transform the graph of the basic absolute value function $f(x) = |x|$.
4. What is the "corner point" of an absolute value function's graph, and how can its coordinates be determined from the equation $g(x) = a|x - h| + k$?
5. When solving an absolute value equation such as $|ax + b| = c$, where c is a positive number, what are the two distinct linear equations that arise?
6. Under what condition does an absolute value equation of the form $|E| = c$ have no solution? Provide an example.
7. Explain the algebraic steps to solve an absolute value inequality of the form $|E| < c$ where $c > 0$. How does this differ from solving $|E| > c$?
8. What does it mean if the graph of an absolute value function $f(x)$ is below the x-axis for a certain interval of x-values? How does this relate to solving $f(x) < 0$?
9. If you are solving an absolute value inequality and you multiply or divide both sides by -1, what critical adjustment must be made to the inequality symbol?
10. In the piecewise definition of $|E|$, where E is an algebraic expression, when is $|E|$ equal to E, and when is it equal to $-E$?
11. How do you solve an absolute value equation?
12. How can you tell whether an absolute value function has two x-intercepts without graphing the function?
13. When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?
14. How can you use the graph of an absolute value function to determine the x-values for which the function values are negative?
15. How do you solve an absolute value inequality algebraically?

HOMEWORK

ALGEBRAIC

16. Describe all numbers x that are at a distance of 4 from the number 8. Express this using absolute value notation.
17. Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number -4. Express this using absolute value notation.
18. Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this using absolute value notation.
19. Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this using absolute value notation.

For the following exercises, solve the equations below and express the answer using set notation.

20. $|x + 3| = 9$
21. $|6 - x| = 5$
22. $|5x - 2| = 11$
23. $|4x - 2| = 11$
24. $2|4 - x| = 7$
25. $3|5 - x| = 5$
26. $3|x + 1| - 4 = 5$
27. $5|x - 4| - 7 = 2$
28. $0 = -|x - 3| + 2$
29. $2|x - 3| + 1 = 2$
30. $|3x - 2| = 7$
31. $|3x - 2| = -7$
32. $\left|\frac{1}{2}x - 5\right| = 11$
33. $\left|\frac{1}{3}x + 5\right| = 14$
34. $-\left|\frac{1}{3}x + 5\right| + 14 = 0$

For the following exercises, find the x- and y-intercepts of the graphs of each function.

35. $f(x) = 2|x + 1| - 10$
36. $f(x) = 4|x - 3| + 4$
37. $f(x) = -3|x - 2| - 1$
38. $f(x) = -2|x + 1| + 6$

For the following exercises, solve each inequality and write the solution in interval notation.

39. $|x - 2| > 10$
40. $2|v - 7| - 4 \geq 42$
41. $|3x - 4| \leq 8$
42. $|x - 4| \geq 8$
43. $|3x - 5| \geq 13$
44. $|3x - 5| \geq -13$
45. $\left|\frac{3}{4}x - 5\right| \geq 7$
46. $\left|\frac{3}{4}x - 5\right| + 1 \leq 16$

GRAPHICAL

For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

47. $y = |x - 1|$
48. $y = |x + 1|$
49. $y = |x| + 1$

For the following exercises, graph the given functions by hand.

50. $y = |x| - 2$
51. $y = -|x|$
52. $y = -|x| - 2$
53. $y = -|x - 3| - 2$
54. $f(x) = -|x - 1| - 2$
55. $f(x) = -|x + 3| + 4$
56. $f(x) = 2|x + 3| + 1$
57. $f(x) = 3|x - 2| + 3$

58. $f(x) = |2x - 4| - 3$
59. $f(x) = |3x + 9| + 2$
60. $f(x) = -|x - 1| - 3$
61. $f(x) = -|x + 4| - 3$
62. $f(x) = \frac{1}{2}|x + 4| - 3$

TECHNOLOGY

63. Use graphing technology to graph $f(x) = 10|x - 2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.
64. Use graphing technology to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

For the following exercises, graph each function using graphing technology. Specify the viewing window.

65. $f(x) = -0.1|0.1(0.2 - x)| + 0.3$
66. $f(x) = 4 \times 10^9 |x - (5 \times 10^9)| + 2 \times 10^9$

EXTENSIONS

For the following exercises, solve the inequality.

67. $|-2x - \frac{2}{3}(x + 1)| + 3 > -1$
68. If possible, find all values of a such that there are no x-intercepts for $f(x) = 2|x + 1| + a$.
69. If possible, find all values of a such that there are no y-intercepts for $f(x) = 2|x + 1| + a$.

REAL-WORLD APPLICATIONS

70. Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.
71. The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.
72. Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.
73. A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.
74. The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

2.7: Inverse Functions

Note to the Instructor (click to expand)

We begin this chapter with defining and finding the inverse of a function. This is a traditional treatment of inverses, including how to graph them and how to find their equations given the original function. We also include domain restriction (as we are about to encounter radical functions).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Arithmetic**
  - Mathematical Operations - Higher-Index Roots: Finding inverses of functions like $f(x) = 5x^3 + 1$ involves using cube roots, as seen in Example 2 where $r = \sqrt[3]{\frac{3V}{2\pi}}$ and $f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$. Understanding higher-index roots is crucial.
- **Graphing Functions**
  - Graphs of Functions: The ability to interpret points on a graph and to understand that the graph of an inverse function is a reflection of the original function's graph across the line $y = x$ is key (Example 6, Example 10, and several homework questions).

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.
- Restrict the domain to find the inverse of a polynomial function.
- Find the inverse of a rational function.

Hawk A.I. Section-Specific Tutor

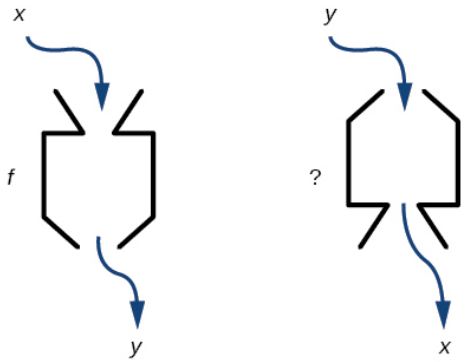
To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure **2.7.1** provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

Figure 2.7.1

Can a function "machine" operate in reverse?



Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

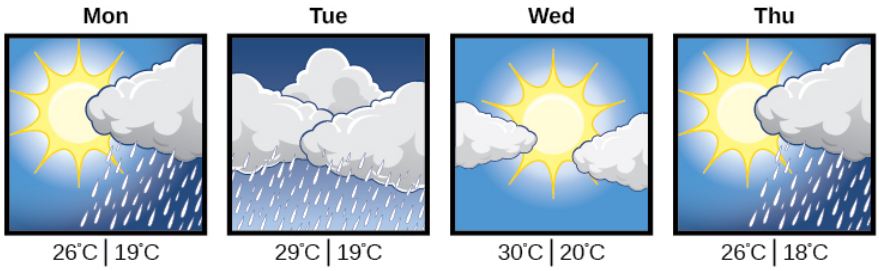
$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ \text{C}.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 2.7.2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

Figure 2.7.2



At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her Algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$\begin{aligned} 26 &= \frac{5}{9}(F - 32) \\ \implies 26 \cdot \frac{9}{5} &= F - 32 \\ \implies F &= 26 \cdot \frac{9}{5} + 32 \approx 79 \end{aligned}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Definition: Inverse Function

For any one-to-one function $y = f(x)$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. The notation f^{-1} is read "f inverse."

Caution

It is important to note that

$$f^{-1}(x) \neq \frac{1}{f(x)},$$

and not all functions have inverses.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x.$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Inverse Function Notation

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x."

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

Theorem

If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}\right) = x.$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Example 2.7.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if $f(2) = 4$, then $f^{-1}(4) = 2$; If $f(5) = 12$, then $f^(-1)(12) = 5$. Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

If we show the coordinate pairs from Example **2.7.1** in a table form, the input and output are clearly reversed. See Table **2.7.1**.

Table 2.7.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Checkpoint 2.7.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Example 2.7.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution

To determine if g is the inverse function of f, we just need to check if g "undoes" f.

$$\begin{aligned} g(f(x)) &= g\left(\frac{1}{x+2}\right) && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x+2}} - 2 && \text{(substituting)} \\ &= x + 2 - 2 && \left(\text{recognizing that } \frac{1}{1/a} = a\right) \\ &= x && \text{(simplifying)} \end{aligned}$$

so

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x} - 2\right) && \text{(substituting)} \\ &= \frac{1}{\left(\frac{1}{x} + 2\right) - 2} && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x}} && \text{(simplifying)} \\ &= x && \left(\text{recognizing that } \frac{1}{1/a} = a\right) \end{aligned}$$

Checkpoint 2.7.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Example 2.7.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution

$$f(g(x)) = f\left(\frac{x}{3}\right) = \left(\frac{x}{3}\right)^3 = \frac{x^3}{27} \neq x.$$

No, the functions are not inverses. The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{1/3}$, that is, the one-third is an exponent, not a multiplier.

Checkpoint 2.7.3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 2.7.3.

Figure 2.7.3

Domain and range of a function and its inverse

Theorem: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$, and the domain of $f(x)$ is the range of $f^{-1}(x)$.

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output **9** from the quadratic function corresponds to the inputs **3** and **–3**. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all!

To put it differently, the quadratic function is not a one-to-one function; it fails the Horizontal Line Test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the Horizontal Line Test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of f^{-1} = $[1, \infty)$.
- The domain of f^{-1} = range of f = $[0, \infty)$.

Q&A

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original

function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Example 2.7.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 2.7.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 2.7.2

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x)=c$	$f(x)=x$	$f(x)=x^2$	$f(x)=x^3$	$f(x)=\frac{1}{x}$
Reciprocal squared	**Cube root**	**Square root**	**Absolute value**	
$f(x)=\frac{1}{x^2}$	$f(x)=\sqrt[3]{x}$	$f(x)=\sqrt{x}$	$f(x)=\lvert x\rvert$	

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0,\infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0,\infty)$.

We can see that the absolute value and reciprocal-squared functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 2.7.4. They both would fail the Horizontal Line Test. However, if a function is restricted to a certain domain so that it passes the Horizontal Line Test, then in that restricted domain, it can have an inverse.

Figure 2.7.4

(a) Absolute value; (b) Reciprocal squared

Checkpoint 2.7.4

The domain of function f is $(1,\infty)$ and the range of function f is $(-\infty,-2)$. Find the domain and range of the inverse function.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 2.7.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 2.7.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

Table 2.7.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Checkpoint 2.7.5

Using Table 2.7.4, find and interpret each of the following:

a. $f(60)$
b. $f^{-1}(60)$

Table 2.7.4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We know that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Example 2.7.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 2.7.5. Find $g(3)$ and $g^{-1}(3)$.

Figure 2.7.5

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3,1)$ tells us that $g(3)=1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x)=3$. By looking for the output value 3 on the vertical axis, we find the point $(5,3)$ on the graph, which means $g(5)=3$, so by definition, $g^{-1}(3)=5$. See Figure 2.7.6.

Figure 2.7.6

Checkpoint 2.7.6

Using the graph in Figure 2.7.6 to answer the following questions.

a. Find $g^{-1}(1)$.
b. Estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula - for example, y as a function of x - we can often find the inverse function by solving to obtain x as a function of y.

How To: Find the Equation of an Invertible Function

Given a function, $f(x)$, we find the inverse (if it exists) by doing the following:

1. Let $y=f(x)$.
2. Solve for x.
3. Interchange x and y.

The result of these operations, if it is a function, is $f^{-1}(x)$.

Example 2.7.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature,

$$C=\frac{5}{9}(F-32).$$

Solution

$$\begin{aligned} C &= \frac{5}{9}(F-32) \\ \Rightarrow \quad \frac{9}{5}C &= F-32 && \left(\text{multiplying both sides by } \frac{9}{5}\right) \\ \Rightarrow \quad \frac{9}{5}C+32 &= F && (\text{adding 32 to both sides}) \end{aligned}$$

By solving in general, we have uncovered the inverse function. If

$$C=h(F)=\frac{5}{9}(F-32),$$

then

$$F=h^{-1}(C)=\frac{9}{5}C+32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Checkpoint 2.7.7

Solve for x in terms of y given $y = \frac{1}{3}(x-5)$.

Example 2.7.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution

$$\begin{aligned} y &= \frac{2}{x-3} + 4 && \text{(starting with the original equation)} \\ \Rightarrow \quad y-4 &= \frac{2}{x-3} && \text{(subtracting 4 from both sides)} \\ \Rightarrow \quad (x-3)(y-4) &= 2 && \text{(multiplying both sides by } (x-3)) \\ \Rightarrow \quad x-3 &= \frac{2}{y-4} && \text{(dividing both sides by } (y-4)) \\ \Rightarrow \quad x &= \frac{2}{y-4} + 3 && \text{(adding 3 to both sides)} \end{aligned}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Note that the domain and range of f in Example 2.7.8 exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 2.7.5.

Table 2.7.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 2.7.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

Solution

$$\begin{aligned} y &= 2 + \sqrt{x-4} && \text{(starting with the original equation)} \\ \Rightarrow \quad y-2 &= \sqrt{x-4} && \text{(subtracting 2 from both sides)} \\ \Rightarrow \quad (y-2)^2 &= x-4 && \text{(squaring both sides)} \\ \Rightarrow \quad (y-2)^2 + 4 &= x && \text{(adding 4 to both sides)} \end{aligned}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

The formula we found for $f^{-1}(x)$ in Example 2.7.9 looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

? Checkpoint 2.7.9

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 2.7.7.

Figure 2.7.7

Quadratic function with domain restricted to $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the Horizontal Line Test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 2.7.8.

Figure 2.7.8

Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

✓ Example 2.7.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 2.7.9, sketch a graph of $f^{-1}(x)$.

Figure 2.7.9

Solution

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 2.7.10.

Figure 2.7.10

The function and its inverse, showing reflection about the identity line

? Checkpoint 2.7.10

Draw graphs of the functions f and f^{-1} from Example 2.7.8.

Q&A

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x.$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

How To: Restrict Domain and Force Invertibility

Given a function that is not one-to-one,

1. Restrict the domain by determining a domain on which the original function is one-to-one (i.e., a domain where the function passes the Horizontal Line Test).
2. Replace $f(x)$ with y.
3. Solve for x interchange x and y.
4. Rename the function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

Example 2.7.11: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f:

a. $f(x) = (x-4)^2$, $x \geq 4$
b. $f(x) = (x-4)^2$, $x \leq 4$

Solution

The original function $f(x) = (x-4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one. See Figure 2.7.11.

Figure 2.7.11

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$\begin{aligned} & y = (x-4)^2 && \text{(letting } y = f(x)\text{)} \\ \Longrightarrow \quad & \pm\sqrt{y} = x-4 && \text{(Extraction of Roots)} \\ \Longrightarrow \quad & 4 \pm \sqrt{y} = x && \text{(adding 4 to both sides)} \\ \Longrightarrow \quad & 4 \pm \sqrt{x} = y && \text{(exchanging } y \text{ and } x\text{)} \end{aligned}$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of x and y for the original $f(x)$, we need to look at the domain of the original function. When we reversed the roles of x and y, this gave us the values y could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

a. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the "+" case:

$$f^{-1}(x) = 4 + \sqrt{x}.$$

b. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the "–" case:

$$f^{-1}(x) = 4 - \sqrt{x}.$$

On the graphs in Figure 2.7.12, we see the original function from Example 2.7.11 graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4, 0)$ is on the graph of f and the coordinate pair $(0, 4)$ is on the graph of f^{-1}. For any coordinate pair, if (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y = x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y = x$.

Figure 2.7.12

Example 2.7.12: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

Solution

We can see this is a parabola with vertex at $(2, -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we repeat what we did in Example 2.7.11.

$$\begin{aligned} & y = (x-2)^2-3 && \text{(substituting } y=f(x)\text{)} \\ \Rightarrow\quad & y+3 = (x-2)^2 && \text{(adding 3 to both sides)} \\ \Rightarrow\quad & \pm\sqrt{y+3} = x-2 && \text{(Extraction of Roots)} \\ \Rightarrow\quad & 2\pm\sqrt{y+3} = x && \text{(adding 2 to both sides)} \\ \Rightarrow\quad & 2\pm\sqrt{x+3} = y && \text{(interchanging } x \text{ and } y\text{)} \end{aligned}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the "+" case. Therefore,

$$f^{-1}(x) = 2+\sqrt{x+3}.$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just have easily opted to restrict the domain on $x \leq 2$, in which case $f^{-1}(x) = 2-\sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 2.7.13. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2,-3)$ is on the graph of f and the coordinate pair $(-3,2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = [2,\infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3,\infty).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.

Figure 2.7.13

? Checkpoint 2.7.12

Find the inverse of the function $f(x) = x^2+1$, on the domain $x \geq 0$.

Solving Applications Involving the Inverse of Polynomial Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

✓ Example 2.7.13: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3.$$

Find the inverse of this function. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi \approx 3.14$.

Solution

Start with the given function for V. Notice that the meaningful domain for the function is $r \geq 0$ since negative radii would not make sense in this context. Also note the range of the function (hence, the domain of the inverse function) is $V \geq 0$. Solve for r in terms of V, using the method outlined previously.

$$\begin{aligned} V &= \frac{2}{3}\pi r^3 \\ \Rightarrow \quad \frac{3}{2}V &= \pi r^3 \\ \Rightarrow \quad \frac{3}{2\pi}V &= r^3 \\ \Rightarrow \quad \sqrt[3]{\frac{3}{2\pi}V} &= r \end{aligned}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi \approx 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &\approx \sqrt[3]{\frac{3(100)}{2(3.14)}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example 2.7.14: Finding the Inverse of a Rational Function

The function $C = \frac{20+0.4n}{100+n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution

We first want the inverse of the function.

$$\begin{aligned} C &= \frac{20+0.4n}{100+n} \\ \Rightarrow \quad C(100+n) &= 20+0.4n \\ \Rightarrow \quad 100C+Cn &= 20+0.4n \\ \Rightarrow \quad 100C-20 &= 0.4n-Cn \\ \Rightarrow \quad 100C-20 &= (0.4-C)n \\ \Rightarrow \quad \frac{100C-20}{0.4-C} &= n \end{aligned}$$

Now evaluate this function for $C = 0.35$ (35%).

$$n = \frac{100(0.35)-20}{0.4-0.35} = \frac{15}{0.05} = 300.$$

We can conclude that 300 mL of the 40% solution should be added.

Checkpoint 2.7.14

Find the inverse of the function $f(x) = \dfrac{x+3}{x-2}$.

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2.7.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Inverse function (concept)
  - Intermediate Algebra - Functions: The Concept of Inverse Functions
- $f^{-1}(x)$ notation
  - Intermediate Algebra - Functions: Inverse Function Notation
- Verifying inverse functions
  - Inverse Functions - Verifying That Two Functions Are Inverse Functions
- One-to-one function
  - One-to-One Functions and Their Inverses - A Brief Review for Precalculus
- Domain and Range of inverse functions
  - Inverse Functions - Finding the Domain and Range of Inverse Functions
- Finding inverse from a formula
  - Inverse Functions - Finding Inverse Functions and Their Graphs
  - Inverse Functions - Finding and Evaluating Inverse Functions
  - Intermediate Algebra - Functions: Finding an Equation for the Inverse Function
- Graph of inverse function
  - Intermediate Algebra - Functions: Graphing the Inverse Function

KEY CONCEPTS

DEFINITIONS

- **Inverse Function:** For a one-to-one function $y = f(x)$, a function $f^{-1}(x)$ is an inverse function of f if $f^{-1}(f(x)) = x$ for all x in the domain of f, and $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} (which is the range of f). The input of the original function becomes the output of the inverse function, and the output of the original function becomes the input of the inverse function.
- **Inverse Function Notation (f^{-1}):** The notation used to represent the inverse of a function f. It is read as "f inverse."
- **Identity Line:** The line $y = x$. The graph of a function and its inverse are reflections of each other across this line.

THEOREMS

- **Inverse Function Property:** If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. (And $f^{-1}(f(x)) = x$ for all x in the domain of f).
- **Domain and Range of Inverse Functions:** The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$, and the domain of $f(x)$ is the range of $f^{-1}(x)$.
- **Condition for Existence of an Inverse Function:** In order for a function to have an inverse, it must be a one-to-one function.
- **Forcing Invertibility by Restricting Domain**: If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

COMMON MISTAKES

- **Confusing Inverse Function Notation with Reciprocal:** The notation $f^{-1}(x)$ for an inverse function does not mean $\frac{1}{f(x)}$. These are generally different concepts.
- **Assuming All Functions Have Inverses:** Not all functions have inverses. A function must be one-to-one to have an inverse function.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions:** When finding an inverse for a function that is not naturally one-to-one (like $f(x) = x^2$), failing to restrict the domain of the original function appropriately will result in an inverse relation that is not a function.
- **Incorrectly Determining the Domain of the Inverse Function:** The domain of f^{-1} is the range of f. Sometimes, after algebraically finding a formula for $f^{-1}(x)$, the apparent domain of this new formula might be larger than the actual range of f. The domain of f^{-1} must be restricted to match the range of f.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions**: Attempting to find an inverse for a function like $f(x) = x^2$ without first restricting its domain to an interval where it is one-to-one. This leads to an inverse relation that is not a function (e.g., $y = \pm\sqrt{x}$).
- **Incorrectly Choosing the Branch of the Inverse**: After restricting the domain of a function (e.g., a quadratic) and finding an inverse involving a square root, selecting the wrong sign (positive or negative) for the square root, which would not correspond to the original restricted domain.
- **Not Verifying if the Original Function is One-to-One**: Proceeding to find an inverse without first confirming that the function (or its restricted version) is one-to-one, for example, by using the Horizontal Line Test.
- **Domain/Range Confusion for Inverse Functions**: Incorrectly stating the domain or range of the inverse function. Remember, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

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2.7.2: HOMEWORK

READING QUESTIONS

1. What does it mean for a function f^{-1} to be the inverse of a function f?
2. How can you algebraically verify if two functions, $f(x)$ and $g(x)$, are inverses of each other?
3. Is the notation $f^{-1}(x)$ the same as $\frac{1}{f(x)}$? Explain.
4. What condition must a function satisfy to have an inverse function? How can this be checked graphically?
5. If a function is not one-to-one, what can sometimes be done to allow an inverse function to be found?
6. How are the domain and range of a function f related to the domain and range of its inverse f^{-1}?
7. If the point (a, b) is on the graph of a one-to-one function f, what corresponding point must be on the graph of f^{-1}?
8. Describe the graphical relationship between a function and its inverse. What line is involved?
9. Outline the algebraic steps to find the formula for an inverse function $f^{-1}(x)$ if you are given the formula for $f(x)$.
10. If $f(x) = \sqrt{x}$ has the domain $[0, \infty)$, its inverse is $f^{-1}(x) = x^2$. What must be the domain of this inverse function $f^{-1}(x)$, and why?
11. Can a function have more than one inverse function over the same domain?
12. If you are given a table of values for a one-to-one function $f(t)$ showing distance traveled in t minutes, how would you find $f^{-1}(d)$ for a given distance d? What would $f^{-1}(d)$ represent?
13. Name a toolkit function (other than the identity function) that is its own inverse.
14. Describe why the Horizontal Line Test is an effective way to determine whether a function is one-to-one?
15. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?
16. Can a function be its own inverse? Explain.
17. Are one-to-one functions either always increasing or always decreasing? Why or why not?
18. How do you find the inverse of a function algebraically?
19. Why is it necessary to restrict the domain of a function like $f(x) = x^2$ before finding its inverse?
20. If you restrict the domain of $f(x) = (x-2)^2 - 3$ to $x \geq 2$, what will be the formula for its inverse $f^{-1}(x)$? What if you restrict the domain to $x \leq 2$?
21. When finding the inverse of a quadratic function, such as $y = ax^2$, you might arrive at $\pm\sqrt{\frac{y}{a}} = x$. How do you decide whether to use the positive or negative root for the inverse function?
22. What is the relationship between the graph of a function and the graph of its inverse with respect to the line $y = x$?
23. Outline the general steps to find the inverse of a polynomial function whose domain has been appropriately restricted.
24. Outline the general steps to find the inverse of a rational function of the form $f(x) = \frac{ax+b}{cx+d}$.
25. If $f(x) = 5x^3 + 1$, and you find its inverse $f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$, how can you verify that these functions are indeed inverses?
26. In the gravel mound problem (Example 5), why is the domain of the volume function $V(r) = \frac{2}{3}\pi r^3$ considered to be $r \geq 0$? What does this imply for the domain of its inverse function?
27. Can all polynomial functions have an inverse if their domain is restricted? Explain why or why not, considering a constant function as an example.
28. When solving for n in terms of C in Example 6 for the function $C = \frac{20+0.4n}{100+n}$, what algebraic steps are involved in isolating n?
29. Explain why we cannot find inverse functions for all polynomial functions.
30. Why must we restrict the domain of a quadratic function when finding its inverse?
31. When finding the inverse of a radical function, what restriction will we need to make?
32. The inverse of a quadratic function will always take what form?

HOMEWORK

ALGEBRAIC

33. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

34. $f(x) = x + 3$
35. $f(x) = x + 5$
36. $f(x) = 2 - x$
37. $f(x) = 3 - x$
38. $f(x) = \frac{x}{x+2}$
39. $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

40. $f(x) = (x+7)^2$
41. $f(x) = (x-6)^2$
42. $f(x) = x^2 - 5$
43. Given $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:
    a. Find $f(g(x))$ and $g(f(x))$.
    b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

44. $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$
45. $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

For the following exercises, find the inverse of the function on the given domain.

46. $f(x) = (x-4)^2$, $[4, \infty)$
47. $f(x) = (x+2)^2$, $[-2, \infty)$
48. $f(x) = (x+1)^2 - 3$, $[-1, \infty)$
49. $f(x) = 2 - \sqrt{3+x}$, $[-3, \infty)$
50. $f(x) = 3x^2 + 5$, $(-\infty, 0]$
51. $f(x) = 12 - x^2$, $[0, \infty)$
52. $f(x) = 9 - x^2$, $[0, \infty)$
53. $f(x) = 2x^2 + 4$, $[0, \infty)$

For the following exercises, find the inverse of the functions.

54. $f(x) = x^3 + 5$
55. $f(x) = 3x^3 + 1$
56. $f(x) = 4 - x^3$
57. $f(x) = 4 - 2x^3$

For the following exercises, find the inverse of the functions.

58. $f(x) = \sqrt{2x+1}$
59. $f(x) = \sqrt{3-4x}$
60. $f(x) = 9 + \sqrt{4x-4}$
61. $f(x) = \sqrt{6x-8} + 5$
62. $f(x) = 9 + 2\sqrt[3]{x}$
63. $f(x) = 3 - \sqrt[3]{x}$
64. $f(x) = \frac{2}{x+8}$
65. $f(x) = \frac{3}{x-4}$
66. $f(x) = \frac{x+3}{x+7}$
67. $f(x) = \frac{x-2}{x+7}$
68. $f(x) = \frac{3x+4}{5-4x}$
69. $f(x) = \frac{5x+1}{2-5x}$
70. $f(x) = x^2 + 2x$, $[-1, \infty)$
71. $f(x) = x^2 + 4x + 1$, $[-2, \infty)$
72. $f(x) = x^2 - 6x + 3$, $[3, \infty)$

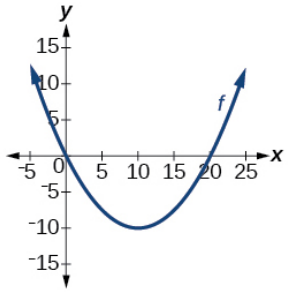
GRAPHICAL

For the following exercises, use graphing technology to determine whether each function is one-to-one.

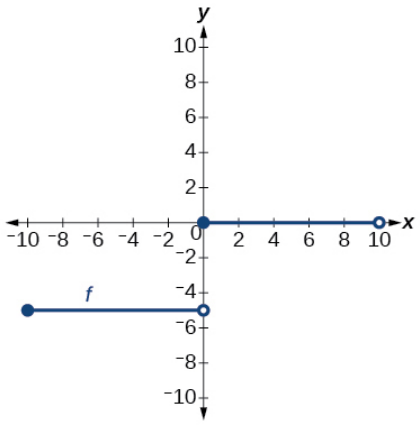
73. $f(x) = \sqrt{x}$
74. $f(x) = \sqrt[3]{3x+1}$
75. $f(x) = -5x + 1$
76. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

77.

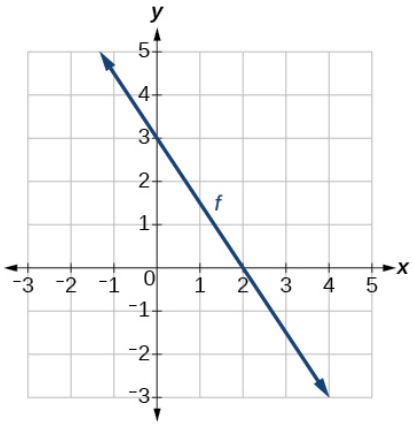


78.



For the following exercises, use the graph of f shown in Figure 2.7.2.1.

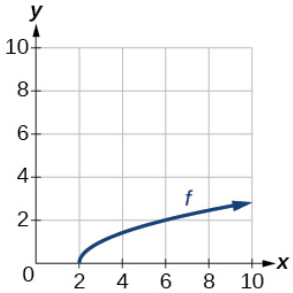
Figure 2.7.2.1



79. Find $f(0)$.
80. Solve $f(x) = 0$.
81. Find $f^{-1}(0)$.
82. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in Figure 2.7.2.2.

Figure 2.7.2.2



83. Sketch the graph of f^{-1}.
84. Find $f(6)$ and $f^{-1}(2)$.
85. If the complete graph of f is shown, find the domain of f.
86. If the complete graph of f is shown, find the range of f.

For the following exercises, find the inverse of the function and graph both the function and its inverse.

87. $f(x) = x^2 + 2,\ x \geq 0$
88. $f(x) = 4 - x^2,\ x \geq 0$
89. $f(x) = (x+3)^2,\ x \geq -3$
90. $f(x) = (x-4)^2,\ x \geq 4$
91. $f(x) = x^3 + 3$
92. $f(x) = 1 - x^3$
93. $f(x) = x^2 + 4x,\ x \geq -2$
94. $f(x) = x^2 - 6x + 1,\ x \geq 3$
95. $f(x) = \frac{2}{x}$
96. $f(x) = \frac{1}{x^2},\ x \geq 0$

For the following exercises, use a graph to help determine the domain of the functions.

97. $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$
98. $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$
99. $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$
100. $f(x) = \sqrt{\frac{x^2-x-20}{x-2}}$
101. $f(x) = \sqrt{\frac{9-x^2}{x+4}}$

NUMERIC

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

102. If $f(6) = 7$, find $f^{-1}(7)$.
103. If $f(3) = 2$, find $f^{-1}(2)$.
104. If $f^{-1}(-4) = -8$, find $f(-8)$.
105. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in Table 2.7.2.1 to evaluate or solve.

Table 2.7.2.1

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

106. Find $f(1)$.
107. Solve $f(x) = 3$.
108. Find $f^{-1}(0)$.
109. Solve $f^{-1}(x) = 7$.
110. Use the tabular representation of f in Table 2.7.2.2 to create a table for $f^{-1}(x)$.

Table 2.7.2.2

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

TECHNOLOGY

For the following exercises, find the inverse function. Then, graph the function and its inverse.

111. $f(x) = \frac{3}{x-2}$
112. $f(x) = x^3 - 1$
113. Find the inverse function of $f(x) = \frac{1}{x-1}$. Use graphing technology to find its domain and range. Write the domain and range in interval notation.

For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with y-coordinates given.

114. $f(x) = x^3 - x - 2,\ y = 1, 2, 3$
115. $f(x) = x^3 + x - 2,\ y = 0, 1, 2$
116. $f(x) = x^3 + 3x - 4,\ y = 0, 1, 2$
117. $f(x) = x^3 + 8x - 4,\ y = -1, 0, 1$
118. $f(x) = x^4 + 5x + 1,\ y = -1, 0, 1$

EXTENSIONS

For the following exercises, find the inverse of the functions with a, b, c positive real numbers.

119. $f(x) = ax^3 + b$
120. $f(x) = x^2 + bx$
121. $f(x) = \sqrt{ax^2 + b}$
122. $f(x) = \sqrt[3]{ax + b}$
123. $f(x) = \frac{ax+b}{x+c}$

REAL-WORLD APPLICATIONS

124. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.
125. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.
126. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

For the following exercises, determine the function described and then use it to answer the question.

127. An object dropped from a height of 200 meters has a height, $h(t)$, in meters after t seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express t as a function of height, h, and find the time to reach a height of 50 meters.
128. An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.
129. The volume, V, of a sphere in terms of its radius, r, is given by $V(r) = \frac{4}{3}\pi r^3$. Express r as a function of V, and find the radius of a sphere with volume of 200 cubic feet.
130. The surface area, A, of a sphere in terms of its radius, r, is given by $A(r) = 4\pi r^2$. Express r as a function of A, and find the radius of a sphere with a surface area of 1000 square inches.
131. A container holds 100 ml of a solution that is 25 ml acid. If n ml of a solution that is 60% acid is added, the function $C(n) = \frac{25+.6n}{100+n}$ gives the concentration, C, as a function of the number of ml

added, n. Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.

132. The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.
133. The volume of a cylinder , V, in terms of radius, r, and height, h, is given by $V = \pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.
134. The surface area, A, of a cylinder in terms of its radius, r, and height, h, is given by $A = 2\pi r^2 + 2\pi r h$. If the height of the cylinder is 4 feet, express the radius as a function of A and find the radius if the surface area is 200 square feet.
135. The volume of a right circular cone, V, in terms of its radius, r, and its height, h, is given by $V = \frac{1}{3}\pi r^2 h$. Express r in terms of V if the height of the cone is 12 inches and find the radius of a cone with volume of 50 cubic inches.
136. Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

CHAPTER OVERVIEW

Chapter 3: Coordinate Trigonometry

3.1: The Trigonometric Functions - Coordinate Definition

3.1.1: Resources and Key Concepts

3.1.2: Homework

3.2: Algebra with Trigonometric Functions

3.2.1: Resources and Key Concepts

3.2.2: Homework

3.3: Introduction to Trigonometric Identities

3.3.1: Resources and Key Concepts

3.3.2: Homework

3.4: Introduction to Proofs in Trigonometry

3.4.1: Resources and Key Concepts

3.4.2: Homework

3.1: The Trigonometric Functions - Coordinate Definition

Note to the Instructor (click to expand)

The definition of the trigonometric functions introduced in this section focuses on trigonometric functions of *any* angle (in degrees) - not just acute, obtuse, or positive angles; however, if given an angle (rather than a coordinate) and asked to evaluate a trigonometric function, the angle will strictly be a *special angle*. Therefore, using technology (e.g., calculators) to evaluate any trigonometric function in this section is *strictly avoided* so as to get the student familiar with these functions at a fundamental level. Simply stated, **students should not require a calculator to evaluate any of the trigonometric functions within this entire chapter!**

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Rationalizing Denominators and Numerators: This skill is required and is discussed in a note within the section regarding conventions for simplifying trigonometric values.
- **An Overview of Functions**
  - The Definition of a Function: This section defines the trigonometric ratios as functions of an angle, so a prior understanding of what a function is (with independent and dependent variables) is necessary.
  - Evaluating Functions: This skill is explicitly tested in the "Skills Refresher" exercises for polynomial, radical, rational, and exponential functions, and the concept is extended to trigonometric functions throughout the section.
  - The Domain and Range of a Function: When introducing the trigonometric functions, domain and range must be considered.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Find the value of a trigonometric function of an angle given a point on the terminal side of the angle.
- Use the coordinate definition of the trigonometric functions to answer a conceptual question about a trigonometric function.
- Determine the quadrants in which an angle could terminate.
- Evaluate the trigonometric functions at the special angles and the quadrantal angles using the coordinate definition of the trigonometric functions.
- Find the value of a trigonometric function given one of the other trigonometric values.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

- Compare the values of trigonometric functions at "non-special" angles without using technology.
- Use the quadrant in which an angle terminates to restrict the sign of a trigonometric function.

Defining Trigonometric Functions in Terms of Coordinates

Armed with our knowledge from the previous chapter, we now define the **trigonometric functions**. We will be viewing the trigonometric functions from three different perspectives in this course - **coordinate trigonometry**, right triangle trigonometry, and unit circle trigonometry.[1] For the most part, each perspective is mathematically equivalent to the others; however, each has an inherent weakness as a standalone perspective of the trigonometric functions. While you can find a favorite perspective and stick with that one for most of the course, understanding all three is the only way to succeed in Trigonometry.

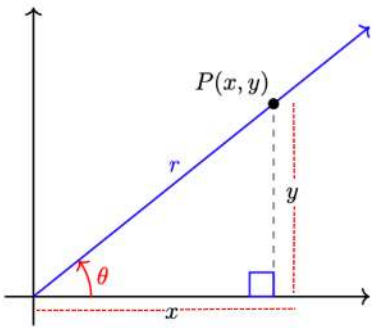
Definition: Trigonometric Functions (Coordinate Definition)

Let θ be an angle in standard position with the point $P(x,y)$ on the terminal side of θ, where P is not the origin. Define $r = \sqrt{x^2+y^2}$. The trigonometric functions of the angle θ are defined as follows:

Function Name	Function Notation		Definition	Function Name	Function Notation		Definition
The sine of θ	$\sin(\theta)$	$=$	$\frac{y}{r}$	The cosecant of θ	$\csc(\theta)$	$=$	$\frac{r}{y}$
The cosine of θ	$\cos(\theta)$	$=$	$\frac{x}{r}$	The secant of θ	$\sec(\theta)$	$=$	$\frac{r}{x}$
The tangent of θ	$\tan(\theta)$	$=$	$\frac{y}{x}$	The cotangent of θ	$\cot(\theta)$	$=$	$\frac{x}{y}$

A few notes are in order. First, the trigonometric functions are simply the ratios of x, y, and r, where $P(x,y)$ is a point on the terminal side of θ and r is the distance from the origin to the point P (see Figure 3.1.1 below).

Figure 3.1.1



Second, because it's the distance from the origin to P, r is always positive. However, x and y can be positive, negative, or zero (but not both zero), depending on the angle θ. For example, in the second quadrant, x is negative, but y is positive, so the cosine and the tangent of angles between $90°$ and $180°$ are negative, but their sines are positive.

Third, referencing Figure 3.1.1, since the tangent function is defined to be $\frac{y}{x}$ and the terminal side of θ is always a ray emanating from the origin, we get that

$$\frac{y}{x} = \frac{\text{rise}}{\text{run}} = \text{slope of the terminal side of } \theta.$$

The interpretation of the value of the tangent function at θ being the slope of the ray that is the terminal side of θ grants us a potent conceptual understanding of the tangent function. Specifically, since a vertical line has an undefined slope, it makes sense that the

tangent function is undefined when $x = 0$. Any ray emanating from the origin and going through a point $P(0, y)$, where $y \neq 0$, will be vertical. While the secant function is also undefined when $x = 0$ (and the cosecant and cotangent functions are undefined when $y = 0$), we will hold off on gaining a conceptual understanding of why until we familiarize ourselves with more of the basics of the trigonometric functions.

Note: Angles are in Standard Position

Unless otherwise stated, all angles from this point forward will be considered angles in standard position.

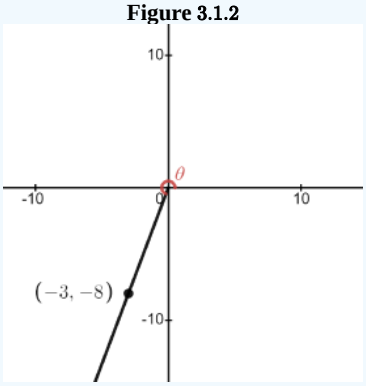
Example 3.1.1

Find the values of the six trigonometric functions if $P(-3, -8)$ is on the terminal side of θ.

Solution

It's a good habit in Trigonometry to start by sketching a graph of the situation.

Figure 3.1.2



We have $x = -3$, $y = -8$, and $r = \sqrt{(-3)^2 + (-8)^2} = \sqrt{9 + 64} = \sqrt{73}$. Therefore,

$$\sin(\theta) = \frac{y}{r} = \frac{-8}{\sqrt{73}} = -\frac{8}{\sqrt{73}} \qquad \csc(\theta) = \frac{r}{y} = \frac{\sqrt{73}}{-8} = -\frac{\sqrt{73}}{8}$$

$$\cos(\theta) = \frac{x}{r} = \frac{-3}{\sqrt{73}} = -\frac{3}{\sqrt{73}} \qquad \sec(\theta) = \frac{r}{x} = \frac{\sqrt{73}}{-3} = -\frac{\sqrt{73}}{3}$$

$$\tan(\theta) = \frac{y}{x} = \frac{-8}{-3} = \frac{8}{3} \qquad \cot(\theta) = \frac{x}{y} = \frac{-3}{-8} = \frac{3}{8}$$

Note: Rationalizing Denominators

In Algebra, we learned to rationalize denominators so we could "simplify" expressions like $-\frac{8}{\sqrt{73}}$; however, in Trigonometry, such manipulations often are pointless. The rule of thumb I will adhere to in this textbook is to rationalize denominators *if* the denominator involves a variable (e.g., $\frac{3}{\sqrt{x-2}}$). In all other cases (specifically, when the numerator and denominator do *not* contain variables), I will only rationalize denominators if there is an "obvious" benefit.

In the case of $-\frac{8}{\sqrt{73}}$, if we tried to rationalize the denominator, we would get

$$-\frac{8}{\sqrt{73}} \cdot \frac{\sqrt{73}}{\sqrt{73}} = -\frac{8\sqrt{73}}{73},$$

which is no better than the original fraction.

Checkpoint 3.1.1

a. Find the equation of the terminal side of the angle in the previous example. (Hint: The terminal side lies on a line that goes through the origin and the point $(-3, -8)$).

b. Show that the point $P'(-6, -16)$ also lies on the terminal side of the angle.

c. Compute the trigonometric ratios for θ using the point P' instead of P.

Answers

a. $y = \frac{8}{3}x$

b. $(-6, -16)$ satisfies $y = \frac{8}{3}x$. That is, the equation $-16 = \frac{8}{3}(-6)$ is true.

c. $r^2 = (-6)^2 + (-16)^2 = 292$, so $r = \sqrt{292} = 2\sqrt{73}$. Then

$$\sin(\theta) = \frac{y}{r} = \frac{-16}{2\sqrt{73}} = -\frac{8}{\sqrt{73}} \qquad \csc(\theta) = \frac{r}{y} = \frac{2\sqrt{73}}{-16} = -\frac{\sqrt{73}}{8}$$

$$\cos(\theta) = \frac{x}{r} = \frac{-6}{2\sqrt{73}} = -\frac{3}{\sqrt{73}} \qquad \sec(\theta) = \frac{r}{x} = \frac{2\sqrt{73}}{-6} = -\frac{\sqrt{73}}{3}$$

$$\tan(\theta) = \frac{y}{x} = \frac{-16}{-6} = \frac{8}{3} \qquad \cot(\theta) = \frac{x}{y} = \frac{-6}{-16} = \frac{3}{8}$$

Example 3.1.2

Find the values of $\cos(\theta)$ and $\tan(\theta)$ if θ is an obtuse angle with $\sin(\theta) = \frac{1}{3}$.

Solution

Because θ is obtuse, the terminal side of the angle lies in the second quadrant, as shown in Figure 3.1.3.

Figure 3.1.3

Because $\sin(\theta) = \frac{1}{3}$, we know that $\frac{y}{r} = \frac{1}{3}$, so we can choose a point P with $y = 1$ and $r = 3$. To find $\cos(\theta)$ and $\tan(\theta)$, we need to know the value of x. From the Pythagorean Theorem,

$$\begin{aligned}
& & x^2 + 1^2 &= 3^2 && \text{(substitute known values)} \\
\xRightarrow{\text{Arithmetic}} & & x^2 + 1 &= 9 && \text{(Laws of Exponents)} \\
\xRightarrow{\cancel{\text{Arithmetic}} \to \text{Algebra}} & & x^2 &= 8 && \text{(subtract 1 from both sides)} \\
\xRightarrow{\cancel{\text{Arithmetic}} \to \text{Algebra}} & & x &= \pm\sqrt{8} && \text{(Extraction of Roots)} \\
\xRightarrow{\cancel{\text{Arithmetic}} \to \text{Algebra}} & & x &= \pm 2\sqrt{2} && \text{(simplifying radicals)}
\end{aligned}$$

Remembering that x is negative in the second quadrant, we get that $x = -2\sqrt{2}$. Thus,

$$\cos(\theta) = \frac{x}{r} = \frac{-2\sqrt{2}}{3} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} = \frac{-1}{2\sqrt{2}}$$

Example 3.1.2 illustrates a common theme in Trigonometry - we will often use the quadrant of the terminal side of an angle to inform our decision on the sign of x- and y-values.

If given a negative trigonometric ratio, as in Checkpoint 3.1.2, it is important to remember that the value of r is always non-negative.[2]

Checkpoint 3.1.2

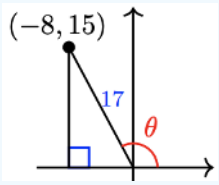
a. Sketch an obtuse angle θ whose cosine is $\frac{-8}{17}$.

b. Find the cosecant and the tangent of θ.

Answers

a.

Figure 3.1.4



b. $\csc(\theta) = \frac{17}{15}, \tan(\theta) = \frac{-15}{8}$

Our use of function notation (e.g., the parentheses in $\sin(\theta) = \frac{y}{r}$) is purposeful. This is because, a given angle θ (which is considered the independent variable) returns a single value for the associated trigonometric ratio (which we consider to be the dependent variable). Thus, by definition, the trigonometric functions are... functions (of angles).

Caution: Notational Abuse

We will sometimes write $\cos\theta$ when we really mean $\cos(\theta)$. This is a common, but accepted, abuse of notation reserved specifically for the trigonometric functions (and, at times, the logarithmic functions).

A common mistake is to think of $\cos\theta$ or $\cos(\theta)$ as a product, cos *times* θ, but this makes no sense, because "cos" by itself has no meaning. Remember that $\cos\theta$ represents a single number, namely the output of the cosine function.

Trigonometric Values at Special Angles

As was mentioned earlier in this textbook, special triangles play a crucial role in Trigonometry. In fact, the angles $30°$, $45°$, and $60°$ are just a few of the so-called **special angles** because we can express the exact values of their trigonometric functions in terms of radicals. In the following example, we explore the values of the trigonometric functions at one of the special angles.

Example 3.1.3

Find the values of all six trigonometric functions of $30°$.

Solution

Like any good Trigonometry student, we start with a sketch.

Figure 3.1.5

Unfortunately, we were not given a value for a point on the terminal side of $\theta = 30°$; however, recall the following figure from our exploration of the special triangles last chapter.

Figure 3.1.6

Rotating this special triangle counterclockwise by $90°$, we can visualize fitting it to our first sketch, almost like a jigsaw puzzle.

Figure 3.1.7

If we let $a = 1$ (a completely arbitrary choice), we get

$$x = \sqrt{3}, \quad y = 1, \text{ and } \quad r = 2.$$

Therefore,

$$\begin{aligned} \sin(30°) &= \frac{y}{r} = \frac{1}{2} & \csc(30°) &= \frac{r}{y} = 2 \\ \cos(30°) &= \frac{x}{r} = \frac{\sqrt{3}}{2} & \sec(30°) &= \frac{r}{x} = \frac{2}{\sqrt{3}} \\ \tan(30°) &= \frac{y}{x} = \frac{1}{\sqrt{3}} & \cot(30°) &= \frac{x}{y} = \sqrt{3} \end{aligned}$$

At this point, you should be asking yourself why we arbitrarily chose $a = 1$ in Example **3.1.3** to "lock in" the value of the point $P\left(\sqrt{3}, 1\right)$. Had we chosen, for example, $a = 5$, we would have gotten

$$x = 5\sqrt{3}, \quad y = 5, \text{ and } \quad r = 10.$$

Therefore,

$$\begin{aligned} \sin(30°) &= \frac{y}{r} = \frac{5}{10} = \frac{1}{2} & \csc(30°) &= \frac{r}{y} = \frac{10}{5} = 2 \\ \cos(30°) &= \frac{x}{r} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2} & \sec(30°) &= \frac{r}{x} = \frac{10}{5\sqrt{3}} = \frac{2}{\sqrt{3}} \\ \tan(30°) &= \frac{y}{x} = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} & \cot(30°) &= \frac{x}{y} = \frac{5\sqrt{3}}{5} = \sqrt{3} \end{aligned}$$

Notice that the values of the trigonometric functions didn't change? In fact, if we had chosen to leave a "as is," we would have arrived at

$$x = a\sqrt{3}, \quad y = a, \text{ and } \quad r = 2a.$$

Therefore,

$$\begin{aligned}
\sin(30°) &= \frac{y}{r} = \frac{a}{2a} = \frac{1}{2} & \csc(30°) &= \frac{r}{y} = \frac{2a}{a} = 2\\
\cos(30°) &= \frac{x}{r} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} & \sec(30°) &= \frac{r}{x} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}\\
\tan(30°) &= \frac{y}{x} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} & \cot(30°) &= \frac{x}{y} = \frac{a\sqrt{3}}{a} = \sqrt{3}
\end{aligned}$$

The choice of a doesn't change the values of the trigonometric functions at all! This demonstration *implies* (but does not prove) that the values of the trigonometric functions do not depend on which point you choose on the terminal side of $30°$ - just as long as that point is *not* the origin.

Note

The result of this last statement is beneficial. If we have angles from our special triangles, we can just set $a = 1$ in those special triangles to get the x- and y-values of the related point.

Checkpoint 3.1.3

Find the values of the six trigonometric functions at $\theta = 45°$.

Answer

$$\begin{aligned}
\sin(45°) &= \frac{1}{\sqrt{2}} & \csc(45°) &= \sqrt{2}\\
\cos(45°) &= \frac{1}{\sqrt{2}} & \sec(45°) &= \sqrt{2}\\
\tan(45°) &= 1 & \cot(45°) &= 1
\end{aligned}$$

We do not need to limit our use of the special angles and special triangles to angles given in the first quadrant. The following example illustrates our tactics if given an angle terminating in a quadrant other than the first, where we can use our knowledge of the special triangles.

Example 3.1.4

Find exact values for all six trigonometric functions of $240°$.

Solution

An angle of $240°$ lies in the third quadrant, as seen in Figure 3.1.8.

Figure 3.1.8

If we want to work with a smaller angle, we consider the angle between the terminal side of $240°$ and the negative x-axis. This can be found by taking the larger angle ($240°$) and subtracting off the smaller angle (the $180°$ angle that gets to the negative x-axis).

$$240° - 180° = 60°$$

This action is illustrated in Figure **3.1.9**.

Figure 3.1.9

That 60° angle between 240° and 180° should be familiar.

Figure 3.1.10

Superimposing this 30°-60°-90° special triangle into our graph, we get Figure **3.1.11**.

Figure 3.1.11

Using our new knowledge that we can let a be any convenient number, we choose $a = 1$. **However**, here is where we need to be careful! If $a = 1$, it is not true that $x = a = 1$. This is because the point $P(x, y)$ on the terminal side of 240° must be in QIII. Therefore, $x = -1$. Likewise, $y = -\sqrt{3}$. No matter what, however, $r > 0$. Thus, $r = 2a = 2$. We now compute the values of the trigonometric functions.

$$\begin{aligned} \sin(240^\circ) &= \frac{y}{r} = -\frac{\sqrt{3}}{2} & \csc(240^\circ) &= \frac{r}{y} = -\frac{2}{\sqrt{3}} \\ \cos(240^\circ) &= \frac{x}{r} = -\frac{1}{2} & \sec(240^\circ) &= \frac{r}{x} = -2 \\ \tan(240^\circ) &= \frac{y}{x} = \sqrt{3} & \cot(240^\circ) &= \frac{x}{y} = \frac{1}{\sqrt{3}} \end{aligned}$$

Checkpoint 3.1.4

Find exact values for the sine, cosine, and tangent of $300°$.

Answer

$$\sin(300°) = \frac{-\sqrt{3}}{2}, \quad \cos(300°) = \frac{1}{2}, \quad \tan(300°) = -\sqrt{3}$$

All special angles between $0°$ and $360°$ are shown in Figure 3.1.12 below.

Figure 3.1.12

The astute student will notice that these can be succinctly written using the following definition.

Definition: Special Angles

The **special angles** are defined to be any angle that can be written as either $\pm 30°k$ or $\pm 45°k$, where $k \in \mathbb{Z}$.

It would be best if you memorized these values or be able to calculate them quickly.

Trigonometric Values at Quadrantal Angles

Evaluating trigonometric functions at quadrantal angles is much easier than at non-quadrantal angles. However, we have to watch out for division by zero.

Example 3.1.5

Find the values of the six trigonometric functions at $-270°$.

Solution

We get Figure 3.1.13 by creating a quick sketch for this situation and remembering that negative angles open clockwise.

Figure 3.1.13

We took the liberty to label a point on the terminal side of $-270°$. In this case, $x = 0$, $y = 1$, and $r = 1$. Therefore,

$$\sin(-270^\circ) = \frac{y}{r} = \frac{1}{1} = 1 \qquad \csc(-270^\circ) = \frac{r}{y} = \frac{1}{1} = 1$$
$$\cos(-270^\circ) = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec(-270^\circ) = \frac{r}{x} = \frac{1}{0} = \text{UNDEFINED}$$
$$\tan(-270^\circ) = \frac{y}{x} = \frac{1}{0} = \text{UNDEFINED} \qquad \cot(-270^\circ) = \frac{x}{y} = \frac{0}{1} = 0$$

Example 3.1.5 illustrates the fact that there are angles for which some of the trigonometric functions are not defined. In terms of functions, we would say that these functions have domain restrictions. We will discuss this in more detail later in this textbook. It suffices to know that the tangent, cotangent, secant, and cosecant functions each have angles for which the functions are undefined. You can see from the definitions of these trigonometric functions where these domain issues occur (either when $x = 0$ or when $y = 0$, depending on the trigonometric function).

? Checkpoint 3.1.5

Find the values of the six trigonometric functions at 540°.

Answer

$$\sin(540^\circ) = 0 \qquad \csc(540^\circ) = \text{UNDEFINED}$$
$$\cos(540^\circ) = -1 \qquad \sec(540^\circ) = -1$$
$$\tan(540^\circ) = 0 \qquad \cot(540^\circ) = \text{UNDEFINED}$$

Caution: Picky Language

Very technically, "$\cot(540^\circ) = \text{UNDEFINED}$" is not correct. It not only sounds odd (read the following sentence out loud, and you will understand what we mean, "This value equals undefined"), but it also is not mathematically accurate. Once you use the equals sign ($=$), you are assigning a value; however, "UNDEFINED" is not a value.

A mathematically accurate response would be, "$\cot(540^\circ)$ is undefined."

Comparing Values of the Trigonometric Functions

Understanding how to compare the values of a trigonometric function for different angles can be beneficial in future math courses. For example, in Calculus, you will need to have an innate understanding that $\sin(50^\circ) < \sin(80^\circ)$. The following example illustrates the thought process.

✓ Example 3.1.6

Without reaching for technology, can you determine which is greater, $\sin(50^\circ)$ or $\sin(80^\circ)$?

Solution

Quickly sketching two angles, $\alpha = 50^\circ$ and $\beta = 80^\circ$, we get the following graph.

Figure 3.1.14

By itself, this graph is not helpful; however, if we recall that the sine of an angle is defined to be $\frac{y}{r}$, where (x, y) is *any* point on the terminal side of the angle and r is the distance from the origin to the point, then we could choose points on the terminal sides of $\alpha = 50°$ and $\beta = 80°$ such that $r = 1$. This gives us $\sin(50°) = \frac{y_1}{r} = \frac{y_1}{1} = y_1$ and $\sin(80°) = \frac{y_2}{r} = \frac{y_2}{1} = y_2$ (see Figure 3.1.15).

Figure 3.1.15

From Figure 3.1.15, we can see that $y_1 < y_2$. Therefore, $\sin(50°) < \sin(80°)$.

? Checkpoint 3.1.6

Without reaching for technology, can you determine which is greater, $\cos(100°)$ or $\cos(170°)$?

Answer

$\cos(100°)$ is greater than $\cos(170°)$

Quadrants and the Signs of the Trigonometric Functions

✓ Example 3.1.7

Give the sign of the sine, cosine, and tangent of the angle.

a. $200°$
b. $300°$

Solutions

a. In standard position, the terminal side of an angle of $200°$ lies in QIII (see Figure 3.1.16(*a*) below). In the third quadrant, $x < 0$ and $y < 0$, but $r > 0$. Thus, $\sin(200°)$ is negative, $\cos(200°)$ is negative, and $\tan(200°)$ is positive.

Figure 3.1.16

b. $300° \in$ QIV, so $x > 0$ and $y < 0$, and $r > 0$. Thus, $\sin(300°)$ is negative, $\cos(300°)$ is positive, and $\tan(300°)$ is negative.

Example **3.1.7** demonstrates a common need in Trigonometry - to determine the sign of a trigonometric function. Luckily, our task is fairly easy.

Theorem: Signs of the Trigonometric Functions

The signs of the trigonometric functions depend on the trigonometric function in question and the quadrant where the terminal side of θ lies.

All trigonometric functions are positive when the terminal side of θ is in the first quadrant.

Sine and the cosecant functions are positive when the terminal side of θ is in the second quadrant.

Tangent and the cotangent functions are positive when the terminal side of θ is in the third quadrant.

Cosine and the secant functions are positive when the terminal side of θ is in the fourth quadrant.

Proof

Since the value of r is always positive (unless the point is the origin), the signs of the trigonometric functions are completely dependent on the signs of the x- and y-coordinates of the point on the terminal side of θ.

The sign of...	... which is...	...depends on the value(s) of...
$\sin(\theta)$	$\frac{y}{r}$	y
$\cos(\theta)$	$\frac{x}{r}$	x
$\tan(\theta)$	$\frac{y}{x}$	x and y
$\csc(\theta)$	$\frac{r}{y}$	y
$\sec(\theta)$	$\frac{r}{x}$	x
$\cot(\theta)$	$\frac{x}{y}$	x and y

Therefore, the signs of the sine and cosecant functions completely depend on the signs of the y-value of the point on the terminal side of θ. Hence, the sine and cosecant are positive in QI and QII. On the other hand, these functions are negative when the angle terminates in either QIII or QIV.

Likewise, the signs of the cosine and secant functions completely depend on the signs of the x-value of the point on the terminal side of θ. Therefore, the cosine and secant functions are positive in QI and QIV - they are negative in QII and QIII.

Finally, since the tangent and cotangent functions rely on the values of both x and y, we must consider what would make these ratios positive and negative. If the values of x and y are both positive or both negative, their ratios will be positive. This occurs in the first and third quadrants. Thus, the tangent and cotangent functions are positive in QI and QIII - they are negative in QII and QIV.

Figure **3.1.17** summarizes these results.

Figure 3.1.17

The bold lettering in the previous theorem is purposeful. Figure **3.1.18** gives the lovely mnemonic, "All Student Take Calculus," to remember that **a**ll trigonometric functions are positive in the first quadrant, the **s**ine and cosecant functions are positive in the second quadrant, the **t**angent and cotangent functions are positive in the third quadrant, and the **c**osine and secant functions are positive in the fourth quadrant.

Figure 3.1.18

Example 3.1.8

Given that $\cos(\theta) = -\frac{3}{5}$ and θ terminates in **QII**, find the values of the remaining trigonometric functions of θ.

Solution

Since we are given $\cos(\theta) = -\frac{3}{5}$, we know the ratio of x to r is $-\frac{3}{5}$. Letting $x = -3$ and $r = 5$, we find y by solving

$$\begin{aligned}
& & r &= \sqrt{x^2 + y^2} && \text{(definition of } r\text{)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 5 &= \sqrt{(-3)^2 + y^2} && \text{(substitute values)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 25 &= 9 + y^2 && \text{(Laws of Exponents and squaring both sides)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 16 &= y^2 && \text{(subtracting 9 from both sides)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \pm 4 &= y && \text{(Extraction of Roots)}
\end{aligned}$$

Using the fact that $\theta \in$ **QIII**, we throw out the positive possibility and set $y = -4$. We now have everything we need!

$$\begin{aligned}
\sin(\theta) &= -\frac{4}{5} & \csc(\theta) &= -\frac{5}{4} \\
\cos(\theta) &= -\frac{3}{5} & \sec(\theta) &= -\frac{5}{3} \\
\tan(\theta) &= \frac{3}{4} & \cot(\theta) &= \frac{4}{3}
\end{aligned}$$

? Checkpoint 3.1.8

In which quadrant(s) is the sine negative but the cosine positive?

Answer

QIV

Proving that Values of the Trigonometric Functions are Independent of P

Up to this point, we have avoided *proving* that the choice of P on the terminal side of θ makes no difference (other than choosing the origin - which is not allowed). We now officially prove the result so we can use it from this point forward.This proves the following theorem.

Theorem: Trigonometric Ratios are Independent of P

Let $P(x,y)$ and $\bar{P}(\bar{x},\bar{y})$ be two distinct points on the terminal side of θ, neither of which are the origin, and let $r=\sqrt{x^2+y^2}$ and $\bar{r}=\sqrt{\bar{x}^2+\bar{y}^2}$. Then

$$\begin{aligned}
\sin(\theta) &= \frac{y}{r} = \frac{\bar{y}}{\bar{r}} & \csc(\theta) &= \frac{r}{y} = \frac{\bar{r}}{\bar{y}}\\
\cos(\theta) &= \frac{x}{r} = \frac{\bar{x}}{\bar{r}} & \sec(\theta) &= \frac{r}{x} = \frac{\bar{r}}{\bar{x}}\\
\tan(\theta) &= \frac{y}{x} = \frac{\bar{y}}{\bar{x}} & \cot(\theta) &= \frac{x}{y} = \frac{\bar{x}}{\bar{y}}
\end{aligned}$$

Proof

Let $P(x,y)$ and $\bar{P}(\bar{x},\bar{y})$ be two distinct points on the terminal side of θ, neither of which are the origin, and let $r=\sqrt{x^2+y^2}$ and $\bar{r}=\sqrt{\bar{x}^2+\bar{y}^2}$ (see the following figure).

Since $\triangle AOP$ and $\triangle \bar{A}O\bar{P}$ are similar,

$$\begin{aligned}
\sin(\theta) &= \frac{y}{r} = \frac{\bar{y}}{\bar{r}} & \csc(\theta) &= \frac{r}{y} = \frac{\bar{r}}{\bar{y}}\\
\cos(\theta) &= \frac{x}{r} = \frac{\bar{x}}{\bar{r}} & \sec(\theta) &= \frac{r}{x} = \frac{\bar{r}}{\bar{x}}\\
\tan(\theta) &= \frac{y}{x} = \frac{\bar{y}}{\bar{x}} & \cot(\theta) &= \frac{x}{y} = \frac{\bar{x}}{\bar{y}}
\end{aligned}$$

Footnotes

[1] Since we are not covering right triangle trigonometry and unit circle trigonometry in this chapter, I chose to only bold coordinate trigonometry.

[2] For those of you who have seen advanced topics in Trigonometry before, you know that the claim, "r is always non-negative" is a bit of an exaggeration. When we introduce the polar coordinate system near the end of this course, we allow r to be negative.

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3.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Trigonometric Functions (Coordinate Definition)
  - Trigonometric Functions of Angles - Review of Trigonometric Functions - The Point Definition

KEY CONCEPTS

DEFINITIONS

- **Trigonometric Functions (Coordinate Definition):** Let θ be an angle in standard position and let $P(x,y)$ be a point on its terminal side. If $r=\sqrt{x^2+y^2}$ is the distance from the origin to P, the six trigonometric functions are defined as the ratios of x, y, and r. For example, $\sin(\theta)=\frac{y}{r}$ and $\cos(\theta)=\frac{x}{r}$.
- **Special Angles:** Angles that can be expressed in the form $\pm 30°k$ or $\pm 45°k$, where k is an integer. The exact values of their trigonometric functions can be found using special triangles.

THEOREMS

- **Signs of the Trigonometric Functions:** The signs of the trigonometric function values depend on the quadrant in which the angle's terminal side lies.
  - **QI:** All functions are positive.
  - **QII:** Sine and cosecant are positive.
  - **QIII:** Tangent and cotangent are positive.
  - **QIV:** Cosine and secant are positive.
- **Trigonometric Ratios are Independent of P:** The values of the trigonometric functions of an angle θ depend only on the angle itself, not on the specific point $P(x,y)$ chosen on the terminal side of θ (provided the point is not the origin).

COMMON MISTAKES

- **Confusing Function Notation with Multiplication:** The notation $\cos(\theta)$ represents a single value, which is the output of the cosine function for the input angle θ. It is not a product of "cos" and "θ". The term "cos" by itself has no mathematical meaning.
- **Incorrectly Stating Undefined Values:** Mathematically, it is more accurate to state that a function "is undefined" at a certain value rather than saying it "equals UNDEFINED". For example, the statement "$\cot(540°)$ is undefined" is preferred over "$\cot(540°) = \text{UNDEFINED}$".
- **Assuming r Can Be Negative:** In the coordinate definition of trigonometric functions, r represents a distance and is always non-negative.
- **Ignoring the Quadrant for Signs:** When finding trigonometric values, the quadrant of the angle is crucial for determining the correct signs of the x and y coordinates. For example, for an obtuse angle (in QII), the x-coordinate must be negative.

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3.1.2: HOMEWORK

READING QUESTIONS

1. Using the variables x, y, and r, what are the definitions for $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$?
2. What is the formula for r in the coordinate definition of the trigonometric functions?
3. The value of $\tan(\theta)$ is equivalent to what geometric property of the angle's terminal side?
4. Why are the values of the trigonometric functions independent of the specific point chosen on the terminal side?
5. Why are some trigonometric functions, like $\tan(90°)$, undefined?
6. What makes an angle a "special angle"?
7. In which quadrant are the sine and cosine functions both negative?
8. What does the mnemonic "All Students Take Calculus" help you remember?
9. If you are given that $\sin(\theta) > 0$ and $\cos(\theta) < 0$, in which quadrant must the terminal side of θ lie?
10. Explain the "rule of thumb" this textbook will use for deciding when to rationalize denominators.

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, evaluate the function.

1. $f(x) = x^2 - 2x$
   a. $f(-3)$
   b. $f(a-3)$
   c. $f(a) - 5$
   d. $f(a) - f(5)$
2. $g(x) = \sqrt{x+4}$
   a. $g(9)$
   b. $g(4h)$
   c. $g(0) + g(1)$
   d. $g(c^2)$
3. $F(x) = \frac{2}{x}$
   a. $F\left(\frac{-1}{2}\right)$
   b. $F\left(\frac{w}{2}\right)$
   c. $F(w+2)$
   d. $F(w) + F(2)$
4. $G(x) = 2^x$
   a. $G(-3)$
   b. $G(a+3)$
   c. $G(a) + G(3)$
   d. $G\left(\frac{3}{2}\right)$

For the following exercises, rationalize the denominator.

5. $\frac{7}{\sqrt{5}}$
6. $\sqrt{\frac{2}{7}}$
7. $\frac{\sqrt{3}}{\sqrt{32}}$
8. $\frac{3}{2\sqrt{5x}}$
9. $\frac{11}{6-\sqrt{3}}$
10. $\frac{\sqrt{3}}{3-\sqrt{5}}$
11. $\frac{\sqrt{5}-2}{\sqrt{3}-\sqrt{5}}$
12. $\frac{-4}{\sqrt[4]{3x^3}}$

For the following exercises, rationalize the numerator.

13. $\frac{11-\sqrt{2}}{6}$
14. $\frac{\sqrt{x}-\sqrt{h}}{\sqrt{x}}$

Answers

1.
   a. 15
   b. $a^2 - 8a + 15$
   c. $a^2 - 2a - 5$
   d. $a^2 - 2a - 15$
2.
   a. $\sqrt{13}$
   b. $2\sqrt{h+1}$
   c. $2 + \sqrt{5}$
   d. $\sqrt{c^2+4}$
3.
   a. -4
   b. $\frac{4}{w}$
   c. $\frac{2}{w+2}$
   d. $\frac{2}{w} + 1$
4.
   a. $\frac{1}{8}$
   b. $8(2^a)$
   c. $2^a + 8$
   d. $2\sqrt{2}$
5. $\frac{7\sqrt{5}}{5}$

6. $\frac{\sqrt{14}}{7}$
7. $\frac{\sqrt{6}}{8}$
8. $\frac{3\sqrt{5x}}{10x}$
9. $\frac{6+\sqrt{3}}{3}$
10. $\frac{3\sqrt{3}+\sqrt{15}}{4}$
11. $\frac{-\sqrt{15}+2\sqrt{3}-5+2\sqrt{5}}{2}$
12. $\frac{-4\sqrt[4]{27x^9}}{3x^3}$
13. $\frac{3}{22+2\sqrt{2}}$
14. $\frac{x-h}{x+\sqrt{hx}}$

HOMEWORK

VOCABULARY CHECK

1. ___ defines new functions in terms of the coordinates of points on the Cartesian Plane and their distance to the origin.
2. In the function notation $f(x)$, x is called the ___ variable.
3. $30°$, $45°$, and $60°$ are examples of ___ angles.
4. A helpful way to remember in which quadrants the trigonometric functions are positive is ASTC. This stands for **A**___ **S**___ **T**___ **C**___.

CONCEPT CHECK

5. We define the trigonometric ratios of any angle by placing the angle in standard position and choosing a point on the terminal side, with $r=\sqrt{x^2+y^2}$.
6. We use the notation $y=f(x)$ to indicate that y is a function of x, that is, x is the input variable and y is the output variable.
7. The trigonometric ratios $\sin\theta$ and $\cos\theta$ are functions of the angle θ.
8. If θ is an angle in standard position, and (x,y) is a point on its terminal side, with $r=\sqrt{x^2+y^2}$, then $\sin(\theta)=\frac{y}{r}$ $\cos(\theta)=\frac{x}{r}$ $\tan(\theta)=\frac{y}{x}$ $\csc(\theta)=\frac{r}{y}$ $\sec(\theta)=\frac{r}{x}$ $\cot(\theta)=\frac{x}{y}$
9. There are always two angles between $0°$ and $360°$ (except for the quadrantal angles) with a given trigonometric ratio.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The angle used in the coordinate definition of the trigonometric functions is measured from the positive y-axis.
11. The value, $\frac{1}{\sqrt{2}}$, is "more correct" than its rationalized form $\frac{\sqrt{2}}{2}$.
12. $\cos\theta$ is another notation for $\cos(\theta)$.
13. $3\sin(30°)=\sin(3\cdot 30°)=\sin(90°)=1$.
14. $\sin(\theta)$ can be undefined.
15. $\tan(\theta)$ can be undefined.
16. All students take Calculus.
17. If P and Q are points on the terminal side of θ, where neither are the origin, and Q is three times the distance from the origin as P, the values of the trigonometric functions defined by Q are three times the values of the same trigonometric functions defined by P.

BASIC SKILLS

For the following exercises, find the exact values of all six trigonometric functions of θ if the given point is on the terminal side of θ.

18. $(3,4)$
19. $\left(-\sqrt{3},1\right)$
20. $(-5,-5)$
21. $(15,-8)$
22. $(2,3)$
23. $\left(-\sqrt{5},1\right)$
24. $(-6,0)$
25. $(0,-2)$
26. (A,B)
27. $(-6t,9t)$ (assume $t>0$)

For the following exercises, use technology to find the approximate values of all six trigonometric functions of θ if the given point is on the terminal side of θ.

28. $(1.27,5.03)$
29. $(3.39,-7.46)$

For the following exercises,
(a) Give the coordinates of point P on the terminal side of the angle.
(b) Find the distance from the origin to point P.
(c) Find $\cos\theta$, $\sin\theta$, and $\tan\theta$.

30.
31.
32.
33.

For the following exercises, determine the $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the θ in the given graph. Assume θ is in standard position.

34.

35.

36.

37.

38. Complete the table with exact values.

θ	30°	45°	60°	120°	135°	150°	210°	225°	240°	300°	315°	330°
$\cos(\theta)$												
$\sin(\theta)$												
$\tan(\theta)$												

39. Fill in the table from memory with exact values. Do you notice any patterns that might help you memorize the values?

θ	0°	30°	45°	60°	90°
$\sin(\theta)$					
$\cos(\theta)$					
$\tan(\theta)$					

40. Fill in the table from memory with decimal approximations to four places.

θ	0°	30°	45°	60°	90°
$\sin(\theta)$					
$\cos(\theta)$					
$\tan(\theta)$					

41. Complete the table with exact values.

θ	0	30°	45°	60°
$\sec(\theta)$				
$\csc(\theta)$				
$\cot(\theta)$				

For the following exercises, choose all values from the list below that are exactly equal to, or decimal approximations for, the given trigonometric ratio. (Try not to use a calculator!)

$\sin 30°$	$\cos 45°$	$\sin 60°$	$\tan 45°$	$\tan 60°$
0.5000	0.5774	0.7071	0.8660	1.0000
$\frac{1}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$

42. $\cos 30°$
43. $\sin 45°$
44. $\tan 30°$
45. $\cos 60°$
46. $\sin 90°$
47. $\cos 0°$

For the following exercises, evaluate. Give exact values.

48. $\sin(30°)$
49. $\cos(0°)$
50. $\tan(45°)$
51. $\sin(60°)$
52. $\csc(30°)$
53. $\sec(0°)$
54. $\cot(45°)$
55. $\csc(60°)$

For the following exercises, draw the angle in standard position, find a point on the terminal side of the angle, and find the value of all six trigonometric functions of the angle.

56. $150°$

57. $-90°$

58. $225°$

59. $180°$

60. $-300°$

61. $90°$

62. $315°$

63. Explain why it makes sense that $\sin 0° = 0$ and $\sin 90° = 1$. Use a figure to illustrate your explanation.

64. Explain why it makes sense that $\cos 0° = 1$ and $\cos 90° = 0$. Use a figure to illustrate your explanation.

For the following exercises, state two possible quadrants where θ could terminate.

65. $\sin(\theta) = \frac{7}{12}$

66. $\cos(\theta) = -\frac{\sqrt{2}}{9}$

67. $\tan(\theta) = -1.2$

68. $\csc(\theta) = -3.87$

69. $\sec(\theta) = \sqrt{2}$

70. $\cot(\theta) = \frac{3}{2}$

For the following exercises, determine in which two quadrants is the statement true.

71. The sine is negative.

72. The cosine is negative.

73. The tangent is positive.

For the following exercises, state the quadrant(s) where θ must terminate.

74. $\sin(\theta) < 0$ and $\cos(\theta) > 0$

75. $\sec(\theta) > 0$ and $\tan(\theta) < 0$

76. $\csc(\theta) < 0$ and $\cot(\theta) < 0$

77. $\sin(\theta)$ and $\cot(\theta)$ have the same sign

For the following exercises, find the values of the remaining trigonometric functions of θ based on the given information.

78. $\sin(\theta) = \frac{21}{29}$, where $\theta \in$ QI

79. $\cos(\theta) = -\frac{3}{5}$, where $\theta \in$ QIII

80. $\tan(\theta) = \frac{15}{8}$, where $\theta \in$ QIII

81. $\csc(\theta) = -\frac{2}{\sqrt{3}}$, where $\tan(\theta) > 0$

82. $\sec(\theta) = \sqrt{2}$, where $\theta \in$ QIV

83. $\cot(\theta) = -\frac{1}{\sqrt{3}}$, where $\theta \in$ QII

84. $\sin(\theta) = -\frac{1}{4}$, where $\cos(\theta) < 0$

85. $\tan(\theta) = \frac{a}{b}$, where both a and b are positive

For the following exercises, find all angles between $0°$ and $360°$ for which the statement is true.

86. $\cos(\theta) = -1$

87. $\sin(\theta) = -1$

88. $\tan(\theta) = -1$

89. Find two angles, $0 \le \theta \le 360°$, with $\sin(\theta) = 0$.

90. Find two angles, $0 \le \theta \le 360°$, with $\cos(\theta) = 0$.

91. Find all angles, $0 \le \theta \le 360°$, with $\sin(\theta) = -1$.

92. Find all angles, $0 \le \theta \le 360°$, with $\cos(\theta) = -1$.

SYNTHESIS QUESTIONS

For the following exercises, determine if the statement is true or false. Justify your response.

93. If $\alpha < \beta$, then $\sin(\alpha) < \sin(\beta)$.

94. If $0 \le \alpha < \beta \le 90°$, then $\sin(\alpha) < \sin(\beta)$.

95. If $0 \le \alpha < \beta \le 90°$, then $\cos(\alpha) < \cos(\beta)$.

96. If $0 \le \alpha < \beta \le 90°$, then $\csc(\alpha) < \csc(\beta)$.

97. Is there an angle θ such that $\cos(\theta) = 3$? Why or why not?

98. Is there an angle θ such that $\csc(\theta) = \frac{1}{5}$? Why or why not?

99. Explain why $|\cos(\theta)| \le 1$ for any angle θ.

100. Is it true that $\tan(\theta) \le 1$ for any angle θ? Explain.

101. Use a sketch to explain why $\cos 90° = 0$.

102. Use a sketch to explain why $\cos 180° = 1$.

For the following exercises, compare the given value with the trigonometric ratios of the special angles to answer the questions.

103. Is the acute angle larger or smaller than $45°$?

   a. $\sin\alpha = 0.7$

   b. $\tan\beta = 1.2$

   c. $\cos\gamma = 0.65$

104. Is the acute angle larger or smaller than $60°$?

   a. $\cos\theta = 0.75$

   b. $\tan\phi = 1.5$

   c. $\sin\psi = 0.72$

For the following exercises, describe what happens to the function values as θ increases from $0°$ to $90°$.

105. $\sin(\theta)$

106. $\cos(\theta)$

107. $\tan(\theta)$

108. $\csc(\theta)$

109. Find two angles, $0 \le \theta \le 360°$, with $\sin(\theta) = \cos(\theta)$.

110. Find two angles, $0 \le \theta \le 360°$, with $\sin(\theta) = -\cos(\theta)$.

111. Explain why the definitions of the trigonometric ratios for a third-quadrant angle (between $180°$ and $270°$) are independent of

the point P chosen on the terminal side. Illustrate with a figure.

112. Explain why the definitions of the trigonometric ratios for a fourth-quadrant angle (between $270°$ and $360°$) are independent of the point P chosen on the terminal side. Illustrate with a figure.
113. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = 3x$ in QI.
114. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = -\frac{1}{3}x$ in QII.
115. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = \frac{4}{5}x$ in QIII.
116. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = -6x$ in QIV.

For the following exercises, evaluate the expression for $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$.

117. $3 + f(30°)$
118. $3f(30°)$
119. $4g(225°) - 1$
120. $-4 + 2g(225°) - 1$
121. $-2f(3\theta)$, for $\theta = 90°$
122. $6f\left(\frac{\theta}{2}\right)$, for $\theta = 90°$
123. $8 - 5g\left(\frac{\theta}{3}\right)$, for $\theta = 360°$
124. $1 - 4g(4\theta)$, for $\theta = 135°$

For the following exercises,
(a) Sketch an angle θ in standard position, $0° \le \theta \le 180°$, with the given properties.
(b) Find expressions for $\cos\theta, \sin\theta$, and $\tan\theta$ in terms of the given variable.

125. $\cos\theta = \frac{x}{3}, x < 0$
126. $\tan\theta = \frac{4}{\alpha}, \alpha < 0$
127. θ is obtuse and $\sin\theta = \frac{y}{2}$
128. θ is obtuse and $\tan\theta = \frac{q}{-7}$
129. θ is obtuse and $\tan\theta = m$
130. $\cos\theta = h$

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3.2: Algebra with Trigonometric Functions

Note to the Instructor (click to expand)

This section focuses on performing our old Algebra (factoring, distribution, powers, etc.) on expressions involving trigonometric functions. The results will not require any trigonometric identities. The point here is for the student to realize that Algebra still works, even though we are not using x.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Laws of Exponents: Once we start working with powers, a mastery of the Laws of Exponents must be assumed.
  - Multiplying and Distributing Polynomial Expressions: This skill is foundational for multiplying expressions containing trigonometric functions, as shown in Examples 3 and 4.
- **Factoring Techniques**
  - Greatest Common Factor: This skill is reviewed and applied to trigonometric expressions in Example 5.
  - Factoring by Grouping: This skill is reviewed and applied to trigonometric expressions in Example 6.
  - Factoring Trinomials: This skill is reviewed and applied to trigonometric expressions in Example 7.
  - Factoring Difference of Squares: This skill is reviewed and applied to trigonometric expressions in Example 8.
  - Factoring Sums and Differences of Cubes: This skill is reviewed and applied to trigonometric expressions in Example 9.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Evaluate expressions involving trigonometric functions.
- Compute powers of trigonometric functions.
- Evaluate products of trigonometric functions.
- Perform factorization on expressions involving trigonometric functions.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

As we move forward, we will need to apply some techniques from Algebra to analyze more complicated trigonometric expressions. Therefore, let's review some algebraic terminology.

Definition: Algebraic Expression

An **algebraic expression** is a well-formed mathematical phrase that combines numbers, variables, and mathematical operations. It does not include an equals sign.

For example, the height of a golf ball is given in feet by the expression $-16t^2 + 64t$, where t is the number of seconds after the ball is hit.

Definition: Evaluate (an algebraic expression)

To **evaluate** an algebraic expression means to substitute a specific value for the variable or variables involved.

Thus, after 1 second, the height of the golf ball is

$$-16(1)^2 + 64(1) = -16 + 64 = 48 \text{ feet},$$

and after 2 seconds, the height is

$$-16(2)^2 + 64(2) = -64 + 128 = 64 \text{ feet}$$

and so on.

Evaluating Trigonometric Expressions

The value of a trigonometric function represents a number, and it may appear as part of an algebraic expression. Expressions containing trigonometric functions can be simplified or evaluated like other algebraic expressions.

Example 3.2.1

Evaluate each expression for $X = 30°$ and $Y = 240°$.

a. $\frac{1}{\sqrt{3}}\tan(Y) + 3\sin(X)$
b. $6\tan(X)\cos(Y)$

Solutions

a. Substituting the values for X and Y, we get the following:

$$\begin{aligned}
&\text{Arithmetic} \to \text{Algebra} \implies \frac{1}{\sqrt{3}}\tan(Y) + 3\sin(X) = \frac{1}{\sqrt{3}}\tan(240°) + 3(\sin 30°) && \text{(substitute)}\\
&\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry} \implies = \frac{1}{\sqrt{3}}(\sqrt{3}) + 3\left(\frac{1}{2}\right) && \text{(evaluating the trigonometric functions)}\\
&\text{Arithmetic} \implies = 1 + \frac{3}{2} && \text{(canceling like factors)}\\
&\text{Arithmetic} \implies = \frac{5}{2} && \text{(addition)}
\end{aligned}$$

b. This expression includes the product of two trigonometric functions.

$$\begin{aligned}
&\text{Arithmetic} \to \text{Algebra} \implies 6\tan(X)\cos(Y) = 6(\tan(30°))(\cos(240°)) && \text{(substitute)}\\
&\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry} \implies = 6\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{2}\right) && \text{(evaluating the trigonometric functions)}\\
&\text{Arithmetic} \implies = -\frac{3}{\sqrt{3}} && \text{(simplify)}\\
&\text{Arithmetic} \to \text{Algebra} \implies = -\sqrt{3} && \text{(rationalizing the denominator)}
\end{aligned}$$

Checkpoint 3.2.1

Evaluate each expression for $X = 30°, Y = 60°$.

a. $4\sin(3X + 45°)$
b. $1 - \cos(4Y)$

Answers

a. $2\sqrt{2}$
b. $\frac{3}{2}$

Caution: Do NOT "Split" Arguments of Functions

In Checkpoint 3.2.1, $\sin(3X + 45°)$ is *not* equal to $\sin(3X) + \sin(45°)$. That is,

$$\sin(90° + 45°) \neq \sin(90°) + \sin(45°)$$

(You can check this for yourself.) Before applying the sine function, we must follow the Order of Operations and evaluate the expression $3X + 45°$ inside parentheses.

Powers of Trigonometric Functions

Compare the two expressions

$$(\cos(\theta))^2 \quad \text{and} \quad \cos(\theta^2)$$

They are not the same.

- The first expression, $(\cos(\theta))^2$, says to compute $\cos(\theta)$ and then square the result.
- $\cos(\theta^2)$, on the other hand, says to square the angle first and then compute the cosine.

For example, if $\theta = 30°$, then

$$(\cos(30°))^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$$

but

$$\cos\left((30°)^2\right) = \cos(900°)$$

We have not delved into the details of how to compute the cosine of such a large angle (yet). Still, it should make sense that if the point $P(x, y)$ is on the terminal side of $\theta = 900°$, then P must be on the terminal side of any angle coterminal with $\theta = 900°$. Specifically, P must be on the terminal side of $900° - 360° = 540°$. Still, this angle is too unwieldy to deal with, so we find another, smaller angle coterminal to $540°$. Luckily, $540° - 360° = 180°$ fits that bill! Hence, if $P(x, y)$ is on the terminal side of $900°$, it must also be on the terminal side of $180°$. Since the point $(-1, 0)$ is on the terminal side of $180°$, we can now compute the cosine of $900°$.

$$\cos\left((30°)^2\right) = \cos(900°) = \cos(180°) = \frac{x}{r} = \frac{-1}{1} = -1.$$

Notice this is not the same value as $\cos(30°)$.

This brings us to a new notation reserved specifically for trigonometric functions.

Notation: Powers of Trigonometric Functions

$\sin^2(\theta)$	means	$(\sin(\theta))^2$	$\csc^2(\theta)$	means	$(\csc(\theta))^2$
$\cos^2(\theta)$	means	$(\cos(\theta))^2$	$\sec^2(\theta)$	means	$(\sec(\theta))^2$
$\tan^2(\theta)$	means	$(\tan(\theta))^2$	$\cot^2(\theta)$	means	$(\cot(\theta))^2$

Example 3.2.2

Evaluate $\sin^2(45°)$.

Solution

$$\begin{aligned} \sin^2(45°) &= (\sin(45°))^2 && \text{(notational definition)} \\ \xrightarrow{\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry}} \quad &= \left(\frac{1}{\sqrt{2}}\right)^2 && \text{(evaluating the trigonometric function)} \\ \xrightarrow{\text{Arithmetic} \to \text{Algebra}} \quad &= \frac{1}{2} && \text{(Laws of Exponents)} \end{aligned}$$

Other powers are written in the same fashion. Thus, for example, $\sin^3(\theta) = (\sin(\theta))^3$.

Checkpoint 3.2.2

Evaluate $\tan^4(60°)$

Answer

9

Caution: Negative Exponents

We have already stated that

$$[\cos(\theta)]^2 = \cos^2(\theta)$$

and

$$[\sin(\theta)]^5 = \sin^5(\theta).$$

However, we will **not** adopt such notation for negative exponents. That is,

$$[\cos(\theta)]^{-1} \neq \cos^{-1}(\theta).$$

The reason for excluding the negative exponents from this "shorthand" notation will become apparent when we discuss inverse trigonometric functions.

Products of Trigonometric Functions

We can multiply together trigonometric expressions, just as we multiply algebraic expressions. Recall that we use the Distributive Law in computing products such as

$$x(3x-2) = 3x^2 - 2x$$

and

$$(x-3)(x+5) = x^2 + 2x - 15.$$

Example 3.2.3

Using the Distributive Law, multiply

$$\cos(t)(3\cos(t)-2).$$

Solution

Think of $\cos(t)$ as a single variable, and multiply by each term inside parentheses. (The algebraic form of the calculation is shown on the right in bold).

$$\text{Arithmetic} \to \text{Algebra} \implies \cos(t)(3\cos(t)-2) = (\cos(t))(3\cos(t)) - (\cos(t)\cdot 2) \quad \textbf{(distribution: } x(3x-2) = x\cdot 3x - x\cdot 2\textbf{)}$$

$$\text{Arithmetic} \to \text{Algebra} \implies = 3\cos^2(t) - 2\cos(t) \quad \textbf{(Commutative Property of Multiplication)}$$

Notice that we write $(\cos(t))^2$ as $\cos^2(t)$.

Checkpoint 3.2.3

Multiply $2\tan(\beta)(4\tan^2(\beta)+\tan(\alpha))$

Answer

$8\tan^3(\beta) + 2\tan(\beta)\tan(\alpha)$

We can also use the Distributive Law to multiply binomials that include trigonometric functions.

Example 3.2.4

Multiply $(4\sin(C)-1)(3\sin(C)+2)$.

Solution

This calculation is similar to the product $(4x-1)(3x+2)$, except that the variable x has been replaced by $\sin(C)$. Compare the calculations for the two products; first the familiar algebraic product:

$$\begin{aligned}(4x-1)(3x+2) &= 4x\cdot 3x + 4x\cdot 2 - 1\cdot 3x - 1\cdot 2\\ &= 12x^2 + 8x - 3x - 2 = 12x^2 + 5x - 2\end{aligned}$$

We compute the product in this example in the same way, but replacing x by $\sin(C)$.

$$\begin{aligned}(4\sin(C)-1)(\sin(C)+2) &= (4\sin(C))(3\sin(C)) + (4\sin(C)\cdot 2) - 1(3\sin(C)) - 1\cdot 2\\ &= 12\sin^2(C) + 5\sin(C) - 2\end{aligned}$$

Checkpoint 3.2.4

Expand $(4\cos(\alpha)+3)^2$

Answer

$16\cos^2(\alpha) + 24\cos(\alpha) + 9$

Factoring Expressions Involving Trigonometric Functions

We can factor trigonometric expressions with the same techniques we use for algebraic expressions. The next few subsections of this topic quickly review all our previous factorization techniques. It is important to note that each of these factoring techniques will be needed at some point during Trigonometry.

Factoring Out a Common Expression

Likely, the first technique you encountered when learning to factor was factoring out the greatest common factor (GCF). I call this factoring out a common expression.

In the next two examples, compare the familiar algebraic factoring with a similar trigonometric expression.

Example 3.2.5

Factor.

a. $6w^2 - 9w$
b. $6\sin^2(\theta) - 9\sin(\theta)$

Solutions

a. We factor out the common factor, $3w$.

$$6w^2 - 9w = 3w(2w - 3)$$

b. We factor out the common factor, $3\sin(\theta)$.

$$6\sin^2(\theta) - 9\sin(\theta) = 3\sin(\theta)(2\sin(\theta) - 3)$$

Checkpoint 3.2.5

Factor.

a. $2a^2 - ab$
b. $2\cos^2(\phi) - \cos(\phi)\sin(\phi)$

Answers

a. $a(2a - b)$
b. $\cos(\phi)(2\cos(\phi) - \sin(\phi))$

Factoring by Grouping

Once you mastered factoring out common expressions from a collection of terms, you moved to factoring by grouping. We only review via example here as we did in the previous topic.

Example 3.2.6

Factor.

a. $10x^2 + 2x - 15xy - 3y$
b. $10\sin^2(\theta) + 2\sin(\theta) - 15\sin(\theta)\cos(\theta) - 3\cos(\theta)$

Solutions

a. Grouping terms, we get

$$\begin{aligned} 10x^2 + 2x - 15xy - 3y &= (10x^2 + 2x) + (-15xy - 3y) && \text{(Grouping)} \\ &= 2x(5x + 1) - 3y(5x + 1) && \text{(Factoring out the GCF from each group)} \\ &= 2x(5x + 1) - 3y(5x + 1) && \text{(Noticing the result has two common factors)} \\ &= (5x + 1)(2x - 3y) && \text{(Factoring out the common factor of } (5x + 1)) \end{aligned}$$

b. The structure of this expression is almost identical to that of the previous problem. Let $x = \sin(\theta)$ and $y = \cos(\theta)$ to get

$$\begin{aligned} 10\sin^2(\theta) + 2\sin(\theta) - 15\sin(\theta)\cos(\theta) - 3\cos(\theta) &= 10x^2 + 2x - 15xy - 3y && \text{(Substitution: } x = \sin(\theta) \text{ and } y = \cos(\theta)) \\ &= (5x + 1)(2x - 3y) && \text{(From part (a))} \\ &= (5\sin(\theta) + 1)(2\sin(\theta) - 3\cos(\theta)) && \text{("Back-substitute:" } \sin(\theta) = x \text{ and } \cos(\theta) = y) \end{aligned}$$

Example **3.2.6b** illustrates a common tactic we will use as we move through mathematics. Namely, when an expression (or equation) looks overly complicated, we make it easier to work with by performing substitutions. In Calculus, this will be tied to a technique called u**-Substitutions** (where we use the variable u instead of x).

? Checkpoint 3.2.6

Factor.

a. $12rt + 9r - 20t - 15$
b. $12\sec(\alpha)\tan(\beta) + 9\sec(\alpha) - 20\tan(\beta) - 15$

Answers

a. $(3r - 5)(4t + 3)$
b. $(3\sec(\alpha) - 5)(4\tan(\beta) + 3)$

Factoring Trinomials

Despite being more complex than factoring a difference of squares (which we will talk about next), the next factorization technique you learned was factoring trinomials.

✓ Example 3.2.7

Factor.

a. $t^2 - 3t - 10$
b. $\tan^2(\alpha) - 3\tan(\alpha) - 10$

Solutions

a. We look for numbers p and q so that $(t + p)(t + q) = t^2 - 3t - 10$. Their product is $pq = -10$, and their sum is $p + q = -3$. By checking the factors of -10 for the correct sum, we find $p = -5$ and $q = 2$. Thus,

$$t^2 - 3t - 10 = (t - 5)(t + 2)$$

b. Now replace t by $\tan(\alpha)$ to find

$$\tan^2(\alpha) - 3\tan(\alpha) - 10 = (\tan(\alpha) - 5)(\tan(\alpha) + 2)$$

? Checkpoint 3.2.7

Factor.

a. $3z^2 - 2z - 1$
b. $3\sin^2(\beta) - 2\sin(\beta) - 1$

Answers

a. $(3z + 1)(z - 1)$
b. $(3\sin(\beta) + 1)(\sin(\beta) - 1)$

Factoring Differences of Squares

In addition to factoring out common expressions and factoring trinomials, special factoring formulas for the Difference of Squares and the Sums and Differences of Cubes will be needed. Recall the following theorem from Algebra.

Theorem: Difference of Squares

For any algebraic expressions F and L,

$$F^2 - L^2 = (F - L)(F + L).$$

Proof

$$\begin{aligned} (F - L)(F + L) &= F^2 + FL - FL - L^2 && \text{(distributing)} \\ &= F^2 - L^2 && \text{(simplifying)} \end{aligned}$$

Example 3.2.8

Factor completely.

$$16\cos^4(\theta) - 1$$

Solution

By this point, you (hopefully) understand what we will do.

$$\begin{aligned}
16\cos^4(\theta) - 1 &= 16x^4 - 1 && \text{(Substitute: } x = \cos(\theta)\text{)} \\
&= (4x^2)^2 - 1^2 && \text{(Laws of Exponents)} \\
&= F^2 - L^2 && \text{(Substitute: } F = 4x^2 \text{ and } L = 1\text{)} \\
&= (F - L)(F + L) && \text{(Difference of Squares)} \\
&= (4x^2 - 1)(4x^2 + 1) && \text{(Resubstitute: } 4x^2 = F \text{ and } 1 = L\text{)} \\
&= ((2x)^2 - 1^2)(4x^2 + 1) && \text{(Laws of Exponents)} \\
&= \left(\overline{F}^2 - \overline{L}^2\right)(4x^2 + 1) && \left(\text{Substitute: } \overline{F} = 2x \text{ and } \overline{L} = 1\right) \\
&= \left(\overline{F} - \overline{L}\right)\left(\overline{F} + \overline{L}\right)(4x^2 + 1) && \text{(Difference of Squares)} \\
&= (2x - 1)(2x + 1)(4x^2 + 1) && \left(\text{Resubstitute: } 2x = \overline{F} \text{ and } 1 = \overline{L}\right) \\
&= (2\cos(\theta) - 1)(2\cos(\theta) + 1)(4\cos^2(\theta) + 1) && \text{(Resubstitute: } \cos(\theta) = x\text{)}
\end{aligned}$$

A couple of notes are in order for Example **3.2.8**. First and foremost, while we took our time with substitutions and being meticulous with our work, you need to reach a level of proficiency in mathematics where you do not need to perform those substitutions to do the factoring. That is, you need to reach the point (and very soon) where you see

$$16\cos^4(\theta) - 1$$

and immediately rewrite it (without substitutions) as

$$\left(4\cos^2(\theta) - 1\right)\left(4\cos^2(\theta) + 1\right) = (2\cos(\theta) - 1)(2\cos(\theta) + 1)\left(4\cos^2(\theta) + 1\right).$$

I cannot express how important it is for you, as the student, to reach that level of mathematical fluency as quickly as possible (truthfully, in a college-level mathematics course, you should already have this level of fluency).

The second note is better mentioned as a cautionary statement.

Caution: $F^2 + L^2$ is NOT Factorable

A common mistake of students in lower levels of Algebra is to think that $F^2 + L^2$ is factorable - this is **not** true (unless we are factoring over the complex number system). That is,

$$4x^2 + 1$$

cannot be factored.[1]

Checkpoint 3.2.8

Factor completely.

$$\sin^4(\alpha) - 81\cos^4(\beta)$$

Answer

$$(\sin(\alpha) - 3\cos(\beta))(\sin(\alpha) + 3\cos(\beta))(\sin^2(\alpha) + 9\cos^2(\beta))$$

Factoring Sums and Differences of Cubes

We have now reached the final factorization method, which we must recall from Algebra.[2]

Theorem: Sums and Differences of Cubes

For any algebraic expressions F and L,

$$\begin{aligned} F^3+L^3 &= (F+L)\left(F^2-FL+L^2\right) \\ F^3-L^3 &= (F-L)\left(F^2+FL+L^2\right) \end{aligned}$$

Proof

$$\begin{aligned} (F+L)\left(F^2-FL+L^2\right) &= F^3-F^2L+FL^2+F^2L-FL^2+L^3 && \text{(distributing)} \\ &= F^3+L^3 && \text{(simplifying)} \end{aligned}$$

and

$$\begin{aligned} (F-L)\left(F^2+FL+L^2\right) &= F^3+F^2L+FL^2-F^2L-FL^2-L^3 && \text{(distributing)} \\ &= F^3-L^3 && \text{(simplifying)} \end{aligned}$$

Example 3.2.9

Factor.

$$27\sin^3(A)-8$$

Solution

This time, we will speed things up a bit.

$$\begin{aligned} 27\sin^3(A)-8 &= 27x^3-8 && \text{(Substitute: } x=\sin(A)\text{)} \\ &= (3x)^3-2^3 && \text{(Laws of Exponents)} \\ &= (3x-2)\left((3x)^2+(3x)(2)+2^2\right) && \text{(Difference of Cubes)} \\ &= (3x-2)\left((9x^2+6x+4\right) \\ &= (3\sin(A)-2)\left((9\sin^2(A)+6\sin(A)+4\right) && \text{(Resubstitute: } \sin(A)=x\text{)} \end{aligned}$$

Checkpoint 3.2.9

Factor.

$$125\csc^3(\theta)+64\cot^3(\phi)$$

Answer

$$(5\csc(\theta)+4\cot(\phi))\left(25\csc^2(\theta)-20\csc(\theta)\cot(\phi)+16\cot^2(\phi)\right)$$

Footnotes

[1] Technically, if we are allowing ourselves to factor over the complex number system (i.e. if we are considering numbers like $2+3i$), then we *can* factor sums of squares; however, this will not be the case for us until nearly the end of this textbook. Even then, most instructors might skip that material.

[2] Believe it or not, other factorization methods exist beyond these basic techniques. For example, in College Algebra (also sometimes referred to as Precalculus), you learn about factoring high-degree polynomials with integer coefficients using the Rational Roots Theorem. There are also critically important methods for factoring expressions involving negative rational exponents - these are encountered frequently in Calculus.

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3.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

KEY CONCEPTS

DEFINITIONS

- **Algebraic Expression:** A mathematical phrase that is a combination of numbers, variables, and mathematical operations, but does not contain an equals sign.
- **Evaluate (an algebraic expression):** The process of substituting a given value for the variable(s) in an algebraic expression and simplifying the result.

THEOREMS

- **Difference of Squares:** For any algebraic expressions F and L, the formula for factoring a difference of squares is $F^2 - L^2 = (F - L)(F + L)$.
- **Sums and Differences of Cubes:** For any algebraic expressions F and L, the formulas for factoring are:
  - Sum of Cubes: $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$
  - Difference of Cubes: $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$

COMMON MISTAKES

- **Do NOT "Split" Arguments of Functions:** A function cannot be distributed across terms inside its argument. For example, $\sin(A + B)$ is not equal to $\sin(A) + \sin(B)$. The expression inside the parentheses must be evaluated first.
- **Incorrect Notation for Negative Exponents:** The shorthand notation $\cos^2(\theta)$ for $(\cos(\theta))^2$ does not apply to negative exponents. Specifically, $[\cos(\theta)]^{-1}$ is not the same as $\cos^{-1}(\theta)$, which represents the inverse cosine function.
- **- Factoring a Sum of Squares:** A sum of two squares, such as $F^2 + L^2$, is not factorable over the real numbers. Attempting to factor it as $(F + L)(F + L)$ or in any other way is incorrect.

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3.2.2: HOMEWORK

READING QUESTIONS

1. What is the difference between an algebraic expression and an equation?
2. Explain the difference between evaluating $(\cos(30°))^2$ and $\cos((30°)^2)$.
3. What does the notation $\sin^2(\theta)$ mean?
4. Why is the notation $\cos^{-1}(\theta)$ not used to represent $\frac{1}{\cos(\theta)}$?
5. When multiplying expressions like $\cos(t)(3\cos(t) - 2)$, what algebraic property is used?
6. What is the purpose of using a substitution, like letting $x = \sin(\theta)$, when factoring a complex trigonometric expression?
7. What is the formula for factoring a difference of squares, $F^2 - L^2$?
8. Is an expression like $\sin^2(\theta) + 9$ factorable over the real numbers? Why or why not?
9. What is the formula for factoring a sum of cubes, $F^3 + L^3$?
10. What is the formula for factoring a difference of cubes, $F^3 - L^3$?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, factor completely.

1. $2x^2 + 5x - 12$
2. $3x^2 - 2x - 8$
3. $12x^2 + x - 1$
4. $6x^2 - 13x - 15$
5. $6x^2 - x - 12$
6. $24x^2 + 10x - 21$
7. $9y^2 - 16$
8. $12m^2 - 75n^2$
9. $81x^4 - 1$
10. $x^8 - 100y^4$
11. $x^3 + 27$
12. $x^6 - 64y^3$
13. $y^9 - 1$
14. $16m^3 - 54y^3$

Answers

1. $(2x - 3)(x + 4)$
2. $(3x + 4)(x - 2)$
3. $(4x - 1)(3x + 1)$
4. $(6x + 5)(x - 3)$
5. $(2x - 3)(3x + 4)$
6. $(6x + 7)(4x - 3)$
7. $(3y - 4)(3y + 4)$
8. $3(2m - 5n)(2m + 5n)$
9. $(3x - 1)(3x + 1)(9x^2 + 1)$
10. $(x^4 - 10y^2)(x^4 + 10y^2)$
11. $(x + 3)(x^2 - 3x + 9)$
12. $(x^2 - 4y)(x^4 + 4x^2y + 16y^2)$
13. $(y - 1)(y^2 + y + 1)(y^6 + y^3 + 1)$
14. $2(2m - 3y)(4m^2 + 6my + 9y^2)$

HOMEWORK

CONCEPT CHECK

1. To evaluate $\cos^2(30°)$, Wyatt used the keystrokes

   COS 30 ∧ 2 ENTER

   and got the answer -1. Were his keystrokes correct? Why or why not?
2. Make up an example to show that $\tan(\theta + \phi) \neq \tan(\theta) + \tan(\phi)$.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

3. Expressions containing trigonometric functions can be simplified or evaluated like other algebraic expressions.
4. We can *solve* expressions like $3\sin(x) + 2\sin(x)$.
5. $\sin(X + Y) = \sin(X) + \sin(Y)$
6. $\cos^2(\theta) = (\cos(\theta))^2$
7. $\sin^{-1}(\theta) = \dfrac{1}{\sin(\theta)}$
8. $64x^2 - 9 = (8x - 3)(8x + 3)$
9. $64x^2 + 9 = (8x + 3)(8x + 3) = (8x + 3)^2$

BASIC SKILLS

For the following exercises, evaluate the expressions, using exact values for the trigonometric ratios.

10. $5\tan 135° + 6\cos 60°$
11. $3\tan 240° + 8\sin 300°$
12. $\sin(15° + 30°)$
13. $\cos(2 \cdot 75°)$
14. $8\cos^2(30°)$
15. $12\sin^2(315°)$
16. $3\tan^2(150°) - \sin^2(45°)$
17. $1 + \tan^2(120°)$

For the following exercises, evaluate the expressions for $x = 30°$, $y = 45°$, and $z = 60°$. Give exact values for your answers.

18. $3\sin x + 5\cos y$

19. $4\tan y + 6\cos y$
20. $-2\tan 3y$
21. $\sin(3x - 2x)$
22. $\cos^2 x + \sin^2 x$
23. $7\sin^2 y + 7\cos^2 y$
24. $\cos x \cos x - \sin x \sin z$
25. $\tan(180° - x)\tan x$

For the following exercises, combine like terms.

26.
   a. $3x^2 - x - 5x^2$
   b. $3\cos^2\theta - \cos\theta - 5\cos^2\theta$
27.
   a. $4x + 5y - y$
   b. $4\cos\theta + 5\sin\theta - \sin\theta$
28.
   a. $-3SC + 7SC$
   b. $-3\sin\theta\cos\theta + 7\sin\theta\cos\theta$
29.
   a. $4SC + 11SC - 17SC$
   b. $4\sin\theta\cos\theta + 11\sin\theta\cos\theta - 17\sin\theta\cos\theta$
30.
   a. $-C^2S^3 + 6C^2S^3$
   b. $-\cos^2\theta\sin^3\theta + 6\cos^2\theta\sin^3\theta$
31.
   a. $7C^2S^2 - (2CS)^2$
   b. $7\cos^2\theta\sin^2\theta - (2\cos\theta\sin\theta)^2$

For the following exercises, simplify the expression, then evaluate.

32. $\cos t + 2\cos t\sin t - 3\cos t$, for $t = 30°$
33. $11\tan w - 4\tan w + 6\tan w\cos w$, for $w = 60°$
34. $7\tan\theta - 4\tan\phi + 3\tan\phi - 6\tan\theta$, for $\theta = 45°, \phi = 120°$
35. $5\sin A + \sin B - 6\sin B + 6\sin A$, for $A = 210°, B = 135°$
36. $-\sin x\cos x - \sin(2x) + 3\sin x\cos x$, for $x = 45°$
37. $4\cos u\sin u + 3\sin 2u - 10\sin u\cos u$, for $u = 270°$

For the following exercises, multiply or expand.

38.
   a. $x(2x - 1)$
   b. $\sin A(2\sin A - 1)$
39.
   a. $q(5r - q)$
   b. $\cos\theta(5\sin\theta - \cos\theta)$
40.
   a. $a(b - 3a)$
   b. $\tan A(\tan B - 3\tan A)$
41.
   a. $3w(2w - z)$
   b. $3\sin\alpha(2\sin\alpha - \sin\beta)$
42.
   a. $(C + 1)(2C - 1)$
   b. $(\cos\phi + 1)(2\cos\phi - 1)$
43.
   a. $(3S - 2)(S + 1)$
   b. $(3\sin B - 2)(\sin B + 1)$
44.
   a. $(a + b)(a - b)$
   b. $(\cos\theta + \cos\phi)(\cos\theta - \cos\phi)$
45.
   a. $(t - w)(t + 4w)$
   b. $(\tan\alpha - \tan\beta)(\tan\alpha + 4\tan\beta)$
46.
   a. $(1 - T)^2$
   b. $(1 - \tan\theta)^2$
47.
   a. $(2 + 3S)^2$
   b. $(2 + 3\sin\theta)^2$
48.
   a. $(T^2 + 2)(T^2 - 2)$
   b. $(\tan^2\theta + 2)(\tan^2\theta - 2)$
49.
   a. $(2c^2 - 3)(2c^2 + 3)$
   b. $(2\cos^2\phi - 3)(2\cos^2\phi + 3)$

For the following exercises, decide whether or not the expressions are equivalent. Justify your response by citing a theorem or providing a counterexample.

50. $\cos(x + y)$, $\cos x + \cos y$
51. $\sin(\theta - \phi)$, $\sin\theta - \sin\phi$
52. $\tan(2A)$, $2\tan A$
53. $\cos\left(\frac{1}{2}\beta\right)$, $\frac{1}{2}\cos\beta$
54. $(\sin\alpha)^2$, $\sin^2\alpha$
55. $(\tan B)^2$, $\tan(B^2)$
56. $\sin 3t + \sin 5t$, $\sin 8t$
57. $3\cos 2x + 5\cos 2x$, $8\cos 2x$
58. $\tan(\theta + 45°) - \tan\theta$, $\tan 45°$
59. $\sin(30° + z) + \sin(30° - z)$, $\sin 60°$
60. Factor each expression, if possible.
   a. $x^2 - 4x$
   b. $x^2 - 4$
   c. $x^2 + 4$

d. x^2+4x+4

e. x^2-4x+4

f. x^2+4x

g. x^2-4x-4

h. $-x^2+4$

For the following exercises, factor.

61.

a. $9m+15n$

b. $9\cos\alpha+15\cos\beta$

62.

a. $12p-20q$

b. $12\sin\theta-20\sin\phi$

63.

a. $5r^2-10qr$

b. $5\tan^2 C-10\tan B\tan C$

64.

a. $2x^2-6xy$

b. $2\sin^2 A-6\sin A\cos A$

65.

a. $9C^2-1$

b. $9\cos^2\beta-1$

66.

a. $25S^2-16S$

b. $25\sin^2\beta-16\sin\beta$

67.

a. $6T^3-8T^2$

b. $6\tan^3 A-8\tan^2 A$

68.

a. $9T^2-15T^3$

b. $9\tan^2 B-15\tan^3 B$

69.

a. t^2-t-20

b. $\tan^2\theta-\tan\theta-20$

70.

a. s^2+5s-6

b. $\sin^2 A+5\sin A-6$

71.

a. $3c^2+2c-1$

b. $3\cos^2 B+2\cos B-1$

72.

a. $8S^2-6S+1$

b. $8\sin^2\phi-6\sin\phi+1$

73.

a. $6S^2-5S-1$

b. $6\sin^2\alpha-5\sin\alpha-1$

74.

a. $T^2-4T-12$

b. $\tan^2\alpha-4\tan\alpha-12$

75.

a. $9C^2-1$

b. $9\cos^2(\theta)-1$

76.

a. $16-9S^2$

b. $16-9\sin^2(\alpha)$

77.

a. $81T^2-16C^2$

b. $81\tan^2(A)-16\cos^2(B)$

78.

a. $81T^4-16C^4$

b. $81\tan^4(A)-16\cos^4(B)$

79.

a. $\frac{1}{9}S^2-\frac{4}{25}C^2$

b. $\frac{1}{9}\sin^2(\phi)-\frac{4}{25}\cos^2(\lambda)$

80.

a. $27C^3-1$

b. $27\cos^3(\theta)-1$

81.

a. $64+27S^3$

b. $64+27\sin^3(\alpha)$

82.

a. $729T^3+64C^3$

b. $729\tan^3(A)+64\cos^3(B)$

83.

a. $729T^6-64C^6$

b. $729\tan^6(A)-64\cos^6(B)$

84.

a. $\frac{1}{27}S^3-\frac{8}{125}C^3$

b. $\frac{1}{27}\sin^3(\phi)-\frac{8}{125}\cos^3(\lambda)$

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3.3: Introduction to Trigonometric Identities

Note to the Instructor (click to expand)

The purpose of this section is to *introduce* the student to identities and contradictions. We officially introduce the Reciprocal, Ratio, and Pythagorean Identities to allow the student a different way to *compute* the value of a trigonometric function given the value of a different trigonometric functions at the same angle - ***proofs are not done by the student here***!

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Multiplying and Dividing Rational Expressions: When working with trigonometric identities, you will often need to perform arithmetic on rational expressions.
  - Adding and Subtracting Rational Expressions: When working with trigonometric identities, you will often need to perform arithmetic on rational expressions.
  - Simplifying Compound Rational Expressions: This skill is required for the "Skills Refresher" exercises in the homework, and the underlying concept is used when deriving the Ratio Identities in Example 3.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Determine if an algebraic equation represents an identity.
- Find the value of a trigonometric function using the Reciprocal, Ratio, and Pythagorean Identities.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

What is an Identity?

Recall that an equation may be true or false, depending on the values of any variables involved. For example, the equation

$$x^2 + 3x = 10$$

is true only if $x = 2$ or $x = -5$. You might have forgotten a bit of language from the past, so let's quickly redefine **conditional equations** and **identities**.

Definition: Conditional Equation

An equation that is true only for certain variable values (and, therefore, is false for others) is called a **conditional equation**.

When you solve a conditional equation, you find the variable values that make the equation true.

Definition: Identity

An equation that is true for all allowable variable values is called an **identity**.

Here are some examples of identities.

$$\begin{aligned} 3(x+y) &= 3x+3y \\ (x+1)^2 &= x^2+2x+1 \end{aligned}$$

In an identity, the statements on either side of the equal sign are called **equivalent expressions**.

Definition: Equivalent Expressions

Two algebraic expressions are called **equivalent** if they have the same value for all allowable variable values.

Example 3.3.1

Which of the following equations are identities?

a. $3s+7s=10s$
b. $5c(c-2s)=5c^2-10cs$
c. $2t-1=3$

Solutions

Many of the algebraic operations you have already learned, such as combining like terms or applying the Distributive Law, produce equivalent expressions.

a. Equation (a) is an identity obtained by combining like terms on the left side.
b. Equation (b) is an identity obtained by applying the Distributive Law on the left side.
c. Equation (c) is not an identity because the equation is valid only for $t=2$.

Checkpoint 3.3.1

Which of the following equations are identities?

a. $(c-s)(c+s)=c^2-s^2$
b. $3t^2=1$
c. $(2c+1)+(s-3)=2c+s-2$

Answers

(a) and (c)

Of course, you wouldn't be asked to *solve* an identity because all values of the variable are solutions. Instead, we use identities to replace one form of an expression with a more useful form. You do this when you solve a quadratic equation by factoring. For example,

$$\begin{aligned} 2x^2-x-1 &= 0 \\ \Longrightarrow \quad (2x+1)(x-1) &= 0 \qquad \text{(factoring)} \end{aligned}$$

Because $(2x+1)(x-1)$ is equivalent to $2x^2-x-1$, we have not changed the equation or its solutions; however, we now can apply the Zero Factor Principle and solve the equation.

Using identities is especially useful when we are working with trigonometric ratios.

Using Trigonometric Ratios in Identities

Because the identity

$$2x^2 - x - 1 = (2x + 1)(x - 1)$$

is true for any value of x, it is true when x is replaced, for instance, by $\cos(\theta)$. This gives us a new identity

$$2\cos^2(\theta) - \cos(\theta) - 1 = (2\cos(\theta) + 1)(\cos(\theta) - 1)$$

Expressions involving the trigonometric functions can be manipulated by the same rules that we use with simple variables (such as the Distributive Law or the Laws of Exponents).

Example 3.3.2

Which of the following equations are identities?

a. $3\sin(\theta) + 7\sin(\theta) = 10\sin(\theta)$
b. $5\cos(\theta)(\cos(\theta) - 2\sin(\theta)) = 5\cos^2(\theta) - 10\cos(\theta)\sin(\theta)$
c. $2\tan(\theta) - 1 = 3$

Solutions

These equations are obtained from the equations in Example **3.3.1** by replacing the variable s with $\sin(\theta)$, the variable c with $\cos(\theta)$, and the variable t with $\tan(\theta)$. Equations (a) and (b) are identities, and equation (c) is not, for the same reasons as before.

Checkpoint 3.3.2

Which of the following equations are identities?

a. $(\cos(\theta) - \sin(\theta))(\cos(\theta) + \sin(\theta)) = \cos^2(\theta) - \sin^2(\theta)$
b. $3\tan^2(\theta) = 1$
c. $(2\cos(\theta) + 1) + (\sin(\theta) - 3) = 2\cos(\theta) + \sin(\theta) - 2$

Answers

(a) and (c)

Caution: Examples are Not Proofs

To show that an equation is *not* an identity, we need only find one value of the variable for which the equation is false (this would create what we call a **contradiction**). However, to show that a given equation *is* an identity, it is not enough to show that the equation is valid for one, or even a dozen, values of the variable. The equation must be true for *all* legitimate values of the variable. Because we cannot check all values of the variable numerically, we must use algebraic methods to *prove* that an equation is an identity.

Ratio Identities

All of the trigonometric functions are related. We begin by considering the relationship between the tangent and the sine and cosine functions (we will find a similar relationship for the cotangent). Complete the following table with exact values.

θ	$\cos(\theta)$	$\sin(\theta)$	$\frac{\sin(\theta)}{\cos(\theta)}$	$\tan(\theta)$
$0°$				
$30°$				
$45°$				
$60°$				

For the angles in the table,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

The question you should have is, "Is this true for all angles?"

The short answer is no, but before we describe why, let's understand why this relationship exists between the tangent, sine, and cosine.

The sine function for an angle θ is defined as $\sin(\theta) = \frac{y}{r}$, where (x, y) is any point on the terminal side of θ (other than the origin) and $r = \sqrt{x^2 + y^2}$. Likewise, the cosine is defined to be $\cos(\theta) = \frac{x}{r}$. If we look at the ratio of the sine to the cosine, we get

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{y/r}{x/r} = \frac{(y/r)}{(x/r)} \cdot \frac{r}{r} = \frac{(y/\cancel{r})}{(x/\cancel{r})} \cdot \frac{\cancel{r}}{\cancel{r}} = \frac{y}{x} = \tan(\theta).$$

A similar argument shows that

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.$$

One essential item to note, however, is that because $\cos(90°) = 0$, the quotient $\frac{\sin(\theta)}{\cos(\theta)}$ is undefined for $\theta = 90°$, and, as such, $\tan(90°)$ is undefined. The same applies for $\theta = 270°$. For all other angles between $0°$ and $360°$, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. A similar argument shows that $\cot(0°)$ and $\cot(180°)$ are undefined.

We are now ready to formalize these statements as a theorem.

Theorem: Ratio Identities

For any angle not coterminal with $90°$ or $270°$,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

For any angle not coterminal with $0°$ or $180°$,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Advice: Memorize These Identities

The Ratio Identities are needed throughout the rest of Mathematics, Science, and Engineering. The need to commit them to memory and their importance cannot be overstated.

Example 3.3.3

If $\sin(\theta) = \frac{12}{13}$ and $\cos(\theta) = -\frac{5}{13}$, find the values of $\tan(\theta)$ and $\cot(\theta)$.

Solution

We apply the Ratio Identities to obtain

$$\begin{aligned}\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{12/13}{-5/13} \\ &= \frac{12/13}{-5/13} \cdot \frac{13}{13} \\ &= \frac{12}{-5} \\ &= -\frac{12}{5}\end{aligned}$$

and

$$\begin{aligned}\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} \\ &= \frac{-5/13}{12/13} \\ &= \frac{-5/13}{12/13} \cdot \frac{13}{13} \\ &= \frac{-5}{12} \\ &= -\frac{5}{12}\end{aligned}$$

? Checkpoint 3.3.3

If $\sin(\theta) = -\frac{1}{5}$ and $\cos(\theta) = -\frac{2\sqrt{6}}{5}$, find the values of $\tan(\theta)$ and $\cot(\theta)$.

Answer

$\tan(\theta) = \frac{1}{2\sqrt{6}}$ and $\cot(\theta) = 2\sqrt{6}$

Reciprocal Identities

We now state one of the more obvious identities. We find the following relationships by comparing the definitions of cosecant, secant, and cotangent to the three fundamental trigonometric functions (sine, cosine, and tangent).

Theorem: Reciprocal Identities

$$\sin(\theta) = \frac{1}{\csc(\theta)} \quad (\text{provided } \csc(\theta) \neq 0) \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \quad (\text{provided } \sin(\theta) \neq 0)$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \quad (\text{provided } \sec(\theta) \neq 0) \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \quad (\text{provided } \cos(\theta) \neq 0)$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \quad (\text{provided } \cot(\theta) \neq 0) \qquad \cot(\theta) = \frac{1}{\tan(\theta)} \quad (\text{provided } \tan(\theta) \neq 0)$$

You likely already figured out the Reciprocal Identities because, for example, $\sin(\theta) = \frac{y}{r}$ and $\csc(\theta) = \frac{r}{y}$ are obvious reciprocals.

Advice: Memorize These Identities

The Reciprocal Identities are needed throughout the rest of Mathematics, Science, and Engineering. The need to commit them to memory and their importance cannot be overstated.

Example 3.3.4

Compute $\cos(\theta)$ given $\sec(\theta) = -\frac{10}{3}$.

Solution

$$\cos(\theta) = \frac{1}{\sec(\theta)} = \frac{1}{-10/3} = -\frac{3}{10}$$

Checkpoint 3.3.4

If $\cot(\theta) = \frac{8}{5}$, find $\tan(\theta)$.

Answer

$\tan(\theta) = \frac{5}{8}$

Caution: Cosecant, Secant, and Cotangent are NOT Inverses

Refrain from calling the cosecant, secant, and cotangent inverse trigonometric ratios. The word "inverse" is very special in Mathematics. While it is true that $\sin(\theta)$ and $\csc(\theta)$ are **multiplicative inverses**, they are not *functional* inverses.

Pythagorean Identities

All of the trigonometric functions are related. We begin by considering the relationship between $\sin(\theta)$ and $\cos(\theta)$. Complete the following table with exact values.

θ	$\cos(\theta)$	$\sin(\theta)$	$\cos^2(\theta)$	$\sin^2(\theta)$	$\cos^2(\theta) + \sin^2(\theta)$
$0°$					
$30°$					
$45°$					
$60°$					
$90°$					

For all the angles in the table,

$$\cos^2\theta + \sin^2\theta = 1.$$

This observation leads to the most important result in all of Trigonometry.

Theorem: The Pythagorean Identity

For any angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Proof

From the coordinate definition of the trigonometric functions, we know that $r = \sqrt{x^2 + y^2}$. Thus,

$$
\begin{aligned}
& r = \sqrt{x^2+y^2} & \\
\Longrightarrow \quad & r^2 = x^2+y^2 & \text{(squaring both sides)} \\
\Longrightarrow \quad & 1 = \frac{x^2}{r^2}+\frac{y^2}{r^2} & \text{(dividing both sides by } r^2\text{)} \\
\Longrightarrow \quad & 1 = \left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2 & \text{(Laws of Exponents)} \\
\Longrightarrow \quad & 1 = (\cos(\theta))^2+(\sin(\theta))^2 & \text{(definitions of the cosine and sine functions)} \\
\Longrightarrow \quad & 1 = \cos^2(\theta)+\sin^2(\theta) & \text{(using a better notation)}
\end{aligned}
$$

From its name, the Pythagorean Identity is intimately related to the Pythagorean Theorem.

Advice: Memorize This Identity

The Pythagorean Identity (also known as the Fundamental Pythagorean Identity) is needed throughout the rest of Mathematics. The need to commit this to memory and its importance cannot be overstated.

Example 3.3.5

Evaluate $\cos^2(27°)+\sin^2(27°)$ without using a calculator.

Solution

Because the Pythagorean Identity, $\cos^2(\theta)+\sin^2(\theta)=1$, is true for any value of θ, it is true in particular for $\theta = 27°$. So $\cos^2(27°)+\sin^2(27°)=1$.

Checkpoint 3.3.5

Evaluate $5\left[\cos^2(12°)+\sin^2(12°)\right]$ without using a calculator.

Answer

5

When we solve more complicated trigonometric equations in later chapters, we will need to simplify trigonometric expressions so that they involve only one of the trigonometric functions. The Pythagorean Identity is useful when we wish to write an equivalent expression for either $\cos^2(\theta)$ or for $\sin^2(\theta)$. The following equivalent forms to the Pythagorean Identity will be helpful throughout Trigonometry.

Corollary: Alternate Forms of the Pythagorean Identity

$$
\begin{aligned}
\sin^2(\theta) &= 1-\cos^2(\theta) \quad \Longrightarrow \quad \sin(\theta) = \pm\sqrt{1-\cos^2(\theta)} \\
\cos^2(\theta) &= 1-\sin^2(\theta) \quad \Longrightarrow \quad \cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}
\end{aligned}
$$

The Pythagorean Identity, along with the Ratio and Reciprocal Identities, show us that all trigonometric functions of an angle are related. If we know the value of one trigonometric function at θ and where θ terminates, we can calculate the remaining trigonometric functions. We do *not* need to find the angle itself to do this. We only need to know which quadrant the angle lies in to determine the correct sign for the trigonometric ratios.

Advice: Memorize These Identities

The alternate forms of the Pythagorean Identity are needed throughout the rest of Mathematics, Science, and Engineering. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example **3.3.6** shows that we do not need to memorize the alternate forms of the Pythagorean Identity as long as we are confident manipulating mathematical expressions.

Example 3.3.6

If $\sin(\theta) = \frac{7}{25}$ and $90° < \theta < 180°$, find $\cos(\theta)$ and $\tan(\theta)$.

Solution

We substitute $\sin(\theta) = \frac{7}{25}$ into the Pythagorean Identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to get

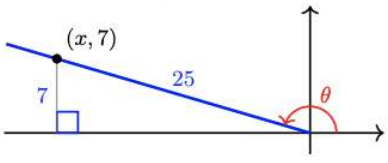
$$\begin{aligned} \cos^2(\theta) + \left(\frac{7}{25}\right)^2 &= 1 && \text{(substituting)} \\ \Longrightarrow \quad \cos^2(\theta) &= 1 - \left(\frac{7}{25}\right)^2 && \text{(isolating } \cos^2(\theta)\text{)} \\ \Longrightarrow \quad \cos^2(\theta) &= 1 - \frac{49}{625} \\ \Longrightarrow \quad \cos^2(\theta) &= \frac{576}{625} \\ \Longrightarrow \quad \cos(\theta) &= \pm\sqrt{\frac{576}{625}} && \text{(taking the square root of both sides)} \\ \Longrightarrow \quad \cos(\theta) &= \pm\frac{24}{25} \end{aligned}$$

Because θ is between $90°$ and $180°$, $\cos(\theta)$ must be negative. Therefore, $\cos(\theta) = -\frac{24}{25}$. To find $\tan\theta$, we use the Ratio Identities to get

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{7/25}{-24/25} = -\frac{7}{24}.$$

We could have solved Example **3.3.6** by creating a quick sketch of θ terminating in QII. Because $\sin(\theta) = \frac{7}{25}$, we can choose a point on the terminal side to have y-coordinate 7 and $r = 25$ (see Figure **3.3.1** below).

Figure 3.3.1



In order to calculate $\cos(\theta)$ and $\tan(\theta)$, we must find the x-coordinate of the point. By the Pythagorean Theorem,

$$\begin{aligned} x^2 + 7^2 &= 25^2 \\ \Longrightarrow \quad x^2 &= 576 && \text{(isolating } x^2\text{)} \\ \Longrightarrow \quad x &= -24 && \text{(taking the square root of both sides)} \end{aligned}$$

Note that x is negative because the point $(x, 7)$ is in the second quadrant. Then

$$\cos(\theta) = \frac{x}{r} = -\frac{24}{25} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} = \frac{7}{-24} = -\frac{7}{24}$$

Checkpoint 3.3.6

a. If $\cos(\theta) = \frac{5}{13}$ and $-90° < \theta < 0°$, find $\sin(\theta)$ and $\tan(\theta)$.
b. Find $\sin(\theta)$ and $\tan(\theta)$ by using an appropriate sketch.

Answers

$$\sin(\theta) = -\frac{12}{13}, \quad \tan(\theta) = -\frac{12}{5}$$

You might have noticed that this subsection is "Pythagorean Identities" (plural). Yet, we only named a single Pythagorean Identity (purposefully titled **The** Pythagorean Identity). We now introduce two additional Pythagorean Identities that will be used frequently throughout Trigonometry.

$$\begin{aligned}
& \cos^2(\theta) + \sin^2(\theta) = 1 && \text{(the Pythagorean Identity)} \\
\Longrightarrow \quad & \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} && \text{(dividing both sides by } \cos^2(\theta)\text{)} \\
\Longrightarrow \quad & 1 + \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 = \left(\frac{1}{\cos(\theta)}\right)^2 && \text{(cancelling like factors and Laws of Exponents)} \\
\Longrightarrow \quad & 1 + (\tan(\theta))^2 = (\sec(\theta))^2 && \text{(Ratio and Reciprocal Identities)} \\
\Longrightarrow \quad & 1 + \tan^2(\theta) = \sec^2(\theta) && \text{(Laws of Exponents)}
\end{aligned}$$

A similar process shows

$$1 + \cot^2(\theta) = \csc^2(\theta).$$

We formalize these statements in a summary theorem.

Theorem: Pythagorean Identities

For any angle θ where the functions are defined,

$$\begin{aligned}
\cos^2(\theta) + \sin^2(\theta) &= 1 \\
1 + \tan^2(\theta) &= \sec^2(\theta) \\
\cot^2(\theta) + 1 &= \csc^2(\theta)
\end{aligned}$$

Advice: Memorize These Identities

Each of the Pythagorean Identities are needed throughout the rest of Mathematics. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

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3.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Pythagorean Identities
  - Fundamental Identities - The Pythagorean Identities: Theorem, Proof, and Examples

KEY CONCEPTS

DEFINITIONS

- **Conditional Equation:** An equation that is true for only a specific set of variable values and is false for others.
- **Identity:** An equation that holds true for all permissible values of the variables involved.
- **Equivalent Expressions:** Two algebraic expressions that produce the same value for all allowable inputs. They form the two sides of an identity.

THEOREMS

- **Ratio Identities:**
  - For any angle θ not coterminal with 90° or 270°, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.
  - For any angle θ not coterminal with 0° or 180°, $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.
- **Reciprocal Identities:**
  - $\sin(\theta) = \frac{1}{\csc(\theta)}$ and $\csc(\theta) = \frac{1}{\sin(\theta)}$
  - $\cos(\theta) = \frac{1}{\sec(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$
  - $\tan(\theta) = \frac{1}{\cot(\theta)}$ and $\cot(\theta) = \frac{1}{\tan(\theta)}$
- **The Pythagorean Identities:** For any angle θ where the functions are defined:
  - $\cos^2(\theta) + \sin^2(\theta) = 1$
  - $1 + \tan^2(\theta) = \sec^2(\theta)$
  - $\cot^2(\theta) + 1 = \csc^2(\theta)$
- **Alternate Forms of the Pythagorean Identity:**
  - $\sin^2(\theta) = 1 - \cos^2(\theta) \implies \sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
  - $\cos^2(\theta) = 1 - \sin^2(\theta) \implies \cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$

COMMON MISTAKES

- **Using Examples as Proofs:** To prove an equation is an identity, you must use algebraic methods. Showing that an equation works for a few numerical examples is not sufficient, as it does not prove it is true for all possible values.
- **Incorrectly Taking Square Roots:** Taking the square root of both sides of an equation like $\cos^2(\theta) + \sin^2(\theta) = 1$ does not result in $\cos(\theta) + \sin(\theta) = 1$. The square root operation does not distribute over addition.
- **Confusing Reciprocals with Inverses:** The cosecant, secant, and cotangent functions are the multiplicative reciprocals of the sine, cosine, and tangent functions, respectively. They are not functional inverses. The term "inverse" has a specific meaning for functions that will be discussed later.

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3.3.2: HOMEWORK

READING QUESTIONS

1. What is the difference between a conditional equation and an identity?
2. To prove that an equation is an identity, is it enough to show that it is true for several different values of the variable? Why or why not?
3. State the Ratio Identity for $\tan(\theta)$.
4. For which angles is the identity for $\tan(\theta)$ not defined?
5. State the three Reciprocal Identities that relate the six trigonometric functions.
6. What is the Fundamental Pythagorean Identity?
7. If you know that $\sin(\theta) = \frac{7}{25}$, why is it also necessary to know the quadrant of θ to find the exact value of $\cos(\theta)$?
8. What are the two Pythagorean Identities that involve the tangent, cotangent, secant, and cosecant functions?
9. Are $\csc(\theta)$ and $\sin(\theta)$ multiplicative inverses or functional inverses?
10. How can you use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to derive the identity involving $\tan^2(\theta)$ and $\sec^2(\theta)$?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, write the expression as a single fraction in simplest form.

1. $\frac{1}{x} + \frac{1}{y}$
2. $1 - \frac{1}{x}$
3. $\frac{1}{x-1} + \frac{1}{x+1}$
4. $x - \frac{x}{x+1}$
5. $\frac{\frac{y}{x} - \frac{x}{y}}{\frac{y}{x} + 1}$
6. $\frac{1 - \frac{x^2}{y^2}}{1 + \frac{x^2}{y^2}}$
7. $\frac{\frac{2a}{x}}{1 - \frac{a^2}{x^2}}$
8. $\frac{\frac{a}{x} + \frac{b}{y}}{1 - \frac{ab}{xy}}$

Answers

1. $\frac{x+y}{xy}$
2. $\frac{x-1}{x}$
3. $\frac{2x}{x^2-1}$
4. $\frac{x^2}{x+1}$
5. $\frac{y-x}{y}$
6. $\frac{y^2-x^2}{y^2+x^2}$
7. $\frac{2ax}{x^2-a^2}$
8. $\frac{ay+bx}{xy-ab}$

HOMEWORK

VOCABULARY CHECK

1. A ___ equation is true only for certain variable values and false for others.
2. An equation that is true for all legitimate values of the variables is called a(n) ___.
3. The expressions on either side of the equal sign in an identity are called ___ because they have the same value for all values of the variable.

CONCEPT CHECK

4. What is the difference between a conditional equation and an identity? Give an example of each.
5. What happens when you try to "solve" an identity?
6. Nhat claims that since $\cos^2(\theta) + \sin^2(\theta) = 1$, we can take the square root of both sides to get the simpler form $\cos(\theta) + \sin(\theta) = 1$. Is she correct? Why or why not?
7. If $\frac{a}{b} = \frac{2}{5}$, is it necessarily true that $a = 2$ and $b = 5$? Explain.
8. If $\tan(\theta) = \frac{3}{11}$, is it true that $\sin(\theta) = 3$ and $\cos(\theta) = 11$? Explain.
9. Explain how you would arrive at the identity

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

given that you know $\cos^2(\theta) + \sin^2(\theta) = 1$.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The reciprocal of the secant is the cosecant.
11. For any angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

12. For any angle θ,

$$\cos(\theta) + \sin(\theta) = 1.$$

13. For any angle θ,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

14. To show that an equation is an identity, we just need to find one value of the unknown that makes the statement true.

BASIC SKILLS

For the following exercises, decide which of the following equations are identities. Explain your reasoning.

15. $(\sqrt{a}+\sqrt{b})^2 = a+b$
16. $\sqrt{a^2-b^2} = a-b$
17. $\dfrac{1}{a+b} = \dfrac{1}{a}+\dfrac{1}{b}$
18. $\dfrac{a+b}{a} = b$
19. $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$
20. $\dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$
21. $(1+\tan\theta)^2 = 1+\tan^2\theta$
22. $\sqrt{1-\sin^2\phi} = 1-\sin\phi$

For the following exercises, evaluate (without using a calculator).

23. $3\cos^2 1.7^\circ + 3\sin^2 1.7^\circ$
24. $4-\cos^2 338^\circ - \sin^2 338^\circ$
25. $\left(\cos^2 20^\circ + \sin^2 20^\circ\right)^4$
26. $\dfrac{18}{\cos^2 17^\circ + \sin^2 17^\circ}$
27. $\dfrac{6}{\cos^2 53^\circ} - 6\tan^2 53^\circ$
28. $\dfrac{1}{\sin^2 102^\circ} - \dfrac{\cos^2 102^\circ}{\sin^2 102^\circ}$

For the following exercises, find exact values for the other trigonometric functions of the given angle.

29. $\cos A = \dfrac{12}{13}$ and $270^\circ < A < 360^\circ$
30. $\sin B = \dfrac{-3}{5}$ and $180^\circ < B < 270^\circ$
31. $\sin\phi = \dfrac{1}{7}$ and $90^\circ < \phi < 180^\circ$
32. $\cos t = \dfrac{-2}{3}$ and $180^\circ < t < 270^\circ$
33. $\tan\theta = -\dfrac{1}{2}$ and $270^\circ < \theta < 360^\circ$
34. $\tan\theta = 2$ and $180^\circ < \theta < 270^\circ$
35. $\tan\theta = \dfrac{3}{4}$ and $180^\circ < \theta < 270^\circ$
36. $\tan\theta = -3$ and $90^\circ < \theta < 180^\circ$
37. $2\cos A + 9 = 8$ and $90^\circ < A < 180^\circ$
38. $25\sin B + 8 = -12$ and $180^\circ < B < 270^\circ$
39. $8\tan\beta + 5 = -11$ and $90^\circ < \beta < 180^\circ$
40. $6(\tan\beta - 4) = -24$ and $90^\circ < \beta < 270^\circ$
41. $\tan^2 C - 3 = 0$ and $90^\circ < C < 180^\circ$ or $270^\circ < C < 360^\circ$
42. $4\cos^2 A - \cos A = 0$ and $0^\circ < A < 180^\circ$

SYNTHESIS QUESTIONS

43. **Proving the Pythagorean Identity.** Prove the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$ by carrying out the following steps. Sketch an angle θ in standard position, and label a point (x, y) on the terminal side, at a distance r from the vertex.
    a. Begin with the equation $\sqrt{x^2+y^2} = r$, and square both sides.
    b. Divide both sides of your equation from part (a) by r^2.
    c. Write the left side of the equation as the sum of the squares of two fractions.
    d. Substitute the appropriate trigonometric ratio for each fraction.
44. **Proving the Ratio Identity.** Prove the Ratio Identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ by carrying out the following steps. Sketch an angle θ in standard position, and label a point (x, y) on the terminal side, at a distance r from the vertex.
    a. Write $\sin\theta$ in terms of y and r, and solve for y.
    b. Write $\cos\theta$ in terms of x and r, and solve for x.
    c. Write $\tan\theta$ in terms of x and y, then substitute your results from parts (a) and (b).
    d. Simplify your fraction in part (c).

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3.4: Introduction to Proofs in Trigonometry

Note to the Instructor (click to expand)

The proofs done by the student in this section only require the Ratio, Reciprocal, and Pythagorean Identities. The material here is also meant to give the student experience in simplifying complex-looking expressions involving trigonometric functions into much cleaner expressions. Algebra involving trigonometric functions (e.g., multiplying expressions or adding rational expressions involving trigonometric functions) is treated. Finally, the student should be able to write trigonometric function in terms of another (using the Pythagorean Identities).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Graphing Functions**
  - Graphs of Functions: This section introduces the technique of graphing both sides of a potential identity to check for plausibility. A basic understanding of function graphs is needed to interpret the results, as shown in Example 4.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Simplify a trigonometric expression.
- Rewrite a trigonometric expression in terms of sines and cosines.
- Simplify an algebraic expression by performing a trigonometric substitution.
- Determine the plausibility of an identity graphically and understand this does not prove an identity (but *can* disprove an identity).
- Prove basic trigonometric identities.

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

We have already been introduced to several fundamental identities in Trigonometry. Namely, the Ratio, Reciprocal, and Pythagorean Identities. We then spent most of that section using those identities to help us evaluate trigonometric functions. In this section, we learn how to use those identities to simplify expressions involving trigonometric functions. We then start dipping our toes into proving identities.

Simplifying Expressions Involving Trigonometric Functions

Recall that an algebraic expression (or mathematical expression) is a combination of symbols that are mathematically "well-formed." The mathematical symbols can include numbers (constants), variables, operations (e.g., addition, subtraction,

multiplication, etc.), functions, brackets, and other grouping symbols to help determine the order of operations. The one symbol that is always missing from an expression is the equals sign (=).

Reminder: Equations versus Expressions

Equations have equals signs - expressions do not.

You **solve** equations, but you **simplify** expressions.

When we simplify an algebraic expression, we obtain a new expression that has the same values as the old one, but is easier to work with. For example, we can apply the Distributive Law and combine like terms to simplify

$$\begin{aligned} 2x(x-6)+3(x+2) &= 2x^2-12x+3x+6 \\ &= 2x^2-9x+6 \end{aligned}$$

The new expression is equivalent to the old one, that is, the expressions have the same value when we evaluate them at any value of x. For instance, you can check that, at $x=3$, the expressions $2x(x-6)+3(x+2)$ and $2x^2-9x+6$ become

$$\begin{aligned} 2(3)(3-6)+3(3+2) &= 6(-3)+3(5) &= -3 \\ 2(3)^2-9(3)+6 &= 18-27+6 &= -3 \end{aligned}$$

To simplify an expression containing trigonometric functions, we treat each function as a single variable. Compare the two calculations below:

$$\begin{array}{ccccc} 8xy & - & 6xy & = & 2xy \\ 8\cos(\theta)\sin(\theta) & - & 6\cos(\theta)\sin(\theta) & = & 2\cos(\theta)\sin(\theta) \end{array}$$

Both calculations are examples of combining like terms. In the second calculation, we treat $\cos(\theta)$ and $\sin(\theta)$ as variables, just as we treat x and y in the first calculation.

In Trigonometry, we are often tasked with simplifying expressions involving trigonometric functions. In doing so, we will use many of our skills from Algebra (e.g., simplifying compound rational expressions, factoring, distributing, etc.) in combination with the identities we have recently discovered (along with those we will soon discover). One strategy for simplifying a trigonometric expression is to reduce the number of different trigonometric functions involved. The following example showcases this process using the Ratio Identities.

Example 3.4.1

a. Simplify: $\cos(\theta)\tan(\theta)+\sin(\theta)$
b. Multiply: $(\cos(\theta)+\sin(\theta))^2$
c. Add: $\frac{\cos(\theta)}{\sin(\theta)}+\frac{\sin(\theta)}{\cos(\theta)}$
d. Simplify: $3\tan(A)+4\tan(A)-2\cos(A)$
e. Simplify: $2-\sin(B)+2\sin(B)$

Solutions

a. We apply the Ratio Identities to replace $\tan(\theta)$ by $\frac{\sin(\theta)}{\cos(\theta)}$ and obtain

$$\begin{aligned} \cos(\theta)\tan(\theta)+\sin(\theta) &= \cos(\theta)\left(\tfrac{\sin(\theta)}{\cos(\theta)}\right)+\sin(\theta) \\ &= \sin(\theta)+\sin(\theta) \\ &= 2\sin(\theta) \end{aligned}$$

b. The instruction to multiply implies we need to square the binomial and simplify.

$$\begin{aligned}(\cos(\theta)+\sin(\theta))^2 &= \cos^2(\theta)+2\cos(\theta)\sin(\theta)+\sin^2(\theta) && \text{(distributing)}\\ &= \cos^2(\theta)+\sin^2(\theta)+2\cos(\theta)\sin(\theta) && \text{(Commutative Property of Addition)}\\ &= 1+2\cos(\theta)\sin(\theta) && \text{(Pythagorean Identity)}\end{aligned}$$

c. While we *could* use the Ratio Identities to make an equivalent expression, the result would be $\cot(\theta)+\tan(\theta)$. At that point, we would be stuck. Instead, let's use the Mathematical Mantra and perform some arithmetic before trying our new Trigonometry skills. Specifically, let's get a common denominator and perform the subtraction.

$$\begin{aligned}\frac{\cos(\theta)}{\sin(\theta)}+\frac{\sin(\theta)}{\cos(\theta)} &= \frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{\cos(\theta)}{\cos(\theta)}+\frac{\sin(\theta)}{\cos(\theta)}\cdot\frac{\sin(\theta)}{\sin(\theta)} && \text{(multiplying each fraction by an expression equivalent to 1 to get common denominators)}\\ &= \frac{\cos^2(\theta)}{\cos(\theta)\sin(\theta)}+\frac{\sin^2(\theta)}{\cos(\theta)\sin(\theta)}\\ &= \frac{\cos^2(\theta)+\sin^2(\theta)}{\cos(\theta)\sin(\theta)} && \text{(adding fractions with like denominators)}\\ &= \frac{1}{\cos(\theta)\sin(\theta)} && \text{(Pythagorean Identity)}\end{aligned}$$

d. Combine like terms.

$$3\tan(A)+4\tan(A)-2\cos(A)=7\tan(A)-2\cos(A)$$

Note that $\tan(A)$ and $\cos(A)$ are not like terms.

e. Combine like terms.

$$2-\sin(B)+2\sin(B)=2+\sin(B)$$

Note that $-\sin(B)$ means $-1\cdot\sin(B)$.

It is often easier to know how a **trigonometric expression** (an expression involving trigonometric functions) will simplify once you try simplification techniques. In Example **3.4.1***a*, most students new to Trigonometry would likely never have looked at $\cos(\theta)\tan(\theta)+\sin(\theta)$ and thought, "Hey, I bet that simplifies down to something nice... like $2\sin(\theta)$." Luckily, as you move forward in Trigonometry (and mathematics), you develop an intuition for when an expression can be simplified; however, cultivating this intuition takes time and experimentation.

? Checkpoint 3.4.1

Simplify each expression.

a. $(1-\cos(\theta))(1+\cos(\theta))$
b. $2\cos(t)-4\cos(w)\sin(w)+3\cos(t)-2\cos(w)$

Answers

a. $\sin^2(\theta)$
b. $5\cos(t)-4\cos(w)\sin(w)-2\cos(w)$

Caution

In Checkpoint **3.4.1***b*, note that $\cos(t)$ and $\cos(w)$ are not like terms. (We can choose values for t and w so that $\cos(t)$ and $\cos(w)$ have different values.)

Rewriting Trigonometric Expressions

It is often necessary, especially in Calculus, to rewrite a trigonometric expression in terms of a single trigonometric function. To do so, we must use identities.

Example 3.4.2

a. Rewrite $\sin(\theta)\cos^2(\theta)$ as an expression involving only sums or differences of powers of $\sin(\theta)$.
b. Rewrite $\cot(\theta)$ in terms of only $\cos(\theta)$.

Solutions

a. Using one of the alternate forms of the Pythagorean Identity, we replace $\cos^2(\theta)$ with $1-\sin^2(\theta)$ to get

$$\begin{aligned} \sin(\theta)\cos^2(\theta) &= \sin(\theta)(1-\sin^2(\theta)) && \text{(Pythagorean Identity)} \\ &= \sin(\theta)-\sin^3(\theta) && \text{(Distributive Law)} \end{aligned}$$

b.

$$\begin{aligned} \cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} && \text{(Ratio Identities)} \\ &= \frac{\cos(\theta)}{\pm\sqrt{1-\cos^2(\theta)}} && \text{(Pythagorean Identities (alternate form))} \\ &= \pm\frac{\cos(\theta)}{\sqrt{1-\cos^2(\theta)}} \end{aligned}$$

If we graph the original expression from Example 3.4.2*a* as $y_1=\sin(x)\cos^2(x)$ and our resulting equivalent expression as $y_2=\sin(x)-\sin^3(x)$, we see that they have the same graph, as shown in Figure 3.4.1 below.[1]

Figure 3.4.1

This should convince us that $\sin(\theta)\cos^2(\theta)$ truly is equivalent to $\sin(x)-\sin^3(x)$.

The results of Example 3.4.2 can be thought of as two new identities,

$$\sin(\theta)\cos^2(\theta)=\sin(\theta)-\sin^3(\theta) \quad \text{and} \quad \cot(\theta)=\pm\frac{\cos(\theta)}{\sqrt{1-\cos^2(\theta)}},$$

however, before you get too concerned with having to memorize these as two more identities, let's be clear:

> *Unless formally stated as a theorem, there is no need to memorize the hundreds of identities we will create, prove, or encounter in Trigonometry.*

This means that the only identities you are responsible for memorizing (so far) are the Reciprocal, Ratio, and Pythagorean Identities.

? Checkpoint 3.4.2

a. Rewrite $\sin^2(\alpha)\cos^2(\alpha)$ as an expression involving only sums or differences of powers of $\cos(\alpha)$.
b. Verify your identity by graphing.

Answers

$\cos^2(\alpha) - \cos^4(\alpha)$

The following example is extremely useful in Calculus II (Integral Calculus).

✓ Example 3.4.3

Simplify the expression

$$\frac{3}{\sqrt{x^2-4}}$$

as much as possible by substituting $2\sec(\theta)$ for x.

Solution

$$\begin{aligned}
\frac{3}{\sqrt{x^2-4}} &= \frac{3}{\sqrt{(2\sec(\theta))^2-4}} && \text{(substituting } 2\sec(\theta) \text{ for } x\text{)}\\
&= \frac{3}{\sqrt{4\sec^2(\theta)-4}} && \text{(Laws of Exponents)}\\
&= \frac{3}{\sqrt{4(\sec^2(\theta)-1)}} && \text{(factoring)}\\
&= \frac{3}{2\sqrt{\sec^2(\theta)-1}} && \text{(simplifying)}\\
&= \frac{3}{2\sqrt{\tan^2(\theta)}} && \text{(Pythagorean Identity: } \tan^2(\theta)=\sec^2(\theta)-1\text{)}\\
&= \frac{3}{2|\tan(\theta)|} && \text{(see comments below)}
\end{aligned}$$

Our solution to Example 3.4.3 needs some clarification to ensure you understand what happened. Most of the work should be understandable; however, two steps might throw you off.

First, from the Pythagorean Identities, we used the fact that

$$1+\tan^2(\theta)=\sec^2(\theta);$$

however, we modified this identity slightly by subtracting 1 from both sides to get

$$\tan^2(\theta)=\sec^2(\theta)-1.$$

The implication of that subtle modification cannot be overstated.

> *Being comfortable with the available identities and willing to manipulate them as needed will play a critical role in your success in Trigonometry.*

The second item that needs our attention is the mathematical equivalence

$$\sqrt{\tan^2(\theta)}=|\tan(\theta)|.$$

A lot of students forget about the absolute values. Let's focus on what is happening here.

Suppose a friend of yours is claiming that the equation $\sqrt{x^2} = x$ is an identity (this is the same as someone saying $\sqrt{\tan^2(\theta)} = \tan(\theta)$). As an astute mathematics student, you know that even though the equation is true for all *positive* values of x, it is false for negative values of x. For example, if $x = -3$, then

$$\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$$

so $\sqrt{x^2} \neq x$. The radical symbol $\sqrt{\ }$ stands for the **principle square root** - that is, the nonnegative square root. Therefore, the left side of the equation, $\sqrt{x^2}$, is never negative. Thus, $\sqrt{x^2}$ cannot equal x when x is a negative number. The equation is false for $x < 0$.

One way to see that $\sqrt{x^2}$ and x are not equivalent is to compare the graphs of $Y_1 = \sqrt{x^2}$ and $Y_2 = x$, shown in Figure 3.4.2 below. You can see that $\sqrt{x^2}$ and x do not have the same value for $x < 0$.

Figure 3.4.2

Coming back to the last step in the solution of Example 3.4.3, we should now feel comfortable saying that $\sqrt{\tan^2(\theta)} \neq \tan(\theta)$ and we should be okay with saying

$$\sqrt{\tan^2(\theta)} = |\tan(\theta)|.$$

Checking the Plausibility of Identities Graphically

Before jumping into how to rigorously prove a trigonometric identity, let's focus on ways to show that a *claimed* identity is *not* an identity.

From the discussion after Example 3.4.3, we can see that, to check whether an equation *might be* an identity, we can compare graphs of Y_1=(left side of the equation) and $Y_2 =$ (right side of the equation). If the two graphs are identical, it is plausible that the equation is an identity. If the two graphs differ, the equation is *not* an identity.

Read that last paragraph again.

> *You cannot use a graph to prove an equation is an identity; however, you can use a graph to demonstrate it is not an identity.*

This is crucial to understand. The following example provides some clarity.

Example 3.4.4

Which of the following equations *might* be identities?

a. $\sin(2\alpha) = 2\sin(\alpha)$
b. $\cos\left(x + \frac{\pi}{180}\right) = \cos(x)$
c. $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$

Solutions

a. Compare the graphs of $y_1 = \sin(2x)$ and $y_2 = 2\sin(x)$. The Desmos graphs for both equations are shown in the figure below.

Figure 3.4.3

Because there are two distinct graphs, the expressions $\sin(2x)$ and $2\sin(x)$ are not equivalent, and consequently, $\sin(2\alpha) = 2\sin(\alpha)$ is not an identity.

b. This time we graph $y_1 = \cos\left(x + \frac{\pi}{180}\right)$ and $y_2 = \cos(x)$.

Figure 3.4.4

Although the graphs appear identical, when we zoom in, we see that the graphs are, indeed, not the same.

Figure 3.4.5

The graphs are so close together that Desmos' resolution does not distinguish them, but zooming in reveals that they are not identical. Because the two graphs differ, the equation $\cos\left(x+\frac{\pi}{180}\right)=\cos(x)$ is not an identity.

c. Letting $y_1=\cos^2\left(\frac{x}{2}\right)$ and $y_2=\frac{1+\cos(x)}{2}$, we get the following graph from Desmos.

Figure 3.4.6

These two graphs *look* identical; however, it is best to zoom in to double-check.

Figure 3.4.7

No matter how much we zoom in, the two graphs *appear* identical. Therefore, we can say that it is *plausible* that $\cos^2\left(\frac{\theta}{2}\right)=\frac{1+\cos(\theta)}{2}$ is an identity.

Example **3.4.4** has a few cautionary tales.

Caution: The Trouble with Technology

1. Example **3.4.4***b* illustrates that graphs can be deceiving: even if two graphs look identical, it is always a good idea to zoom in or check some numerical values.
2. The related equation is *not* an identity if the two graphs are different.
3. Graphs cannot be used to prove that an equation is an identity.

Checkpoint 3.4.4

Use graphs to decide which of the following equations *might* be identities.

a. $\cos(2\theta)=2\cos(\theta)$

b. $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$

c. $\cos\left(\theta^2\right) = \cos^2(\theta)$

Answers

(b)

Proving Identities

We have proved a trigonometric identity when we show that one trigonometric expression is equivalent to another. In Example **3.4.1***a* we proved that the equation

$$\cos(\theta)\tan(\theta) + \sin(\theta) = 2\sin(\theta)$$

is an identity; it is valid for all values of θ (as long as the tangent function is defined).

A common strategy for proving an identity is to transform one side of the equation using equivalent expressions until it is identical to the other side. To help us choose the transformations at each step of the proof, we try to match the algebraic form of the final expression.

Example 3.4.5

Prove the identity

$$1 + \tan^2(t) = \frac{1}{\cos^2(t)}.$$

Solution

By manipulating the *left side* of the equation, we will show that the expression $1 + \tan^2(t)$ is equivalent to $\frac{1}{\cos^2(t)}$. First, we use the Ratio Identities to write the expression in terms of sines and cosines:

$$1 + \tan^2(t) = 1 + \left(\frac{\sin(t)}{\cos(t)}\right)^2 = 1 + \frac{\sin^2(t)}{\cos^2(t)}$$

Next, we notice that the right side of the proposed identity has only one term, so we combine the terms on the *left side*. So that the fractions have the same denominator, we write 1 as $\frac{\cos^2(t)}{\cos^2(t)}$.

$$1 + \frac{\sin^2(t)}{\cos^2(t)} = \frac{\cos^2(t)}{\cos^2(t)} + \frac{\sin^2(t)}{\cos^2(t)} = \frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)}$$

Finally, we apply the Pythagorean Identity to the numerator.

$$\frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)}$$

Thus, $1 + \tan^2(t) = \frac{1}{\cos^2(t)}$, and the identity is proved.

When you write out the proof of an identity, your goal is to transform the expression on one side of the identity into the expression on the other, showing one step of the calculation on each line of your proof. You can justify each step to the right of the calculation. The proof of the identity in the previous example would look like this:

$$\begin{aligned}
\text{LHS} &= 1+\tan^2(t) \\
&= 1+\left(\frac{\sin(t)}{\cos(t)}\right)^2 && \text{(Ratio Identities)} \\
&= 1+\frac{\sin^2(t)}{\cos^2(t)} && \text{(Laws of Exponents)} \\
&= \frac{\cos^2(t)}{\cos^2(t)}+\frac{\sin^2(t)}{\cos^2(t)} && \text{(Get a common denominator to combine fractions)} \\
&= \frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)} && \text{(adding fractions)} \\
&= \frac{1}{\cos^2(t)} && \text{(Pythagorean Identity)} \\
&= \text{RHS}
\end{aligned}$$

For now, we will focus on transforming the left side of an equation into the right side; however, as we move forward in Trigonometry, we will adopt the rule of thumb of transforming the more "complicated" side of the equation into the other side.

Example 3.4.6

Prove the identity.

$$(\sin(\theta)-\cos(\theta))^2-1=-2\sin(\theta)\cos(\theta)$$

Solution

$$\begin{aligned}
\text{LHS} &= (\sin(\theta)-\cos(\theta))^2-1 \\
&= \sin^2(\theta)-2\cos(\theta)\sin(\theta)+\cos^2(\theta)-1 && \text{(distributing)} \\
&= \sin^2(\theta)+\cos^2(\theta)-1-2\cos(\theta)\sin(\theta) && \text{(Commutative Property of Addition)} \\
&= 1-1-2\cos(\theta)\sin(\theta) && \text{(Pythagorean Identity)} \\
&= -2\cos(\theta)\sin(\theta) && \text{(simplifying)} \\
&= \text{RHS}
\end{aligned}$$

Checkpoint 3.4.6

Prove the identity $2\cos^2(x)-1=1-2\sin^2(x)$

Answer

$2\cos^2(x)-1=2(1-\sin^2(x))=2-2\sin^2(x)-1=1-2\sin^2(x)$

Before we leave this section, it's important to note that you will be proving many identities in Trigonometry. It takes PRACTICE! We have introduced the idea of a proof early so that we spend a good deal of time sharpening those skills and giving advice on tactics as we move forward. For now, the best advice is to do all the proofs in the homework section - even if your instructor does not assign them.

Footnotes

[1] If you tried to graph these functions using your graphing calculator or another graphing technology and didn't get graphs similar to those in Figure **3.4.1**, it is likely because your graphing device is in **degree mode**. The graphs we create in Trigonometry require a mode called **radian mode**. The meaning of these modes will be explained later.

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3.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Proving Identities
  - Fundamental Identities - Tactics for Proving Trigonometric Identities
- Trigonometric Substitution and Simplifying Trigonometric Expressions
  - Fundamental Identities - The Pythagorean Identities: Theorem, Proof, and Examples

KEY CONCEPTS

COMMON MISTAKES

- **Confusing Expressions with Equations:** Expressions do not have equals signs and are simplified, whereas equations have equals signs and are solved.
- **Treating Different Angle Variables as Like Terms:** Trigonometric functions with different angle variables, such as $\cos(t)$ and $\cos(w)$, are not like terms and cannot be combined.
- **Forgetting Absolute Value with Square Roots:** The square root of a squared expression is its absolute value, not just the expression itself. For example, $\sqrt{\tan^2(\theta)} = |\tan(\theta)|$, not just $\tan(\theta)$.
- **The Trouble with Technology:**
  - Graphs can be deceiving; two different graphs may appear identical unless you zoom in.
  - If the graphs of the two sides of an equation are different, the equation is not an identity.
  - You cannot use a graph to prove that an equation is an identity, only to show it is plausible or to disprove it.

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3.4.2: HOMEWORK

READING QUESTIONS

1. What is the difference between simplifying an expression and solving an equation?
2. What is a common first strategy for simplifying a trigonometric expression that involves multiple trigonometric functions?
3. In the expression $5\cos(t) - 2\cos(w)$, can the two terms be combined? Why or why not?
4. Explain the difference between $\sqrt{x^2}$ and x. Why is this important when simplifying expressions like $\sqrt{\tan^2(\theta)}$?
5. Can you use a graph to prove that an equation is an identity? Explain your answer.
6. How can you use a graph to prove that an equation is *not* an identity?
7. What does it mean if the graphs of the two sides of an equation appear to be identical?
8. What is the general strategy described in this section for proving a trigonometric identity?
9. In Example 1b, what algebraic process is used to multiply $(\cos(\theta) + \sin(\theta))^2$?
10. What fundamental identity is used to simplify $\cos^2(\theta) + \sin^2(\theta)$ to 1 in Example 1c?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, simplify the expression.

1. $\frac{\frac{1}{x}}{\frac{1}{y}}$
2. $\frac{1}{x} - \frac{1}{y}$
3. $\frac{1}{x} - x$
4. $\frac{y}{x} + \frac{1}{y}$
5. $\frac{x}{y} - \frac{y}{x}$

For the following exercises, multiply.

6. $(x-y)^2$
7. $(x-1)(x+1)$
8. $(x-3)(x+5)$

Answer

1. $\frac{y}{x}$
2. $\frac{y-x}{xy}$
3. $\frac{(1-x)(1+x)}{x}$
4. $\frac{y^2+x}{xy}$
5. $\frac{(x-y)(x+y)}{xy}$
6. $x^2 - 2xy + y^2$
7. $x^2 - 1$
8. $x^2 + 2x - 15$

HOMEWORK

VOCABULARY CHECK

1. A(n) ___ is a combination of symbols that are mathematically "well-formed."
2. You ___ equations and ___ expressions.

CONCEPT CHECK

3. Explain how to use graphs to disprove an identity.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

4. To check whether an equation is an identity, we can compare graphs of the left side of the equation and the right side of the equation. If the two graphs agree, the equation is an identity.
5. To prove an identity, we write one side of the equation in equivalent forms until it is identical to the other side of the equation.

BASIC SKILLS

For the following exercises, use identities to rewrite each expression.

6. $\cos(\theta)$ as an expression in $\sin(\theta)$ only
7. $\sin(\theta)$ as an expression in $\cos(\theta)$ only
8. $\cot(\theta)$ as an expression in $\sin(\theta)$ only
9. $\csc(\theta)$ as an expression in $\cos(\theta)$ only

For the following exercises, use graphs to decide which of the following equations *might be* identities.

10. $\sin 2t = 2\sin t$
11. $\cos\theta + \sin\theta = 1$
12. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
13. $\frac{\tan^2 x}{1+\tan^2 x} = \sin^2 x$
14. $\tan x + \frac{1}{\tan x} = \sin x \cos x$

For the following exercises, write each expression in terms of $\sin(\theta)$ and $\cos(\theta)$ and simplify, if possible.

15. $\csc(\theta)\tan(\theta)$
16. $\frac{\sec(\theta)}{\csc(\theta)}$
17. $\frac{\sec(\theta)}{\tan(\theta)}$
18. $\frac{\cot(\theta)}{\tan(\theta)}$

19. $\sec(\theta)+\tan(\theta)$

20. $\sin(\theta)\tan(\theta)+\sec(\theta)$

21. $\csc(\theta)-\cot(\theta)\cos(\theta)$

For the following exercises, add or subtract the expressions and simplify completely. Leave your answers in terms of sine or cosine.

22. $\cos(\theta)-\dfrac{1}{\sin(\theta)}$

23. $\dfrac{1}{\sin(\theta)}-\dfrac{1}{\cos(\theta)}$

24. $\tan(\theta)+\csc(\theta)$

25. $\dfrac{1}{\cos^2\beta}-\dfrac{\sin^2\beta}{\cos^2\beta}$

26. $\dfrac{1}{\sin^2\phi}-\dfrac{1}{\tan^2\phi}$

For the following exercises, multiply and simplify completely.

27. $(\sec(\theta)-1)(\sec(\theta)+1)$

28. $(\sin(\theta)+3)(\sin(\theta)+5)$

29. $(\cos(\theta)+1)(\cos(\theta)-1)$

30. $(\sin(\theta)-\cos(\theta))^2$

31. $(\sin(\theta)-9)^2$

For the following exercises, simplify the expression completely. Write your answers in terms of a single trigonometric function or a constant.

32. $\cos^2\alpha(1+\tan^2\alpha)$

33. $\cos^3\phi+\sin^2\phi\cos\phi$

34. $\tan^2 A-\tan^2 A\sin^2 A$

35. $\cos^2 B\tan^2 B+\cos^2 B$

36. $\dfrac{1-\cos^2 z}{\cos^2 z}$

37. $\dfrac{\sin t}{\cos t\tan t}$

For the following exercises, use identities to rewrite each expression.

38. $2-\cos^2\theta+2\sin\theta$ as an expression in $\sin\theta$ only

39. $3\sin^2 B+2\cos B-4$ as an expression in $\cos B$ only

40. $\cos^2\phi-2\sin^2\phi$ as an expression in $\cos\phi$ only

41. $\cos^2\phi\sin^2\phi$ as an expression in $\sin\phi$ only

For the following exercises, use the given substitution and completely simplify the result.

42. $\sqrt{25-x^2}$; substituting $5\sin(\theta)$ for x

43. $\sqrt{x^2+1}$; substituting $\tan(\theta)$ for x

44. $\sqrt{x^2-9}$; substituting $3\sec(\theta)$ for x

45. $\sqrt{81-9x^2}$; substituting $3\sin(\theta)$ for x

46. $\sqrt{36x^2+1296}$; substituting $6\tan(\theta)$ for x

For the following exercises, show that the equation is an identity by transforming the left side into the right side.

47. $\cos(\theta)\csc(\theta)\tan(\theta)=1$

48. $\dfrac{\cos(\theta)}{\sec(\theta)}=\cos^2(\theta)$

49. $\dfrac{\csc(\theta)\tan(\theta)}{\sec(\theta)}=1$

50. $\cos(\theta)\cot(\theta)+\sin(\theta)=\csc(\theta)$

51. $\tan^2(\theta)+1=\sec^2(\theta)$

52. $\csc(\theta)\tan(\theta)-\cos(\theta)=\dfrac{\sin^2(\theta)}{\cos(\theta)}$

53. $(1+\sin w)(1-\sin w)=\cos^2 w$

54. $(\cos\theta-1)(\cos\theta+1)=-\sin^2\theta$

55. $(\cos\theta-\sin\theta)^2=1-2\sin\theta\cos\theta$

56. $\sin^2 x-\cos^2 x=1-2\cos^2 x$

57. $\sec(\theta)(\sin(\theta)+\cos(\theta))=\tan(\theta)+1$

58. $\tan\theta\cos\theta=\sin\theta$

59. $\dfrac{\sin\mu}{\tan\mu}=\cos\mu$

60. $\cos^4 x-\sin^4 x=\cos^2 x-\sin^2 x$

61. $1-2\cos^2 v+\cos^4 v=\sin^4 v$

62. $\dfrac{\sin u}{1+\cos u}=\dfrac{1-\cos u}{\sin u}$
Hint. Multiply the numerator and denominator of the left side by $1-\cos u$.

63. $\dfrac{\sin v}{1-\sin v}=\dfrac{\tan v(1+\sin v)}{\cos v}$
Hint. Multiply the numerator and denominator of the left side by $1+\sin v$.

For the following exercises, rewrite tangents in terms of sines and cosines, then simplify each side separately (without adding, subtracting, multiplying, or dividing between sides). Finally, show the equation is an identity by transforming the left side into the right side. (These problems involve simplifying complex fractions. See the Algebra Refresher to review this skill.)

64. $\dfrac{\tan\alpha}{1+\tan\alpha}=\dfrac{\sin\alpha}{\sin\alpha+\cos\alpha}$

65. $\dfrac{1-\tan u}{1+\tan u}=\dfrac{\cos u-\sin u}{\cos u+\sin u}$

66. $\dfrac{1+\tan^2\beta}{1-\tan^2\beta}=\dfrac{1}{\cos^2\beta-\sin^2\beta}$

67. $\tan^2 v-\sin^2 v=\tan^2 v\sin^2 v$

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CHAPTER OVERVIEW

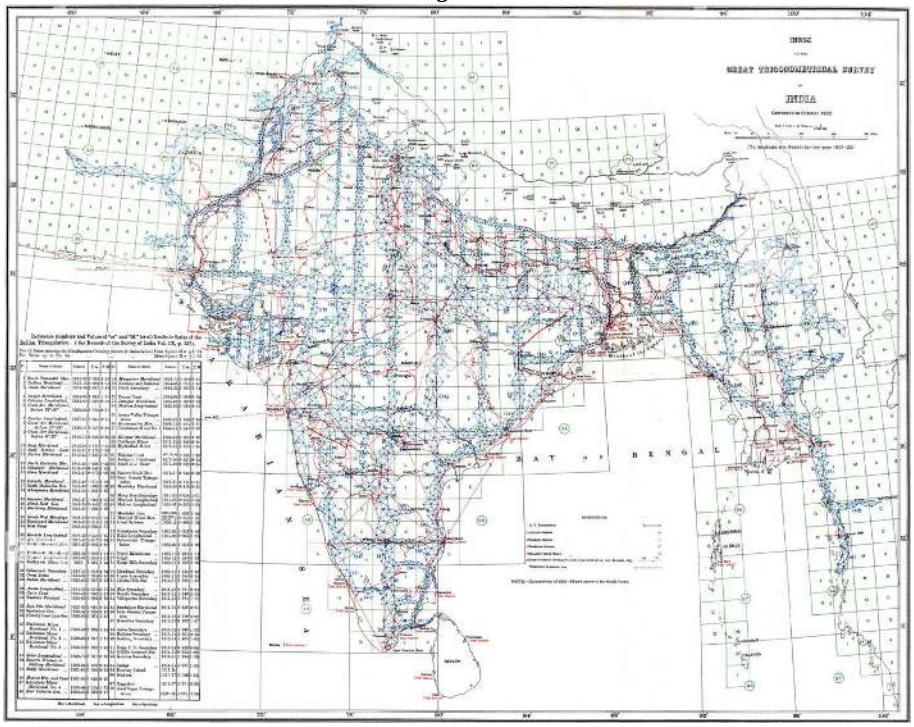
Chapter 4: Right Triangle Trigonometry

How would one measure the distance to an inaccessible object, such as a ship at sea? In the 6th century BC, the Greek philosopher Thales estimated the distances to ships at sea using **triangulation**, a method for calculating distances by forming triangles. Using Trigonometry and the measured length of just one side, the lengths of the other sides can be calculated.

Triangulation has been used to compute distances ever since. In the 16th century, mapmakers used triangulation to position distant places accurately. As new methods in navigation and astronomy required greater precision, the idea of a survey using chains of triangles was developed.

In 1802, the East India Company embarked on the Great Trigonometrical Survey of India, which aimed to measure the entire Indian subcontinent with scientific precision.

Figure 1



The surveyors began by measuring a baseline near Madras. The baseline was the only distance they measured; all other distances were calculated from it using measured angles. Each calculated distance became the base side of another triangle used to calculate the distance to another point, which in turn started another triangle. Eventually this process formed a chain of triangles connecting the origin point to other locations.

Because of the size of the area to be surveyed, the surveyors did not triangulate the whole of India. Instead, they created a ”gridiron” of triangulation chains running from north to south and east to west, as seen on the survey map above.

The survey, completed in 1871, calculated the heights of the Himalayan giants: Everest, K2, and Kanchenjunga. It also provided one of the first accurate measurements of a section of an arc of longitude.

Today, triangulation is used for many purposes, including surveying, navigation, metrology, astrometry, binocular vision, and location of earthquakes.

4.1: The Trigonometric Functions - Right Triangle Definition

4.1.1: Resources and Key Concepts

4.1.2: Homework
4.2: Technology and Right Triangle Trigonometry
4.2.1: Resources and Key Concepts
4.2.2: Homework
4.3: Solving Right Triangles
4.3.1: Resources and Key Concepts
4.3.2: Homework
4.4: Applications Involving Right Triangles
4.4.1: Resources and Key Concepts
4.4.2: Homework

4.1: The Trigonometric Functions - Right Triangle Definition

Note to the Instructor (click to expand)

This section introduces the student to the Right Triangle definition of the trigonometric functions. As such, all angles will be acute (and in degrees); however, if given an angle (rather than the sides of a triangle) and asked to evaluate a trigonometric function, the angle will strictly be an acute *special angle*. Therefore, using technology (e.g., calculators) to evaluate any trigonometric function in this section is *strictly avoided* so as to get the student familiar with right triangle trigonometry at a fundamental level.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: The Pythagorean Theorem is essential in this section for finding the length of a missing side in a right triangle, which is a required first step in many problems (e.g., Example 1).
- **Solving Equations**
  - Solving Literal Equations: This skill is needed for many problems in Trigonometry.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Find the value of a trigonometric function of an unknown angle in a right triangle given at least two sides of the triangle.
- Compute the values of the remaining trigonometric functions of an unknown angle within a right triangle given one trigonometric ratio.
- Use the Ratio, Reciprocal, and Pythagorean Identities, along with Right Triangle Trigonometry, to compute trigonometric values.
- Find the exact value of a trigonometric function for a special angle.
- Use the Cofunction Identities to find the value of a trigonometric function.
- Use exact values to simplify an expression involving trigonometric functions.
- Recognize and understand the constraints on the values of certain trigonometric functions.

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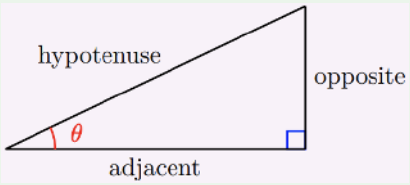
Introduction to Right Triangle Trigonometry

Our initial foray into Trigonometry involved defining the trigonometric functions in terms of points on the Cartesian plane and the distances from those points to the origin. In this chapter, we focus on Trigonometry outside of a coordinate system. Specifically, we redefine the trigonometric functions in terms of ratios of sides of a right triangle.

Definition: Trigonometric Functions (Right Triangle Definition)

If θ is one of the angles in a right triangle, where the side opposite θ is called "opposite," the side adjacent to θ is called "adjacent," and the hypotenuse is named as such, we define the trigonometric functions of θ as follows:

Function		Ratio	Function		Ratio
$\sin(\theta)$	$=$	$\frac{\text{opposite}}{\text{hypotenuse}}$	$\csc(\theta)$	$=$	$\frac{\text{hypotenuse}}{\text{opposite}}$
$\cos(\theta)$	$=$	$\frac{\text{adjacent}}{\text{hypotenuse}}$	$\sec(\theta)$	$=$	$\frac{\text{hypotenuse}}{\text{adjacent}}$
$\tan(\theta)$	$=$	$\frac{\text{opposite}}{\text{adjacent}}$	$\cot(\theta)$	$=$	$\frac{\text{adjacent}}{\text{opposite}}$



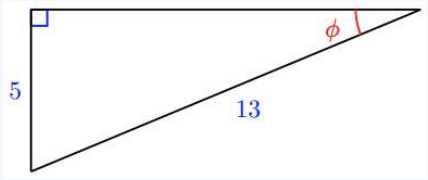
While this definition looks different than our coordinate definition of the trigonometric functions, it is, in fact, the same definition if we restrict the coordinates in our original definition to the first quadrant. While it might seem limiting for us to create a new definition of the trigonometric functions by restricting our original definition, this new definition serves a greater purpose - it simplifies Trigonometry immensely. It allows us to separate the triangle from the Cartesian coordinate system.

Example 4.1.1

Find all trigonometric ratios of the labeled angle in each triangle.

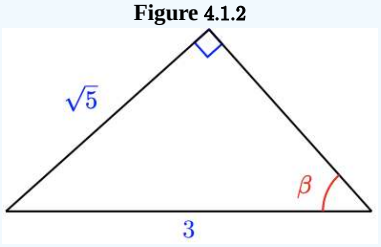
a.

Figure 4.1.1



b.

Figure 4.1.2



Solutions

a. The side opposite angle ϕ is 5 and the hypotenuse is 13. Let x be the length of the side adjacent to ϕ. To find x, we invoke the Pythagorean Theorem.

$$
\begin{aligned}
x^2 + 5^2 &= 13^2 && \text{(Pythagorean Theorem)} \\
\implies x^2 + 25 &= 169 && \text{(simplifying)} \\
\implies x^2 &= 144 && \text{(subtracting 25 from both sides)} \\
\implies x &= \pm 12 && \text{(Extraction of Roots)} \\
\implies x &= 12 && (x \text{ must be positive - it's a side length})
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sin(\phi) &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} & \csc(\phi) &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{13}{5} \\
\cos(\phi) &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} & \sec(\phi) &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{13}{12} \\
\tan(\phi) &= \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12} & \cot(\phi) &= \frac{\text{adjacent}}{\text{opposite}} = \frac{12}{5}
\end{aligned}
$$

b. The side opposite angle β is $\sqrt{5}$ and the hypotenuse is 3. Labeling the side adjacent to β as x, we get

$$
\begin{aligned}
x^2 + \left(\sqrt{5}\right)^2 &= 3^2 && \text{(Pythagorean Theorem)} \\
\implies x^2 + 5 &= 9 && \text{(simplifying)} \\
\implies x^2 &= 4 && \text{(subtracting 5 from both sides)} \\
\implies x &= 2 && \text{(Extraction of Roots)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sin(\beta) &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} & \csc(\beta) &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{3}}{\sqrt{5}} \\
\cos(\beta) &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{3} & \sec(\beta) &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{2} \\
\tan(\beta) &= \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{5}}{2} & \cot(\phi) &= \frac{\text{adjacent}}{\text{opposite}} = \frac{2}{\sqrt{5}}
\end{aligned}
$$

The result of the computation to find the missing side in Example **4.1.1** part (a) reveals one of the **Pythagorean triples**, $(5, 12, 13)$.

Definition: Pythagorean Triple

A **Pythagorean triple** consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$.

There are an infinite number of Pythagorean triples, but the common ones found in Trigonometry textbooks are $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$. Triangles having sides lengths matching Pythagorean triples get a special name.

Definition: Pythagorean Triangle

A right triangle whose sides form a Pythagorean triple is called a **Pythagorean triangle**.

It can be proved that every Pythagorean triangle must be a right triangle. For more information, see this Wikipedia page.

? Checkpoint 4.1.1

Find the sine of the labeled angle in the triangle below. Round the answer to 4 decimal places.

Figure 4.1.3

Answer

$\sin(\alpha) \approx 0.7442$

The following example illustrates a necessary skill for us as we move forward in Trigonometry.

Example 4.1.2

Write algebraic expressions for the six trigonometric ratios of the angle θ if the hypotenuse has length 2 and the side adjacent to θ has length D.

Solution

We begin by finding the missing side, which we shall call x.

$$\begin{aligned} & D^2 + x^2 = 2^2 && \text{(Pythagorean Theorem)} \\ \Longrightarrow\ & D^2 + x^2 = 4 \\ \Longrightarrow\ & x^2 = 4 - D^2 \\ \Longrightarrow\ & x = \sqrt{4 - D^2} \end{aligned}$$

We can now fill in the values of the trigonometric ratios.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{4-D^2}}{2} \qquad \csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{4-D^2}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{D}{2} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{D}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{4-D^2}}{D} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{D}{\sqrt{4-D^2}}$$

Checkpoint 4.1.2

If the hypotenuse of a right triangle is h and the side opposite θ is L, write expressions in h and L for all six trigonometric functions of θ.

Answer

$$\sin(\theta) = \frac{L}{h} \qquad \csc(\theta) = \frac{h}{L}$$

$$\cos(\theta) = \frac{\sqrt{h^2 - L^2}}{h} \qquad \sec(\theta) = \frac{h}{\sqrt{h^2 - L^2}}$$

$$\tan(\theta) = \frac{L}{\sqrt{h^2 - L^2}} \qquad \cot(\theta) = \frac{\sqrt{h^2 - L^2}}{L}$$

Caution: Right Triangle Trigonometry Works For... Right Triangles

To calculate the trigonometric function of an angle within a triangle, it is crucial that the triangle be a *right* triangle. For example, in the triangle below, $\sin(\alpha) = \frac{4}{7}$, because $\triangle ABC$ is a right triangle. It is *not* true that $\sin(\alpha) = \frac{5}{7}$, or $\sin(\alpha) = \frac{6}{7}$.

Figure 4.1.4

Revisiting the Fundamental Identities

Our previous identities (the Ratio, Reciprocal, and Pythagorean Identities) still hold when using the right triangle definition of the trigonometric functions. For example, notice that

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\frac{\text{hypotenuse}}{\text{opposite}}} = \frac{1}{\csc(\theta)}.$$

We leave it to the reader to show that all other Reciprocal Identities also hold.

Likewise,

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\sin(\theta)}{\cos(\theta)}.$$

Again, we leave it to the reader to finish showing that both Ratio Identities still hold.

Finally, it is left as a homework exercise to show that the Pythagorean Identity still holds with this new definition.

Example 4.1.3

If θ is an acute angle in a right triangle and $\tan(\theta) = \frac{15}{8}$, compute the remaining trigonometric ratios of θ.

Solution

It is best to sketch the right triangle with θ and where the side adjacent to θ has length 8 and the side opposite θ has length 15.

Figure 4.1.5

Using the Pythagorean Theorem, we get

$$\begin{aligned} 8^2 + 15^2 &= h^2 \\ \Longrightarrow \quad 289 &= h^2 \\ \Longrightarrow \quad 17 &= h \end{aligned}$$

Hence,

$$\sin(\theta) = \frac{15}{17} \Longrightarrow \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{15/17} = \frac{17}{15}$$

$$\cos(\theta) = \frac{8}{17} \Longrightarrow \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{8/17} = \frac{17}{8}$$

$$\tan(\theta) = \frac{15}{8} \Longrightarrow \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{15/8} = \frac{8}{15}$$

Checkpoint 4.1.3

If β is an acute angle in a right triangle and $\csc(\beta) = \frac{2}{\sqrt{3}}$, compute the remaining trigonometric ratios of β.

Answer

$$\sin(\beta) = \frac{\sqrt{3}}{2} \qquad \csc(\beta) = \frac{2}{\sqrt{3}}$$

$$\cos(\beta) = \frac{1}{2} \qquad \sec(\beta) = 2$$

$$\tan(\beta) = \sqrt{3} \qquad \cot(\beta) = \frac{1}{\sqrt{3}}$$

Example 4.1.4

If $\sin(\theta) = \frac{3}{7}$ and $\cos(\theta) = \frac{2\sqrt{10}}{7}$, where θ is an acute angle within a right triangle, find the value of $\cot(\theta)$ without drawing a triangle.

Solution

$$\begin{aligned} \cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} \\ &= \frac{2\sqrt{10}/7}{3/7} \\ &= \frac{2\sqrt{10}}{3} \end{aligned}$$

Checkpoint 4.1.4

If λ is an acute angle in a right triangle, $\tan(\lambda) = \frac{2}{3}$, and $\cos(\lambda) = \frac{3}{\sqrt{13}}$, find $\sin(\lambda)$ using only the Ratio Identities.

Answer

$\sin(\lambda) = \frac{2}{\sqrt{13}}$

Right Triangle Trigonometry and the Special Angles

Now that we have a right triangle definition for the trigonometric functions, we can use our special triangles to help us quickly evaluate the trigonometric functions at $30°$, $45°$, and $60°$ angles. These angles are foundations upon which we build the rest of Trigonometry.

Table 4.1.1: *Fundamental Trigonometric Ratios for the Special Angles*

Angle	Sine	Cosine	Tangent
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

The exact values for these trigonometric ratios should be committed to memory. A great way to remember them is to know the two special triangles in Figure **4.1.6** below. Writing down the three trigonometric ratios for the special angles from these triangles is simple.

Figure 4.1.6: *Left: The* $45° - 45° - 90°$ *special triangle, Right: The* $30° - 60° - 90°$ *special triangle*

For example, we can quickly compute $\tan(60°)$ and $\sec(30°)$ by sketching the $30°$-$60°$-$90°$ triangle and finding the ratios of the relevant sides.

$$\sec(30°) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}} \qquad \tan(60°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

These are *exact* values for the trigonometric ratios, but we can also find decimal approximations. Use a calculator to verify the following approximate values.

		exact value		approximation
$\sec(30°)$	$=$	$\frac{2}{\sqrt{3}}$	$\approx$	1.1547
$\tan(60°)$	$=$	$\sqrt{3}$	$\approx$	1.7321

Caution: Approximations are NOT Exact

It is essential to understand the difference between *exact* and *approximate* values. The decimal approximations above are rounded off. Even if a calculator shows ten or twelve digits, the values are not precisely correct - although they are adequate for most practical calculations.

Computational Note: To Rationalize or Not To Rationalize... That is the Question

Did you notice that we left $\sec(30°) = \frac{2}{\sqrt{3}}$ with a radical in its denominator? This is quite common in Trigonometry. While one can rationalize the denominator to get

$$\sec(30°) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

the need to do so with numerical expressions (expressions *not* involving variables) is an antiquated "throwback" to the days when calculators did not exist. As such, unless it truly simplifies an expression, we will be okay with leaving radicals in the denominators of numerical expressions for the rest of mathematics.

Example 4.1.5

Matterhorn Chocolate bars are sold in boxes shaped like triangular prisms. The two triangular ends of a box are equilateral triangles with 8-centimeter sides. When stacking the boxes on a shelf, the second level will be a triangle's altitude above the bottom level. Find the exact length of the triangle's altitude, h.

Solution

We have seen this question before (near the beginning of the textbook). The altitude divides the triangle into two $30° - 60° - 90°$ right triangles, as shown in Figure 4.1.7 below.

Figure 4.1.7

The altitude is adjacent to the $30°$ angle and the hypotenuse of the right triangle is 8 centimeters. Thus, using the fact that we already know $\cos(30°) = \frac{\sqrt{3}}{2}$, we get

$$\begin{aligned} \cos(30°) &= \frac{\text{adjacent}}{\text{hypotenuse}} && \text{(Fill in the values.)} \\ \Longrightarrow \quad \frac{\sqrt{3}}{2} &= \frac{h}{8} \\ \Longrightarrow \quad h &= 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3} && \text{(multiplying both sides by 8)} \end{aligned}$$

The altitude is *exactly* $4\sqrt{3}$ centimeters long. From this exact answer, we can find approximations to any degree of accuracy we like. You can check that $4\sqrt{3} \approx 6.9282$, so the altitude is approximately 6.9 centimeters long.

Checkpoint 4.1.5

Show that $\cos^2(60°) + \sin^2(60°) = 1$.

Answer

$$\cos^2(60°) + \sin^2(60°) = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$$

Example 4.1.6

Let $x = 60°$ and $y = 45°$. Evaluate.

$$\sqrt{3}\csc(x) - \frac{5}{\sqrt{2}}\sec(y) + \sqrt{6}\tan(90° - x)\sin(y)$$

Solution

Using our two special triangles in Figure 4.1.6, we get

$$\begin{aligned} \sqrt{3}\csc(x) - \frac{5}{\sqrt{2}}\sec(y) + \sqrt{6}\tan(90° - x)\sin(y) &= \sqrt{3}\csc(60°) - \frac{5}{\sqrt{2}}\sec(45°) + \sqrt{6}\tan(90° - 60°)\sin(45°) \\ &= \sqrt{3}\csc(60°) - \frac{5}{\sqrt{2}}\sec(45°) + \sqrt{6}\tan(30°)\sin(45°) \\ &= \cancel{\sqrt{3}}\left(\frac{2}{\cancel{\sqrt{3}}}\right) - \left(\frac{5}{\cancel{\sqrt{2}}}\right)\left(\frac{\cancel{\sqrt{2}}}{1}\right) + \left(\cancel{\sqrt{6}}\right)\left(\frac{1}{\cancel{\sqrt{3}}}\right)\left(\frac{1}{\cancel{\sqrt{2}}}\right) \\ &= 2 - 5 + 1 \\ &= -2 \end{aligned}$$

Checkpoint 4.1.6

Evaluate and write as a single expression.

$$3\cos(45°) - 2\cot(30°)$$

Answer

$$\frac{3-2\sqrt{6}}{\sqrt{2}}$$

Cofunction Identities

The following definition may answer a burning question you have had since being introduced to trigonometric functions.

Definition: Cofunction

The sine and cosine are called **cofunctions**. Likewise, the tangent and cotangent, as well as the secant and cosecant, are called cofunctions.

We say the sine is the cofunction of the cosine, and the cosine is the cofunction of the sine. The same holds for tangent and cotangent, and secant and cosecant. Now that we have a definition for cofunctions, what's the big deal?

The right triangle definition of the trigonometric functions reveals a new identity that was not so obvious when working with the coordinate definition of trigonometric functions. Consider the right triangle in Figure 4.1.8 below.

Figure 4.1.8

By the Triangle Sum, $\angle A + \angle B + \angle C = 180°$; however, $\angle C = 90°$. Therefore, $\angle A + \angle B = 90°$. That is, $\angle A = 90° - \angle B$. By the right triangle definition,

$$\sin(\angle A) = \frac{\text{Opposite } A}{\text{Hypotenuse}} = \frac{\text{Adjacent } B}{\text{Hypotenuse}} = \cos(\angle B) = \cos(90° - \angle A)$$

$$\sec(\angle A) = \frac{\text{Hypotenuse}}{\text{Adjacent } A} = \frac{\text{Hypotenuse}}{\text{Opposite } B} = \csc(\angle B) = \csc(90° - \angle A)$$

$$\tan(\angle A) = \frac{\text{Opposite } A}{\text{Adjacent } A} = \frac{\text{Adjacent } B}{\text{Opposite } B} = \cot(\angle B) = \cot(90° - \angle A)$$

At this point, I hope you can see a budding relationship between a function and its cofunction. The following theorem summarizes the results (the proof of which will be held off until we have more mathematics to work with).

Theorem: Cofunction Identities

The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. That is,

$$\sin(\theta) = \cos(90° - \theta) \quad \text{and} \quad \cos(\theta) = \sin(90° - \theta)$$

$$\sec(\theta) = \csc(90° - \theta) \quad \text{and} \quad \csc(\theta) = \sec(90° - \theta)$$

$$\tan(\theta) = \cot(90° - \theta) \quad \text{and} \quad \cot(\theta) = \tan(90° - \theta)$$

Example 4.1.7

Evaluate (by hand).

$$\sin^2(13°) + \sin^2(77°)$$

Solution

$$\begin{aligned} \sin^2(13°) + \sin^2(77°) &= \sin^2(13°) + (\sin(77°))^2 \\ &= \sin^2(13°) + (\cos(90° - 77°))^2 \\ &= \sin^2(13°) + (\cos(13°))^2 \\ &= \sin^2(13°) + \cos^2(13°) \\ &= 1 \end{aligned}$$

Checkpoint 4.1.7

Write the function in terms of its cofunction.

a. $\sin(13°)$
b. $\cot(60°)$
c. $\sec(y)$

Answers

a. $\sin(13°) = \cos(90° - 13°) = \cos(77°)$
b. $\cot(60°) = \tan(90° - 60°) = \tan(30°)$
c. $\sec(y) = \csc(90° - y)$

Observations of Values for Trigonometric Functions Using Right Triangle Trigonometry

We have now spent a decent amount of time with the trigonometric functions - both defined in terms of coordinates on the Cartesian plane and in terms of ratios of sides of right triangles. It's now time to formalize some observations.

Trigonometric Functions are Invariant When Scaling Right Triangles

The trigonometric function of an angle is the same for any right triangle, no matter the size or orientation of the triangle. Another way to say this is the values of the trigonometric functions are **invariant** under scaling of the right triangle. For example, Figure 4.1.9 (below) shows three right triangles, each with a $50°$ angle. Although the sides of the triangle may be bigger or smaller, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is always the same for that angle because the triangles are similar. Therefore, $\sin(50°)$ is the same for all three right triangles.

Figure 4.1.9

Similarly, the values of all the other trigonometric functions at $50°$ remain constant when we scale the triangles to be larger or smaller.

Some Trigonometric Functions Have Limited Ranges

Consider the right triangle definition of the trigonometric functions. Since the hypotenuse is the longest side of a right triangle, the ratios

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

will always be less than 1.

Moreover, if we think about the coordinate definition of the trigonometric functions,

$$\sin(\theta) = \frac{y}{r} \quad \text{and} \quad \cos(\theta) = \frac{x}{r}.$$

Since $r = \sqrt{x^2 + y^2} \geq \sqrt{y^2} = |y|$ and $r = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |x|$,

$$|y| \leq r \implies \frac{|y|}{r} \leq 1 \implies \left|\frac{y}{r}\right| \leq 1 \implies |\sin(\theta)| \leq 1 \implies -1 \leq \sin(\theta) \leq 1$$

and

$$|x| \leq r \implies \frac{|x|}{r} \leq 1 \implies \left|\frac{x}{r}\right| \leq 1 \implies |\cos(\theta)| \leq 1 \implies -1 \leq \cos(\theta) \leq 1$$

The only times the sine or cosine become 1 or -1 is when the given angle is a quadrantal angle.

Example 4.1.8

If θ is an angle within a right triangle, is it possible for $\sec(\theta) = \frac{1}{2}$? Explain.

Solution

$\sec(\theta)$ is the ratio of the hypotenuse to the adjacent side of θ; however, the hypotenuse of a right triangle is always longer than the other sides. Therefore, the ratio $\frac{\text{hypotenuse}}{\text{adjacent}}$ must be greater than 1 (by the discussion before this example, the secant can attain a value of 1 only if the given angle is a quadrantal angle).

Checkpoint 4.1.8

Can $\csc(\theta)$ ever be 2?

Answer

Yes!

Using Special Angles to Approximate Trigonometric Functions at Other Angles

In a right triangle, as the angle θ increases, $\sin(\theta)$ increases but $\cos(\theta)$ decreases. Figure 4.1.10 (below) demonstrates why this is true. In each right triangle, the hypotenuse has the same length. However, as the angle increases, the opposite side gets longer, and the adjacent side gets shorter.

Figure 4.1.10

We can use the special angles as benchmarks for estimating and mental calculation. For example, we know that $\sin(60°) \approx 0.8660$, so if $\sin(\theta) = 0.95$ for some unknown angle θ within a right triangle, we know that $\theta > 60°$, because as θ increases, the sine of θ

increases as well.

Example 4.1.9

If $\cos(\alpha) > \frac{\sqrt{3}}{2}$, where α is an angle within a right triangle, what can we say about α?

Solution

As an angle increases from $0°$ to $90°$, its cosine decreases (why?). Now, $\cos(30°) = \dfrac{\sqrt{3}}{2}$, so if $\cos(\alpha) > \frac{\sqrt{3}}{2}$, then α must be less than $30°$.

Checkpoint 4.1.9

If $1 < \tan(\beta) < \sqrt{3}$, where β is one of the angles in a right triangle, what can we say about β?

Answer

$45° < \beta < 60°$

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4.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Right Triangle Trigonometry
  - Right Triangle Trigonometry - Trigonometric Functions – A Right Triangle Approach
  - Trigonometry of Right Triangles - Review of Trigonometric Functions - A Right Triangle Approach
- Special Angles
  - Right Triangle Trigonometry - Evaluating Trigonometric Functions for the Special Acute Angles
  - Trigonometry of Right Triangles - Review of Special Triangles in Trigonometry
- Cofunction Identities
  - Right Triangle Trigonometry - The Cofunction Identities

KEY CONCEPTS

DEFINITIONS

- **Trigonometric Functions (Right Triangle Definition):** For an angle θ in a right triangle, the trigonometric functions are defined as ratios of the side opposite θ, the side adjacent to θ, and the hypotenuse. For example, $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$.
- **Pythagorean Triple:** A set of three positive integers, a, b, and c, such that $a^2 + b^2 = c^2$.
- **Pythagorean Triangle:** A right triangle whose side lengths form a Pythagorean triple.
- **Cofunction:** Pairs of trigonometric functions consisting of sine and cosine, tangent and cotangent, and secant and cosecant. For example, the cofunction of sine is cosine.

THEOREMS

- **Cofunction Identities:** The value of a trigonometric function of an angle is equal to the value of its cofunction at the complement of the angle. For example, $\sin(\theta) = \cos(90° - \theta)$ and $\tan(\theta) = \cot(90° - \theta)$.

COMMON MISTAKES

- **Applying Right Triangle Trigonometry to Non-Right Triangles:** The right triangle definitions of trigonometric functions are only valid for angles within a right triangle. They cannot be applied to obtuse or acute triangles in general.
- **Confusing Exact and Approximate Values:** An exact value, like $\frac{\sqrt{3}}{2}$, is precise. A decimal approximation, like 0.8660, is a rounded value. For theoretical work, exact values are required.
- **Rationalizing Numerical Denominators Unnecessarily:** While rationalizing is sometimes useful, expressions like $\frac{2}{\sqrt{3}}$ are perfectly acceptable in trigonometry, and rationalizing them to $\frac{2\sqrt{3}}{3}$ is often an unnecessary step.

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4.1.2: HOMEWORK

READING QUESTIONS

1. In a right triangle, what are the definitions of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ in terms of the opposite side, adjacent side, and hypotenuse?
2. What is a Pythagorean triple?
3. Can the right triangle definitions of the trigonometric functions be applied to an obtuse triangle?
4. If you are given the lengths of the opposite and adjacent sides in a right triangle, what theorem must you use to find the length of the hypotenuse?
5. What is the cofunction of the tangent function?
6. What does the Cofunction Identity theorem state about $\sin(20°)$?
7. Why is the value of $\cos(\theta)$ always less than or equal to 1 for any acute angle θ?
8. Is it possible for $\sec(\theta)$ to equal $\frac{1}{2}$ for an angle θ in a right triangle? Explain why or why not.
9. What happens to the value of $\sin(\theta)$ as θ increases from 0° to 90°?
10. What is the difference between an exact value, like $\sqrt{2}$, and an approximate value, like 1.414?

SKILLS REFRESHER

Review the following skills you will need for this section.

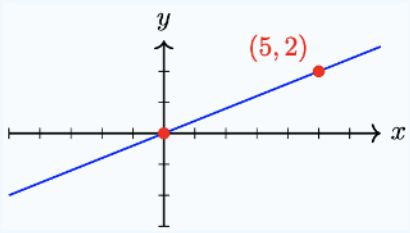
? SKILLS REFRESHER

For the following exercises, write two more ratios equivalent to the given fraction.

1. $\frac{10}{4}$
2. $\frac{6}{8}$
3. 0.6
4. 1.5

For the following exercises, compute the slope of the line.

5.



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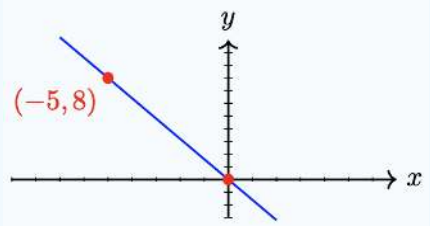

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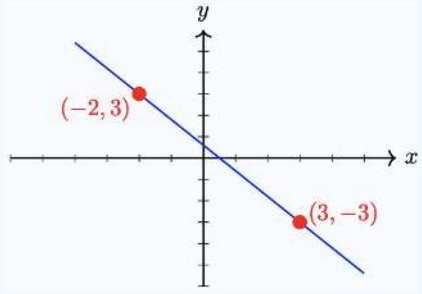

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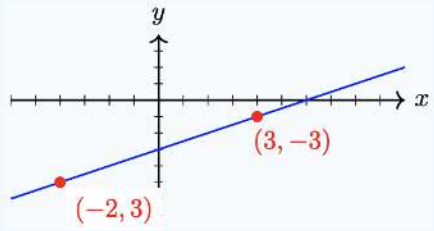


For the following exercises, solve.

9. $\frac{12}{x} = 48$
10. $\frac{60}{x} = 80$

For the following exercises, solve the formula for the specified variable.

11. $C = 2\pi r$ for r (Circumference of a Circle)
12. $r = \frac{d}{t}$ for t
13. $P = 2a + b - c$ for a
14. $x = \frac{2}{3}y + 10$ for y
15. $S = 2\pi rh + 2\pi r^2$ for h (Surface Area of a Right Circular Cylinder)

Answers

(Many answers are possible for 1-4.)

1. $\frac{5}{2}, \frac{20}{8}$
2. $\frac{3}{4}, \frac{12}{16}$
3. $\frac{3}{5}, \frac{12}{20}$
4. $\frac{6}{4}, \frac{12}{8}$
5. $\frac{2}{5}$
6. $\frac{-8}{5}$
7. $\frac{-6}{5}$
8. $\frac{1}{2}$
9. $\frac{1}{4}$
10. $\frac{3}{4}$
11. $r = \frac{C}{2\pi}$

12. $t = \frac{d}{r}$
13. $a = \frac{P-b+c}{2}$
14. $y = \frac{3}{2}(x - 10)$
15. $h = \frac{S-2\pi r^2}{2\pi r}$

HOMEWORK

VOCABULARY CHECK

1. The sequences of natural numbers, (3, 4, 5) and (7, 24, 25), are both examples of ___.
2. Just like the tangent and cotangent, the secant and cosecant are called ___.

CONCEPT CHECK

3. Sketch a figure that illustrates why $\cos 25°$ is the same for every right triangle with a $25°$ angle.
4. Sketch a figure that illustrates why $\cos\theta$ decreases as θ increases from $0°$ to $90°$.
5. Linda says that $\sin\theta = \frac{4}{7}$ in the figure. Is she correct? Why or why not?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. Given a triangle having θ as one of its interior angles, the tangent of the angle θ is the ratio of the side opposite θ to the side adjacent to θ.
7. $\sin(60°) = 0.8660$
8. The value of a trigonometric function of an angle is equal to the value of its cofunction at the complement of the angle.

BASIC SKILLS

For the following exercises, fill in the table.

9.

	sin	cos	tan
θ			
ϕ			

10.

	sin	cos	tan
θ			
ϕ			

11.

	sin	cos	tan
θ			
ϕ			

12.

	sin	cos	tan
θ			
ϕ			

13. In each of the figures for the previous exercises, what is the relationship between the angles θ and ϕ?
14. Study the tables for the previous problems. What do you notice about the values of the sine and cosine for the angles θ and ϕ? Explain why this is true.
15. There is a relationship between the tangent, the sine, and the cosine of any angle. Study the tables for the previous problems to discover this relationship. Write your answer as an equation.

For the following exercises,
(a) Find the length of the unknown side.
(b) Find the sine, cosine, and tangent of θ. Round your answers to four decimal places.

16.

17.

18.

19.

20.

21.

For the following exercises, find exact values for the six trigonometric ratios of the angle θ.

22.

23.

24.

25.

For the following exercises,
(a) Sketch and label the sides of a right triangle with angle θ.
(b) Sketch and label another right triangle with angle θ and longer sides.

26. $\cos\theta = \frac{3}{5}$

27. $\tan\theta = \frac{7}{2}$

28. $\tan\theta = \frac{11}{4}$

29. $\sin\theta = \frac{4}{9}$

30. $\sin\theta = \frac{1}{9}$

31. $\cos\theta = \frac{7}{8}$

For the following exercises, use a trigonometric ratio to write an equation for x in terms of θ.

32.

33.

34.

35.

36.

37.

For the following exercises, write algebraic expressions for the six trigonometric ratios of the angle θ.

38.

39.

40.

41.

For the following exercises, write the expression in terms of sine and cosine, and simplify.

42. $\frac{\csc t}{\cot t}$

43. $\frac{\tan v}{\sec v}$

44. $\sec \beta - \tan \beta$

45. $\cot \alpha + \csc \alpha$

For the following exercises, choose all values from the list below that are exactly equal to, or decimal approximations for, the given trigonometric ratio. (Try not to use a calculator!)

$\sin 30°$	$\cos 45°$	$\sin 60°$	$\tan 45°$	$\tan 60°$
0.5000	0.5774	0.7071	0.8660	1.0000
$\frac{1}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$

46. $\cos 30°$

47. $\sin 45°$

48. $\tan 30°$

49. $\cos 60°$

For the following exercises, evaluate. Give exact values.

50. $\sin(30°)$

51. $\tan(45°)$

52. $\sin(60°)$

53. $\csc(30°)$

54. $\cot(45°)$

55. $\csc(60°)$

SYNTHESIS QUESTIONS

For the following exercises, evaluate.

56. $4\cot(60°) + 2\sec(45°)$

57. $\frac{1}{2}\csc(30°) - \frac{1}{4}\cot(30°)$

For the following exercises, solve the equation assuming that θ is an acute angle in a right triangle.

58. $3\csc(\theta) + 2 = 8$

59. $-2\sec(\theta) + 8 = 6$

60. $\sqrt{2}\sec(\theta) = 2$

61. $\sqrt{3}\cot(\theta) = 1$

62.

a. Use the figure to explain what happens to $\tan\theta$ as θ increases, and why.

b. Use the figure to explain what happens to $\cos\theta$ as θ increases, and why.

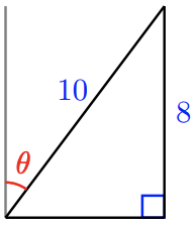
For the following exercises, explain why the trigonometric ratio is *not* correct.

63. $\sin\theta = \frac{5}{9}$

64. $\tan\theta = \frac{4}{7}$

65. $\cos\theta = \frac{21}{20}$

66. $\sin\theta = \frac{8}{10}$



For the following exercises, sketch and label a right triangle, then fill in the blank.

67.

a. If $\sin\theta = 0.2358$, then $\cos(90° - \theta) =$_____.

b. If $\cos\alpha = \frac{3}{11}$, then _____ $(90° - \alpha) = \frac{3}{11}$.

c. If $\sin 42° = n$, then $\cos$_____$= n$.

d. If $\cos 13° = z$, then $\sin$_____$= z$.

68.

a. If $\cos\beta = \frac{2}{\sqrt{7}}$, then $\sin(90° - \beta) =$_____.

b. If $\sin\phi = 0.693$, then _____$-(90° - \phi) = 0.693$.

c. If $\cos 87° = p$, then $\sin$_____$= p$.

d. If $\sin 59° = w$, then $\cos$_____$= w$.

69.

a. If $\sin\phi = \frac{5}{13}$ and $\cos\phi = \frac{12}{13}$, then $\tan\phi =$_____.

b. If $\cos\beta = \frac{1}{\sqrt{10}}$, and $\sin\beta = \frac{3}{\sqrt{10}}$, then $\tan\beta =$_____.

c. If $\tan B = \frac{2}{\sqrt{5}}$ and $\cos B = \frac{\sqrt{5}}{3}$, then $\sin B =$_____.

d. If $\sin W = \sqrt{\frac{3}{7}}$ and $\tan W = \frac{\sqrt{3}}{2}$, then $\cos W =$_____.

70.

a. If $\cos\theta = \frac{2}{\sqrt{10}}$ and $\sin\theta = \sqrt{\frac{3}{5}}$, then $\tan\theta =$_____.

b. If $\sin\alpha = \frac{\sqrt{2}}{4}$, and $\cos\alpha = \frac{\sqrt{14}}{4}$, then $\tan\alpha =$_____.

c. If $\tan A = \frac{\sqrt{7}}{3}$ and $\cos A = \frac{3}{4}$, then $\sin A =$_____.

d. If $\sin V = \sqrt{\frac{10}{5}}$ and $\tan V = \frac{2}{5}$, then $\cos V =$_____.

71. Explain why the cosine of a $73°$ angle is always the same, no matter what size triangle the angle is in. Illustrate your explanation with a sketch.

72.

a. What is the slope of the line through the origin and point P?

b. What is the tangent of the angle θ?

c. On the same grid, sketch an angle whose tangent is $\frac{8}{5}$

73. Which of the following numbers are equal to $\cos 45°$?

a. $\frac{\sqrt{2}}{2}$

b. $\frac{1}{\sqrt{2}}$

c. $\frac{2}{\sqrt{2}}$

d. $\sqrt{2}$

74. Which of the following numbers are equal to $\tan 30°$?

a. $\sqrt{3}$

b. $\frac{1}{\sqrt{3}}$

c. $\frac{\sqrt{3}}{3}$

d. $\frac{3}{\sqrt{3}}$

75. Which of the following numbers are equal to $\tan 60°$?

a. $\sqrt{3}$

b. $\frac{1}{\sqrt{3}}$

c. $\frac{\sqrt{3}}{3}$

d. $\frac{3}{\sqrt{3}}$

76. Which of the following numbers are equal to $\sin 60°$?

a. $\frac{3}{\sqrt{2}}$

b. $\frac{\sqrt{3}}{2}$

c. $\frac{\sqrt{2}}{3}$

d. $\frac{2}{\sqrt{3}}$

CHALLENGE PROBLEMS

77. The diagram shows a unit circle. Find six line segments whose lengths are, respectively, $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

78. The figure shows a unit circle and an angle θ in standard position. Each of the six trigonometric ratios for θ is represented by the length of a line segment in the figure. Find the line segment for each ratio, and explain your choice.

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4.2: Technology and Right Triangle Trigonometry

Note to the Instructor (click to expand)

We finally reach the moment where we are evaluating trigonometric functions at non-special *acute* angles. As such, technology (e.g., Desmos or a calculator) will be necessary. This will also be needed to convert a Decimal Degree angle to DMS (and vice versa), as well as to find the *acute* angle within a right triangle given the sides of the triangle.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Use a calculator to approximate the value of a trigonometric function.
- Perform addition and subtraction on angles expressed in DMS.
- Convert between DMS form and DD form.
- Use a calculator to approximate an acute angle given the value of a trigonometric function.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Using a Calculator to Compute the Values of Trigonometric Functions

The trigonometric values for non-special angles were originally collected into tables and are available on scientific calculators (and other, more recent, technology). For example, let's find the sine of $50°$.

Note: Change to Degree Mode

Before evaluating any trigonometric function, ensure your technology is set to the correct "mode." In Trigonometry, the two standard modes are **degree mode** and **radian mode**. At this point, we will only be working with angles measured in degrees, so you must only concern yourself with degree mode. If using a scientific or graphing calculator, refer to your instruction manual on setting the calculator to degree mode. The scientific calculator in Desmos is set to degree mode by default, so you do not have to make any adjustments if using that technology; however, the *graphing* calculator in Desmos defaults to radian mode. To change Desmos' graphing calculator to degree mode, open the "Graph Settings" and select "Degrees."

Example 4.2.1

Approximate $\sin(50°)$.

Scientific and Graphing Calculators

Press the **SIN** button. Your calculator will display

$$\sin($$

with an open parenthesis as the prompt to enter an angle. Type **50** and press **ENTER** (since your calculator should already be in degree mode, it will interpret the **50** as $50°$). The calculator returns **0.7660444431** (please verify).

Desmos

Using the scientific calculator in Desmos, type **sin(50)** and hit **ENTER** (since the scientific calculator in Desmos defaults to degree mode, it will interpret the **50** as $50°$). Desmos returns **0.7660444431** (please verify).

It is important to note that the values returned by the technology used in Example **4.2.1** are not exact values; it turns out that the sine of $50°$ is an irrational number, and your calculator shows as many digits as its display will allow (not all sine values are as "nice" as the sine of $30°$). Unless otherwise instructed, we round to four decimal places, so we write

$$\sin(50°) \approx 0.7660.$$

Take note of using the approximation sign, $\approx$, instead of the equal sign - this is important!

Checkpoint 4.2.1

a. Use a calculator to complete the table, rounding answers to four decimal places.

θ	$0°$	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$	$70°$	$80°$	$89°$
$\tan(\theta)$										

\)

b. What do you notice about the values of $\tan(\theta)$ as θ increases from $0°$ to $89°$? If the values of $\tan(\theta)$ are plotted against the values of θ, will the graph be a straight line? Why or why not?

Answers

a.

θ	$0°$	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$	$70°$	$80°$	$90°$
$\tan(\theta)$	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713	57.2900

b. The values of $\tan(\theta)$ increase seemingly without bound as θ increases from $0°$ to $89°$. The graph will not be a line because the slopes between successive points are not constant.

Example 4.2.2

Use technology to compute the value of each trigonometric function.

a. $\cos(23.7°)$
b. $\tan(72.13°)$
c. $\sin^2(10°)$

Solutions

a. $\cos(23.7°) \approx 0.9156625933 \approx 0.9157$
b. $\tan(72.13°) \approx 3.101611189 \approx 3.1016$
c. To compute $\sin^2(10°)$, we need to recall that

$$\sin^2(10°) = [\sin(10°)]^2.$$

Therefore, we start by getting the approximation $\sin(10°) \approx 0.1736481777$. Using as many of these digits as possible, we then square the result to get

$$\sin^2(10°) \approx (0.1736481777)^2 \approx 0.03015368961 \approx 0.0304.$$

Caution: Save Rounding for the Final Answer

Example **4.2.2***c* demonstrates a significant point that bears repeating - **rounding is for display purposes only!** Never round a result and then use that rounded result for other computations.

For example, had we rounded our approximation of $\sin(10°)$ to be 0.1736 and then squared, we would have incorrectly stated $\sin^2(10°) \approx 0.0301$. While the difference between this result and the more accurate result of 0.0304 may seem insignificant, these rounding errors can propagate in such a way as to make massive differences between answers.

Checkpoint 4.2.2

a. Use a calculator to complete the table. Round the values of sine and cosine to four decimal places.

θ	$0°$	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$	$70°$	$80°$	$90°$
$\sin(\theta)$										
$\cos(\theta)$										

b. Is there a pattern to the values of sine and cosine? Why this is true? (Hint: If one (non-right) angle of a right triangle measures x degrees, how big is the other angle? Sketch the triangle and label each angle's opposite and adjacent sides.)

Answers

a.

θ	$0°$	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$	$70°$	$80°$	$90°$
$\sin(\theta)$	0	0.1737	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848	1
$\cos(\theta)$	1	0.9849	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1737	0

b. The cosine of θ is equal to the sine of the complement of θ, or $\cos(\theta) = \sin(90 - \theta)$. This result confirms the Cofunction Identity from the previous section.

Now that we have access to technology to approximate the value of trigonometric functions at non-special angles, we can revisit simplifying and evaluating trigonometric expressions.

Example 4.2.3

Simplify, and evaluate for $z = 40°$.

$$3\sin(z) - \sin(z)\tan(z) + 3\sin(z)$$

Solution

We can combine like terms to get

$$6\sin(z) - \sin(z)\tan(z)$$

Because $40°$ is not one of the angles for which we know exact trig values, we use a calculator to evaluate the expression. It is *very* important to make sure we save any rounding until the very end of the computation. Therefore, we enter the following keystrokes into our calculator (or other technology):

$$6 \quad \times \quad \text{SIN} \quad 40 \quad) \quad - \quad \text{SIN} \quad 40 \quad) \quad \times \quad \text{TAN} \quad 40 \quad \text{ENTER}$$

Doing so, we get approximately 3.3174.

Checkpoint 4.2.3

Simplify, and evaluate for $x = 25°, y = 70°$.

$$3\cos(x) + \cos(y) - 2\cos(y) + \cos(x)$$

Answer

3.2832

Unlike the fundamental trigonometric functions, calculators do not have keys for the cosecant, secant, and cotangent functions. Instead, we calculate their values as reciprocals.

Example 4.2.4

Use technology to approximate $\sec(47°)$ to three decimal places.

Solution

Enter

$$1 \div \cos 47) \text{ ENTER}$$

to obtain $\sec(47°) \approx 1.466$. Or we can calculate $\cos(47°)$ first, and then use the reciprocal key:

$$\cos 47) \text{ ENTER } x^{-1} \text{ ENTER}$$

Checkpoint 4.2.4

Use technology to approximate $\csc(82°)$ to three decimal places.

Answer

1.010

Small Angles: Degrees, Minutes, and Seconds

The traditional angular measurement system we have covered is called the **Decimal Degree** (**DD**) system. Decimal Degrees offer simplicity and ease of use, making them a preferred choice for everyday applications like GPS navigation. However, it's essential to recognize that they are not a one-size-fits-all solution. In scenarios where precision is paramount, such as offshore navigation or land surveying, a slightly modified **Degree-Minute-Second** (**DMS**) system shines.

The DMS system divides degrees into smaller units called **minutes** and **seconds**. One minute is $\frac{1}{60}$ of a degree, and 1 second is $\frac{1}{60}$ of a minute, or $\frac{1}{3600}$ of a degree. We use the following notation for minutes and seconds.

Definition: Minutes and Seconds

- One minute: $1' = \dfrac{1°}{60}$
- One second: $1'' = \dfrac{1'}{60} = \dfrac{1°}{3600}$

To convert a measure of $42.125°$ to the DMS system, we start by noting that $42.125° = 42° + 0.125°$. Converting the partial amount of degrees to minutes, we find $0.125°\left(\frac{60'}{1°}\right) = 7.5' = 7' + 0.5'$. Converting the partial amount of minutes to seconds gives $0.5'\left(\frac{60''}{1'}\right) = 30''$. Putting it all together yields

$$\begin{aligned} 42.125° &= 42° + 0.125° \\ &= 42° + 7.5' \\ &= 42° + 7' + 0.5' \\ &= 42° + 7' + 30'' \\ &= 42°\,7'\,30'' \end{aligned}$$

On the other hand, to convert $117°\,15'\,45''$ to decimal degrees, we first compute $15'\left(\frac{1°}{60'}\right) = \frac{1}{4}°$ and $45''\left(\frac{1°}{3600''}\right) = \frac{1}{80}°$. Then we find

$$\begin{aligned} 117°\,15'\,45'' &= 117° + 15' + 45'' \\ &= 117° + \tfrac{1}{4}° + \tfrac{1}{80}° \\ &= \tfrac{9381}{80}° \\ &= 117.2625° \end{aligned}$$

Example 4.2.5

Let $\alpha = 111.371^\circ$ and $\beta = 37^\circ\, 28'\, 17''$.

a. Convert α to the DMS system. Round your answer to the nearest second.
b. Convert β to Decimal Degrees. Round your answer to the nearest thousandth of a degree.
c. Sketch α and β.
d. Find a supplementary angle for α.
e. Find a complementary angle for β.

Solutions

a. To convert α to the DMS system, we start with $111.371^\circ = 111^\circ + 0.371^\circ$. Next we convert $0.371^\circ \left(\frac{60'}{1^\circ}\right) = 22.26'$. Writing $22.26' = 22' + 0.26'$, we convert $0.26' \left(\frac{60''}{1'}\right) = 15.6''$. Hence,

$$\begin{aligned} 111.371^\circ &= 111^\circ + 0.371^\circ \\ &= 111^\circ + 22.26' \\ &= 111^\circ + 22' + 0.26' \\ &= 111^\circ + 22' + 15.6'' \\ &= 111^\circ\, 22'\, 15.6'' \end{aligned}$$

Rounding to seconds, we obtain $\alpha \approx 111^\circ\, 22'\, 16''$.

b. To convert β to decimal degrees, we convert $28' \left(\frac{1^\circ}{60'}\right) = \frac{7}{15}^\circ$ and $17'' \left(\frac{1^\circ}{3600'}\right) = \frac{17}{3600}^\circ$. Putting it all together, we have

$$\begin{aligned} 37^\circ\, 28'\, 17'' &= 37^\circ + 28' + 17'' \\ &= 37^\circ + \frac{7}{15}^\circ + \frac{17}{3600}^\circ \\ &= \frac{134897}{3600}^\circ \\ &\approx 37.471^\circ \end{aligned}$$

c. To sketch α, we first note that $90^\circ < \alpha < 180^\circ$. If we divide this range in half, we get $90^\circ < \alpha < 135^\circ$, and once more, we have $90^\circ < \alpha < 112.5^\circ$. This gives us a pretty good estimate for α, as shown below. Proceeding similarly for β, we find $0^\circ < \beta < 90^\circ$, then $0^\circ < \beta < 45^\circ$, $22.5^\circ < \beta < 45^\circ$, and lastly, $33.75^\circ < \beta < 45^\circ$.

Figure 4.2.1

d. To find a supplementary angle for α, we seek an angle θ so that $\alpha + \theta = 180^\circ$. We get $\theta = 180^\circ - \alpha = 180^\circ - 111.371^\circ = 68.629^\circ$.

e. To find a complementary angle for β, we seek an angle γ so that $\beta + \gamma = 90^\circ$. We get $\gamma = 90^\circ - \beta = 90^\circ - 37^\circ\, 28'\, 17''$. While we could reach for the calculator to obtain an approximate answer, let's do something more mathematically engaging. We first rewrite $90^\circ = 90^\circ\, 0'\, 0'' = 89^\circ\, 60'\, 0'' = 89^\circ\, 59'\, 60''$. In essence, we are "borrowing" $1^\circ = 60'$ from the degree place, and then "borrowing" $1' = 60''$ from the minutes place. This yields, $\gamma = 90^\circ - 37^\circ\, 28'\, 17'' = 89^\circ\, 59'\, 60'' - 37^\circ\, 28'\, 17'' = 52^\circ\, 31'\, 43''$.

Checkpoint 4.2.5

a. Convert $10° 45'$ to DD.
b. Convert $27.25°$ to DMS.

Answers

a. $10.75°$
b. $27° 15'$

Calculator Note

Most scientific and graphing calculators have keys to enter angles in DMS format. You may also have keys or commands that let you convert from one format to another. Consult your calculator's manual to see how this is done with your particular model.

Some argue that DMS coordinates belong to a bygone era, overshadowed by the convenience of Decimal Degrees and digital mapping tools. However, this couldn't be further from the truth. DMS coordinates remain a cornerstone in various industries, including maritime, aviation, surveying, and astronomy, where precision and tradition go hand in hand. These coordinates have stood the test of time, adapting to modern technology while retaining their significance.

Another misconception is that DMS coordinates have limited utility, primarily confined to specialized fields. While DMS coordinates are prevalent in aviation, maritime, and surveying, they also have broader applications in geocaching, emergency services, and outdoor recreational activities. Their precision and familiarity make them a versatile choice in various contexts.

Sometimes, you need to add or subtract angles in the DMS system.

Example 4.2.6

Perform the arithmetic.

a. $23° 16' 47'' + 17° 50' 24''$
b. $23° 16' 47'' - 17° 50' 24''$

Solutions

a. When adding angles in the DMS system, it is best to perform the arithmetic as we did in elementary school. This is because we will be "carrying" between measures (i.e., carrying $60''$ to $1'$, and $60'$ to $1°$).

$$\begin{array}{rrrr} & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & & 71'' \end{array} \quad \Longrightarrow \quad \begin{array}{rrrr} & & & 1' \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & & 11'' \end{array} \qquad \text{(Carrying the } 60'' \text{ to } 1'\text{)}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & & & 1' \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & 67' & 11'' \end{array}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & & 1^\circ & 1' \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & 7' & 11'' \end{array} \qquad \text{(Carrying the } 60' \text{ to } 1^\circ\text{)}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & & 1^\circ & 1' \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & 41^\circ & 7' & 11'' \end{array}$$

Therefore, the sum is $41^\circ\, 7'\, 11''$.

b. When subtracting angles in the DMS system, it is best to perform the arithmetic like we did in elementary school. This is because we will be "borrowing" between measures (i.e., borrowing $1'$ to $60''$, and 1° to $60'$).

$$\begin{array}{rrrr} & 23^\circ & 16' & 47'' \\ - & 17^\circ & 50' & 24'' \\ \hline & & & 23'' \end{array} \quad \Longrightarrow \quad \begin{array}{rrrr} & 22^\circ & 76' & 47'' \\ - & 17^\circ & 50' & 24'' \\ \hline & & & 23'' \end{array} \qquad \text{(Borrowing the } 1^\circ \text{ to } 60'\text{)}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & 22^\circ & 76' & 47'' \\ - & 17^\circ & 50' & 24'' \\ \hline & 5^\circ & 26' & 23'' \end{array}$$

Therefore, the difference is $5^\circ\, 26'\, 23''$.

? Checkpoint 4.2.6

a. Add $53^\circ\, 34'\, 12''$ and $10^\circ\, 25'\, 55''$.

b. Subtract $23^\circ\, 12'$ from 77°.

Answers

a. $64^\circ\, 0'\, 7''$

b. $53^\circ\, 48'$

Computing the value of a trigonometric function where the angle has been given in the DMS system is as simple as first converting the angle to the DD system.

Example 4.2.7

Evaluate $\cot(10^\circ\,45')$.

Solution

In Checkpoint 4.2.5, you found $10^\circ\,45' = 10.75^\circ$. Thus,

$$\cot(10^\circ\,45') = \cot(10.75^\circ) = \frac{1}{\tan(10.75^\circ)} \approx 5.2672.$$

Checkpoint 4.2.7

Evaluate $\sec(23^\circ\,16'\,47'')$.

Answer

≈ 1.0886

Finding an Acute Angle Using a Calculator

While watching her niece at the playground, Nhat wonders how steep the slide is. She takes out her measuring tape and finds the slide to be 77 inches high over a horizontal distance of 136 inches, as shown below.

Figure 4.2.2

Nhat knows that one way to describe the steepness of an incline is to calculate its slope, which, in this case, is

$$\frac{\Delta y}{\Delta x} = \frac{77}{136} \approx 0.5662.$$

However, she wants to know what angle the slide makes with the horizontal. Thinking about it for a moment, she realizes that the slope she had just calculated is also the tangent of the angle she wants (can you see this from the figures above?). This begs the question: If we know the tangent of an angle, can we find the angle? The answer is yes!

Locate the key labeled TAN^{-1} on your calculator; it is probably the second function above the TAN key. Enter

$$\text{2nd} \quad \text{TAN} \quad 0.5662$$

to find that

$$\tan^{-1}(0.5662) \approx 29.5185^\circ.$$

This means that 29.5185° is the approximate angle whose tangent is 0.5662. We read the notation, $\tan^{-1}(0.5662) \approx 29.5185^\circ$ as "**inverse tangent** of 0.5662 is approximately 29.5185 degrees."

Note: Use Degree Mode

Until later in this course, you should make sure your calculator is set to "degree" mode. If you do not do this, your answers, while correct in a different angular measure system, will be incorrect.

Caution: Do Not Use Rounded Values for Calculations

Before describing the inverse tangent, we must note the flaw in our work. We *approximated* the value of the fraction, $\frac{77}{136}$, and then used that approximation to approximate the angle - this is very bad form. In truth, we should not have approximated $\frac{77}{136}$, but instead just computed

$$\tan^{-1}\left(\frac{77}{136}\right) \approx 29.5175°.$$

Do you see that the returned angle from the inverse tangent is slightly different from our calculations using the rounded approximation of the fraction?

The moral of the story is that, whenever possible, **do not use rounded or approximate values to get yet another approximation** - this leads to a propagation of rounding errors.

When we find $\tan^{-1}$ of a number, we find an angle whose tangent is that number. Similarly, $\sin^{-1}$ and $\cos^{-1}$ are read as "**inverse sine**" and "**inverse cosine**." They find an angle with the given sine or cosine.

Example 4.2.8

Find the angle whose sine is 0.6834.

Solution

Enter

$$\text{2nd} \quad \text{SIN} \quad 0.6834$$

into your calculator to find

$$\sin^{-1}(0.6834) \approx 43.11°.$$

So $43.11°$ is the angle whose sine is *approximately* 0.6834. Alternatively, we can say that

$$\sin(43.11°) \approx 0.6834.$$

You can check the last equation on your calculator.

Note: Equivalent Statements

In Example 4.2.8, the two equations

$$\sin(43.11°) \approx 0.6834 \quad \text{and} \quad \sin^{-1}(0.6834) \approx 43.11°$$

say the same thing in different ways.

Caution: Inverse Trigonometric Functions are NOT Reciprocals

The notation $\sin^{-1}(x)$ does *not* mean $\frac{1}{\sin(x)}$!

We indeed use negative exponents to indicate reciprocals of numbers, for example, $a^{-1} = \frac{1}{a}$ and $3^{-1} = \frac{1}{3}$. However, "sin" by itself is not a variable.

- $\sin^{-1}(x)$ means "the angle whose sine is x"
- $\frac{1}{\sin(x)}$ means "the reciprocal of the sine of angle x"

(You may recall that $f^{-1}(x)$ denotes the inverse function for $f(x)$. We will study inverse trigonometric functions in a later chapter.)

? Checkpoint 4.2.8

Write the following fact in two different ways: $68°$ is the angle whose cosine is 0.3746.

Answer

$\cos(68°) = 0.3746$ or $\cos^{-1}(0.3746) = 68°$

Note: Inputs and Outputs of Inverse Trigonometric Functions

Without going into too much detail (yet), it is essential to note that an inverse trigonometric function's input (a.k.a., the argument) is a ratio. The output is an angle.

✓ Example 4.2.9

Find the acute angle for which the equation is true. Round answers to the nearest tenth.

a. $\cos(\beta) = 0.28$
b. $\tan(\kappa) = 1.93$

Solutions

a. We use the following button sequence:

2nd COS 0.28 ENTER.

This returns 73.73979529. Hence, $\beta \approx 73.7°$.

b. Using the sequence

2nd TAN 1.93 ENTER,

we get 62.60975803. Hence, $\kappa \approx 62.6°$.

? Checkpoint 4.2.9

Find the acute angle for which $\tan(\theta) = 1.23$. Round your answer to the nearest tenth.

Answer

$\theta \approx 50.9°$

Finding acute angles is slightly more complex when the equation involves the cosecant, secant, or cotangent. A good rule of thumb is to rewrite the equation in terms of the fundamental trigonometric functions using the Reciprocal Identities.

✓ Example 4.2.10

Suppose x is an acute angle within a right triangle. Solve each equation. Round answers to two decimal places.

a. $\sec(x) = 8.104$
b. $3\cot(x) - 2 = 7$

Solutions

a. We first rewrite the equation as

$$\frac{1}{\cos(x)} = 8.104 \implies \cos(x) = \frac{1}{8.104}$$

We then enter

2nd COS 1 ÷ 8.104

into the calculator to find

$$\cos^{-1}\left(\frac{1}{8.104}\right) \approx 82.91°.$$

Therefore, the solution to $\sec(x) = 8.104$ is $x \approx 82.91°$.

b. The equation handed to us needs to be simplified into the form $\cot(\text{angle}) = \text{number}$ before we can find the angle.

$$\begin{aligned} 3\cot(x) - 2 &= 7 \\ \Longrightarrow \quad 3\cot(x) &= 9 \\ \Longrightarrow \quad \cot(x) &= 3 \end{aligned}$$

We now rewrite the cotangent as the tangent using the Reciprocal Identities.

$$\cot(x) = 3 \implies \frac{1}{\tan(x)} = 3 \implies \tan(x) = \frac{1}{3}$$

Finally, we grab our technology and enter

$$\text{2nd} \quad \text{TAN} \quad 1 \quad \div \quad 3$$

into the calculator to find

$$\tan^{-1}\left(\frac{1}{3}\right) \approx 18.43°$$

Therefore, the solution to $3\cot(x) - 2 = 7$ is $x \approx 18.43°$.

Checkpoint 4.2.10

Let ω be an acute angle within a right triangle. Solve $2\csc(\omega) + 3 = 11$. Round the answer to two decimal places.

Answer

$\omega \approx 14.48°$

Activities

Activity: Trigonometric Ratios

A. Using Ratios and Proportions

Recall that two related quantities or variables are proportional if their ratio is always the same.

1. a. On any given day, the cost of filling up your car's gas tank is proportional to the number of gallons of gas you buy. For each purchase below, compute the ratio

$$\frac{\text{total cost of gasoline}}{\text{number of gallons}}$$

Gallons of Gas Purchased	Total Cost	$\frac{\text{Dollars}}{\text{Gallon}}$
5	\$14.45	
12	\$34.68	
18	\$52.02	

b. Write an equation that you can solve to answer the question: How much does 21 gallons of gas cost? Use the ratio $\frac{\text{Dollars}}{\text{Gallon}}$ in your equation.

c. Write an equation that you can solve to answer the question: How many gallons of gas can you buy for \$46.24? Use the ratio $\frac{\text{Dollars}}{\text{Gallon}}$ in your equation.

2. A recipe for coffee cake calls for $\frac{3}{4}$ cup of sugar and $1\frac{3}{4}$ cup of flour.
   a. What is the ratio of sugar to flour? Write your answer as a common fraction, then give a decimal approximation rounded to four places.

   For parts (b) and (c) below, write an equation that you can solve to answer the question. Use the ratio $\frac{\text{Amount of sugar}}{\text{Amount of flour}}$ your equation.
   b. How much sugar should you use if you use 4 cups of flour? Compute your answer in two ways: write the ratio as a common fraction and then write the ratio as a decimal approximation. Are your answers the same?
   c. How much flour should you use if you use 4 cups of sugar? Compute your answer in two ways: write the ratio as a common fraction and then write the ratio as a decimal approximation. Are your answers the same?
3. You are making a scale model of the Eiffel Tower, which is 324 meters tall and 125 meters wide at its base.

Figure 4.2.3

   a. Compute the ratio of the base width to the tower's height. Round your answer to four decimal places.

   Use your ratio to write equations and answer the questions below:
   b. If the base of your model is 8 inches wide, how tall should the model be?
   c. If you make a larger model that is 5 feet tall, how wide will the base be?

B. Similar Triangles

1. Recall that two triangles are similar if their corresponding sides are proportional. The corresponding angles of similar triangles are equal.
   a. What is the ratio of the two given sides in each triangle? Are the corresponding sides of the three triangles proportional? How do we know that $\alpha = \beta = \gamma$?

Figure 4.2.4

   b. Find the hypotenuse of each right triangle.
   c. Use the sides of the appropriate triangle to compute $\sin\alpha, \sin\beta$, and $\sin\gamma$. Round your answers to four decimal places. Does the sine of an angle depend on the lengths of its sides?
   d. How do you know that the triangle below is similar to the three triangles in part (a)? Write an equation using the ratio from part (c) to find x.

Figure 4.2.5

2. In the three right triangles below, the angle θ is the same size.

Figure 4.2.6

a. Use the first triangle to calculate $\cos\theta$. Round your answer to four decimal places.
b. In the second triangle, explain why $\frac{x}{4.3} = \frac{10}{13}$. Write an equation using your answer to part (a) and solve it to find x.
c. Write and solve an equation to find z in the third triangle.

3. a. Use your calculator to find the value of $\frac{h}{2.4}$. (Hint: Which trigonometric ratio should you use?) What is the length of side h?

Figure 4.2.7

b. What is the value of $\frac{6}{w}$ for the triangle below? Write an equation and solve for w.

Figure 4.2.8

c. Write an equation and solve it to find x in the triangle above.

? Activity: Obtuse Angles

An obtuse angle has measure between $90°$ and $180°$, exclusively. In this activity, we will investigate the trigonometric functions of an obtuse angle.

Figure 4.2.9

a. Draw an angle θ in standard position with the point $P(6, 4)$ on its terminal side.

b. Find r, the distance from the origin to P.
c. Calculate all six trigonometric functions of θ. Give both exact answers and decimal approximations rounded to four places.
d. Use the inverse cosine key on your calculator to find θ. Use your calculator to verify the values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ that you found in part (c).
e. Draw another angle ϕ in standard position with the point $Q(-6, 4)$ on its terminal side. Explain why ϕ is the supplement of θ. (Hint: Consider the right triangles formed by drawing vertical lines from P and Q).
f. Can you use the right triangle definition of the trigonometric functions to compute the sine and cosine of ϕ? Why or why not?
g. Calculate all six trigonometric functions of ϕ. How are these trigonometric values of ϕ related to the trigonometric values of θ?
h. Explain why θ and ϕ have the same sine but different cosines.
i. Use the inverse cosine key on your calculator to find ϕ. Use your calculator to verify the values of $\sin(\phi)$, $\cos(\phi)$, and $\tan(\phi)$ that you found in part (g).
j. Compute $180° - \phi$. What answer should you expect to get?

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4.2.1: RESOURCES AND KEY CONCEPTS

KEY CONCEPTS

DEFINITIONS

- **Minutes and Seconds:** A system for measuring fractions of a degree. One minute ($1'$) is defined as $\frac{1}{60}$ of a degree. One second ($1''$) is defined as $\frac{1}{60}$ of a minute, or $\frac{1}{3600}$ of a degree.

COMMON MISTAKES

- **Using the Wrong Calculator Mode:** Before evaluating any trigonometric function, you must ensure your calculator is set to the correct mode (degree mode for this section). Using the wrong mode will result in incorrect answers.
- **Rounding Intermediate Steps:** Rounding should only be done for the final answer. If you round a value and then use that rounded value in another calculation, the rounding error can propagate and lead to a significantly different and less accurate final result.
- **Confusing Inverse Functions with Reciprocals:** The notation $\sin^{-1}(x)$ represents the inverse sine, which finds the angle whose sine is x. It does not mean the reciprocal, $\frac{1}{\sin(x)}$, which is $\csc(x)$.
- **Using Rounded Values in Inverse Functions:** When finding an angle, you should use the exact ratio if possible. For example, to find the angle whose tangent is $\frac{77}{136}$, you should compute $\tan^{-1}\left(\frac{77}{136}\right)$ directly rather than first approximating $\frac{77}{136}$ as a decimal and then taking the inverse tangent of the rounded decimal.

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4.2.2: HOMEWORK

READING QUESTIONS

1. What calculator mode must you be in to evaluate trigonometric functions of angles given in degrees?
2. When a calculator displays a value for $\sin(50°)$, is this an exact value or an approximation?
3. What does the text mean by "rounding is for display purposes only"?
4. How many minutes are in one degree? How many seconds are in one minute?
5. Explain the process of converting an angle from Decimal Degree (DD) form to Degrees-Minutes-Seconds (DMS) form.
6. How do you calculate the value of $\sec(47°)$ on a calculator that does not have a SEC button?
7. What does the notation $\tan^{-1}(0.5662)$ represent?
8. What is the input for an inverse trigonometric function (an angle or a ratio)? What is the output?
9. Explain the difference in meaning between $\sin^{-1}(x)$ and $(\sin(x))^{-1}$.
10. When performing addition of angles in DMS form, what value do you "carry" when the sum of the seconds is 60 or more?

SKILLS REFRESHER

Review the following skills you will need for this section.

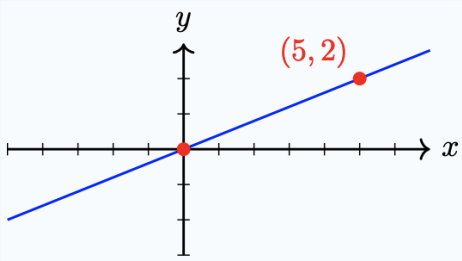
? SKILLS REFRESHER

For the following exercises, write two more ratios equivalent to the given fraction.

1. $\frac{10}{4}$
2. $\frac{6}{8}$
3. 0.6
4. 1.5

For the following exercises, compute the slope of the line.

5.



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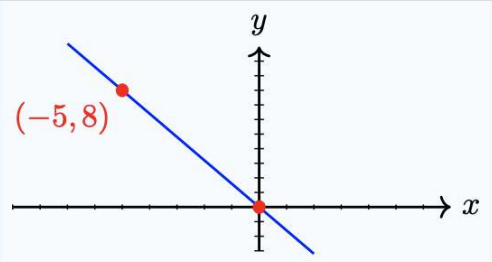

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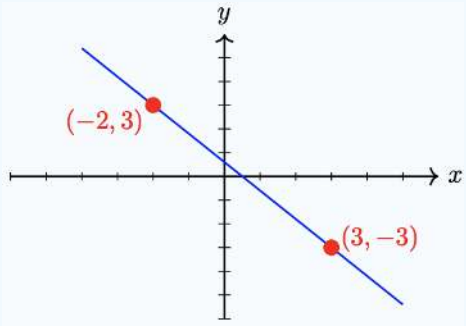

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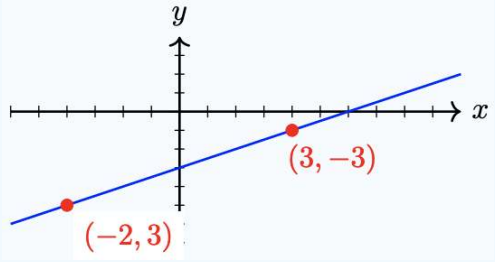


For the following exercises, solve.

9. $\frac{12}{x} = 48$
10. $\frac{60}{x} = 80$

Answers

(Many answers are possible for 1-4.)

1. $\frac{5}{2}, \frac{20}{8}$
2. $\frac{3}{4}, \frac{12}{16}$
3. $\frac{3}{5}, \frac{12}{20}$
4. $\frac{6}{4}, \frac{12}{8}$
5. $\frac{2}{5}$
6. $\frac{-8}{5}$
7. $\frac{-6}{5}$
8. $\frac{1}{2}$
9. $\frac{1}{4}$
10. $\frac{3}{4}$

HOMEWORK

VOCABULARY CHECK

1. Calculators and other technology have two modes that deal with the values of trigonometric functions - ___ mode and ___ mode. At this point, we only use ___ mode.

CONCEPT CHECK

2. Explain how you would convert an angle given in Decimal Degrees (DD) to Degrees-Minutes-Seconds (DMS).
3. Does your calculator give you the exact decimal values for the trigonometric ratios of acute angles?
4. $1°$ is equal to ___ minutes(s).
5. $1°$ is equal to ___ second(s).
6. $1'$ is equal to ___ degree(s).
7. $1'$ is equal to ___ seconds(s).
8. $1''$ is equal to ___ degree(s).
9. $1''$ is equal to ___ minutes(s).
10. The SIN, COS, and TAN keys on a calculator (or other technology) returns

    the value of the trigonometric function an angle

    (circle one) when you input

    the value of the trigonometric function an angle

    (circle one).
11. The SIN^{-1}, COS^{-1}, and TAN^{-1} keys on a calculator (or other technology) returns

    the value of the trigonometric function an angle

    (circle one) when you input

    the value of the trigonometric function an angle

    (circle one).
12. Min and Paige try to approximate $2.1\tan(37°)$. Min types the following into her calculator

    $$2.1 \quad \times \quad \text{TAN} \quad 37 \quad \text{ENTER},$$

    and the calculator returns approximately 1.5825; however, Paige types

    $$\text{TAN} \quad 37 \quad \times \quad 2.1 \quad \text{ENTER}$$

    into her calculator and gets an approximation of 4.5864. They each have their calculators set to Degree Mode, so that's not the issue. Who has the wrong approximation? If she cannot change the order of what she types into her calculator, but *can* add or remove a button press, what can she do to fix her error?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

13. $\cot^{-1}(\theta) = \dfrac{1}{\cot(\theta)} = \tan(\theta)$.
14. $\cos^{-1}(\theta) = \sin(\theta)$

BASIC SKILLS

For the following exercises, evaluate. Round answers to 3 decimal places.

15. $\sin(53°)$
16. $\cos(18°)$
17. $\tan(5°)$
18. $\sin(85°)$
19. $\csc(27°)$
20. $\sec(8°)$
21. $\cot(65°)$
22. $\csc(11°)$
23.
    a. Fill in the table for values of $\tan\theta$. Round your answers to four decimal places.

θ	$0°$	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$	$70°$	$80°$
$\tan\theta$									

    b. Fill in the table for values of $\tan\theta$. Round your answers to three decimal places.

θ	$81°$	$82°$	$83°$	$84°$	$85°$	$86°$	$87°$	$88°$	$89°$
$\tan\theta$									

    c. What happens to $\tan\theta$ as θ increases?

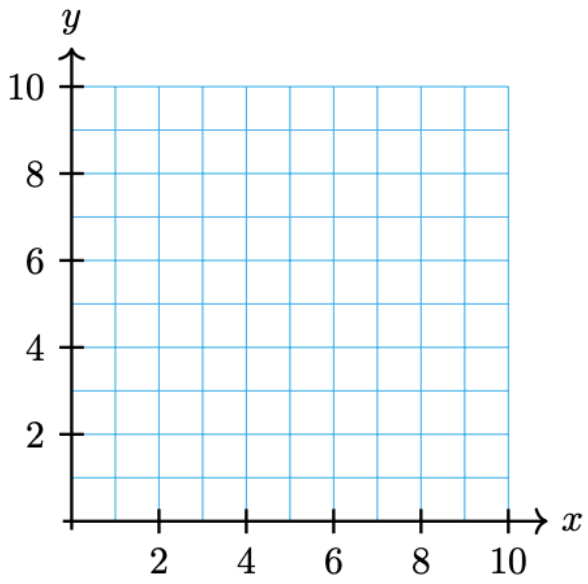
    d. What value does your calculator give for $\tan 90°$? Why?
24.
    a. Use your calculator to fill in a table of values for $\cos\theta$, rounded to hundredths.

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
$\cos\theta$							

    b. If you plotted the points in your table, would they lie on a straight line? Why or why not?
25.
    a. Use your calculator to complete the table. Rounded your answers to hundredths

θ	$14°$	$22°$	$35°$	$42°$	$58°$	$78°$
$\tan\theta$						

    b. Use the values of $\tan\theta$ to sketch all the angles listed in the table. Locate the vertex of each angle at the origin, and the initial side along the positive x-axis.



For the following exercises, evaluate the expressions using a calculator.

26.

a. $\sin(10° + 40°)$

b. $\sin 10° + \sin 40°$

c. $\sin 10° \cos 40° + \cos 10° \sin 40°$

27.

a. $\cos(20° + 50°)$

b. $\cos 20° + \cos 50°$

c. $\cos 20° \cos 50° - \sin 20° \sin 50°$

28.

a. $\cos(2 \cdot 24°)$

b. $2 \cos 24°$

c. $2\cos^2(24°) - 1$

29.

a. $\cos(3 \cdot 19°)$

b. $3 \cos 19°$

c. $4\cos^3(19°) - 3\cos 19°$

30.

a. $\cos^2(17°) + \sin^2(17°)$

b. $\sec^2(86°) - \tan^2(86°)$

c. $\csc^2(32.1°) - \cot^2(32.1°)$

31.

a. $\sin^2(25°) + \cos^2(25°)$

b. $\dfrac{1}{\cos^2(80°)} - \tan^2(80°)$

c. $\dfrac{1}{\sin^2(8°)} - \cot^2(8°)$

For the following exercises, convert the angles into the DMS system. Round each of your answers to the nearest second.

32. $18.9°$

33. $6.78°$

34. $55.17°$

35. $31.425°$

36. $70.214°$

37. $159.84°$

38. $220.43°$

39. $116.323°$

For the following exercises, convert the angles into decimal degrees. Round each of your answers to three decimal places.

40. $25° 15'$

41. $13° 50'$

42. $89° 12' 9''$

43. $32° 10' 12''$

44. $502° 35''$

45. $237° 58' 43''$

For the following exercises, find the complement of the given angle.

46. $27° 45' 24''$

47. $88° 10' 36''$

48. $2° 27' 13''$

For the following exercises, find the supplement of the given angle.

49. $27° 45' 24''$

50. $88° 10' 36''$

51. $113° 7''$

For the following exercises, perform the arithmetic.

52. $88° 10' 36'' - 27° 45' 24''$

53. $88° 10' 36'' + 27° 45' 24''$

54. $113° 7'' - 2° 27' 13''$

55. $113° 7'' + 2° 27' 13''$

For the following exercises, evaluate the trigonometric function.

56. $\sin(27° 45' 24'')$

57. $\cos(55° 12')$

58. $\tan(32° 10' 40'')$

59. $\csc(7° 7' 50'')$

60. $\sec(19° 57'')$

61. $\cot(27° 55')$

62. $\sin(5° 55')$

63. $\tan(89° 40' 1'')$

For the following exercises, evaluate the expression.

64. $\sin^{-1} 0.2$

65. $\cos^{-1} 0.8$

66. $\tan^{-1} 1.5$

67. $\tan^{-1} 2.5$

68. $\cos^{-1} 0.2839$

69. $\sin^{-1} 0.4127$

70. Evaluate the expressions, and explain what each means.

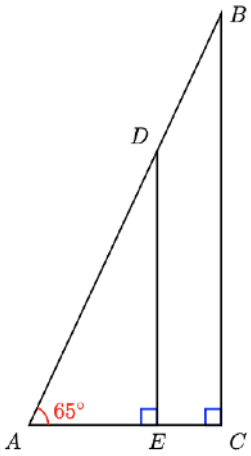
$$\sin^{-1}(0.6), \quad (\sin 6°)^{-1}$$

71. Evaluate the expressions, and explain what each means.

$$\cos^{-1}(0.36), \quad (\cos 36°)^{-1}$$

SYNTHESIS QUESTIONS

72. Here are two right triangles with a $65°$ angle.



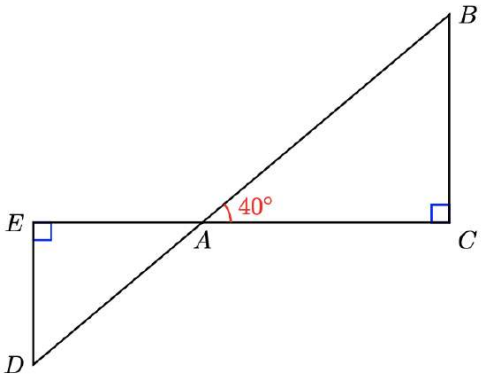
a. Measure the sides AB and BC with a ruler. Use the lengths to estimate $\sin 65°$.

b. Measure the sides AD and DE with a ruler. Use the lengths to estimate $\sin 65°$.

c. Use your calculator to look up $\sin 65°$. Compare your answers. How close were your estimates?

73. Use the figure in the previous problem to calculate two estimates each for the cosine and tangent of $65°$. Compare your estimates to your calculator's values for $\cos 65°$ and $\tan 65°$.

74. Here are two right triangles with a $40°$ angle.



a. Measure the sides AB and AC with a ruler. Use the lengths to estimate $\cos 40°$.

b. Measure the sides AD and AE with a ruler. Use the lengths to estimate $\cos 40°$.

c. Use your calculator to look up $\cos 40°$. Compare your answers. How close were your estimates?

75. Use the figure in Problem 74 to calculate two estimates each for the cosine and tangent of $40°$. Compare your estimates to your calculator's values for $\sin 40°$ and $\tan 40°$.

For the following exercises, write two different equations for the statement.

76. The cosine of $15°$ is 0.9659.

77. The sine of $70°$ is 0.9397.

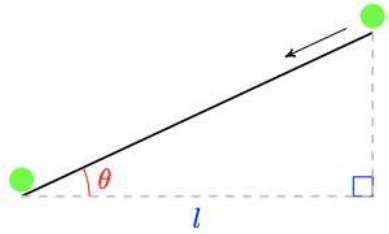
78. The angle whose tangent is 3.1445 is $65°$.

79. The angle whose cosine is 0.0872 is $85°$.

APPLICATIONS

80. When a plane is tilted by an angle θ from the horizontal, the time required for a ball starting from rest to roll a horizontal distance of l feet on the plane is

$$t = \sqrt{\frac{l}{8}\csc(2\theta)} \text{ seconds}$$



a. How long, to the nearest 0.01 second, will it take the ball to roll 2 feet horizontally on a plane tilted by $12°$?

b. Solve the formula for l in terms of t and θ.

81. After a heavy rainfall, the depth, D, of the runoff flow at a distance x feet from the watershed down a slope at angle α is given by

$$D = (kx)^{0.6}(\cot(\alpha))^{0.3} \text{ inches}$$

where k is a constant determined by the surface roughness and the intensity of the runoff.

a. How deep, to the nearest 0.01 inch, is the runoff 100 feet down a slope of $10°$ if $k = 0.0006$?

b. Solve the formula for x in terms of D and α.

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4.3: Solving Right Triangles

Note to the Instructor (click to expand)

This section focuses on solving right triangles. As such, all angles will be acute. In addition, applications and concept exercises require the student to find missing angles and/or sides within a right triangle; however, more robust applications involving angles of elevation or depression, navigation, and problems involving more than a single right triangle in a three-dimensional environment are reserved for the next section.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Simplifying Radical Expressions: This skill is needed for the "Skills Refresher" exercises in the homework.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Solve a right triangle for a missing side.
- Solve a right triangle for a missing angle.
- Solve an application using Right Triangle Trigonometry.
- Use Right Triangle Trigonometry in a robust geometry problem.

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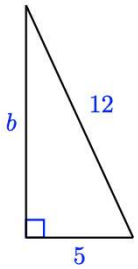
With the Pythagorean Theorem, we can find one side of a right triangle if we know the other two sides. For example, the missing side in Figure **4.3.1** can be found as follows

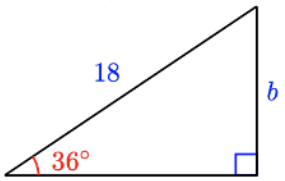
$$\begin{aligned} & b^2 + 5^2 &&= 12^2 \\ \implies & b^2 + 25 &&= 144 \\ \implies & b^2 &&= 119 \\ \implies & b &&= \sqrt{119} \end{aligned}$$

Figure 4.3.1



However, is there a way to find the length of side b in Figure 4.3.2, given that we only know the length of one side of the right triangle and one of the non-right angles?

Figure 4.3.2



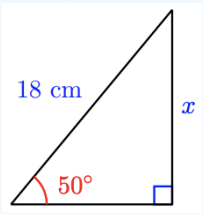
Using a Trigonometric Function to Find an Unknown Side

When finding the length of a side of a right triangle, we must be given at least two pieces of information. If we are given two side lengths (as in Figure 4.3.1), we use the Pythagorean Theorem. If, on the other hand, we are given an angle and a side length, we determine the trigonometric function relating the missing side and the provided information.

Example 4.3.1

Find the side length opposite the $50°$ angle in Figure 4.3.3.

Figure 4.3.3



Solution

In this triangle, the trigonometric function that relates the missing side to the given information is the sine.

$$\sin(50°) = \frac{\text{opposite}}{\text{hypotenuse}}$$

We use a calculator to find an approximate value for the sine of $50°$, filling in the lengths of the hypotenuse and the opposite side to get

$$0.7660 \approx \frac{x}{18}$$

We solve for x to find

$$x \approx 18(0.7660) = 13.788$$

To two decimal places, the length of the opposite side is approximately 13.79 centimeters.

Caution: Save Rounding for Display Purposes Only

In the previous example, even though we showed only four decimal places in $\sin(50°)$, you should not round off intermediate steps in a calculation - the answer loses accuracy with each rounding. The following keystrokes can be used to avoid entering an extended approximation for $\sin(50°)$:

$$18 \quad \times \quad \text{SIN} \quad 50 \quad \text{ENTER}$$

The calculator returns 13.78879998.

As an additional warning, recall that *most* scientific and graphing calculators return $\text{SIN}($ when you press the SIN key. If you intend to perform the multiplication in the opposite order (i.e., you intend to compute $\sin(50°) \times 18$), you must close the parentheses on the argument of the sine. That is, you would use the following keystrokes in your calculator:

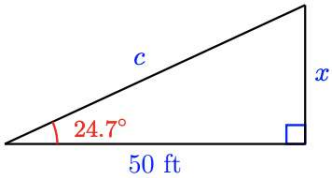
SIN 50) × 18.

Without the closing parenthesis, the calculator computes $\sin(50 \times 18°)$, which is not equivalent to $\sin(50°) \times 18$.

Checkpoint 4.3.1

a. Use the tangent ratio to find x in Figure 4.3.4.

Figure 4.3.4



b. Use the sine ratio to find the hypotenuse, c, of the triangle.

c. Use the Pythagorean Theorem to find the hypotenuse of the triangle. Is the answer the same with both methods? Explain why the calculations might give (slightly) different answers.

Answers

a. $x \approx 23$ ft

b. $c \approx 55$ ft

c. The answers agree when rounded to units. Rounding during calculation can cause the results to differ.

Solving Triangles

A triangle has six parts: three sides and three angles. In a right triangle, we know that one of the angles is $90°$. If given enough information, we can find the measures of the remaining angles within the right triangle and the lengths of the sides. This is called **solving the triangle**.

Definition: Solving a Triangle

Solving a triangle means finding all the missing side lengths and angle measures, given some initial information about the triangle.

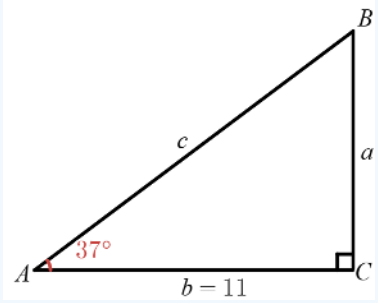
Example 4.3.2

In the right triangle $\triangle ABC$, $\angle A = 37°$ and (non-hypotenuse) side $b = 11$. Solve the triangle.

Solution

Since $\angle A = 37°$, side a is not the hypotenuse of the right triangle. Moreover, we were told that side b is not the hypotenuse. Thus, c must be the hypotenuse and $\angle C = 90°$. Creating a rough sketch of $\triangle ABC$, we get

Figure 4.3.5



Using the Triangle Sum, we quickly compute $\angle B = 90° - 37° = 53°$. We now concentrate on finding the missing sides by using the trigonometric functions and the given information. The tangent relates $37°$ to sides a and b. Therefore,

$$\tan(37°) = \frac{a}{11} \implies a = 11\tan(37°) \approx 8.2891.$$

Finally, we use the cosine to relate $37°$ to the given side b and the hypotenuse c.

$$\cos(37°) = \frac{11}{c} \implies c = \frac{11}{\cos(37°)} \approx 13.7735.$$

Traditional Notation: The Right Angle of $\triangle ABC$

Given the right triangle $\triangle ABC$, we adopt the convention used throughout most math textbooks that $\angle C$ is the $90°$ angle and side c is the hypotenuse.

Checkpoint 4.3.2

Sketch a right triangle with one angle of $37°$ and the side adjacent to that angle of length 5 centimeters. Without doing the calculations, list the steps used to solve the triangle.

Answer

Use $\tan(37°)$ to find the opposite side. Use $\cos(37°)$ to find the hypotenuse. Subtract $37°$ from $90°$ to find the third angle.

In Example 4.3.2 and Checkpoint 4.3.2, we were given an angle and a side. Using those two pieces of information (along with the fact that one of the other angles in the right triangle is $90°$), we were able to solve the triangles completely. What if we were not given any angles (other than the right angle) but two side lengths instead?

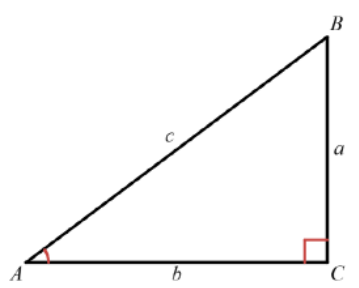
Example 4.3.3

Solve $\triangle ABC$ given that $a = 3$ and $c = 5$.

Solution

It's always best to start by drawing a triangle.

Figure 4.3.6



We can use the Pythagorean Theorem to compute the length of side b, or we could recognize that this is a Pythagorean Triple. No matter how we compute it, $b = 4$. Now, we need to find the missing angles. Luckily, we were given that this is a right triangle, so once we find either $\angle A$ or $\angle B$, we can use the Triangle Sum to find the other angle. Let's go ahead and find $\angle A$. Since $\sin(\angle A) = \frac{a}{c} = \frac{3}{5}$, we use a calculator to compute $\angle A = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.8699°$. Thus,

$$\angle B = 180° - \angle C - \angle A \approx 180° - 90° - 36.8699° \approx 53.1301°$$

It's often best to organize all of this information in a table like so:

$\angle A$	$\approx$	$36.8699°$	a	$=$	3
$\angle B$	$\approx$	$53.1301°$	b	$=$	4
$\angle C$	$=$	$90°$	c	$=$	5

Checkpoint 4.3.3

Solve the right triangle $\triangle ABC$ given $b = 1.12$ and $c = 3.25$. Round your *displayed* answers to the nearest hundredth.

Answer

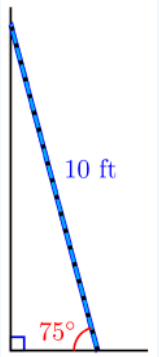
$$\begin{aligned} \angle A &\approx 69.84^\circ & a &\approx 3.05 \\ \angle B &\approx 20.16^\circ & b &= 1.12 \\ \angle C &= 90^\circ & c &= 3.25 \end{aligned}$$

The power of Trigonometry comes from its usefulness in helping us solve physical applications.

Example 4.3.4

A 10-foot ladder is leaning against a wall. If the ladder's base is too far from the wall, the base can slide out. If the ladder base is too close to the wall, there's a risk that the ladder could tip over backward. The leaning ladder should make a 75° angle with the ground for safety reasons.

Figure 4.3.7



a. How far should the base of the ladder be from the wall?
b. How far up the wall will the top of the ladder reach?

Solutions

a. The distance between the ladder's base and the wall is the side adjacent to the 75° angle. We can find the side adjacent to the 75° angle using the cosine ratio.

$$\begin{aligned} \cos(75^\circ) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \implies 0.2588 &\approx \frac{b}{10} \\ \implies 10(0.2588) &\approx b \\ \implies b &\approx 2.588 \end{aligned}$$

The ladder's base should be about 2.6 feet from the wall.

b. We could use the Pythagorean Theorem to find how far up the wall the ladder will reach. However, using the given information rather than the calculated values to find the other unknown parts is better. We will use the sine ratio.

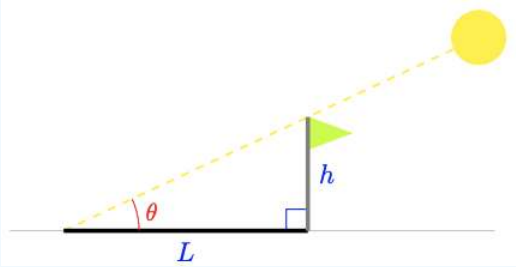
$$\begin{aligned} \sin(75^\circ) &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \implies 0.9659 &\approx \frac{a}{10} \\ \implies 10(0.9659) &\approx a \\ \implies a &\approx 9.659 \end{aligned}$$

The ladder will reach about 9.7 feet.

Checkpoint 4.3.4

The length, L, of the shadow cast by a flagpole on a sunny day depends on the height, h, of the flagpole and the angle, θ, that the sun's rays make with the ground (see Figure 4.3.8).

Figure 4.3.8



a. Write an expression for the length, L, of the shadow cast by a flagpole of height h when the sun makes an angle of θ from the ground.
b. Find the length (to the nearest 0.01 meter) of the shadow cast by a 3-meter flagpole when the sun makes an angle of $20°$ from the ground.

Answers

a. $L = h\cot(\theta)$
b. The shadow is about 8.24 meters long.

Introducing Right Triangles to Non-Right Triangles

While the trigonometric functions are defined in terms of ratios of sides of a right triangle, they are useful on many occasions when a non-right triangle is involved.

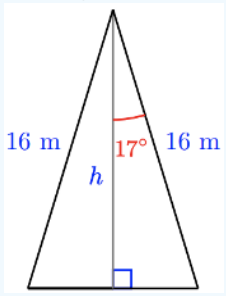
Example 4.3.5

The vertex angle of an isosceles triangle is $34°$, and the equal sides are 16 meters long. Find the altitude of the triangle.

Solution

The triangle described is *not* a right triangle. However, the altitude of an isosceles triangle bisects the vertex angle. It divides the triangle into two congruent right triangles, as shown in Figure 4.3.9. The 16-meter side becomes the hypotenuse of the right triangle. The altitude, h, of the original triangle is the side adjacent to the $17°$ angle.

Figure 4.3.9



Which of the three trigonometric ratios is helpful in this problem? The cosine is the ratio that relates the hypotenuse and the adjacent side. Therefore, we will begin with the equation

$$\cos(17°) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

We use a calculator to find $\cos 17^\circ$ and fill in the lengths of the sides.

$$0.9563 \approx \frac{h}{16}$$

Solving for h gives

$$h \approx 16(0.9563) = 15.3008$$

The altitude of the triangle is about 15.3 meters long.

? Checkpoint 4.3.5

Another isosceles triangle has base angles of 72° and equal sides of length 6.8 centimeters. Find the size of the base.

Answer

4 cm

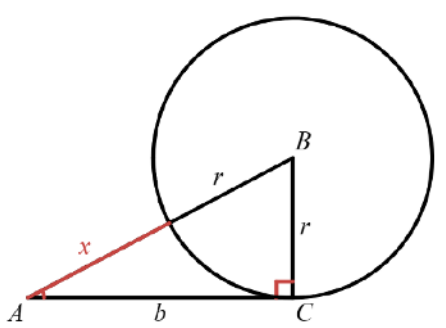
Thought-Provoking Right Triangle Geometry

Part of your excursion into Trigonometry (and higher-level mathematics) is to think critically and problem-solve. One way to engage your imagination and thinking skills is through abstract geometry problems that are often fun!

✔ Example 4.3.6

Figure $4.3.10$ shows a circle of radius r and a right triangle with side a along the radius of the circle and side b tangent to the circle.

Figure 4.3.10



If $\angle B = 50^\circ$ and $b = 10$ inches, find the radius of the circle and the partial length of the hypotenuse, x, that sits outside the circle. Round your answers to the nearest tenth.

Solution

Throughout Trigonometry, you will be faced with thought-provoking questions. The beautiful thing about right triangle trigonometry is that you typically need to find a trigonometric function that relates the given information to the missing information. In this case, we are given $\angle B$ and side b. We are initially asked for the radius of the circle, which is the same as the side length a of the right triangle. The trigonometric function relating $\angle B$ to sides a and b is the tangent.

$$\tan(\angle B) = \frac{b}{a} = \frac{b}{r}.$$

Therefore,

$$\tan(50^\circ) = \frac{10}{r} \implies r = \frac{10}{\tan(50^\circ)}.$$

Hence, the radius of the circle is $\frac{10}{\tan(50°)} \approx 8.4$ inches.

Now we need to find the length x. Again, we want to find a trigonometric function that relates the information to what we seek. Unfortunately, we are looking for only *part* of the hypotenuse of the right triangle; however, the *entire* hypotenuse is $x+r$, and we just found r. The sine is the trigonometric function that relates $\angle B$ to side b and the hypotenuse.

$$\sin(\angle B) = \frac{b}{x+r} \implies x+r = \frac{b}{\sin(\angle B)} \implies x = \frac{b}{\sin(\angle B)} - r.$$

Substituting the given values for $\angle B$ and side b, along with the **exact** value for r, we get

$$x = \frac{10}{\sin(50°)} - \frac{10}{\tan(50°)} \approx 4.66 \text{ inches.}$$

We continually harp on this but notice that we used the *exact* value of r when doing our final computation in Example 4.3.6. This is because we always want to avoid using rounded values for subsequent computations.

Checkpoint 4.3.6

Let $r = 5$ and $\angle A = 24°$ in Figure 4.3.10 (Example 4.3.6), find x. Round your answer to the nearest tenth.

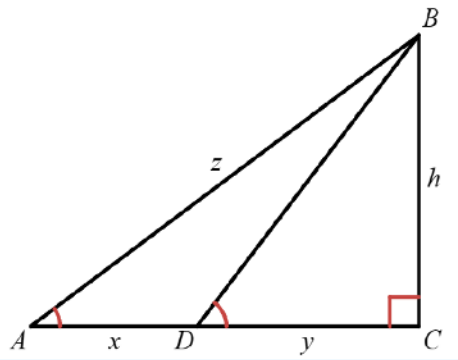
Answer

$x \approx 7.3$

Example 4.3.7

In Figure 4.3.11, $x = 5$, $\angle A = 25°$, and $\angle BDC = 48°$. Find y.

Figure 4.3.11



Solution

It is helpful to redraw only $\triangle BCD$ and then $\triangle ABC$, noticing that they share the common side h. For $\triangle BCD$, we only have one piece of given information, $\angle BDC$; however, we need to find y, and it will be useful to work with h since both triangles share that side (I call this the "link" between the triangles). The trigonometric function relating these is the tangent.

$$\tan(\angle BDC) = \frac{h}{y} \implies y\tan(\angle BDC) = h.$$

Hence,

$$h = y\tan(48°).$$

Moving to $\triangle ABC$, since we were given x and $\angle A$, it seems best to use the tangent again to get h involved. Therefore,

$$\tan(\angle A) = \frac{h}{x+y} \implies (x+y)\tan(\angle A) = h.$$

That is,

$$h = (5 + y)\tan(25°.)$$

We now equate these two versions of h to each other (since they both equal h, they are both equal to each other).

$$\begin{aligned} y\tan(48°) = (5 + y)\tan(25°) &\implies y\tan(48°) = 5\tan(25°) + y\tan(25°) \\ &\implies y\tan(48°) - y\tan(25°) = 5\tan(25°) \\ &\implies y(\tan(48°) - \tan(25°)) = 5\tan(25°) \\ &\implies y = \frac{5\tan(25°)}{\tan(48°) - \tan(25°)} \\ &\implies y \approx 3.62 \end{aligned}$$

? Checkpoint 4.3.7

Let $\angle A = 24°$, $y = 12$, and $\angle BDC = 62°$ in Figure 4.3.11 (Example 4.3.7). Find x. Round your answer to the nearest tenth.

Answer

$x \approx 38.7$

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4.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Trigonometric Applications
  - Trigonometry of Right Triangles - Review of Trigonometric Functions - A Right Triangle Approach
  - Right Triangle Trigonometry - Applications of Right Triangle Trigonometry (Part 1)
  - Trigonometry of Right Triangles - Review of Applications of Right Triangle Trigonometry (Part 1)
  - Inverse Trigonometric Functions - Applications of Right Triangle Trigonometry (Part 2)

KEY CONCEPTS

DEFINITIONS

- **Solving a Triangle:** The process of finding the measures of all unknown side lengths and angles of a triangle, based on some given initial information.

COMMON MISTAKES

- **Save Rounding for Display Purposes Only:** You should not round off intermediate steps in a calculation. Using a rounded value in a subsequent calculation introduces and propagates errors, leading to a less accurate final answer. Use your calculator's stored values or the exact expressions whenever possible.
- **Using Calculated Values Instead of Given Values:** When solving a triangle, it is better to use the original given information for each calculation rather than a value you calculated in a previous step. This helps avoid carrying forward any rounding errors.
- **Incorrect Calculator Entry:** Be careful with the order of operations on your calculator. For example, to compute $\sin(50°) \times 18$, you may need to close the parenthesis after the 50, as in `SIN(50) * 18`. Without the parenthesis, the calculator might compute $\sin(50° \times 18)$, which is incorrect.

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4.3.2: HOMEWORK

READING QUESTIONS

1. What does it mean to "solve a triangle"?
2. If you are given one acute angle and the length of the hypotenuse in a right triangle, which trigonometric function would you use to find the length of the side opposite the given angle?
3. If you are given the lengths of the two legs of a right triangle, how do you find the measure of one of the acute angles?
4. Why is it generally better to use the given information for each calculation, rather than using values you have already calculated?
5. What is the first step you should always take when starting to solve a triangle, especially in an application problem?
6. How can right triangle trigonometry be used to find the altitude of an isosceles triangle that is not a right triangle?
7. In a right triangle $\triangle ABC$ with right angle at C, if you are given sides a and c, what inverse trigonometric function would you use to find $\angle A$?
8. The text warns that when calculating $\sin(50°) \times 18$, you might need to enter `SIN(50) * 18` into your calculator. What might the calculator compute if you enter `SIN(50 * 18)` instead?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, simplify.

1. $\sqrt{2}\sqrt{2}$
2. $\frac{3}{\sqrt{3}}$
3. $\sqrt{8}$
4. $\sqrt{\frac{3}{4}}$

For the following exercises, rationalize the denominator.

5. $\frac{1}{\sqrt{2}}$
6. $\frac{2}{\sqrt{3}}$
7. $\frac{6}{\sqrt{3}}$
8. $\frac{4}{\sqrt{8}}$

Answers

1. 2
2. $\sqrt{3}$
3. $2\sqrt{2}$
4. $\frac{\sqrt{3}}{2}$
5. $\frac{\sqrt{2}}{2}$
6. $\frac{2\sqrt{3}}{3}$
7. $2\sqrt{3}$
8. $\sqrt{2}$

HOMEWORK

VOCABULARY CHECK

1. Given enough initial information about a triangle, we can find the lengths of all remaining sides and angles. This process is known as ___.

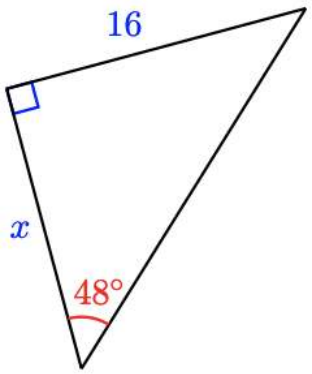
CONCEPT CHECK

2. How many parts of a right triangle (including the right angle) do you need to know to solve it?
3. Why is it better to use the given values when solving a triangle than the ones you calculated?
4. What is the $\sin^{-1}$ (or $\cos^{-1}$ or $\tan^{-1}$) button on the calculator used for?
5. If you knew one acute angle and the side opposite that angle, which trigonometric ratio would you use to find the hypotenuse of a right triangle?

BASIC SKILLS

For the following exercises, use one of the three trigonometric ratios to find the unknown side of the triangle. Round your answer to hundredths.

6.



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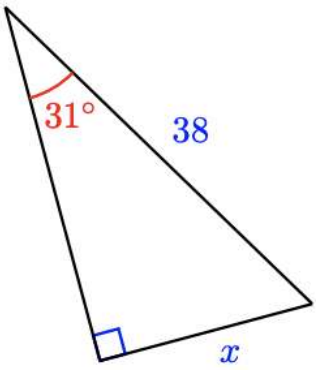

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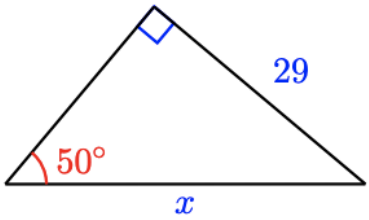

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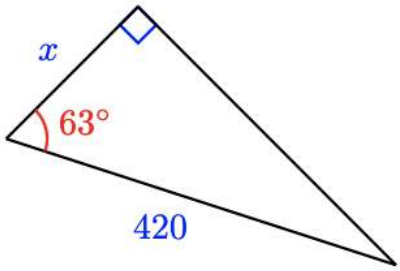

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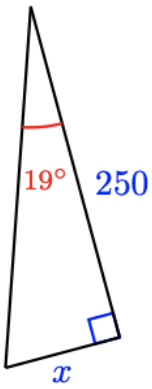

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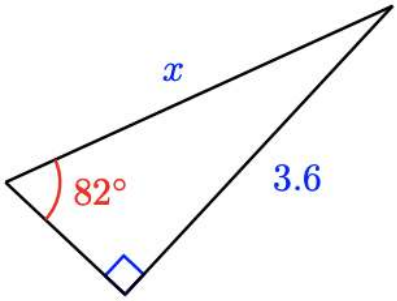
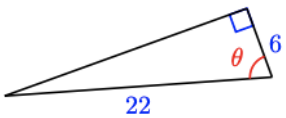


For the following exercises, find the labeled angle. Round your answer to tenths of a degree.

12.



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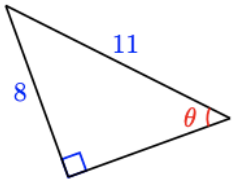

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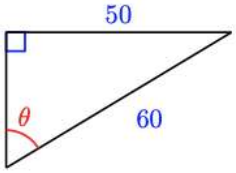

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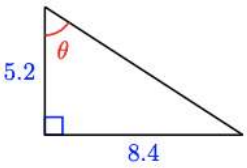

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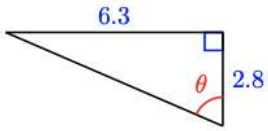

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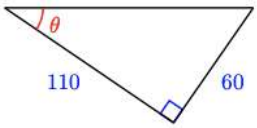
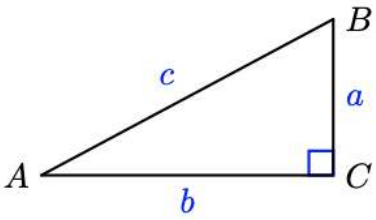


In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) List the steps you would use to solve the triangle without doing the calculations.

18. $B = 53.7°, b = 8.2$
19. $B = 80°, a = 250$
20. $A = 25°, b = 40$
21. $A = 15°, c = 62$
22. $A = 64.5°, c = 24$
23. $B = 44°, b = 0.6$

For the following exercises, solve the triangle. Round answers to hundredths.

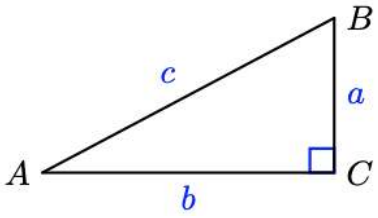
24.

25.

26.

27.

In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) Solve the triangle.

28. $A = 42°, c = 26$

29. $B = 28°, c = 6.8$

30. $B = 33°, a = 300$

31. $B = 79°, a = 116$

32. $A = 12°, a = 4$

33. $A = 50°, a = 10$

For the following exercises, sketch and label a right triangle with the given properties. Solve the triangle.

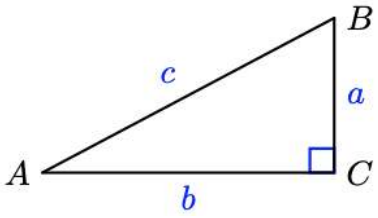
34. One angle is $40°$, and the side opposite that angle is 8 inches

35. One angle is $65°$, and the side adjacent to that angle is 30 yards

36. One angle is $28°$, and the hypotenuse is 56 feet

37. One leg is 15 meters, and the hypotenuse is 18 meters

In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) Solve the triangle.

38. $a = 18, b = 26$

39. $a = 35, b = 27$

40. $b = 10.6, c = 19.2$

41. $a = 88, c = 132$

For the following exercises, solve the triangle. Give your answers as exact values.

42.

43.

44.

45.

46.

47.

48.

49.

50.

51.

SYNTHESIS QUESTIONS

For the following exercises, find the triangle's altitude. Round your answer to two decimal places.

52.

53.

54.

55.

For the following exercises, find the chord length AB. Round your answer to two decimal places.

56.

57.

58. Triangle ABC is equilateral, and its angle bisectors meet at point P. The sides of $\triangle ABC$ are 6 inches long. Find the length of AP.

59. Find an exact value for the perimeter of each parallelogram.

a.

b.

c.

60.

a. Find the area of the outer square.

b. Find the dimensions and the area of the inner square.

c. What is the ratio of the area of the outer square to the area of the inner square?

61.

a. Find the area of the inner square.

b. Find the dimensions and the area of the outer square.

c. What is the ratio of the area of the outer square to the area of the inner square?

APPLICATIONS

For the following exercises,
(a) Sketch a right triangle that illustrates the situation. Label your sketch with the given information.
(b) Choose the appropriate trigonometric ratio, write an equation, and solve the problem.

62. **Distance Across a Canyon.** To measure the distance across a canyon, Roy first sights an interesting rock directly opposite on the other side. He then walks 200 yards down the canyon's rim and sights the rock again, this time at an angle of $18.5°$ from the canyon rim. What is the width of the canyon?

63. **Engineering Accessibility.** Ramps for wheelchairs should be no steeper than an angle of $6°$. How much horizontal distance should a ramp that rises 5 feet in height be allowed?

64. **Cable Length.** A cable car on a ski lift traverses a horizontal distance of 1800 meters at an angle of $38°$. How long is the cable?

65. **Cable Angle.** The gondola cable for the ski lift at Snowy Peak is 2458 yards long and climbs 1860 feet. What angle with the horizontal does the cable make?

66. **Construction.** Loi is building the loft on her summer cottage. At its central point, the height of the loft is 8 feet, and the roof's pitch should be $24°$. How long should the rafters be?

67. **Leaning Tower of Pisa.** The Leaning Tower of Pisa, if standing perfectly vertical, would have a height of 55 meters - unfortunately, the tower is leaning (thus, its famous name). An object dropped from the top of the tower lands 4.8 meters from the base of the tower. At what angle from the horizontal does the tower lean?

68. **Mining.** A mining company locates a vein of minerals at a depth of 32 meters. However, a layer of granite is directly above the minerals, so they decide to drill at an angle, starting 10 meters from their original location. At what angle from the horizontal should they drill?

69. **Chichen Itza.** The Mayan pyramid of El Castillo in Chichen Itza, Mexico, has 91 steps, each 26 cm high and 30 cm deep.

    a. What angle does the side of the pyramid make with the horizontal?

    b. What is the distance up the face of the pyramid, from the base to the top platform?

70. **Ferris Wheel.** The first Ferris Wheel was built for the 1893 Chicago World's Fair. It had a diameter of 250 feet, and the boarding platform at the wheel's base was 14 feet above the ground. What would be your height above the ground if you boarded the wheel and rotated through an angle of $50°$?

71. **The Physics of Light.** The distance that sunlight must travel to pass through a layer of Earth's atmosphere depends on the thickness of the atmosphere and the sun's angle.

    a. Write an expression for the distance, d, that sunlight travels through a layer of atmosphere of thickness h.

    b. Find the distance (to the nearest mile) that sunlight travels through a 100-mile layer of atmosphere when the sun is $40°$ above the horizon.

72. **Railroad Engineering.** In railroad design, the degree of curvature of a section of track is the angle subtended by a chord 100 feet long.

    a. Use the figure to write an expression for the radius, r, of a curve whose degree of curvature is θ. (Hint: The bisector of the angle θ is perpendicular to the chord.)

    b. Find the radius of a curve whose degree of curvature is $43°$.

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4.4: Applications Involving Right Triangles

Note to the Instructor (click to expand)

This section expands on the applications and methods introduced in the previous section. The applications treated here cover angles of elevation and depression, navigation, and right triangle applications in three-dimensions. As is the case with everything in this chapter, all angles are acute.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Linear Functions**
  - Equations of Lines: This skill is needed for homework problems that involve finding the equation of a line given its angle of inclination.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Solve an application problem involving an angle of elevation or depression.
- Determine the angle of inclination (or declination) of a line.
- Solve a navigation problem involving bearing or heading.
- Solve a compound right triangle application.

Hawk A.I. Section-Specific Tutor

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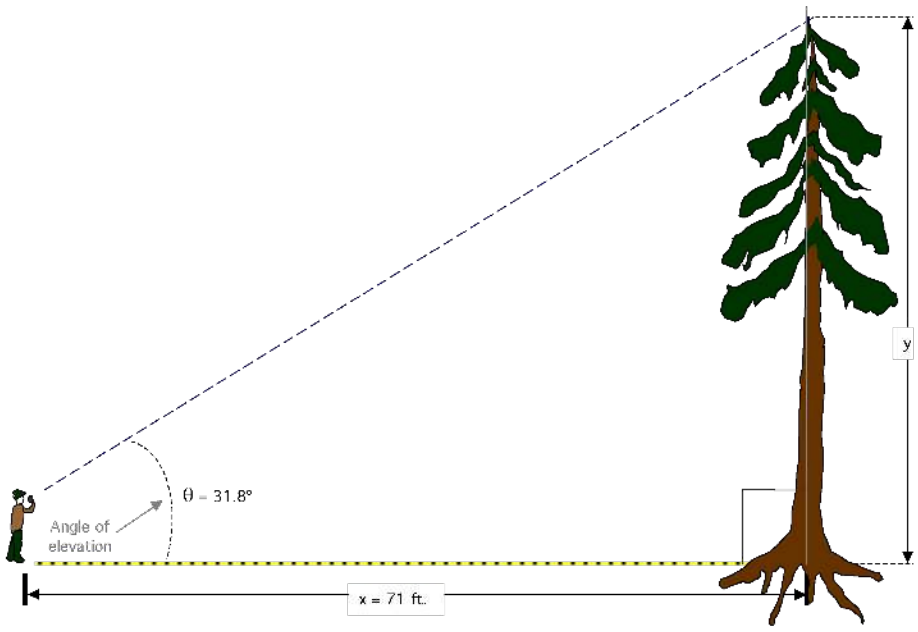
To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

We now venture into practical applications of right triangle trigonometry.[1] The concepts introduced in this section are used throughout the rest of this text and are the most common applications (although not the only applications) within Trigonometry.

Angles of Elevation or Depression

Suppose you want to measure the height of a tall tree without climbing it (see the figure below).

Figure 4.4.1



We can measure the distance to the tree's base and use a surveying tool called a theodolite to measure the angle between the ground and our line of sight to the top of the tree. This angle is called the **angle of elevation**.

Definition: Angle of Elevation

The **angle of elevation** (or **angle of inclination**) is the angle formed between a horizontal line and the line of sight when an observer looks upwards at an object. It represents the amount an observer would need to elevate their gaze (or line of sight) from a horizontal plane to see an object above them.

Since the height we want is the side opposite the angle of elevation, and we already measured the distance to the base of the tree, we could relate the quantities using the tangent function.

Example 4.4.1

Find the height of a cliff if the angle of elevation to the top of the cliff is $58°$ at a distance of 300 feet from the base of the cliff.

Solution

As always, we begin by sketching the situation.

Figure 4.4.2

We know the angle of elevation, $58°$, and the side length adjacent to that angle, 300 feet. We use the tangent function to find the cliff's height (which is opposite the given angle). Letting h be the height of the cliff, we get

$$\tan(58°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{300} \implies h = 300\tan(58°) \approx 480 \text{ feet.}$$

Checkpoint 4.4.1

The tallest living tree is a coastal redwood named Hyperion, measuring 378.1 feet tall. From a distance of 100 feet from the base of the tree, what is the angle of elevation to the top of the tree? Round the answer to the nearest degree.

Answer

$75°$

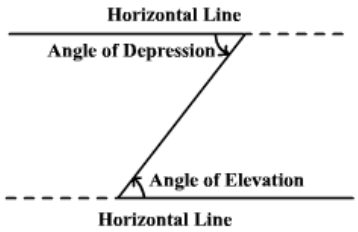
The angle of elevation has a "sister" concept called the **angle of depression**.

Definition: Angle of Depression

The **angle of depression** (or **angle of declination**) is the angle formed between a horizontal line and the line of sight when an observer looks downwards at an object. It represents the amount an observer would need to lower their gaze (or line of sight) from a horizontal plane to see an object below them.

The two concepts are illustrated in Figure 4.4.3.

Figure 4.4.3



Example 4.4.2

Phuong takes a helicopter tour through the Grand Canyon. At one point, the pilot hovers 500 feet from the canyon wall. Phuong, being a prepared mathematician, takes out her protractor and measures the angle of elevation to Pima Point (the top of the canyon wall) as $67°$ and the angle of depression to the canyon floor as $82°$. To the nearest foot, what is the elevation gain from the canyon floor to Pima Point?

Solution

We start by sketching the situation.

Figure 4.4.4

We must find the value of $h_1 + h_2$. The trigonometric function relating $67°$ to h_1 and 500 is the tangent. Thus,

$$\tan(67°) = \frac{h_1}{500} \implies 500\tan(67°) = h_1.$$

Similarly,

$$\tan(82°) = \frac{h_2}{500} \implies 500\tan(82°) = h_2.$$

Therefore, the elevation gain from the canyon floor to Pima Point is

$$h_1 + h_2 = 500\tan(67°) + 500\tan(82°) \approx 4,736 \text{ feet}.$$

Checkpoint 4.4.2

On August 7th, 1974, French street performer Philippe Petit began the most famous high-wire walk in history, calmly traversing the space between the former Twin Towers (in New York City) at an elevation of 1,350 feet without any safety gear. Not long after stepping out onto the thin wire, the angle of depression to the base of one building was $88°$, and the angle of depression to the base of the other building was $86°$. How many feet was his tightrope walk?

Answer

Roughly 141.5 feet.

Angles of Lines

As a rare moment in Mathematics, we have an alternate definition of **angle of inclination** when it comes to lines.

Definition: Angle of Inclination of a Line

The **angle of inclination of a line** is the measure of the angle in the positive direction from the positive x-axis to the line.

Figure 4.4.5 demonstrates the angle of inclination of a line.

Figure 4.4.5

It is important to note that the angle of inclination to the line is measured from where the line crosses the x-axis. For example, the angle of inclination in Figure 4.4.6 is the *positive* angle, β, between the x-axis (in the positive direction) and the line, and this happens to be an obtuse angle.

Figure 4.4.6

Recall that the slope of a line is given by the ratio

$$\frac{\text{change in } y}{\text{change in } x}$$

as we move from one point to another on the line. So, if we create a right triangle by dropping a perpendicular segment from the line to the x-axis, the ratio of sides $\frac{\text{opposite}}{\text{adjacent}}$ gives the slope of the line.

Theorem: The Tangent and the Angle of Inclination of a Line

If the slope of the line is m, then the angle of inclination of the line, α, is related to the slope by the formula

$$\tan(\alpha) = m$$

where $0° \leq \alpha \leq 180°$.

Example 4.4.3

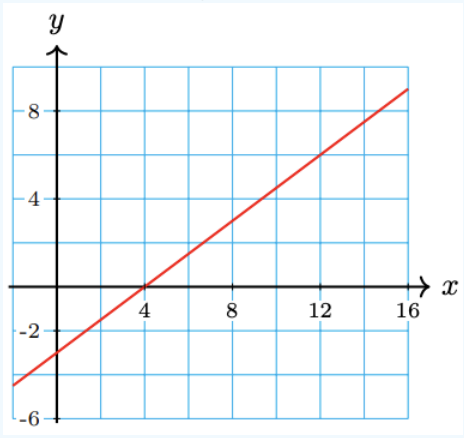
Find the angle of inclination of the line $y = \frac{3}{4}x - 3$.

Solution

The slope of the line is $\frac{3}{4}$. Therefore,

$$\begin{aligned} \tan(\alpha) &= \tfrac{3}{4} \\ \Longrightarrow \quad \alpha &= \tan^{-1}\left(\tfrac{3}{4}\right) \approx 36.9° \end{aligned}$$

Figure 4.4.7



The angle of inclination is approximately $36.9°$.

Checkpoint 4.4.3

Find the angle of inclination of the line in Figure 4.4.8.

Figure 4.4.8: $y = -\frac{6}{5}x + 2$

Answer

$\approx 129.8°$

Navigation Problems

Up to this point, we have presented two definitions for the trigonometric functions. Coordinate trigonometry allows us to work with any angle (positive, negative, acute, or obtuse) as long as it is in standard position. Right triangle trigonometry will enable us to

work with acute angles within a right triangle without a coordinate system. With navigation problems, we officially begin to marry the two definitions into a unified approach to Trigonometry.

Heading

Navigational directions for vessels or aircraft are often given as a **heading**.

Definition: Heading (navigation)

In aviation and nautical navigation, the **heading** of a vessel or aircraft is the compass direction measured clockwise from due north in which the craft's bow or nose is pointed.

For example, a heading of $110°$ is shown in Figure 4.4.9.

Figure 4.4.9

However, as shown in Figure 4.4.10, this heading generates a terminal side shared with the angle $-20°$ in standard position.

Figure 4.4.10

While there is a method for converting a heading to an angle in standard position, it is much better to get used to making a sketch of the given situation and superimposing a right triangle with hypotenuse along the terminal side of the given heading angle and adjacent side along the x-axis, as shown in Figure 4.4.11.

Figure 4.4.11

Once we have the superimposed right triangle, we can use right triangle trigonometry to answer all questions about this situation.

Example 4.4.4

Piyali leaves the airport at a heading of $245°$ and flies 60 miles. How far south of the airport is she at that time?

Solution

Sketching a heading angle of $245°$, we get

Figure 4.4.12

We now superimpose a right triangle on this graph, where the hypotenuse is on the terminal side of the heading angle and the adjacent side is on the x-axis.

Figure 4.4.13

The key to using this superimposed right triangle is to use our *deductive* skills to determine the angle between the hypotenuse (terminal side of the heading angle) and the adjacent side (x-axis). Had the heading angle been $270°$, Piyali's plane would be heading directly west along the negative x-axis (the adjacent side of the right triangle). This means there must be a small angle of $25°$ between the adjacent side of the right triangle and the hypotenuse. Said differently, to get to the $245°$ heading, we could have turned clockwise from due north by $270°$ and then backed off by $25°$. Figure 4.4.14 shows the resulting angle within the superimposed right triangle.

Figure 4.4.14

Piyali's plane flew along the hypotenuse of this right triangle for 60 miles. To determine how far south she is from the airport (located at the origin), we need to compute the length of the side opposite the $25°$ angle. The trigonometric function relating this angle, the opposite side, and the hypotenuse is the sine.

$$\sin(25°) = \frac{\text{opposite}}{60} \implies \text{opposite} = 60\sin(25°) \approx 25.36.$$

Hence, Piyali is about 25.4 miles south of the airport.

? Checkpoint 4.4.4

Lap leaves the airport and flies 150 miles at a heading of $132°$. How far east of the airport is he at that time?

Answer

111.5 miles

Bearing

Land-based navigation often uses **bearing** instead of heading.

Definition: Bearing (navigational)

In land-based navigation, the **bearing** (also called the **quadrant bearing**) of an object (person, vehicle, etc.) represents the direction that object is facing or moving. The notation for bearing is a letter followed by an acute angle followed by a letter. These are known as **Meridian - Angle - Direction**.

- **Meridian:** The first letter indicates whether the angle is measured from North (N) or South (S), whichever is closer to the direction.
- **Angle:** This is the acute angle measured from the chosen meridian to the direction of interest.
- **Direction:** The second letter indicates whether the angle is measured towards the East (E) or West (W).

An example of a bearing is $\text{S } 20° \text{ W}$ (read as, "south by 20 degrees west"). Conceptually, this means the following: initially face south, then turn by $20°$ in the westerly direction. Thus, a bearing describes a line as heading north or south and deflected some degrees toward the east or west. Since a bearing always starts with a north or south facing and then a turn toward east or west, bearings will always have an acute angle.

Figure 4.4.15 illustrates some directions given in bearings.

Figure 4.4.15

When working with directions given in bearings, it is crucial that you understand the angle given will always be between the y-axis (the north-south line) and the terminal side of the angle. This is the only time in Trigonometry you will see such a convention. As long as you keep your thinking cap on and do not try to memorize your way through Trigonometry, you should be fine (especially if you get in the habit of superimposing right triangles on your sketches as we did with headings).

Example 4.4.5

Roy is hiking on the Pacific Crest Trail. He leaves his campsite at 6:00 A.M., walking at a bearing of **N 38° W**. His average hiking speed is 3.4 miles per hour. At 9:00 A.M., he decides to eat breakfast at Scissor's Crossing. At this time, how far north is he from the campsite?

Solution

Quickly sketching the situation, we get Figure **4.4.16**.

Figure 4.4.16

We superimpose a right triangle on this figure, where the hypotenuse is along the terminal side of the bearing angle, and one side is on the x-axis.[2] The result is Figure **4.4.17**.

Figure 4.4.17

Notice that we already included that Roy has hiked a total of $3 \times 3.4 = 10.2$ miles by 9:00 A.M. Also, from the superimposed right triangle, we can see that the trigonometric function relating the hypotenuse to the vertical distance and the $53°$ angle (which is the complement of the given $37°$ angle) is the sine function. Hence,

$$\sin(53°) = \frac{y}{10.2} \implies y = 10.2\sin(53°) \approx 8.1461.$$

Therefore, Roy is approximately 8 miles north of his campsite at 9:00 A.M.

? Checkpoint 4.4.5

Ivan leaves the airport and flies 150 miles at a bearing of $\mathrm{S}\,48°\,\mathrm{E}$. How far east of the airport is he at that time?

Answer

111.5 miles

✓ Example 4.4.6

Mount Whitney is 5.86 miles due east of the Pacific Crest Trail, labeled as point P. The John Muir Trail, labeled as point J, is 1.32 miles due south of Mount Whitney. You hike from Mount Whitney to point J on the John Muir Trail. Once at point J, you decide to hike to point P on the Pacific Crest Trail. What is the bearing in which you must hike *from* point J on the John Muir Trail *to* point P on the Pacific Crest Trail, and how many miles will your hike be? (For simplicity, we will ignore the fact that there are significant elevation gains and losses between each of these points.)

Figure 4.4.18

Solution

With bearing problems, it is beneficial to draw a small set of axes at the point we are moving *from*, as seen in Figure 4.4.19.

Figure 4.4.19

We then draw a line connecting points J and P, as follows.

Figure 4.4.20

From the given image, the bearing *from* point J *to* point P has the form $\mathrm{N}\,\theta°\,\mathrm{W}$. The tangent is the trigonometric function that relates the given sides of the right triangle and the angle θ. Hence,

$$\tan(\theta) = \frac{5.86}{1.32} \implies \theta = \tan^{-1}\left(\frac{5.86}{1.32}\right) \approx 77.3°.$$

Hence, to hike from point J on the John Muir Trail to point P on the Pacific Crest Trail, we must maintain an approximate bearing of $\mathbf{N}\,77.3^\circ\,\mathbf{W}$. The distance covered during that hike (if there was a direct trail between points J and P) is the hypotenuse of the right triangle.

$$d = \sqrt{5.86^2 + 1.32^2} \approx 6 \text{ miles}.$$

? Checkpoint 4.4.6

Sutter County Airport is 28.4 miles due north of Sacramento International Airport. If Placerville Airport is 45.3 miles due east of Sacramento International Airport, what is the bearing from Sutter County Airport to Placerville Airport, and what is the distance between these two airports?

Answer

$\mathbf{S}\,57.9^\circ\,\mathbf{E}$, and 53.5 miles.

In Examples 4.4.5 and 4.4.6, we were given a single destination for a bearing. We were essentially told, "Go this direction for so many miles." However, an important application within navigation occurs when you have been given "stop-over points" where the bearing is changed. The following example illustrates, in great detail, how to manage these problems.

✔ Example 4.4.7

Roy has made significant progress along the Pacific Crest Trail and has reached the Sierra Nevada range. Last night, he made camp at Junction A. In the morning, he breaks camp at 6:00 A.M. and hikes at a bearing of $\mathbf{N}\,42^\circ\,\mathbf{E}$. After two hours, he reaches Junction B, where the trail turns 90° in the bearing of $\mathbf{N}\,48^\circ\,\mathbf{W}$. After another hour, he runs into Junction C, which is a "shortcut" trail that directly connects his current location back to his original campsite at Junction A.

Normally, Roy would never consider "back-tracking" as it's a complete waste of time and energy; however, seeing the "shortcut" trail sparks his memory that he left his trash bag back at his campsite (Junction A)! Being a conscientious hiker, he decides to burn the time and return to ensure he "leaves no trace" on the trail.

If Roy hikes at an average pace of 3.4 miles per hour, what is the distance between Junction C and Junction A, and what is his bearing from Junction C to Junction A?

Solution

This is an excellent example for several reasons. First, in the "real world," the application of mathematics often occurs when you have a lot of extraneous information and "word noise" in a problem. It would be best to get used to simplifying the language in a situation and removing extraneous bits. Here's a great rewording:

"The bearing from A to B is $\mathbf{N}\,42^\circ\,\mathbf{E}$ and the distance is 6.8. The bearing from B to C is $\mathbf{N}\,48^\circ\mathbf{W}$ and the distance is 3.4. Find the distance between A and C, and the bearing from C to A."

Now that we have removed as much extraneous wording as possible let's draw a picture of the situation. We begin with a simple sketch of the bearing from A, including that Roy hiked for 2 hours at 3.4 miles per hour.

Figure 4.4.21

It's important to note that we drew the north-south and east-west axes at the initial point in Figure **4.4.21**. Once we have drawn the angle and its terminal side, we place B at the tip of that terminal side (see Figure **4.4.22**. We also draw in another set of north-south and east-west axes.

Figure 4.4.22

The next "stop-over" point, at the end of that last 3.4-mile hike, is C. Plotting that point and drawing in a final set of north-south and east-west axes, we arrive at Figure **4.4.23** below.

Figure 4.4.23

Take a moment to compare Figures **4.4.22** and **4.4.23**. The latter figure has more information than just an extra set of axes - we have included alternate interior angles! As you will soon see, this information will be critical in helping us finish the problem. Speaking of which, let's draw a line connecting C to A.

Figure 4.4.24

Figure 4.4.24 is the most important for you to be able to draw. It would be best if you had it to complete this problem successfully. The triangle we have drawn is a right triangle. We should immediately know this from the clue that Roy turns $90°$ once he arrives at Junction B.[3] The distance back to A from C is the length, d, of the hypotenuse of the right triangle. Hence,

$$d = \sqrt{6.8^2 + 3.4^2} \approx 7.6 \text{ miles}.$$

To find the bearing, we first must recognize that, from C, Roy must face south and turn through a small angle in the westerly direction to face A. We need that little sliver of a green angle. Focusing only on the right triangle, we can see that

$$\tan(\angle ACB) = \frac{6.8}{3.4} = 2 \implies \angle ACB = \tan^{-1}(2) \approx 63.4°.$$

However, if we call that little sliver of an angle α, then $\angle ACB = \alpha + 48°$. Therefore,

$$63.4° \approx \alpha + 48° \implies \alpha \approx 15.4°.$$

Thus, Roy's bearing from C to A is $\text{S } 15.4° \text{ W}$.

Note: Place Small Sets of Axes at Each Stop-Over Point

As Example **4.4.7** illustrates, bearing problems where direction changes occur are more manageable if you get into the good habit of drawing small "north-south, east-west" axes at each stop-over point.

Checkpoint 4.4.7

A ship leaves port bearing **S 32° E**. It travels in this direction for 1 mile and changes its bearing to **S 58°W**. It then travels an additional 10 miles. At this point, the captain remembers he left his phone back at port (and he wanted to post a ton of videos of this trip on his social media platforms). Find the direct distance and the bearing from this point back to port so that the "social-influencer" captain can get his phone. (Round answers to the nearest tenth.)

Answer

10.0 miles at a bearing of **N 52.3° E**.

Compound Right Triangle Applications

An application that requires you to use more than a single right triangle is called a **compound right triangle application** (don't bother looking for that phrase elsewhere, we just made it up to label applications that fit into this category). These entertaining applications often require you to think in three dimensions, and they help build your ability to break larger problems into small pieces.

Example 4.4.8

Camille is worried about her neighbor's tree eventually falling and hitting her house. Unfortunately, her neighbor has refused to allow her to pay someone to measure the exact height of the tree, so Camille resorts to her top-notch math skills. Since she cannot step onto her neighbor's property, Camille stands on her property, facing the tree, and finds the angle of elevation to the top of the tree to be **38°**. She then turns **90°** and walks for 32 feet (staying on her property the entire time). At this point, she turns to face her neighbor's tree and measures the angle between the path she just walked and her neighbor's tree to be **50°**. Use this information and Figure **4.4.25** to find the tree's height.

Figure 4.4.25

Solution

This is a compound right triangle problem because the situation requires *at least* two right triangles. The key to solving such problems is to recognize that these triangles share a common side. If we call this shared side s, we can "separate" the triangles, as seen in Figure 4.4.26.

Figure 4.4.26

In both cases, we can use the tangent function to relate the given angle and the two sides so that

$$\tan(38°) = \frac{h}{s} \quad \text{and} \quad \tan(50°) = \frac{s}{32}.$$

We now use the shared side, s, as the common link between the triangles. That is, solve both equations for s.

$$s = \frac{h}{\tan(38°)} \quad \text{and} \quad s = 32\tan(50°).$$

Therefore,

$$\frac{h}{\tan(38°)} = 32\tan(50°).$$

We solve this final equation for h to find that the height of the tree is

$$h = 32\tan(50°)\tan(38°) \approx 29.8 \text{ feet}.$$

? Checkpoint 4.4.8

Sang's boat loses power while on a boating trip out on the ocean. She radios her position to a pair of radar stations 43 miles apart along the coastline. The radio stations are directly north and south of each other. The bearing of her boat from the southern radio station is $\text{N}\,54°\,\text{W}$, while the bearing from the northern radio station is $\text{S}\,72°\,\text{W}$. To the nearest mile, how far from the coastline is Sang's boat?

Answer

41 miles

Footnotes

[1] While we have encountered applications before this section, the applications presented now involve new language that gets used outside of Trigonometry.

[2] You would not be incorrect if you superimposed a right triangle with a side along the y-axis. If you do this, you will use the cosine of $37°$ to determine how far north Roy is from his campsite; *however*, superimposing right triangles with sides along the y-axis will eventually lead to some confusion and mistakes. It's best to nip that bad habit in the bud and always superimpose our right triangles with sides along the x-axis.

[3] Another way we could have recognized $\angle ABC$ as being a right angle is using the fact that $\angle ABC + 42° + 48° = 180°$.

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4.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Angle of Elevation and Depression
  - Trigonometry of Right Triangles - Review of Applications of Right Triangle Trigonometry (Part 1)
- Compound Right Triangle
  - Right Triangle Trigonometry - Applications of Right Triangle Trigonometry (Part 1)

KEY CONCEPTS

DEFINITIONS

- **Angle of Elevation (or Angle of Inclination):** The angle formed between a horizontal line and the line of sight when an observer looks upwards at an object.
- **Angle of Depression (or Angle of Declination):** The angle formed between a horizontal line and the line of sight when an observer looks downwards at an object.
- **Angle of Inclination of a Line:** The measure of the angle in the positive direction from the positive x-axis to the line.
- **Heading (navigation):** The compass direction in which a vessel or aircraft is pointed, measured clockwise from due north.
- **Bearing (navigational):** The direction of an object, given as an acute angle measured from a north-south line toward either east or west. It is also called a quadrant bearing.

THEOREMS

- **The Tangent and the Angle of Inclination of a Line:** If a line has a slope of m, then its angle of inclination, α, is related to the slope by the formula $\tan(\alpha) = m$, where $0° \leq \alpha < 180°$.

COMMON MISTAKES

- **Confusing Heading and Bearing:** A heading is a single angle measured clockwise from due north. A bearing is an acute angle measured from the north-south line toward the east or west.
- **Incorrectly Orienting Triangles in Navigation Problems:** When superimposing a right triangle onto a navigation sketch, it is best to always place one of the legs on a horizontal (east-west) axis to avoid confusion, rather than on a vertical (north-south) axis.
- **Forgetting to Place Axes at Stop-Over Points:** In navigation problems with multiple parts or changes in direction, it is much easier to manage the geometry if you draw a new set of small north-south, east-west axes at each point where a turn is made.

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4.4.2: HOMEWORK

READING QUESTIONS

1. What is the difference between an angle of elevation and an angle of depression?
2. How is the angle of inclination of a line defined?
3. How is the slope of a line, m, related to its angle of inclination, α?
4. In navigation, what is a "heading"? From what direction and in what manner is it measured?
5. What are the three components of a navigational "bearing"?
6. What is the primary difference between how a heading is measured and how a bearing is measured?
7. In a multi-part navigation problem, what is a helpful drawing strategy to use at each "stop-over" point?
8. What defines a "compound right triangle application"?
9. In a compound right triangle problem, what is the key to solving the problem?

HOMEWORK

VOCABULARY CHECK

1. The angle between the horizon and the line of sight above the horizon is called the angle of ___, also known as the angle of ___.
2. A(n) ___ is an angle measured clockwise from due north.
3. A(n) ___ is a direction given in the form north or south by a certain number of degrees east or west (denoted, for example, by $\mathrm{S}\,13^\circ\,\mathrm{W}$).

CONCEPT CHECK

4. If the angle of inclination of a line is greater than 45°, what can you say about its slope?
5. What is the bearing that shares a terminal side with the standard position angle $\theta = 45^\circ$?
6. What is the bearing that shares a terminal side with the standard position angle $\theta = 120^\circ$?
7. What is the bearing that shares a terminal side with the standard position angle $\theta = -150^\circ$?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

8. An angle of depression is measured from due south to an individual's line of sight below the horizon.
9. The angle given in a bearing must be acute.
10. The angle given in a heading must be acute.
11. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 120°.
12. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -120°.
13. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 60°.
14. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -60°.
15. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 240°.
16. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -240°.

BASIC SKILLS

For the following exercises, sketch the situation using a right triangle and label each angle.

17. The angle of elevation from the ground to the top of a power pole is 50°.
18. The angle of depression from the eye of a hiker on a cliff to the bottom of the canyon is 85°.
19. Two people are looking up at a hot air balloon. They are positioned on opposite sides of the ballon. The angle of elevation to the ballon for one person is 57°, and the angle of elevation to the ballon for the other person is 30°.

For the following exercises, sketch a diagram illustrating the given bearing.

20. The bearing from Sacramento to South Lake Tahoe is $\mathrm{N}\,72^\circ\,\mathrm{E}$.
21. The bearing from New York City to Philadelphia is $\mathrm{S}\,48^\circ\,\mathrm{W}$.
22. The bearing from the Southern Terminus of the Pacific Crest Trail to the Northern Terminus is $\mathrm{N}\,10^\circ\,\mathrm{W}$.
23. The bearing from Lincoln Airport to Placerville Airport is $\mathrm{S}\,69^\circ\,\mathrm{E}$.

For the following exercises, find the angle of inclination of the line.

24. $y = \frac{5}{4}x - 3$
25. $y = 6 + \frac{2}{9}x$
26. $y = -2 - \frac{3}{8}x$
27. $y = \frac{-7}{2}x + 1$

For the following exercises, find an equation for the line passing through the given point with an angle of inclination α.

28. $(3, -5), \alpha = 28^\circ$
29. $(-2, 6), \alpha = 67^\circ$
30. $(-8, 12), \alpha = 112^\circ$
31. $(-4, -1), \alpha = 154^\circ$

SYNTHESIS QUESTIONS

32. The slope of a line is a function of its angle of inclination, $m = f(\alpha)$. Complete the table and sketch a graph of the function.

α	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
m													

a. What happens to the slope of the line as α increases toward $90°$?

b. What happens to the slope of the line as α decreases toward $90°$?

33. The angle of inclination of a line is a function of its slope, $\alpha = g(m)$. Complete the table and sketch a graph of the function.

m	-20	-10	-5	-2	-1	-0.75	-0.5	-0.25	0
α									

m	0.25	0.5	0.75	1	2	5	10	20
α								

a. What happens to the angle of inclination as the slope increases toward infinity?

b. What happens to the angle of inclination as the slope decreases toward negative infinity?

APPLICATIONS

ANGLES OF ELEVATION AND ANGLES OF DEPRESSION

For the following exercises,
(a) Sketch a right triangle that illustrates the situation. Label your sketch with the given information.
(b) Choose the appropriate trigonometric ratio, write an equation, and solve the problem.

34. **Estimating Height.** The birdhouse in Phuong's front yard is 12 feet tall, and its shadow at 4 P.M. is 15 feet 4 inches long. What is the angle of elevation of the sun at 4 P.M.?

35. **Estimating the Angle of Elevation.** If a 65.2-foot sculpture casts a shadow 42.1 feet long, find the sun's angle of elevation to the nearest tenth of a degree.

36. **Air Traffic Control.** To measure the height of cloud cover, airport controllers fix a searchlight to shine a vertical beam on the clouds. The searchlight is 120 yards from the office. A technician in the office measures the angle of elevation to the light on the cloud cover at $54.8°$. What is the height of the cloud cover?

37. **Salvage Crew.** A salvage ship is searching for the wreck of a pirate vessel on the ocean floor. Using sonar, they locate the wreck at an angle of depression of $36.2°$. The depth of the ocean at their location is 260 feet. How far should they move to be directly above the wrecked vessel?

38. **Volcanology.** Cinder cones are the most common type of volcano in the world. They may look like an idealized depiction of a volcano as they are steep, conical hills that usually have a prominent crater at the top. Maha decides to hike to the top of Capulin Volcano, a cinder cone in New Mexico. Once at the rim, she measures the angle of depression to the surrounding plain to be $26.3°$. Her pedometer shows that her hike up to the top of Capulin Volcano covered 2,939 feet. Assuming the hike up the volcano was along a straight path, what was her elevation gain?

39. **Meteorology.** The radio signal from a weather balloon indicates that it is 1500 meters from a meteorologist on the ground. The angle of elevation to the balloon is $48°$. What is the balloon's altitude?

40. **Ancient Warfare.** According to Chinese legend, around 200 BC, the general Han Xin used a kite to determine the distance from his location to an enemy palace. He then dug a secret tunnel which emerged inside the palace. When the kite was directly above the palace, its angle of elevation was $27°$ and the string to the kite was 1850 feet long. How far did Han Xin's troops have to dig?

41. **Ocean Rescue.** A hot air balloon was forced far off course, and the occupants had to make a forced landing at sea. They sent out a distress signal before taking the balloon down, and, very luckily, a United States Lifesaving Association boat was nearby. The distress signal the boat receives gives the balloon's altitude before descending as 1,128 feet. Immediately, an observer on the boat could see the balloon and measure the angle of elevation to the balloon as $22.1°$. How far would the boat travel to reach any survivors if the balloon went directly (vertically) down and crashed into the water?

42. **Airplane Navigation.** An airplane begins its descent when its altitude is 10 kilometers. The angle of descent should be $3°$ from horizontal.

a. How far from the airport (measured along the ground) should the airplane begin its descent?

b. How far will the airplane travel on its descent to the airport?

43. **Estimating Height.** Linda is on the upper floor of a tall building. She measures the angle of elevation to the building across the street to be $38°$ and the angle of depression to the base of that building as $65°$. If she knows the two buildings are 50 feet apart, how tall is the building across the street?

44. **Height of a Flagpole.** To estimate the height of a flagpole, Kim positions herself due south of the pole, and Matt stands due east of the pole. If they are the same distance from the pole and 50 feet from each other, find the height of the pole if the angle of elevation from the ground to the top of the pole at each person's position is $49°$.

45. **Estimating Height - D&D Edition.** Deitrich Wyvernfang bravely stands in defense against a massive stone giant. He is waiting for the precise moment when the creature steps within reach of his trusty glaive; however, as with most stone giants, the beast is being measured in its approach and purposefully keeps its distance. In the intervening moments, Hoot de'La Temp, Deitrich's trusty but enigmatic companion, is overwhelmed with a momentary curiosity. In his studies of all creatures magical and curious, Hoot had never read of a stone giant this tall.

Always taking advantage of the moment, Hoot places his hand

on Deitrich's shoulder, casually saying, "Hold the giant off for me. This is a rare opportunity to measure the stature of such a creature in its non-deceased form." Deitrich's surprised, wide-eyed glance at Hoot is all the wizard needs to know that his friend must share his excitement about such an opportunity. Kneeling, Hoot quickly utters the arcane words for his *calculate* spell and measures the angle of elevation to the stone giant's head to be $52° 34'$. Standing, he nods at Deitrich (who, as Hoot thinks about it, now looks disgusted for some reason) and says, "Keep him still for just a few moments longer."

Hoot walks directly away from the stone giant, keeping Deitrich between himself and the dangerous creature. Each step he takes is a measured 3-foot distance. He smiles warmly as his friend decides to help him by barking "But... But... But..." in time with his steps - what a great companion. After ten such steps, Hoot turns, kneels, and casts his *calculate* spell again. This time, the angle of elevation to the now confused giant's head measures $23° 31'$. Standing, the wizard shouts victoriously, "Got it! That beast is precisely ___ feet tall! Well, to the nearest tenth of a foot, that is. You may slay him now."

What is the number of feet that Hoot estimated the height of this stone giant to be?

46. **Neighborly Dispute.** Nam's neighbors have built a massive pile of junk in their yard topped with a standard door frame. He would like to report them to the county for the eyesore, but the county will only give the neighbor a citation if the structure is more than 30 feet tall. Unfortunately, Nam's neighbors are not allowing him to measure the height of the junk pile on their property, so he resorts to his mathematics training. From a point on the ground (level with the base of the junk pile), Nam measures the angle of elevation to the top of the entire pile to be $37°$. From that same point, he measures the angle of elevation to the base of the door frame (on the top of the junk pile) to be $32°$. Assuming the door frame is made to industry standards, it should be 80 inches tall. How tall is the entire pile of junk (in feet)? Can the county cite the neighbor?
47. **Estimating Distance.** To find the distance across a ravine, Roy takes some measurements from a small airplane. When he is a short distance from the ravine at an altitude of 500 feet, he finds that the angle of depression to the near side of the ravine is $56°$, and the angle of depression to the far side is $32°$. What is the width of the ravine? (Hint: First find the horizontal distance from Roy to the near side of the ravine.)
48. **Radius of the Earth.** A communications satellite is in a low earth orbit (LOE) at an altitude of 400 km. From the satellite, the angle of depression to the earth's horizon is $19.728°$. Use this information to calculate the radius of the planet.
49. **Sunlit Area.** The window in Nhat's office is 4 feet wide and 5 feet tall. The bottom of the window is 3 feet from the floor. When the sun is at an angle of elevation of $64°$, what is the area of the sunny spot on the floor?

HEADING PROBLEMS

For the following exercises, a ship sails from the seaport on the given heading for the given distance.
(a) Make a sketch showing the ship's current location relative to the seaport.
(b) How far east or west of the seaport is the ship's present location? How far north or south?

50. $36°, 26$ miles
51. $124°, 80$ km
52. $230°, 120$ km
53. $318°, 75$ miles
54. $285°, 32$ km
55. $192°, 260$ miles
56. A private plane flies 2.5 hours at 110 mph on a heading of $45°$. It then turns and continues another 1.5 hours at the same speed but on a bearing of $135°$.
    a. How far is the plane from its starting point at the end of this time?
    b. What is the heading from the pilot's starting location to her ending location?

BEARING PROBLEMS

57. **Ship Navigation.** A ship travels on a course of bearing $\text{N } 62° \text{ E}$ for 83.1 miles. How many miles north and how many miles east has the boat traveled?
58. **Ship Navigation.** A ship leaves port and travels 20 miles in the direction $\text{S } 29° \text{ W}$. It then turns $90°$ and travels another 14.4 miles in the direction $\text{N } 61° \text{ W}$. At that time, how far is the ship from port, and what is its bearing back to the port?
59. **Hiking Navigation.** Mike parks his car on a road that runs east-west. From this road, he sets out on a straight trail at a bearing of $\text{N } 65° \text{ E}$ and hikes for 25 miles to a water cache. After a short break, Mike turns onto another straight trail with a bearing of $\text{S } 25° \text{ E}$ for an additional 5 miles to a lunch spot.
    a. To the nearest tenth of a mile, how far was Mike from the east-west road when he reached the water cache?
    b. By the time Mike reaches the lunch spot, what is the bearing from the lunch spot back to his car?
    c. How far away (in terms of direct distance) is Mike's car from the lunch spot?
60. ***Locate Object* - D&D Edition.** The Bandits, as the heroes are known, are searching for a *wand of water control* that has been lost to the overgrowth of the Amedio Jungle. Hoot de'La Temp, the group's enigmatic wizard, casts a *locate object* spell multiple times to find the missing wand. Unfortunately, this spell only reveals the *direction* of an object - not its distance (another issue is that the spell can only find an object once the object is within 1,000 feet of the wizard). Frustrated and tired, the Bandits begin

trekking back to town; however, after a short time, Hoot decides to give the spell one more try.

"Wait! I sense the wand - it's that way," Hoot exclaims, pointing deeper into the jungle.

Anlie, the Bandit's gruff ranger quickly snaps, "How far? Not all the way back to that group of gnolls, I hope."

"I don't know. This magic only gives me direction - not distance," Hoot responds, frustratedly shaking his head. "This particular issue concerns the bending of the Gygaxian weaves with those of the natural weaves present in the surrounding region. I have often thought there might be a better..."

Cutting the wizard off, Anlie tersely states, "I don't need a magic lesson, Hoot. A simple, 'I don't know' works."

The ranger looks up through the jungle canopy, searching for signs only someone with her skill could observe. "Okay, you're pointing roughly north by $45°$ east." She looks around, spotting a path with less vegetation, "How much time do you have left on that spell?"

Hoot, unruffled by the interruption, shrugs, "Seven minutes." Pausing, he adds, "And 36 seconds?"

Rolling her eyes, Anlie quickly moves down the trail. "Follow me."

After a few minutes of hiking through the jungle, Anlie turns to Hoot. "Now point toward the wand."

Furrowing his brow and giving the ranger an appraising look, Hoot absentmindedly raises his arm to the side, pointing his index finger in the perceived direction of the magic item.

Anlie looks at the wizard accusingly, "Are you sure?"

Feigning a look of innocence, the wizard responds, "Nobody can be absolutely certain about these things, but I have the utmost confidence that it's roughly out there," wiggling his finger.

Sighing, Anlie looks around, spending a minute looking for those hidden signs only she seems to see. "Okay, you're pointing north by $45°$ west. It's 1,050 feet back to where we started in the direction north by $78°$ west."

"Are *you* sure," the wizard asks with a grin.

Giving him a flat stare, Anlie sighs again, "Yes, and I know how far away the wand is."

How far away is the *wand of water control* from their current position?

61. **Distance.** From two observation points 25 miles apart on a straight shoreline running north and south, the bearings to a boat at sea are $\mathrm{N}\,26°\,\mathrm{W}$ and $\mathrm{S}\,56°\,\mathrm{W}$. What is the shortest distance from the boat to the shore?

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CHAPTER OVERVIEW

Chapter 5: Polynomial and Rational Functions

35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by Mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how Mathematics can be used in such applications.

5.1: Complex Numbers

5.1.1: Resources and Key Concepts

5.1.2: Homework

5.2: Quadratic Functions

5.2.1: Resources and Key Concepts

5.2.2: Homework

5.3: Power Functions and Polynomial Functions

5.3.1: Resources and Key Concepts

5.3.2: Homework

5.4: Graphs of Polynomial Functions

5.4.1: Resources and Key Concepts

5.4.2: Homework

5.5: Dividing Polynomials

5.5.1: Resources and Key Concepts

5.5.2: Homework

5.6: Zeros of Polynomial Functions

5.6.1: Resources and Key Concepts

5.6.2: Homework

5.7: Rational Functions

5.7.1: Resources and Key Concepts

5.7.2: Homework

5.8: Revisiting Inverses
5.8.1: Resources and Key Concepts
5.8.2: Homework

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5.1: Complex Numbers

Note to the Instructor (click to expand)

This required topic is needed for students to work with imaginary solutions of quadratic equations (coming up in the next section).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Sets and Numbers**
  - Properties of Real Numbers: Operations like addition and multiplication of complex numbers are defined by extending properties of real numbers, such as the distributive property used in $(a+bi)(c+di)$.
- **Simplifying Expressions**
  - Laws of Exponents: Used to understand $i^2 = (\sqrt{-1})^2 = -1$ and for simplifying higher powers of i as shown in the proof of the "Powers of the Imaginary Unit" theorem (e.g., $i^{4q+r} = (i^4)^q \cdot i^r$). This is also needed for homework problems.
  - Simplifying Radical Expressions: Skills in simplifying radicals like $\sqrt{25}$ are needed when expressing square roots of negative numbers, for instance, $\sqrt{-25} = \sqrt{25}\sqrt{-1} = 5i$. Homework problems (like $\sqrt{-2} \cdot \sqrt{-8}$) require expressing terms with i before multiplication.
  - Rationalizing Denominators and Numerators: The technique of multiplying the numerator and denominator by a conjugate to eliminate radicals from a denominator is directly analogous to multiplying by the complex conjugate to ensure a real-valued denominator when dividing complex numbers, as demonstrated in Example 7. This is also needed for homework problems.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Express square roots of negative numbers as multiples of i.
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

Hawk A.I. Section-Specific Tutor

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The study of Mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. For example, we still have no solution to equations such as $x^2 + 4 = 0$. Our best guesses might be 2 or -2. But if we test 2 in this equation, it does not work. If we test -2, it does not work. If we want to have a solution for this equation, we will have to go farther than we have so far. After all, to this point we have described the square root of a negative number as undefined. Fortunately, there is another system of numbers that provides solutions to problems such as these. In this section, we will explore this number system and how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of i

We know how to find the square root of any positive real number. In a similar way, we can find the square root of a negative number. The difference is that the root is not real. We begin our exploration of this new beast by defining the **unit imaginary number**.

Definition: Unit Imaginary Number

The **unit imaginary number**, denoted as i, is defined as the square root of negative 1. That is,

$$\sqrt{-1} = i.$$

The unit imaginary number is often referred to as the **imaginary unit**. This definition leads directly to the following theorem.

Theorem: The Square of the Imaginary Unit

$$i^2 = -1$$

Proof

$$i^2 = (i)^2 = \left(\sqrt{-1}\right)^2 = -1$$

We can write the square root of any negative number as a multiple of i. Consider the square root of -25.

$$\sqrt{-25} = \sqrt{25 \cdot (-1)} = \sqrt{25}\sqrt{-1} = 5i.$$

We use $5i$ and not $-5i$ because the principal root of 25 is the positive root.

By defining the imaginary unit, we have access to an entirely new type of number - the **complex number**.

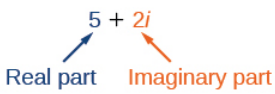
Definition: Complex Number

A **complex number** is a number that can be written in the form $a + bi$ where

- a is called the **real part**, and
- b is called the **imaginary part** of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and b is not equal to 0, the complex number is called an **imaginary number**. The form, $a + bi$ is called the **standard form** of the complex number.

Complex numbers are a combination of real and imaginary numbers. For example, $5 + 2i$ is a complex number. The real part is 5, and it happens that 5 is a real number. The imaginary part is 2, and $2i$ is called an imaginary number.



Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number!

If we were to gather all of the complex numbers in existence, we would have a completely new number system.

Definition: Complex Number System

The set of all numbers of the form $a+bi$ is called the **complex number system** (symbolically denoted as $\mathbb{C}$).

Example 5.1.1: Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

Solution

$$\sqrt{-9}=\sqrt{9}\sqrt{-1}=3i$$

In standard form, this is $0+3i$.

Checkpoint 5.1.1

Express $\sqrt{-24}$ in standard form.

Plotting a Complex Number on the Complex Plane

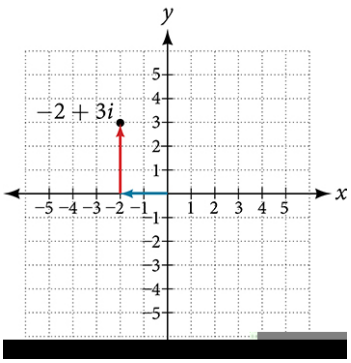
We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. Two components means we need two number lines - one for the real part and one for the imaginary part. Thus is borne the **complex plane**.

Definition: Complex Plane

The **complex plane** is a coordinate system in which the horizontal axis represents the real part of a complex number and the vertical axis represents the imaginary part. In this sense, we define the horizontal axis to be the **real axis** and the vertical axis to be the **complex axis**. Complex numbers are the points on the plane, expressed as ordered pairs (a,b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

Let's consider the number $-2+3i$. The real part of the complex number is -2 and the imaginary part is 3. We plot the ordered pair $(-2,3)$ to represent the complex number $-2+3i$ as shown in Figure 5.1.1.

Figure 5.1.1



Example 5.1.2: Plotting a Complex Number on the Complex Plane

Plot the complex number $3-4i$ on the complex plane.

Solution

The real part of the complex number is 3, and the imaginary part is -4. We plot the ordered pair $(3,-4)$ as shown in Figure 5.1.2.

Figure 5.1.2

Checkpoint 5.1.2

Plot the complex number $-4-i$ on the complex plane.

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers; however, we need to define what it means to add and subtract complex numbers.

Definition: Addition and Subtraction of Complex Numbers

Given two complex numbers, $z_1 = a+bi$ and $z_2 = c+di$, we define addition and subtraction as follows:

Addition:

$$z_1 + z_2 = (a+bi)+(c+di) = (a+c)+(b+d)i.$$

Subtraction:

$$z_1 - z_2 = (a+bi)-(c+di) = (a-c)+(b-d)i.$$

That is, to add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

Example 5.1.3: Adding Complex Numbers

Add $3-4i$ and $2+5i$.

Solution

We add the real parts and add the imaginary parts.

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i && \text{(regrouping)}\\ &= (3+2)+(-4+5)i && \text{(substituting)}\\ &= 5+i && \text{(simplifying)}\end{aligned}$$

Checkpoint 5.1.3

Subtract $2+5i$ from $3–4i$.

Multiplying Complex Numbers

When defining multiplication of complex numbers, we borrow heavily from binomial products.

Definition: Multiplication of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define multiplication as follows:

$$\begin{aligned} z_1 \cdot z_2 &= (z_1)(z_2) \\ &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + (ad+bc)i + bc(-1) \\ &= (ac-bd) + (ad+bc)i \end{aligned}$$

While it might be tempting, you should *never* memorize the result of this definition. Instead, treat multiplying complex numbers like multiplying binomials.

Let's begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for example,

$$\begin{aligned} 3(6+2i) &= (3\cdot 6) + (3\cdot 2i) && \text{Distribute.} \\ &= 18 + 6i && \text{Simplify.} \end{aligned}$$

Example 5.1.4: Multiplying a Complex Number by a Real Number

Find the product $4(2+5i)$.

Solution

Distribute the 4.

$$\begin{aligned} 4(2+5i) &= (4\cdot 2) + (4\cdot 5i) && \text{(distributing)} \\ &= 8 + 20i && \text{(simplifying)} \end{aligned}$$

Checkpoint 5.1.4

Find the product $-4(2+6i)$.

Example 5.1.5: Multiplying a Complex Number by a Complex Number

Multiply $(4+3i)(2-5i)$.

Solution

$$\begin{aligned} (4+3i)(2-5i) &= (4\cdot 2) - (4\cdot 5i) + (3i\cdot 2) - (3i\cdot 5i) && \text{(distributing)} \\ &= 8 - 20i + 6i - 15i^2 && \text{(simplifying)} \\ &= 8 - 14i - 15(-1) && \text{(combining like terms, and } i^2 = -1\text{)} \\ &= 8 - 14i + 15 && \text{(simplifying)} \\ &= 23 - 14i && \text{(simplifying)} \end{aligned}$$

? Checkpoint 5.1.5

Multiply $(3-4i)(2+3i)$.

Dividing Complex Numbers

When building a definition for division of complex numbers, we must come to some agreement concerning allowable forms of complex numbers. At this time, a complex number in standard form must look like $a+bi$, where both a and b are real numbers. Thus, it is safe to claim that a and b, if fractions, cannot have complex numbers in their denominators. Said differently, we require that any fraction must have a real-number denominator. This is so important, it bears repeating.

> Any fraction must have a real-valued denominator.

Therefore, when dividing a complex number by another complex number, we need to find a number by which we can multiply the numerator and the denominator that will force the denominator to be a pure real number. That is, given $z_1 = a+bi$ and $z_2 = c+di$, we need to find a number, let's call this k for now, such that

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{(a+bi)}{(c+di)} \cdot \frac{k}{k} = \frac{k(a+bi)}{k(c+di)},$$

where $k(c+di)$ is a pure real number.

A great choice for k is $c-di$. This is becuase

$$k(c+di) = (c-di)(c+di) = c^2 + cdi - cdi - d^2i^2 = c^2 + \cancel{cdi} - \cancel{cdi} - d^2(-1) = c^2 + d^2.$$

Since both c and d are real numbers, this result is a real number. Our choice for k has a special name (and is *incredibly* important as we move through the rest of Mathematics).

Definition: Complex Conjugate

The **complex conjugate** of $z = a+bi$ is defined to be $\bar{z} = a-bi$. Likewise, the complex conjugate of $z = a-bi$ is defined to be $\bar{z} = a+bi$.

Prior to this definition, we proved the following theorem.

Theorem: Products of Complex Conjugates

Let $z = a+bi$, so that its complex conjugate is $\bar{z} = a-bi$. Then

$$z \cdot \bar{z} = a^2 + b^2.$$

Proof

Let $z = a+bi$. Then

$$\begin{aligned} z \cdot \bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - \cancel{abi} + \cancel{abi} - b^2(-1) \\ &= a^2 + b^2 \end{aligned}$$

We now have everything we need to create a definition for division of complex numbers that wil result in denominators being real numbers.

Definition: Division of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define division as follows:

$$\begin{aligned} z_1 \div z_2 &= \frac{z_1}{z_2} \\ &= \frac{a+bi}{c+di} \\ &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac-adi+bci-bdi^2}{c^2+d^2} \\ &= \frac{ac+(-ad+bc)i-bd(-1)}{c^2+d^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i \\ &= r+si \end{aligned}$$

where r and s are real numbers.

As was mentioned with multiplication of complex numbers, it is best not to memorize this resulting formula. You should be able to just "do the math."

Example 5.1.6: Finding Complex Conjugates

Find the complex conjugate of each number.

a. $2+i\sqrt{5}$
b. $-\frac{1}{2}i$

Solution

a. The number is already in the form $a+bi$. The complex conjugate is $a-bi$, or $2-i\sqrt{5}$.
b. We can rewrite this number in the form $a+bi$ as $0-\frac{1}{2}i$. The complex conjugate is $a-bi$, or $0+\frac{1}{2}i$. This can be written simply as $\frac{1}{2}i$.

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by i.

Example 5.1.7: Dividing Complex Numbers

Divide $(2+5i)$ by $(4-i)$.

Solution

We begin by writing the problem as a fraction:

$$\frac{2+5i}{4-i}.$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}.$$

To multiply two complex numbers, we expand the product as we would with polynomials.

$$\begin{aligned}
\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} &= \frac{8+2i+20i+5i^2}{(4)^2+(-1)^2} && \text{(distributing and using the fact that } (a+bi)(a-bi)=a^2+b^2) \\
&= \frac{8+22i-5}{16+1} && \text{(simplifying)} \\
&= \frac{3+22i}{17} && \text{(simplifying)} \\
&= \frac{3}{17}+\frac{22}{17}i && \text{(separating the real and imaginary parts)}
\end{aligned}$$

Note that this expresses the quotient in standard form.

Example 5.1.8: Substituting a Complex Number into a Polynomial Function

Let $f(x)=x^2-5x+2$. Evaluate $f(3+i)$.

Solution

$$\begin{aligned}
f(3+i) &= (3+i)^2-5(3+i)+2 && \text{(substituting)} \\
&= (9+6i+i^2)-15-5i+2 && \text{(distributing)} \\
&= 9+6i+(-1)-15-5i+2 && (i^2=-1) \\
&= -5+i && \text{(combining like terms)}
\end{aligned}$$

Checkpoint 5.1.8

Let $f(x)=2x^2-3x$. Evaluate $f(8-i)$.

Example 5.1.9: Substituting an Imaginary Number in a Rational Function

Let $f(x)=\frac{2+x}{x+3}$. Evaluate $f(10i)$.

Solution

$$
\begin{aligned}
f(10i) &= \frac{2+10i}{10i+3} && \text{(substituting)}\\
&= \frac{2+10i}{3+10i} && \text{(rewriting the denominator in standard form)}\\
&= \frac{(2+10i)}{(3+10i)} \cdot \frac{(3-10i)}{(3-10i)} && \text{(multiplying by the conjugate of the denominator)}\\
&= \frac{6-20i+30i-100i^2}{(3)^2+(10)^2} && \text{(distributing)}\\
&= \frac{6-20i+30i-100(-1)}{(3)^2+(10)^2} && (i^2=-1)\\
&= \frac{106+10i}{109} && \text{(simplifying)}\\
&= \frac{106}{109}+\frac{10}{109}i && \text{(separating the real and imaginary parts)}
\end{aligned}
$$

? Checkpoint 5.1.9

Let $f(x)=x+1x-4$. Evaluate $f(-i)$.

Simplifying Powers of i

The powers of i are cyclic. Let's look at what happens when we raise i to increasing powers.

$$
\begin{aligned}
i^1 &= i\\
i^2 &= -1\\
i^3 &= i^2 \cdot i = -1 \cdot i = -i\\
i^4 &= i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1\\
i^5 &= i^4 \cdot i = 1 \cdot i = i
\end{aligned}
$$

We can see that when we get to the fifth power of i, it is equal to the first power. As we continue to multiply i by itself for increasing powers, we will see a cycle of 4. Let's examine the next 4 powers of i.

$$
\begin{aligned}
i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1\\
i^7 &= i^6 \cdot i = i^2 \cdot i = i^3 = -i\\
i^8 &= i^7 \cdot i = i^3 \cdot i = i^4 = 1\\
i^9 &= i^8 \cdot i = i^4 \cdot i = i^5 == i
\end{aligned}
$$

Theorem: Powers of the Imaginary Unit

Let $p \in \mathbb{W}$ (the Whole Number system) and r be the remainder when p is divided by 4. Then

$$i^p = i^r.$$

Proof

Let $p \in \mathbb{W}$ and r be the remainder when p is divided by 4. Then r is either 0, 1, 2, or 3. The Divison Algorithm tells us that

$$\frac{p}{4} = q + \frac{r}{4},$$

where q is the quotient and r is the remainder. Multiplying both sides by 4, we get

$$p = 4q + r.$$

Hence,

$$\begin{aligned} i^p &= i^{4q+r} && \text{(substituting)} \\ &= i^{4q} \cdot i^r && \text{(Laws of Exponents)} \\ &= (i^4)^q \cdot i^r && \text{(Laws of Exponents)} \\ &= (1)^q \cdot i^r && (i^4 = 1) \\ &= 1 \cdot i^r \\ &= i^r \end{aligned}$$

Therefore, all whole-numbered powers of i can be simplified to

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = (-1) \cdot i = -i \end{aligned}$$

Example 5.1.10: Simplifying Powers of i

Evaluate i^{35}.

Solution

Since $i^4 = 1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35 = 4 \cdot 8 + 3$. Therefore,

$$i^{35} = i^{4 \cdot 8 + 3} = i^{4 \cdot 8} \cdot i^3 = (i^4)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i.$$

Q&A

Can we write i^{35} in other helpful ways?

As we saw in Example 5.1.10, we reduced i^{35} to i^3 by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of i^{35} may be more useful. Table 5.1.1 shows some other possible factorizations.

Table 5.1.1

Factorization of i^{35}	$i^{34} \cdot i$	$i^{33} \cdot i^2$	$i^{31} \cdot i^4$	$i^{19} \cdot i^{16}$
Reduced form	$(i^2)^{17} \cdot i$	$i^{33} \cdot (-1)$	$i^{31} \cdot 1$	$i^{19} \cdot (i^4)^4$
Simplified form	$(-1)^{17} \cdot i$	$-i^{33}$	i^{31}	i^{19}

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

5.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Complex numbers
  - Arithmetic - Sets - Complex Numbers - Definitions, Theorem, and Example
- Complex plane
  - Arithmetic - Sets - Complex Numbers - The Complex Plane and Plotting Complex Numbers
- Adding and Subtracting complex numbers
  - Arithmetic - Sets - Complex Numbers - Adding and Subtracting Complex Numbers
- Multiplying complex numbers
  - Arithmetic - Sets - Complex Numbers - Multiplying Complex Numbers
- Dividing complex numbers
  - Arithmetic - Sets - Complex Numbers - Dividing Complex Numbers
- Powers of i
  - Arithmetic - Sets - Complex Numbers - Simplifying Powers of i

KEY CONCEPTS

DEFINITIONS

- **Unit Imaginary Number**: The number, denoted as i, which is defined as the square root of negative one ($\sqrt{-1} = i$).
- **Complex Number**: A number that can be expressed in the form $a + bi$, where a and b are real numbers.
  - **Real Part**: For a complex number $a + bi$, a is the real part.
  - **Imaginary Part**: For a complex number $a + bi$, b is the imaginary part.
  - **Imaginary Number**: A complex number $a + bi$ where $a = 0$ and $b \neq 0$.
  - **Standard Form**: The representation of a complex number as $a + bi$.
- **Complex Number System**: The entire set of numbers of the form $a + bi$, symbolically denoted as $\mathbb{C}$.
- **Complex Plane**: A coordinate system used for graphing complex numbers, where the horizontal axis (real axis) represents the real part and the vertical axis (complex axis) represents the imaginary part.
  - **Real Axis**: The horizontal axis in the complex plane.
  - **Complex Axis**: The vertical axis in the complex plane.
- **Addition of Complex Numbers**: If $z_1 = a + bi$ and $z_2 = c + di$, their sum is $z_1 + z_2 = (a + c) + (b + d)i$.
- **Subtraction of Complex Numbers**: If $z_1 = a + bi$ and $z_2 = c + di$, their difference is $z_1 - z_2 = (a - c) + (b - d)i$.
- **Multiplication of Complex Numbers**: If $z_1 = a + bi$ and $z_2 = c + di$, their product is $z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$. (Treat this like binomial multiplication rather than memorizing the formula.)
- **Complex Conjugate**: For a complex number $z = a + bi$, its complex conjugate is $\bar{z} = a - bi$. Similarly, the complex conjugate of $z = a - bi$ is $\bar{z} = a + bi$.
- **Division of Complex Numbers**: For two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ (where $z_2 \neq 0$), their quotient $\frac{z_1}{z_2}$ is found by multiplying the numerator and denominator by the conjugate of the denominator, resulting in $\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$. (Perform the multiplication by the conjugate rather than memorizing this formula.)

THEOREMS

- **The Square of the Imaginary Unit**: $i^2 = -1$.
- **Products of Complex Conjugates**: Let $z = a + bi$, so that its complex conjugate is $\bar{z} = a - bi$. Then $z \cdot \bar{z} = a^2 + b^2$.
- **Powers of the Imaginary Unit**: Let $p \in \mathbb{W}$ (the Whole Number system) and r be the remainder when p is divided by 4. Then $i^p = i^r$. (This implies $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$).

COMMON MISTAKES

- Memorizing the formula for the product of two complex numbers instead of treating the multiplication like that of two binomials. The text advises: "you should never memorize the result of this definition. Instead, treat multiplying complex numbers like multiplying binomials."
- Leaving a complex number in the denominator of a fraction when expressing a complex number. The text states: "Any fraction must have a real-valued denominator."
- Memorizing the formula for the division of two complex numbers instead of understanding and applying the process of multiplying the numerator and denominator by the complex conjugate of the denominator. The text advises: "it is best not to memorize this resulting formula. You should be able to just 'do the math.'"

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5.1.2: HOMEWORK

READING QUESTIONS

1. What is the definition of the imaginary unit i? What is the value of i^2?
2. How do you express the square root of a negative number, like $\sqrt{-49}$, in terms of i?
3. What are the two main components of a complex number when written in standard form $a+bi$?
4. Describe the complex plane. What does the horizontal axis represent, and what does the vertical axis represent?
5. How do you add two complex numbers, for example, $(3-2i)+(1+5i)$?
6. How do you subtract two complex numbers, for example, $(5+i)-(2-3i)$?
7. Explain the process of multiplying two complex numbers, such as $(2+3i)(1-i)$. What crucial substitution is made involving i?
8. What is the complex conjugate of a complex number $a+bi$? How is it found?
9. What is the result when a complex number is multiplied by its complex conjugate? (e.g., $(a+bi)(a-bi)$)
10. Why is the complex conjugate used when dividing complex numbers? What is the goal?
11. Outline the steps to divide $\frac{2+5i}{4-i}$, as shown in Example 5.1.2.7.
12. What are the first four powers of i (i.e., i^1, i^2, i^3, i^4)?
13. How can you simplify a higher power of i, such as i^{35}, using the cyclic nature of its powers? (Refer to Example 5.1.2.10)
14. If a complex number is $0+3i$, what is its real part and what is its imaginary part? Is it considered a pure imaginary number?
15. Explain how to add complex numbers.
16. What is the basic principle in multiplication of complex numbers?
17. Give an example to show the product of two imaginary numbers is not always imaginary.
18. What is a characteristic of the plot of a real number in the complex plane?

HOMEWORK

ALGEBRAIC

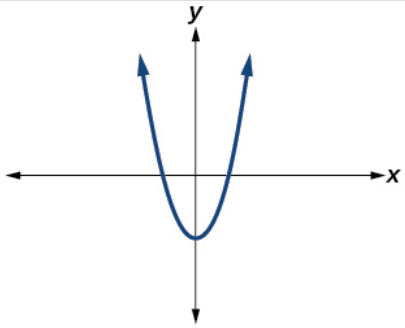
For the following exercises, evaluate the algebraic expressions.

19. If $f(x)=x^2+x-4$, evaluate $f(2i)$.
20. If $f(x)=x^3-2$, evaluate $f(i)$.
21. If $f(x)=x^2+3x+5$, evaluate $f(2+i)$.
22. If $f(x)=2x^2+x-3$, evaluate $f(2-3i)$.
23. If $f(x)=\frac{x+1}{2-x}$, evaluate $f(5i)$.
24. If $f(x)=\frac{1+2x}{x+3}$, evaluate $f(4i)$.

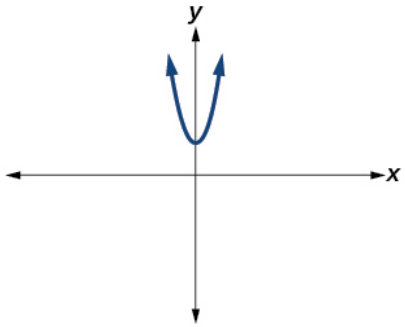
GRAPHICAL

For the following exercises, determine the number of real and nonreal solutions for each quadratic function shown.

25.



26.



For the following exercises, plot the complex numbers on the complex plane.

27. $1-2i$
28. $-2+3i$
29. i
30. $-3-4i$

NUMERIC

For the following exercises, perform the indicated operation and express the result as a simplified complex number.

31. $(3+2i)+(5-3i)$
32. $(-2-4i)+(1+6i)$
33. $(-5+3i)-(6-i)$
34. $(2-3i)-(3+2i)$
35. $(-4+4i)-(-6+9i)$
36. $(2+3i)(4i)$
37. $(5-2i)(3i)$
38. $(6-2i)(5)$
39. $(-2+4i)(8)$
40. $(2+3i)(4-i)$
41. $(-1+2i)(-2+3i)$
42. $(4-2i)(4+2i)$
43. $(3+4i)(3-4i)$
44. $\frac{3+4i}{2}$
45. $\frac{6-2i}{3}$
46. $\frac{-5+3i}{2i}$
47. $\frac{6+4i}{i}$
48. $\frac{2-3i}{4+3i}$
49. $\frac{3+4i}{2-i}$
50. $\frac{2+3i}{2-3i}$

51. $\sqrt{-9}+3\sqrt{-16}$
52. $-\sqrt{-4}-4\sqrt{-25}$
53. $\frac{2+\sqrt{-12}}{2}$
54. $\frac{4+\sqrt{-20}}{2}$
55. i^8
56. i^{15}
57. i^{22}

TECHNOLOGY

For the following exercises, use a calculator to help answer the questions.

58. Evaluate $(1+i)^k$ for $k=4$, 8, and 12. Predict the value if $k=16$.
59. Evaluate $(1-i)^k$ for $k=2$, 6, and 10. Predict the value if $k=14$.
60. Evaluate $(1+i)^k-(1-i)^k$ for $k=4$, 8, and 12. Predict the value if $k=16$.
61. Show that a solution of $x^6+1=0$ is $\frac{\sqrt{3}}{2}+\frac{1}{2}i$.
62. Show that a solution of $x^8-1=0$ is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

EXTENSIONS

For the following exercises, evaluate the expressions, writing the result as a simplified complex number.

63. $\frac{1}{i}+\frac{4}{i^3}$
64. $\frac{1}{i^{11}}-\frac{1}{i^{21}}$
65. $i^7(1+i^2)$
66. $i^{-3}+5i^7$
67. $\frac{(2+i)(4-2i)}{(1+i)}$
68. $\frac{(1+3i)(2-4i)}{(1+2i)}$
69. $\frac{(3+i)^2}{(1+2i)^2}$
70. $\frac{3+2i}{2+i}+(4+3i)$
71. $\frac{4+i}{i}+\frac{3-4i}{1-i}$
72. $\frac{3+2i}{1+2i}-\frac{2-3i}{3+i}$

5.2: Quadratic Functions

Note to the Instructor (click to expand)

This *very dense* sections covers everything the student will need to know about quadratic functions. Plan to spend a bit of time here as the topics go through graphing quadratics, optimization problems, the discriminant, solving equations that are quadratic-in-form, solving absolute value equations involving quadratics and quadratic inequalities, and graphing the absolute value of a quadratic function. Again, this is a *very* dense section.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Equations**
  - Solving Absolute Value Equations: The section extends this to solving absolute value equations involving quadratic expressions, like $|x^2 - 5x| = 6$ in Example 15, which builds upon basic absolute value equation solving.
- **Quadratic Functions**
  - Vertex Form and Completing the Square: This section uses vertex form $f(x) = a(x - h)^2 + k$ extensively and mentions completing the square as a method to convert from general to vertex form (Example 4). Prior knowledge is assumed.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.
- Use the discriminant to determine the number of solutions to a quadratic equation.
- Solve equations that are quadratic-in-form.
- Solve a quadratic inequality using the graphical and sign chart methods.
- Solve an absolute value equation involving a quadratic.
- Graph the absolute value of a quadratic function.

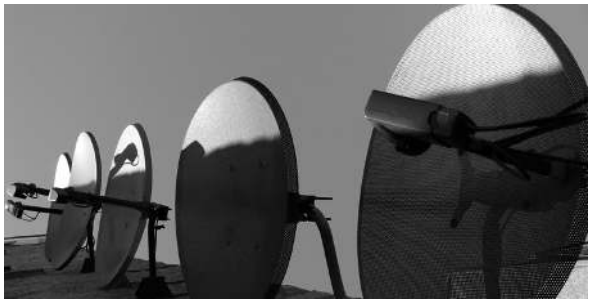
Hawk A.I. Section-Specific Tutor

COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Figure 5.2.1

An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)



Curved antennas, such as the ones shown in Figure **5.2.1**, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

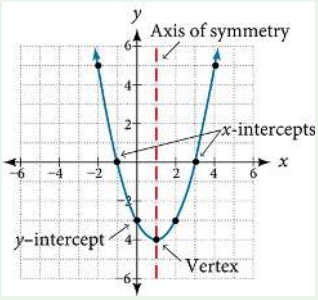
It is assumed that you have already worked with quadratic functions in your prerequisite Algebra course. Therefore, a lot of the theory is mentioned in brief.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a **parabola**.

Definition: Parabola

The shape in the following figure is called a **parabola**. Every parabola has one extreme point called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the parabola. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in following figure.



The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros** of the quadratic function, the values of x at which $y = 0$.

Definition: Zeros of Functions

The value c in the domain of the function f is called a **zero** of f if $f(c) = 0$.

While we try our best in Mathematics to avoid multiple definitions for the same thing, you will sometimes hear someone say "**roots**" instead of zeros.

Definition: Roots of Eqautions

If c solves a given equation, we say that c is a **root** of the equation.

The reason for the ambiguity is that, if c is a zero of the function f, then $f(c) = 0$. That is, given the *equation* $f(x) = 0$, c would be a solution because $f(c) = 0$. Hence, zeros of a function *are* roots of the associated equation (and vice versa).

In any case, I will try my best to use the word "zeros" when speaking of functions and "roots" when speaking of solutions to equations; however, to most professors, either is acceptable.

Example 5.2.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 5.2.2.

Figure 5.2.2

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$, so this is the y-intercept.

Quadratic Functions in General Form

Recall from Algebra that the parabola is the graph of a quadratic function. There are two forms for quadratic functions (and polynomials, in general) - **general form** and **standard form**. We begin our review with the general form of a quadratic function.

Definition: General Form of a Quadratic Function

The **general form** of a quadratic function is

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers and $a \neq 0$.

It was shown in your prerequisite Algebra course that the parabola opens upward if $a > 0$. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry. To do so, we first reintroduce a core theorem in all Algebra courses.

Theorem: Quadratic Formula

The roots of the equation

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof

Suppose a, b, and c are real numbers, where $a \neq 0$. Then

$$\begin{aligned}
& ax^2 + bx + c = 0 && \\
\Rightarrow\ & ax^2 + bx = -c && \text{(subtracting } c \text{ from both sides)} \\
\Rightarrow\ & x^2 + \frac{b}{a}x = -\frac{c}{a} && \text{(dividing both sides by } a, \text{ where } a \neq 0) \\
\Rightarrow\ & x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \left(\text{completing the square on the left side by adding } \left(\frac{b}{2a}\right)^2 \text{ to both sides}\right) \\
\Rightarrow\ & \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{(factoring the left side)} \\
\Rightarrow\ & \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} && \text{(Laws of Exponents)} \\
\Rightarrow\ & \left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} && \text{(getting common denominators by multiplying numerator and denominator of the first term by } 4a) \\
\Rightarrow\ & \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} && \text{(combining fractions with like denominators)} \\
\Rightarrow\ & x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && \text{(Extraction of Roots)} \\
\Rightarrow\ & x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2\sqrt{a^2}} && \text{(Properties of Radicals)} \\
\Rightarrow\ & x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \left(\sqrt{a^2} = \pm a, \text{ but this gets 'absorbed' into the } \pm \text{ out front}\right) \\
\Rightarrow\ & x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} && \left(\text{subtracting } \frac{b}{2a} \text{ from both sides}\right) \\
\Rightarrow\ & x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{(combining fractions with like denominators)}
\end{aligned}$$

With the Quadratic Formula in hand, we now can find the equation for the **axis of symmetry**.

Theorem: Axis of Symmetry for a Quadratic Function

The vertical line, $x = -\frac{b}{2a}$, is the axis of symmetry for the graph of the quadratic function $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

Proof

To avoid complications, we only submit the proof for two cases: the quadratic function has one zero or it has two real zeros (the case for no real zeros can be derived geometrically from the latter case).

Case 1 (1 Zero): Suppose the quadratic function has only one zero. Then the Quadratic Formula must return $x = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}$ (if the radicand in the

numerator was not 0, the Quadratic Formula would return *two* zeros, $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$). Hence, the vertex of the parabola (where the axis of symmetry crosses) occurs when $x = -\frac{b}{2a}$. That is, the axis of symmetry is $x = -\frac{b}{2a}$.

Case 2 (2 Real Zeros): Suppose the quadratic function has two real zeros. Then the Quadratic Formula shows us that these occur at $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. Since a parabola is symmetric about its axis of symmetry, the axis of symmetry must be halfway between these two x-values. Averaging the zeros for our function, we get

$$\begin{aligned}
\frac{\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}}{2} &= \frac{\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}}{2} \cdot \frac{2a}{2a} && \text{(simplifying the compound rational expression)} \\
&= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{4a} && \text{(distributing)} \\
&= \frac{-2b}{4a} && \text{(simplifying)} \\
&= -\frac{b}{2a} && \text{(simplifying)}
\end{aligned}$$

Hence, the axis of symmetry is $x = -\frac{b}{2a}$.

Figure 5.2.3 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense, because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

Figure 5.2.3

Since the vertex always occurs where the axis of symmetry intersects the parabola, we immediately get the following theorem.

Theorem: Coordinates of the Vertex

The vertex of the quadratic function $f(x) = ax^2 + bx + c$ is located at the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Proof

Since the vertex occurs where the axis of symmetry intersects the graph of the quadratic function, the x-coordinate of this intersection is $x = -\frac{b}{2a}$. The corresponding y-coordinate is $f\left(-\frac{b}{2a}\right)$.

The general form of a quadratic function is commonly encountered in applications and Physics. While most Algebra textbooks focus on the second form (standard form), you should strive to understand both forms.

Quadratic Functions in Standard Form

The general form of a quadratic can be incredibly useful; however, we cannot escape the need to talk about a quadratic function's alternate identity - the **standard form**.

Definition: Standard Form of a Quadratic Function

The **standard form** of a quadratic function is

$$f(x) = a(x-h)^2 + k,$$

where a, h, and k are real numbers and $a \neq 0$. This is also known as **vertex form**.

The standard form of a quadratic function has a very appealing feature.

Theorem: Vertex of a Quadratic Function in Standard Form

The vertex of the quadratic function given in the standard form

$$f(x) = a(x-h)^2 + k$$

is (h, k).

That is, if given a quadratic function in standard form, you have little-to-no work involved in finding its vertex.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 5.2.4 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

Figure 5.2.4

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 5.2.5 is the graph of this basic function.

Figure 5.2.5

From our previous work with transformations of functions, if $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 5.2.4, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 5.2.4, $h < 0$, so the graph is shifted 2 units to the left. Again, from transformations, we know that the magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph *appears* to become narrower (this is an illusion), and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph *appears* to become wider (another illusion), but in fact there is a vertical compression. In Figure 5.2.4, $|a| > 1$, so the graph *appears* narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2 + k &= ax^2 + bx + c \\ \implies \quad ax^2 - 2ahx + (ah^2 + k) &= ax^2 + bx + c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah = b \implies h = -\frac{b}{2a}.$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal, we get

$$\begin{aligned} ah^2 + k &= c \\ \implies \quad k &= c - ah^2 \\ &= c - a\left(-\frac{b}{2a}\right)^2 \\ &= c - \frac{b^2}{4a}. \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h) = k$.

Example 5.2.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 5.2.6 as a transformation of $f(x) = x^2$, and then expand the formula and simplify terms to write the equation in general form.

Figure 5.2.6

Solution

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. Substituting the coordinates of any point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$\begin{aligned} -1 &= a(0+2)^2 - 3 \\ \Rightarrow \quad 2 &= 4a \\ \Rightarrow \quad \frac{1}{2} &= a \end{aligned}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} g(x) &= \frac{1}{2}(x+2)^2 - 3 \\ &= \frac{1}{2}(x+2)(x+2) - 3 \\ &= \frac{1}{2}(x^2 + 4x + 4) - 3 \\ &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Checkpoint 5.2.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 5.2.7. Find an equation for the path of the ball. Does the shooter make the basket?

Figure 5.2.7

(credit: modification of work by Dan Meyer)

Example 5.2.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}.$$

The vertical coordinate of the vertex will be at

$$k = f(h) = f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}.$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. Therefore, using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}.$$

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs (k) and where it occurs (x).

If given a quadratic function in general form, most students are going to use the formula $-\frac{b}{2a}$ to find the x-value of the vertex (and substitute this into the original quadratic function to find the corresponding y-value for the vertex). This is perfectly fine, and it will always lead you to the vertex of a given quadratic function; however, another approach, which is inevitably going to be necessary in this course, is to **complete the square** like you did in your Elementary Algebra course (and as we did in the proof of the Quadratic Formula).

Example 5.2.4: Rewriting in Standard Form Using Completing the Square

Rewrite the quadratic function $f(x) = 2x^2 - 6x + 7$ in standard (vertex) form by Completing the Square.

Solution

Since Completing the Square is an older skill, we will just use it here rather than explaining the entire process.

$$\begin{aligned}
f(x) &= 2x^2 - 6x + 7 \\
&= 2(x^2 - 3x) + 7 && \text{(factoring the lead coefficient from both variable terms)} \\
&= 2\left(x^2 - 3x + \left(\frac{-3}{2}\right)^2\right) + 7 - 2\left(\frac{-3}{2}\right)^2 && \left(\text{Completing the Square and offsetting the } 2\left(\frac{-3}{2}\right)^2 \text{ term with } -2\left(\frac{-3}{2}\right)^2\right) \\
&= 2\left(x - \frac{3}{2}\right)^2 + 7 - 2\left(\frac{9}{4}\right) && \text{(factoring and simplifying)} \\
&= 2\left(x - \frac{3}{2}\right)^2 + \frac{14}{2} - \frac{9}{2} && \text{(common denominator)} \\
&= 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} && \text{(simplifying)}
\end{aligned}$$

Important Skill: Completing the Square

The purpose of being introduced to the process of Completing the Square in your Algebra courses was to prove the Quadratic Formula. We now see that this same process can be used to rewrite a quadratic function in vertex form; however, these are not the only uses of Completing the Square. We will use this skill again in this course when working with conic sections. In Calculus II, you will find many opportunities to use this very same skill when manipulating a seemingly impossible-to-work-with expression into something more simple.

Checkpoint 5.2.4

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

At the beginning of Calculus, the difference quotient is used on several different types of functions; however, one of the most common encounters is with quadratic functions.

Example 5.2.5: Revisiting the Difference Quotient

Find the difference quotient for the function.

$$f(x) = -3x^2 + 2x + 1$$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(-3(x+h)^2+2(x+h)+1)-(-3x^2+2x+1)}{h} && \text{(substituting)}\\
&= \frac{-3(x^2+2hx+h^2)+2x+2h+1+3x^2-2x-1}{h} && \text{(distributing)}\\
&= \frac{-3x^2-6hx-3h^2+2x+2h+1+3x^2-2x-1}{h} && \text{(distributing)}\\
&= \frac{-6hx-3h^2+2h}{h} && \text{(simplifying)}\\
&= \frac{h(-6x-3h+2)}{h} && \text{(factoring)}\\
&= -6x-3h+2 && \text{(cancelling like factors)}
\end{aligned}$$

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers; however, because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Theorem: Domain and Range of a Quadratic Function

The domain of any quadratic function is all real numbers.

The range of a quadratic function written in general form, $f(x)=ax^2+bx+c$, with a positive a value is $f(x)\geq f\left(-\frac{b}{2a}\right)$, or $\left[f\left(-\frac{b}{2a}\right),\infty\right)$.

The range of a quadratic function written in general form with a negative a value is $f(x)\leq f\left(-\frac{b}{2a}\right)$, or $\left(-\infty,f\left(-\frac{b}{2a}\right)\right]$.

The range of a quadratic function written in standard form, $f(x)=a(x-h)^2+k$, with a positive a value is $f(x)\geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x)\leq k$.

The theorem above is only provided for completeness - it should not be memorized.

Example 5.2.6: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x)=-5x^2+9x-1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h=-\frac{b}{2a}=-\frac{9}{2(-5)}=\frac{9}{10}.$$

The maximum value is given by $f(h)$.

$$f\left(\frac{9}{10}\right)=-5\left(\frac{9}{10}\right)^2+9\left(\frac{9}{10}\right)-1=\frac{61}{20}.$$

The range is $\left(-\infty,\frac{61}{20}\right]$.

Checkpoint 5.2.6

Find the domain and range of $f(x)=2\left(x-\frac{4}{7}\right)^2+\frac{8}{11}$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in Figure 5.2.8.

Figure 5.2.8

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 5.2.7: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solutions

Let's use a diagram such as Figure 5.2.9 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

Figure 5.2.9

a. We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L:

$$W = 80 - 2L.$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so\[A = LW =L (80-2L) = 80L-2 L^2. \nonumber \]This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is\[A(L)=-2 L^2 +80L. \nonumber \]

b. The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$\begin{aligned} h &= -\frac{80}{2(-2)} \\ &= 20 \end{aligned} \qquad \text{and} \qquad \begin{aligned} k &= A(20) \\ &= 80(20) - 2(20)^2 \\ &= 800 \end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length

40 feet.

Example 7 also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 5.2.10.

Figure 5.2.10

Example 5.2.8: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation

$$\text{Revenue} = pQ.$$

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} = \frac{-5,000}{2} = -2,500.$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$\begin{aligned} Q &= -2500p + b \\ \Longrightarrow \quad 84,000 &= -2500(30) + b \qquad \text{(substituting)} \\ \Longrightarrow \quad b &= 159,000 \qquad \text{(isolating the variable)} \end{aligned}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\begin{aligned} \text{Revenue} &= pQ \\ \Longrightarrow \quad \text{Revenue} &= p(-2,500p + 159,000) \\ \Longrightarrow \quad \text{Revenue} &= -2,500p^2 + 159,000p \end{aligned}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} = 31.8.$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) = 2,528,100.$$

As before, Example 5.2.8 could also be solved by graphing the quadratic as in Figure 5.2.11. We can see the maximum revenue on a graph of the quadratic function.

Figure 5.2.11

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 5.2.12 that the number of x-intercepts can vary depending upon the location of the graph.

Figure 5.2.12

Number of x-intercepts of a parabola

Example 5.2.9: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 = -2.$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$. That is, we want to solve

$$0 = 3x^2 + 5x - 2.$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2).$$

Thus, $x = \frac{1}{3}$ or $x = -2$. So the x-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

By graphing the function, we can confirm that the graph crosses the y-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 5.2.13

Figure 5.2.13

Example 9 illustrates a strength of having the general form of a quadratic function - the y-intercept is the constant term. That is, given $f(x) = ax^2 + bx + c$, the y-intercept is $f(0) = c$. This is only true, however, when the quadratic is given in general form.

Another lesson to be learned from Example 5.2.9 is that to solve $ax^2 + bx + c = 0$, we don't have to always resort to the Quadratic Formula. In fact, whenever you need to solve a quadratic equation, you should spend about 30 seconds or so trying to factor the given quadratic before plunging into the algorithm called the Quadratic Formula.

As Figure 5.2.12 illustrates, there are some quadratic functions that do not have x-intercepts. Algebraically, this occurs when the given quadratic function is **prime**. Just like a prime number, a prime quadratic is not factorable (across the real number system). That is, no matter how ingenious we are, we will never be able to write a prime quadratic in the form $(c_1x + d_1)(c_2x + d_2)$ (where c_1, c_2, d_1, and d_2 are real numbers).

In Mathematics, we always strive to develop methods to reduce our workload. It would be nice to know ahead of time that the quadratic function $f(x) = ax^2 + bx + c$ is prime. Knowing this would allow us to skip trying to find any x-intercepts. Luckily, we have such a method.

The x-intercepts of a quadratic function occur when $ax^2 + bx + c = 0$. We know from the Quadratic Formula that the solutions to this equation are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. However, the x-axis is a *real number* axis. That is, only real numbers exist along the x-axis. Hence, our x-intercepts *cannot* be imaginary numbers.

From our previous work, we know we have an imaginary number if our expression contains the square root of a negative number. Therefore, we don't want the radicand in the numerator of the Quadratic Formula to be negative. That is, we require $b^2 - 4ac \geq 0$. This expression is so important in Mathematics, it deserves its own name.

Definition: Discriminant

The **discriminant** of the quadratic function $f(x) = ax^2 + bx + c$ is defined to be

$$D = b^2 - 4ac.$$

From our discussion above, if the discriminant is negative, the quadratic function will not have any x-intercepts. Likewise, if the discriminant is negative, the equation $ax^2 + bx + c = 0$ will not have real solutions.

If the discriminant is 0, then (from the Quadratic Formula) the x-intercepts for the quadratic function would occur at

$$x = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

Since this is a single value, a discriminant of 0 implies that the graph of the quadratic function has only one x-intercept (as seen in the middle frame of Figure 5.2.12).

When the discriminant is positive, the quadratic function has two real roots, which means its graph crosses the x-axis twice.

Theorem: Zeros of a Quadratic Function and the Discriminant

Let

$$f(x) = ax^2 + bx + c.$$

The graph of f has

- two x-intercepts if the discriminant is positive,
- one x-intercept if the discriminant is 0, and
- no x-intercepts if the discriminant is negative.

Another way to say this is the equation, $ax^2 + bx + c = 0$, has

- two real roots if the discriminant is positive,
- one real root if the discriminant is 0, and
- two non-real roots if the discriminant is negative.

Example 5.2.10: Using the Discriminant to Save Time

Find the y- and x-intercepts of the quadratic $f(x) = -3x^2 + 4x - 2$.

Solution

Since the quadratic function is already in general form, finding the y-intercept is as simple as looking for the constant term. In this case, $(0, -2)$ is the y-intercept.

As for any x-intercepts, per my advice earlier, we spend roughly 30 seconds trying to factor the function, but have no success. Before running straight to the full Quadratic Formula, let's only compute one part - the discriminant (heck, we need to anyway if we are eventually going to use the Quadratic Formula).

$$b^2 - 4ac = 4^2 - 4(-3)(-2) = 16 - 24 = -8 < 0.$$

Since the discriminant is negative, the function doesn't have any x-intercepts.

It is important to clarify that finding the x-intercepts (or roots) of a quadratic function and *solving* a quadratic equation are often, but not always, the same thing. How are these processes different? Since the horizontal intercepts of a graph can only be real numbers, we stop our search for such intercepts if the discriminant of the quadratic function is negative. *In contrast*, the solutions to a random quadratic equation *can* be imaginary, and therefore the discriminant can be negative.

Example 5.2.11: Solving a Quadratic Equation

Find all the solutions of the quadratic equation.

$$0 = -3x^2 + 4x - 2$$

Solution

From Example 5.2.10, we know the discriminant is -8. Thus, the solutions of this equation are going to be imaginary. An *incredibly* important piece of advice is to start our solution process by multiplying both sides of the given equation by -1. We do this because the original lead coefficient is negative, which only serves to complicate using the Quadratic Formula. Therefore, let's focus on solving

$$0 = 3x^2 - 4x + 2.$$

Note that the discriminant,

$$b^2 - 4ac = (-4)^2 - 4(3)(2) = 16 - 24 = -8$$

doesn't change when multiplying both sides of the quadratic equation by -1. Now we can solve.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{(Quadratic Formula)} \\
&= \frac{-(-4) \pm \sqrt{\text{discriminant}}}{2(3)} && \text{(substituting)} \\
&= \frac{4 \pm \sqrt{-8}}{6} && \text{(substituting and simplifying)} \\
&= \frac{4 \pm \sqrt{4 \cdot -2}}{6} \\
&= \frac{4 \pm 2\sqrt{-2}}{6} && \text{(simplifying the radical)} \\
&= \frac{4 \pm 2\sqrt{-1}\sqrt{2}}{6} \\
&= \frac{4 \pm 2i\sqrt{2}}{6} && \left(\sqrt{-1} = i\right) \\
&= \frac{4}{6} \pm \frac{2i\sqrt{2}}{6} && \text{(splitting the fraction)} \\
&= \frac{2}{3} \pm \frac{i\sqrt{2}}{3} && \text{(simplifying fractions)} \\
&= \frac{2}{3} \pm \frac{\sqrt{2}}{3}i && \text{(rewriting in the standard form of a complex number)}
\end{aligned}$$

Example 5.2.12: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero; however, because this quadratic is not easily factorable, we start by checking the discriminant to see if there are any x-intercepts at all.

$$b^2 - 4ac = 4^2 - 4(2)(-4) = 16 + 32 = 48 > 0.$$

Thus, there will be two x-intercepts. Our time computing the discriminant is not wasted since we will need its value when solving the equation $2x^2 + 4x - 4 = 0$. Use the Quadratic Formula, we get the following:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{(Quadratic Formula)} \\
&= \frac{-4 \pm \sqrt{\text{discriminant}}}{2(2)} && \text{(substituting)} \\
&= \frac{-4 \pm \sqrt{48}}{4} && \text{(substituting)} \\
&= \frac{-4 \pm \sqrt{16 \cdot 3}}{4} \\
&= \frac{-4 \pm 4\sqrt{3}}{4} && \text{(simplifying the radical expression)} \\
&= \frac{4(-1 \pm \sqrt{3})}{4} && \text{(factoring out the GCF from the numerator)} \\
&= \frac{\cancel{4}^{1}(-1 \pm \sqrt{3})}{\cancel{4}^{1}} && \text{(canceling like factors)} \\
&= -1 \pm \sqrt{3}
\end{aligned}$$

The graph has x-intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

Again, we can check our work for Example 5.2.12 by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 5.2.14.

Figure 5.2.14

? Checkpoint 5.2.12

In a separate Checkpoint, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Solving Quadratic Inequalities

We now turn our attention to **quadratic inequalities**.

Definition: Quadratic Inequality

A **quadratic inequality** is an inequality that contains a quadratic expression. This inequality can be written in any of the following forms:

$$ax^2 + bx + c < 0 \qquad ax^2 + bx + c \leq 0$$
$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c \geq 0$$

If we consider $f(x) = x^2 - x - 6$, then solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of x for which the graph of $y = f(x) = x^2 - x - 6$ (the parabola) is below the graph of $y = 0$ (the x-axis). See Figure 5.2.15.

Figure 5.2.15

We can see that the graph of f dips below the x-axis between its two x-intercepts. The zeros of f are $x = -2$ and $x = 3$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all x in $(-2, 3)$, and $f(x) > 0$ for all x in $(3, \infty)$. Thus, the solution to $x^2 - x - 6 < 0$ is all x in the interval $(-2, 3)$.

The method introduced above (graphing the quadratic function and determining when the graph satisfies the inequality) is called the **graphical method** of solving a quadratic inequality. This is the method I prefer to teach in my courses because it simultaneously reinforces graphing and it is *much* faster (in the long run) than the alternative method. Before discussing the alternative method, let's look at a couple examples.

Example 5.2.13: Solving Quadratic Inequalities Using the Graphing Method

Solve the following inequalities using the graphing method.

a. $2x^2 \leq 3 - x$
b. $x^2 - 2x > 1$

Solutions

a. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. Therefore, we want to know when the function $f(x) = 2x^2 + x - 3$ is at or below the x-axis. Since the lead coefficient is positive, we know the parabola representing f opens upward. We just need to find the zeros of f by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. A simple and quick sketch yields the following:

By our sketch, we can see $f(x)$ is at or below the x-axis on the interval $\left[-\frac{3}{2}, 1\right]$. This is the solution to our inequality.

b. Once again, we rewrite $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, our parabola is opening upward. We spend a moment sketching the graph of f and asking ourselves when the parabola is strictly above the x-axis (because we are solving $f(x) > 0$).

So, we see that the interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$ solves our inequality.

The alternative method is called the **sign chart method** of solving a quadratic inequality. Like the graphical method, we start by finding the x-intercepts. We then mark those intercepts on a number line. For example, considering our original inequality $x^2 - x - 6 < 0$, we would create the sign chart in Figure 5.2.16.

Figure 5.2.16

Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(-2, 3)$.

An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis. This allows us to determine the sign of *all* of the function values on a given interval by testing the function at just *one* value in the interval. We use this idea in the following example.

Example 5.2.14: Solving Quadratic Inequalities Using the Sign Chart Method

Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

a. $2x^2 \leq 3 - x$
b. $x^2 - 2x > 1$

Solutions

a. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $(-\infty, -\frac{3}{2})$, $(-\frac{3}{2}, 1)$ and $(1, \infty)$. For the interval $(-\infty, -\frac{3}{2})$, we choose $x = -2$; for $(-\frac{3}{2}, 1)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $(-\infty, -\frac{3}{2})$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $(-\frac{3}{2}, 1)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$.

Since we are solving $2x^2 + x - 3 \leq 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$. Checking the sign diagram, we see this is $(-\frac{3}{2}, 1)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $[-\frac{3}{2}, 1]$.

To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$).

b. Once again, we re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$ and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $(-\infty, 1 - \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$.

To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x values where the graph of g is above the graph of h.

Applications Involving Quadratic Functions

It should come as no surprise that quadratic functions appear in a lot of applications - from business to physics. You will continue to encounter them in such contexts when you get to Calculus. Therefore, let's take a look at a couple common applications.

Example 5.2.15: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

a. When does the ball reach the maximum height?
b. What is the maximum height of the ball?
c. When does the ball hit the ground?
d. When will the ball be above 104 feet?

Solutions

a. The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2} = 2.5.$$

The ball reaches a maximum height after 2.5 seconds.

b. To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H\left(-\frac{b}{2a}\right) = H(2.5) = -16(2.5)^2 + 80(2.5) + 40 = 140.$$

The ball reaches a maximum height of 140 feet.

c. We need to solve

$$0 = -16t^2 + 80t + 40$$

which has the same solution as

$$0 = 16t^2 - 80t - 40.$$

Since the coefficients are all divisible by 8, we divide both sides by 8 to get

$$0 = 2t^2 - 10t - 5.$$

Try as you might, the quadratic on the right side doesn't factor. Therefore, we use the Quadratic Formula. Before going any further, let's compute the discriminant (if it's negative, then we can stop because this would mean the ball would never touch the ground).

$$\text{discriminant} = b^2 - 4ac = (-10)^2 - 4(2)(-5) = 100 + 40 = 140 > 0.$$

Thus, we continue with the Quadratic Formula.

$$\begin{aligned} t &= \frac{-b \pm \sqrt{\text{discriminant}}}{2a} && \text{(Quadratic Formula)} \\ &= \frac{10 \pm \sqrt{140}}{2(2)} && \text{(substituting)} \\ &= \frac{10 \pm \sqrt{140}}{4} && \text{(simplifying)} \end{aligned}$$

Because this is an application and the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t \approx 5.458 \quad \text{or} \quad t \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 5.2.17.

Figure 5.2.17

d. This question is akin to asking us to solve $-16t^2 + 80t + 40 > 104$. We starting by getting 0 on one side:

$$-16t^2 + 80t - 64 > 0.$$

Again, it would be better to get that lead coefficient to be positive, so we multiply both sides by -1 (remember that the inequality flips):

$$16t^2 - 80t + 64 < 0.$$

We then divide both sides by the GCF of each of the coefficients (which is 16):

$$t^2 - 5t + 4 < 0.$$

We want to know when the outputs of this parabola are below the t-axis. We know this parabola is opening upward and the left side factors to $(t-4)(t-1)$. This implies the roots are $t = 1$ and $t = 4$. This means that our solution is the interval $(1, 4)$ (see Figure 5.2.18).

Figure 5.2.18

It's important to note that Figure 5.2.18 in Example 5.2.15 doesn't represent the path of the ball. We arrived at the graph in Figure 5.2.18 by manipulating an inequality based on the function for the path of the ball's travel.

? Checkpoint 5.2.15

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solving Absolute Value Equations Involving Quadratics

We finish this section with two synthesis synthesis topics:

1. Solving equations involving the composition of the absolute value function and a quadratic function, and
2. Graphing the absolute value of a quadratic function.

Both of these topics will seem more complex than our previous work because you are combining previously-learned skills (called synthesizing skills). The crucial thing to remember is the piecewise definition of the absolute value (repeated below because, yes, it's that important).

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

We will be combining this knowledge with our graphing skills for quadratic functions.

Example 5.2.16: Solve Absolute Value Equations Involving Quadratic Expressions

Solve each absolute value equation.

a. $|2x^2 + x - 3| = 0$
b. $|2x^2 + x - 3| = -6$
c. $|x^2 - 5x| = 6$

Solution

a. As with any synthesis question, you need to think critical more than with basic questions. In this case, you are being asked when the left side is zero. The only time the absolute value is zero is when its argument is zero. So this question boils down to solving

$$2x^2 + x - 3 = 0.$$

In Example 5.2.14, we found the zeros of this function to be $x = -\frac{3}{2}$ and $x = 1$. Thus, the solutions to this absolute value equation are $x = -\frac{3}{2}$ and $x = 1$.

b. Again, we have a synthesis question where thinking critically ahead of time saves time. We are being asked, "When is an absolute value equal to a negative number?" We know that this cannot happen. Thus, this absolute value equation does not have a solution. In Mathematics, we often say the solution **DNE** (does not exist).

c. We finally arrive at an example which requires more than superficial thinking. Using the definition of the absolute value, we see that the left side has the equivalent form

$$|x^2-5x| = \begin{cases} x^2-5x, & \text{if } x^2-5x \geq 0 \\ -(x^2-5x), & \text{if } x^2-5x < 0 \end{cases}$$

So, we need to determine when $x^2-5x=0$ because this is where the "pivot point" of the piecewise definition occurs. We factor the equation to $x(x-5)=0$ and note that the solutions are $x=0$ and $x=5$. Since $y=x^2-5x$ is a parabola opening upward, and we just found out that the zeros of this function are 0 and 5, this function is at or above the horizontal axis on the interval $(-\infty,0]\cup[5,\infty)$. Thus, it is below the horizontal axis on $(0,5)$. We modify our piecewise definition accordingly.

$$|x^2-5x| = \begin{cases} x^2-5x, & \text{if } x \in (-\infty,0]\cup[5,\infty) \\ -(x^2-5x), & \text{if } x \in (0,5) \end{cases}$$

Thus, our original equation becomes the following:

Misplaced '#'

Hence, our solutions to the original absolute value equation are $x=-1,2,3$ and 6.

It is very important to check that the solutions from Example 5.2.16 occur in the proper intervals. That is, we arrives at the solutions $x=-1$ and $x=6$ whenever $x \in (-\infty,0]\cup[5,\infty)$. Both of these solutions exist in these intervals, so we are fine. I leave it to you to check that the second solution set exists within the interval $(0,5)$.

We conclude this section with a graph of a more complicated absolute value function.

Example 5.2.17: Graph the Absolute Value of a Quadratic Function

Graph $f(x)=|x^2-x-6|$.

Solution

Using the definition of absolute value, we have

$$f(x) = \begin{cases} (x^2-x-6), & \text{if } x^2-x-6 \geq 0 \\ -(x^2-x-6), & \text{if } x^2-x-6 < 0 \end{cases}$$

We begin by graphing $y=g(x)=x^2-x-6$ using the intercepts and the vertex. To find the x-intercepts, we solve $x^2-x-6=0$. Factoring gives $(x-3)(x+2)=0$ so $x=-2$ or $x=3$. Hence, $(-2,0)$ and $(3,0)$ are x-intercepts. The y-intercept $(0,-6)$ is found by setting $x=0$. To plot the vertex, we find

$$x=-\frac{b}{2a}=-\frac{-1}{2(1)}=\frac{1}{2}, \text{ and } y=\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)-6=-\frac{25}{4}=-6.25.$$

Plotting, we get the parabola on the left side of Figure 5.2.19.

To obtain points on the graph of $y=f(x)=|x^2-x-6|$, we can take points on the graph of $g(x)=x^2-x-6$ and apply the absolute value to each of the y values on the parabola. We see from the graph of g that for $x \leq -2$ or $x \geq 3$, the y-values on the parabola are greater than or equal to zero (since the graph is on or above the x-axis), so the absolute value leaves these portions of the graph alone. For x between -2 and 3, however, the y-values on the parabola are negative. For example, the point $(0,-6)$ on $y=x^2-x-6$ would result in the point $(0,|-6|)=(0,-(-6))=(0,6)$ on the graph of $f(x)=|x^2-x-6|$. Proceeding in this manner for all points with x-coordinates between -2 and 3 results in the graph on the right side of Figure 5.2.19.

Figure 5.2.19

If we take a step back and look at the graphs of g and f in the Example 5.2.17, we notice that to obtain the graph of f from the graph of g, we reflect a *portion* of the graph of g about the x-axis. We can see this analytically by substituting $g(x)=x^2-x-6$ into the formula for $f(x)$ and calling to mind the concept of reflections from transformations of functions.

$$f(x) = \begin{cases} g(x), & \text{if} \quad g(x) \geq 0 \\ -g(x), & \text{if} \quad g(x) < 0 \end{cases}$$

The function f is defined so that when $g(x)$ is negative (i.e., when its graph is below the x-axis), the graph of f is its refection across the x-axis. This is a general template to graph functions of the form $f(x) = |g(x)|$. From this perspective, the graph of $f(x) = |x|$ can be obtained by reflecting the portion of the line $g(x) = x$ which is below the x-axis back above the x-axis creating the characteristic "*vee*" shape.

Focus on Calculus - Solving Equations that are Quadratic-in-Form

You have already solved quite a few quadratic equations throughout your academic journey in Mathematics; however, as you approach Calculus, you will need to focus your attention on pattern recognition instead of expecting equations identical in form to $ax^2 = bx + c = 0$. We must begin to recognize that

$$a\blacksquare^2 + b\blacksquare + c = 0$$

is nearly this same equation.

Definition: Quadratic-in-Form

An equation that can be written in the form

$$a\,[f(x)]^2 + bf(x) + c = 0$$

is called an equation that is **quadratic-in-form**.

It turns out that any equation that is quadratic-in-form can be solved using methods similar to those from our Elementary and Intermediate Algebra courses. Let's illustrate the process with a few examples.

Example 5.2.18: Solving an Equation that is Quadratic-in-Form

Solve $3(3x-1)^2 + 13(3x-1) - 10 = 0$.

Solution

Many students would immediately distribute things out to get

$$27x^2 + 21x - 20 = 0$$

and try to factor; however, this is not taking advantage of the form of the original equation.

Notice that there are some common expressions within the original equation - namely $3x - 1$. If we let $u = 3x - 1$, we could rewrite the equation as

$$3u^2 + 13u - 10 = 0.$$

Hopefully, you agree that this equation looks much easier to solve. This type of substitution is formally called a **u-substitution**. The choice of the variable u is arbitrary, and you are encouraged to use whatever variable you want (other than the variable in the original equation).

We can factor the left side of this new equation to get

$$(3u - 2)(u + 5) = 0.$$

Therefore, $u = \frac{2}{3}$ or $u = -5$.

Since the original problem involved the variable x, we need to **resubstitute** $u = 3x - 1$ to arrive at

$$\begin{array}{rrcl} & u & = \frac{3}{2} \quad \text{and} \quad u = -5 & \\ \Longrightarrow & 3x - 1 & = \frac{3}{2} \quad \text{and} \quad 3x - 1 = -5 & \text{(resubstituting)} \\ \Longrightarrow & 3x & = \frac{3}{2} + 1 \quad \text{and} \quad 3x = -5 + 1 & \text{(adding 1 to both sides)} \\ \Longrightarrow & 3x & = \frac{5}{2} \quad \text{and} \quad 3x = -4 & \text{(simplifying)} \\ \Longrightarrow & x & = \frac{5}{6} \quad \text{and} \quad x = -\frac{4}{3} & \text{(dividing both sides by 3)} \end{array}$$

Thus, the solutions to the original equation are $x = \frac{5}{6}$ and $x = -\frac{4}{3}$.

You will be using the u-substitution process illustrated in Example **5.2.18** often throughout Calculus. As such, it's best to start looking for patterns like this from this point forward.

Example 5.2.19: Solving an Equation that is Quadratic-in-Form

Solve $x^4 - 3x^2 = 45 + x^2$.

Solution

Since this equation has several terms involving powers of x, it is best to collect all information on one side (this tends to "reveal" the type of equation we are working with). Subtracting 45 and x^2 from both sides, we get

$$x^4 - 4x^2 - 45 = 0.$$

The key observation here is that the exponent on x in the first term is *twice* that of the exponent on x in the second term. That is, this has the same form as a quadratic equation. Letting $u = x^2$, we get

$$\begin{aligned} x^4 - 4x^2 - 45 &= (x^2)^2 - 4x^2 - 45 && \text{(Laws of Exponents: Power Law)} \\ &= u^2 - 4u - 45 && \text{(substituting } u = x^2\text{)} \end{aligned}$$

Thus, we instead focus on solving

$$\begin{aligned} u^2 - 4u - 45 &= 0 \\ \implies (u-9)(u+5) &= 0 \qquad \text{(factoring)} \end{aligned}$$

Hence, $u = 9$ or $u = -5$. However, we are supposed to be solving for x, so we resubstitute $u = x^2$.

$$\implies x^2 = 9 \quad \text{or} \quad x^2 = -5.$$

While we can easily solve the first equation to get $x = \pm 3$, the second equation doesn't have a solution (over the real number system) because there isn't a real number whose square becomes a negative number. Thus, the solutions to the original equation are $x = -3$ and $x = 3$.

5.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Completing the square
  - Intermediate Algebra - Completing the Square (Part A)
  - Intermediate Algebra - Completing the Square (Part B)
- Graphing quadratic functions in standard/vertex form
  - Intermediate Algebra - Everything About Graphing Parabolas
  - Intermediate Algebra - Graphing Parabolas in Vertex Form
- Graphing quadratic functions in general form
  - Quadratic Functions - Understanding How Graphs of Parabolas are Related to Their Quadratic Functions
  - Intermediate Algebra - Graphing Quadratic Equations (Part A)
  - Intermediate Algebra - Graphing Quadratic Equations (Part B)
- Solving quadratic equations
  - Intermediate Algebra - Introduction to Quadratic Equations
  - Algebra in Review: Equations - Quadratic (Zero Factor Principle)
  - Intermediate Algebra - Solving Quadratic Equations by Factoring (Part A)
  - Intermediate Algebra - Solving Quadratic Equations by Factoring (Part B)
  - Intermediate Algebra - Quadratic Formula (Part A)
  - Intermediate Algebra - Quadratic Formula (Part B)
  - Intermediate Algebra - Quadratic Formula (Part C)
  - Intermediate Algebra - Quadratic Formula (Part D)
- Solving equations that are quadratic-in-form
  - Equations - Overview - A Review for College Algebra: Solving Equations That Are Quadratic-in-Form
- Solving quadratic inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Quadratic Inequalities
  - Intermediate Algebra - Quadratic Inequalities (Part A)
  - Intermediate Algebra - Quadratic Inequalities (Part B)
  - Intermediate Algebra - Quadratic Inequalities (Part C)
  - Intermediate Algebra - Quadratic Inequalities (Part D)
- Applications involving quadratic functions
  - Quadratic Functions - Determining the Maximum and Minimum Values of Quadratic Functions
  - Intermediate Algebra - Quadratic Applications
  - Optimization Problems - Maximizing Revenue the Non-Calculus Way
- CAPSTONE!!!
  - Algebra in Review: Capstone - Quadratics

KEY CONCEPTS

DEFINITIONS

- **Quadratic Function:** A function of degree two, which can be written in the general form $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$.
- **Parabola:** The U-shaped curve that is the graph of a quadratic function.
- **Vertex (of a parabola):** The point on the parabola where the function reaches its maximum or minimum value. It is also the turning point of the graph.
- **Axis of Symmetry (of a parabola):** A vertical line passing through the vertex of a parabola, dividing it into two mirror images. Its equation is $x = h$, where h is the x-coordinate of the vertex.
- **General Form (of a quadratic function):** $f(x) = ax^2 + bx + c$.
- **Standard Form (or Vertex Form of a quadratic function):** $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex of the parabola.
- **Zeros (or Roots or x-intercepts of a quadratic function):** The values of x for which $f(x) = 0$. These are the points where the parabola intersects the x-axis.
- **y-intercept (of a quadratic function):** The point where the parabola intersects the y-axis, found by evaluating $f(0)$. In general form, this is $(0, c)$.
- **Discriminant:** For a quadratic equation $ax^2 + bx + c = 0$ (or function $f(x) = ax^2 + bx + c$), the discriminant is the value $D = b^2 - 4ac$. It helps determine the nature and number of real roots (x-intercepts).
- **Prime Quadratic:** A quadratic expression that cannot be factored into linear factors with real coefficients. This occurs when its discriminant is negative.
- **u-substitution:** A technique used to solve equations that are "quadratic-in-form" by substituting a variable (often u) for an expression, transforming the equation into a standard quadratic form.

THEOREMS

- **Axis of Symmetry and Vertex (from General Form):** For a quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the line $x = -\frac{b}{2a}$. The vertex is at $(-\frac{b}{2a}, f(-\frac{b}{2a}))$.
- **Vertex (from Standard Form):** For a quadratic function $f(x) = a(x - h)^2 + k$, the vertex is (h, k).
- **Relationship between General and Standard Forms:**
  - Given $f(x) = ax^2 + bx + c$, then $h = -\frac{b}{2a}$ and $k = c - ah^2 = c - a(-\frac{b}{2a})^2 = c - \frac{b^2}{4a}$. More simply, $k = f(h)$.
- **Domain and Range of a Quadratic Function:**
  - The domain of any quadratic function is all real numbers, $(-\infty, \infty)$.
  - If $a > 0$, the range is $[k, \infty)$, where k is the y-coordinate of the vertex.

- If $a < 0$, the range is $(-\infty, k]$, where k is the y-coordinate of the vertex.

- **Zeros and the Discriminant:** For a quadratic function $f(x) = ax^2 + bx + c$, let $D = b^2 - 4ac$ be the discriminant.
  - If $D > 0$, the function has two distinct real zeros (two x-intercepts).
  - If $D = 0$, the function has one real zero (one x-intercept; the vertex is on the x-axis).
  - If $D < 0$, the function has no real zeros (no x-intercepts); the zeros are complex conjugates.
- **Quadratic Formula:** The solutions to the quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

COMMON MISTAKES TO AVOID

- **Confusing h in Standard Form:** In $f(x) = a(x - h)^2 + k$, if the form is, for example, $f(x) = a(x + 3)^2 + k$, misidentifying h as 3 instead of -3.
- **Errors in Completing the Square:** Forgetting to factor out the leading coefficient a from the x^2 and x terms before completing the square.
- **Factoring Errors:** Incorrectly factoring a quadratic trinomial when trying to find x-intercepts or solve inequalities.
- **Solving Quadratic Inequalities:** Forgetting to flip the inequality sign if multiplying/dividing the entire inequality by a negative number.
- **Equations Quadratic-in-Form:** After solving for the substituted variable (e.g., u), forgetting to substitute back to solve for the original variable (e.g., x).
- **Absolute Value Equations with Quadratics:** Not considering both positive and negative cases for the expression inside the absolute value (e.g., for $|Q(x)| = c$, solving only $Q(x) = c$).
- **Graphing $|Q(x)|$:** Reflecting the entire graph of $Q(x)$ instead of only the parts below the x-axis.

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5.2.2: HOMEWORK

READING QUESTIONS

1. What is the shape of the graph of a quadratic function called?
2. What are the key features of a parabola? (List at least three)
3. What is the general form of a quadratic function? What is the standard (or vertex) form?
4. How can you determine from the general form $f(x) = ax^2 + bx + c$ whether the parabola opens upward or downward?
5. What is the formula for the x-coordinate of the vertex (and the axis of symmetry) when a quadratic function is in general form?
6. If a quadratic function is in standard form $f(x) = a(x-h)^2 + k$, what are the coordinates of its vertex?
7. How does the value of a in $f(x) = a(x-h)^2 + k$ affect the width and direction of opening of the parabola compared to $y = x^2$?
8. What is the domain of any quadratic function? How is the range determined?
9. What is the discriminant of a quadratic function, and what does it tell you about the x-intercepts of the parabola?
10. When would you use the Quadratic Formula to find the x-intercepts of a parabola?
11. Describe the process of u-substitution for solving an equation that is quadratic-in-form, like $x^4 - 4x^2 - 45 = 0$.
12. Explain the graphical method for solving a quadratic inequality like $x^2 - x - 6 < 0$.
13. Explain the sign chart method for solving a quadratic inequality.
14. When solving an absolute value equation involving a quadratic, such as $|x^2 - 5x| = 6$, what are the two main cases you need to consider?
15. How do you graph a function of the form $f(x) = |Q(x)|$, where $Q(x)$ is a quadratic function? What happens to the parts of the graph of $Q(x)$ that are below the x-axis?
16. Explain the advantage of writing a quadratic function in standard form.
17. How can the vertex of a parabola be used in solving real world problems?
18. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.
19. What is another name for the standard form of a quadratic function?
20. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

HOMEWORK

ALGEBRAIC

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

21. $f(x) = x^2 - 12x + 32$
22. $g(x) = x^2 + 2x - 3$
23. $f(x) = x^2 - x$
24. $f(x) = x^2 + 5x - 2$
25. $h(x) = 2x^2 + 8x - 10$
26. $k(x) = 3x^2 - 6x - 9$
27. $f(x) = 2x^2 - 6x$
28. $f(x) = 3x^2 - 5x - 1$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

29. $y(x) = 2x^2 + 10x + 12$
30. $f(x) = 2x^2 - 10x + 4$
31. $f(x) = -x^2 + 4x + 3$
32. $f(x) = 4x^2 + x - 1$
33. $h(t) = -4t^2 + 6t - 1$
34. $f(x) = \frac{1}{2}x^2 + 3x + 1$
35. $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

36. $f(x) = (x-3)^2 + 2$
37. $f(x) = -2(x+3)^2 - 6$
38. $f(x) = x^2 + 6x + 4$
39. $f(x) = 2x^2 - 4x + 2$
40. $k(x) = 3x^2 - 6x - 9$

For the following exercises, solve the equations over the complex numbers.

41. $x^2 = -25$
42. $x^2 = -8$
43. $x^2 + 36 = 0$
44. $x^2 + 27 = 0$
45. $x^2 + 2x + 5 = 0$
46. $x^2 - 4x + 5 = 0$
47. $x^2 + 8x + 25 = 0$
48. $x^2 - 4x + 13 = 0$
49. $x^2 + 6x + 25 = 0$
50. $x^2 - 10x + 26 = 0$
51. $x^2 - 6x + 10 = 0$
52. $x(x-4) = 20$
53. $x(x-2) = 10$
54. $2x^2 + 2x + 5 = 0$
55. $5x^2 - 8x + 5 = 0$
56. $5x^2 + 6x + 2 = 0$
57. $2x^2 - 6x + 5 = 0$
58. $x^2 + x + 2 = 0$
59. $x^2 - 2x + 4 = 0$

For the following exercises, use the vertex (h,k) and a point on the graph (x,y) to find the general form of the equation of the quadratic function.

60. $(h,k) = (2,0)$, $(x,y) = (4,4)$
61. $(h,k) = (-2,-1)$, $(x,y) = (-4,3)$
62. $(h,k) = (0,1)$, $(x,y) = (2,5)$
63. $(h,k) = (2,3)$, $(x,y) = (5,12)$
64. $(h,k) = (-5,3)$, $(x,y) = (2,9)$

65. $(h,k)=(3,2)$, $(x,y)=(10,1)$
66. $(h,k)=(0,1)$, $(x,y)=(1,0)$
67. $(h,k)=(1,0)$, $(x,y)=(0,1)$

GRAPHICAL

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

68. $f(x)=x^2-2x$
69. $f(x)=x^2-6x-1$
70. $f(x)=x^2-5x-6$
71. $f(x)=x^2-7x+3$
72. $f(x)=-2x^2+5x-8$
73. $f(x)=4x^2-12x-3$

For the following exercises, write the equation for the graphed function.

74.

75.

76.

77.

78.

79.

NUMERIC

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

80.

x	–2	–1	0	1	2
y	5	2	1	2	5

81.

x	–2	–1	0	1	2
y	1	0	1	4	9

82.

x	–2	–1	0	1	2
y	–2	1	2	1	–2

83.

x	–2	–1	0	1	2
y	–8	–3	0	1	0

84.

x	–2	–1	0	1	2
y	8	2	0	2	8

TECHNOLOGY

For the following exercises, use a calculator to find the answer.

85. Graph on the same set of axes the functions $f(x)=x^2$, $f(x)=2x^2$, and $f(x)=\frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?
86. Graph on the same set of axes $f(x)=x^2$, $f(x)=x^2+2$, $f(x)=x^2+5$, and $f(x)=x^2-3$. What appears to be the effect of adding a constant?
87. Graph on the same set of axes $f(x)=x^2$, $f(x)=(x-2)^2$, $f(x)=(x-3)^2$, and $f(x)=(x+4)^2$. What appears to be the effect of adding or subtracting those numbers?
88. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x)=-\frac{32}{(80)^2}x^2+x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use graphing technology to determine the height of the object when it has traveled 100 feet away horizontally.
89. A suspension bridge can be modeled by the quadratic function $h(x)=.0001x^2$ with $-2000\le x\le 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use graphing technology to estimate how far from the center does the bridge have a height of 100 feet.

EXTENSIONS

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

90. Vertex $(1,-2)$, opens up.
91. Vertex $(-1,2)$ opens down.
92. Vertex $(-5,11)$, opens down.
93. Vertex $(-100,100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

94. Contains $(1,1)$ and has shape of $f(x)=2x^2$. Vertex is on the y-axis.
95. Contains $(-1,4)$ and has the shape of $f(x)=2x^2$. Vertex is on the y-axis.
96. Contains $(2,3)$ and has the shape of $f(x)=3x^2$. Vertex is on the y-axis.
97. Contains $(1,-3)$ and has the shape of $f(x)=-x^2$. Vertex is on the y-axis.
98. Contains $(4,3)$ and has the shape of $f(x)=5x^2$. Vertex is on the y-axis.
99. Contains $(1,-6)$ has the shape of $f(x)=3x^2$. Vertex has x-coordinate of -1.

REAL-WORLD APPLICATIONS

100. Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.

101. Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.
102. Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.
103. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?
104. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?
105. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = 45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.
106. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.
107. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?
108. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?
109. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

5.3: Power Functions and Polynomial Functions

Note to the Instructor (click to expand)

The purpose of the material in this section is to introduce the vocabulary needed to work with polynomial functions. Students are introduced to power functions, polynomials, and end behavior **but** are not asked to graph at this time.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.
- Compute the Difference Quotient for power and polynomial functions.

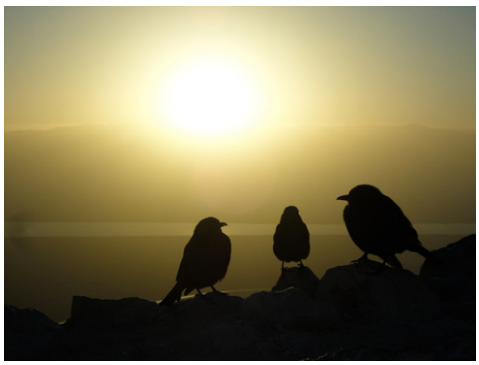
Hawk A.I. Section-Specific Tutor

COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Figure 5.3.1

credit: Jason Bay, Flickr



Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table **5.3.1**.

Table 5.3.1

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function called a **power function**.

Definition: Power Function

A **power function** is a function that can be represented in the form

$$f(x) = kx^p,$$

where k and p are real numbers. k is known as the **coefficient**.

As an example, consider functions for area or volume. The function for the area of a circle with radius r is $A(r) = \pi r^2$ and the function for the volume of a sphere with radius r is $V(r) = \frac{4}{3}\pi r^3$. Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Q&A

Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an **exponential function**, not a power function. We will deal with these types of functions later.

Example 5.3.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions. The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$, respectively. The quadratic and cubic functions are power functions with whole number powers. The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$. The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Checkpoint 5.3.1

Which functions are power functions?

$$\begin{aligned} f(x) &= 2x^2 \cdot 4x^3 \\ g(x) &= -x^5 + 5x^3 - 4x \\ h(x) &= \tfrac{2x^5-1}{3x^2+4} \end{aligned}$$

The ever-important Difference Quotient is often used in Calculus when working with power function.

Example 5.3.2: Revisiting the Difference Quotient

Compute the Difference Quotient for the given function.

$$f(x) = x^4$$

Solution

Recall that the Difference Quotient for the function f is defined to be

$$\frac{f(x+h)-f(x)}{h}.$$

Therefore, in this case, we have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^4 - x^4}{h} && \text{(substituting)}\\
&= \frac{\left((x+h)^2\right)^2 - x^4}{h} && \text{(Laws of Exponents)}\\
&= \frac{\left((x+h)(x+h)\right)^2 - x^4}{h}\\
&= \frac{\left(x^2+2hx+h^2\right)^2 - x^4}{h} && \text{(distributing)}\\
&= \frac{\left(x^2+2hx+h^2\right)\left(x^2+2hx+h^2\right) - x^4}{h}\\
&= \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4 - x^4}{h} && \text{(distributing)}\\
&= \frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} && \text{(combining like terms)}\\
&= \frac{h\left(4x^3+6x^2h+4xh^2+h^3\right)}{h} && \text{(factoring out the GCF from the numerator)}\\
&= \frac{\cancel{h}^{1}\left(4x^3+6x^2h+4xh^2+h^3\right)}{\cancel{h}^{1}} && \text{(canceling like factors)}\\
&= 4x^3+6x^2h+4xh^2+h^3 && \text{(simplifying)}
\end{aligned}$$

In Example 5.3.2, we could have taken the alternative approach of using factoring techniques to simplify the numerator.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^4 - x^4}{h} && \text{(substituting)}\\
&= \frac{\left((x+h)^2\right)^2 - \left(x^2\right)^2}{h} && \text{(Laws of Exponents)}\\
&= \frac{\left((x+h)^2 - x^2\right)\left((x+h)^2 + x^2\right)}{h} && \text{(Difference of Squares)}\\
&= \frac{\left((x+h) - x\right)\left((x+h) + x\right)\left((x+h)^2 + x^2\right)}{h} && \text{(Difference of Squares)}\\
&= \frac{(h)\left(2x+h\right)\left(x^2+2hx+h^2+x^2\right)}{h} && \text{(simplifying)}\\
&= \frac{\cancel{h}^{1}\left(2x+h\right)\left(x^2+2hx+h^2+x^2\right)}{\cancel{h}^{1}} && \text{(canceling like factors)}\\
&= \left(2x+h\right)\left(2x^2+2hx+h^2\right) && \text{(simplifying)}\\
&= 4x^3+6x^2h+4xh^2+h^3 && \text{(distributing)}
\end{aligned}$$

While both methods are valid and have their place, a student well-prepared for Calculus will inevitably want to try the second method.

Identifying End Behavior of Power Functions

Figure 5.3.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

Figure 5.3.2

Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, \quad f(x) \to \infty.$$

Figure 5.3.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

Figure 5.3.3

Odd-power functions

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 5.3.2 we see that functions of the form $f(x) = x^n$, where n is even, are symmetric about the y-axis. In Figure 5.3.3, we see that functions of the form $f(x) = x^n$, where n is odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form, we write

$$\text{as } x \to -\infty, \quad f(x) \to -\infty \quad \text{and as } x \to \infty, \quad f(x) \to \infty.$$

The behavior of the graph of a function as the input values get very negative ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).

Figure 5.3.4 shows the end behavior of power functions in the form $f(x) = kx^n$, where n is a non-negative integer depending on the power and the constant.

Figure 5.3.4

Example 5.3.3: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. Therefore, as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure 5.3.5.

Figure 5.3.5

Example 5.3.4: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 5.3.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty;\ \text{as } x \to \infty,\ f(x) \to -\infty.$$

Figure 5.3.6

We can check our work by using the table feature on a graphing utility.

Table 5.3.2

x	$f(x)$
–10	1,000,000,000
–5	1,953,125
0	0
5	–1,953,125
10	–1,000,000,000

We can see from Table 5.3.2 that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Checkpoint 5.3.4

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius, r, of the spill depends on the number of weeks, w, that have passed. This relationship is linear.

$$r(w) = 24 + 8w.$$

We can combine this with the formula for the area A of a circle, $A(r) = \pi r^2$. Composing these functions gives a formula for the area in terms of weeks.

$$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2.$$

Distributing gives the formula

$$A(w) = 576\pi + 384\pi w + 64\pi w^2.$$

This formula is an example of a **polynomial function**.

Definition: Polynomial Function

A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0,$$

where $n \in \mathbb{N}$. This is called the **general form of a polynomial function**. Each a_i is a **coefficient** and can be any real number. Each product $a_i x_i$ is called a **term** of the function. The largest n for which $a_n \neq 0$ is called the **degree** of the polynomial function. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Example 5.3.5: Identifying Polynomial Functions

Which of the following are polynomial functions?

$$\begin{aligned} f(x) &= 2x^3 \cdot 3x + 4 \\ g(x) &= -x(x^2 - 4) \\ h(x) &= 5\sqrt{x} + 2 \end{aligned}$$

Solution

The first two functions are examples of polynomial functions because they can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Example 5.3.6: Revisiting the Difference Quotient

Find and simplify the Difference Quotient for the following function.

$$P(x) = 2x^3 + x - 4$$

Solution

$$\begin{aligned}
\frac{P(x+h)-P(x)}{h} &= \frac{2(x+h)^3+(x+h)-4-(2x^3+x-4)}{h} && \text{(substituting)}\\
&= \frac{2(x+h)^3+(x+h)-4-2x^3-x+4}{h} && \text{(distributing the negative)}\\
&= \frac{2(x^3+3x^2h+3xh^2+h^3)+x+h-4-2x^3-x+4}{h} && \text{(distributing)}\\
&= \frac{2x^3+6x^2h+6xh^2+2h^3+x+h-4-2x^3-x+4}{h} && \text{(distributing)}\\
&= \frac{6x^2h+6xh^2+2h^3+h}{h} && \text{(combining like terms)}\\
&= \frac{h\left(6x^2+6xh+2h^2+1\right)}{h} && \text{(factoring out the GCF from the numerator)}\\
&= \frac{\cancel{h}^{1}\left(6x^2+6xh+2h^2+1\right)}{\cancel{h}^{1}} && \text{(canceling like factors)}\\
&= 6x^2+6xh+2h^2+1 && \text{(simplifying)}
\end{aligned}$$

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power (this is what we call, in the definition of the polynomial function, **general form**).

Example 5.3.7: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$\begin{aligned}
f(x) &= 3+2x^2-4x^3\\
g(t) &= 5t^5-2t^3+7t\\
h(p) &= 6p-p^3-2
\end{aligned}$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Checkpoint 5.3.7

Identify the degree, leading term, and leading coefficient of the polynomial $f(x)=4x^2-x^6+2x-6$.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its **end behavior**. Recall the following definition.

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x\to\infty$ or $x\to-\infty$).

To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate

the graph. We state this in the following theorem along with a formal proof of the result.

Theorem: End Behavior of a Polynomial Function

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Then the end behavior of the graph of $P(x)$ will match the end behavior of the graph of $y = a_n x^n$.

Proof

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Then

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1}x^{n-1}}{a_n x^n} + \frac{a_{n-2}x^{n-2}}{a_n x^n} + \cdots + \frac{a_2 x^2}{a_n x^n} + \frac{a_1 x}{a_n x^n} + \frac{a_0}{a_n x^n}\right) && \text{(factoring out } a_n x^n\text{)} \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \frac{a_{n-2}}{a_n x^2} + \cdots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right) && \text{(Laws of Exponents)} \end{aligned}$$

Since $a_n \neq 0$, the fractions $\frac{a_k}{a_n}$ are finite constants. For notational efficiency, let's denote $\frac{a_k}{a_n}$ as b_k. Therefore,

$$P(x) = a_n x^n \left(1 + b_{n-1} \cdot \frac{1}{x} + b_{n-2} \cdot \frac{1}{x^2} + \cdots + b_2 \cdot \frac{1}{x^{n-2}} + b_1 \cdot \frac{1}{x^{n-1}} + b_0 \frac{1}{x^n}\right).$$

(The next step is proved in Calculus, but can be understood here as a strong possibility.) As long as the power on x is not negative, as $x \to \pm\infty$, $\frac{1}{x^p} \to 0$. Since all the powers on x in

$$P(x) = a_n x^n \left(1 + b_{n-1} \cdot \frac{1}{x} + b_{n-2} \cdot \frac{1}{x^2} + \cdots + b_2 \cdot \frac{1}{x^{n-2}} + b_1 \cdot \frac{1}{x^{n-1}} + b_0 \frac{1}{x^n}\right)$$

are not negative, all terms involving x in the denominator must tend to 0 as $x \to \pm\infty$. That is, as $x \to \pm\infty$, $P(x) \to a_n x^n (1 + 0 + 0 + \cdots + 0 + 0 + 0) = a_n x^n$. Hence, the end behavior of $P(x)$ is $y = a_n x^n$.

Table **5.3.3** illustrates some examples.

Table 5.3.3

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example 5.3.8: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure **5.3.7**.

Figure 5.3.7

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very negative, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to -\infty;\ \text{as } x \to \infty,\ f(x) \to \infty.$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

? Checkpoint 5.3.8

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 5.3.8.

Figure 5.3.8

✓ Example 5.3.9: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty;\ \text{as } x \to \infty,\ f(x) \to -\infty.$$

? Checkpoint 5.3.9

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. At first glance, most people think we are immediately going to find the **turning points** of the function.

Definition: Turning Point

A **turning point** for the graph of a function is a point at which the function values change from increasing to decreasing or decreasing to increasing.

Unfortunately, determining the exact locations of the turning points requires Calculus (however, we will soon be able to determine the maximum number of turning points for any given polynomial!).

While any further discussion of turning points must be placed on hold, we can (at the very least) concern ourselves with the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept, $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure 5.3.9.

Figure 5.3.9

✓ Example 5.3.10: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the y- and x -intercepts.

Solution

The y-intercept occurs when the input is zero so substitute 0 for x.

Illegal pream-token (f)

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\implies \quad x-2=0 \quad \text{or} \quad x+1=0 \quad \text{or} \quad x-4=0$$
$$\implies \quad x=2 \quad \text{or} \quad x=-1 \quad \text{or} \quad x=4$$

The x-intercepts are $(2,0)$, $(-1,0)$, and $(4,0)$.

We can see these intercepts on the graph of the function shown in Figure 5.3.10.

Figure 5.3.10

Example 5.3.11: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x)=x^4-4x^2-45$, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4-4(0)^2-45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0,-45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$f(x)=x^4-4x^2-45=(x^2-9)(x^2+5)=(x-3)(x+3)(x^2+5).$$

Therefore, we solve

$$0=(x-3)(x+3)(x^2+5).$$

This mean $x-3=0$ or $x+3=0$ ($x^2+5=0$ would require $x^2=-5$, which has no real solution). Hence, $x=3$ or $x=-3$. The x-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 5.3.11. We can see that the function is even because $f(x)=f(-x)$.

Figure 5.3.11

Checkpoint 5.3.11

Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y- and x-intercepts.

Comparing Smooth and Continuous Graphs

We already know that the degree of a polynomial function helps us to determine its end behavior; however, the degree also helps us to determine the number of x-intercepts and, as such, a rough approximation to the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n zeros, or x-intercepts. Therefore, the graph of the polynomial function of degree n must have at most n–1 turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

Theorem: Number of Turning Points for a Polynomial Function

A polynomial of degree n will have, at most, $n-1$ turning points and n horizontal intercepts.

Proof

This proof must wait until Calculus.

The proof of this theorem requires more mathematical "technology" than we currently have access to, so we hold off until Calculus.

While not technically true, we will, for the sake of simplicity, define a **continuous function** to be one that has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Example 5.3.12: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution

The polynomial has a degree of 10, so there are at most 10 x-intercepts and at most $10 - 1 = 9$ turning points.

Checkpoint 5.3.12

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$.

Example 5.3.13: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure **5.3.12** based on its intercepts and turning points?

Figure 5.3.12

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure **5.3.13**.

Figure 5.3.13

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Checkpoint 5.3.13

What can we conclude about the polynomial represented by the graph shown in Figure **5.3.14** based on its intercepts and turning points?

Figure 5.3.14

Example 5.3.14: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0,0)$.

The x-intercepts are found by determining the zeros of the function.

$$0 = -4x(x+3)(x-4) \implies x = 0 \text{ or } x+3 = 0 \text{ or } x-4 = 0 \implies x = 0 \text{ or } x = -3 \text{ or } x = 4.$$

The x-intercepts are $(0,0)$, $(-3,0)$, and $(4,0)$.

The degree is 3 so the graph has at most 2 turning points.

Checkpoint 5.3.14

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

5.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Power function
  - Intermediate Algebra - Integer Exponents: Power Functions
- End behavior
  - Power Functions and Polynomial Functions - Identifying End Behavior of Polynomial Functions
  - Polynomial Functions and Their Graphs - End Behavior of Polynomials
- Polynomial function
  - Polynomial Functions - The Concept
- Zeros (of a polynomial)
  - Graphs of Polynomial Functions - Using Factoring to Find Zeros of Polynomial Functions

KEY CONCEPTS

DEFINITIONS

- **Power Function**: A function that can be represented in the form $f(x) = kx^p$, where k and p are real numbers, and k is known as the coefficient.
- **End Behavior of a Function**: The way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).
- **Polynomial Function**: A function that can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $n \in \mathbb{N}$ (a non-negative integer) and the coefficients a_i are real numbers.
- **General Form of a Polynomial Function**: The form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, typically with terms arranged in descending order of power.
- **Coefficient (of a term in a polynomial)**: The real number factor multiplying the variable part of a term (e.g., in a_ix^i, a_i is the coefficient).
- **Term (of a polynomial function)**: Each product a_ix^i in the polynomial function.
- **Degree (of a polynomial function)**: The largest exponent n for which the coefficient a_n is not zero.
- **Leading Term (of a polynomial function)**: The term containing the highest power of the variable (the term with the highest degree).
- **Leading Coefficient (of a polynomial function)**: The coefficient of the leading term.
- **Turning Point (of a polynomial graph)**: A point at which the function values change from increasing to decreasing or decreasing to increasing.
- **Continuous Function (informal definition)**: A function whose graph has no breaks; it can be drawn without lifting the pen from the paper.
- **Smooth Curve (informal definition)**: A graph that has no sharp corners.

THEOREMS

- **End Behavior of a Polynomial Function**: Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Then the end behavior of the graph of $P(x)$ will match the end behavior of the graph of its leading term $y = a_nx^n$.
- **Number of Turning Points for a Polynomial Function**: A polynomial of degree n will have, at most, $n - 1$ turning points and n horizontal intercepts (zeros).

COMMON MISTAKES

- **Confusing Power Functions with Exponential Functions**: A power function has a variable base and a fixed power (e.g., x^2), while an exponential function has a fixed base and a variable power (e.g., 2^x). The function $f(x) = 2^x$ is not a power function.
- **Incorrectly Identifying the Degree or Leading Coefficient**: When a polynomial is not written in standard general form (descending powers), care must be taken to identify the term with the highest power to determine the degree and leading coefficient.
- **Errors in Determining End Behavior**: Misinterpreting the combined effect of the sign of the leading coefficient and the parity (even/odd) of the degree of the polynomial on its end behavior.
- **Assuming Maximum Number of Turning Points/Intercepts is Always Achieved**: A polynomial of degree n has *at most* $n - 1$ turning points and *at most* n x-intercepts. It does not necessarily have exactly that many.

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5.3.2: HOMEWORK

READING QUESTIONS

1. What is the general form of a power function? Give an example of a toolkit function that is a power function.
2. Is $f(x) = 2^x$ a power function? Why or why not?
3. How does the end behavior of an even-degree power function $f(x) = kx^n$ (where n is even) differ based on the sign of k?
4. How does the end behavior of an odd-degree power function $f(x) = kx^n$ (where n is odd) differ based on the sign of k?
5. What is the general form of a polynomial function? What restriction is placed on the exponents of the variable?
6. Define the "degree" of a polynomial.
7. Define the "leading term" and "leading coefficient" of a polynomial.
8. How does the leading term of a polynomial function relate to its end behavior?
9. What is a "turning point" of a polynomial graph?
10. What is the maximum number of x-intercepts a polynomial of degree n can have?
11. What is the maximum number of turning points a polynomial of degree n can have?
12. How do you find the y-intercept of a polynomial function given in general form?
13. If a polynomial function is given in factored form, like $f(x) = (x-2)(x+1)(x-4)$, how can you find its x-intercepts?
14. Are the graphs of polynomial functions smooth and continuous? What do these terms mean informally?
15. When computing the difference quotient for $f(x) = x^4$, what algebraic techniques might be useful for expanding $(x+h)^4$?
16. Explain the difference between the coefficient of a power function and its degree.
17. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?
18. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.
19. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?
20. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

HOMEWORK

ALGEBRAIC

For the following exercises, identify the function as a power function, a polynomial function, or neither.

21. $f(x) = x^5$
22. $f(x) = (x^2)^3$
23. $f(x) = x - x^4$
24. $f(x) = \frac{x^2}{x^2-1}$
25. $f(x) = 2x(x+2)(x-1)^2$
26. $f(x) = 3^{x+1}$

For the following exercises, find the degree and leading coefficient for the given polynomial.

27. $-3x^4$
28. $7 - 2x^2$
29. $-2x^2 - 3x^5 + x - 6$
30. $x(4 - x^2)(2x + 1)$
31. $x^2(2x - 3)^2$

For the following exercises, determine the end behavior of the functions.

32. $f(x) = x^4$
33. $f(x) = x^3$
34. $f(x) = -x^4$
35. $f(x) = -x^9$
36. $f(x) = -2x^4 - 3x^2 + x - 1$
37. $f(x) = 3x^2 + x - 2$
38. $f(x) = x^2(2x^3 - x + 1)$
39. $f(x) = (2 - x)^7$

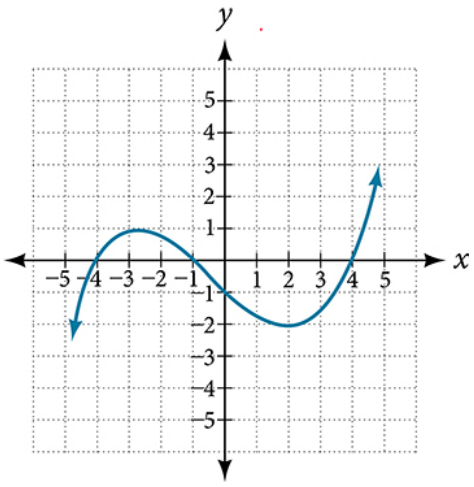
For the following exercises, find the intercepts of the functions.

40. $f(t) = 2(t-1)(t+2)(t-3)$
41. $g(n) = -2(3n-1)(2n+1)$
42. $f(x) = x^4 - 16$
43. $f(x) = x^3 + 27$
44. $f(x) = x(x^2 - 2x - 8)$
45. $f(x) = (x+3)(4x^2 - 1)$

GRAPHICAL

For the following exercises, determine the least possible degree of the polynomial function shown.

46.



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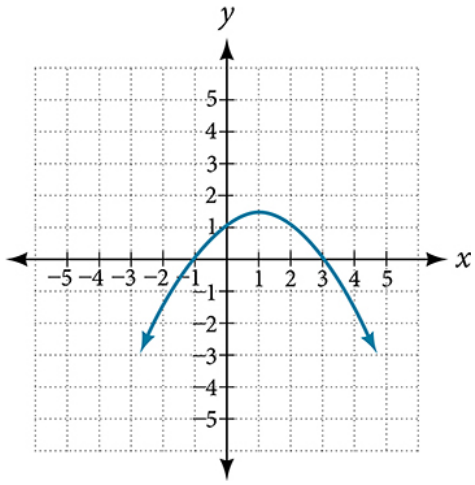

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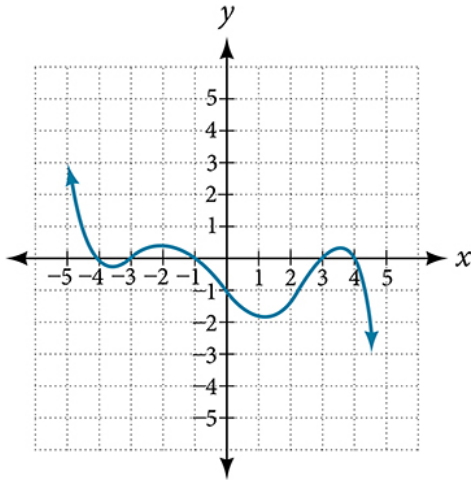

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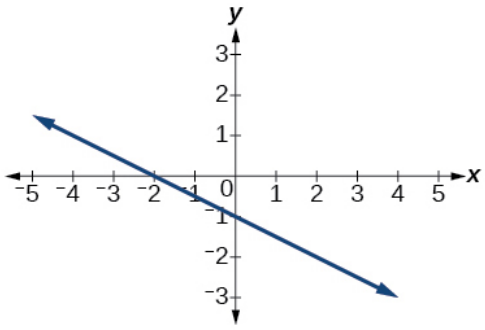

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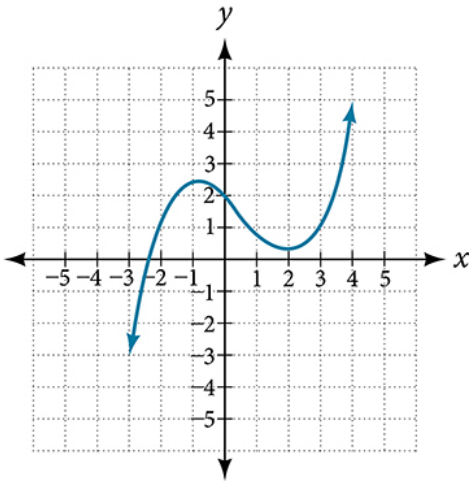

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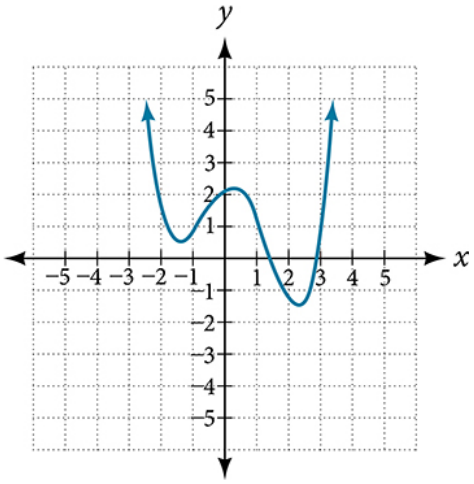

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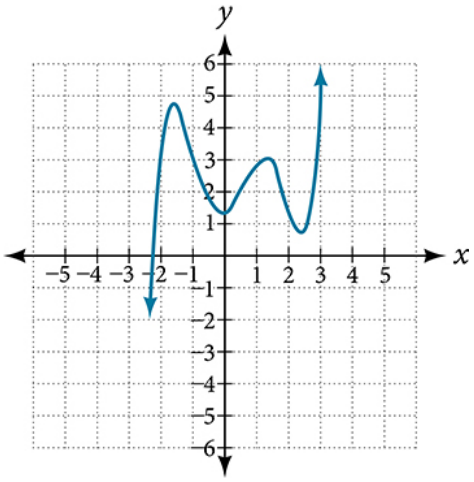
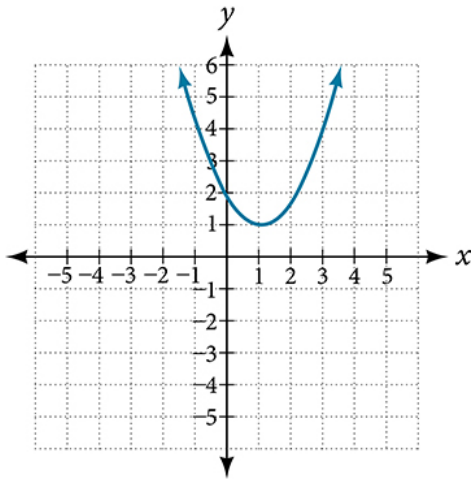
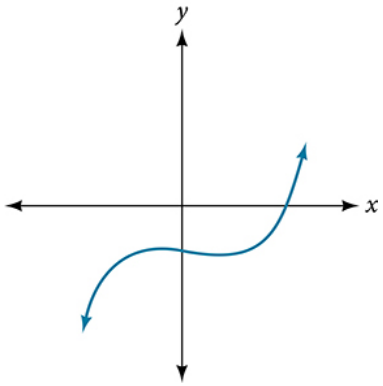




For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

54.



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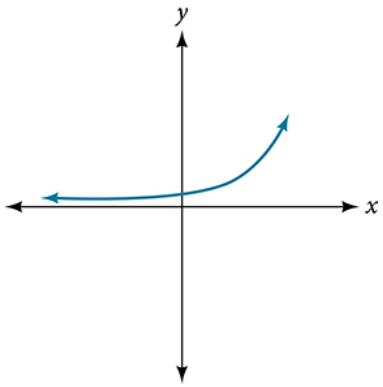

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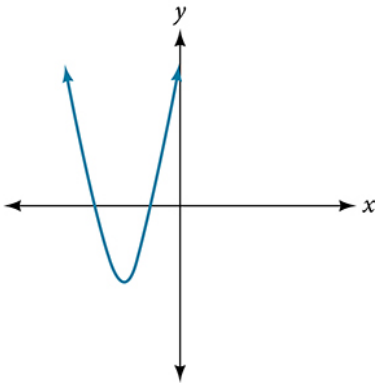

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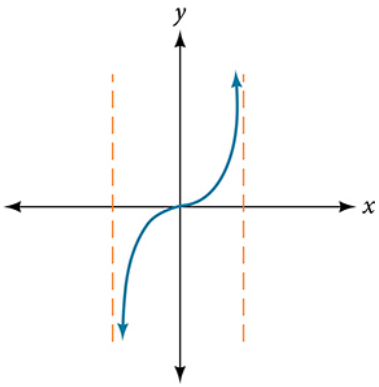

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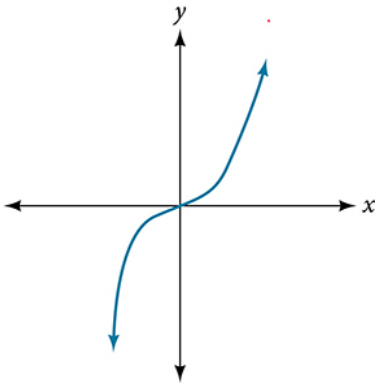

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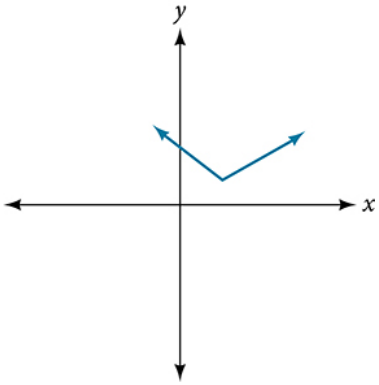
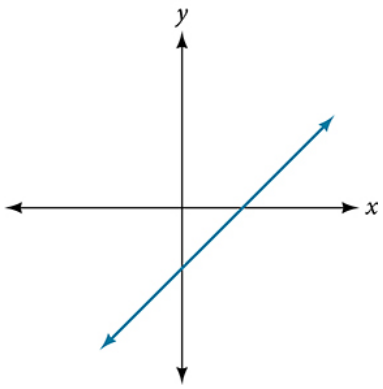




NUMERIC

For the following exercises, make a table to confirm the end behavior of the function.

61. $f(x) = -x^3$
62. $f(x) = x^4 - 5x^2$
63. $f(x) = x^2(1-x)^2$
64. $f(x) = (x-1)(x-2)(3-x)$
65. $f(x) = \frac{x^5}{10} - x^4$

TECHNOLOGY

For the following exercises, graph the polynomial functions using graphing technology. Based on the graph, determine the intercepts and the end behavior.

66. $f(x) = x^3(x-2)$
67. $f(x) = x(x-3)(x+3)$
68. $f(x) = x(14-2x)(10-2x)$
69. $f(x) = x(14-2x)(10-2x)^2$
70. $f(x) = x^3 - 16x$
71. $f(x) = x^3 - 27$
72. $f(x) = x^4 - 81$
73. $f(x) = -x^3 + x^2 + 2x$
74. $f(x) = x^3 - 2x^2 - 15x$
75. $f(x) = x^3 - 0.01x$

EXTENSIONS

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

76. The y-intercept is $(0,-4)$. The x-intercepts are $(-2,0)$, $(2,0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.
77. The y-intercept is $(0,9)$. The x-intercepts are $(-3,0)$, $(3,0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to -\infty$.
78. The y-intercept is $(0,0)$. The x-intercepts are $(0,0)$, $(2,0)$. Degree is 3. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.
79. The y-intercept is $(0,1)$. The x-intercept is $(1,0)$. Degree is 3. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to -\infty$.
80. The y-intercept is $(0,1)$. There is no x-intercept. Degree is 4. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.

REAL-WORLD APPLICATIONS

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

81. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.
82. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.
83. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.
84. An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.
85. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

5.4: Graphs of Polynomial Functions

Note to the Instructor (click to expand)

This section introduces multiplicities of zeros for polynomials and goes over the process of *accurately* graphing a polynomial. Make sure you focus on the local behavior of the function near a zero with multiplicities greater than 2 (discussing the "flattening" as the multiplicity increases). Solving polynomial inequalities using graphs and graphing absolute values of polynomial functions are also required topics. I have included the IVT as an ***optional topic*** for those who want to discuss it.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Equations**
  - Solving Polynomial Equations: While this section focuses on finding zeros (which is equivalent to solving $P(x) = 0$), prior experience with solving polynomial equations by factoring is beneficial.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.
- Graph the absolute value of a polynomial function.
- Solve a polynomial inequality (using a graph).

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 5.4.1.

Table 5.4.1

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

The revenue can be modeled by the polynomial function $R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$, where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the

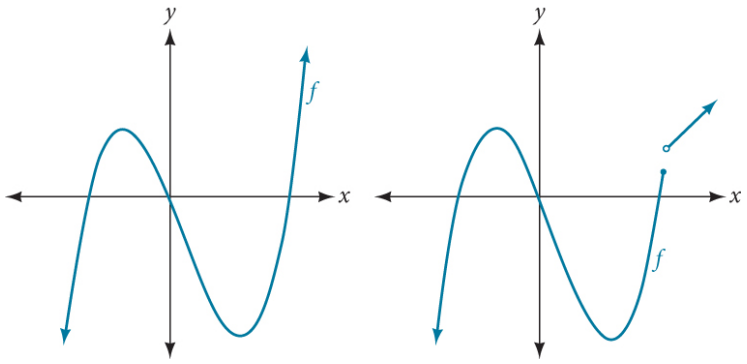
revenue for the company increasing? Over which intervals is the revenue for the company decreasing?

These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous (again, as mention in the previous section, this is not technically true; however, we will use this flawed definition of continuity until we reach Calculus). Figure **5.4.1** shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

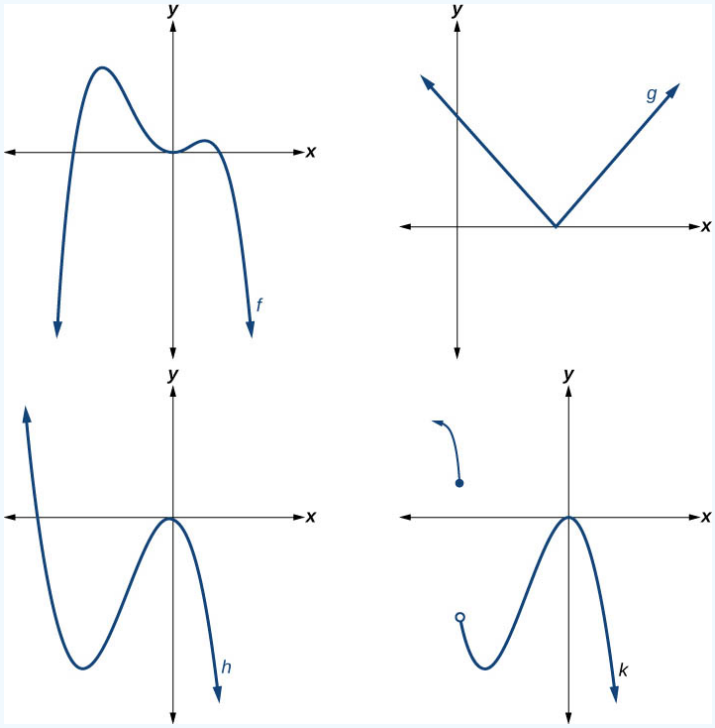
Figure 5.4.1



Example 5.4.1: Recognizing Polynomial Functions

Which of the graphs in Figure **5.4.2** represents a polynomial function?

Figure 5.4.2



Solution

The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Q&A

Do all polynomial functions have as their domain all real numbers?

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple Quadratic Formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

Example 5.4.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$\begin{aligned}
& x^6 - 3x^4 + 2x^2 = 0 \\
\Rightarrow\quad & x^2(x^4 - 3x^2 + 2) = 0 && \text{(factoring out the GCF)} \\
\Rightarrow\quad & x^2(u^2 - 3u + 2) = 0 && \text{(substituting } u = x^2 \text{ in the trinomial)} \\
\Rightarrow\quad & x^2(u-2)(u-1) = 0 && \text{(factoring the trinomial)} \\
\Rightarrow\quad & x^2(x^2-2)(x^2-1) = 0 && \text{(resubstituting } u = x^2) \\
\Rightarrow\quad & x^2(x^2-2)(x-1)(x+1) = 0 && \text{(factoring a Difference of Squares)}
\end{aligned}$$

Thus,

$$\begin{array}{lllll}
x^2 = 0 & \text{or} & x^2 - 2 = 0 & \text{or} & x - 1 = 0 \quad \text{or} \quad x + 1 = 0 \\
\Rightarrow\ x = 0 & & x^2 = 2 & & x = 1 \qquad x = -1 \\
\Rightarrow\ x = 0 & & x = \pm\sqrt{2} & & x = 1 \qquad x = -1
\end{array}$$

This gives us five x-intercepts: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$, and $(-\sqrt{2},0)$ (see Figure 5.4.3). We can see that this is an even function.

Figure 5.4.3

Example 5.4.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 \\ \Longrightarrow \quad x^2(x-5) - 1(x-5) &= 0 \qquad \text{(factoring by grouping)} \\ \Longrightarrow \quad (x-5)(x^2-1) &= 0 \qquad \text{(factor out the GCF)} \\ \Longrightarrow \quad (x-5)(x-1)(x+1) &= 0 \qquad \text{(factoring the Difference of Squares)} \end{aligned}$$

Hence, $x = 5$, or $x = 1$, or $x = -1$. There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$ (see Figure 5.4.4).

Figure 5.4.4

Example 5.4.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution

The y-intercept can be found by evaluating $g(0)$.

$$g(0) = (0-2)^2(2(0)+3) = 12.$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \implies x = 2 \text{ or } x = -\frac{3}{2}.$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial. For example, Figure 5.4.5 shows that our answers for Example 5.4.4 are correct.

Figure 5.4.5

Example 5.4.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 5.4.6, it appears that there are x-intercepts at $x = -3$, -2, and 1.

Figure 5.4.6

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$h(x) = x^3 + 4x^2 + x - 6 = (x + 3)(x + 2)(x - 1).$$

Checkpoint 5.4.5

Find the y- and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off.

Suppose, for example, we graph the function $f(x) = (x+3)(x-2)^2(x+1)^3$. Notice in Figure 5.4.7 that the behavior of the function at each of the x-intercepts is different.

Figure 5.4.7

Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.

The x-intercept at $x = -3$ is the solution of equation $(x+3) = 0$. The graph passes directly through the x-intercept at $x = -3$. We say the factor is **linear** (has a degree of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a **single zero** because the zero corresponds to a single factor of the function.

The x-intercept at $x = 2$ is the repeated solution of equation $(x-2)^2 = 0$. The graph touches the axis at the intercept and changes direction. We say the factor is **quadratic** (degree 2), so the behavior near the intercept is like that of a quadratic - it bounces off of the horizontal axis at the intercept. Since $(x-2)^2 = (x-2)(x-2)$, the factor is repeated. This leads us to an important definition that will be useful when it comes to polynomial functions.

Definition: Multiplicity of a Zero

If a polynomial contains a factor of the form $(x-h)^p$, and if this is the greatest such factor of $(x-p)$, we say that $x = h$ is a **zero of multiplicity p**.

Hence, the zero associated with $x = 2$ has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept at $x = -1$ is the repeated solution of factor $(x+1)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. We say this factor is **cubic** (degree 3), so the behavior near the intercept is like that of a cubic - with the same "S"-shape near the intercept as the toolkit function $f(x) = x^3$. By definition, this is a zero of multiplicity 3.

Now that we know what it means for a zero to have multiplicity p, it is natural to ask if there is any ramification of the multiplicity of a zero.

Theorem: Local Behavior Near a Zero

If $x = h$ is a zero of multiplicity p for a given polynomial, then the local behavior of the graph of the polynomial near $x = h$ matches the behavior of $y = \pm(x-h)^p$, where the sign of the coefficient is determined by other characteristics of the given polynomial.

In truth, most people (myself included) would say that the behavior of the polynomial near to a zero of multiplicity p matches the graph of $y = x^p$, but possibly reflected about the x-axis. What we *mean* is that the behavior of the graph near to the zero matches the look of $y = x^p$ near the origin.

For zeros with even multiplicities, the graphs *touch* or are **tangent** to the x-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the x-axis. See Figure 5.4.8 for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

Figure 5.4.8

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis. For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Caution: Graph Accurately Near Zeros!

You will eventually be asked to make accurate sketches of polynomial functions. Part of this skill is to show that you know the impact of high multiplicity on a zero. The higher the multiplicity, the flatter the polynomial graph is near to the zero - be prepared to show this.

Finally, you will inevitably notice the following fact.

Theorem: Sum of Multiplicities and the Degree of a Polynomial

The degree of a polynomial factored into the product of linear and irreducible quadratic factors is sum of the multiplicities of the linear factors and the twice the sum of the multiplicities of the irreducible quadratic factors. That is, if

$$P(x) = (a_1x + b_1)^{m_1}(a_2x + b_2)^{m_2} \cdots (a_kx + b_k)^{m_k}(c_1x^2 + d_1)^{p_1}(c_2x^2 + d_2)^{p_2} \cdots (c_nx^2 + d_n)^{p_n},$$

then the degree of $P(x)$ is

$$m_1 + m_2 + \cdots + m_k + 2(p_1 + p_2 + \cdots p_n).$$

Example 5.4.6: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure 5.4.9 to identify the zeros of the function and their possible multiplicities.

Figure 5.4.9

Solution

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of -3 could have multiplicity 2, 4, or 6 (remember, we were told this polynomial has degree 6, so the sum of the multiplicities of the zeros must be 6).

The next zero occurs at $x = -1$. The graph looks almost linear at this point, so this is likely a zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the x-axis, so the multiplicity of the zero must be odd. From the shape of the curve near to $x = 4$, we know the multiplicity cannot be 1 (the graph is not *locally linear* near $x = 4$). Hence, $x = 4$ must be a zero of multiplicity either 3 or 5.

Now that we have an idea of the *possible* multiplicities, let's build a table and use a little deductive logic to identify the exact multiplicities.

Zero	Possible Multiplicities
-3	$2, 4, 6$
-1	1
4	$3, 5$

Since the sum of the multiplicities must be 6, and we have three zeros, the maximum multiplicity for any given zero is 3. This cleans up our table quite a bit:

Zero	Possible Multiplicities
-3	2
-1	1
4	3

Thus, we have identified the zeros are their exact multiplicities.

? Checkpoint 5.4.6

Use the graph of the function of degree 7 in Figure **5.4.10** to identify the zeros of the function and their multiplicities.

Figure 5.4.10

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

will either ultimately rise or fall as x increases or decreases without bound. This is because, for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very negative inputs, say –100 or -1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing power functions, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure **5.4.11** summarizes all four cases.

Figure 5.4.11

Understanding the Relationship Between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's *local behavior*. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure **5.4.12**. The graph has three turning points.

Figure 5.4.12

This function is a 4th degree polynomial function and has 3 turning points. This should be familiar because, as mentioned in the previous section, the maximum number of turning points of a polynomial function is always one less than the degree of the function.

Example 5.4.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

1. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
2. $f(x) = -(x-1)^2(1+2x^2)$

Solutions

1. First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$. Identify the degree of the polynomial function. This polynomial function is of degree 5. The maximum number of turning points is $5 - 1 = 4$.
2. First, identify the leading term of the polynomial function if the function were expanded. Looking at the given function, $f(x) = -(x-1)^2(1+2x^2)$, we can see that the factor $(x-1)^2$ will contribute x^2 (if we multiplied it out) and the factor $1 + 2x^2$ contributes $2x^2$. Therefore, the leading term is

$$a_n x^n = -(x^2)(2x^2) = -2x^4.$$

Therefore, this polynomial function is of degree 4. As such, the maximum number of turning points is $4 - 1 = 3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

How To: Graph a polynomial function

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetric about the y-axis. That is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetric about the origin. That is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example 5.4.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis like a line at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$f(0) = -2(0+3)^2(0-5) = -2 \cdot 9 \cdot (-5) = 90.$$

The y- intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 5.4.13.

Figure 5.4.13

To sketch this, we consider that:

- As $x \to -\infty$, $f(x) \to \infty$. So we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $\pm f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of the x-axis, so the function must start increasing.

  At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure 5.4.14.

Figure 5.4.14

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure 5.4.15.

Figure 5.4.15

As $x \to \infty$, the function $f(x) \to -\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 5.4.16, and verify that the resulting graph looks like our sketch in Figure 5.4.15.

Figure 5.4.16

The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$

Checkpoint 5.4.8

Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

While you will not be graphing the absolute value of polynomial functions in Calculus, you *will* be graphing the absolute value of other functions. Therefore, it's a good idea to keep that skill fresh (it's a simple skill once you know how to graph the non-absolute value function).

Example 5.4.9: Revisiting the Graphs of Absolute Value Functions

Graph

$$f(x) = \left|-0.9(x+2)^3(x-1)^6(x-5)^5\right|.$$

Solution

As we have done in previous sections, we initially concern ourselves with the *non-absolute value* function

$$f(x) = -0.9(x+2)^3(x-1)^6(x-5)^5.$$

We list the roots and their multiplicities, as well as the behavior of the graph near that root:

Root	Multiplicity	Behavior
-2	3	like x^3
1	6	like x^6 (a very flat parabola)
5	5	like x^5 (a flat cubic)

Next, we note that the degree of the polynomial is $3+6+5=14$, so the end behavior is like that of x^2 *except* the leading coefficient is negative. Therefore, the end behavior is more like $-x^2$. That is, as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to -\infty$. Reading from left to right, the graph of f should

- start low
- increase until it crosses the x-axis at $x=-2$ like a cubic
- climb for a bit, but then turn to fall back to the x-axis to "bounce" like a very flat parabola at $x=1$
- climb for a bit, but then turn to fall back to the x-axis to cross at $x=5$ like a flat cubic
- it then continues to fall forever.

Figure 5.4.17 shows the graph of the non-absolute value function.

Figure 5.4.17

To graph the absolute value of this function, we take all the function outputs (y-values) that are negative and we force them to be positive. In effect, this keeps the part of the graph from Figure 5.4.17 that is already above the x-axis in place; however, all the parts that are below the x-axis get reflected about the y-axis to become positive (above the x-axis). See Figure 5.4.18.

Figure 5.4.18

Solving Polynomial Inequalities

We finish this section with a topic that might not seem related to graphing polynomial functions - solving polynomial inequalities.

There are those who insist on students solving inequalities using sign charts, and then there are those who want to reinforce a student's graphing skills. If you cannot tell, I am in the latter camp. To solve *any* inequality, you could move all terms to one side of the inequality, graph the resulting expression on the non-zero side, and use that graph to solve the inequality. The key to being able to do this is know how to graph that non-zero expression.

Let's see this in an example.

Example 5.4.10

Solve

$$3x^3 - 4x \geq 6x^2 - 8.$$

Solution

Let's start by getting all terms to one side.

$$\begin{aligned} & 3x^3 - 4x \geq 6x^2 - 8 \\ \Longrightarrow \quad & 3x^3 - 6x^2 - 4x + 8 \geq 0 \end{aligned}$$

Now we want to graph the expression on the left side. That is, let's graph

$$g(x) = 3x^3 - 6x^2 - 4x + 8.$$

Since this is a polynomial, we would like to find the roots and their multiplicities. Let's try factoring by grouping:

$$\begin{aligned} g(x) &= 3x^3 - 6x^2 - 4x + 8 \\ &= 3x^2(x-2) - 4(x-2) \\ &= (x-2)(3x^2 - 4) \end{aligned}$$

Hence, this polynomial has the following roots (and multiplicities):

Root	Multiplicity
2	1
$\frac{2}{\sqrt{3}}$	1
$-\frac{2}{\sqrt{3}}$	1

Since $g(x)$ is a cubic, we know it "starts low and ends high." A quick sketch yields the graph in Figure **5.4.19**.

Figure 5.4.19

Remember, we are being asked to solve $3x^3 - 6x^2 - 4x + 8 \geq 0$. This is equivalent to asking, "When is the graph of $g(x) = 3x^3 - 6x^2 - 4x + 8$ at or above the x-axis?" The answer to this question is simple once you have the graph. We can see from Figure **5.4.19**, that $g(x) \geq 0$ when

$$x \in \left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right] \cup [2, \infty).$$

Using the Intermediate Value Theorem (optional topic)

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x-axis and another point at $x = b$ lies below the x-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$. This means that we are assured there is a solution c where $f(c) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure **5.4.20** shows that there is a zero between a and b.

Figure 5.4.20

Using the Intermediate Value Theorem to show there exists a zero.

Theorem: Intermediate Value Theorem (for Polynomials)

Let f be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Proof

This proof must wait until Calculus.

To be honest, the Intermediate Value Theorem applies to any continuous function (not just polynomials); however, that extra bit of power to the theorem is reserved for Calculus (where the Intermediate Value Theorem becomes incredibly useful).

Example 5.4.11: Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution

As a start, evaluate $f(x)$ at the integer values $x = 1$, 2, 3, and 4. See Table 5.4.2.

Table 5.4.2

x	1	2	3	4
$f(x)$	5	0	–3	2

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4. Thus, we have shown that there are *at least* two real zeros between $x = 1$ and $x = 4$.

We can verify this result by looking at the graph of the function in Figure 5.4.21.

Figure 5.4.21

Checkpoint 5.4.11

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from Calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. We restate the definition from earlier in this textbook below for completeness.

Definition: Local Extrema

A function f has a **local maximum** at $x = b$ in an open interval (a, c) if $f(b) \geq f(x)$ for all $x \neq b$ in the interval. Likewise, f has a **local minimum** at $x = b$ in (a, c) if $f(b) \leq f(x)$ for all $x \neq b$ in the interval.

If f has more than one local maximum on a given interval, we call the collection of these maximums the **local maxima**. Likewise, if f has more than one local minimum on a given interval, we call the collection of these minimums the **local minima**.

Together, local maxima and minima are called **local extrema** (or **local extreme values**).

Some texts use the word "relative" instead of "local." We will use the latter throughout this text.

Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a global maximum or a global minimum. Again, we restate the definition from earlier in this text below for completeness.

Definition: Absolute Extrema

A function f has an **absolute maximum** at $x = b$ if $f(b) \geq f(x)$ for all x in the domain of f. Likewise, a function f has an **absolute minimum** at $x = b$ if $f(b) \leq f(x)$ for all x in the domain of f.

If f has more than one local absolute maximum, we call the collection of these maximums the **absolute maxima**. Likewise, if f has more than one absolute minimum, we call the collection of these minimums the **absolute minima**.

Together, absolute maxima and minima are called **absolute extrema**.

Some texts use the word "global" instead of "absolute." We will use the latter throughout this text.

We can see the difference between local and global extrema in Figure 5.4.22.

Figure 5.4.22

Q&A

Do all polynomial functions have a global minimum or maximum?

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

Example 5.4.12: Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution

We will start this problem by drawing a picture like that in Figure 5.4.23, labeling the width of the cut-out squares with a variable, w.

Figure 5.4.23

Notice that after a square is cut out from each end, it leaves a $(14 - 2w)$ cm by $(20 - 2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

$$V(w) = (20 - 2w)(14 - 2w)w = 280w - 68w^2 + 4w^3.$$

Notice, since the factors are w, $20–2w$, and $14–2w$, the three zeros are 0, 10, and 7, respectively. Because a height of 0 cm is not reasonable, we consider the only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0 < w < 7$. Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure 5.4.24. We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem - values from 0 to 7.

Figure 5.4.24

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure 5.4.25.

Figure 5.4.25

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

? Checkpoint 5.4.12

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

5.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Graphs of polynomial functions
  - Graphs of Polynomial Functions - Graphing Polynomial Functions
  - Polynomial Functions and Their Graphs - Graphing Polynomials Using End Behavior and Zeros
- Zeros (and their multiplicities) of polynomial functions
  - Graphs of Polynomial Functions - Identifying Zeros and Their Multiplicities
  - Polynomial Functions and Their Graphs - Zeros of Functions and Their Multiplicities
- Factoring polynomials
  - Graphs of Polynomial Functions - Using Factoring to Find Zeros of Polynomial Functions
- Polynomial modeling
  - Polynomial Functions - Applications

KEY CONCEPTS

DEFINITIONS

- **Zeros of a Polynomial Function**: The values of x for which $f(x) = 0$. These correspond to the x-intercepts of the graph.
- **Multiplicity of a Zero**: If a polynomial contains a factor of the form $(x-h)^p$, and if this is the greatest such factor of $(x-p)$, we say that $x = h$ is a zero of multiplicity p.
- **Single Zero**: A zero of multiplicity 1. The graph passes directly through the x-axis at a single zero.

THEOREMS

- **Local Behavior Near a Zero**: If $x = h$ is a zero of multiplicity p for a given polynomial, then the local behavior of the graph of the polynomial near $x = h$ matches the behavior of $y = \pm(x-h)^p$, where the sign of the coefficient is determined by other characteristics of the given polynomial.
- **Sum of Multiplicities and the Degree of a Polynomial**: The sum of the multiplicities of the factors for a polynomial is the degree of the polynomial function.
- **Intermediate Value Theorem (for Polynomials)**: Let f be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

COMMON MISTAKES

- **Incorrectly Identifying Multiplicity**: Miscounting the multiplicity of a zero, especially when a factor is repeated or when a zero comes from a factor like $(x^2 - k)$ which might yield two distinct zeros each of multiplicity 1.
- **Misinterpreting Behavior at Zeros based on Multiplicity**: Confusing whether the graph crosses or touches/bounces at an x-intercept based on the multiplicity of the zero (odd multiplicity means cross, even multiplicity means touch/bounce).
- **Errors in Determining End Behavior**: Incorrectly identifying the leading term (especially if the polynomial is not in standard form) or misapplying the rules for end behavior based on the degree and sign of the leading coefficient.
- **Exceeding Maximum Number of Turning Points**: Sketching a graph with more turning points than $n - 1$, where n is the degree of the polynomial.

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5.4.2: HOMEWORK

READING QUESTIONS

1. What are two key visual characteristics of the graphs of polynomial functions of degree 2 or more?
2. What are the "zeros" of a polynomial function $f(x)$, and how do they relate to the graph's x-intercepts?
3. Describe three methods mentioned for finding the zeros of a polynomial function.
4. What does the "multiplicity" of a zero tell us about the behavior of the graph of a polynomial function at that x-intercept?
   - What happens if the multiplicity is odd?
   - What happens if the multiplicity is even?
5. How does the graph's behavior near an x-intercept change for higher even or odd multiplicities?
6. What determines the end behavior of a polynomial function's graph?
7. What is the maximum number of turning points a polynomial function of degree n can have?
8. List the general steps suggested for sketching the graph of a polynomial function.
9. What does the Intermediate Value Theorem state about a polynomial function f if $f(a)$ and $f(b)$ have opposite signs?
10. If you know the x-intercepts and their multiplicities for a polynomial, how can you write a general formula for that polynomial function? What additional piece of information is needed to find the specific stretch factor a?
11. What is the difference between a local maximum/minimum and a global maximum/minimum for a polynomial function?
12. Do all polynomial functions have a global minimum or maximum? Explain.
13. How do you graph the function $g(x) = |f(x)|$ if you know the graph of $f(x)$, where $f(x)$ is a polynomial function?
14. Describe a graphical approach to solving a polynomial inequality like $P(x) > 0$.
15. What is the difference between an x-intercept and a zero of a polynomial function f?
16. If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?
17. Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.
18. Explain how the factored form of the polynomial helps us in graphing it.
19. If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

HOMEWORK

ALGEBRAIC

For the following exercises, find the x- or t-intercepts of the polynomial functions.

20. $C(t) = 2(t-4)(t+1)(t-6)$
21. $C(t) = 3(t+2)(t-3)(t+5)$
22. $C(t) = 4t(t-2)^2(t+1)$
23. $C(t) = 2t(t-3)(t+1)^2$
24. $C(t) = 2t^4 - 8t^3 + 6t^2$
25. $C(t) = 4t^4 + 12t^3 - 40t^2$
26. $f(x) = x^4 - x^2$
27. $f(x) = x^3 + x^2 - 20x$
28. $f(x) = x^3 + 6x^2 - 7x$
29. $f(x) = x^3 + x^2 - 4x - 4$
30. $f(x) = x^3 + 2x^2 - 9x - 18$
31. $f(x) = 2x^3 - x^2 - 8x + 4$
32. $f(x) = x^6 - 7x^3 - 8$
33. $f(x) = 2x^4 + 6x^2 - 8$
34. $f(x) = x^3 - 3x^2 - x + 3$
35. $f(x) = x^6 - 2x^4 - 3x^2$
36. $f(x) = x^6 - 3x^4 - 4x^2$
37. $f(x) = x^5 - 5x^3 + 4x$

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

38. $f(x) = x^3 - 9x$, between $x = -4$ and $x = -2$.
39. $f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.
40. $f(x) = x^5 - 2x$, between $x = 1$ and $x = 2$.
41. $f(x) = -x^4 + 4$, between $x = 1$ and $x = 3$.
42. $f(x) = -2x^3 - x$, between $x = -1$ and $x = 1$.
43. $f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

For the following exercises, find the zeros and give the multiplicity of each.

44. $f(x) = (x+2)^3(x-3)^2$
45. $f(x) = x^2(2x+3)^5(x-4)^2$
46. $f(x) = x^3(x-1)^3(x+2)$
47. $f(x) = x^2(x^2+4x+4)$
48. $f(x) = (2x+1)^3(9x^2-6x+1)$
49. $f(x) = (3x+2)^5(x^2-10x+25)$
50. $f(x) = x(4x^2-12x+9)(x^2+8x+16)$
51. $f(x) = x^6 - x^5 - 2x^4$
52. $f(x) = 3x^4 + 6x^3 + 3x^2$
53. $f(x) = 4x^5 - 12x^4 + 9x^3$
54. $f(x) = 2x^4(x^3 - 4x^2 + 4x)$
55. $f(x) = 4x^4(9x^4 - 12x^3 + 4x^2)$

GRAPHICAL

For the following exercises, graph the polynomial functions. Note x- and y-intercepts, multiplicity, and end behavior.

56. $f(x) = (x+3)^2(x-2)$
57. $g(x) = (x+4)(x-1)^2$
58. $h(x) = (x-1)^3(x+3)^2$
59. $k(x) = (x-3)^3(x-2)^2$
60. $m(x) = -2x(x-1)(x+3)$

61. $n(x) = -3x(x+2)(x-4)$

For the following exercises, use the graph to identify zeros and multiplicity.

62.

63.

64.

65.

TECHNOLOGY

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

66. $f(x) = x^3 - x - 1$
67. $f(x) = 2x^3 - 3x - 1$
68. $f(x) = x^4 + x$
69. $f(x) = -x^4 + 3x - 2$
70. $f(x) = x^4 - x^3 + 1$

REAL-WORLD APPLICATIONS

For the following exercises, write the polynomial function that models the given situation.

71. A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.
72. Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.
73. A square has sides of 12 units. Squares $x+1$ by $x+1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.
74. A cylinder has a radius of $x+2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.
75. A right circular cone has a radius of $3x+6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius r and height h.

5.5: Dividing Polynomials

Note to the Instructor (click to expand)

The entire purpose of this section is to introduce the student to synthetic division, which is preferable to know prior to moving forward with polynomial theory.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Dividing Polynomials - Long Division: While this section teaches/reviews polynomial long division, prior exposure from a foundational Algebra course would be beneficial.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Figure 5.5.1

Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m.[1] We can easily find the volume using elementary Geometry.

$$V = l \cdot w \cdot h = 61.5 \cdot 40 \cdot 30 = 73,800$$

So the volume is 73,800 cubic meters (m^3). Suppose we knew the volume, length, and width. We could divide to find the height.

$$h = \frac{V}{l \cdot w} = \frac{73,800}{61.5 \cdot 40} = 30.$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose

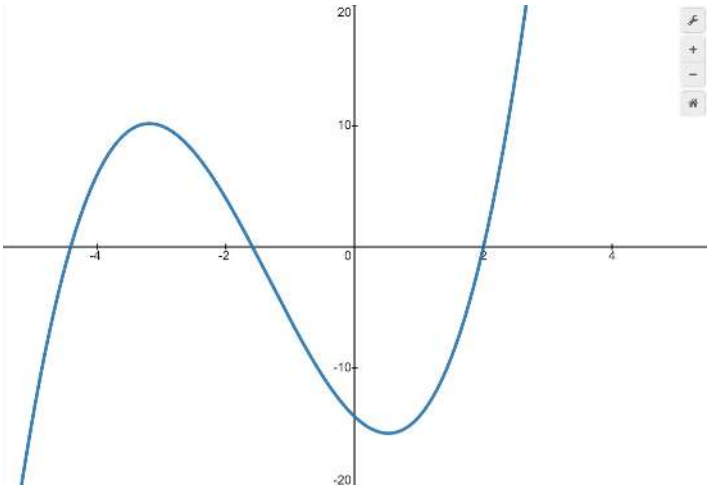
the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all the factoring techniques we learned in Intermediate Algebra, this equation foils us at every turn. If we graph f using Desmos, we get the graph in Figure 5.5.2.

Figure 5.5.2

$f(x) = x^3 + 4x^2 - 5x - 14$



The graph suggests that the function has three zeros, one of which is $x = 2$. It's easy to show that $f(2) = 0$, but the other two zeros seem less friendly. Even though we could click on the curve in Desmos to find decimal approximations for these, we seek an analytical method (a method by hand) to find the remaining zeros exactly.

Since f is a polynomial function and $f(2) = 0$, it should seem realistic that $(x - 2)$ is a factor of f. That is, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. Therefore, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)q(x)$, where $q(x)$ is some other polynomial. If it even exists, how could we find such a $q(x)$? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$
\begin{array}{rr|l}
 & & x^2 + 6x + 7 \\
\hline
x - 2 & \big) & x^3 + 4x^2 - 5x - 14 \\
 & - & (x^3 - 2x^2) \\
\hline
 & & 6x^2 - 5x \\
 & - & (6x^2 - 12x) \\
\hline
 & & 7x - 14 \\
 & - & (7x - 14) \\
\hline
 & & 0
\end{array}
$$

Thus, $\frac{x^3+4x^2-5x-14}{x-2} = x^2 + 6x + 7$. This means $x^3 + 4x^2 - 5x - 14 = (x - 2)(x^2 + 6x + 7)$, so to find the zeros of f, we now solve $(x - 2)(x^2 + 6x + 7) = 0$. We get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. The point of this section is to prepare us to be able to find as many zeros of a polynomial as possible by using division. We begin with a friendly reminder of what we can expect when we divide polynomials.

Theorem: Division Algorithm

Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. Then there exist two unique polynomials, $q(x)$ and $r(x)$, such that

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)},$$

where either $r(x) = 0$ or the degree of r is strictly less than the degree of d. By multiplying both sides of this equation by $d(x)$, we get the equivalent form

$$p(x) = q(x)d(x) + r(x).$$

Each of these polynomials is given a special name:

- $d(x)$ is called the **divisor**
- $p(x)$ is called the **dividend**
- $q(x)$ is called the **quotient**
- $r(x)$ is called the **remainder**

For example, we just performed long division and discovered that $\frac{x^3+4x^2-5x-14}{x-2} = x^2 + 6x + 7$, where the remainder is 0. In this equation, the dividend is $x^3 + 4x^2 - 5x - 14$, the divisor is $x - 2$, the quotient is $x^2 + 6x + 7$, and (as was just mentioned) the remainder is 0.

Example 5.5.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution

$$\begin{array}{r|rrrrr} & & & 5x & - & 2 \\ \hline x+1 & 5x^2 & + & 3x & - & 2 \\ & -\ (5x^2 & + & 5x) & & \\ \hline & & & -2x & - & 2 \\ & - & & (-2x & - & 2) \\ \hline & & & & & 0 \end{array}$$

The quotient is $5x - 2$. The remainder is 0. We write the result as

$$\frac{5x^2 + 3x - 2}{x + 1} = 5x - 2$$

or

$$5x^2 + 3x - 2 = (x + 1)(5x - 2).$$

The division problem in Example 5.5.1 had a remainder of 0 and the result was that our original polynomial (the dividend) had a factor of $(x + 1)$ (our divisor). This is a preview of an important theorem introduced in the next section.

Example 5.5.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Solution

$$
\begin{array}{r|l}
 & \quad 2x^2 + 5x - 7 \\
\cline{2-2}
3x-2 & \big) \; 6x^3 + 11x^2 - 31x + 15 \\
 & -(6x^3 - 4x^2) \\
\cline{2-2}
 & \quad 15x^2 - 31x \\
 & -(15x^2 - 10x) \\
\cline{2-2}
 & \quad -21x + 15 \\
 & -(-21x + 14) \\
\cline{2-2}
 & \quad 1
\end{array}
$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2} = 2x^2+5x-7+\frac{1}{3x-2}.$$

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1 = 6x^3+11x^2-31x+15.$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

? Checkpoint 5.5.2

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Focus on Calculus - Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and can be quite cumbersome. As such, we hunt for a better method. Fortunately, people like Ruffini and Horner have already blazed this trail. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1 (although, it can be adjusted to divide by linear factors like $3x-2$)).

Let's look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the negative signs.

$$
\begin{array}{r|l}
 & \quad x^2 + 6x + 7 \\
\cline{2-2}
x-2 & \big) \; x^3 + 4x^2 - 5x - 14 \\
 & -x^3 + 2x^2 \\
\cline{2-2}
 & \quad 6x^2 - 5x \\
 & -6x^2 + 12x \\
\cline{2-2}
 & \quad 7x - 14 \\
 & -7x + 14 \\
\cline{2-2}
 & \quad 0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them.

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 6x & + & 7 \\
\hline
x-2 & x^3 & + & 4x^2 & - & 5x & - & 14 \\
 & \boxed{-x^3} & + & 2x^2 & & & & \\
\hline
 & & & 6x^2 & - & 5x & & \\
 & & & \boxed{-6x^2} & + & 12x & & \\
\hline
 & & & & & 7x & - & 14 \\
 & & & & & \boxed{-7x} & + & 14 \\
\hline
 & & & & & & & 0
\end{array}
$$

Our algorithm ensures this is always the case, so we can omit them without losing any information. Also, note that the terms we "bring down" (namely the $-5x$ and -14) aren't essential to recopy, so we omit them, too.

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 6x & + & 7 \\
\hline
x-2 & x^3 & + & 4x^2 & - & 5x & - & 14 \\
 & & & 2x^2 & & & & \\
\hline
 & & & 6x^2 & & & & \\
 & & & & & 12x & & \\
\hline
 & & & & & 7x & & \\
 & & & & & & & 14 \\
\hline
 & & & & & & & 0
\end{array}
$$

Now, let's move things up a bit and, for reasons that will become clear in a moment, copy the x^3 into the last row.

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 6x & + & 7 \\
\hline
x-2 & x^3 & + & 4x^2 & - & 5x & - & 14 \\
 & & & 2x^2 & & 12x & & 14 \\
\hline
 & x^3 & & 6x^2 & & 7x & & 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. Take the time to work back through the original division problem. You will find that this is exactly the way we determined the quotient polynomial. This means we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{r|rrrrrrr}
\underline{-2} & x^3 & + & 4x^2 & - & 5x & - & 14 \\
 & & & 2x^2 & & 12x & & 14 \\
\hline
 & x^3 & & 6x^2 & & 7x & & 0
\end{array}
$$

We've streamlined things quite a bit, but we can still do more. Let’s take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$
\begin{array}{r|rrrr}
\underline{2} & 1 & 4 & -5 & -14 \\
 & & 2 & 12 & 14 \\
\hline
 & 1 & 6 & 7 & 0
\end{array}
$$

We have constructed a synthetic division tableau for this polynomial division problem. Let's rework our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then, "bring down" the first coefficient of the dividend.

$$\begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was "brought down" to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|cccc} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third-degree polynomial and divided by a first-degree polynomial, so the quotient is a second-degree polynomial. Hence the quotient is x^2+6x+7. The number in the box is the remainder.

Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x-c$. It is important to note that it works only for these kinds of divisors (you'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1). Also, take note that when a polynomial (of degree at least 1) is divided by $x-c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

How To: Use synthetic division to divide a polynomial by a binomial of the form $x-k$.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 0, the next number from the right has degree 1, and so on.

Example 5.5.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2-3x-36$ by $x-3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x+12$. The remainder is 0. So $x-3$ is a factor of the original polynomial.

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36.$$

Example 5.5.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3+10x^2-6x-20$ by $x+2$.

Solution

The binomial divisor is $x+2$ so $k=-2$. Add each column, multiply the result by -2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} \underline{-2} & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2+2x-10$. The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in Figure 5.5.3 shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.

Figure 5.5.3

Example 5.5.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4+10x^3+7x^2-6$ by $x-1$.

Solution

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} \underline{1}\,| & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3+x^2+8x+8+\frac{2}{x-1}$.

Checkpoint 5.5.5

Use synthetic division to divide $3x^4+18x^3-3x+40$ by $x+7$.

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Example 5.5.6: Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4-3x^3-33x^2+54x$. The length of the solid is given by $3x$ and the width is given by $x-2$. Find the height of the solid.

Solution

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure 5.5.4.

Figure 5.5.4

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$\begin{aligned} V &= l\cdot w\cdot h \\ \implies 3x^4-3x^3-33x^2+54x &= 3x\cdot(x-2)\cdot h \end{aligned}$$

To solve for h, first divide both sides by $3x$.

$$\begin{aligned} \frac{3x\cdot(x-2)\cdot h}{3x} &= \frac{3x^4-3x^3-33x^2+54x}{3x} \\ \implies (x-2)h &= x^3-x^2-11x+18 \end{aligned}$$

Now solve for h using synthetic division.

$$h=\frac{x^3-x^2-11x+18}{x-2}$$

$$\begin{array}{r|rrrr} \underline{2}\,| & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is x^2+x-9 and the remainder is 0. The height of the solid is x^2+x-9.

Checkpoint 5.5.6

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

5.5.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Synthetic division
  - Dividing Polynomials - Synthetic Division

KEY CONCEPTS

DEFINITIONS

- **Dividend**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $p(x)$ is the dividend (the polynomial being divided).
- **Divisor**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $d(x)$ is the divisor (the polynomial by which another polynomial is divided).
- **Quotient**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $q(x)$ is the quotient (the result of the division, excluding the remainder).
- **Remainder**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $r(x)$ is the remainder (the part left over after division if the divisor does not divide the dividend exactly).
- **Synthetic Division**: A shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1 (e.g., $x - k$).

THEOREMS

- **Division Algorithm**: Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. Then there exist two unique polynomials, $q(x)$ and $r(x)$, such that $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

COMMON MISTAKES

- **Sign Errors in Long Division**: When subtracting partial products in polynomial long division, errors often occur if the signs of all terms in the polynomial being subtracted are not correctly changed.
- **Misalignment of Terms in Long Division**: Not aligning like terms vertically during the long division process, which can lead to errors in subtraction or in bringing down the next term.
- **Incorrectly Handling Missing Terms**: When setting up long division or synthetic division, failing to include a zero coefficient as a placeholder for any missing terms (e.g., if a cubic polynomial has no x^2 term, a $0x^2$ should be used).
- **Errors in Synthetic Division Setup**:
  - Using the wrong value for k when dividing by $x - k$. For example, if dividing by $x + 2$, then $k = -2$, not 2.
  - Incorrectly listing the coefficients of the dividend.
- **Arithmetic Errors in Synthetic Division**: Making mistakes in the multiply-add steps of synthetic division.
- **Misinterpreting the Result of Synthetic Division**: Incorrectly determining the degree of the quotient (it is one less than the degree of the dividend when dividing by a linear factor).
- **Using Synthetic Division for Non-Linear Divisors**: Attempting to use synthetic division when the divisor is not a linear factor of the form $x - k$ (or adaptable to it). Synthetic division is a specialized method.

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5.5.2: HOMEWORK

READING QUESTIONS

1. What are the four main components in the Division Algorithm for polynomials: $f(x) = d(x)q(x) + r(x)$? Define each term.
2. When performing long division of polynomials, how do you know when the division process is complete? What condition must the remainder satisfy?
3. If the remainder of a polynomial division is 0, what does this imply about the relationship between the divisor, the dividend, and the quotient?
4. Describe the first step in setting up polynomial long division (e.g., dividing $2x^3 - 3x^2 + 4x + 5$ by $x + 2$).
5. In polynomial long division, after multiplying the current term of the quotient by the entire divisor, what is the next arithmetic operation performed?
6. What is synthetic division, and for what special case of divisor is it typically used?
7. In synthetic division, if you are dividing by $x - k$, what value representing the divisor do you write outside the division symbol? What if you are dividing by $x + k$?
8. Describe the process of bringing down the leading coefficient and the subsequent multiply-add steps in synthetic division.
9. How do you interpret the bottom row of numbers in a completed synthetic division problem to find the quotient and remainder?
10. If a polynomial dividend is missing a term (e.g., no x^2 term in a cubic polynomial), how should this be handled when setting up either long division or synthetic division?
11. How can you check your work after performing polynomial division?
12. Can polynomial division be used to find unknown dimensions of a geometric solid if its volume and some other dimensions are given as polynomials?
13. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?
14. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

HOMEWORK

ALGEBRAIC

For the following exercises, use long division to divide. Specify the quotient and the remainder.

15. $(x^2 + 5x - 1) \div (x - 1)$
16. $(2x^2 - 9x - 5) \div (x - 5)$
17. $(3x^2 + 23x + 14) \div (x + 7)$
18. $(4x^2 - 10x + 6) \div (4x + 2)$
19. $(6x^2 - 25x - 25) \div (6x + 5)$
20. $(-x^2 - 1) \div (x + 1)$
21. $(2x^2 - 3x + 2) \div (x + 2)$
22. $(x^3 - 126) \div (x - 5)$
23. $(3x^2 - 5x + 4) \div (3x + 1)$
24. $(x^3 - 3x^2 + 5x - 6) \div (x - 2)$
25. $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

For the following exercises, use synthetic division to find the quotient.

26. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$
27. $(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$
28. $(6x^3 - 10x^2 - 7x - 15) \div (x + 1)$
29. $(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$
30. $(9x^3 - 9x^2 + 18x + 5) \div (3x - 1)$
31. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$
32. $(-6x^3 + x^2 - 4) \div (2x - 3)$
33. $(2x^3 + 7x^2 - 13x - 3) \div (2x - 3)$
34. $(3x^3 - 5x^2 + 2x + 3) \div (x + 2)$
35. $(4x^3 - 5x^2 + 13) \div (x + 4)$
36. $(x^3 - 3x + 2) \div (x + 2)$
37. $(x^3 - 21x^2 + 147x - 343) \div (x - 7)$
38. $(x^3 - 15x^2 + 75x - 125) \div (x - 5)$
39. $(9x^3 - x + 2) \div (3x - 1)$
40. $(6x^3 - x^2 + 5x + 2) \div (3x + 1)$
41. $(x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1)$
42. $(x^4 - 3x^2 + 1) \div (x - 1)$
43. $(x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3)$
44. $(x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2)$
45. $(x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2)$
46. $(x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5)$
47. $(x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3)$
48. $(4x^4 - 2x^3 - 4x + 2) \div (2x - 1)$
49. $(4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1)$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

50. $x - 2$, $4x^3 - 3x^2 - 8x + 4$
51. $x - 2$, $3x^4 - 6x^3 - 5x + 10$
52. $x + 3$, $-4x^3 + 5x^2 + 8$
53. $x - 2$, $4x^4 - 15x^2 - 4$
54. $x - \frac{1}{2}$, $2x^4 - x^3 + 2x - 1$
55. $x + \frac{1}{3}$, $3x^4 + x^3 - 3x + 1$

GRAPHICAL

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

56. Factor is $x^2 - x + 3$

57. Factor is $x^2 + 2x + 4$

58. Factor is $x^2 + 2x + 5$

59. Factor is $x^2 + x + 1$

60. Factor is x^2+2x+2

For the following exercises, use synthetic division to find the quotient and remainder.

61. $\frac{4x^3-33}{x-2}$
62. $\frac{2x^3+25}{x+3}$
63. $\frac{3x^3+2x-5}{x-1}$
64. $\frac{-4x^3-x^2-12}{x+4}$
65. $\frac{x^4-22}{x+2}$

TECHNOLOGY

For the following exercises, use technology to answer the questions.

66. Consider $\frac{x^k-1}{x-1}$ with $k=1$, 2, and 3. What do you expect the result to be if $k=4$?
67. Consider $\frac{x^k+1}{x+1}$ for $k=1$, 3, and 5. What do you expect the result to be if $k=7$?
68. Consider $\frac{x^4-k^4}{x-k}$ for $k=1$, 2, and 3. What do you expect the result to be if $k=4$?
69. Consider $\frac{x^k}{x+1}$ with $k=1$, 2, and 3. What do you expect the result to be if $k=4$?
70. Consider $\frac{x^k}{x-1}$ with $k=1$, 2, and 3. What do you expect the result to be if $k=4$?

EXTENSIONS

For the following exercises, use synthetic division to determine the quotient involving a complex number.

71. $\frac{x+1}{x-i}$
72. $\frac{x^2+1}{x-i}$
73. $\frac{x+1}{x+i}$
74. $\frac{x^2+1}{x+i}$
75. $\frac{x^3+1}{x-i}$

REAL-WORLD APPLICATIONS

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

76. Length is $x+5$, area is $2x^2+9x-5$.
77. Length is $2x+5$, area is $4x^3+10x^2+6x+15$
78. Length is $3x–4$, area is $6x^4-8x^3+9x^2-9x-4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

79. Volume is $12x^3+20x^2-21x-36$, length is $2x+3$, width is $3x-4$.
80. Volume is $18x^3-21x^2-40x+48$, length is $3x–4$, width is $3x–4$.
81. Volume is $10x^3+27x^2+2x-24$, length is $5x–4$, width is $2x+3$.
82. Volume is $10x^3+30x^2-8x-24$, length is 2, width is $x+3$.

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

83. Volume is $\pi(25x^3-65x^2-29x-3)$, radius is $5x+1$.
84. Volume is $\pi(4x^3+12x^2-15x-50)$, radius is $2x+5$.
85. Volume is $\pi(3x^4+24x^3+46x^2-16x-32)$, radius is $x+4$.

5.6: Zeros of Polynomial Functions

Note to the Instructor (click to expand)

This very dense section contains the Factor and Remainder theorems along with the Rational Roots Theorem (which I called the Rational Zero Theorem and was too lazy to change back to the Rational Roots Theorem). You also cover the Fundamental Theorem of Algebra, Complete Factorization Theorem, and the Conjugate Zeros Theorem. Like I said, this is a dense section. Expect to spend at least two class sessions here. This is necessary theory for Calculus.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Arithmetic**
  - Factors, Prime Factorizations, and Least Common Multiples: Understanding factors of integers is crucial for applying the Rational Zeros Theorem, which involves factors of the constant term and the leading coefficient.
- **Factoring Techniques**
  - Factoring Sums and Differences of Cubes: While not explicitly shown in examples, this technique could be relevant for factoring certain types of cubic polynomials that might arise.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zeros Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Solve real-world applications of polynomial equations.

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as well as sheet cakes to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**.

Theorem: Remainder Theorem

If a polynomial $f(x)$ of degree greater than or equal to one is divided by $x-k$, the remainder is the value $f(k)$.

Proof

Let $f(x)$ be a polynomial of degree one or greater. The Division Algorithm states there exists a quotient, $q(x)$, and remainder, $r(x)$, such that

$$\frac{f(x)}{x-k} = q(x) + \frac{r(x)}{x-k}.$$

Multiplying both sides by $x-k$, we get

$$f(x) = q(x)(x-k) + r(x).$$

We now use this to evaluate $f(k)$.

$$f(k) = q(k)(k-k) + r(k) = q(k)\cdot 0 + r(k) = r(k).$$

Hence, $f(k) = r(k)$. However, the remainder must be of a degree strictly less than that of the divisor. Since the divisor, $x-k$, is of degree one, the remainder must be of degree zero. That is, $r(k) = r$ is constant. Thus,

$$f(k) = r = \text{the remainder when } f(x) \text{ is divided by } x-k.$$

The Remainder Theorem gives us an alternative (and sometimes faster) way to evaluate polynomial functions. Let's see how this works.

Example 5.6.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate

$$f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7 \text{ at } x = 2.$$

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x-2$.

$$\begin{array}{r|rrrrr} \underline{2} & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array}$$

The remainder is 25. Therefore, $f(2) = 25$.

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ \Longrightarrow \quad f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \\ &= 6(16) - (8) - 15(4) + 2(2) - 7 \\ &= 96 - 8 - 60 + 4 - 7 \\ &= 25 \end{aligned}$$

If you pay attention to the amount of work required to evaluate $f(2)$ in Example 5.6.1 using traditional methods, you should notice that it's actually longer than using the Remainder Theorem. Moreover, synthetic division only requires the operations of addition

and multiplication, whereas our traditional function evaluation method requires addition, multiplication, and powers.

Checkpoint 5.6.1

Use the Remainder Theorem to evaluate

$$f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2 \text{ at } x = -3.$$

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors.

Theorem: Factor Theorem

Let $f(x)$ be a polynomial function of degree greater than or equal to one. k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

Proof

Let $f(x)$ be a polynomial function of degree greater than or equal to one.

($\Longrightarrow$): Suppose k is a zero of $f(x)$. Then $f(k) = 0$. By the Remainder Theorem, if $f(x)$ is divided by $x-k$, then

$$\frac{f(x)}{x-k} = q(x) + \frac{r(x)}{x-k} = q(x) + \frac{f(k)}{x-k} = q(x) + \frac{0}{x-k} = q(x).$$

Multiplying both sides by $x-k$, we get

$$f(x) = q(x)(x-k).$$

Hence, $x-k$ is a factor of $f(x)$.

($\Longleftarrow$): Suppose $x-k$ is a factor of $f(x)$. Then

$$f(x) = (x-k)g(x),$$

for some polynomial $g(x)$. Therefore,

$$f(k) = (k-k)g(k) = 0 \cdot g(k) = 0.$$

Thus, $f(k) = 0$, which means k is a zero of $f(x)$.

We have been "dancing around" the Factor Theorem for a couple of sections now; however, it wasn't until now that we formally stated the result. As we will soon see, a polynomial of degree n with complex-numbered coefficients will have n complex-numbered zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

Example 5.6.2: Using the Factor Theorem to Solve a Polynomial Equation

Show that $(x+2)$ is a factor of $x^3 - 6x^2 - x + 30$. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

Solution

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)\left(x^2-8x+15\right).$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5).$$

By the Factor Theorem, the zeros of x^3-6x^2-x+30 are -2, 3, and 5.

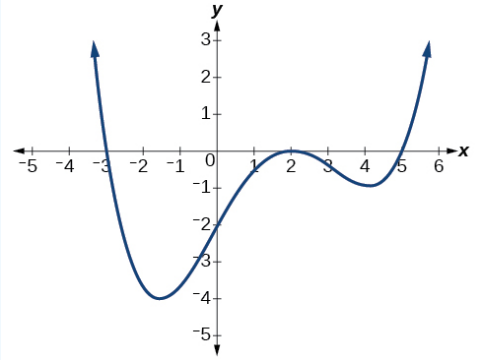
Checkpoint 5.6.2

Use the Factor Theorem to find the zeros of $f(x)=x^3+4x^2-4x-16$ given that $(x-2)$ is a factor of the polynomial.

Example 5.6.3: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 5.6.1.

Figure 5.6.1



Solution

This graph has three x-intercepts: $x=-3$, 2, and 5. The y-intercept is located at $(0,-2)$. At $x=-3$ and $x=5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x=2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x)=a(x+3)(x-2)^2(x-5).$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0,-2)$, to solve for a.

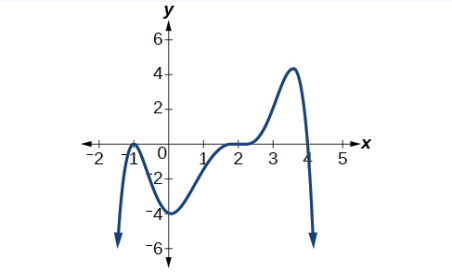
$$\begin{aligned} & f(0) = a(0+3)(0-2)^2(0-5) && \text{(substituting)} \\ \Longrightarrow\quad & -2 = a(0+3)(0-2)^2(0-5) && \text{(substituting)} \\ \Longrightarrow\quad & -2 = -60a && \text{(simplifying)} \\ \Longrightarrow\quad & \frac{1}{30} = a && \text{(dividing both sides by } -60) \end{aligned}$$

The graphed polynomial appears to represent the function $f(x)=\frac{1}{30}(x+3)(x-2)^2(x-5)$.

Checkpoint 5.6.3

Given the graph shown in Figure 5.6.2, write a formula for the function shown.

Figure 5.6.2



Using the Rational Zeros Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zeros Theorem** (also called the **Rational Roots Theorem**) helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial. Before stating the Rational Zeros Theorem, let's motivate *why* it works with a simple example.

Consider a quadratic function with two zeros, $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0 and then construct the original quadratic function absent its stretching factor.

$$\begin{aligned}
& x - \frac{2}{5} = 0 \quad \text{or} \quad x - \frac{3}{4} = 0 && \text{(setting each factor equal to 0)} \\
\Longrightarrow \quad & 5x - 2 = 0 \quad \text{or} \quad 4x - 3 = 0 && \text{(multiplying both sides of the equation to eliminate fractions)} \\
\Longrightarrow \quad & f(x) = (5x - 2)(4x - 3) && \text{(creating the quadratic function, multiplying the factors)} \\
\Longrightarrow \quad & f(x) = 20x^2 - 23x + 6 && \text{(expanding the polynomial)} \\
\Longrightarrow \quad & f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)
\end{aligned}$$

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

If we used this single example to infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient, we would be (somewhat) citing the Rational Zeros Theorem. However, as we know, examples are not proofs!

Theorem: Rational Zeros Theorem

Suppose

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is a polynomial of degree n with $n \geq 1$, and a_0, a_1, ..., a_n are *integers*. If r is a *rational* zero of f, then r must be of the form $\pm \frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

Proof

Suppose c is a zero of f and $c = \frac{p}{q}$ in lowest terms. This means p and q have no common factors. Since $f(c) = 0$, we have

$$f(c) = f\left(\frac{p}{q}\right) = a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \ldots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

That is,

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \ldots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides of this equation by q^n, we clear the denominators to get

$$a_n p^n + a_{n-1} p^{n-1} q + \ldots + a_1 p q^{n-1} + a_0 q^n = 0.$$

Keeping the term $a_n p^n$ on the left side and moving everything else to the right side, we get

$$a_n p^n = -a_{n-1} p^{n-1} q - \ldots - a_1 p q^{n-1} - a_0 q^n.$$

However, each term on the right side has a factor of q in common. Therefore,

$$a_n p^n = \left(-a_{n-1} p^{n-1} - \ldots - a_1 p q^{n-2} - a_0 q^{n-1}\right) q.$$

Recall that p, q, and a_k are integers (for all k). Moreover, $n \geq 1$. This shows that $-a_{n-1} p^{n-1} - \ldots - a_1 p q^{n-2} - a_0 q^{n-1}$ must be an integer (can you see why?). Hence, $a_n p^n$ is an integer multiple of q. Moreover, since $a_n p^n = \left(a_n p^{n-1}\right) p$, $a_n p^n$ is also an integer multiple of p. However, p and q have no common factors. Thus, a_n must be the integer multiple of q. This shows that q is a factor of the leading coefficient a_n.

If we rearrange the equation

$$a_n p^n + a_{n-1} p^{n-1} q + \ldots + a_1 p q^{n-1} + a_0 q^n = 0$$

as

$$a_0 q^n = -a_n p^n - a_{n-1} p^{n-1} q - \ldots - a_1 p q^{n-1},$$

we can play the same game and conclude a_0 is a multiple of p, and we have the result we desire.

When the leading coefficient is 1, the Rational Zeros Theorem simplifies to stating that the possible rational zeros are the factors of the constant term.

Caution: The Rational Zeros Theorem Requires Integer Coefficients

An often overlooked requirement to using the Rational Zeros Theorem is that the coefficients of the polynomial must be integers. Do not make the mistake of trying to use the Rational Zeros Theorem on polynomials with coefficients of the form $\frac{3}{7}$, $\sqrt{2}$, or $3 + 2i$.

Example 5.6.4: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$

Solution

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, -4, and the factors of the leading coefficient, 2.

The factors of -4 are $p = \pm 1, \pm 2, \pm 4$.

The factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of -4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{1}{2} \quad \frac{p}{q} = \pm\frac{2}{1}, \pm\frac{2}{2} \quad \frac{p}{q} = \pm\frac{4}{1}, \pm\frac{4}{2}.$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}.$$

Example 5.6.5: Using the Rational Zeros Theorem to Find Rational Zeros

Use the Rational Zeros Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

Solution

The Rational Zeros Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } 1}{\text{factor of } 2} \end{aligned}$$

The factors of 1 are ± 1 and the factors of 2 are ± 1 and ± 2. The possible values for $\frac{p}{q}$ are ± 1 and $\pm \frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

$$\begin{aligned} f(-1) &= 2(-1)^3 + (-1)^2 - 4(-1) + 1 = 4 \\ f(1) &= 2(1)^3 + (1)^2 - 4(1) + 1 = 0 \\ f\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 1 = 3 \\ f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 1 = -\frac{1}{2} \end{aligned}$$

Of those, -1, $-\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

Checkpoint 5.6.5

Use the Rational Zeros Theorem to find the rational zeros of

$$f(x) = x^3 - 5x^2 + 2x + 1.$$

Finding the Zeros of Polynomial Functions

The Rational Zeros Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use synthetic division repeatedly to determine all of the zeros of a polynomial function.

Example 5.6.6: Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Solution

First, note that the coefficients of the given polynomial function are all integers - remember, we cannot use the Rational Zeros Theorem if this is not the case. The Rational Zeros Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of -1 and q is a factor of 4.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } -1}{\text{factor of } 4} \end{aligned}$$

The factors of -1 are ± 1 and the factors of 4 are $\pm 1, \pm 2$, and ± 4. The possible values for $\frac{p}{q}$ are ± 1, $\pm \frac{1}{2}$, and $\pm \frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

$$\begin{array}{r|rrrr} \underline{1} & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)\left(4x^2+4x+1\right).$$

The quadratic is a perfect square. Therefore, $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

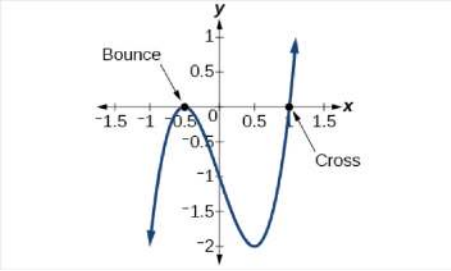
We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x+1=0 \implies x=-\frac{1}{2}.$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Investigating Example $5.6.6$ a little more, we look at the graph of the function f in Figure $5.6.3$. Notice, at $x=-0.5$, the graph bounces off the x-axis, indicating the even multiplicity $(2,4,6,...)$ for the zero -0.5. At $x=1$, the graph crosses the x-axis, indicating the odd multiplicity $(1,3,5,...)$ for the zero $x=1$.

Figure 5.6.3



Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Theorem: Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued coefficients, then $f(x)$ has at least one complex-valued zero.

Proof

The proof of this theorem is beyond the scope of lower-division mathematics.

The Fundamental Theorem of Algebra is an example of an "existence theorem" in Mathematics. It guarantees the existence of at least one complex-valued zero but gives us no algorithm to use in finding it. It took mathematicians hundreds of years to prove the theorem in its full generality, and some of that history is recorded here. Note that the Fundamental Theorem of Algebra applies to not only polynomial functions with real coefficients but to those with complex number coefficients as well.

Suppose f is a polynomial function of degree four, and $f(x)=0$. The Fundamental Theorem of Algebra states that there is at least one complex-valued solution, call it z_1. By the Factor Theorem, we can write $f(x)$ as a product of $x-z_1$ and a polynomial quotient. Since $x-z_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it z_2. So we can write the polynomial quotient as a

product of $x - z_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them, and each one will yield a factor of $f(x)$.

In truth, "a polynomial with complex-valued coefficients" means that the coefficients can be any number we can imagine - from 2 to $\sqrt{2}$ to $\frac{1}{2}$ to $2 + 2i$. Therefore, most people write the Fundamental Theorem of Algebra as, "If $f(x)$ is a polynomial of degree $n \geq 1$, then $f(x)$ has at least one complex zero," thereby omitting the needless "with complex-valued coefficients."

Although the Fundamental Theorem of Algebra is a major theorem in Mathematics, it is the following corollary that makes the theorem useful.

Corollary: Complete Factorization Theorem

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued coefficients, then $f(x)$ has exactly n linear factors over the complex number system. That is,

$$f(x) = a(x - z_1)(x - z_2)...(x - z_n),$$

where z_1, z_2, ..., z_n are complex numbers and $a \in \mathbb{R}$.

Therefore, $f(x)$ has n complex-valued zeros if we allow for multiplicities.

Proof

The proof of this theorem is done using mathematical induction and the Fundamental Theorem of Algebra. Since this topic is usually covered *prior* to learning proofs by mathematical induction, the proof is omitted.

As with the Fundamental Theorem of Algebra, the Complete Factorization Theorem can be shortened to, "If $f(x)$ is a polynomial of degree $n \geq 1$, then $f(x)$ has exactly n linear factors over the complex number system."

Q & A

Does every polynomial have at least one imaginary zero?

No. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

Example 5.6.7: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of

$$f(x) = 3x^3 + 9x^2 + x + 3.$$

Solution

The Rational Zeros Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } 3}{\text{factor of } 3} \end{aligned}$$

The factors of 3 are ± 1 and ± 3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ± 3, ± 1, and $\pm \frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with -3.

$$\begin{array}{r|rrrr} -3 & 3 & 9 & 1 & 3 \\ & & -9 & 0 & -3 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

Dividing by $(x + 3)$ gives a remainder of 0, so -3 is a zero of the function. The polynomial can be written as

$$(x+3)\left(3x^2+1\right).$$

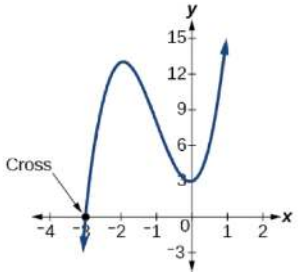
We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$\begin{aligned} 3x^2+1 &= 0 \\ \Longrightarrow \quad x^2 &= -\frac{1}{3} \\ \Longrightarrow \quad x &= \pm\sqrt{-\frac{1}{3}} \\ &= \pm\frac{i\sqrt{3}}{3} \end{aligned}$$

The zeros of $f(x)$ are -3 and $\pm\frac{i\sqrt{3}}{3}$.

Figure 5.6.4 shows the graph of the function f from Example 5.6.7. Notice that, at $x=-3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x=-3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. Either the multiplicity of $x=-3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x=-3$ is three. Either way, our result is correct.

Figure 5.6.4



? Checkpoint 5.6.7

Find the zeros of

$$f(x)=2x^3+5x^2-11x+4.$$

Using the Complete Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the Fundamental Theorem of Algebra, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The Complete Factorization Theorem tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Theorem: Conjugate Zeros Theorem

If the polynomial function f has *real* coefficients and a complex zero of the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero of f.

Proof

The proof requires several theorems from complex numbers. These are mentioned within the proof. If $z=a+bi$ is a zero of a polynomial function $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, then

$$
\begin{aligned}
f(\overline{z}) &= a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1\overline{z} + a_0 && \text{(substituting)} \\
&= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \cdots + a_1\overline{z} + a_0 && \left(\overline{z^n} = (\overline{z})^n\right) \\
&= \overline{a_n}\overline{z^n} + \overline{a_{n-1}}\overline{z^{n-1}} + \cdots + \overline{a_1}\overline{z} + \overline{a_0} && (\text{if } a_i \in \mathbb{R}, \text{ then } a_i = \overline{a_i}) \\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && (\overline{z_1}\cdot\overline{z_2} = \overline{z_1\cdot z_2}) \\
&= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} && \left(\overline{z_1} + \overline{z_2} + \cdots + \overline{z_k} = \overline{z_1 + z_2 + \cdots + z_k}\right) \\
&= \overline{f(z)} \\
&= \overline{0} && (z \text{ is a zero of } f(x)) \\
&= 0 && \left(\overline{k} = k \text{ for real numbers } k\right)
\end{aligned}
$$

Hence, $\overline{z}$ (the conjugate) is also a zero of $f(x)$.

Example 5.6.8: Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of -3, 2, and i, such that $f(-2) = 100$.

Solution

Because $x = i$ is a zero, by the Complex Conjugate Theorem $x = -i$ is also a zero. The polynomial must have factors of $(x + 3)$, $(x - 2)$, $(x - i)$, and $(x + i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$
\begin{aligned}
f(x) &= a(x+3)(x-2)(x-i)(x+i) \\
&= a\left(x^2 + x - 6\right)\left(x^2 + 1\right) \\
&= a\left(x^4 + x^3 - 5x^2 + x - 6\right)
\end{aligned}
$$

We need to find a to ensure $f(-2) = 100$. Substitute $x = -2$ and $f(2) = 100$ into $f(x)$.

$$
\begin{aligned}
100 &= a\left((-2)^4 + (-2)^3 - 5(-2)^2 + (-2) - 6\right) \\
\implies 100 &= a(-20) \\
\implies -5 &= a
\end{aligned}
$$

So the polynomial function is

$$f(x) = -5\left(x^4 + x^3 - 5x^2 + x - 6\right)$$

or

$$f(x) = -5x^4 - 5x^3 + 25x^2 - 5x + 30.$$

In Example 5.6.8, we found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

Q & A

If $2 + 3i$ were given as a zero of a polynomial with real coefficients, would $2 - 3i$ also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

? Checkpoint 5.6.8

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1) = 10$.

It's probably good to summarize the theorems and corollaries we have encountered so far in this section, along with their requirements.

Table 5.6.1

Theorem or Corollary	Requirement	Result
Rational Zeros Theorem	Coefficients must be integers	Rational zeros of the polynomial are always ratios of factors of the constant term and the leading coefficient.
Fundamental Theorem of Algebra	Coefficients can be any number	The polynomial has at least one complex zero.
Complete Factorization Theorem	Coefficients can be any number	The polynomial can be factored into n linear factors over the complex number system.
Conjugate Zeros Theorem	Coefficients must be real numbers	If $a + bi$ is a zero, then $a - bi$ is also a zero.

Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

✓ Example 5.6.9

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as well as sheet cakes to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

Solution

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh$. We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w + 4$. We were given that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w$. Let's write the volume of the cake in terms of width of the cake.

$$\begin{aligned} V &= (w+4)(w)\left(\frac{1}{3}w\right) \\ &= \frac{1}{3}w^3 + \frac{4}{3}w^2 \end{aligned}$$

Substitute the given volume into this equation.

$$\begin{aligned} 351 &= \frac{1}{3}w^3 + \frac{4}{3}w^2 && \text{(substituting)} \\ \Longrightarrow \quad 1053 &= w^3 + 4w^2 && \text{(multiplying both sides by 3)} \\ \Longrightarrow \quad 0 &= w^3 + 4w^2 - 1053 && \text{(subtracting 1053 from both sides)} \end{aligned}$$

The Rational Zeros Theorem tells us that the possible rational zeros are ± 1, ± 3, ± 9, ± 13, ± 27, ± 39, ± 81, ± 117, ± 351, and ± 1053. We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a

cake, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check $x = 1$.

$$\begin{array}{r|rrrr} \underline{1} & 1 & 4 & 0 & -1053 \\ & & 1 & 5 & 5 \\ \hline & 1 & 5 & 5 & -1048 \end{array}$$

Since 1 is not a solution, we will check $x = 3$.

$$\begin{array}{r|rrrr} \underline{3} & 1 & 4 & 0 & -1053 \\ & & 3 & 21 & 63 \\ \hline & 1 & 7 & 21 & -990 \end{array}$$

Since 3 is not a solution either, we will test $x = 9$.

$$\begin{array}{r|rrrr} \underline{9} & 1 & 4 & 0 & -1053 \\ & & 9 & 117 & 1053 \\ \hline & 1 & 13 & 117 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

$$l = w + 4 = 9 + 4 = 13 \text{ and } h = \frac{1}{3}w = \frac{1}{3}(9) = 3.$$

>The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches.

? Checkpoint 5.6.9

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

5.6.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Remainder and Factor Theorems
  - Dividing Polynomials - The Remainder and Factor Theorems
- Rational Zeros Theorem (Rational Roots Theorem)
  - Real Zeros of Polynomials - Motivating the Rational Roots Theorem
  - Real Zeros of Polynomials - Proof of the Rational Roots Theorem
  - College Algebra - Polynomial Functions - Rational Roots Theorem - Example
  - Real Zeros of Polynomials - Examples Using the Rational Roots Theorem
- Fundamental Theorem of Algebra
  - Complex Zeros and the Fundamental Theorem of Algebra - Intro to the Fundamental Theorem of Algebra
  - Complex Zeros and the Fundamental Theorem of Algebra - Using the Fundamental Theorem of Algebra
- Complete Factorization Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Complete Factorization Theorem
- Complex Conjugate Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Zeros and Conjugate Zeros Theorems

KEY CONCEPTS

THEOREMS

- **Remainder Theorem**: If a polynomial $f(x)$ of degree greater than or equal to one is divided by $x-k$, the remainder is the value $f(k)$.
- **Factor Theorem**: Let $f(x)$ be a polynomial function of degree greater than or equal to one. k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.
- **Rational Zeros Theorem (Rational Roots Theorem)**: Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is a polynomial of degree $n\geq 1$, and $a_0,a_1,\ldots,a_n$ are integers. If $r=\pm\frac{p}{q}$ is a rational zero of f, in lowest terms, then p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.
- **Fundamental Theorem of Algebra**: If $f(x)$ is a polynomial of degree $n\geq 1$ with complex-valued coefficients, then $f(x)$ has at least one complex-valued zero.
- **Complete Factorization Theorem (Corollary to Fundamental Theorem of Algebra)**: If $f(x)$ is a polynomial of degree $n\geq 1$ with complex-valued coefficients, then $f(x)$ has exactly n linear factors over the complex number system. That is, $f(x)=a(x-z_1)(x-z_2)\ldots(x-z_n)$, where $z_1,z_2,\ldots,z_n$ are complex numbers (the zeros) and $a\in\mathbb{R}$. Therefore, $f(x)$ has n complex-valued zeros if we allow for multiplicities.
- **Conjugate Zeros Theorem (Complex Conjugate Theorem)**: If the polynomial function f has real coefficients and a complex zero of the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero of f.

COMMON MISTAKES

- **Misapplying the Rational Zeros Theorem**: Attempting to use the Rational Zeros Theorem when the polynomial coefficients are not integers.
- **Stopping After Finding One Zero**: When a zero is found using the Rational Zeros Theorem and synthetic division, forgetting to use the quotient (depressed polynomial) to find the remaining zeros.
- **Not Accounting for Multiplicity of Zeros**: When factoring or listing zeros, not indicating the multiplicity if a zero is repeated.
- **Forgetting Complex Conjugate Zeros**: If a polynomial has real coefficients and a complex zero $a+bi$ is found or given, forgetting that its conjugate $a-bi$ must also be a zero.
- **Incorrectly Constructing a Polynomial from Zeros**: Forgetting the leading coefficient a when constructing a polynomial from its zeros.

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5.6.2: HOMEWORK

READING QUESTIONS

1. What does the Remainder Theorem state about dividing a polynomial $f(x)$ by $x-k$? How can it be used to evaluate $f(k)$?
2. Explain the Factor Theorem. How does it relate the zeros of a polynomial to its factors?
3. What is the Rational Zeros Theorem used for? What conditions must the coefficients of the polynomial satisfy to use this theorem?
4. If $\frac{p}{q}$ is a possible rational zero of $f(x)=a_nx^n+\cdots+a_0$, what do p and q represent?
5. According to the Fundamental Theorem of Algebra, how many complex zeros does a polynomial of degree $n>0$ have (counting multiplicities)?
6. What does the Complete Factorization Theorem say about factoring a polynomial of degree $n>0$?
7. What is the Complex Conjugate Theorem? If $2+3i$ is a zero of a polynomial with real coefficients, what other zero must it have?
8. If you are asked to find a polynomial of a certain degree with given zeros, what is the general approach using its factors? What role does the leading coefficient a play?
9. When using synthetic division to test for zeros from the list generated by the Rational Zeros Theorem, what remainder value indicates that a number is indeed a zero?
10. If you find a zero k of a polynomial $f(x)$, how can you use this information to find other zeros? (Hint: consider the quotient polynomial).
11. Describe a use for the Remainder Theorem.
12. Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.
13. What is the difference between rational and real zeros?
14. If synthetic division reveals a zero, why should we try that value again as a possible solution?

HOMEWORK

ALGEBRAIC

For the following exercises, use the Remainder Theorem to find the remainder.

15. $(x^4-9x^2+14)\div(x-2)$
16. $(3x^3-2x^2+x-4)\div(x+3)$
17. $(x^4+5x^3-4x-17)\div(x+1)$
18. $(-3x^2+6x+24)\div(x-4)$
19. $(5x^5-4x^4+3x^3-2x^2+x-1)\div(x+6)$
20. $(x^4-1)\div(x-4)$
21. $(3x^3+4x^2-8x+2)\div(x-3)$
22. $(4x^3+5x^2-2x+7)\div(x+2)$

For the following exercises, use the given factor and the Factor Theorem to find all real zeros for the given polynomial function.

23. $f(x)=2x^3-9x^2+13x-6;\quad x-1$
24. $f(x)=2x^3+x^2-5x+2;\quad x+2$
25. $f(x)=3x^3+x^2-20x+12;\quad x+3$
26. $f(x)=2x^3+3x^2+x+6;\quad x+2$
27. $f(x)=-5x^3+16x^2-9;\quad x-3$
28. $x^3+3x^2+4x+12;\quad x+3$
29. $4x^3-7x+3;\quad x-1$
30. $2x^3+5x^2-12x-30;\quad 2x+5$

For the following exercises, use the Rational Zeros Theorem to find all real zeros.

31. $x^3-3x^2-10x+24=0$
32. $2x^3+7x^2-10x-24=0$
33. $x^3+2x^2-9x-18=0$
34. $x^3+5x^2-16x-80=0$
35. $x^3-3x^2-25x+75=0$
36. $2x^3-3x^2-32x-15=0$
37. $2x^3+x^2-7x-6=0$
38. $2x^3-3x^2-x+1=0$
39. $3x^3-x^2-11x-6=0$
40. $2x^3-5x^2+9x-9=0$
41. $2x^3-3x^2+4x+3=0$
42. $x^4-2x^3-7x^2+8x+12=0$
43. $x^4+2x^3-9x^2-2x+8=0$
44. $4x^4+4x^3-25x^2-x+6=0$
45. $2x^4-3x^3-15x^2+32x-12=0$
46. $x^4+2x^3-4x^2-10x-5=0$
47. $4x^3-3x+1=0$
48. $8x^4+26x^3+39x^2+26x+6$

For the following exercises, find all complex solutions (real and non-real).

49. $x^3+x^2+x+1=0$
50. $x^3-8x^2+25x-26=0$
51. $x^3+13x^2+57x+85=0$
52. $3x^3-4x^2+11x+10=0$
53. $x^4+2x^3+22x^2+50x-75=0$
54. $2x^3-3x^2+32x+17=0$

NUMERIC

For the following exercises, list all possible rational zeros for the functions.

55. $f(x)=x^4+3x^3-4x+4$
56. $f(x)=2x^3+3x^2-8x+5$
57. $f(x)=3x^3+5x^2-5x+4$
58. $f(x)=6x^4-10x^2+13x+1$
59. $f(x)=4x^5-10x^4+8x^3+x^2-8$

For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

60.

61.

62.

63.

64.

For the following exercises, use the graphs to write a polynomial function of least degree.

65.

66.

67.

TECHNOLOGY

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

68. $f(x) = 6x^3 - 7x^2 + 1$
69. $f(x) = 4x^3 - 4x^2 - 13x - 5$
70. $f(x) = 8x^3 - 6x^2 - 23x + 6$
71. $f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$
72. $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

EXTENSIONS

For the following exercises, use the given information about the polynomial graph to write the equation.

73. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. y-intercept at $(0, -4)$.
74. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. y-intercept at $(0, 6)$.
75. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. y-intercept at $(0, 9)$.
76. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. y-intercept at $(0, -3)$.
77. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$.
78. Degree 3. Zeros at $x = 4$, $x = 3$, and $x = 2$. y-intercept at $(0, -24)$.
79. Degree 3. Zeros at $x = -3$, $x = -2$ and $x = 1$. y-intercept at $(0, 12)$.
80. Degree 5. Roots of multiplicity 2 at $x = -3$ and $x = 2$ and a root of multiplicity 1 at $x = -2$. y-intercept at $(0, 4)$.
81. Degree 4. Roots of multiplicity 2 at $x = \frac{1}{2}$ and roots of multiplicity 1 at $x = 6$ and $x = -2$. y-intercept at $(0, 18)$.
82. Double zero at $x = -3$ and triple zero at $x = 0$. Passes through the point $(1, 32)$.

For the following exercises, construct a polynomial function of least degree possible using the given information.

83. Real roots: –1, 1, 3 and $(2, f(2)) = (2, 4)$
84. Real roots: –1, 1 (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$
85. Real roots: $-2, \frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$
86. Real roots: $-\frac{1}{2}, 0, \frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$
87. Real roots: –4, –1, 1, 4 and $(-2, f(-2)) = (-2, 10)$

REAL-WORLD APPLICATIONS

For the following exercises, find the dimensions of the box described.

88. The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.
89. The length, width, and height are consecutive whole numbers. The volume is 120 cubic inches.
90. The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.
91. The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.
92. The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

For the following exercises, find the dimensions of the right circular cylinder described.

93. The radius is 3 inches more than the height. The volume is 16π cubic inches.
94. The height is one less than one half the radius. The volume is 72π cubic meters.
95. The radius and height differ by one meter. The radius is larger and the volume is 48π cubic meters.
96. The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.
97. The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

5.7: Rational Functions

Note to the Instructor (click to expand)

This section covers rational functions and their graphs. You should use the Calculus "arrow" notation as often as possible so that it is not foreign to students once they get into Calculus. Solving rational inequalities by using graphs is also covered (and is a great reinforcement of the need to graph).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Equations**
  - Solving Rational Equations: While not directly solving equations in this section, the process of finding x-intercepts (setting numerator to zero) and vertical asymptotes/holes (setting denominator to zero) involves solving polynomial equations derived from the rational function.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the Difference Quotient of rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.
- Solve a rational inequality by graphing.

Hawk A.I. Section-Specific Tutor

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Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x.

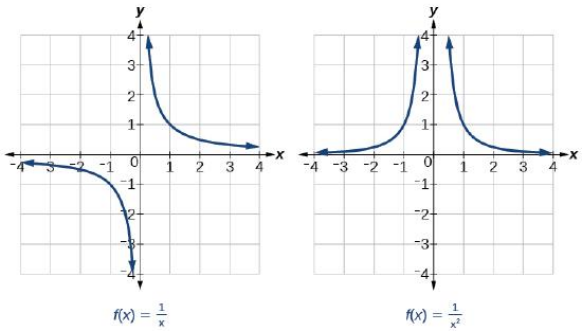
The average cost function, which yields the average cost per item for x items produced, is $f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$. Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore **rational functions**, which have variables in the denominator.

Focus on Calculus - Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Let's examine these graphs and notice some of their features.

Figure 5.7.1
Graphs of Toolkit Functions



Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

1. On the left branch of the graph, the curve approaches the x-axis ($y = 0$) as $x \to -\infty$.
2. As the graph approaches $x = 0$ from the left ($x \to 0^-$), the curve drops ($f(x) \to -\infty$), but as we approach zero from the right ($x \to 0^+$), the curve rises ($f(x) \to \infty$).
3. Finally, on the right branch of the graph, the curves approaches the x-axis ($y = 0$) as $x \to \infty$.

To summarize, we use arrow notation to show that x or $f(x)$ is approaching a particular value. See Table 5.7.1.

Table 5.7.1

Defining Arrow Notation

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 5.7.2.

Table 5.7.2

Numerical Investigation of $f(x) = \frac{1}{x}$ as $x \to 0^-$.

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

We would indicate this behavior using the arrow notation: "as $x \to 0^-$, $f(x) \to -\infty$."

On the other hand, as the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 5.7.3.

Table 5.7.3

Numerical Investigation of $f(x) = \frac{1}{x}$ as $x \to 0^+$.

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

This is interpreted using the following arrow notation: "as $x \to 0^+$, $f(x) \to \infty$." See Figure **5.7.2**.

Figure 5.7.2

Graphical Investigation of $f(x) = \frac{1}{x}$.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See Figure **5.7.3**.

Figure 5.7.3

Showing the vertical asymptote, $x = 0$, for the function $f(x) = \frac{1}{x}$.

Definition: Vertical Asymptote

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a from either the left or the right. We write

$f(x) \to \infty$ as $x \to a^-$ if the graph increases without bound as x approaches a from the left,

$f(x) \to -\infty$ as $x \to a^-$ if the graph decreases without bound as x approaches a from the left,

$f(x) \to \infty$ as $x \to a^+$ if the graph increases without bound as x approaches a from the right,

and

$f(x) \to -\infty$ as $x \to a^+$ if the graph decreases without bound as x approaches a from the right.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. Likewise, as the values of x approach negative infinity, the function values approach 0. See Figure 5.7.4. Symbolically, using arrow notation, we write: "as $x \to \infty$, $f(x) \to 0$," and "as $x \to -\infty$, $f(x) \to 0$."

Figure 5.7.4

Graphical Investigation of $f(x) = \frac{1}{x}$.

Based on this overall behavior (and the graph), we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 5.7.5.

Figure 5.7.5

Showing the vertical and horizontal asymptotes for the function $f(x) = \frac{1}{x}$.

Definition: Horizontal Asymptote

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$f(x) \to b$ as $x \to \infty$ if the graph approaches the line $y = b$ as x increases without bound

and

$f(x) \to b$ as $x \to -\infty$ if the graph approaches the line $y = b$ as x decreases without bound.

Example 5.7.1: Using Arrow Notation

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 5.7.6.

Figure 5.7.6

A transformation of $y = \frac{1}{x}$.

Solution

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$. We can see that as $x \to 2^-$, $f(x) \to -\infty$, and as $x \to 2^+$, $f(x) \to \infty$. Moreover, as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. Likewise, as the inputs increase without bound, the graph levels off at 4. That is, $f(x) \to 4$ as $x \to \pm\infty$.

? Checkpoint 5.7.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

✓ Example 5.7.2: Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2}.$$

The graph of the shifted function is displayed in Figure 5.7.7.

Figure 5.7.7

The graph of $f(x) = \frac{1}{x+2} + 3$.

As $x \to -2^-$, $f(x) \to -\infty$. Therefore, the graph has a vertical asymptote at $x = -2$. We could have arrived at this same conclusion by considering that $f(x) \to \infty$ as $x \to -2^+$.

As $x \to \pm\infty$, $f(x) \to 3$. Thus, the graph has a horizontal asymptote at $y = 3$.

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

? Checkpoint 5.7.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solving Applied Problems Involving Rational Functions

In Example 5.7.2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Definition: Rational Function

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0}, \text{ where } Q(x) \neq 0.$$

Example 5.7.3: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

Solution

Let t be the number of minutes since the tap opened. Since the water flows in at 10 gallons per minute, and the sugar is poured in at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\text{water:} \quad W(t) = 100 + 10t \text{ in gallons}$$
$$\text{sugar:} \quad S(t) = 5 + 1t \text{ in pounds}$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5+t}{100+10t}.$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t = 12$.

$$C(12) = \frac{5+12}{100+10(12)} = \frac{17}{220}.$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0) = \frac{5+0}{100+10(0)} = \frac{1}{20}.$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$, the concentration is greater after 12 minutes than at the beginning.

In Example 5.7.3, we found the horizontal asymptote by dividing the leading coefficient in the numerator by the leading coefficient in the denominator, which led to $y = \frac{1}{10} = 0.1$. This means the concentration, C, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

? Checkpoint 5.7.3

There are $1,200$ freshmen and $1,500$ sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

While not a true "application," finding the Difference Quotient (as we have been stating over and over again) is something you will repeatedly need to do in Calculus. Let's revisit that topic right now.

Example 5.7.4: Revisiting the Difference Quotient

Find the Difference Quotient of the following function.

$$R(t) = \frac{2t-1}{3t+7}$$

Solution

$$\begin{aligned}
\frac{R(t+h)-R(t)}{h} &= \frac{\frac{2(t+h)-1}{3(t+h)+7} - \frac{2t-1}{3t+7}}{h} && \text{(substituting)}\\
&= \frac{\frac{2t+2h-1}{3t+3h+7} - \frac{2t-1}{3t+7}}{h} && \text{(distributing)}\\
&= \frac{\frac{2t+2h-1}{3t+3h+7} - \frac{2t-1}{3t+7}}{h} \cdot \frac{(3t+3h+7)(3t+7)}{(3t+3h+7)(3t+7)} && \text{(simplifying the compound rational expression)}\\
&= \frac{(2t+2h-1)(3t+7)-(2t-1)(3t+3h+7)}{h(3t+3h+7)(3t+7)} && \text{(distributing)}\\
&= \frac{6t^2+14t+6ht+14h-3t-7-(6t^2+6ht+14t-3t-3h-7)}{h(3t+3h+7)(3t+7)} && \text{(distributing (numerator only))}\\
&= \frac{6t^2+14t+6ht+14h-3t-7-6t^2-6ht-14t+3t+3h+7}{h(3t+3h+7)(3t+7)} && \text{(distributing the negative (numerator only))}\\
&= \frac{17h}{h(3t+3h+7)(3t+7)} && \text{(combining like terms)}\\
&= \frac{17\cancel{h}^{1}}{\cancel{h}^{1}(3t+3h+7)(3t+7)} && \text{(canceling like factors)}\\
&= \frac{17}{(3t+3h+7)(3t+7)} && \text{(simplifying)}
\end{aligned}$$

Caution: Know When Not to Distribute

With difference quotients, you will commonly need to multiply the numerator **and** denominator of an expression by the LCD of all fractions within a compound fraction (a.k.a. simplifying compound fractions). When doing so, knowing what to distribute and what not to distribute is imperative.

Simplifying Compound Fractions

The entire point of multiplying the numerator **and** denominator of your compound fraction by the LCD of all the minor fractions is to "get rid of" denominators in those minor fractions; however, with difference quotients, it is common that the denominator of the entire (major) fraction is only h. In this case, **do not** distribute out the denominator! Doing so will complicate the mathematics, and you will lose visibility of factors that cancel.

Finding the Domains of Rational Functions

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Theorem: Domain of a Rational Function

The domain of a rational function includes all real numbers except those that cause the denominator of the unsimplified rational function to equal zero.

Caution: Domain is Always Considered Prior to Simplifications

A common mistake is for students to simplify or otherwise perform Algebra on an expression and *then* state the domain. In this simplification process, you might "remove" a domain issue that originally existed. Therefore, you need to state the domain *prior* to simplifying the function.

For example, if $f(x) = \frac{x+2}{(x+2)(x+3)}$, you would need to state the domain of the function *prior* to simplifying. In this case, $x = -2$ and $x = -3$ cause division by zero. Therefore, the domain is $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

Had you simplified the function first, you would have arrived at

$$f(x) = \frac{x+2}{(x+2)(x+3)} = \frac{1}{x+3},$$

and you would have (incorrectly) stated the domain as $(-\infty, -3) \cup (-3, \infty)$. In truth, $f(x)$ is *not* equal to $\frac{1}{x+3}$ because, if it were, we would be able to evaluate $f(-2)$. However, looking at the original function, evaluating f at $x = -2$ causes division by zero!

Example 5.7.5: Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$\begin{aligned} x^2 - 9 &= 0 \\ \Longrightarrow \quad x^2 &= 9 \\ \Longrightarrow \quad x &= \pm 3 \end{aligned}$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$. Writing this in interval notation, we say the domain of f is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

A graph of the function from Example 5.7.5 is shown in Figure 5.7.8. It confirms that the function is not defined when $x = \pm 3$.

Figure 5.7.8

Graph of $f(x) = \frac{x+3}{x^2+9}$.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Checkpoint 5.7.5

Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

Example 5.7.6: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$\Rightarrow \quad x = -2, 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 5.7.9 confirms the location of the two vertical asymptotes.

Figure 5.7.9

Graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Removable Discontinuities

Occasionally, a graph will contain a hole.

Definition: Removable Discontinuity

A single point where the graph of a function is not defined, indicated by an open circle on the graph, is called a **removable discontinuity**.

For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be rewritten by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}.$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $x-3$ is *not* a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 5.7.10.

Figure 5.7.10

Showcasing the removable discontinuity for the graph of the function $f(x) = \frac{x^2-1}{x^2-2x-3}$.

Definition: Removable Discontinuity of a Rational Function

A removable discontinuity for the graph of a rational function occurs at $x = a$ if

1. a is a zero for a factor in the denominator that is common with a factor in the numerator, and
2. after simplification, a is no longer a zero of the denominator.

Proof

The most elegant proof of this theorem occurs in Calculus.

An clean way to summarize this theorem is to first factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This x-value leads to a removable discontinuity if the multiplicity of the factor in the numerator is greater than or equal to that in the denominator. If the multiplicity of the factor is greater in the denominator, then there is still an asymptote at that value.

Example 5.7.7: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}.$$

Notice that there is a common factor in the numerator and the denominator, $x–2$. The zero for this factor is $x = 2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x + 2$. The zero for this factor is $x = –2$. Therefore, the vertical asymptote is $x = –2$. See Figure 5.7.11.

Figure 5.7.11

Graph of $k(x) = \frac{x-2}{x^2-4}$ showing the vertical asymptote and removable discontinuity.

The graph of this function will have the vertical asymptote at $x = –2$, but at $x = 2$ the graph will have a hole.

Checkpoint 5.7.7

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Identifying End Behavior of Rational Functions

As we have discussed, horizontal asymptotes help describe the behavior of a graph as the input gets infinitely large (either positive or negative); however, it is possible that the graph of a function doesn't tend to a horizontal line as $x \to \pm\infty$. In fact, there are an infinite number of cases in which graphs tend to non-horizontal lines or, in more advanced cases, other higher-degreed functions. Therefore, rather than speaking of identifying "horizontal asymptotes" of graphs, it's best to keep things general and speak of the **end behavior** of a graph.

Definition: End Behavior

The **end behavior** of a graph is a description, usually as a function, of what the function values of the graph tend to approach as the inputs increase or decrease without bound.

For example, the function $f(x) = 3x + 1 + \frac{1}{x}$ is the solid curve graphed in Figure 5.7.12.

Figure 5.7.12

Graph of $f(x) = 3x + 1 + \frac{1}{x}$ showcasing a slant asymptote.

As $x \to \pm\infty$, the graph quickly begins to "mimic" the dashed line, which happens to have equation $y = 3x + 1$. Why does this happen? The third term of the $f(x)$, which is $\frac{1}{x}$, tends to 0 (it tends to disappear) as $x \to \pm\infty$. This forces $f(x)$ to eventually "mimic" what *doesn't* disappear (specifically, $y = 3x + 1$). Hence, we would say that the end behavior of $f(x) = 3x + 1 + \frac{1}{x}$ is $y = 3x + 1$. This type of end behavior is called a **slant asymptote** because it is a line.

With this stated, we can see that a horizontal asymptote is a *type* of end behavior, but it is not the *only* type. Our goal at this point is to discover the types of end behaviors a rational function might have.

Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. There are three distinct outcomes when checking for the end behavior of a rational function; however, let's define a new notation that will simplify some of our discussion.

Definition: Polynomial Degree Notation

The degree of a polynomial $f(x)$ is denoted $\deg(f(x))$.

For example, $\deg(1 - x^{17} + x) = 17$.

With that definition, we are ready to explore the end behavior possibilities of rational functions. We begin by letting $f(x) = \frac{N(x)}{D(x)}$, where N and D are polynomials and $D(x) \neq 0$.

Case 1 ($\deg(D(x)) > \deg(N(x))$)

Consider the function $f(x) = \frac{4x+2}{x^2+4x-5}$. A little Algebra goes a long way:

$$\begin{aligned} \frac{4x+2}{x^2+4x-5} &= \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \cdot \frac{4x+2}{x^2+4x-5} && \text{(multiplying numerator and denominator by } \tfrac{1}{x^2}\text{)} \\ &= \frac{\frac{4}{x} + \frac{2}{x^2}}{1 + \frac{4}{x} - \frac{5}{x^2}} && \text{(distributing)} \end{aligned}$$

As $x \to \pm\infty$, all expressions of the form $\frac{\text{constant}}{x}$ and $\frac{\text{constant}}{x^2}$ will tend to 0. Thus, as $x \to \pm\infty$, $f(x) \to \frac{0+0}{1+0+0} = \frac{0}{1} = 0$. That is, $y = 0$ is a horizontal asymptote for $f(x)$. See Figure 5.7.13. Note that this graph crosses the horizontal asymptote.

Figure 5.7.13

Horizontal Asymptote $y=0$ *when* $f(x)=\frac{N(x)}{D(x)}$, *where* $D(x)\neq 0$ *and* $\deg(N(x))<\deg(D(x))$.

Case 2 $\deg(D(x))<\deg(N(x))$ **(by 1)**

In this textbook, we will only consider the possibility of the numerator being one degree more than the denominator. Your instructor might spend a few minutes of class time to discuss what happens when the difference of degrees between numerator and denominator is greater than 1, but that is not a major topic in Calculus.

Consider the function $f(x)=\frac{3x^2-2x+1}{x-1}$. Since we can think of this as an improper fraction (where the numerator might be somewhat divisible by the denominator), we perform synthetic division to arrive at

$$\frac{3x^2-2x+1}{x-1}=3x+1+\frac{2}{x-1}.$$

As $x\to\pm\infty$, the expression $\frac{2}{x-1}\to 0$. Therefore, our function will "mimic" $y=3x+1$. That is, $f(x)\to 3x+1$ as $x\to\pm\infty$. Thus, the line $y=3x+1$ is a slant asymptote of $f(x)$ (see Figure **5.7.14**).

Figure 5.7.14

Slant Asymptote when $f(x)=\frac{N(x)}{D(x)}$, *where* $D(x)\neq 0$ *and* $\deg(N(x))>\deg(D(x))$ *by 1.*

Case 3 $\deg(D(x))=\deg(N(x))$

Consider the function $f(x)=\frac{3x^2+2}{x^2+4x-5}$. Performing some Algebra, we get

$$\begin{aligned}\frac{3x^2+2}{x^2+4x-5} &= \frac{\frac{1}{x^2}}{\frac{1}{x^2}}\cdot\frac{3x^2+2}{x^2+4x-5} && \text{(multiplying numerator and denominator by } \tfrac{1}{x^2}\text{)}\\ &= \frac{3+\frac{2}{x^2}}{1+\frac{4}{x}-\frac{5}{x^2}} && \text{(distributing)}\end{aligned}$$

As $x\to\pm\infty$, all expressions of the form $\frac{\text{constant}}{x}$ and $\frac{\text{constant}}{x^2}$ will tend to 0. Thus, as $x\to\pm\infty$, $f(x)\to\frac{3+0}{1+0-0}=\frac{3}{1}=3$. That is, $y=3$ is a horizontal asymptote for $f(x)$. See Figure **5.7.15**. Again, note that this graph crosses the horizontal asymptote.

Figure 5.7.15

Horizontal Asymptote when $f(x) = \frac{N(x)}{D(x)}$, *where* $D(x) \neq 0$ *and* $\deg(N(x)) = \deg(D(x))$.

Notice that, while the graph of a rational function will never cross a *vertical* asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph of a *rational function* will have at most one horizontal (or slant) asymptote. As an aside, there exist *non-rational* functions that have two horizontal or slant asymptotes (and there are some that have both a horizontal *and* a slant asymptote); however, we are only focusing our attention on rational functions for this discussion.

Theorem: End Behavior of a Rational Function

Let $f(x) = \frac{N(x)}{D(x)}$ be a rational function. Then

- $f(x)$ has a horizontal asymptote of $y = 0$ if $\deg(D(x)) > \deg(N(x))$
- $f(x)$ has a horizontal asymptote of $y = \frac{a}{b}$ if $\deg(D(x)) = \deg(N(x))$, where a is the lead coefficient of $N(x)$ and b is the lead coefficient of $D(x)$
- $f(x)$ has a slant or oblique asymptote if $\deg(D(x)) < \deg(N(x))$. The asymptote is a slant asymptote if $\deg(N(x)) = \deg(D(x)) + 1$; otherwise, it is an oblique asymptote. In either case, the equation of the asymptote is the quotient after dividing $N(x)$ by $D(x)$.

Example 5.7.8: Identifying Horizontal and Slant Asymptotes

For the functions below, identify the horizontal or slant asymptote.

a. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
b. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$
c. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will let $p(x)$ represent the numerator and $q(x)$ the denominator.

a. The degree of p is equal to the degree of q (3) so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
b. The degree of p is greater than the degree of q by 1. Therefore, this rational function has a slant asymptote. To find the equation of the slant asymptote, we divide $\frac{x^2 - 4x + 1}{x + 2}$. This is easily done using synthetic division.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

The quotient is $x - 6$ and the remainder is 13. Thus, the equation of the slant asymptote is $y = x - 6$.
c. The degree of p is less than the degree of q, so there is a horizontal asymptote at $y = 0$.

Example 5.7.9: Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \frac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty \implies C(t) \to \frac{1}{10}.$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 5.7.10: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1$,-2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 5.7.16.

Figure 5.7.16

Graph of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Checkpoint 5.7.10

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Like all functions, a rational function will have a y-intercept when the input is zero, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero. Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero; however, since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 5.7.11: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the y-intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5} = -0.6.$$

The x-intercepts will occur when the function is equal to zero:

$$\begin{aligned} 0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \\ \Longrightarrow \quad 0 &= (x-2)(x+3) \qquad \text{(a fraction is zero only when the numerator is zero)} \end{aligned}$$

This happens when $x = 2$ or when $x = -3$. Therefore, the y-intercept is $(0, -0.6)$, and the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 5.7.17.

Figure 5.7.17

Graph of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Checkpoint 5.7.11

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Graphing Rational Functions

In Example 5.7.10, we see that the numerator of a rational function reveals the x-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 5.7.18.

Figure 5.7.18

Graph of the toolkit function $y = \frac{1}{x}$.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure **5.7.19**.

Figure 5.7.19

Graph of the toolkit function $y = \frac{1}{x^2}$.

For example, the graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure **5.7.20**.

Figure 5.7.20

Graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph bounces (like a parabola), consistent with the quadratic nature of the factor.

- At the x-intercept $x = 3$ corresponding to the $(x - 3)$ factor of the numerator, the graph passes through the axis (like a line) as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x + 3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x - 2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of $f(x) = \frac{1}{x}$.

How To: Graph a rational function

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 5.7.12: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3.$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0, 3)$ and x-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 5.7.21.

Figure 5.7.21

An unfinished graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$ (the middle portion).

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 5.7.22. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

Figure 5.7.22

Graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

? Checkpoint 5.7.12

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an x-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of x-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and $xi \neq v_j$ for all i and j, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_m}},$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

Example 5.7.13: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 5.7.23.

Figure 5.7.23

Solution

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 5.7.24.

Figure 5.7.24

The original graph with asymptotes drawn in for guidance.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}.$$

To find the stretch factor, we can use another clear point on the graph, such as the y-intercept $(0, -2)$.

$$\begin{aligned} -2 &= a\frac{(0+2)(0-3)}{(0+1)(0-2)^2} \\ \Rightarrow \quad -2 &= a\frac{-6}{4} \\ \Rightarrow \quad a &= \frac{-8}{-6} \\ \Rightarrow \quad a &= \frac{4}{3} \end{aligned}$$

This gives us a final function of

$$f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}.$$

Revisiting Solving Inequalities

As we did with polynomials, we can solve inequalities involving rational expressions using our graphing skills. Again, many instructors would have you solve rational inequalities using sign charts; however, I find solving via graphing (1) is faster, (2) leads to a more natural understanding of what the solution to a rational inequality means, and (3) reinforces graphing skills. Let's see how this is done via an example.

Example 5.7.14: Solving a Rational Inequality

Find all solutions to the inequality.

$$\frac{6x^2+14x-34}{x+4} > 5x-9$$

Solution

We begin as we did when solving polynomial inequalities - we move all terms to one side and simplify as much as possible.

$$\begin{aligned} & \frac{6x^2+14x-34}{x+4} > 5x-9 \\ \Rightarrow \quad & \frac{6x^2+14x-34}{x+4} - (5x-9) > 0 && \text{(subtracting } 5x-9 \text{ from both sides)} \\ \Rightarrow \quad & \frac{6x^2+14x-34}{x+4} - \frac{(5x-9)(x+4)}{x+4} > 0 && \text{(rewriting with a common denominator)} \\ \Rightarrow \quad & \frac{6x^2+14x-34}{x+4} - \frac{5x^2+11x-36}{x+4} > 0 && \text{(distributing)} \\ \Rightarrow \quad & \frac{6x^2+14x-34-(5x^2+11x-36)}{x+4} > 0 && \text{(combining fractions)} \\ \Rightarrow \quad & \frac{6x^2+14x-34-5x^2-11x+36}{x+4} > 0 && \text{(distributing)} \\ \Rightarrow \quad & \frac{x^2+3x+2}{x+4} > 0 && \text{(combining like terms)} \\ \Rightarrow \quad & \frac{(x+2)(x+1)}{x+4} > 0 && \text{(factoring the trinomial)} \end{aligned}$$

It's important to note that you would have done all of this work if you were going to solve the inequality using sign charts or graphing - there is no difference in "grunt work" up to this point.

As we did with polynomial inequalities, we graph the expression on the left side. That is, we want to sketch $f(x) = \frac{(x+2)(x+1)}{x+4}$. Our goal is to identify when the graph of f is *above* the x-axis (because we are solving $f(x) > 0$).

- **Horizontal Intercepts:** The numerator of f is zero at $x = -2$ and $x = -1$. Therefore, the graph has x-intercepts at $(-2, 0)$ and $(-1, 0)$. Both of these zeros have multiplicity 1, so the graph will pass through each like a line.
- **Vertical Intercept:** $f(0) = \frac{2}{4} = \frac{1}{2}$, so the y-intercept is $\left(0, \frac{1}{2}\right)$.
- **Vertical Asymptote:** The denominator is zero when $x = -4$, which is not a zero shared with the numerator. Therefore, f has a vertical asymptote at $x = -4$. Additionally, the degree of the factor, $x + 4$, in the denominator is 1, so the vertical asymptote shares the behavior of $\frac{1}{x}$.
- **Horizontal/Slant Asymptote:** Since the numerator of f has degree 2 and the denominator has degree 1, there will be a slant asymptote. To find the equation of the slant asymptote, it's best to use the unfactored form, $f(x) = \frac{x^2+3x+2}{x+4}$, and divide.

$$\begin{array}{r|rrr} -4 & 1 & 3 & 2 \\ & & -4 & 4 \\ \hline & 1 & -1 & 6 \end{array}$$

Hence, the quotient is $x-1$ and the remainder is 6. Therefore, the equation of the slant asymptote is $y=x-1$.

We now have enough information to sketch the graph. Starting at the y-intercept (which is slightly above the x-axis), we have a choice, start sketching to the left or right. For no better reason than random choice, I will choose to sketch to the right. Does the graph rise or fall? If we started drawing to the right and had the graph fall, it would eventually intersect the x-axis; however, we do not have any x-intercepts greater than 0. Thus, the graph climbs and eventually tends to the line $y=x-1$ (see Figure 5.7.25).

Going back to the y-intercept, we now start sketching the curve to the left. Since there is an x-intercept at $x=-1$, our curve falls (as we draw to the left), goes through $(-1,0)$, and (since there is another x-intercept) climbs to strike the x-axis at $x=-2$. It goes through this intercept and we are now above the x-axis. If we continue to draw to the left (again, see Figure 5.7.25), our curve would approach the vertical asymptote, $x=-4$. We have a choice - have the graph rise to infinity as $x\to -4^+$, or have the graph fall to negative infinity as $x\to -4^+$. Since we do not have any more x-intercepts, the choice is out of our hands. The function grows without bound (to infinity) as $x\to -4^+$.

As stated in the list above, the behavior of the function near to the vertical asymptote mimics that of $\frac{1}{x}$, so we know $f(x)\to -\infty$ as $x\to -4^-$. That is, since the graph of f is rising on one side of $x=-4$, it must be falling on the other side. Finally, as we continue drawing to the left, the function tends to the slant asymptote.

Figure 5.7.25

Graph of $y=\frac{(x+2)(x+1)}{x+4}$

Trust me when I say that you will be able to sketch a function like this in a couple of minutes once you practice these types of problems. Do not freak out.

Our ultimate goal was to find out when $f(x)>0$. Now that we have the graph, we can see this occurs when $x\in(-4,-2)\cup(-1,\infty)$.

5.7.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Rational function
  - Rational Functions - A Brief Review of Rational Functions
- Vertical asymptote
  - Rational Functions - Identifying Vertical Asymptotes of Rational Functions
  - Rational Functions - Asymptotic Behavior of Rational Functions
- Horizontal asymptote
  - Rational Functions - Identifying Horizontal Asymptotes of Rational Functions
- Slant asymptote (oblique asymptote)
  - Rational Functions - Identifying Slant Asymptotes of Rational Functions
- Domain of a rational function
  - Rational Functions - Finding the Domain of Rational Functions
- Graphing rational functions
  - Rational Functions - Graphing Rational Functions
  - Rational Functions - Graphing Rational Functions
  - Algebra Lecture: Graphing Rational Functions
- Solving rational inequalities
  - Rational Functions - Solving Rational Inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Rational Inequalities

KEY CONCEPTS

DEFINITIONS

- **Arrow Notation**: A symbolic way to describe the behavior of a function as its input x approaches a certain value or infinity. Examples: $x \to a^-$ (approaches a from the left), $x \to \infty$ (approaches infinity), $f(x) \to L$ (the output approaches L).
- **Vertical Asymptote**: A vertical line $x = a$ where the graph of a function tends toward positive or negative infinity as the inputs approach a from either the left or the right.
- **Horizontal Asymptote**: A horizontal line $y = b$ where the graph of a function approaches the line as the inputs increase or decrease without bound ($x \to \infty$ or $x \to -\infty$).
- **Rational Function**: A function that can be written as the quotient of two polynomial functions, $f(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$.
- **Removable Discontinuity (Hole)**: A single point where the graph of a function is not defined, indicated by an open circle on the graph. It occurs in a rational function at $x = a$ if $(x - a)$ is a factor of both the numerator and the denominator, and after simplification, a is no longer a zero of the denominator.
- **End Behavior (of a rational function's graph)**: A description, usually as a function (e.g., horizontal or slant asymptote), of what the function values of the graph tend to approach as the inputs increase or decrease without bound.
- **Polynomial Degree Notation ($deg(f(x))$)**: Denotes the degree of a polynomial $f(x)$.
- **Slant Asymptote (Oblique Asymptote)**: A non-horizontal, non-vertical line that the graph of a rational function approaches as $x \to \infty$ or $x \to -\infty$. It occurs when the degree of the numerator is exactly one more than the degree of the denominator.

THEOREMS

- **Domain of a Rational Function**: The domain of a rational function includes all real numbers except those that cause the denominator of the unsimplified rational function to equal zero.
- **Vertical Asymptotes of Rational Functions**: Vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.
- **Removable Discontinuity of a Rational Function**: A removable discontinuity for the graph of a rational function occurs at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator, and after simplification, a is no longer a zero of the denominator.
- **End Behavior of a Rational Function $f(x) = \frac{N(x)}{D(x)}$**:
  - If $deg(D(x)) > deg(N(x))$, then $f(x)$ has a horizontal asymptote at $y = 0$.
  - If $deg(D(x)) = deg(N(x))$, then $f(x)$ has a horizontal asymptote at $y = \frac{a}{b}$, where a is the lead coefficient of $N(x)$ and b is the lead coefficient of $D(x)$.
  - If $deg(N(x)) = deg(D(x)) + 1$, then $f(x)$ has a slant (oblique) asymptote whose equation is the quotient after dividing $N(x)$ by $D(x)$ (ignoring the remainder).

COMMON MISTAKES

- **Incorrectly Identifying Domain Before Simplification**: Finding the domain of a rational function *after* simplifying it by canceling common factors. The domain must be determined from the original, unsimplified function.
- **Confusing Holes with Vertical Asymptotes**: If a factor $(x - a)$ cancels from the numerator and denominator, there is a hole at $x = a$, not a vertical asymptote, provided a is not still a zero of the simplified denominator.
- **Errors in Determining Horizontal/Slant Asymptotes**:
  - Making errors in calculating the ratio of leading coefficients for horizontal asymptotes when degrees are equal.
  - Errors in polynomial division when finding slant asymptotes.
- **Assuming Graphs Cannot Cross Horizontal/Slant Asymptotes**: While graphs of rational functions never cross their

vertical asymptotes, they can cross their horizontal or slant asymptotes.

- **Errors in Simplifying Difference Quotients for Rational Functions**: Mistakes in finding a common denominator or in distributing terms when simplifying the complex fraction that arises.
- **Solving Rational Inequalities Incorrectly**: Making errors when finding critical points (zeros of numerator and denominator) or when testing intervals, or misinterpreting the graph if using the graphical method.

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5.7.2: HOMEWORK

READING QUESTIONS

1. What is a rational function? Give its general form.
2. Explain what arrow notation like $x \to a^+$ and $f(x) \to -\infty$ means.
3. What is a vertical asymptote of a rational function? How do you typically find the location(s) of vertical asymptotes from the function's equation?
4. What is a horizontal asymptote? Describe the three cases for determining the horizontal asymptote of a rational function based on the degrees of the numerator and denominator.
5. Under what condition does a rational function have a slant asymptote instead of a horizontal asymptote? How is its equation found?
6. What is a removable discontinuity (or hole) in the graph of a rational function? How is it identified from the function's equation?
7. How do you find the domain of a rational function?
8. How are the x-intercepts of a rational function typically found? What about the y-intercept?
9. Describe how the multiplicity of a factor in the denominator affects the behavior of the graph near a vertical asymptote (odd vs. even multiplicity).
10. When writing the equation of a rational function from its graph, what information do the x-intercepts provide? What about the vertical asymptotes? How can the stretch factor a be determined?
11. When solving a rational inequality graphically, like $\frac{P(x)}{Q(x)} > 0$, what are you looking for on the graph of $f(x) = \frac{P(x)}{Q(x)}$?
12. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?
13. What is the fundamental difference in the graphs of polynomial functions and rational functions?
14. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?
15. Can a graph of a rational function have no vertical asymptote? If so, how?
16. Can a graph of a rational function have no x-intercepts? If so, how?

HOMEWORK

ALGEBRAIC

For the following exercises, find the domain of the rational functions.

17. $f(x) = \frac{x-1}{x+2}$
18. $f(x) = \frac{x+1}{x^2-1}$
19. $f(x) = \frac{x^2+4}{x^2-2x-8}$
20. $f(x) = \frac{x^2+4x-3}{x^4-5x^2+4}$

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

21. $f(x) = \frac{4}{x-1}$
22. $f(x) = \frac{2}{5x+2}$
23. $f(x) = \frac{x}{x^2-9}$
24. $f(x) = \frac{x}{x^2+5x-36}$
25. $f(x) = \frac{3+x}{x^3-27}$
26. $f(x) = \frac{3x-4}{x^3-16x}$
27. $f(x) = \frac{x^2-1}{x^3+9x^2+14x}$
28. $f(x) = \frac{x+5}{x^2-25}$
29. $f(x) = \frac{x-4}{x-6}$
30. $f(x) = \frac{4-2x}{3x-1}$

For the following exercises, find the x-and y-intercepts for the functions.

31. $f(x) = \frac{x+5}{x^2+4}$
32. $f(x) = \frac{x}{x^2-x}$
33. $f(x) = \frac{x^2+8x+7}{x^2+11x+30}$
34. $f(x) = \frac{x^2+x+6}{x^2-10x+24}$
35. $f(x) = \frac{94-2x^2}{3x^2-12}$

For the following exercises, describe the local and end behavior of the functions.

36. $f(x) = \frac{x}{2x+1}$
37. $f(x) = \frac{2x}{x-6}$
38. $f(x) = \frac{-2x}{x-6}$
39. $f(x) = \frac{x^2-4x+3}{x^2-4x-5}$
40. $f(x) = \frac{2x^2-32}{6x^2+13x-5}$

For the following exercises, find the slant asymptote of the functions.

41. $f(x) = \frac{24x^2+6x}{2x+1}$
42. $f(x) = \frac{4x^2-10}{2x-4}$
43. $f(x) = \frac{81x^2-18}{3x-2}$
44. $f(x) = \frac{6x^3-5x}{3x^2+4}$
45. $f(x) = \frac{x^2+5x+4}{x-1}$

GRAPHICAL

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

46. The reciprocal function shifted up two units.
47. The reciprocal function shifted down one unit and left three units.
48. The reciprocal squared function shifted to the right 2 units.
49. The reciprocal squared function shifted down 2 units and right 1 unit.

For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

50. $p(x) = \frac{2x-3}{x+4}$
51. $q(x) = \frac{x-5}{3x-1}$
52. $s(x) = \frac{4}{(x-2)^2}$
53. $r(x) = \frac{5}{(x+1)^2}$
54. $f(x) = \frac{3x^2-14x-5}{3x^2+8x-16}$
55. $g(x) = \frac{2x^2+7x-15}{3x^2-14x+15}$

56. $a(x) = \frac{x^2+2x-3}{x^2-1}$
57. $b(x) = \frac{x^2-x-6}{x^2-4}$
58. $h(x) = \frac{2x^2+x-1}{x-4}$
59. $k(x) = \frac{2x^2-3x-20}{x-5}$
60. $w(x) = \frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$
61. $z(x) = \frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

For the following exercises, write an equation for a rational function with the given characteristics.

62. Vertical asymptotes at $x = 5$ and $x = -5$, x-intercepts at $(2, 0)$ and $(-1, 0)$, y-intercept at $(0, 4)$
63. Vertical asymptotes at $x = -4$ and $x = -1$, x-intercepts at $(1, 0)$ and $(5, 0)$, y-intercept at $(0, 7)$
64. Vertical asymptotes at $x = -4$ and $x = -5$, x-intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$
65. Vertical asymptotes at $x = -3$ and $x = 6$, x-intercepts at $(-2, 0)$ and $(1, 0)$, Horizontal asymptote at $y = -2$
66. Vertical asymptote at $x = -1$, Double zero at $x = 2$, y-intercept at $(0, 2)$
67. Vertical asymptote at $x = 3$, Double zero at $x = 1$, y-intercept at $(0, 4)$

For the following exercises, use the graphs to write an equation for the function.

68.

69.

70.

71.

72.

73.

74.

75.

NUMERIC

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

76. $f(x) = \frac{1}{x-2}$
77. $f(x) = \frac{x}{x-3}$
78. $f(x) = \frac{2x}{x+4}$
79. $f(x) = \frac{2x}{(x-3)^2}$
80. $f(x) = \frac{x^2}{x^2+2x+1}$

TECHNOLOGY

For the following exercises, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.

81. $f(x) = \frac{2}{x+1}$
82. $f(x) = \frac{4}{2x-3}$
83. $f(x) = \frac{2}{(x-1)(x+2)}$
84. $f(x) = \frac{x+2}{(x-1)(x-4)}$
85. $f(x) = \frac{(x+3)^2}{(x-1)^2(x+1)}$

EXTENSIONS

For the following exercises, identify the removable discontinuity.

86. $f(x) = \frac{x^2-4}{x-2}$
87. $f(x) = \frac{x^3+1}{x+1}$
88. $f(x) = \frac{x^2+x-6}{x-2}$
89. $f(x) = \frac{2x^2+5x-3}{x+3}$
90. $f(x) = \frac{x^3+x^2}{x+1}$

REAL-WORLD APPLICATIONS

For the following exercises, express a rational function that describes the situation.

91. A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.
92. A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

For the following exercises, use the given rational function to answer the question.

93. The concentration C of a drug in a patient's bloodstream t hours after injection in given by $C(t) = \frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?
94. The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C(t) = \frac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

95. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.
96. A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.

97. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.
98. A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.
99. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

5.8: Revisiting Inverses

3.02: Revisiting Inverses

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5.8.1: Resources and Key Concepts

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5.8.2: HOMEWORK

3.2.02: Homework

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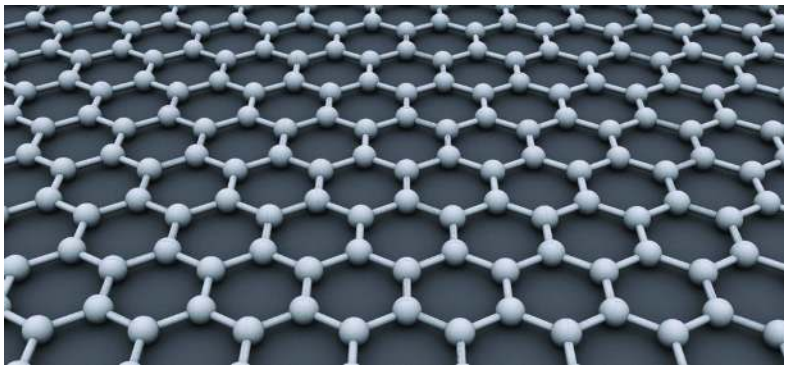
CHAPTER OVERVIEW

Chapter 6: Radical Functions

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

Graphene is an incredible strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: "AlexanderAIUS" / Wikimedia Commons)



6.1: Radical Expressions Encountered in Calculus

6.1.1: Resources and Key Concepts

6.1.2: Homework

6.2: Rational Exponent Expressions Encountered in Calculus

6.2.1: Resources and Key Concepts

6.2.2: Homework

6.3: Radical Equations

6.3.1: Resources and Key Concepts

6.3.2: Homework

6.4: Radical Functions

6.4.1: Resources and Key Concepts

6.4.2: Homework

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6.1: Radical Expressions Encountered in Calculus

Note to the Instructor (click to expand)

This is kind of a "catch-all" section for simplifying radical expressions (at least the kinds of simplifications needed for Calculus). The focus is on roots of powers and rationalizing (both topics, of which, are incredibly important in Calculus). It is critical that you spend time to do some difference quotient problems here involving radical functions.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Properties of Radicals and Simplifying Expressions Involving Roots: This entire Corequisite Codex section is foundational for the current textbook section.
  - Rationalizing Denominators and Numerators: This is a key skill explicitly taught and used in this section, particularly in the context of Difference Quotients.
  - Adding, Subtracting, and Multiplying Radical Expressions: These skills are prerequisite for simplifying expressions before or after rationalization, or when working with difference quotients.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Simplify expressions of the form $\sqrt[n]{a^n}$.
- Rationalize denominators and numerators.
- Compute the Difference Quotient of radical functions.

Hawk A.I. Section-Specific Tutor

COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

We dealt with inverses of polynomial functions (over restricted domains) at the end of the previous chapter. At that time, we discovered that such inverses are often **radical functions**. Other than the square root and cubed root functions, however, radical functions (as a class of functions) are not very common in Calculus. On the other hand, manipulating functions and expressions involving radicals (and, alternatively, expressions involving rational exponents) occurs so frequently in Calculus that we would benefit from improving on our base skills.

Simplify Variable Expressions of the Form $\sqrt[n]{a^n}$

There are many subtleties in Mathematics when dealing with radical expressions and, unfortunately, almost all of these occur in Calculus. The first such subtlety involves the nth root of an expression raised to the nth power. The following theorem reminds us of the complication.

Theorem: Roots of Powers

For any natural numer $n \geq 2$,

$$\sqrt[n]{a^n} = \begin{cases} a, & \text{if } n \text{ is odd} \\ |a|, & \text{if } n \text{ is even} \end{cases}$$

An easy way to remember this theorem is to know that even-indexed radicals only return non-negative values. Therefore, $\sqrt[10]{a^{10}}$, for example, must always return $|a|$ and $\sqrt[10]{a^{20}} = \sqrt[10]{(a^2)^{10}}$ always returns $|a^2|$. Of course, after forcing the absolute values on the result, we need to use some logic to simplify. In the latter case, we know that $a^2 \geq 0$. Therefore, $|a^2| = a^2$.

Let's try some examples to solidify this concept.

Example 6.1.1

Simplify each expression.

a. $\sqrt{x^2}$
b. $\sqrt[3]{n^3}$
c. $\sqrt[4]{p^4}$
d. $\sqrt[5]{y^5}$

Solutions

a. Since the index, 2, is even, $\sqrt{x^2} = \sqrt[2]{x^2} = |x|$.
b. This is an odd indexed root so there is no need for an absolute value sign.

$$\sqrt[3]{m^3} = m.$$

c. Since the index is even $\sqrt[4]{p^4} = |p|$.
d. Since the index is odd, $\sqrt[5]{y^5} = y$.

Checkpoint 6.1.1

Simplify:

a. $\sqrt{b^2}$
b. $\sqrt[3]{w^3}$
c. $\sqrt[4]{m^4}$
d. $\sqrt[5]{q^5}$

Answers

a. $|b|$
b. w
c. $|m|$
d. q

What about square roots of higher powers of variables? The Power Property of the Laws of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}.$$

Looking now at the square root, we get

$$\sqrt{a^{2m}}.$$

Since $(a^m)^2 = a^{2m}$,

$$\sqrt{(a^m)^2}.$$

Since n is even, $\sqrt[n]{a^n} = |a|$. Therefore,

$$|a^m| \implies \sqrt{a^{2m}} = |a^m|.$$

We apply this concept in the next example.

Example 6.1.2

Simplify:

a. $\sqrt{x^6}$
b. $\sqrt{y^{16}}$
c. $\sqrt[3]{y^{18}}$
d. $\sqrt[4]{z^8}$

e. $\sqrt{16n^2}$
f. $-\sqrt{81c^2}$
g. $\sqrt[3]{64p^6}$
h. $\sqrt[4]{16q^{12}}$

Solutions

a. Since $(x^3)^2 = x^6$,

$$\sqrt{(x^3)^2}.$$

Since the index is even, this becomes $|x^3|$. Hence,

$$\sqrt{x^6} = |x^3|.$$

b. Since $(y^8)^2 = y^{16}$,

$$\sqrt{(y^8)^2}.$$

Since the index is even, this becomes $|y^8|$. In this case the absolute value sign is not needed as y^8 is not negative. Therefore,

$$\sqrt{y^{16}} = y^8$$

c. Since $(y^6)^3 = y^{18}$,

$$\sqrt[3]{(y^6)^3}.$$

Since 3 is odd, this becomes y^6.

d. Since $(z^2)^4 = z^8$,

$$\sqrt[4]{(z^2)^4}.$$

Since z^2 is positive, we do not need an absolute value sign. Therefore, this simplifies to z^2

e. Since $(4n)^2 = 16n^2$,

$$\sqrt{(4n)^2}.$$

Since the index is even, this reduces to $4|n|$.

f. Since $(9c)^2 = 81c^2$,

$$-\sqrt{(9c)^2}.$$

Since the index is even, this becomes $-9|c|$.

g. Rewrite $64p^6$ as $(4p^2)^3$. Therefore, we have

$$\sqrt[3]{(4p^2)^3}.$$

Taking the cube root, we get

$$4p^2.$$

h. Rewrite the radicand as a fourth power.

$$\sqrt[4]{(2q^3)^4}.$$

Take the fourth root to get

$$2|q^3|.$$

? Checkpoint 6.1.2

Simplify:

a. $\sqrt{y^{18}}$
b. $\sqrt{z^{12}}$
c. $\sqrt[4]{u^{12}}$

d. $\sqrt[3]{v^{15}}$
e. $\sqrt{64x^2}$
f. $-\sqrt{100p^2}$
g. $\sqrt[3]{27x^{27}}$
h. $\sqrt[4]{81q^{28}}$

Answers

a. $|y^9|$
b. z^6
c. $|u^3|$
d. v^5
e. $8|x|$
f. $-10|p|$
g. $3x^9$
h. $3|q^7|$

Rationalizing Numerators and Denominators

By now, you have probably figured out that finding the Difference Quotient of a function is a big deal in Calculus. After all, we have covered the Difference Quotient in multiple sections up to this point. It turns out that in order to find the Difference Quotient of a radical function, we need to be able to rationalize denominators (and, sometimes, numerators) of fractions involving radicals. To do this, we need to generalize a definition we gave previously.

Definition: Conjugate

The **conjugate** of $a+b$ is $a-b$. Likewise, the conjugate of $a-b$ is $a+b$.

When we encountered this concept in the previous chapter, it was in reference to complex conjugates; however, we need not restrict the definition to complex-valued numbers.

Recall from Algebra, when the denominator of a fraction is a sum or difference involving at least one square root, we use **conjugate multiplication** to **rationalize the denominator**.

$$\begin{array}{l|l} (a-b)(a+b) & (2-\sqrt{5})(2+\sqrt{5}) \\ = a^2-b^2 & = 2^2-(\sqrt{5})^2 \\ & = 4-5 \\ & = -1 \end{array}$$

As seen above, when we multiply a binomial that includes a square root by its conjugate, the product has no square roots. This process is so important and common in Calculus, we should codify it with a theorem.

Theorem: Conjugate Multiplication

Given algebraic expressions a and b, the product of $a+b$ with its conjugate, $a-b$, is always a^2-b^2.

Proof

$$\begin{aligned} (a+b)(a-b) &= a^2-ab+ab-b^2 && \text{(distributing)} \\ &= a^2-b^2 && \text{(simplifying)} \end{aligned}$$

Of course, this is simply the Difference of Squares factorization theorem stated in reverse, so it should not be a surprising result.

Example 6.1.3

Simplify: $\frac{5}{2-\sqrt{3}}$

Solution

$$\begin{aligned}
\frac{5}{2-\sqrt{3}} &= \frac{5}{\left(2-\sqrt{3}\right)} \cdot \frac{\left(2+\sqrt{3}\right)}{\left(2+\sqrt{3}\right)} && \text{(multiplying numerator and denominator by the conjugate of the denominator)} \\
&= \frac{5\left(2+\sqrt{3}\right)}{2^2-\left(\sqrt{3}\right)^2} && \text{(distributing (denominator only))} \\
&= \frac{5\left(2+\sqrt{3}\right)}{4-3} && \text{(simplifying)} \\
&= \frac{5\left(2+\sqrt{3}\right)}{1} && \text{(simplifying)} \\
&= 5\left(2+\sqrt{3}\right) && \text{(simplifying)}
\end{aligned}$$

Notice we did not distribute the **5** in the answer of Example **6.1.3**. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

? Checkpoint 6.1.3

Simplify: $\frac{3}{1-\sqrt{5}}$.

Answer

$-\frac{3(1+\sqrt{5})}{4}$

✓ Example 6.1.4

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution

$$\begin{aligned}
\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}} &= \frac{\sqrt{3}}{\left(\sqrt{u}-\sqrt{6}\right)} \cdot \frac{\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}+\sqrt{6}\right)} && \text{(multiplying numerator and denominator by the conjugate of the denominator)} \\
&= \frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2} && \text{(distributing (denominator only))} \\
&= \frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{u-6} && \text{(simplifying)}
\end{aligned}$$

? Checkpoint 6.1.4

Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer

$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$

As with simplifying compound fractions, you must instinctively know when to rationalize a numerator or denominator throughout Calculus (without explicit instruction from your instructor).

✓ Example 6.1.5: Revisiting the Difference Quotient

Find the Difference Quotient of the following function.

$$f(x)=\frac{1}{\sqrt{x}}$$

Simplify this expression as much as possible, and make sure your final simplification does not have a factor of h in the denominator.

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h} && \text{(substituting)}\\
&= \frac{\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)}{h}\cdot\frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} && \text{(multiplying numerator and denominator by the LCD)}\\
&= \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} && \text{(distributing (numerator only))}
\end{aligned}$$

Now that the compound fraction has been simplified, we are faced with an expression that *still* has the h in the denominator. Remember, a hidden, underlying reason will compel us to somehow remove that through creative uses of Algebra. One such use is conjugate multiplication!

$$\begin{aligned}
\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} &= \frac{\left(\sqrt{x}-\sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{\left(\sqrt{x}+\sqrt{x+h}\right)}{\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(multiplying numerator and denominator by the conjugate of the numerator)}\\
&= \frac{\left(\sqrt{x}\right)^2-\left(\sqrt{x+h}\right)^2}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(distributing (numerator only))}\\
&= \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(simplifying)}\\
&= \frac{x-x-h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(distributing)}\\
&= \frac{-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(simplifying)}\\
&= \frac{-\cancel{h}^{1}}{\cancel{h}^{1}\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(canceling like factors)}\\
&= \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(simplifying)}
\end{aligned}$$

Our final expression is no longer a compound fraction, nor does it have that pesky h in the denominator!

In the solution to Example **6.1.5**, there were a couple of times when we did *not* (and, technically, could not) distribute the denominator. We were forced to multiply by the LCD because of the fractions in the *numerator*. Remember: only use distribution on the offending piece (the numerator in both cases).

? Checkpoint 6.1.5

Find and simplify the difference quotient for the following function.

$$r(x)=\sqrt{x}$$

Answer

$\frac{1}{\sqrt{x+h}+\sqrt{x}}$

Caution: To Distribute or Not To Distribute... That is the Question

With difference quotients, you will commonly need to multiply the numerator **and** denominator of an expression by either

a. the LCD of all fractions within a compound fraction (a.k.a. simplifying compound fractions), or
b. the conjugate of either the numerator *or* the denominator (a.k.a. rationalizing).

When doing so, knowing what to distribute and what not to distribute is imperative.

Simplifying Compound Fractions

The entire point of multiplying the numerator **and** denominator of your compound fraction by the LCD of all the minor fractions is to "get rid of" denominators in those minor fractions; however, with difference quotients, it is common that the denominator of the entire

(major) fraction is only h. In this case, **do not** distribute out the denominator! Doing so will complicate the mathematics, and you will lose visibility of factors that cancel.

Rationalizing

In a difference quotient, the entire point of multiplying the numerator **and** denominator by the conjugate of either the numerator *or* the denominator is to clear radicals. This will only happen with the conjugate pairs. So, definitely distribute the conjugate pairs, but **do not** distribute the non-conjugates.

For example, in Example 6.1.5, we multiplied the numerator **and** denominator by the conjugate of the numerator. We then distributed the *numerator* but did not bother with the distribution in the denominator. This is because distribution in the numerator cleared the radicals, but distribution in the denominator would do nothing other than make a mess.

6.1.1: Resources and Key Concepts

Resources

Videos

- Rationalizing denominators and numerators
  - Expressions - Radicals - Simplifying by Rationalizing Denominators and Numerators: A Review
  - Radicals and Rational Exponents - A Review of Rationalizing Denominators
  - Intermediate Algebra - Radical Expressions: Rationalizing Denominators

Key Concepts

Definitions

- **Conjugate (of an expression involving square roots)**: The conjugate of $a+b$ is $a-b$. Likewise, the conjugate of $a-b$ is $a+b$. This definition is extended from complex conjugates to expressions involving square roots (e.g., the conjugate of $2-\sqrt{3}$ is $2+\sqrt{3}$).
- **Radical Functions**: Functions involving roots (e.g., square root, cube root).

Theorems

- **Roots of Powers**: For any natural number $n \geq 2$, $\sqrt[n]{a^n} = \begin{cases} a, & \text{if } n \text{ is odd} \\ |a|, & \text{if } n \text{ is even} \end{cases}$.
- **Conjugate Multiplication**: Given algebraic expressions a and b, the product of $a+b$ with its conjugate, $a-b$, is always a^2-b^2. (This is essentially the difference of squares factorization.)

Common Mistakes

- **Forgetting Absolute Value with Even Roots**: When simplifying an even-indexed root of a variable raised to an even power, forgetting to use absolute value signs. For example, $\sqrt{x^6} = |x^3|$, not just x^3, unless it's known that $x \geq 0$.
- **Incorrectly Applying Absolute Value with Odd Roots**: Using absolute value signs when simplifying odd-indexed roots. For odd roots, $\sqrt[n]{a^n} = a$.
- **Errors in Rationalizing**:
  - Multiplying only the numerator or only the denominator by the conjugate, instead of both.
  - Incorrectly multiplying out the conjugate pair (e.g., not getting a^2-b^2).
  - Distributing unnecessarily in the non-conjugate part, which might obscure common factors that could cancel.
- **Errors in Simplifying Compound Fractions (in Difference Quotients)**:
  - Not multiplying by the correct LCD of all minor fractions.
  - Incorrectly distributing or canceling terms when simplifying the compound fraction.
- **Not Simplifying Radicals Completely**: Leaving perfect n^{th} powers inside an nth root (e.g., $\sqrt{16n^2}$ should be simplified to $4|n|$).

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6.1.2: HOMEWORK

This page is a draft and is under active development.

READING QUESTIONS

1. What is the result of simplifying $\sqrt[n]{a^n}$ when n is an even natural number? What if n is an odd natural number?
2. When simplifying $\sqrt{x^6}$, why is the result $|x^3|$ and not just x^3 (unless $x \geq 0$)?
3. What is the conjugate of the expression $\sqrt{u} - \sqrt{6}$?
4. When rationalizing a denominator (or numerator) that is a binomial involving a square root, what do you multiply the numerator and denominator by?
5. What is the product of an expression like $(a - b)$ and its conjugate $(a + b)$?
6. When simplifying a difference quotient for a radical function like $f(x) = \frac{1}{\sqrt{x}}$, what is the primary algebraic technique used to eliminate the radicals from the numerator after initial simplification?
7. In the context of simplifying difference quotients involving compound fractions with radicals, when should you distribute terms in the denominator, and when should you avoid it?
8. Simplify $\sqrt[4]{(2q^3)^4}$, assuming q can be any real number.
9. Why is it important to rationalize the numerator when working with difference quotients of radical functions in Calculus?
10. If you are simplifying $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ by multiplying the numerator and denominator by $\sqrt{x+h} + \sqrt{x}$, which part (numerator or denominator) should you fully multiply out (distribute), and which part should you generally leave in factored form? Why?

HOMEWORK

A textbook homework assignment needs to be created for this section.

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6.2: Rational Exponent Expressions Encountered in Calculus

Note to the Instructor (click to expand)

The purpose of this *very short* section is to reintroduce students to rational exponents and to acquaint them with factoring expressions having rational exponents. These occur somewhat frequently in Calculus.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Simplifying Expressions Involving Positive Rational Exponents: Direct prerequisite for the expressions encountered.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Factor and simplify common expressions found in Calculus involving rational powers

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

While you will be presented with radical expressions in Calculus, more often than not these will be written with rational exponents. In fact, it is advantageous in Calculus to use rational exponents instead of radical notation.

When working with expressions in Calculus involving rational exponents, there are three skills we need to review (or introduce). In order of frequency and importance, these are:

1. understanding the meaning and impact of a negative exponent,
2. factoring out the GCF from an expression, and
3. knowing how to multiply the numerator and denominator of an expression by a unit fraction to simplify.

The first of these is addressed with a quick reminder:

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad \frac{1}{a^{-n}} = a^n,$$

We will be using this Law of Exponents frequently in this course and in Calculus.

The second might *seem* silly to mention (because you likely have been factoring out the GCF from algebraic expressions since elementary school); however, a simple change in the *look* of an expression throws many students into a downward spiral. The critical concept to remember when factoring out the GCF from an algebraic expression is that factoring is a division process. That is, when we factor $3x - 6$, we *divide out* the 3 from both $3x$ and -6. Hence,

$$3x - 6 = 3\left(\frac{3x}{3} - \frac{6}{3}\right) = 3(x - 2).$$

Of course, most people skip that middle step. With this being said, we can focus on factoring more complex algebraic expressions.

When factoring the GCF out of an algebraic expression, we were taught to factor out the smallest-powered versions of the shared factors. For example, given $a^M b^n + a^m b^N$, where $m < M$ and $n < N$, we identify the common factors as a (to a power) and b (to a power). Since the smallest power on a is m and the smallest power on b is n, we factor out $a^m b^n$.

$$a^M b^n + a^m b^N = a^m b^n \left(\frac{a^M b^n}{a^m b^n} + \frac{a^m b^N}{a^m b^n} \right) = a^m b^n \left(a^{M-m} + b^{N-n} \right).$$

To memorize this formula is preposterous, so we will focus on the concept rather than the rote memorization.

Example 6.2.1

Simplify the expression by factoring:

$$3(1+x)^{1/3} - x(1+x)^{-2/3}$$

Solution

Both terms in the expression share the factor $(1+x)$; however, the second term has the smallest-powered version of this factor. Therefore, we factor out $(1+x)^{-2/3}$.

$$\begin{aligned} 3(1+x)^{1/3} - x(1+x)^{-2/3} &= (1+x)^{-2/3} \left(\frac{3(1+x)^{1/3}}{(1+x)^{-2/3}} - \frac{x(1+x)^{-2/3}}{(1+x)^{-2/3}} \right) && \text{(factoring out the GCF (which is just division))} \\ &= (1+x)^{-2/3} \left(3(1+x)^{1/3+2/3} - x(1+x)^{-2/3+2/3} \right) && \text{(Laws of Exponents)} \\ &= (1+x)^{-2/3} \left(3(1+x) - x(1+x)^0 \right) && \text{(simplifying)} \\ &= (1+x)^{-2/3} \left(3(1+x) - x \right) && \text{(Laws of Exponents)} \\ &= (1+x)^{-2/3} \left(3+2x \right) && \text{(distributing and combining like terms)} \end{aligned}$$

If you pay close attention, Example **6.2.1** reveals a shortcut to factoring out the smallest-powered version of each shared factor: factoring out $(1+x)^{-2/3}$ ends up subtracting $-\frac{2}{3}$ from the remaining exponents. That is, it *adds* the $\frac{2}{3}$ to those exponents. If you pay attention and do a few examples of your own, this trick speeds up factoring problems like this significantly.

Another thing to note from Example **6.2.1** is that we left our answer with a negative exponent - this is perfectly fine (and actually preferred in Calculus).

The next example demonstrates that we always try to factor completely, if possible.

Example 6.2.2

Factor:

$$x\left(2x+7\right)^{1/3} + 5\left(2x+7\right)^{-2/3}$$

Solution

$$\begin{aligned} x\left(2x+7\right)^{1/3} + 5\left(2x+7\right)^{-2/3} &= \left(2x+7\right)^{-2/3} \left(x\left(2x+7\right)^{3/3} + 5\left(2x+7\right)^0 \right) && \text{(factoring out the GCF)} \\ &= \left(2x+7\right)^{-2/3} \left(x\left(2x+7\right) + 5 \right) && \text{(simplifying)} \\ &= \left(2x+7\right)^{-2/3} \left(2x^2 + 7x + 5 \right) && \text{(distributing)} \\ &= \left(2x+7\right)^{-2/3} \left(2x+5 \right)\left(x+1 \right) && \text{(factoring the trinomial)} \end{aligned}$$

Checkpoint 6.2.2

Factor completely.

$$2x(4x-3)^{1/2} - 5(4x-3)^{-1/2}$$

Importance of Concept

This type of expression commonly occurs as the result of the **Product Rule for Differentiation** in

Calculus.

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Expressions like those in Examples **6.2.1** and **6.2.2** commonly occur as the result of the Product Rule in Differential Calculus (Calculus I). You will have need to simplify those without any prompting from your instructor.

Another style of algebraic expression that frequently occurs in Calculus is in the following example.

Example 6.2.3

Simplify the expression.

$$\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18}$$

Solution

There are two approaches to simplifying expressions like these. The first is to mimic what we did in Examples **6.2.1** and **6.2.2**.

Method #1

$$\begin{aligned}
\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18} &= \frac{(18+x)^{-1/2}\left(2(18+x)^{2/2}-x(18+x)^{0}\right)}{x+18} && \text{(factoring the GCF from the numerator)}\\
&= \frac{(18+x)^{-1/2}(2(18+x)-x)}{x+18} && \text{(simplifying)}\\
&= \frac{(18+x)^{-1/2}(36+2x-x)}{x+18} && \text{(distributing)}\\
&= \frac{(18+x)^{-1/2}(36+x)}{x+18} && \text{(combining like terms)}\\
&= \frac{(36+x)}{(18+x)^{1/2}(x+18)} && \text{(Laws of Exponents)}\\
&= \frac{36+x}{(18+x)^{3/2}} && \text{(Laws of Exponents)}
\end{aligned}$$

While this method works, and there is nothing wrong with it, it is helpful to show another approach.

Method #2

This approach will use the fract that a^{-n} is the fraction $\frac{1}{a^n}$

$$\begin{aligned}
\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18} &= \frac{2(18+x)^{1/2}-x\frac{1}{(18+x)^{1/2}}}{x+18} && \text{(Laws of Exponents)}\\
&= \frac{\left(2(18+x)^{1/2}-x\frac{1}{(18+x)^{1/2}}\right)}{(x+18)}\cdot\frac{(18+x)^{1/2}}{(18+x)^{1/2}} && \text{(simplifying the compound fraction)}\\
&= \frac{2(18+x)^{2/2}-x}{(x+18)^{3/2}} && \text{(distributing)}\\
&= \frac{2(18+x)-x}{(x+18)^{3/2}} && \text{(simplifying)}\\
&= \frac{36+2x-x}{(x+18)^{3/2}} && \text{(distributing)}\\
&= \frac{36+x}{(x+18)^{3/2}} && \text{(combining like terms)}
\end{aligned}$$

While both approaches in Example **6.2.3** look like they take the same amount of work, the second (Method #2) is often a little faster. Expressions of the form in Example **6.2.3** are the result of the Quotient Rule in Differential Calculus. As before, you will have need to simplify these without any prompting from your instructor.

? Checkpoint 6.2.3

Simplify the expression. Fully simplify into a single expression with positive exponents.

$$\frac{8(1+x)^{1/3} - x(1+x)^{-2/3}}{(1+x)^{2/3}}$$

Importance of Concept

This type of expression commonly occurs as the result of the **Quotient Rule for Differentiation** in **Calculus**.

Question Help: Video Written Example

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6.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Factoring with rational exponents
  - Factoring Polynomials - Factoring Expressions with Rational or Negative Exponents
  - Algebraic Expressions - Factoring Expressions with Negative Rational Exponents
  - Intermediate Algebra - Rational Exponents: Factoring/Distributing Expressions

KEY CONCEPTS

COMMON MISTAKES

- **Errors with Negative Exponents**: Incorrectly interpreting or manipulating negative exponents, such as $a^{-n} \neq -a^n$ or misapplying rules when factoring.
- **Incorrectly Identifying the GCF with Rational/Negative Exponents**: When factoring expressions like $3(1+x)^{1/3} - x(1+x)^{-2/3}$, choosing the wrong power for the GCF $(1+x)$. The rule is to factor out the term with the smallest (most negative or least positive) exponent.
- **Errors in Subtracting Exponents During Factoring**: When factoring out a GCF like A^m from A^M, the remaining exponent is $M-m$. Mistakes often occur with signs, especially if m is negative (e.g., $1/3-(-2/3)=1/3+2/3$).
- **Distributing Unnecessarily in Denominators**: When simplifying complex fractions or expressions resulting from the quotient rule, distributing terms in the overall denominator can make further simplification or cancellation harder. It's often better to keep denominators factored.
- **Algebraic Errors in Simplifying Compound Fractions**: When multiplying by LCD to clear minor denominators, making errors in distribution or combining terms.
- **Not Factoring Completely**: After an initial factoring step, failing to see if any resulting polynomial factors (like a quadratic) can be factored further.

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6.2.2: HOMEWORK

This page is a draft and is under active development.

READING QUESTIONS

1. What does a negative exponent, such as a^{-n}, signify? How can it be rewritten with a positive exponent?
2. When factoring out the Greatest Common Factor (GCF) from an algebraic expression where terms have different powers of a common base (e.g., $x^{1/3}$ and $x^{-2/3}$), which power of the common base should you factor out?
3. If you factor $(1+x)^{-2/3}$ from $3(1+x)^{1/3}$, what is the exponent of $(1+x)$ in the remaining factor inside the parentheses? Show the subtraction of exponents.
4. When simplifying an expression like $\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18}$ by first factoring out the GCF from the numerator, what is a common GCF to choose?
5. Describe an alternative method for simplifying an expression like $\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18}$ that involves treating negative exponents as fractions.
6. Why is it often preferred in Calculus to leave answers with negative exponents rather than converting them to fractions?
7. When simplifying expressions that arise from the Quotient Rule in Calculus, such as the one in Example 3, why is it generally advisable not to distribute terms in the final overall denominator?
8. What is the critical concept to remember when factoring out a GCF from an algebraic expression? (Hint: "factoring is a _____ process")

HOMEWORK

The textbook homework has not been written at this time.

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6.3: Radical Equations

Note to the Instructor (click to expand)

This entire section is focused on solving equations involving radicals and rational exponents. While not terribly common in Calculus, these occur enough that exposure is critical.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Solve radical equations involving one or two radicals.
- Solve equations involving rational exponents.
- Solve equations that are quadratic-in-form involving radicals or rational exponents.
- Use radicals in applications

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

As is the theme in College Algebra, we begin this section with the assumption that you already have exposure to solving simple radical equations. We still spend a few moments recalling old techniques, but quickly jump into solving more complex radical equations.

Solving Radical Equations Containing One Radical

We start our trip down memory lane by recalling an old definition.

Definition: Radical Equation

An equation in which unknowns (variables) are part of a radicand is called a **radical equation**.

When solving these equations, our goal is to isolate the radical. At that point, our strategy will be to raise both sides of the equation to the power of the index. This will, in turn, eliminate the radical; however, we have something to watch out for when solving radical equations!

Caution: Solving Equations Containing Even-Indexed Radicals

Care must be taken when working with expressions and equations involving even-indexed radicals. This is due to the nature of such beasts. Remember, the argument of an even-indexed radical cannot be negative, nor can an even-indexed radical return a negative value.

If, while solving a radical equation, we obtain a "solution" that would otherwise cause an even-indexed radical to be negative or the argument of the radical to be negative, then we know this "solution" will not work. In such cases we call this an **extraneous solution** as we did when we solved rational equations.

Definition: Extraneous Solution

An **extraneous solution**, also known as a **spurious solution**, is a root of a transformed (manipulated) equation that is not a root of the original equation because it was excluded from the domain of the original equation.

As the definition suggests, extraneous solutions are always a concern when functions within an equation have implicit domain restrictions (e.g., rational and radical functions).

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

Example 6.3.1: Solving a radical equation involving a single radical expression

Solve: $\sqrt{5n-4}-9=0$.

Solution

$$\begin{aligned} \sqrt{5n-4}-9 &= 0 \\ \implies \sqrt{5n-4} &= 9 && \text{(isolating the radical by adding 9 to both sides)} \\ \implies \left(\sqrt{5n-4}\right)^2 &= 9^2 && \text{(squaring both sides. \textbf{Warning:} This forces us to check our final solution)} \\ \implies 5n-4 &= 81 && \text{(simplifying)} \\ \implies 5n &= 85 && \text{(adding 4 to both sides)} \\ \implies n &= 17 && \text{(dividing both sides by 5)} \end{aligned}$$

Now we **must** to check the solution. When doing so, you need to go back to the original equation to check your answer.

$$\sqrt{5(17)-4}-9=\sqrt{85-4}-9=\sqrt{81}-9=9-9=0.\checkmark$$

Thus, $n=17$ is the solution to the equation.

? Checkpoint 6.3.1

Solve: $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

It is important to remember the following advice:

> *Whenever you raise both sides of an equation to an even power, you must check your final solutions.*

In truth, the reason why we (as mathematicians) ask you to do this is to catch any inconsistencies that might have slipped past you. For example, suppose we were asked to solve $\sqrt{3x-1}-7=-9$. We would start by isolating the radical to get $\sqrt{3x-1}=-2$. If you are paying attention (and not asleep at the "mathematical wheel"), you would notice that you have an even-indexed radical equaling a negative number - **this cannot happen!** At this point, you would stop the solution process and state that the solution doesn't exist (some people use the notation "DNE" for "does not exist").

However, if you didn't notice the negative on the right side, you would continue to solve by squaring both sides to get

$$3x-1=4 \implies 3x=5 \implies x=\frac{5}{3}.$$

As long as we develop the habit to check our solutions whenever we square both sides of an equation, we have a second safety net.

$$\sqrt{3\left(\frac{5}{3}\right)-1}-7=\sqrt{5-7}-7=\sqrt{-2}-7$$

which is not a real number. As such, it cannot become -9.

I often try to get my students to stop at each step of the solution process to think about whether or not the statement before them makes sense, but checking your work at the end is your second opportunity to catch the logical flaw.

✔ Example 6.3.2

Solve: $\sqrt{9k-2}+1=0$.

Solution

$$\begin{aligned} \sqrt{9k-2}+1 &= 0 \\ \implies \sqrt{9k-2} &= -1 && \text{(isolating the radical by subtracting 1 from both sides)} \end{aligned}$$

At this point, if we were paying attention, we could stop because a square root is never equal to -1 (at least, not over the real numbers).

Having recognized the square root equaling a negative number in the last step of Example **6.3.2** saved us time. Had we not seen that issue, we would have continued on to get

$$\begin{aligned}
\sqrt{9k-2}+1 &= 0 \\
\Longrightarrow \quad \sqrt{9k-2} &= -1 && \text{(isolating the radical by subtracting 1 from both sides)} \\
\Longrightarrow \quad 9k-2 &= 1 && \text{(squaring both sides)} \\
\Longrightarrow \quad 9k &= 3 && \text{(adding 2 to both sides)} \\
\Longrightarrow \quad k &= \frac{1}{3} && \text{(dividing both sides by 9)}
\end{aligned}$$

Because we squared both sides, we go back to the original equation and check our answer.

$$\sqrt{9\left(\frac{1}{3}\right)-2}+1=\sqrt{3-2}+1=\sqrt{1}+1=1+1=2\neq 0.$$

This is a **contradiction**. Hence, $k=\frac{1}{3}$ is an extraneous solution.

? Checkpoint 6.3.2

Solve: $\sqrt{2r-3}+5=0$.

Answer

no solution

✓ Example 6.3.3

Solve: $\sqrt{p-1}+1=p$.

Solution

$$\begin{aligned}
\sqrt{p-1}+1 &= p \\
\Longrightarrow \quad \sqrt{p-1} &= p-1 && \text{(isolating the radical by subtracting 1 from both sides)} \\
\Longrightarrow \quad \left(\sqrt{p-1}\right)^2 &= (p-1)^2 && \text{(squaring both sides)} \\
\Longrightarrow \quad p-1 &= p^2-2p+1 && \text{(simplifying and distributing)} \\
\Longrightarrow \quad 0 &= p^2-3p+2 && \text{(subtracting } p \text{ and adding 1 to both sides)} \\
\Longrightarrow \quad 0 &= (p-2)(p-1) && \text{(factoring)}
\end{aligned}$$

Therefore, our *candidate* solutions are $p=2$ or $p=1$. However, because we squared both sides, we need to check these results (using the original equation).

$p=2$				$p=1$			
$\Longrightarrow$	$\sqrt{2-1}+1$	$\stackrel{?}{=}$	2	$\Longrightarrow$	$\sqrt{1-1}+1$	$\stackrel{?}{=}$	1
$\Longrightarrow$	$\sqrt{1}+1$	$\stackrel{?}{=}$	2	$\Longrightarrow$	$\sqrt{0}+1$	$\stackrel{?}{=}$	1
$\Longrightarrow$	$1+1$	$\stackrel{?}{=}$	2	$\Longrightarrow$	$0+1$	$\stackrel{?}{=}$	1
$\Longrightarrow$	2	$=$	$2\checkmark$	$\Longrightarrow$	1	$=$	$1\checkmark$

Therefore, the solutions are $p=1$ and $p=2$.

? Checkpoint 6.3.3

Solve: $\sqrt{x-2}+2=x$.

Answer

$x=2, x=3$

It should be obvious that when the index of the radical is n, we eventually raise both sides of the radical equation to the nth power.

Example 6.3.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution

$$\begin{aligned}
& \sqrt[3]{5x+1}+8 = 4 && \\
\Longrightarrow \quad & \sqrt[3]{5x+1} = -4 && \text{(isolating the radical by subtracting 4 from both sides)} \\
\Longrightarrow \quad & \left(\sqrt[3]{5x+1}\right)^3 = (-4)^3 && \text{(raising both sides to the third power)} \\
\Longrightarrow \quad & 5x+1 = -64 && \text{(simplifying)} \\
\Longrightarrow \quad & 5x = -65 && \text{(subtracting 1 from both sides)} \\
\Longrightarrow \quad & x = -13 && \text{(dividing both sides by 5)}
\end{aligned}$$

It's always a good idea to check your answers when solving equations in Mathematics; however, it's only truly required (at least for now) when we have raised both sides of an equation to an even power (e.g., when solving radical equations) or when we have multiplied both sides of an equation by an expression containing a variable (e.g., when solving a rational equation). This is why we did not bother checking our solution in Example 6.3.4. We leave it to the reader to check that $x=-13$ works.

Checkpoint 6.3.4

Solve: $\sqrt[3]{4x-3}+8=5$

Answer

$x=-6$

Solving Equations Containing Rational Exponents

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical equation. Since $a^{m/n}=\sqrt[n]{a^m}$, we first raise each side of the equation to the power of the denominator of the rational exponent. Once that is done, we take the mth root of both sides of the equation (or raise both sides to the $\frac{1}{m}{}^{\text{th}}$ power).

Example 6.3.5

Solve: $(3x-2)^{1/4}+3=5$.

Solution

$$\begin{aligned}
& (3x-2)^{1/4}+3 = 5 && \\
\Longrightarrow \quad & (3x-2)^{1/4} = -2 && \text{(subtracting 3 from both sides)} \\
\Longrightarrow \quad & \left((3x-2)^{1/4}\right)^4 = (-2)^4 && \text{(raising both sides to the fourth power)} \\
\Longrightarrow \quad & (3x-2)^{4/4} = 16 && \text{(Laws of Exponents)} \\
\Longrightarrow \quad & (3x-2)^1 = 16 && \text{(simplifying)} \\
\Longrightarrow \quad & 3x-2 = 16 && \text{(simplifying)} \\
\Longrightarrow \quad & 3x = 18 && \text{(adding 2 to both sides)} \\
\Longrightarrow \quad & x = 6 && \text{(dividing both sides by 3)}
\end{aligned}$$

Since we raised both sides to an even power, we need to check for extraneous solutions. Going back to the original equation and substituting $x=6$, we get

$$(3(6)-2)^{1/4}+3=(18-2)^{1/4}+3=(16)^{1/4}+3=\sqrt[4]{16}+3=2+3=5\checkmark.$$

Checkpoint 6.3.5

Solve: $(9x+9)^{1/4}-2=1$

Answer

$x=8$

As with radical equations, care must be taken when solving equations containing rational exponents. We must remember all of the techniques we previously learned. For example, in the equation $a^{4/7} = 1$, we start by raising both sides to the 7th power to get

$$a^4 = 1^7 = 1.$$

However, we now have the variable a raised to an even power. Extraction of Roots tells us that $a = \pm 1$. This is important!

Example 6.3.6

Solve: $3\left(\frac{1}{2}x+1\right)^{2/3} - 5 = 7$

Solution

$$\begin{aligned}
& 3\left(\frac{1}{2}x+1\right)^{2/3} - 5 = 7 \\
\Longrightarrow\quad & 3\left(\frac{1}{2}x+1\right)^{2/3} = 12 && \text{(begin isolating the rational exponent expression by adding 5 to both sides)} \\
\Longrightarrow\quad & \left(\frac{1}{2}x+1\right)^{2/3} = 4 && \text{(continue isolating the rational exponent expression by dividing both sides by 3)} \\
\Longrightarrow\quad & \left(\left(\frac{1}{2}x+1\right)^{2/3}\right)^3 = (4)^3 && \text{(raise both sides to the third power)} \\
\Longrightarrow\quad & \left(\frac{1}{2}x+1\right)^{6/3} = 64 && \text{(Laws of Exponents)} \\
\Longrightarrow\quad & \left(\frac{1}{2}x+1\right)^{2} = 64 && \text{(simplifying)} \\
\Longrightarrow\quad & \frac{1}{2}x+1 = \pm 8 && \text{(Extraction of Roots)} \\
\Longrightarrow\quad & \frac{1}{2}x = -1 \pm 8 && \text{(subtracting 1 from both sides)} \\
\Longrightarrow\quad & x = -2 \pm 16 && \text{(multiplying both sides by 2)}
\end{aligned}$$

Thus, the solutions are $x = -2 - 16 = -18$ and $x = -2 + 16 = 14$.

Checkpoint 6.3.6

Solve: $6(3x-5)^{3/2} - 4 = 2$

Answer

$x = 2$

Solving Radical Equations with Two Radicals

If we have a radical equation containing two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first. In the next example, when one radical is isolated, the second radical is also isolated.

Example 6.3.7

Solve: $\sqrt[3]{4x-3} - \sqrt[3]{3x+2} = 0$.

Solution

$$\begin{aligned}
& \sqrt[3]{4x-3} - \sqrt[3]{3x+2} = 0 \\
\Longrightarrow\quad & \sqrt[3]{4x-3} = \sqrt[3]{3x+2} && \text{(isolating one of the radicals (which actually isolates the other in this example))} \\
\Longrightarrow\quad & \left(\sqrt[3]{4x-3}\right)^3 = \left(\sqrt[3]{3x+2}\right)^3 && \text{(raising both sides to the third power)} \\
\Longrightarrow\quad & 4x-3 = 3x+2 && \text{(simplifying)} \\
\Longrightarrow\quad & x = 5 && \text{(subtracting } 3x \text{ and adding 3 to both sides)}
\end{aligned}$$

The solution is $x = 5$. We leave it to you to show that 5 checks.

Checkpoint 6.3.7

Solve: $\sqrt[3]{7x+1} = \sqrt[3]{2x-5}$.

Answer

$x = -\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we "rinse and repeat" by isolating the remaining radical and raising both sides of the equation to the power of the index again.

Example 6.3.8

Solve: $\sqrt{m}+1 = \sqrt{m+9}$.

Solution

$$\begin{aligned}
& \sqrt{m}+1 = \sqrt{m+9} && \\
\Rightarrow\ & (\sqrt{m}+1)^2 = \left(\sqrt{m+9}\right)^2 && \text{(since a radical was already isolated, square both sides)} \\
\Rightarrow\ & (\sqrt{m}+1)(\sqrt{m}+1) = m+9 && \text{(simplifying)} \\
\Rightarrow\ & m+2\sqrt{m}+1 = m+9 && \text{(distributing)} \\
\Rightarrow\ & 2\sqrt{m} = 8 && \text{(begin isolating the remaining radical by subtracting both } m \text{ and 1 from both sides)} \\
\Rightarrow\ & \sqrt{m} = 4 && \text{(dividing both sides by 2)} \\
\Rightarrow\ & (\sqrt{m})^2 = 4^2 && \text{(squaring both sides)} \\
\Rightarrow\ & m = 16 && \text{(simplifying)}
\end{aligned}$$

Since we squared both sides during our solution process, we need to check that our solution works in the original equation.

$$\begin{aligned}
& \sqrt{16}+1 \stackrel{?}{=} \sqrt{16+9} \\
\Rightarrow\ & 4+1 \stackrel{?}{=} \sqrt{25} \\
\Rightarrow\ & 5 = 5\checkmark
\end{aligned}$$

Therefore, the solution is $m = 16$.

Checkpoint 6.3.8

Solve: $3-\sqrt{x} = \sqrt{x-3}$.

Answer

$x = 4$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation. This procedure will now work for any radical equation.

How To: Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials (as in in the next example). Remember the pattern in $(a+b)^2 = a^2+2ab+b^2$ or $(a-b)^2 = a^2-2ab+b^2$.

Example 6.3.9

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution

$$\begin{aligned}
&\sqrt{q-2}+3 &&= \sqrt{4q+1} && \\
\Longrightarrow\quad &\left(\sqrt{q-2}+3\right)^2 &&= \left(\sqrt{4q+1}\right)^2 && \text{(squaring both sides since one radical is already isolated)}\\
\Longrightarrow\quad &\left(\sqrt{q-2}+3\right)\left(\sqrt{q-2}+3\right) &&= 4q+1 && \text{(simplifying)}\\
\Longrightarrow\quad &q-2+6\sqrt{q-2}+9 &&= 4q+1 && \text{(distributing)}\\
\Longrightarrow\quad &q+7+6\sqrt{q-2} &&= 4q+1 && \text{(combining like terms)}\\
\Longrightarrow\quad &6\sqrt{q-2} &&= 3q-6 && \text{(begin isolating the remaining radical by subtracting } q \text{ and 7 from both sides)}\\
\Longrightarrow\quad &2\sqrt{q-2} &&= q-2 && \text{(noticing that all coefficients are share a common divisor, we divide both sides by 3)}\\
\Longrightarrow\quad &\left(2\sqrt{q-2}\right)^2 &&= (q-2)^2 && \text{(instead of dividing by 2 (which would introduce fractions), we square both sides)}\\
\Longrightarrow\quad &2^2\left(\sqrt{q-2}\right)^2 &&= (q-2)(q-2) && \text{(Laws of Exponents)}\\
\Longrightarrow\quad &4(q-2) &&= q^2-4q+4 && \text{(simplifying and distributing)}\\
\Longrightarrow\quad &4q-8 &&= q^2-4q+4 && \text{(distributing)}\\
\Longrightarrow\quad &0 &&= q^2-8q+12 && \text{(subtracting } 4q \text{ and adding 8 to both sides)}\\
\Longrightarrow\quad &0 &&= (q-6)(q-2) && \text{(factoring the trinomial)}
\end{aligned}$$

Therefore, our *candidate* solutions are $q=6$ and $q=2$. We leave it to the reader to check that neither is extraneous.

Checkpoint 6.3.9

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer

$x=5$

Equations involving rational exponents are fertile territory to run into equations that are quadratic-in-form. Not only must you recognize their form, but you also must watch out for all the pitfalls mentioned throughout this section concerning checking your solutions.

Example 6.3.10: Revisiting Solving Equations Quadratic-in-Form

Solve: $x-2\sqrt{x}-8=0$.

Solution

This is a radical equation that can be written in quadratic form. If we let $u=\sqrt{x}$, then $u^2=\left(\sqrt{x}\right)^2=x$ and we can write

$$\begin{aligned}
&x-2\sqrt{x}-8 &&= 0 && \\
\Longrightarrow\quad &u^2-2u-8 &&= 0 && \text{(substituting)}\\
\Longrightarrow\quad &(u-4)(u+2) &&= 0 && \text{(factoring the trinomial)}
\end{aligned}$$

Therefore, $u=4$ or $u=-2$. Since $u=\sqrt{x}$, we resubstitute to get $\sqrt{x}=4$ or $\sqrt{x}=-2$; however, square roots are never negative. Hence, we throw out that possibility and focus on $\sqrt{x}=4 \implies x=16$. Checking this solution in the *original* equation, we get

$$16-2\sqrt{16}-8=16-2(4)-8=16-8-8=0.\checkmark$$

Thus, the solution to the equation is $x=16$.

Checkpoint 6.3.10

Solve: $\sqrt{x}-\sqrt[4]{x}-2=0$

Answer

$x=16$

While Example 6.3.10 is interesting, the following example showcases a more frequent occurrence.

Example 6.3.11: Revisiting Solving Equations Quadratic-in-Form

Solve: $x^{-2/3} - 3x^{-1/3} - 10 = 0$.

Solution

If we let $u = x^{-1/3}$, then $u^2 = \left(x^{-1/3}\right)^2 = x^{-2/3}$ and we can write

$$\begin{aligned} & x^{-2/3} - 3x^{-1/3} - 10 = 0 \\ \Rightarrow \quad & u^2 - 3u - 10 = 0 && \text{(substituting)} \\ \Rightarrow \quad & (u-5)(u+2) = 0 && \text{(factoring the trinomial)} \\ \Rightarrow \quad & \left(x^{-1/3} - 5\right)\left(x^{-1/3} + 2\right) = 0 && \text{(resubstituting)} \end{aligned}$$

Therefore, $x^{-1/3} = 5$ or $x^{-1/3} = -2$.

$$\begin{array}{rrcl|rcll} & x^{-1/3} & = & 5 & x^{-1/3} & = & -2 & \\ \Rightarrow & \frac{1}{x^{1/3}} & = & 5 & \frac{1}{x^{1/3}} & = & -2 & \text{(Laws of Exponents)} \\ \Rightarrow & x^{1/3} & = & \frac{1}{5} & x^{1/3} & = & -\frac{1}{2} & \text{(if } a = b, \text{ where neitehr are zero, then } \frac{1}{a} = \frac{1}{b}\text{)} \\ \Rightarrow & \left(x^{1/3}\right)^3 & = & \left(\frac{1}{5}\right)^3 & \left(x^{1/3}\right)^3 & = & \left(-\frac{1}{2}\right)^3 & \text{(raising both sides to the third power)} \\ \Rightarrow & x^{3/3} & = & \frac{1^3}{5^3} & x^{3/3} & = & -\frac{1^3}{2^3} & \text{(Laws of Exponents)} \\ \Rightarrow & x & = & \frac{1}{125} & x & = & -\frac{1}{8} & \text{(simplifying)} \end{array}$$

The domain of the original equation only has the restriction that $x \neq 0$ (can you see why?). Furthermore, the solution process involved cubing, not squaring. Squaring can introduce extraneous solutions, but cubing does not, so a check is not necessary. We leave it to the reader to check that these solutions work in the original equation.

Applications Involving Radicals

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. The following Problem Solving Strategy is incredibly helpful as you progress through College Algebra and into Calculus.

Problem Solving Strategy

1. **Read** the problem carefully and get clarification on words or concepts that are foreign to you.
2. **Understand** what is being stated and asked by rephrasing statements into language you connect with. Draw a figure to represent the situation, if possible.
3. **Label** the given and wanted information. The wanted information should have a variable representing it.
4. **Translate** into an equation relating the given and wanted information by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good Algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **State** the answer to the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The following (non-memorizable) formula allows us to determine how long it will take a fallen object to hit the ground.

Theorem: Time for a Dropped Object to Strike the Ground

On Earth, if an object is dropped from a height of h feet (where h is not too large) and if we ignore air resistance, the time (in seconds) it will take to reach the ground is approximated by using the formula $t = \frac{\sqrt{h}}{4}$.

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

$$
\begin{aligned}
t &= \frac{\sqrt{h}}{4} \\
\Longrightarrow \quad t &= \frac{\sqrt{64}}{4} && \text{(substituting)} \\
\Longrightarrow \quad t &= \frac{8}{4} && \text{(simplifying)} \\
\Longrightarrow \quad t &= 2 && \text{(simplifying)}
\end{aligned}
$$

Therefore, it would take **2** seconds for an object dropped from a height of **64** feet to reach the ground (ignoring deceleration from air resistance).

Don't Memorize!

The previous theorem is (for most instructors) a non-memorizable theorem. It's just a shortcut to approximate the time it takes for an object to fall a certain distance (and, in truth, it is only partially accurate for short falls).

Example 6.3.12

Kaitlyn dropped her sunglasses from a bridge **400** feet above a river. How many seconds does it take for the sunglasses to reach the river?

Solution

1. **Read** the problem.
2. **Understand** the situation. We are looking for the time it takes for Kaitlyn's sunglasses to reach the river.
3. **Label** the given and wanted information. Let t represent the unknown time it will take the sunglasses to fall to the river.

Given	Want
$h = 400$	t

4. **Translate** into an equation. The recent theorem allows us to relate h and t by the formula

$$t = \frac{\sqrt{h}}{4}.$$

Substituting in the given information, we get

$$t = \frac{\sqrt{400}}{4}.$$

5. **Solve** the equation.

$$
\begin{aligned}
t &= \frac{\sqrt{400}}{4} \\
&= \frac{20}{4} && \text{(simplifying)} \\
&= 5 && \text{(simplifying)}
\end{aligned}
$$

6. **Check** the solution.

$$
\begin{aligned}
5 &\stackrel{?}{=} \frac{\sqrt{400}}{4} \\
\Longrightarrow \quad 5 &\stackrel{?}{=} \frac{20}{4} \\
\Longrightarrow \quad 5 &= 5 \checkmark
\end{aligned}
$$

Thus, **5** is correct *mathematically*. It also makes sense logically (for example, if the answer was negative, that would not make sense).

7. **State** the answer. It takes **5** seconds for Kaitlyn's sunglasses to reach the river.

Checkpoint 6.3.12

A window washer dropped a squeegee from a platform **196** feet above the sidewalk. How many seconds does it take for the squeegee to reach the sidewalk?

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Theorem: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car (in miles per hour) before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Again, this formula is not memorizable *and* it is only accurate under special circumstances (e.g., it is not accurate at all if the roadway was covered in ice).

Example 6.3.13

After a car accident, the skid marks for one car measured 190 feet. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

1. **Read!**
2. **Understand:** We are looking for the initial speed of the car before the brakes were applied.
3. **Label:** Let s represent the unknown speed of the car and d represent the length of the skid.

Given	Want
$d = 190$	s

4. **Translate:**

$$s = \sqrt{24d} = \sqrt{24(190)}.$$

5. **Solve:**

$$\begin{aligned} s &= \sqrt{24(190)} \\ &= \sqrt{4560} && \text{(simplifying)} \\ &\approx 67.5277\ldots && \text{(approximating)} \\ &\approx 67.5 && \text{(rounding to the nearest tenth)} \end{aligned}$$

6. **Check:**

$$\begin{aligned} 67.5 &\overset{?}{\approx} \sqrt{24(190)} \\ \implies 67.5 &\overset{?}{\approx} \sqrt{4560} \\ \implies 67.5 &\approx 67.5277\ldots \checkmark \end{aligned}$$

Thus, 67.5 is correct *mathematically*. It also makes sense logically that the car could have been going 67.5 miles per hour.

7. **State:** The speed of the car before the brakes were applied was roughly 67.5 miles per hour.

Checkpoint 6.3.13

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7 mph

6.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Solving radical equations
  - Equations - Overview - A Review for College Algebra: Solving Radical Equations
  - Intermediate Algebra - Radical Equations: Solving

KEY CONCEPTS

DEFINITIONS

- **Radical Equation**: An equation in which unknowns (variables) are part of a radicand.
- **Extraneous Solution (Spurious Solution)**: A root of a transformed (manipulated) equation that is not a root of the original equation because it was excluded from the domain of the original equation or introduced by operations like squaring both sides.
- **Contradiction**: A statement that is both true and false simultaneously (e.g., arriving at $2 \neq 0$ after assuming a solution exists and checking it).

COMMON MISTAKES

- **Forgetting to Check for Extraneous Solutions**: When raising both sides of an equation to an even power (e.g., squaring), it is crucial to check all candidate solutions in the original equation, as this process can introduce extraneous solutions.
- **Incorrectly Isolating the Radical**: Failing to isolate the radical term on one side of the equation before raising both sides to a power. This leads to more complicated expressions that still contain radicals.
- **Errors in Raising Both Sides to a Power**: Not raising the entire side to the power, especially if it's a binomial (e.g., if $\sqrt{A} = B + C$, then $A = (B + C)^2$, not $B^2 + C^2$).
- **Errors with Rational Exponents**: Incorrectly applying powers to "undo" rational exponents (e.g., for $x^{m/n} = k$, raise both sides to n/m).
- **Not Recognizing Equations Quadratic-in-Form**: Failing to see that a radical or rational exponent equation can be solved using u-substitution if it's quadratic-in-form.

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6.3.2: HOMEWORK

READING QUESTIONS

1. What is a radical equation?
2. What is the primary strategy for solving a radical equation containing one radical?
3. Why is it crucial to check your solutions when solving radical equations, especially if you've raised both sides to an even power? What kind of solutions might arise?
4. If, after isolating an even-indexed radical, you find it is equal to a negative number (e.g., $\sqrt{x} = -2$), what can you conclude about the real solutions to the equation?
5. When solving an equation with a rational exponent, like $(3x-2)^{1/4}+3=5$, what is the first step after isolating the term with the rational exponent?
6. If an equation contains two radicals, what is the general approach to solving it?
7. When solving an equation like $x-2\sqrt{x}-8=0$, what substitution could you make to transform it into a quadratic equation?
8. What does the term "extraneous solution" mean in the context of solving radical equations?
9. If you are solving an equation and raise both sides to an odd power (e.g., cubing both sides), is it still mandatory to check for extraneous solutions for the same reasons as with even powers?
10. Describe the general steps for solving a radical equation as outlined in the "How To: Solve a Radical Equation" box.

HOMEWORK

In the following exercises, solve.

11. $\sqrt{5x-6}=8$
12. $\sqrt{4x-3}=7$
13. $\sqrt{5x+1}=-3$
14. $\sqrt{3y-4}=-2$
15. $\sqrt[3]{2x}=-2$
16. $\sqrt[3]{4x-1}=3$
17. $\sqrt{2m-3}-5=0$
18. $\sqrt{2n-1}-3=0$
19. $\sqrt{6v-2}-10=0$
20. $\sqrt{12u+1}-11=0$
21. $\sqrt{4m+2}+2=6$
22. $\sqrt{6n+1}+4=8$
23. $\sqrt{2u-3}+2=0$
24. $\sqrt{5v-2}+5=0$
25. $\sqrt{u-3}+3=u$
26. $\sqrt{v-10}+10=v$
27. $\sqrt{r-1}=r-1$
28. $\sqrt{s-8}=s-8$
29. $\sqrt[3]{6x+4}=4$
30. $\sqrt[3]{11x+4}=5$
31. $\sqrt[3]{4x+5}-2=-5$
32. $\sqrt[3]{9x-1}-1=-5$
33. $(6x+1)^{1/2}-3=4$
34. $(3x-2)^{1/2}+1=6$
35. $(8x+5)^{1/3}+2=-1$
36. $(12x-5)^{1/3}+8=3$
37. $(12x-3)^{1/4}-5=-2$
38. $(5x-4)^{1/4}+7=9$
39. $\sqrt{x+1}-x+1=0$
40. $\sqrt{y+4}-y+2=0$
41. $\sqrt{z+100}-z=-10$
42. $\sqrt{w+25}-w=-5$
43. $3\sqrt{2x-3}-20=7$
44. $2\sqrt{5x+1}-8=0$
45. $2\sqrt{8r+1}-8=2$
46. $3\sqrt{7y+1}-10=8$

In the following exercises, solve.

47. $\sqrt{3u+7}=\sqrt{5u+1}$
48. $\sqrt{4v+1}=\sqrt{3v+3}$
49. $\sqrt{8+2r}=\sqrt{3r+10}$
50. $\sqrt{10+2c}=\sqrt{4c+16}$
51. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$
52. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$
53. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$
54. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$
55. $\sqrt{a}+2=\sqrt{a+4}$
56. $\sqrt{r}+6=\sqrt{r+8}$
57. $\sqrt{u}+1=\sqrt{u+4}$
58. $\sqrt{x}+1=\sqrt{x+2}$
59. $\sqrt{a+5}-\sqrt{a}=1$
60. $-2=\sqrt{d-20}-\sqrt{d}$
61. $\sqrt{2x+1}=1+\sqrt{x}$
62. $\sqrt{3x+1}=1+\sqrt{2x-1}$
63. $\sqrt{2x-1}-\sqrt{x-1}=1$
64. $\sqrt{x+1}-\sqrt{x-2}=1$
65. $\sqrt{x+7}-\sqrt{x-5}=2$
66. $\sqrt{x+5}-\sqrt{x-3}=2$

In the following exercises, solve. Round approximations to one decimal place.

67. **Landscaping.** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
68. **Landscaping.** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
69. **Gravity.** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
70. **Gravity.** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the

Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

71. **Accident investigation.** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.
72. **Accident investigation.** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

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6.4: Radical Functions

Note to the Instructor (click to expand)

Typical textbooks do not have a full chapter dedicated to radical functions and equations involving radicals; however, this topic just seemed disjoint enough to warrant its own chapter. In this section, we officially introduce the concept of the radical function. While they have seen most of this before, it's best to spend some extra time here on domains of radical functions as well as difference quotients. Finally, getting a basic idea of the shapes of the nth root functions is useful for Calculus.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Inequalities**
  - Solving Quadratic Inequalities: Used to determine the domain of functions with even-indexed roots where the radicand is quadratic (e.g., finding where $-21x^2 + 2x + 8 \geq 0$ in Example 6).

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

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Evaluate a Radical Function

In this section we will extend our previous work with functions to *officially* include radicals. A lot of the material in this section will feel like a review of material from early in the course; however, after a few brief definitions, theorems, and examples at this "review" level, we will increase the level to be a bit more challenging.

Recall from Algebra that if a function is defined by a radical expression, we call it a **radical function**. In particular,

- the square root function is $f(x) = \sqrt{x}$.
- the cube root function is $f(x) = \sqrt[3]{x}$.

For completeness, we include the official definition of a radical function here.

Definition: Radical Function

A **radical function** is a function that is defined by a radical expression.

Let's take a moment to review how to evaluate radical functions.

Example 6.4.1

For the function $f(x) = \sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solutions

a. To evaluate $f(5)$, substitute 5 for x.

$$\begin{aligned} f(5) &= \sqrt{2 \cdot 5 - 1} && \text{(substituting)} \\ &= \sqrt{10-1} && \text{(simplifying)} \\ &= \sqrt{9} && \text{(simplifying)} \\ &= 3 && \text{(simplifying)} \end{aligned}$$

b. We follow the same process as in part a.

$$\begin{aligned} f(-2) &= \sqrt{2 \cdot (-2) - 1} && \text{(substituting)} \\ &= \sqrt{-4-1} && \text{(simplifying)} \\ &= \sqrt{-5} && \text{(simplifying)} \end{aligned}$$

Since the square root of a negative number is not a real number, the function does not have a (real) value at $x = -2$.

Example $6.4.1b$ illustrates an important point - when working with functions (radical or not), we are assuming that the number system we're interested in is the real number system. Therefore, we do not want imaginary numbers as function values.

Checkpoint 6.4.1

For the function $f(x) = \sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answers

a. $f(6) = 4$
b. no value at $x = 0$

Given a cube root function, we follow the same procedure.

Example 6.4.2

For the function $g(x) = \sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solutions

a. We substitute 14 into $g(x)$.

$$\begin{aligned} g(14) &= \sqrt[3]{14-6} && \text{(substituting)} \\ &= \sqrt[3]{8} && \text{(simplifying)} \\ &= 2 && \text{(simplifying)} \end{aligned}$$

b. As we did with part a, we substitute.

$$\begin{aligned} g(-2) &= \sqrt[3]{-2-6} && \text{(substituting)} \\ &= \sqrt[3]{-8} && \text{(simplifying)} \\ &= -2 && \text{(simplifying)} \end{aligned}$$

As Example **6.4.2***b* shows, one of the major differences between a square root function and a cube root function is that we can evaluate the cube root of a negative number. In fact, as you (hopefully) remember from Algebra, the domain of a radical function with an odd index is all real numbers, while the domain of an even-indexed radical function is restricted so that the radicand (the argument of the radical function) is non-negative. We will write this as a theorem in a little bit.

? Checkpoint 6.4.2

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answers

a. $g(4) = 2$
b. $g(1) = -1$

Again, as you should recall from Algebra, radical functions need not be restricted to square roots and cube roots.

✓ Example 6.4.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solutions

a.

$$\begin{aligned} f(4) &= \sqrt[4]{5 \cdot 4 - 4} && \text{(substituting)} \\ &= \sqrt[4]{20-4} && \text{(simplifying)} \\ &= \sqrt[4]{16} && \text{(simplifying)} \\ &= 2 && \text{(simplifying)} \end{aligned}$$

b.

$$\begin{aligned} f(4) &= \sqrt[4]{5 \cdot (-12) - 4} && \text{(substituting)} \\ &= \sqrt[4]{-60-4} && \text{(simplifying)} \\ &= \sqrt[4]{-64} && \text{(simplifying)} \end{aligned}$$

Since the fourth root of a negative number is not a real number, the function does not have a (real) value at $x = -12$.

? Checkpoint 6.4.3

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answers

a. $f(4) = 2$
b. $f(-1) = 1$

The Domain of a Radical Function

To find the domain of a radical function, we use our properties of radicals from Algebra. For a radical with an even index, we said the radicand had to be greater than or equal to zero. This is because even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Theorem: Properties of $\sqrt[n]{a}$

When n is an even number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

Therefore, to find the domain of a radical function with an even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number. Again, for completeness, we include this statement as a theorem.

Theorem: Domain of a Radical Function

When the index of the radical is even, the domain of the radical function consists of all real numbers for which the radicand is greater than or equal to zero. When the index of the radical is odd, the domain of a radical function is all real numbers.

Example 6.4.4

Find the domain of the function, $f(x) = \sqrt{3x - 4}$. Write the domain in interval notation.

Solution

Since the function, $f(x) = \sqrt{3x - 4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

$$\begin{aligned} & 3x - 4 \geq 0 && \text{(setting the radicand greater than or equal to 0)} \\ \Rightarrow \quad & 3x \geq 4 && \text{(adding 4 to both sides)} \\ \Rightarrow \quad & x \geq \frac{4}{3} && \text{(dividing both sides by 3)} \end{aligned}$$

The domain of $f(x) = \sqrt{3x - 4}$ is all values $x \geq \frac{4}{3}$, and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Checkpoint 6.4.4

Find the domain of the function, $f(x) = \sqrt{6x - 5}$. Write the domain in interval notation.

Answer

$\left[\frac{5}{6}, \infty\right)$

Up to this point in this section, we really have not stated anything new. Everything we have covered should feel like a repeat from your previous Algebra classes. Now we begin to improve upon these older skills to introduce examples more relevant to Calculus.

Example 6.4.5

Find the domain of the function, $g(x) = \sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution

Since the function, $g(x) = \sqrt{\frac{6}{x-1}}$, has a radical with an even index, we know the radicand must be greater than or equal to 0. Moreover, since the numerator is a nonzero constant, the radicand cannot be zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive (since the numerator is positive). Therefore, we set $x - 1 > 0$ and solve. This yields $x > 1$. So far, our *candidate* domain is $(1, \infty)$.

An additional consideration when dealing with functions involving fractions is that the denominator cannot be zero. Therefore, we know $x \neq 1$; however, since 1 is not in the interval $(1, \infty)$, we don't have to worry about this extra restriction.

Putting this together we get the domain is $(1, \infty)$.

Checkpoint 6.4.5

Find the domain of the function, $f(x) = \sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer

$(-3, \infty)$

Example 6.4.6

Find the domain of the function, $h(x) = \sqrt[6]{-21x^2 + 2x + 8}$. Write the domain in interval notation.

Solution

The even index on this radical function tells us that the radicand must be greater than or equal to zero. Therefore, we need to solve

$$-21x^2 + 2x + 8 \geq 0.$$

Proceeding as we did when solving quadratic inequalities, we factor this quadratic to get

$$(-3x + 2)(7x + 4) \geq 0.$$

We then sketch a graph of the quadratic expression on the left side of the inequality to find when the curve is at or above the x-axis.

We can see from the graph that the inequality is satsified when $-\frac{4}{7} \leq x \leq \frac{2}{3}$. Hence, the domain of the radical function is $[-\frac{4}{7}, \frac{2}{3}]$.

? Checkpoint 6.4.6

Find the domain of the function, $f(x) = \sqrt{x^3 + x^2 - 4x - 4}$. Write the domain in interval notation.

Answer

$[-2, -1] \cup [2, \infty)$

Example 6.4.6 and Checkpoint 6.4.6 illustrate the need to keep our skills with the material we previously covered in this course sharp. This is the way of Calculus - once you have learned a skill, it is likely going to come into play at some point in the future.

✔ Example 6.4.7: Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$.

Solution

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(x+2)(x-3)}{x-1} \geq 0$. To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, we will continue to reinforce the graphing skills we learned previously and choose to sketch the rational function (see Figure 6.4.1).

Figure 6.4.1

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2, 1) \cup [3, \infty)$.

Domains of Non-Radical Functions Containing Radicals

In Calculus, you often work with functions that *contain* radical expressions *in addition* to containing non-radical expressions. Technically speaking, these are not called radical functions - they are simply referred to as compositions of functions (a catchall phrase encompassing most functions). Examples of these styles of functions are:

- $f(x) = 3x\sqrt[3]{2 - x}$
- $g(x) = \sqrt{2 - \sqrt[4]{x + 3}}$
- $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$

Being able to identify the domains of functions like these is going to be a necessary skill for success in Calculus.

To find the domains of functions like these, we use a method akin to the Order of Operations - working from the innermost function to the outermost. It is often easier to use the method of sign charts when finding the domains of composite functions.

When creating a sign chart for a given function, we are looking for when a denominator becomes zero, when an even-indexed radical has a negative as its radicand, or when the argument of a logarithm (which we haven't encountered yet) is zero or negative. There are other, more rare, cases where domain issues occur, but we will approach those only as needed.

Let's spend some time investigating how such domains can be determined.

Example 6.4.8

State the domains of the following functions.

a. $f(x) = 3x\sqrt[3]{2-x}$
b. $g(x) = \sqrt{2-\sqrt[4]{x+3}}$
c. $k(x) = \frac{2x}{\sqrt{x^2-1}}$

Solutions

a. As far as domain is concerned, $f(x)$ has no denominators and no even roots, which means there are no domain issues. Hence, the domain of this function is $(-\infty, \infty)$.

b. In $g(x) = \sqrt{2-\sqrt[4]{x+3}}$, we have two radicals both of which are even-indexed. To satisfy the innermost radical, $\sqrt[4]{x+3}$, we require $x+3 \geq 0$ or $x \geq -3$. Thus, our *candidate* domain of $g(x)$ is $[-3, \infty)$. This is the largest possible domain for the function g; however, we have more to consider, and this domain might shrink.

"Zooming out" in our function, we see we need $2-\sqrt[4]{x+3} \geq 0$. While it may be tempting to write this as $2 \geq \sqrt[4]{x+3}$ and take both sides to the fourth power, there are times when this technique will produce erroneous results. Instead, we solve $2-\sqrt[4]{x+3} \geq 0$ using a sign chart. We take a moment to find out when $2-\sqrt[4]{x+3} = 0$.

$$\begin{aligned} 2-\sqrt[4]{x+3} &= 0 & \\ \Longrightarrow \quad 2 &= \sqrt[4]{x+3} & \text{(adding } \sqrt[4]{x+3} \text{ to both sides)} \\ \Longrightarrow \quad 2^4 &= \left(\sqrt[4]{x+3}\right)^4 & \text{(raising both sides to the fourth power)} \\ \Longrightarrow \quad 16 &= x+3 & \text{(simplifying)} \\ \Longrightarrow \quad 13 &= x & \text{(subtracting 3 from both sides)} \end{aligned}$$

One thing to be careful of throughout all of Mathematics is raising both sides of an equation to an even power - this *sometimes* introduces extraneous solutions. Therefore, we need to check to see if $x = 13$ is an extraneous solution. Substituting $x = 13$ into the expression $\sqrt{2-\sqrt[4]{x+3}}$ does check since

$$2-\sqrt[4]{x+3} = 2-\sqrt[4]{13+3} = 2-\sqrt[4]{16} = 2-2 = 0.$$

Remember, we are currently trying to figure out when $2-\sqrt[4]{x+3} \geq 0$. We make a sign chart with marked with $x = 13$ (where the expression on the left side of the inequality is zero) and, since the *candidate* domain is $[-3, \infty)$, we start our number line at $x = -3$.

We find $2-\sqrt[4]{x+3} \geq 0$ on $[-3, 13]$ so this is the domain of g.

c. To find the domain of k, we have both an even root and a denominator to concern ourselves with. To satisfy the square root, $x^2-1 \geq 0$. Setting $r(x) = x^2-1$, we find the zeros of r to be $x = \pm 1$, and we find the sign chart of r to be

We find $x^2-1 \geq 0$ for $(-\infty, -1] \cup [1, \infty)$. This is our *candidate* domain, which might shrink as we look further into the problem.

To keep the denominator of $k(x)$ away from zero, we set $\sqrt{x^2-1} = 0$. We leave it to the reader to verify the solutions are $x = \pm 1$, both of which must be excluded from the domain. Hence, the domain of k is $(-\infty, -1) \cup (1, \infty)$.

Graphing Higher-Indexed Radical Functions

We already learned the graphs of the square root and cube root functions, however, we have yet to graph higher-indexed radical functions.

It is worth remarking that we could define $f(x) = \sqrt[n]{x}$ functionally as the inverse of $g(x) = x^n$ with the stipulation that when n is even, the domain of g is restricted to $[0, \infty)$. From what we know about $g(x) = x^n$ from our investigations of polynomials and their inverses, we can produce the graphs of $f(x) = \sqrt[n]{x}$ by reflecting the graphs of $g(x) = x^n$ across the line $y = x$. Below are the graphs of $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$. The point $(0, 0)$ is indicated as a reference. The axes are hidden so we can see the vertical steepening near $x = 0$ and the horizontal flattening as $x \to \infty$.

The odd-indexed radical functions also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to \pm\infty$. In the exercises, you'll have a chance to graph some basic radical functions using transformations.

Example 6.4.9

Sketch a graph of

$$f(x) = -\frac{1}{2}\sqrt[6]{2x+8} + 1.$$

Solution

We know the base graph of this function (it is in the images above), so we just need to apply the transformations. We begin by modifying the function to get it into an appropriate form.

$$f(x) = -\frac{1}{2}\sqrt[6]{2x+8} + 1 = -\frac{1}{2}\sqrt[6]{2(x+4)} + 1.$$

Reading from left to right, the base graph of $y = \sqrt[6]{x}$:

- gets reflected about the x-axis ($y = -\sqrt[6]{x}$)
- gets compressed vertically by a factor of $\frac{1}{2}$ ($y = -\frac{1}{2}\sqrt[6]{x}$)
- gets compressed horizontally by a factor of $\frac{1}{2}$ ($y = -\frac{1}{2}\sqrt[6]{2x}$)
- gets shifted 4 units to the left ($y = -\frac{1}{2}\sqrt[6]{2(x+4)}$)
- gets shifted 1 unit up ($y = -\frac{1}{2}\sqrt[6]{2(x+4)} + 1$)

The graph of $f(x)$ is shown below.

Applications Involving Radicals

Radical functions often arise in Calculus as a result of a process called **optimization**. The following example illustrates one such optimization problem (but we are allowing ourselves technology because we don't have Calculus yet).

Example 6.4.10

Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs $\$15$ per mile to run cable along the road and $\$20$ per mile to run cable off of the road.

a. Express the total cost C of connecting the junction box to the post as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the post. Determine a reasonable applied domain for the problem.
b. Use graphing technology to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off the road?

Solution

a. The cost is broken into two parts: the cost to run cable along Route 117 at $\$15$ per mile, and the cost to run it off road at $\$20$ per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $C = 15x + 20z$.

   Our next goal is to determine z as a function of x. The diagram suggests we can use the Pythagorean Theorem to get

$$y^2 + 30^2 = z^2.$$

   But we also see $x + y = 50$, so that $y = 50 - x$. Hence,

$$z^2 = (50 - x)^2 + 900.$$

   Solving for z, we obtain

$$z = \pm\sqrt{(50-x)^2+900}.$$

Since z represents a distance, we choose $z = \sqrt{(50-x)^2+900}$ so that our cost as a function of x only is given by

$$C(x) = 15x + 20\sqrt{(50-x)^2+900}.$$

From the context of the problem, we have the applied domain of $x \in [0, 50]$.

b. Graphing $y = C(x)$ using Desmos, we find the relative minimum (which is also the absolute minimum in this case) to two decimal places to be $(15.98, 1146.86)$.

Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is $\$1146.86$. The ability to stream live SasquatchCasts? Priceless.

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6.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

KEY CONCEPTS

DEFINITIONS

- **Radical Function**: A function that is defined by a radical expression.

THEOREMS

- **Properties of $\sqrt[n]{a}$ (Regarding Real Numbers)**:
  - When n is an even number and $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
  - When n is an even number and $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**:
  - When the index of the radical is even, the domain of the radical function consists of all real numbers for which the radicand is greater than or equal to zero.
  - When the index of the radical is odd, the domain of a radical function consists of all real numbers (assuming the radicand itself is defined for all real numbers).

COMMON MISTAKES

- **Ignoring Domain Restrictions for Even-Indexed Radicals**: Attempting to evaluate an even-indexed radical function for an input that makes the radicand negative, or failing to correctly identify the domain by setting the radicand ≥ 0.
- **Forgetting Denominator Restrictions within Radicands**: When the radicand of an even-indexed root is a rational expression, not only must the entire rational expression be ≥ 0, but the denominator of that rational expression also cannot be zero.

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6.4.2: HOMEWORK

READING QUESTIONS

1. What is a radical function? Give an example of a square root function and a cube root function.
2. When evaluating radical functions, why is it important to consider the index of the radical (even vs. odd) if the radicand might be negative?
3. What is the primary condition that the radicand of an even-indexed radical function must satisfy for the function to yield a real number?
4. What is the domain of an odd-indexed radical function like $f(x) = \sqrt[3]{x-2}$?
5. How do you find the domain of an even-indexed radical function like $f(x) = \sqrt{3x-4}$?
6. If the radicand of an even-indexed radical function is a rational expression, such as $f(x) = \sqrt{\frac{x+2}{x-1}}$, what conditions must be met to determine the domain?
7. Describe the general shape of the graph of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.
8. How do transformations (shifts, stretches, reflections) affect the graph of a basic radical function like $y = \sqrt{x}$?
9. When finding the domain of a function like $g(x) = \sqrt[4]{2 - \sqrt[4]{x+3}}$, which radical's domain requirements should you address first?
10. What does it mean if, when evaluating a radical function for a specific input, you find the radicand of an even root is negative (e.g., $f(-2)$ for $f(x) = \sqrt{2x-1}$)?

HOMEWORK

In the following exercises, evaluate each function.

1. $f(x) = \sqrt{4x-4}$, find
   a. $f(5)$
   b. $f(0)$
2. $f(x) = \sqrt{6x-5}$, find
   a. $f(5)$
   b. $f(-1)$
3. $g(x) = \sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
4. $g(x) = \sqrt{3x+1}$, find
   a. $g(8)$
   b. $g(5)$
5. $F(x) = \sqrt{3-2x}$, find
   a. $F(1)$
   b. $F(-11)$
6. $F(x) = \sqrt{8-4x}$, find
   a. $F(1)$
   b. $F(-2)$
7. $G(x) = \sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
8. $G(x) = \sqrt{4x+1}$, find
   a. $G(11)$
   b. $G(2)$
9. $g(x) = \sqrt[3]{2x-4}$, find
   a. $g(6)$
   b. $g(-2)$
10. $g(x) = \sqrt[3]{7x-1}$, find
    a. $g(4)$
    b. $g(-1)$
11. $h(x) = \sqrt[3]{x^2-4}$, find
    a. $h(-2)$
    b. $h(6)$
12. $h(x) = \sqrt[3]{x^2+4}$, find
    a. $h(-2)$
    b. $h(6)$
13. For the function $f(x) = \sqrt[4]{2x^3}$, find
    a. $f(0)$
    b. $f(2)$
14. For the function $f(x) = \sqrt[4]{3x^3}$, find
    a. $f(0)$
    b. $f(3)$
15. For the function $g(x) = \sqrt[4]{4-4x}$, find
    a. $g(1)$
    b. $g(-3)$
16. For the function $g(x) = \sqrt[4]{8-4x}$, find
    a. $g(-6)$
    b. $g(2)$

In the following exercises, find the domain of the function and write the domain in interval notation.

17. $f(x) = \sqrt{3x-1}$
18. $f(x) = \sqrt{4x-2}$
19. $g(x) = \sqrt{2-3x}$
20. $g(x) = \sqrt{8-x}$
21. $h(x) = \sqrt{\frac{5}{x-2}}$
22. $h(x) = \sqrt{\frac{6}{x+3}}$
23. $f(x) = \sqrt{\frac{x+3}{x-2}}$
24. $f(x) = \sqrt{\frac{x-1}{x+4}}$
25. $g(x) = \sqrt[3]{8x-1}$
26. $g(x) = \sqrt[3]{6x+5}$
27. $f(x) = \sqrt[3]{4x^2-16}$
28. $f(x) = \sqrt[3]{6x^2-25}$
29. $F(x) = \sqrt[4]{8x+3}$
30. $F(x) = \sqrt[4]{10-7x}$
31. $G(x) = \sqrt[5]{2x-1}$
32. $G(x) = \sqrt[5]{6x-3}$

In the following exercises, find the domain of the function, graph the function, and use the graph to determine the range.

33. $f(x) = \sqrt{x+1}$
34. $f(x) = \sqrt{x-1}$
35. $g(x) = \sqrt{x+4}$
36. $g(x) = \sqrt{x-4}$
37. $f(x) = \sqrt{x}+2$
38. $f(x) = \sqrt{x}-2$
39. $g(x) = 2\sqrt{x}$
40. $g(x) = 3\sqrt{x}$
41. $f(x) = \sqrt{3-x}$
42. $f(x) = \sqrt{4-x}$
43. $g(x) = -\sqrt{x}$
44. $g(x) = -\sqrt{x}+1$
45. $f(x) = \sqrt[3]{x+1}$
46. $f(x) = \sqrt[3]{x-1}$
47. $g(x) = \sqrt[3]{x+2}$
48. $g(x) = \sqrt[3]{x-2}$
49. $f(x) = \sqrt[3]{x}+3$
50. $f(x) = \sqrt[3]{x}-3$
51. $g(x) = \sqrt[3]{x}$
52. $g(x) = -\sqrt[3]{x}$
53. $f(x) = 2\sqrt[3]{x}$
54. $f(x) = -2\sqrt[3]{x}$

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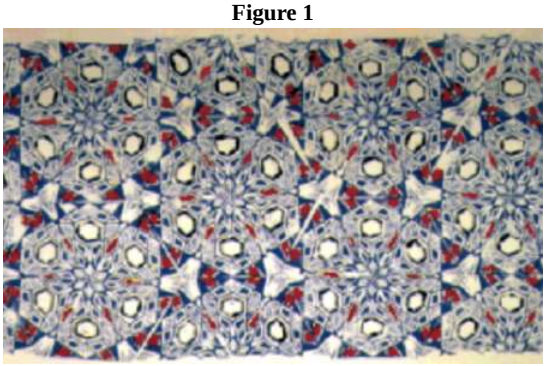
CHAPTER OVERVIEW

Chapter 7: Radian Measure and the Circular Functions

Figure 1



Have you ever wondered why we divide the circle into 360 degrees? Nobody knows the answer, but it may have started around 600 BCE with the Babylonians.

The Babylonians lived between the Tigris and Euphrates rivers in present-day Turkey and Syria. They kept written records using a stylus to press cuneiform, or wedge-shaped, symbols into wet clay tablets, which were then baked in the sun. Thousands of these tablets have survived and give us detailed information about the mathematical practices of the time.

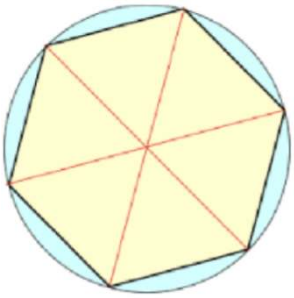
Figure 2

The Babylonians used a base 60 number system because the number 60 has many factors. They did not invent decimal fractions, so they found it challenging to deal with remainders when dividing. But 60 can be divided evenly by 2, 3, 4, 5, and 6, which made calculations with common fractions much more manageable.

We still have traces of their base 60 system: there are 60 seconds in a minute and 60 minutes in an hour.

In Geometry, Babylonian mathematicians used the corner of an equilateral triangle as their basic unit of angular measure. Naturally, they divided that angle into 60 smaller angles. Now, suppose the corners of six equilateral triangles are placed together. In that case, they form a complete circle, so there are six times 60, or 360, degrees of arc in a circle.

Figure 3

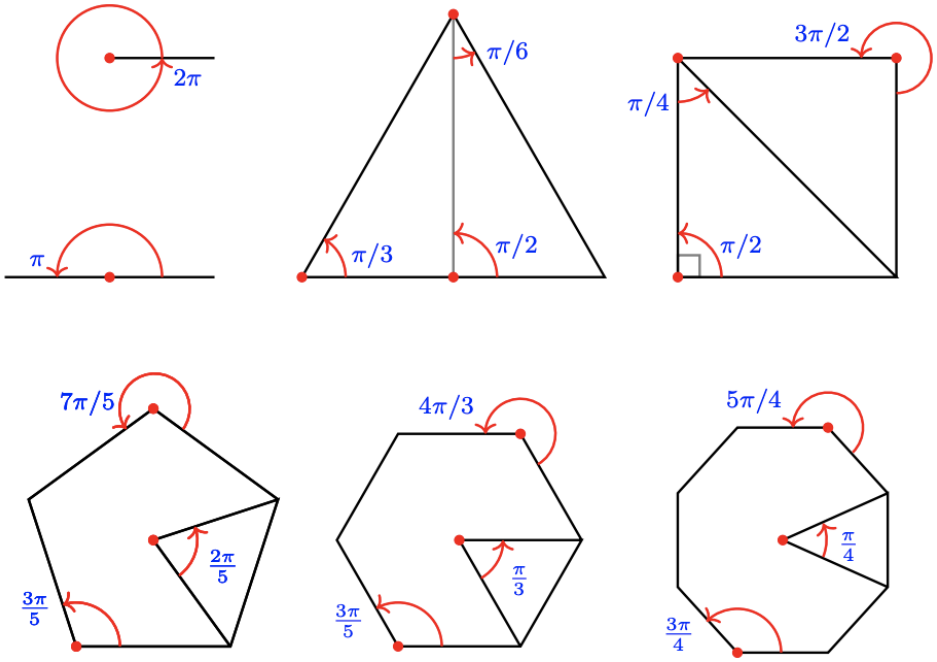


During the reign of Nebuchadnezzar, using the tools and technology available to them, Babylonian astronomers calculated that a complete year numbered 360 days. This made dividing the circle into 360 degrees even more helpful.

The number 360 is not fundamental to the nature of a circle. If ancient civilizations had defined the whole circle as some other number of degrees, we'd probably be using that number today.

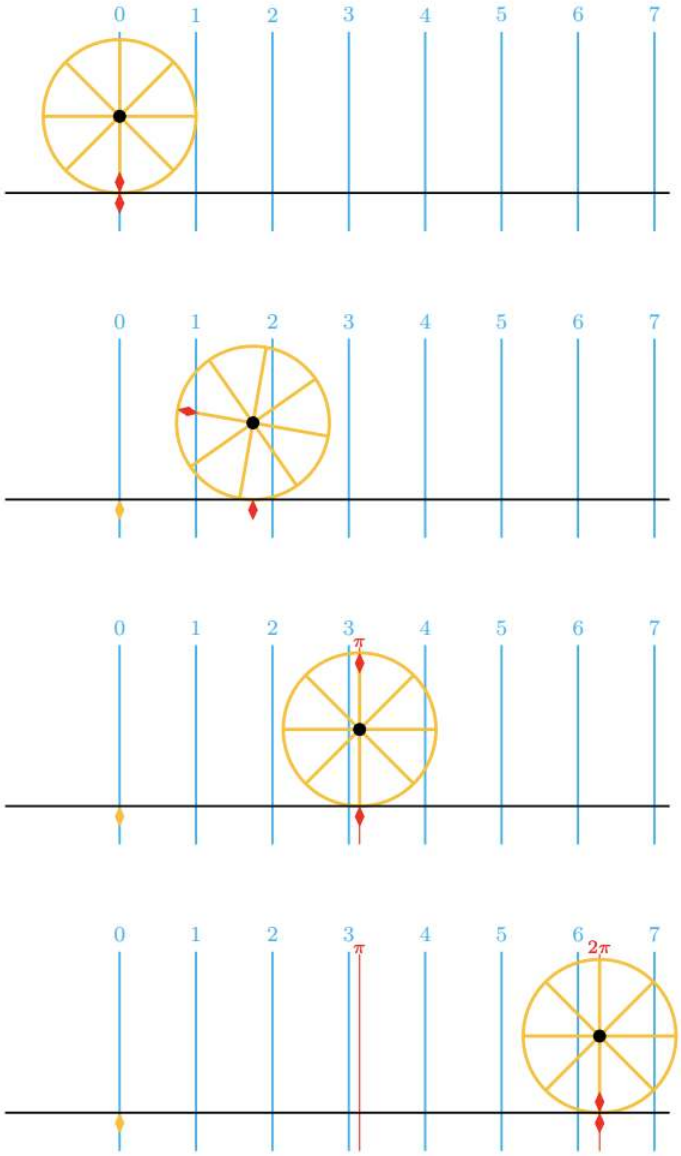
But why do we need another, different way to measure angles? In this chapter, we'll study **radian measure**, which initially may seem awkward and unnatural. As a hint, consider that although 360 is not fundamental to circles, the number π is.

Figure 4



Radians connect the measure of an angle with the arc length it cuts out on a circle.

Figure 5



Imagine a circle with a radius of 1 unit rolling along a straight line. The circumference of a circle is 2π times the length of its radius, so when the circle has turned through one complete revolution, it has traveled a distance of 2π units. When the circle has turned through half a revolution, it has traveled a distance of π units.

In radian measure, the angle through which the circle has turned is equal to the distance the circle has traveled or, put another way, the length of the arc that has unrolled on the line.

Roger Cotes, an English mathematician who worked closely with Isaac Newton, developed the idea of radian measure. He described the radian in everything but name and recognized its naturalness as a unit of angular measure.

The term **radian** first appeared in print on June 5, 1873, on an exam written by James Thomson, the brother of Lord Kelvin, at Queen's College in Belfast.

In Calculus and most other branches of mathematics beyond practical Geometry, angles are nearly always measured in radians. Because radians arise naturally when dealing with circles, essential relationships are expressed more concisely in radians. In particular, results involving trigonometric functions are more straightforward when the variables are expressed in radians.

7.1: Reference Angles

7.1.1: Resources and Key Concepts

7.1.2: Homework
7.2: Radian Measure and Arc Length
7.2.1: Resources and Key Concepts
7.2.2: Homework
7.3: The Trigonometric Functions - Unit Circle Definition
7.3.1: Resources and Key Concepts
7.3.2: Homework
7.4: Velocities and Areas of Sectors
7.4.1: Resources and Key Concepts
7.4.2: Homework

7.1: Reference Angles

Note to the Instructor (click to expand)

In this section, we return to considering *any* angle (not just acute angles) - related to the *special angles* or not (although the focus is primarily on angles related to special angles). This is where we tie Coordinate Trigonometry to Right Triangle Trigonometry via the use of reference triangles and reference angles. We increase our calculator usage in two ways:

1. We finally have the need to compute trigonometric functions at any angle (e.g., $-137°$). While we can do so by direct calculator usage, we should not miss the opportunity to show that, for example, $\sin(-137°)$ is nearly the same as the sine of the reference angle for $-137°$. This foreshadows how we will be working with Trigonometry from this point forward.
2. We need to find *non-acute* angles satisfying given trigonometric ratios (e.g., for what θ is $\sin(\theta) = -0.682$?).

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Identify the reference angle for an angle given in standard position.
- Evaluate a trigonometric function at a non-acute angle using the reference angle.
- Use technology to approximate the value of a trigonometric function at a non-acute angle.
- Determine an angle given the quadrant in which the angle terminates and the value of a trigonometric function.

Hawk A.I. Section-Specific Tutor

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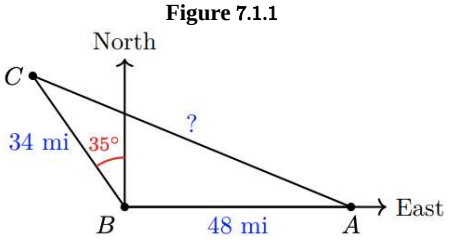
The town of Avery lies 48 miles due east of Baker, and Clio is 34 miles in the bearing $\mathrm{N}\,35°\,\mathrm{W}$ from Baker. How far is it from Avery to Clio?

We know how to solve right triangles using the trigonometric ratios; however, the triangle formed by the three towns is *not* a right triangle because it includes an obtuse angle of $125°$ at B, as shown in Figure 7.1.1.

Figure 7.1.1



Recall that we call this type of triangle an oblique triangle. Later in this course, we learn how to solve oblique triangles, but first, we must be able to find the values of the trigonometric functions for obtuse angles.

Reference Angles

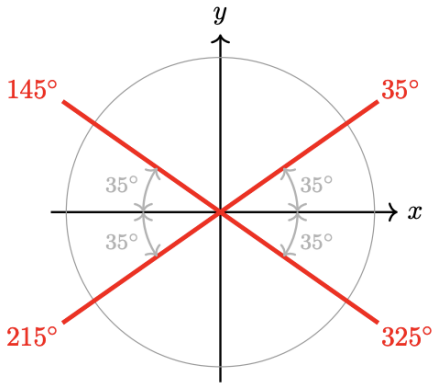
We now introduce an essential tool for evaluating trigonometric functions and solving trigonometric equations involving non-acute angles.

Definition: Reference Angle

The **reference angle** for an angle θ given in standard form is the acute angle formed between the terminal side of θ and the x-axis. By convention, the reference angle is always considered positive (even if it opens in a clockwise direction). It is denoted with the "hat" notation, $\hat{\theta}$.

Any acute angle θ is the reference angle for four angles between $0°$ and $360°$, one in each quadrant. The figure below shows the four angles in standard position whose reference angle is $35°$. Note that each angle is found by measuring $35°$ from the x-axis in the appropriate quadrant and that the four angles together make a "bow-tie" shape.

Figure 7.1.2



Example 7.1.1

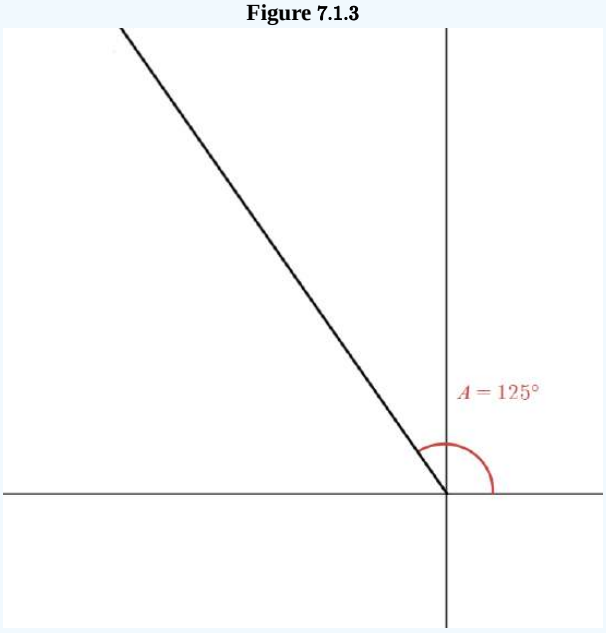
Find and label the reference angle for the given angle.

a. $A = 125°$
b. $B = 240°$
c. $\alpha = 310°$
d. $\beta = -315°$
e. $t = 820°$

Solutions

a. We begin by sketching the angle and its terminal side.

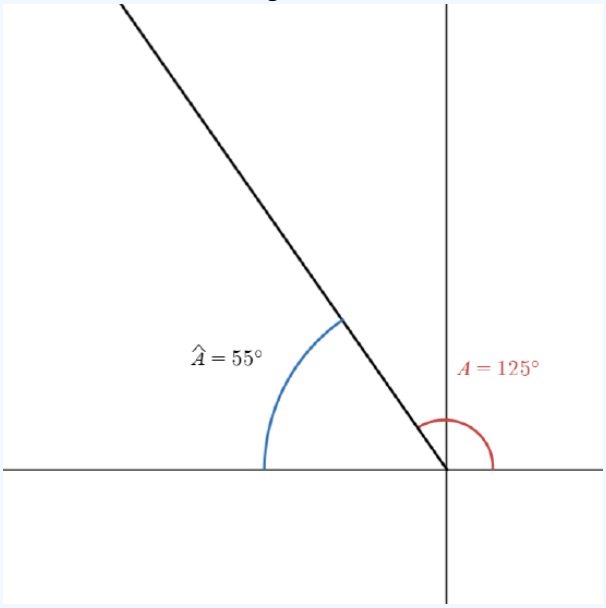
Figure 7.1.3



https://math.libretexts.org/@go/page/197523

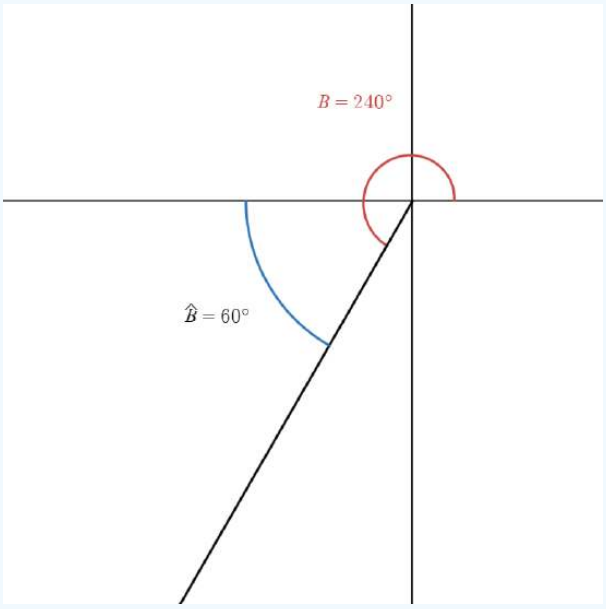
Since the terminal side of A is in QII, the smallest angle that gets us directly to the x-axis can be found by imagining we rotated to the negative x-axis ($180°$) and then backed off by $\hat{A}$ to get to $A = 125°$. Mathematically, this is $180° - 125° = 55°$. Hence, the reference angle for A is $\hat{A} = 55°$.

Figure 7.1.4



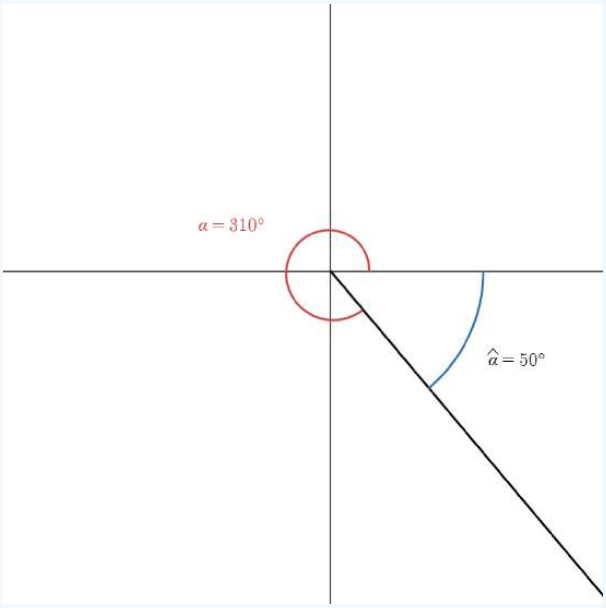
b. Again, sketch the original angle and its terminal side. $B \in$ QIII, so the smallest angle to the x-axis can be found by imagining we rotated to the negative x-axis ($180°$) and then added the reference angle $\hat{B}$ to get to $B = 240°$. That is, $180° + \hat{B} = 240°$. Therefore, $\hat{B} = 240° - 180° = 60°$ (see Figure 7.1.5).

Figure 7.1.5



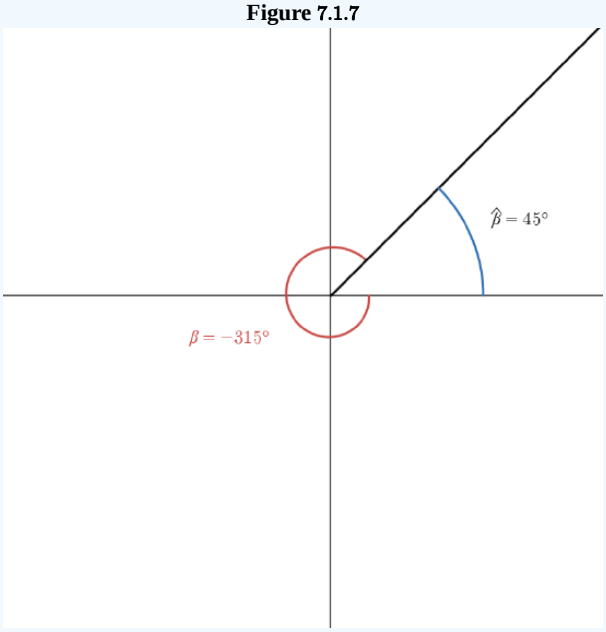
c. A quick sketch shows that $\alpha \in$ QIV. To find the reference angle, we imagine rotating in a full circle ($360°$) and backing off by the angle $\hat{\alpha}$ to arrive at $310°$. Therefore, $360° - \hat{\alpha} = 310°$. Hence, $360° - 310° = 50° = \hat{\alpha}$ (see Figure 7.1.6).

Figure 7.1.6



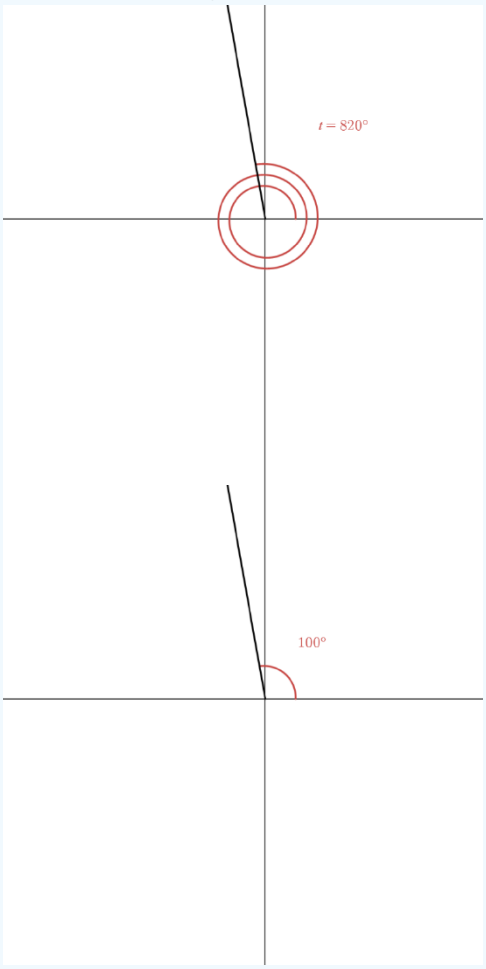
d. β being a negative angle doesn't change the tactic too much. We sketch $\beta = -315°$ and its terminal side. Since β terminates in QI, we can imagine finding the reference angle by rotating a full revolution in the negative direction ($-360°$) and backing off by $\hat{\beta}$ to arrive at $\beta = -315°$. "Backing off" in this instance means "adding back" $\hat{\beta}$. Therefore, we want to compute $-360° + \hat{\beta} = -315°$. Solving this equation yields $\hat{\beta} = 45°$ (see Figure 7.1.7).

Figure 7.1.7



e. When an angle is outside of the interval $(-360°, 360°)$, a little extra thought is involved (but not much more). Sketching the angle and its terminal side, it's easy to notice that we can find a coterminal angle to $t = 820°$ that is between $0°$ and $360°$. Subtracting $720°$, we arrive at the coterminal angle $100°$ (see Figure 7.1.8).

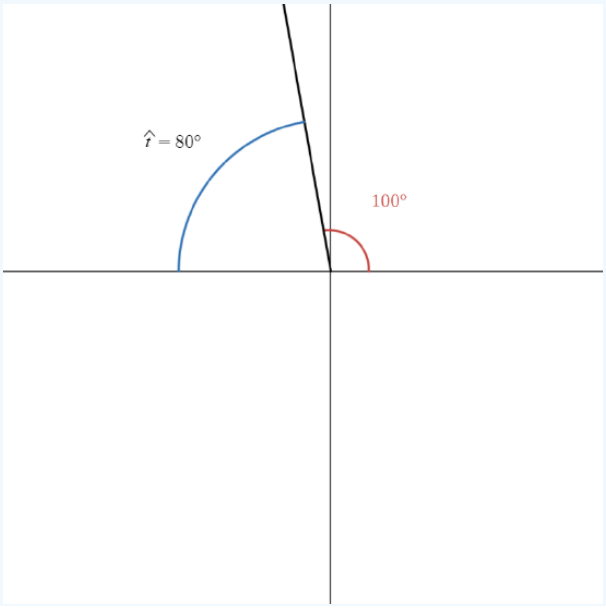
Figure 7.1.8



We are now back to the simpler case of a quadrant II angle. To find the reference angle, we imagine we rotated to the negative x-axis ($180°$) and then backed off by $\hat{t}$ to arrive at $100°$. Mathematically, this means

$$180° - \hat{t} = 100° \implies \hat{t} = 80°.$$

Figure 7.1.9



While you should strive for understanding in mathematics over memorization, some students initially find the following guidelines helpful until they get used to reference angles. Trust me when I say you want to use the following guideline only for a while - it's too bulky.

Let θ be an angle in standard position.

1. If $0° < \theta < 360°$, let $t = \theta$.
2. If θ is not between $0°$ and $360°$, then let t be the angle coterminal to θ that *is* between $0°$ and $360°$.

$$\begin{aligned}
&\text{If} \quad t \text{ terminates in QI}, \quad \text{then} \quad \hat{\theta} = t \\
&\text{If} \quad t \text{ terminates in QII}, \quad \text{then} \quad \hat{\theta} = 180° - t \\
&\text{If} \quad t \text{ terminates in QIII}, \quad \text{then} \quad \hat{\theta} = t - 180° \\
&\text{If} \quad t \text{ terminates in QIV}, \quad \text{then} \quad \hat{\theta} = 360° - t
\end{aligned}$$

Caution: Reference Angles are Acute

By definition, reference angles are acute angles. This means they are *not* $0°$ or $90°$. In fact, quadrantal angles do not have reference angles.

Checkpoint 7.1.1

Draw the angle and state its reference angle.

a. $\alpha = 80°$
b. $\beta = 130°$
c. $\kappa = 225°$
d. $\gamma = 330°$
e. $\lambda = -125°$

Answers

a. $\hat{\alpha} = 80°$
b. $\hat{\beta} = 50°$
c. $\hat{\kappa} = 45°$
d. $\hat{\gamma} = 30°$
e. $\hat{\lambda} = 55°$

Evaluating Trigonometric Functions Using Reference Angles

We can use reference angles to help us evaluate a trigonometric function at a non-acute angle θ. To do so, we construct a **reference triangle** for θ.

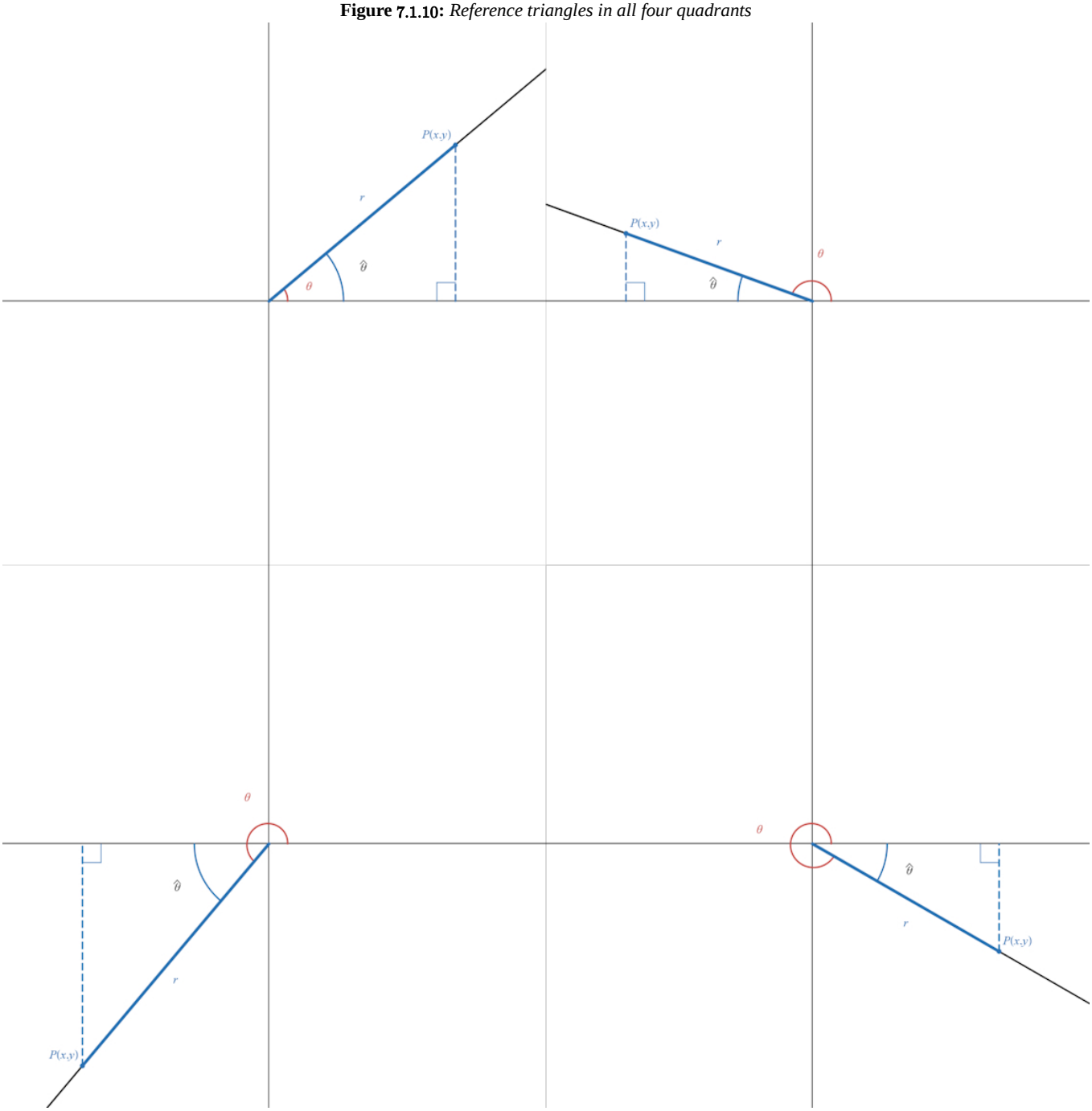
Definition: Reference Triangle

A **reference triangle** is a right triangle formed by drawing a perpendicular from a point on the terminal side of an angle (in standard position) to the x-axis.

A reference triangle can easily be drawn by choosing a point P on the terminal side of θ, and then drawing a line from P perpendicular to the x-axis.

The figure below shows angles θ in all four quadrants and the reference angle, $\hat{\theta}$, for each. Study the figures, and make sure you understand how to find the reference angle in each quadrant.

Figure 7.1.10: *Reference triangles in all four quadrants*



Summary

These observations may help you remember the formulas:

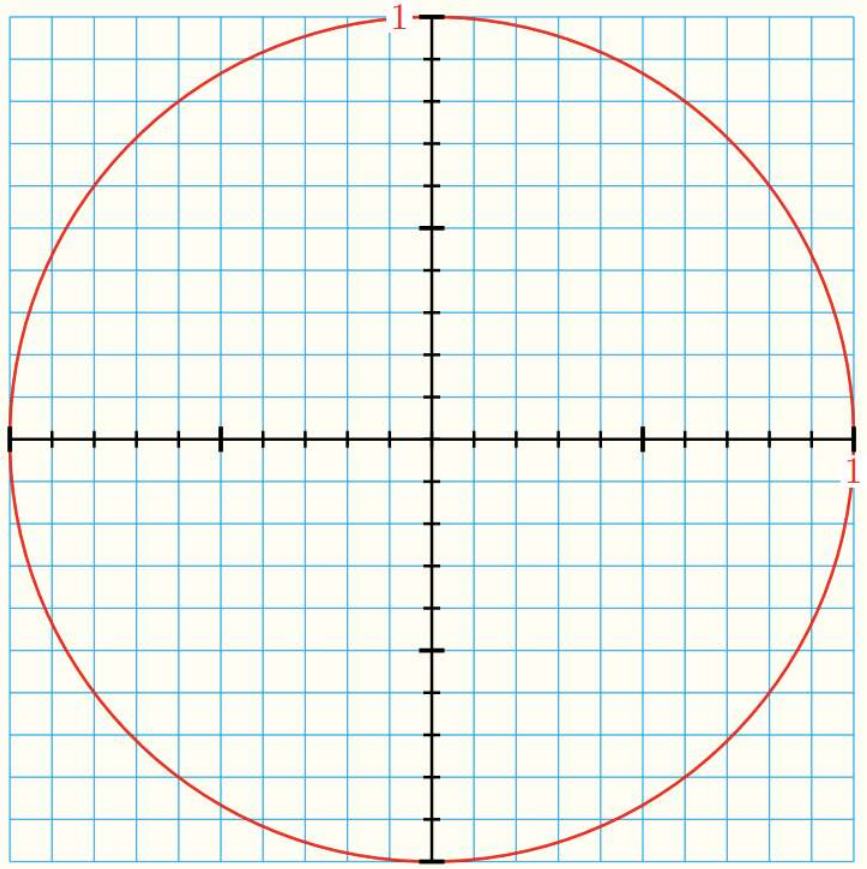
- The right triangle formed in this way always lies between the terminal side and the x-axis.
- The positive acute angle formed between the terminal side and the x-axis is the reference angle, and the right triangle is the reference triangle.

Activity

The following activity reveals the usefulness of reference angles and reference triangles when evaluating trigonometric functions of non-acute angles.

Activity: Reference Angles

Figure 7.1.11



a. Use a protractor to draw an angle of $56°$ in standard position. Draw its reference triangle.
b. Use your calculator to find the sine and cosine of $56°$, rounded to two decimal places. Label the sides of the reference triangle with their lengths.
c. What are the coordinates of the point P where your angle intersects the unit circle?
d. Draw the reflection of your reference triangle across the y-axis so that you have a congruent triangle in the second quadrant.
e. You now have the reference triangle for a second-quadrant angle in standard position. What is that angle?
f. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point Q where the angle intersects the unit circle.
g. Draw the reflection of your triangle from part (d) across the x-axis so that you have a congruent triangle in the third quadrant.
h. You now have the reference triangle for a third-quadrant angle in standard position. What is that angle?
i. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point R where the angle intersects the unit circle.
j. Draw the reflection of your triangle from part (g) across the y-axis so that you have a congruent triangle in the fourth quadrant.
k. You now have the reference triangle for a fourth-quadrant angle in standard position. What is that angle?
l. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point where the angle intersects the unit circle.
m. **Generalize:** All four of your angles have the same reference angle, $56°$. For each quadrant, write a formula for the angle whose reference angle is θ.
    i. Quadrant I:
    ii. Quadrant II:

iii. Quadrant III:
iv. Quadrant IV:

Reference Angle Theorem

In all four versions of Figure **7.1.10**, coordinate trigonometry states that

$$\begin{aligned} \sin(\theta) &= \frac{y}{r} & \csc(\theta) &= \frac{r}{y} \\ \cos(\theta) &= \frac{x}{r} & \sec(\theta) &= \frac{r}{x} \\ \tan(\theta) &= \frac{y}{x} & \cot(\theta) &= \frac{x}{y} \end{aligned}$$

However, it is also true that in all four versions of Figure **10**, the side of the reference triangle opposite $\hat{\theta}$ has length $|y|$, the side of the reference triangle adjacent to $\hat{\theta}$ has length $|x|$, and the hypotenuse of the reference triangle has a length of r. Therefore, by right triangle trigonometry,

$$\begin{aligned} \sin\left(\hat{\theta}\right) &= \frac{|y|}{r} & \csc\left(\hat{\theta}\right) &= \frac{r}{|y|} \\ \cos\left(\hat{\theta}\right) &= \frac{|x|}{r} & \sec\left(\hat{\theta}\right) &= \frac{r}{|x|} \\ \tan\left(\hat{\theta}\right) &= \frac{|y|}{|x|} & \cot\left(\hat{\theta}\right) &= \frac{|x|}{|y|} \end{aligned}$$

Thus, the values of the trigonometric functions of the angle θ match the values of the trigonometric functions of $\hat{\theta}$, except, possibly, for a difference in sign. The sign of the trigonometric function's value can be determined by the quadrant in which the terminal side of the angle lies. This result is summarized in the following theorem.

Theorem: Reference Angle Theorem

The value of a trigonometric function of any angle is equal to that of the function at its reference angle, except for sign. The quadrant determines the sign of the function.

In the previous chapter, we used a calculator to approximate the values of the trigonometric functions at *acute* angles. Your calculator can easily find the value of a trigonometric function at *any* angle. We showcase this in the following example.

Example 7.1.2

a. Find the reference angle for $200°$.
b. Sketch $200°$ and its reference angle in standard position, along with the reference triangle. Verify that both angles have the same trigonometric values, up to sign.

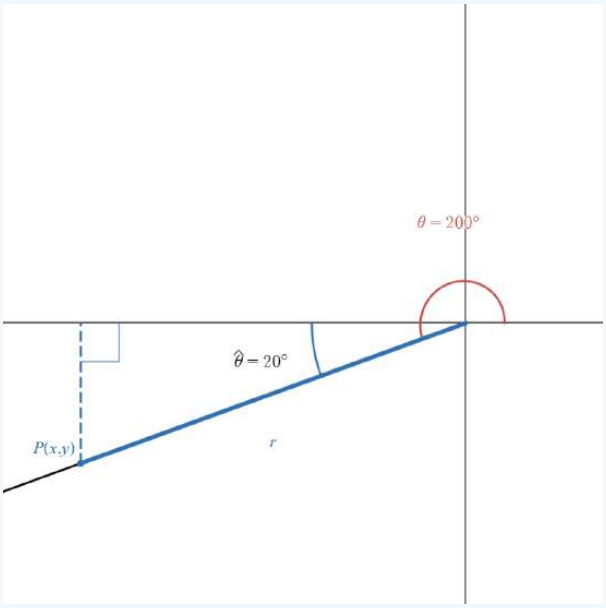
Solutions

a. In standard position, an angle of $200°$ lies in the third quadrant. Therefore, we rotate to $180°$ and add the reference to get to $200°$. Therefore, the reference angle is

$$200° - 180° = 20° = \hat{\theta}$$

b. Both angles are shown in Figure **7.1.12**.

Figure 7.1.12



You can use your calculator to verify the following values.

$$\begin{aligned} \sin(20°) &\approx 0.3420 & \sin(200°) &\approx -0.3420 \\ \cos(20°) &\approx 0.9397 & \cos(200°) &\approx -0.9397 \\ \tan(20°) &\approx 0.3640 & \tan(200°) &\approx 0.3640 \end{aligned}$$

Note that the results in Example 7.1.2 align with what we said previously - the sign of the trigonometric function's value can be determined by the quadrant in which the terminal side of the angle lies. Thus, to compute $\sin(200°)$, we could note that $\theta \in$ QIII and the reference angle is $20°$. Since the sine is negative in QIII, we get

$$\begin{aligned} \sin(200°) &= \boxed{?}\sin(20°) && \left(\sin(\theta) \text{ is equal to } \sin\left(\hat{\theta}\right), \text{ except for, possibly, the sign}\right) \\ &= -\sin(20°) && (\text{Since } 200° \in \text{QIII, sine is negative}) \end{aligned}$$

From this point forward in Trigonometry, the following Trigonometry Mantra will be helpful.

> *Trigonometry is about two things - reference angles and quadrants.*

While this mantra oversimplifies the subject, most of Trigonometry is easier once you know the reference angle and quadrant for a given angle.

? Checkpoint 7.1.2

a. Find the reference angle for $285°$.

b. Sketch $285°$ and its reference angle in standard position, along with the reference triangle. Verify that both angles have the same trigonometric ratios, up to sign.

Answers

a. $75°$

b. $\sin(285°) = -\sin(75°) \approx -0.9659, \cos(285°) = \cos(75°) \approx 0.2588, \tan(285°) = -\tan(75°) \approx -3.7321$

When combined with the Reference Angle Theorem, the following corollary improves the speed with which we can evaluate trigonometric functions.

Corollary: Rotational Symmetry

For any integer n,

$$\begin{aligned} \sin(\theta+360°n) &= \sin(\theta) & \csc(\theta+360°n) &= \csc(\theta) \\ \cos(\theta+360°n) &= \cos(\theta) & \sec(\theta+360°n) &= \sec(\theta) \\ \tan(\theta+360°n) &= \tan(\theta) & \cot(\theta+360°n) &= \cot(\theta) \end{aligned}$$

This result is from our previous discussion that the value of a trigonometric function is the same for all angles coterminal with the original given angle.

An Additional Note About the Tangent

As we will investigate later, it turns out that

$$\tan(\theta+180°n)=\tan(\theta) \quad \text{and} \quad \cot(\theta+180°n)=\cot(\theta)$$

for any integer n; however, this is not needed for our current discussion.

Let's see how this corollary, along with focusing on reference angles and quadrants, speeds up the evaluation of trigonometric functions whose reference angles are the special angles.

Example 7.1.3

Find the exact value of each trigonometric function.

a. $\cos(225°)$
b. $\cot(120°)$
c. $\tan(-210°)$
d. $\sin(690°)$

Solutions

a. We quickly note that $225° \in$ QIII and the reference angle is $\hat{\theta}=45°$. Thus,

$$\begin{aligned} \cos(225°) &= \boxed{?}\cos(45°) && \text{(Reference Angle Theorem)} \\ &= -\cos(45°) && \text{(cosine is negative in QIII)} \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

b. $120° \in$ QII and the reference angle is $\hat{\theta}=60°$. Thus,

$$\begin{aligned} \cot(120°) &= \boxed{?}\cot(60°) && \text{(Reference Angle Theorem)} \\ &= -\cot(60°) && \text{(cotangent is negative in QII)} \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

c. While we could solve this problem by rotating clockwise by $210°$, let's use the corollary.

$$\begin{aligned} \tan(-210°) &= \tan(-210°+360°) && \text{(corollary)} \\ &= \tan(150°) \\ &= \boxed{?}\tan(30°) && \text{(Reference Angle Theorem)} \\ &= -\tan(30°) && (150° \in \text{QII and tangent is negative in QII}) \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

d.

$$\begin{aligned} \sin(690^\circ) &= \sin(690^\circ - 360^\circ) && \text{(corollary)} \\ &= \sin(330^\circ) \\ &= \boxed{?}\sin(30^\circ) && \text{(Reference Angle Theorem)} \\ &= -\sin(30^\circ) && \text{(330° ∈ QIV and sine is negative in QIV)} \\ &= -\frac{1}{2} \end{aligned}$$

? Checkpoint 7.1.3

Find the exact value of each trigonometric function (without touching a calculator).

a. $\sin(210^\circ)$
b. $\sec(-780^\circ)$
c. $\sin(405^\circ)$

Answers

a. $-\frac{1}{2}$
b. 2
c. $\frac{1}{\sqrt{2}}$

Finding Non-Acute Angles

We now turn our attention back to finding angles given values of trigonometric functions. Last chapter, we were able to find θ such that $\tan(\theta) = 0.5662$. At that time, we used the TAN^{-1} button on our calculator to find $\tan^{-1}(0.5662) \approx 29.5185^\circ$. At that time, however, we were only working with right triangle trigonometry, and the angles we were interested in were all acute. When working with non-acute angles, the inverse trigonometric functions can get tricky.

To avoid the trickiness associated with inverse trigonometric functions (and to give us more time with Trigonometry before diving into the more profound theory of the inverse trigonometric functions), it's best to think of them as returning *reference* angles. Since reference angles are always acute, this interpretation works with what we did in the last chapter; however, as we have seen in this section, we are now dealing with angles that are not necessarily acute.

> *For now, think of the inverse trigonometric functions as returning reference angles.*

We must evaluate the inverse trigonometric function at a *positive* value to return a reference angle. The reason for this cannot be fully justified at this time. Still, we *can* say that, for example, $\sin^{-1}(-0.31)$ returns an angle θ so that $\sin(\theta) = -0.31$; however, we know that the values of *all* trigonometric functions are *positive* in QI. Since we want the returned angle to be a reference angle, it must be acute (and, therefore, in the first quadrant). Hence, there is no possibility for a reference angle to be returned if the argument of our inverse trigonometric function is negative.

If we agree that inverse trigonometric functions evaluated at positive numbers return reference angles, we can ask for solutions to equations involving trigonometric functions.

For example, find θ if

$$\sin(\theta) = -0.31,$$

where $\theta \in$ QIII.

First, it should make sense that θ could be in the third quadrant because the sine at θ is negative, and the sine is negative in the third quadrant (as well as in the fourth quadrant). Remember, Trigonometry is about two things - reference angles and quadrants. We have already been given a quadrant. We evaluate the inverse sine function at $|-0.31| = 0.31$ to find the reference angle. Hence, the reference angle is

$$\hat{\theta} = \sin^{-1}(|-0.31|) = \sin^{-1}(0.31) \approx 18.1°.$$

The angle in QIII with reference angle $\hat{\theta} \approx 18.1°$ is $\theta = 180° + \hat{\theta} \approx 180° + 18.1° = 198.1°$.

Summary

To find the angle θ so that the trigonometric function of θ is a given ratio:

1. **Reference Angle.** Use the inverse trigonometric function of the absolute value of the given ratio to determine the reference angle, $\hat{\theta}$.
2. **Quadrant.** Use the given information within the problem to "lock down" the quadrant in which θ terminates.
3. **Actual Angle.** Use the reference angle and quadrant from (1) and (2) to determine θ.

The second step in the summary above assumes you are given extra information to help determine the quadrant in which θ terminates. While this will be true in this section, we will eventually shed that requirement. We will begin to find *all* angles that make a given trigonometric equation valid.

Example 7.1.4

Find θ to the nearest tenth of a degree, if $0° < \theta < 360°$.

a. $\cos(\theta) = -0.8457$, where $\theta \in$ QIII.
b. $\tan(\theta) = -1.9122$, where $\sin(\theta) < 0$.
c. $\csc(\theta) = 2.1174$, where $\cos(\theta) < 0$.

Solutions

a. **Reference Angle.** We start by finding the reference angle using the absolute value of the given ratio.

$$\hat{\theta} = \cos^{-1}(|-0.8457|) = \cos^{-1}(0.8457) \approx 32.3°$$

Quadrant. Luckily, we were explicitly given the quadrant in which θ terminates as QIII.
Actual Angle. Thus,

$$\theta = 180° + \hat{\theta} \approx 180° + 32.3° = 212.3°.$$

b. **Reference Angle.**

$$\hat{\theta} = \tan^{-1}(|-1.9122|) = \tan^{-1}(1.9122) \approx 62.4°$$

Quadrant. This example requires a little extra thought than part (a). We are given that both the angle's sine and tangent are negative. The sine is negative in quadrants III and IV, while the tangent is negative in quadrants II and IV. The quadrant where both of these conditions are true is quadrant IV.
Actual Angle. Hence,

$$\theta = 360° - \hat{\theta} \approx 360° - 62.4° = 297.6°.$$

c. **Reference Angle.** Since our calculator (or other technology) does not have an inverse cosecant button, we begin by rewriting the given equation using the Ratio Identities.

$$\csc(\theta) = 2.1174 \implies \frac{1}{\sin(\theta)} = 2.1174 \implies \frac{1}{2.1174} = \sin(\theta)$$

Therefore,

$$\hat{\theta} = \sin^{-1}\left(\frac{1}{2.1174}\right) \approx 28.2°.$$

Quadrant. The cosecant is positive in the same quadrants the sine is positive - namely, QI and QII. The cosine is negative in QII and QIII. Therefore, the quadrant where θ terminates will be the shared quadrant, which is QII.
Actual Angle. Thus,

$$\theta = 180^\circ - \hat{\theta} \approx 180^\circ - 28.2^\circ = 151.8^\circ.$$

Checkpoint 7.1.4

a. Solve $\sec(\theta) = -1.8$, where $90^\circ < \theta < 180^\circ$. Round your answer to the nearest tenth of a degree.
b. Find the sine of θ given that $\tan(\theta) = -2$, $\cos(\theta) > 0$, and $0^\circ < \theta < 360^\circ$. Round your answer to four decimal places.

Answers

a. $\theta \approx 123.7^\circ$
b. $\sin(\theta) \approx -0.8944$

Of course, if given a "special" trigonometric ratio, you should not reach for a calculator.

Example 7.1.5

Find α if $\cos(\alpha) = -\frac{\sqrt{3}}{2}$ and $\alpha \in$ QII, where $0^\circ < \alpha < 360^\circ$.

Solution

Reference Angle. It is crucial that you recognize the special ratios. Sketching the 30°-60°-90° triangle, we see that the reference angle for which $\cos(\alpha) = \left|-\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2}$ is $\hat{\alpha} = 30^\circ$.
Quadrant. $\alpha \in$ QII
Actual Angle. Thus,

$$\alpha = 180^\circ - \hat{\alpha} = 180^\circ - 30^\circ = 150^\circ.$$

Checkpoint 7.1.5

Solve $\tan(\beta) = -\sqrt{3}$ given $0^\circ < \beta < 360^\circ$ and $\cos(\beta) > 0$.

Answer

$\beta = 300^\circ$

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7.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Reference Angle
  - Angles - Reference Angles

KEY CONCEPTS

DEFINITIONS

- **Reference Angle:** For an angle θ in standard position, the reference angle, denoted $\hat{\theta}$, is the positive acute angle formed between the terminal side of θ and the x-axis.
- **Reference Triangle:** A right triangle formed by drawing a perpendicular line from a point on the terminal side of an angle in standard position to the x-axis.

THEOREMS

- **Reference Angle Theorem:** The value of a trigonometric function of any angle is equal to the value of that same function at its reference angle, except possibly for a difference in sign. The sign is determined by the quadrant in which the angle's terminal side lies.
- **Corollary: Rotational Symmetry:** For any integer n, the value of a trigonometric function is the same for all coterminal angles. For example, $\sin(\theta + 360°n) = \sin(\theta)$.

COMMON MISTAKES

- **Reference Angles are Acute:** By definition, reference angles are always acute (strictly between 0° and 90°). Quadrantal angles do not have reference angles.
- **Measuring from the y-axis:** A reference angle is always measured from the terminal side to the x-axis, never to the y-axis.
- **Negative Reference Angles:** Reference angles are always considered positive, even if the rotation to the x-axis is clockwise.
- **Using Negative Inputs for Inverse Functions:** When using an inverse trigonometric function to find a reference angle, the input must be the positive (absolute) value of the trigonometric ratio.

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7.1.2: HOMEWORK

READING QUESTIONS

1. What is a reference angle?
2. How is a reference triangle for an angle θ constructed?
3. Are reference angles ever negative or greater than 90°?
4. What is the reference angle for 150°?
5. What is the reference angle for 300°?
6. Explain the Reference Angle Theorem in your own words.
7. What two pieces of information are needed to evaluate a trigonometric function at a non-acute angle without a calculator?
8. If you are trying to find an angle θ in Quadrant III where $\sin(\theta) = -0.5$, what is the first step in finding the reference angle using a calculator?
9. Once you find the reference angle $\hat{\theta}$, how do you find the actual angle θ if it terminates in Quadrant II?
10. Once you find the reference angle $\hat{\theta}$, how do you find the actual angle θ if it terminates in Quadrant IV?

HOMEWORK

VOCABULARY CHECK

1. The ___ for θ is the positive acute angle formed between the terminal side of θ and the x-axis.

CONCEPT CHECK

2. Camille wants an obtuse angle, θ, that satisfies $\sin(\theta) = 0.3$. Ivan presses some buttons on his calculator and reports that $\theta \approx 17.46°$. Explain Ivan's error and give a correct approximation of θ accurate to two decimal places.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

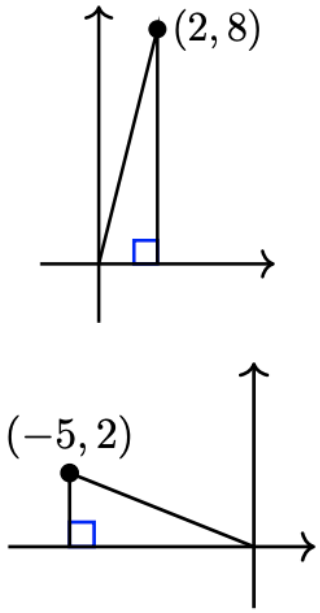
3. To construct a reference triangle for an angle, we must choose a point P on the terminal side of θ and draw a line from point P perpendicular to the y-axis.
4. The trigonometric ratios of any angle are equal to the ratios of its reference angle, except for sign. The sign of the ratio is determined by the quadrant.

BASIC SKILLS

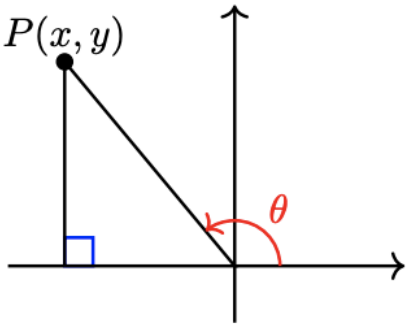
5. Give the lengths of the legs and the hypotenuse of each right triangle.

   a.

   b.



6. Explain why the length of the horizontal leg of the right triangle is $-x$.



For the following exercises, find the reference angle for the given angle. Make a sketch showing the angle, the reference angle, and the reference triangle.

7. $\theta = 100°$
8. $A = 125°$
9. (MyOpenMath) $\beta = 216°$
10. $\theta = 194° 15'$
11. $\alpha = 242°$
12. $\phi = 297°$
13. $\lambda = 137° 50'$
14. $\theta = 336°$

For the following exercises, find three angles between $90°$ and $360°$ having the given reference angle, and sketch all four angles on the same grid.

15. $15°$
16. $26°$
17. $40°$

18. $50°$

19. $68°$

20. $75°$

For the following exercises, find the exact value of the trigonometric function using reference angles and quadrants (do not use a calculator).

21. $\cos(120°)$

22. $\sin(135°)$

23. $\tan(150°)$

24. $\sec(225°)$

25. $\sin(240°)$

26. $\cot(210°)$

27. $\csc(330°)$

28. $\cos(315°)$

29. $\sin(300°)$

30. $\cos(-150°)$

31. $\sin(-210°)$

32. $\sec(-315°)$

33. $\sin(585°)$

34. $\cos(840°)$

35. $\tan(-585°)$

For the following exercises, use the values below to find the value of the trigonometric function. Do not use a calculator!

$\cos(23°) \approx 0.9205 \quad \sin(46°) \approx 0.7193 \quad \tan(78°) \approx 4.7046$

36. $\cos(157°)$

37. $\sin(226°)$

38. $\csc(314°)$

39. $\sec(203°)$

40. $\tan(258°)$

41. $\cot(282°)$

42. $\sin(-134°)$

43. $\cos(-383°)$

44. On the circle in the figure, all angles are shown in standard position. Find the measure in degrees of the angles labeled (a) - (i) and state their reference angles.

45.

a. Draw three angles, one in each quadrant except the first, whose reference angle is $60°$.

b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

46.

a. Draw three angles, one in each quadrant except the first, whose reference angle is $30°$.

b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

47.

a. Draw three angles, one in each quadrant except the first, whose reference angle is $45°$.

b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

48. Complete the table with exact values.

θ	$30°$	$45°$	$60°$	$120°$	$135°$	$150°$	$210°$	$225°$	$240°$	$300°$	$315°$	$330°$
$\sin(\theta)$												
$\cos(\theta)$												
$\tan(\theta)$												
$\csc(\theta)$												
$\sec(\theta)$												
$\cot(\theta)$												

For the following exercises, find a second angle between $0°$ and $360°$ with the given trigonometric ratio.

49. $\sin(75°)$

50. $\cos(32°)$

51. $\tan(84°)$

52. $\csc(16°)$

53. $\sec(47°)$

54. $\cot(56°)$

For the following exercises, use the given information to find an approximation of θ to the nearest degree if $0° \le \theta < 360°$.

55. $\tan(\theta) = 8.1443$ with $\theta \in$ QIII

56. $\sin(\theta) = 0.7880$ with $\theta \in$ QII

57. $\cos(\theta) = 0.9205$ with $\theta \in$ QIV

58. $\tan(\theta) = -3.4874$ with $\theta \in$ QII

59. $\sin(\theta) = -0.9962$ with $\theta \in$ QIV

60. $\cos(\theta) = -0.0349$ with $\theta \in$ QII

61. $\sin(\theta) = 0.8381$ where $\cos(\theta) > 0$

62. $\tan(\theta) = -2.1728$ where $\sin(\theta) > 0$

63. $\cos(\theta) = -0.2155$ where $\tan(\theta) > 0$

64. $\sin(\theta) = -0.1179$ where $\tan(\theta) < 0$

65. $\cos(\theta) = 0.1818$ where $\csc(\theta) < 0$

66. $\tan(\theta) = 1.7531$ where $\sec(\theta) < 0$

67. $\csc(\theta) = 2.8549$ where $\cos(\theta) < 0$

68. $\sec(\theta) = -1.3888$ where $\csc(\theta) > 0$

69. $\cot(\theta) = -0.9411$ where $\cos(\theta) < 0$

For the following exercises, use the given information to find the exact value of θ if $0° \le \theta < 360°$. Do not use a calculator!

70. $\tan(\theta) = \sqrt{3}$ with $\theta \in$ QIII

71. $\sin(\theta) = \frac{1}{2}$ with $\theta \in$ QII

72. $\cos(\theta) = \frac{1}{\sqrt{2}}$ with $\theta \in$ QIV

73. $\tan(\theta) = -1$ with $\theta \in$ QII

74. $\sin(\theta) = -\frac{\sqrt{3}}{2}$ with $\theta \in$ QIV

75. $\cos(\theta) = -\frac{1}{2}$ with $\theta \in$ QII
76. $\sin(\theta) = \frac{1}{\sqrt{2}}$ where $\cos(\theta) > 0$
77. $\tan(\theta) = -\frac{1}{\sqrt{3}}$ where $\sin(\theta) > 0$
78. $\cos(\theta) = -\frac{\sqrt{3}}{2}$ where $\tan(\theta) > 0$
79. $\sin(\theta) = -\frac{1}{2}$ where $\tan(\theta) < 0$
80. $\cos(\theta) = \frac{1}{2}$ where $\csc(\theta) < 0$
81. $\tan(\theta) = 1$ where $\sec(\theta) < 0$
82. $\csc(\theta) = 2$ where $\cos(\theta) < 0$
83. $\sec(\theta) = -\frac{2}{\sqrt{3}}$ where $\csc(\theta) > 0$
84. $\cot(\theta) = -\sqrt{3}$ where $\cos(\theta) < 0$

SYNTHESIS QUESTIONS

For the following exercises, find exact values for all solutions between 0° and 360°.

85. $\cos(\theta) = -\cos(24°)$
86. $\tan(\theta) = -\tan(9°)$
87. $\sin(\theta) = -\sin(66°)$
88. $\cos(\theta) = -\cos(78°)$
89. $\tan(\theta) = -\tan(31°)$
90. $\sin(\theta) = -\sin(42°)$
91. Later we will be able to show that $\sin(18°) = \frac{\sqrt{5}-1}{4}$. What is the exact value of $\sin(162°)$?
92. Later we will be able to show that $\cos(36°) = \frac{\sqrt{5}+1}{4}$. What is the exact value of $\cos(144°)$?

For the following exercises,
(a) Sketch an angle in standard position with the given properties.
(b) Find $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.
(c) Find the angle θ, rounded to tenths of a degree.

93. The point (−5, 12) is on the terminal side.
94. The point (12, 9) is on the terminal side.
95. $\cos(\theta) = -0.8$
96. $\cos(\theta) = \frac{5}{13}$
97. $\cos(\theta) = \frac{3}{11}$
98. $\cos(\theta) = -\frac{5}{6}$
99. $\tan(\theta) = -\frac{1}{6}$
100. $\tan(\theta) = \frac{9}{5}$
101. $\tan(\theta) = 4$
102. $\tan(\theta) = -1$

For the following exercises,
(a) Find the sine and cosine of the angle.
(b) Sketch the supplement of the angle in standard position. (Use congruent triangles.)
(c) Find the sine and cosine of the supplement.
(d) Find the angle and its supplement, rounded to the nearest degree.

103.
104.
105.
106.

For the following exercises, refer to the following table of values.

θ	15°	25°	65°	75°	105°	115°	155°	165°
$\cos(\theta)$								
$\sin(\theta)$								
$\tan(\theta)$								

107. Use your calculator to fill in the table. Round values to four decimal places.
108. For each angle θ in the table, the angle $180° - \theta$ is also in the table.
    a. What is true about $\sin\theta$ and $\sin(180° - \theta)$?
    b. What is true about $\cos\theta$ and $\cos(180° - \theta)$?
    c. What is true about $\tan\theta$ and $\tan(180° - \theta)$?

The following exercises preview the concept of solving trigonometric equations (a topic reserved for later in the course). For each equation, find two different angles between 0° and 360° that satisfy the equation. Round your answers to the nearest 0.1°.

109. $\sin(\theta) = 0.7$
110. $\sin(\theta) = 0.1$
111. $\frac{\sin(\theta)}{6} = 0.14$
112. $\frac{5}{\sin(\theta)} = 6$
113. $4.8 = \frac{3.2}{\sin(\theta)}$
114. $1.5 = \frac{\sin(\theta)}{0.3}$

For the following exercises, fill in the blanks with complements or supplements.

115. If $\sin(57°) = q$, then $\sin(\_\_) = q$, and $\cos(\_\_)$ and $\cos(\_\_)$ both equal q.
116. If $\sin(18°) = w$, then $\sin(\_\_) = w$, $\cos(\_\_) = w$, and $\cos(\_\_) = -w$.
117. If $\cos(74°) = m$, then $\cos(\_\_) = -m$, and $\sin(\_\_)$ and $\sin(\_\_)$ both equal m.
118. If $\cos(36°) = t$, then $\cos(\_\_) = -t$, and $\sin(\_\_)$ and $\sin(\_\_)$ both equal t.
119.
    a. Sketch the line $y = \frac{3}{4}x$.
    b. A point on the line with a positive x-coordinate.
    c. The line $y = \frac{3}{4}x$ makes an angle with the positive x-axis. What is that angle?
    d. Repeat parts (a) through (c) for the line $y = \frac{-3}{4}x$, except find a point with a *negative* x-coordinate.

120.

a. Sketch the line $y = \frac{5}{3}x$.

b. Find a point on the line with a positive x-coordinate.

c. The line $y = \frac{5}{3}x$ makes an angle with the positive x-axis. What is that angle?

d. Repeat parts (a) through (c) for the line $y = \frac{-5}{3}x$, except find a point with a *negative* x-coordinate.

For the following exercises,
(a) Find the coordinates of point P. Round to two decimal places.
(b) Find the sides BC and PC of $\triangle PCB$.
(c) Find side PB.

121.

122.

CHALLENGE PROBLEMS

123. Why are the sines of supplementary angles equal, but the cosines are not? What about the tangents of supplementary angles?

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7.2: Radian Measure and Arc Length

Note to the Instructor (click to expand)

This section introduces radian measure and, as a natural consequence of the definition, arc length. All of the previous work we did evaluating trigonometric functions (including using reference angles and quadrants) is quickly repeated using radian measure.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Find the radian measure of the central angle given the radius and arc length.
- Convert an angle from degrees to radians or vice versa.
- Evaluate a trigonometric function using radians.
- Identify the reference angle for a given angle measured in radians.
- Solve an application involving arc length.

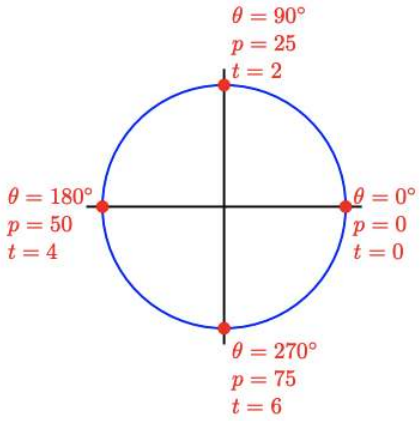
Hawk A.I. Section-Specific Tutor

COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Imagine riding on a Ferris wheel with a radius of 100 feet, where each rotation takes eight minutes. We can use angles in a standard position to describe your location as you travel around the wheel. Figure **7.2.1** shows the locations indicated by $\theta = 0°$, $90°$, $180°$, and $270°$.

Figure 7.2.1

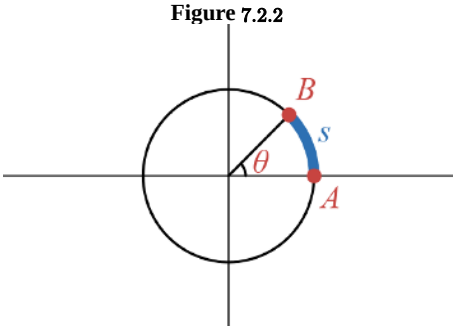


However, degrees are not the only way to specify a location on a circle. We could use a percent of one complete revolution[1] and label the same locations by $p = 0$, $p = 25$, $p = 50$, and $p = 75$. Another option is to use the time elapsed so that, for this example, we would have $t = 0$, $t = 2$, $t = 4$, and $t = 6$ minutes (see Figure **7.2.1**).

Another helpful method uses distance traveled, or arc length, along the circle. In Figure **7.2.1**, how far have you traveled around the Ferris wheel at each of the locations shown, assuming you hopped onto the ride at $\theta = 0°$?

Before we consider that question, let's agree on some vocabulary. To help us understand the vocabulary, let's consider Figure **7.2.2**.

Figure 7.2.2



We begin by naming a portion of the edge of a circle.

Definition: Arc and Arc Length

An **arc** along a circle is a portion of the circumference of a circle. It's the curved line connecting two points on the circle's edge. The length of this curve is called the **arc length**.

In Figure 7.2.2, the arc is in blue and its length, which we labeled as s, is called the arc length.

It should be natural to think that any arc along the circumference of a circle is related to an angle "opening" at the center of the circle. This angle (which is labeled as θ in Figure 7.2.2) gets its own name.

Definition: Central Angle

An angle with vertex at the center of a circle is called a **central angle** of the circle.

Thus, for the given arc in Figure 7.2.2, the related central angle is θ. If the arc represents a distance traveled, we sometimes refer to such an angle as the **angle of displacement**.

Finally, we need to introduce the conventional language used throughout most of Mathematics when it comes to relating the central angle to the arc.

Definition: Subtend

A central angle whose initial and terminal sides meet the endpoints of an arc is said to **subtend** (or **span**) the arc.

This definition means that the angle θ in Figure 7.2.2 subtends an arc of length s.

Arc Length

Recall that the circumference of a circle is given by the formula

$$C = 2\pi r. \tag{7.2.1}$$

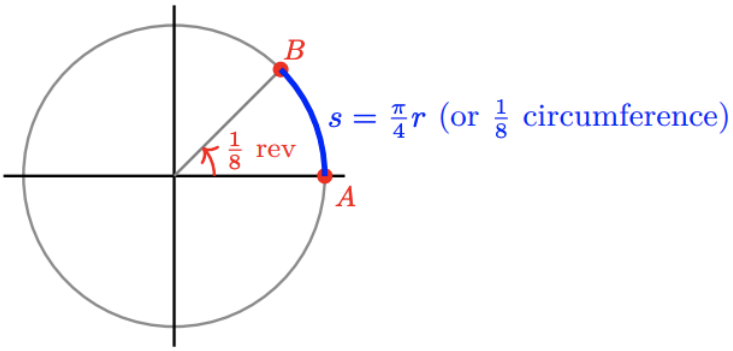
Another way to say this is that the circumference of a circle is proportional to its radius - increasing the radius of the circle increases the circumference and vice versa.

According to Equation 7.2.1, if we walk around the entire circumference of a circle, the distance we travel is the product of 2π and the radius, or *approximately* 6.28 times the radius (by the way, just like π, 2π gets its own special name - the Greek letter tau. That is, we define $\tau = 2\pi$). If we walk only part of the way around the circle, then the distance we travel depends also on the angle of displacement.

For example, an angle of $45°$ is $\frac{1}{8}$ of a complete revolution, so the arc length, s, from point A to point B in Figure 7.2.3 is $\frac{1}{8}$ of the circumference. Therefore,

$$s = \frac{1}{8}(2\pi r) = \frac{\pi}{4}r.$$

Figure 7.2.3



Similarly, the angle of displacement from point A to point C in Figure 7.2.4 is $\frac{3}{4}$ of a complete revolution, so the arc length along the circle from A to C, shown below, is

$$s = \frac{3}{4}(2\pi r) = \frac{3\pi}{2}r.$$

Figure 7.2.4

In general, the length of the arc spanned by an angle is proportional to the size of the angle for a given circle.

Theorem: Arc Length Formula (Part 1)

The length of the arc, s, in a circle of radius r, spanned by a fraction of a revolution, p, is

$$s = p \cdot (2\pi r).$$

The Ferris wheel in the introduction has circumference

$$C = 2\pi(100) \approx 628 \text{ feet},$$

so in half a revolution, you travel approximately 314 feet around the edge. In one-quarter revolution, you travel approximately 157 feet.

To indicate the same four locations from Figure 7.2.1 on the wheel by distance traveled, we would use $s = 0$, $s \approx 157$, $s \approx 314$, and $s \approx 471$, as shown below.

Figure 7.2.5

Example 7.2.1

What length of arc is spanned by an angle of $120°$ on a circle of radius 12 centimeters?

Solution

Because $\frac{120°}{360°} = \frac{1}{3}$, an angle of $120°$ is $\frac{1}{3}$ of a complete revolution, as shown below.

Figure 7.2.6

Using the arc length formula with $r = 12$, we find that

$$s = \frac{1}{3}(2\pi \cdot 12) = \frac{2\pi}{3} \cdot 12 = 8\pi \text{cm}$$

or about 25.1 cm.

Checkpoint 7.2.1

How far have you traveled around the edge of a Ferris wheel of radius 100 feet when you have turned through an angle of $150°$?

Answer

Approximately 261.8 ft

Measuring Angles in Radians

If you think about measuring arc length, you will see that the degree measure of the spanning angle is not as important as the fraction of one revolution it covers. This observation suggests a new unit of measurement for angles, one that is better suited to calculations involving arc length. We'll make one change in our formula for arc length from

$$\textbf{arc length} = (\textbf{fraction of one revolution}) \cdot (2\pi r)$$

to

$$\textbf{arc length} = (\textbf{fraction of one revolution} \cdot 2\pi) \cdot r.$$

You might be asking yourself why we made such a silly change since these two formulas state the same thing (just with the parentheses in different spots).

The subtle difference between these two interpretations of arc length leads to an entirely new angular measurement system. We'll call the quantity in parentheses, (fraction of one revolution $\times 2\pi$), the **radian measure** of the angle that spans the arc.

Definition: Radian Measure

The **radian measure** of an angle is given by

$$\textbf{fraction of one revolution } \times 2\pi.$$

For example, one complete revolution ($360°$) is equal to 2π radians, and one-quarter revolution ($90°$) is equal to $\frac{1}{4}(2\pi) = \frac{\pi}{2}$ radians. Figure 7.2.7 shows the radian measure of the quadrantal angles.

Figure 7.2.7

Note: Radian Measure is Universal

Unlike degree measure, which depends on a human-created construct dependent on either the number of days for the Earth to rotate around the sun or the base-60 numbering system used by the Babylonians (depending on which historical viewpoint of the creation of the degree measurement system you believe), radian measure makes sense on *any* planet and within *any* numbering system. That is, if an alien species were to arrive on Earth, radian measures for angles are likely something we would both have in common.

Example 7.2.2

What is the radian measure of an angle of $120°$?

Solution

An angle of $120°$ is $\frac{120°}{360°} = \frac{1}{3}$ of a complete revolution, as we saw in the previous example. Thus, an angle of $120°$ has a radian measure of $\frac{1}{3}(2\pi)$, or $\frac{2\pi}{3}$.

Checkpoint 7.2.2

What fraction of a revolution is π radians? How many degrees is that?

Answer

Half a revolution, $180°$

Radian measure does not have to be expressed in multiples of π. Remember that $\pi \approx 3.14$, so one complete revolution is approximately 6.28 radians, and one-quarter revolution is $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, or about 1.57 radians. Figure 7.2.8 shows decimal approximations for the quadrantal angles.

Figure 7.2.8

Table 7.2.1: *Quadrantal Angles in Both Degrees and Radians*

Degrees	Radians: Exact Values	Radians: Decimal Approximations
$0°$	0	0
$90°$	$\frac{\pi}{2}$	1.57
$180°$	π	3.14
$270°$	$\frac{3\pi}{2}$	4.71
$360°$	2π	6.28

Memorize the Exact Radian Values of Common Angles

Because they are "benchmarks" for comparing angles, you should memorize the exact values of the angles in the previous table in radians!

Example 7.2.3

In which quadrant would you find an angle of 2 radians? An angle of 5 radians?

Solution

Look at Figure 7.2.5. The second quadrant includes angles between $\frac{\pi}{2}$ and π, or approximately 1.57 and 3.14 radians, so 2 radians lies in the second quadrant. An angle of 5 radians is between 4.71 and 6.28, or between $\frac{3\pi}{2}$ and 2π radians, so it lies in the fourth quadrant.

Checkpoint 7.2.3

Draw a circle centered at the origin and sketch (in standard position) angles of approximately 3 radians, 4 radians, and 6 radians.

Answer

Figure 7.2.9

It turns out that measuring angles in radians is helpful for many applications besides calculating arc lengths, so we need to start thinking in radians. To help that process, we'll first learn to convert between degrees and radians.

Converting Between Degrees and Radians

It is not difficult to convert the measure of an angle in degrees to its measure in radians or vice versa. One complete revolution is equal to 2π radians or to $360°$, so

$$2\pi \text{ radians } = 360°.$$

Dividing both sides of this equation by 2π radians or by $360°$ gives us an **angle measure conversion theorem**:

Theorem: Angle Measure Conversion Theorem

$$2\pi \text{ radians} = 360^\circ$$

$$\implies 1 = \frac{180^\circ}{\pi \text{ radians}} \quad \text{(dividing both sides by } 2\pi \text{ radians)}$$

and

$$\implies \frac{\pi \text{ radians}}{180^\circ} = 1 \quad \text{(dividing both sides instead by } 360^\circ\text{)}$$

Example 7.2.4

a. Convert 3 radians to degrees.
b. Convert 3 degrees to radians.

Solutions

a.

$$\begin{aligned} 3 \text{ radians} &= 3 \text{ radians} \times (1) \\ &= 3 \cancel{\text{ radians}} \times \left(\frac{180^\circ}{\pi \cancel{\text{ radians}}}\right) \\ &= \frac{540^\circ}{\pi} \\ &\approx 171.9^\circ \end{aligned}$$

b.

$$\begin{aligned} 3^\circ &= 3^\circ \times (1) \\ &= 3\cancel{^\circ} \times \left(\frac{\pi \text{ radians}}{180\cancel{^\circ}}\right) \\ &= \frac{\pi}{60} \text{ radians} \\ &\approx 0.05 \text{ radians} \end{aligned}$$

Technically speaking, a radian is what is known as a **unitless measure**. This is because, from the definition of radian measure, a radian is the product of a fraction of a revolution (no units) and 2π (also, no units). Therefore, it is acceptable to answer Example 7.2.4*b* with "approximately 0.05" instead of "approximately 0.05 radians." Moreover, there is no need to include the word "radian" when the context of the situation implies the constant is an angular measure (and the angular measure is not in degrees). For example, the solution to 7.2.4*a* could have been written as

$$\begin{aligned} 3 &= 3 \times (1) \\ &= 3 \times \left(\frac{180^\circ}{\pi}\right) \\ &= \frac{540^\circ}{\pi} \\ &\approx 171.9^\circ \end{aligned}$$

Checkpoint 7.2.4

a. Convert 60° to radians. Give both an exact answer and an approximation to three decimal places.
b. Convert $\frac{3\pi}{4}$ to degrees.

Answers

a. $\frac{\pi}{3} \approx 1.047$
b. $135°$

From our angle conversion theorem, we also learn that

$$1 \text{ radian } = \frac{180°}{\pi} \approx 57.3°.$$

Figure 7.2.10

So while $1°$ is a relatively small angle, 1 radian is much larger (nearly $60°$, in fact). However, this is reasonable because there are only a little more than 6 radians in an entire revolution. An angle of 1 radian is shown above.

We'll soon see that, for many applications, it is easier to work entirely in radians. In fact, in the study of Calculus, you will find that radian measure *must* be used instead of degree measure.

For reference, Figure 7.2.1 shows a radian protractor.

Figure 7.2.11

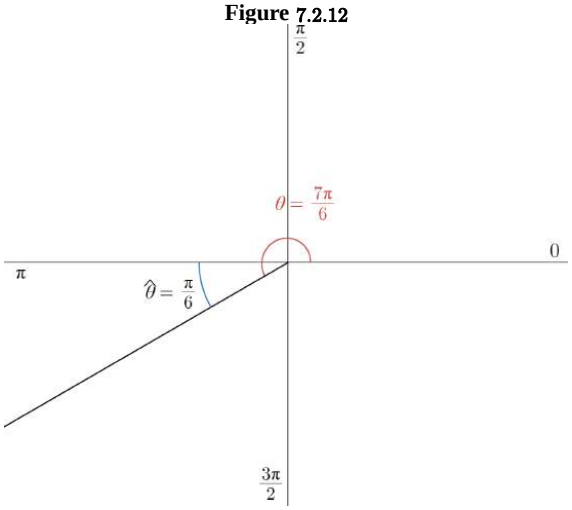
Quadrants and Reference Angles in Radians

Radian measure takes a bit to get used to, but you will soon find it easier to work with than degree measure. Understanding the acute special angles and quadrantal angles in radian measure, we now ask how to quickly determine in which quadrant an angle terminates. We also want to know the reference angle for a given angle in radian measure. We will focus on angles of the form $\frac{n\pi}{d}$ for this discussion.[2]

A beneficial habit to get into when given an angle in radian measure of the form $\frac{n\pi}{d}$ is to think in multiples of the denominator, d. This is because $\frac{d\pi}{d} = \pi$, $\frac{2d\pi}{d} = 2\pi$, $\frac{3d\pi}{d} = 3\pi$, and so on. If you sketch these angles, they terminate on either the positive or negative x-axis - this will be useful.

For example, if asked to find the quadrant in which $\theta = \frac{7\pi}{6}$ terminates, we consider that θ is *slightly* more than $\frac{6\pi}{6} = \pi$ (which is equivalent to $180°$). In fact, it is only $\frac{1\pi}{6}$ more than $\frac{6\pi}{6} = \pi$. Thus, to get to θ, we rotate *past* π and into quadrant three by the small amount of $\frac{\pi}{6}$ radians. This small extra angle of rotation is the reference angle for θ. That is, the reference angle for $\theta = \frac{7\pi}{6}$ is $\hat{\theta} = \frac{\pi}{6}$ (see Figure 7.2.12).

Figure 7.2.12



This concept of using multiples of the denominator need not be limited to positive angles. For example, if $\theta = -\frac{9\pi}{5}$, we imagine rotating in the negative direction to $-\frac{5\pi}{5} = -\pi$; however, this is nowhere near $-\frac{9\pi}{5}$. Therefore, we continue until we get to $-\frac{10\pi}{5} = -2\pi$. At this point, we will have rotated back to the positive x-axis, but we will have rotated *slightly* too far. Therefore, we back off (in the positive direction) by $\frac{1\pi}{5}$ to arrive at $-\frac{9\pi}{5}$, which terminates in the first quadrant. Hence, $\theta = -\frac{9\pi}{5} \in \text{QI}$ with reference angle $\hat{\theta} = \frac{\pi}{5}$ (see Figure **7.2.13**).

Figure 7.2.13

Example 7.2.5

State the quadrant in which $\beta = \frac{5\pi}{3}$ terminates, state its reference angle and sketch.

Solution

$\beta = \frac{5\pi}{3}$ is almost all the way to $\frac{6\pi}{3} = 2\pi$, but we have to back off by $\frac{\pi}{3}$. Hence, $\frac{5\pi}{3} \in \text{QIV}$ and $\hat{\beta} = \frac{\pi}{3}$ (see Figure **7.2.14**).

Figure 7.2.14

Checkpoint 7.2.5

State the quadrant in which $\alpha = -\frac{17\pi}{16}$ terminates, state its reference angle, and sketch.

Answer

$\alpha \in$ QII and $\hat{\alpha} = \frac{\pi}{16}$

Arc Length Formula

We now return to our calculation of arc length and see the first instance in which measuring angles in radians is useful. To calculate an arc length, we need only multiply the circle's radius by the radian measure of the spanning angle, θ. Look again at our formula for arc length:

$$\text{arc length } = (\text{fraction of one revolution } \times 2\pi) \cdot r$$

The quantity in parentheses, "fraction of one revolution $\times 2\pi$," is the measure of the spanning angle in radians. Thus, if θ is measured in radians, we have the following formula for arc length, s.

Theorem: Arc Length Formula

On a circle of radius r, the length s of an arc spanned by an angle θ in radians is

$$s = r\theta.$$

Caution: The Radian Requirement

It is crucial to remember that the arc length formula, $s = r\theta$, is only valid if θ is in radians.

Thus, there is a special relationship between arc length and radians. An angle of 1 radian spans an arc on a circle equal to the radius of the circle, as shown in Figure **7.2.15**. Likewise, the length of any arc is just the product of the measure of its spanning angle in radians and the radius of the circle.

Figure 7.2.15

So, for instance, we can use the formula to find the arc length spanned by an angle of **2.5** radians on a circle of radius six inches:

$$s = r\theta = (6 \text{ inches})\,(2.5) = 15 \text{ inches}.$$

Note that we didn't use the word "radians" when replacing θ with **2.5**. This is because, as we mentioned previously, radians is a unitless measure. Therefore, when performing **unit analysis** (which is the act of including units when performing computations to

understand the meaning of the result of a series of computations), we leave the word "radians" off the angle.

We can also use the formula in the form $\theta = \frac{s}{r}$ to find an angle that spans a given arc. For example, an arc length equal to one radius determines a central
angle of one radian, or approximately $57.3°$.

In the following example, we use the arc length formula to compute a change in latitude on the Earth's surface.

Example 7.2.6

The Earth's radius is roughly 3,960 miles. If you travel 500 miles due north, how many degrees of latitude will you traverse? (Latitude is measured in degrees north or south of the equator.)

Solution

We think of the distance of 500 miles as an arc length on the surface of the Earth, as shown below. Substituting $s = 500$ and $r = 3960$ into the arc length formula gives

$$\begin{aligned} 500\,\text{miles} &= 3960\,\text{miles} \times \theta \\ \Longrightarrow \quad \frac{500\ \cancel{\text{miles}}}{3960\ \cancel{\text{miles}}} &= \theta \\ \Longrightarrow \quad \theta &\approx 0.1263\ \text{radians} \end{aligned}$$

Figure 7.2.16

To convert the angle measure to degrees, we multiply by the conversion factor $\frac{180°}{\pi}$ to get

$$0.1263\left(\frac{180°}{\pi}\right) \approx 7.23°$$

Your latitude has changed by about $7.23°$.

Checkpoint 7.2.6

The distance around the face of a large clock from 2 to 3 is five feet. What is the radius of the clock?

Answer

9.55 feet

Example 7.2.7

The London Eye, located on the South Bank of the River Thames in London, is Europe's tallest Ferris wheel. In fact, when it first opened in 2000, it was the *world's* tallest Ferris wheel. It has a diameter of 120 meters. Interestingly, the London Eye rotates slowly enough that it never stops - riders easily step on and off at the wheel's bottom platform while the wheel is in motion.

Figure 7.2.17

Loi boards the London Eye to take in an elevated city view. If Loi has moved 80 meters along the circumference of the wheel, how many degrees has the London Eye rotated?

Solution

Parsing this question into language associated with our recent material, we have a circle of diameter 120 meters, which means the radius of the circle is 60 meters. Loi is moving through an arc of length 80 meters along the edge of this circle. Our task is to find the central angle that subtends this arc. In the formula

$$s = r\theta,$$

$s = 80$ meters and $r = 60$ meters. Therefore,

$$80 \text{ meters} = (60 \text{ meters})\,\theta \implies \frac{80 \cancel{\text{ meters}}}{60 \cancel{\text{ meters}}} = \frac{4}{3} = \theta.$$

However, this angle is in *radians*. We must convert this to degrees.

$$\frac{4}{3} \times \frac{180^\circ}{\pi} \approx 76.4^\circ.$$

Hence, Loi moves through an approximate angle of 76.4°.

? Checkpoint 7.2.7

A central angle in a circle of radius 6 cm cuts off an arc of length 24 cm. What is the radian measure of the angle? What is the degree measure?

Answer

$4 \text{ radians} \approx 229^\circ$

Trigonometric Functions of Angles in Radians

Measuring angles in radians has other applications besides calculating arc length. One such use is evaluating trigonometric functions of angles in radians. Many applications in Trigonometry (and beyond) use radians for input values rather than degrees. For this reason, it is crucial to know the trigonometric values for the special angles, which you learned in degrees previously, when the angles are given in radians. Fortunately, the value of a trigonometric function is the same whether the angle is measured in radians or degrees.

For example, $\frac{\pi}{3}$ radians is the same as 60°, because $\frac{\pi}{3} \cdot \frac{180^\circ}{\pi} = 60^\circ$, so

$$\sin\left(\frac{\pi}{3}\right) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

Table 7.2.2 shows the radian measures for the special angles and the quadrantal angles, as well as the exact values of the sine, cosine, and tangent at these angles.

Table 7.2.2

θ in Degrees	θ in Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined
$180°$	π	0	-1	0
$270°$	$\frac{3\pi}{2}$	-1	0	undefined
$360°$	2π	0	1	0

You should memorize these function values and be able to use them to find trigonometric values for the special angles in all four quadrants, as they will appear frequently throughout the rest of Trigonometry.

Example 7.2.8

Compute the value of $\csc\left(\frac{\pi}{3}\right)$.

Solution

Our identities still hold - working in radians does not change that fact. Therefore, by the Reciprocal Identities,

$$\csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}.$$

Checkpoint 7.2.8

Compute the value of $\cot\left(\frac{\pi}{4}\right)$.

Answer

1

As before, we use reference angles to define the trigonometric functions for angles greater than $90°$.

Example 7.2.9

Give exact values for the following.

a. $\tan\left(\frac{2\pi}{3}\right)$

b. $\cos\left(\frac{5\pi}{4}\right)$

Solutions

a. The reference angle for $\theta = \frac{2\pi}{3}$ is $\hat{\theta} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$, and the tangent is negative in the second quadrant. (See the figure below.) Thus, $\tan\left(\frac{2\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$.

Figure 7.2.18

b The reference angle for $\theta = \frac{5\pi}{4}$ is $\hat{\theta} = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$, and the cosine is negative in the third quadrant, so $\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Figure 7.2.19

? Checkpoint 7.2.9

Give exact values for the following.

a. $\sin\left(\frac{5\pi}{6}\right)$

b. $\tan\left(\frac{7\pi}{4}\right)$

Answers

a. $\frac{1}{2}$

b. -1

Footnotes

[1] The word "revolution" will be used extensively from this point forward. Some texts use "rotation."

[2] Angles in radians need not have a π attached to them, but it does help when determining the quadrant in which the angle terminates. Therefore, we restrict ourselves to angles of the form $\frac{n\pi}{d}$ for this short discussion.

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7.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Radian Measure
  - Angles - Angle Measure: Radians
- Arc Length
  - Angles - Arc Length
  - Angle Measure - Arc Length
- Degree to Radian Conversion
  - Angles - Relating Degrees to Radians
- Reference Angles in Radians
  - Angles - Reference Angles

KEY CONCEPTS

DEFINITIONS

- **Arc and Arc Length:** An arc is the portion of the circumference of a circle between two points on its edge. The arc length is the distance along this curved portion.
- **Central Angle:** An angle whose vertex is at the center of a circle.
- **Subtend:** A central angle is said to subtend (or span) an arc if its sides pass through the endpoints of the arc.
- **Radian Measure:** The radian measure of an angle is defined as the product of its fraction of a complete revolution and 2π.

THEOREMS

- **Angle Measure Conversion Theorem:** The relationship between radians and degrees can be used to create conversion factors: $1 = \frac{180^\circ}{\pi \text{ radians}}$ and $1 = \frac{\pi \text{ radians}}{180^\circ}$.
- **Arc Length Formula:** On a circle of radius r, the length s of an arc spanned by a central angle θ is given by $s = r\theta$, provided that θ is measured in radians.

COMMON MISTAKES

- **The Radian Requirement:** The arc length formula $s = r\theta$ is only valid if the angle θ is measured in radians. Using an angle in degrees will produce an incorrect result.
- **Confusing Radians and Degrees:** Be mindful of the units for a given angle. An angle of 1 radian is approximately 57.3°, whereas an angle of 1° is very small. Do not use the two interchangeably.
- **Thinking Radians Must Include π:** While many common angles in radians are expressed as multiples of π (e.g., $\frac{\pi}{4}$), an angle can be measured in radians without π (e.g., 2 radians).

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7.2.2: HOMEWORK

READING QUESTIONS

1. What is the difference between an arc and an arc length?
2. What defines a central angle?
3. How is the radian measure of an angle defined based on a fraction of a revolution?
4. How many radians are in one full revolution (360°)?
5. What is the formula for arc length, s, if the central angle θ is measured in radians and the radius is r?
6. What is the most important requirement for the angle θ when using the arc length formula $s = r\theta$?
7. What is the conversion factor used to convert an angle from degrees to radians?
8. What is the conversion factor used to convert an angle from radians to degrees?
9. In which quadrant does an angle of 5 radians terminate?
10. What is the reference angle for $\theta = \frac{5\pi}{3}$?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, use the appropriate conversion factor to convert units.

1. $\frac{1 \text{ mile}}{1.609 \text{ kilometers}} = 1$
   a. 10 miles = _____ km
   b. 50 km = _____ miles
2. $\frac{1 \text{ acre}}{0.405 \text{ hectare}} = 1$
   a. 40 acres = _____ hectares
   b. 5 hectares = _____ acres
3. $\frac{1 \text{ horsepower}}{746 \text{ watts}} = 1$
   a. 250 horsepower = _____ watts
   b. 1000 watts = _____ horsepower
4. $\frac{1 \text{ troy ounce}}{480 \text{ grains}} = 1$
   a. 0.5 troy oz = _____ grains
   b. 100 grains = _____ troy oz

For the following exercises, determine how many degrees are in each fraction of one complete revolution.

5. $\frac{1}{4}$
6. $\frac{1}{5}$
7. $\frac{1}{6}$
8. $\frac{1}{8}$

Answers

1.
   a. 16.09 km
   b. 31.08 mi
2.
   a. 16.2 hectares
   b. 12.35 acres
3.
   a. 186,500 watts
   b. 1.34 horsepower
4.
   a. 240 grains
   b. 0.21 troy oz
5. $90°$
6. $72°$
7. $60°$
8. $45°$

HOMEWORK

VOCABULARY CHECK

1. Unlike degree measure, radian measure is a ___ measure (i.e., radian measure does not require a measurement symbol).
2. An angle with vertex at the center of the circle is called a(n) ___.
3. A central angle whose sides meet the endpoints of an arc is said to ___ the arc. Another way to say this is the angle ___ the arc.
4. The ___ length is the distance along the edge of a circle subtended by a central angle θ.
5. The product of 2π and the fraction of one revolution cut out by the angle θ is named the ___ measure of the angle.

CONCEPT CHECK

6. The length of a circular arc depends on what two variables?
7. Define the radian measure of an angle.
8. What is the conversion factor from radians to degrees?
9. On a unit circle, the length of an arc is equal to what other quantity?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The distance we travel around a circle of diameter d is proportional to the angle of displacement. In fact,

$$\text{arc length} = (\text{fraction of one revolution}) \cdot (\pi d)$$

11. We measure angles in radians when we work with arc length.
12. An arc length equal to one radius determines a central angle of one degree.

13. An angle given in radian measure must be written as $k\pi$, where k is a real number.
14. $\frac{180^\circ}{\pi \text{ radians}} = 1$
15. On a circle of radius r, the length s of an arc spanned by an angle θ is $s = r\theta$.

BASIC SKILLS

16.

Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Degrees									

a. Convert each angle to degrees.

b. Sketch each angle on a circle like this, and label in radians.

17.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
Degrees													

a. Convert each angle to degrees.

b. Sketch each angle on a circle like this, and label in radians.

For the following exercises, express each fraction of one complete rotation in degrees and radians.

18.
a. $\frac{1}{3}$
b. $\frac{2}{3}$
c. $\frac{4}{3}$
d. $\frac{5}{3}$

19.
a. $\frac{1}{5}$
b. $\frac{2}{5}$
c. $\frac{3}{5}$
d. $\frac{4}{5}$

20.
a. $\frac{1}{8}$
b. $\frac{3}{8}$
c. $\frac{5}{8}$
d. $\frac{7}{8}$

21.
a. $\frac{1}{12}$
b. $\frac{1}{6}$
c. $\frac{5}{12}$
d. $\frac{5}{6}$

For the following exercises, label each angle in standard position with radian measure (assume the angles shown are special angles).

22. Rotate counter-clockwise from 0.

23. Rotate clockwise from 0.

For the following exercises, give a decimal approximation to hundredths for each angle in radians.

24.
a. $\frac{\pi}{6}$
b. $\frac{5\pi}{6}$
c. $\frac{7\pi}{6}$
d. $\frac{11\pi}{6}$

25.
a. $\frac{\pi}{4}$
b. $\frac{\pi}{4}$
c. $\frac{5\pi}{4}$
d. $\frac{7\pi}{4}$

26. Locate and label each angle from Problem 24 on the unit circle below. (The circle is marked off in tenths of a radian.)

27. Locate and label each angle from Problem 25 on the unit circle below. (The circle is marked off in tenths of a radian.)

For the following exercises, choose the best decimal approximation for each angle in radians from the list below. Do not use a calculator; use the fact that π is a little greater than 3.

$$0.52, 0.79, 2.09, 2.36, 2.62, 3.67, 5.24, 5.50$$

28. $\frac{2\pi}{3}$
29. $\frac{\pi}{4}$
30. $\frac{5\pi}{6}$
31. $\frac{5\pi}{3}$
32. $\frac{\pi}{6}$
33. $\frac{7\pi}{4}$
34. $\frac{3\pi}{4}$
35. $\frac{7\pi}{6}$

For the following exercises, state in which quadrant each angle lies.

36.
a. $\frac{\pi}{4}$
b. $\frac{3\pi}{4}$
c. $\frac{5\pi}{4}$
d. $\frac{7\pi}{4}$

37.

a. $\frac{\pi}{6}$

b. $\frac{5\pi}{6}$

c. $\frac{7\pi}{6}$

d. $\frac{11\pi}{6}$

38.

a. $\frac{\pi}{3}$

b. $\frac{2\pi}{3}$

c. $\frac{4\pi}{3}$

d. $\frac{5\pi}{3}$

39.

a. $\frac{\pi}{5}$

b. $\frac{8\pi}{7}$

c. $\frac{25\pi}{13}$

d. $\frac{47\pi}{48}$

For the following exercises, complete the table.

40. (MyOpenMath)

Degrees	18°	60°	90°
Radians			

41. (MyOpenMath)

Radians	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
Degrees			

42. (MyOpenMath)

Radians	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
Degrees			

43.

Radians	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
Degrees			

For the following exercises, convert to radians. Write the answer in exact form and then give approximations rounded to the nearest hundredths.

44. (MyOpenMath)

a. 120°

b. 420°

c. -180°

d. -225°

45. (MyOpenMath)

a. 75°

b. 236°

c. -327°

d. $143^\circ\,40'$

46.

a. 138°

b. 194°

c. 342°

d. $281^\circ\,15'$

For the following exercises, convert to degrees. Round answers to the nearest tenth.

46. (MyOpenMath)

a. $-\frac{11\pi}{15}$

b. $\frac{2\pi}{7}$

c. $-\frac{6\pi}{5}$

47.

a. 0.8

b. 3.5

c. 5.1

48.

a. 1.1

b. 2.6

c. 4.6

For the following exercises, state the quadrant in which the angle terminates and give the reference angle (in radians).

48. $\frac{\pi}{3}$

49. (MyOpenMath) $\frac{3\pi}{4}$

50. $\frac{9\pi}{8}$

51. $\frac{21\pi}{22}$

52. (MyOpenMath) $-\frac{5\pi}{6}$

53. (MyOpenMath) $-\frac{5\pi}{3}$

54. (MyOpenMath) $\frac{11\pi}{4}$

55. $-\frac{15\pi}{7}$

56. 2.4

57. -3.1

For the following exercises, find an angle in each quadrant, rounded to tenths, with the same reference angle as the angle given in radians.

58. 5.8

59. 2.9

60. 3.7

61. 5.1

62. 1.8

63. 4.4

For the following exercises, use the arc length formula to answer the questions. Round answers to hundredths.

64. (MyOpenMath) Find the arc length spanned by an angle of 2 radians on a circle of radius 2 miles.
65. Find the arc length spanned by an angle of $80°$ on a circle of radius 4 inches.
66. Find the arc length spanned by an angle of $200°$ on a circle of radius 18 feet.
67. Find the radius of a circle if an angle of $250°$ spans an arc length of 18 meters.
68. Find the radius of a circle if an angle of $20°$ spans an arc length of 0.5 kilometers.
69. (MyOpenMath) Find the angle, in radians, subtended by an arc length of 7 feet on a circle of radius 10 feet.
70. (MyOpenMath) Find the angle, in degrees, subtended by an arc length of 7 feet on a circle of radius 6 feet.
71. Find the angle, in radians, subtended by an arc length of 28 centimeters on a circle of diameter 20 centimeters.
72. Find the angle, in radians, subtended by an arc length of 1.6 yards on a circle of diameter 2 yards.

For the following exercises, find the exact value of the trigonometric function. Do not use a calculator!

70. $\sin\left(\frac{\pi}{3}\right)$
71. $\cos\left(\frac{\pi}{6}\right)$
72. $\tan\left(\frac{\pi}{4}\right)$
73. $\csc\left(\frac{\pi}{4}\right)$
74. $2\sec\left(\frac{\pi}{3}\right)$
75. $\cot\left(\frac{\pi}{6}\right)$
76. $-\sin\left(\frac{5\pi}{6}\right)$
77. $\cos\left(\frac{4\pi}{3}\right)$
78. $\tan\left(\frac{11\pi}{6}\right)$
79. $3\csc\left(-\frac{4\pi}{3}\right)$
80. $\sec\left(-\frac{3\pi}{4}\right)$
81. $-\pi\cot\left(-\frac{13\pi}{6}\right)$
82. $\sin\left(\frac{\pi}{2}\right)$
83. $\cos(\pi)$
84. $\tan\left(\frac{3\pi}{2}\right)$

SYNTHESIS QUESTIONS

For the following exercises, find an angle between 0 and 2π that is coterminal with the given angle.

85. (MyOpenMath) $\frac{29\pi}{10}$

For the following exercises, find the exact value of the expression for $x = \frac{\pi}{6}$.

85. $\sin(2x)$
86. $\cos(3x)$
87. $\frac{1}{3}\sin(5x)$
88. $6 - \sin(x)$
89. $\frac{2}{3} + \tan\left(\frac{3}{2}x\right)$
90. $\sin(\pi - x)$
91. $\cos(\pi + 2x)$
92. $3\sin\left(2x + \frac{\pi}{3}\right)$
93. $-2 + \frac{3}{4}\cos\left(7x - \frac{\pi}{2}\right)$
94.
    a. Sketch a circle of radius 4 units, and mark the positions of 1, 2, 3, 4, 5, and 6 radians on the circle.
    b. On a circle of radius 4 feet, find the arc length determined by each angle in radians.

θ	1	2	3	4	5	6
s						

    c. Graph θ against s. What is the slope of the graph?
    d. If you double the angle, what happens to the arc length? What happens if you triple θ?
95.
    a. Sketch several concentric circles with increasing radii, and draw an angle of 2 radians through all of them.
    b. Find the arc length determined by an angle of 2 radians on circles of given radius.

r	1	2	3	4	5	6
s						

    c. Graph s against r. What is the slope of the graph?
    d. If you double the radius r, what happens to the arc length? What happens if you triple r?
96. Recall that to calculate a fraction of a revolution in degrees, we divide the angle by $360°$. For example, $90°$ is $\frac{1}{4}$ of a revolution because $\frac{90°}{360°} = \frac{1}{4}$.
    a. Write an expression that gives the fraction of a revolution for an angle θ in radians.
    b. Use your expression to calculate what fraction of a revolution is represented by each of the following angles: $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{3}$, $\theta = \frac{7\pi}{6}$.

APPLICATIONS

For the following exercises, use the arc length formula to answer the questions.

97. **Big Ben.**
    a. Through how many radians does the minute hand of a clock sweep between 9:05 pm and 9:30 pm?

b. The dial of Big Ben's clock in London is 23 feet in diameter. How long is the arc traced by the minute hand between $9:05\text{pm}$ and $9:30\text{pm}$?

98. **Floral Clock.** The largest clock ever constructed was the Floral Clock in the garden of the 1904 World's Fair in St. Louis. The hour hand was 50 feet long, the minute hand was 75 feet long, and the radius of the clock's face was 112 feet.

    a. If you started at the 12 and walked 500 feet clockwise around the clock face, through how many radians would you walk?

    b. If you started your walk at noon, how long would it take the minute hand to reach your position? How far did the tip of the minute hand move in its arc?

99. **The Foucault Pendulum.** In 1851 Jean-Bernard Foucault demonstrated the rotation of the earth with a pendulum installed in the Pantheon in Paris. Foucault's pendulum consisted of a cannonball suspended on a 67 meter wire, and it swept out an arc of 8 meters on each swing. Through what angle did the pendulum swing? Give your answer in radians and then in degrees, rounded to the nearest hundredth.

100. **Linear Distance.** A wheel with radius 40 centimeters is rolled a distance of 1000 centimeters on a flat surface. Through what angle has the wheel rotated? Give your answer in radians and then in degrees, rounded to one decimal place.

101. **Distance Between Cities.** Cosumnes River College (Sacramento, California) and Stony Brook University (Stony Brook, New York) are 2,544 miles apart along the surface of the Earth. Assuming that the radius of the Earth is 3,963 miles, find the radian measure of the central angle having vertex at the center of the Earth that has Cosumnes River College on one side and Stony Brook University on the other.

102. **Distance on the Surface of the Earth (MyOpenMath).** Find the distance along an arc on the surface of the earth that subtends a central angle of 10 minutes (recall, 1 minute = 1/60 degree). For this exercise, assume the radius of the earth is 3960 miles.

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7.3: The Trigonometric Functions - Unit Circle Definition

Note to the Instructor (click to expand)

We wrap up our trio of trig definitions with the Unit Circle definition. I deliberately chose to introduce this definition last because we (human beings) tend to stick with the first or second way we learn something. Since the Unit Circle definition of the trigonometric functions is a completely memorizable definition/method, I consider it the worst to introduce early. Most of your time in this section will be spent on convincing the student that trigonometric functions can also be defined in terms of arc length along the unit circle (hence, the phrase "circular functions"). While I do not spend an insane amount of time on this material, understanding the definition presented in this section (along with its relationship to the area of the sector with arc length t) is important for the development of the hyperbolic functions in Calculus I.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Evaluate a trigonometric function using the unit circle
- Find the value of a trigonometric function given a point on the unit circle.
- Use a calculator to approximate the value of a trigonometric function for an angle in radians.
- Use the unit circle to answer a conceptual question about a trigonometric function.

Hawk A.I. Section-Specific Tutor

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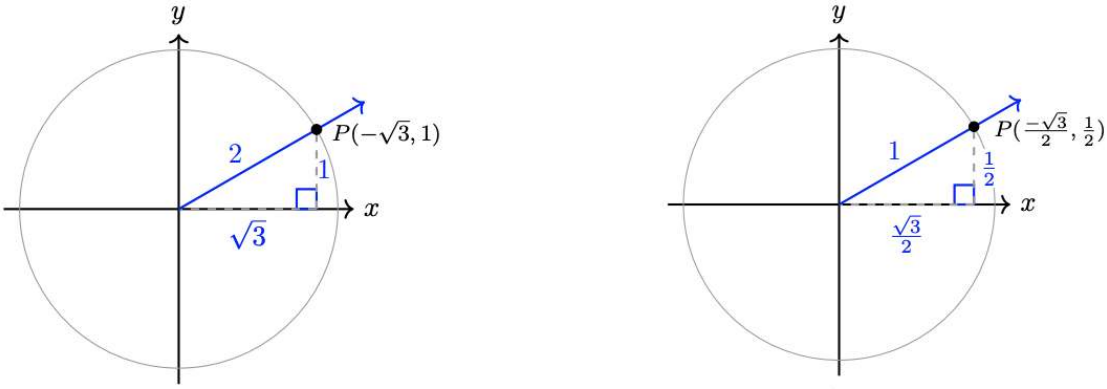
We now venture into the third and final definition of the trigonometric functions. Many students have already encountered this definition from their high school Pre-Calculus course; however, there are three reasons I have held off on introducing the trigonometric functions via this definition until now:

1. In my experience, learning the unit circle definition of the trigonometric functions before the right triangle definition leads to more memorization and a less accurate understanding of the trigonometric functions.
2. Unit circle trigonometry is theoretically more challenging to grasp. It has two interpretations - one in terms of angles and the other in terms of arc lengths (and, as such, most students who learn it before right triangle trigonometry tend to have a weak grasp of the subject).
3. Unit circle trigonometry relies on radian measure.

Connecting the Sine and Cosine Functions to the Unit Circle

The left image in Figure **7.3.1** below shows an angle of $30°$ in standard position in a circle of radius 2. The hypotenuse of its reference triangle is the radius of the circle, so the legs of the triangle have lengths 1 and $\sqrt{3}$. The coordinates of the point P where the terminal side meets the circle are thus $(\sqrt{3}, 1)$.[1]

Figure 7.3.1



The right image in Figure **7.3.1** is a unit circle. In this figure, the hypotenuse of the reference triangle for $30°$ has length 1. What are the coordinates of the point P where the terminal side meets the circle? Each side of this triangle is half the length of the sides of the similar triangle in the left image, so the coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.[2]

You should recognize the coordinates of the point P as $(\cos(30°), \sin(30°))$. In fact, from the coordinate definition of the trigonometric functions, we should not be surprised to see that, given the point (x, y) on the unit circle on the terminal side of the angle θ,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{1} = x \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{1} = y$$

We have discovered an important property of unit circles.

Lemma

Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then

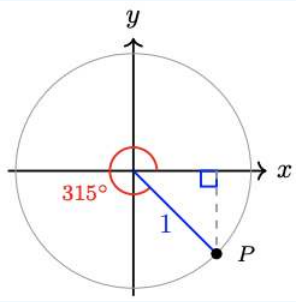
$$\cos(\theta) = x \quad \text{and} \quad \sin(\theta) = y.$$

While this lemma is not entirely profound, it will lead to a more significant concept. Before we get there, let's use the lemma in an example.

Example 7.3.1

Find the coordinates of point P on the unit circle below.

Figure 7.3.2



Solution

The coordinates of P are given by

$$x = \cos(315°), \quad y = \sin(315°).$$

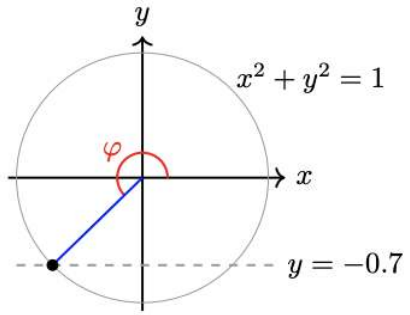
The reference angle for $315°$ is $360° - 315° = 45°$, so the trigonometric functions of $315°$ are the same as the functions for $45°$, up to sign.

Because $315°$ is in the fourth quadrant, we have $\cos(315°) = \frac{1}{\sqrt{2}}$ and $\sin(315°) = -\frac{1}{\sqrt{2}}$. Thus, the coordinates of P are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

? Checkpoint 7.3.1

Find the sine, cosine, and tangent of the angle φ shown below.

Figure 7.3.3



Answer

$\sin(\varphi) = -0.7000, \cos(\varphi) \approx -0.7141, \tan(\varphi) \approx 0.9802$

A Reminder of Radian Measure and Arc Length

For the remainder of this chapter, we will focus on using radians instead of degrees. This change will greatly increase the utility of the trigonometric functions. To illustrate this, and as a refresher of previously learned material, recall that the radius of a unit circle is 1. Therefore, the arc length formula becomes $s = r\theta = (1)\theta = \theta$. Thus, on a *unit circle*, an arc of length 1 determines a central angle of 1 radian. Moreover, the measure of an angle in radians is equal to the length of the arc it spans. In other words, if we walk along the edge of a circle of radius 1, each time we travel through an arc length equivalent to one radius, we have rotated through an angle of one radian.

✓ Example 7.3.2

You are walking on a trail around a circular pond with a radius of one mile. You have hiked for four miles. Sketch your location now.

Solution

The pond is a unit circle, so you have traversed an angle in radians equal to the arc length traveled, 4 miles. An angle of 4 radians is in the middle of the third quadrant relative to your starting point, more than halfway but less than three-quarters around the pond.

Figure 7.3.4

? Checkpoint 7.3.2

An ant walks around the rim of a circular birdbath of a radius of 1 foot. How far has the ant walked when it has turned through an angle of $210°$?

Answer

3.67 ft

Trigonometric Functions of Real Numbers and the Unit Circle Definition

We now venture into a more theoretical area of Trigonometry. For this discussion, we will denote the arc length of a circle subtended by the angle θ with the letter t instead of s (for no better reason than tradition).

Consider the unit circle shown in Figure **7.3.5** below, and the angle θ determined by the arc of length t. The radian measure of θ is the same as the length t of the arc. For example, in this figure $\theta = t = 2$.

Figure 7.3.5: *Traversing the unit circle*

Therefore, as long as we restrict our conversation to the unit circle, instead of thinking of the sine or cosine as a function of an *angle*, θ, we can change our perspective and think of these as functions of the *length of an arc*, t, along the unit circle. Since the arc length is not an angle but instead a simple real number, we are evaluating the trigonometric function at a real number.

This brings us to our third and final definition of the trigonometric functions.

Definition: Trigonometric Functions (Unit Circle Definition)

Let $P(x, y)$ be a point on the unit circle, and let t be the arc length from the point $(1, 0)$ to P along the circumference of the unit circle. The trigonometric functions of the real number t (also called the **circular functions** in this context) are defined as follows:

Function		Ratio	Function		Ratio
$\sin(t)$	$=$	y	$\csc(t)$	$=$	$\frac{1}{y}$
$\cos(t)$	$=$	x	$\sec(t)$	$=$	$\frac{1}{x}$
$\tan(t)$	$=$	$\frac{y}{x}$	$\cot(t)$	$=$	$\frac{x}{y}$

We call the trigonometric functions of arc length the circular functions because they are defined by the coordinates of points on a unit circle.

We can think of the definition this way: to find the sine or cosine of a real number t, we draw an arc of length t on a unit circle and then find the sine or cosine of the angle θ determined by the arc.

Note: The Difference is All in Your Head

In practice, there is no difference between finding the value of a trigonometric function at the number 2 and the value at the angle of 2 radians. In each case, we set the calculator to Radian mode and evaluate, for example,

$$\text{COS}\,2 \approx -0.4161 \quad \text{and} \quad \text{SIN}\,2 \approx 0.9093.$$

So, in some sense, a trigonometric function of a radian is the same as a trigonometric function of a real number. This result is so important that it bears repeating:

> *To find the value of a trigonometric function of a real number t, we find the value of the trigonometric function of the angle t radians.*

Example 7.3.3

Let $t = \frac{2\pi}{3}$ be the length of the arc along the unit circle from the point $(1, 0)$ to $P(x, y)$. Evaluate each of the following functions.

a. $\sin(t)$
b. $\cot(t)$

Solutions

a. Since $t = \frac{2\pi}{3}$, we are being asked to evaluate $\sin\left(\frac{2\pi}{3}\right)$. As stated before, a trigonometric function's value at a real number (an arc length along the unit circle) and the equivalent radian angle are the same. Thus,

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

b. As in part a., we just evaluate

$$\cot\left(\frac{2\pi}{3}\right) = -\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}.$$

Checkpoint 7.3.3

Evaluate $\sec(t)$, where $t = \frac{5\pi}{4}$ is the arc length along the unit circle from the point $(1, 0)$.

Answer

$-\sqrt{2}$

Example 7.3.4

Find the coordinates of the terminal point, P, of an arc of length t starting at $(1, 0)$ on a unit circle.

a. $t = \frac{\pi}{2}$
b. $t = \frac{5\pi}{6}$

Solutions

a. On a unit circle, the coordinates of P are $\left(\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right)$, so

$$x = \cos\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad y = \sin\left(\frac{\pi}{2}\right) = 1$$

The coordinates of P are $(0, 1)$, as shown below.

Figure 7.3.6

(We could also observe that an arc of length $\frac{\pi}{2}$ is one-quarter of a unit circle, so the point P sits at the twelve o'clock position on the circle.)

b. The coordinates of P are $\left(\cos\left(\frac{5\pi}{6}\right), \sin\left(\frac{5\pi}{6}\right)\right)$, so

$$x = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$y = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

Thus, the coordinates of P are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, as shown below.

Figure 7.3.7

Checkpoint 7.3.4

Find the coordinates of the terminal point, P, of an arc of length t starting at (1,0) on a unit circle.

a. $t = \pi$
b. $t = \frac{\pi}{3}$

Answers

a. $(-1, 0)$

b. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Using a Calculator to Compute the Values of Trigonometric Functions

Before moving forward, let's talk about using your calculator's Radian mode. In the previous section, we evaluated the trigonometric functions at the special radian angles (i.e., multiples of $\pi/6$, $\pi/4$, and $\pi/3$). We now focus on evaluating the trigonometric functions at non-special radian angles (and, therefore, non-special real numbers).

Note: Change to Radian Mode

If you use a scientific or graphing calculator, refer to your instruction manual on setting it to radian mode. The scientific calculator in Desmos is set to degree mode by default, so if you are using that technology, you need to click the "DEG" toggle so that "RAD" is showing; however, the *graphing* calculator in Desmos defaults to radian mode.

Setting the calculator to radian mode allows us to avoid converting from radians to degrees to evaluate trigonometric functions. First, change the calculator setting from Degree mode to Radian mode. Then enter

$$\text{SIN}\,\pi \div 3$$

and the calculator will return 0.8660254038. You can check that this number is a decimal approximation for $\frac{\sqrt{3}}{2}$.

Example 7.3.5

Use your calculator to find the sine and cosine of the following angles in radians. Round your answers to four decimal places.

a. $\theta = \frac{7\pi}{4}$
b. $\theta = 3.5$

Solutions

a. With your calculator in radian mode, enter

$$\text{COS}\,7\,\pi \div 4$$

and

$$\text{SIN}\,7\,\pi \div 4$$

to find

$$\cos\left(\frac{7\pi}{4}\right) \approx 0.7071067812$$
$$\sin\left(\frac{7\pi}{4}\right) \approx -0.7071067812$$

Rounding to four decimal places gives $\cos\left(\frac{7\pi}{4}\right) \approx 0.7071$ and $\sin\left(\frac{7\pi}{4}\right) \approx -0.7071$.

b. Your calculator will give you the values

$$\cos(3.5) \approx -0.9364566873$$
$$\sin(3.5) \approx -0.3507832277$$

Rounding to four places, we have $\cos(3.5) \approx -0.9365$ and $\sin(3.5) \approx -0.3508$. Note that 3.5 radians is a third-quadrant angle, so the signs of the trigonometric values make sense.

Checkpoint 7.3.5

Use your calculator to find the tangents of the following real numbers. Round your answers to four decimal places.

a. $t = \frac{5\pi}{12}$
b. $t = 5.2$

Answers

a. 3.7321
b. −1.8856

We can now use the trigonometric functions to model periodic behavior as functions of *time* instead of angles. Indeed, we now have the ability to use any variable as the argument of a trigonometric function.

Example 7.3.6

The sunset time in Stockholm, Sweden, on the nth day of the year can be modeled by

$$T = 3.11 \sin(0.017n - 1.38) + 6.03$$

where T is given in hours after noon. Find the sunset time on January 1 (day $n = 1$) and on July 1 (day $n = 182$).

Solution

Evaluate the function for $n = 1$ to find

$$T = 3.11 \sin(0.017(1) - 1.38) + 6.03 \approx 2.99.$$

On January 1, sunset in Stockholm occurs about 2.99 hours after noon, or at 2:59 pm.

Evaluate the function at $n = 182$ to find

$$T = 3.11 \sin(0.017(182) - 1.38) + 6.03 \approx 9.11.$$

On July 1, sunset occurs about 9.11 hours after noon, or at 9:07 pm. (Actually, 10:07 pm, because of daylight saving time.)

Checkpoint 7.3.6

Variable stars are important in astronomy because they are used to estimate distances. Their magnitude, or brightness, varies periodically and can be modeled by trigonometric functions. The star T Herculis reached its minimum magnitude on December 27, 2004, and t days later its magnitude is approximately

$$M = 10.2 - 2.2 \cos(0.038t)$$

a. What was the magnitude of T Herculis on December 27, 2004?
b. What was the magnitude of T Herculis on December 27, 2006 (730 days later)?

Answers

a. 8
b. 12.09

Example 7.3.7

Use the graph of the unit circle shown below to estimate $\cos(2.5)$, $\sin(2.5)$, and $\tan(2.5)$.

Figure 7.3.8

Solution

The circle is scaled in units of 0.1 radians, and an arc of 2.5 radians in standard position has its terminal point, P, in the second quadrant. The coordinates of P are approximately $(-0.8, 0.6)$, so we have

$$\cos(2.5) \approx -0.8 \quad \text{and} \quad \sin(2.5) \approx 0.6.$$

To find $\tan(2.5)$, we calculate $\frac{y}{x}$.

$$\tan(2.5) = \frac{y}{x} \approx \frac{0.6}{-0.8} = -0.75.$$

Checkpoint 7.3.7

Use the graph of the unit circle in the previous example to estimate $\cos(4.2)$, $\sin(4.2)$, and $\tan(4.2)$.

Answer

Approximately −0.49, −0.87, and 1.78, respectively.

Activities

Activity 7.3.1: Special Angles in Radians

Figure 7.3.9

1. Use the unit circle to estimate the sine, cosine, and tangent of each arc of given length.
   a. 0.6
   b. 2.3
   c. 3.5
   d. 5.3
2. Use the unit circle to estimate two solutions to each equation.
   a. $\cos(t) = 0.3$
   b. $\sin(t) = 0.7$
3. Sketch the angle on the unit circle. Find the reference angle in radians, rounded to two decimal places, and sketch the reference triangle.
   a. 1.8
   b. 5.2
   c. 3.7

4. Give a decimal approximation to two places for each angle, then the degree measure of each.

Radians	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
Decimal Approx.													
Degrees													

Radians	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$\frac{23\pi}{12}$	2π
Decimal Approx.													
Degrees													

5. On the unit circle above, plot the endpoint of each arc in standard position.
   a. $\frac{\pi}{3}$
   b. $\frac{7\pi}{6}$
   c. $\frac{7\pi}{4}$

We have already introduced the idea that, given an arc length t along the unit circle, $\cos(t)$ is the x-value of the point where the arc length terminates and $\sin(t)$ is the y-value; however, there are other interesting geometric interpretations of the circular functions of real number. To understand these interpretations better, the following activity should be explored independently or during class.

? Activity 7.3.2: Geometric Interpretations of the Circular Functions

Part 1

Figure 7.3.10 shows a unit circle centered at O. The line through AC is tangent to the circle at A. (Recall that a tangent to a circle is perpendicular to the radius that meets it.)

Figure 7.3.10

a. What is the radius of the circle?
b. Find line segments whose lengths are $\cos(\theta)$ and $\sin(\theta)$.
c. Explain why $\triangle OAC$ is similar to $\triangle OBD$.
d. Find a line segment in $\triangle OAC$ whose length is $\tan(\theta)$. Explain why your choice is correct.
e. Find a line segment in $\triangle OAC$ whose length is $\sec(\theta)$. Explain why your choice is correct.

Part 2

Figure 7.3.11 is another unit circle, with tangent BD.

Figure 7.3.11

a. What are the measures of the angles $\angle OCD$ and $\angle OCB$?
b. Explain why $\triangle OCB$ is similar to $\triangle OAC$.
c. Find a line segment in $\triangle OCB$ whose length is $\tan(\theta)$. Explain why your choice is correct.

d. Find a line segment in $\triangle OCB$ whose length is $\sec(\theta)$. Explain why your choice is correct.
e. Explain why $\triangle OCB$ is similar to $\triangle DCO$.
f. Find a line segment in $\triangle DCO$ whose length is $\csc(\theta)$. Explain why your choice is correct.

Domain and Range of the Circular Functions

By interpreting the trigonometric functions in terms of the unit circle definition (as circular functions), we can discuss their domains and ranges.

If t is the length of the arc along the unit circle, then we can see that, no matter how large or small, negative or positive, we make t, the arc length will terminate at a point $P(x, y)$ on the unit circle. Since $x = \cos(t)$ and $y = \sin(t)$, the cosine and sine functions are defined for all real numbers.

Moreover, whenever $x = 0$, we know that $\sec(t) = \frac{1}{x}$ and $\tan(t) = \frac{y}{x}$ are both undefined. We know the values of x on the unit circle become 0 precisely when $t = \pm\frac{\pi}{2}$ or $\pm\frac{3\pi}{2}$ or $\pm\frac{5\pi}{2}$. In general, these two functions are undefined at $t = \frac{\pi}{2} + k\pi$, where k is an integer. A similar argument reveals the cosecant and tangent functions to be undefined when $tk\pi$ (again, where k is an integer). The following theorem summarizes these results.

Theorem: Domains of the Circular Functions

Let t be a real number and k be an integer. The domains of the circular functions are

$\sin(t)$ and $\cos(t)$	All real numbers, or $(-\infty, \infty)$
$\tan(t)$ and $\sec(t)$	All real numbers except $t = \dfrac{\pi}{2} + k\pi$
$\cot(t)$ and $\csc(t)$	All real numbers except $t = k\pi$

While there is a way to write the domains of tangent, cotangent, secant, and cosecant in interval notation, it's a bit complex.

The activity, Geometric Interpretations of the Circular Functions, can be beneficial in visualizing the ranges of the circular functions. For example, Figure **7.3.10** shows us that the values of the cosine and sine functions (which are the x and y coordinates of points along the unit circle) are always between -1 and 1, inclusive. Thus, the range of these functions is $[-1, 1]$.

Since the values of the tangent function can be interpreted as the slope of the ray emanating from the origin through the point $P(x, y)$ on the unit circle, we can easily see that the tangent can get as large (both positively and negatively) as we want. This is because the slopes of these line can get as steep as we want (and they can even be zero). Hence, the range of the tangent function is $(-\infty, \infty)$.

The following theorem states the ranges of all of the circular functions. While not incredibly useful now, they will become increasingly important as we move forward, and we should introduce them early so you can get used to them.

Theorem: Ranges of the Circular Functions

Let t be a real number. The ranges of the circular functions are

$\sin(t)$ and $\cos(t)$	$[-1, 1]$
$\tan(t)$ and $\cot(t)$	$(-\infty, \infty)$
$\sec(t)$ and $\csc(t)$	$(-\infty, -1] \cup [1, \infty)$

Example 7.3.8

Is it possible for $\sec(t) = -\frac{1}{2}$?

Solution

Since the range of the secant function is $(-\infty, -1] \cup [1, \infty)$, and $-\frac{1}{2}$ is not in this interval, it is *not* possible for $\sec(t) = -\frac{1}{2}$.

Checkpoint 7.3.8

For each function, state a value the function could return and a value it could not.

a. $\cos(t)$
b. $\csc(t)$

Answers

a. 2 is not a possible output, but 1/2 is.
b. 1/2 is not a possible output, but 2 is.

Footnotes

[1] You should check that these coordinates satisfy the equation of the circle, $x^2 + y^2 = 4$.

[2] Once again, you should check that these coordinates satisfy the equation of the circle, $x^2 + y^2 = 1$.

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7.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- The Unit Circle
  - The Unit Circle - The Unit Circle
- Unit Circle Definition of the Trigonometric Functions
  - Unit Circle - Trigonometric Functions: A Unit Circle Approach
  - Unit Circle - Evaluating Trigonometric Functions of Non-Acute, Special Angles
  - Unit Circle - Evaluating Trigonometric Functions at Axial Angles
  - The Unit Circle - Special Triangles
  - The Unit Circle - The Reference Number
  - Trigonometry Mini-Lecture: Graph the Unit Circle with a TI Graphing Calculator & Find Trig Values

KEY CONCEPTS

DEFINITIONS

- **Trigonometric Functions (Unit Circle Definition):** Let t be a real number representing the arc length from the point (1, 0) to a point $P(x, y)$ on the unit circle. The trigonometric functions of t, also called circular functions, are defined by the coordinates of P. For example, $\cos(t) = x$ and $\sin(t) = y$.

THEOREMS

- **Lemma:** Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then $\cos(\theta) = x$ and $\sin(\theta) = y$.
- **Domains of the Circular Functions:** Let t be a real number and k be an integer. The domains of the circular functions are:
  - $\sin(t)$ and $\cos(t)$: All real numbers, or $(-\infty, \infty)$
  - $\tan(t)$ and $\sec(t)$: All real numbers except $t = \frac{\pi}{2} + k\pi$
  - $\cot(t)$ and $\csc(t)$: All real numbers except $t = k\pi$
- **Ranges of the Circular Functions:** Let t be a real number. The ranges of the circular functions are:
  - $\sin(t)$ and $\cos(t)$: $[-1, 1]$
  - $\tan(t)$ and $\cot(t)$: $(-\infty, \infty)$
  - $\sec(t)$ and $\csc(t)$: $(-\infty, -1] \cup [1, \infty)$

COMMON MISTAKES

- **Using the Wrong Calculator Mode:** When working with the unit circle definition, angles and arc lengths are measured in radians. You must ensure your calculator is set to Radian mode, not Degree mode, to get correct approximations.
- **Confusing Angle Measure with Arc Length:** While the numerical value of an angle in radians is equal to the arc length it subtends on a unit circle, it is important to remember they are conceptually different. An angle is an amount of rotation, while an arc length is a distance.

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7.3.2: HOMEWORK

READING QUESTIONS

1. What is the radius of a unit circle?
2. If the terminal side of an angle θ intersects the unit circle at a point $P(x, y)$, what do the x- and y-coordinates represent?
3. On a unit circle, what is the relationship between the radian measure of a central angle and the length of the arc it subtends?
4. When trigonometric functions are defined based on the arc length on a unit circle, what are they sometimes called?
5. To find the value of a trigonometric function for a real number t, what do you do?
6. What calculator mode should be used when evaluating trigonometric functions for angles given in radians or for real numbers?
7. What is the domain of the sine function?
8. What is the domain of the tangent function?
9. What is the range of the cosine function?
10. What is the range of the secant function?
11. Is it possible for $\sin(t)$ to equal 2? Why or why not?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, simplify.

1. a. $\frac{2}{3}+\frac{1}{6}$
   b. $\frac{2x}{3}+\frac{x}{6}$
2. a. $\frac{3}{4}-\frac{5}{8}$
   b. $\frac{3n}{4}-\frac{5n}{8}$
3. a. $2-\frac{3}{4}$
   b. $2b-\frac{3b}{4}$
4. a. $1+\frac{3}{8}$
   b. $m+\frac{3m}{8}$
5. a. $\frac{5}{6}-\frac{7}{6}$
   b. $\frac{5q}{6}-\frac{7q}{6}$
6. a. $\frac{2}{3}-\frac{5}{3}$
   b. $\frac{2p}{3}-\frac{5p}{3}$

Answers

1. a $\frac{5}{6}$, b $\frac{5x}{6}$
2. a $\frac{1}{8}$, b $\frac{n}{8}$
3. a $\frac{5}{4}$, b $\frac{5b}{4}$
4. a $\frac{11}{8}$, b $\frac{11m}{8}$
5. a $\frac{-1}{6}$, b $\frac{-q}{6}$
6. a -1, b $-p$

HOMEWORK

VOCABULARY CHECK

1. A circle of radius 1 having center at the origin is called a(n) ___.
2. When considered as functions of arc length, we call the trigonometric functions ___ functions because they are defined by the coordinates of points on a unit circle.

CONCEPT CHECK

3. Explain why $\cos\theta \leq 1$ for any angle θ.
4. How many solutions are there to the equation $\cos(\theta) = 0.4$ between $0°$ and $360°$?
5. On a unit circle, the ___ measure of the central angle and the ___ length it cuts off are the same value.
6. List the trigonometric functions having a domain of all real numbers.
7. List the trigonometric functions having a range of all real numbers.
8. Sketch a figure on a unit circle to illustrate each equation.
   a. $\sin\frac{3\pi}{4} = 0.7071$
   b. $\cos 2.5 = -0.8011$
9. Write down the multiples of $\frac{\pi}{12}$ from 0 to 2π. Reduce each fraction.
10. Sketch arcs in standard position on a unit circle with the following lengths.

$$s=\frac{\pi}{2},\quad s=\frac{3\pi}{4},\quad s=\frac{\pi}{3},\quad s=\frac{2\pi}{3}$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

11. Let $P(x, y)$ be a point on the unit circle determined by an arc of length t from the point $(1, 0)$ to P. Then the coordinates of P are given by

$$x=\sin(t),\quad y=\cos(t).$$

12. If $\beta < \alpha$, then $\sin(\beta) < \sin(\alpha)$.

13. The circular functions are functions of real numbers. As such, we can set our calculator to *either* degree *or* radian mode to get approximations of these functions.
14. The domain of the tangent function includes all real numbers *except* for $\frac{\pi}{2}$.
15. The range of the sine and cosine functions is -1 to 1, inclusive.

BASIC SKILLS

For the following exercises, each point on the unit circle is the terminal point of a special angle in standard position. Give exact values for the radian measure, t, of the angle, and the coordinates (x, y) of the point.

16.

	a	b	c	d
t				
x				
y				

17.

	a	b	c	d
t				
x				
y				

18.

	a	b	c	d
t				
x				
y				

19.

	a	b	c	d
t				
x				
y				

20. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
$\frac{5\pi}{6}$			

21. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{7\pi}{6}$			
$\frac{5\pi}{4}$			
$\frac{4\pi}{3}$			

22. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{5\pi}{3}$			
$\frac{7\pi}{4}$			
$\frac{11\pi}{6}$			

For the following exercises, evaluate each function.

23. $\sin(45°)$
24. $\cos(150°)$
25. $\cot(270°)$
26. $\tan(240°)$
27. $\sec(315°)$
28. $\csc(210°)$
29. $\cos\left(\frac{\pi}{6}\right)$
30. $\sin\left(\frac{3\pi}{2}\right)$
31. $\tan\left(\frac{2\pi}{3}\right)$
32. $\sec\left(\frac{7\pi}{6}\right)$
33. $\csc\left(\frac{5\pi}{4}\right)$
34. $\cot\left(\frac{4\pi}{3}\right)$

For the following exercises, use the unit circle to find all values of θ between 0 and 2π for which the given equation is true.

35. $\sin(\theta) = \frac{\sqrt{3}}{2}$
36. $\cos(\theta) = -\frac{\sqrt{3}}{2}$
37. $\tan(\theta) = -1$
38. $\csc(\theta) = -2$
39. $\sec(\theta) = 1$
40. $\cot(\theta) = \sqrt{3}$
41. Estimate the x-coordinate of the points on the unit circle designated by each angle and complete the table. (Hint: Use symmetry in the second, third, and fourth quadrants.)

Angle	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
x-coordinate										

Angle	100°	110°	120°	130°	140°	150°	160°	170°	180°
x-coordinate									

Angle	190°	200°	210°	220°	230°	240°	250°	260°	270°
x-coordinate									

Angle	280°	290°	300°	310°	320°	330°	340°	350°	360°
x-coordinate									

42. Estimate the y-coordinate of the points on the unit circle designated by each angle and complete the table. (Hint: Use symmetry in the second, third, and fourth quadrants.)

Angle	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
x-coordinate										

Angle	100°	110°	120°	130°	140°	150°	160°	170°	180°
x-coordinate									

Angle	190°	200°	210°	220°	230°	240°	250°	260°	270°
x-coordinate									

Angle	280°	290°	300°	310°	320°	330°	340°	350°	360°
x-coordinate									

For the following exercises, use the unit circle to estimate the sine, cosine, and tangent of each arc of given length.

43. a. 0.4
    b. 1.2
    c. 2

44. a. 0.8
    b. 2.6
    c. 4

45. a. 2.8
    b. 3.5
    c. 5

46. a. 3
    b. 4.3
    c. 5.5

For the following exercises, use the unit circle to estimate two numbers with the given trigonometric value.

47. $\cos t = 0.3$
48. $\sin t = 0.1$
49. $\sin t = -0.7$
50. $\cos t = -0.6$
51. $\tan t = \dfrac{-4}{9}$
52. $\tan t = \dfrac{8}{6}$

Each of the following exercises describes an arc in standard position on the unit circle. In which quadrant does the terminal point of the arc lie?

53. $\sin s > 0, \quad \cos s < 0$
54. $\sin s < 0, \quad \cos s > 0$
55. $\cos t < 0, \quad \tan t < 0$
56. $\cos t > 0, \quad \tan t < 0$
57. $\sin x < 0, \quad \tan x > 0$
58. $\sin x > 0, \quad \tan x < 0$

For the following exercises, without a calculator, decide whether the quantity is positive or negative.

59. $\cos 2.7$
60. $\sin 4.1$
61. $\tan 3.8$
62. $\tan 5.4$
63. $\sin 2.2$
64. $\cos 4.9$

Starting at $(1, 0)$, you walk s units around a unit circle. For the following exercises, sketch a unit circle showing your position. What are your coordinates?

65. $s = 2.5$
66. $s = 4.3$
67. $s = 8.5$
68. $s = 11$

SYNTHESIS QUESTIONS

For the following exercises, place the trigonometric values in order from smallest to largest. Use the figure to help you, but try not to use a calculator!

69. $\sin 0.5, \quad \sin 1.5, \quad \sin 2.5, \quad \sin 3.5$
70. $\cos 1.6, \quad \cos 2.6, \quad \cos 3.6, \quad \cos 5.6$
71. $\cos 2, \quad \cos 3, \quad \cos 4, \quad \cos 5$
72. $\sin 2.8, \quad \sin 3.8, \quad \sin 4.8, \quad \sin 5.8$

For the following exercises, the figure shows an arc of length t, and the coordinates of its terminal point. Find the terminal point of each related arc given below, and give its sine, cosine, and tangent.

a $2\pi - t$

b $\pi - t$

c $\pi + t$

73.

74.

75.

76.

77. Use the figure to help you fill in the blanks.

    a. As θ increases from $0°$ to $90°$, $f(\theta) = \sin\theta$ _____ from _____ to _____.

    b. As θ increases from $90°$ to $180°$, $f(\theta) = \sin\theta$ _____ from _____ to _____.

    c. As θ increases from $180°$ to $270°$, $f(\theta) = \sin\theta$ _____ from _____ to _____.

    d. As θ increases from $270°$ to $360°$, $f(\theta) = \sin\theta$ _____ from _____ to _____.

78. Use the figure to help you fill in the blanks.

a. As θ increases from $0°$ to $90°, f(\theta) = \cos\theta$ _____ from _____ to _____.

b. As θ increases from $90°$ to $180°, f(\theta) = \cos\theta$ _____ from _____ to _____.

c. As θ increases from $180°$ to $270°, f(\theta) = \cos\theta$ _____ from _____ to _____.

d. As θ increases from $270°$ to $360°, f(\theta) = \cos\theta$ _____ from _____ to _____.

For the following exercises, use the grid provided below.

79. Draw two different angles α and β in standard position whose sine is 0.6.

a. Use a protractor to measure α and β.

b. Find the reference angles for both α and β. Draw in the reference triangles.

80. Draw two different angles θ and ϕ in standard position whose sine is -0.8.

a. Use a protractor to measure θ and ϕ.

b. Find the reference angles for both θ and ϕ. Draw in the reference triangles.

For the following exercises, use the grid provided below.

81. Draw two different angles α and β in standard position whose cosine is 0.3.

a. Use a protractor to measure α and β.

b. Find the reference angles for both α and β. Draw in the reference triangles.

82. Draw two different angles θ and ϕ in standard position whose cosine is −0.4.

a. Use a protractor to measure θ and ϕ.

b. Find the reference angles for both θ and ϕ. Draw in the reference triangles.

For the following exercises, find all angles between $0°$ and $360°$ for which the statement is true.

83.

a. $\cos\theta = -1$

b. $\sin\theta = -1$

c. $\tan\theta = -1$

84.

a. Find two angles, $0 \le \theta \le 360°$, with $\sin\theta = 0$.

b. Find two angles, $0 \le \theta \le 360°$, with $\cos\theta = 0$.

For the following exercises, find two points on the unit circle with the given coordinate. Sketch the approximate location of the points on the circle. (Hint: what is the equation for the unit circle?)

85. $x = 0.2$

86. $x = -0.6$

87. $y = -0.35$

88. $y = 0.7$

89. $x = \dfrac{-\sqrt{3}}{2}$

90. $y = \dfrac{1}{\sqrt{2}}$

For the following exercises, find the angle in radians between 0 and 2π determined by the terminal point on the unit circle. Round your answer to hundredths.

91. (−0.1782, 0.9840)

92. (−0.8968, −0.4425)

93. (0.8855, −0.4646)

94. (0.9801, 0.1987)

95. (−0.7659, −0.6430)

96. (0.9602, −0.2794)

97.

a. Sketch a unit circle and the line $y = x$. Find the coordinates of the two points where the line and the circle intersect.

b. State your answers to part (a) using trigonometric functions.

98.

a. Sketch a unit circle and the line $y = -x$. Find the coordinates of the two points where the line and the circle intersect.

b. State your answers to part (a) using trigonometric functions.

99.

a. Sketch a line that passes through the origin and the point (8,3). What is the slope of the line?

b. What is the angle of inclination of the line in radians, measured from the positive x-axis?

100.

a. Sketch a line that passes through the origin and the point (3,8). What is the slope of the line?

b. What is the angle of inclination of the line in radians, measured from the positive x-axis?

For the following exercises, find an equation for the line with the given angle of inclination passing through the given point.

101. $\alpha = \dfrac{\pi}{3}, (4, 2)$

102. $\alpha = \dfrac{5\pi}{6}, (-6, 3)$

103. $\alpha = 2.4, (5, -8)$

104. $\alpha = 0.6, (-2, -3)$

105. If $\sec(\theta) = \dfrac{x}{4}$ for $0 < \theta < \dfrac{\pi}{2}$, find an expression for $\ln|\sec(\theta) + \tan(\theta)|$ in terms of x.

APPLICATIONS

106. The sunrise time in Wellington, New Zealand, on the nth day of the year can be modeled by

$$S = 1.93\sin(0.016n - 1.13) + 6.14$$

where S is given in hours after midnight. Find the sunrise time on January 1 (day $n = 1$), April 1 (day $n = 91$), July 1 (day $n = 182$), and on October 1 (day $n = 274$).

107. The variable star RT Cygni reached its maximum magnitude on May 22, 2004, and t days later its magnitude is given by

$$M = 9.55 - 2.25\cos(0.033t)$$

Find the magnitude of RT Cygni on days $t = 0$, $t = 48$, $t = 95$, $t = 142$, and $t = 190$. (Note that smaller values of M denote brighter magnitudes.)

108. City Park features a circular jogging track with a radius of 1 mile centered on the open-air bandstand. You start jogging on the track 1 mile due east of the bandstand and proceed counterclockwise. What are your coordinates relative to the bandstand when you jogged five miles?

109. Silver Reservoir is a circular man-made lake with a radius of 1 kilometer. If you start at the easternmost point on the reservoir and walk counterclockwise for 4 kilometers, how far south of your initial position are you?

Use the results of Problem 110 for Problems 111 - 113.

110. Use similar triangles to show that the coordinates of a point P determined by angle θ on a circle of radius r are $x = r\cos\theta, y = r\sin\theta$. (See the figure below.)

111. The Astrodome in Houston has a diameter of 710 feet. If you start at the easternmost point and walk counterclockwise around its perimeter for 250 feet, how far north of your starting point are you?

112. The Barringer meteor crater near Winslow, Arizona, is 1182 meters in diameter. You start at the easternmost point on the crater's rim and walk counterclockwise around the edge. After walking for 1 kilometer, what is your position relative to your starting point?

113. One of the most intriguing features of Stonehenge is the position of the four Station Stones.

They form the corners of a rectangle inscribed in the Aubrey Circle on the perimeter of the henge, which has a diameter of 288 feet. A line from the center of the circle and perpendicular to the long edge of the rectangle points through the Slaughter Stones at the entrance of the henge and out to the Heel Stone. If you stood in the circle's center on the summer solstice, you would see the sun rise directly over the Heel Stone.

a. The sun rises $48.6°$ east of north on the summer solstice at Stonehenge. If the positive y-axis points north, find the coordinates of the henge entrance relative to its center.

b. The northernmost station stone is located $66.6°$ of arc counterclockwise from the entrance. Find its coordinates relative to the center of the henge.

CHALLENGE PROBLEMS

114. Use similar triangles to show that the coordinates of point P on the unit circle shown below are $(\cos t, \sin t)$.

115. Use similar triangles to show that $ST = \tan t$.

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7.4: Velocities and Areas of Sectors

Note to the Instructor (click to expand)

There is no subtlety to what is in this section or why I placed it here - you need radian measure to discuss either of the topics in this section, so it could not occur until this chapter. I suppose we could have covered this material directly after introducing radian measure (and prior to the Unit Circle definition of the trigonometric functions); however, I happened to place it here before thinking about that order. Feel free to make the switch if you feel so inclined (it actually might make the connection between the Unit Circle definition of the trigonometric functions and the area of the sector subtended by θ more digestible).

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Calculate the area of a sector formed by a central angle.
- Solve an application involving sector area.
- Calculate the linear velocity of a point moving with uniform circular motion.
- Calculate the angular velocity of a point moving with uniform circular motion.
- Convert from linear velocity to angular velocity and vice versa.
- Solve application problems involving linear and angular velocity.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Areas of Sectors

Now that we have paired angles with real numbers via radian measure, a world of applications awaits us.

We recently discovered the relationship between the length, s, of the arc subtended by an angle θ (in radians) in a circle of radius r to be

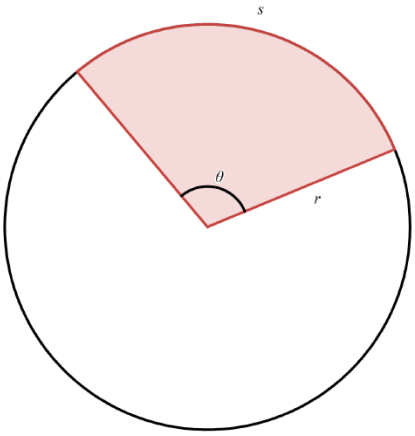
$$s = r\theta.$$

We now investigate the area of the **sector** formed by this central angle θ.

Definition: Sector (of a circle)

The region of a circle bounded by the initial and terminal sides of a central angle θ and the arc subtended by θ is called a **sector of the circle** (or, simply, **sector**).

Figure 7.4.1: *The area of a sector of a circle of radius r subtended by the angle θ*



To find the area of a sector, recall that the *total area* of a circle of radius r is πr^2; however, the shaded region in Figure \(7.4.1 \) is not the entire circle. In fact, it is a specific *portion* of this circle. In terms of angular measure, we have only shaded the region covered by an angle rotated through θ out of 2π radians of a full circle. That is, the portion of the circle we have shaded is

$$A = \frac{\theta}{2\pi} \text{ of the area of the full circle} = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta.$$

This proves the following theorem.

Theorem: Area of a Sector

If θ (in radians) is the central angle in a circle of radius r, then the area of the sector subtended by θ is

$$A = \frac{1}{2} r^2 \theta.$$

Finding the areas of sectors becomes increasingly essential as you move into Calculus (specifically, Integral Calculus). Be sure to spend the time now to master this relatively straightforward concept.

Example 7.4.1

Find the area of the sector subtended by a central angle of $120°$ in a circle of radius 4.

Solution

The most common mistake when using the formula for the area of a sector is not working in radian measure. Since $120° = \frac{2\pi}{3}$, we find the area of the sector to be

$$A = \frac{1}{2} \cdot (4)^2 \left(\frac{2\pi}{3}\right) = \frac{16\pi}{3}.$$

Checkpoint 7.4.1

If the area of a sector of a circle having radius 12 meters is 100 square meters, find the measure of the central angle (in radians).

Answer

$\frac{25}{18}$ radians

Example 7.4.2: The Midnight Sun Golf Challenge

In certain parts of the world near the Arctic, such as Norway, the sun doesn't set during part of the summer. This phenomenon is known as the Midnight Sun. A golf course in Norway takes advantage of this by hosting a special event: The Midnight Sun Golf Challenge. During this event, golfers can play as many holes as they want throughout the night.

The golf course has a unique challenge on their signature par 3 hole, which features a circular green with a radius of 15 meters. If a golfer lands their ball within a particular marked sector of the green, where the central angle is 30 degrees, they win a prize. To determine the likelihood of winning, calculate the area of the sector where golfers must land their ball.

Solution

We are given $\theta = 30^\circ = \frac{\pi}{6}$ radians and $r = 15$ meters. Therefore, the area of the sector is

$$A = \frac{1}{2} \cdot (15)^2 \frac{\pi}{6} = \frac{225\pi}{12} \approx 59 \text{ square meters}$$

As an aside, the probability that the golfer's ball would land in this sector (given the ball lands on the green and that all areas are equally likely for the ball to land in) would be the ratio of this sector area to the area of the entire green, which is

$$\frac{A_{\text{sector}}}{A_{\text{green}}} = \frac{225\pi/12}{\pi(15)^2} = \frac{1}{12} \approx 0.083 = 8.3\%.$$

This should make sense because 30° is one-twelfth of a full rotation of a circle.

Checkpoint 7.4.2

A circular park has a sector fenced off for dogs. The sign posted by Sacramento County states that dogs can enjoy 6,000 square feet of play area; however, Paige finds this claim rather dubious. Unafraid about how foolish she might look, Paige takes a very large tape measure to the park and measures the radius of the park to be 120 feet. She then measures the length of fencing along the arc of the dog park area to be roughly 84 feet. Is the sign claiming that dogs have 6,000 square feet of play area correct?

Answer

No. The given measurements yield a central angle of 0.7 radians. Thus, the area is actually

$$A = \frac{1}{2}(120)^2(0.7) = 5,040 \text{ square feet.}$$

Linear Motion

We have already used the variable s to denote the arc length in a circle of radius r subtended by a central angle θ. We might not have mentioned that this same variable is used throughout Mathematics and Physics to denote spatial position (hence, the letter s). For example, in Algebra and Calculus, you might represent the height of a projectile (relative to the ground) at any time t by the function

$$s(t) = -16t^2 + 20t + 5.$$

Note that we have used the words spatial position and height to represent the same thing. In truth, "height" only applies if an object rises or falls relative to some point in space. "Spatial position," on the other hand, is a bit more flexible and allows us to consider the distance an object is away from a centralized point. Another word commonly used in place of spatial position is **displacement**.

Once we have a function for an object's spatial position (or displacement) at a given time t, we can expand our investigation of the object's motion to include its velocity. In Physics, the **average velocity** (sometimes called the **average linear velocity**) of an object, denoted $\overline{v}$ and read as "v-bar," is defined as the average rate of change of the spatial position of the object with respect to time. As a result, we have

$$\overline{v} = \frac{\text{displacement}}{\text{time}} = \frac{s}{t}.$$

The quantity $\overline{v}$ has units of $\frac{\text{length}}{\text{time}}$ and conveys two ideas: the direction in which the object is moving and how fast the position of the object is changing. For example, if we define movement to the right as positive and movement to the left as negative, an object moving 10 feet to the right in 5 seconds has an average velocity over that period of

$$\overline{v} = \frac{+10 \text{ feet}}{5 \text{ seconds}} = 2 \text{ feet per second.}$$

In this situation, we have written a plus sign on the 10 feet because the object is moving to the right by 10 feet over this period. On the other hand, if the object moving to the left at 10 feet for every 5 seconds, its average velocity over that period would be

$$\overline{v} = \frac{-10 \text{ feet}}{5 \text{ seconds}} = -2 \text{ feet per second.}$$

If we could somehow measure the velocity of an object at a specific moment in time (rather than over an interval of time), we could call that the **instantaneous velocity**, v. The instantaneous velocity of an object is often just called the **velocity** of the object.

It is a common mistake for people to confuse velocity with speed. The **speed** of an object is the magnitude of its instantaneous velocity. That is,

$$\text{speed} = |v| .$$

Note that speed is *never* negative, while velocity (and average velocity) can be negative (denoting both speed and direction).

✔ Example 7.4.3

Ants are incredibly fast, especially when you consider their size! The Saharan silver ant (Cataglyphis bombycina) holds the record for the fastest speed ever measured in an ant. It can move at a breathtaking 168.3 inches in five seconds. Find the average velocity of this ant.

Solution

Let $s = 168.3$ inches and $t = 5$ seconds. Then the average velocity of the Saharan silver ant is

$$\overline{v} = \frac{168.3 \text{ inches}}{5 \text{ seconds}} = 33.66 \text{ inches per second.}$$

? Checkpoint 7.4.3

Kenyan athlete Kelvin Kiptum set a men's world record time of 2:00:35 on October 8, 2023, at the 2023 Chicago Marathon (26.2 miles). Find his average velocity for the race in miles per hour. Round your answer to the nearest hundredth.

Answer

13.04 miles per hour.

What if we wanted to know how fast the Kenyan athlete in Checkpoint 7.4.3 was running at exactly 1 hour and 12 minutes into the marathon? That is, what if we wanted to know his instantaneous velocity at precisely 1 hour and 12 minutes into the marathon? We know that he had an *average linear velocity* of approximately 13.04 miles per hour; however, this does not mean he ran at 13.04 miles per hour the entire race. There were many moments when he ran faster than this and others when he ran slower. Since his velocity was not constant throughout the marathon, we would be hard-pressed to compute his instantaneous velocity at 1 hour and 12 minutes into the race - the answer to this type of question is the purview of Calculus.

Since we do not have Calculus at our disposal, if we ever want to answer a question about instantaneous velocity (again, which we call velocity), we must assume that the object's velocity is constant throughout the motion. Then, and only then, it can be shown that the average linear velocity is the same as the instantaneous velocity, v. For the remainder of our discussion, we will make such an assumption. That is, from this sentence forward in this section, we will assume that the object, person, car, particle, or thing we are considering is moving at a constant speed. Hence, its average velocity is equivalent to its instantaneous velocity at any given moment. Again, we will simplify the wording to "velocity."

Circular Motion - Linear Velocity

Our next excursion into the realm of applications comes through circular motion. Suppose an object is moving as pictured below along a circular path of radius r from the point P to the point Q in an amount of time t.

Figure 7.4.2

Like before, s represents a displacement so that $s > 0$ means the object is traveling in a counter-clockwise direction, and $s < 0$ indicates movement in a clockwise direction. Note that with this convention, the formula we used to define radian measure, namely $s = r\theta$, still holds since a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation.

Again, just as before, the quantity

$$\overline{v} = \frac{s}{t}$$

is called the average linear velocity. This is despite the fact that s is a length along the curved edge of a circle. This is because if the path of the object were "uncurled" from a circle to form a line segment, then the average velocity of the object on that line segment would be the same as the average velocity on the circle.

Example 7.4.4

As of 2024, the world's largest Ferris wheel is the Dubai Eye. It opened in 2021 on tiny Bluewaters Island in the Arabian Gulf. The diameter of the wheel is 820 feet, and a single rotation lasts 38 minutes. Assuming the wheel never stops turning, what is a rider's (instantaneous) linear velocity on the Dubai Eye?

Solution

We need to compute how far along the edge of the circle formed by the Dubai Eye that a rider moves through in a certain amount of time. Luckily, we were told a rider completes a full circuit in 38 minutes. Therefore, we let $t = 38$ minutes. Moreover, the distance traveled in that time is the same as the circumference of the circle. Here, we can use the faithful formula,

$$C = 2\pi r.$$

The diameter of the Dubai Eye is 820 feet, so the radius is 410 feet. Thus, a rider travels a distance of

$$s = C = 2\pi(410\,\text{feet}) = 820\pi\,\text{feet}.$$

We now use our velocity formula,

$$v = \frac{s}{t} = \frac{820\pi\,\text{feet}}{38\,\text{minutes}} \approx 67.8\,\text{feet per minute}.$$

For reference, this is a glacial 0.8 miles per hour.

Example 7.4.4 showcases that computing the linear velocity of an object along a circular path is very much the same as computing the velocity along a line, we just compute $\frac{\text{distance}}{\text{time}} = \frac{s}{t}$.

Checkpoint 7.4.4

Gum gets stuck to the outer edge of your bike tire. Suppose the tire rotates so that the gum travels along the circular path of its edge 5 inches in 2 seconds. What is the linear velocity of the gum? (By the way, this is also the linear speed of the tire.)

Answer

$v = 2.5$ inches per second

Circular Motion - Angular Velocity

Dividing both sides of $s = r\theta$ by t, we get a very powerful relation:

$$\frac{s}{t} = \frac{r\theta}{t} \implies v = r \cdot \frac{\theta}{t}$$

Definition: Angular Velocity

The quantity $\frac{\theta}{t}$ is called the **average angular velocity** of an object undergoing circular motion. Its units are radians per unit time. If the rate of change of the angle θ with respect to time is constant, then it is simply called the **angular velocity**, and it is denoted by ω (omega). Therefore,

$$\omega = \frac{\theta}{t}.$$

Example 7.4.5

A point on a circle rotates through $38°$ in 3 hours. What is the angular velocity of the point?

Solution

Remember that the formula, $s = r\theta$, requires θ to be in radians. This is the same θ as the angle in $\omega = \frac{\theta}{t}$. Therefore, we need to convert the given angle to radians.

$$\omega = \frac{38°}{3\,\text{hours}} = \frac{38\pi\,\text{radians}}{540\,\text{hours}} \approx 0.2211\,\text{radians per hour}.$$

Checkpoint 7.4.5

Suppose a point on a circle rotates through $300°$ in two hours. Find the *exact* value of the angular velocity of the point.

Answer

$\frac{5\pi}{6}$ radians per hour

Circular Motion - Relating Linear and Angular Velocities

We now formalize our recent statement.

Theorem: Linear and Angular Veclocity Formula

For an object moving on a circular path of radius r with constant angular velocity ω, the (linear) velocity of the object is given by

$$v = r\omega.$$

We need to talk about units here. The units of v are $\frac{\text{length}}{\text{time}}$, the units of r are length only, and the units of ω are $\frac{\text{radians}}{\text{time}}$. Thus the left hand side of the equation $v = r\omega$ has units $\frac{\text{length}}{\text{time}}$, whereas the right hand side has units $\text{length} \cdot \frac{\text{radians}}{\text{time}} = \frac{\text{length} \cdot \text{radians}}{\text{time}}$. The supposed

contradiction in units is resolved by remembering that radians are a dimensionless quantity and angles in radian measure are identified with real numbers so that the units $\frac{\text{length} \cdot \text{radians}}{\text{time}}$ reduce to the units $\frac{\text{length}}{\text{time}}$. We are long overdue for an example.

Example 7.4.6

Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle which completes one revolution in (approximately) 24 hours. The path traced out by the point during these 24 hours is the latitude of that point. Cosumnes River College is at $38^\circ\, 27'\, 14.07''$ north latitude, and it can be shown that the radius of the Earth at this latitude is approximately 3958 miles. Find the linear velocity, in miles per hour, of Cosumnes River College as the world turns.

Solution

To use the formula $v = r\omega$, we first need to compute the angular velocity ω. The Earth makes one revolution in 24 hours. One revolution is 2π radians, so

$$\omega = \frac{2\pi \text{ radians}}{24 \text{ hours}} = \frac{\pi}{12 \text{ hours}},$$

where, once again, we are using the fact that radians are real numbers and are dimensionless. (For simplicity's sake, we are also assuming that we are viewing the rotation of the Earth as counter-clockwise, so $\omega > 0$.) Hence, the linear velocity is

$$v = 3958 \text{ miles} \cdot \frac{\pi}{12 \text{ hours}} \approx 1036 \frac{\text{miles}}{\text{hour}}$$

Checkpoint 7.4.6

How fast is a bike moving, in miles per hour, if its rear tire has a radius of 15 inches and the angular velocity of the rear tire is 20 radians per second?

Answer

≈ 17 miles per hour

It is worth noting that the quantity $\frac{1 \text{ revolution}}{24 \text{ hours}}$ in Example $7.4.6$ is called the **ordinary frequency** of the motion.

Definition: Ordinary Frequency

When an object is undergoing circular motion, the number of cycles (or revolutions) per unit time is called the **ordinary frequency** and is usually denoted by the variable f.

The higher the ordinary frequency, the faster the motion. This is because the circular motion must go through more cycles per unit time. An ordinary frequency that you are familiar with is a radio station's frequency. For example, if you listen to 93.7 on the FM dial, that means you are listening to a signal broadcasting at a frequency of 93.7 MHz. This means 93.7 million Hertz. A Hertz is defined to be 1 cycle per second. Therefore, 93.7 MHz means you are listening to a station broadcasting at 93.7 million cycles per second.

The **angular frequency**, on the other hand, tells us how many radians an object turns through in a given time.

Definition: Angular Frequency

When an object is undergoing circular motion, the number of radians it turns through per unit time is called the **angular frequency** and is usually denoted by the variable ω.

Since there are 2π radians per cycle and f cycles per unit time, the angular frequency is $2\pi f$. However, angular frequency is radians per unit time, and so is angular velocity. Thus, angular frequency and angular velocity mean the same thing! That is, $\omega = 2\pi f$.

One final "measuring stick" for circular motion is very important - the **period** of the circular motion.

Definition: Period (of circular motion)

When an object is undergoing circular motion, we define $T = \frac{1}{f}$ to be the **period** of the motion. It is the amount of time it takes for the object to complete one cycle of the motion.

In the scenario of Example **7.4.6**, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period's time. The distance of the object to the center of rotation is the radius of the circle, r, and is the "magnification factor," which relates ω and v. We will have more to say about frequencies and periods later in this text.

We close out this section with a beautiful application requiring great attention to detail.

Example 7.4.7

A bicycle with a 27-inch-diameter rear wheel has its gears set so that the chainring has a 4-inch diameter and the rear sprocket has a 3-inch diameter (see image below). The cyclist pedals at 250 rpm. What is the speed of the bicycle in miles per hour?

Figure 7.4.3

Solution

In a complex problem like this, it's best to collect the given data by object. That is, we will have data for the chainring, the sprocket, and the rear wheel. We might not use all of these objects in our computations, but just listing them out is often helpful.

	Chainring	Sprocket	Wheel
Radius	$r_c = 2$ inches	$r_s = 1.5$ inches	$r_w = 13.5$ inches
LinearVelocity	$v_c = ?$	$v_s = ?$	$v_w = ?$
AngularVelocity	$\omega_c = ?$	$\omega_s = ?$	$\omega_w = ?$

Before we get into computations, a little bit of logic is helpful. First, how quickly the sprocket is spinning should match how quickly the rear wheel is spinning (in angles per unit time). Therefore, $\omega_s = \omega_w$. Additionally, we were told the cyclist is pedaling at 250 rpm (revolutions per minute). Thus, the chainring (which the pedal is attached to) is spinning at 250 rpm. We need to convert this to "radians per minute" using the fact that there are 2π radians per revolution. Therefore,

$$250\,\text{rpm} = \frac{250\,\text{revolutions}}{\text{minute}} = \frac{250\ \cancel{\text{revolutions}}}{\text{minute}} \cdot \frac{2\pi\,\text{radians}}{\cancel{\text{revolution}}} = \frac{500\pi\,\text{radians}}{\text{minute}}.$$

Notice that we kept the word "radians" in this computation. It's not necessary to do this, but it allows us to easily interpret the meaning of the quantity we are working with. At this point, let's update our table.

	Chainring	Sprocket	Wheel
Radius	$r_c = 2$ inches	$r_s = 1.5$ inches	$r_w = 13.5$ inches
LinearVelocity	$v_c = ?$	$v_s = ?$	$v_w = ?$
AngularVelocity	$\omega_c = 500\pi \frac{\text{radians}}{\text{minute}}$	$\omega_s = \omega_w$	$\omega_w = \omega_s$

We now use the completed information about the chainring (the radius and angular velocity) to find the linear velocity of a point on the edge of the chainring.

$$v_c = r_c\omega_c = (2\,\text{inches})(500\pi\,\text{radians/minute}) = 1000\pi\,\frac{\text{inches}}{\text{minute}}$$

Remember, radians is a unitless measure. Since a point along the edge of the chainring is just a link in the chain connecting the chainring to the sprocket, this will also be the linear velocity of a link moving along the edge of the sprocket. Thus, our table gets updated to the following.

	Chainring	Sprocket	Wheel
Radius	$r_c = 2\,\text{inches}$	$r_s = 1.5\,\text{inches}$	$r_w = 13.5\,\text{inches}$
LinearVelocity	$v_c = 1000\pi \dfrac{\text{inches}}{\text{minute}}$	$v_s = 1000\pi \dfrac{\text{inches}}{\text{minute}}$	$v_w = ?$
AngularVelocity	$\omega_c = 500\pi \dfrac{\text{radians}}{\text{minute}}$	$\omega_s = \omega_w$	$\omega_w = \omega_s$

We now can find the angular velocity of the sprocket.

$$v_s = r_s\omega_s \implies \omega_s = \frac{v_s}{r_s} = \frac{1000\pi \cancel{\text{ inches}}/\text{minute}}{1.5 \cancel{\text{ inches}}} = \frac{2000\pi}{3}\frac{\text{radians}}{\text{minute}}$$

Again, this angular velocity is exactly the same as the angular velocity of the wheel. Thus, our table becomes

	Chainring	Sprocket	Wheel
Radius	$r_c = 2\,\text{inches}$	$r_s = 1.5\,\text{inches}$	$r_w = 13.5\,\text{inches}$
LinearVelocity	$v_c = 1000\pi \dfrac{\text{inches}}{\text{minute}}$	$v_s = 1000\pi \dfrac{\text{inches}}{\text{minute}}$	$v_w = ?$
AngularVelocity	$\omega_c = 500\pi \dfrac{\text{radians}}{\text{minute}}$	$\omega_s = \dfrac{2000\pi}{3}\dfrac{\text{radians}}{\text{minute}}$	$\omega_w = \dfrac{2000\pi}{3}\dfrac{\text{radians}}{\text{minute}}$

We finally arrive at the penultimate computation - the linear velocity of the wheel.

$$v_w = r_w\omega_w = (13.5\,\text{inches})\left(\frac{2000\pi}{3}\frac{\text{radians}}{\text{minute}}\right) = 9000\pi\frac{\text{inches}}{\text{minute}}$$

Did you know the word "penultimate" means "second-to-last?" This means we have one more step (the ultimate step?). First, the linear velocity of a point on the wheel gives the velocity of the bicycle; however, this is in inches per minute and we were asked to find the answer in miles per hour. This is a nice **unit analysis** problem. Let's convert the distance units first.

$$9000\pi\frac{\cancel{\text{inches}}}{\text{minute}}\cdot\frac{\cancel{\text{foot}}}{12\cancel{\text{ inches}}}\cdot\frac{\text{mile}}{5280\cancel{\text{ feet}}} = \frac{9000\pi}{(12)(5280)}\frac{\text{miles}}{\text{minute}}$$

Now let's convert the time units.

$$\frac{9000\pi}{(12)(5280)}\frac{\text{miles}}{\cancel{\text{minute}}}\cdot\frac{60\cancel{\text{ minutes}}}{\text{hour}} = \frac{(9000\pi)(60)}{(12)(5280)}\,\text{mph} \approx 26.8\,\text{mph}$$

? Checkpoint 7.4.7

If the chainring of a bicycle is 150 mm in diameter, the sprocket is 80 mm in diameter, and the wheel is 700 mm in diameter, how fast would one need to pedal, in revolutions per minute, in order to maintain a speed of 24 kilometers per hour?

Answer

Approximately 97 rpm.

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7.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Area of a Sector
  - Angle Measure - Area of a Sector
- Circular Motion
  - Angle Measure - Circular Motion

KEY CONCEPTS

DEFINITIONS

- **Sector (of a circle):** The region of a circle that is bounded by a central angle's initial and terminal sides and the arc it subtends.
- **Average Velocity:** The average rate of change of an object's spatial position with respect to time.
- **Instantaneous Velocity:** The velocity of an object at a specific moment in time. Often simply called "velocity."
- **Speed:** The magnitude of an object's instantaneous velocity, given by the formula $\text{speed} = |v|$.
- **Angular Velocity:** The rate of change of an angle with respect to time, denoted by ω. It is calculated as $\omega = \frac{\theta}{t}$, where θ is in radians.
- **Ordinary Frequency:** The number of cycles or revolutions an object completes per unit of time, denoted by f.
- **Angular Frequency:** The number of radians an object turns through per unit of time. It is equivalent to angular velocity, ω.
- **Period (of circular motion):** The amount of time it takes for an object in circular motion to complete one full cycle. It is the reciprocal of the ordinary frequency, $T = \frac{1}{f}$.

THEOREMS

- **Area of a Sector:** If θ (in radians) is the central angle in a circle of radius r, then the area of the sector subtended by θ is $A = \frac{1}{2}r^2\theta$.
- **Linear and Angular Velocity Formula:** For an object moving on a circular path of radius r with constant angular velocity ω, the linear velocity of the object is given by $v = r\omega$.

COMMON MISTAKES

- **Using Degrees in Formulas:** The formulas for the area of a sector ($A = \frac{1}{2}r^2\theta$) and the relationship between linear and angular velocity ($v = r\omega$) both require the angle θ and the angular velocity ω to be in radian-based units. Using degrees will lead to incorrect answers.
- **Confusing Velocity and Speed:** Velocity can be negative, indicating direction, while speed is the magnitude of velocity and is never negative.

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7.4.2: HOMEWORK

READING QUESTIONS

1. What is a sector of a circle?
2. What is the formula for the area of a sector, and what unit must the central angle be in for the formula to be valid?
3. What is the difference between linear velocity and angular velocity?
4. What is the formula that relates linear velocity (v) and angular velocity (ω) for an object moving in a circle of radius r?
5. What are the typical units for angular velocity?
6. What is the difference between ordinary frequency and angular frequency?
7. What is the period of circular motion, and how does it relate to ordinary frequency?
8. If a point on a circle rotates through an angle of 38° in 3 hours, what must you do first before calculating its angular velocity?
9. Why is it acceptable to say that the units of $r\omega$ are $\frac{\text{length}}{\text{time}}$ even though the units of ω are $\frac{\text{radians}}{\text{time}}$?

HOMEWORK

VOCABULARY CHECK

1. The region bounded by the initial and terminal sides of a central angle θ and the edge of the circle is called a ___ of a circle.
2. A word commonly used in place of spatial position is ___.
3. The ___ of a particle is the average rate of change of its position with respect to a change in time. If we could take an instantaneous "snapshot" of this form of motion, we would call it the ___ of the particle.
4. The ___ of a particle is the magnitude of its velocity.

CONCEPT CHECK

5. Write each statement as a mathematical formula.

   a. The length of the arc, s, is equal to:

   $$\text{(the fraction of a revolution)} \times \text{(circumference of the circle)}$$

   b. The area of the sector, A, is equal to:

   $$\text{(the fraction of a revolution)} \times \text{(area of the circle)}$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. To compute the area of a sector, we multiply the given angle by half of the square of the circle's radius.
7. Given two circles, one of radius r_1 and the other of radius r_2, particles on each circle moving at a constant angular velocity, ω, move at the same linear velocity.

BASIC SKILLS

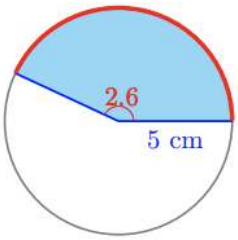
For the following exercises, θ is the central angle in a circle of radius r. In each case, find the area of the sector.

8. $\theta = \frac{2\pi}{3}$, $r = 7$ yards
9. $\theta = \frac{7\pi}{6}$, $r = \sqrt{2}$ m
10. $\theta = \frac{6\pi}{7}$, $r = \sqrt{3}$ miles
11. $\theta = 3$, $r = \frac{1}{2}$
12. $\theta = 5$, $r = 4.1$ cm
13. $\theta = 60°$, $r = 2$ feet
14. $\theta = 330°$, $r = \pi$ inches
15. $\theta = 32°$, $r = 8$ parsecs
16. (MyOpenMath) $\theta = 212°$, $r = 0.88$ leagues

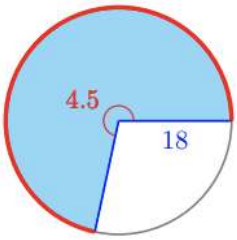
For the following exercises, find the area of the shaded sector.

17.

a.



b.



DETERMINING AREAS OF SECTORS

18. An arc of length 8 feet is cut off by a central angle of $2\pi/3$ radians. Find the area of the resulting sector.
19. A central angle of 4 radians cuts off an arc of length 16 meters. Find the area of the resulting sector.
20. (MyOpenMath) Calculate the area of a sector with a radius of 6 cm and a central angle of $30°$.
21. A sector has a radius of 4 meters and a central angle of $45°$. Find its area.
22. (MyOpenMath) If the radius of a sector is 7.4 inches and the central angle is $225°$, what is the area of the sector?
23. Determine the area of a sector with a radius of 8 feet and central angle $2\pi/3$.
24. Calculate the area of a sector with a 3-meter radius and central angle $5\pi/6$.

DETERMINING THE RADIUS OF A SECTOR

25. If the area of a sector is 25 cm^2 and the central angle is $45°$, find the radius.
26. Determine the radius of a sector with an area of 30 m^2 and a central angle of 60 degrees.
27. A sector has an area of 14 in^2 and central angle $\pi/6$. What is its radius?
28. What is the radius of a sector with an area of 78 cm^2 and a central angle of 120 degrees?
29. What is the radius of a sector if the area is 100 cm^2 and the central angle is 250 degrees?

DETERMINING THE CENTRAL ANGLE OF A SECTOR

30. Find the radian measure of the central angle of a sector with a radius of 5 cm and an area of 20 cm^2.
31. Determine the radian measure of the central angle if a sector has a radius of 3 meters and an area of 15 m^2.
32. What is the degree measure of the central angle of a sector with a radius of 7 inches and an area of 24 in^2?
33. Calculate the degree measure of the central angle for a sector with a radius of 10 feet and an area of 50 ft^2.
34. Calculate the degree measure of the central angle for a sector with a radius of 11 cm and an area of 30.25 cm^2.

DETERMINING ANGULAR VELOCITY

35. **Wheel Rotation.** If a wheel rotates 300 times in one minute, find its angular velocity in radians per second.
36. **Merry-go-Round.** How many complete revolutions does a merry-go-round make in 10 minutes if it has an angular velocity of 0.5 radians per second?
37. **Mixing Machine.** A mixing blade in a blender rotates at 1200 revolutions per minute. Calculate its angular velocity in radians per second.
38. **Computer Hard Drive.** The spindle of a computer hard drive spins at 7200 RPM. What is the angular velocity of the spindle in radians per second?
39. **Electric Generator.** An electric generator's rotor spins at 3600 revolutions per minute. Determine the rotor's angular velocity in radians per second.
40. **Laundry Machine.** During the spin cycle, a washing machine drum rotates at 800 revolutions per minute. Calculate the angular velocity in radians per second.
41. **Vehicle's Alternator.** The alternator in a car operates at 6000 RPM when the engine is at full speed. Find the angular velocity of the alternator in radians per second.

CONVERTING BETWEEN LINEAR AND ANGULAR VELOCITIES

42. **Bike Tire.** A bicycle tire has a radius of 35 cm. If the tire rotates at an angular velocity of 2 radians per second, calculate the linear velocity of a point on the tire's rim.
43. **Circular Machinery.** A part of a machine moves in a circle of radius 15 cm at a constant speed. If the linear velocity of the part is 90 cm/s, calculate its angular velocity in radians per second.
44. **Ceiling Fan Speed.** The blade of a ceiling fan is 0.7 meters long from the center to the tip. If the tip of the blade moves at a linear velocity of 4.5 meters per second, calculate the angular velocity of the blade in radians per second.
45. **Amusement Park Ride.** A merry-go-round has a radius of 5 meters. If a horse on the edge of the ride moves at a linear velocity of 3 meters per second, find the angular velocity of the merry-go-round in radians per second.
46. **Industrial Flywheel.** An industrial flywheel has a radius of 2 meters. If it is required to achieve a linear speed of 10 meters per second at its rim, determine the necessary angular velocity in radians per second.
47. **Athletics Track Runner.** A runner is jogging along a circular track with a radius of 25 meters, maintaining a constant linear speed of 2.5 meters per second. Calculate the runner's angular velocity in radians per second.
48. **Spinning Top.** A spinning top has a radius of 8 cm. If the linear velocity of a point on its edge is 1.6 meters per second during a spin, what is the angular velocity of the top in radians per second?
49. **Windmill Blades.** A windmill's blade has an angular velocity of 0.8 radians per second and each blade is 3 meters long. Calculate the linear velocity at the tip of a blade.
50. **Rotating Sign.** A circular sign rotates at an angular velocity of 2 radians per second. If the radius of the sign is 1.5 meters, what is the linear velocity of the edge of the sign?
51. **Gear in a Machine.** A gear in a factory machine has a radius of 20 cm and rotates with an angular velocity of 5 radians per second. Determine the linear velocity of a point on the edge of the gear.
52. **Carousel Horse.** A carousel horse moves in a circular path with a radius of 4 meters. If the angular velocity of the carousel is 1 radian per second, find the linear velocity of the horse.
53. **Bicycle Pedaling Speed.** A bicycle's pedal arm is 17.5 cm long, and the cyclist is pedaling at an angular velocity of 4 radians per second. Calculate the linear velocity at the end of the pedal arm.

SYNTHESIS QUESTIONS

54. Calculate the radius of a sector whose area is one-fifth of the area of a circle with a radius of 10 cm.
55. Determine the radius of a sector if its area is equivalent to half the area of a circle with a radius of 14 meters.
56. Two circular saw blades are spinning, one at 120 RPM and the other at 180 RPM. If both blades have a diameter of 30 cm, which blade has a higher linear velocity at the rim, and by how much?

APPLICATIONS

SECTOR AREA APPLICATIONS

57. **Land Allocation.** A farmer plans to allocate a portion of his circular field for a special crop. The field has a radius of 100 meters. If he uses a 60-degree sector for this crop, calculate the area that will be cultivated.
58. **Clock Face Design.** A designer is creating a new clock face where each hour marker is represented by a different colored sector. The clock has a radius of 12 cm. Calculate the area of one hour sector assuming a simple 12-hour clock.
59. **Architectural Design.** An architect designs a circular window with decorative sectors. Each sector has a central angle of 45 degrees and the window has a diameter of 2 meters. Calculate the area of one decorative sector.
60. **City Planning.** A city planner needs to design a roundabout garden that occupies a 38-degree sector of a circle with a radius of 20 meters. Calculate the area available for planting.
61. **Pizza Slicing.** A pizza place offers a unique pizza slice shaped as a sector. The pizza has a diameter of 40 cm. If the central angle for each slice is 20 degrees, find the area of one slice of pizza.
62. **Sports Fields.** A sports complex includes a sector-shaped running track portion used for sprinting drills. If the track follows a 40-degree sector of a circle with a radius of 100 meters, calculate the area of this track section.
63. **Public Park Design.** A public park is designed with a circular area for a water fountain, surrounded by flower beds laid out in sectors. If the circular area has a radius of 15 meters and each sector for the flower beds is 30 degrees, find the area of one flower bed.
64. **Non-Uniform Rainfall.** A circular region with a radius of 20 km receives rainfall. However, the rain is not uniformly distributed. If a 120-degree sector receives twice the amount of rain per square kilometer as the rest of the region, calculate the total rainfall if the sector received 600 mm.
65. **Engineering Design.** An engineer is designing a gear in the shape of a sector. The gear must mesh with another gear with an area three times larger but with the same arc length. If the radius of the smaller gear is 15 cm, determine the central angle of the sector for the smaller gear.
66. **Advanced Pizza Slicing.** A circular pizza is sliced such that each consecutive slice has a central angle 10 degrees greater than the previous one, starting from 10 degrees. Calculate the area of the 5th slice if the pizza has a radius of 30 cm.
67. **Sector-based Cooling Zones.** A circular mechanical part is designed with cooling zones in the form of sectors. Each zone dissipates heat differently. If one zone with a 60-degree angle dissipates heat at 100 watts per square meter and the radius of the part is 30 cm, find the total heat dissipation for this zone.
(*Hint:* Heat dissipation is the total number of watts of heat released in a given area.)

VELOCITY APPLICATIONS

68. Find the angular velocity of the following objects.
    a. The London Eye Ferris wheel, which makes one revolution every 20 minutes.
    b. An old-fashioned long-playing record, which revolved $33\frac{1}{3}$ times in 60 seconds.
69. **Conveyor Belt.** A conveyor belt moves at a linear velocity of 1.5 meters per second. If the driving drum has a radius of 0.5 meters, calculate the angular velocity of the drum in radians per second.
70. **Bike Wheel Spin.** A bicycle wheel has a radius of 0.4 meters. If the wheel is spinning with an angular velocity of 5 radians per second, what is the linear velocity of a point on the rim?
71. **Rollerblade Wheel.** A rollerblade wheel has a radius of 40 mm. If a rollerblader is moving at a linear speed of 10 km/h, find the angular velocity of the wheels in radians per second.
72. **Ferris Wheel.** A Ferris wheel has a radius of 25 meters and rotates at a linear velocity of 2 meters per second at its rim. Calculate its angular velocity in radians per second.
73. **Tire Rotation (MyOpenMath).** A truck with 20-inch-diameter tires is traveling at 50 miles per hour.
    a. Find the angular velocity of the tire in radians per minute.
    b. How many revolutions per minute do the tires make?
74. **Tire Rotation.** A car tire has a radius of 35 cm. If the car is traveling at 80 km/h, find the angular velocity of the tire in radians per second.
75. **Potter's Wheel Speed.** If the edge of a potter's wheel moves at a linear velocity of 15 m/s and the wheel has a radius of 1.2 meters, determine the angular velocity in radians per second.
76. **Drill Bit Rotation.** A drill bit has a radius of 1 cm. If it needs to achieve a linear velocity of 20 m/s at its tip, calculate the required angular velocity in radians per second.
77. **Record Player.** If a point on a record player is moving with a linear velocity of 0.5 m/s and the record has a radius of 15 cm, what is the angular velocity in radians per second?
78. **Electric Fan.** An electric fan has blades that are each 0.25 meters long. If a point on the tip of a blade moves with a linear velocity of 3 m/s, what is the angular velocity of the blades in radians per second?
79. **Satellite Dish Rotation.** A satellite dish has a radius of 2 meters. If it rotates such that a point on its edge moves at a linear velocity of 0.1 m/s, calculate the angular velocity in radians per second.
80. **Fan Speed.** A ceiling fan has blades that are each 0.75 meters long. If the fan spins at 200 revolutions per minute (RPM), calculate the angular velocity in radians per second.
81. **Merry-Go-Round.** Children are riding on a merry-go-round that has a radius of 2 meters and makes one complete rotation every 4 seconds. Find the angular velocity in radians per second.

82. **Driving Speed and Wheel Rotation.** A car with wheels of radius 30 cm is traveling at a speed of 60 km/h. Calculate the angular velocity of the wheels in radians per second.
83. **Bicycle Dynamics.** A bicycle's front wheel has a diameter of 70 cm and rotates at 150 RPM while cruising. Determine the linear velocity of the bicycle.
84. **Angular Velocity of a Clock's Second Hand.** Calculate the angular velocity in radians per hour of the second hand of a clock.
85. **Disk Rotation.** A disk in a laboratory experiment rotates at a speed of 600 RPM. Calculate its angular velocity in radians per second.
86. **Jogging on a Track.** A jogger runs around a circular track with a diameter of 200 meters at a speed of 8 km/h. Compute the angular velocity of the jogger in radians per hour.
87. **Wheelchair Wheel Rotations.** A wheelchair has wheels of diameter 60 cm. If a person moves at a speed of 4 km/h, determine the angular velocity of the wheels in radians per second.
88. **Spinner Toy.** A spinner toy rotates 4000 times in one minute. Calculate its angular velocity in radians per second.
89. **Record Player.** A record player spins a record at 33 revolutions per minute. Find the angular velocity in radians per second and the linear velocity at a point 10 cm from the center.
90. **Dryer.** Clothes dryers draw 3.5 times as much power as washing machines, so newer machines have been engineered for greater efficiency. A vigorous spin cycle reduces the time needed for drying, and some front-loading models spin at a rate of 1500 rotations per minute.
    a. If the radius of the drum is 11 inches, how far do your socks travel in one minute?
    b. How fast are your socks traveling during the spin cycle?
91. **Hubble Telescope.** The Hubble telescope is in orbit around the earth at an altitude of 600 kilometers, and completes one orbit in 97 minutes.
    a. How far does the telescope travel in one hour? (The radius of the earth is 6400 kilometers.)
    b. What is the speed of the Hubble telescope?
92. **Energy-Producing Windmill.** The first large windmill used to generate electricity was built in Cleveland, Ohio in 1888. Its sails were 17 meters in diameter, and moved at 10 rotations per minute. How fast did the ends of the sails travel?
93. **Largest Windmill.** The largest windmill operating today has wings 54 meters in length. To be most efficient, the tips of the wings must travel at 50 meters per second. How fast must the wings rotate?
94. **Planetary Motion.** Assume a simplified model where a planet moves in a circular orbit around its star with a constant angular velocity. If the radius of the orbit is 1.5×10^6 kilometers and it completes one orbit in 215 days, find the linear velocity of the planet.
95. **Wheel Rotation.** A car's tire has a diameter of 60 cm. If the car is traveling at a speed of 72 km/h, calculate the angular velocity of the tire in radians per second.
96. **CD Player Speed.** A CD rotates at 500 revolutions per minute (RPM) when playing. What is the linear velocity at a point 3 cm from the center of the CD?
97. **Angular Speed of a Wind Turbine.** A wind turbine blade is 45 meters long, and it completes one rotation every three seconds. Calculate the angular velocity in radians per second and the linear velocity at the tip of the blade.
98. **Bicycle Wheel Dynamics.** A bicycle wheel has a radius of 35 cm. If the bicycle is moving at a speed of 15 km/h, find the wheel's angular velocity in radians per second.
99. **Roller Coaster Speed.** A roller coaster moves in a vertical loop of radius 12 meters at a speed of 36 km/h at the highest point of the loop. Calculate the angular velocity at this point.
100. **Flywheel Operation.** A flywheel in a machine spins at 1800 RPM. If the radius of the flywheel is 20 cm, calculate the linear velocity at the edge of the flywheel.
101. **Satellite Orbiting.** A satellite orbits the Earth in a circular path with a radius of 10,000 km at a constant speed of 7 km/s. Determine the angular velocity of the satellite in radians per hour.
102. **Speed of a Ferris Wheel.** A Ferris wheel with a diameter of 100 meters takes 4 minutes to make one complete revolution. Calculate the linear velocity and angular velocity of a cabin at the edge of the Ferris wheel.
103. **Potter's Wheel.** A potter's wheel moves from rest to an angular speed of 120 RPM in 5 seconds. Calculate the angular acceleration and the linear velocity at a point 15 cm from the center at the end of 5 seconds.
104. **Circular Saw.** A circular saw blade has a radius of 15 cm and rotates at a speed of 3000 RPM. Find the angular velocity in radians per second and the linear velocity at the perimeter of the blade.

CHALLENGE PROBLEMS

104. If the radius of a sector is doubled while the central angle remains the same, by what factor does the area increase?

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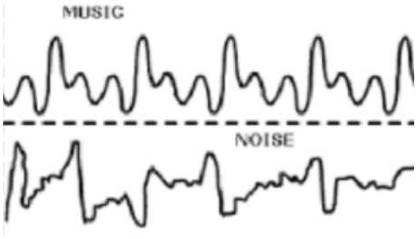
CHAPTER OVERVIEW

Chapter 8: Graphs of the Trigonometric Functions

Musical instruments produce sound by the vibration of a string on a violin, guitar, or piano, or a column of air in a brass or woodwind instrument. The vibration causes a periodic variation in air pressure that is heard as sound. Periodic vibrations create a pleasing, musical sound, while random vibrations sound like noise.

Figure 1

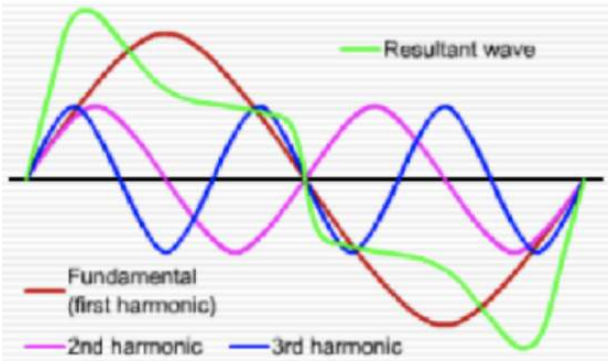


There are three distinguishing characteristics of musical notes: loudness, pitch, and timbre.

- The **loudness** of a note is measured by the magnitude, or amplitude, of the changes in air pressure.
- The **pitch** of the note is determined by its frequency, the number of times its basic pattern is repeated each second. For example, a note with pitch 440 hertz repeats its pattern 440 times per second. Humans can hear frequencies that range roughly from 20 to 18,000 hertz.
- The **timbre** of the note allows us to tell a piano note from a guitar note with the same loudness and pitch.

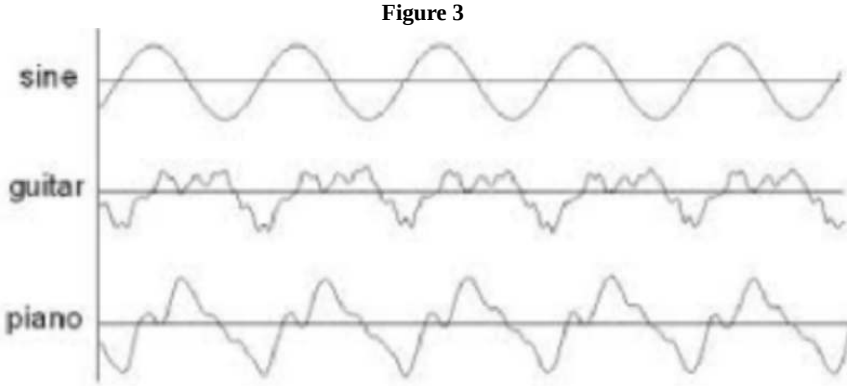
The simplest pressure vibration is produced by a tuning fork. The graph of its pressure function resembles a sine curve, and the corresponding sound is called a **pure tone**. When a musical instrument produces a pure tone, not only is its frequency produced, but integer multiples of the **fundamental frequency**, called **harmonics**, are produced as well. Sine curves of different frequencies combine to create the timbre of the note.

Figure 2



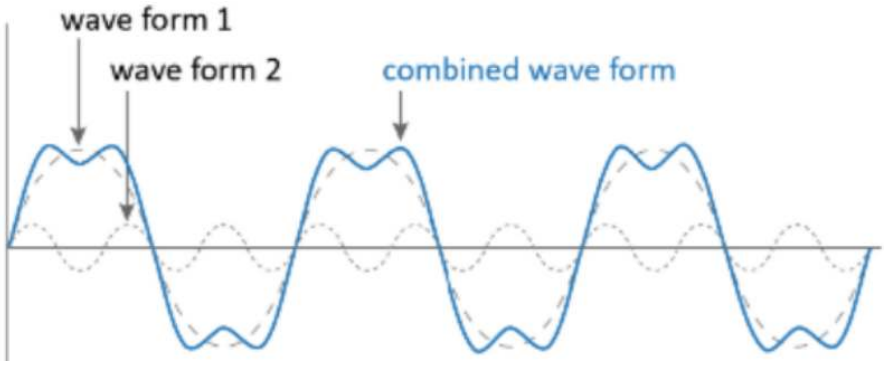
If two musical instruments play the same note, the notes have the same pitch, but they sound different because the amplitudes of each of the harmonics is different for the two instruments.

Figure 3



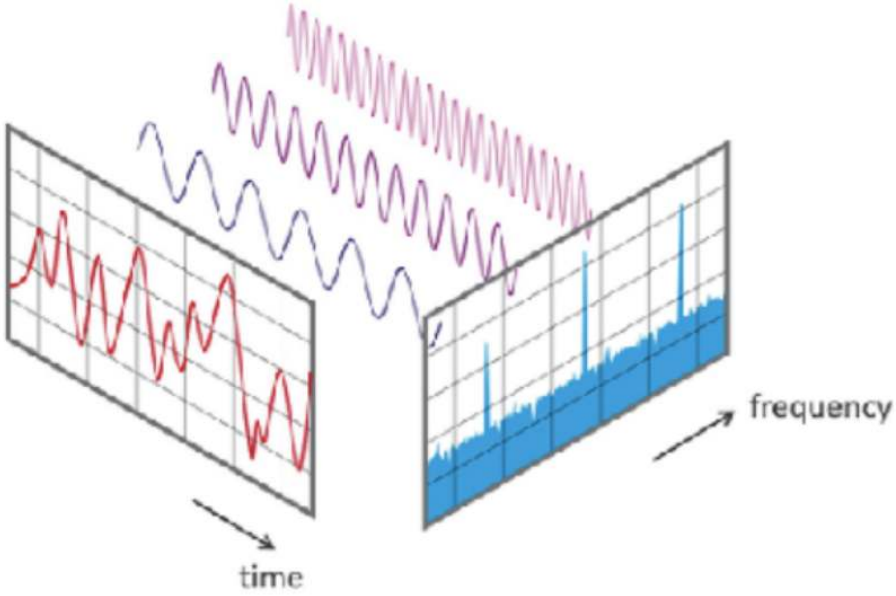
The French mathematician Joseph Fourier, who lived from 1768 to 1830, discovered that any periodic wave can be written as the sum of a number of sines and cosines with differing amplitudes.

Figure 4



Fourier created a method for determining the frequencies and amplitudes of the simpler waves that make up a more complicated periodic function. Applying the Fourier transform to a sampled musical note reveals which component frequencies are present in the note. The same sound can then be re-synthesized by including those frequency components.

Figure 5



Today, Fourier analysis is the foundation of signal processing, not only for audio waves, but for radio waves, light waves, seismic waves, and even images. By analyzing a compound waveform, the ”noise” components can be isolated and removed in order to smooth out the signal.

Fourier analysis is also used in X-ray crystallography to reconstruct a crystal structure from its diffraction pattern, and in nuclear magnetic resonance spectroscopy to determine the mass of ions from the frequency of their motion in a magnetic field.

8.1: Base Graphs of the Fundamental Trigonometric Functions

8.1.1: Resources and Key Concepts

8.1.2: Homework

8.2: Non-Rigid Transformations

8.2.1: Resources and Key Concepts

8.2.2: Homework

8.3: Rigid Transformations

8.3.1: Resources and Key Concepts

8.3.2: Homework

8.4: Sinusoidal Models

8.4.1: Resources and Key Concepts

8.4.2: Homework

8.5: Harmonic Motion

8.5.1: Resources and Key Concepts

8.5.2: Homework

8.6: Graphs of the Remaining Trigonometric Functions

8.6.1: Resources and Key Concepts

8.6.2: Homework

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8.1: Base Graphs of the Fundamental Trigonometric Functions

Note to the Instructor (click to expand)

I chose to only introduce the graphs of the sine, cosine, and tangent because these are the foundational graphs for the remainder of trigonometry. The graphs of the remaining trigonometric functions are held off until the very end of the chapter. Additionally, this section introduces the symmetry of these functions. Therefore, we can spend some time doing proofs involving symmetry and these functions.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Sets and Numbers**
  - Intervals and Interval Notation: This notation is used to state the domain and range of the trigonometric functions.
  - Sets and Set Notation: Set-builder notation is used to describe the domain of the tangent function.
- **Graphing Functions**
  - Symmetry: The concepts of even and odd functions, which correspond to y-axis and origin symmetry, are introduced and used to define the Symmetry Identities.
- **Rational Functions**
  - Asymptotes: The concept of a vertical asymptote is essential for understanding and graphing the tangent function.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Sketch the base graphs of the fundamental trigonometric functions.
- Determine whether the graph of an equation involving trigonometric functions is even, odd, or neither.
- Evaluate a trigonometric function using the Symmetry Identities.

Hawk A.I. Section-Specific Tutor

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

When you were learning Algebra, you were introduced to variables, expressions involving variables, equations involving variables, and graphs of those equations (in that order). In Trigonometry, this order is slightly modified. We have learned about trigonometric functions and manipulated expressions involving those functions; however, before solving equations involving those functions, it's best to be able to visualize their graphs.

This section focuses on the base graphs of the **fundamental trigonometric functions**.

Definition: Fundamental Trigonometric Functions

The sine, cosine, and tangent are called the **fundamental trigonometric functions**.

The graphs of the remaining trigonometric functions will be introduced after we have a few sections to get used to the graphs of the fundamental trigonometric functions.

Motivating Periodic Functions of Angles

Imagine that you are riding on a Ferris wheel. As the wheel turns, your height above the ground increases and decreases. This pattern repeats each time the Ferris wheel makes a complete rotation. This type of motion is called **periodic motion**.

Definition: Periodic Motion

When a body repeats its movement along a path in a regular and predictable manner over a specific time interval, its movement is called **periodic motion**.

In Mathematics, we seek to model periodic motion with a function.

Definition: Periodic Function

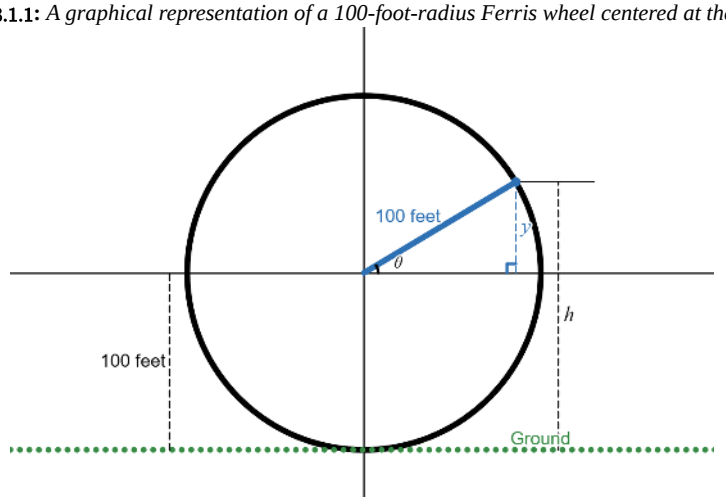
A mathematical function modeling periodic motion is called a **periodic function**. A periodic function repeats its values at regular intervals, meaning its graph has a pattern that repeats itself over a specific horizontal distance or time interval.

We use periodic functions to model phenomena that exhibit cyclical behavior, such as the height of tides, seasonal growth patterns in plants and animals, radio waves, and planetary motion.

Let's create a mathematical model for a ride on a Ferris wheel. Suppose the Ferris wheel has a radius of 100 feet and rotates counter-clockwise. Our model will be a function that describes the rider's height above the ground as they ride the wheel. To graph the Ferris wheel function, we must first specify the input and output variables and then choose a coordinate system to display their values.

While developing the model, let's place the origin at the center of the Ferris wheel (see Figure 8.1.1 below).

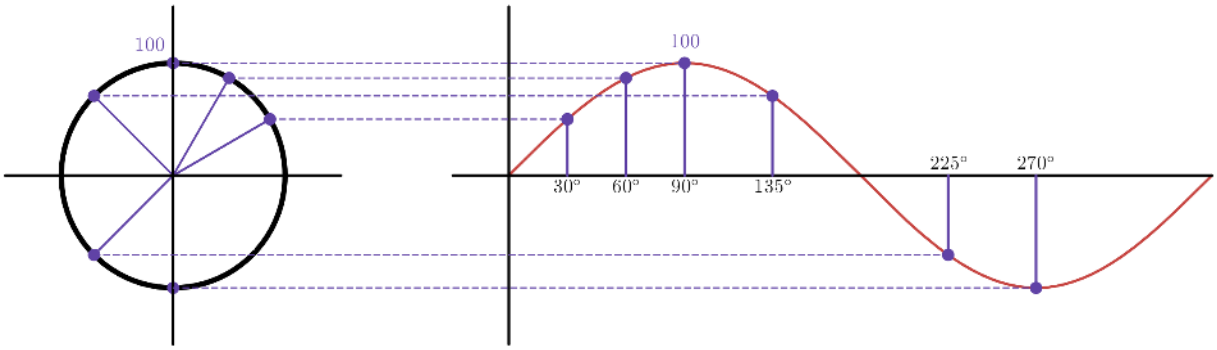
Figure 8.1.1: *A graphical representation of a 100-foot-radius Ferris wheel centered at the origin.*



While this is not the most realistic placement for us to imagine (it makes us think that the bottom of the Ferris wheel is below ground level), this temporary placement will reveal an important characteristic of the rider's motion. The line from the origin to the rider's position on the wheel makes an angle, θ, with the horizontal. This angle will be the independent variable for the function. The height of the rider, h, is also a variable and is related to the y-coordinate of the rider's position. Figure 8.1.1 shows that $h = y + 100$ because the center of the wheel is 100 feet above the ground.

Let's focus on the rider's height as a function of angle. Therefore, we'll first graph y as the dependent variable instead of h. As the angle θ increases from $0°$ to $90°$, the rider's y-coordinate increases from 0 to 100. At that point, they are at the top of the wheel. Then, as θ increases from $90°$ to $180°$, the rider's y-coordinate decreases from 100 back to 0 (see Figure 8.1.2).

Figure 8.1.2: *"Unwinding" the circle to get a visual model of the height function.*



Finally, as θ increases from $180°$ to $360°$, the rider's y-coordinate decreases from 0 to -100 and then increases from -100 back to 0. At that point, the rider has made one complete rotation on the Ferris wheel. If she goes around again, θ increases from $360°$ to $720°$, and the graph of her y-coordinate will repeat the pattern of the first rotation. Again, Figure 8.1.2 shows how the rider's y-coordinate is plotted as a function of the angle θ.

Look back at Figure 8.1.1 and notice that $\sin(\theta) = \frac{y}{100}$. Therefore,

$$y = 100\sin(\theta).$$

For example, when $\theta = 30°$, the y-coordinate of the rider is

$$y = 100\sin(30°) = 100\left(\frac{1}{2}\right) = 50$$

and the rider's height above the ground is $h = y + 100 = 150$ feet. In general, then, h is given as a function of θ by

$$h = y + 100 = 100\sin(\theta) + 100.$$

This is our model for a rider's height on a Ferris wheel as a function of angle.

The graph on the right side of Figure 8.1.2 is called a **sine curve**.

Definition: Sine Curve

A smooth, wave-like curve that oscillates (moves up and down) periodically and which can be derived from either the sine or cosine function is called a **sine curve**. A curve of this nature is often called **sinusoidal**,

While it might seem odd that we use the blanket phrase "sine curve" when describing a graph derived from the sine *or* the cosine, this consolidation will make sense shortly.

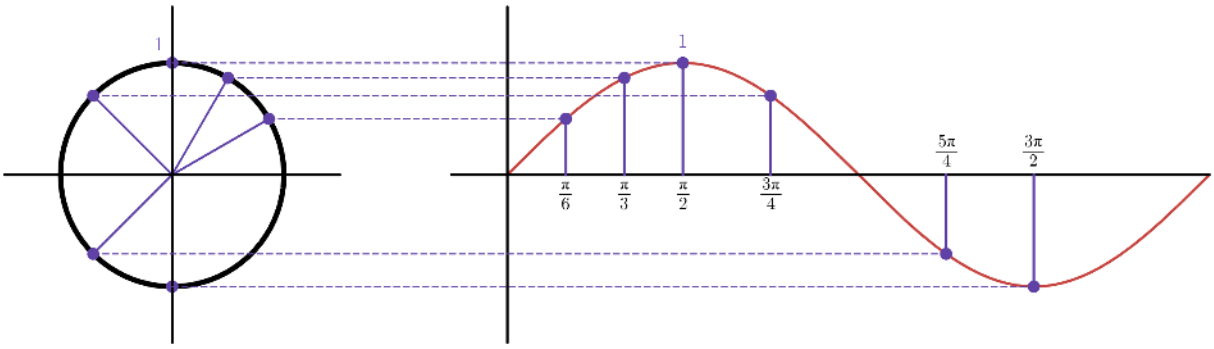
An Aside: Degrees or Radians?

Before we go any further, we need to make one thing clear:

> *Figure 8.1.2 is the first and the last time you will see a graph of the sine (or any other trigonometric) function in terms of degrees.*

While there are several reasons that we endeavor only to graph trigonometric functions using radians, the two most impactful are the relation to the unit circle and scale. When we derived the unit circle definition of the trigonometric functions, we did so using radian measure. If we redraw Figure 8.1.2 using radians as the angular measure (and the unit circle instead of a circle of radius 100), we get Figure 8.1.3.

Figure 8.1.3: *Drawing the sine curve using the unit circle.*



Note that the scale on the vertical axis (namely, that the largest and smallest y-values are 1 and -1, respectively) and the scale on the horizontal axis (from $x = 0$ to $x = 2\pi$) are more closely aligned than had we kept the angular measure in degrees.

Graphing the Sine Function

There are two main ways students initially learn to graph the sine function - by plotting points or by tracing the y-values along the unit circle. The latter is what we did in Figure 8.1.3. The former is not the ideal method, but it's a good idea to showcase it right now so we can link the computed values of the sine function to points on a graph.

Point-Plotting Approach

Up to this point, we have mostly let sine be a function of the angle θ; however, we recently discovered (via our discussion of circular functions) that the trigonometric functions are also functions of real numbers. Therefore, we should feel fine working with $f(x) = \sin(x)$. The independent variable, x, could represent an angle, the arc length along a circle, or time - our application determines the context.

Example 8.1.1

Sketch a portion of the graph of $y = \sin(x)$ on the interval $[0, 2\pi]$ by making a table of values.

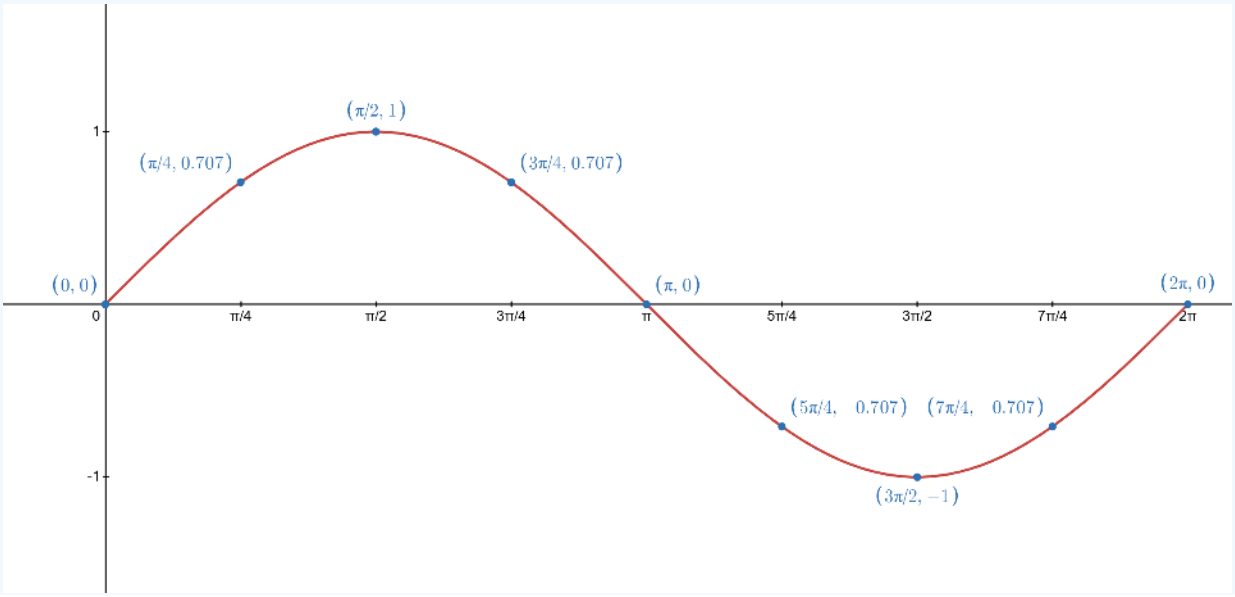
Solution

To graph $f(x) = \sin(x)$ without relating to the unit circle, we need to list out several values of the sine function.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin(x)$ (exact)	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0
$\sin(x)$ (approximate)	0	0.707	1	0.707	0	-0.707	-1	-0.707	0

Plotting these values and connecting them with a smooth curve, we get the following:

Figure 8.1.4

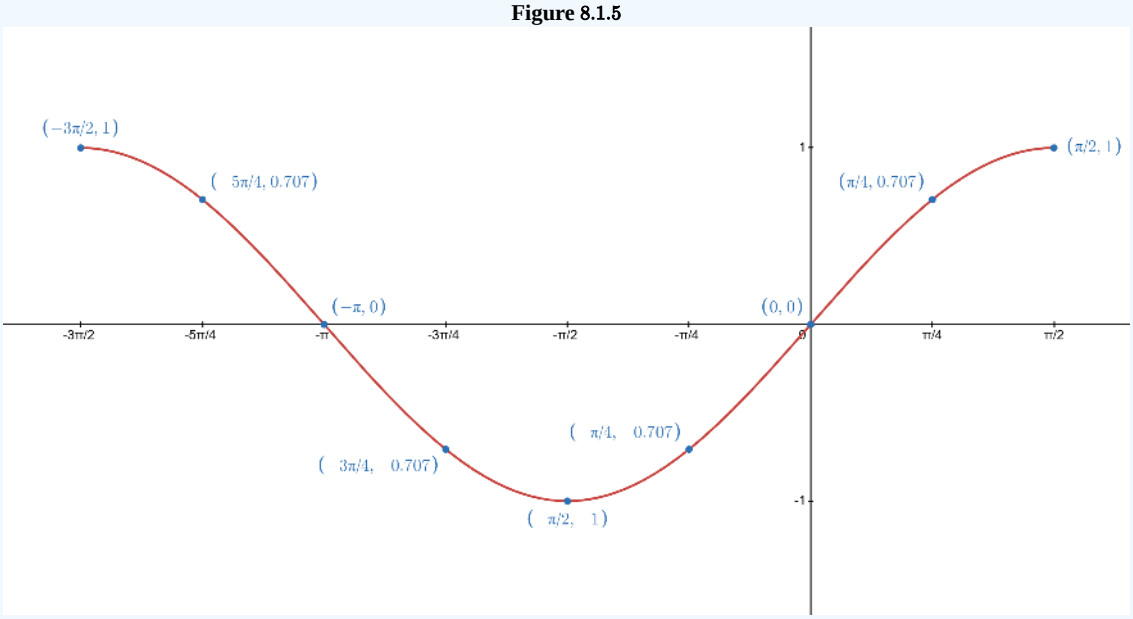


Checkpoint 8.1.1

Sketch of portion of the graph of $y = \sin(x)$ on the interval $\left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$ by making a table of values.

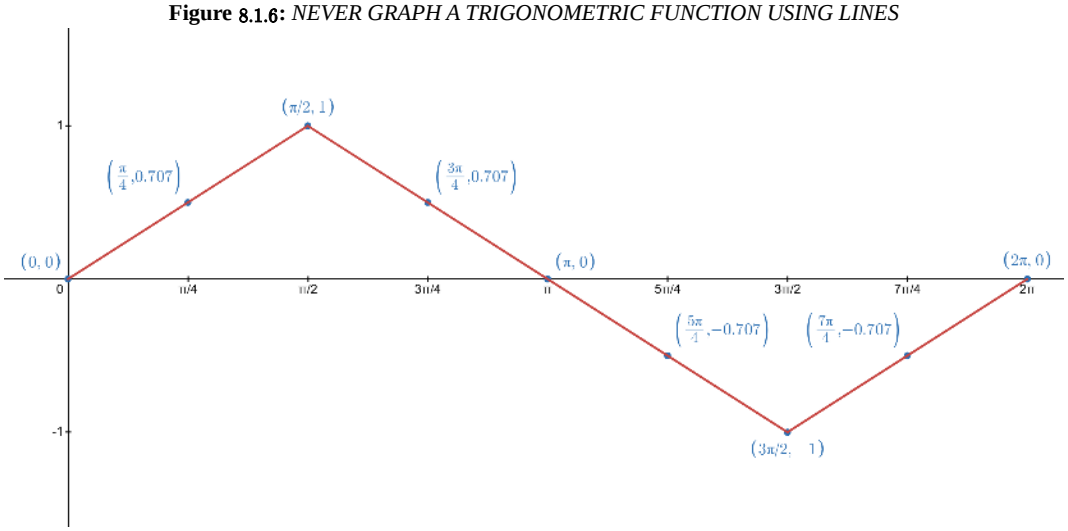
Answer

Figure 8.1.5



There are several issues with plotting points to obtain the sine function graph. The first is the amount of time spent computing values. We will be graphing the sine curve throughout the rest of our mathematical journey, and if we need to plot points every time we graph, we will be wasting time and effort. The second issue has to do with "connecting the dots." Those who have no idea of the curvature of the sine graph end up connecting the points with lines and arrive at something like Figure 8.1.6.

Figure 8.1.6: *NEVER GRAPH A TRIGONOMETRIC FUNCTION USING LINES*



I hope you can see the difference between the correct graph of the sine function from Example 8.1.1 and the incorrect graph represented in Figure 8.1.6.

The third issue with graphing using point-plotting involves building a natural understanding of the sine curve. Many features, like domain and range, slip out of the student's grasp, who consistently requires the "connect the dots" method when graphing trigonometric functions.

Finally, the fourth issue occurs when needing to extend the graph of the sine function beyond the interval $[0, 2\pi]$. As Exercise 8.1.1 shows, and as we shall see in more concrete form shortly, the sine curve oscillates in both directions (as $x \to \infty$ and as $x \to -\infty$). If we needed to plot points to extend the graph, we would be plotting and computing for a long time.

With all these considerations in mind, we will avoid using point-plotting when graphing trigonometric functions (other than introducing the base graphs of the trigonometric functions).

Unit Circle Approach

We have already motivated the sine curve using the unit circle in Figure 8.1.3; however, let's spend a few more moments solidifying our understanding of the relationship between the unit circle and the sine curve.

Recall, $\sin(t)$ is the value of the y-coordinate for the point P on the unit circle, where t is simultaneously the arc length from $(1, 0)$ to the point P and the angle subtended by this arc. The following Interactive Element shows how the sine curve is created by tracing these y-coordinates as the angle (or arc length) t increases from 0 to 2π (if you only have access to the PDF of this textbook, then please go to the following link: https://www.desmos.com/calculator/aq5r4ka4dd)

We now can easily "extend" the graph of the sine function by allowing t to go beyond 2π (https://www.desmos.com/calculator/imbgdopjtz).

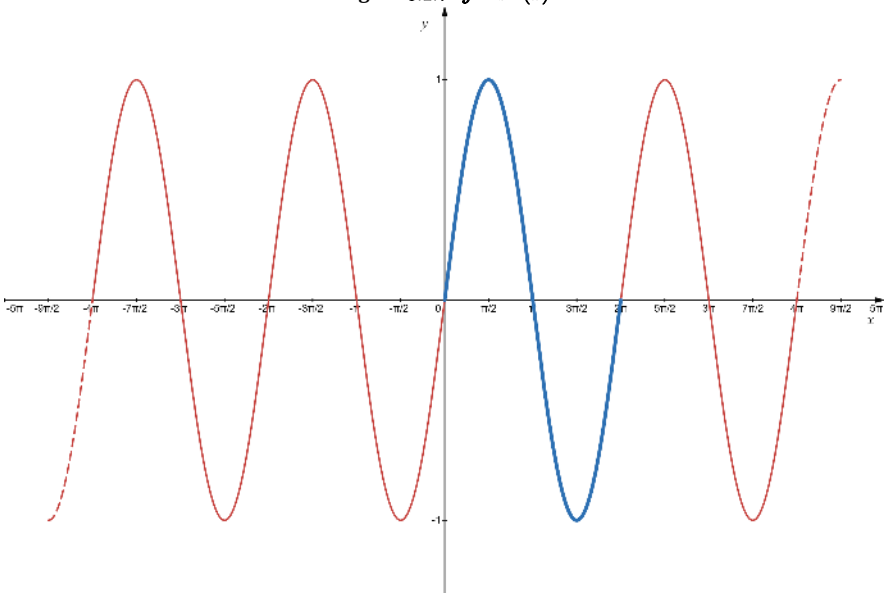
Basic Features of the Sine Curve

Now that we know how to graph the base sine curve, we can dive into the details of this function.

Domain and Range

We now have enough knowledge to graph the *entire* sine curve. Figure 8.1.7 shows the general sine curve along with a single cycle (in blue) on the interval $[0, 2\pi]$.

Figure 8.1.7: $y = \sin(x)$



Since the sine curve oscillates forever in both directions, it should be evident that the sine function has an output for every input. Therefore, the domain of the sine function is all real numbers. Moreover, we can see from the graph that the values of the sine function oscillate between -1 and 1 (inclusive). Hence, the range must be $[-1, 1]$. We summarize these results in the following theorem.

Theorem: Domain and Range of the Sine Function

The domain of $f(x) = \sin(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Caution: Is it $\sin(x)$ or $\sin(t)$?

It is common for those learning Trigonometry to need clarification about the meaning of the argument of the sine function at this point. When we speak of the unit circle, we are speaking of the graph of all solutions, (x, y), to the equation

$$x^2 + y^2 = 1.$$

As was mentioned in a previous section, the x-coordinate of a point on the unit circle is defined to be $\cos(t)$ and the y-coordinate is defined to be $\sin(t)$, where t is simultaneously the arc length along the unit circle to the point in question and the angle subtended by that arc length.

Unfortunately, we will now be graphing $f(x) = \sin(x)$. This value of x differs from the related x-coordinate on the unit circle! You can (and should) think of the x in $f(x) = \sin(x)$ as a "dummy variable." If you prefer to replace it with θ or t, please feel free to do so; however, we will continue to write sine as a function of x to get you used to this often confusing convention.

Period

Another observation is that the sine function repeats its pattern every 2π units along the x-axis.

Definition: Period

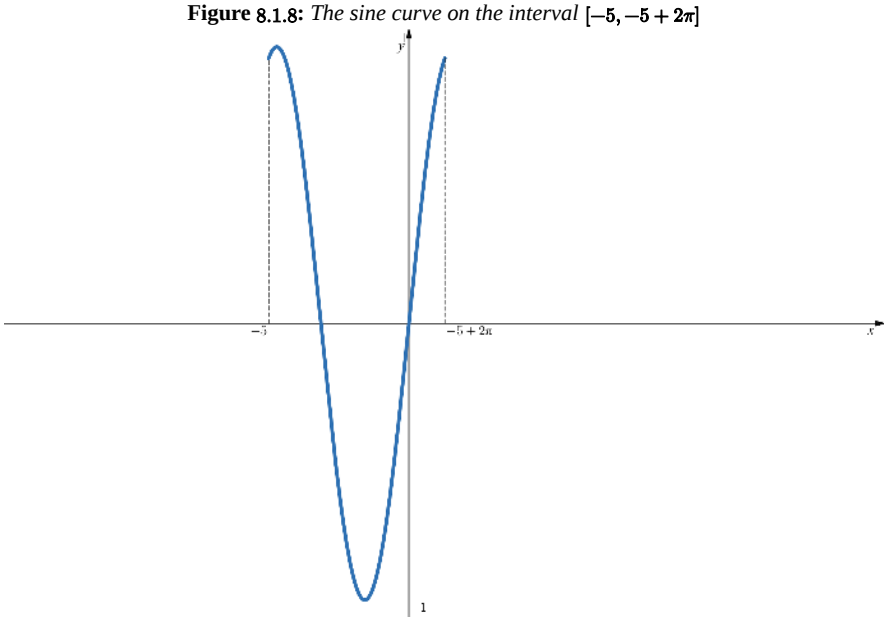
For any function $f(x)$, the smallest positive number k for which

$$f(x + k) = f(x)$$

for all x in the domain of f is called the **period** of $f(x)$.

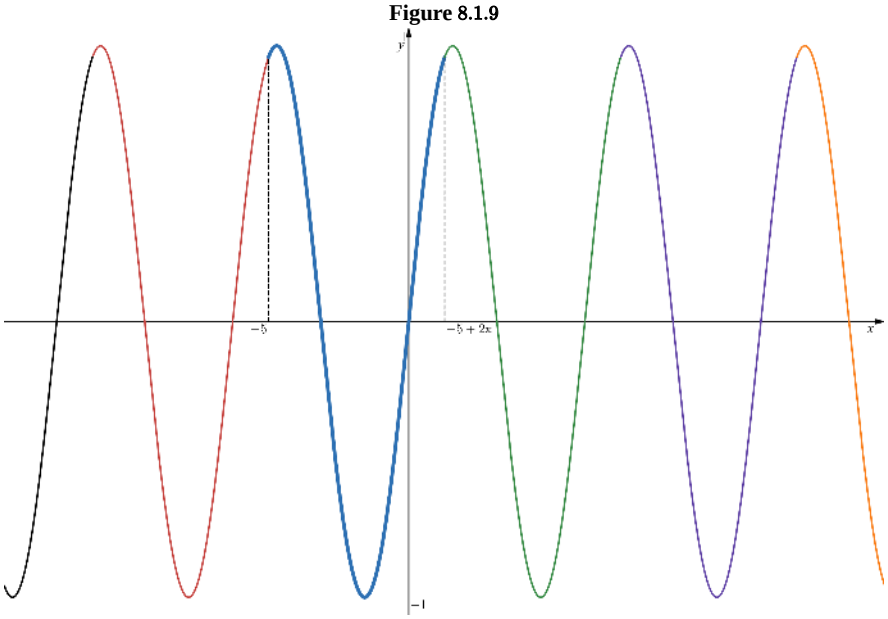
If we take any interval of length 2π along the x-axis, we will see that the corresponding section of the sine curve over that interval can be "copied-and-pasted" an infinite number of times to create the entire sine curve. For example, while not convenient, we could consider the interval $[-5, -5 + 2\pi]$ along the x-axis, as shown in Figure 8.1.8 below.

Figure 8.1.8: *The sine curve on the interval* $[-5, -5 + 2\pi]$



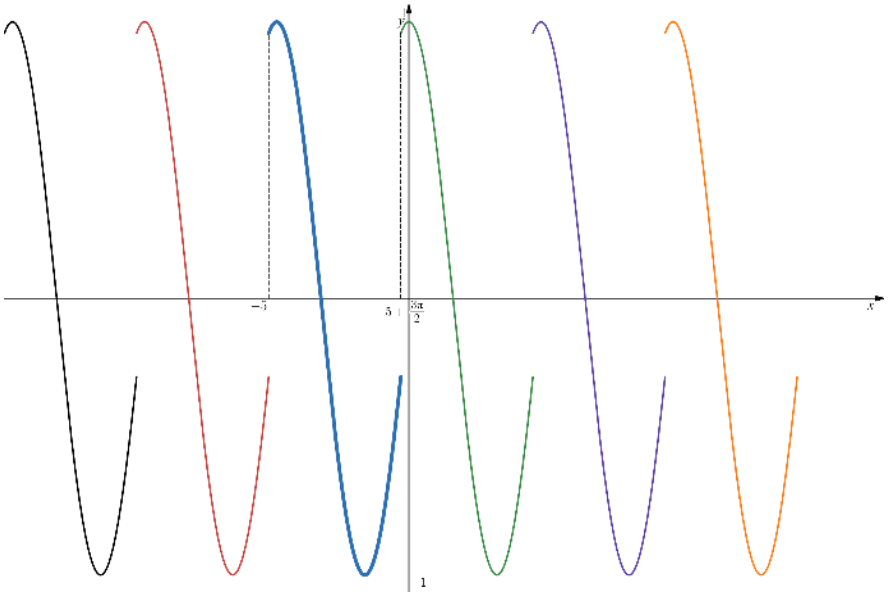
"Copying-and-pasting" that segment an infinite number of times in both directions, we rebuild the entire sine curve.

Figure 8.1.9



Moreover, if we had selected a smaller interval along the x-axis, say $[-5, -5 + \frac{3\pi}{2}]$, we would not have captured enough of the behavior of the sine curve to appropriately recreate the entire curve by "copying-and-pasting" the smaller section.

Figure 8.1.10



Hence, by the definition of the period of a function, the following theorem should make sense.

Theorem: Period of the Sine Function

The period of $f(x) = \sin(x)$ is 2π. This is also known as the **natural period** of the sine function.

While the graph of the sine function (and all of the other trigonometric functions) extends in both directions forever (see Figure 8.1.7), the section of the graph starting at $x = 0$ and ending at $x = 2\pi$ is going to be very important as we move forward.

Definition: Principal Cycle (of a trigonometric function)

The **principal cycle** of a trigonometric function is the interval starting at $x = 0$ and terminating at $x = p$, where p is the period of the trigonometric function.

Thus, by definition, we get the following theorem.

Theorem: Principal Cycle of the Sine Function

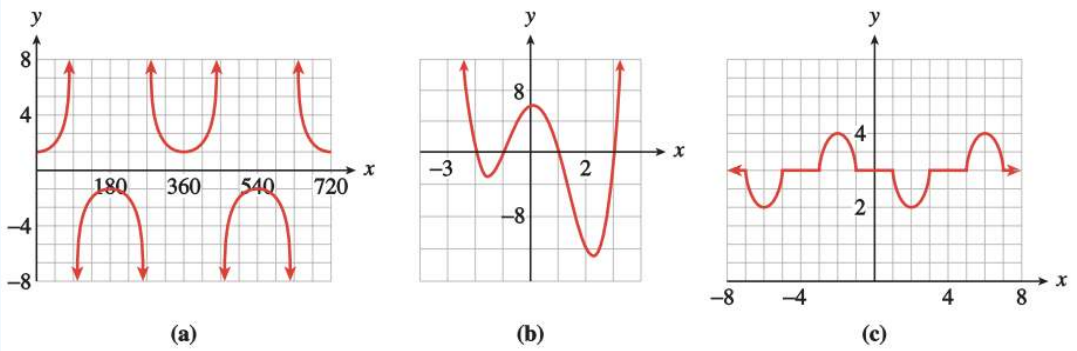
The principal cycle of the sine function is $[0, 2\pi]$.

It is important to note that there are other periodic functions besides the sine function.

Example 8.1.2

Which of the functions shown below are periodic? If the function is periodic, give its period.

Figure 8.1.11



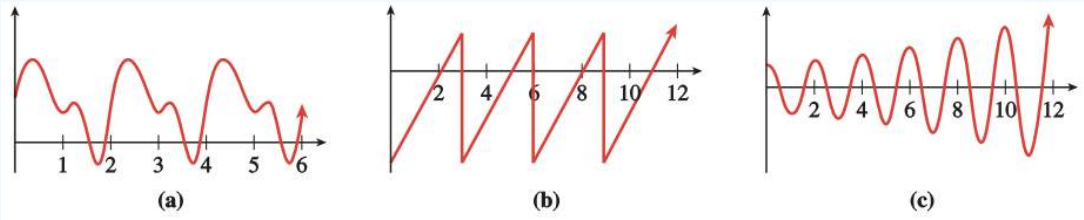
Solutions

a. This graph is periodic with period 360.
b. This graph is not periodic.
c. This graph is periodic with period 8.

? Checkpoint 8.1.2

Which of the functions shown below are periodic? If the function is periodic, give its period.

Figure 8.1.12



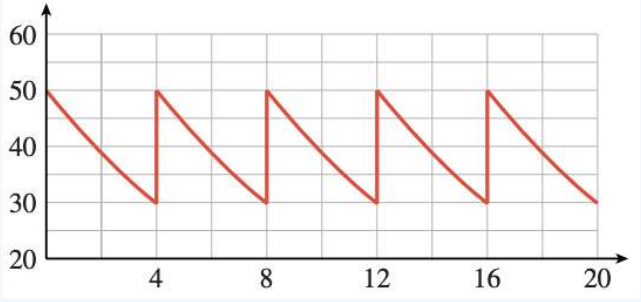
Answers

a. Period: 2
b. Period: 3
c. Not periodic

✓ Example 8.1.3

A patient receives regular doses of medication to maintain a certain level of the drug in his body. After each dose, the patient's body eliminates a certain percentage of the medication before the next dose. The graph shows the amount of the drug, in milliliters, in the patient's body as a function of time in hours.

Figure 8.1.13



a. How much of the medication is administered with each dose?
b. How often is the medication administered?
c. What percent of the drug is eliminated from the body between doses?

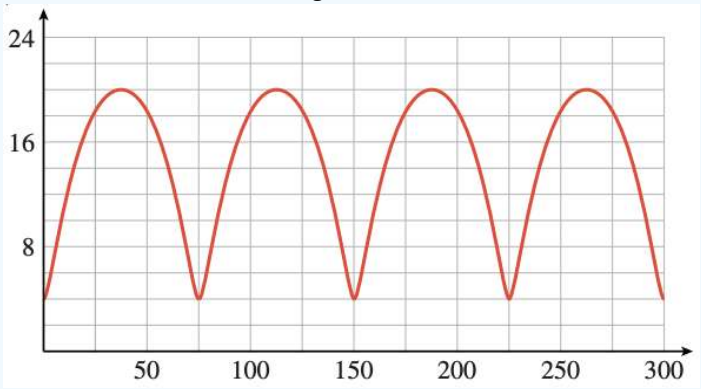
Solutions

a. The medication level increases from 30 ml to 50 ml at each cycle of the graph, so 20 ml of medication is administered at each dose.
b. The medication level peaks sharply every four hours after administering each new dose.
c. The medication level declines by 20 ml between doses, or $\frac{20}{50} = 0.6$, or 60.

Checkpoint 8.1.3

You are sitting on your front porch late one evening, and you see a light coming down the road tracing out the path shown below, with distances in inches. You realize you are seeing a bicycle light fixed on a bike's front wheel.

Figure 8.1.14



a. Approximately what is the period of the graph?
b. How far above the ground is the light?
c. What is the diameter of the bicycle wheel?

Answers

a. 75 in
b. 4 in
c. 24 in

Amplitude

The sine curve (and, as we shall soon see, the cosine curve) has a feature almost exclusively unique among the graphs of all other functions - **amplitude**.

Definition: Amplitude (of a function)

Let M and m be a function's greatest and least values, respectively. The **amplitude** of the graph of the function is defined to be

$$\text{Amplitude} = \frac{1}{2}|M - m|.$$

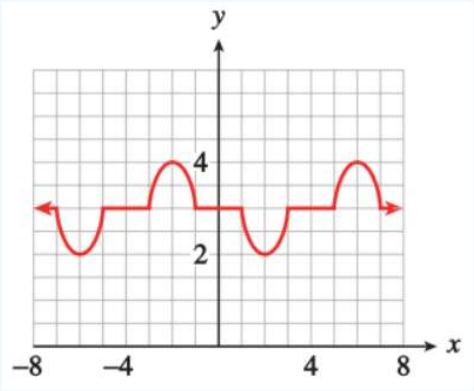
That is, the amplitude is half the distance between the highest and lowest values of the function.

Example 8.1.4

Determine the amplitude of each function.

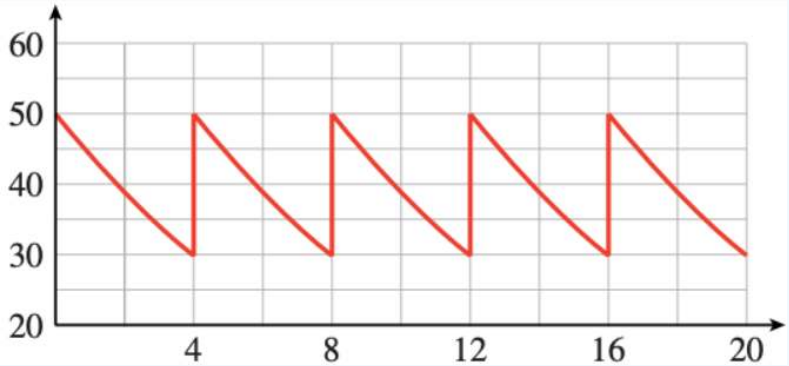
a.

Figure 8.1.15



b.

Figure 8.1.16



Solutions

a. The maximum value for the graph of this function is $M = 4$ and the minimum is $m = 2$. Therefore, the amplitude of the graph is

$$\text{Amplitude} = \frac{1}{2}|4 - 2| = 1.$$

b. The maximum value is $M = 50$ and the minimum is $m = 30$. Thus,

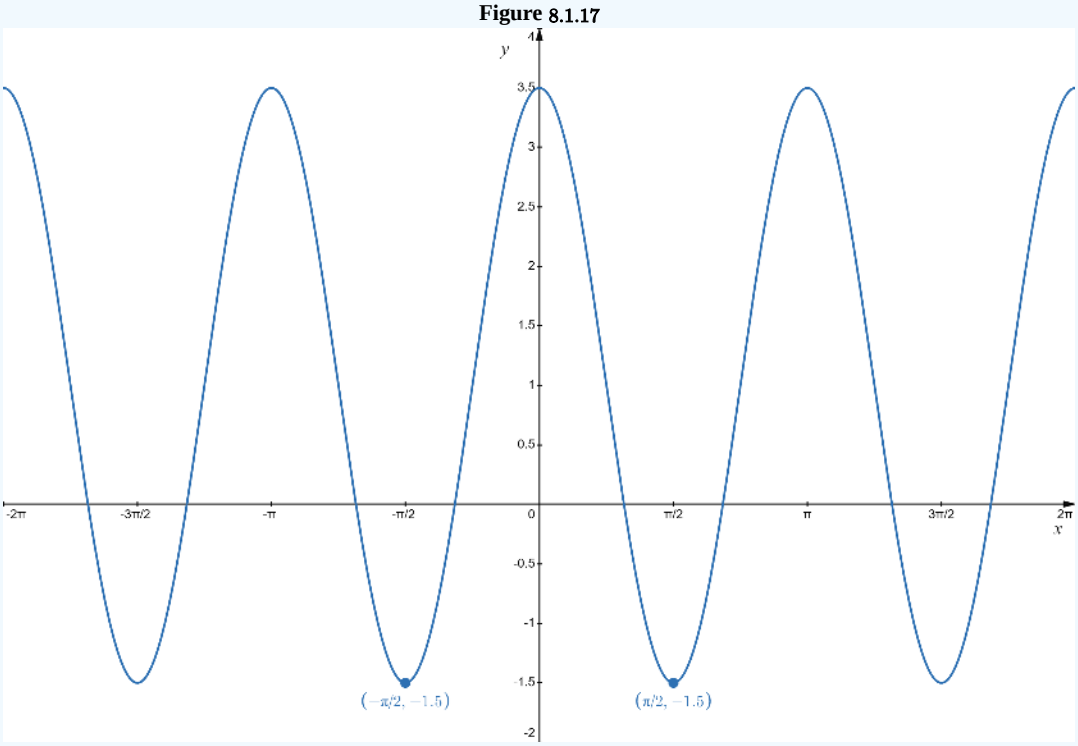
$$\text{Amplitude} = \frac{1}{2}|50 - 30| = 10.$$

? Checkpoint 8.1.4

Determine the amplitude and period of the function.

https://math.libretexts.org/@go/page/197561

Figure 8.1.17



Answers

Period: π
Amplitude: 2.5

The concept of amplitude is generally reserved for the sine and cosine functions; however, it occasionally gets referenced when working with other periodic functions.

Theorem: Amplitude of the Sine Function

The amplitude of $f(x) = \sin(x)$ is 1.

Proof

The maximum value of $f(x) = \sin(x)$ is $M = 1$ and the minimum value is $m = -1$. Hence,

$$\text{Amplitude} = \frac{1}{2}|1 - (-1)| = 1.$$

Zeros

We know that the graph of $f(x) = \sin(x)$ crosses the x-axis when $\sin(x) = 0$; however, using the intuitive visualization of the sine function being the y-coordinates of points along the edge of the unit circle, we can see (from Figure 8.1.3, Example 8.1.1, or either of the Interactive Elements above) that this occurs when $x = 0, \pi, 2\pi, -\pi, -2\pi, \ldots$. The following theorem provides a much better way to write this.

Theorem: Zeros of the Sine Function

The zeros of $f(x) = \sin(x)$ occur at $x = \pi k$, where $k \in \mathbb{Z}$.

As a reminder, the notation, $\in$, means "belongs to" and the notation, $\mathbb{Z}$, means "the set of all integers." Therefore, as long as $k \in \mathbb{Z}$, $x = \pi k$ is shorthand for

$$x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \cdots.$$

We will be using this notation extensively from this point forward in Mathematics.

Graphing the Cosine Function

Now that we have set the foundation for all the concepts we need, our discussion of the cosine function graph will be shorter than that of the sine function. We begin using the worst method for graphing a function - point-plotting.

Point-Plotting Approach

Example 8.1.5

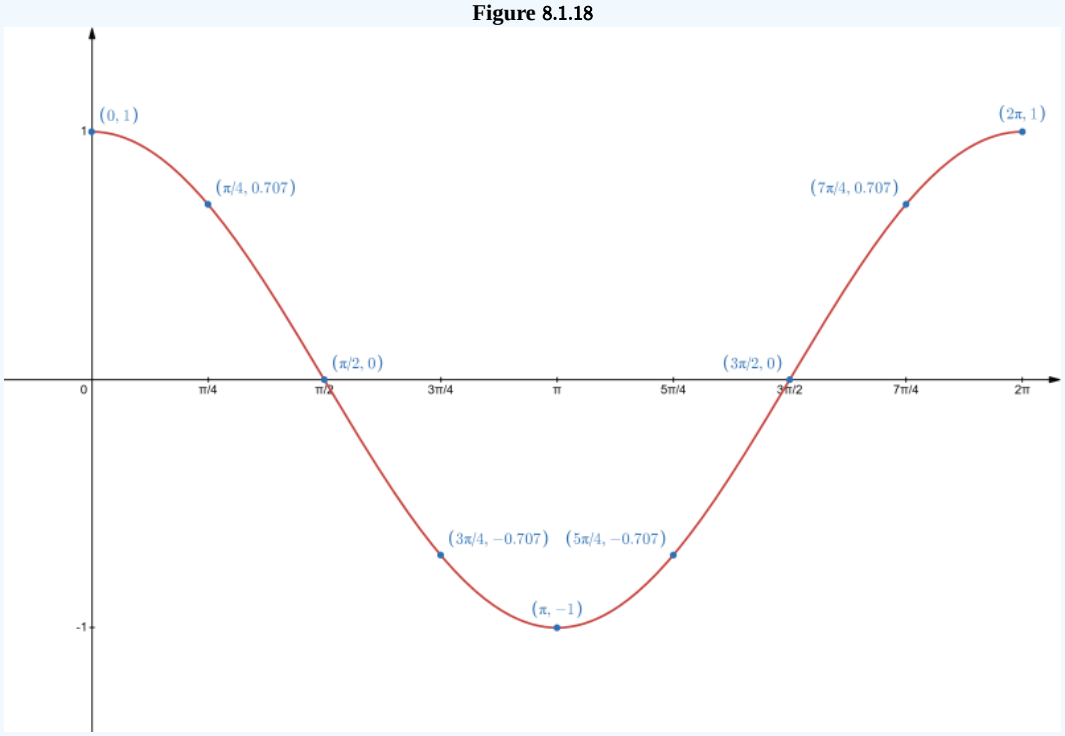
Sketch a portion of the graph of $y = \cos(x)$ on the interval $[0, 2\pi]$ by making a table of values.

Solution

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos(x)$ (exact)	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
$\cos(x)$ (approximate)	1	0.707	0	-0.707	-1	-0.707	0	0.707	1

Plotting these values and connecting them with a smooth curve, we get the following:

Figure 8.1.18



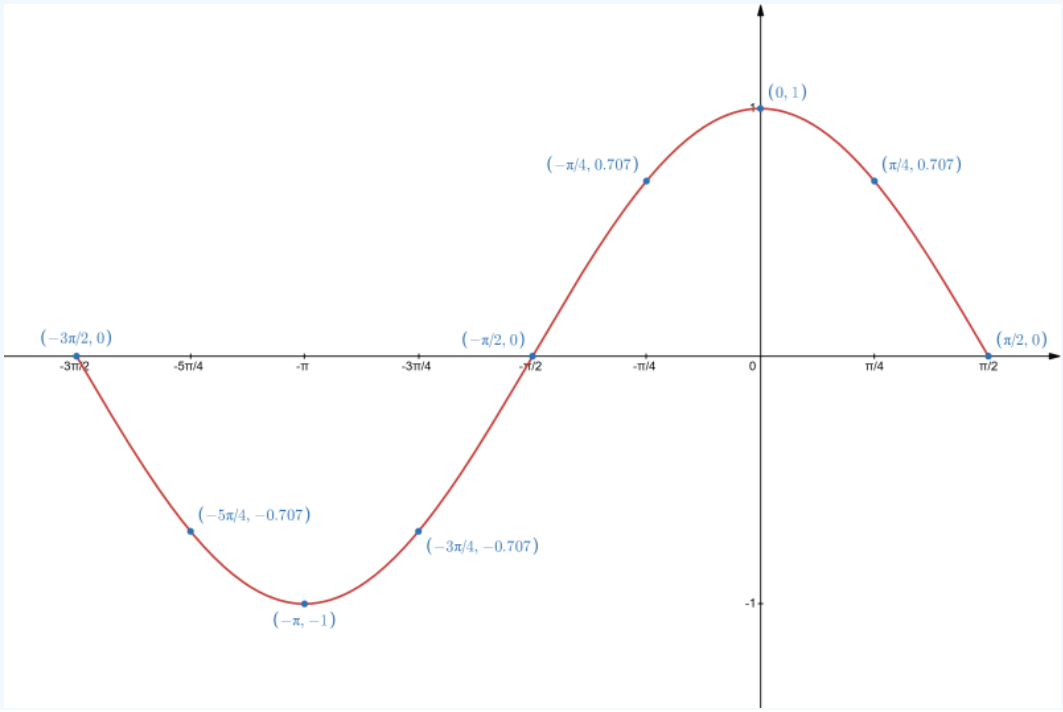
Checkpoint 8.1.5

Sketch of portion of the graph of $y = \cos(x)$ on the interval $[-\frac{3\pi}{2}, \frac{\pi}{2}]$ by making a table of values.

Answer

https://math.libretexts.org/@go/page/197561

Figure 8.1.19

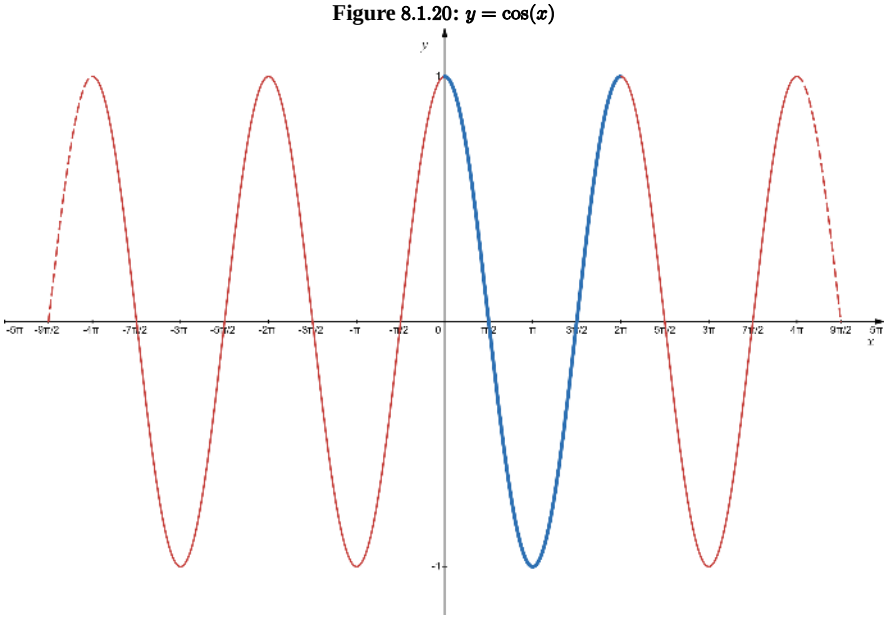


Unit Circle Approach

Relating the cosine curve to the unit circle is similar to what we did with the sine curve; however, we must remember that $\cos(t)$ is the x-coordinate of the points on the unit circle. This makes the visual a little more challenging, but you can play with the following Interactive Element (https://www.desmos.com/calculator/rr51fjmchw) to help cement the concept.

As before, we show the general cosine curve along with a single cycle (in blue) on the interval $[0, 2\pi]$ in Figure 8.1.20 below.

Figure 8.1.20: $y = \cos(x)$



It is no coincidence that the graph in Figure 8.1.20 has many characteristics resembling the graph of the sine function (Figure 8.1.7). The sine and cosine functions are highly related, so their graphs are similar (but not the same). We will explore this similarity later.

Domain, Range, Period, Amplitude, and Zeros

As with the graph of the sine function, the graph of the cosine function reveals the domain and range of the cosine, its period, and the amplitude.

Theorem: Domain and Range of the Cosine Function

The domain of $f(x) = \cos(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Theorem: Period of the Cosine Function

The period of $f(x) = \cos(x)$ is 2π. Thus, the natural period of the cosine function is 2π.

Theorem: Principal Cycle of the Cosine Function

The principal cycle of the cosine function is $[0, 2\pi]$.

Theorem: Amplitude of the Cosine Function

The amplitude of $f(x) = \cos(x)$ is 1.

Finally, we know that the graph of $f(x) = \cos(x)$ crosses the x-axis when $\cos(x) = 0$. This happens when the x-coordinates of points along the edge of the unit circle become 0. This occurs when the angular argument of the cosine function is $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, $\pm\frac{7\pi}{2}$, ... The following theorem summarizes this result.

Theorem: Zeros of the Cosine Function

The zeros of $f(x) = \cos(x)$ occur at $x = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$.

There are other ways you could write this set of zeros. For example,

$$x = \frac{\pi}{2} + \pi k = \frac{\pi}{2} + \frac{2\pi k}{2} = \frac{\pi + 2\pi k}{2} = \frac{\pi}{2}(1 + 2k) = (2k + 1)\frac{\pi}{2}.$$

I find that students remember the form given in the theorem the best.

Graphing the Tangent Function

The tangent function is the more abstract graph of the fundamental trigonometric functions. Therefore, we will spend much time deriving the graph and discussing how it is generated. In the end, students will gravitate to either the conceptual approach (which uses the graphs of the sine and cosine to help develop the tangent curve) or the unit circle approach (which uses the slope of the terminal side of the angle). Understanding both approaches to graphing the tangent curve is best because both will be needed in future mathematics courses.

Before we get to those "headier" approaches to graphing the tangent function, we start, as we did with the sine curve, by plotting points.

Point-Plotting Approach

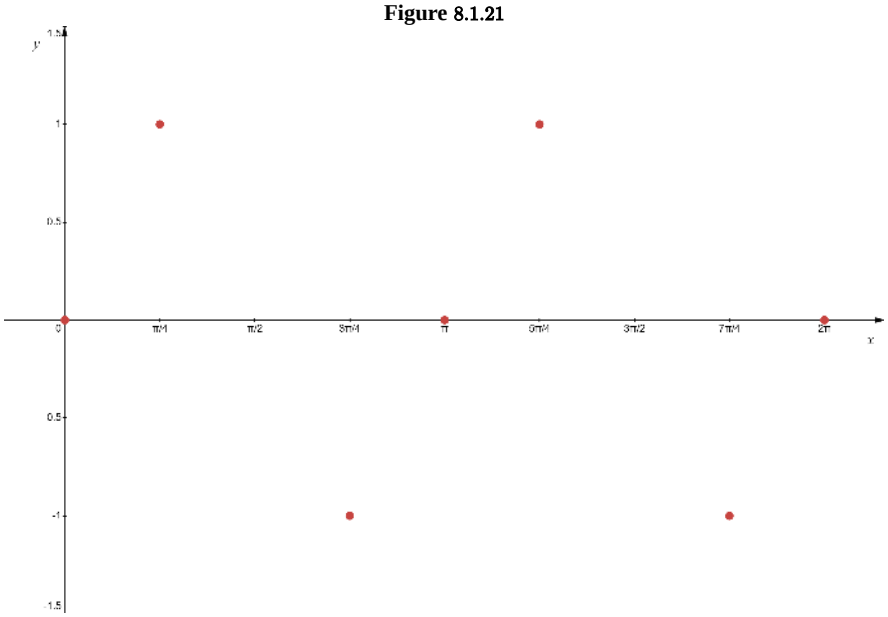
As with the sine and cosine functions, we begin by making a table of values and plotting points. If we decide to use the same "test points" to build our table of values, something odd happens.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\tan(x)$	0	1	Undefined	-1	0	1	Undefined	-1	0

The fact that the tangent function is undefined at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ should not be surprising. By the Ratio Identity, $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and we have just mentioned that the cosine has zeros at $x = \frac{\pi}{2} + \pi k$. That is, the tangent function encounters division by zero at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

What does this mean about the graph of the tangent function? If we decide to plot the points in our list where the tangent *is* defined, all we would get is the following graph.

Figure 8.1.21



We are now faced with the question, "How do we connect these points?"

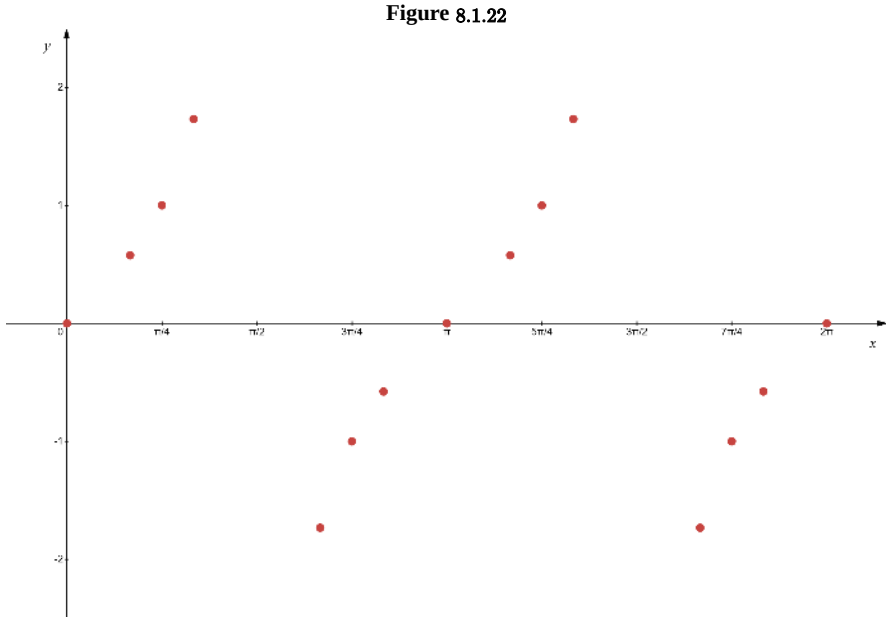
This is one of the many weaknesses of graphing via point-plotting - we are often left guessing how the graph behaves between the points.

In these situations, most people plot more points to see if the true behavior of the graph reveals itself. If we make our table more robust by including every special angle from 0 to 2π, we get the following:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3}{\frac{\pi}{4}}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5}{\frac{\pi}{4}}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7}{\frac{\pi}{4}}$	$\frac{11}{\frac{\pi}{6}}$	$\frac{2}{\pi}$
$\tan(x)$ (exact)	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
$\tan(x)$ (approximate)	0	0.577	1	1.732	Undefined	-1.732	-1	0.577	0	0.577	1	1.732	Undefined	-1.732	-1	0.577	0

However, as detailed and exhaustive as this might seem, the graphical result is only slightly better.

Figure 8.1.22

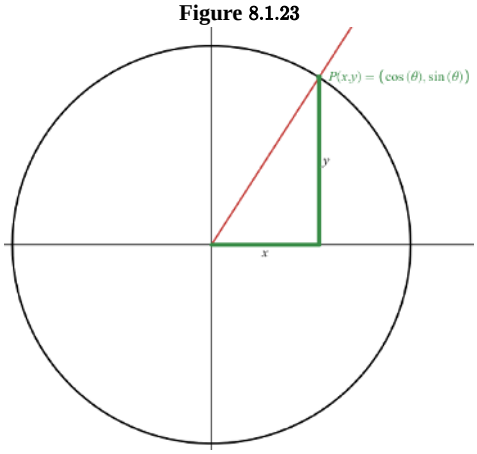


While Figure 8.1.22 begins to reveal some of the patterns of the tangent graph, it's not quite revealing everything we need. Therefore, we must throw in the proverbial towel and try a different method of graphing the tangent function.

Unit Circle Approach

From the unit circle definition of the trigonometric functions, if $P(x, y)$ is a point on the unit circle, then $\tan(\theta) = \frac{y}{x}$, where θ is the angle from the positive x-axis to the ray connecting the origin to P (see Figure 8.1.23).

Figure 8.1.23



Therefore, $\tan(\theta)$ is the slope of the ray connecting the origin to P. When $\theta = 0$, this slope is 0. As θ increases from 0, the slope of this ray also increases; however, as θ approaches $\frac{\pi}{2}$, something interesting happens - the slope of the ray increases without bound. Eventually, at $\theta = \frac{\pi}{2}$, the ray becomes vertical; however, we know from Algebra that vertical lines have undefined slopes. As such, this results in a vertical asymptote in our tangent function graph. The following Interactive Element (https://www.desmos.com/calculator/tv71jqbwil) displays the resulting curve. I recommend you spend a while truly trying to understand what is going on here.

Conceptual Approach

To truly understand what is going on with the tangent function graph, we need to understand the nature of a ratio when its denominator is approaching zero. Consider the values of the three functions, $f(x) = \frac{1}{x}$, $g(x) = \frac{13}{x}$, and $h(x) = -\frac{2}{x^2}$, as the values of x approach 0.

$f(x)=\frac{1}{x}$ as $x\to 0^+$	$g(x)=\frac{13}{x}$ as $x\to 0^-$	$h(x)=-\frac{2}{x^2}$ as $x\to 0^-$
$\frac{1}{0.1}=10$	$\frac{13}{-0.1}=-130$	$-\frac{2}{(-0.1)^2}=-200$
$\frac{1}{0.01}=100$	$\frac{13}{-0.01}=-1300$	$-\frac{2}{(-0.01)^2}=-20000$
$\frac{1}{0.001}=1000$	$\frac{13}{-0.001}=-13000$	$-\frac{2}{(-0.001)^2}=-2000000$
$\frac{1}{0.0001}=10000$	$\frac{13}{-0.0001}=-130000$	$-\frac{2}{(-0.0001)^2}=-200000000$

Before we continue, it's important to clarify the notations 0^+ and 0^-.

Definition

The notation $x\to a^+$ is read as "x is approaching a from the right side of a," and it means that x is strictly greater than a and is getting closer and closer to a.

The notation $x\to a^-$ is read as "x is approaching a from the left side of a," and it means that x is strictly less than a and is getting closer and closer to a.

In both cases, x will never take on the value of a.

Hence, $x\to 0^+$ means that x is getting close to 0, but is still greater than 0, and $x\to 0^-$ means x is getting close to 0, but is still less than 0.

In all three functions from the table, the denominators are approaching zero. In the first and third columns, the denominators approach 0^+ (zero from the right). In the second column, the denominators approach 0^- (zero from the left). Notice that, in all three cases, the smaller the denominator gets, the larger the value of the fraction becomes. Understanding this property is critical to conceptually understanding vertical asymptotes. I have summarized this result below.

> If the numerator is approaching a **nonzero** constant and the denominator is approaching (but never taking on the value of) 0, the ratio $\frac{N(x)}{D(x)}$ approaches either $+\infty$ or $-\infty$. The sign of the result is determined by comparing the signs of the numerator and denominator of the ratio of functions as x is approaching a.

Thus, a vertical asymptote for a function results from what happens as its numerator approaches some nonzero constant while its denominator approaches (but does not become) 0.

We have already mentioned that the tangent function is undefined when cosine is 0 and that this occurs precisely when $x=\frac{\pi}{2}+k\pi$, where $k\in\mathbb{Z}$. If we use the knowledge from the conversation above, we will be able to determine if the tangent has vertical asymptotes at these values (and the behavior of the tangent curve near these vertical asymptotes).

Let's focus our efforts on what happens as the x-values approach $\frac{\pi}{2}$. Since $\tan(x)=\frac{\sin(x)}{\cos(x)}$, we start by sketching both the sine and cosine functions near $x=\frac{\pi}{2}$.

Figure 8.1.24

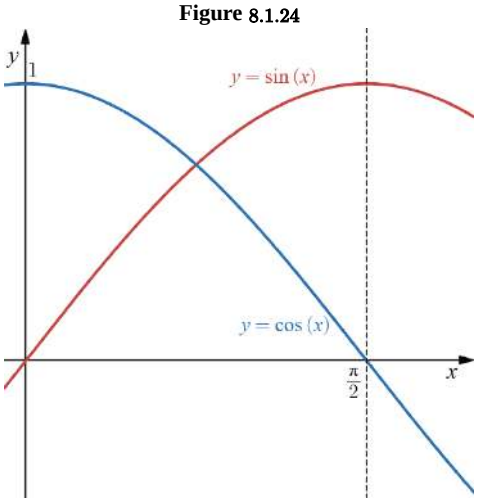
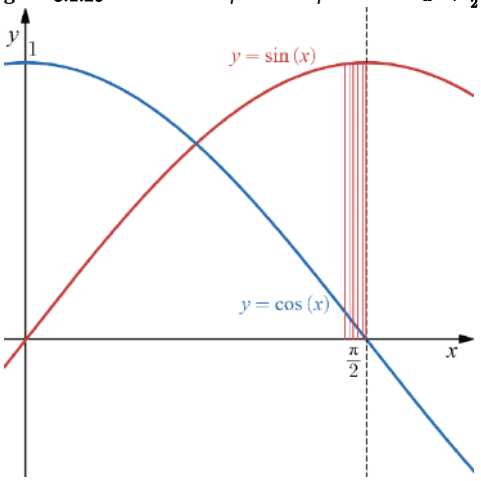


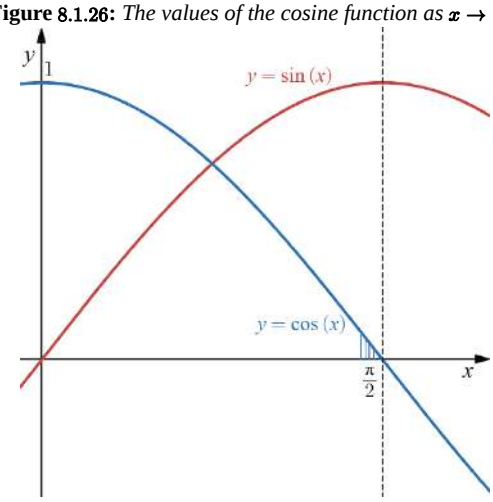
Figure **8.1.25** shows the values of the sine as $x \to \frac{\pi}{2}^-$.

Figure 8.1.25: *The values of the sine function as* $x \to \frac{\pi}{2}^-$



Likewise, Figure **8.1.26** shows the values of cosine as $x \to \frac{\pi}{2}^-$.

Figure 8.1.26: *The values of the cosine function as* $x \to \frac{\pi}{2}^-$



From a graphical investigation, as x approaches $\frac{\pi}{2}$ from the left,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \text{ approaches } \frac{1}{0^+} \text{ which, in turn, approaches } +\infty. \tag{8.1.1}$$

https://math.libretexts.org/@go/page/197561

We now consider what happens as $x \to \frac{\pi}{2}^+$. Consider Figures 8.1.27 and 8.1.28.

Figure 8.1.27: *The values of the sine function as* $x \to \frac{\pi}{2}^+$

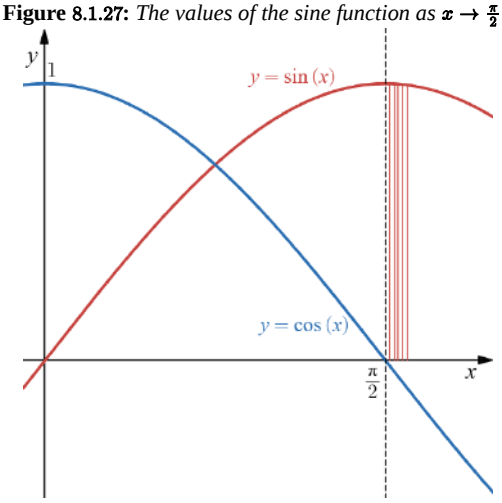
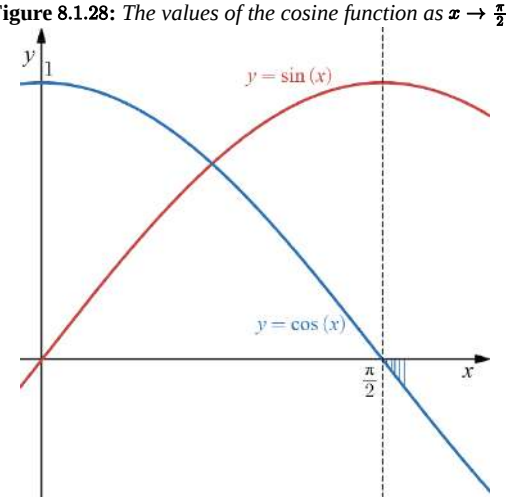


Figure 8.1.28: *The values of the cosine function as* $x \to \frac{\pi}{2}^+$

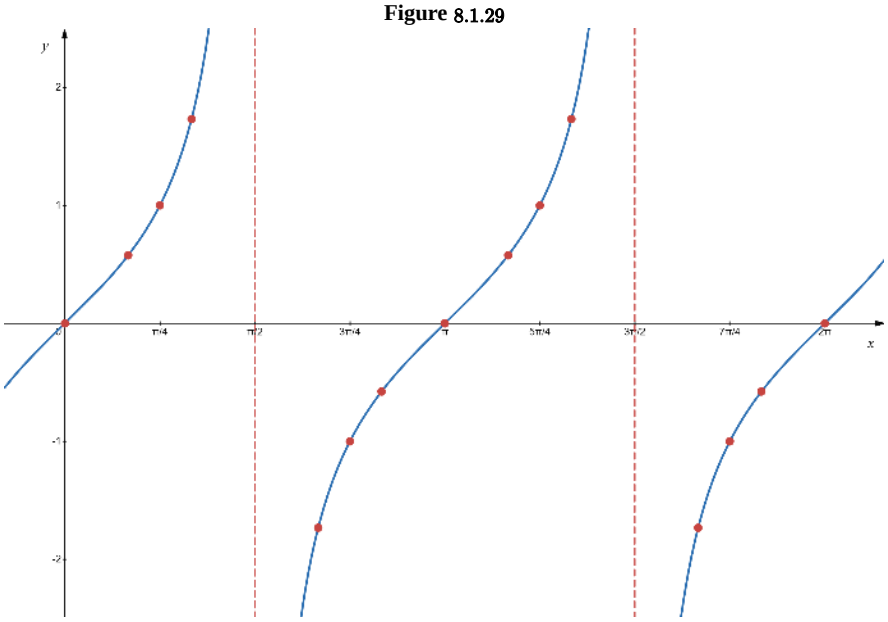


In this case, we can see the sine function is approaching 1, but the cosine is approaching 0 from the negative side of 0. Hence,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \text{ approaches } \frac{1}{0^-} \text{ which, in turn, approaches } -\infty. \tag{8.1.2}$$

Combining our point-plotting in Figure 8.1.22 with the results from 8.1.1 and 8.1.2, we get the following graph.

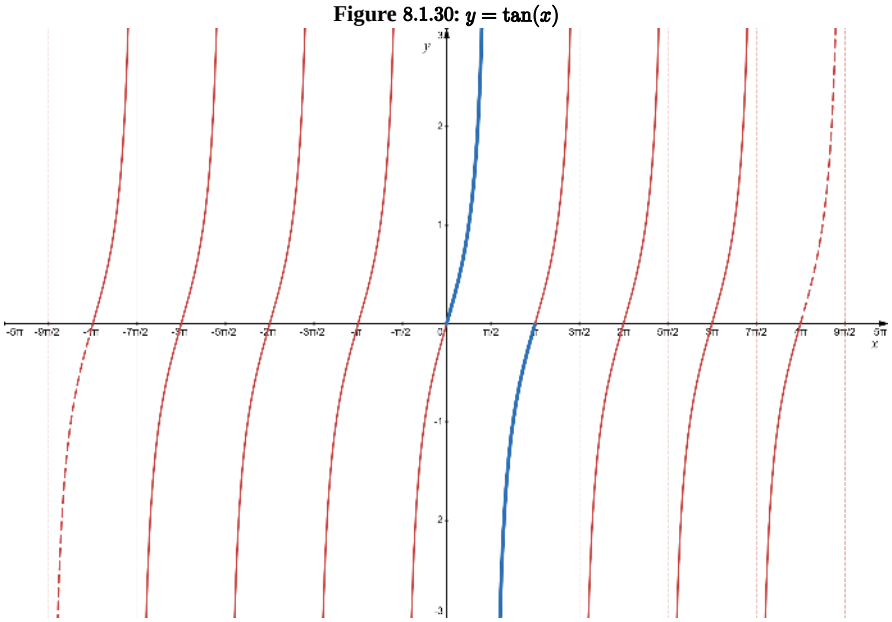
Figure 8.1.29



Notice that we took the liberty to draw in the vertical asymptote at $x = \frac{3\pi}{2}$.

We now have enough knowledge to graph the *entire* tangent curve. Figure 8.1.30 shows the general tangent curve along with a single cycle (in blue) on the interval $[0, \pi]$.

Figure 8.1.30: $y = \tan(x)$



Domain, Range, Period, and Zeros

There are a few essential features to mention about the tangent curve. First, the pattern repeats every π units. Therefore, the period of the tangent function is π. The domain is missing the values of x for which the cosine is zero, and the range is all real numbers. Finally, the zeros of the tangent function occur when the sine function has zeros. All of these facts are summarized below.

Theorem: Domain and Range of the Tangent Function

The domain of the tangent function is $\{x \mid x \neq \frac{\pi}{2} + \pi k\}$, where $k \in \mathbb{Z}$. The range is $(-\infty, \infty)$.

Theorem: Period of the Tangent Function

The period of $f(x) = \tan(x)$ is π. Thus, the natural period of the tangent function is π.

Theorem: Principal Cycle of the Tangent Function

The principal cycle of the tangent function is $[0, \pi]$.

Theorem: Zeros of the Tangent Function

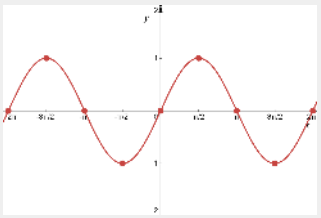
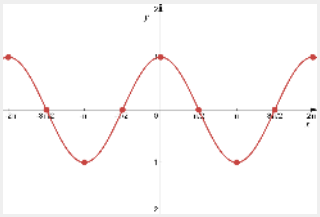
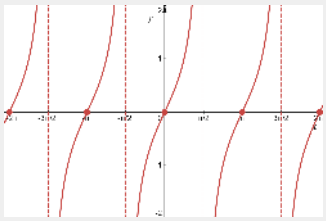
The zeros of the tangent function occur at $x = \pi k$, where $k \in \mathbb{Z}$.

In Figure 8.1.30, we highlighted the single period on the interval $[0, \pi]$. Most textbooks use the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$; however, our choice of highlighted interval will serve an important purpose in the next section.

Also, note that we didn't state the amplitude of the tangent function. This is because the tangent function does not have an amplitude. To have an amplitude, a function must have a greatest and least value; however, the range of the tangent is $(-\infty, \infty)$. Therefore, it is **unbounded** (i.e., it does not have a greatest or least value). As such, it cannot have an amplitude.

Summary of the Graphs of the Fundamental Trigonometric Functions

Before we continue, let's summarize what we have discovered about the graphs of the fundamental trigonometric functions.

	Sine Function	**Cosine Function**	**Tangent Function**
Graph			
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	All $x \neq \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$
Range	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$
Amplitude	1	1	Not defined
Natural Period	2π	2π	π
Principal Cycle	$[0, 2\pi]$	$[0, 2\pi]$	$[0, \pi]$
Zeros	$x = \pi k,\ k \in \mathbb{Z}$	$x = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$	$x = \pi k,\ k \in \mathbb{Z}$

Finally, when asked to graph the sine or cosine functions, we only need to know their values at the quadrantal angles (these correspond to the "dots" on the graphs of the sine and cosine curves above). This concept will lead to a graphing method for transformations of trigonometric functions in our next section.

Symmetry Identities

Now that we have the base graphs of the fundamental trigonometric functions, it's natural to discuss their graphs' symmetric behavior.

Recall from Algebra the following definition.

Definition: Even and Odd Functions

The function f is called an **even function** if for every input x in the domain of f

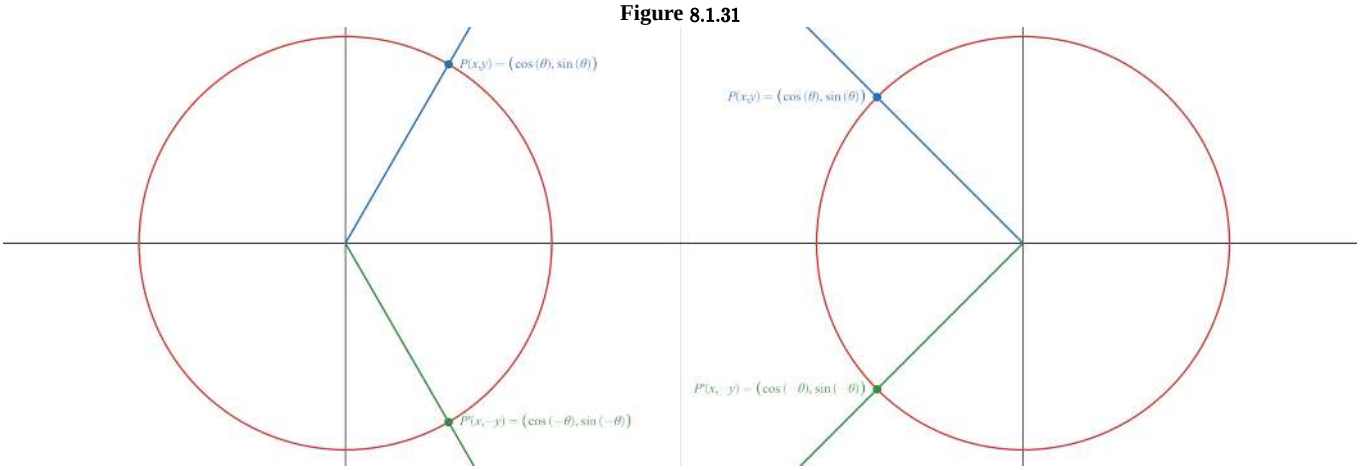
$$f(-x) = f(x).$$

The function f is called an **odd function** if for every input x in the domain of f

$$f(-x) = -f(x).$$

Figure 8.1.31 is the key to proving the symmetric behavior of the trigonometric functions.

Figure 8.1.31



Note that, in both graphs in Figure 8.1.31, the point P on the terminal side of θ is a reflection across the x-axis of the point P' on the terminal side of $-\theta$. Hence, each point's x-coordinate remains the same, but the y-coordinates are opposite in sign. Therefore, no matter what quadrant in which θ terminates,

$$x = \cos(\theta) \qquad \text{and} \qquad x = \cos(-\theta).$$

Thus,

$$\cos(\theta) = \cos(-\theta),$$

and so cosine is an even function.

Moreover,

$$y = \sin(\theta) \qquad \text{and} \qquad -y = \sin(-\theta).$$

The equation on the right side can be solved for y to become $y = -\sin(-\theta)$. Hence, $\sin(\theta) = -\sin(-\theta)$. Stated more clearly,

$$\sin(-\theta) = -\sin(\theta).$$

Thus, sine is an odd function.

From these two results, the proofs of the symmetric behaviors of the remaining trigonometric functions are much easier. For example,

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta).$$

Thus, tangent is an odd function. The rest of the proofs are left as homework exercises.

These results are summarized in the following theorem.

Theorem: Symmetry Identities

The sine, cosecant, tangent, and cotangent are odd functions. The cosine and secant are even functions.

Of course, we could have guessed that the sine and tangent are odd functions because their graphs are symmetric about the origin. Similarly, we could have guessed the cosine is an even function because its graph is symmetric about the y-axis.

Advice: Memorize These Identities

The Symmetry Identities are needed throughout the rest of Trigonometry and Calculus. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 8.1.6

Is the function even, odd, or neither?

$$f(x) = x\sin\left(x^3 - x\right) + x^2$$

Solution

$$\begin{aligned} f(-x) &= (-x)\sin\left((-x)^3 - (-x)\right) + (-x)^2 \\ &= -x\sin\left(-x^3 + x\right) + x^2 \\ &= -x\sin\left(-(x^3 - x)\right) + x^2 \\ &= x\sin\left(x^3 - x\right) + x^2 && \text{(sine is an odd function)} \\ &= f(x) \end{aligned}$$

Thus, f is an even function.

Checkpoint 8.1.6

Determine if the function is even, odd, or neither.

$$g(x) = x^3\cos(x) - \tan(x)$$

Answer

Odd

Finally, we can take advantage of the symmetric behavior of a trigonometric function to speed up evaluations.

Example 8.1.7

Evaluate each function using symmetry.

a. $\sec\left(-\frac{5\pi}{6}\right)$
b. $\sin\left(-\frac{5\pi}{4}\right)$

Solutions

a.

$$\begin{aligned} \sec\left(-\frac{5\pi}{6}\right) &= \sec\left(\frac{5\pi}{6}\right) && \text{(secant is even)} \\ &= \frac{1}{\cos\left(\frac{5\pi}{6}\right)} && \text{(Reciprocal Identity)} \\ &= \frac{1}{-\cos\left(\frac{\pi}{6}\right)} && \text{(reference angle and quadrant)} \\ &= -\frac{1}{\sqrt{3}/2} \\ &= -\frac{2}{\sqrt{3}} \end{aligned}$$

b.

$$\begin{aligned} \sin\left(-\frac{5\pi}{4}\right) &= -\sin\left(\frac{5\pi}{4}\right) && \text{(sine is odd)} \\ &= \sin\left(\frac{\pi}{4}\right) && \text{(reference angle and quadrant)} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Checkpoint 8.1.7

Evaluate $\cot\left(-\frac{2\pi}{3}\right)$ using symmetry.

Answer

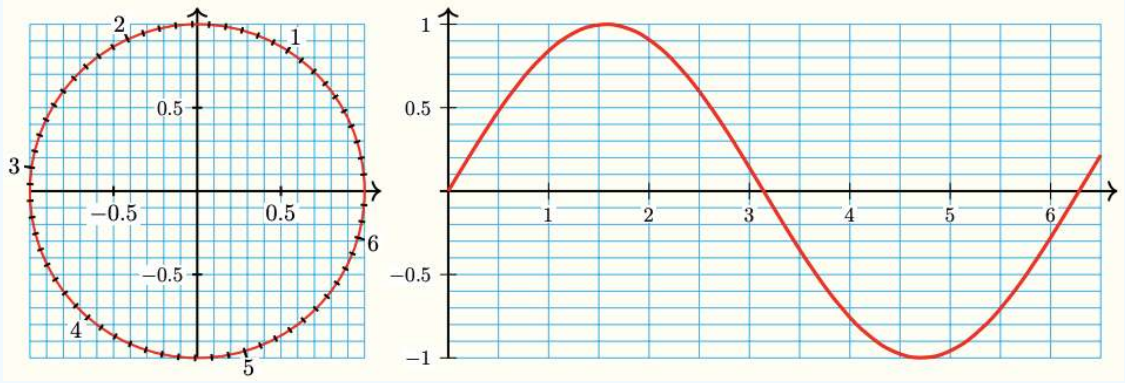
$\frac{1}{\sqrt{3}}$

Evaluating trigonometric functions using symmetry can be thought of as an alternative, not a replacement, to how we have been evaluating them up to this point. I recommend practicing using symmetry for evaluation because it can speed up computations.

Activity

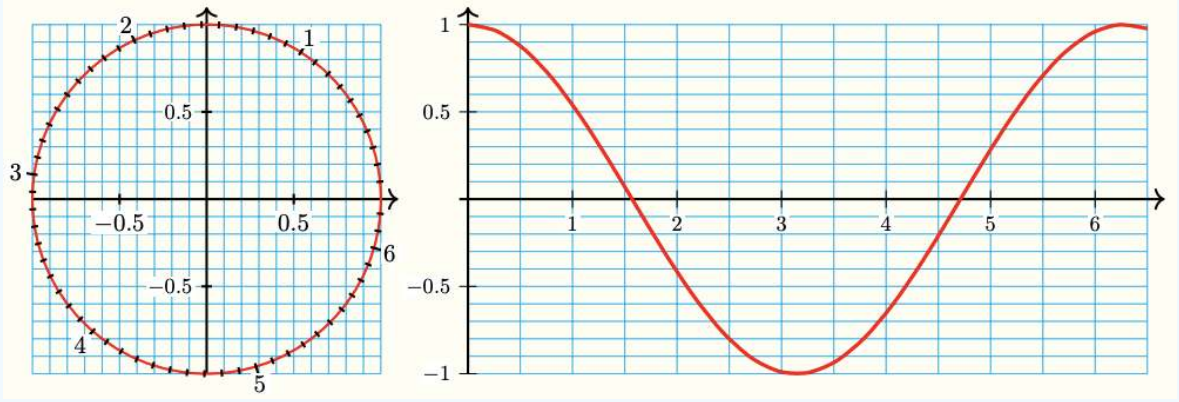
Activity: Solving Basic Trigonometric Equations

1.



a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin(x) = 0.65$. Show your solutions on the graph.
b. Use the unit circle to estimate two solutions of the equation $\sin(x) = 0.65$. Show your solutions on the circle.
c. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin(x) = -0.2$. Show your solutions on the graph.
d. Use the unit circle to estimate two solutions of the equation $\sin(x) = -0.2$. Show your solutions on the circle.

2.



a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos(x) = 0.15$. Show your solutions on the graph.
b. Use the unit circle to estimate two solutions of the equation $\cos(x) = 0.15$. Show your solutions on the circle.
c. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos(x) = -0.4$. Show your solutions on the graph.
d. Use the unit circle to estimate two solutions of the equation $\cos(x) = -0.4$. Show your solutions on the circle.

Solving Trigonometric Equations Using Graphs

Now that we have a basic understanding of the graphs of the fundamental trigonometric functions, we can begin to visualize the solutions to basic trigonometric equations.

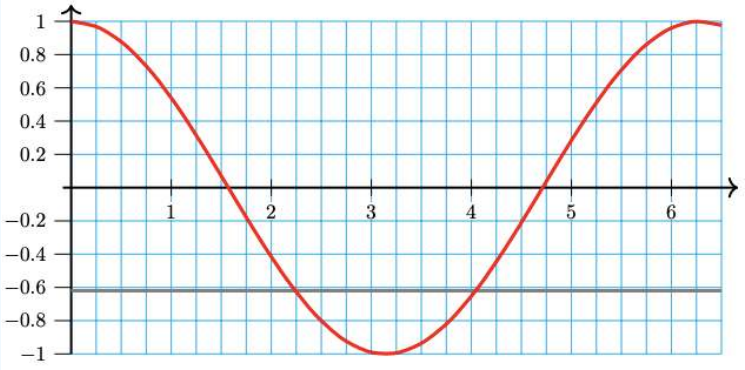
Example 8.1.8

Use a graph to find all solutions of $\cos(t) = -0.62$ between 0 and 2π.

Answer

Figure 8.1.32 shows a graph of $y = \cos(t)$, with t in radians, and the horizontal line $y = -0.62$. The solutions of the equation are the t-coordinates of the intersection points, and these *appear* to be about $t = 2.25$ and slightly over 4.0, perhaps 4.05. You can check that these values roughly satisfy the equation; it is difficult to read the graph with any greater accuracy.

Figure 8.1.32



Caution: Graphs Are a Terrible Method for Solving Equations

Example 8.1.8 showcases the issue with using a graph to solve *any* equation - you get very rough approximations. In truth, the solutions to $\cos(t) = -0.62$ on the interval $[0, 2\pi]$ are approximately 2.2395 and 4.0436 (rounded to four decimal places). There is virtually no way to gain this level of accuracy using a hand-drawn graph.

An interesting observation comes from the fact that the sine and cosine curves oscillate forever in both directions. Had we not restricted our answers in Example 8.1.8 to lie between 0 and 2π, we would have encountered an infinite number of times where the cosine curve intersects the line $y = -0.62$. Thus, the equation $\cos(t) = -0.62$ has an infinite number of solutions when considering all real numbers instead of restricting ourselves to the interval $[0, 2\pi]$.

Checkpoint 8.1.8

Use a graph to find the two angles between 0 and 2π that satisfy $\sin(t) = -0.85$.

Answer

Approximately 4.14 and 5.25

Footnotes

[1] Despite calling this definition rigorous, it is not the most stringent definition of a vertical asymptote - you encounter the true definition of a vertical asymptote in Calculus.

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8.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Graphs of the Sine and Cosine Functions
  - Trigonometric Graphs - Review of Graphs of Sine and Cosine Functions
- Graphs of the Tangent Function
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Tangent and Cotangent

KEY CONCEPTS

DEFINITIONS

- **Fundamental Trigonometric Functions:** The sine, cosine, and tangent functions.
- **Periodic Motion:** Movement that repeats along a path in a regular and predictable manner over a specific time interval.
- **Periodic Function:** A function that models periodic motion by repeating its values at regular intervals.
- **Sine Curve (Sinusoidal):** A smooth, wave-like curve that oscillates periodically and can be derived from either the sine or cosine function.
- **Period:** For a function $f(x)$, the period is the smallest positive number k such that $f(x+k)=f(x)$ for all x in the domain of f.
- **Principal Cycle (of a trigonometric function):** The portion of the graph of a trigonometric function on the interval starting at $x=0$ and ending at $x=p$, where p is the period of the function.
- **Amplitude (of a function):** Half the distance between the function's maximum (M) and minimum (m) values, calculated as $\frac{1}{2}|M-m|$.
- **Even Function:** A function f is even if $f(-x)=f(x)$ for every x in its domain. Its graph is symmetric with respect to the y-axis.
- **Odd Function:** A function f is odd if $f(-x)=-f(x)$ for every x in its domain. Its graph is symmetric with respect to the origin.

THEOREMS

- **Domain and Range of the Sine Function:** The domain of $f(x)=\sin(x)$ is $(-\infty,\infty)$ and the range is $[-1,1]$.
- **Period of the Sine Function:** The period of $f(x)=\sin(x)$ is 2π.
- **Principal Cycle of the Sine Function:** The principal cycle of the sine function is $[0,2\pi]$.
- **Amplitude of the Sine Function:** The amplitude of $f(x)=\sin(x)$ is 1.
- **Zeros of the Sine Function:** The zeros of $f(x)=\sin(x)$ occur at $x=\pi k$, where $k\in\mathbb{Z}$.
- **Domain and Range of the Cosine Function:** The domain of $f(x)=\cos(x)$ is $(-\infty,\infty)$ and the range is $[-1,1]$.
- **Period of the Cosine Function:** The period of $f(x)=\cos(x)$ is 2π.
- **Principal Cycle of the Cosine Function:** The principal cycle of the cosine function is $[0,2\pi]$.
- **Amplitude of the Cosine Function:** The amplitude of $f(x)=\cos(x)$ is 1.
- **Zeros of the Cosine Function:** The zeros of $f(x)=\cos(x)$ occur at $x=\frac{\pi}{2}+\pi k$, where $k\in\mathbb{Z}$.
- **Domain and Range of the Tangent Function:** The domain of the tangent function is $\{x|x\neq\frac{\pi}{2}+\pi k\}$, where $k\in\mathbb{Z}$. The range is $(-\infty,\infty)$.
- **Period of the Tangent Function:** The period of $f(x)=\tan(x)$ is π.
- **Principal Cycle of the Tangent Function:** The principal cycle of the tangent function is $[0,\pi]$.
- **Zeros of the Tangent Function:** The zeros of the tangent function occur at $x=\pi k$, where $k\in\mathbb{Z}$.
- **Symmetry Identities:** The sine, cosecant, tangent, and cotangent functions are odd. The cosine and secant functions are even.

COMMON MISTAKES

- **Graphing in Degrees:** From this point forward, graphs of trigonometric functions will use radians for the horizontal axis, not degrees. This is critical for maintaining proper scale and for applications in calculus.
- **Connecting Plotted Points with Straight Lines:** The graphs of trigonometric functions are smooth curves. Never connect plotted points with straight line segments.
- **Confusing the Argument of a Function with a Coordinate:** The input variable for a function like $y=\sin(x)$ is a dummy variable representing an angle or real number. It should not be confused with the x-coordinate of a point on the unit circle.
- **Using Graphs to Solve Equations:** Using a graph to find the solution to an equation provides only a rough approximation and is not a valid method for finding exact solutions.
- **Using Graphs as Proofs:** A graph cannot be used to prove an equation is an identity, as it may be deceiving. However, a graph can be used to show an equation is *not* an identity if the graphs of the two sides are different.

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8.1.2: HOMEWORK

READING QUESTIONS

1. What are the three fundamental trigonometric functions?
2. What is a periodic function, and what does its "period" represent?
3. What is the period of the sine function? What is the period of the tangent function?
4. What is the "amplitude" of a function?
5. What is the amplitude of the cosine function? Why does the tangent function not have an amplitude?
6. What is the domain and range of the sine function?
7. What is the domain and range of the tangent function?
8. What is the "principal cycle" of the sine function?
9. At what values of x does the graph of $y = \cos(x)$ have x-intercepts?
10. At what values of x does the graph of $y = \tan(x)$ have vertical asymptotes?
11. What does it mean for a function to be "even"? Which of the fundamental trigonometric functions are even?
12. What does it mean for a function to be "odd"? Which of the fundamental trigonometric functions are odd?

HOMEWORK

VOCABULARY CHECK

1. Which trigonometric functions are called the *fundamental* trigonometric functions?
2. The smallest distance (along the x-axis) for which a function starts a repeating pattern (that continues forever) is called the ___ of the function.
3. The ___ for a function is found by taking one-half the sum of the function's maximum and minimum values.
4. If the graph of a function is symmetric about the origin, we say that the function is ___.

CONCEPT CHECK

5. List the values from 0 to 2π by multiples of $\frac{\pi}{4}$.
6. List the values from 0 to 2π by multiples of $\frac{\pi}{6}$.
7. List several ways in which the graph of $y = \tan(\theta)$ is different from the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$.
8. Sketch two examples of a function with period 8: one that is sinusoidal and one that is not.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

9. The amplitude of a function is the vertical distance between the maximum and minimum values (if these exist).
10. $\tan(\theta + 180°) = \tan\theta$ for any value of θ.
11. If $f(x + p) = f(x)$ for all x, then the constant p is called the period of the function.
12. The tangent function has period π. It is undefined at odd multiples of $\frac{\pi}{2}$ and is increasing on each interval of its domain.
13. Functions that have graphs shaped like sines or cosines are called sinusoidal.
14. The cosine graph looks just like the sine graph except flipped upside down.
15. The period of a sine or cosine function is the distance between horizontal intercepts.
16. Periodic functions are used to model phenomena that exhibit cyclical behavior.

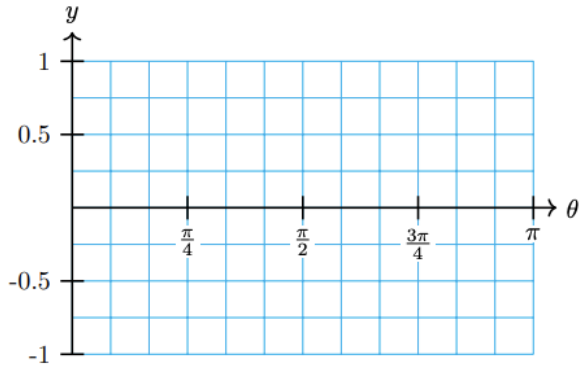
BASIC SKILLS

17.

a. Use your calculator to complete the table of values. Round values to hundredths.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\cos\theta$													

b. Sketch a graph of $y = \cos\theta$ on the grid.

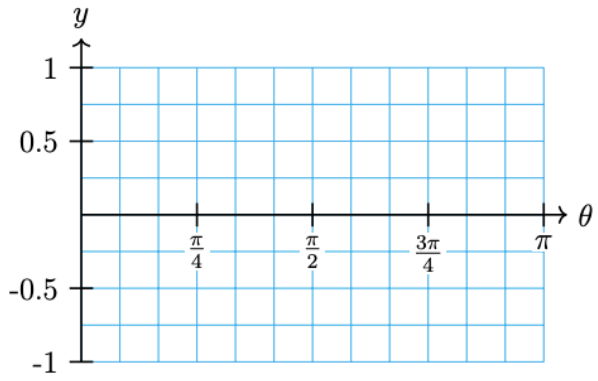


18.

a. Use your calculator to complete the table of values. Round values to hundredths.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\sin\theta$												

b. Sketch a graph of $y = \sin\theta$ on the grid.



19.

a. Sketch a graph of $y = \cos x$ using the guide points in the table below.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$					

b. State the domain and range of the function $y = \cos x$.

20.

a. Sketch a graph of $y = \tan x$ using the guide points in the table below.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$\cos x$					

b. State the domain and range of the function $y = \tan x$.

21. Sketch a graph of $y = \sin x$ on each grid.

22. Sketch a graph of $y = \cos x$ on each grid.

23.

a. Sketch a graph of $f(x) = \sin x$ where x is a real number.

b. State the domain and range of $f(x) = \sin x$.

24.

a. Sketch a graph of $g(x) = \cos x$ where x is a real number.

b. State the domain and range of $g(x) = \cos x$.

25.

a. Sketch a graph of $h(x) = \tan x$ where x is a real number.

b. State the domain and range of $h(x) = \tan x$.

26. Sketch $f(x) = \cos x$ and $g(x) = \sin x$ on the same grid.

27. The graph shows your height as a function of angle as you ride the Ferris wheel. For each location A - E on the Ferris wheel, mark the corresponding point on the graph.

28. The graph shows your height as a function of angle as you ride the Ferris wheel. For each location F - J on the graph, mark the corresponding point on the Ferris wheel.

29. The graph shows the horizontal displacement of your foot from the center of the chain gear as you pedal a bicycle. For each location K - O on the chain gear, mark the corresponding point on the graph.

30. The graph shows the horizontal displacement of your foot from the center of the chain gear as you pedal a bicycle. For each location P - T on the graph, mark the corresponding point on the chain gear.

For the following exercises, use the figures below. Show your solutions on the graphs.

31.

a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin x = 0.65$.

b. Use the unit circle to estimate two solutions of the equation $\sin x = 0.35$.

32.

a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin x = -0.2$.

b. Use the unit circle to estimate two solutions of the equation $\sin x = -0.6$.

For the following exercises, use the figures below. Show your solutions on the graphs.

33.

a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos x = -0.4$.

b. Use the unit circle to estimate two solutions of the equation $\cos x = -0.8$.

34.

a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos x = 0.15$.

b. Use the unit circle to estimate two solutions of the equation $\cos x = 0.55$.

For the following exercises, use the graph of $y = \tan x$ to estimate two solutions to the equation.

35. $\tan x = 4$
36. $\tan x = 7$
37. $\tan x = -0.5$
38. $\tan x = -2.5$
39. $\sin x = 5.5 \cos x$
40. $\sin x = -4.5 \cos x$
41. $-2 \sin x = 3 \cos x$
42. $2 \sin x = \cos x$

For the following exercises, find all solutions between 0 and 2π. Round to two decimal places. Sketch your solutions on a unit circle.

43. $\cos t = 0.74$
44. $\sin t = 0.58$
45. $\tan t = 1.6$
46. $\tan x = -0.6$
47. $\sin x = -0.72$
48. $\cos x = -0.48$

Which of the graphs in the following exercises are periodic? If the graph is periodic, give its period.

49.

50.

51.

52.

53.

a. Is the function shown periodic? If so, what is its period? If not, explain why not.

b. Compute the difference between the maximum and minimum function values. Sketch in the midline of the graph.

c. Find the smallest positive value of k for which $f(x) = f(x + k)$ for all x.

d. Find the smallest positive values of a and b for which $f(b) - f(a)$ is a maximum.

For the following exercises, use symmetry to help in evaluating the given trigonometric function.

54. $\cos(-30°)$
55. $\sin\left(-\frac{5\pi}{6}\right)$
56. $\tan(-135°)$
57. $\sec\left(-\frac{5\pi}{3}\right)$
58. $\csc(-315°)$
59. $\cot\left(-\frac{7\pi}{6}\right)$

SYNTHESIS QUESTIONS

60. An ant is walking clockwise around the face of a sundial in the garden. The sundial is a circle with a 12-inch diameter, and the ant makes one circuit of the sundial in 24 seconds.

a. Sketch the sundial and a coordinate system with its origin at the center of the dial. Draw the y-axis to align with the gnomon (pointer) of the sundial.

b. Suppose we start timing the ant when it is at the tip of the gnomon. Complete the table showing the ant's location as an angle in standard position and its y-coordinate at that time.

t	0	2	4	6	8	10	12	14	16	18	20	22	24
θ	90°	60°											
$y = f(t)$	6	$3\sqrt{3}$											

c. Sketch a graph of the ant's y-coordinate as a function of time.

d. If the ant makes a second trip around the sundial, what will the graph look like from $t = 24$ to $t = 48$? Explain the statement $f(t + 24) = f(t)$.

61. Repeat the previous exercise, but make a table and graph of the ant's x-coordinate as a function of time.

t	0	2	4	6	8	10	12	14	16	18	20	22	24
θ	90°	60°											
$x = g(t)$	0	3											

62. Jorge is standing at the point $(5, 0)$ on the square shown. If he walks around the square in the counter-clockwise direction, his y-coordinate is a function of the distance, d, he has walked.

a. What is the y-coordinate of Jorge's position when he has walked 2 units? 5 units? 8 units? (Start filling in the table in part (c).)

b. After Jorge reaches the upper right corner of the square, he will turn left and start walking along the top of the square. What is his y-coordinate from $d = 5$ to $d = 15$?

c. Fill in the rest of the table with the y-coordinates of Jorge's position when he has walked a distance d.

d	0	2	5	8	10	12	15	18	20	22	25	28	30	32	35	38	40
y																	

d. Plot a graph of Jorge's coordinate as a function of the distance he has walked.

63. Repeat the previous exercise, except consider Jorge's x-coordinate as a function of the distance he has walked.

d	0	2	5	8	10	12	15	18	20	22	25	28	30	32	35	38	40
x																	

a. Refer to the previous exercise. Suppose you know where Jorge is on the square at some given time. Can you predict where he will be after he has walked another 40 units?

b. Let $y = f(d)$. What does your answer say about $f(d)$ and $f(d + 40)$ for any positive value of d?

c. How does the graph of f for $0 \le d \le 40$ compare with the graph for $40 \le d \le 80$?

d. Describe how the graph of f would continue for $0 \le d \le 400$.

64. Refer to the previous exercise. Sketch a graph of Jorge's x-coordinate, $x = g(d)$, for $0 \le d \le 120$.

PROOFS

For the following exercises, you may assume that sine is an odd function and cosine is an even function.

65. Prove that tangent is an odd function.

66. Prove that secant is an even function.

67. Prove that cosecant is an odd function.

68. Prove that cotangent is an odd function.

For the following exercises, prove the identity.

69. $\cot(-\theta)\sin(-\theta) = \cos(\theta)$

70. $\dfrac{\cos(-x)\sec(x)}{\tan(-x)} = -\cot(x)$

71. $\sin(-y)\sec(-y) = \tan(-y)$

72. $\sin(-\theta)\sec(-\theta)\cot(-\theta) = 1$

73. $1 = 2\cos(x)\sec(-x) + \tan(-y)\cot(y)$

74. $\dfrac{\tan(-\gamma)}{\sin(\gamma)} = -\sec(\gamma)$

75. $\tan(A) - \dfrac{1}{\tan(-A)} = \dfrac{\sec(-A)}{\sin(A)}$

76. $\sec(z) + \tan(z)\sin(-z) = \cos(-z)$

77. $\sin(-\phi)[\csc(-\phi) + \sec(-\phi)] = 1 + \tan(-\phi)$

APPLICATIONS

For the following exercises, sketch a periodic function that models the situation.

78. At a ski slope, the lift chairs take 5 minutes to travel from the bottom, at an elevation of 3000 feet, to the top, at elevation 4000 feet. The cable supporting the ski lift chairs is a loop turning on pulleys at a constant speed. At the top and bottom, the chairs are at a constant elevation for a few seconds to allow skiers to get on and off.

a. Sketch a graph of $h(t)$, the height of one chair at time t. Show at least two complete trips.

b. What is the period of $h(t)$?

79. The heater in Wyatt's house doesn't have a thermostat; it runs on a timer. It uses 300 watts when it is running. Wyatt sets the heater to run from 6 am to noon, and again from 4 pm to 10 pm in the evening.

a. Sketch a graph of $P(t)$, the power drawn by the heater as a function of time. Show at least two days of heater use.

b. What is the period of $P(t)$?

80. Piyali adds water to her fish pond once a week to keep the depth at 30 centimeters. During the week, the water evaporates at a constant rate of 0.5 centimeters per day.

a. Sketch a graph of $D(t)$, the depth of the water as a function of time. Show at least two weeks.

b. What is the period of $D(t)$?

81. Roy's miniature schnauzer, Sophie, is very energetic and bounces excitedly at dinner time. Sophie can jump 30 inches high, and each jump takes him 0.8 seconds.

a. Sketch a graph of Sophie's height, $D(t)$, as a function of time. Show at least two jumps.

b. What is the period of $h(t)$?

82. Each situation describes a periodic function. Match each situation with the appropriate graph.

a. When the heart contracts, blood pressure in the arteries rises rapidly to a peak (systolic blood pressure) and then falls off quickly to a minimum (diastolic blood pressure). Blood pressure is a function of time.

b. After an injection is given to a patient, the amount of the drug present in his bloodstream decreases over time. The patient receives injections at regular intervals to restore the drug level to the prescribed level. The amount of the drug present is a function of time.

c. The monorail shuttle train between the north and south terminals at Gatwick Airport departs from the south terminal every 12 minutes. The distance from the train to the south terminal is a function of time.

d. Ron gets a haircut every two weeks. The length of his hair is a function of time.

83. Match each of the following situations with an appropriate graph below.

a. The number of hours of daylight in Salt Lake City varies from a minimum of 9.6 hours on the winter solstice to a maximum of 14.4 hours on the summer solstice.

b. A weight is 6.5 feet above the floor, suspended from the ceiling by a spring. The weight is pulled down to 5 feet above the floor and released, rising past 6.5 feet in 0.5 seconds before attaining its maximum height of feet. Neglecting the effects of friction, the height of the weight will continue to oscillate between its minimum and maximum height.

c. The voltage used in U.S. electrical current changes from 155V to −155V and back 60 times each second.

d. Although the moon is spherical, what we see from earth looks like a disk, sometimes only partly visible. The percentage of the moon's disk that is visible varies between 0 (at new moon) to 100 (at full moon).

84. The table shows sunrise and sunset times in Los Angeles on the fifteenth of each month.

Month	Oct	Nov	Dec	Jan	Feb	Mar
Sunrise	5:58	6:26	6:51	6:59	6:39	6:04
Sunset	17:20	16:50	16:45	17:07	17:37	18:01

Month	Apr	May	Jun	Jul	Aug	Sep
Sunrise	5:22	4:52	4:42	4:43	5:15	5:37
Sunset	18:25	18:48	19:07	19:05	18:40	18:00

a. Use the left-hand grid to plot the sunrise times and sketch a sinusoidal graph through the points.

b. Use the right-hand grid to plot the sunset times and sketch a sinusoidal graph through the points.

85.

a. Use the data from Problem 65 to complete the table with the hours of sunlight in Los Angeles on the fifteenth of each month.

Month	Oct	Nov	Dec	Jan	Feb	Mar
Hours of Daylight						

Month	Apr	May	Jun	Jul	Aug	Sep
Hours of Daylight						

b. Plot the daylight hours and sketch a sinusoidal graph through the points.

86. Many people who believe in astrology also believe in biorhythms. The graph shows an individual's three biorhythms, physical, emotional, and intellectual, for 36 days, from $t = 0$ on September 30 to November 5.

a. Find the dates of highest and lowest activity for each biorhythm during the month of October.

b. Find the period of each biorhythm in days.

c. On the day of your birth, all three biorhythms are at their maximum. How old will you be before all three are again at the maximum level?

87. The path of a satellite orbiting above the earth makes a sinusoidal graph on a map of the earth, with its midline at the equator. On the map on the next page, sketch a graph for a satellite that orbits the earth every 90 minutes, and strays no farther than 4000 km from the equator. (One degree of latitude is equal to 111 kilometers.) The satellite passes over the spot $0°$ latitude and $0°$ longitude at time $t = 0$. Label a scale on the equator to serve as a time axis for your graph.

88. The graph shows arterial blood pressure, measured in millimeters of mercury (mmHg), as a function of time.

a. What are the maximum (systolic) and minimum (diastolic) pressures? The pulse pressure is the difference of systolic and diastolic pressures. What is the pulse pressure?

b. The mean arterial pressure is the diastolic pressure plus one-third of the pulse pressure. Calculate the mean arterial pressure, and draw a horizontal line on the graph at that pressure.

c. The blood pressure graph repeats its cycle with each heartbeat. What is the heart rate, in beats per minute, of the person whose blood pressure is shown in the graph?

89. Here is a tide chart for Los Angeles for the week of December 17–23, 2000. The horizontal axis shows time in hours, with $t = 12$ corresponding to noon on December 17. The vertical axis shows the height of the tide in feet above mean sea level.

a. High tides occurred at 3:07 am and 2:08 pm on December 17, and low tides at 8:41 am and 9:02 pm. Estimate the heights of the high and low tides on that day.

b. Is tide height a periodic function of time? Use the information from part (a) to justify your answer.

c. Make a table showing approximate times and heights for the high tides throughout the week. Make a similar table for the low tides.

d. Describe the trend in the heights of the high tides over the week. Describe the trend in the heights of the low tides.

e. What is the largest height difference between consecutive high and low tides during the week shown? When does it occur?

90. The apparent magnitude of a star is a measure of its brightness as seen from earth. Smaller values of apparent magnitude correspond to brighter stars. The graph below, called a light curve, shows the apparent magnitude of the star Algol as a function of time. Algol is actually a system of two stars, a bright principal star and its dimmer companion, in orbit around each other. As each star passes in front of the other it eclipses some of the light that reaches earth.

a. The light curve is periodic. What is its period?

b. What is the range of apparent magnitudes of the Algol system?

c. Explain the large and small dips in the light curve. What is happening to cause the dips?

91. Some stars, called Cepheid variable stars, appear to pulse, getting brighter and dimmer periodically. The graph shows the light curve for the star Delta Cephei.

a. What is the period of the graph?

b. What is the range of apparent magnitudes for Delta Cephei?

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8.2: Non-Rigid Transformations

Note to the Instructor - Why Non-Rigid Transformations First?

The choice to cover non-rigid transformations (amplitude and period) before the rigid transformations is purposeful. Traditionally, when performing transformations, it is best to save the rigid transformations for the end.

Learning Objectives

- Determine the amplitude of a sinusoidal function.
- Determine the period of a sine, cosine, or tangent function.
- Graph a sine, cosine, or tangent function having a non-standard amplitude or period.
- Determine the equation of the graph of a trigonometric function.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The material in the next few sections relies heavily on your prerequisite graphing skills from Algebra. As such, we will try our best to "backfill" some of that information here.[1] We begin by reintroducing the definition of a **transformation** of the graph of a function.

Definition: Transformation (of the graph of a function)

A **transformation** of the graph of a function is an alteration of the graph's orientation (by reflections across axes), scale, and position through various operations.

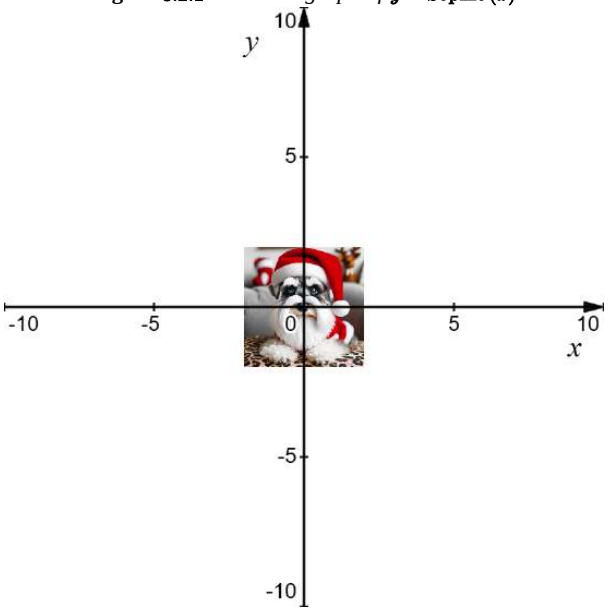
What you may *not* have learned in your Algebra course is that these transformations can be categorized into two distinct groups - **rigid** and **non-rigid transformations**.

Definition: Rigid Transformation

A **rigid transformation** keeps the orientation and scale of the original function's graph but changes its position.

You likely remember rigid transformations as **shifts**. As a reminder of the basic idea of a rigid transformation, let's consider the adorable "base graph" of my dog, $y = \text{Sophie}(x)$ in Figure 8.2.1.[2]

Figure 8.2.1: *The base graph of* $y = \text{Sophie}\,(x)$

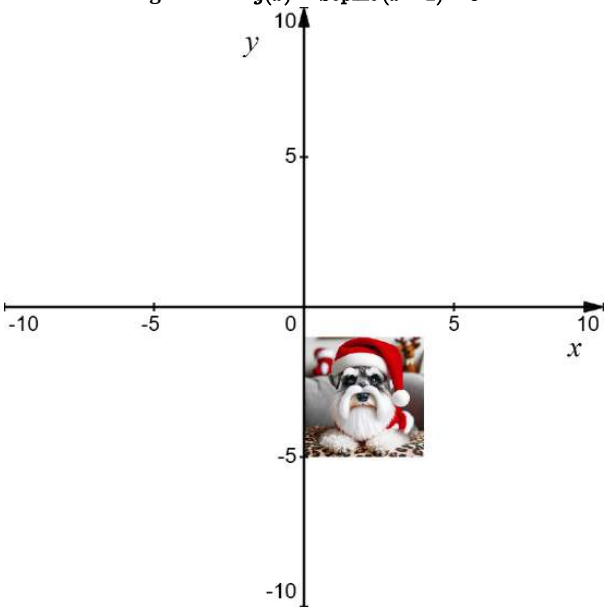


From Algebra, we know that if we wanted to shift the graph of Sophie 2 units to the right and 3 units down, the resulting function would be

$$g(x) = \text{Sophie}\,(x-2) - 3.$$

The orientation and scale of the graph of this new "function" (seen in Figure 8.2.2) would be the same as the base graph of the "Sophie" function, but the position is different.

Figure 8.2.2: $g(x) = \text{Sophie}\,(x-2) - 3$



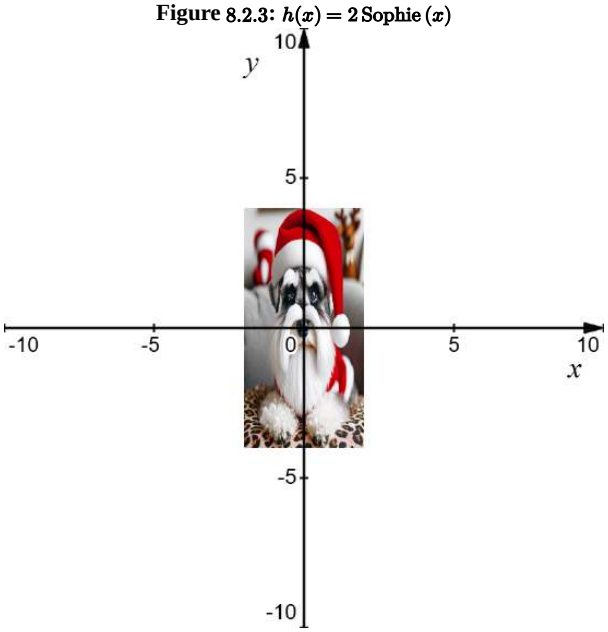
In general, given a function $f(x)$, we can obtain the graph of $g(x) = f(x + C) + D$ by *subtracting* C from all of the x-coordinates of the graph of $f(x)$ and *adding* D to all the y-coordinates.[3]

Definition: Non-rigid Transformation

A **non-rigid transformation** retains the graph's original center position but scales the graph vertically or horizontally or changes the graph's orientation by reflecting the graph about a line.

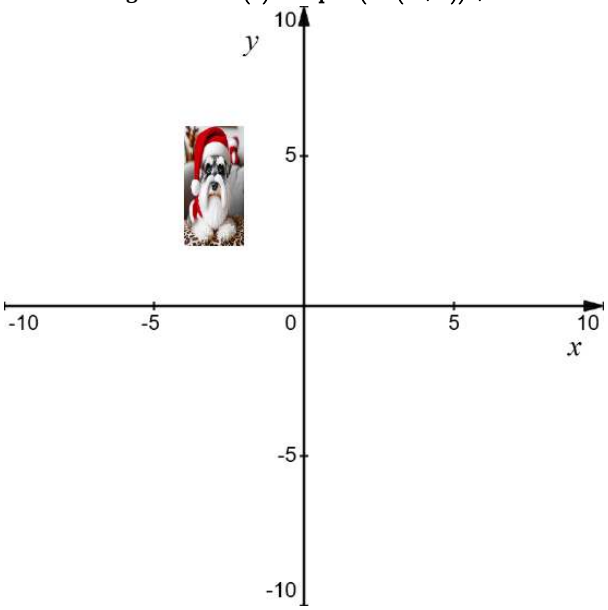
For example, if we took the base graph of our "Sophie function" from Figure **8.2.1** and decided to scale it vertically (also known as stretching) by a factor of 2, we would need to multiply all the y-coordinates of the points in the base graph by 2. From Algebra, we know this is achieved by multiplying the function by 2 to get $h(x) = 2\,\text{Sophie}\,(x)$. The graph of this new image would look like the following:

Figure 8.2.3: $h(x) = 2\,\text{Sophie}\,(x)$



We also learned (again, from Algebra) to combine transformations to get a new function from an old one. For example, we could take $\text{Sophie}\,(x)$, scale it horizontally by a factor of $1/2$ (also known as compressing), reflect it about the y-axis, shift it 3 units left, and shift it 4 units up. The result of all of these manipulations is the function $k(x) = \text{Sophie}\,(-2\,(x+3)) + 4$, and it looks like the following image:

Figure 8.2.4: $k(x) = \text{Sophie}\,(-2\,(x+3)) + 4$



In this section, we will perform these same transformations on trigonometric functions; however, because these functions are periodic, we can only capture a small section of their graphs. The process we introduce to apply these transformations and the names we define during this development will be used throughout the rest of Trigonometry and Calculus.

Our ultimate goal is to graph a trigonometric function of the form

$$f(x) = A\,\text{trig}\,(Bx + C) + D, \tag{8.2.1}$$

where "trig" could be any of the trigonometric functions.[4] The unknown values, A, B, C, and D, are called **parameters**.

Definition: Parameter

A **parameter** is a variable that influences the behavior or output of a mathematical object (like a function or equation) but is held constant within a specific context or scenario. It's a quantity that helps define or constrain a system, function, or dataset.

We will review what happens to the graph of the base trigonometric function when we adjust these parameters.

Amplitude and the Graph of $y = A\,\text{trig}(x)$

We start our investigation of the effect of the parameters in Equation 8.2.1 by setting $B = 1$, $C = 0$, and $D = 0$. Therefore, we are interested in the effect of A on the graph of

$$f(x) = A\,\text{trig}\,(x)\,.$$

If $A > 0$, then from Algebra, we know that the graph of $y = Ag(x)$ is a vertical stretch (if $A > 1$) or compression (if $0 < A < 1$) of the graph of g; however, does this still hold in Trigonometry? The following example might lead us to the answer.

Example 8.2.1

Make a table of values and sketch the graph of each function. Compare your graph to the function's base graph.

a. $f(x) = 3\sin(x)$
b. $g(x) = \frac{1}{5}\cos(x)$
c. $h(x) = \pi\tan(x)$

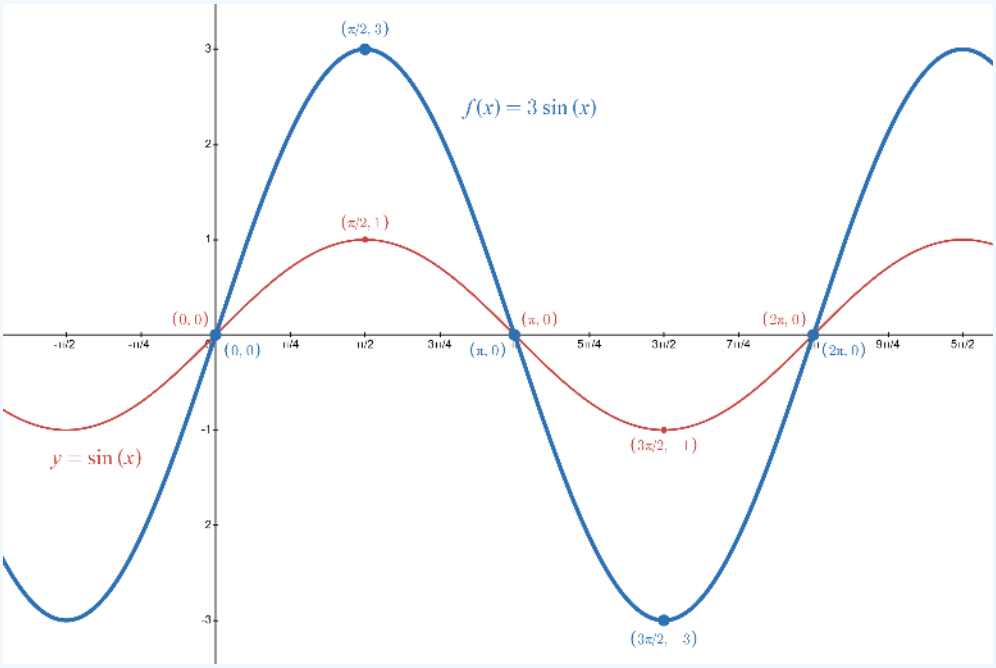
Solutions

a. Now that we know the basic shape of the sine function, we only need to make a table of the quadrantal angles.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x)$	0	3	0	-3	0

Plotting these points and graphing the base sine function, we get the following:

Figure 8.2.5



We can immediately see that the graph of $f(x) = 3\sin(x)$ is a vertical stretch of the base graph by a factor of 3. That is, all y-values of $f(x) = 3\sin(x)$ are three times that of the y-values for $y = \sin(x)$. Judging from this single example, it *looks like* the role of A in $f(x) = A\,\text{trig}(x)$ is to multiply the y-values of the base trigonometric function by A. However, looks can be deceiving, and this is *not* a proof. Let's do the second example and see if our insight holds.

b. Again, we use our knowledge of the basic shape of the cosine graph, along with a table of quadrantal angles.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$g(x)$	$\frac{1}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$

Plotting these points and graphing the base cosine function, we get the following:

Figure 8.2.6

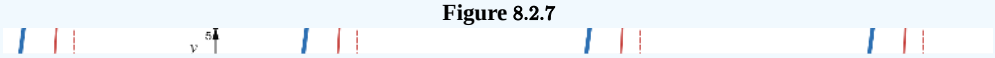
As before, we see that the graph of $g(x) = \frac{1}{5}\cos(x)$ is a vertical compression of the base graph by a factor of $1/5$.[5] That is, all y-values of $g(x) = \frac{1}{5}\cos(x)$ are one-fifth of the y-values for $y = \cos(x)$.

c. The tangent function always requires special consideration because it is undefined at certain quadrantal angles. Therefore, our approach from parts (a) and (b) needs to be modified. Again, this is where having a strong grasp of the base graph for the tangent function comes into play. Since the natural period of the tangent is π, we will plot coordinates for $x = 0, \pi/4, 3\pi/4,$ and π (completely avoiding the domain issue by skipping over $x = \pi/2$).

x	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π
$h(x)$	0	π	$-\pi$	0

Plotting these points and graphing the base tangent function, we get the following:

Figure 8.2.7



Again, $h(x) = \pi\tan(x)$ is a vertical stretch of the base graph by a factor of π.[6]

Example 8.2.1 verifies that our experience in Algebra still holds in Trigonometry; however, we artificially restricted the values of A to be positive. What happens if $A < 0$? Again, from Algebra, we know the graph of $y = -g(x)$ would be a reflection of the graph of the function g about the x-axis. Does this still hold in Trigonometry?

? Checkpoint 8.2.1

Make a table of values and sketch the graph of each function. Compare your graph to the function's base graph.

a. $f(x) = -2\cos(x)$
b. $g(x) = -\frac{1}{2}\sin(x)$

Answer

a.

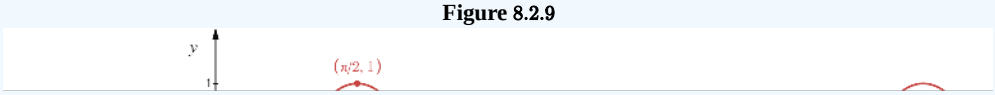
https://math.libretexts.org/@go/page/197564

Figure 8.2.8

The coefficient of -2 multiplies all y-values of the base graph by -2. This effectively causes a vertical stretch (by a factor of 2) and a reflection about the x-axis.

b.

Figure 8.2.9



The coefficient of $-1/2$ multiplies all y-values of the base graph by $-1/2$, effectively compressing the graph vertically by a factor of $1/2$ and reflecting it about the x-axis.

The following theorem summarizes these results.

Theorem: Vertical Scaling

The graph of

$$f(x) = A\,\text{trig}\,(x)$$

vertically scales the graph of $y = \text{trig}\,(x)$ by a factor of $|A|$.

- If $|A| > 1$, it is a vertical stretch by a factor of $|A|$.
- If $0 < |A| < 1$, it is a vertical compression by a factor of $|A|$.

Additionally, if $A < 0$, the graph is reflected about the horizontal axis.

In either case, the y-coordinates of points on the graph of $y = \text{trig}\,(x)$ are multiplied by A to become the new points on the graph of $f(x) = A\,\text{trig}\,(x)$.

Note: Vertical Scaling Changes the Range

Except for the tangent function (whose range is all real numbers), the range of a trigonometric function will change under vertical scaling. For example, in Example 8.2.1*b* the range of $g(x) = \frac{1}{5}\cos(x)$ is $\left[-\frac{1}{5}, \frac{1}{5}\right]$.

When we introduced the definition of amplitude, we also mentioned that while the *base* sine and cosine functions have amplitudes of 1, the tangent does not have an amplitude because its range is unbounded. We now introduce the amplitude of *any* sinusoidal function.

Example 8.2.2

Compute the amplitude of each function.

a. $f(x) = 3\sin(x)$
b. $g(x) = \frac{1}{5}\cos(x)$

Solutions

a. From Example 8.2.1*a*, we see the greatest and least values of the graph are 3 and -3, respectively. Thus, the amplitude of $f(x) = 3\sin(x)$ is

$$\text{Amplitude} = \frac{1}{2}|3 - (-3)| = \frac{1}{2}|6| = 3.$$

b. From Example 8.2.1*b*, we see the greatest and least values of the graph are $1/5$ and $-1/5$, respectively. Thus, the amplitude of $g(x) = \frac{1}{5}\cos(x)$ is

$$\text{Amplitude} = \frac{1}{2}\left|\frac{1}{5} - \left(-\frac{1}{5}\right)\right| = \frac{1}{2}\left|\frac{2}{5}\right| = \frac{1}{5}.$$

Example 8.2.2 provides an observation that will allow us to state the amplitude of a sinusoidal function quickly.

Theorem: Amplitude of a Sinusoidal Function

The amplitude of $f(x) = A\cos(x)$ or $g(x) = A\sin(x)$ is $|A|$.

Checkpoint 8.2.2

State the amplitude of each function.

a. $f(x) = -2\cos(x)$
b. $g(x) = -\frac{1}{2}\sin(x)$
c. $h(x) = 12\tan(x)$

Answer

a. **2**

b. $\frac{1}{2}$

c. Trick question... the tangent function does not have an amplitude.

If you are using the digital version of this textbook, feel free to use the following Interactive Element to help solidify the concept of amplitude and the effect of A on the graph of $f(x) = A\sin(x)$.

Period

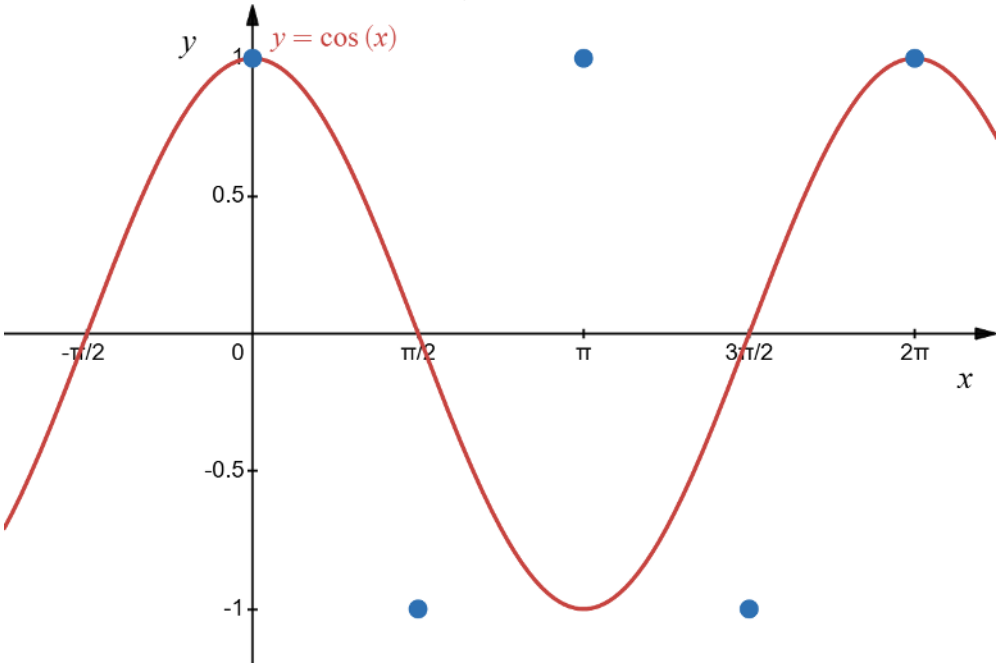
The next parameter in Equation 8.2.1 is B. Very technically, you should have already encountered graphs of functions of the form $y = g(Bx)$ in Algebra, but I find that many people are confounded by what is truly happening and, very honestly, you don't see the true effect of that transformation until Trigonometry. Therefore, we are going to cover it in detail here. To determine this parameter's effects on the trigonometric function's graph, we take an example and build a table of values.

Suppose we are asked to graph $y = \cos(2x)$. By building a table of values using the quadrantal angles, we get the following:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos(2x)$	1	-1	1	-1	1

Graphing the base cosine curve along with these points yields Figure 8.2.10.

Figure 8.2.10

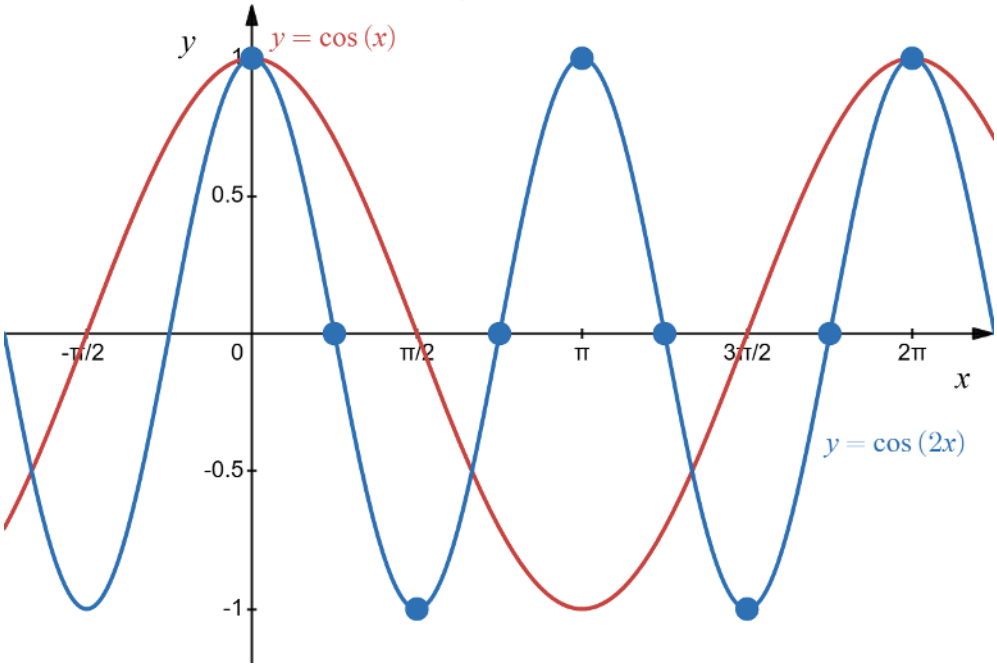


The issue should be readily apparent - we need more points to plot. The quadrantal angles provide too coarse of a picture. Let's add some values to our table.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \cos(2x)$	1	0	-1	0	1	0	-1	0	1

Plotting these additional points (and connecting them with a nice sinusoidal curve), we get the following figure.

Figure 8.2.11



Before moving on, we need to mention a few things about Figure 8.2.11. First, recall that the principal cycle of the cosine function is $[0, 2\pi]$. This means that standard cosine function goes through a single cycle on that interval; however, $y = \cos(2x)$ goes through *two* complete cycles on the same interval. This is a beautiful way of thinking about the effect of B on the graph of $y = A\,\text{trig}\,(Bx)$. B tells us how many "cycles" of the graph get squeezed into the principal cycle of the base function. Another interpretation is that B tells us how many times faster the curve is "oscillating" than the base curve.[7] In this case, $y = \cos\,(2x)$ oscillates twice as fast as $y = \cos\,(x)$. This being said, we can safely conjecture that B modifies the function's period (how long it takes to complete one cycle).

The second item worth mentioning is that you will usually only be asked to graph a single cycle of a trigonometric function, and this single cycle will be based on the principal cycle. Hand-drawn (and computer-generated) graphs look crowded when more than one cycle is sketched. In general, your instructor will know you just copied a computer-generated graph of a given trigonometric function if you include several cycles.

This brings us to our third and most important observation. For any trigonometric function, we only need five special x-values to obtain an accurate sketch of a single cycle (again, based on the principal cycle). For simplicity, we label these x_0, x_1, x_2, x_3, and x_4. These are

- x_0**:** the beginning x-value
- x_4**:** the ending x-value
- x_2**:** the value of x directly between the beginning and ending x-values
- x_1 **and** x_3**:** the two values of x between the beginning and middle, and middle and ending x-values

As they say, a picture is worth a thousand words, so let's visualize this using $y = \cos\,(2x)$.

Figure 8.2.12

Each of the x-values in Figure **8.2.12** is called a **key numbers** of a single graph cycle.

Definition: Key Numbers

The **key numbers** of a cycle for a trigonometric function are the x-values of the of the initial, final, middle, and quarter-points in the principal cycle. These can be found by partitioning the cycle into four equally-spaced subintervals.

To showcase how the key numbers help graph trigonometric functions, suppose we are asked to graph the principal cycle of the base cosine function, $y = \cos(\theta)$. Again, the principal cycle for the cosine function is $0 \leq \theta \leq 2\pi$. Partitioning this interval into four equally-spaced subintervals requires taking the length of this interval (2π) and dividing by four to get

$$\frac{2\pi}{4} = \frac{\pi}{2}.$$

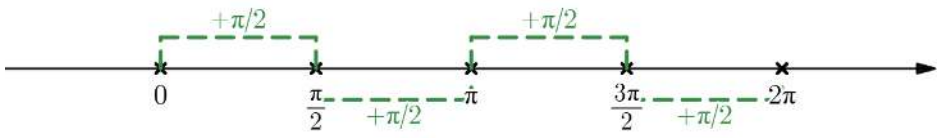
We will call the length of this subinterval the **step size** for this function.

Definition: Step Size

The distance between any two consecutive key numbers is called the **step size**.

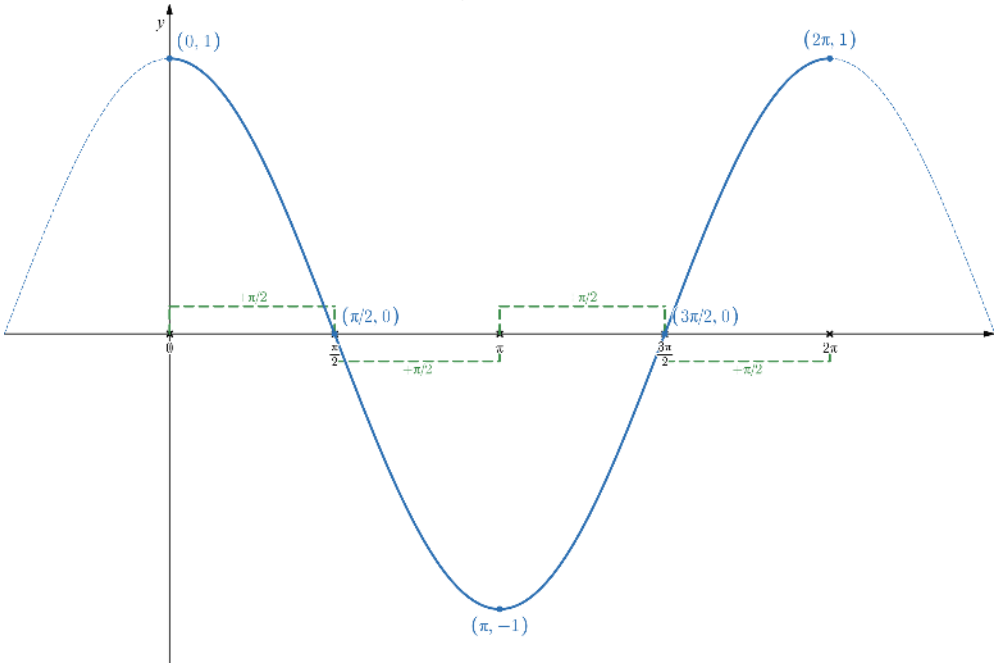
To graph, we mark the first x-value in the *principal cycle*, which is $x = 0$. Hence, $x_0 = 0$. From there, we "mark by steps" until we reach the end of the single cycle (see the figure below).

Figure 8.2.13



Since we know the basic shape of the cosine graph, we can complete the graph using these key numbers.

Figure 8.2.14



Before going into an example, let's amend a definition we created in the previous section. We know that the *natural period* of the cosine function (and sine function) is 2π; however, what is the period of the modified function $y = \cos(2x)$?

Recall that a single cycle of $y = \cos(\theta)$ occurs on $0 \le \theta \le 2\pi$. Replacing θ with $2x$, we have $0 \le 2x \le 2\pi$. Dividing all three sides of this inequality by 2, we get $0 \le x \le \pi$. That is, $y = \cos(2x)$ goes through a single cycle on the interval $[0, \pi]$. Hence, the period of $y = \cos(2x)$ is π. Generalizing this observation leads us to the following theorem.

Theorem: Period of a Trigonometric Function

Let p be the natural period of a trigonometric function. Then the period of $y = A\,\text{trig}(Bx)$ is $\frac{p}{B}$.

Thus, the period of $y = A\cos(Bx)$ is $2\pi/B$, and the period of $y = A\tan(Bx)$ is π/B (remember, the natural period of the tangent is π).

While we are at it, let's create a formula to help us compute the step size.

Theorem: Step Size Formula

The step size for a trigonometric function is $\frac{\text{Period}}{4}$.

The step size simplifies the process of finding the key numbers. By starting with the first key number, x_0, we add the step size to arrive at the second key number, x_1. To this, we add the step size to get the third key number, x_2. And so on.

We are overdue for some examples.

Example 8.2.3

Sketch a single cycle of each function, stating its amplitude (if applicable), period, and step size.

a. $f(x) = \cos(2x)$
b. $g(x) = -3\sin\left(\frac{x}{3}\right)$
c. $y = \frac{1}{5}\tan(-4x)$

Solutions

a. The process we illustrate here (and that we build on moving forward) is a bit algorithmic but very helpful in graphing trigonometric functions. It's based on the "template" trigonometric function

$$y = A\,\text{trig}\,(Bx + C) + D.$$

In this section, $C = D = 0$, so we are only dealing with functions of the form

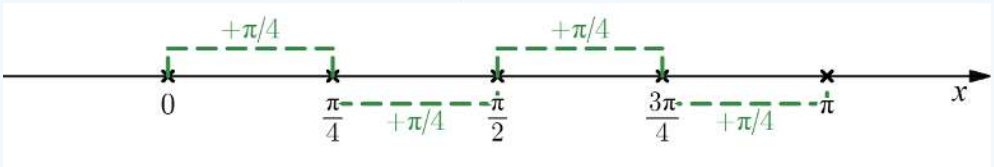
$$y = A\,\text{trig}\,(Bx).$$

Specifically, for this problem, we have $f(x) = 1\cos(2x)$. We read this from left to right.

A	$=$	1	$\Longrightarrow$	Amplitude	$=$	1
B	$=$	2	$\Longrightarrow$	Period	$=$	$\dfrac{\text{natural period}}{B} = \dfrac{2\pi}{2} = \pi$
				Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi}{4}$

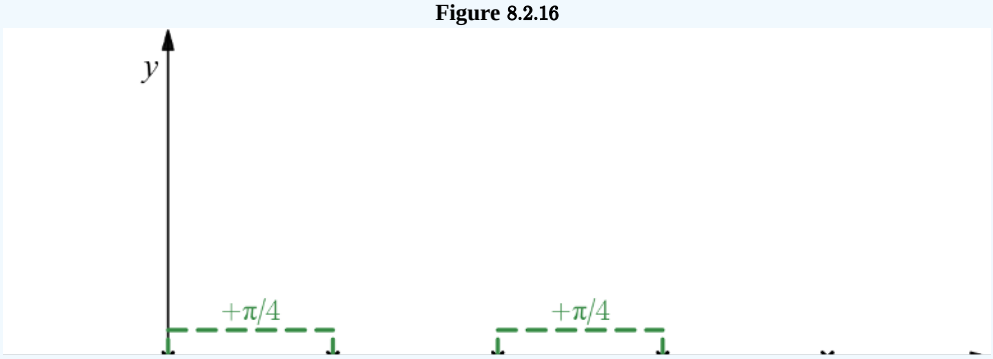
We can now sketch an accurate graph of a single cycle. The initial x-value in the principal cycle of all trigonometric functions is $x = 0$. To find the next key number, we add the step size. See the figure below.

Figure 8.2.15



The length of this interval is π, which matches the period, so we are on track for success. Since the initial key number is at $x = 0$, we now know where the y-axis is, so let's draw that in.

Figure 8.2.16



The function we are graphing is the *positive* cosine, so we can fill in the basic shape of a single cycle.

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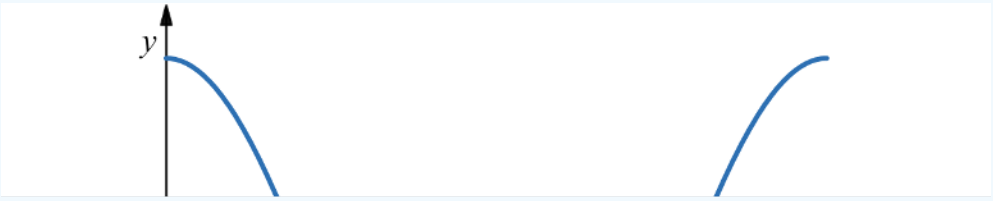
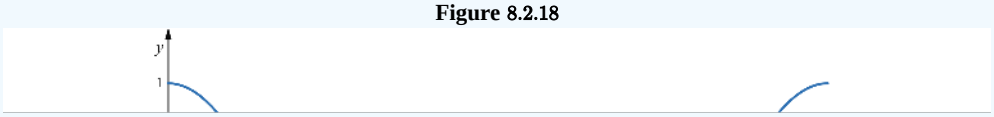
Figure 8.2.17



Finally, we use the amplitude to label the y-axis. Since the amplitude is 1, the maximum value is 1 and the minimum value of the function is -1.

Figure 8.2.18



I know I said that, unless asked, you should refrain from graphing more than a single cycle, however, it is nice to let your reader know that this pattern is cyclic. We showcase this by extending the graph for a short distance in both directions.

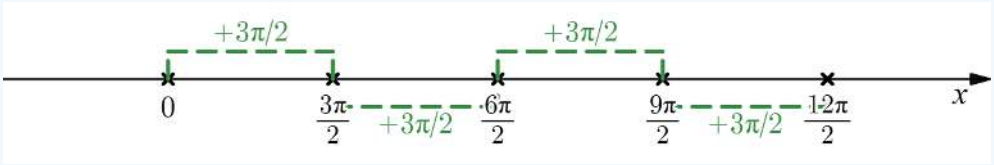
Figure 8.2.19

b. We have $g(x) = -3\sin\left(\frac{x}{3}\right) = -3\sin\left(\frac{1}{3}x\right)$. Reading this from left-to-right, we get the following.

A	$=$	-3	$\Longrightarrow$	Amplitude	$=$	3
B	$=$	$\frac{1}{3}$	$\Longrightarrow$	Period	$=$	$\frac{\text{natural period}}{B} = \frac{2\pi}{1/3} = 6\pi$
				Step Size	$=$	$\frac{\text{Period}}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Remembering that the initial x-value for the principal cycle is $x = 0$, we find all of the key numbers and label the x-axis.

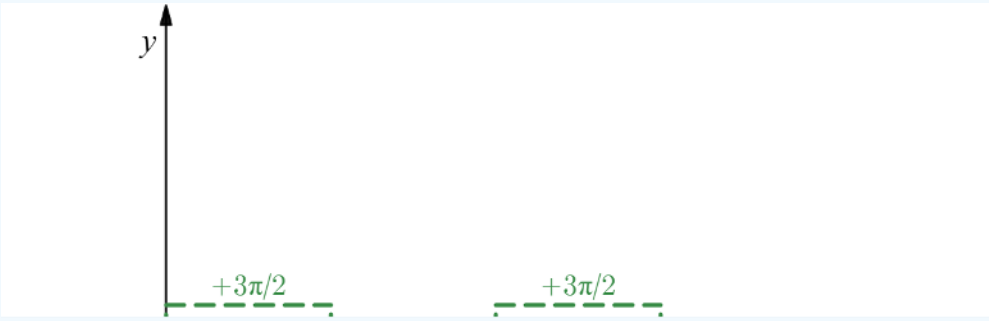
Figure 8.2.20



Again, we see the length of this interval is $\frac{12\pi}{2} = 6\pi$, which is the period. Therefore, we must be doing something right! Placing the y-axis appropriately, we get the following.

https://math.libretexts.org/@go/page/197564

Figure 8.2.21



Here is where paying attention to the original function is essential (technically, paying attention is always important) - the amplitude of the function is 3, but the sine function has been reflected about the x-axis because $A < 0$. Therefore, we label the y-axis with the maximum and minimum values of 3 and -3 but draw our sine function reflected about the x-axis.

Figure 8.2.22

c. The tangent function might seem like the oddball of the trigonometric functions. Still, the process of sketching a single cycle remains the same, except that the tangent does not have an amplitude. One difference you should notice with this specific example is that the coefficient on x is *negative*. This is where remembering the Symmetry Identities pays off. Since tangent is an odd function,

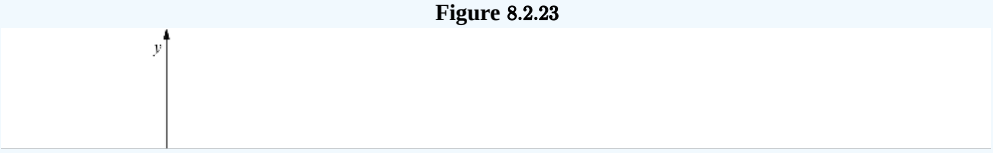
$$y = \frac{1}{5}\tan(-4x) = -\frac{1}{5}\tan(4x).$$

Now that we have taken care of the negative argument in the function, we get to work. It is essential to remember that the natural period of the tangent function is π.

$B = 4 \implies$	Period $=$	$\dfrac{\text{natural period}}{B} = \dfrac{\pi}{4}$
	Step Size $=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi/4}{4} = \dfrac{\pi}{16}$

Again, the initial x-value in the principal cycle is $x = 0$. To find the next key number, we add the step size. See the figure below.

Figure 8.2.23



We know what the base graph of the tangent looks like, so we know the beginning and ending key numbers (0 and $4\pi/16$) are going to be the zeros of the function and the middle key number ($2\pi/16$) is the vertical asymptote; however, what do we do with $\pi/16$ and $3\pi/16$? Up to this point, we have not bothered with the coefficient of the function, $-1/5$. Here is where it comes into play. If we were to evaluate the given function at $\pi/16$, we would get

$$-\frac{1}{5}\tan\left(4\left(\frac{\pi}{16}\right)\right) = -\frac{1}{5}\tan\left(\frac{\pi}{4}\right) = -\frac{1}{5}(1) = -\frac{1}{5}.$$

That is, the excess key numbers, $\pi/16$ and $3\pi/16$, give us two convenient points to plot on the graph, and the value at these points is $\pm A$. Sketching a tangent function that has been reflected about the x-axis and plotting those two points, we get the following graph.

Figure 8.2.24

Did you notice I didn't bother simplifying fractions when I labeled the key numbers in Examples **8.2.3***b* and **8.2.3***c*? This was purposeful and is a technique we will use moving forward.

Note: Changes in the Period Affect the Domain of the Tangent

While the domain of the sine and cosine functions remain all real numbers no matter the period, the tangent's domain changes when we change the period. For example, in Example **8.2.3***c* the domain of the tangent does not include $\frac{\pi}{8}$ (whereas, in the base tangent function, $\frac{\pi}{8}$ is in the domain). This is because a change in the period of a function causes **horizontal scaling** (stretching or compressing along the horizontal axis). When this scaling occurs, any vertical asymptotes the function might have also move.

Checkpoint 8.2.3

Sketch a single cycle of $y = -\frac{1}{2}\cos(7x)$, stating its amplitude, period, and step size.

Answer

Amplitude: $\frac{1}{2}$
Period: $\frac{2\pi}{7}$

Step Size: $\frac{\pi}{14}$

As before, the following Interactive Element (accessible through the digital version of the textbook) can help to solidify the concept of period and the effect of B on the graph of $f(x) = A\sin(Bx)$.

Summary of Graphing Trigonometric Functions Using Non-Rigid Transformations

Summary of Graphing $f(x) = \overline{A}\,\text{trig}\left(\overline{B}x\right)$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x\right):$$

Precondition Stage

We begin by cleaning up the trigonometric function and ensuring that the coefficient of x is positive. This is done using the Symmetry Identities. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\text{trig}\,(Bx).$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line, mark the initial key number as 0, and mark the remaining key numbers by successively adding the **step size** to the previous key number.

- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or
  - label the remaining key points on the graph (if the given function is the tangent).

To be clear, let's do a final example. This time, however, let's practice extending the graph as well.

Example 8.2.4

Sketch the graph of each function over the given interval.

a. $g(x) = -\frac{1}{2}\cos\left(-\frac{2}{5}x\right)$ for $-10\pi \le x \le 10\pi$

b. $h(x) = 3\tan(6x)$ for $-\frac{\pi}{12} \le x \le \frac{\pi}{2}$

Solutions

a. **Precondition Stage:** Since the coefficient of x is negative, we use the Symmetry Identities to rewrite the given function. Cosine is an even function. Therefore, $\cos(-\theta) = \cos(\theta)$. Hence, $g(x)$ can be rewritten as

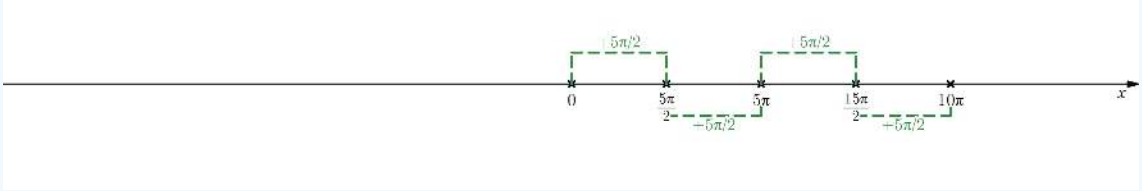
$$g(x) = -\frac{1}{2}\cos\left(\frac{2}{5}x\right).$$

Information Stage: Since the given function is sinusoidal, it has an amplitude, and its natural period is 2π.

A	$=$	$-\frac{1}{2}$	$\Longrightarrow$	Amplitude	$=$	$\lvert A\rvert = \frac{1}{2}$
B	$=$	$\frac{2}{5}$	$\Longrightarrow$	Period	$=$	$\frac{\text{Natural Period}}{B} = \frac{2\pi}{2/5} = 10\pi$
			$\Longrightarrow$	Step Size	$=$	$\frac{\text{Period}}{4} = \frac{10\pi}{4} = \frac{5\pi}{2}$

Graphing Stage: Drawing a horizontal line (the x-axis), marking the initial key number ($x = 0$), and marking the remaining key numbers using the step size, we get the following.

Figure 8.2.25

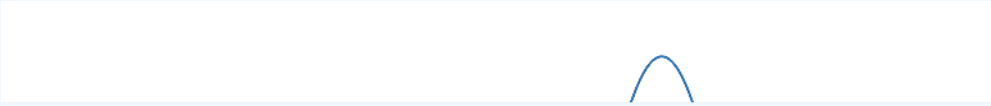


Notice that I left quite a bit of open space on the left half of the number line. This is because we will eventually extend the single cycle of our cosine function so that our graph covers the interval $[-10\pi, 10\pi]$. We take a moment to make sure the distance from the initial key number ($x = 0$) to the final key number ($x = 10\pi$) is the same as the period. In this case, the distance is

$$|10\pi - 0| = 10\pi = \text{Period},$$

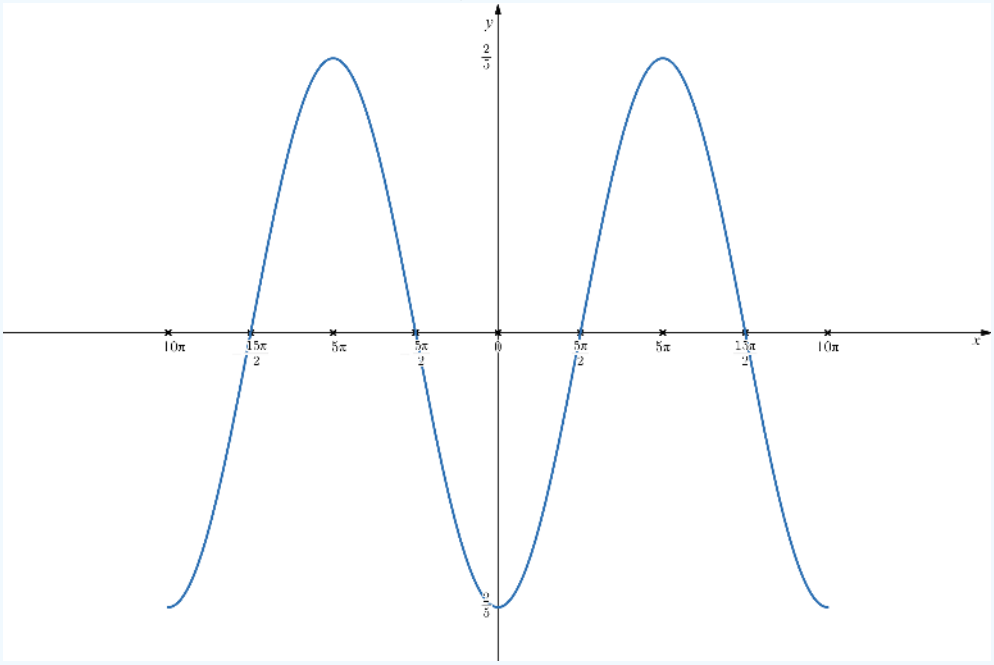
so we are on the right track. We now sketch a single cycle of the *reflected* cosine (because $A < 0$).

Figure 8.2.26



Finally, we draw in the y-axis at $x = 0$, label the maximum and minimum values using the amplitude, and extend the graph to cover the interval $[-10\pi, 10\pi]$.

Figure 8.2.27



b. **Precondition Stage:** Since the coefficient of x is already positive, we do not need to clean up the trigonometric function.

Information Stage: Since h is a tangent function, it does not have an amplitude, and its natural period is π.

$B = 6$	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B}$	$= \dfrac{\pi}{6}$
	$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi/6}{4}$	$= \dfrac{\pi}{24}$

Graphing Stage: Starting at $x = 0$ and marking the key points, we get the following number line (which has been extended because of the interval over which we are asked to graph).

Figure 8.2.28

Note that the distance between the beginning key number and the ending key number is

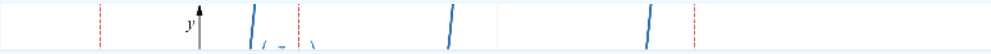
$$\left|\frac{4\pi}{24} - 0\right| = \frac{4\pi}{24} = \frac{\pi}{6} = \text{Period}.$$

Thus, we are good to go! We now graph a single cycle of the tangent function. Remember that there is a vertical asymptote in the middle of the cycle (at $x = 2\pi/24$).

Figure 8.2.29

From here, we draw the y-axis at $x = 0$, label the points on the curve at $x = \pi/24$ and $x = 3\pi/24$, and extend the graph to cover the requested interval.

Figure 8.2.30



Finding Equations Given Graphs

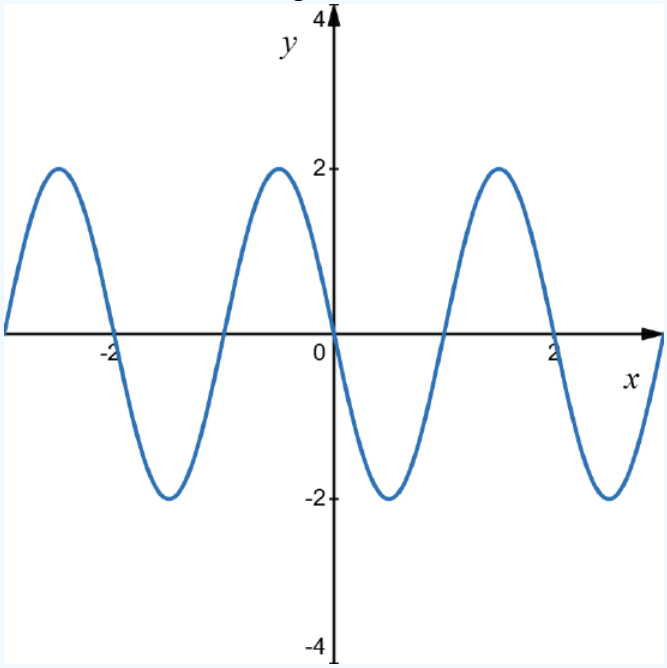
We wrap up this section by inverting our process. Instead of graphing a function given its equation, we find the equation for a function given its graph. This process is similar to what we introduced in the summary above.

Example 8.2.5

Find the equation of the graphed function.

a.

Figure 8.2.31



b.

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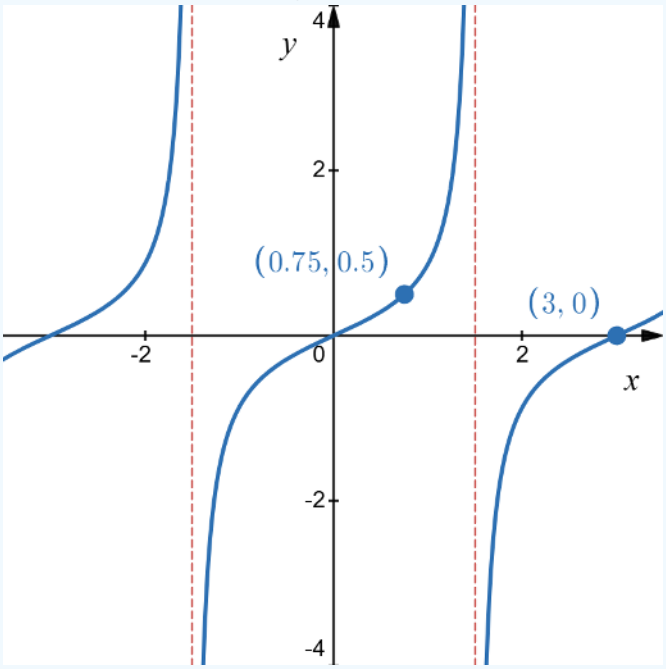
Figure 8.2.32



Solutions

a. Our first task is to identify the base function. The shape of this function is sinusoidal. Moreover, because it goes through the origin, it is a sine function (although, technically, it can be written as a cosine function as well). However, it has been reflected about the x-axis, so we know that $A < 0$. We can also see that the amplitude is 2. Thus, $A = -2$ so that

$$f(x) = -2\sin(Bx).$$

All we need to do is identify B. We know that B is *associated* with the period of the function, but it is not *equal* to the period. The relationship between B and the period is

$$\text{Period} = \frac{\text{Natural Period}}{B}.$$

However, the natural period of the sine function is 2π, so

$$\text{Period} = \frac{2\pi}{B} \implies B = \frac{2\pi}{\text{Period}}.$$

The graph shows that the function completes one full cycle every 2 units. Hence, the period is 2. Substituting that into our formula, we get

$$B = \frac{2\pi}{2} = \pi.$$

Thus, our function is

$$f(x) = -2\sin(\pi x).$$

b. The base function here is the tangent. So we know we are dealing with a function of the form

$$g(x) = A\tan(Bx).$$

The value of A will be a bit trickier to get here because the tangent does not have an amplitude; however, all is not lost. We instead start by finding B. As in part (a),

$$\text{Period} = \frac{\text{Natural Period}}{B}$$

and we know the natural period of the tangent function is π. Therefore,

$$\text{Period} = \frac{\pi}{B} \implies B = \frac{\pi}{\text{Period}}.$$

We can see that a single cycle gets completed in 3 units; thus, the period of this function is 3. Hence,

$$B = \frac{\pi}{3}.$$

Now we have

$$g(x) = A\tan\left(\frac{\pi}{3}x\right).$$

To find the value of A, we recall that any point on the graph of an equation must also satisfy the equation. In particular, because the point $\left(\frac{3}{4}, \frac{1}{2}\right)$ lies on this tangent curve, when $x = \frac{3}{4}$, it must be the case that $y = \frac{1}{2}$. That is,

$$\frac{1}{2} = A\tan\left(\frac{\pi}{3}\cdot\frac{3}{4}\right) \implies \frac{1}{2} = A\tan\left(\frac{\pi}{4}\right) \implies \frac{1}{2} = A\cdot 1 \implies A = \frac{1}{2}.$$

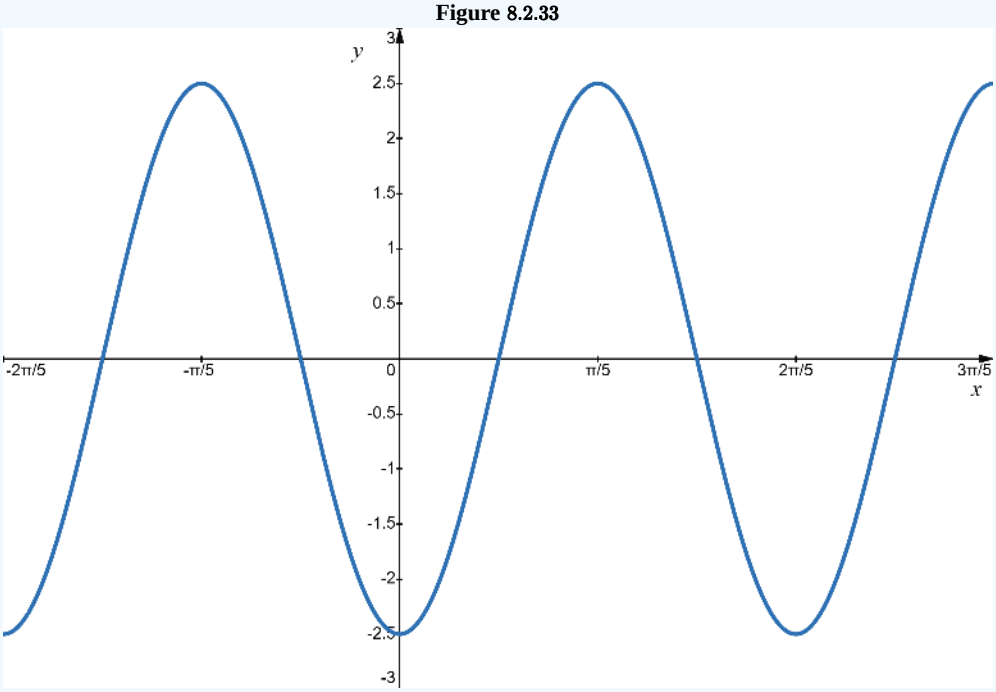
Thus, the equation of this tangent function is

$$g(x) = \frac{1}{2}\tan\left(\frac{\pi}{3}x\right).$$

? Checkpoint 8.2.5

Find the equation of the function graphed below.

Figure 8.2.33



Answer

$$f(x) = -\frac{5}{2}\cos(5x)$$

Footnotes

[1] If you need more of a review of transformations of functions than what we cover here, it's a good idea to make your instructor aware.

[2] Technically, this is *not* a function (Sophie doesn't pass the Vertical Line Test); however, using an image usually conveys the point of transformations to students more efficiently than traditional functions.

[3] Again, it is imperative that you have already had total exposure to these concepts from Algebra. What we present here is *not* instruction but a reminder (with a cute dog).

[4] However, we will stay with the fundamental trigonometric functions for a while until we get used to transformations of trigonometric functions.

[5] Some authors and instructors will say, "$g(x)$ is a compression of the graph of $y = \cos(x)$ by a factor of 5." I have always found this language a bit confusing. Therefore, I will adhere to the convention in this textbook of stating "a vertical compression by a factor of $\frac{1}{5}$." The fact that the word "compression" is involved gives the context that the y-values of the original function are being "squished" to make them smaller than before.

[6] Don't let the coefficient π fool you - it's just a number greater than 1 (approximately 3.142).

[7] I placed "oscillating" in quotes because this interpretation doesn't work for the tangent function. That is, $y = \tan(2x)$ doesn't oscillate twice as fast as the base tangent function because the tangent doesn't oscillate.

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8.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Graphing Functions**
  - Transformations: This section is entirely about applying non-rigid transformations (stretches, compressions, and reflections) to the base graphs of trigonometric functions. A prior understanding of how parameters alter a base graph is essential.

VIDEOS

- Transformations
  - Trigonometric Graphs - Review of Graphs of Sine and Cosine Functions
  - Trigonometric Graphs - Review of Identifying the Equation for a Sinusoidal Function from a Graph
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions
  - Graphs of the Other Trigonometric Functions - Identifying Equations of Trig Functions from Graphs

KEY CONCEPTS

DEFINITIONS

- **Transformation (of the graph of a function):** An alteration of a graph's orientation, scale, and/or position through various mathematical operations.
- **Rigid Transformation:** A transformation that changes a graph's position (e.g., shifts) but keeps its original orientation and scale.
- **Non-rigid Transformation:** A transformation that keeps the graph's original center position but changes its scale (stretching or compressing) or orientation (reflecting).
- **Parameter:** A variable that influences the output or behavior of a function but is held constant within a specific context. In this section, A and B in $y = A\text{trig}(Bx)$ are parameters.
- **Key Numbers:** The x-values of the initial, final, middle, and quarter-points of a single cycle of a trigonometric function's graph.
- **Step Size:** The horizontal distance between any two consecutive key numbers.

THEOREMS

- **Vertical Scaling:** The graph of $f(x) = A\text{trig}(x)$ vertically scales the graph of $y = \text{trig}(x)$ by a factor of $|A|$. If $|A| > 1$, it is a vertical stretch. If $0 < |A| < 1$, it is a vertical compression. If $A < 0$, the graph is also reflected about the horizontal axis.
- **Amplitude of a Sinusoidal Function:** The amplitude of $f(x) = A\cos(x)$ or $g(x) = A\sin(x)$ is $|A|$.
- **Period of a Trigonometric Function:** If p is the natural period of a trigonometric function, then the period of $y = A\text{trig}(Bx)$ is $\frac{p}{|B|}$. For sine and cosine, the period is $\frac{2\pi}{|B|}$. For tangent, the period is $\frac{\pi}{|B|}$.
- **Step Size Formula:** The step size for graphing a trigonometric function is given by the formula $\frac{\text{Period}}{4}$.

COMMON MISTAKES

- **Confusing Period and the Parameter B:** The period of a function like $y = \sin(Bx)$ is not B. The period is calculated by dividing the natural period of the base function by $|B|$.
- **Using the Wrong Natural Period:** The natural period of sine and cosine is 2π, while the natural period of tangent is π. Using the wrong natural period when calculating the new period is a common error.
- **Ignoring a Negative Coefficient on the Angle:** When graphing a function like $y = \tan(-4x)$, you must first use the Symmetry Identities to handle the negative. Since tangent is an odd function, $y = \tan(-4x) = -\tan(4x)$. Graphing without this first step will lead to an incorrect orientation.
- **Forgetting Tangent Has No Amplitude:** The concept of amplitude applies to sinusoidal functions (sine and cosine) because they have maximum and minimum values. The tangent function is unbounded, so it does not have an amplitude.

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8.2.2: HOMEWORK

READING QUESTIONS

1. What is the difference between a rigid and a non-rigid transformation?
2. In the function $y = A\sin(Bx)$, what effect does the parameter A have on the graph? What happens if A is negative?
3. In the function $y = A\cos(Bx)$, what effect does the parameter B have on the graph?
4. What is the amplitude of the function $y = -5\cos(x)$?
5. What is the period of the function $y = \sin(3x)$?
6. What is the period of the function $y = \tan(2x)$?
7. What are the five "key numbers" used to sketch one cycle of a trigonometric graph?
8. How do you calculate the "step size" for a graph?
9. When graphing $y = \cos(-2x)$, what property of the cosine function allows you to rewrite it first?
10. Why does the tangent function not have an amplitude?

HOMEWORK

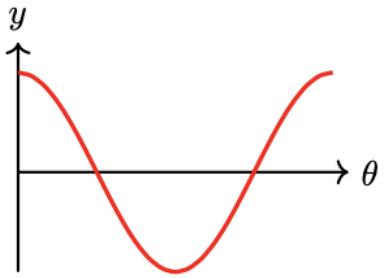
VOCABULARY CHECK

1. To ___ the graph of a function, we mean to stretch or compress the graph horizontally or vertically.
2. A ___ transformation keeps the function's original scale and orientation but changes its position.
3. A ___ transformation keeps the function's original central position but changes its scale or orientation.
4. If a graph is symmetric about the y-axis, we say it is an ___ function.
5. An___ function is one whose graph is symmetric about the origin.
6. If $A > 1$, then the graph of $y = A\sin(Bx)$ is ___ vertically by a factor of ___.
7. If $0 < A < 1$, then the graph of $y = A\tan(Bx)$ is ___ vertically by a factor of ___.

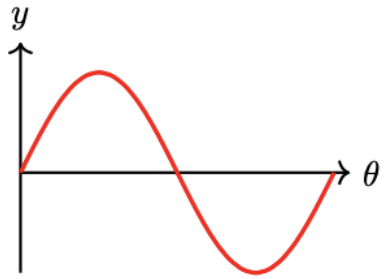
CONCEPT CHECK

8. What is the range of $f(x) = A\sin(Bx)$?
9. What is the effect of $A < 0$ on the graph of the function $g(x) = A\tan(Bx)$?
10. What is the period of $h(x) = A\cos(Bx)$?
11. What is the period of $k(x) = A\tan(Bx)$?
12. When graphing $y = \cos(-Bx)$, we first rewrite the function as $y =$ ___ because cosine is an ___ function.
13. When graphing $y = \tan(-Bx)$, we first rewrite the function as $y =$ ___ because tangent is an ___ function.
14. One of these graphs is $y = A\sin k\theta$, and the other is $y = A\cos k\theta$. Explain how you know which is which.

a.



b.



True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

15. The natural period of the cosine function is 2π.
16. The natural period of the tangent function is 2π.
17. Dividing the period by 4 gives us the step size.
18. Dividing the period of the function by 4 partitions a single cycle of the trigonometric function into four subintervals.
19. There are four key numbers.

BASIC SKILLS

20. Write an equation for a sine function with amplitude 6.
21. Write an equation for a cosine function with amplitude $\frac{1}{2}$.
22. Write an equation for a sine function with period $\frac{\pi}{2}$.
23. Write an equation for a cosine function with period 4π.

For the following exercises, state the amplitude (if applicable) and period of the graph.

24. $y = -\cos(4x)$
25. $y = -\sin(3x)$
26. $y = -5\sin\left(\frac{x}{3}\right)$
27. $y = 6\cos\left(\frac{x}{2}\right)$
28. $y = \tan(2x)$
29. $y = \tan(4x)$

For the following exercises,
(i) Graph the function.
(ii) State the amplitude and period of the function.

30. $y = 3\cos(\theta)$

31. $y = 4\sin(\theta)$

32. $y = \cos(3\theta)$

33. $y = \sin(2\theta)$

For the following exercises, sketch one cycle of each graph by hand and label scales on the axes.

34.

a. $y = \cos x$

b. $y = \cos 4x$

c. $y = -\cos x$

35.

a. $y = \sin x$

b. $y = \sin 3x$

c. $y = -\sin 3x$

36.

a. $y = \sin x$

b. $y = \sin \frac{x}{3}$

c. $y = -5\sin \frac{x}{3}$

37.

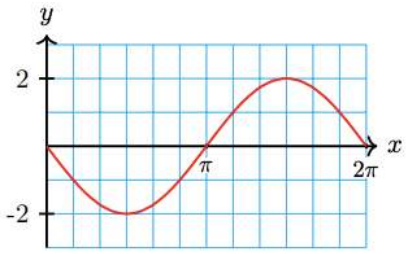
a. $y = \cos x$

b. $y = \cos \frac{x}{2}$

c. $y = 6\cos \frac{x}{2}$

For the following exercises, write an equation for the graph using sine or cosine.

38.



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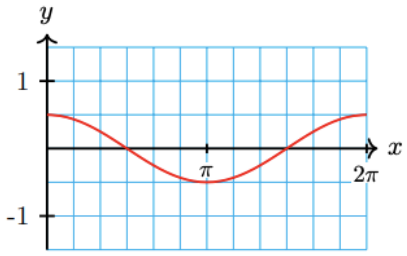

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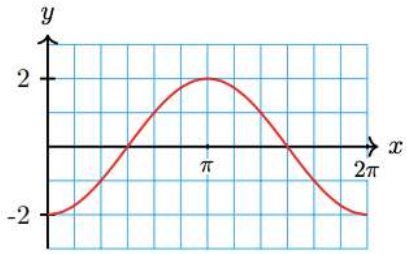

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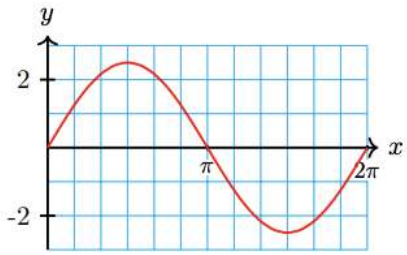
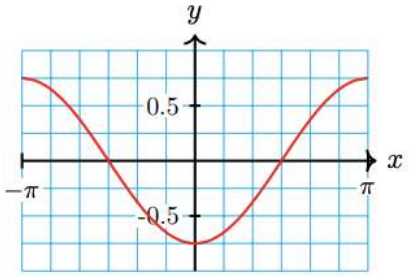


42.



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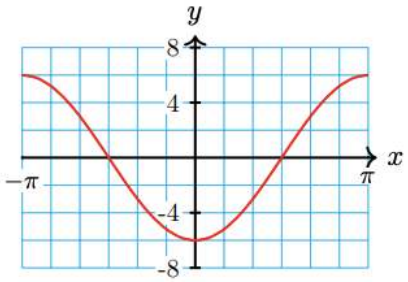
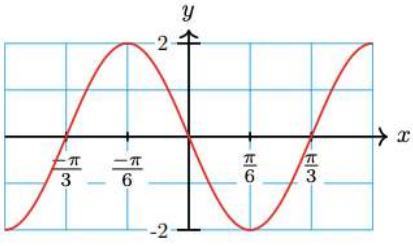


For the following exercises,
(i) State the graph's amplitude and period.
(ii) Write an equation for the graph using sine or cosine.

44.

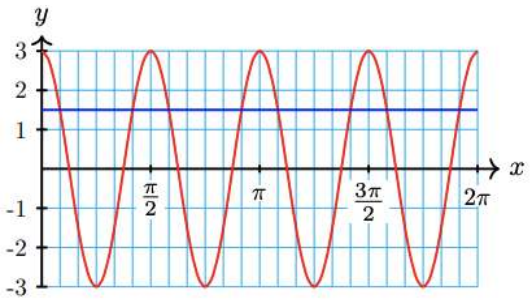


45.

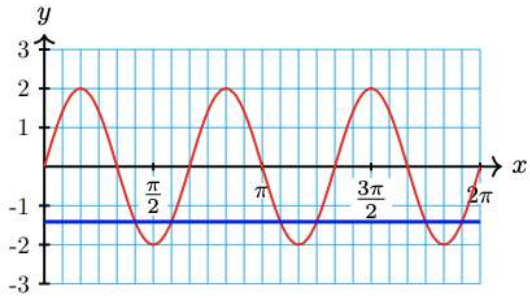
SYNTHESIS QUESTIONS

For the following exercises, use the graph to find all solutions between 0 and 2π.

46. $3\cos(4x) = 1.5$



47. $2\sin(3x) = -\sqrt{2}$



For the following exercises,
(i) Use technology to graph the function for $0 \le x \le 2\pi$.
(ii) Use the graph from part (i) to approximate all solutions between 0 and 2π. Round your answers to hundredths.

48.

a. $f(x) = 3\sin(2x)$

b. $3\sin(2x) = -1.5$

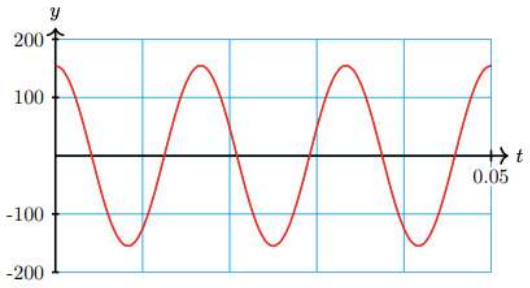
49.

a. $g(x) = -2\cos(3x)$

b. $-2\cos(3x) = 1$

APPLICATIONS

50. **Electric Current.** The following graph illustrates that the voltage used in U.S. electrical current changes from 155V to -155V and back 60 times each second. Write an equation for this graph.



51. **Electric Current.** If I represents the intensity of the current in an alternating circuit and t represents time, then

$$I(t) = 50\sin(120\pi t),$$

where I is measured in amperes and t is measured in seconds. Find the maximum value of I and the time it takes for I to go through one complete cycle.

52. **Sound Waves.** The oscillations in air pressure representing the sound wave for a tone at the standard pitch of C can be modeled by the equation

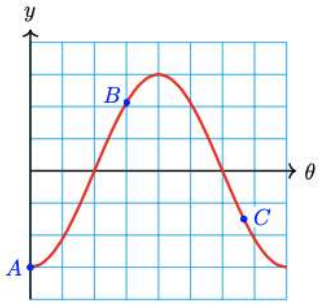
$$S(t) = 0.05\sin(524\pi),$$

where S is the sound pressure in pascals after t seconds. Sketch a single cycle of the graph of this function.

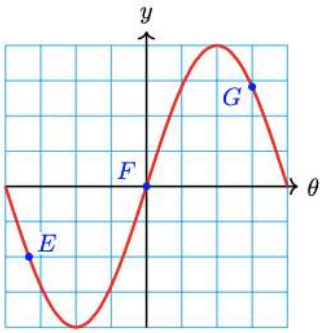
CHALLENGE PROBLEMS

For the following exercises, give the coordinates of the points on the graph.

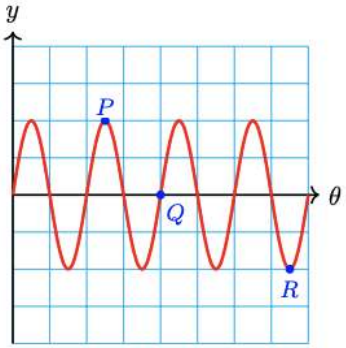
53. $f(\theta) = -3\cos(\theta)$



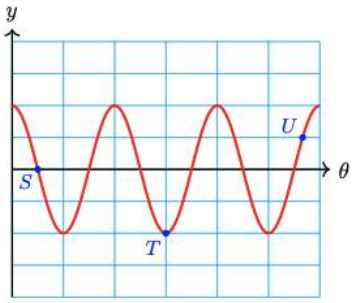
54. $f(\theta) = 4\sin(\theta)$



55. $f(\theta) = \sin(4\theta)$



56. $f(\theta) = \cos(3\theta)$



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8.3: Rigid Transformations

Note to the Instructor - Don't Let the Name Fool You

While the rigid transformations are introduced in this section, the majority of the instruction will likely be focused on graphing trigonometric functions requiring *all* transformations. There isn't another section on graphing transformations of the fundamental trigonometric functions, so make sure students really have practice graphing lots of transformations here.

Learning Objectives

- Graph fundamental trigonometric functions having vertical transformations.
- Graph fundamental trigonometric functions having horizontal transformations.
- Graph fundamental trigonometric functions having combinations of all transformations.
- Given the graph, write the corresponding trigonometric function.
- Evaluate given sinusoidal models for applications.
- Build a model for an application based on the graph.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

In the previous section, we stated that our ultimate goal (when it comes to graphing the trigonometric functions) is to graph any trigonometric function of the form

$$f(x) = A\,\text{trig}\,(Bx + C) + D. \tag{8.3.1}$$

Now that we know the effect of the parameters A and B on the graph of the fundamental trigonometric functions, it's time to turn our attention to the parameters C and D.

Phase Shifts (also known as Horizontal Shifts)

Continuing with alphabetical order, let's focus on the effect of the parameter C; however, let's make things simple (for now). Recall[1], the graph of the function $y = g(x + C)$ is a horizontal shift of the graph of g. It moves the graph to the *left* C units if $C > 0$, and *right* C units if $C < 0$.[2] Most students in Algebra accept this as truth and memorize the result; however, as you are taking a college-level Trigonometry course, it might serve you well to understand *why* this is true.

Click here to delve into why the graph of $y = g(x + C)$ is a horizontal shift of the graph of $g(x)$

Well, hello there! I am glad you clicked me! This means you are a natural science person because you investigate truths. Let's start with a simple statement.

Let $g(x)$ be a function.

Pretty simple, right?

Because g is a function, each input leads to a single output. Let's grab an *arbitrary* value from the domain of g and call it x_1 (you can call it "happy face" or "star," but x_1 is easier to write). Again, since g is a function, $g(x_1)$ must return a single value. Let's call this output y_1.

So far, all we have said is that g is a function and that $g(x_1) = y_1$; however, this last statement means the point $(x_1, g(x_1))$ is on the graph of g.

Now, your friend comes along and says, "Hey, I need to graph $g(x+C)$. All I have been given is that C is positive."

You pause and think for a moment. Turning to your friend, you say, "I don't know what that graph looks like, but I know that when I plug in x_1 to g, I get out y_1."

Your friend smirks and says, "Okay, let's plug x_1 into $g(x+C)$ to find its output!"

You watch him work for a few moments until he gets stuck. He ends up with $g(x_1+C)$, and you look at him quizzically, "I don't know what the value of g is at x_1+C. I only know its value at x_1."

Then it hits you!

"You did it wrong! You want to evaluate $g(x_1)$. You need to let the argument of your function, which is $x+C$, be x_1."

You start doing the math,

$$x+C=x_1 \implies x=x_1-C.$$

You pause... thinking.

"So, to get the argument of $g(x+C)$ to be x_1, we need to plug in x_1-C for x."

Thinking even harder, you have the epiphany.

"With the original function, we plugged in x_1 and got out y_1. With $g(x+C)$ we needed to plug in x_1-C to get $g(x_1-C+C)$ $=g(x_1)=y_1$. That is, the original point, x_1, gets shifted to the left C units!"

Since our point, $(x_1, g(x_1))$ was arbitrary, all points on the graph of g experience the same shift. Hence, the graph of $g(x+C)$, where $C>0$, is the graph of g shifted C units to the left.

Example 8.3.1

Make a table of values and sketch a graph of $y=-2\cos\left(x+\frac{\pi}{3}\right)$.

Solution

From Algebra, we expect the graph to be shifted $\frac{\pi}{3}$ units to the left compared to the graph of $y=\cos(x)$.

As we commonly try to do, we want the argument of $\cos\left(x+\frac{\pi}{3}\right)$ (which is the expression $x+\frac{\pi}{3}$) to be quadrantal angles.

x	$x+\frac{\pi}{3}$	$\cos\left(x+\frac{\pi}{3}\right)$	$y=2\cos\left(x+\frac{\pi}{3}\right)$
	0		
	$\frac{\pi}{2}$		
	π		
	$\frac{3\pi}{2}$		
	2π		

The second column of this table is what we desire $x+\frac{\pi}{3}$ to be. For $x+\frac{\pi}{3}$ to become 0, we solve

$$x+\frac{\pi}{3}=0 \implies x=-\frac{\pi}{3}.$$

Likewise, for $x+\frac{\pi}{3}$ to become $\frac{\pi}{2}$, we solve

$$x+\frac{\pi}{3}=\frac{\pi}{2} \implies x=\frac{\pi}{6}.$$

We continue this process to fill out the first column of the table.

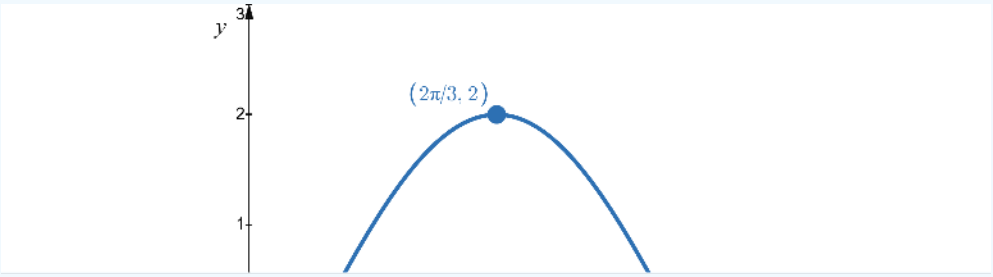
x	$x+\frac{\pi}{3}$	$\cos\left(x+\frac{\pi}{3}\right)$	$y=2\cos\left(x+\frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	0		
$\frac{\pi}{6}$	$\frac{\pi}{2}$		
$\frac{2\pi}{3}$	π		
$\frac{7\pi}{6}$	$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	2π		

Finally, we fill out the third and fourth columns using the second column as the arguments.

x	$x+\frac{\pi}{3}$	$\cos\left(x+\frac{\pi}{3}\right)$	$y=2\cos\left(x+\frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	0	1	-2
$\frac{\pi}{6}$	$\frac{\pi}{2}$	0	0
$\frac{2\pi}{3}$	π	-1	2
$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	0	0
$\frac{5\pi}{3}$	2π	1	-2

We now plot the x-values (in the first column) and their corresponding function values (in the fourth column) to make the graph. One cycle of the graph is shown below. As we expected,the graph is shifted $\frac{\pi}{3}$ units to the left compared to the graph of $y=-2\cos(x)$.

Figure 8.3.1



Checkpoint 8.3.1

Complete the table and sketch a graph of $y = -3\sin\left(x - \frac{\pi}{6}\right)$. (Do you expect the graph to be shifted to the left or the right, compared to the graph of $y = -3\sin(x)$?)

x	$x - \frac{\pi}{6}$	$\sin\left(x - \frac{\pi}{6}\right)$	$y = -3\sin\left(x - \frac{\pi}{6}\right)$
	0		
	$\frac{\pi}{2}$		
	π		
	$\frac{3\pi}{2}$		
	2π		

Answer

x	$x - \frac{\pi}{6}$	$\sin\left(x - \frac{\pi}{6}\right)$	$y = -3\sin\left(x - \frac{\pi}{6}\right)$
$\frac{\pi}{6}$	0	0	0
$\frac{2\pi}{3}$	$\frac{\pi}{2}$	1	-3
$\frac{7\pi}{6}$	π	0	0
$\frac{5\pi}{3}$	$\frac{3\pi}{2}$	-1	3
$\frac{13\pi}{6}$	2π	0	0

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Figure 8.3.2

Example **8.3.1** and Checkpoint **8.3.1** should convince you that our previous experience from Algebra with horizontal shifts of functions still works with trigonometric functions; however, let's solidify that concept to drive the point home.

We know that all the fundamental trigonometric functions have principal cycles occurring on the interval $0 \le \theta \le p$, where θ is the argument and p is the function's natural period. This means that the modified function, $f(x) = \text{trig}\,(x + C)$, goes through that same cycle when $0 \le x + C \le p$. Subtracting C from all three sides, we discover that this modified trigonometric function goes through the same single cycle on the interval $-C \le x \le p - C$. That is, the starting point for the principal cycle has been shifted from 0 to $-C$. This shift in the starting point from the principal cycle is known as the **phase shift** of the trigonometric function. This definition is so important that we should codify it.

Definition: Phase Shift

The **phase shift** of a trigonometric function is where the original x-value from the principal cycle (which is $x = 0$) has been shifted.

Moreover, we just proved the following result.

Lemma: Phase Shift Formula for $y = \text{trig}(x + C)$

The phase shift for a trigonometric function of the form $y = \text{trig}(x + C)$ is $-C$.

Thus, the phase shift for the function in Example **8.3.1** is $-C = -\frac{\pi}{3}$ and the phase shift for the function in Checkpoint **8.3.1** is $-C = -\left(-\frac{\pi}{6}\right) = \frac{\pi}{6}$.

If you're wondering why we called that previous statement a lemma instead of a theorem, it is because a lemma is a stepping stone to a bigger theorem, and the bigger theorem (which we will get to shortly) is going to supercede that lemma. Before we get there, however, we still have some work to do.

We now can expertly graph a trigonometric function of the form

$$f(x) = A\,\text{trig}\,(x + C)\,,$$

but what happens when the coefficient of x is not 1? That is, if we are given

$$f(x) = A\,\text{trig}\,(Bx + C)\,,$$

is the phase shift still $-C$?

The answer to this question is an adamant **no** (and this is why we only wrote that last conjecture as a lemma).

Using our previous derivation, the modified function, $f(x) = A\,\text{trig}\,(Bx + C)$, goes through its principal cycle when $0 \le Bx + C \le p$, where p is the natural period of the trigonometric function. Subtracting C from all three sides, we get $-C \le Bx \le p - C$. *Assuming* $B > 0$, we can divide all three sides by B to find that this modified trigonometric function goes through the same single cycle on the interval $-\frac{C}{B} \le x \le \frac{p-C}{B}$. That is, the starting point for the principal cycle has been shifted from 0 to $-\frac{C}{B}$. Therefore, the phase shift is $-\frac{C}{B}$. The following theorem summarizes all of this into a nice, clean statement.

Theorem: Phase Shift Formula

The phase shift of the function $y = A\,\text{trig}\,(Bx + C)$ is $-\frac{C}{B}$.

If $B < 0$, we use the Symmetry Identities to force it to become positive before attempting to find the phase shift.

Caution: C is Not Always the Phase Shift

When considering the function

$$f(x) = A\,\text{trig}\,(Bx + C)\,,$$

it is crucial to remember that B is ***not*** the period and C is ***not necessarily*** the phase shift.[3]

- $\text{Period} = \dfrac{\text{natural period}}{B}$
- $\text{Phase Shift} = -\dfrac{C}{B}$

Example 8.3.2

State the phase shift of each function.

a. $y = 2\tan(6x - \pi)$
b. $y = \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right)$

Solutions

a. In this case, $B = 6$ and $C = -\pi$. Hence, the phase shift for this function is

$$-\frac{C}{B} = -\frac{-\pi}{6} = \frac{\pi}{6}.$$

b. As in the previous section, we must use the Symmetry Identities when $B < 0$. When using these identities, be sure to factor the negative off *both* B and C.

$$\begin{aligned} y &= \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right) \\ &= \sqrt{5}\sin\left(-\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)\right) \\ &= -\sqrt{5}\sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right) \qquad \text{(sine is an odd function)} \end{aligned}$$

Hence, the phase shift is

$$-\frac{C}{B} = -\frac{-\pi/4}{\pi/2} = \frac{1}{2}.$$

Checkpoint 8.3.2

State the phase shift of $y = -2\cos\left(2x - \frac{\pi}{3}\right)$

Answer

$\frac{\pi}{6}$

Some instructors (including myself) introduce an alternative way of manipulating the argument of the trigonometric function $f(x) = A\,\text{trig}\,(Bx + C)$. If you factor B from all terms in the argument, your function turns into

$$f(x) = A\,\text{trig}\left(B\left(x + \frac{C}{B}\right)\right).$$

In this form, the phase shift reveals itself because $\frac{C}{B} = -\left(-\frac{C}{B}\right)$ and so

$$f(x) = A\,\text{trig}\left(B\left(x - \left(-\frac{C}{B}\right)\right)\right).$$

We will include this in an example or two, just in case your instructor likes this form.

We now modify our graphing tactic from the previous section to include this new transformation.

Summary of Graphing $f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right)$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right):$$

Precondition Stage

We begin by cleaning up the trigonometric function and ensuring that the coefficient of x is positive. This is done using the Symmetry Identities. Again, as a warning, if $\overline{B} < 0$, you will factor a negative off both terms in the argument of the trigonometric function. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\,\text{trig}\,(Bx + C).$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$
- $\text{Phase Shift} = -\dfrac{C}{B}$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line and mark the initial key number as the **phase shift**.
- Mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or
  - label the remaining key points on the graph (if the given function is the tangent).

Example 8.3.3

Graph a single cycle of the given trigonometric function, labeling key numbers, the y-axis, stating the period, step size, amplitude, and phase shift.

a. $y = 2\tan(6x - \pi)$
b. $y = \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right)$

Solutions

a. **Precondition Stage:** Since $B = 6 > 0$, there is nothing to precondition; however, if we wanted to factor the 6 from both terms in the argument, we would get

$$y = 2\tan\left(6\left(x - \frac{\pi}{6}\right)\right),$$

which reveals the phase shift to be $\frac{\pi}{6}$.
Information Stage: This is not a sinusoidal function, so we will not have an amplitude, and the natural period is π.

B	$=$	6	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{6}$
			$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi/6}{4} = \dfrac{\pi}{24}$
C	$=$	$-\pi$	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{-\pi}{6} = \dfrac{\pi}{6} = \dfrac{4\pi}{24}$

It's a good idea to get the phase shift and the step size to have the same denominator, as you will add the step size to the phase shift in the graphing stage.
Graphing Stage: Graphing a number line (the x-axis), we plot the phase shift and, from there, mark the remaining key numbers using the step size.

Figure 8.3.3

The distance between the initial key number and the final key number is

$$\left|\frac{8\pi}{24} - \frac{4\pi}{24}\right| = \frac{4\pi}{24} = \frac{\pi}{6} = \text{Period}$$

so we have done well so far. We now sketch a single cycle of the unreflected tangent function, drawing in the vertical asymptote halfway through the cycle.

Figure 8.3.4

We can see that the initial key number is positive and, therefore, to the right of the y-axis. Consequently, we draw in the y-axis where we think it should be. Simultaneously, let's plot the points corresponding to the key numbers $\frac{5\pi}{24}$ and $\frac{7\pi}{24}$.

Figure 8.3.5

This completes one full cycle of the graph of the given function (although we decided to show some extra cycles, it's not a requirement).

b. **Precondition Stage:** Since $B = -\frac{\pi}{2} < 0$, we need to factor a negative from both terms in the argument of the function.

$$\begin{aligned} y &= \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right) \\ &= \sqrt{5}\sin\left(-\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)\right) \\ &= -\sqrt{5}\sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right) \qquad \text{(sine is an odd function)} \end{aligned}$$

Moreover, if your instructor prefers the factored form, we could factor the $\frac{\pi}{2}$ from both terms in the argument to get

$$y = -\sqrt{5}\sin\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right).$$

Doing this reveals the phase shift to be $\frac{1}{2}$.

Information Stage: This is a sinusoidal function, so we will have an amplitude, and the natural period is 2π.

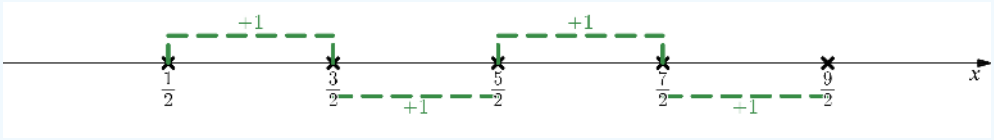
A	$=$	$-\sqrt{5}$	$\Longrightarrow$	Amplitude	$=$	$\lvert A\rvert = \lvert-\sqrt{5}\rvert = \sqrt{5}$
B	$=$	$\frac{\pi}{2}$	$\Longrightarrow$	Period	$=$	$\frac{\text{Natural Period}}{B} = \frac{2\pi}{\pi/2} = 4$
			$\Longrightarrow$	Step Size	$=$	$\frac{\text{Period}}{4} = \frac{4}{4} = 1$
C	$=$	$-\frac{\pi}{4}$	$\Longrightarrow$	Phase Shift	$=$	$-\frac{C}{B} = -\frac{-\pi/4}{\pi/2} = \frac{1}{2}$

Graphing Stage: It's important to remember that we are graphing the "cleaned" function

$$y = -\sqrt{5}\sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right).$$

Graphing the x-axis, plotting the phase shift, and marking the subsequent key numbers by increasing by the step size, we get the following:

Figure 8.3.6



We then graph the *reflected* sine function.

Figure 8.3.7

Finally, we place the y-axis appropriately, label it using the amplitude, and label the maximum and minimum points on the sine curve (this is something we didn't do before, but some instructors expect it, so we want to showcase good behavior).

Figure 8.3.8

Checkpoint 8.3.3

Graph $y = -2\cos\left(2x + \frac{\pi}{3}\right)$ and state the period, amplitude, step size, and phase shift.

Answer

Amplitude: 2
Period: π
Step Size: $\frac{\pi}{4}$
Phase Shift: $-\frac{\pi}{6}$

Figure 8.3.9

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As we have done previously, you can use the following Interactive Element to help visualize the effect of C on the graph of the sine function.

Vertical Shifts and the Midline

We finish our discussion of transformations with the parameter D in the function

$$f(x) = A\,\text{trig}\,(Bx + C) + D.$$

Of all the transformations from your Algebra class, this is the one most students are pretty good with. Recall, the graph of $y = g(x) + D$ is the graph of g shifted up (if $D > 0$) or down (if $D < 0$) $|D|$ units. That is, the "$+D$" is just adding D to all the y-values of the original function.

The only new concept here is the idea of a **midline** for a trigonometric function.

Definition: Midline

The **midline** of a trigonometric function is a horizontal line that represents the average value of the function. For sinusoidal functions, it is situated halfway between the maximum and minimum points. The midline for all base graphs of the trigonometric functions is $y = 0$.

The midline is essentially the center line around which the graph of the trigonometric function oscillates (for the case of the sinusoidal curves). Once a vertical shift occurs, this "hidden" axis appears, and it is handy when graphing trigonometric functions.

For example, consider the function

$$y = -2\sin(\pi x) + 3.$$

From the previous section, we should be able to quickly sketch *most* of this function, but the "$+3$" at the end is new. We know the base function is the sine, the fact that the coefficient of the sine is negative means the base graph has been reflected about the x-axis, the amplitude (because the graph is sinusoidal) is $|-2| = 2$, the period is $\frac{2\pi}{B} = \frac{2\pi}{\pi} = 2$, and the step size is $\frac{\text{Period}}{4} = \frac{2}{4} = \frac{1}{2}$. If this is all we had, then the graph would look like the following:

Figure 8.3.10: $y = -2\sin(\pi x)$

Since our given function is $y = -2\sin(\pi x) + 3$, we need to add three to all of the outputs in Figure 8.3.10.[4] As we redraw our graph and shift up three units, it is helpful to move up the hidden midline.

Figure 8.3.11: $y = -2\sin(\pi x) + 3$

Let's update our graphing tactic a final time.

Summary of Graphing $f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right) + D$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right) + D:$$

Precondition Stage

- If the trigonometric function is not written in the order listed above (e.g. if you are given $y = 3 - 2\sin(5 - x)$), rewrite it so it matches the form above - this is critical!
- Clean up the trigonometric function using the Symmetric Identities to ensure the coefficient of x is positive. Again, as a warning, if $\overline{B} < 0$, you will factor a negative off both terms in the argument of the trigonometric function. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\,\text{trig}\,(Bx + C) + D.$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$
- $\text{Phase Shift} = -\dfrac{C}{B}$
- $\text{Vertical Shift} = D$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line and label it $y = D$. This is your **vertical shift**.
- Mark the initial key number as the **phase shift**

- Mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or
  - label the remaining key points on the graph (if the given function is the tangent).

Example 8.3.4

Graph each function. State the amplitude (if applicable), period, step size, phase shift, and vertical shift. Your graph should have five key points (or four key points and an asymptote) labeled, as well as the midline.

a. $y = -5 + 5\sin\left(3\pi x + \frac{\pi}{2}\right)$
b. $y = \frac{1}{3}\tan(2x - 1) + 1$

Solutions

a. **Precondition Stage:** Rewriting the given function into proper form, we get

$$y = 5\sin\left(3\pi x + \frac{\pi}{2}\right) - 5.$$

Information Stage:

A	$=$	5	$\Longrightarrow$	Amplitude	$=$	5
B	$=$	3π	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{2\pi}{3\pi} = \dfrac{2}{3}$
			$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{2/3}{4} = \dfrac{1}{6}$
C	$=$	$\dfrac{\pi}{2}$	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{\pi/2}{3\pi} = -\dfrac{1}{6}$
D	$=$	-5	$\Longrightarrow$	Vertical Shift	$=$	-5

That last line tells us the midline is $y = -5$.
Graphing Stage: We start by graphing a horizontal line and labeling it $y = -5$.

Figure 8.3.12

We then plot five equally-spaced points, labeling the first using the phase shift, and adding the step size to reach each subsequent point.

Figure 8.3.13

Double-checking the length of the interval, we see that

$$\left|\frac{3}{6} - \left(-\frac{1}{6}\right)\right| = \frac{4}{6} = \frac{2}{3} = \text{Period}.$$

So far, so good! The function we want to graph is an *unreflected* sine function, so let's use these key numbers to sketch.

Figure 8.3.14

From here, we can see where $x = 0$ is, so we know where the y-axis goes.

Figure 8.3.15

Since the midline is at $y = -5$ and the amplitude is 5, we can label the y-axis by adding and subtracting the amplitude from the midline.

Figure 8.3.16

Finally, now that we see where $y = 0$, we can draw the x-axis.

Figure 8.3.17

That is a complete cycle of the graph of $y = -5 + 5\sin\left(3\pi x + \frac{\pi}{2}\right)$.

b. **Precondition Stage:** There is nothing to precondition here.
Information Stage:

$B = 2$	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{2}$		
	$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi/2}{4} = \dfrac{\pi}{8}$		
$C = -1$	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{-1}{2} = \dfrac{1}{2}$		
$D = 1$	$\Longrightarrow$	Vertical Shift	$=$	1		

This is a case where the phase shift ($\frac{1}{2}$) and the step size ($\frac{\pi}{8}$) are not easily added together. Please pay close attention to how we handle this during the graphing stage.

Graphing Stage: Graph the midline, plot five equally-spaced points, call the first point the phase shift, and label the remaining using the step size.

Figure 8.3.18

We now graph the *unreflected* tangent function and graph the vertical asymptote in the middle of the cycle. Moreover, we place the y-axis.

Figure 8.3.19

We can see that two of our key points are "labeled" ($\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2} + \frac{4\pi}{8}, 1\right)$) and we have the asymptote labeled ($x = \frac{1}{2} + \frac{2\pi}{8}$); however, we have two extra key numbers that need to be used. Despite the tangent function not having an amplitude, $|A| = \frac{1}{3}$ still serves a purpose. It tells us how far away from the midline the points corresponding to those unlabeled key numbers should be. That is, the y-value associated with $x = \frac{1}{2} + \frac{\pi}{8}$ is $1 + \frac{1}{3} = \frac{4}{3}$. Likewise, the y-value for $x = \frac{1}{2} + \frac{3\pi}{8}$ is $1 - \frac{1}{3} = \frac{2}{3}$. We label these points and draw in the x-axis (appropriately).

Figure 8.3.20

? Checkpoint 8.3.4

Graph the function. State the amplitude (if applicable), period, step size, phase shift, and vertical shift. Your graph should have five key points (or four key points and an asymptote) labeled, as well as the midline.

$$f(x) = 3 - \frac{7}{5}\tan\left(2x - \frac{\pi}{2}\right)$$

Answer

Period: $\frac{\pi}{2}$
Step Size: $\frac{\pi}{8}$
Phase Shift: $\frac{\pi}{4}$
Vertical Shift: 3

Figure 8.3.21

Finding Equations Given Graphs

Finding the equation of a trigonometric function given its graph is only slightly more complex when there has been a phase shift or vertical shift. The key to success with these problems is identifying two points on the curve. This is *usually* enough to find the equation.

Example 8.3.5

Find the equation of the graphed function.

a.

Figure 8.3.22

b.

Figure 8.3.23

Solutions

a. When rebuilding an equation from a graph, it's best to figure out the information using the order with which we graph. That is, try to identify the vertical shift, phase shift, period (which might require finding the step size), and amplitude (if applicable). This is more of a "rule of thumb," so there might be times you do things out of order.

Vertical Shift: Since the maximum and minimum of this sinusoidal function are $-\frac{1}{2}$ and $-\frac{3}{2}$, respectively, the midline must sit directly between these two values. That is, the vertical shift is the *average* of these.

$$\text{Vertical Shift} = D = \frac{-\frac{1}{2} + \left(-\frac{3}{2}\right)}{2} = -1.$$

Phase Shift: Given the graph of a sinusoidal function and asked to find the equation, you have *a lot* of options. For example, the given graph could be interpreted as a sine function shifted slightly to the left (or slightly to the right and reflected about $y = -1$); however, try to use the given information to make the process easier. We will assume the leftmost point we were given, $\left(-\frac{5\pi}{18}, -\frac{3}{2}\right)$, is the phase shift. Thus,

$$\text{Phase Shift} = -\frac{C}{B} = -\frac{5\pi}{18}. \tag{8.3.2}$$

While this does not immediately give us B or C, it does tell us that we are dealing with a reflected cosine function. We will use Equation 8.3.2 in a moment.

Period or Step Size: Assuming the cycle begins at $x = -\frac{5\pi}{18}$, we can see that the curve is *halfway* through a full cycle by $x = \frac{\pi}{18}$. That is, half the period is $\left|-\frac{5\pi}{18} - \frac{\pi}{18}\right| = \frac{6\pi}{18} = \frac{\pi}{3}$. Therefore, the full period is $\frac{2\pi}{3}$, and so

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{3} \implies B = 3.$$

We can now go back to Equation 8.3.2 and let $B = 3$ to get

$$-\frac{C}{3} = -\frac{5\pi}{18} \implies C = \frac{5\pi}{6}.$$

Amplitude: The amplitude is the distance from the midline to the maximum (or minimum). In this case, the amplitude is

$$\text{Amplitude} = |A| = \frac{1}{2}.$$

Putting this altogether, and remembering that the function is a *reflected* cosine function, we get

$$f(x) = -\frac{1}{2}\cos\left(3x + \frac{5\pi}{6}\right) - 1.$$

It's *crucial* for me to state that there are an infinite number of current answers when it comes to stating the equation of a trigonometric function. If checking your work against a friend's work and if your friend has a different function, you should graph both equations to see if they are equivalent.

b. **Vertical Shift**: If you imagine a horizontal midline for this tangent-like curve, it would be at $y = 1$. Thus,

$$D = 1.$$

Phase Shift: Remember that the principal cycle of the tangent function starts at the midline (x-axis). Thus, we will assume the first point, $(1, 1)$, is where the cycle of this shifted function starts. This means the function is a reflected tangent function and

$$\text{Phase Shift} = -\frac{C}{B} = 1. \tag{8.3.3}$$

Period or Step Size: The tangent is a nice function to deal with sometimes because we can see the period from the vertical asymptotes. The distance between these matches the period of the given tangent function. Therefore,

$$\text{Period} = \frac{\pi}{B} = |2.5 - (-0.5)| = 3 \implies B = \frac{\pi}{3}.$$

Pluggin this back into Equation 8.3.3, we get

$$-\frac{C}{\pi/3} = 1 \implies C = -\frac{\pi}{3}.$$

Amplitude: Since we are dealing with a tangent function, there is no amplitude, but we still need to determine the coefficient of the tangent function. Luckily, we were given a second point, $\left(\frac{7}{4}, -1\right)$. So far we have

$$g(x) = A\tan(Bx + C) + D = A\tan\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + 1.$$

Letting the output be -1 and $x = \frac{7}{4}$, we get

$$\begin{aligned}
-1 &= A\tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right) + 1 \\
\implies -2 &= A\tan\left(\frac{7\pi}{12} - \frac{4\pi}{12}\right) \\
\implies -2 &= A\tan\left(\frac{3\pi}{12}\right) \\
\implies -2 &= A\tan\left(\frac{\pi}{4}\right) \\
\implies -2 &= A(1) \\
\implies -2 &= A
\end{aligned}$$

Hence,

$$g(x) = -2\tan\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + 1.$$

? Checkpoint 8.3.5

Find the equation of the graphed function.

Figure 8.3.24

Answer

$y = 3\sin\left(2x + \frac{\pi}{4}\right) - 1$

Revisiting the Cofunction Identities

Recall the Cofunction Identities (which we rewrite here in radian measure).

Theorem: Cofunction Identities (radians)

The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. That is,

$$\begin{aligned} \sin(\theta) &= \cos\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad \cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) \\ \sec(\theta) &= \csc\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad \csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right) \\ \tan(\theta) &= \cot\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad \cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right) \end{aligned}$$

At the time, we could only prove these identities for acute angles. We are still not at a point where we can prove these for all angles, but we can now *visualize* these identities. For example, let's consider the function $y = \cos\left(\frac{\pi}{2} - x\right)$. The Cofunction Identities tell us that this should be the same as $y = \sin(x)$. Let's manipulate that cosine function a little.

$$\begin{aligned} y &= \cos\left(\tfrac{\pi}{2} - x\right) \\ &= \cos\left(-\left(-\tfrac{\pi}{2} + x\right)\right) \\ &= \cos\left(-\left(x - \tfrac{\pi}{2}\right)\right) \\ &= \cos\left(x - \tfrac{\pi}{2}\right) \qquad \text{(cosine is an even function)} \end{aligned}$$

So, that cosine function is just like a regular cosine function, but with a phase shift of $\frac{\pi}{2}$. Graphing this, we get the following:

Figure 8.3.25

However, this is *exactly* the graph of $y = \sin(x)$. Therefore, we feel confident in stating that $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$. You can use the same technique to convince yourself that the other Cofunction Identities are true (go ahead, it's a great exercise!).

Introduction to Sinusoidal Models

Many natural phenomena can be modeled with sinusoidal functions. In this subsection, we will work with applications where we have either been given the explicit sinusoidal model or the graph that models a situation. In the next section, we will learn to build sinusoidal models from language.

Example 8.3.6

The formula

$$D(t) = 12.25 - 5.25\cos\left(\frac{\pi}{6}t\right)$$

gives the number of hours of daylight in Glasgow, Scotland, as a function of time, in months after January 1.

a. Graph the function by hand.
b. State the midline, amplitude, and period of the graph.
c. How many hours of daylight does Glasgow enjoy on the longest day of the year?
d. In which months does Glasgow experience less than 8 hours of light daily?

Solutions

a. We begin by rewriting the function as

$$D(t) = -5.25\cos\left(\frac{\pi}{6}t\right) + 12.25.$$

The amplitude is $A = 5.25$, the period is $\frac{2\pi}{\pi/6} = 12$. Therefore, the step size is 3. Since $C = 3$, there is no phase shift. Finally, the vertical shift is 12.25. Graphing the reflected cosine starting at $t = 0$, stepping by 3, with an amplitude of 5.25, we get the following graph.

Figure 8.3.26

b. The midline of the graph is D = 12.25 hours, the amplitude is 5.25 hours, and the period is 12 months.
c. On the longest day of the year, there are 12.25 + 5.25 = 17.5 hours of daylight in Glasgow.

d. From the graph, we see that $D(t) < 8$ for t between 0 and 2, or in January and February. $D(t)$ is also less than 8 for a short time between $t = 11$ and $t = 12$ at the end of December.

Checkpoint 8.3.6

The pistons in an automobile engine move up and down in the cylinders. If t is in milliseconds, the distance from the top of the piston to the top of the cylinder is given in centimeters by

$$D = 7 + 6\sin(4\pi t)$$

a. Graph the function by hand.
b. State the midline, amplitude, and period of the graph.
c. Find the largest and the smallest clearance between the piston and the top of the cylinder.

Answer

a.
b. Midline: $D = 7$, amplitude: 6 , period: 0.5millisec
c. Largest: $13\ \text{cm}$, smallest: $1\ \text{cm}$

Example 8.3.7

Sunspots are dark regions on the Sun first observed by Galileo in 1610. The number of sunspots is not constant but varies with a period of approximately 11 years, called the solar cycle. Solar activity is directly related to this cycle and can cause disturbances in the Earth's upper atmosphere. Planning for satellite orbits and space missions requires knowledge of solar activity levels years in advance.

The figure below shows sunspot data for the last solar cycle, from July 1985 through June 1996, and a sinusoidal function that models the data.

Figure 8.3.27

The period of the solar cycle shown is 10.8 years. Use this information and the graph to write a formula for the model.

Solution

The graph has the shape of $y = -\cos(x)$. It appears to have midline $y = 75$ and amplitude 75, so $|A| = 75$ and $A = -75$. To find B, we compute

$$\text{Period} = \frac{2\pi}{B} \implies B = \frac{2\pi}{\text{Period}} = \frac{2\pi}{10.8} \approx 0.58$$

Thus, a formula for the model is

$$y = -75\cos(0.58x + C)] + 75.$$

If we are trying to fit the red curve, the phase shift would be 0 and so $C = 0$.Hence,

$$y = -75\cos(0.58x) + 75.$$

Checkpoint 8.3.7

The figure below shows the sunspot data for the solar cycle that began in July 1996, and a curve of best fit calculated by NASA. This curve is not sinusoidal, but has a similar shape.

Figure 8.3.28

a. The minimum sunspot number occurred in October 1996. Use the graph to estimate the period, midline, and amplitude of a sinusoidal function that approximates the data.
b. Write a function, $s(t)$, that approximates the data, where t is the number of years since July 1996.

c. Use your function to predict the sunspot number in January 2025.

Answer

a. Period: 10 years, midline: $y = 61.5$, amplitude: 53

b. $s(t) = 61.5 - 53\cos\left(\frac{\pi}{5}t - \frac{\pi}{20}\right)$

c. ≈ 18.6

Footnotes

[1] From Algebra.

[2] You were likely taught this in a slightly different form. It is traditional to teach horizontal shifts using $g(x - C)$ (notice the minus instead of the plus sign?). When introduced this way, C causes a shift to the *right* when $C > 0$ and a shift to the *left* when $C < 0$.

[3] While C is not necessarily the phase shift, it is called the **phase angle** in physics.

[4] "Outputs" is a loose word for "function values" or "y-values."

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8.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Graphing Trigonometric Functions
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions
- Sinusoidal Models
  - Trigonometric Graphs - Technology and Graphs of Trigonometric Functions (Part 1)
  - Graphs of the Other Trigonometric Functions - An Application Involving the Tangent Function
- Finding Equations from Graphs
  - Trigonometric Graphs - Review of Identifying the Equation for a Sinusoidal Function from a Graph

KEY CONCEPTS

DEFINITIONS

- **Phase Shift:** The horizontal shift of a trigonometric function. It represents the x-value to which the starting point of the principal cycle (originally at $x = 0$) has been moved.
- **Midline:** A horizontal line that represents the average value of a trigonometric function. For sinusoidal functions, it is the line situated halfway between the maximum and minimum points.

THEOREMS

- **Lemma: Phase Shift Formula for y = trig(x + C):** The phase shift for a trigonometric function of the form $y = \text{trig}(x + C)$ is $-C$.
- **Phase Shift Formula:** The phase shift of the function $y = A\text{trig}(Bx + C)$ is $-\frac{C}{B}$.
- **Cofunction Identities (radians):** The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. For example, $\sin(\theta) = \cos(\frac{\pi}{2} - \theta)$.

COMMON MISTAKES

- **Confusing C with the Phase Shift:** In the function $y = A\text{trig}(Bx + C)$, the phase shift is $-\frac{C}{B}$, not simply $-C$. The value of B affects the horizontal shift.
- **Ignoring Negative B-values:** Before determining the period or phase shift, if the parameter B is negative, you must first use the Symmetry Identities to make the coefficient of x positive. Forgetting this step will result in an incorrect phase shift.
- **Incorrectly Identifying the Midline:** The midline is the average of the maximum and minimum values, not simply the lowest point of the graph. It is the horizontal line the graph oscillates around.

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8.3.2: HOMEWORK

READING QUESTIONS

1. What is a phase shift and what does it represent on a graph?
2. What is the midline of a trigonometric function?
3. For the general function $f(x) = A\text{trig}(Bx + C) + D$, which parameter determines the midline?
4. What is the formula for the phase shift of the function $y = A\sin(Bx + C)$?
5. How can you find the midline of a sinusoidal graph if you know its maximum and minimum values?
6. When graphing a function like $y = \sin(x + \frac{\pi}{3})$, does the graph shift to the left or to the right?
7. Before finding the phase shift for a function like $y = \cos(-2x + \pi)$, what is the necessary first step?
8. The text shows that the graph of $y = \cos(\frac{\pi}{2} - x)$ is identical to the graph of $y = \sin(x)$. What transformation of the cosine graph produces the sine graph?
9. When finding the equation of a sinusoidal function from its graph, why is it helpful to identify the vertical shift first?
10. When finding the period of a tangent function from its graph, what graphical feature can you measure the distance between?

HOMEWORK

VOCABULARY CHECK

1. The ___ tells us the x-value of where the original initial number in the principal cycle has moved.
2. Given the function $f(x) = A\ \text{trig}\,(Bx + C) + D$, the graph of the equation $y = D$ is called the ___ of the trigonometric function.
3. Given the function $f(x) = A\ \text{trig}\,(Bx + C) + D$, D represents the ___.

CONCEPT CHECK

4. Write an equation for a cosine function with midline -5.
5. Write an equation for a sine function with midline 2.
6. Which of the following functions are the same? Explain your answer.

$$f(x) = \cos\left(2x - \frac{\pi}{3}\right), \quad g(x) = (\cos 2x) - \frac{\pi}{3}, \quad h(x) = \cos 2x - \cos\frac{\pi}{3}$$

7. Which of the following functions are the same? Explain your answer.

$$f(x) = \sin\left(2x - \frac{\pi}{3}\right), \quad g(x) = \sin\left[2\left(x - \frac{\pi}{6}\right)\right], \quad h(x) = 2\sin\left(x - \frac{\pi}{6}\right)$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

8. The phase shift is also known as the horizontal shift.
9. The amplitude of $f(x) = -3\tan(2x - 1)$ is 3.
10. The phase shift of the function $f(x) = -3\tan(2x - 1)$ is 1.
11. The phase shift of the function $f(x) = -3\tan(2x - 1)$ is -1.
12. The functions $y = 2\cos(\pi x - 4)$ and $y = -3\sin\left(\pi\left(x - \frac{4}{\pi}\right)\right)$ have the same phase shift.

BASIC SKILLS

For the following exercises, calculate the phase shift.

13. $y = -2\sin\left(3x - \frac{\pi}{4}\right)$
14. $y = 6\cos\left(2x + \frac{3\pi}{4}\right)$
15. $y = 12\cos\left(\frac{t}{3} + 0.8\right)$
16. $y = -5\sin\left(\frac{t}{4} - 0.6\right)$

For the following exercises, state the amplitude, period, and midline of the graph.

17. $y = -3 + 2\sin(x)$
18. $y = 4 - 3\cos(x)$
19. $y = 1 - \cos(\pi x)$
20. $y = 2 + \sin(2\pi x)$

For the following exercises,
(a) Sketch the graphs of f and g on the same axes.
(b) State the phase shift from f to g.
(c) Find all values of x for which $g(x) = 1$ for $-\pi \le x \le \pi$.
(d) Find all values of x for which $g(x) = 0$ for $-\pi \le x \le \pi$.

21. $f(x) = \sin(x)$ and $g(x) = \sin\left(x - \frac{\pi}{3}\right)$.
22. $f(x) = \cos(x)$ and $g(x) = \cos\left(x + \frac{\pi}{3}\right)$.
23. $f(x) = \tan(x)$ and $g(x) = \tan\left(x + \frac{\pi}{4}\right)$.
24. $f(x) = \tan(x)$ and $g(x) = \tan\left(x - \frac{\pi}{2}\right)$.

For the following exercises,
(a) What are the amplitude and phase shift of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 0$ for $0 \le x \le 2\pi$.

25. $f(x) = -2\cos\left(x + \frac{\pi}{6}\right)$
26. $f(x) = -6\sin\left(x - \frac{2\pi}{3}\right)$
27. $f(x) = \frac{1}{2}\tan\left(x + \frac{\pi}{5}\right)$
28. $f(x) = -\frac{3}{7}\cos\left(x - \frac{\pi}{4}\right)$

For the following exercises,
(a) What are the period and phase shift of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 1$ for $0 \le x \le 2\pi$.
(d) Find all values of x for which $f(x) = 0$ for $0 \le x \le 2\pi$.

29. $f(x) = \cos\left(2x - \frac{\pi}{3}\right)$
30. $f(x) = \sin\left(3x + \frac{\pi}{2}\right)$
31. $f(x) = \sin\left(\pi x + \frac{\pi}{3}\right)$
32. $f(x) = \cos\left(\pi x - \frac{\pi}{3}\right)$

For the following exercises,
(a) What are the midline, period, phase shift, and amplitude of

the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 1$ for $0 \le x \le 2\pi$.
(d) Find all values of x for which $f(x) = k$ for $0 \le x \le 2\pi$, where k is stated within the problem.

33. $y = 3\sin\left(\frac{x}{2} - \frac{\pi}{6}\right) + 4,\ k = 4$

34. $y = 2\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) - 1,\ k = -1$

For the following exercises,
(a) State the amplitude.
(b) State the period.
(c) State the step size.
(d) State the phase shift.
(e) State the vertical shift.
(f) Use this information to create an accurate graph of a single cycle of the trigonometric function. Be sure to label any asymptotes and all key points.

35. $y = 2 - 5\cos(2t)$

36. $y = -2 + 4\sin(3t)$

37. $y = 1 + 3\cos\left(\frac{t}{2}\right)$

38. $y = -2 - 3\sin\left(\frac{t}{4}\right)$

39. $y = 4 + 2\tan(3x)$

40. $y = 3 - \tan\left(\frac{x}{4}\right)$

41. $y = 1 - 2\tan\left(\frac{x}{3}\right)$

42. $y = 2\sin(2x - \pi) + 1$

43. $y = -1 + 3\cos(2x + \pi)$

44. $y = -2 - \tan\left(3x - \frac{\pi}{2}\right)$

45. $y = \frac{3}{2} - \frac{1}{2}\cos(3x + \pi)$

46. $y = -\frac{1}{3}\sin\left(\pi x + \frac{3\pi}{2}\right) + 2$

47. $y = -3\tan(2x - 1) + 1$

48. $y = -4\cos(\pi x - 3) - 4$

For the following exercises, state the equation of the function.

49.

50.

51.

52.

53.

54.

55.

56.

For the following exercises, find a formula for the graph of the given function in terms of
(a) sine
(b) cosine

57.

58.

59.

60.

61.

62.

63.

64.

For the following exercises, find and graph a single cycle of a sinusoidal function with the given properties.

65. Amplitude of 2, a period of 3, and is shifted 4 units to the left and 5 units upwards compared with the sine function.

66. Amplitude of 3, a period of 24, and is shifted 2 units to the right and 4 units upwards compared with the cosine function.

67. Amplitude of 5, a period of 360, its midline at $y = 12$, and passes through (0,7).

68. Amplitude of 50, a period of 30, its midline at $y = 50$, and passes through (0,100).

For the following exercises, label the scales on the axes for the graph.

69. $y = 3 - 4\sin(2x)$

70. $y = 2\cos(5x) + 2$

71. $y = \frac{1}{2}\sin(3x) + \frac{3}{2}$

72. $y = \frac{2}{5}\cos(6x) + \frac{4}{5}$

73. $50 - 30\sin\left(\frac{x}{4}\right)$

74. $25\cos\left(\frac{x}{3}\right) + 15$

75. $y = 4\sin(\pi x) - 3$

76. $y = \frac{1}{2}\cos\left(\frac{\pi x}{2}\right) + 2$

SYNTHESIS QUESTIONS

For the following exercises, use the graph to find all solutions between 0 and 2π.

77. $2 + 3\sin(2x) = 0.5$

78. $2 + 4\cos(2x) = 4$

79. $-3 + \tan(3x) = -2$

80. $2 + \tan(4x) = 3$

For the following exercises,
(i) Use a calculator to graph the function for $0 \le x \le 2\pi$.
(ii) Use the intersect feature to find all solutions between 0 and 2π. Round your answers to hundredths.

81.

a. $h(x) = 2 - 4\cos\left(\frac{x}{4}\right)$

b. $2 - 4\cos\left(\frac{x}{4}\right) = 0$

82.

a. $H(x) = 3 + 2\sin\left(\frac{x}{2}\right)$

b. $3 + 2\sin\left(\frac{x}{2}\right) = 5$

83.

a. $G(x) = -1 + 3\cos(3x)$

b. $-1 + 3\cos(3x) = 1$

84.

a. $F(x) = 4 - 3\sin(2x)$

b. $4 - 3\sin(2x) = 2.5$

For the following exercises, give the coordinates of the points on the graph.

85. $f(\theta) = -3 + \cos\theta$

86. $f(\theta) = 1 + \sin\theta$

APPLICATIONS

87. The tide in Yorktown is approximated by the function

$$h(t) = 1.4 - 1.4\cos(0.51t)$$

measured in feet above low tide, where t is the number of hours since the last low tide.

a. What are the midline, period, and amplitude?

b. Graph $h(t)$ for two periods, labeling the points that correspond to high tide and low tide.

c. If the last low tide occurred at 5:00 am, predict when the next high and low tides will occur.

88. When observed from Earth, the moon looks like a disk that is partially visible and partially in shadow. The percentage of the disk that is visible can be approximated by

$$V(t) = 50\cos(0.21t) + 50$$

where t is the number of days since the last full moon.

a. Graph $V(t)$ in the window

$$X_{\min} = 0, X_{\max} = 30$$

$$Y_{\min} = 0, Y_{\max} = 120$$

b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to full moon, half moon, and new moon. (New moon occurs when the part of the moon receiving sunlight is facing directly away from the Earth.)

d. At what times during the lunar month is 25% of the moon visible? Mark those points on your graph.

e. During which days is less than 50% of the moon visible? Mark the corresponding points on your graph.

89. The tide in Malibu is approximated by the function

$$h(t) = 2.5 - 2.5\cos(0.5t)$$

measured in feet above low tide, where t is the number of hours since the last low tide.

a. Graph $h(t)$ in the window

$$X_{\min} = 0, X_{\max} = 25$$

$$Y_{\min} = 0, Y_{\max} = 6$$

b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to high tide and low tide.

d. How high is high tide, and at what times does it occur?

e. At what times during the 25-hour period is the tide 4 feet above low tide? Mark those points on your graph.

f. Kathie walks along the beach only when the tide is below 1 foot. Find the intervals on your graph when the tide is below 1 foot.

90. The average daily high temperature in the town of Beardsley, Arizona is approximated by the function

$$T(d) = 85.5 - 19.5\cos(0.0175d - 0.436)$$

where the temperature is measured in degrees Fahrenheit, and d is the number of days since January 1.

a. Graph $T(d)$ in the window

$$X_{\min} = 0, X_{\max} = 365$$

$$Y_{\min} = 0, Y_{\max} = 110$$

b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to highest and lowest average temperature.

d. What is the hottest day and what is its average temperature? What is the day with the lowest average temperature, and what is that temperature?

e. At what times during the year are average high temperatures above $90°$? Mark those points on your graph.

91. A weight is suspended from the ceiling on a spring. The weight is pushed straight up, compressing the spring, then released. The height of the weight above the ground is given by the function

$$H(t) = 2 + 0.5\cos(6t)$$

where the height is measured in meters, and t is the number of seconds since the mass was released.

a. Graph $H(t)$ in the window

$$X_{\min} = 0, X_{\max} = 3$$

$$Y_{\min} = 0, Y_{\max} = 3$$

b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to highest and lowest positions of the mass.

d. How high is highest point, and when is that height attained during the first 3 seconds?

e. How high is lowest point, and when is that height attained during the first 3 seconds?

f. Find the intervals during the first 3 seconds when the mass is less than 2 meters above the ground.

For the following exercises, write an equation for the sinusoidal function whose graph is shown.

92. The number of hours of daylight in Salt Lake City varies from a minimum of 9.6 hours on the winter solstice to a maximum of 14.4 hours on the summer solstice. Time is measured in months, starting at the winter solstice.

93. A weight is 6.5 feet above the floor, suspended from the ceiling by a spring. The weight is pulled down to 5 feet above the floor and released, rising past 6.5 feet in 0.5 seconds before attaining its maximum height of feet. The weight oscillates between its minimum and maximum height.

94. Although the moon is spherical, what we see from Earth looks like a disk, sometimes only partly visible. The percentage of the moon's disk that is visible varies between 0 (at new moon) to 100 (at full moon), over a 28-day cycle.

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8.4: Sinusoidal Models

Note to the Instructor - What Counts as a Sinusoidal Model?

This section covers any type of application or data-driven model that can be described using a sine or cosine function. In an effort to make this material a little more "bite-sized," I broke out the simple and damped harmonic motion material into their own section (the next section). For now, it's important that the students understand the concepts of angular and ordinary frequency. Playing with data in Desmos can be very useful when it comes to instruction for this section.

Learning Objectives

- Build a sinusoidal model from an application.
- Build a sinusoidal model from data.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Rewriting the Coordinate Definition

We begin this section with a brief interlude. Recall the coordinate definition of the trigonometric functions, which is presented below for our discussion.

Definition: Trigonometric Functions (Coordinate Definition)

Let θ be an angle in standard position with the point $P(x,y)$ on the terminal side of θ, where P is not the origin. Define $r = \sqrt{x^2+y^2}$. The trigonometric functions of the angle θ are defined as follows:

Function Name	Function Notation		Definition	Function Name	Function Notation		Definition
The sine of θ	$\sin(\theta)$	$=$	$\frac{y}{r}$	The cosecant of θ	$\csc(\theta)$	$=$	$\frac{r}{y}$
The cosine of θ	$\cos(\theta)$	$=$	$\frac{x}{r}$	The secant of θ	$\sec(\theta)$	$=$	$\frac{r}{x}$
The tangent of θ	$\tan(\theta)$	$=$	$\frac{y}{x}$	The cotangent of θ	$\cot(\theta)$	$=$	$\frac{x}{y}$

If we hold r to be constant and plot all points a distance of r units from the origin, then we will have graphed a circle of radius r centered at the origin. By the coordinate definition of the trigonometric functions, if the point (x,y) is on this circle and is on the terminal side of θ, then

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}.$$

We have reached a point in this text where writing these formulas in an alternate form is more useful. Solving the formulas for x and y, respectively, we get the following:

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta). \tag{8.4.1}$$

The formulas in 8.4.1 apply to any point on the terminal side of θ on a circle of radius r. Understanding these formulas is essential for our success moving forward in Trigonometry.

Sinusoidal Models

Now that we have a solid understanding of the graphs for the fundamental trigonometric functions, it's time we focused on their applications. In the same way exponential functions can be used to model a wide variety of phenomena in nature, the cosine and sine functions can be used to model their fair share of natural behaviors. In particular, sinusoidal waves are common in physics, engineering, computer science, signal processing, and many other fields. For consistency, we will focus on building applications using the sine function.[1] Moreover, we switch to the independent variable t (instead of x or θ) since the applications in this section are time-dependent.

Before diving into the applications, let's summarize some important facts and definitions about the sine function. Let

$$f(t) = A\sin(Bt + C) + D. \tag{8.4.2}$$

From the material so far in this chapter, we know that $|A|$ is the amplitude, the period is $\frac{2\pi}{B}$, the phase shift is $-\frac{C}{B}$, and the vertical shift (or midline) is D.

Along with knowing these formulas, it is helpful to remember what these quantities mean in context. As we discuss these contextual meanings, we will inevitably introduce more definitions (mostly from physics) that we will occasionally be using throughout the remainder of this text.

- The amplitude measures the sine wave's maximum **displacement** (change in position) from its midline. Therefore, if the midline of a sinusoidal function is $y = 120$ and the amplitude is 30, the maximum displacement of the sinusoidal wave is 30.
- The period is the length of time it takes to complete one cycle of the sinusoid. For a slightly oversimplified example, the signal your antenna is receiving from a broadcasting radio station is a sinusoidal wave. If this wave takes 1.0152×10^{-8} seconds to complete a single cycle, then its period is 1.0152×10^{-8}.
  - A more physically interesting, and important, interpretation related to the period is the ordinary frequency of the sinusoid (often just called the frequency). This was introduced previously but not really used, so let's recall some information about the frequency. This is the number of cycles the sinusoid goes through in a given unit of time (generally, per second). Since the period is in seconds per cycle and frequency is cycles per second, it should make sense that period and frequency are reciprocals. That is, if f is the frequency of a sinusoid and T is its period,

    $$f = \frac{1}{T} \quad \text{and} \quad T = \frac{1}{f}.$$

    Thus, the frequency of the radio station mentioned above is

    $$f = \frac{1}{T} = \frac{1}{1.0152 \times 10^{-8}} \approx 98,502,758 \text{ cycles per second.}$$

    The unit for "cycles per second" is called the **Hertz** (**Hz**). Therefore, the radio station is operating at roughly 98.5 million Hertz, or 98.5 MHz (megahertz).[2]
  - Another concept we introduced previously is the angular frequency. In terms of Equation 8.4.2, the angular frequency is defined as the value of B. This tells how many cycles are completed over an interval of length 2π.
- The phase shift represents how much of a "head start" the sinusoid has over the unshifted sine function.
  - The **phase angle** (also called the **phase**), which was only mentioned in a footnote in the previous section, is the value of C. It is the angle corresponding to $t = 0$. Since we have not officially written the definition for this, it is provided below.

Definition: Phase Angle

The **phase angle** (also called the **phase**) is defined to be the value of C in the sinusoidal functions $A\sin(Bx + C) + D$ and $A\cos(Bx + C) + D$.

All of this information is summarized below.

Summary: Sine Curve Properties

- The **amplitude** is $|A|$
- The **angular frequency** is B and the **ordinary frequency** is $f = \frac{B}{2\pi}$
- The **period** is $T = \frac{2\pi}{B} = \frac{1}{f}$
- The **phase angle** is C and the **phase shift** is $-\frac{C}{B}$
- The vertical shift or **midline** is D

From this point forward, when dealing with applications, we will use the terms *angular frequency*, *frequency*, and *phase* when possible. Therefore, it's best to make a note of these definitions so you don't forget them.

Our first foray into sinusoidal motion puts these definitions and the equations in 8.4.1 to good use.

Example 8.4.1

The Giant Wheel at Cedar Point is a circle with a diameter of 128 feet, which sits on an 8-foot-tall platform, making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming that the riders are at the edge of the circle, find a sinusoid that describes the height of the passengers above the ground t seconds after they pass the point on the wheel closest to the ground.

Solution

As is most often the case, we begin by sketching the situation below. We will assume a counter-clockwise rotation.[3]

Figure 8.4.1

We were told the diameter of the circle is 128 feet. Therefore, its radius is 64 feet. For simplicity, and to introduce a coordinate system to the situation, let's graph this circle centered at *the origin*. We will raise it up eventually, but this initial sketch can be very helpful.

Figure 8.4.2

Our eventual goal is to find the height of a rider, which is related to the y-coordinates of this circle. However, we know from 8.4.1 that

$$y = r\sin(\theta) = 64\sin(\theta).$$

One thing you must get used to is that our sinusoidal models are functions of time - not angles. Therefore, we will trade θ out for $Bt + C$ - this is the most general argument of the sine function and something we have been working with recently. Hence, the height of a rider on the Great Wheel will be related to the function

$$64\sin(\theta) = 64\sin(Bt + C).$$

Since the Great Wheel is not partially underground, we need to lift this function up. Specifically, we need to shift the center of our circle so that the bottom of the Great Wheel is 8 feet above the ground (which is the x-axis in our figure above). Thus, we need to move this circle up 64 feet plus an additional 8 feet, for a total of 72 feet.

Figure 8.4.3

Therefore, we are considering a sinusoidal model of the form

$$h(t) = 64\sin(Bt + C) + 72.$$

There are only two things left to consider - the angular frequency and the phase angle of the sinusoidal model.[4]

The angular frequency is B, which is know is related to the period, T, by the formula

$$T = \frac{2\pi}{B}$$

$$\implies B = \frac{2\pi}{T}. \tag{8.4.3}$$

The period is the time it takes for the sinusoidal function to complete one cycle. For this application, the period is the time it takes the Giant Wheel to perform *one* full revolution. We are told the Giant Wheel goes through *two* full revolutions in 2 minutes and 7 seconds. Therefore, it takes 127 seconds to go through *two* revolutions and half that to go through *one*. Hence, the period is

$$\text{Period} = T = \frac{\text{time}}{\text{one revolution}} = \frac{127}{2} \text{ seconds per revolution.}$$

Substituting this into 8.4.3, we get

$$B = \frac{2\pi}{T} = \frac{2\pi}{127/2} = \frac{4\pi}{127}.$$

The sinusoidal model is really taking shape now.

$$h(t) = 64\sin\left(\frac{4\pi}{127}t + C\right) + 72.$$

The height of a rider at $t = 0$ is 8. Therefore, letting $t = 0$ and $h = 8$, we get

$$8 = 64\sin(C) + 72 \implies -1 = \sin(C) \implies C = \frac{3\pi}{2}.$$

With this, we have found the equation of motion for the rider,

$$h(t) = 64\sin\left(\frac{4\pi}{127}t + \frac{3\pi}{2}\right) + 72.$$

We can check the reasonableness of our answer by graphing $y = h(t)$ over the interval $\left[0, \frac{127}{2}\right]$.

A few remarks about Example 8.4.1 are in order. First, note that the amplitude of 64 in our answer corresponds to the radius of the Giant Wheel. This means that passengers on the Giant Wheel never stray more than 64 feet vertically from the center of the Wheel, which makes sense. Another way to say this (using our new definitions) is that the maximum displacement of a rider on the Great Wheel is 64 feet.

The second thing to note is our process of finding the value of the phase angle. Trigonometry is a fun topic because there are usually several ways to do something, and not one is better than another. In fact, if I were investigating this problem and were not concerned about instructing you on new language, I would have found the value of C using a different approach. To figure out the phase angle, we could have sketched the path of a rider on the Great Wheel from the time they enter $(t = 0)$.

Figure 8.4.4

Notice that this sketch looks more like a reflected cosine than a sine; however, we made our choice to work with sine, and, darn it, we are going to stick with it. The actual sine function starts one-fourth of the way through this cycle. That is, at time $\frac{1}{4}\left(\frac{127}{2}\right) = \frac{127}{8}$. Hence, the sinusoidal model has a phase shift of $\frac{127}{8}$. Using the relation $\textbf{Phase Shift} = -\frac{C}{B}$, we get

$$\begin{aligned} \textbf{Phase Shift} &= -\frac{C}{B} \\ \Longrightarrow \quad \frac{127}{8} &= -\frac{C}{4\pi/127} \\ \Longrightarrow \quad \frac{127}{8} &= -\frac{127C}{4\pi} \\ \Longrightarrow \quad -\frac{4\pi}{8} &= C \\ \Longrightarrow \quad -\frac{\pi}{2} &= C \end{aligned}$$

Thus, our sinusoidal model is

$$h(t) = 64\sin\left(\frac{4\pi}{127}t - \frac{\pi}{2}\right) + 72.$$

Now, you might say, "Hey, that's not what we got in Example 8.4.1!" I beg to differ. Remember, the value of $\sin(\theta)$ is the same as the value of $\sin(\theta - 2\pi)$. If you subtract 2π from the argument in the answer to Example 8.4.1, you would get $h(t) = 64\sin\left(\frac{4\pi}{127}t - \frac{\pi}{2}\right) + 72$.

The third and final thing to mention is the phase shift of our answer, which is $\frac{127}{8} = 15.875$. This represents the "time delay" (in seconds) we introduce by starting the motion at the point P as opposed to the point Q. Said differently, passengers who "start" at P take 15.875 seconds to "catch up" to the point Q.

? Checkpoint 8.4.1

Outside temperature over a day can be modeled as a sinusoidal function. Suppose you know the high temperature for the day is 78 degrees and the low temperature of 42 degrees occurs at 6 AM. Assuming t is the number of hours since midnight, find an equation for the temperature, T, in terms of t.

Answer

$T(t) = 60 + 18\sin\left(\frac{\pi}{12}t - \pi\right)$

Our next example shows how to deal with data that is sinusoidal in nature.

Example 8.4.2

According to the U.S. Naval Observatory website, the number of hours, H, of daylight that Fairbanks, Alaska received on the 21st day of the nth month of 2009 is given below. Here, $t = 1$ represents January 21, 2009, $t = 2$ represents February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

a. Find a sinusoid that models these data and use graphing technology to graph your answer along with the data.
b. Compare your answer to part (a) to one obtained using the regression feature of your graphing technology.

Solutions

a. We plot the data below (using Desmos' table feature) to get a feel for it.

Figure 8.4.5

The data certainly appear sinusoidal,[5] but when it comes down to it, manually fitting a sinusoid to data is not an exact science. We do our best to find the constants A, B, C and D so that the function

$$H(t) = A\sin(Bt + C) + D$$

closely matches the data. We first go after the vertical shift D whose value determines the midline. In a typical sinusoid, the value of D is the average of the maximum and minimum values. So here we take

$$D = \frac{3.3 + 21.8}{2} = 12.55.$$

Next is the amplitude, $|A|$, which is the displacement from the midline to the maximum (and minimum) values. We find

$$|A| = |21.8 - 12.55| = 9.25.$$

As long as we know we are going to choose our sine function to be positive, we have

$$H(t) = 9.25\sin(Bt + C) + 12.55.$$

Next, we go after the angular frequency, B. Since the data collected is over the span of a year (12 months), we take the period to be $T = 12$ months.[6] This means

$$B = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}.$$

The last quantity to find is the phase, C. It is easier in this case to find the phase *shift*, $-\frac{C}{B}$. Since we picked $A > 0$, the phase shift corresponds to the first value of t with $H(t) = 12.55$ (the midline value). Here, we choose $t = 3$, since the

output of 12.4 is closer to 12.55 than the next value, 15.9, which corresponds to $t = 4$. Hence, $-\frac{C}{B} = 3$, so $C = -3B = -3\left(\frac{\pi}{6}\right) = -\frac{\pi}{2}$. Thus, we have

$$H(t) = 9.25 \sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 12.55.$$

Below is a graph of our data with the curve $y = H(t)$.

Figure 8.4.6

b. Using the following command in Desmos, we get

Figure 8.4.7

While both models seem to fit the data reasonably, Desmos' model is the better fit.

? Checkpoint 8.4.2

The table shows the number of hours of daylight in Glasgow, Scotland on the first of each month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Daylight Hours	7.1	8.7	10.7	13.1	15.3	17.2	17.5	16.7	13.8	11.5	9.2	7.5

a. Sketch a sinusoidal graph of daylight hours as a function of time, with $t = 1$ in January.
b. Estimate the period, amplitude, and midline of the graph.

Answers

a. Plot the data points and fit a sinusoidal curve by eye, as shown below.

Figure 8.4.8

b. The period of the graph is 12 months. The midline is approximately $y = 12.25$, and the amplitude is approximately 5.25.

Footnotes

[1] If you want to build models based on the cosine function, you are welcome to use the Cofunction Identities we encountered previously to transform our sine models into cosine models.

[2] If you live in Sacramento, California, this is the frequency for the radio station 98.5 FM KRXQ.

[3] Otherwise, we could observe the motion of the wheel from the other side.

[4] I warned you that we were going to be using that new language. Remember, "angular frequency" is the value of B and the "phase angle" is the value of C.

[5] Okay, it appears to be a "V" shape. Just humor me for a moment.

[6] Even though the data collected lies in the interval $[1, 12]$, which has a length of 11, we need to think of the data point at $t = 1$ as a representative sample of the amount of daylight for every day in January. That is, it represents $H(t)$ over the interval $[0, 1]$. Similarly, $t = 2$ is a sample of $H(t)$ over $[1, 2]$, and so forth.

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8.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Sinusoidal Models
  - Graphs of the Sine and Cosine Functions - Applications Involving Sinusoidal Functions
  - Trigonometric Graphs - Review of Applications Involving Sinusoidal Functions

KEY CONCEPTS

DEFINITIONS

- **Trigonometric Functions (Coordinate Definition):** For an angle θ in standard position, a point $P(x, y)$ on its terminal side at a distance r from the origin has coordinates given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- **Maximum Displacement:** In the context of a sinusoidal function, the maximum displacement is the amplitude, which measures the wave's maximum change in position from its midline.
- **Ordinary Frequency (f):** The number of cycles a sinusoid completes in a given unit of time. It is the reciprocal of the period ($f = \frac{1}{T}$).
- **Hertz (Hz):** The unit for ordinary frequency, equivalent to one cycle per second.
- **Angular Frequency (ω or B):** The value of the parameter B in a sinusoidal function. It is related to the ordinary frequency by the formula $\omega = 2\pi f$.
- **Phase Angle:** The value of the parameter C in the sinusoidal function $f(t) = A\sin(Bt + C) + D$. It corresponds to the angle at $t = 0$.

THEOREMS

- **Sine Curve Properties:** For a function of the form $f(t) = A\sin(Bt + C) + D$:
  - The amplitude is $|A|$.
  - The angular frequency is B.
  - The ordinary frequency is $f = \frac{B}{2\pi}$.
  - The period is $T = \frac{2\pi}{B} = \frac{1}{f}$.
  - The phase angle is C.
  - The phase shift is $-\frac{C}{B}$.
  - The vertical shift, or midline, is D.

COMMON MISTAKES

- **Confusing Period and Frequency:** The period is the time it takes to complete one cycle (e.g., seconds per cycle), while frequency is the number of cycles completed in a unit of time (e.g., cycles per second). They are reciprocals of each other.
- **Confusing Angular Frequency and Ordinary Frequency:** The angular frequency is the parameter B in the sinusoidal model, while the ordinary frequency f must be calculated using $f = B/2\pi$.
- **Confusing Phase Angle and Phase Shift:** The phase angle is the parameter C, while the phase shift is a horizontal translation of the graph calculated as $-C/B$.
- **Assuming Only One Correct Model:** For a given sinusoidal graph, there are infinitely many correct equations that can model it. A cosine function can be used instead of a sine function, and different phase shifts can produce the same graph.

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8.4.2: HOMEWORK

READING QUESTIONS

1. If a point (x, y) is on the terminal side of an angle θ on a circle of radius r, what are the formulas for x and y in terms of r and θ?
2. In the sinusoidal model $f(t) = A\sin(Bt + C) + D$, what does each parameter (A, B, C, D) represent in terms of the graph's properties?
3. What is the difference between ordinary frequency and angular frequency?
4. How is the period of a sinusoidal function related to its ordinary frequency?
5. What is the difference between a phase angle and a phase shift?
6. If a sinusoidal model has a maximum value of 50 and a minimum value of 10, what are its amplitude and midline?
7. If a Ferris wheel completes 2 revolutions in 120 seconds, what is its period in seconds?
8. Using the period from the previous question, what would be the angular frequency, B, for a sinusoidal model of the Ferris wheel's motion?
9. When building a sinusoidal model from data, what feature of the data helps you determine the vertical shift (midline)?
10. When building a sine model $(A > 0)$ from data, what feature of the data helps you determine the phase shift?

HOMEWORK

VOCABULARY CHECK

1. Change in position is also known as ___.
2. The period of a function measures the number of ___ per ___.
3. The ordinary frequency (or, simply, frequency) of a sinusoid measures the number of ___ per ___.
4. The unit for frequency is the ___.
5. In the function

$$f(t) = A\sin(Bt + C) + D,$$

B is called the ___, C is called the ___, and $-\frac{C}{B}$ is called the ___.

CONCEPT CHECK

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

The following questions concern the sinusoidal function

$$f(t) = A\sin(Bt + C) + D.$$

6. B is the period
7. C is the phase shift
8. $y = D$ is the midline
9. $-\frac{C}{B}$ is the phase shift
10. If T is the period of the sinusoid and f is the frequency, $f = \frac{1}{T}$.

BASIC SKILLS

For the following exercises,
(a) Estimate the amplitude, period, and midline of a sinusoidal function that fits the data.
(b)Write a formula for the function.

11.

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5
$f(t)$	5.2	4.26	2	-0.26	-1.2	-0.26	2	4.26	5.2	4.26	2

12.

s	0	1	2	3	4	5	6	7	8	9	10
$g(s)$	3	2.58	2.4	2.58	3	3.42	3.6	3.42	3	2.58	2.4

13.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$H(x)$	5	7.9	9.8	9.8	7.9	5	2.1	0.2	0.2

14.

t	0	0.79	1.57	2.36	3.14	3.93	4.71	5.50	6.28
$V(t)$	1	-0.17	-3	-5.8	-7	-5.3	-3	-0.17	1

APPLICATIONS

15. The sounds we hear are made up of mechanical waves. The note "A" above the note "middle C" is a sound wave with ordinary frequency

$$f = 440 \text{ Hertz } = 440\frac{\text{cycles}}{\text{second}}.$$

Find a sinusoid which models this note, assuming that the amplitude is 1 and the phase shift is 0.

16. The average daily high temperature in Fairbanks, Alaska can be approximated by a sinusoidal function with a period of 12 months. The low temperature of $-1.6°$ occurs in January, and the high temperature of $72.3°$ in July.
    a. What are the midline, period, and amplitude?
    b. Write a formula for the average daily high temperature $T(m)$, where m is the number of months since January.
    c. What is the displacement of the temperature?
    d. What is the ordinary frequency?
    e. What is the angular frequency?
    f. Graph $T(m)$ for two periods, labeling the points that correspond to highest and lowest average temperature.
17. Depending on its phase, the moon looks like a disk that is partially visible and partially in shadow. The visible fraction ranges from 0% to 100% and can be approximated by a sinusoidal function $V(t)$, where t is the number of days since the last full moon. The time between successive full moons (a lunar month) is 29.5 days.

a. What are the period, midline, and amplitude of $V(t)$?

b. Write a formula for $V(t)$.

c. What is the ordinary frequency?

d. What is the angular frequency?

e. Graph your function over two periods, labeling the points that correspond to full moon, half moon, and new moon.

18. The height of a child's toy suspended at the end of a spring is approximated by a sinusoidal function. The toy's height ranges between 200 centimeters and 260 centimeters above the ground, and it completes one up-and-down cycle every 0.8 seconds.

a. What are the midline, period, and amplitude?

b. Let $h(t)$ be the height of the toy in centimeters, where $t = 0$ seconds corresponds to a time when the object was at the midline and moving upwards. Graph $h(t)$ for two periods, labeling the points that correspond to the high and low positions of the toy.

c. What is the ordinary frequency of $h(t)$?

d. What is the angular frequency of $h(t)$?

e. When does the toy reach its maximum height the second time?

19. The voltage, V, in an alternating current source has amplitude $220\sqrt{2}$ and ordinary frequency $f = 60$ Hertz. Find a sinusoid which models this voltage. Assume that the phase is 0.

20. The London Eye is a popular tourist attraction in London, England and is one of the largest Ferris Wheels in the world. It has a diameter of 135 meters and makes one revolution (counterclockwise) every 30 minutes. It is constructed so that the lowest part of the Eye reaches ground level, enabling passengers to simply walk on and off the ride. Find a sinusoid which models the height h of the passenger above the ground in meters t minutes after they board the Eye at ground level.

21. Repeat the previous problem, but find a sinusoid which models the horizontal displacement, x, of the passenger from the center of the Eye in meters t minutes after they board the Eye. Here we take $x(t) > 0$ to mean the passenger is to the right of the center, while $x(t) < 0$ means the passenger is to the left of the center.

22. The average daily maximum temperature in Stockholm, Sweden is 30°F in January and 72°F in July.

a. Sketch a sinusoidal graph of $S(t)$, the average maximum temperature in Stockholm as a function of time, for one year.

b. Give the period, midline, and amplitude of your graph.

c. Write an equation for the function.

23. The average daily maximum temperature in Riyadh, Saudi Arabia is 86°F in January and 113°F in July.

a. Sketch a sinusoidal graph of $R(t)$, the average maximum temperature in Riyadh as a function of time, for one year.

b. Give the period, midline, and amplitude of your graph.

c. Write an equation for the function.

24. The height of the tide in Cabot Cove can be approximated by a sinusoidal function. At 5 am on July 23, the water level reached its high mark at the 20-foot line on the pier, and at 11 am, the water level was at its lowest at the 4-foot line.

a. Sketch a graph of $W(t)$, the water level as a function of time, from 5 am on July 23 to 5 am on July 24.

b. Give the period, midline, amplitude of your graph.

c. State the phase angle and phase shift.

d. State the ordinary frequency and angular frequency.

e. Write an equation for the function.

25. The population of mosquitoes at Marsh Lake is a sinusoidal function of time. The population peaks around June 1 at about 6000 mosquitoes per square kilometer, and is smallest on December 1, at 1000 mosquitoes per square kilometer.

a. Sketch a graph of $M(t)$, the number of mosquitoes as a function of the month, where $t = 0$ on June 1.

b. Give the period, midline, amplitude of your graph.

c. State the phase angle and phase shift.

d. State the ordinary frequency and angular frequency.

e. Write an equation for the function.

26. The paddlewheel on the Delta Queen steamboat is 28 feet in diameter, and is rotating once every ten seconds. The bottom of the paddlewheel is 4 feet below the surface of the water.

a. The ship's logo is painted on one of the paddlewheel blades. At $t = 0$, the blade with the logo is at the top of the wheel. Sketch a graph of the logo's height above the water as a function of t.

b. Give the period, midline, amplitude of your graph.

c. State the phase angle and phase shift.

d. State the ordinary frequency and angular frequency.

e. Write an equation for the function.

27. Min's bicycle wheel is 24 inches in diameter, and she has a light attached to the spokes 10 inches from the center of the wheel. It is dark, and she is cycling home slowly from work. The bicycle wheel makes one revolution every second.

a. At $t = 0$, the light is at its highest point the bicycle wheel. Sketch a graph of the light's height as a function of t.

b. Give the period, midline, amplitude of your graph.

c. State the phase angle and phase shift.

d. State the ordinary frequency and angular frequency.

e. Write an equation for the function.

28. You are performing the yo-yo trick, "Around the World," in which the yo-yo is thrown so it sweeps out a vertical circle. Suppose the yo-yo string is 28 inches and it completes one revolution in 3 seconds. If the closest the yo-yo ever gets to the ground is 2 inches, find a sinusoid which models the height, h, of the yo-yo above the ground in inches t seconds after it leaves its lowest point.

29. The table below lists the average temperature of Lake Erie as measured in Cleveland, Ohio on the first of the month for each month during the years 1971 – 2000. For example, $t = 3$ represents the average of the temperatures recorded for Lake Erie on every March 1 for the years 1971 through 2000.

Month Number, t	1	2	3	4	5	6	7	8	9	10	11	12
Temperature (°F), T	36	33	34	38	47	57	67	74	73	67	56	46

a. Fit a sinusoid to these data.

b. Using graphing technology, graph your model along with the data set to judge the reasonableness of the fit.

c. Use the model you found in part (a) to predict the average temperature recorded for Lake Erie on April 15th and September 15th during the years 1971–2000.

d. Compare your results to those obtained using graphing technology.

30. The fraction of the moon illuminated at midnight Eastern Standard Time on the tth day of June, 2009 is given in the table below.

Day of June, t	3	6	9	12	15	18	21	24	27	30
Fraction Illuminated, F	0.81	0.98	0.98	0.83	0.57	0.27	0.04	0.03	0.26	0.58

a. Fit a sinusoid to these data.

b. Using graphing technology, graph your model along with the data set to judge the reasonableness of the fit.

c. Use the model you found in part (a) to predict the fraction of the moon illuminated on June 1, 2009.

d. Compare your results to those obtained using graphing technology.

31. Consider the pendulum below. Ignoring air resistance, the angular displacement of the pendulum from the vertical position, θ, can be modeled as a sinusoid.

The amplitude of the sinusoid is the same as the initial angular displacement, θ_0, of the pendulum and the period of the motion is given by

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum and $g \approx 9.8\frac{\text{m}}{\text{s}^2}$ is the acceleration due to gravity.

a. Find a sinusoid which gives the angular displacement θ as a function of time, t. Arrange things so $\theta(0) = \theta_0$.

b. Suppose Sang's Seth-Thomas antique schoolhouse clock needs a period of $T = \frac{1}{2}$ second and she can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should she set the pendulum?

c. Continuing with part (b) and assuming the initial displacement of the pendulum is $15°$, find a sinusoid which models the displacement of the pendulum, θ, as a function of time, t, in seconds.

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8.5: Harmonic Motion

Note to the Instructor - Harmonic Motion and Differential Equations

The material in this section could easily be taught when covering the material in the previous section. I broke it into its own section because I didn't want to have a massive sinusoidal model section. Also, harmonic motion is a big topic in Differential Equations (nearly a full chapter's worth) and having a single section for it here might make it stand out more for the student.

Learning Objectives

- Build a model for simple harmonic motion.
- Build a model for damped harmonic motion.

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Hawk A.I. Section-Specific Tutor

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Harmonic motion is a form of periodic motion, but there are factors to consider that differentiate the two types. Applications involving general periodic motion cycle through their periods with no outside interference. This type of motion is what we have dealt with up to this point in Trigonometry. In terms of sinusoidal functions, we think of this as **oscillatory motion**.

Definition: Oscillatory Motion

Oscillatory motion is a repetitive back-and-forth or to-and-fro movement of an object around a central point or equilibrium position. This movement is characterized by a regular pattern of variation in a quantity, like displacement, velocity, or acceleration, over time.

Examples of oscillatory motion include temperature measurements throughout the day, the elevation of a rider on a Ferris wheel, and sound waves. In all of these examples, the motion is periodic. Still, there is no physical force requiring the measured quantity to tend back to the midline of the periodic motion.

Harmonic motion, on the other hand, requires a restorative force.

Definition: Harmonic Motion

Harmonic motion is a repetitive movement back and forth through an equilibrium point. It is characterized by a restorative force that's proportional to the displacement from equilibrium and acts in the opposite direction.

As a reminder from Physics, a force can be considered a push or pull. Thus, an object goes through harmonic motion if it oscillates (much like periodic motion) but is constantly being pulled by a force back to the midline of the oscillatory motion. Harmonic motion is crucial in Differential Equations, Engineering, and Physics. Harmonic motion includes the motion of springs, gravitational force, and magnetic force. It can further be broken into two major types - **simple harmonic motion** (also called **undamped harmonic motion**) and **damped harmonic motion**.[1]

Solving Exponential Equations

The material in this section relies heavily on your knowledge of exponential and logarithmic functions, the Laws of Logarithms, and your ability to solve exponential equations. Since it might have been a while since you have seen such topics, let's take a moment to review the "nuts and bolts" needed for the material at hand.

Definition: Exponential Function

For any real number x, an **exponential function** is a function with the form

$$f(x) = ab^x,$$

where a is a non-zero real number called the **initial value** and b is any positive real number, called the **base**, such that $b \neq 1$. This function is also referred to as the **exponential function base b**.

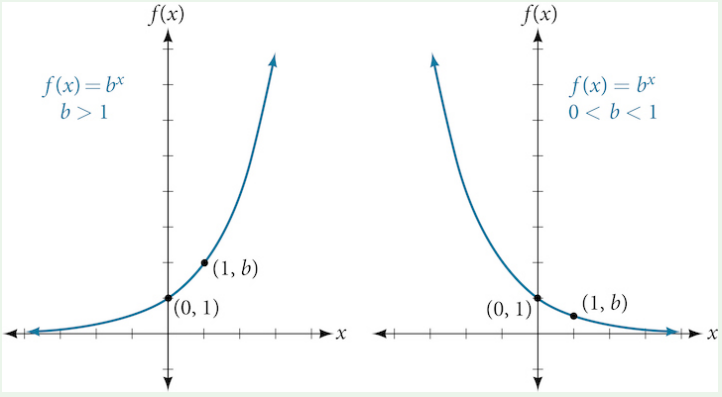
Many exponential functions appear in population models, where the function values rely on time. Therefore, you will commonly encounter exponential functions in the form $P(t) = P_0 \cdot b^t$, where $P(0) = P_0 \cdot b^0 = P_0$ and, as such, P_0 is called the **initial population**.

While there is a chapter's-worth of theory and material we could cover when talking about exponential functions (and the interested student *should* take a College Algebra course to explore such theory), for this section we just need to recall two pieces of theory. First up is the graph of the exponential function.

Theorem: Properties of the Graphs of Exponential Functions

An exponential function with the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$
- Secondary key point: $(1, b)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$



The leftmost graph in the theorem above is of significant value when we discuss damped harmonic motion near the end of this section.

The second piece of theory we must recall from Algebra is how to solve a simple exponential equation of the form $k = e^x$, where $k > 0$.

To solve such an equation, we need to remember that logarithms and exponentials are inverses to one another. More specifically,

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x.$$

Therefore, to solve $k = e^x$, we apply the natural logarithmic function to both sides of the equation to get the following:

$$\begin{aligned} k &= e^x \\ \implies \ln(k) &= \ln(e^x) \\ \implies \ln(k) &= x. \end{aligned}$$

Simple Harmonic Motion

The "prototype" motion, called **simple harmonic motion**, involves a restorative force but assumes that the motion will continue forever.

Definition: Simple Harmonic Motion

Simple harmonic motion is a special type of harmonic motion resulting from a sinusoidal oscillation which continues indefinitely. We call this type of harmonic motion a **non-driven**, **undamped** motion.

There are two words in this definition that you might be unfamiliar with.

A **driven** motion is one that experiences an additional pulse of energy. Think of this as the energy a parent gives to a child when they are one a swing. The child's motion on the swing as they go back and forth is harmonic motion. Every once in a while, the parent gives an extra push to the child so that their harmonic motion speeds back up - they are *driving* the harmonic motion with extra energy (in fact, the child swinging her legs also adds a bit of extra energy to keep her in motion).

In **non-driven** harmonic motion, the parent would not be pushing the child (and the child would not be swinging her legs to gain more "swing energy"). You can imagine what would happen in this scenario - the swing would oscillate back and forth, but with less magnitude each time. Eventually, the child would stop swinging.

But *why* would the child stop swinging?

In reality, there is almost always a **damping** force - a force that slows the oscillatory motion. In the case of the child on the swing, the damping force is due to friction (specifically, the resistive force of friction due to air and likely the friction on the chains connected to the swing set.

Undamped motion, on the other hand, never experiences loss of energy due to friction or other external factors. As such, simple harmonic motion (being both undamped and non-driven) is not the most realistic type of harmonic motion; however, it's a necessary starting point to understand the robust forms.

Imagine a weighted object hanging on a spring. When that object is not disturbed, we say that the object is at rest or in **equilibrium**. If the object is pulled down and then released, the spring's force (called the spring's **restorative force**) pulls the object back toward equilibrium, and harmonic motion begins. The restorative force of the spring is directly proportional to the object's displacement from its equilibrium point.[2]

Theorem: Simple Harmonic Motion Model

The motion of a particle undergoing simple harmonic motion is modeled using a sinusoidal function. It can be written as either

$$s(t) = A\sin(\omega t + \phi) \quad \text{or} \quad s(t) = A\cos(\omega t + \phi),$$

where the frequency of the motion is $f = \frac{1}{\text{Period}} = \frac{\omega}{2\pi}$ cycles per unit time.

The position of a particle relative to the ground is often unimportant when building a model for simple harmonic motion. This is because we are only concerned with modeling the particle's motion - not the height. Moreover, all our previous training with graphs of sinusoidal functions still holds here. The only difference is that we are replacing B with ω, and C with ϕ. This is mainly because ω and ϕ are the typical notations used in Physics.

Finally, the sinusoidal nature of these phenomena observed in physics is just that - an observation. We have not, and cannot, prove this result in Trigonometry. You do, however, prove this result in Differential Equations.

Example 8.5.1

The displacement of a mass suspended by a spring is modeled by the function

$$s(t) = 1.5 \sin(6\pi t),$$

where $s(t)$ is measured in cm and t in seconds.

a. Find the amplitude, period, and frequency of the motion of the mass.
b. Sketch a graph of the displacement of the mass.

Solutions

a. From our previous experience, the amplitude is $|1.5| = 1.5$, the period is $\frac{2\pi}{\omega} = \frac{2\pi}{6\pi} = \frac{1}{3}$ seconds per cycle, and the frequency is $f = \frac{1}{\text{Period}} = 3$ cycles per second.
b.

Figure 8.5.1

Checkpoint 8.5.1

A mass suspended from a spring is pulled 3 cm below its resting position. The mass is released at time $t = 0$ and allowed to oscillate. After $\frac{1}{3}$ second, it is observed that the mass returns to its lowest position. Find a function to model this motion relative to its initial resting position.

Answer

$s(t) = -3\cos(6\pi t)$

Damped Harmonic Motion

In reality, a pendulum does not swing back and forth forever, and an object on a spring does not bounce up and down forever. Eventually, the pendulum stops swinging, the object stops bouncing, and both return to equilibrium. As mentioned previously, periodic motion in which a damping force (typically friction) acts is known as **damped harmonic motion**.

Definition: Damped Harmonic Motion

Damped harmonic motion results from a sinusoidal oscillation that decreases in amplitude over time due to energy dissipation. This is typically caused by frictional forces or other dissipative forces that remove energy from the system.

In Physics, various formulas account for the damping factor on the moving object. Some of these are Calculus-based formulas that involve derivatives. We will use damping functions based on decreasing exponential functions (as these are the most common damping functions).

Theorem: Damped Harmonic Motion

The motion of a particle undergoing damped harmonic motion is modeled using a **quasi-sinusoidal function**. The displacement of the particle can be written as either

$$s(t) = A\,e^{-ct}\sin(\omega t + \phi) \quad \text{or} \quad s(t) = A\,e^{-ct}\cos(\omega t + \phi),$$

where c is called the **damping factor**, $\frac{2\pi}{\omega}$ is called the **quasi-period**, and $\frac{1}{\text{Quasi-Period}} = \frac{\omega}{2\pi}$ is called the **quasi-frequency**.

The words quasi-period and quasi-frequency reflect the fact that these models are not true sinusoidal models.

Conceptually (and graphically), the decreasing exponential function forces a boundary on the model's position values (outputs). The following Interactive Element should provide more understanding than I could write in a few paragraphs.

Example 8.5.2

Model the equations that fit the two scenarios and use graphing technology to graph the functions: Two mass-spring systems exhibit damped harmonic motion at a quasi-frequency of 0.5 cycles per second. Both have an initial displacement of 10 cm. The first has a damping factor of 0.5, and the second has a damping factor of 0.1.

Solutions

At time $t = 0$, the displacement is the maximum of 10 cm, which calls for the cosine function. The cosine function will apply to both models. We are given the quasi-frequency, $f = \frac{\omega}{2\pi}$, of 0.5 cycles per second. Thus,

$$\begin{aligned} \frac{\omega}{2\pi} &= 0.5 \\ \Longrightarrow \quad \omega &= \pi \end{aligned}$$

The first spring system has a damping factor of $c = 0.5$. Following the general model for damped harmonic motion, we have

$$s_1(t) = 10e^{-0.5t}\cos(\pi t).$$

Figure 8.5.2

The second spring system has a damping factor of $c = 0.1$ and can be modeled as

$$s_2(t) = 10e^{-0.1t}\cos(\pi t).$$

Figure 8.5.3

Notice the differing effects of the damping constant. The function's local maximum and minimum values with the damping factor $c = 0.5$ decrease much more rapidly than the function with $c = 0.1$.

? Checkpoint 8.5.2

Find and graph a function of the form $s(t) = Ae^{-ct}\cos(\omega t)$ that models the information given.

a. $A = 20$, $c = 0.05$, and the quasi-period is 4
b. $A = 2$, $c = 1.5$, and the quasi-frequency is 3

Answers

a. $s(t) = 20e^{-0.05t}\cos\left(\frac{\pi}{2}t\right)$.

Figure 8.5.4

b. $s(t) = 2e^{-1.5t}\cos(6\pi t)$.

Figure 8.5.5

Example 8.5.3

Find and graph a function of the form $s(t) = Ae^{-ct}\sin(\omega t)$ that models the information given.

a. $A = 7$, $c = 10$, and the quasi-period is $p = \frac{\pi}{6}$
b. $A = 0.3$, $c = 0.2$, and the quasi-frequency is $f = 20$

Solutions

Calculate the value of ω and substitute the known values into the model.

a. As the quasi-period is $\frac{2\pi}{\omega}$, we have

$$\begin{aligned} \frac{\pi}{6} &= \frac{2\pi}{\omega} \\ \Longrightarrow \quad \omega\pi &= 6(2\pi) \\ \Longrightarrow \quad \omega &= 12 \end{aligned}$$

The damping factor is given as 10 and A is 7. Thus, the model is $s(t) = 7e^{-10t}\sin(12t)$.

Figure 8.5.6

b. As the quasi-frequency is $\frac{\omega}{2\pi}$, we have

$$\begin{aligned} 20 &= \frac{\omega}{2\pi} \\ \Longrightarrow \quad 40\pi &= \omega \end{aligned}$$

The damping factor is given as 0.2 and $A = 0.3$. The model is $s(t) = 0.3e^{-0.2t}\sin(40\pi t)$.

Figure 8.5.7

A comparison of Examples 8.5.3*a* and 8.5.3*b* illustrates how we choose between the sine or cosine functions to model sinusoidal criteria. We see that the cosine function is at the maximum displacement when $t = 0$, and the sine function is at the equilibrium point when $t = 0$. For example, consider the equation

$$y = 20e^{-0.05t}\cos\left(\frac{\pi}{2}t\right)$$

from Checkpoint 8.5.2. We can see from the graph that when $t = 0$, $y = 20$, which is the initial amplitude. Check this by setting $t = 0$ in the cosine equation:

$$\begin{aligned} y &= 20e^{-0.05(0)}\cos\left(\frac{\pi}{2}\cdot 0\right) \\ &= 20(1)(1) \\ &= 20 \end{aligned}$$

Using the sine function yields

$$\begin{aligned} y &= 20e^{-0.05(0)}\sin\left(\frac{\pi}{2}\cdot 0\right) \\ &= 20(1)(0) \\ &= 0 \end{aligned}$$

Thus, cosine is the correct function.

? Checkpoint 8.5.3

Write the equation for damped harmonic motion given $A = 10$, $c = 0.5$, and the quasi-period is $p = 2$, where the initial value of the function is $s(0) = -10$.

Answer

$s(t) = -10e^{-0.5t}\cos(\pi t)$

✔ Example 8.5.4

A spring measuring 10 inches in natural length is compressed by 5 inches and released. It oscillates once every 3 seconds, and its amplitude decreases by 30% every second. Find an equation that models the position of the spring t seconds after being released.

Solution

The amplitude begins at 5 inches and deceases by 30% each second. We will make an "amplitude function" as an intermediate step. We can write the amplitude portion of the function as

$$\text{Amp}(t) = Ae^{-ct}.$$

We know that the amplitude is 5 when $t = 0$. Hence,

$$5 = Ae^{-c(0)} \implies 5 = Ae^0 \implies 5 = A(1) \implies 5 = A.$$

Therefore, our amplitude function becomes

$$\text{Amp}(t) = 5e^{-ct}.$$

We also know the amplitude drops by 30% when $t = 1$. "Dropping by 30%" is the same as saying the amplitude at $t = 1$ is 70% (100% minus 30%) of its original value at $t = 1$. Thus, we want to solve $0.7(5) = 5e^{-c(1)}$.

$$\begin{aligned} & 0.7(5) = 5e^{-c(1)} & \\ \implies \quad & 0.7 = e^{-c} & \text{(dividing)} \\ \implies \quad & \ln(0.7) = -c & \text{(taking the log of both sides)} \\ \implies \quad & -\ln(0.7) = c & \text{(solving for } c\text{)} \\ \implies \quad & c \approx -0.357 & \end{aligned}$$

Let's address the quasi-period. The spring completes a cycle every 3 seconds. This is the quasi-period, and we can use the formula to find ω.

$$3 = \frac{2\pi}{\omega}$$

$$\Rightarrow \quad \omega = \frac{2\pi}{3}$$

The natural length of 10 inches is the midline. We will use the cosine function since the spring starts at maximum displacement. This portion of the equation is represented as

$$y = \cos\left(\frac{2\pi}{3}t\right) + 10.$$

Finally, we put both functions together. Our model for the position of the spring at t seconds is given as

$$y \approx 5e^{-0.357t}\cos\left(\frac{2\pi}{3}t\right) + 10.$$

The approximation sign is needed because the coefficient of t is an approximation.

Figure 8.5.8

Example 8.5.5

A guitar string is plucked and vibrates in a damped harmonic motion. The string is pulled and displaced 2 cm from its resting position. After 3 seconds, the displacement measures 1 cm. Find the damping constant.

Solution

Following our tactic from Example 8.5.4, we recognize that the initial displacement, 2 cm, represents A. Therefore, the amplitude function for our model is

$$\text{Amp}(t) = 2e^{-ct}.$$

It is stated that after 3 seconds, maximum displacement measures one-half of its original value. Therefore, we have the equation

$$\frac{2}{2} = 3\,e^{-c(3)}.$$

Solving for c, we get the following:

$$\begin{aligned}
& \frac{2}{2} = 2\,e^{-c(3)} && \\
\Rightarrow \quad & 1 = 2e^{-3c} && \text{(simplifying)} \\
\Rightarrow \quad & \frac{1}{2} = e^{-3c} && \text{(dividing both sides by 2)} \\
\Rightarrow \quad & \ln\left(\frac{1}{2}\right) = -3c && \text{(taking the ln of both sides)} \\
\Rightarrow \quad & -\frac{\ln\left(\frac{1}{2}\right)}{3} = c && \text{(dividing both sides by } -3\text{)} \\
\Rightarrow \quad & \frac{\ln\left(\frac{1}{2}\right)^{-1}}{3} = c && \text{(Laws of Logarithms)} \\
\Rightarrow \quad & \frac{\ln(2)}{3} = c && \text{(Laws of Exponents)}
\end{aligned}$$

Thus, the damping constant is $c = \frac{\ln(2)}{3}$.

Footnotes

[1] Within both simple and damped harmonic motion, you also encounter free and driven (or forced) motion. These topics are reserved for Differential Equations; however, you can think of **free motion** as motion without extra external forces being applied. **Driven** (or **forced**) **motion** occurs when the object gains or loses kinetic energy due to an extra external force being applied.

For example, if we attach a mass to a spring, allow it to stretch to equilibrium, pull down slightly once it has reached equilibrium, and then let go, the motion of the mass will be free, damped motion. It starts oscillating, and we do not add any other forces to its movement; however, the spring constantly tries to pull (or push) the mass back to equilibrium. As time goes on, the amplitude of the oscillations decreases as the mass settles to the equilibrium position.

On the other hand, if we attach a mass to a spring, allow it to stretch to equilibrium, pull down slightly once it has reached equilibrium, let go, and then give it an extra push each time it reaches its lowest point (kind of like pushing a child on a swing), the motion is driven, damped motion. It's still damped because we are trying to be realistic by assuming that, when left alone, the amplitude of the oscillations should get smaller; however, it is driven because we add energy to the mass by pushing it each time it reaches its lowest point.

[2] From Algebra, F is **directly proportional** to x if there exists a constant, k, such that $F = kx$. k is called the **constant of proportionality**.

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8.5.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Exponential Functions**
  - The Basics of Exponential Functions: This section introduces damped harmonic motion, which is modeled using an exponential decay factor. Understanding the behavior of exponential functions is crucial.
  - Graphing Exponential Functions: Visualizing the exponential decay that bounds the graph of a damped harmonic motion function is a key concept in this section.
- **Logarithmic Functions**
  - The Basics of Logarithmic Functions: Since the Laws of Logarithms are needed, it is necessary to understand the fundamentals of logarithms.
  - Laws of Logarithms: These laws are used when solving for the damping constant in application problems, as seen in Example 5.
- **Solving Equations**
  - Solving Exponential Equations: This skill is required to find the damping constant, c, from an amplitude function of the form $A(t) = A_0e^{-ct}$, as demonstrated in Example 4.

VIDEOS

- Simple Harmonic Motion
  - Modeling Harmonic Motion - Simple Harmonic Motion
- Damped Harmonic Motion
  - Modeling Harmonic Motion - Damped Harmonic Motion

KEY CONCEPTS

DEFINITIONS

- **Oscillatory Motion:** A repetitive back-and-forth or to-and-fro movement of an object around a central point or equilibrium position.
- **Harmonic Motion:** A repetitive movement back and forth through an equilibrium point, characterized by a restorative force that is proportional to the displacement from equilibrium and acts in the opposite direction.
- **Exponential Function:** A function of the form $f(x) = ab^x$, where a is the non-zero initial value and b is a positive real number (not equal to 1) called the base.
- **Simple Harmonic Motion:** A special type of harmonic motion resulting from a sinusoidal oscillation that continues indefinitely without any loss of energy. It is also known as non-driven, undamped motion.
- **Damped Harmonic Motion:** A sinusoidal oscillation that decreases in amplitude over time due to energy dissipation, typically caused by frictional or other dissipative forces.

THEOREMS

- **Properties of the Graphs of Exponential Functions:** An exponential function of the form $f(x) = b^x$ (where $b > 0, b \neq 1$) is one-to-one, has a horizontal asymptote at $y = 0$, a domain of $(-\infty, \infty)$, a range of $(0, \infty)$, and a y-intercept at $(0, 1)$.
- **Simple Harmonic Motion Model:** The motion of a particle undergoing simple harmonic motion is modeled by a sinusoidal function, which can be written as either $s(t) = A\sin(\omega t + \phi)$ or $s(t) = A\cos(\omega t + \phi)$, where $f = \frac{\omega}{2\pi}$ is the frequency of the motion.
- **Damped Harmonic Motion:** The displacement of a particle undergoing damped harmonic motion is modeled using a quasi-sinusoidal function, which can be written as either $s(t) = Ae^{-ct}\sin(\omega t + \phi)$ or $s(t) = Ae^{-ct}\cos(\omega t + \phi)$. In these models, c is the damping factor, $\frac{2\pi}{\omega}$ is the quasi-period, and $\frac{\omega}{2\pi}$ is the quasi-frequency.

COMMON MISTAKES

- **Confusing Harmonic and Oscillatory Motion:** While all harmonic motion is oscillatory, not all oscillatory motion is harmonic. Harmonic motion specifically requires a restorative force that pulls an object back toward an equilibrium position.
- **Confusing Angular Frequency with Ordinary Frequency:** In the context of harmonic motion, ω represents the angular frequency. The ordinary frequency, f, which measures cycles per unit time (Hertz), must be calculated using the formula $f = \frac{\omega}{2\pi}$.
- **Choosing the Incorrect Model Function:** When modeling harmonic motion, a cosine function is typically used if the motion starts at maximum displacement ($t = 0$). A sine function is used if the motion starts at the equilibrium point ($t = 0$).
- **Ignoring the Damping Factor:** Forgetting that the amplitude in damped harmonic motion is not constant but is a decreasing exponential function, $A(t) = A_0e^{-ct}$, will lead to an incorrect model.

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8.5.2: HOMEWORK

READING QUESTIONS

1. What is the key feature that distinguishes harmonic motion from general oscillatory motion?
2. What is the difference between simple harmonic motion and damped harmonic motion?
3. In the model for simple harmonic motion, $s(t) = A\sin(\omega t + \phi)$, what do A and ω represent?
4. How is the ordinary frequency, f, related to the angular frequency, ω?
5. What is the general form of the function that models damped harmonic motion?
6. In the damped harmonic motion model, what is the term for the parameter c?
7. What is a "quasi-period"?
8. When modeling the motion of a spring, what initial condition suggests using a cosine function?
9. When modeling the motion of a spring, what initial condition suggests using a sine function?
10. To solve for the damping constant c in an amplitude function like $A(t) = A_0e^{-ct}$, what mathematical function is typically used?

HOMEWORK

VOCABULARY CHECK

1. The force associated with an object's tendency to return to equilibrium is called the ___ force.
2. Of the two types of harmonic motion discussed in this section, ___ harmonic motion is more realistic as it considers friction.
3. While an object undergoing simple harmonic motion has a period and frequency, an object undergoing damped harmonic motion has a ___ and ___.
4. Another name for simple harmonic motion is ___ harmonic motion.

CONCEPT CHECK

5. Explain the effect of a damping factor on the graphs of harmonic motion functions.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. A particle undergoing harmonic motion will theoretically oscillate forever.
7. A particle undergoing damped harmonic motion will theoretically oscillate forever.
8. The frequency of the particle undergoing damped harmonic motion described by

$$s(t) = 2e^{-0.01t}\sin\left(\frac{2\pi}{3}t\right)$$

is $f = \frac{\omega}{2\pi} = \frac{2\pi/3}{2\pi} = \frac{1}{3}$.

BASIC SKILLS

For the following exercises, find the amplitude, period, and frequency of the given function.

9. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function

$$h(t) = 14\sin(120\pi t),$$

where t is measured in seconds.

10. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function

$$h(t) = 8\sin(6\pi t),$$

where t is measured in seconds.

11. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function

$$h(t) = 11\sin(12\pi t),$$

where t is measured in seconds.

12. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function

$$h(t) = 4\cos(2\pi t),$$

where t is measured in seconds.

13. The displacement, $h(t)$, in centimeters, of a mass suspended by a spring is modeled by the function

$$h(t) = -5\cos(60\pi t),$$

where t is measured in seconds.

APPLICATIONS

For the following exercises, construct an equation that models the described behavior.

14. A spring attached to the ceiling is pulled 10 cm down from equilibrium and released. The amplitude decreases by 15% each second. The spring oscillates 18 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.
15. A spring attached to the ceiling is pulled 7 cm down from equilibrium and released. The amplitude decreases by 11% each second. The spring oscillates 20 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.
16. A spring attached to the ceiling is pulled 17 cm down from equilibrium and released. After 3 seconds, the amplitude has decreased to 13 cm. The spring oscillates 14 times each second. Find a function that models the distance, D, the end of the spring

is from equilibrium in terms of seconds, t, since the spring was released.

17. A spring attached to the ceiling is pulled 19 cm down from equilibrium and released. After 4 seconds, the amplitude has decreased to 14 cm. The spring oscillates 13 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

For the following exercises, create a function modeling the described behavior. Then, calculate the desired result using a calculator.

18. A spring attached to a ceiling is pulled down 11 cm from equilibrium and released. After 2 seconds, the amplitude has decreased to 6 cm. The spring oscillates 8 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

19. A spring attached to a ceiling is pulled down 21 cm from equilibrium and released. After 6 seconds, the amplitude has decreased to 4 cm. The spring oscillates 20 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

20. Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 8 times per second, was initially pulled down 32 cm from equilibrium, and the amplitude decreases by 50% each second. The second spring, oscillating 18 times per second, was initially pulled down 15 cm from equilibrium and after 4 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

21. Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 14 times per second, was initially pulled down 2 cm from equilibrium, and the amplitude decreases by 8% each second. The second spring, oscillating 22 times per second, was initially pulled down 10 cm from equilibrium and after 3 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

22. A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

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8.6: Graphs of the Remaining Trigonometric Functions

Note to the Instructor - Avoid the Temptation to Speed

The graphs of the rest of the trigonometric functions are covered in this section. Try your best to avoid treating this material like an afterthought. The graphs of the secant and cosecant spring up in odd places in Calculus, so the students need to be aware that this material, while held off for the end of the chapter, is critical to understand.

Learning Objectives

- Graph the cosecant, secant, and cotangent functions, along with transformations of these functions.
- Determine the equation of a trigonometric function given its graph.

Hawk A.I. Section-Specific Tutor

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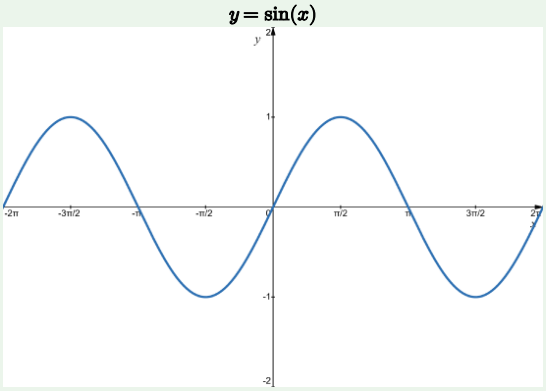
A Summary of the Graphs of the Fundamental Trigonometric Functions

Before we go too far, let's summarize what we know about the fundamental trigonometric functions.

Theorem: Properties of the Graph of the Sine Function

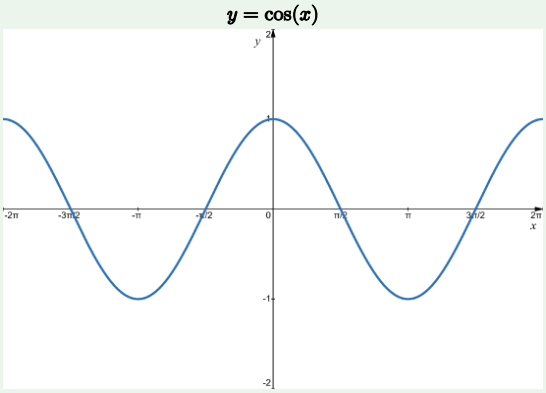
$y = \sin(x)$



- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = k\pi$, where $k \in \mathbb{Z}$

Theorem: Properties of the Graph of the Cosine Function

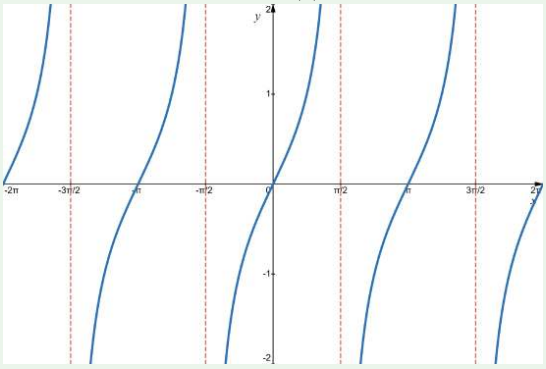
$y = \cos(x)$



- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Even (symmetric about the y-axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

Theorem: Properties of the Graph of the Tangent Function

$y = \tan(x)$



- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A
- **Zeros:** $x = k\pi$, where $k \in \mathbb{Z}$

The principal cycle of the tangent is $[0, \pi]$, even though the tangent is undefined halfway through the cycle. It's just the cycle we use as the "standard" for our base graph, and it is okay that the tangent is undefined at some point within the cycle. The principal cycle is *not* the domain.

An Interlude - Vertical Asymptotes

We now turn our attention to the graphs of the remaining trigonometric functions - $\csc(x)$, $\sec(x)$, and $\cot(x)$. In each of these cases, we must understand the conceptual approach to graphing we developed in when we introduced the graphs of the fundamental trigonometric functions.

Recall, if given a fraction where the numerator is held constant and the denominator is *approaching* zero (but not equal to zero), then the value of the fraction is approaching either $+\infty$ or $-\infty$. To drive the point home, let's demonstrate this behavior both numerically and graphically.

For the numerical demonstration, you can create any rational function you want where the numerator is a constant and the denominator is an algebraic expression. All we require is that the denominator approaches zero as the values of x approach some value. For example, suppose we choose the numerator to be 7 and the denominator to be $6x^2+11x+3$. Then the function we are working with is

$$R(x)=\frac{7}{6x^2+11x+3}.$$

We now want the values of x to approach some number that causes division by zero. To reveal what causes division by zero for this function, we factor the denominator as follows:

$$R(x)=\frac{7}{(3x+1)(2x+3)}.$$

We can see this function would have division by zero at $x=-\frac{1}{3}$ and $x=-\frac{3}{2}$.

Let's build two tables of values - the first to see what happens to the function as $x\to-\frac{3}{2}^-$, and the second to see what happens as $x\to-\frac{3}{2}^+$.[1] For convenience, we will rewrite $-\frac{3}{2}$ in its decimal form, -1.5.

x	Denominator $=(3x+1)(2x+3)$	$R(x)$
-2	5	1.4
-1.6	0.76	9.2105263
-1.55	0.365	19.178082
-1.51	0.0706	99.150142
-1.505	0.03515	199.14651
-1.501	0.007006	999.14359

x	Denominator $=(3x+1)(2x+3)$	$R(x)$
-1	-2	-3.5
-1.4	-0.64	-10.9375
-1.45	-0.335	-20.895522
-1.49	-0.0694	-100.86455
-1.495	-0.03485	-200.86083
-1.499	-0.006994	-1000.8579

In the first table, as x is approaching -1.5 from the left, the denominator of $R(x)$ is approaching zero from the right. Simultaneously, the function values are becoming larger and larger positive numbers. That is,

$$\text{As } x\to-\frac{3}{2}^-,\ R(x)\to\infty.$$

Similarly, the second table demonstrates that, as x is approaching -1.5 from the right, the denominator of $R(x)$ approaches zero from the left. This causes the function values to become larger and larger negative numbers. We would say,

$$\text{As } x\to-\frac{3}{2}^+,\ R(x)\to-\infty.$$

For the graphical demonstration, we quickly sketch the graph of $R(x)$ around $x=-\frac{3}{2}$.[2]

Figure 8.6.1

As you can see from the graph, the y-values of the function (also called the function values) become more and more positive as the x-values get closer and closer to -1.5 from the left. Likewise, the function values become increasingly more negative as the x-values approach -1.5 from the right.

The moral of the story can be stated as follows:

> *If the numerator is a nonzero constant and the denominator is tending to (but not becoming) zero, the ratio*
>
> $$\frac{\text{numerator}}{\text{denominator}}$$
>
> *approaches either $+\infty$ or $-\infty$. The sign can be determined by comparing the signs of the numerator and denominator as the denominator approaches zero. Graphically, this means there is a vertical asymptote at this value of x.*

Graphing the Cosecant Function

We know that $\csc(x) = \frac{1}{\sin(x)}$. Therefore, by the preceding discussion, as the denominator tends to zero, the overall ratio (which is the cosecant) will tend to $+\infty$ or $-\infty$. The zeros of the sine function occur at $x = k\pi$, where $k \in \mathbb{Z}$ (see Figure 8.6.2). Let's start by focusing on the zero at $x = 0$.

Figure 8.6.2: $y = \sin(x)$

From Figure 8.6.2, we can see as $x \to 0^-$, the y-values of $\sin(x)$ are approaching 0^-. Therefore,

$$\text{as } x \to 0^-, \frac{1}{\sin(x)} \to \frac{1}{0^-} \to -\infty.$$

It is *negative* infinity because the numerator is *positive*, but the denominator is negative (albeit, very close to zero). Similarly, as $x \to 0^+$, the y-values of $\sin(x)$ are approaching 0^+. Therefore,

$$\text{as } x \to 0^+, \frac{1}{\sin(x)} \to \frac{1}{0^+} \to +\infty = \infty.$$

Again, we know the result is *positive* infinity because the numerator is positive and the denominator is positive. Sketching this behavior, we get the following:

Figure 8.6.3: *Beginning the sketch of* $y = \csc(x)$

It's hard to see in Figure 3, but we drew a vertical asymptote at $x = 0$ (the y-axis). A similar investigation shows that

$$\text{as } x \to \pi^-, \ \frac{1}{\sin(x)} \to \frac{1}{0^+} \to \infty$$

and

$$\text{as } x \to \pi^+, \ \frac{1}{\sin(x)} \to \frac{1}{0^-} \to -\infty.$$

Using this to fill in a little more of the graph, we get the following:

Figure 8.6.4: *Getting closer to the graph of* $y = \csc(x)$

From this, we expect the cosecant function to have vertical asymptotes at these zeros of $\sin(x)$ (due to the division by zero). Furthermore, we expect the cosecant to be positive when its reciprocal (the sine) is positive, and negative when the reciprocal is negative. This gets us very close to the base graph of the cosecant.

Figure 8.6.5: *Almost there*

To finish out the graph of the base cosecant function, we will build a table of values for the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and use this to extrapolate the features for the rest of the graph.

x	$\csc(x)$
$\frac{\pi}{6}$	2
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{2}{\sqrt{3}}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{2}{\sqrt{3}}$
$\frac{3\pi}{4}$	$\sqrt{2}$
$\frac{5\pi}{6}$	2

Plotting these values, we get the following:

Figure 8.6.6: *So close!*

We can now see the pattern for the entire graph of the cosecant. Filling in the blanks, we get our final graph.

Figure 8.6.7: *The base graph of the cosecant,* $y = \csc(x)$

I have left a very lightly sketched sine function in this graph. This is because when you are graphing the cosecant, you will inevitably need to start by graphing its reciprocal.

Just as we did with the fundamental trigonometric functions, we now summarize the properties of the cosecant function (with the help of its graph as a visual aid).

Theorem: Properties of the Graph of the Cosecant Function

$y = \csc(x)$

- **Domain:** All $x \neq k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

Graphing the Secant Function

The derivation of the graph of the secant function is similar to that of the cosecant function. It is left for you as an exercise.

Theorem: Properties of the Graph of the Secant Function

$y = \sec(x)$

- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Even (symmetric about the y-axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

As with the graph of the cosecant, we usually lightly graph the reciprocal of the secant (which is the cosine) to help us in graphing the secant.

Graphing the Cotangent Function

As with the cosecant and the secant, to graph the cotangent function, we will use its reciprocal. That is, because $\cot(x) = \frac{1}{\tan(x)}$, we will first graph the tangent and discuss what happens as the tangent function tends to zero.

Figure 8.6.8: $y = \tan(x)$

Figure 8.6.8 shows the base graph of the tangent function. Using the same arguments as before, we can get the behavior of the reciprocal function, $\frac{1}{\tan(x)} = \cot(x)$, near the zeros of the tangent.

Figure 8.6.9: *Almost done graphing the base graph of the cotangent*

Notice in Figure 8.6.9 that the vertical asymptotes are not in the same places as the vertical asymptotes for the tangent. This is because the tangent is undefined when the cosine function is zero (at $x = \frac{\pi}{2} + k\pi$) and the cotangent is undefined when the sine is zero (at $x = k\pi$). Evaluating the cotangent function for a few special values between $x = 0$ and $x = \pi$ should give us the pattern of those missing middle regions.

x	$\cot(x)$
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	-1

Plotting these points, filling out the rest of the curve with the same pattern, and erasing the underlying tangent graph, we get the following:

Figure 8.6.10: *The base graph of the cotangent,* $y = \cot(x)$

As before, we list all of the properties in a convenient theorem.

Theorem: Properties of the Graph of the Cotangent Function

$y = \cot(x)$

- **Domain:** All $x \neq k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A

- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

Transformations of the Reciprocal Functions

Graphing transformations of the reciprocal functions is the same as graphing the transformations of the fundamental trigonometric functions. In fact, to graph transformations of the cosecant and secant functions, we first fully graph the transformations of their related fundamental functions. After that, graphing the transformed cosecant and secant functions is a somewhat trivial move. Let's see this in action.

Example 8.6.1

Graph a single cycle of each function. State all key characteristics of the graph (including domain, range, period, phase shift, etc.).

a. $f(x) = -1.2 + 2.8\sec\left(-3x - \frac{\pi}{2}\right)$
b. $g(x) = 2\cot(2x - 2) + 1$

Solutions

a. Just as we have done previously, we rewrite it in the better form

$$f(x) = 2.8\sec\left(-3x - \frac{\pi}{2}\right) - 1.2$$

Using the fact that secant is an even function, we clean this up a little more to the following:

$$f(x) = 2.8\sec\left(-\left(3x + \frac{\pi}{2}\right)\right) - 1.2 = 2.8\sec\left(3x + \frac{\pi}{2}\right) - 1.2$$

Now that we are finished "preconditioning" the function, it's time we discussed how we are going to graph it. With the secant and cosecant, we perform the intermediate step of graphing their reciprocal functions, the cosine and sine, respectively. So, let's graph

$$y = 2.8\cos\left(3x + \frac{\pi}{2}\right) - 1.2.$$

We list out all the essential information.

A	$=$	2.8	$\Longrightarrow$	Amplitude	$=$	2.8
B	$=$	3	$\Longrightarrow$	Period	$=$	$\frac{\text{Natural Period}}{B} = \frac{2\pi}{3}$
			$\Longrightarrow$	Step Size	$=$	$\frac{\text{Period}}{4} = \frac{2\pi/3}{4} = \frac{\pi}{6}$
C	$=$	$\frac{\pi}{2}$	$\Longrightarrow$	Phase Shift	$=$	$-\frac{C}{B} = -\frac{\pi/2}{3} = -\frac{\pi}{6}$
D	$=$	-1.2	$\Longrightarrow$	Vertical Shift	$=$	-1.2

As usual, we graph and label the midline, label the initial number as the phase shift, and step to the other key numbers using the step size.

Figure 8.6.11

Note that the distance between the first and last key number is

$$\left|\frac{3\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{2\pi}{3} = \text{Period},$$

so we have done something right! We then graph the non-reflected cosine, draw in and label the y-axis, and use that to place the x-axis.

Figure 8.6.12

Now comes the new part. Wherever the cosine function crosses the midline, the original function is undefined. This gives us our vertical asymptotes at $x = 0$, $x = \pm\frac{2\pi}{6} = \frac{\pi}{3}$, $x = \pm\frac{4\pi}{6} = \frac{2\pi}{3}$, and so on. Graphing these vertical asymptotes allows us to graph the secant.

Figure 8.6.13

Notice that I didn't remove the sketch of the cosine. This is perfectly fine (and some instructors expect you to leave it). Finally, we should state as much about this function as we can. The domain (by looking at our spectacular graph) is all $x \neq \frac{\pi}{3}k$, where $k \in \mathbb{Z}$. The range (again, from the graph) is $(-\infty, -4] \cup [1.6, \infty)$.[3] The only thing that we cannot state at this point are the zeros of this function (luckily, I didn't ask for them).

b. Before we start, I need to be clear and state that the secant and cosecant functions are the only ones where we must first graph their reciprocal functions. The base graph for the cotangent is simple enough to handle without bothering with the tangent.

Now, there isn't any preconditioning necessary for this function, so we go ahead and jump right into the other stages. Just like the tangent, the cotangent does not have an amplitude and its natural period is π.

B	$=$	2	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{2}$
			$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi/2}{4} = \dfrac{\pi}{8}$
C	$=$	-2	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{-2}{2} = 1$
D	$=$	1	$\Longrightarrow$	Vertical Shift	$=$	1

Graphing and labeling the midline, plotting the phase shift as the first key number, and stepping to each subsequent key number using the step size, we get the following:

Figure 8.6.14

Because the phase shift was a rational number and the step size was an irrational number, we could not easily add them. Leaving them in the form above is preferred. If you look at the base graph for the cotangent, its principal cycle occurs from $x = 0$ to $x = \pi$. This cycle starts and ends with a vertical asymptote, decreases from left to right, and crosses the midline halfway through. That's precisely what we are going to do here.

Figure 8.6.15

Now we place the y-axis one unit to the left of the phase shift and the x-axis one unit down from the midline. How do I know how far one unit is? It's more of an art of approximation.

Figure 8.6.16

Finally, we *should* label the two points on the curve at $x = 1 + \frac{\pi}{8}$ and $x = 1 + \frac{3\pi}{8}$. The good news is that the y-values of these points vary from the midline by $|A|$. Thus, the corresponding y-values are $y = 1 + 2 = 3$ and $y = 1 - 2 = -1$.

Figure 8.6.17

The domain of the function is all $x \neq 1 + \frac{\pi}{2}k$, where $k \in \mathbb{Z}$. The range is all real numbers, $\mathbb{R}$.

? Checkpoint 8.6.1

Graph the function. Clearly state all key information (including domain and range).

$$y = 3 - 4\csc\left(-\pi x + \frac{2}{3}\pi\right)$$

Answer

Amplitude of the related sine function: 4
Period: 2
Step Size: $\frac{1}{2}$
Phase Shift: $\frac{2}{3}$
Vertical Shift: 3
Domain: All $x \neq \frac{2}{3} + k$, where $k \in \mathbb{Z}$
Range: $(-\infty, 1] \cup [7, \infty)$

Figure 8.6.18

Finding Equations Given Graphs

Finding the equation of a reciprocal function given its graph requires a little bit more work than with the fundamental trigonometric functions.

Example 8.6.2

Find the equation of the graphed function.

a.

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Figure 8.6.19

b.

Figure 8.6.20

Solutions

a. As with the sine and cosine functions, graphs of the cosecant and secant are so similar that you could use either as an equation for the given graph (they would just be shifts of one another). A good rule of thumb when building trigonometric equations from graphs is to read the graph from left to right. The piece of information (whether a point or an equation of an asymptote) you have been given will set the beginning of the cycle for the trigonometric function. Therefore, in this case, we will assume the cycle of the trigonometric function starts at $x = \frac{3}{4}$. With the cosecant and secant, it's beneficial to draw a sinusoid "helper" function. Looking at the given graph, we can see that the midline should be at $y = -\frac{1}{2}$. Drawing this in allows us to sketch our "helper" function.

Figure 8.6.21

We now find the equation of the "helper" function. It is definitely a sine function starting at $x = \frac{3}{4}$, so the phase shift is

$$-\frac{C}{B} = \frac{3}{4}.$$

The midline is $y = -\frac{1}{2}$, so the vertical shift is

$$D = -\frac{1}{2}.$$

Additionally, the amplitude is the distance between the midline and the maximum or minimum values of the sinusoid. Since the maximum is $y = 0$ and the midline is $y = -\frac{1}{2}$, the amplitude is $\frac{1}{2}$. Moreover, since this sine function is not reflected,

$$A = \frac{1}{2}.$$

The only thing left to find is the period. This would give us an equation to find B, which in turn would allow us to find C. The cycle begins at $x = \frac{3}{4}$ and three-quarters of the way through the cycle, we are given the point $\left(\frac{21}{4}, -1\right)$. That is, the distance between $\frac{3}{4}$ and $\frac{21}{4}$ is three-quarters of the period.

$$\begin{aligned} \frac{3}{4} \text{ of the Period} &= \left|\frac{3}{4} - \frac{21}{4}\right| \\ &= \left|-\frac{18}{4}\right| \\ &= \frac{9}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Period} &= \left(\frac{9}{2}\right)\left(\frac{4}{3}\right) \\ &= 6 \end{aligned}$$

Since the period is $\frac{2\pi}{B}$,

$$\frac{2\pi}{B} = 6 \implies \frac{2\pi}{6} = B \implies \frac{\pi}{3} = B.$$

Earlier we stated that $-\frac{C}{B} = \frac{3}{4}$, so

$$-\frac{C}{\pi/3} = \frac{3}{4} \implies C = -\frac{\pi}{4}.$$

Thus, the "helper" function is

$$y = A\sin(Bx + C) + D = \frac{1}{2}\sin\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

Finally, the *actual* function is the reciprocal of the sine. That is, the function in the initial graph is

$$y = \frac{1}{2}\csc\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

b. Again, you *could* write the equation for this in terms of the (reflected) tangent function; however, we will write our equation in terms of the cotangent. Sometimes, you have to "eyeball" the features of a graph. In this example, the midline *looks* to be $y = 4$, so we will say $D = 4$. The phase shift could be $-\frac{\pi}{6}$ or $+\frac{\pi}{6}$, but the extra point they gave us is in the interval $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$, so we will choose our phase shift to be $-\frac{\pi}{6}$. This means

$$-\frac{C}{B} = -\frac{\pi}{6}.$$

Furthermore, the period is easy to spot on this graph. From vertical asymptote to vertical asymptote, the distance is $\frac{2\pi}{6} = \frac{\pi}{3}$. Therefore, the period is $\frac{\pi}{3}$ and, as such,

$$\frac{\pi}{B} = \frac{\pi}{3} \implies B = 3.$$

With this, we can find C.

$$-\frac{C}{B} = -\frac{\pi}{6} \implies \frac{C}{3} = \frac{\pi}{6} \implies C = \frac{\pi}{2}.$$

We are close to the finish line. Our function is

$$y = A \cot\left(3x + \frac{\pi}{2}\right) + 4.$$

Now we use the given data point to help us find the value of A.

$$\begin{aligned} 4.1 &= A \cot\left(3\left(-\frac{\pi}{12}\right) + \frac{\pi}{2}\right) + 4 \\ \implies 0.1 &= A \cot\left(-\frac{\pi}{4} + \frac{\pi}{2}\right) \\ \implies 0.1 &= A \cot\left(\frac{\pi}{4}\right) \\ \implies 0.1 &= A\,(1) \\ \implies 0.1 &= A \end{aligned}$$

Thus, the function is

$$y = 0.1 \cot\left(3x + \frac{\pi}{2}\right) + 4.$$

? Checkpoint 8.6.2

Find the equation of the graphed function.

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Figure 8.6.22

Answer

$y = 2\sec\left(3x + \frac{\pi}{5}\right) + 1$

Footnotes

[1] Recall, from section 5.1, the notation $x \to a^-$ means we are allowing the values of x to approach a, but from the left side of a. Likewise, $x \to a^+$ means x is approaching a, but from the right side of a. In both cases, we never allow x to become a.

[2] This is (hopefully) a skill you have from your prerequisite Algebra course; however, you might be taking this course in Trigonometry prior to taking a course in College Algebra. If this is the case, you will review how to graph rational functions once you take that course.

[3] With transformed functions, the graph is the easiest way to find the domain and range. This is a *great* reason to know how to graph these functions.

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8.6.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Graphs of Cosecant and Secant
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Secant and Cosecant
  - Graphs of the Other Trigonometric Functions - Graphing Secant and Cosecant Functions
- Graph of Cotangent
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Tangent and Cotangent
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions

KEY CONCEPTS

THEOREMS

- **Properties of the Graph of the Cosecant Function:**
  - Domain: All $x \neq k\pi$, where $k \in \mathbb{Z}$
  - Range: $(-\infty, -1] \cup [1, \infty)$
  - Symmetry: Odd (symmetric about the origin)
  - Principal Cycle: $[0, 2\pi]$
  - Natural Period: 2π
  - Amplitude: N/A
  - Zeros: None
- **Properties of the Graph of the Secant Function:**
  - Domain: All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
  - Range: $(-\infty, -1] \cup [1, \infty)$
  - Symmetry: Even (symmetric about the y-axis)
  - Principal Cycle: $[0, 2\pi]$
  - Natural Period: 2π
  - Amplitude: N/A
  - Zeros: None
- **Properties of the Graph of the Cotangent Function:**
  - Domain: All $x \neq k\pi$, where $k \in \mathbb{Z}$
  - Range: $(-\infty, \infty)$
  - Symmetry: Odd (symmetric about the origin)
  - Principal Cycle: $[0, \pi]$
  - Natural Period: π
  - Amplitude: N/A
  - Zeros: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

COMMON MISTAKES

- **Forgetting the Reciprocal Relationship:** The easiest way to graph the cosecant and secant functions is to first lightly sketch the graph of their reciprocal functions (sine and cosine, respectively). The key features of the reciprocal graph (zeros, maximums, minimums) determine the key features of the cosecant/secant graph (asymptotes, minimums, maximums).
- **Misplacing Asymptotes:** The vertical asymptotes for $y = \csc(x)$ and $y = \cot(x)$ occur where $\sin(x) = 0$. The vertical asymptotes for $y = \sec(x)$ occur where $\cos(x) = 0$. Confusing these will lead to incorrect graphs.
- **Incorrectly Transforming Asymptotes:** When graphing a transformed function like $y = A\sec(Bx + C) + D$, the vertical asymptotes are located where the "helper" function, $y = A\cos(Bx + C) + D$, crosses its midline, $y = D$.

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8.6.2: HOMEWORK

READING QUESTIONS

1. How is the graph of the cosecant function related to the graph of the sine function?
2. Where do the vertical asymptotes of the secant function occur?
3. What is the natural period of the cotangent function?
4. What is the domain of the cosecant function?
5. What is the range of the secant function?
6. Which of the reciprocal trigonometric functions are odd? Which are even?
7. What is the recommended first step when graphing a transformed cosecant or secant function?
8. Why do the cosecant, secant, and cotangent functions not have an amplitude?
9. The zeros of the sine function correspond to what features on the graph of the cosecant function?
10. The local maximums on the graph of the sine function correspond to what features on the graph of the cosecant function?

HOMEWORK

CONCEPT CHECK

1. Describe the general shape of the graph of the cosecant function.
2. What is the relationship between the sine function and the cosecant function? How does this relationship affect their graphs?
3. Describe the general shape of the graph of the secant function.
4. What is the relationship between the cosine function and the secant function? How does this relationship affect their graphs?
5. Describe the general shape of the graph of the cotangent function.
6. What is the relationship between the tangent function and the cotangent function? How does this relationship affect their graphs?

BASIC SKILLS

7. Determine the period, phase shift, and midline of the function $y = 3\sec(2x - \pi) + 1$.
8. Determine the period, phase shift, and midline of the function $y = 2\csc(x + \frac{\pi}{4}) - 3$.
9. Determine the period, phase shift, and midline of the function $y = -\cot\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 4$.
10. Determine the period, phase shift, and midline of the function $y = 4\sec\left(\frac{1}{3}x + \frac{\pi}{6}\right) - 2$.
11. Determine the period, phase shift, and midline of the function $y = -2\csc(3x - \pi) + 1$.
12. Determine the period, phase shift, and midline of the function $y = 3\cot\left(4x + \frac{\pi}{2}\right) - 5$.
13. Graph $y = \sec(x - \frac{\pi}{2}) + 2$.
14. Graph $y = 2\csc\left(\frac{\pi}{2}x\right) - 1$.
15. Graph $y = -\cot\left(x + \frac{\pi}{4}\right) + 3$.
16. Graph $y = 3\sec(2x) - 4$.
17. Graph $y = -\csc\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 2$.
18. Graph $y = \cot\left(\frac{3}{2}x + \frac{\pi}{6}\right) - 1$.
19. Graph $y = -2\sec\left(\frac{1}{3}x - \frac{\pi}{4}\right) + 5$.
20. Graph $y = 4\csc(3x) - 3$.
21. Graph $y = \cot(2x) + 4$.
22. Graph $y = -\sec\left(\frac{2}{3}x + \frac{\pi}{2}\right) - 2$.
23. Graph $y = \csc\left(\frac{\pi}{3}x - \frac{\pi}{6}\right) + 1$.
24. Graph $y = -3\cot\left(\frac{1}{2}x + \frac{\pi}{3}\right) + 2$.

For the following exercises, assume that the related sinusoidal function has an amplitude of 1.

25. Given the graph of a secant function that has a period of π, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift up by 2 units, determine the equation of the graph.
26. Given the graph of a cosecant function that has a period of 2π, a phase shift of $\frac{\pi}{6}$ to the left, and a vertical shift down by 3 units, determine the equation of the graph.
27. Given the graph of a reflected secant function that has a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{2}$ to the left, and a vertical shift down by 4 units, determine the equation of the graph.
28. Given the graph of a reflected cosecant function that has a period of 4π, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift up by 5 units, determine the equation of the graph.
29. Given the graph of a secant function with a period of 2π, a vertical shift up by 1 unit, and where the related sinusoidal function has an amplitude of 3, determine the equation of the graph.
30. Given the graph of a cosecant function with a period of π, a vertical shift down by 4 units, and where the related sinusoidal function has an amplitude of 2, determine the equation of the graph.
31. Given the graph of a reflected secant function with a period of $\frac{\pi}{3}$, a phase shift of $\frac{\pi}{2}$ to the left, a vertical shift down by 3 units, and where the related sinusoidal function has an amplitude of 1.2, determine the equation of the graph.
32. Given the graph of a reflected cosecant function with a period of $\frac{\pi}{4}$, a phase shift of $\frac{\pi}{6}$ to the right, a vertical shift up by 2 units, and where the related sinusoidal function has an amplitude of 2, determine the equation of the graph.

For the following exercises, find the equation of the function in terms of cosecant, secant, or cotangent.

33.

34.

35.

SYNTHESIS QUESTIONS

36. Given the function $y = 3\csc(2x - \frac{\pi}{2}) + 4$, find all asymptotes within the interval $[0, 2\pi]$. Explain the process of identifying these asymptotes.

37. Given the function $y = 2\sec\left(\frac{1}{2}x - \frac{\pi}{3}\right) - 1$, find all asymptotes within the interval $[-2\pi, 2\pi]$. Describe the method used to determine these asymptotes.

38. For the function $y = -3\csc(4x + \frac{\pi}{4}) + 2$, determine all asymptotes within the interval $[0, \pi]$. Explain your reasoning and the steps taken to find these asymptotes.

39. Given the function $y = \sec(3x - \frac{\pi}{6}) - 2$, find the equations of all vertical asymptotes within the interval $[-\pi, \pi]$. Describe the procedure used to identify these asymptotes.

40. For the function $y = -\csc\left(\frac{3}{2}x + \frac{\pi}{3}\right) + 5$, find the equations of all vertical asymptotes within the interval $[-\pi, \pi]$. Detail the steps and logic used to locate these asymptotes.

41. Consider the function $y = 4\sec\left(2x - \frac{\pi}{2}\right) + 1$. Identify all vertical asymptotes within the interval $[0, 2\pi]$ and explain how you determined their locations.

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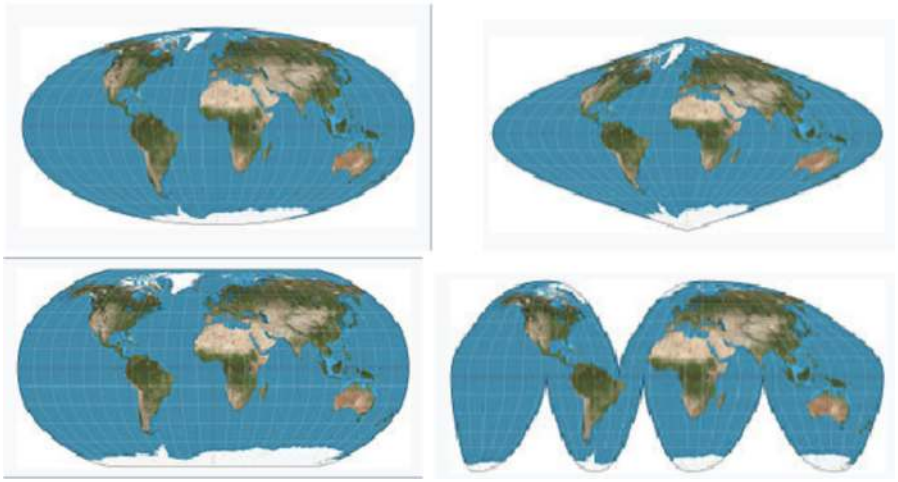
CHAPTER OVERVIEW

Chapter 9: Analytic Trigonometry

Mapmakers have always faced an unavoidable challenge: It is impossible to translate the surface of a sphere onto a flat map without some form of distortion. Over the years, a variety of map projections have been developed to suit different uses.

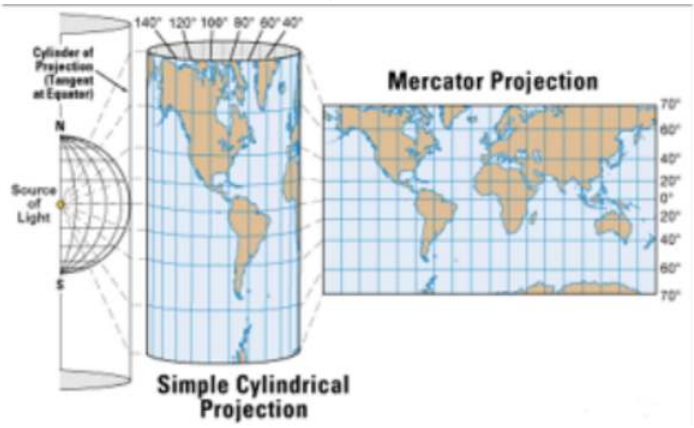
Figure 1



The sixteenth century was an age of discovery, when explorers and merchants began sailing to distant and previously unknown lands. But at that time there was no reliable technology for navigation. Although more regions of the world were being mapped more accurately, a flat map by itself was not enough to help a sailor in the middle of the ocean.

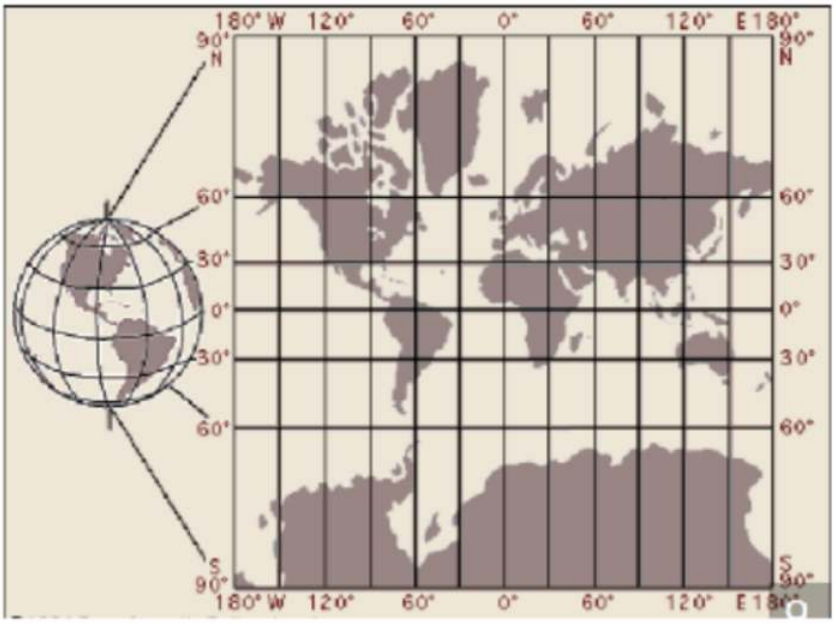
In 1569, the Flemish cartographer Gerardus Mercator published a new map using what is known as a cylindrical projection. To imagine how a Mercator projection works, picture shining a light through a glass globe onto a piece of paper rolled into a cylinder and wrapped around the globe. The cylinder is tangent to the globe at its equator.

Figure 2



Notice how the latitude lines are farther apart the farther you get from the Equator. This projection distorts the size of objects as the latitude increases, so that Greenland and Antarctica appear much larger than they actually are.

Figure 3



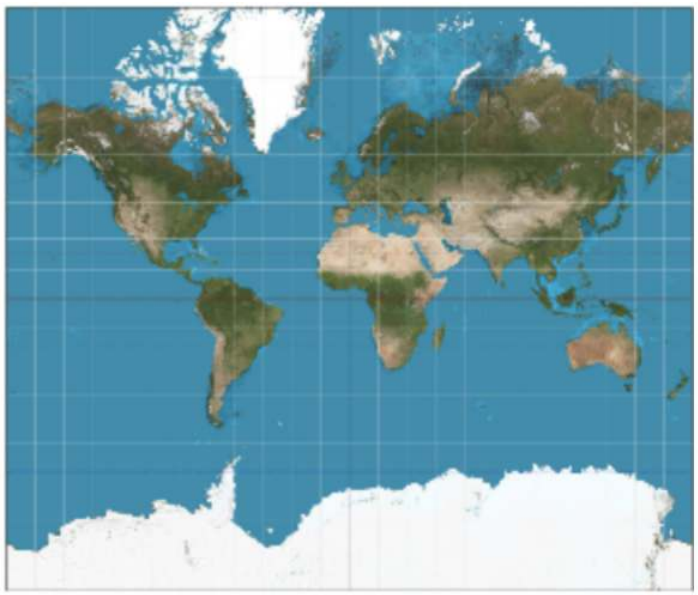
But the Mercator projection map is ideally suited for navigation, because any straight line on the map is a line of constant true bearing. If a navigator measures the bearing on the map from his location to his destination, he can set his ship's compass for the same bearing and maintain that course.

However, the Mercator projection does not preserve distances. On a globe, circles of latitude (also known as parallels) get smaller as they move away from the Equator towards the poles. Thus, in the Mercator projection, when a globe is "unwrapped" on to a rectangular map, the parallels need to be stretched to the length of the Equator. Mercator had to increase the scale of his map gradually as it moved away from the equator, so that the latitude lines appear equal in length to the equator.

The horizontal scale factor at any latitude must be inversely proportional to lengths on that latitude. Because the radius of the circle of latitude θ is $R\cos\theta$, the corresponding parallel on the map must be stretched by a factor of $\frac{1}{\cos\theta}$. And because the secant is the reciprocal of the cosine, the scale factor is proportional to the secant of the latitude.

A variety of "equal area" projections have been developed in modern times, but the Mercator projection is still widely used, in classrooms and atlases. For many people, it represents our image of the world.

Figure 4



9.1: Basic Trigonometric Identities and Proof Techniques
9.1.1: Resources and Key Concepts
9.1.2: Homework
9.2: Sum and Difference Identities
9.2.1: Resources and Key Concepts
9.2.2: Homework
9.3: Double-Angle Identities
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9.3.2: Homework
9.4: Half-Angle and Power Reduction Identities
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9.5: Sum-to-Product and Product-to-Sum Identities
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9.5.2: Homework

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9.1: Basic Trigonometric Identities and Proof Techniques

Note to the Instructor - It's a Brief Section

This section might seem awkward to you and your students if you don't buy into the purpose of its existence. The point of this material is to clarify what identities and contradictions are. Use this time to summarize the identities the students already know and focus on proof tactics. **Be very clear that performing mathematical operations "between sides of the equality" is illegal!** Doing such tom-foolery is assuming that the identity is true to begin with.

Learning Objectives

- Prove a trigonometric equation is an identity.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

We already introduced the concept of an identity early on in this textbook. At the time, we had only recently been introduced to the trigonometric functions; however, we had enough knowledge to understand and use the Reciprocal, Ratio, and Pythagorean Identities. Since then, we have encountered two other sets of identities - the Symmetry Identities and the Cofunction Identities. For convenience, the following tables summarize all of these basic identities and some of their more common equivalent forms.

Reciprocal Identities	
Basic Identity	***Alternate Form***
$\sin(\theta) = \frac{1}{\csc(\theta)}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\cos(\theta) = \frac{1}{\sec(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$
$\tan(\theta) = \frac{1}{\cot(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

Ratio Identities	
Basic Identity	***Alternate Form***
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities	
Basic Identity	***Alternate Forms***
$\cos^2(\theta) + \sin^2(\theta) = 1$	$\cos^2(\theta) = 1 - \sin^2(\theta)$ and $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$
	$\sin^2(\theta) = 1 - \cos^2(\theta)$ and $\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
$1 + \tan^2(\theta) = \sec^2(\theta)$	$\tan^2(\theta) = \sec^2(\theta) - 1$
$\cot^2(\theta) + 1 = \csc^2(\theta)$	$\cot^2(\theta) = \csc^2(\theta) - 1$

Symmetry Identities	
Basic Identity	***Alternate Form***
$\sin(-\theta) = -\sin(\theta)$	$\csc(-\theta) = -\csc(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\sec(-\theta) = \sec(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\cot(-\theta) = -\cot(\theta)$

Cofunction Identities	
Basic Identity	*Alternate Form*
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$
$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$
$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

Students who focus on memorization over understanding tend to panic at this point. They look at the preceding tables as huge lists of things to memorize. While it is true that you will have to recall these identities without looking them up or having them on a note card, it is crucial you understand most of these identities can be derived from the basic definitions of Trigonometry - this will not only shorten the list of what you feel you need to memorize, it will also give you a stronger foundation for success in Mathematics.

For example, the Pythagorean Identities are built from the unit circle definition of the trigonometric functions, the Ratio Identities, and the Reciprocal Identities. The unit circle definition tells us that any point, (x, y), on the unit circle can be represented as $(\cos(\theta), \sin(\theta))$, where θ is the angle in standard position whose terminal side contains the point (x, y). Since the equation of the unit circle is $x^2 + y^2 = 1$, this gives us the main Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. The alternate forms come from algebraic manipulations.

If we divide both sides of the main Pythagorean Identity by $\cos^2(\theta)$, we get

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \quad \Longrightarrow \quad \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta) \quad \Longrightarrow \quad 1 + \tan^2(\theta) = \sec^2(\theta).$$

A similar approach proves $\cot^2(\theta) + 1 = \csc^2(\theta)$.

The Symmetry Identities can be recalled very quickly from our graphical understanding of the trigonometric functions - cosine and its reciprocal are the only two trigonometric functions whose graphs are symmetric about the y-axis, so they are even functions. The rest of the trigonometric functions have graphs symmetric about the origin (so they are odd functions).

The Cofunction Identities are the only set of identities that we have not proved so far; however, we will remedy that in the next section.

Proving Identities

We have used identities to manipulate trigonometric expressions into different forms and to prove other identities. The remainder of this section reinforces the latter skill (which, in turn, will strengthen the former).

We start with a tactic to help us prove trigonometric identities.

How To... Prove a Trigonometric Identity

1. **Complex to Simple:** It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. **Algebra Before Trigonometry:** Follow the Mathematical Mantra. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. **Trigonometric Identities:** Always look for fundamental trigonometric identities - these can often be used as a key to proving a given statement. As we move forward, we will learn new identities to add to our list of fundamental identities.
4. **Keep an "Eye on the Prize:"** Always take note of the side you are not currently working on. It might contain clues as to what you should do next.
5. **Switch to Sines and Cosines:** If these steps do not yield the desired result, try converting all terms to sines and cosines.
6. **Switch Sides:** If you get stuck manipulating one side of the statement, move to the other side and work on that for a while.

It's important to note that the preceding "How To" is a "rule of thumb." As with most rules of thumb, there are exceptions to what to do and when to do it. For example, you might be given a statement to prove where *both* sides appear to be equal in complexity. In this case, you just choose a side to start with based on other factors. Likewise, you might find that switching trigonometric functions to sines and cosines makes things too messy. You might look for fundamental identities involving the reciprocal functions in these cases.

In the end, becoming proficient in proving trigonometric identities is all about practice, and the method is entirely heuristic. Despite having a tactic for tackling proofs of trigonometric identities, most of your understanding and mastery will come from proving them on your own.

Example 9.1.1

Verify $\tan(\theta)\cos(\theta) = \sin(\theta)$.

Solution

We will start on the left side, as it is the more complicated side:

LHS			RHS
$\tan(\theta)\cos(\theta)$			$\sin(\theta)$
$=$	$\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta)$	(Ratio Identity)	
$=$	$\sin(\theta)$		
$=$	RHS		

In the solution to Example **9.1.1**, we created a table where the first column contained the left-hand side (and its manipulations) and the second column contained the right-hand side (and, if we needed, it would have had manipulations of the right-hand side as well). The vertical bar between these two sides is purposeful. When trying to prove an identity, you are trying to show, through logical, deductive steps, that one side of the identity is equivalent to the other side. Until, or unless, you have proven these two sides are equivalent, you cannot assume they are the same. Therefore, you cannot treat it as an equation. This means you cannot add, subtract, multiply, divide, or square both sides because you assume the equation is true once you do so.

To be clear, when proving the identity in Example **9.1.1**, it would be illegal for us to do the following

$$\begin{aligned} & & \text{LHS} &= \text{RHS} \\ \implies & & \tan(\theta)\cos(\theta) &= \sin(\theta) \\ \implies & & \left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta) &= \sin(\theta) \\ \implies & \cancel{\cos(\theta)}\cdot\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cos(\theta) &= \cos(\theta)\cdot\sin(\theta) \\ \implies & & \sin(\theta)\cos(\theta) &= \sin(\theta)\cos(\theta) \end{aligned}$$

Doing this assumes that the two statements (the left side and the right side) are equal from the beginning.

Caution: Do not cross the line

When proving any identity (whether in Trigonometry or other Mathematics courses), treat each side separately and do not perform operations *between* sides. Do not add, subtract, multiply, or divide both sides by the same expression. Moreover, do not square both sides or perform any other operations between sides simultaneously.

Checkpoint 9.1.1

Verify the identity $\csc(\theta)\cos(\theta)\tan(\theta) = 1$.

Example 9.1.2

Prove the identity.

a. $[1+\sin(x)][1+\sin(-x)] = \cos^2(x)$
b. $\sin^2(\theta) = \frac{\sec^2(\theta)-1}{\sec^2(\theta)}$
c. $\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos(\theta)+\sin(-\theta)$
d. $[1-\cos^2(x)][1+\cot^2(x)] = 1$
e. $\frac{\cos(\theta)}{1+\sin(\theta)} = \frac{1-\sin(\theta)}{\cos(\theta)}$

Solutions

a. Working on the left side of the equation, we have

LHS			RHS
$[1+\sin(x)][1+\sin(-x)]$			$\cos^2(x)$
$=$	$[1+\sin(x)][1-\sin(x)]$	(Symmetry Identity)	
$=$	$1-\sin^2(x)$	(distribution)	
$=$	$\cos^2(x)$	(Pythagorean Identity)	
$=$	RHS		

b. This is a case where the right side is more complex. Let's start there.

LHS	RHS	
$\sin^2(\theta)$	$\frac{\sec^2(\theta)-1}{\sec^2(\theta)}$	
	$= \frac{\sec^2(\theta)}{\sec^2(\theta)} - \frac{1}{\sec^2(\theta)}$	(arithmetic)
	$= 1 - \frac{1}{\sec^2(\theta)}$	(arithmetic)
	$= 1-\cos^2(\theta)$	(Reciprocal Identity)
	$= \sin^2(\theta)$	(Pythagorean Identity)
	$=$ LHS	

c. It should be obvious that the left side is the more complicated side in this case; however, notice that both sides have negative angular arguments for some of their trigonometric functions. Let's start by cleaning those up on both sides.

LHS		RHS
$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}$		$\cos(\theta)+\sin(-\theta)$
		$= \cos(\theta)-\sin(\theta)$ (Symmetry Identity)
$= \frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)}$	(it's best to rewrite powers first)	
$= \frac{[-\sin(\theta)]^2-[\cos(\theta)]^2}{-\sin(\theta)-\cos(\theta)}$	(Symmetry Identities)	
$= \frac{[\sin(\theta)]^2-[\cos(\theta)]^2}{-\sin(\theta)-\cos(\theta)}$	(squaring the negative)	
$= -\frac{[\sin(\theta)]^2-[\cos(\theta)]^2}{\sin(\theta)+\cos(\theta)}$	(factoring the negative from the denominator)	
$= -\frac{[\sin(\theta)-\cos(\theta)][\sin(\theta)+\cos(\theta)]}{\sin(\theta)+\cos(\theta)}$	(Difference of Squares)	
$= -\frac{[\sin(\theta)-\cos(\theta)]\cancel{[\sin(\theta)+\cos(\theta)]}}{\cancel{\sin(\theta)+\cos(\theta)}}$	(canceling like factors)	
$= -[\sin(\theta)-\cos(\theta)]$		
$= -\sin(\theta)+\cos(\theta)$	(distribution)	
$= \cos(\theta)-\sin(\theta)$	(Commutative Property of Addition)	
$=$ RHS		

d. Starting with the left side, we get the following:

	LHS		RHS
	$[1-\cos^2(x)]\,[1+\cot^2(x)]$		1
$=$	$[1-\cos^2(x)]\left[1+\dfrac{\cos^2(x)}{\sin^2(x)}\right]$	(rewriting in terms of sines and cosines)	
$=$	$[1-\cos^2(x)]\left[\dfrac{\sin^2(x)}{\sin^2(x)}+\dfrac{\cos^2(x)}{\sin^2(x)}\right]$	(getting a common denominator)	
$=$	$[1-\cos^2(x)]\left[\dfrac{\sin^2(x)+\cos^2(x)}{\sin^2(x)}\right]$	(adding fractions)	
$=$	$[1-\cos^2(x)]\left[\dfrac{1}{\sin^2(x)}\right]$	(Pythagorean Identity)	
$=$	$[\sin^2(x)]\left[\dfrac{1}{\sin^2(x)}\right]$	(Pythagorean Identity)	
$=$	$\dfrac{\sin^2(x)}{\sin^2(x)}$		
$=$	1		
$=$	**RHS**		

e. Both sides of this identity are equally complex. Let's start with the left side and see how far we get. This example illustrates our need to "keep our eyes on the prize." If you want to somehow transform the left side into the right, we will eventually need to get a $1-\sin(\theta)$ in the numerator of the left side. That tells us where to start with this proof.

	LHS		RHS
	$\dfrac{\cos(\theta)}{1+\sin(\theta)}$		$\frac{1-\sin(\theta)}{\cos(\theta)}$
$=$	$\dfrac{\cos(\theta)\,[1-\sin(\theta)]}{1-\sin^2(\theta)}$	(distribution)	
$=$	$\dfrac{\cos(\theta)\,[1-\sin(\theta)]}{\cos^2(\theta)}$	(Pythagorean Identity)	
$=$	$\dfrac{\cancel{\cos(\theta)}\,[1-\sin(\theta)]}{\cos^{\cancel{2}}(\theta)}$	(cancel like factors)	
$=$	$\dfrac{1-\sin(\theta)}{\cos(\theta)}$		
$=$	**RHS**		

Example **9.1.2*d*** is a great example of a proof where making a different choice from the start can change the proof dramatically. For example, most people new to proofs would likely have distributed the expressions on the left side to get

	LHS		RHS
	$[1-\cos^2(x)]\,[1+\cot^2(x)]$		1
$=$	$1+\cot^2(x)-\cos^2(x)-\cos^2(x)\cot^2(x)$	(distribution)	
$=$	$1-\cos^2(x)+\cot^2(x)-\cos^2(x)\cot^2(x)$	(Commutative Property of Addition)	
$=$	$\sin^2(x)+\cot^2(x)-\cos^2(x)\cot^2(x)$	(Pythagorean Identity)	
$=$	$\sin^2(x)+\cot^2(x)\,[1-\cos^2(x)]$	(factor)	
$=$	$\sin^2(x)+\cot^2(x)\sin^2(x)$	(Pythagorean Identity)	
$=$	$\sin^2(x)+\dfrac{\cos^2(x)}{\sin^2(x)}\sin^2(x)$	(Ratio Identity)	
$=$	$\sin^2(x)+\dfrac{\cos^2(x)}{\cancel{\sin^2(x)}}\cancel{\sin^2(x)}$	(cancel like factors)	
$=$	$\sin^2(x)+\cos^2(x)$		
$=$	1	(Pythagorean Identity)	
$=$	**RHS**		

The moral of the story here is that there are often many ways to prove an identity. With practice, you begin to see the fastest routes through the proof.

? Checkpoint 9.1.2

Prove the identity.

a. $\frac{\cot(\theta)}{\csc(\theta)} = \cos(\theta)$

b. $\frac{\sin^2(\theta)-1}{\tan(\theta)\sin(\theta)-\tan(\theta)} = \frac{\sin(\theta)+1}{\tan(\theta)}$

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9.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Proof Techniques
  - Fundamental Identities - Tactics for Proving Trigonometric Identities
  - A Review of the Basic Trigonometric Identities - Proving Trigonometric Identities (Part 2)

KEY CONCEPTS

COMMON MISTAKES

- **Do Not Cross the Line:** When proving an identity, you must treat each side of the equation separately. Do not add, subtract, multiply, divide, or perform any other operation on both sides simultaneously, as this assumes the identity is already true.
- **Giving Up on One Side:** If you get stuck while manipulating one side of an identity, you should try working with the other side to see if you can meet in the middle or gain insight into the necessary steps.
- **Not Using Sines and Cosines:** When other algebraic or identity-based strategies fail, a reliable technique is to convert all trigonometric functions in the expression into their sine and cosine equivalents.
- **Building Instead of Simplifying:** It is almost always easier to start with the more complex side of an identity and simplify it to match the other side, rather than trying to build up the simpler side.

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9.1.2: HOMEWORK

READING QUESTIONS

1. What are the five main families of basic trigonometric identities summarized in this section?
2. What is the first step recommended in the "How To" guide for proving a trigonometric identity?
3. When proving an identity, why is it considered "illegal" to perform an operation on both sides of the equals sign?
4. If you are stuck while trying to prove an identity, what are two strategies the text suggests trying?
5. What does it mean to "keep an eye on the prize" when proving an identity?
6. In the proof of $[1+\sin(x)][1+\sin(-x)]=\cos^2(x)$, which two fundamental identities were used?
7. What is the general strategy for proving an identity if you are unsure where to begin?
8. Explain why it is generally easier to start with the more complex side of an identity.

HOMEWORK

CONCEPT CHECK

1. We showed that $\sin(\theta)=\pm\sqrt{1-\cos^2(\theta)}$ for all θ. Give an example of an angle θ such that $\sin(\theta)=-\sqrt{1-\cos^2(\theta)}$.
2. We showed that $\cos(\theta)=\pm\sqrt{1-\sin^2(\theta)}$ for all θ. Give an example of an angle θ such that $\cos(\theta)=-\sqrt{1-\sin^2(\theta)}$.
3. A student claims that $\cos(\theta)+\sin(\theta)=1$ is an identity because $\cos(0)+\sin(0)=1+0=0$. How would you respond to this student?

PROOFS

For the following exercises, prove the identity. Assume all quantities are defined.

4. $\tan(\theta)+\tan(-\theta)=0$
5. $\cos^2(\theta)-\sin^2(\theta)=2\cos^2(\theta)-1$
6. $\cos^2(\theta)-\sin^2(\theta)=1-2\sin^2(\theta)$
7. $\dfrac{\sin^2(\theta)}{1-\sin^2(\theta)}=\tan^2(\theta)$
8. $\sin^4(\theta)-\cos^4(\theta)=\sin^2(\theta)-\cos^2(\theta)$
9. $\sin^2(x)+\tan^2(x)+\cos^2(x)=\sec^2(x)$
10. $\cos^4(\theta)-\sin^4(\theta)=1-2\sin^2(\theta)$
11. $\dfrac{\sin(x)+\cos(x)}{\cos(x)}=\tan(x)+1$
12. $\dfrac{\cos^2(\theta)}{1+\sin(\theta)}=1-\sin(\theta)$
13. $\dfrac{\sin^2(x)}{1+\cos(x)}=1-\cos(x)$
14. $\dfrac{\cos(x)}{1+\sin(x)}=\sec(x)-\tan(x)$
15. $\dfrac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)}=\dfrac{\tan(x)-1}{1+\tan(x)}$
16. $\dfrac{1-\tan(\theta)}{1+\tan(\theta)}=\dfrac{\cot(\theta)-1}{\cot(\theta)+1}$
17. $\dfrac{1}{\tan(x)+\cot(x)}=\sin(x)\cdot\cos(x)$
18. $\dfrac{\tan(x)+\sec(x)-1}{\tan(x)-\sec(x)+1}=\dfrac{1+\sin(x)}{\cos(x)}$
19. $\dfrac{1+\sin(x)}{\cos(x)}=\sec(x)+\tan(x)$
20. $\dfrac{\sin(x)-\cos(x)}{\cos(-x)-\sin(-x)}=\dfrac{-1-\tan(-x)}{1+\tan(x)}$
21. $\dfrac{\sin(-x)}{1+\cos(x)}=\dfrac{\cos(-x)-1}{\sin(x)}$
22. $\dfrac{\sec(x)-1}{\sec(x)+1}=\dfrac{\sin^2(x)}{(\cos(x)+1)^2}$
23. $\dfrac{\sec(x)-1-\tan(-x)}{\tan(x)-\sec(-x)+1}=\dfrac{1-\sin(-x)}{\cos(x)}$
24. $\dfrac{\cos(-x)}{1-\sin(-x)}=\sec(-x)+\tan(-x)$
25. $\dfrac{1-\sin(-x)}{\cos(x)}=\sec(-x)-\tan(-x)$
26. $\sin(3\pi-2\theta)=-\sin(2\theta-3\pi)$
27. $\dfrac{\cos(x)}{1-\sin(x)}=\sec(x)+\tan(x)$
28. $\dfrac{1+\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)}=\csc\left(\frac{\pi}{2}-x\right)+\tan(x)$
29. $\dfrac{1-\cos(x)}{\cos(x)}=\dfrac{\sin^2(x)}{\cos(x)(1+\cos(x))}$
30. $\dfrac{\sin^2(x)}{\cos(x)}=\tan(x)\cdot\sin(x)$
31. $\dfrac{\sin(-x)-\cos(x)}{\cos(x)}=\tan(-x)-1$
32. $\dfrac{1-2\cos^2(\theta)}{\sin(\theta)\cos(\theta)}=\tan(\theta)-\cot(\theta)$
33. $\cos\left(-\dfrac{\pi}{4}-5t\right)=\cos\left(5t+\dfrac{\pi}{4}\right)$
34. $\dfrac{\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)+1}=\dfrac{1-\sin\left(\frac{\pi}{2}-x\right)}{\sin(x)}$
35. $\dfrac{1-\tan^2(\theta)}{1-\cot^2(\theta)}=1-\sec^2(\theta)$
36. $\dfrac{\cos(-x)}{1+\sin(-x)}=\sec(x)-\tan(-x)$
37. $\dfrac{1+\cos(x)}{\sin(x)}=\csc(x)+\cot(x)$
38. $\dfrac{1-\sin^2(x)}{\cos(x)}=\cos(x)$
39. $\dfrac{1-\sin(x)}{1+\sin(x)}=\dfrac{\sec(x)-\tan(x)}{\sec(x)+\tan(x)}$
40. $\dfrac{\sec(x)-1}{\sec(x)+1}=\dfrac{1-\cos(x)}{1+\cos(x)}$
41. $\dfrac{1+\sin(x)}{\cos(x)}=\dfrac{\cos(x)}{1-\sin(x)}$
42. $\tan(-t^2+1)=-\tan(t^2-1)$
43. $\dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)}=\csc\left(\frac{\pi}{2}-x\right)-\tan(x)$
44. $\dfrac{\sin(x)-\sin\left(\frac{\pi}{2}-x\right)}{\sin(x)+\cos(x)}=\dfrac{\cot\left(\frac{\pi}{2}-x\right)-1}{1+\tan(x)}$

45. $\frac{\cot\left(\frac{\pi}{2}-B\right)+\sec(B)-1}{\tan(B)-\csc\left(\frac{\pi}{2}-B\right)+1}=\frac{1-\sin(-B)}{\cos(B)}$
46. $\frac{\sin(\phi)}{1+\cos(\phi)}=\frac{1-\cos(\phi)}{\sin(\phi)}$
47. $\frac{\cos^2(\theta)}{1-\sin(\theta)}=1+\sin(\theta)$
48. $\csc(-\theta-5)=-\csc(\theta+5)$
49. $\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\sin(\alpha)}=\csc\left(\frac{\pi}{2}-\alpha\right)-\tan(\alpha)$
50. $\frac{1+\sin(-\theta)}{1+\sin(\theta)}=\frac{\sec(-\theta)+\tan(-\theta)}{\sec(-\theta)+\tan(\theta)}$
51. $\frac{1+\cos(-A)}{\sin(-A)}=-\csc(A)+\cot(-A)$
52. $\frac{\cos(-\phi)-1}{\cos(\phi)}=\frac{\sin^2(-\phi)}{\cos(\phi)(1+\cos(\phi))}$
53. $\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1-\sin(\theta)}=\csc\left(\frac{\pi}{2}-\theta\right)+\cot\left(\frac{\pi}{2}-\theta\right)$
54. $\frac{\sec(-\alpha)-1}{\sec(\alpha)+1}=\frac{\sin^2(-\alpha)}{(\cos(-\alpha)+1)^2}$
55. $\sec(-6t)=\sec(6t)$
56. $-\frac{\cos^2(-A)}{1+\sin(-A)}=-1+\sin(-A)$
57. $\frac{\tan(\theta)+\tan(\phi)}{\cot(\theta)+\cot(\phi)}=\tan(\theta)\tan(\phi)$
58. $\frac{\sin^2(-\alpha)}{\cos(-\alpha)}=-\tan(\alpha)\cdot\sin(-\alpha)$
59. $\frac{\sec(y)-1}{\sec(y)+1}=\frac{1-\cos(-y)}{1+\cos(-y)}$
60. $\frac{\sin^2\left(\frac{\pi}{2}-A\right)}{1-\sin(A)}=\frac{\cos^2(A)}{1-\cos\left(\frac{\pi}{2}-A\right)}$
61. $\frac{1+\sin\left(\frac{\pi}{2}-B\right)}{\sin(B)}=\sec\left(\frac{\pi}{2}-B\right)+\cot(B)$
62. $\frac{\sin\left(\frac{\pi}{2}-\phi\right)-1}{\sin\left(\frac{\pi}{2}-\phi\right)}=\frac{\sin^2(\phi)}{\sin\left(\frac{\pi}{2}-\phi\right)\left(1+\sin\left(\frac{\pi}{2}-\phi\right)\right)}$
63. $\cot(9-7\theta)=-\cot(7\theta-9)$
64. $\frac{\csc\left(\frac{\pi}{2}-B\right)-1}{\sec(B)+1}=\frac{\cos^2\left(\frac{\pi}{2}-B\right)}{\left(\sin\left(\frac{\pi}{2}-B\right)+1\right)^2}$
65. $\frac{1-\cos(\alpha)}{\sin^2(\alpha)}=\frac{1}{1+\cos(\alpha)}$

CHALLENGE PROBLEMS

66. Suppose that you are given a system of two equations of the following form:

$$\begin{aligned} A\cos(\phi) &= B\nu_1 - B\nu_2\cos(\theta) \\ A\sin(\phi) &= B\nu_2\sin(\theta) \end{aligned}$$

Show that

$$A^2 = B^2\left(\nu_1^2+\nu_2^2-2\nu_1\nu_2\cos(\theta)\right).$$

67. Suppose that a point with coordinates

$$(x,y)=\left(a(\cos(\psi)-\epsilon),a\sqrt{1-\epsilon^2}\sin(\psi)\right)$$

is a distance $r>0$ from the origin, where $a>0$ and $0<\epsilon<1$. Use $r^2=x^2+y^2$ to show that $r=a(1-\epsilon\cos(\psi))$.

(Note: These coordinates arise in the study of elliptical orbits of planets.)

68. Show that each trigonometric function can be written solely in terms of the sine function.

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9.2: Sum and Difference Identities

Note to the Instructor - Prove Things!

Do not succumb to laziness! Prove one of the Sum and Difference Identities. It's important that students see the logic necessary to derive such foundational theorems.

Learning Objectives

- Use the Sum and Difference Identities to find the exact value of a trigonometric function.
- Prove an equation is an identity using the Sum and Difference Identities.
- Simplify an expression using the Sum and Difference Identities.
- Use the Sum and Difference Identities to graph an equation.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

We have come a long way in Trigonometry. We have defined the trigonometric functions (in terms of angles, ratios of sides of right triangles, and real numbers), proved trigonometric identities with them, used them to solve applications, graphed them, and evaluated them by hand at the special angles. If we needed to evaluate a trigonometric function at a non-special angle, however, we must use technology. But, is this always the case?

For example, if we know the sine of $27°$, can we find the sine of $2(27°) = 54°$ without using a calculator? Or, if we know $\cos(\alpha)$ and $\cos(\beta)$, can we calculate $\cos(\alpha + \beta)$?

The Sum and Difference Identities for the Cosine

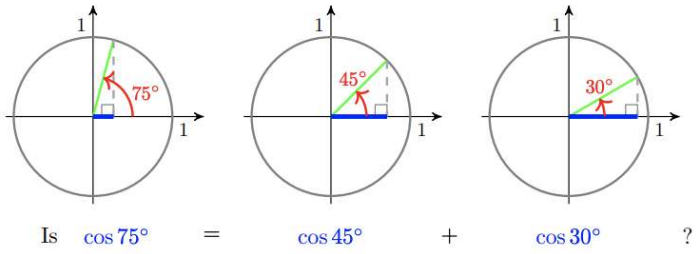
All the identities that relate the trigonometric ratios of different angles are derived from the Sum and Difference Identities (which really should be named the "Sum and Difference of Angles Identities," but that's too long to write). Let's see why we need these formulas.

Is it true that

$$\cos(\alpha + \beta) = \cos(\alpha) + \cos(\beta)?$$

We can test this hypothesis by evaluating both expressions for some specific values of α and β, say $\alpha = 45°$ and $\beta = 30°$, as shown below.

Figure 9.2.1



Taking a unit circle approach, you should be able to see that $\cos(75°)$ (which is the x-value of the point on the terminal side of $75°$) is in fact smaller than either $\cos(45°)$ or $\cos(30°)$. Therefore, it cannot be true that $\cos(75°)$ is equal to $\cos(45°) + \cos(30°)$.

Example 9.2.1

Verify that $\cos(45° + 30°)$ is not equal to $\cos(45°) + \cos(30°)$.

Solution

We know that $\cos(45°) = \frac{1}{\sqrt{2}} \approx 0.7071$ and $\cos(30°) = \frac{\sqrt{3}}{2} \approx 0.8660$. Thus, $\cos(45°) + \cos(30°) \approx 1.5731$. For $\cos(45° + 30°) = \cos(75°)$, we use a calculator to get the approximation 0.2588. Therefore, the two expressions are not equal.

Checkpoint 9.2.1

Show that $\sin(60° + 30°)$ is not equal to $\sin(60°) + \sin(30°)$.

Answer

$\sin(60° + 30°) = 1$, but $\sin(60°) + \sin(30°) = \frac{\sqrt{3}}{2} + \frac{1}{2}$

Caution: You Cannot "Split" a Trigonometric Function Across its Arguments

Because the values of the expressions in the Example 9.2.1 and Checkpoint 9.2.1 are different, it is *not* true that $\cos(\alpha + \beta)$ is equal to $\cos(\alpha) + \cos(\beta)$ for all angles α and β, or that $\sin(\alpha + \beta)$ is equal to $\sin(\alpha) + \sin(\beta)$.

This is a general property of almost all functions. That is, if f is a function, it is generally the case that

$$f(a \pm b) \neq f(a) \pm f(b).$$

It turns out that there is a relationship between the trigonometric ratios for $\alpha \pm \beta$ and the trigonometric ratios of α and β.

Theorem: Sum and Difference Identities for the Cosine

For angles α and β,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta).$$

Proof

Let α and β be angles in standard position and let α_0 and β_0 be the angles coterminal to α and β, respectively, each of which measure between 0 and 2π radians. Moreover, without loss of generality, assume that $\alpha_0 > \beta_0$ (if this assumption were not true, we would just relabel the variables accordingly).

Since α_0 is coterminal with α, $\alpha = \alpha_0 + 2\pi m$ for some $m \in \mathbb{Z}$. Likewise, $\beta = \beta_0 + 2\pi n$ for some $n \in \mathbb{Z}$. Therefore,

$$\cos(\alpha - \beta) = \cos(\alpha_0 + 2\pi m - (\beta_0 + 2\pi n)) = \cos(\alpha_0 - \beta_0 + 2\pi(m - n)).$$

Since m and n are both integers, $m - n$ is also an integer. Moreover, cosine has a period of 2π means that $\cos(\alpha_0 - \beta_0 + 2\pi(m - n)) = \cos(\alpha_0 - \beta_0)$. Thus,

$$\cos(\alpha - \beta) = \cos(\alpha_0 - \beta_0).$$

All of this is to reduce our work to the angles α_0 and β_0. We now draw and label the points P and Q on the unit circle which are on the terminal sides of α_0 and β_0, respectively.

The image on the right side of this figure is a rotation of the image on the left clockwise by β_0 radians. Since the angles $\angle POQ$ and $\angle AOB$ are congruent, the distance between P and Q is equal to the distance between A and B. The Distance Formula yields

$$\begin{aligned}
& \text{Disatnce between } P \text{ and } Q && = && \text{Distance between } A \text{ and } B && \\
\Longrightarrow\ & \sqrt{(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2} && = && \sqrt{(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2} && \text{(Distance Formula)} \\
\Longrightarrow\ & (\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2 && = && (\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 && \text{(squaring both sides)} \\
\Longrightarrow\ & \begin{array}{c}\cos^2(\alpha_0) - 2\cos(\alpha_0)\cos(\beta_0) + \cos^2(\beta_0) \\ + \sin^2(\alpha_0) - 2\sin(\alpha_0)\sin(\beta_0) + \sin^2(\beta_0)\end{array} && = && \begin{array}{c}\cos^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0) + 1 \\ + \sin^2(\alpha_0 - \beta_0)\end{array} && \text{(distributing)} \\
\Longrightarrow\ & \begin{array}{c}\cos^2(\alpha_0) + \sin^2(\alpha_0) \\ -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) \\ + \cos^2(\beta_0) + \sin^2(\beta_0)\end{array} && = && \begin{array}{c}\cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) \\ -2\cos(\alpha_0 - \beta_0) \\ +1\end{array} && \text{(Commutative Property of Addition)} \\
\Longrightarrow\ & \begin{array}{c}1 \\ -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) \\ +1\end{array} && = && \begin{array}{c}1 \\ -2\cos(\alpha_0 - \beta_0) \\ +1\end{array} && \text{(Pythagorean Identity)} \\
\Longrightarrow\ & 2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) && = && 2 - 2\cos(\alpha_0 - \beta_0) && \text{(simplifying)} \\
\Longrightarrow\ & -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) && = && -2\cos(\alpha_0 - \beta_0) && \text{(subtracting 2 from both sides)} \\
\Longrightarrow\ & \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0) && = && \cos(\alpha_0 - \beta_0) && \text{(dividing both sides by } -2\text{)}
\end{aligned}$$

Combining this result with the fact that $\cos(\alpha - \beta) = \cos(\alpha_0 - \beta_0)$, we have the identity

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta).$$

To get the sum identity for cosine, we use this result along with the Symmetry Identities.

$$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))=\cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

Hence,

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

The $\pm$ and $\mp$ signs on the left and right side of this identity are always opposite.

Advice: Memorize These Identities

The Sum and Difference Identities for the cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Notice that to find the cosine of $\alpha+\beta$ we must know (or be able to find) both trigonometric ratios for both and α and β.

The Sum and Difference Identities can be used to find exact values for trigonometric ratios of various angles.

Example 9.2.2

Find an exact value for $\cos(105°)$.

Solution

We can write $105°$ as the sum of two special angles: $105°=60°+45°$. Now apply the sum of angles identity for the cosine.

$$\begin{aligned}\cos(60°+45°) &= \cos(60°)\cos(45°)-\sin(60°)\sin(45°)\\ &= \frac{1}{2}\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Thus, $\cos 105°=\frac{\sqrt{2}-\sqrt{6}}{4}$. You can check that your calculator gives the same decimal approximation of about -0.2588 for both $\cos 105°$ and $\frac{\sqrt{2}-\sqrt{6}}{4}$.

Checkpoint 9.2.2

Suppose that $\sin(\theta)=\frac{-5}{13}$ and $\cos(\theta)=\frac{12}{13}$. Find an exact value for $\cos\left(\frac{\pi}{4}+\theta\right)$.

Answer

$$\frac{17}{13\sqrt{2}}=\frac{17\sqrt{2}}{26}$$

Proving the Cofunction Identities

We have been using the Cofunction Identities for quite a while, but have yet to see a proof. This is usually unforgivable in Mathematics; however, I wanted you to have some experience with these identities and, at the time, we didn't have the "mathematical technology" (read as, the knowledge) to prove them - now, we do. Let's prove $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$.

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta) && \text{(Difference Identity for the Cosine)}\\ &= (0)\cos(\theta)+(1)\sin(\theta) && \text{(evaluating trig functions)}\\ &= \sin(\theta)\end{aligned}$$

The remaining Cofunction Identities are left for you to prove as exercises.

The Sum and Difference Identities for the Sine

Theorem: Sum and Difference Identities for the Sine

For angles α and β,

$$\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta).$$

Proof

To prove the Sum and Difference Identities for the sine, we make use of the Cofunction Identities.

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\tfrac{\pi}{2}-(\alpha+\beta)\right) && \text{(Cofunction Identities)}\\
&= \cos\left(\left(\tfrac{\pi}{2}-\alpha\right)-\beta\right) && \text{(Associative Property of Addition)}\\
&= \cos\left(\tfrac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\tfrac{\pi}{2}-\alpha\right)\sin(\beta) && \text{(Difference Identity for Cosine)}\\
&= \sin(\alpha)\cos(\beta)+\sin\left(\tfrac{\pi}{2}-\alpha\right)\sin(\beta) && \text{(Cofunction Identities)}\\
&= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) && \text{(Cofunction Identities)}
\end{aligned}$$

Thus, we get the identity.

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta).$$

A similar proof shows that

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta).$$

Notice that the sine function "keeps its sign." That is, if we are given $\alpha+\beta$, the result will be a sum of functions. If given $\alpha-\beta$, we will have a difference of functions.

Advice: Memorize These Identities

The Sum and Difference Identities for the sine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 9.2.3

Suppose that $\sin(\theta)=0.6$ and $\cos(\theta)=-0.8$. Find an exact value for $\sin\left(\theta+\frac{2\pi}{3}\right)$.

Solution

Recall that $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{2\pi}{3}\right)=\frac{-1}{2}$. Substituting these values into the sum formula for sine, we find

$$\begin{aligned}
\sin\left(\theta+\frac{2\pi}{3}\right) &= \sin(\theta)\cos\left(\frac{2\pi}{3}\right)+\cos(\theta)\sin\left(\frac{2\pi}{3}\right)\\
&= 0.6\left(\frac{-1}{2}\right)+(-0.8)\left(\frac{\sqrt{3}}{2}\right)\\
&= \frac{3}{5}\cdot\frac{-1}{2}+\frac{-4}{5}\cdot\frac{\sqrt{3}}{2}\\
&= \frac{-3-4\sqrt{3}}{10}
\end{aligned}$$

Example 9.2.4

Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$.

Solution

We need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is $\frac{19\pi}{12}=\frac{16\pi}{12}+\frac{3\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$. We get

$$\begin{aligned}
\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)\\
&= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\
&= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\
&= \frac{-\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

? Checkpoint 9.2.4

Evaluate $\sin\left(\frac{7\pi}{12}\right)$ exactly.

Answer

$$\frac{\sqrt{3}+1}{2\sqrt{2}}$$

✓ Example 9.2.5

If α is a Quadrant II angle with $\sin(\alpha)=\frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta)=2$, find $\sin(\alpha-\beta)$.

Solution

This is one of my favorite styles of problems. There are two main ways to find the solution - using identities or using right triangle trigonometry. I will cover both methods here, but I must confess - I highly prefer the right triangle approach.

Using Identities

In order to find $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$. To find $\cos(\alpha)$, we use the Pythagorean Identity $\cos^2(\alpha)+\sin^2(\alpha)=1$. Since $\sin(\alpha)=\frac{5}{13}$, we have $\cos^2(\alpha)+\left(\frac{5}{13}\right)^2=1$, or $\cos(\alpha)=\pm\frac{12}{13}$. Since α is a Quadrant II angle, $\cos(\alpha)=-\frac{12}{13}$. We now set about finding $\cos(\beta)$ and $\sin(\beta)$. We have several ways to proceed, but the Pythagorean Identity $1+\tan^2(\beta)=\sec^2(\beta)$ is a quick way to get $\sec(\beta)$, and hence, $\cos(\beta)$. With $\tan(\beta)=2$, we get $1+2^2=\sec^2(\beta)$ so that $\sec(\beta)=\pm\sqrt{5}$. Since β is a Quadrant III angle, we choose $\sec(\beta)=-\sqrt{5}$ so $\cos(\beta)=\frac{1}{\sec(\beta)}=\frac{1}{-\sqrt{5}}=-\frac{\sqrt{5}}{5}$. We now need to determine $\sin(\beta)$. We could use The Pythagorean Identity $\cos^2(\beta)+\sin^2(\beta)=1$, but we opt instead to use a quotient identity. From $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, we have $\sin(\beta)=\tan(\beta)\cos(\beta)$ so we get $\sin(\beta)=(2)\left(-\frac{\sqrt{5}}{5}\right)=-\frac{2\sqrt{5}}{5}$. We now have all the pieces needed to find $\sin(\alpha-\beta)$:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ &= \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right)-\left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right)\\ &= -\frac{29\sqrt{5}}{65}\end{aligned}$$

Using Right Triangle Trigonometry

For this method, we remind ourselves that Trigonometry is mostly about quadrants and reference angles. We draw $\alpha\in$ QII and $\beta\in$ QIII and their corresponding reference angles.

Figure 9.2.2

Neither of these graphs are drawn to scale, nor do they need to be. Since $\sin(\alpha)=\frac{5}{13}$, we wisely choose the terminal point so that its y-value is 5 and its distance to the origin is 13. Looking at the reference triangle, this should remind you that sine is opposite over hypotenuse. Likewise, since $\tan(\beta)=2=\frac{2}{1}$, we choose the terminal point of β to have a y-value of -2 (remember, $\beta\in$ QIII) and an x-value of -1. Again, the tangent is opposite over adjacent, so the tangent of $\hat{\beta}$ is $\frac{-2}{-1}=2$.

Figure 9.2.3

Finally, we use the Pythagorean Theorem to help fill in the missing sides (taking into account the quadrants).

Figure 9.2.4

From these figures, we read off the information we need.

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)=\frac{5}{13}\cdot\frac{-1}{\sqrt{5}}-\frac{-12}{13}\cdot\frac{-2}{\sqrt{5}}=\frac{-5+(-24)}{13\sqrt{5}}=-\frac{29}{13\sqrt{5}}.$$

? Checkpoint 9.2.5

If α is a Quadrant II angle with $\sin(\alpha)=\frac{3}{5}$, and β is a Quadrant III angle with $\cos(\beta)=-\frac{5}{13}$, find $\cos(\alpha-\beta)$.

Answer

$-\frac{56}{65}$

The Sum and Difference Identities for the Tangent

There are also sum and difference formulas for the tangent. The proof for the tangent versions involves using the Quotient Identities to rewrite the tangent in terms of the sine and cosine, which we leave for you in the homework area.

Theorem: Sum and Difference Identities for the Tangent

For angles α and β where the following formula is defined,

$$\tan(\alpha\pm\beta)=\frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$$

There is a subtle restriction in the last theorem that the two before it didn't have. Since the tangent function does not have a domain consisting of all real numbers, the Sum and Difference Identities for the tangent have exceptions whenever α or β are odd multiples of $\frac{\pi}{2}$; however, this theorem is rarely used and, as such, this exception is not going to be encountered (at least, not at this level).

Example 9.2.6

Find an exact value for $\tan(75°)$.

Solution

We observe that $75° = 45° + 30°$, so $\tan(75°) = \tan(45° + 30°)$. We can apply the sum formula for tangent.

$$\begin{aligned}\tan(45° + 30°) &= \frac{\tan(45°) + \tan(30°)}{1 - \tan(45°)\tan(30°)} \\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\left(\frac{1}{\sqrt{3}}\right)} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

Checkpoint 9.2.6

Evaluate $\tan\left(\frac{\pi}{12}\right)$ exactly.

Answer

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Revisiting Proofs of Identities

Every time we introduce a new set of major identities (like the Sum and Difference Identities), we get to practice proofs again. In some sense, proofs begin to get much easier once we learn identities like the Sum and Difference Identities. This is because one of the first things we should look for when trying to prove an identity is the existence of other identities within the claim. That is, we look for Pythagorean Identities, Symmetry Identities, and so on. Now we have a new set of identities to look for.

Example 9.2.7

Prove the following identity.

$$\frac{\sin(\alpha - \beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha) - \tan(\beta)$$

Solution

The left side is the more complex side, so we start there. Also, we immediately spot our newly-discovered Sum and Difference Identities, so we know where to begin.

$$\begin{aligned}\frac{\sin(\alpha - \beta)}{\cos(\alpha)\cos(\beta)} &= \frac{\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} && \text{(Sum and Difference Identities)} \\ &= \frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} && \text{(property of fractions)} \\ &= \frac{\sin(\alpha)\,\cancel{\cos(\beta)}}{\cos(\alpha)\,\cancel{\cos(\beta)}} - \frac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)} && \text{(cancel like factors)} \\ &= \frac{\sin(\alpha)}{\cos(\alpha)} - \frac{\sin(\beta)}{\cos(\beta)} \\ &= \tan(\alpha) - \tan(\beta) && \text{(Ratio Identities)}\end{aligned}$$

Checkpoint 9.2.7

Prove.

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$$

Revisiting Graphs of Trigonometric Functions

The Sum and Difference Identities allow us to expand the set of trigonometric functions we can graph. Up to this point, we have only been able to graph functions of the form

$$f(x) = A\,\text{trig}\,(Bx + C) + D.$$

However, now we can graph certain *combinations* of trigonometric functions. Specifically, those having the form of the Sum and Difference Identities.

Example 9.2.8

Graph the function.

$$f(x) = \frac{1}{2}\cos(2x)\cos(3x) - \frac{1}{2}\sin(2x)\sin(3x)$$

Solution

Prior to this moment, there would have been no possibility for us to graph this function by hand; however, we now know, from the Sum and Difference Identities, that

$$f(x) = \frac{1}{2}\cos(2x)\cos(3x) - \frac{1}{2}\sin(2x)\sin(3x) = \frac{1}{2}\left[\cos(2x)\cos(3x) - \sin(2x)\sin(3x)\right] = \frac{1}{2}\cos(2x+3x) = \frac{1}{2}\cos(5x).$$

The amplitude of this cosine function is $\frac{1}{2}$, the period is $\frac{2\pi}{5}$, the step size is $\frac{\pi}{10}$, the phase shift is 0, and the vertical shift is 0. Putting this together, we get the following graph.

Figure 9.2.5

Checkpoint 9.2.8

Graph the function.

$$g(x) = -2\sin(\pi x)\cos\left(\frac{\pi}{2}x\right) + 2\sin\left(\frac{\pi}{2}x\right)\cos(\pi x)$$

Answer

Figure 9.2.6

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9.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Sum and Difference Identities
  - Sum and Difference Identities - The Sum and Difference Identities: Theorem and Proof
  - Sum and Difference Identities - Evaluating Trigonometric Functions of Non-Special Angles
  - Sum and Difference Identities - Using the Sum and Difference Identities
  - Sum and Difference Identities - Proving Trigonometric Identities (Part 3)

KEY CONCEPTS

THEOREMS

- **Sum and Difference Identities for the Cosine:** For any angles α and β:
  - $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$
  - $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$
- **Sum and Difference Identities for the Sine:** For any angles α and β:
  - $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
  - $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$
- **Sum and Difference Identities for the Tangent:** For any angles α and β where the functions are defined:
  - $\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$
  - $\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

COMMON MISTAKES

- **You Cannot "Split" a Trigonometric Function Across its Arguments:** A very common mistake is to assume that a trigonometric function distributes over a sum or difference. For example, $\cos(\alpha+\beta)$ is not equal to $\cos(\alpha)+\cos(\beta)$. The Sum and Difference Identities must be used instead.
- **Confusing the Signs in the Identities:** The signs in the Sum and Difference Identities can be tricky. For the cosine identities, the sign in the middle of the expression is the opposite of the operation in the argument (e.g., $\cos(\alpha+\beta)$ has a minus sign). For the sine identities, the sign in the middle matches the operation in the argument (e.g., $\sin(\alpha+\beta)$ has a plus sign).

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9.2.2: HOMEWORK

READING QUESTIONS

1. Is $\cos(A+B)$ equivalent to $\cos(A)+\cos(B)$? Explain why or why not.
2. What is the Sum Identity for the cosine function?
3. What is the Difference Identity for the sine function?
4. How can the Sum and Difference Identities be used to find an exact value for $\cos(105°)$?
5. This section proves the Cofunction Identity $\cos(\frac{\pi}{2}-\theta)=\sin(\theta)$. What primary identity was used to begin this proof?
6. What is the Sum Identity for the tangent function?
7. If you need to find $\sin(\alpha-\beta)$ and you are only given $\sin(\alpha)$ and $\tan(\beta)$ (along with their quadrants), what other trigonometric values must you find first?
8. How can the Sum and Difference Identities be used to simplify an expression like $\cos(2x)\cos(3x)-\sin(2x)\sin(3x)$ into a single trigonometric function?

SKILLS REFRESHER

? SKILLS REFRESHER

For the following exercises,
(a) Compute $f(2), f(3)$, and $f(2+3)$.
(b) Determine if it is true that $f(a+b)=f(a)+f(b)$ whenever the function values are defined.

1. $f(x)=3x+2$
2. $f(x)=x^2$
3. $f(x)=\sqrt{x}$
4. $f(x)=\frac{1}{x}$
5. $f(x)=|x|$
6. $f(x)=2^x$

Answers

1. a. 8, 11, 17
   b. Not true
2. a. 4, 9, 25
   b. Not true
3. a. $\sqrt{2},\sqrt{3},\sqrt{5}$
   b. Not true
4. a. $\frac{1}{2},\frac{1}{3},\frac{1}{5}$
   b. Not true
5. a. 2, 3, 5
   b. Not true
6. a. 4, 8, 32
   b. Not true

HOMEWORK

CONCEPT CHECK

1. Explain why $f(a+b)=f(a)+f(b)$ is not a valid application of the Distributive Law.
2. Nhat says that $\sin(\theta+\frac{\pi}{6})=\frac{1}{2}+\sin(\theta)$. Is she correct? Explain.
3. Piyali says that $\tan(\theta+\frac{\pi}{4})=\frac{1+\tan(\theta)}{1-\tan(\theta)}$. Is she correct? Explain.

BASIC SKILLS

4. Show that $\cos(45°+45°)$ is not equal to $\cos(45°)+\cos(45°)$.
5. Show that $\tan(60°-30°)$ is not equal to $\tan(60°)-\tan(30°)$.
6. Use your calculator to verify that $\tan(87°-29°)$ is not equal to $\tan(87°)-\tan(29°)$.
7. Use your calculator to verify that $\cos(52°+64°)$ is not equal to $\cos(52°)+\cos(64°)$.
8. Use graphs to show that $\sin(x-\frac{\pi}{6})$ is not equivalent to $\sin(x)-\sin(\frac{\pi}{6})$.
9. Use graphs to show that $\tan(x+\frac{\pi}{4})$ is not equivalent to $\tan(x)+\tan(\frac{\pi}{4})$.

For the following exercises, use the Sum and Difference Identities to find the exact value. You may have need of the Ratio, Reciprocal, or Symmetry Identities as well.

10. $\cos(15°)$
11. $\sin(15°)$
12. $\tan(15°)$
13. $\cos(75°)$
14. $\cos(165°)$
15. $\sin(165°)$
16. $\sec(165°)$
17. $\sin(105°)$
18. $\csc(195°)$
19. $\cot(255°)$
20. $\tan(375°)$
21. $\cos\left(\frac{13\pi}{12}\right)$
22. $\sin\left(\frac{11\pi}{12}\right)$
23. $\tan\left(\frac{13\pi}{12}\right)$
24. $\cos\left(\frac{7\pi}{12}\right)$

25. $\tan\left(\frac{17\pi}{12}\right)$
26. $\sin\left(\frac{\pi}{12}\right)$
27. $\cot\left(\frac{11\pi}{12}\right)$
28. $\csc\left(\frac{5\pi}{12}\right)$
29. $\sec\left(-\frac{\pi}{12}\right)$
30. $\sin\left(\frac{29\pi}{12}\right)$

For the following exercises, find the exact value of the expression without using a calculator.

31. $\cos(-10°)\cos(35°)+\sin(-10°)\sin(35°)$
32. $\cos\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)-\sin\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$
33. $\sin\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$
34. $\sin(80°)\cos(55°)+\cos(80°)\sin(55°)$

For the following exercises, find exact values for the trigonometric ratios. (Do not use a calculator!)

35. Suppose $\cos(\alpha)=\frac{3}{5}, \sin(\alpha)=\frac{4}{5}, \cos(\beta)=\frac{5}{13}$, and $\sin(\beta)=\frac{-12}{13}$. Evaluate the following.
    a. $\cos(\alpha+\beta)$
    b. $\sin(\alpha+\beta)$
    c. $\tan(\alpha+\beta)$
36. Suppose $\cos(\alpha)=\frac{-2}{3}, \sin(\alpha)=\frac{\sqrt{5}}{3}, \cos(\beta)=\frac{\sqrt{3}}{2}$, and $\sin(\beta)=\frac{-1}{2}$. Evaluate the following.
    a. $\cos(\alpha-\beta)$
    b. $\sin(\alpha-\beta)$
    c. $\tan(\alpha-\beta)$
37. If $\tan(t)=\frac{3}{4}$ and $\tan(s)=\frac{-7}{24}$, find exact values for the following.
    a. $\tan(s+t)$
    b. $\tan(s-t)$
38. If $\tan(x)=-3$ and $\tan(y)=-5$, find exact values for the following.
    a. $\tan(x+y)$
    b. $\tan(x-y)$
39. Suppose $\cos(\theta)=\frac{15}{17}$ and $\sin(\phi)=\frac{3}{5}$, where θ and ϕ are in quadrant I. Evaluate the following.
    a. $\cos(\theta+\phi)$
    b. $\tan(\theta-\phi)$
40. Suppose $\cos(\theta)=\frac{15}{17}$, where θ is in quadrant IV, and $\sin(\phi)=\frac{3}{5}$, where ϕ is in quadrant II. Evaluate the following.
    a. $\sin(\theta-\phi)$
    b. $\tan(\theta+\phi)$
41. If $\sin(\alpha)=\frac{12}{13}, \frac{\pi}{2}<\alpha<\pi$, and $\cos(\beta)=\frac{-3}{5}, \pi<\beta<\frac{3\pi}{2}$, find exact values for the following.
    a. $\sin(\alpha+\beta)$
    b. $\cos(\alpha+\beta)$
    c. $\tan(\alpha+\beta)$
    d. Sketch the angles α, β and $\alpha+\beta$.
42. If $\cos(\alpha)=\frac{3}{8}, \frac{3\pi}{2}<\alpha<2\pi$, and $\sin(\beta)=\frac{-1}{4}, \pi<\beta<\frac{3\pi}{2}$, find exact values for the following.
    a. $\sin(\alpha-\beta)$
    b. $\cos(\alpha-\beta)$
    c. $\tan(\alpha-\beta)$
    d. Sketch the angles α, β and $\alpha-\beta$.
43. If α is a Quadrant IV angle with $\cos(\alpha)=\frac{\sqrt{5}}{5}$, and $\sin(\beta)=\frac{\sqrt{10}}{10}$, where $\frac{\pi}{2}<\beta<\pi$, find
    a. $\cos(\alpha+\beta)$
    b. $\sin(\alpha+\beta)$
    c. $\tan(\alpha+\beta)$
    d. $\cos(\alpha-\beta)$
    e. $\sin(\alpha-\beta)$
    f. $\tan(\alpha-\beta)$
44. If $\csc(\alpha)=3$, where $0<\alpha<\frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta)=-7$, find
    a. $\cos(\alpha+\beta)$
    b. $\sin(\alpha+\beta)$
    c. $\tan(\alpha+\beta)$
    d. $\cos(\alpha-\beta)$
    e. $\sin(\alpha-\beta)$
    f. $\tan(\alpha-\beta)$
45. If $\sin(\alpha)=\frac{3}{5}$, where $0<\alpha<\frac{\pi}{2}$, and $\cos(\beta)=\frac{12}{13}$ where $\frac{3\pi}{2}<\beta<2\pi$, find
    a. $\sin(\alpha+\beta)$
    b. $\cos(\alpha-\beta)$
    c. $\tan(\alpha-\beta)$
46. If $\sec(\alpha)=-\frac{5}{3}$, where $\frac{\pi}{2}<\alpha<\pi$, and $\tan(\beta)=\frac{24}{7}$, where $\pi<\beta<\frac{3\pi}{2}$, find
    a. $\csc(\alpha-\beta)$
    b. $\sec(\alpha+\beta)$
    c. $\cot(\alpha+\beta)$
47. If $\sin(\theta)=0.2$ and $\cos(\theta)>0$, find $\sin\left(\theta+\frac{\pi}{3}\right)$.
48. If $\cos(\theta)=0.6$ and $\sin(\theta)<0$, find $\cos\left(\theta+\frac{3\pi}{4}\right)$.

For the following exercises, use the Sum and Difference Identities to expand each expression. Simplify your answers completely.

49. $\sin(\theta - 270°)$

50. $\cos(270° + \theta)$

51. $\cos\left(t + \frac{\pi}{6}\right)$

52. $\sin\left(t - \frac{2\pi}{3}\right)$

53. $\tan\left(\beta - \frac{\pi}{6}\right)$

54. $\tan\left(\phi + \frac{\pi}{4}\right)$

55. Use the Sum Identities for sine and cosine to derive a formula for each expression. Then use graphs to verify your formula.

    a. $\cos(\theta + 90°)$

    b. $\sin(\theta + 90°)$

56. Use the Sum Identities for sine and cosine to derive a formula for each expression. Then use graphs to verify your formula.

    a. $\cos(\theta + \pi)$

    b. $\sin(\theta + \pi)$

57. Use the Difference Identities for sine and cosine to derive formulas for the following.

    a. $\cos\left(\theta - \frac{\pi}{2}\right)$

    b. $\sin\left(\theta - \frac{\pi}{2}\right)$

For the following exercises, graph one cycle of the function by hand.

58. $f(x) = \sin(4x)\cos(3x) - \cos(4x)\sin(3x)$

59. $f(x) = \sin(x)\cos(2x) + \cos(2x)\sin(x)$

60. $f(x) = \frac{1}{5}\cos(5x)\cos(2x) + \frac{1}{5}\sin(5x)\sin(2x)$

61. $f(x) = 2\cos(x)\cos(2x) - 2\sin(x)\sin(2x)$

SYNTHESIS QUESTIONS

For the following exercises, simplify the function using the Sum and Difference Identities.

62. $f(x) = \sin\left(x + \frac{\pi}{2}\right)$

63. $f(x) = \cos\left(x - \frac{\pi}{4}\right)$

64. $f(x) = \tan(\pi - x)$

65. $f(x) = \sin\left(\frac{\pi}{6} - x\right)$

66. $f(x) = \cos\left(x + \frac{11\pi}{12}\right)$

67. $f(x) = \cos\left(\frac{2\pi}{3} - x\right)$

For the following exercises,
(a) Use graphs to decide if the equation is an identity.
(b) If the equation is not an identity, find a value of the variable that makes the equation false.

68. $\sin\left(\frac{\pi}{2} + \beta\right) = 1 + \sin(\beta)$

69. $\cos\left(\frac{\pi}{3} - \beta\right) = \cos\left(\beta - \frac{\pi}{3}\right)$

70. $\sin(A + 180°) = -\sin(A)$

71. $\cos\left(\phi + \frac{\pi}{3}\right) = \frac{1}{2} + \cos(\phi)$

72. $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin(x) + \cos(x))$

73. $2\cos\left(x - \frac{\pi}{6}\right) = \sin(x) + \sqrt{3}\cos(x)$

74. $\sin\left(x - \frac{\pi}{3}\right) + \cos\left(x + \frac{\pi}{6}\right) = 0$

75.

    a. Use a graphing device to draw the graph of $f(x) = \sin(x) + \cos(x)$ using $-\pi \le x \le 2\pi$ and $-2 \le y \le 2$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

    b. Use one of the sum identities to rewrite the expression $\sin\left(x + \frac{\pi}{4}\right)$. Then use the values of $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$ to further rewrite the expression.

    c. Use the result from part (b) to show that the function $f(x) = \sin(x) + \cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

76.

    a. Use a graphing device to draw the graph of $g(x) = \sin(x) + \sqrt{3}\cos(x)$ using $-\pi \le x \le 2\pi$ and $-2.5 \le y \le 2.5$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

    b. Use one of the sum identities to rewrite the expression $\sin\left(x + \frac{\pi}{3}\right)$. Then use the values of $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ to further rewrite the expression.

    c. Use the result from part (b) to show that the function $g(x) = \sin(x) + \sqrt{3}\cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

PROOFS

77. Use the Difference Identities for sine and cosine to prove the statement.

    a. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$

    b. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$

For the following exercises, prove the identity.

78. $\cos(\theta - \pi) = -\cos(\theta)$

79. $\sin(\pi - \theta) = \sin(\theta)$

80. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

81. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

82. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

83. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

84. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$

85. $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$

86. $\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

87. $\frac{\tan(\alpha + \beta)}{\tan(\alpha - \beta)} = \frac{\sin(\alpha)\cos(\alpha) + \sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha) - \sin(\beta)\cos(\beta)}$

88. $\frac{\sin(t + h) - \sin(t)}{h} = \cos(t)\left(\frac{\sin(h)}{h}\right) + \sin(t)\left(\frac{\cos(h) - 1}{h}\right)$

89. $\frac{\cos(t + h) - \cos(t)}{h} = \cos(t)\left(\frac{\cos(h) - 1}{h}\right) - \sin(t)\left(\frac{\sin(h)}{h}\right)$

90. $\frac{\tan(t+h)-\tan(t)}{h}=\left(\frac{\tan(h)}{h}\right)\left(\frac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

91. $\cot(A+B)=\frac{\cot(A)\cot(B)-1}{\cot(A)+\cot(B)}$

92. $\cot(A-B)=\frac{\cot(A)\cot(B)+1}{\cot(B)-\cot(A)}$

93. $\tan(\theta+45°)=\frac{1+\tan(\theta)}{1-\tan(\theta)}$

94. $\frac{\cos(A+B)}{\sin(A)\cos(B)}=\cot(A)-\tan(B)$

95. $\cot(A)+\cot(B)=\frac{\sin(A+B)}{\sin(A)\sin(B)}$

96. $\frac{\sin(A-B)}{\sin(A+B)}=\frac{\cot(B)-\cot(A)}{\cot(B)+\cot(A)}$

97. $\cos(A+B+C) = \cos(A)\cos(B)\cos(C) - \cos(A)\sin(B)\sin(C) - \sin(A)\cos(B)\sin(C) - \sin(A)\sin(B)\cos(C)$

98. $\tan(A+B+C) = \frac{\tan(A)+\tan(B)+\tan(C)-\tan(A)\tan(B)\tan(C)}{1-\tan(B)\tan(C)-\tan(A)\tan(C)-\tan(A)\tan(B)}$

99. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.

100. Verify the Difference Identities for sine and tangent.

CHALLENGE PROBLEMS

101. Prove the identity

$$\sin(\theta)+\cos(\theta)=\sqrt{2}\sin(\theta+45°).$$

Explain why this shows that

$$-\sqrt{2}\le\sin(\theta)+\cos(\theta)\le\sqrt{2}$$

for all angles θ. For which θ between $0°$ and $360°$ would $\sin(\theta)+\cos(\theta)$ be the largest?

For the following exercises, verify the Addition and Subtraction Identities for acute angles.

102. The figure below shows a right triangle inscribed in a rectangle.

a. Label the legs l_1 and l_2 of the right triangle with their lengths.

b. Explain why $\theta_1=\beta$ and $\theta_2=\alpha+\beta$. Label the diagram with these angles.

c. Label the legs s_1 and s_2 of the bottom triangle with their lengths.

d. Label the legs s_3 and s_4 of the top left triangle with their lengths.

e. Label the legs s_5 and s_6 of the top right triangle with their lengths.

f. Use the fact that the opposite sides of a rectangle are equal to state the addition formulas for sine and cosine.

103. The figure below shows a right triangle inscribed in a rectangle.

a. Label the legs l_1 and l_2 of the right triangle with their lengths.

b. Explain why $\theta_1=\beta$ and $\theta_2=\alpha-\beta$. Label the diagram with these angles.

c. Label the legs s_1 and s_2 of the bottom triangle with their lengths.

d. Label the legs s_3 and s_4 of the top left triangle with their lengths.

e. Label the legs s_5 and s_6 of the top right triangle with their lengths.

f. Use the fact that the opposite sides of a rectangle are equal to state the subtraction formulas for sine and cosine.

104. Follow the steps to prove the difference of angles formula for cosine,

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$$

a. Write an expression for $(AB)^2$, the square of the distance between the points A and B, using the Law of Cosines for $\triangle AOB$.

b. Write another expression for $(AB)^2$ using the Distance Formula and the coordinates of A and B.

c. Equate the two expressions for $(AB)^2$ you obtained in (a) and (b). Simplify the equation to obtain $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

105. For any positive angles A, B, and C such that $A+B+C=90°$, show that

$$\tan(A)\tan(B)+\tan(B)\tan(C)+\tan(C)\tan(A)=1.$$

106. Prove the identity

$$\sin(A+B)\cos(B)-\cos(A+B)\sin(B)=\sin(A).$$

Note that the right side depends only on A, while the left side depends on both A and B.

107. A line segment of length $r>0$ from the origin to the point (x,y) makes an angle α with the positive x-axis, so that $(x,y)=(r\cos(\alpha),r\sin(\alpha))$, as in the figure below. What are the endpoint's new coordinates (x',y') after a counterclockwise rotation by an angle β? Your answer should be in terms of r, α, and β.

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9.3: Double-Angle Identities

Note to the Instructor - Standard, but Important

There is not much I can clarify here that you don't already know. I suppose the only thing that I should repeat (from the text) is that the Sum and Difference Identities, and the Double-Angle Identities must be memorized (at least for the sine and cosine). Do *not* fall into the trap of allowing students to have these on a note card during exams - they will not have such luxury in Calculus.

Learning Objectives

- Use a Double-Angle Identity to find the exact value of a trigonometric function.
- Prove an identity using a Double-Angle Identity.
- Use a Double-Angle Identity to graph an equation.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The Double Angle Identities

Several other useful identities can be derived from the Sum and Difference Identities. The Double-Angle Identities are a special case of the Sum Identities, where $\alpha = \beta$.

Theorem: Double-Angle Identities

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 2\cos^2(\theta) - 1 \\
&= 1 - 2\sin^2(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)}
\end{aligned}$$

Proof

Deriving the Double-Angle Identity for sine begins with the Sum Identity,

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}
\sin(\theta + \theta) &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\
\Longrightarrow \quad \sin(2\theta) &= 2\sin(\theta)\cos(\theta)
\end{aligned}$$

Deriving the Double-Angle Identity for cosine gives us three options. First, starting from the Sum Identity,

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}
\cos(\theta + \theta) &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) \\
\Longrightarrow \quad \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)
\end{aligned}$$

Using the Pythagorean Identities, we can expand this Double-Angle Identity for cosine and get two more variations. The first variation is

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= (1 - \sin^2(\theta)) - \sin^2(\theta) \\ &= 1 - 2\sin^2(\theta)\end{aligned}$$

The second variation is

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - (1 - \cos^2(\theta)) \\ &= 2\cos^2(\theta) - 1\end{aligned}$$

Similarly, to derive the Double-Angle Identity for the tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \\ \Longrightarrow \quad \tan(\theta + \theta) &= \frac{\tan(\theta) + \tan(\theta)}{1 - \tan(\theta)\tan(\theta)} \\ \Longrightarrow \quad \tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)}\end{aligned}$$

Advice: Memorize The Sine and Cosine Versions of These Identities

The Double-Angle Identities for the sine and cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated. The tangent version is less common and can be derived from the sine and cosine versions if needed.

You can also justify the identities to yourself by graphing both sides of the formula to see that the graphs are identical.

Caution: Don't Factor Out of Functions!

The first thing we can learn from the Double-Angle Identities is that $\sin(2\theta)$ is not equal to $2\sin(\theta)$! You can check this very easily by choosing a value for θ, say $45°$. Then

$$\sin(2\theta) = \sin(90°) = 1$$

but

$$2\sin(\theta) = 2\sin(45°) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

and $\sqrt{2} \neq 1$. Similarly, $\cos(2\theta) \neq 2\cos(\theta)$ and $\tan(2\theta) \neq 2\tan(\theta)$.

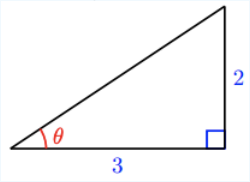
Finding Exact Values of Trigonometric Functions Involving Double Angles

We now reach a point in Trigonometry where we are told the value of a trigonometric function of θ, but we are interested in the value of a trigonometric function of 2θ. In these instances, the Double-Angle Identities (which, again, are special sub-cases of the Sum and Difference Identities) are what we need.

Example 9.3.1

Find $\sin(2\theta)$ for the angle θ shown in Figure 9.3.1.

Figure 9.3.1



Solution

We start by using the Pythagorean Theorem to find the hypotenuse of the triangle.

$$c^2 = 2^2 + 3^2 = 13$$

so $c = \sqrt{13}$. Thus, $\cos(\theta) = \frac{3}{\sqrt{13}}$ and $\sin(\theta) = \frac{2}{\sqrt{13}}$. Now we can use these values in the Double-Angle Identity to find $\sin(2\theta)$.

$$\begin{aligned} \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ &= 2\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right) \\ &= \frac{12}{13} \end{aligned}$$

Checkpoint 9.3.1

Find $\cos(2\theta)$ and $\tan(2\theta)$ for the angle θ shown in Example 9.3.1.

Answer

$\cos(2\theta) = \frac{5}{13}, \quad \tan(2\theta) = \frac{12}{5}$

Example 9.3.2

Given that $\cot(\theta) = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solutions

If we draw a triangle to reflect the information given, we can find the values needed to help us. We are given $\cot(\theta) = -\frac{3}{4}$, such that θ is in quadrant II. The cotangent of an angle is equal to the adjacent side over the opposite side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. We use the Pythagorean Theorem to find the length of the hypotenuse:

$$\begin{aligned} (-4)^2 + (3)^2 &= c^2 \\ \Longrightarrow \quad 16 + 9 &= c^2 \\ \Longrightarrow \quad 25 &= c^2 \\ \Longrightarrow \quad c &= 5 \end{aligned}$$

Now we can draw the situation appropriately.

Figure 9.3.2: *Not drawn to scale!*

When drawn like this, you can use the coordinate definition of the trigonometric functions or the right triangle definition using $\hat{\theta}$ and the fact that θ terminates in QII - it's your choice.

a. Let's begin by writing the Double-Angle Identity for sine.

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

We see that we to need to find $\sin(\theta)$ and $\cos(\theta)$. Based on our drawing, we see that the hypotenuse equals 5, so $\sin(\theta) = \frac{4}{5}$, and $\cos(\theta) = -\frac{3}{5}$. Substitute these values into the equation and simplify.Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the Double-Angle Identity for cosine.

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Again, substitute the values of the sine and cosine into the equation and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2\\ &= \frac{9}{25} - \frac{16}{25}\\ &= -\frac{7}{25}\end{aligned}$$

c. Write the Double-Angle Identity for tangent.

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$$

In this formula, we need the tangent. Using the Reciprocal Identity, we have $\tan(\theta) = -\frac{4}{3}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} \\ &= \frac{-\frac{8}{3}}{1-\frac{16}{9}} \\ &= -\frac{8}{3}\left(-\frac{9}{7}\right) \\ &= \frac{24}{7}\end{aligned}$$

A note about Example **9.3.2c** is in order. As we have seen throughout Trigonometry, several ways to approach a problem usually exist. It's easy to get so intent on using one method with a problem that we blind ourselves to simpler approaches. For example, we could have used the Ratio Identities to get

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-24/25}{-7/25} = \frac{24}{7}.$$

? Checkpoint 9.3.2

Given $\sin(\alpha) = \frac{5}{8}$, with θ in quadrant I, find $\cos(2\alpha)$.

Answer

$$\cos(2\alpha) = \frac{7}{32}$$

We can work with algebraic expressions instead of numerical values for the trigonometric ratios.

✔ Example 9.3.3

Use Figure **9.3.3** to express $\cos(2\phi)$ in terms of a.

Figure 9.3.3

Solution

We use the Pythagorean Theorem to find an expression for the third side of the triangle.

$$\begin{aligned} a^2 + b^2 &= 3^2 \\ \Longrightarrow \quad b &= \sqrt{9-a^2}\end{aligned}$$

Now we can write expressions for the sine and cosine of ϕ.

$$\cos(\phi) = \frac{a}{3} \quad \text{and} \quad \sin(\phi) = \frac{\sqrt{9-a^2}}{3}$$

Finally, we substitute these expressions into the Double-Angle Identity.

$$\begin{aligned}\cos(2\phi) &= \cos^2(\phi) - \sin^2(\phi) \\ &= \left(\frac{a}{3}\right)^2 - \left(\frac{\sqrt{9-a^2}}{3}\right)^2 \\ &= \frac{a^2}{9} - \frac{9-a^2}{9} \\ &= \frac{2a^2-9}{9}\end{aligned}$$

? Checkpoint 9.3.3

For the triangle in Example 9.3.2, find expressions for $\sin(2\phi)$ and $\tan(2\phi)$.

Answer

$$\sin(2\phi) = \frac{2a\sqrt{9-a^2}}{9}, \quad \tan(2\phi) = \frac{2a\sqrt{9-a^2}}{2a^2-9}$$

✓ Example 9.3.4

Find an expression for $\cos(2\beta)$ if you know that $\cos(\beta) = \frac{x}{4}$.

Solution

We use the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$.

$$\begin{aligned} \cos(2\beta) &= 2\cos^2(\beta) - 1 \\ &= 2\left(\frac{x}{4}\right)^2 - 1 \\ &= \frac{2x^2}{16} - 1 \\ &= \frac{2x^2 - 16}{16} \\ &= \frac{x^2 - 8}{8} \end{aligned}$$

? Checkpoint 9.3.4

Find an expression for $\cos(2\alpha)$ if you know that $\sin(\alpha) = \frac{6}{w}$.

Answer

$$\frac{w^2 - 76}{w^2}$$

Revisiting Proofs of Identities

Establishing identities using the Double-Angle Identities is performed using the same steps we used previously.

✓ Example 9.3.5

Verify the following identity.

$$1 + \sin(2\theta) = (\sin(\theta) + \cos(\theta))^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned} \text{RHS} &= (\sin(\theta)+\cos(\theta))^2 \\ &= \sin^2(\theta)+2\sin(\theta)\cos(\theta)+\cos^2(\theta) && \text{(distribution)} \\ &= (\sin^2(\theta)+\cos^2(\theta))+2\sin(\theta)\cos(\theta) && \text{(Commutative and Associative Properties of Addition)} \\ &= 1+2\sin(\theta)\cos(\theta) && \text{(Pythagorean Identities)} \\ &= 1+\sin(2\theta) && \text{(Double-Angle Identities)} \\ &= \text{LHS} \end{aligned}$$

The process in Example **9.3.5** is not complicated, as long as we recall the perfect square formula from algebra:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$

where $a = \sin(\theta)$ and $b = \cos(\theta)$. Part of being successful in Mathematics is the ability to recognize patterns. While the terms or symbols may change, the Algebra remains consistent.

? Checkpoint 9.3.5

Verify the identity.

$$\cos^4(\theta) - \sin^4(\theta) = \cos(2\theta).$$

Answer

$$\cos^4(\theta) - \sin^4(\theta) = (\cos^2(\theta) - \sin^2(\theta))(\cos^2(\theta) + \sin^2(\theta)) = \cos(2\theta) \cdot 1 = \cos(2\theta)$$

✓ Example 9.3.6

Prove the identity.

$$\tan(2\theta) = \frac{2}{\cot(\theta) - \tan(\theta)}$$

Solution

Despite the right side looking more complicated, the obvious Double-Angle Identity on the left side tells us that we should at least start on that side and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned} \text{LHS} &= \tan(2\theta) \\ &= \frac{2\tan(\theta)}{1-\tan^2(\theta)} && \text{(Double-Angle Identity)} \\ &= \frac{2\tan(\theta)\left(\frac{1}{\tan(\theta)}\right)}{(1-\tan^2(\theta))\left(\frac{1}{\tan(\theta)}\right)} && \text{(multiply by a term that results in desired numerator)} \\ &= \frac{2}{\frac{1}{\tan(\theta)} - \frac{\tan^2(\theta)}{\tan(\theta)}} \\ &= \frac{2}{\cot(\theta) - \tan(\theta)} && \text{(Reciprocal Identity)} \\ &= \text{RHS} \end{aligned}$$

In Example **9.3.6**, if we had chosen to work on the right side, we would have worked backward to arrive at the equivalency. While this is possible (and not too tricky), I like taking advantage of obvious identities. There are usually several paths to a desired result

when using the identities to simplify a trigonometric expression or solve a trigonometric equation.

Checkpoint 9.3.6

Prove the identity.

$$\cos(2\theta)\cos\theta = \cos^3(\theta) - \cos(\theta)\sin^2(\theta)$$

Answer

$$\cos(2\theta)\cos(\theta) = (\cos^2(\theta) - \sin^2(\theta))\cos(\theta) = \cos^3(\theta) - \cos(\theta)\sin^2(\theta)$$

Revisiting Graphs of Trigonometric Functions

As with the previous section, we use this opportunity to expand the types of trigonometric functions we can graph.

Example 9.3.7

Graph the function.

$$f(x) = -8\cos^2(3x) + 4$$

Solution

Careful observation and a little pre-factoring will reveal that this function contains a Double-Angle Identity.

$$\begin{aligned} f(x) &= -8\cos^2(3x) + 4 \\ &= -4(2\cos^2(3x) - 1) && \text{(factoring out the GCF)} \\ &= -4\cos(6x) && \text{(Double-Angle Identity)} \end{aligned}$$

This is a reflected cosine function with amplitude 4, period $\frac{\pi}{3}$, step size $\frac{\pi}{12}$, and no phase shift or vertical shift.

Figure 9.3.4

Checkpoint 9.3.7

Graph the function.

$$g(x) = \frac{6\tan\left(\frac{\pi}{4}x\right)}{1-\tan^2\left(\frac{\pi}{4}x\right)}$$

Answer

$g(x) = 3\tan\left(\frac{\pi}{2}x\right)$.

Figure 9.3.5

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9.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Double-Angle Identities
  - Double-Angle Identities - Theorem and Proof
  - Double-Angle Identities - Using the Double-Angle Identities
  - Double-Angle Identities - Proving Trigonometric Identities (Part 4)

KEY CONCEPTS

THEOREMS

- **Double-Angle Identities:**
  - $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
  - $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
  - $\cos(2\theta) = 2\cos^2(\theta) - 1$
  - $\cos(2\theta) = 1 - 2\sin^2(\theta)$
  - $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

COMMON MISTAKES

- **Don't Factor Out of Functions:** A common mistake is to treat a trigonometric function as if it distributes over its argument. For example, $\sin(2\theta)$ is not equal to $2\sin(\theta)$. The Double-Angle Identities must be used to correctly evaluate or expand such expressions.
- **Ignoring Simpler Approaches:** It is easy to get so focused on using one specific method that you miss a simpler path. For example, after finding $\sin(2\theta)$ and $\cos(2\theta)$, the easiest way to find $\tan(2\theta)$ is often to use the Ratio Identity rather than the Double-Angle Identity for tangent.

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9.3.2: HOMEWORK

READING QUESTIONS

1. How are the Double-Angle Identities derived from the Sum and Difference Identities?
2. What is the Double-Angle Identity for $\sin(2\theta)$?
3. List the three different forms of the Double-Angle Identity for $\cos(2\theta)$.
4. If you are given the value of $\cos(\theta)$ and need to find $\cos(2\theta)$, which of the three forms of the cosine double-angle identity would be the most direct to use?
5. Provide a numerical example that demonstrates that $\sin(2\theta) \neq 2\sin(\theta)$.
6. If you need to find the value of $\sin(2\theta)$, what two pieces of information about θ must you know?
7. How can a Double-Angle Identity be used to help graph a function like $f(x) = -8\cos^2(3x) + 4$?
8. When proving an identity, what should you look for if you see an expression like $\cos(2x)$?

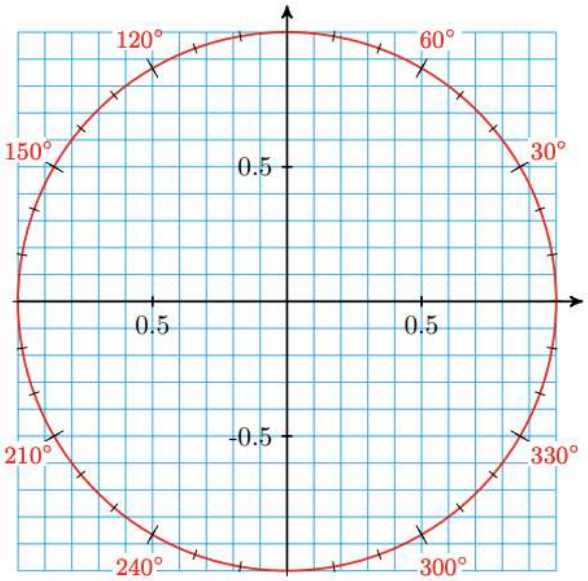
HOMEWORK

CONCEPT CHECK

1. Provide an example to show that doubling an angle does not double its sine or cosine.

BASIC SKILLS

For the following exercises, use the unit circle to estimate trigonometric values. Then verify with your calculator.



2. Does $\sin(2 \cdot 90°) = 2\sin(90°)$?
3. Does $\cos(2 \cdot 25°) = 2\cos(25°)$?
4. Does $\tan(2 \cdot 70°) = 2\tan(70°)$?
5. Does $\tan(2 \cdot 100°) = 2\tan(100°)$?

For the following exercises, verify that each statement is true

6. $\sin(90°) = 2\sin(45°)\cos(45°)$
7. $\sin(60°) = 2\sin(30°)\cos(30°)$
8. $\cos(60°) = \cos^2(30°) - \sin^2(30°)$
9. $\tan(60°) = \dfrac{2\tan(30°)}{1 - \tan^2(30°)}$

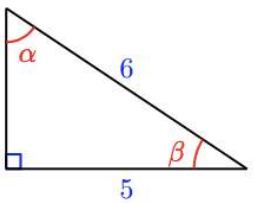
For the following exercises, determine if the statement is true or false. Explain your answer.

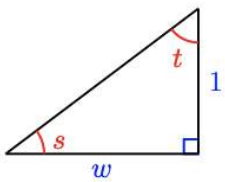
10. If $\cos(\alpha) = 0.32$, then $\cos(2\alpha) = 0.64$.
11. If $\sin(\phi) = r$, then $\sin(2\phi) = 2r$.

For the following exercises, use the Double Angle Identities to simplify the expression. Leave your answer in exact form.

12. $2\sin(34°)\cos(34°)$
13. $\cos^2\left(\frac{\pi}{10}\right) - \sin^2\left(\frac{\pi}{10}\right)$
14. $1 - 2\sin^2\left(\frac{\pi}{16}\right)$
15. $2\cos^2(18°) - 1$
16. $\cos^2(3\theta) - \sin^2(3\theta)$
17. $2\sin(2\alpha)\cos(2\alpha)$
18. $2\sin(5t)\cos(5t)$
19. $\cos^2(4w) - \sin^2(4w)$
20. $\dfrac{2\tan(64°)}{1 - \tan^2(64°)}$
21. $\dfrac{2\tan\left(\frac{\pi}{5}\right)}{1 - \tan^2\left(\frac{\pi}{5}\right)}$
22. $2\cos^2(2\beta) - 1$
23. $1 - 2\sin^2(6s)$
24. $\cos^2(28°) - \sin^2(28°)$
25. $2\cos^2(37°) - 1$
26. $1 - 2\sin^2(17°)$
27. $\cos^2(37°) - \sin^2(37°)$
28. $\cos^2(9x) - \sin^2(9x)$
29. $\cos^2(6x) - \sin^2(6x)$
30. $4\sin(8x)\cos(8x)$
31. $6\sin(5x)\cos(5x)$

For the following exercises, use the figures to find the trigonometric ratios.





32.

a. $\sin(\alpha)$

b. $\cos(\alpha)$

c. $\tan(\alpha)$

d. $\sin(2\alpha)$

e. $\cos(2\alpha)$

f. $\tan(2\alpha)$

33.

a. $\sin(\beta)$

b. $\cos(\beta)$

c. $\tan(\beta)$

d. $\sin(2\beta)$

e. $\cos(2\beta)$

f. $\tan(2\beta)$

34.

a. $\sin(s)$

b. $\cos(s)$

c. $\tan(s)$

d. $\sin(2s)$

e. $\cos(2s)$

f. $\tan(2s)$

35.

a. $\sin(t)$

b. $\cos(t)$

c. $\tan(t)$

d. $\sin(2t)$

e. $\cos(2t)$

f. $\tan(2t)$

36. Suppose $\cos(\theta) = \frac{12}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Compute exact values for the following.

a. $\sin(\theta)$

b. $\sin(2\theta)$

c. $\cos(2\theta)$

d. $\tan(2\theta)$

e. Sketch the angles θ and 2θ.

37. Suppose $\sin(\phi) = \frac{5}{6}$ and $\frac{\pi}{2} < \phi < \pi$. Compute exact values for the following.

a. $\cos(\phi)$

b. $\sin(2\phi)$

c. $\cos(2\phi)$

d. $\tan(2\phi)$

e. Sketch the angles ϕ and 2ϕ.

38. If $\tan(u) = -4$ and $270° < u < 360°$, find exact values for the following.

a. $\tan(2u)$

b. $\cos(2u)$

c. $\sin(2u)$

39. If $\tan(v) = \frac{2}{3}$ and $180° < v < 270°$, find exact values for the following.

a. $\tan(2v)$

b. $\cos(2v)$

c. $\sin(2v)$

For the following exercises, use the given information about θ to find the exact values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

40. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$

41. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$

42. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

43. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$

44. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$

45. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

46. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$

47. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$

48. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$

49. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

50. Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas

a. $\cos(\theta) = \sqrt{1-x^2}$

b. $\sin(2\theta) = 2x\sqrt{1-x^2}$

c. $\cos(2\theta) = 1 - 2x^2$

51. Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas

a. $\cos(\theta) = \frac{1}{\sqrt{x^2+1}}$

b. $\sin(\theta) = \frac{x}{\sqrt{x^2+1}}$

c. $\sin(2\theta) = \frac{2x}{x^2+1}$

d. $\cos(2\theta) = \frac{1-x^2}{x^2+1}$

52. If $\sin(\theta) = \frac{x}{2}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\cos(2\theta)$ in terms of x.

53. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

For the following exercises, graph one cycle of the given function by hand.

54. $f(x) = 4\sin(3x)\cos(3x)$

55. $g(x) = \frac{2}{3}\cos^2\left(\frac{\pi}{3}x\right) - \frac{1}{2}$

56. $h(x) = -\frac{5}{7}\sin\left(\frac{1}{3}x\right)\cos\left(\frac{1}{3}x\right)$

57. $k(x) = \frac{3\tan\left(\frac{1}{2}x\right)}{2 - 2\tan^2\left(\frac{1}{2}x\right)}$

58. $m(x) = 2\cos^2(5x) - 2\sin^2(5x)$

PROOFS

For the following exercises, prove the given identity for any right triangle $\triangle ABC$, where C is a right angle.

59. $\sin(A-B)=\cos(2B)$

60. $\cos(A-B)=\sin(2A)$

61. $\sin(2A)=\dfrac{2ab}{c^2}$

62. $\cos(2A)=\dfrac{b^2-a^2}{c^2}$

63. $\tan(2A)=\dfrac{2ab}{b^2-a^2}$

64. Prove the Double-Angle Identity $\sin(2\theta)=2\sin(\theta)\cos(\theta)$.

65. Prove the Double-Angle Identity $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$.

For the following exercises,
(a) Use graphs to decide if the equation is an identity.
(b) If the equation is not an identity, find a value of the variable that makes the equation false.

66. $\cos(4\theta)=4\cos(\theta)$

67. $\cos(2x)=(\cos(x)+\sin(x))(\cos(x)-\sin(x))$

For the following exercises, prove the identity. Assume all quantities are defined.

68. $(\cos(\theta)+\sin(\theta))^2=1+\sin(2\theta)$

69. $(\cos(\theta)-\sin(\theta))^2=1-\sin(2\theta)$

70. $\dfrac{\sin(2\theta)}{\sin(\theta)}-\dfrac{\cos(2\theta)}{\cos(\theta)}=\sec(\theta)$

71. $\dfrac{\sin(3\theta)}{\sin(\theta)}-\dfrac{\cos(3\theta)}{\cos(\theta)}=2$

72. $\tan(2\theta)=\dfrac{1}{1-\tan(\theta)}-\dfrac{1}{1+\tan(\theta)}$

73. $\csc(2\theta)=\dfrac{\cot(\theta)+\tan(\theta)}{2}$

74. $(\sin^2 x-1)^2=\cos(2x)+\sin^4 x$

75. $\sin(2x)=\dfrac{2\tan(x)}{1+\tan^2(x)}$

76. $\tan(2x)=\dfrac{2\sin(x)\cos(x)}{2\cos^2(x)-1}$

77. $\cot(x)-\tan(x)=2\cot(2x)$

78. $\dfrac{\sin(2\theta)}{1+\cos(2\theta)}=\tan(\theta)$

79. $\cos(2\alpha)=\dfrac{1-\tan^2(\alpha)}{1+\tan^2(\alpha)}$

80. $\dfrac{1+\cos(2t)}{\sin(2t)-\cos(t)}=\dfrac{2\cos(t)}{2\sin(t)-1}$

81. $\sin(3x)=3\sin(x)\cos^2(x)-\sin^3(x)$

82. $\cos(3x)=\cos^3(x)-3\sin^2(x)\cos(x)$

83. $8\sin^4(\theta)=\cos(4\theta)-4\cos(2\theta)+3$

84. $8\cos^4(\theta)=\cos(4\theta)+4\cos(2\theta)+3$

85. $\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$

86. $\sin(4\theta)=4\sin(\theta)\cos^3(\theta)-4\sin^3(\theta)\cos(\theta)$

87. $\cos(4\theta)=8\cos^4(\theta)-8\cos^2(\theta)+1$

88. $\cos(8\theta)=128\cos^8(\theta)-256\cos^6(\theta)+160\cos^4(\theta)-32\cos^2(\theta)+1$

89. $\sec(2\theta)=\dfrac{\cos(\theta)}{\cos(\theta)+\sin(\theta)}+\dfrac{\sin(\theta)}{\cos(\theta)-\sin(\theta)}$

90. $\dfrac{1}{\cos(\theta)-\sin(\theta)}+\dfrac{1}{\cos(\theta)+\sin(\theta)}=\dfrac{2\cos(\theta)}{\cos(2\theta)}$

91. $\dfrac{1}{\cos(\theta)-\sin(\theta)}-\dfrac{1}{\cos(\theta)+\sin(\theta)}=\dfrac{2\sin(\theta)}{\cos(2\theta)}$

92. $\dfrac{\cos^2(\psi)}{\cos^2(\theta)}=\dfrac{1+\cos(2\psi)}{1+\cos(2\theta)}$

CHALLENGE PROBLEMS

93. Some trigonometry textbooks used to claim (incorrectly) that $\sin(\theta)+\cos(\theta)=\sqrt{1+\sin(2\theta)}$ was an identity. Give an example of a specific angle θ that would make that equation false. Is $\sin(\theta)+\cos(\theta)=\pm\sqrt{1+\sin(2\theta)}$ an identity? Justify your answer.

94. In general, what is the largest value that $\sin(\theta)\cos(\theta)$ can take? Justify your answer.

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9.4: Half-Angle and Power Reduction Identities

Note to the Instructor - Repeated Advice

The Sum and Difference, Double-Angle, and Half-Angle Identities must be memorized (at least for the sine and cosine). Do *not* fall into the trap of allowing students to have these on a note card during exams - they will not have such luxury in Calculus.

Learning Objectives

- Use the Power Reduction Formulas to rewrite the power of a trigonometric function in terms of single powers.
- Use a Half-Angle Identity to find the exact value of a trigonometric function.
- Prove an identity using a Half-Angle Identity
- Use the Half-Angle Identities to graph a function.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The Power Reduction Identities

The Double-Angle Identities can be used to derive the Power Reduction Identities**,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are essential in higher-level math courses, Calculus in particular. Also called the Power Reduction Formulas (or, simply, the Reduction Formulas), these form three identities and are easily derived from the Double-Angle Identities.[1]

Theorem: Power Reduction Identities

$$\begin{aligned} \sin^2(\theta) &= \frac{1-\cos(2\theta)}{2} \\ \cos^2(\theta) &= \frac{1+\cos(2\theta)}{2} \\ \tan^2(\theta) &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \end{aligned}$$

Proof

We can use two of the three Double-Angle Identities for cosine to derive the Power Reduction Identities for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2(\theta)$. Solving for $\sin^2(\theta)$, we get:

$$\begin{aligned} \cos(2\theta) &= 1 - 2\sin^2(\theta) \\ \implies 2\sin^2(\theta) &= 1 - \cos(2\theta) \\ \implies \sin^2(\theta) &= \frac{1-\cos(2\theta)}{2} \end{aligned}$$ Power Reduction Identity for Sine

Next, we use the formula $\cos(2\theta) = 2\cos^2(\theta) - 1$. Solving for $\cos^2(\theta)$, we get:

$$\begin{aligned} \cos(2\theta) &= 2\cos^2(\theta) - 1 \\ \implies 1 + \cos(2\theta) &= 2\cos^2(\theta) \\ \implies \cos^2(\theta) &= \frac{1+\cos(2\theta)}{2} \end{aligned}$$ Power Reduction Identity for Cosine

The last Reduction Identity is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}
\tan^2(\theta) &= \frac{\sin^2(\theta)}{\cos^2(\theta)} \\
&= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{(substitute the Reduction Identities)} \\
&= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} && \text{Power Reduction Identity for Tangent}
\end{aligned}$$

Advice: Memorize the Sine and Cosine Versions of These Identities

The Power Reduction Identities for the sine and cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated. The tangent version is less common and can be derived from the sine and cosine versions if needed.

Using the Reduction Identities to Rewrite an Expression Without Powers Greater Than 1

The Power Reduction Identities are frequently used in Calculus to make an otherwise impossible problem possible. This is because working with higher powers of trigonometric functions can be problematic in Integral Calculus.

Example 9.4.1

Write an equivalent expression for $\cos^4(x)$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the Reduction Identity for cosine twice.

$$\begin{aligned}
\cos^4(x) &= \left(\cos^2(x)\right)^2 \\
&= \left(\frac{1+\cos(2x)}{2}\right)^2 && \text{(Reduction Identity)} \\
&= \frac{1}{4}\left(1+2\cos(2x)+\cos^2(2x)\right) && \text{(distribution)} \\
&= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos(4x)}{2}\right) && \text{(distribution and Reduction Identity)} \\
&= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) && \text{(distribution)} \\
&= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x) && \text{(combining like terms)}
\end{aligned}$$

Checkpoint 9.4.1

Write an equivalent expression for $\sin^4(3x)$ that does not involve any powers of sine or cosine greater than 1.

Answer

$$\frac{3}{8}-\frac{1}{2}\cos(6x)+\frac{1}{8}\cos(12x)$$

Revisiting Proofs of Identities

The Power Reduction Identities can come in handy when trying to prove identities.

Example 9.4.2

Prove

$$\sin^3(2x) = \frac{1}{2}\sin(2x)\left[1-\cos(4x)\right].$$

Solution

We will work on simplifying the left side of the equation.

$$\begin{aligned}
\text{LHS} &= \sin^3(2x) \\
&= \sin(2x)\cdot\sin^2(2x) \\
&= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] && \text{(Power Reduction Identity)} \\
&= \sin(2x)\left(\frac{1}{2}\right)\left[1-\cos(4x)\right] \\
&= \frac{1}{2}\sin(2x)[1-\cos(4x)] \\
&= \text{RHS}
\end{aligned}$$

In Example 9.4.2, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. This is because the Power Reduction Identity states

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}.$$

We let $\theta = 2x$, so $2\theta = 4x$.

Checkpoint 9.4.2

Prove that

$$10\cos^4(x) = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x).$$

Answer

$$\begin{aligned}
10\cos^4(x) &= 10\left(\cos^2(x)\right)^2 \\
&= 10\left[\frac{1+\cos(2x)}{2}\right]^2 && \text{(Power Reduction Identity)} \\
&= \frac{10}{4}\left[1+2\cos(2x)+\cos^2(2x)\right] && \text{(distribution)} \\
&= \frac{10}{4} + \frac{10}{2}\cos(2x) + \frac{10}{4}\left(\frac{1+\cos(4x)}{2}\right) && \text{(distribution and Power Reduction Identity)} \\
&= \frac{10}{4} + \frac{10}{2}\cos(2x) + \frac{10}{8} + \frac{10}{8}\cos(4x) \\
&= \frac{30}{8} + 5\cos(2x) + \frac{10}{8}\cos(4x) \\
&= \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)
\end{aligned}$$

The Half-Angle Identities

The next set of identities can be quickly derived from the Power Reduction Identities by replacing θ with $\frac{\alpha}{2}$ and taking the square root of both sides. These are known as the Half-Angle Identities.

Theorem: Half-Angle Identities

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}$$

Proof

The Half-Angle Identity for sine is derived as follows:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \qquad \text{(Power Reduction Identity)}$$

$$\implies \sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \qquad \left(\text{substituting } \frac{\alpha}{2} \text{ for } \theta\right)$$

$$= \frac{1-\cos(\alpha)}{2}$$

$$\implies \sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}} \qquad \text{(taking the square root of both sides)}$$

The derivations of the Half-Angle Identities for cosine and tangent are left for the reader.

Note that the Half-Angle Identities are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Instead, it depends on the quadrant in which $\frac{\theta}{2}$ terminates.

Using the Half-Angle Identities to Evaluate Trigonometric Functions

Example 9.4.3

Find the exact value of $\sin(15°)$.

Solution

Since $15° = \frac{30°}{2}$, we use the Half-Angle Identity for sine:

$$\sin(15°) = \sin\left(\frac{30°}{2}\right)$$

$$= +\sqrt{\frac{1-\cos(30°)}{2}} \qquad (\text{since } 15° \in \text{QI},\ \sin(15°) \text{ is positive})$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

Remember that we can check the answer with a graphing calculator.

? Checkpoint 9.4.3

Find the exact value of $\cos\left(\frac{\pi}{8}\right)$

Answer

$$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

When using a Half-Angle Identity, you must address the $\pm$ sign. To do this, you consider the quadrant within which the *half-angle* terminates. For example, suppose you are given $\cos(\theta) = \frac{5}{7}$, where $\theta \in$ QIV. If you are then asked to find $\cos\left(\frac{\theta}{2}\right)$, you should start by determining the quadrant within which $\frac{\theta}{2}$ lies because that will determine the sign of $\cos\left(\frac{\theta}{2}\right)$. We use the fact that $\theta \in$ QIV to do this. This implies

$$\begin{aligned} & \frac{3\pi}{2} < \theta < 2\pi \\ \Rightarrow \quad & \frac{3\pi}{4} < \frac{\theta}{2} < \pi \end{aligned}$$

Hence, $\frac{\theta}{2} \in$ QII. We then use the Half-Angle Identity and the given information to finish the problem.

$$\begin{aligned} \cos\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{1+\cos(\theta)}{2}} \\ &= -\sqrt{\frac{1+\frac{5}{7}}{2}} \\ &= -\sqrt{\frac{7+5}{14}} \\ &= -\sqrt{\frac{12}{14}} \\ &= -\sqrt{\frac{6}{7}} \end{aligned}$$

✓ Example 9.4.4

Given that $\tan(\alpha) = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solutions

Using the given information, we draw the triangle shown below.

Figure 9.4.1

Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate

$$\sin(\alpha) = -\frac{8}{17}$$

and

$$\cos(\alpha) = -\frac{15}{17}.$$

Before we start, we must remember that if α is in quadrant III, then $\pi < \alpha < \frac{3\pi}{2}$, so $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II.

a. To find $\sin\left(\frac{\alpha}{2}\right)$, we begin by writing the Half-Angle Identity for sine. Then, we substitute the value of the cosine we found from the triangle above and simplify.

$$\begin{aligned}
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos(\alpha)}{2}} && \text{(Half-Angle Identity)} \\
&= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm\sqrt{\frac{32}{34}} \\
&= \pm\sqrt{\frac{16}{17}} \\
&= \pm\frac{4}{\sqrt{17}} \\
&= \frac{4}{\sqrt{17}} && \left(\frac{\alpha}{2} \in \text{QII}\right)
\end{aligned}$$

We choose the positive value of $\sin\left(\frac{\alpha}{2}\right)$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\left(\frac{\alpha}{2}\right)$, we will write the Half-Angle Identity for cosine, substitute the value of the cosine we found from the triangle, and simplify.

$$\begin{aligned}
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos(\alpha)}{2}} && \text{(Half-Angle Identity)} \\
&= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{2}{17}}{2}} \\
&= \pm\sqrt{\frac{2}{34}} \\
&= \pm\sqrt{\frac{1}{17}} \\
&= -\frac{1}{\sqrt{17}} && \left(\frac{\alpha}{2} \in \text{QII}\right)
\end{aligned}$$

We choose the negative value of $\cos\left(\frac{\alpha}{2}\right)$ because the angle is in quadrant II and the cosine is negative in quadrant II.

c. To find $\tan\left(\frac{\alpha}{2}\right)$, we write the Half-Angle Identity for the tangent. Again, we substitute the value of the cosine we found from the triangle and simplify.

$$\begin{aligned}
\tan\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}} && \text{(Half-Angle Identity)} \\
&= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} \\
&= \pm\sqrt{\frac{32}{2}} \\
&= -\sqrt{16} && \left(\frac{\alpha}{2} \in \text{QII}\right) \\
&= -4
\end{aligned}$$

We choose the negative value of $\tan\left(\frac{\alpha}{2}\right)$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

? Checkpoint 9.4.4

Given that $\sin(\alpha) = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Answer

$$-\frac{2}{\sqrt{5}}$$

The following example requires us to recall how to find an acute angle using the inverse trigonometric function. For a review of this skill, please see section 3.2.

Example 9.4.5

A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan(\theta) = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high-level competition, and $\tan(\theta) = \frac{5}{3}$ for high competition, we can find $\cos(\theta)$ from the right triangle and the Pythagorean Theorem so that we can use the Half-Angle Identities (see the figure below).

Figure 9.4.2

We see that $\cos(\theta) = \frac{3}{\sqrt{34}}$. We can use the Half-Angle Identity for tangent:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}.$$

Since $\tan(\theta)$ is in the first quadrant, so is $\tan\left(\frac{\theta}{2}\right)$.

$$\begin{aligned}\tan\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1-\frac{3}{\sqrt{34}}}{1+\frac{3}{\sqrt{34}}}} \\ &= \sqrt{\frac{\sqrt{34}-3}{\sqrt{34}+3}} \\ &\approx 0.57\end{aligned}$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7°$. So, the angle of the ramp for novice competition is $\approx 29.7°$.

Revisiting Graphs of Trigonometric Functions

Example 9.4.6

Graph the function.

$$y = 3\sin^2\left(\frac{x}{2}\right)$$

Solution

We begin by recognizing this as the square of a Half-Angle Identity.

$$\begin{aligned}y &= 3\sin^2\left(\frac{x}{2}\right) \\ &= 3\left[\pm\sqrt{\frac{1-\cos(x)}{2}}\right]^2 \\ &= 3\left(\frac{1-\cos(x)}{2}\right) \\ &= -\frac{3}{2}\cos(x) + \frac{3}{2}\end{aligned}$$

We see this is a reflected cosine function with amplitude $\frac{3}{2}$, period 2π, step size $\frac{\pi}{2}$, phase shift 0, and vertical shift $\frac{3}{2}$. Using all this information, we obtain the following graph.

Figure 9.4.3

Checkpoint 9.4.6

Graph the function.

$$y = -2\cos^2(3x) - 1$$

Answer

Figure 9.4.4

Footnotes

[1] I have been using the convention in this textbook to call all our theorems based on the trigonometric functions by the term "identities." That is, we have learned the Pythagorean Identities, the Ratio Identities, the Symmetry Identities, the Double-Angle Identities, and so on. You will also hear of these referred to as "formulas." That is, a student from another class might say, "Double-Angle Formulas." This is fine. The only reason for my convention is that the word "identity" has more meaning. It implies that the statement is always true. On the other hand, the word "formula" has always seemed a bit more circumstantial (i.e., not always true) to me. In the context of the theorems we have presented so far, you could use either terminology and you would still be correct.

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9.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

KEYWORDS

- Power Reduction Identities
  - Double-Angle Identities - Power-Reduction Formulas and Their Use
- Half-Angle Identities
  - Half-Angle Identities - Theorem and Proof
  - Half-Angle Identities - Evaluating Trig Functions of Non-Special Angles Using Half-Angle Identities
  - Half-Angle Identities - Using the Half-Angle Identities
  - Half-Angle Identities - Proving Trigonometric Identities (Part 5)

KEY CONCEPTS

THEOREMS

- **Power Reduction Identities:**
  - $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
  - $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$
  - $\tan^2(\theta) = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$
- **Half-Angle Identities:**
  - $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
  - $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
  - $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}$

COMMON MISTAKES

- **Forgetting the $\pm$ Sign:** The Half-Angle Identities for sine and cosine include a $\pm$ symbol. This does not mean both signs are correct. You must determine the correct sign (positive or negative) based on the quadrant in which the half-angle $\frac{\theta}{2}$ terminates.
- **Determining the Sign from the Wrong Angle:** The sign for a half-angle identity, like $\sin(\frac{\theta}{2})$, depends on the quadrant of $\frac{\theta}{2}$, not the quadrant of the original angle θ.
- **Incorrectly Applying Power Reduction:** The Power Reduction Identities are used to rewrite even powers of sine and cosine in terms of the first power of cosine. They cannot be used directly on odd powers.

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9.4.2: HOMEWORK

READING QUESTIONS

1. What is the primary purpose of the Power Reduction Identities?
2. From which identities are the Power Reduction Identities derived?
3. What is the Power Reduction Identity for $\sin^2(\theta)$?
4. How are the Half-Angle Identities derived from the Power Reduction Identities?
5. What is the Half-Angle Identity for $\cos(\frac{\theta}{2})$?
6. What does the $\pm$ symbol in the Half-Angle Identities signify?
7. How do you determine whether to use the positive or negative root when applying a Half-Angle Identity?
8. If θ is in Quadrant IV, in which quadrant must $\frac{\theta}{2}$ lie?
9. How can you use the Half-Angle Identities to help graph a function like $y = 3\sin^2(\frac{x}{2})$?
10. To find the exact value of $\sin(15°)$ using a Half-Angle Identity, what angle would you use for θ?

HOMEWORK

BASIC SKILLS

For the following exercises, rewrite the expression without exponents.

1. $\cos^2(5x)$
2. $\cos^2(6x)$
3. $\sin^4(8x)$
4. $\sin^4(3x)$
5. $\cos^2(x)\sin^4(x)$
6. $\cos^4(x)\sin^2(x)$

For the following exercises, use the Half-Angle Identities to find the exact value.

7. $\cos(75°)$
8. $\sin(105°)$
9. $\cos(67.5°)$
10. $\sin(157.5°)$
11. $\tan(112.5°)$
12. $\cos\left(\frac{7\pi}{12}\right)$
13. $\sin\left(\frac{\pi}{12}\right)$
14. $\sin\left(\frac{\pi}{24}\right)$
15. $\cos\left(\frac{\pi}{24}\right)$
16. $\sin\left(\frac{9\pi}{8}\right)$
17. $\tan\left(\frac{3\pi}{8}\right)$
18. $\cos\left(\frac{\pi}{8}\right)$
19. $\sin\left(\frac{5\pi}{8}\right)$
20. $\tan\left(\frac{7\pi}{8}\right)$
21. $\sin\left(-\frac{\pi}{8}\right)$
22. $\sin\left(\frac{13\pi}{24}\right)$

For the following exercises, use the given information about θ to find the exact values of

- $\sin\left(\frac{\theta}{2}\right)$
- $\cos\left(\frac{\theta}{2}\right)$
- $\tan\left(\frac{\theta}{2}\right)$

23. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$
24. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$
25. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$
26. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$
27. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$
28. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$
29. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$
30. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$
31. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$
32. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

SYNTHESIS PROBLEMS

33. Let θ be a Quadrant III angle with $\cos(\theta) = -\frac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\frac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \frac{7\pi}{2}$ and then assuming $\pi < \theta < \frac{3\pi}{2}$ and computing $\sin\left(\frac{\theta}{2}\right)$ in both cases.

For the following exercises, graph *two* cycles of the function.

34. $y = 8\sin^2\left(\frac{x}{2}\right)$
35. $y = 2\cos^2\left(\frac{x}{2}\right)$
36. $y = -2\cos^2\left(\frac{x}{2}\right) + 4$
37. $y = \frac{1}{3}\cos^2\left(\frac{x}{2}\right) - \frac{1}{6}$

PROOFS

For the following exercises, prove the identity.

38. $\tan\left(\frac{1}{2}\theta\right) = \csc(\theta) - \cot(\theta)$
39. $\tan^2\left(\frac{1}{2}\theta\right) = \frac{\tan(\theta) - \sin(\theta)}{\tan(\theta) + \sin(\theta)}$
40. $\sin^2\left(\frac{\theta}{2}\right) = \frac{\csc(\theta) - \cot(\theta)}{2\csc(\theta)}$
41. $2\sin^2\left(\frac{\theta}{2}\right) = \frac{\sin^2(\theta)}{1 + \cos(\theta)}$
42. $2\cos^2\left(\frac{\theta}{2}\right) = \frac{\sin^2(\theta)}{1 - \cos(\theta)}$

43. $\cos^2\left(\frac{\theta}{2}\right) = \frac{\tan(\theta) + \sin(\theta)}{2\tan(\theta)}$

44. $\sec^2\left(\frac{\theta}{2}\right) = \frac{2\sec(\theta)}{\sec(\theta) + 1}$

45. $\csc^2\left(\frac{\theta}{2}\right) = \frac{2\sec(\theta)}{\sec(\theta) - 1}$

46. $4\sin^4(\theta) = 1 - 2\cos(2\theta) + \cos^2(2\theta)$

47. $\cos^4(\theta) = \frac{1}{4} + \frac{\cos(2\theta)}{2} + \frac{\cos^2(2\theta)}{4}$

CHALLENGE QUESTIONS

48. Without using your calculator, show that $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

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9.5: Sum-to-Product and Product-to-Sum Identities

Note to the Instructor - Don't Skip, but Don't Over-Accentuate

Students need to know these identities exist - they are used infrequently in Calculus, and having knowledge of their existence is assumed; however, do not force students to memorize these.

Learning Objectives

- Rewrite a product of sines and cosines as a sum or a difference.
- Express a sum or difference of sines and cosines as a product.
- Prove an identity using the Sum-to-Product and Product-to-Sum Identities.

Hawk A.I. Section-Specific Tutor

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You should commit to memory the identities we have covered up to this point. Specifically, you need to be able to recall the following identities without access to a note card.

Reciprocal Identities	
Basic Identity	***Alternate Form***
$\sin(\theta) = \frac{1}{\csc(\theta)}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\cos(\theta) = \frac{1}{\sec(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$
$\tan(\theta) = \frac{1}{\cot(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

Ratio Identities	
Basic Identity	***Alternate Form***
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities	
Basic Identity	***Alternate Forms***
$\cos^2(\theta) + \sin^2(\theta) = 1$	$\cos^2(\theta) = 1 - \sin^2(\theta)$ and $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$
	$\sin^2(\theta) = 1 - \cos^2(\theta)$ and $\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
$1 + \tan^2(\theta) = \sec^2(\theta)$	$\tan^2(\theta) = \sec^2(\theta) - 1$
$\cot^2(\theta) + 1 = \csc^2(\theta)$	$\cot^2(\theta) = \csc^2(\theta) - 1$

Symmetry Identities	
Basic Identity	*Alternate Form*
$\sin(-\theta) = -\sin(\theta)$	$\csc(-\theta) = -\csc(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\sec(-\theta) = \sec(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\cot(-\theta) = -\cot(\theta)$

Cofunction Identities	
Basic Identity	*Alternate Form*
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$
$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$
$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

Sum and Difference of Angles Identities
$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

Double-Angle Identities
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta)\end{aligned}$

Power Reduction Identities
$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$
$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$

Half-Angle Identities
$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$
$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$

You should note that the tangent identities are not listed for our recent formulas. This is because they are rarely used and, if needed, can be derived from the identities for sine and cosine.

The Sum-to-Product and Product-to-Sum Identities

Our next batch of identities, the Product-to-Sum Identities,[1] can be easily verified by expanding each of the right-hand sides per the Sum and Difference Identities (we leave the details as exercises). They are of particular use in Calculus, and we list them here for reference.

Theorem: Product-to-Sum Identities

For angles α and β,

$$\begin{aligned}\sin(\alpha)\sin(\beta) &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]\\ \cos(\alpha)\cos(\beta) &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]\\ \sin(\alpha)\cos(\beta) &= \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]\end{aligned}$$

The final set of identities in this chapter, the Sum-to-Product Identities, are related to the Product-to-Sum Identities. These are easily verified using the Product-to-Sum Identities, and as such, their proofs are left as exercises.

Theorem: Sum-to-Product Identities

For angles α and β,

$$\begin{aligned}\sin(\alpha)\pm\sin(\beta) &= 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)\\ \cos(\alpha)+\cos(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\ \cos(\alpha)-\cos(\beta) &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\end{aligned}$$

Do Not Memorize These Identities!

I cannot think of any situation where committing these identities to memory is beneficial. It's enough to know they exist and when they are needed.

Example 9.5.1

a. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
b. Write $\sin(4\theta)\cos(2\theta)$ as a sum.
c. Write $\sin(\theta)-\sin(3\theta)$ as a product.

Solutions

a. Identifying $\alpha=2\theta$ and $\beta=6\theta$, we find

$$\begin{aligned}\cos(2\theta)\cos(6\theta) &= \frac{1}{2}[\cos(2\theta-6\theta)+\cos(2\theta+6\theta)]\\ &= \frac{1}{2}\cos(-4\theta)+\frac{1}{2}\cos(8\theta)\\ &= \frac{1}{2}\cos(4\theta)+\frac{1}{2}\cos(8\theta),\end{aligned}$$

where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta)=\cos(4\theta)$.

b. Write the formula for the product of sine and cosine. Then, substitute the given values into the formula and simplify.

$$\begin{aligned}\sin(\alpha)\cos(\beta) &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\ \Longrightarrow\quad \sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)]\\ &= \frac{1}{2}[\sin(6\theta)+\sin(2\theta)]\end{aligned}$$

c. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta), \end{aligned}$$

where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$.

The reader is reminded that all of the identities presented in this section, which regard the circular functions as functions of angles (in radian measure), apply equally well to the circular (trigonometric) functions considered as functions of real numbers.

? Checkpoint 9.5.1

a. Write the product as a sum or difference.

$$\sin(x + y)\cos(x - y)$$

b. Write the sum as a product.

$$\sin(3\theta) + \sin(\theta)$$

Answer

a. $\frac{1}{2}(\sin(2x) + \sin(2y))$

b. $2\sin(2\theta)\cos(\theta)$

Using the Sum-to-Product and Product-to-Sum Identities to Evaluate Trigonometric Functions

There are times when obtaining exact values of trigonometric functions is easier when using the Sum-to-Product and Product-to-Sum Identities. Again, you are not expected to memorize these identities, but you *are* expected to be able to recognize when they might be of use.

✓ Example 9.5.2

Find the exact value of $\cos(15°) - \cos(75°)$.

Solution

We could tackle this problem by using a Half-Angle Identity for the first term and a Sum of Angles Identity for the second; however, recognizing that this is a difference of cosines also gives us the clue that we could use a Sum-to-Product Identity.

$$\begin{aligned} \cos(15°) - \cos(75°) &= -2\sin\left(\frac{15° + 75°}{2}\right)\sin\left(\frac{15° - 75°}{2}\right) \quad \text{(Sum-to-Product Identity)} \\ &= -2\sin(45°)\sin(-30°) \\ &= -2\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

? Checkpoint 9.5.2

Find the exact value of $\cos\left(\frac{11\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$.

Answer

$\frac{-2 - \sqrt{3}}{4}$

Revisiting Proofs of Identities

While proofs of identities that require Sum-to-Product or Product-to-Sum Identities are no more challenging than proofs involving the other trigonometric identities, using these newer identities less frequently makes recognizing the need for them more difficult. Try your best to remember that when all else fails, you might need to use a Sum-to-Product or Product-to-Sum Identity.

Example 9.5.3

Prove the identity.

$$\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} = -\tan(t)$$

Solution

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}
\text{LHS} &= \frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} \\
&= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)} \qquad \text{(Sum-to-Product Identities)} \\
&= \frac{-2\sin(3t)\sin(t)}{2\sin(3t)\cos(t)} \\
&= -\frac{\sin(t)}{\cos(t)} \\
&= -\tan(t) \\
&= \text{RHS}
\end{aligned}$$

Footnotes

[1] These are also known as the Prosthaphaeresis Formulas and have a rich history. I recommend that you conduct some research on them as your schedule allows.

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9.5.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Sum-to-Product Identities and Product-to-Sum Identities
  - The Product to Sum and Sum to Product Formulas - Theorem, Proof, and Examples

KEY CONCEPTS

THEOREMS

- **Product-to-Sum Identities:** For any angles α and β:
  - $\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
  - $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
  - $\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
- **Sum-to-Product Identities:** For any angles α and β:
  - $\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$
  - $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
  - $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

COMMON MISTAKES

- **Unnecessary Memorization:** It is not beneficial to commit the Sum-to-Product and Product-to-Sum identities to memory. It is sufficient to know that they exist and to be able to recognize when they might be useful.
- **Forgetting Negative Signs or Coefficients:** It is easy to forget the negative sign in the formula for $\cos(\alpha) - \cos(\beta)$ or the coefficient of $\frac{1}{2}$ in the Product-to-Sum identities. Always double-check the formula when applying it.

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9.5.2: HOMEWORK

READING QUESTIONS

1. How can the Product-to-Sum identities be verified?
2. What is the Product-to-Sum identity used to rewrite a product of two cosines, $\cos(\alpha)\cos(\beta)$, as a sum?
3. What is the Sum-to-Product identity used to rewrite a difference of two sines, $\sin(\alpha)-\sin(\beta)$, as a product?
4. According to the text, is it beneficial to memorize the Sum-to-Product and Product-to-Sum identities?
5. What two different methods are mentioned in Example 2 for finding the exact value of $\cos(15°)-\cos(75°)$?
6. In Example 1, what previously learned identity is used to rewrite $\cos(-4\theta)$ as $\cos(4\theta)$?
7. When might you need to use a Sum-to-Product or Product-to-Sum identity?

HOMEWORK

BASIC SKILLS

For the following exercises, write the given product as a sum.

1. $\cos(3\theta)\cos(5\theta)$
2. $\sin(2\theta)\sin(7\theta)$
3. $\sin(9\theta)\cos(\theta)$
4. $\cos(2\theta)\cos(6\theta)$
5. $\sin(3\theta)\sin(2\theta)$
6. $\cos(\theta)\sin(3\theta)$

For the following exercises, write the given sum as a product.

7. $\cos(3\theta)+\cos(5\theta)$
8. $\sin(2\theta)-\sin(7\theta)$
9. $\cos(5\theta)-\cos(6\theta)$
10. $\sin(9\theta)-\sin(-\theta)$
11. $\sin(\theta)+\cos(\theta)$
12. $\cos(\theta)-\sin(\theta)$

For the following exercises, find the exact values of the trigonometric expressions.

13. $2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{3}\right)$
14. $\sin\left(\frac{\pi}{12}\right)-\sin\left(\frac{7\pi}{12}\right)$
15. $\cos\left(\frac{5\pi}{12}\right)+\cos\left(\frac{7\pi}{12}\right)$

PROOFS

16. Verify the Product to Sum Identities.
17. Verify the Sum to Product Identities.

For the following exercises, prove the identity.

18. $\frac{\sin(x)+\sin(y)}{\cos(x)+\cos(y)}=\tan\left(\frac{x+y}{2}\right)$
19. $\frac{\sin(x)-\sin(y)}{\cos(x)-\cos(y)}=-\cot\left(\frac{x+y}{2}\right)$
20. $\sin(x)+\sin(2x)+\sin(3x)=4\cos\left(\frac{x}{2}\right)\cos(x)\sin\left(\frac{3x}{2}\right)$
21. $\cos(x)+\cos(2x)+\cos(3x)=4\cos\left(\frac{x}{2}\right)\cos(x)\cos\left(\frac{3x}{2}\right)-1$
22. $\frac{\cos(x)-\cos(3x)}{\sin(3x)-\sin(x)}=\tan(2x)$
23. $-\tan(4x)=\frac{\cos(3x)-\cos(5x)}{\sin(3x)-\sin(5x)}$
24. $32\sin^2(\theta)\cos^4(\theta)=2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)$
25. $32\sin^4(\theta)\cos^2(\theta)=2-\cos(2\theta)-2\cos(4\theta)+\cos(6\theta)$

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CHAPTER OVERVIEW

Chapter 10: Trigonometric Equations

10.1: The Fundamental Inverse Trigonometric Functions

10.1.1: Resources and Key Concepts

10.1.2: Homework

10.2: The Remaining Inverse Trigonometric Functions

10.2.1: Resources and Key Concepts

10.2.2: Homework

10.3: Solving Trigonometric Equations - Algebraic Techniques

10.3.1: Resources and Key Concepts

10.3.2: Homework

10.4: Solving Trigonometric Equations Using Identities

10.4.1: Resources and Key Concepts

10.4.2: Homework

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10.1: The Fundamental Inverse Trigonometric Functions

Note to the Instructor - Haven't We Seen This Before?

When writing parts of this text, I was trying my best to stick with the mantra, "Don't introduce it, unless they are going to use it." While we did introduce the inverse trigonometric functions back when we covered Right Triangle Trigonometry, it was strictly from a "button-mashing on a calculator" perspective for *positive* ratios. Hence, from a student perspective, those pesky buttons on their calculator (or the arc-functions in Desmos) only returned angles between **0** and **90°** (or, when we revisited them after learning about radian measure, between 0 and $\frac{\pi}{2}$).

We had another visit of the inverse trigonometric functions when we discussed reference angles. At that time, I was very careful to tip-toe around the truth of inverse trigonometric functions and just told the reader to evaluate inverse trigonometric functions at absolute values of the given ratios. In doing so, I mentioned that this would return a reference angle.

This section dives into the deep waters and discusses what inverse trigonometric functions truly are for the big three (sine, cosine, and tangent). We need this theory before diving into solving trigonometric equations.

Learning Objectives

- Find the exact value of an inverse trigonometric function.
- Use technology to approximate the value of an inverse trigonometric function.
- Evaluate compositions involving trigonometric functions and inverse trigonometric functions.
- Simplify expressions involving trigonometric functions and inverse trigonometric functions.

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Hawk A.I. Section-Specific Tutor

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We previously introduced the idea of finding an *acute* angle given the value of a trigonometric function. We learned to use the calculator keys $\text{SIN}^{-1}, \text{COS}^{-1}$, and TAN^{-1} to find approximate values of θ when we know either $\sin(\theta), \cos(\theta)$, or $\tan(\theta)$. For example, if we know that $\cos(\theta) = 0.3$, then the *acute* angle θ is approximated by

$$\theta = \cos^{-1}(0.3) \approx 1.2661 \text{ radians}.$$

If you consider the initial piece of information, namely that $\cos(\theta) = 0.3$, this implies that we can use the $\text{SIN}^{-1}, \text{COS}^{-1}$, and TAN^{-1} keys to solve trigonometric equations (as long as the angle we are interested in is acute), just as we use square roots to solve quadratic equations.

Up to this point, that's all we needed to know about the inverse trigonometric functions; however, we will soon be solving more complex trigonometric equations, requiring a deeper discussion of them.

Does Every Function Have an Inverse?

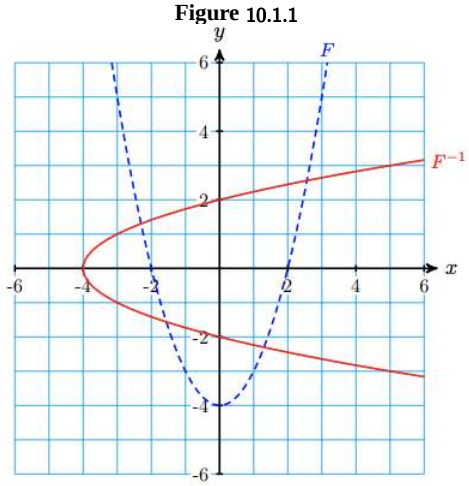
Consider the function $F(x) = x^2 - 4$. First, we'll find a formula for the inverse. We let $y = F(x)$ and solve for x

$$\begin{aligned} y &= x^2 - 4 \\ \Rightarrow \quad y + 4 &= x^2 \\ \Rightarrow \quad \pm\sqrt{y+4} &= x \end{aligned}$$

We then interchange x and y to get $y = \pm\sqrt{x+4}$.

The graphs of F and its inverse, $F^{-1}(x) = \pm\sqrt{x+4}$, are shown in Figure **10.1.1**. You can see that although F is a function, its inverse is not because the graph of the inverse does not pass the Vertical Line Test.[1]

Figure 10.1.1



From Algebra, for a function, f, to have an inverse that is also a function, we require f to pass the Horizontal Line Test. Recall that a function that passes the Horizontal Line Test is called one-to-one.

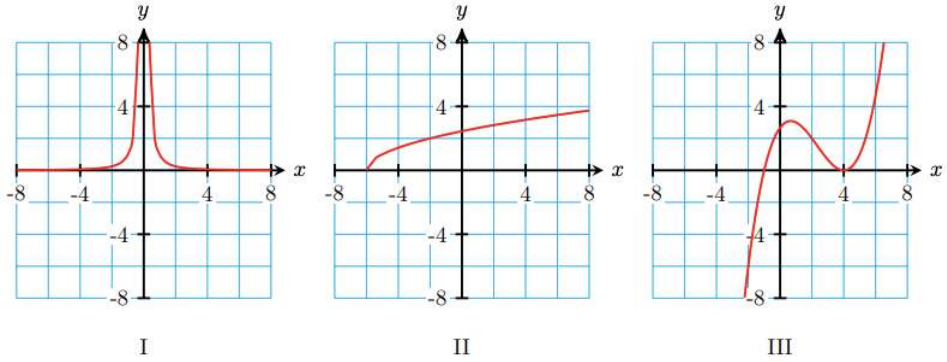
At this point, I want to be *very* clear about a few points:

1. You can always find the inverse of a function graphically. All you need to do is go through each point on the graph of the function and swap the x- and y-coordinates. In doing so, the domain of your new graph is the old function's range, and the new graph's range is the old function's domain.
2. It might be impossible to find the equation representing the inverse. For example, $f(x) = e^x - 3x^2 + \sin(x)$ is definitely a function, but solving $y = e^x - 3x^2 + \sin(x)$ for x is not going to happen.
3. The inverse of a function need not be a function itself. We have just seen that $F(x) = x^2 - 4$ is an example of a function with an inverse that is *not* a function.
4. If a function f passes the Horizontal Line Test, then the function is called one-to-one and its inverse is also a function that we call the inverse function of f.

Example 10.1.1

a. Which of the graphs below represent functions?
b. Which of the functions are one-to-one?
c. Which of these functions have inverse functions?

Figure 10.1.2



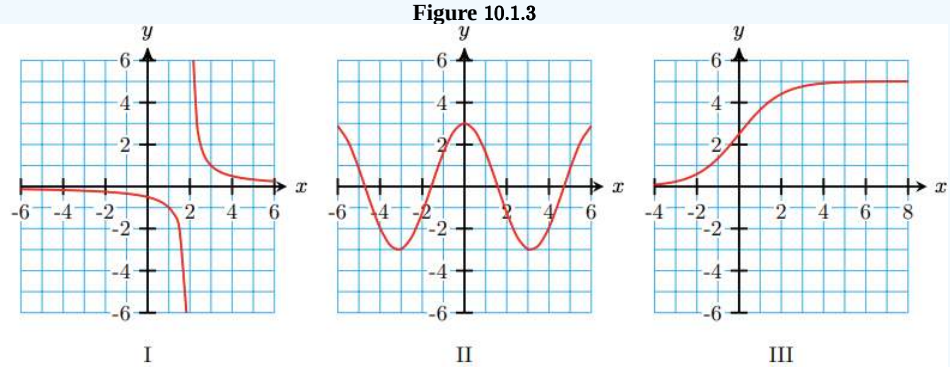
Solutions

a. All three graphs pass the Vertical Line Test, so all three represent functions.
b. Only the function represented by Graph II passes the Horizontal Line Test, so it is the only one-to-one function.
c. Only function II is one-to-one, so it is the only function that has an inverse function.

? Checkpoint 10.1.1

Which of the functions below are one-to-one?

Figure 10.1.3



Answer

I and III

Restricting the Domain: The Inverse Sine Function

Sometimes, it is so crucial that the inverse be a function that we are willing to sacrifice part of the original function to achieve this result.

Look again at the graph of $F(x) = x^2 - 4$. If we use only nonnegative x-values for the domain, we create a new function,

$$f(x) = x^2 - 4, \quad x \geq 0.$$

The graph of this new function is shown as a dashed curve in Figure 10.1.4.

Figure 10.1.4

The new function is one-to-one, and its inverse, $f^{-1}(x) = \sqrt{x+4}$, is also a function.[2]

We say that we have **restricted the domain** of the original function, and we will use this technique to define inverse functions for the trigonometric functions.

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

The sine function is *not* one-to-one; there are many angles that have the same sine value. In order to define its inverse function, we must restrict the domain of the sine to an interval on which the y-values do not repeat. But there are many candidates for such an interval; which one do we choose?

The most useful interval starts at $x = 0$ and moves as far as we can in either direction until the y-values begin to repeat. By doing so, we obtain the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as shown in Figure 10.1.5. This piece of the function includes all of the original range values, from -1 to 1.

Figure 10.1.5

Because the sine function is one-to-one in this domain, its inverse is a function.

The graph of the inverse sine function, $y = \sin^{-1}(x)$, is shown in Figure 10.1.6. Its domain is the same as the range of the sine function, namely $-1 \leq x \leq 1$, and its range is our restricted domain for the sine, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Figure 10.1.6

Definition: The Inverse Sine Function

The function $f(x) = \sin^{-1}(x)$ is defined as follows:

$$\sin^{-1}(x) = \theta \quad \text{if and only if} \quad \sin(\theta) = x \quad \text{and} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

In other words, $\sin^{-1}(x)$ is the angle in radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, whose sine is x. There are many angles with a given sine value x, but only one of these angles can be $\sin^{-1}(x)$. This is why the calculator's SIN^{-1} key only returns angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.[3]

- If $x = \sin(\theta)$ is positive, the inverse sine function delivers a first quadrant angle, $0 \leq \theta \leq \frac{\pi}{2}$. This should make sense because all trigonometric functions are positive in the first quadrant.
- If $x = \sin(\theta)$ is negative, the inverse sine function delivers a fourth quadrant angle, $-\frac{\pi}{2} \leq \theta \leq 0$.

As the next example illustrates, when dealing with inverse trigonometric functions, we focus on finding the *quadrant* and the *reference angle* for the function.

Example 10.1.2

Simplify each expression without using a calculator.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
c. $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$
d. $\sin^{-1}(\sin(\pi))$
e. $\sin(\sin^{-1}(0.2))$

Solutions

a. By definition, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$ is an angle whose sine is $-\frac{\sqrt{3}}{2}$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. That is, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$ means $\sin(\theta) = -\frac{\sqrt{3}}{2}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. While you might be able to quickly state the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ right now, we are going to take our time showcasing the thought process so that we can work with more complicated problems.

Quadrant: Because $\sin(\theta) = -\frac{\sqrt{3}}{2}$ is negative, we further restrict θ to the interval $[-\frac{\pi}{2}, 0]$. In fact, since the value of the sine function is not ± 1 or 0, we know θ is not a quadrantal angle, so $\theta \in (-\frac{\pi}{2}, 0)$. Hence, $\theta \in \text{QIV}$.

Reference Angle: Since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, the reference angle for θ is $\hat{\theta} = \frac{\pi}{3}$.

Now that we have the quadrant and reference angle, we can state the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. $\theta \in (-\frac{\pi}{2}, 0)$ with a reference angle of $\hat{\theta} = \frac{\pi}{3}$ means

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}.$$

b. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \theta$ means $\sin(\theta) = -\frac{1}{\sqrt{2}}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Quadrant: Since the sine is negative, we need $\theta \in (-\frac{\pi}{2}, 0)$.

Reference Angle: We know the reference angle for $\sin(\theta) = -\frac{1}{\sqrt{2}}$ is $\hat{\theta} = \frac{\pi}{4}$.

A reference angle of $\frac{\pi}{4}$ in the interval $(-\frac{\pi}{2}, 0)$ means

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}.$$

c. As promised, the method we used in parts (a) and (b) can be used for more complicated problems (like this one).

$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \theta$ means $\sin(\theta) = \sin\left(\frac{5\pi}{6}\right)$ where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. A lot of people would say that θ must therefore be $\frac{5\pi}{6}$; however, $\frac{5\pi}{6}$ is not in the interval between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Quadrants and reference angles to the rescue!

Quadrant: Since $\sin(\theta) = \sin\left(\frac{5\pi}{6}\right)$ is positive (because $\frac{5\pi}{6} \in$ QII), $\theta \in (0, \frac{\pi}{2})$. Hence, $\theta \in$ QI.

Reference Angle: Since $\sin(\theta) = \sin\left(\frac{5\pi}{6}\right)$, θ must have the same reference angle as $\frac{5\pi}{6}$. That is, $\hat{\theta} = \frac{\pi}{6}$.

So, we know $\theta \in$ QI with a reference angle of $\frac{\pi}{6}$. Hence,

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \frac{\pi}{6}.$$

d. $\sin^{-1}(\sin(\pi)) = \theta$ means $\sin(\theta) = \sin(\pi)$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

This is one of those moments where quadrants and reference angles will not work. This is because π is a quadrantal angle. In these cases, we simply evaluate $\sin(\pi) = 0$. Therefore, $\sin^{-1}(\sin(\pi)) = \sin^{-1}(0)$. We want an angle whose sine is 0, and which lies in the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. This angle is 0. Therefore, $\sin^{-1}(\sin(\pi)) = 0$.[4]

e. We finally reach an example where the inverse trigonometric function is *the argument* of another function. Our approach will be the same as before - we start by focusing on the inverse trigonometric function.

$\sin^{-1}(0.2) = \theta$ means $\sin(\theta) = 0.2$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Believe it or not, we have actually found the value of $\sin(\sin^{-1}(0.2))$. Reread the statement above, and then follow the next line:

$$\sin\left(\sin^{-1}(0.2)\right) = \sin(\theta) = 0.2.$$

This illustrates the fact that

$$\sin\left(\sin^{-1}(r)\right) = r$$

as long as $r \in [-1, 1]$.

The process we performed in Example 10.1.2 is crucial to understand. It reinforces the statement, "Trigonometry is mostly about quadrants and reference angles."

When evaluating $\sin^{-1}(r)$, where r is some number (often a ratio), we let the inverse equal θ (this is to remind us that we are looking for an angle). That is, we begin by writing $\sin^{-1}(r) = \theta$. In *most* cases, we are looking for a *reference angle*, $\hat{\theta}$, such that $\sin\left(\hat{\theta}\right) = |r|$. We use the sign of r to inform us as to which *quadrant* θ is in. Once we have figured out these two details, we easily state the value of θ (remembering the requirement that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$).

Example 10.1.2*d* showcases the exception to the "find the reference angle and quadrant" rule. If r is 0, 1, or -1, then we want an angle such that $\sin(\theta)$ is equal to that value; however, these values are from quadrantal angles (0, $\frac{\pi}{2}$, and $-\frac{\pi}{2}$, respectively). As such, we will not have a reference angle to find.

Finally, while Example 10.1.2*e* illustrates that $\sin\left(\sin^{-1}(r)\right) = r$ as long as $r \in [-1, 1]$, Examples 10.1.2*c* and *d* give us the following caution.

Caution: $\sin^{-1}(\sin(\theta))$

$\sin^{-1}(\sin(\theta))$ is not necessarily θ.

The following theorem codifies the properties of the inverse sine function.

Theorem: Properties of the Inverse Sine Function

Properties of $f(x) = \sin^{-1}(x)$:

- $\theta = \sin^{-1}(x)$ if and only if $\sin(\theta) = x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- Domain: $[-1, 1]$
- Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\sin(\sin^{-1}(x)) = x$ for all $x \in [-1, 1]$
- $\sin^{-1}(\sin(x)) = x$ provided $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Checkpoint 10.1.2

Simplify each expression without using a calculator.

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
c. $\sin(\sin^{-1}(-0.389))$

Answers

a. $\frac{\pi}{3}$
b. $\frac{\pi}{3}$
c. -0.389

The Inverse Cosine and Inverse Tangent Functions

The cosine and tangent functions are also periodic, so, just as with the sine function, to define their inverse functions we must restrict their domains to intervals where they are one-to-one. The graph of cosine is shown in Figure 10.1.7.

Figure 10.1.7

Once again, the choice of these intervals is arbitrary. Suppose we start at $\theta = 0$ on the cosine graph. In that case, we can move in only one direction, either right or left, without encountering repeated y-values. As shown in Figure 10.1.7, we choose to move in the positive direction to obtain the interval $0 \le \theta \le \pi$. In this domain, the inverse of cosine is a function (see Figure 8).

Figure 10.1.8

Definition: Inverse Cosine Function

The function $f(x) = \cos^{-1}(x)$ is defined as follows:

$$\cos^{-1}(x) = \theta \quad \text{if and only if} \quad \cos(\theta) = x \quad \text{and} \quad 0 \le \theta \le \pi.$$

The range of the inverse cosine function is $0 \le y \le \pi$, so it delivers angles in the first and second quadrants. (Compare to the inverse sine, whose outputs are angles in the first or fourth quadrants.)

Theorem: Properties of the Inverse Cosine Function

Properties of $f(x) = \cos^{-1}(x)$:

- $\theta = \cos^{-1}(x)$ if and only if $\cos(\theta) = x$ where $0 \le \theta \le \pi$
- Domain: $[-1, 1]$
- Range: $[0, \pi]$
- $\cos(\cos^{-1}(x)) = x$ for all $x \in [-1, 1]$
- $\cos^{-1}(\cos(x)) = x$ provided $x \in [0, \pi]$

Example 10.1.3

Simplify each expression without using a calculator.

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$

Solutions

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$ means $\cos(\theta)=-\frac{\sqrt{3}}{2}$, where $\theta\in[0,\pi]$.

Quadrant: Since $\cos(\theta)=-\frac{\sqrt{3}}{2}$ is negative, we further restrict θ so that $\theta\in\left(\frac{\pi}{2},\pi\right)$. Thus, $\theta\in$ QII.

Reference Angle: To find the reference angle, we ask ourselves what acute angle, $\hat{\theta}$, results in

$$\cos\left(\hat{\theta}\right)=\left|-\frac{\sqrt{3}}{2}\right|=\frac{\sqrt{3}}{2}.$$

From our special triangles, we know this is $\hat{\theta}=\frac{\pi}{6}$.

The angle in the second quadrant with this reference angle is $\theta=\frac{5\pi}{6}$, so

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}.$$

b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)=\theta$ means $\cos(\theta)=\cos\left(\frac{7\pi}{4}\right)$, where $\theta\in[0,\pi]$.

Quadrant: Because $\frac{7\pi}{4}\in$ QIV, we know that $\cos(\theta)=\cos\left(\frac{7\pi}{4}\right)$ is positive. Therefore, $\theta\in\left(0,\frac{\pi}{2}\right)$.

Reference Angle: Since $\cos(\theta)=\cos\left(\frac{7\pi}{4}\right)$, the reference angle for θ must be equal to the reference angle for $\frac{7\pi}{4}$. That is, $\hat{\theta}=\frac{\pi}{4}$.

We have a reference angle of $\frac{\pi}{4}$ in Quadrant I, this means

$$\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)=\frac{\pi}{4}.$$

Let's take a moment to highlight an important takeaway from Example 10.1.3*b*. First, we know that the inverse trigonometric functions *return* angles. What part (b) of 10.1.3 shows is that, given $\cos^{-1}(\cos(\alpha))=\theta$, the reference angle for both θ and α are the same. That is, we had $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)=\theta$. The reference angle for $\frac{7\pi}{4}$ is $\frac{\pi}{4}$, which is exactly the reference angle $\hat{\theta}$.

This same principle holds for $\sin^{-1}(\sin(\alpha))=\theta$. Therefore, given $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$, we know this will return an angle θ such that $\hat{\theta}=\frac{\pi}{3}$.

The reason for this common reference angle stems from the definition of the inverse trigonometric function,

$$\sin^{-1}(\sin(\alpha))=\theta \quad \text{if and only if} \quad \sin(\theta)=\sin(\alpha).$$

Therefore, $\hat{\alpha}$ must equal $\hat{\theta}$.

Checkpoint 10.1.3

Simplify $\cos^{-1}(-1)$ without using a calculator.

Answer

π

Finally, consider the graph of the tangent function. To choose a convenient interval on which the tangent is one-to-one, we start at $x = 0$ and move as far as we can in either direction along the x-axis. In this way, we obtain one cycle of the graph on the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, as shown in Figure 10.1.9.

Figure 10.1.9

The range of the tangent on that interval includes all real numbers. Consequently, the domain of the inverse tangent function consists of all real numbers, and its range is the interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The graph of the inverse tangent function is shown below. Its outputs are angles in the first and fourth quadrants (see Figure 10.1.10.

Figure 10.1.10

Note: The Inverse Tangent Graph

While you don't see the inverse sine and inverse cosine graphs in Mathematics beyond this point, the graph of the inverse tangent function is vital for Calculus concepts. This is a good one to be able to derive quickly.

Definition: The Inverse Tangent Function

The function $f(x) = \tan^{-1}(x)$ is defined as follows:

$$\tan^{-1}(x) = \theta \quad \text{if and only if} \quad \tan(\theta) = x \qquad \text{and} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Theorem: Properties of the Inverse Tangent Function

Properties of $f(x) = \tan^{-1}(x)$:

- $\theta = \tan^{-1}(x)$ if and only if $\tan(\theta) = x$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- Domain: $(-\infty, \infty)$
- Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$
- As $x \to -\infty$, $\tan^{-1}(x) \to -\frac{\pi}{2}$, and as $x \to \infty$, $\tan^{-1}(x) \to \frac{\pi}{2}$
- $\tan(\tan^{-1}(x)) = x$ for all $x \in \mathbb{R}$
- $\tan^{-1}(\tan(x)) = x$ provided $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Example 10.1.4

Simplify each expression without using a calculator.

a. $\tan^{-1}(-\sqrt{3})$

b. $\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)$

Solutions

a. $\tan^{-1}(-\sqrt{3}) = \theta$ means $\tan(\theta) = -\sqrt{3}$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Quadrant: Since $\tan(\theta)$ is negative, we further restrict the angle to $\theta \in (-\frac{\pi}{2}, 0)$, or Quadrant IV.

Reference Angle: Since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, $\hat{\theta} = \frac{\pi}{3}$.

Combining our results, we get

$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}.$$

b. $\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right) = \theta$ means $\tan(\theta) = \tan\left(\frac{7\pi}{4}\right)$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Quadrant: Because $\frac{7\pi}{4}$ is in Quadrant IV, $\tan\left(\frac{7\pi}{4}\right)$ is negative. Hence, $\tan(\theta)$ is negative. As such, we further

restrict the angle to $\theta \in (-\frac{\pi}{2}, 0)$.

Reference Angle: Since $\tan(\theta) = \tan\left(\frac{7\pi}{4}\right)$, both arguments must share the same reference angle. Therefore, $\hat{\theta} = \frac{\pi}{4}$.

Thus,

$$\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right) = -\frac{\pi}{4}.$$

Checkpoint 10.1.4

Simplify $\tan^{-1}(-1)$ without using a calculator.

Answer

$$-\frac{\pi}{4}$$

Summary of Inverse Trigonometric Functions

Summarizing the inverse trigonometric functions and their properties before moving forward is helpful.

Summary: Inverse Trigonometric Functions

Key Concepts

Trigonometric functions take angles as arguments and return ratios. Inverse trigonometric functions take ratios as arguments and return (restricted) angles. That is,

$$\text{trig}(\text{angle}) = \text{ratio} \quad \text{and} \quad \text{trig}^{-1}(\text{ratio}) = \text{angle}.$$

To drive this point home, the rest of this summary uses r for "ratio" and θ for the angle.

Basic Definitions

$$\sin^{-1}(r) = \theta \quad \text{if and only if} \quad \sin(\theta) = r, \quad \text{where} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\cos^{-1}(r) = \theta \quad \text{if and only if} \quad \cos(\theta) = r, \quad \text{where} \quad 0 \le \theta \le \pi$$

$$\tan^{-1}(r) = \theta \quad \text{if and only if} \quad \tan(\theta) = r, \quad \text{where} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Domains and Ranges

Function	Domain	Range	Function	Domain	Range
$r = \sin(\theta)$	$\theta \in (\infty, \infty)$	$r \in [-1, 1]$	$\theta = \sin^{-1}(r)$	$r \in [-1, 1]$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$r = \cos(\theta)$	$\theta \in (\infty, \infty)$	$r \in [-1, 1]$	$\theta = \cos^{-1}(r)$	$r \in [-1, 1]$	$\theta \in [0, \pi]$
$r = \tan(\theta)$	$\theta \neq \frac{\pi}{2} + \pi k$	$r \in (-\infty, \infty)$	$\theta = \tan^{-1}(r)$	$r \in (-\infty, \infty)$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Invertibility Properties

$$\begin{aligned}
\sin\left(\sin^{-1}(x)\right) &= x \qquad \text{for } -1 \le x \le 1 \\
\cos\left(\cos^{-1}(x)\right) &= x \qquad \text{for } -1 \le x \le 1 \\
\tan\left(\tan^{-1}(x)\right) &= x \qquad \text{for } -\infty < x < \infty \\
\sin^{-1}\left(\sin(x)\right) &= x \qquad \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}\left(\cos(x)\right) &= x \qquad \text{only for } 0 \le x \le \pi \\
\tan^{-1}\left(\tan(x)\right) &= x \qquad \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Advice for Use

When asked to find the value of an inverse trigonometric function, $\text{trig}^{-1}(r)$, we start by recalling that the inverse trigonometric functions return angles. Therefore, we write

$$\text{trig}^{-1}(r) = \theta.$$

Our goal is to find *an appropriately restricted angle*, θ, such that

$$\text{trig}(\theta) = r.$$

A good piece of advice is to find the *reference angle*, $\hat{\theta}$, by determining the value of

$$\text{trig}^{-1}(|r|) = \hat{\theta}$$

(Recall that *all* inverse trigonometric functions return angles in quadrant I if the argument of the inverse is positive.)

Once you have the reference angle, you can use the sign of r along with the knowledge of where your inverse trigonometric function returns angles to determine the value of $\theta = \text{trig}^{-1}(r)$.

The Invertibility Properties are tricky for most students. Just remember that the inverse trigonometric functions return restricted angles. That's why, for example, $\cos^{-1}(\cos(x))$ is not necessarily x.

Using a Calculator to Evaluate Inverse Trigonometric Functions

We have been working with the inverse trigonometric buttons on your calculator for quite some time; however, it is important to clarify how they are used once you encounter negative ratios as arguments of inverse trigonometric functions.

Example 10.1.5

Evaluate each of the following using a calculator. Your answers should be in radians.

a. $\sin^{-1}(0.97)$
b. $\cos^{-1}(-0.561)$
c. $\tan^{-1}(-3)$

Solutions

a. This is a straightforward use of the calculator. At this point, you should be familiar enough with this to know we enter $\text{SIN}^{-1}\,0.97$ in the calculator to get approximately 1.3252 radians; however, we are going to use this to link back to the *quadrant* and *reference angle* concept we have been building throughout this section.

$\sin^{-1}(0.97) = \theta$ means $\sin(\theta) = 0.97$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Quadrant: Since the sine is positive, we further restrict our angle to $\theta \in (0, \frac{\pi}{2})$, or Quadrant I.

Reference Angle: The reference angle can be found by using your calculator (in radian mode) to evaluate $\sin^{-1}(|0.97|) \approx 1.3252$.

So our angle is in Quadrant I with a reference angle of $\hat{\theta} \approx 1.3252$. This means,

$$\sin^{-1}(0.97) \approx 1.3252.$$

b. $\cos^{-1}(-0.561) = \theta$ means $\cos(\theta) = -0.561$, where $\theta \in [0, \pi]$

Quadrant: Since $\cos(\theta)$ is negative, we can guarantee that $\theta \in$ QII (more specifically, $\theta \in (\frac{\pi}{2}, \pi)$).

Reference Angle: To find the *reference angle* for θ, we compute $\hat{\theta} = \cos^{-1}(|-0.561|) = \cos^{-1}(0.561) \approx 0.9752$ radians.

Now that we have a quadrant (II) and a reference angle (0.9752), we find the actual angle by subtracting the reference angle from π. Thus,

$$\cos^{-1}(-0.561) \approx \pi - 0.9752 \approx 2.1664.$$

c. $\tan^{-1}(-3) = \theta$ means $\tan(\theta) = -3$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Quadrant: Since the tangent is negative, we can restrict this interval to $\theta \in (-\frac{\pi}{2}, 0)$, or Quadrant IV.

Reference Angle: The reference angle is

$$\hat{\theta} = \tan^{-1}(|-3|) = \tan^{-1}(3) \approx 1.2490.$$

Thus,

$$\tan^{-1}(-3) \approx -1.2490.$$

I want to be clear, your calculator would have returned the answers to Example 10.1.5*b* and 10.1.5*c* if you had just entered $\text{COS}^{-1}\ -0.561$ and $\text{TAN}^{-1}\ -3$, respectively; however, as we move forward, your calculator will become less reliable when it comes to giving you the answer you're looking for. This is why I recommend evaluating the inverse trigonometric functions at their arguments' absolute value and treating the result as a reference angle. You honestly cannot go wrong if you do this.

? Exercise 10.1.5

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

1.9823 radians or $113.578°$

Alternate Notations

The inverse sine function, $\sin^{-1}(x)$, is also called the **arcsine function** and denoted by $\arcsin(x)$. (This terminology reminds us that the output of the inverse sine function is an angle, or the arc on a unit circle determined by that angle, as shown in Figure 10.1.11.)

Figure 10.1.11

Similarly, the inverse cosine function is sometimes denoted by $\arccos(x)$, and the inverse tangent function by $\arctan(x)$.

✓ Example 10.1.6

Simplify each expression.

a. $\cos\left(\arccos\left(\frac{1}{3}\right)\right)$
b. $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right)$

Solutions

a. $\arccos\left(\frac{1}{3}\right) = \theta$ means $\cos(\theta) = \frac{1}{3}$, where $0 \le \theta \le \pi$. Since $\frac{1}{3} \in [-1, 1]$, we can use the Invertibility Property to state

$$\cos\left(\arccos\left(\frac{1}{3}\right)\right) = \cos(\theta) = \frac{1}{3}.$$

b. $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right) = \theta$ means $\sin(\theta) = \sin\left(\frac{3\pi}{2}\right)$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Noticing that $\frac{3\pi}{2}$ is a quadrantal angle, we go ahead and simplify the right side of that last equation.

$$\sin(\theta) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Therefore, we require an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that the sine of that angle is -1. This would be $-\frac{\pi}{2}$.Thus,

$$\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right) = -\frac{\pi}{2}.$$

Checkpoint 10.1.6

Simplify each expression.

a. $\tan\left(\arctan\left(-\frac{5}{2}\right)\right)$
b. $\arctan\left(\tan\left(-\frac{5\pi}{6}\right)\right)$

Answers

a. $-\frac{5}{2}$
b. $\frac{\pi}{6}$

Note: Do Arc Functions Return Angles?

Up to this point, we have been very clear that the inverse trigonometric functions return angles; however, very technically, the arcfunctions return *real numbers* - not angles. This is why many instructors prefer you to answer in radians when dealing with arcfunctions rather than degrees. Fortunately, this is a subtlety that you don't have to worry about. You can assume the arcfunctions, like the inverse trigonometric functions, return angles if that helps you remember them.

Simplifying Expressions Involving Different Trigonometric Functions

The key to simplifying expressions involving inverse trigonometric functions is to remember that the inverse sine, cosine, or tangent of a number can be treated as an angle. It can often clarify the computations if we assign a name such as θ or ϕ to the inverse trigonometric value. Additionally, drawing triangles on the Cartesian coordinate system can be immensely helpful.

Example 10.1.7

Evaluate $\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$.

Solution

Notice this is a mixture of trigonometric functions. Our process changes only slightly from before.

$\cos^{-1}\left(\frac{3}{4}\right) = \theta$ means $\cos(\theta) = \frac{3}{4}$, where $\theta \in [0, \pi]$.

Quadrant: Since the cosine of θ is positive, we know $\theta \in$ QI. *Because we have a mixture of trigonometric functions*, it's best to draw a reference triangle in QI.

Figure 10.1.12

We now can use this reference triangle to answer the question (i.e., there is no need to find the reference angle).

$\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$ simplifies to $\sin(\theta)$. Using the Pythagorean Theorem and the definition for sine, we get

$$\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right) = \sin(\theta) = \frac{\sqrt{7}}{4}.$$

? Checkpoint 10.1.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$.

Answer

$$\frac{3}{\sqrt{13}}$$

We can verify the results of the previous example using a calculator, but the same technique can be applied to simplify similar expressions involving variables.

✓ Example 10.1.8

Simplify $\tan\left(\sin^{-1}(x)\right)$, assuming that $-1 < x < 0$.

Solution

Let $\theta = \sin^{-1}(x)$, so that $\sin(\theta) = x = \frac{x}{1}$, and the expression $\tan\left(\sin^{-1}(x)\right) = \tan(\theta)$. Since $-1 < x < 0$, sine is negative. Therefore, the inverse sine returns an angle in QIV. Specifically, $-\frac{\pi}{2} \leq \theta \leq 0$. Drawing a reference triangle in QIV, we get

Figure 10.1.13

Now we the fact that $\tan(\theta)$ is opposite over hypotenuse to get

$$\tan\left(\sin^{-1}(x)\right) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}.$$

The answer to Example 10.1.8, $\frac{x}{\sqrt{1-x^2}}$, *looks* positive, but we know this cannot be the case because the tangent should be negative in QIV. Looks can be deceiving! While the denominator is definitely positive, the numerator, x, is negative. Therefore, the answer is, in fact, negative!

? Checkpoint 10.1.8

Simplify $\sin\left(\tan^{-1}(z)\right)$, assuming that $z \geq 0$.

Answer

$$\frac{z}{\sqrt{1+z^2}}$$

Simplifying Expressions Involving Identities

Now that we have learned all of the major identities in Trigonometry, we can use them with the inverse trigonometric functions. In doing so, we will demonstrate our fluency with inverse trigonometric functions and identities.

✓ Example 10.1.9

a. Rewrite the following as an algebraic expression of x and state the domain on which the equivalence is valid.

$$f(x) = \cos(2\arcsin(x))$$

b. Find the exact value of the expression.

$$\sin\left(\cos^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{5}\right)\right)$$

Solution

a. We start by writing $\theta = \arcsin(x)$ so that θ lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) = x$. We want to express $\cos(2\arcsin(x)) = \cos(2\theta)$ in terms of x. Since $\cos(2\theta)$ is defined everywhere, we get no additional restrictions on θ. We have three choices for rewriting $\cos(2\theta)$:

$$\cos^2(\theta) - \sin^2(\theta), \quad 2\cos^2(\theta) - 1, \quad \text{and} \quad 1 - 2\sin^2(\theta).$$

Since we know $x = \sin(\theta)$, it is easiest to use the last form:

$$\cos(2\arcsin(x)) = \cos(2\theta) = 1 - 2\sin^2(\theta) = 1 - 2x^2.$$

To find the restrictions on x, we consider our substitution, $\theta = \arcsin(x)$. Since $\arcsin(x)$ is defined only for $-1 \leq x \leq 1$, the function, $f(x) = \cos(2\arcsin(x)) = 1 - 2x^2$, is valid only on $[-1, 1]$.

b. Let $\alpha = \cos^{-1}\left(-\frac{1}{3}\right)$ and $\beta = \sin^{-1}\left(\frac{2}{5}\right)$. Then we can rewrite the expression as

$$\sin\left(\cos^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{5}\right)\right) = \sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

where

$$\cos(\alpha) = -\frac{1}{3}, \quad 0 \leq \alpha \leq \pi$$

and

$$\sin(\beta) = \frac{2}{5}, \quad -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}.$$

Since $\cos(\alpha)$ is negative, we immediately know $\alpha \in$ QII. Moreover, $\sin(\beta)$ being positive means that $\beta \in$ QI. Drawing reference triangles, we get the following figures:

Figure 10.1.14

We now evaluate the equivalent expression, $\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, using these reference triangles.

$$\begin{aligned}
\sin\left(\cos^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{5}\right)\right) &= \sin(\alpha + \beta) \\
&= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
&= \left(\frac{2\sqrt{2}}{3}\right)\left(\frac{\sqrt{21}}{5}\right) + \left(-\frac{1}{3}\right)\left(\frac{2}{5}\right) \\
&= \frac{2\sqrt{42}}{15} - \frac{2}{15} \\
&= \frac{2\sqrt{42} - 2}{15}
\end{aligned}$$

? Checkpoint 10.1.9

Find the exact value of the expression.

$$\sin\left(2\cos^{-1}\left(-\frac{1}{3}\right)\right)$$

Answer

$$-\frac{4\sqrt{2}}{9}$$

Modeling with Inverse Functions

The inverse trigonometric functions are used to model situations in which an angle is described in terms of one of its trigonometric ratios.

Example 10.1.10

The bottom of a 3-meter-tall tapestry on a chateau wall is at your eye level. The angle θ subtended vertically by the tapestry changes as you approach the wall.

a. Express your distance from the wall, d, as a function of θ.
b. Express θ as a function of d.

Solutions

a. As shown below, we sketch the triangle formed by the tapestry and the lines of sight to its bottom and top. From the triangle we see that $\tan(\theta) = \frac{3}{d}$, so $d = \frac{3}{\tan(\theta)}$.

Figure 10.1.15

b. Because $\tan(\theta) = \frac{3}{d}, \theta = \tan^{-1}\left(\frac{3}{d}\right)$.

In Example 10.1.10, we didn't have to worry about the quadrant in which θ belongs. Most (but not all) applications share this quality. If we are looking at the top of a tapestry on a wall, and if the bottom of the tapestry is at eye level, no matter how close or how far we are away from the tapestry, the angle of inclination from eye level to the top of the tapestry will always be acute (i.e., a quadrant I angle).

Checkpoint 10.1.10

The tapestry from the previous example includes a 2-meter-tall unicorn with its feet at the bottom of the tapestry.

a. Express α, the angle subtended vertically by the unicorn, as a function of d, your distance to the tapestry.
b. Express β, the angle subtended by the portion of the tapestry above the unicorn, as a function of d. (See the figure below).

Figure 10.1.16

Answers

a. $\alpha = \tan^{-1}\left(\frac{2}{d}\right)$
b. $\beta = \tan^{-1}\left(\frac{3}{d}\right) - \tan^{-1}\left(\frac{2}{d}\right)$

Footnotes

[1] Recall that if a graph passes the Vertical Line Test, there is only one y-value for each value of x.

[2] We could also have used only negative x-values for the domain, or some smaller interval, just as long as the resulting function is one-to-one.

[3] In this course, we have yet to consider inverse trigonometric functions returning angles other than acute (and, therefore, positive) ones.

[4] It is also true that $\sin(\pi) = 0$, but π is not in the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

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10.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **An Overview of Functions**
  - Function Composition and Domain: This skill is necessary for evaluating and simplifying composite functions involving trigonometric and inverse trigonometric functions, such as $\sin(\arccos(x))$.
  - Inverse Functions: This section is built on the concept of inverse functions. Finding a formula for an inverse, using the Horizontal Line Test to determine if a function has an inverse function, and understanding the relationship between the domain and range of a function and its inverse are all critical prerequisite skills. These are reviewed in the "Skills Refresher" exercises.
  - Restricting Domains and Forcing Invertibility: This section's primary theoretical basis is the idea of restricting the domains of the non-invertible trigonometric functions to create new functions that are one-to-one and thus have inverse functions.

VIDEOS

- Inverse Trigonometric Functions
  - Implicit Differentiation - Derivatives of Inverse Trig Functions: Theorem, Proof, and Example
  - Inverse Trigonometric Functions - Defining the Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Evaluating Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Applications of Right Triangle Trigonometry (Part 2)

KEY CONCEPTS

DEFINITIONS

- **The Inverse Sine Function:** The function $f(x) = \sin^{-1}(x)$ (or $\arcsin(x)$) is defined as $\sin^{-1}(x) = \theta$ if and only if $\sin(\theta) = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- **The Inverse Cosine Function:** The function $f(x) = \cos^{-1}(x)$ (or $\arccos(x)$) is defined as $\cos^{-1}(x) = \theta$ if and only if $\cos(\theta) = x$ and $0 \leq \theta \leq \pi$.
- **The Inverse Tangent Function:** The function $f(x) = \tan^{-1}(x)$ (or $\arctan(x)$) is defined as $\tan^{-1}(x) = \theta$ if and only if $\tan(\theta) = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

THEOREMS

- **Lemma:** Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then $\cos(\theta) = x$ and $\sin(\theta) = y$.
- **Forcing Invertibility by Restricting Domain:** If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.
- **Properties of the Inverse Sine Function:** For $f(x) = \sin^{-1}(x)$:
  - Domain: $[-1, 1]$
  - Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  - $\sin(\sin^{-1}(x)) = x$ for all x in $[-1, 1]$
  - $\sin^{-1}(\sin(x)) = x$ provided x is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- **Properties of the Inverse Cosine Function:** For $f(x) = \cos^{-1}(x)$:
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\cos(\cos^{-1}(x)) = x$ for all x in $[-1, 1]$
  - $\cos^{-1}(\cos(x)) = x$ provided x is in $[0, \pi]$
- **Properties of the Inverse Tangent Function:** For $f(x) = \tan^{-1}(x)$:
  - Domain: $(-\infty, \infty)$
  - Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$
  - $\tan(\tan^{-1}(x)) = x$ for all x in $\mathbb{R}$
  - $\tan^{-1}(\tan(x)) = x$ provided x is in $(-\frac{\pi}{2}, \frac{\pi}{2})$

COMMON MISTAKES

- **Inverse Composition Property:** Be cautious with compositions like $\sin^{-1}(\sin(\theta))$. This expression is not necessarily equal to θ. It is only equal to θ if θ is within the restricted range of the inverse sine function, $[-\frac{\pi}{2}, \frac{\pi}{2}]$. A similar caution applies to inverse cosine and inverse tangent.
- **Inverse Trigonometric Functions are NOT Reciprocals:** The notation $\sin^{-1}(x)$ denotes the inverse sine function, not the reciprocal of sine. The reciprocal of $\sin(x)$ is $\csc(x)$.
- **Forgetting Domain and Range Restrictions:** The inverse trigonometric functions have specific, restricted ranges. For example, the output of $\arccos(x)$ must be an angle between 0 and π, inclusive. Forgetting these restrictions is a common source of error.

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10.1.2: HOMEWORK

READING QUESTIONS

1. What graphical test is used to determine if a function is one-to-one?
2. Why is it necessary to restrict the domain of the sine function before we can define its inverse function?
3. What is the restricted domain of the cosine function that is used to define the inverse cosine function?
4. What is the domain of the function $y = \arcsin(x)$?
5. What is the range of the function $y = \arccos(x)$?
6. What is the range of the function $y = \arctan(x)$?
7. The expression $\sin^{-1}(0.5)$ represents an angle. In which quadrant does this angle lie?
8. The expression $\cos^{-1}(-0.5)$ represents an angle. In which quadrant does this angle lie?
9. Is the statement $\sin^{-1}(\sin(3\pi)) = 3\pi$ true or false? Explain your reasoning.
10. Is the statement $\sin(\sin^{-1}(0.5)) = 0.5$ true or false? Explain your reasoning.
11. What is the key difference between the meaning of $\sin^{-1}(x)$ and $(\sin(x))^{-1}$?
12. When simplifying an expression like $\cos(\arctan(x))$, what is a helpful first step?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, find the inverse of the function, if it exists.

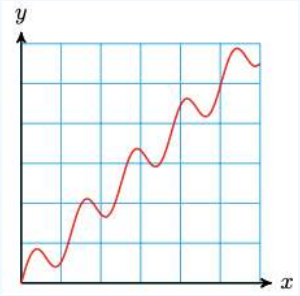
1. $f(x) = 2x^2 - 16x + 1$
2. $g(x) = (x-2)^3$
3. $f(x) = \dfrac{3x-1}{5x-1}$

For the following exercises,
(a) Find a formula for the inverse function.
(b) State the domain and range of the inverse function.
(c) Graph the function and its inverse on the same grid.

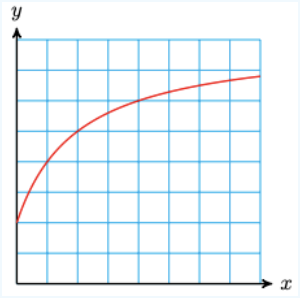
4. $f(x) = \frac{1}{2}x - 4$
5. $g(x) = 3x + 6$
6. $F(x) = 2 + \frac{1}{x}$
7. $G(x) = \frac{1}{x+5}$
8. $h(x) = \sqrt{x+2}$
9. $H(x) = 3 + \sqrt[3]{x}$

For the following exercises, which functions have an inverse function? Explain your answer.

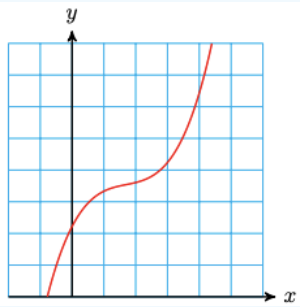
10.



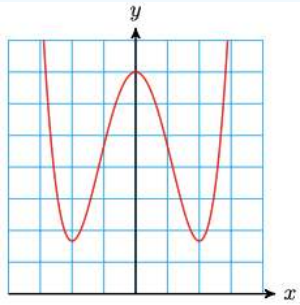
11.



12.



13.



For the following exercises, use technology to graph the function and decide if it has an inverse function.

14. $f(x) = \sin(2x) - \cos(x)$
15. $g(x) = 4e^{-(x/4)^2}$
16. $G(x) = \sqrt{25 - x^2}$
17. $F(x) = \ln(x^3 + 8)$

Answers

1. No inverse exists for this function.
2. $g^{-1}(x) = \sqrt[3]{x} + 2$

3. $\frac{x-1}{5x-3}$
4.
   a. $f^{-1}(x) = 2x + 8$
   b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$
   c.
5.
   a. $g^{-1}(x) = \frac{1}{3}x - 2$
   b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$
   c.
6.
   a. $F^{-1}(x) = \frac{1}{x-2}$
   b. Dom: $x \neq 0$ Rge: $y \neq 2$
   c.
7.
   a. $G^{-1}(x) = \frac{1}{x} - 5$
   b. Dom: $x \neq -5$ Rge: $y \neq 0$
   c.
8.
   a. $h^{-1}(x) = x^2 - 2$
   b. Dom: $x \geq 0$ Rge: $y \geq -2$
   c.
9.
   a. $H^{-1}(x) = (x-3)^3$
   b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$
   c.
10. No. This graph does not pass the Horizontal Line Test.
11. Yes. This graph passes the Horizontal Line Test.
12. Yes. This graph passes the Horizontal Line Test.
13. No. This graph does not pass the Horizontal Line Test.
14. No.
15. No.
16. No.
17. Yes.

HOMEWORK

CONCEPT CHECK

1. The following is a table of values defining a function f. Make a table of values for f^{-1}.

x	-3	-2	0	1	4
$f(x)$	6	3	1	0	-1

2. What does it mean for a function to be one-to-one? Give an example.
3. Why do we restrict the domains of the trigonometric functions when we define their inverse functions?
4. Which of the following expressions is undefined? Why?
   a. $\cos^{-1}(0)$
   b. $\arctan(-2)$
   c. $\arcsin(-2)$
5. Use your calculator to evaluate $\sin(118°)$, then evaluate $\sin^{-1}$ of your answer. Explain the result.
6.
   a. Write x as a function of θ.
   b. Write θ as a function of x.
7.
   a. Write x as a function of θ.
   b. Write θ as a function of x.

BASIC SKILLS

For the following exercises,
(a) Evaluate each pair of angles to the nearest 0.1 radian, and show that they are supplements.
(b) Sketch both angles.
(c) Find the sine of each angle.

8. $\theta = \cos^{-1}\left(\frac{3}{4}\right), \phi = \cos^{-1}\left(-\frac{3}{4}\right)$
9. $\theta = \cos^{-1}\left(\frac{1}{5}\right), \phi = \cos^{-1}\left(-\frac{1}{5}\right)$
10. $\theta = \cos^{-1}(0.1525), \quad \phi = \cos^{-1}(-0.1525)$
11. $\theta = \cos^{-1}(0.6825), \quad \phi = \cos^{-1}(-0.6825)$

For the following exercises, use a calculator to evaluate. Round your answers to the nearest tenth of a radian.

12. $\sin^{-1}(0.2838)$
13. $\tan^{-1}(4.8972)$
14. $\cos^{-1}(0.6894)$
15. $\arccos(-0.8134)$
16. $\arctan(-1.2765)$
17. $\arcsin(-07493)$

For the following exercises, give exact values.

18. $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
19. $\tan^{-1}(-1)$
20. $\sin^{-1}\left(-\frac{1}{2}\right)$
21. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
22. $\arctan\left(\frac{1}{\sqrt{3}}\right)$
23. $\arcsin(-1)$
24. $\arccos(-1)$
25. $\arctan\left(\sqrt{3}\right)$
26. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
27. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
28. $\arctan\left(-\sqrt{3}\right)$

29. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
30. $\arccos(0)$
31. $\arctan(0)$
32. $\arcsin(0)$
33. $\arccos\left(\frac{1}{2}\right)$
34. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$
35. $\arcsin\left(\frac{1}{2}\right)$
36. $\arccos\left(\frac{\sqrt{2}}{2}\right)$
37. $\arctan(1)$
38. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
39. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
40. $\arcsin(1)$
41. $\arccos(1)$

For the following exercises, without using a calculator, find the exact value or state that it is undefined.

42. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$
43. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$
44. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$
45. $\sin(\arcsin(-0.42))$
46. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$
47. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$
48. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$
49. $\cos\left(\arccos\left(\frac{5}{13}\right)\right)$
50. $\cos(\arccos(-0.998))$
51. $\cos(\arccos(\pi))$
52. $\tan(\arctan(-1))$
53. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$
54. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$
55. $\tan(\arctan(0.965))$
56. $\tan(\arctan(3\pi))$

For the following exercises, find the exact value without using a calculator.

57. $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$
58. $\tan\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$
59. $\cos\left[\tan^{-1}(-2)\right]$
60. $\sin\left[\tan^{-1}\left(-\frac{3}{\sqrt{5}}\right)\right]$
61. $\sin\left[\cos^{-1}\left(-\frac{2\sqrt{6}}{7}\right)\right]$
62. $\cos\left[\sin^{-1}\left(-\frac{2}{7}\right)\right]$
63. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$
64. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$
65. $\sin(\arctan(-2))$
66. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$
67. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$
68. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$
69. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$
70. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$
71. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$
72. $\cot(\arctan(0.25))$
73. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$
74. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$
75. $\sec(\arctan(10))$
76. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$
77. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

For the following exercises, simplify the expression and state the domain on which the equivalence is valid.

78. $\tan\left(\cos^{-1}(x)\right)$
79. $\cos\left(\sin^{-1}(h)\right)$
80. $\sin(\arccos(x))$
81. $\cos(\arctan(x))$
82. $\tan(\arcsin(x))$
83. $\sec(\arctan(x))$
84. $\csc(\arccos(x))$
85. $\sin\left(\tan^{-1}(2t)\right)$
86. $\tan\left(\sin^{-1}(3b)\right)$

For the following exercises, find the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$
88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$
89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$
90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$
91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$
92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$
93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$
94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$
95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$
96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$
98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$
99. $\arctan\left(\tan\left(\pi\right)\right)$
100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

For the following exercises, complete the table of values and sketch the function.

102.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos^{-1}x$									

103.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1}x$									

104.

x	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\tan^{-1}x$							

105. Use a graphing calculator to answer each of the following questions. Then explain the results.
   a. Does $\cos^{-1}(x)=\frac{1}{\cos(x)}$?
   b. Does $\sin^{-1}(x)=\frac{1}{\sin(x)}$?
   c. Does $\tan^{-1}(x)=\frac{1}{\tan(x)}$?

106.
   a. Sketch a graph of $y=\cos^{-1}(x)$, and label the scales on the axes.
   b. Use transformations to sketch graphs of $y=2\cos^{-1}(x)$ and $y=\cos^{-1}(2x)$.
   c. Does $2\cos^{-1}(x)=\cos^{-1}(2x)$?

107.
   a. Sketch a graph of $y=\sin^{-1}(x)$, and label the scales on the axes.
   b. Use transformations to sketch graphs of $y=\frac{1}{2}\sin^{-1}(x)$ and $y=\cos^{-1}\left(\frac{1}{2}x\right)$.
   c. Does $\frac{1}{2}\sin^{-1}(x)=\cos^{-1}\left(\frac{1}{2}x\right)$?

108.
   a. Sketch a graph of $y=\tan^{-1}(x)$, and label the scales on the axes.
   b. Use your calculator to graph $y=\frac{\sin^{-1}(x)}{\cos^{-1}(x)}$ on a suitable domain.
   c. Does $\tan^{-1}(x)=\frac{\sin^{-1}(x)}{\cos^{-1}(x)}$?

109.
   a. Use your calculator to sketch $y=\sqrt[3]{x}$ and $\tan^{-1}(x)$ on $[-10,10]$.
   b. Describe the similarities and differences in the two graphs.

For the following exercises, find the exact value for the expression.

110. $\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$
111. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$
112. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$
113. $\sin\left(2\arctan\left(2\right)\right)$
114. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$
115. $\sin\left(\frac{\arctan(2)}{2}\right)$
116. $\sin\left(2\tan^{-1}(4)\right)$
117. $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$
118. $\tan\left(2\cos^{-1}\left(\frac{3}{4}\right)\right)$
119. $\sin\left(2\cos^{-1}\left(-\frac{4}{5}\right)\right)$
120. $\tan\left(2\sin^{-1}\left(\frac{1}{3}\right)\right)$
121. $\cos\left(2\tan^{-1}\left(\frac{3}{2}\right)\right)$
122. Let $\alpha=\cos^{-1}\left(-\frac{4}{5}\right),\beta=\sin^{-1}\left(\frac{5}{13}\right)$. Find exact values for the following.
   a. $\cos(\alpha+\beta)$
   b. $\sin(\alpha+\beta)$
   c. $\cos(\alpha-\beta)$
   d. $\sin(\alpha-\beta)$
123. Let $\alpha=\sin^{-1}\left(-\frac{15}{17}\right),\beta=\tan^{-1}\left(\frac{4}{3}\right)$. Find exact values for the following.
   a. $\cos(\alpha+\beta)$
   b. $\sin(\alpha+\beta)$
   c. $\cos(\alpha-\beta)$
   d. $\sin(\alpha-\beta)$
124. Find an exact value for $\sin\left(\tan^{-1}\left(\frac{3}{4}\right)+\sin^{-1}\left(-\frac{4}{5}\right)\right)$.
125. Find an exact value for $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)+\sin^{-1}\left(-\frac{3}{5}\right)\right)$.

For the following exercises, rewrite the expression as an algebraic expression without trigonometric functions.

126. $\sin\left(2\tan^{-1}(x)\right)$
127. $\cos\left(2\sin^{-1}(x)\right)$
128. $\sin\left(2\cos^{-1}(w)\right)$
129. $\cos\left(2\tan^{-1}(w)\right)$
130. $\sin(\arccos(2x))$
131. $\sin\left(\arccos\left(\frac{x}{5}\right)\right)$
132. $\cos\left(\arcsin\left(\frac{x}{2}\right)\right)$
133. $\cos\left(\arctan\left(3x\right)\right)$
134. $\sin(2\arcsin(7x))$
135. $\sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right)$
136. $\cos(2\arcsin(4x))$

137. $\sec(\arctan(2x))\tan(\arctan(2x))$

138. $\sin(\arcsin(x)+\arccos(x))$

139. $\cos(\arcsin(x)+\arctan(x))$

140. $\tan(2\arcsin(x))$

141. $\sin\left(\frac{1}{2}\arctan(x)\right)$

142. If $x=5\sin(\theta), 0°<\theta<90°$, express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of x.

143. If $x-1=2\cos(\theta), 0°<\theta<90°$, express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of x.

144. If $x=3\tan(\theta)$, write $\theta+\frac{1}{4}\sin(2\theta)$ in terms of x.

145. If $x=5\cos(\theta)$, write $\frac{\theta}{2}-\cos(2\theta)$ in terms of x.

146. If $\sin(\theta)=\frac{x}{2}$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, find an expression for $\theta+\sin(2\theta)$ in terms of x.

147. If $\tan(\theta)=\frac{x}{7}$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, find an expression for $\frac{1}{2}\theta-\frac{1}{2}\sin(2\theta)$ in terms of x.

148. If $\sec(\theta)=\frac{x}{4}$ for $0<\theta<\frac{\pi}{2}$, find an expression for $4\tan(\theta)-4\theta$ in terms of x.

149.

a. For what values of x is the function $f(x)=\sin(\arcsin(x))$ defined?

b. Is $\sin(\arcsin(x))=x$ for all x where it is defined? If not, for what values of x is the equation false?

c. For what values of x is the function $g(x)=\arcsin(\sin(x))$ defined?

d. Is $\arcsin(\sin(x))=x$ for all x where it is defined? If not, for what values of x is the equation false?

150.

a. For what values of x is the function $f(x)=\cos(\arccos(x))$ defined?

b. Is $\cos(\arccos(x))=x$ for all x where it is defined? If not, for what values of x is the equation false?

c. For what values of x is the function $g(x)=\arccos(\cos(x))$ defined?

d. Is $\arccos(\cos(x))=x$ for all x where it is defined? If not, for what values of x is the equation false?

SYNTHESIS QUESTIONS

For the following exercises, state the domain of the function.

151. $f(x)=\arcsin(5x)$

152. $f(x)=\arccos\left(\frac{3x-1}{2}\right)$

153. $f(x)=\arcsin(2x^2)$

154. $f(x)=\arccos\left(\frac{1}{x^2-4}\right)$

155. $f(x)=\arctan(4x)$

APPLICATIONS

For the following exercises, sketch a figure to help you model each problem.

156. Ivan is watching the launch of a satellite at Cape Canaveral. The viewing area is 500 yards from the launch site. The angle of elevation, θ, of Ivan's line of sight increases as the booster rocket rises.

a. Write a formula for the height, h, of the rocket as a function of θ.

b. Write a formula for θ as a function of h.

c. Evaluate the formula in part (b) for $h=1000$, and interpret the result.

157. Camille's house lies under the flight path from the city airport, and commercial airliners pass overhead at an altitude of 15,000 feet. As Camille watches an airplane recede, its angle of elevation, θ, decreases.

a. Write a formula for the horizontal distance, d, to the airplane as a function of θ.

b. Write a formula for θ as a function of d.

c. Evaluate the formula in part (b) for $d=20,000$, and interpret the result.

158. While driving along the interstate, you approach an enormous 50-foot-wide billboard that sits just beside the road. Your viewing angle, θ, increases as you get closer to the billboard.

a. Write a formula for your distance, d, from the billboard as a function of θ.

b. Write a formula for θ as a function of d.

c. Evaluate the formula in part (b) for $d=200$, and interpret the result.

159. Sang is walking along the bank of a straight river toward a 20-meter-long bridge over the river. Let θ be the angle subtended horizontally by Sang's view of the bridge.

a. Write a formula for Sang's distance from the bridge, d, as a function of θ.

b. Write a formula for θ as a function of d.

c. Evaluate the formula in part (b) for $d=500$, and interpret the result.

160. Wyatt is viewing a 4-meter tall painting whose base is 1 meter above his eye level.

a. Write a formula for α, the angle subtended from Wyatt's eye level to the bottom of the painting, when he stands x meters from the wall.

b. Write a formula for β, the angle subtended by the painting, in terms of x.

c. Evaluate the formula in part (b) for $x=5$, and interpret the result.

161. A 5-foot-tall mirror is positioned so that its bottom is 1.5 feet below Kim's eye level.

a. Write a formula for α, the angle subtended by the section of the mirror below Kim's eye level, when she stands x feet from the mirror.

b. Write a formula for θ, the angle subtended by the entire mirror, in terms of x.

c. Evaluate the formula in part (b) for $x = 10$, and interpret the result.

CHALLENGE PROBLEMS

162. Use your calculator to graph $y = \sin^{-1}(x) + \cos^{-1}(x)$.

a. State the domain and range of the graph.

b. Explain why the graph looks as it does.

163. Use your calculator to graph $y = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$.

a. State the domain and range of the graph.

b. Explain why the graph looks as it does.

164. Show that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ for $-1 \le x \le 1$.

165. Use the figure of a unit circle to answer the following.

a. Write an expression for the area of the shaded sector in terms of θ.

b. How are θ and t related in the figure? (Hint: Write an expression for $\sin(\theta)$.)

c. Combine your answers to (a) and (b) to write an expression for the area of the sector in terms of t.

166. Use the figure of a unit circle to answer the following.

a. Write an expression for the height of the shaded triangle in terms of t. (Hint: Use the Pythagorean Theorem.)

b. Write an expression for the area of the triangle in terms of t.

c. Combine your answers to (b) and to the previous problem to write an expression for the area bounded above by the upper semicircle, below by the x-axis, on the left by the y-axis, and on the right by $x = t$, when $0 \le t \le 1$.

167. Use the following picture and the series of exercises on the next page to show that

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$

a. Clearly $\triangle AOB$ and $\triangle BCD$ are right triangles because the line through O and A and the line through C and D are perpendicular to the x-axis. Use the distance formula to show that $\triangle BAD$ is also a right triangle (with $\angle BAD$ being the right angle) by showing that the sides of the triangle satisfy the Pythagorean Theorem.

b. Use $\triangle AOB$ to show that $\alpha = \arctan(1)$

c. Use $\triangle BAD$ to show that $\beta = \arctan(2)$

d. Use $\triangle BCD$ to show that $\gamma = \arctan(3)$

e. Use the fact that O, B and C all lie on the x-axis to conclude that $\alpha + \beta + \gamma = \pi$. Thus $\arctan(1) + \arctan(2) + \arctan(3) = \pi$.

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10.2: The Remaining Inverse Trigonometric Functions

Note to the Instructor - Where's the "Stewart Calculus" Definition?

For those who don't know, the Stewart Calculus textbook and, subsequently, Cosumnes River College's OER Calculus textbook use a different range definition for the secant and cosecant functions than introduced in this section. The redefinition that occurs in Calculus is to make some developments in Calculus II work more smoothly; however, I chose to stick with the conventional definitions in this text so students going to other campuses (or for those who are using the online homework) don't have to worry about things.

Learning Objectives

- Find the exact value of a non-fundamental inverse trigonometric function.
- Use technology to approximate the value of a non-fundamental inverse trigonometric function.
- Evaluate compositions involving trigonometric functions and non-fundamental inverse trigonometric functions.
- Simplify expressions involving trigonometric functions and non-fundamental inverse trigonometric functions.

Hawk A.I. Section-Specific Tutor

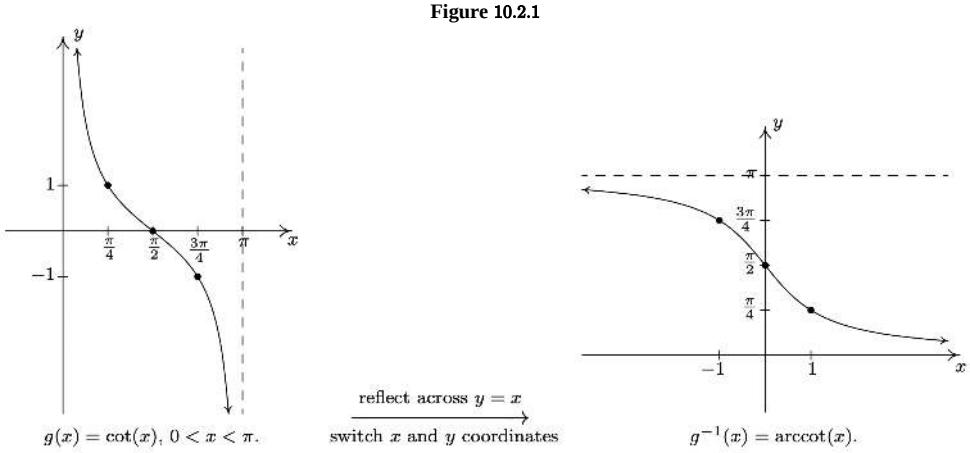
Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

In the previous section, we completely ignored the inverses of the secant, cosecant, and cotangent. This is not because they are unimportant - in fact, the inverses of the secant and cosecant are essential parts of Integral Calculus (also known as Calculus II). We have delayed their introduction because it's best to master the inverses of the fundamental trigonometric functions before jumping into the remaining inverse trigonometric functions.

The Inverse Cotangent Function

Like the rest of the trigonometric functions, the cotangent is not naturally one-to-one. Therefore, we restrict the domain of $g(x) = \cot(x)$ to its principal cycle on $(0, \pi)$ to obtain a one-to-one segment of the cotangent. The vertical asymptotes, $x = 0$ and $x = \pi$, of the graph of $g(x) = \cot(x)$ become the horizontal asymptotes, $y = 0$ and $y = \pi$, of the graph of $g^{-1}(x) = \operatorname{arccot}(x)$ (which, recall, is a reflection of the graph of $y = \cot(x)$ about the line $y = x$).

Figure 10.2.1



The definition of the arccotangent follows the same pattern as the definitions of our other inverse trigonometric functions.

Definition: Inverse Cotangent Function

The function $f(x) = \cot^{-1}(x)$ is defined as follows:

$$\cot^{-1}(x) = \theta \quad \text{if and only if} \quad \cot(\theta) = x \quad \text{and} \quad 0 < \theta < \pi.$$

Again, I want to be clear: Very technically, the function $\cot^{-1}(x)$ returns an *angle* and $\operatorname{arccot}(x)$ returns a *real number*. This is true for all inverse trigonometric functions. Therefore, it is valid for $\cot^{-1}(x)$ to return an angle in degree measure or radian measure; however, it is technically not valid for $\operatorname{arccot}(x)$ to return a number in degree measure (as this is an angle). Since radians is a unitless measure, a radian is just a real number, so

$\text{arccot}(x)$ can return a number in radian measure. The moral of the story is that you are always safe to work in radian measure when dealing with inverse trigonometric functions, but working in degree measure is a little risky.

Theorem: Properties of the Inverse Cotangent Function

Properties of $f(x) = \cot^{-1}(x)$

- $\theta = \cot^{-1}(x)$ if and only if $\cot(\theta) = x$ where $0 < \theta < \pi$
- Domain: $(-\infty, \infty)$
- Range: $(0, \pi)$
- As $x \to -\infty$, $\cot^{-1}(x) \to \pi^-$, and as $x \to \infty$, $\cot^{-1}(x) \to 0^+$
- $\cot(\cot^{-1}(x)) = x$ for all $x \in \mathbb{R}$
- $\cot^{-1}(\cot(x)) = x$ provided $0 < x < \pi$

Example 10.2.1

a. Find the exact values of the following.

i. $\text{arccot}(-\sqrt{3})$

ii. $\cot(\text{arccot}(-5))$

b. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

$$\cos(\text{arccot}(2x))$$

Solutions

a. i. $\text{arccot}(-\sqrt{3}) = \theta$ means $\cot(\theta) = -\sqrt{3}$, where $\theta \in (0, \pi)$.

Quadrant: Since the cotangent is negative, we further restrict the angle to $\theta \in (\frac{\pi}{2}, \pi)$, or Quadrant II.

Reference Angle: The reference angle for the cotangent that gives $\cot\left(\hat{\theta}\right) = \sqrt{3}$ is $\hat{\theta} = \frac{\pi}{6}$.

Given the quadrant annd reference angle, we get

$$\text{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}.$$

ii. It is helpful to remember that $\text{trig}\left(\text{trig}^{-1}(x)\right) = x$ for all of the trigonometric functions as long as x is in the domain of trig^{-1} (which is the range of trig). Therefore, $\cot(\text{arccot}(-5)) = \cot\left(\cot^{-1}(-5)\right) = -5$ because -5 is definitely in the domain of the inverse cotangent.

Another way to think about this is to let $\theta = \text{arccot}(-5)$. Then θ belongs to the interval $(0, \pi)$ and $\cot(\theta) = -5$. Hence, $\cot(\text{arccot}(-5)) = \cot(\theta) = -5$.

b. To get started, we let $\theta = \text{arccot}(2x)$ so that $\cot(\theta) = 2x$ where $0 < \theta < \pi$. We then sketch two reference triangles, one in QI and the other in QII.

Figure 10.2.2

In terms of θ,

$$\cos(\operatorname{arccot}(2x)) = \cos(\theta).$$

Therefore, we can use our reference triangles to evaluate $\cos(\theta)$. Looking at the reference triangle in QI, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$. Likewise, the reference triangle from the QII sketch yields $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$. Hence, it appears that

$$\cos(\operatorname{arccot}(2x)) = \cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}.$$

Being good mathematicians, we take a moment to reflect and make sure our logic is solid. What might give us pause is the fact that our second sketch, where $\theta \in$ QII, leads to cosine being negative. Still, our result, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$, looks to be positive; however, looks can be deceiving!

The expression, $2x$, in the second sketch, *is*, in fact, negative. The negativity of it is "hidden" within the variable x. That is, if the x-value of the point on the upper-left vertex of that reference triangle is -4, then $2x = -4 \implies x = -2$. Hence, when $\theta \in$ QII, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$ is negative (which is what we want).

We now concern ourselves with the domain of the function. We know that the domain of the arccotangent is all real numbers. Hence, the domain of $\operatorname{arccot}(2x)$ is also all real numbers. Moreover, the range of the arccotangent is $(0, \pi)$. Since the outputs of $\operatorname{arccot}(2x)$ are the inputs of $\cos(\operatorname{arccot}(2x))$ and the cosine can take any value as an input, the domain remains unchanged. That is, the domain of $y = \cos(\operatorname{arccot}(2x))$ is $(-\infty, \infty)$.

? Checkpoint 10.2.1

Find the exact value of the expression.

$$\sin\left(\cot^{-1}\left(-\frac{5}{2}\right)\right)$$

Answer

$$\frac{2}{\sqrt{29}}$$

The Inverse Secant and Cosecant Functions

The last two functions to invert are secant and cosecant. The principal cycle for each is highlighted in the figure below.

Figure 10.2.3

It is clear from the graph of secant that we cannot find one single, continuous piece of its graph which covers its entire range of $(-\infty,-1]\cup[1,\infty)$, and which restricts the domain of the function so that it is one-to-one. The same is true for cosecant. Thus, in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1,\infty)$, and another piece to cover the bottom, namely $(-\infty,-1]$. To make the definitions of the inverse secant and cosecant functions align with our previous work, we restrict their domains to coincide with the restrictions on cosine and sine, respectively.

It is critical to note that there are two ways to restrict the domains of the secant and cosecant functions. The method shown in this text aligns with all other texts in Trigonometry and Precalculus; however, there is a *much* better choice of domain restriction for these functions that makes Calculus (specifically, Integral Calculus) work much more smoothly. Despite this, we give the traditional definition here because it aligns with all of our previous work.

When defining the arccosine, we restricted the cosine to the interval $[0,\pi]$; however, such a restriction needs to be modified for the secant because the secant is undefined at $x=\frac{\pi}{2}$. Thus, we restrict the domain of the secant to $[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$ to help us define the arcsecant.

Figure 10.2.4

This gives us the following definition.

Definition: The Inverse Secant Function

The function $f(x)=\sec^{-1}(x)$ is defined as follows:

$$\sec^{-1}(x)=\theta \quad \text{if and only if} \quad \sec(\theta)=x \quad \text{and} \quad \theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right].$$

Likewise, we restricted the sine function to the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ when defining the arcsine, but such a restriction will not work for the arccosecant because the cosecant is undefined at $x=0$. Therefore, we restrict $g(x)=\csc(x)$ to $[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$ to define the arccosecant.

Figure 10.2.5

Note that, for both arcsecant and arccosecant, the domain is $(-\infty,-1]\cup[1,\infty)$. We can rewrite this using set-builder notation as $\{x \mid |x|\geq 1\}$. This is often done in Calculus textbooks, so we include it here for completeness.

Definition: The Inverse Cosecant Function

The function $f(x)=\csc^{-1}(x)$ is defined as follows:

$$\csc^{-1}(x)=\theta \quad \text{if and only if} \quad \csc(\theta)=x \quad \text{and} \quad \theta\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right].$$

Using these definitions, we get the following properties of the arcsecant and arccosecant functions.

Theorem: Properties of the Inverse Secant and Inverse Cosecant Functions

Properties of $f(x)=\operatorname{arcsec}(x)$

- $\sec^{-1}(x)=\theta$ if and only if $\sec(\theta)=x$ where $\theta\in[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
- Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
- Range: $[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
- As $x\to-\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{+}$, and as $x\to\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{-}$
- $\sec(\operatorname{arcsec}(x))=x$ provided $|x|\geq 1$
- $\operatorname{arcsec}(\sec(x))=x$ provided $0\leq x<\frac{\pi}{2}$ or $\frac{\pi}{2}<x\leq\pi$

Properties of $f(x)=\operatorname{arccsc}(x)$

- $\csc^{-1}(x)=\theta$ if and only if $\csc(\theta)=x$ where $\theta\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
- Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
- Range: $[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$
- As $x\to-\infty$, $\operatorname{arccsc}(x)\to 0^{-}$, and as $x\to\infty$, $\operatorname{arccsc}(x)\to 0^{+}$
- $\csc(\operatorname{arccsc}(x))=x$ provided $|x|\geq 1$
- $\operatorname{arccsc}(\csc(x))=x$ provided $-\frac{\pi}{2}\leq x<0$ or $0<x\leq\frac{\pi}{2}$
- Additionally, arccosecant is an odd function.

Example 10.2.2

a. Find the exact values of the following.

i. $\sec^{-1}(2)$
ii. $\operatorname{arccsc}(-2)$
iii. $\operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right)$
iv. $\cot\left(\operatorname{arccsc}\left(-3\right)\right)$

b. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

i. $\tan(\operatorname{arcsec}(x))$
ii. $\cos(\operatorname{arccsc}(4x))$

Solutions

a.

i. Let $\theta = \sec^{-1}(2)$. Then $\sec(\theta) = 2$, where $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. Of these two intervals, the only one where secant is positive is $[0, \frac{\pi}{2})$. Thus, $\theta \in$ QI. Moreover, $\sec\left(\hat{\theta}\right) = 2$ implies $\cos\left(\hat{\theta}\right) = \frac{1}{2}$. Hence, $\hat{\theta} = \frac{\pi}{3}$. Therefore,

$$\sec^{-1}(2) = \theta = \frac{\pi}{3}.$$

ii. We now start to streamline our arguments.

$$\begin{aligned}
& \operatorname{arccsc}(-2) = \theta && && \text{(inverse trigonometric functions return angles,)} \\
\Longrightarrow\ & \csc(\theta) = -2, && \text{where } \theta \in \left[-\tfrac{\pi}{2}, 0\right) \cup \left(0, \tfrac{\pi}{2}\right] && \text{(but the angle is restricted)} \\
\Longrightarrow\ & \theta \in \left[-\tfrac{\pi}{2}, 0\right) && && \text{(cosecant is negative)} \\
\text{and}\ & \hat{\theta} = \frac{\pi}{6} && && \\
\Longrightarrow\ & \operatorname{arccsc}(-2) = -\frac{\pi}{6} && && \left(\text{reference angle of } \tfrac{\pi}{6} \text{ in the interval } \left[-\tfrac{\pi}{2}, 0\right)\right)
\end{aligned}$$

iii. Our choice of non-standard angle for this example is purposeful. It's to force you to think about the process instead of focusing solely on computing.

$$\begin{aligned}
& \operatorname{arcsec}\left(\sec\left(\tfrac{8\pi}{7}\right)\right) = \theta, && \text{where } \theta \in \left[0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right] && \text{(inverse trigonometric functions return restricted angles)} \\
\Longrightarrow\ & \sec(\theta) = \sec\left(\tfrac{8\pi}{7}\right) && &&
\end{aligned}$$

Therefore, the reference angle for θ matches the reference angle for $\frac{8\pi}{7}$. That is,

$$\hat{\theta} = \frac{\pi}{7}.$$

We now need to determine in which quadrant θ belongs. Since $\frac{8\pi}{7} \in$ QIII, $\sec\left(\frac{8\pi}{7}\right)$ is negative. Moreover, since $\sec(\theta) = \sec\left(\frac{8\pi}{7}\right)$, $\sec(\theta)$ must be negative. Hence, $\theta \in (\frac{\pi}{2}, \pi]$. Thus, $\theta = \frac{6\pi}{7}$. This means that

$$\operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right) = \theta = \frac{6\pi}{7}.$$

iv. As usual, we begin by addressing the fact that the inverse trigonometric function is an angle.

$$\begin{aligned}
& \operatorname{arccsc}(-3) = \theta, && \text{where } \theta \in \left[-\tfrac{\pi}{2}, 0\right) \cup \left(0, \tfrac{\pi}{2}\right] && \text{(inverse trigonometric functions return restricted angles)} \\
\Longrightarrow\ & \csc(\theta) = -3 && &&
\end{aligned}$$

However, we are interested in

$$\cot(\operatorname{arccsc}(-3)) = \cot(\theta).$$

Since the cosecant is negative, we know that $\theta \in [-\frac{\pi}{2}, 0)$.Let sketch a reference triangle in QIV to help us determine $\cot(\theta)$.

Figure 10.2.6

From the reference triangle, we see that

$$\cot\left(\operatorname{arccsc}\left(-3\right)\right) = \cot\left(\theta\right) = -2\sqrt{2}.$$

b.

i. As per usual, let $\theta = \operatorname{arcsec}(x)$ so that $\sec\left(\theta\right) = x$, where we restrict θ to the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. We then sketch two reference triangles, one for $\theta \in [0, \frac{\pi}{2})$ and the other for $\theta \in (\frac{\pi}{2}, \pi]$.

Figure 10.2.7

We now use these reference triangles to help us determine the value of $\tan(\operatorname{arcsec}(x)) = \tan\left(\theta\right)$. In this case, we run into a very interesting problem. In Quadrant I, $\tan\left(\theta\right) = \sqrt{x^2 - 1}$; however, in Quadrant II, $\tan\left(\theta\right) = -\sqrt{x^2 - 1}$. Pausing for a moment, let's think about this before going further. Does this make sense?

We know tangent is supposed to be positive in Quadrant I. Since $\sec\left(\theta\right) = x$, we know either $x \geq 1$ or $x \leq -1$. Luckily, $\sqrt{x^2 - 1}$ is definitely positive for these values of x (unless $x = \pm 1$, but that would mean $\tan\left(\theta\right) = 0 \implies \theta = 0$ which is not in Quadrant I). So the statement, $\tan\left(\theta\right) = \sqrt{x^2 - 1}$ when $\theta \in (0, \frac{\pi}{2})$ makes sense.

In Quadrant II, the tangent is supposed to be negative. It's easy to see that $-\sqrt{x^2 - 1}$ is negative whenever $\theta \in (\frac{\pi}{2}, \pi)$. Thus, the statement, $\tan\left(\theta\right) = -\sqrt{x^2 - 1}$ when $\theta \in (\frac{\pi}{2}, \pi)$ also makes sense. Therefore,

$$\tan(\theta) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } 0 \leq \theta < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

Now we need to determine what these conditions on θ mean for x. Since $x = \sec(\theta)$, when $0 \leq \theta < \frac{\pi}{2}$, $x \geq 1$, and when $\frac{\pi}{2} < \theta \leq \pi$, $x \leq -1$. Since we encountered no further restrictions on θ, the equivalence below holds for all x in $(-\infty, -1] \cup [1, \infty)$.

$$\tan(\operatorname{arcsec}(x)) = \begin{cases} \sqrt{x^2-1}, & \text{if } x \geq 1 \\ -\sqrt{x^2-1}, & \text{if } x \leq -1 \end{cases}$$

ii. While I always use the graphing tactic, you do not have to think geometrically to work with inverse trigonometric functions. You can rely solely on identities.

For example, to simplify $\cos(\operatorname{arccsc}(4x))$, we start by letting $\theta = \operatorname{arccsc}(4x)$. Then $\csc(\theta) = 4x$ for $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

We now set about finding an expression for $\cos(\operatorname{arccsc}(4x)) = \cos(\theta)$. Since $\cos(\theta)$ is defined for all θ, we do not encounter any additional restrictions on θ. From $\csc(\theta) = 4x$, we get $\sin(\theta) = \frac{1}{4x}$, so to find $\cos(\theta)$, we can make use of the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. Substituting $\sin(\theta) = \frac{1}{4x}$ gives

$$\cos^2(\theta) + \left(\frac{1}{4x}\right)^2 = 1.$$

Solving, we get

$$\cos(\theta) = \pm\sqrt{\frac{16x^2-1}{16x^2}} = \pm\frac{\sqrt{16x^2-1}}{4|x|}.$$

Since $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, we know $\cos(\theta) \geq 0$. Therefore, we choose

$$\cos(\theta) = \frac{\sqrt{16-x^2}}{4|x|}.$$

(The absolute values here are necessary, since x could be negative.) To find the values for which this equivalence is valid, we look back at our original substitution, $\theta = \operatorname{arccsc}(4x)$. Since the domain of $\operatorname{arccsc}(x)$ requires its argument x to satisfy $|x| \geq 1$, the domain of $\operatorname{arccsc}(4x)$ requires $|4x| \geq 1$. This implies $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$. Since we had no additional restrictions on θ, the equivalence

$$\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4|x|}$$

holds for all x in $(-\infty, -\frac{1}{4}] \cup [\frac{1}{4}, \infty)$.

? Checkpoint 10.2.2

Rewrite the function as one that does not involve trigonometric functions and state the domain.

$$f(x) = \sin\left(\sec^{-1}(2x)\right)$$

(Your answer might be a piecewise function or it might involve absolute values.)

Answer

There are two ways to write this function:

$$f(x) = \frac{\sqrt{4x^2-1}}{2|x|},$$

or

$$f(x) = \begin{cases} \dfrac{\sqrt{4x^2-1}}{2x}, & \text{if } x \geq \dfrac{1}{2} \\ -\dfrac{\sqrt{4x^2-1}}{2x}, & \text{if } x < -\dfrac{1}{2} \end{cases}$$

The domain of this function is all $x \in (-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)$.

Calculators and the Non-Fundamental Inverse Trigonometric Functions

In the sections to come, we will need to approximate the values of the inverse trigonometric functions. We have already discussed how to use technology to approximate values of the arcsine, arccosine, and arctangent functions; however, if we are asked for an approximation of the arccotangent, arcsecant, or arccosecant, how is this to be done when our calculators do not have these functions?

Example 10.2.3

a. Use a calculator to approximate the following values to four decimal places.

i. $\cot^{-1}(2)$

ii. $\operatorname{arcsec}(5)$

iii. $\operatorname{arccot}(-2)$

iv. $\csc^{-1}\left(-\frac{3}{2}\right)$

b. Find the domain and range of

$$f(x) = \frac{\pi}{2} - \sec^{-1}\left(\frac{x}{5}\right).$$

Solutions

a.

i. $\cot^{-1}(2) = \theta$ means $\cot(\theta) = 2$, where $\theta \in (0, \pi)$.

Quadrant: Since the cotangent is positive, we know $\theta \in (0, \frac{\pi}{2})$.

Reference Angle: 2 is not a special ratio for the cotangent, so we will need technology to help us find the reference angle; however, our calculators do not have an inverse cotangent button. What do we do? The answer is quite simple - we use the Reciprocal Identities. Since $\cot(\theta) = 2$, $\tan(\theta) = \frac{1}{2}$. To compute the reference angle, we evaluate

$$\hat{\theta} = \tan^{-1}\left(\left|\frac{1}{2}\right|\right) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636$$

Now that we have a reference angle (≈ 0.4636) and a quadrant (QII), we get

$$\cot^{-1}(2) \approx 0.4636.$$

ii. $\operatorname{arcsec}(5) = \theta$ means $\sec(\theta) = 5$, where $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

Quadrant: Since the secant is positive, we further restrict the angle to $\theta \in (0, \frac{\pi}{2})$ (Quadrant I).

Reference Angle: Again, since our technology does not have an inverse secant button, we use the Reciprocal Identities.

$$\sec(\theta) = 5 \implies \cos(\theta) = \frac{1}{5}.$$

Now we use a calculator to compute

$$\hat{\theta} = \cos^{-1}\left(\left|\frac{1}{5}\right|\right) = \cos^{-1}\left(\frac{1}{5}\right) \approx 1.3694.$$

Using our reference angle (≈ 1.3694) and quadrant (QI), we get

$$\operatorname{arcsec}(5) \approx 1.3694.$$

iii. $\operatorname{arccot}(-2) = \theta$ means $\cot(\theta) = -2$, where $\theta \in (0, \pi)$.

Quadrant: Since $\cot(\theta) < 0$, we know $\frac{\pi}{2} < \theta < \pi$. That is, $\theta \in$ QII.

Reference Angle: Once again, we do not have an appropriate inverse trigonometric function on the calculator, so we use the Reciprocal Identities.

$$\cot(\theta) = -2 \implies \tan(\theta) = -\frac{1}{2}.$$

Thus, we compute the reference angle as follows:

$$\hat{\theta} = \tan^{-1}\left(\left|-\frac{1}{2}\right|\right) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636.$$

We have a reference angle (≈ 0.4636) and a quadrant (QII). To find the actual angle, we subtract our reference angle from π to get

$$\operatorname{arccot}(-2) = \theta = \pi - \hat{\theta} = \pi - \arctan\left(\frac{1}{2}\right) \approx 2.6779.$$

iv. $\csc^{-1}\left(-\frac{3}{2}\right) = \theta$ means $\csc(\theta) = -\frac{3}{2}$, where $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

Quadrant: Since $\csc(\theta) < 0$, we further restrict the angle to the interval $\theta \in [-\frac{\pi}{2}, 0)$ (Quadrant IV).

Reference Angle: We do not have an appropriate inverse trigonometric button on our calculator for the cosecant. Therefore, we must use the Reciprocal Identities.

$$\csc(\theta) = -\frac{3}{2} \implies \sin(\theta) = -\frac{2}{3}.$$

Hence,

$$\hat{\theta} = \sin^{-1}\left(\left|-\frac{2}{3}\right|\right) = \sin^{-1}\left(\frac{2}{3}\right) \approx 0.7297.$$

Now that we have a reference angle (≈ 0.7297) and a quadrant (QIV), we get

$$\csc^{-1}\left(-\frac{3}{2}\right) \approx -0.7297.$$

b. The domain of the arcsecant is normally $(-\infty, -1] \cup [1, \infty)$. That is, if we were given $\sec^{-1}(r)$, we know that allowable values of r are $r \in (-\infty, -1] \cup [1, \infty)$. In this case, however, the argument of the arcsecant is $\frac{x}{5}$. Hence, $\frac{x}{5} \in (-\infty, -1] \cup [1, \infty)$. Rewriting this in interval notation will be helpful.

$$\begin{aligned} & \frac{x}{5} \leq -1 \quad \text{or} \quad \frac{x}{5} \geq 1 \\ \implies \quad & x \leq -5 \quad \text{or} \quad x \geq 5 \end{aligned}$$

Thus, the domain of f is $(-\infty, -5] \cup [5, \infty)$.

We know the range of the arcsecant is normally $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. Given the function $f(x) = \frac{\pi}{2} - \sec^{-1}\left(\frac{x}{5}\right)$, the only transformation that affects the range is the addition of $\frac{\pi}{2}$ to the output values. Thus, the range of f is $[\frac{\pi}{2}, \pi) \cup (\pi, \frac{3\pi}{2}]$.

? Checkpoint 10.2.3

Use a calculator to approximate $\operatorname{arcsec}\left(-\frac{7}{3}\right)$.

Answer

≈ 2.0137

The inverse trigonometric functions are typically found in applications whenever the measure of an angle is required. The following example presents one such scenario.

✓ Example 10.2.4

The roof on the house below has a "6/12 pitch." This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.

Figure 10.2.8

Solution

If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet. We find the angle of inclination, labeled θ below, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$. Since θ is an acute angle, we can use the arctangent function and we find $\theta = \arctan\left(\frac{1}{2}\right)$ radians $\approx 26.56°$.

Figure 10.2.9

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10.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Defining the Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Evaluating Inverse Trigonometric Functions

KEY CONCEPTS

DEFINITIONS

- **Inverse Cotangent Function:** The function $f(x) = \cot^{-1}(x)$ or $\operatorname{arccot}(x)$ is defined as $\operatorname{arccot}(x) = \theta$ if and only if $\cot(\theta) = x$ and $0 < \theta < \pi$.
- **The Inverse Secant Function:** The function $f(x) = \sec^{-1}(x)$ or $\operatorname{arcsec}(x)$ is defined as $\operatorname{arcsec}(x) = \theta$ if and only if $\sec(\theta) = x$ and $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
- **The Inverse Cosecant Function:** The function $f(x) = \csc^{-1}(x)$ or $\operatorname{arccsc}(x)$ is defined as $\operatorname{arccsc}(x) = \theta$ if and only if $\csc(\theta) = x$ and $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

THEOREMS

- **Properties of the Inverse Cotangent Function:**
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - As $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^-$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^+$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(x)) = x$ provided $0 < x < \pi$
- **Properties of the Inverse Secant and Inverse Cosecant Functions:**
  - **arcsec(x):** Domain is $(-\infty, -1] \cup [1, \infty)$; Range is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
  - **arccsc(x):** Domain is $(-\infty, -1] \cup [1, \infty)$; Range is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(x)) = x$ provided $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
  - $\operatorname{arccsc}(\csc(x)) = x$ provided $x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$
  - Arccosecant is an odd function.

COMMON MISTAKES

- **Forgetting Domain and Range Restrictions:** The inverse reciprocal trigonometric functions have very specific ranges. For example, $\operatorname{arccot}(x)$ only returns angles in Quadrants I and II, while $\operatorname{arcsec}(x)$ returns angles in Quadrants I and II (excluding $\frac{\pi}{2}$). Forgetting these restrictions is a primary source of error, especially when simplifying compositions like $\operatorname{arcsec}(\sec(\frac{8\pi}{7}))$.
- **Calculator Usage:** Calculators do not have dedicated buttons for arccot, arcsec, or arccsc. To evaluate these, you must use the Reciprocal Identities first. For example, to find $\operatorname{arccot}(2)$, you must evaluate $\arctan(\frac{1}{2})$.
- **Ignoring the Sign of x in Algebraic Expressions:** When simplifying an expression like $\tan(\operatorname{arcsec}(x))$, the result may depend on whether x is positive or negative, as this determines the quadrant of the angle $\theta = \operatorname{arcsec}(x)$. The result can be a piecewise function.

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10.2.2: HOMEWORK

READING QUESTIONS

1. Why must the domain of the cotangent function be restricted to define its inverse?
2. What is the restricted domain (principal cycle) used to define the inverse cotangent function?
3. What is the domain and range of the $\operatorname{arccot}(x)$ function?
4. Why is the domain of the $\operatorname{arcsec}(x)$ and $\operatorname{arccsc}(x)$ functions the set $(-\infty,-1]\cup[1,\infty)$?
5. What is the range of the $\operatorname{arcsec}(x)$ function? Why is $\frac{\pi}{2}$ excluded?
6. How would you use a calculator to find the value of $\operatorname{arccot}(5)$?
7. Is the statement $\operatorname{arcsec}(\sec(x))=x$ true for all values of x? Explain.
8. When simplifying an expression like $\sin(\operatorname{arccot}(x))$, what is a helpful first step?
9. What is the key difference between the range of $\arccos(x)$ and $\operatorname{arcsec}(x)$?

HOMEWORK

BASIC SKILLS

For the following exercises, find the exact value.

1. $\operatorname{arccot}\left(\sqrt{3}\right)$
2. $\operatorname{arccot}\left(\frac{1}{\sqrt{3}}\right)$
3. $\operatorname{arcsec}\left(\sqrt{2}\right)$
4. $\operatorname{arccsc}\left(\sqrt{2}\right)$
5. $\operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right)$
6. $\operatorname{arccsc}\left(\frac{2}{\sqrt{3}}\right)$
7. $\operatorname{arcsec}(2)$
8. $\operatorname{arccsc}(2)$
9. $\operatorname{arccot}(1)$
10. $\operatorname{arcsec}(1)$
11. $\operatorname{arccsc}(1)$
12. $\operatorname{arccot}(0)$
13. $\operatorname{arccot}\left(-\sqrt{3}\right)$
14. $\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)$
15. $\operatorname{arcsec}(-2)$
16. $\operatorname{arccsc}(-2)$
17. $\operatorname{arcsec}\left(-\sqrt{2}\right)$
18. $\operatorname{arccsc}\left(-\sqrt{2}\right)$
19. $\operatorname{arcsec}\left(-\frac{2}{\sqrt{3}}\right)$
20. $\operatorname{arccsc}\left(-\frac{2}{\sqrt{3}}\right)$
21. $\operatorname{arcsec}(-1)$
22. $\operatorname{arccot}(-1)$
23. $\operatorname{arccsc}(-1)$

For the following exercises, find the exact value or state that it is undefined.

24. $\cot(\operatorname{arccot}(1))$
25. $\cot\left(\operatorname{arccot}\left(-\sqrt{3}\right)\right)$
26. $\cot\left(\operatorname{arccot}\left(-\frac{7}{24}\right)\right)$
27. $\cot(\operatorname{arccot}(-0.001))$
28. $\cot\left(\operatorname{arccot}\left(\frac{17\pi}{4}\right)\right)$
29. $\sec(\operatorname{arcsec}(2))$
30. $\sec(\operatorname{arcsec}(-1))$
31. $\sec\left(\operatorname{arcsec}\left(\frac{1}{2}\right)\right)$
32. $\sec(\operatorname{arcsec}(0.75))$
33. $\sec(\operatorname{arcsec}(117\pi))$
34. $\csc\left(\operatorname{arccsc}\left(\sqrt{2}\right)\right)$
35. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$
36. $\csc\left(\operatorname{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$
37. $\csc(\operatorname{arccsc}(1.0001))$
38. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$

For the following exercises, find the exact value or state that it is undefined.

39. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$
40. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$
41. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$
42. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$
43. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$
44. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$
45. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$
46. $\operatorname{arccot}(\cot(\pi))$
47. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$
48. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$
49. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$
50. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$
51. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$
52. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$
53. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$
54. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

55. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

For the following exercises, find the exact value or state that it is undefined.

56. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right)$
57. $\sin\left(\operatorname{arccsc}\left(-3\right)\right)$
58. $\cos\left(\operatorname{arccot}\left(3\right)\right)$
59. $\cos\left(\operatorname{arcsec}\left(5\right)\right)$
60. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$
61. $\tan\left(\operatorname{arccot}\left(12\right)\right)$
62. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right)$
63. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$
64. $\csc\left(\operatorname{arccot}\left(9\right)\right)$

SYNTHESIS QUESTIONS

For the following exercises, find the exact value or state that it is undefined.

65. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$
66. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right)$
67. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right)$
68. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

For the following exercises, find the domain of the given function. Write your answers in interval notation.

69. $f(x)=\operatorname{arccot}\left(\frac{2x}{x^2-9}\right)$
70. $f(x)=\arctan(\ln(2x-1))$
71. $f(x)=\operatorname{arccot}(\sqrt{2x-1})$
72. $f(x)=\operatorname{arcsec}(12x)$
73. $f(x)=\operatorname{arccsc}(x+5)$
74. $f(x)=\operatorname{arcsec}\left(\frac{x^3}{8}\right)$
75. $f(x)=\operatorname{arccsc}\left(e^{2x}\right)$

APPLICATIONS

76. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut, it touches level ground 360 feet from the tower's base. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.
77. At Cliffs of Insanity Point, The Great Sasquatch Canyon is 7117 feet deep. From that point, a fire is seen at a location known to be 10 miles away from the base of the sheer canyon wall. What angle of depression is made by the line of sight from the canyon edge to the fire? Express your answer using degree measure rounded to one decimal place.
78. Shelving is being built at the Utility Muffin Research Library, which is to be 14 inches deep. An 18-inch rod will be attached to the wall and the underside of the shelf at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?
79. A parasailor is being pulled by a boat on Lake Ippizuti. The cable is 300 feet long, and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.
80. A tag-and-release program to study the Sasquatch population of the eponymous Sasquatch National Park has begun. From a 200-foot-tall tower, a ranger spots a Sasquatch lumbering through the wilderness directly toward the tower. Let θ denote the angle of depression from the top of the tower to a point on the ground. If the range of the rifle with a tranquilizer dart is 300 feet, find the smallest value of θ for which the corresponding point on the ground is in range of the rifle. Round your answer to the nearest hundredth of a degree.

CHALLENGE PROBLEMS

81. Show that $\operatorname{arcsec}(x)=\arccos\left(\frac{1}{x}\right)$ for $|x|\geq 1$.
82. Show that $\operatorname{arccsc}(x)=\arcsin\left(\frac{1}{x}\right)$ for $|x|\geq 1$.

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10.3: Solving Trigonometric Equations - Algebraic Techniques

Note to the Instructor - Start Simple

The equations in this section include everything *except for* those requiring trigonometric identities. Hence, students need to recall how to solve linear, quadratic, and rational equations. Moreover, they will need to recall what it means to factor out a GCF and the implications of dividing both sides of an equation by an algebraic expression (something I just tell students to not do, but to each their own).

Learning Objectives

- Solve a simple trigonometric equation.
- Use techniques from algebra to solve a trigonometric equation.

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The "Non-Uniqueness" of Solutions to Trigonometric Equations

Throughout Algebra, you constantly looked for "the" solution to an equation. For example, you might have been asked to solve

$$3x - \frac{1}{2} = 10 \quad \text{or} \quad 2e^{2x-1} = 5 \quad \text{or} \quad \sqrt{5x-1} = 3.$$

In each of these cases, the equation has a single solution; however, this was not the extent of your "equation solving" in Algebra. You also solved equations like

$$x^2 + 5x + 6 = 0 \quad \text{and} \quad \log_2\left(5x^2 - 1\right) = 2.$$

These last two equations each have two solutions. We now explore equations that often have an infinite number of solutions.

Definition: Trigonometric Equation

A **trigonometric equation** is an equation that involves trigonometric functions of a variable, typically an angle.

For example,

$$2\sec(x) - 1 = 0, \quad \sin^2(x) = 1, \quad \text{and} \quad 2\cos^2(x) = \sin(x) + 1$$

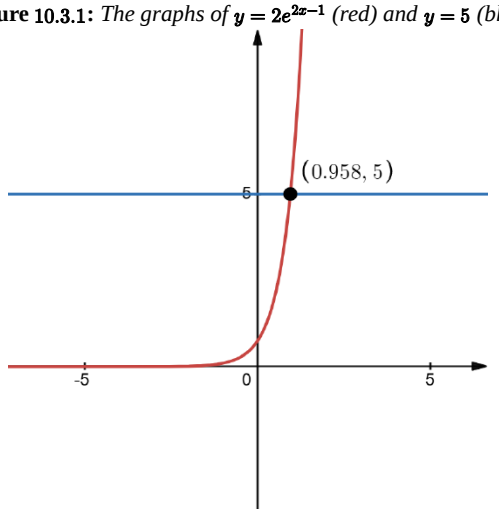
are all trigonometric equations.

Before we discuss how to solve trigonometric equations, it's best to recall the graphical meaning of solutions of equations.

In Algebra, when you were asked to solve an equation like $2e^{2x-1} = 5$, you learned all the mathematical manipulations - divide by 5, take the natural logarithm of both sides, add 1, then divide by 2. This results in the solution $x = \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{1}{2}$; however, before you learned how to procedurally work to an answer to such an equation, you learned a way to interpret this solution graphically.

When asked to solve an equation, one interpretation is that you are being asked to find the x-values for which the graph of the left side of the equation meets the graph of the right side. For example, when asked to solve $2e^{2x-1} = 5$, we could view the graphs of $y = 2e^{2x-1}$ and $y = 5$ to visualize the solution.

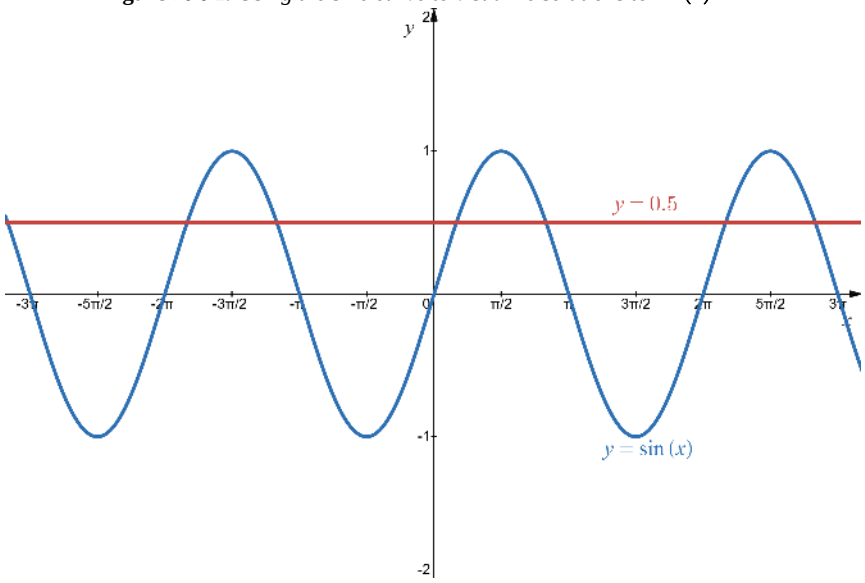
Figure 10.3.1: *The graphs of* $y = 2e^{2x-1}$ *(red) and* $y = 5$ *(blue)*



We can see in Figure 10.3.1 that these graphs meet at $x \approx 0.958 \approx \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{1}{2}$.

Although not the best way (or even a good way) to solve an equation, the visual we get from the example above justifies why trigonometric equations often have infinite solutions. For example, Figure 10.3.2 illustrates that $\sin(x) = 0.5$ has five solutions on the interval $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$. These solutions correspond to the points on the unit circle where $y = 0.5$. Since the sine function oscillates in both directions forever, we can easily imagine that the sine wave intersects the line $y = 0.5$ an infinite number of times.

Figure 10.3.2: *Using the sine curve to visualize solutions to* $\sin(x) = 0.5$



Despite knowing there are an infinite number of solutions to $\sin(x) = 0.5$, we know from our recent discussions of the inverse trigonometric functions that $x = \sin^{-1}(0.5)$ will only give us one of these solutions. Therefore, we must create a method to obtain *all* solutions to a given trigonometric equation.

Solving Basic Trigonometric Equations

The keys to finding all solutions to a trigonometric equation can be summarized in two fundamental concepts from Trigonometry: quadrants and reference angles. We will begin with an example to illustrate our thought process.

Example 10.3.1

Find all degree solutions to the equation

$$\sin(\theta) = -0.6428,$$

and state the solutions occurring on the interval $[0°, 360°]$. Round your answers to the nearest tenth of a degree.

Solution

Since -0.6428 is not a special ratio, we will need to use technology to help us find the solution; however, technology should be reserved for the last step in any mathematical procedure.
Quadrants: Since $\sin(\theta)$ is negative, we know that θ is either in QIII or QIV.
Reference Angle: To find the reference angle, we compute

$$\sin^{-1}(|-0.6428|) = \sin^{-1}(0.6428) \approx 40.0°.$$

So we have a reference angle, $\hat{\theta} \approx 40.0°$, and *two* quadrants, QIII and QIV.
Solutions Associated with Quadrant III: The solution in QIII having reference angle $\hat{\theta} \approx 40.0°$ is

$$\theta = 180° + \hat{\theta} \approx 220.0°.$$

Since $\sin(\theta) = \sin(\theta + 360°k)$ (where $k \in \mathbb{Z}$), any angle coterminal to $220.0°$ will also lead to a solution of our equation. Thus, the general solution associated with Quadrant III is

$$\theta \approx 220.0° + 360°k, \quad \text{where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant IV: Taking a lesson from the previous paragraph, the solution in QIV having reference angle $\hat{\theta} \approx 40.0°$ is

$$\theta = 360° - \hat{\theta} + 360°k \approx 320.0° + 360°k, \quad \text{where } k \in \mathbb{Z}.$$

Now that we have all solutions, let's restrict them to only the solutions occurring on the interval $[0°, 360°]$. To do this, we consider both sets of infinite solutions,

$$\theta \approx 220.0° + 360°k$$
$$\theta \approx 320.0° + 360°k,$$

and we let $k = 0$ (any other value of k will result in angles outside of the requested interval). Hence, the solutions on $[0°, 360°]$ are approximately $220.0°$ and $320.0°$.

A few notes about Example 10.3.1 are in order. First, our equation had solutions derived from two different quadrants. This should not be a surprise when you consider Figure 10.3.2. The sine curve should attain a y-value of -0.6428 twice through any single cycle.

Second, since the given ratio, -0.6428, was not a special ratio, we had to grab a calculator to get an approximation of the reference angle by way of the arcsine. Moreover, we had to change our calculator to degree mode. You must remain aware of the angular measurement system in which you are being asked to work. You will frequently switch between degree and radian mode when solving trigonometric equations.

Third, we were told to round to the nearest tenth of a degree. If you are using online homework and need to answer with an approximation, pay attention to the level of rounding they request. If they don't state it explicitly, assume four decimal places (as this works well for most online homework systems).

Checkpoint 10.3.1

Solve the equation $\sin(\theta) = -0.4$ for angles between $0°$ and $360°$. Round your answers to the nearest tenth of a degree.

Answer

$203.6°, 336.4°$

Example 10.3.2

Find all solutions to the given equation, then state the solutions occurring on the interval $[0, 2\pi]$. Round your answers to the nearest tenth.

a. $\sin(x) = \frac{1}{2}$
b. $\cos(x) = -0.6$
c. $\tan(x) = -\sqrt{3}$
d. $\sec(x) = 3.7$

Solutions

a. Since $\frac{1}{2}$ is a special ratio for the sine function, we do not bother (and should not bother) grabbing a calculator.
Quadrants: Since $\sin(x) = \frac{1}{2}$ is positive, we know that x is either in QI or QII.
Reference Angle: From our special triangles, we know that $\hat{x} = \frac{\pi}{6}$.
Solutions Associated with Quadrant I: The angle in Quadrant I with reference angle $\frac{\pi}{6}$ is $x = \frac{\pi}{6}$. Any angle coterminal with this will also be a solution to the given equation, so our set of solutions associated with Quadrant I is

$$x = \frac{\pi}{6} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant II: The angle in Quadrant II with reference angle $\frac{\pi}{6}$ is $\frac{5\pi}{6}$. Thus,

$$x = \frac{5\pi}{6} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Finally, the solutions in the interval $[0, 2\pi]$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

b. -0.6 is not a special ratio for cosine, so we will eventually need to reach for technology to solve this equation.
Quadrants: We are given a negative cosine function, and we know this can only occur if the argument of the cosine terminates in either QII or QIII.
Reference Angle: To find the reference angle associated with the equation $\cos(x) = -0.6$, we approximate $\cos^{-1}(|-0.6|) = \cos^{-1}(0.6)$. It's important to note that we are looking for radian solutions to this equation, so we need to switch our calculator into radian mode.

$$\hat{x} = \cos^{-1}(0.6) \approx 0.927295218$$

Notice that I wrote down all of the digits to the approximation the calculator returned. This is because we should save our rounding until the very end of the problem. We want to work with as many digits as possible but *display* a solution rounded to the nearest tenth to the reader.
Solutions Associated with Quadrant II: The angle in Quadrant II having reference angle $\hat{x}$ is $\pi - \hat{x}$. Again, we list all coterminal angles as well.

$$x = \pi - \hat{x} + 2\pi k \approx 2.2 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant III: The angle in Quadrant III having reference angle $\hat{x}$ is $\pi + \hat{x}$. Therefore,

$$x = \pi + \hat{x} + 2\pi k \approx 4.1 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x \approx 2.2$ and $x \approx 4.1$.

c. $\frac{\sqrt{3}}{1}$ is definitely a special ratio for the tangent function, so we will abstain from calculator usage here.
Quadrants: The given tangent function is negative, so we know the argument of the function terminates in either QII or QIV.
Reference Angle: From our special triangles, $\hat{x} = \frac{\pi}{3}$.
Solutions Associated with Quadrant II: The angle in Quadrant II having reference angle $\hat{x}$ is $\pi - \hat{x}$. Therefore,

$$x = \pi - \hat{x} + 2\pi k = \pi - \frac{\pi}{3} + 2\pi k = \frac{2\pi}{3} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant IV: The angle in Quadrant IV having reference angle $\hat{x}$ is $2\pi - \hat{x}$. Therefore,

$$x = 2\pi - \hat{x} + 2\pi k = 2\pi - \frac{\pi}{3} + 2\pi k = \frac{5\pi}{3} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

d. 3.7 is not a special ratio for the secant, so we know we will need a calculator at some point.
Quadrants: The given secant function is positive. Therefore, the argument terminates in either QI or QIV.
Reference Angle: Since our calculator doesn't have an inverse secant button, we rewrite $\sec(x) = 3.7$ using the Reciprocal Identities as

$$\cos(x) = \frac{1}{3.7}.$$

Therefore,

$$\hat{x} = \cos^{-1}\left(\left|\frac{1}{3.7}\right|\right) = \cos^{-1}\left(\frac{1}{3.7}\right) \approx 1.297122589.$$

Solutions Associated with Quadrant I: The angle in Quadrant I having reference angle $\hat{x}$ is $\hat{x}$. Thus,

$$x = \hat{x} + 2\pi k \approx 1.3 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant IV: The angle in Quadrant IV having reference angle $\hat{x}$ is $2\pi - \hat{x}$. Therefore,

$$x = 2\pi - \hat{x} + 2\pi k \approx 5.0 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x \approx 1.3$ and $x \approx 5.0$.

Note: Combining Solutions for the Tangent

We found the set of all solutions to Example $10.3.2c$ to be

$$x = \frac{2\pi}{3} + 2\pi k$$

and

$$x = \frac{5\pi}{3} + 2\pi k.$$

However, the tangent and cotangent are special functions in the world of Trigonometry in that their natural periods are π. That is, when finding their solutions, we can add π instead of 2π.

For example, when solving $\tan(x) = -\sqrt{3}$, we could have written our solution associated with Quadrant II as

$$x = \frac{2\pi}{3} + \pi k, \text{ where } k \in \mathbb{Z}.$$

In doing so, we capture all of the Quadrant IV solutions as well because, when $k = 1$, $x = \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$. This is precisely our Quadrant IV solution!

Thus, from this point forward, when solving trigonometric equations involving the tangent and cotangent functions, we will use this strategy to "combine" our solutions into a single statement.

? Checkpoint 10.3.2

Find exact solutions to the given equation, and then state the solutions occurring on the interval $[0, 2\pi]$.

$$\cos(x) = -\frac{1}{\sqrt{2}}$$

Answer

$$\text{Solutions associated with QII}: \quad x = \frac{3\pi}{4} + 2\pi k, \text{ where } k \in \mathbb{Z}$$

$$\text{Solutions associated with QIII}: \quad x = \frac{5\pi}{4} + 2\pi k \text{ where } k \in \mathbb{Z}$$

On $[0, 2\pi]$, the solutions are $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$.

Solving Trigonometric Equations Requiring Algebraic Techniques

The first Ferris wheel was built for the Chicago World's Fair in 1893. It had a diameter of 250 feet and could carry 2160 people in 36 carriages. From the top of the wheel, passengers could see into four states. After loading all the passengers, the wheel made one revolution in nine minutes.

If you are in the bottom carriage of the Ferris wheel at the start of its revolution, your height after t seconds is given by

$$h = f(t) = 139 - 125\cos\left(\frac{2t}{3}\right).$$

For how long are you more than 240 feet above the ground?

Figure **10.3.3** shows a graph of the height function and a horizontal line at $h = 240$.

Figure 10.3.3

The graph shows that $h = 240$ at approximately 215 seconds and 325 seconds into the ride. Your height is more than 240 feet between those two times, or for about 110 seconds. To find a more precise solution, we can use algebraic methods. As an example, we'll solve the slightly simpler equation

$$139 - 125\cos(\theta) = 240.$$

We'll look for all solutions for θ between $0°$ and $360°$. We begin by isolating the trigonometric ratio on one side of the equation.

$$\begin{aligned} & 139 - 125\cos(\theta) = 240 \\ \Longrightarrow \quad & -125\cos(\theta) = 101 && \text{(subtract 139 from both sides)} \\ \Longrightarrow \quad & \cos(\theta) = -0.808 && \text{(divide both sides by } -125) \end{aligned}$$

We have solved equations like this one before: we use the inverse cosine to solve for θ. Remember that there are two angles between $0°$ and $360°$ that have a cosine of -0.808, one in the second quadrant and one in the third quadrant. The reference angle is

$$\hat{\theta} = \cos^{-1}(|-0.808|) = \cos^{-1}(0.808) \approx 36.09901626°.$$

Thus, the solution (rounded to the nearest tenth of a degree) in the second quadrant is

$$\theta = 180° - \hat{\theta} \approx 143.9°$$

and the solution in the third quadrant is

$$\theta = 180° + \hat{\theta} \approx 216.1°.$$

It's important to note that we have not discussed how to solve the original equation, $240 = 139 - 125\cos\left(\frac{2t}{3}\right)$. Instead, we considered the slightly simpler version, $240 = 139 - 125\cos(\theta)$. We will explore the original version a little bit later in this section.

In general, our old techniques of solving algebraic equations for an unknown still hold when solving trigonometric equations.

Example 10.3.3

Solve the equation $4\csc(\theta) - 1 = 7$ for $0° \leq \theta \leq 360°$.

Solution

We begin by isolating the trigonometric ratio.

$$\begin{aligned} & 4\csc(\theta) - 1 = 7 \\ \Longrightarrow \quad & 4\csc(\theta) = 8 && \text{(add 1 to both sides)} \\ \Longrightarrow \quad & \csc(\theta) = 2 && \text{(divide both sides by 4)} \\ \Longrightarrow \quad & \sin(\theta) = \frac{1}{2} && \text{(Reciprocal Identity)} \end{aligned}$$

Quadrants: Sine is positive in QI and QII.
Reference Angle: $\frac{1}{2}$ is a special ratio for the sine function. Using our knowledge of the special triangles, we get the reference angle $\hat{\theta} = 30°$.
Now that we know the angles terminate in QI and QII with reference angle $\hat{\theta} = 30°$, we can confidently state the solutions are $\theta = 30°$ and $\theta = 150°$.

Checkpoint 10.3.3

Solve the equation $5\tan(\theta) + 7 = 2$ for $0 \leq \theta \leq 2\pi$.

Answer

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$

Example 10.3.3 and Checkpoint 10.3.3 illustrate an **implied angle measure**. That is, in Example 10.3.3, the statement, "for $0° \leq \theta \leq 360°$" implies our answer should be in degrees. Likewise, the statement, "for $0 \leq \theta \leq 2\pi$," in Checkpoint 10.3.3, implies our answer should be in radians.

Example 10.3.4

Solve the equation $3\tan(\beta) + 1 = -8$ for $0 \leq \beta \leq 2\pi$. Round your solutions to three decimal places.

Solution

First, we isolate the trigonometric ratio.

$$\begin{aligned} & 3\tan(\beta) + 1 = -8 \\ \Longrightarrow \quad & 3\tan(\beta) = -9 && \text{(subtract 1 from both sides)} \\ \Longrightarrow \quad & \tan(\beta) = -3 && \text{(divide both sides by 3)} \end{aligned}$$

There are two angles with tangent -3, one in the second quadrant and one in the fourth quadrant. The reference angle is $\hat{\beta} = \tan^{-1}(3) \approx 1.249045772$. Thus, we get the following:

$$\begin{aligned} \text{Solution Associated with QII}: \quad & \beta = \pi - \hat{\beta} \approx 1.893 \\ \text{Solution Associated with QIV}: \quad & \beta = 2\pi - \hat{\beta} \approx 5.034 \end{aligned}$$

It should not be surprising in Example 10.3.4 that the solutions are π radians apart. As we mentioned previously, the solutions of an equation $\tan(\theta) = k$ always differ by π radians (or $180°$). Once we have found one solution, we can add π to find the other solution between 0 and 2π.

? Checkpoint 10.3.4

Solve the equation $3\sec(C) - 6 = -2$, for $0° \leq C \leq 360°$. Round your solutions to three decimal places.

Answer

$C = 41.410°$ or $C = 318.590°$

✓ Example 10.3.5

Solve $\frac{\sin(\beta)}{5} - 3 = 1$ for $0 \leq \beta \leq 2\pi$

Solution

We begin by isolating $\sin(\beta)$.

$$\begin{aligned} \frac{\sin(\beta)}{5} - 3 &= 1 \\ \Longrightarrow \quad \frac{\sin(\beta)}{5} &= 4 && \text{(add 3 to both sides)} \\ \Longrightarrow \quad \sin(\beta) &= 20 && \text{(multiply both sides by 5)} \end{aligned}$$

Because $\sin(\beta)$ is never greater than 1, there is no angle β whose sine is 20. The equation has no solution.

? Checkpoint 10.3.5

Solve $2\sec(\theta) - 5 = -4$ for $0° \leq \theta \leq 360°$.

Answer

No solution.

Example 10.3.5 and Checkpoint 10.3.5 show us that not all trigonometric equations have solutions. Just as you did in Algebra, you should pause after each step to check the result for realism. Forming this into a habit can save you time and frustration in the future.

The equations in Examples 10.3.3 through 10.3.5 are **linear-in-form**. That is, if you substitute a variable, say t, for the trigonometric function, the equation would look like a linear equation in each case. The initial steps to solve the equation mimic those of a linear equation.

For example, the equation in Example 10.3.4 could be written as $3t + 1 = -8$. Solving for t, we get $t = -3$; however, t was a substitution for $\tan(\beta)$. Thus, $\tan(\beta) = -3$.

This tactic of simplifying the look of an equation using a substitution is beneficial as you move forward in Mathematics. While we will not use it in every example, Example 10.3.6 (below) shows how it can be helpful.

✓ Example 10.3.6

Solve $4\tan^2(\theta) + 3 = 15$ for $0 \leq \theta \leq 2\pi$.

Solution

We start with the substitution $t = \tan(\theta)$ and solve $4t^2 + 3 = 15$ for t.

$$\begin{aligned} 4t^2 + 3 &= 15 \\ \Longrightarrow \quad 4t^2 &= 12 && \text{(subtract 3 from both sides)} \\ \Longrightarrow \quad t^2 &= 3 && \text{(divide both sides by 4)} \\ \Longrightarrow \quad t &= \pm\sqrt{3} && \text{(Extraction of Roots)} \end{aligned}$$

Resubstituting $\tan(\theta)$ for t, we get

$$\tan(\theta) = \pm\sqrt{3}.$$

There are two angles between 0 and 2π with tangent $\sqrt{3}$ and two angles with tangent $-\sqrt{3}$, making four solutions to this equation - one in each quadrant! Moreover, we know the reference angle such that $\tan\left(\hat{\theta}\right) = \sqrt{3}$ is $\hat{\theta} = \frac{\pi}{3}$.

Recall that tangent is the special function with period π. Therefore, we only need to list the solutions associated with QI and QII and add integer multiples of π to get all other solutions.

$$\text{Solutions associated with QI}: \quad \theta = \frac{\pi}{3} + \pi k, \text{ where } k \in \mathbb{Z}$$

$$\text{Solutions associated with QII}: \quad \theta = \pi - \hat{\theta} + \pi k = \pi - \frac{\pi}{3} + \pi k = \frac{2\pi}{3} + \pi k, \text{ where } k \in \mathbb{Z}$$

To restrict this infinite set of solutions to those in the interval $[0, 2\pi]$, we let $k = 0$ and $k = 1$ to get

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ and } \frac{5\pi}{3}.$$

? Checkpoint 10.3.6

Solve the equation $2\sin^2(\theta) = 1$ for $0° \le \theta \le 360°$.

Answer

$\theta = 45°, \theta = 135°, \theta = 225°$, or $\theta = 315°$

Example 10.3.6 is **quadratic-in-form** because, with the substitution $t = \tan(\theta)$, it can be written as the quadratic equation $4t^2 + 3 = 15$.

Definition: Quadratic-in-Form

An equation that can be written in the form

$$a[f(x)]^2 + bf(x) + c = 0$$

is called an equation that is **quadratic-in-form**.

From this point forward in Mathematics, you will be using your techniques from Algebra to solve trigonometric equations that are quadratic-in-form. If you are rusty with that material, I highly recommend reviewing it.

✓ Example 10.3.7

Solve $2\cos^2(\theta) - \cos(\theta) - 1 = 0$ for $0 \le \theta \le 2\pi$.

Solution

Compare this equation to the algebraic equation $2t^2 - t - 1 = 0$. Our equation has the same algebraic form but with t replaced by $\cos(\theta)$.

We can solve the algebraic equation by factoring $2t^2 - t - 1$ as $(2t + 1)(t - 1)$, and we'll use the same strategy on the trigonometric equation.

$$\begin{aligned} 2\cos^2(\theta) - \cos(\theta) - 1 &= 0 \\ \implies (2\cos(\theta) + 1)(\cos(\theta) - 1) &= 0 \qquad \text{(factor)} \end{aligned}$$

By the Zero Factor Property, this means

$$\begin{aligned} 2\cos(\theta)+1 &= 0 \quad \text{or} \quad \cos(\theta)-1 = 0 \\ \Longrightarrow \quad \cos(\theta) &= -\frac{1}{2} \quad \text{or} \quad \cos(\theta) = 1 \end{aligned}$$

Now we solve each equation for θ.

For $\cos(\theta) = -\frac{1}{2}$: We know cosine is negative in QII and QIII and that $\hat{\theta} = \frac{\pi}{3}$. Thus,

$$\text{Solutions associated with QII}: \quad \theta = \pi - \hat{\theta} + 2\pi k = \pi - \frac{\pi}{3} + 2\pi k = \frac{2\pi}{3} + 2\pi k$$

$$\text{Solutions associated with QIII}: \quad \theta = \pi + \hat{\theta} + 2\pi k = \pi + \frac{\pi}{3} + 2\pi k = \frac{4\pi}{3} + 2\pi k$$

For $\cos(\theta) = 1$: This will result in a quadrantal angle. Visualizing the graph of the cosine, we can (hopefully) see that the cosine attains a value of 1 at $x = 0$ and every integer multiple of 2π after that. That is,

$$\theta = 0 + 2\pi k = 2\pi k.$$

Combining all of these solution sets and restricting our answers to those between 0 and 2π, we get

$$\theta = 0,\ \theta = \frac{2\pi}{3},\ \theta = \frac{4\pi}{3},\ \text{or}\ \theta = 2\pi.$$

? Checkpoint 10.3.7

Solve $9\sin^2(\theta) - 3\sin(\theta) = 2$ for $0° \le \theta \le 360°$. Round your answers to the nearest degree.

Answer

$\theta = 42°, \theta = 138°, \theta = 199°$, or $\theta = 341°$

Solving Trigonometric Equations with Non-Standard Arguments

If n is an integer, what can we say about the solutions of the equation

$$\sin(nx) = k?$$

For example, how many solutions are there for the equation $\sin(2x) = 0.5$? Truthfully, we already know there will be an infinite number of solutions; however, on the interval from 0 to 2π, how many solutions will this equation have?

As was stated earlier in the text, the coefficient of x can be thought of as the number of oscillations over an interval of length equal to the natural period of the function. That is, $\sin(2x)$ goes through two full oscillations on any interval of length 2π. Compare Figure 10.3.4 (below) to Figure 10.3.2.

Figure 10.3.4: *Visualizing solutions to* $\sin(2x) = 0.5$

You can see there are twice as many solutions on the same interval as that of $\sin(x) = 0.5$. This visual helps us wrap our brains around the fact that the number of solutions to such equations over an interval like $[0, 2\pi]$ might be more (or less) than two.

Example 10.3.8

Find all solutions of $\sin(2x) = 0.5$ between 0 and 2π.

Solution

Treat the argument as if it were a single variable. The sine is positive, meaning the angle, $2x$, must terminate in **QI** or **QII**. Additionally, we know that the reference angle is $\widehat{2x} = \frac{\pi}{6}$. At this point, let's pause to make sure we understand what just happened. I am treating the argument of the sine, which is $2x$, as a single variable - sort of like calling it θ. Once I find a reference angle, I have found it for that single variable, which is $2x$. Like we traditionally do, we use the "hat" notation to denote the reference angle; however, in this case, the reference angle is for $2x$ - not x. Therefore, our reference angle notation is $\widehat{2x}$.

Continuing as usual,

$$\text{Solutions associated with QI}: \quad 2x = \frac{\pi}{6} + 2\pi k$$

$$\text{Solutions associated with QII}: \quad 2x = \frac{5\pi}{6} + 2\pi k$$

where $k \in \mathbb{Z}$. We finally arrive at the reason why I write "Solutions **associated with**" rather than "Solutions **in**." Our goal is to solve for x. This requires us to divide both equations by 2 (including dividing the $2\pi k$). Therefore, we get

$$\text{Solutions associated with QI}: \quad x = \frac{\pi}{12} + \pi k$$

$$\text{Solutions associated with QII}: \quad x = \frac{5\pi}{12} + \pi k$$

where $k \in \mathbb{Z}$. Do you notice that the "Solutions associated with QII" doesn't start with a Quadrant II angle? That's perfectly fine! The solution angles have been adjusted by the coefficient of x so they might not belong to the original quadrant they were derived from. This is why I use the words "associated with."

We now need to restrict our answers to the interval $[0, 2\pi]$. Allowing $k = 0$, we get $x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$ - both of which are in the requested interval. Trying $k = 1$, we get $x = \frac{13\pi}{12}$ and $x = \frac{17\pi}{12}$. Again, both of these solutions are still within the requested

interval. Trying $k = 2$, however, gives $x = \frac{25\pi}{12}$ and $x = \frac{29\pi}{12}$. Neither of these are in the requested interval. Thus, the solutions on $[0, 2\pi]$ are

$$x = \frac{\pi}{12}, x = \frac{5\pi}{12}, \frac{13\pi}{12}, \text{ or } x = \frac{17\pi}{12}.$$

Notice in Example **10.3.8**, that we had *twice* the number of solutions we would typically expect. Do you see why?

? Checkpoint 10.3.8

a. Sketch a graph of $y = \cos(2x)$ for $0 \leq \theta \leq 2\pi$.
b. Find exact values for all solutions of $\cos(2x) = -\frac{1}{\sqrt{2}}$ between 0 and 2π.

Answer

a.

b. $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

As we observed earlier, equations involving the tangent function are easier to solve because there is only one solution in each cycle of the graph. Once we have found one solution, we can find all the others by adding multiples of the period.

✔ Example 10.3.9

Find all solutions of $\tan(2t) = -1$ between 0 and 2π.

Solution

Again, treating the argument, $2t$, as a single variable, we know the tangent is negative in **QII** and **QIV**; however, the tangent is that special case (along with the cotangent) where we can say the second solution is just π plus the first solution. So, we only need to concentrate on the solution in the second quadrant. The reference angle is $\widehat{2t} = \frac{\pi}{4}$. This is all we need!

$$\text{Solutions associated with QII}: \quad 2t = \frac{3\pi}{4} + \pi k$$

where $k \in \mathbb{Z}$. We then divide both sides of the equation by 2 to get

$$\text{Solutions associated with QII}: \quad t = \frac{3\pi}{8} + \frac{\pi}{2}k$$

where $k \in \mathbb{Z}$. Just so you can see a different approach, let's build a table to get all the solutions between 0 and 2π.

k	$\frac{3\pi}{8} + \frac{\pi}{2}k$
0	$\frac{3\pi}{8}$
1	$\frac{7\pi}{8}$
2	$\frac{11\pi}{8}$
3	$\frac{15\pi}{8}$

If we move to $k = 4$, the solutions will be outside of the requested interval. Thus, our solutions are $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$ and $\frac{15\pi}{8}$. The following figure gives a visual of these solutions.

Figure 10.3.5

Checkpoint 10.3.9

Find all solutions of $\tan(2x) = \sqrt{3}$ between 0 and 2π.

Answer

$\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

The tactics shown in Examples **10.3.8** and **10.3.9** work for more complicated equations.

Example 10.3.10

Solve $\sin(2x + 1.5) = -0.3$ for $0 \le x \le 2\pi$. Round your answer to four decimal places.

Solution

Again, we treat the entire argument as a single variable. The sine is negative in QIII and QIV. Since -0.3 is not a special ratio for the sine, we will have to use technology. The reference angle is given by

$$\widehat{2x+1.5} = \sin^{-1}(|-0.3|) = \sin^{-1}(0.3) \approx 0.304692654.$$

Therefore,

$$\text{Solutions associated with QIII}: \quad 2x + 1.5 = \pi + \widehat{2x+1.5} + 2\pi k$$

$$\text{Solutions associated with QIV}: \quad 2x + 1.5 = 2\pi - \widehat{2x+1.5} + 2\pi k$$

where $k \in \mathbb{Z}$. Despite having an approximation for the reference angle, $\widehat{2x+1.5}$, I like to leave the equations in exact form (without decimal approximations) until I get to a final answer. Subtracting 1.5 from both sides of each equation and dividing by 2, we get the following:

$$\text{Solutions associated with QIII}: \quad x = \frac{\pi}{2} + \frac{\widehat{2x+1.5}}{2} + \pi k - 0.75$$

$$\text{Solutions associated with QIV}: \quad x = \pi - \frac{\widehat{2x+1.5}}{2} + \pi k - 0.75$$

where $k \in \mathbb{Z}$. At this time, I will use technology to find an approximation for these solutions.

$$\text{Solutions associated with QIII}: \quad x = 0.973142654 + \pi k$$

$$\text{Solutions associated with QIV}: \quad x = 2.239246327 + \pi k$$

where $k \in \mathbb{Z}$. Let's build a table like we did in Example **10.3.9**.

k	$0.973142654 + \pi k$	$2.239246327 + \pi k$
0	0.973142654	2.239246327
1	4.114735308	5.38083898

Going to $k = 2$ gives solutions outside of the requested interval. Thus, the solutions on $[0, 2\pi]$ are 0.9731, 2.2392, 4.1147, and 5.3808.

Checkpoint 10.3.10

Solve $4\cos(3x - 0.5) = -3.2$ for $0 \le x \le 2\pi$.

Answer

1.00, 1.43, 3.09, 3.52, 5.19, 5.62

Applications

Trigonometric equations often arise in the study of periodic models.

Example 10.3.11

A piston is pumping vertically at a rate of 1000 cycles per second. The distance between its lowest and highest position is 16 centimeters.

a. Suppose the piston is at its midline position at $t = 0$ and moving downwards. Find a formula for the sinusoidal function $h(t)$ that gives the piston's height.
b. Use the formula to find the first two times the piston is 14 centimeters above its lowest position.

Solutions

a. Because the piston starts at its midline and moves down, we will write $h(y)$ as a sine function:

$$h(t) = -A\sin(Bt) + D.$$

The amplitude is $A = \frac{16}{2} = 8$, and the midline is $D = 8$. The period is $\frac{1}{1000}$, so $\frac{B}{1000} = 2\pi$, and $B = 2000\pi$. Thus,

$$h(t) = -8\sin(2000\pi t) + 8.$$

The graph is shown in Figure 10.3.6.

Figure 10.3.6

b. With $h(t) = 14$, we have

$$\begin{aligned} -8\sin(2000\pi t) + 8 &= 14 \\ \implies -8\sin(2000\pi t) &= 6 \qquad \text{(subtract 8 from both sides)} \\ \implies \sin(2000\pi t) &= -\frac{3}{4} \qquad \text{(divide both sides by } -8\text{)} \end{aligned}$$

Sine is negative in QIII and QIV. Moreover, we can compute the reference angle as follows:

$$\widehat{2000\pi t} = \sin^{-1}\left(\left|-\frac{3}{4}\right|\right) = \sin^{-1}\left(\frac{3}{4}\right) \approx 0.848062079.$$

We want the first two *positive* solutions to this equation.

$$\text{Solutions associated with QIII}: \quad 2000\pi t = \pi + \sin^{-1}\left(\frac{3}{4}\right) + 2\pi k \implies t = \frac{1}{2000} + \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$$

$$\text{Solutions associated with QIV}: \quad 2000\pi t = 2\pi - \sin^{-1}\left(\frac{3}{4}\right) + 2\pi k \implies t = \frac{1}{1000} - \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$$

where $k \in \mathbb{Z}$. Building a quick table, we get the following:

k	$\frac{1}{2000} + \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$	$\frac{1}{1000} - \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$
0	0.000634973	0.000865027
1	0.001634973	0.000965027
⋮	⋮	⋮

From our table, we can see the first two times the piston is at a height of 14 centimeters are approximately $t = 0.000865$ and $t = 0.000635$ seconds.

Checkpoint 10.3.11

A lever on an oil well is pumping vertically at a rate of 10 cycles per minute. The distance between its lowest position at the ground and its highest position is 1.8 meters.

a. Suppose the lever is at its midline position at $t = 0$ and moving downwards. Find a formula for the sinusoidal function $h(t)$ that gives the lever's height.
b. Find the first two times the lever is 1 meter above its lowest position.

Answer

a. $h(t) = -0.9\sin(20\pi t) + 0.9$
b. 0.0518 minutes and 0.9823 minutes

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10.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Solving Equations**
  - Solving Absolute Value Equations: This skill is needed for the "Skills Refresher" exercises.
  - Solving Quadratic Equations: This section requires solving equations that are quadratic-in-form. The "Skills Refresher" also includes basic quadratic equations.
  - Solving Equations Quadratic-in-Form: This section uses this skill frequently.
  - Solving Polynomial Equations: The Zero-Factor Property, used for solving factored polynomial equations, is a key technique for solving factored trigonometric equations (Example 7).
  - Solving Rational Equations: This skill is needed for the "Skills Refresher" exercises.
  - Solving Logarithmic Equations: This skill is needed for the "Skills Refresher" exercises.

VIDEOS

- Solving Trigonometric Equations
  - Solving Trigonometric Equations - The Concepts Behind Solving Trigonometric Equations
  - Solving Trigonometric Equations - Basic Equations

KEY CONCEPTS

DEFINITIONS

- **Trigonometric Equation:** An equation that involves one or more trigonometric functions of a variable.
- **Quadratic-in-Form:** An equation that can be written in the form $a[f(x)]^2 + b[f(x)] + c = 0$, where $f(x)$ is some function.

COMMON MISTAKES

- **Forgetting to Find All Solutions:** Trigonometric equations typically have an infinite number of solutions due to the periodic nature of the functions. Forgetting to add multiples of the period ($+2\pi k$ or πk) will result in an incomplete solution set.
- **Stopping After the Inverse Function:** Using an inverse trigonometric function on a calculator provides only one solution (the principal value). You must use that angle as a reference angle to find all solutions in the required quadrants.
- **Incorrectly Handling Non-Standard Arguments:** When solving an equation like $\sin(2x) = k$, the solutions found are for the argument $2x$. A common mistake is forgetting the final step of dividing by 2 to solve for x.
- **Not Checking for Extraneous or Invalid Solutions:** It is important to check for realism. For example, an equation that simplifies to $\sin(\beta) = 20$ has no solution, since the range of the sine function is $[-1, 1]$.
- **Algebraic Errors:** Common algebraic mistakes, such as incorrectly applying the zero-factor principle or dividing by a variable that could be zero, can lead to incorrect or incomplete solutions.

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10.3.2: HOMEWORK

READING QUESTIONS

1. Why do trigonometric equations typically have an infinite number of solutions?
2. What are the two fundamental concepts from trigonometry that are the keys to finding all solutions to a trigonometric equation?
3. If you are solving $\cos(\theta) = -0.6$, in which two quadrants will the solutions lie?
4. After using a calculator to find a reference angle, how do you find the general solution for all angles coterminal with that solution?
5. What does it mean for a trigonometric equation to be "quadratic-in-form"?
6. What algebraic technique is used to solve an equation like $2\cos^2(\theta) - \cos(\theta) - 1 = 0$?
7. When solving an equation involving $\tan(x)$ or $\cot(x)$, how does finding the general solution differ from finding it for sine or cosine?
8. When solving an equation with a non-standard argument, such as $\sin(2x) = 0.5$, what should you treat as the "angle" when finding the reference angle?
9. How does the coefficient on the variable, like the '2' in $\sin(2x)$, affect the number of solutions within the interval $[0, 2\pi]$?
10. Why does an equation like $\sin(x) = 5$ have no solution?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, solve the equation.

1. $(2x-6)^2 = 4$
2. $\sqrt{2x-6} = 4$
3. $\dfrac{1}{2x-6} = 4$
4. $|2x-6| = 4$
5. $\log 2x - 6 = 4$
6. $2^{2x-6} = 4$

For the following exercises, the "solution" contains an error. Find the error, and supply a correct solution.

7.

$$\begin{aligned} 3x^2 - 5x &= 0 \\ \Rightarrow \quad 3x^2 &= 5x \\ \Rightarrow \quad 3x &= 5 \\ \Rightarrow \quad x &= \frac{5}{3} \end{aligned}$$

8.

$$\begin{aligned} 4x^2 - 5x &= 12 \\ \Rightarrow \quad x^2 &= 3 \\ \Rightarrow \quad x &= \sqrt{3} \end{aligned}$$

9.

$$\begin{aligned} (2x+1)^2 &= 4 \\ \Rightarrow \quad 2x+1 &= 2 \\ \Rightarrow \quad 2x &= 1 \\ \Rightarrow \quad x &= \frac{1}{2} \end{aligned}$$

10.

$$\begin{aligned} 2x^2 - 4x &= 3 \\ \Rightarrow \quad 2x(x-2) &= 3 \\ \Rightarrow \quad 2x = 3 \quad &\text{or} \quad x - 2 = 3 \\ \Rightarrow \quad x = \frac{3}{2} \quad &\text{or} \quad x = 5 \end{aligned}$$

Answers

1. 2, 4
2. 11
3. $\dfrac{25}{8}$
4. 1, 5
5. 5003
6. 4
7. We can't divide by x. $x = 0, \dfrac{5}{3}$
8. 3 has two square roots. $x = \pm\sqrt{3}$
9. 4 has two square roots. $x = \dfrac{-3}{2}, \dfrac{1}{2}$
10. One side must be zero to use the zero-factor principle. No real solutions.

HOMEWORK

CONCEPT CHECK

1. How many solutions between $0°$ and $360°$ does the equation $\cos(\theta) = k$ have for each value of k between -1 and 0?
2. How many solutions between $0°$ and $360°$ does the equation $\cos(\theta) = k$ have for each value of k greater than 1?
3. How many solutions between $0°$ and $360°$ does the equation $\sin^2(\theta) = k$ have for each value of k between -1 and 0?
4. If $x = 0.68$ is one solution of the equation $\cos(x) = c$, what is other? Illustrate on a unit circle.
5. If $t = 2.45$ is one solution of the equation $\sin(t) = k$, what is the other? Illustrate on a unit circle.
6. If $\theta = 1.73$ is one solution of the equation $\tan(\theta) = C$, what is the other?

7. Explain why the equation $\cos(nx) = k, \quad 0 < k < 1$, has $2n$ solutions between 0 and 2π.
8. How many solutions are there to the equation $\cos(\theta) = 0.4$? How many solutions are there between 0 and 2π?
9. Find all angles between $0°$ and $360°$ for which the statement is true.
   a. $\cos(\theta) = -1$
   b. $\sin(\theta) = -1$
   c. $\tan(\theta) = -1$
10.
   a. Find two angles, $0 \le \theta \le 360°$, with $\sin(\theta) = 0$.
   b. Find two angles, $0 \le \theta \le 360°$, with $\cos(\theta) = 0$.
11.
   a. Find two angles, $0 \le \theta \le 360°$, with $\sin(\theta) = \cos(\theta)$.
   b. Find two angles, $0 \le \theta \le 360°$, with $\sin(\theta) = -\cos(\theta)$.

BASIC SKILLS

For the following exercises,
(a) Evaluate the expression at the given values of the variable.
(b) Give one solution to the equation.

12.
   a. $\sin(\theta) + \cos(\theta); \quad \theta = 0°, 30°, 45°, 60°$
   b. $\sin(\theta) + \cos(\theta) = \sqrt{2}$
13.
   a. $\sin^2(\alpha) - \cos(\alpha); \quad \alpha = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$
   b. $\sin^2(\alpha) - \cos(\alpha) = 1$
14.
   a. $\sin(\beta) + 2\cos^2(\beta); \quad \beta = 210°, 225°, 240°, 270°$
   b. $\sin(\beta) + 2\cos^2(\beta) = -1$
15.
   a. $3\cos^2(\phi) - \sin^2(\phi); \quad \phi = \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}$
   b. $3\cos^2(\phi) - \sin^2(\phi) = 2$

For the following exercises, find all degree solutions (in exact form) and then state the solutions for $0° \le \theta \le 360°$.

16. $7\sin(\theta) + 11 = 11$
17. $3 = 5 - 4\cos(\theta)$
18. $3\tan(\theta) = \sqrt{3}$
19. $8\csc(\theta) + 5 = -11$
20. $9\sec(\theta) + 15 = 6$
21. $6\cot(\theta) + 21 = 15$
22. $0 = \sqrt{2} + 2\sin(\theta)$
23. $\sqrt{3}\cos(\theta) = -\frac{3}{2}$
24. $\cos^2(\theta) - 1 = 0$
25. $1 - \sin^2(\theta) = 0$
26. $3\csc^2(\theta) - 4 = 0$
27. $0 = 4 - 2\sec^2(\theta)$
28. $1 - \tan^2(\theta) = 0$
29. $0 = 2\cot^2(\theta) - 6$
30. $2\sin^2(\theta) + \sin(\theta) - 1 = 0$
31. $4 - \cos(\theta) = 4\cos^2(\theta) + 3\cos(\theta) + 5$
32. $2\sin^2(\theta) - 11\sin(\theta) = -5$
33. $2 = \csc^2(\theta) - \csc(\theta)$
34. $\sec^2(\theta) + 3\sec(\theta) + 2 = 0$
35. $\sin(4x) = -1$
36. $\cos(3t) = 0$
37. $5\tan(2q) = 0$
38. $6\sin(4w) = -3\sqrt{2}$
39. $4\cos(3\phi) = -2$

For the following exercises, find all radian solutions (in exact form) and then state the solutions for $0 \le \theta \le 2\pi$.

40. $\sqrt{2}\sin(\theta) + 9 = 8$
41. $\tan(x) - 2\sin(x)\tan(x) = 0$
42. $13 = 13 - 4\cos(\theta)$
43. $4\tan(\theta) - 2 = -6$
44. $8\csc(\theta) + 5 = -3$
45. $2\cos^2(t) + \cos(t) = 1$
46. $\sec(x)\sin(x) - 2\sin(x) = 0$
47. $-6\sec(\theta) + 4 = -8$
48. $3\cot(\theta) + \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$
49. $0 = 4\sqrt{2} + 8\sin(\theta)$
50. $\sqrt{3}\cos(\theta) = -\frac{3}{2}$
51. $\sin^2(\theta) - 1 = 0$
52. $1 - \cos^2(\theta) = 0$
53. $4\sec^2(\theta) - 3 = 0$
54. $0 = 1.5 - 0.75\csc^2(\theta)$
55. $1 - \cot^2(\theta) = 0$
56. $0 = -6 + 18\tan^2(\theta)$
57. $\tan^5(x) = \tan(x)$
58. $2\sin^2(\theta) + 3\sin(\theta) = -1$
59. $3 = 6\cos^2(\theta) - 3\cos(\theta)$
60. $8\sin^2(x) + 6\sin(x) + 1 = 0$
61. $\tan^2(x) = -1 + 2\tan(-x)$
62. $2\cos^2(\theta) + 7\cos(\theta) + 3 = 0$
63. $-2 = \csc^2(\theta) - 3\csc(\theta)$
64. $\sec^2(\theta) - 2 = -\sec(\theta)$
65. $\sqrt{3}\tan(2\alpha) = 3$
66. $2\sin(2\beta) = 1$
67. $-6\cos(2\theta) = 6$
68. $3\tan(3w) = \sqrt{3}$

69. $2\tan(3u) = -2$
70. $\frac{5}{7}\tan(\pi x) + 11 = 11$
71. $8\sin\left(2\theta - \frac{\pi}{6}\right) = -4$
72. $6\cos\left(3\theta - \frac{\pi}{2}\right) = -3\sqrt{2}$
73. $3\tan\left(\frac{w}{2} + \frac{\pi}{4}\right) + 4 = 1$
74. $50\sec\left(\frac{t}{\pi} + 6\right) - 10 = -110$
75. $2\tan(2\pi\beta) + 5 = 3$
76. $160\sin(\pi\phi - 1) + 10 = 90$
77. $2\cos\left(3t + \frac{\pi}{4}\right) = \sqrt{3}$
78. $2 - \tan\left(2x - \frac{\pi}{3}\right) = 2$
79. $7\csc\left(\frac{\phi}{2} + \frac{3\pi}{4}\right) + 3 = -4$

For the following exercises, solve the equation for $0° \le \theta \le 360°$. Round your answers to two decimal places.

80. $4 = 6\sin(\theta)$
81. $3 = 5\cos(\theta)$
82. $\frac{1}{2}\tan(\theta) - 1 = -3$
83. $7\csc(\theta) + 2 = -16$
84. $3\sec(\theta) - 2 = 5$
85. $2 = 5 - \frac{1}{3}\cot(\theta)$
86. $6\cos^2(\theta) = 2$
87. $2 - 7\sin^2(\phi) = 1$
88. $-4\tan^2(\theta) + 5 = -4$
89. $8 - 2\sec^2(\theta) = -2$
90. $3 = 9 - 2\csc^2(\theta)$
91. $9\cot^2(\theta) - 9 = -5$
92. $5\sin^2(\theta) + \sin(\theta) = 0$
93. $10\cos^2(\theta) - 7\cos(\theta) + 1 = 0$
94. $4\tan^2(\theta) = \tan(\theta)$
95. $\cot^2(\theta) - 2\cot(\theta) = 15$
96. $5\sin(3\theta) - 3 = -4$
97. $9\cos(2\theta) + 1 = 6$
98. $8\tan(4t) + 1 = -11$
99. $6\csc(2r) + 2 = 13$

For the following exercises, solve the equation for $0 \le \theta \le 2\pi$. Round your answers to two decimal places.

100. $5\sin(\theta) - 4 = -6$
101. $-4 + 3\cos(\theta) = -2$
102. $-2\tan(\theta) + 5 = 6$
103. $-4 = 5 - \csc(\theta)$
104. $3\sec(\theta) - 2 = -10$
105. $\frac{1}{3}\cot(\theta) + 1 = -2$
106. $5\sin^2(\theta) = 2$
107. $3 - 6\cos^2(\phi) = -1$
108. $4 - 4\tan^2(\theta) = -5$
109. $4 - \sec^2(\theta) = -1$
110. $-1 + 2\csc^2(\theta) = 5$
111. $\frac{9}{2}\cot^2(\theta) - 1 = 1$
112. $\sin^2(\theta) + 8\sin(\theta) + 7 = 0$
113. $\cos^2(\theta) - 4\cos(\theta) + 3 = 0$
114. $\tan^2(\theta) - 5\tan(\theta) + 6 = 0$
115. $6\cot^2(\theta) - \cot(\theta) - 1 = 0$
116. $150\sin(3s) = 27$
117. $7\cos(2t) - 3 = 2$
118. $3 = 3\tan(4x) + 4$
119. $2 - 8\sec(3t) = -10$
120. $9\sin\left(\frac{\pi}{2}(t-1)\right) + 5 = -1$
121. $3 - 5\cos(2\phi - 1) = 6$
122. $14\tan(\pi\beta - 4) + 31 = 10$
123. $0.3\csc\left(\frac{\pi}{2}(\alpha + 0.1)\right) + 0.4 = 1.9$
124. $2\sec\left(\frac{\pi}{4}\left(t + \frac{1}{4}\right)\right) + 3 = -7$
125. $6\cot\left(\frac{\pi}{3}(\theta - 1)\right) + 4 = 5$
126. $120\sin\left(\frac{\pi}{3}(t - 0.2)\right) + 21 = -3$
127. $16\cos(3t - 1) + 4 = -8$
128. $23 - 24\tan(\pi x + 2) = 17$
129. $3\csc\left(3w - \frac{\pi}{3}\right) + 1 = 6$
130. $12\sec\left(\frac{\pi}{2}(t + 0.3)\right) - 7 = 9$
131. $8\cot\left(4t - \frac{\pi}{3}\right) - 24 = 1$

PROOFS

132. What is the angle in the first quadrant with reference angle θ? What is the angle in the third quadrant with reference angle θ? What is the difference of these angles? Explain how this fact shows that the solutions of $\tan(\theta) = k$, for $k > 0$, differ by π.
133. What is the angle in the second quadrant with reference angle θ? What is the angle in the fourth quadrant with reference angle θ? What is the difference of these angles? Explain how this fact shows that the solutions $\tan(\theta) = k$, for $k < 0$, differ by π.

APPLICATIONS

134. A sinusoidal function can approximate the population of deer in Marquette County over a typical year. The population reached a maximum of 50,000 deer on September 1st and a minimum of 42,000 on March 1st.
    a. Write a formula for the function $P(t)$ that gives the deer population on the first of each month, where $t = 0$ is September 1.
    b. When is the deer population 45,000? Give exact expressions and approximations rounded to two decimal places.
    c. Graph your function over one period, and label the points that correspond to a deer population of 45,000. Is the population greater or less than 45,000 between the two solutions?

135. The percent of the moon visible from Earth is a sinusoidal function ranging from 0% to 100%, with a period of 29.5 days.
    a. Write a formula for the function $f(t)$ that gives the percent of the moon that is visible, if a new moon (0% visible) occurs at $t = 0$ days.
    b. When is 25% of the moon visible? Give approximations rounded to two decimal places.
    c. Graph your function over one period, and label the points corresponding to a quarter moon. Is more or less than 25% of the moon visible between the two solutions you found in part (b)?
136. A Ferris wheel has a diameter of 20 meters and completes one revolution every 60 seconds. Jorge is at the lowest position of the Ferris wheel, 1 meter above the ground, at $t = 0$ seconds.
    a. Write a formula for the function $h(t)$ that gives Jorge's altitude in meters after t seconds.
    b. When is Jorge at an altitude of 18 meters during his first revolution? Give exact expressions and approximations rounded to two decimal places.
    c. Graph your function over one period, labeling the points that correspond to an altitude of 18 meters. Is Jorge above or below 18 meters between the two solutions you found in part (b)?
137. High tides occur every 12.2 hours at Point Lookout. The depth of the water at the end of Roy's dock is 2.6 meters at high tide and 1.8 meters at low tide.
    a. Write a formula for the function $d(t)$ that gives the depth of the water t hours after last night's high tide.
    b. When is the water at the end of the dock 2 meters deep? Give exact approximations rounded to two decimal places.
    c. Graph your function over one period, labeling the points that correspond to a depth of 2 meters. Is the water depth greater or less than 2 meters between the two solutions you found in part (b)?

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10.4: Solving Trigonometric Equations Using Identities

Note to the Instructor - The Remaining Types of Equations

The equations in this section require students to use trigonometric identities. Additionally, students will learn how to rewrite a sum or difference of sine and cosine as a single sinusoidal function. If you don't recall, this is a necessary technique for Differential Equations.

Learning Objectives

- Use an identity to solve a trigonometric equation.
- Rewrite a sum or difference of sine and cosine as a single sinusoidal function.
- Use technology to approximate solutions to equations involving trigonometric functions.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Having a solid foundation of how trigonometric equations are solved, we now integrate our knowledge of trigonometric identities to further expand our equation-solving abilities.

Solving Equations Involving Different Trigonometric Functions Having the Same Arguments

The equations encountered in the previous section were ideal in that each featured only one trigonometric function, and all occurrences of that function within the equation shared the same argument. That is, we only encountered equations like the following:

$$\begin{aligned} \sin(\theta) &= -0.6428 \\ \tan(2x) &= \sqrt{3} \\ 4\tan^2(\theta) + 3 &= 15 \\ 2\cos^2(\theta) - \cos(\theta) - 1 &= 0 \end{aligned}$$

Each of these equations features only one trigonometric function. In the fourth equation, that trigonometric function (the cosine) occurs twice, but in both cases, the arguments are identical (θ).

How would we solve a trigonometric equation if it contains two *different* trigonometric functions? For example, if we were tasked with solving

$$\sec^2(x) = \tan(x) + 3,$$

what would we do?

Ultimately, the answer is that we use a mixture of the trigonometric identities learned in this course and algebraic manipulations learned in Algebra. The worst thing I could do at this point is throw a bunch of examples at you showing how to solve specific equations - that approach will make the *single* equation-solving tactic seem like a *dozen* different tactics. To avoid this, let's start by reviewing how we solved trigonometric equations in the previous section and codify this into a tactic.

When we were given an equation of the form

$$A\,\text{trig}(Bx + C) + D = E,$$

we first isolated the trigonometric function (using techniques from Algebra) to a form like

$$\text{trig}(Bx + C) = c,$$

where c is a real number. I also suggested a "substitution" technique if the argument of the trigonometric function wasn't a simple variable like θ or x. Let's suppose we use that technique to let $u = Bx + C$. Then our equation would simplify down to

$$\text{trig}(u) = c.$$

At this point, we used the sign of c to help us determine the quadrants and reference angle associated with our solution. Once those two pieces of information were found, we stated the answers.

This review leads to the following tactic for solving "basic" trigonometric equations.

How To... Solve a "Basic" Trigonometric Equation

Given *any* trigonometric equation of the form

$$\text{trig}(u) = c,$$

where "trig" could be sin, cos, tan, csc, sec, or cot, do the following:

1. If the value of c is such that u is a quadrantal angle, skip to Step 5.
2. Use the sign of c to determine in which quadrants u terminates.
3. Find the reference angle, $\hat{u}$.
4. Using the reference angle from Step 3, write the solutions *associated* with the quadrants listed in Step 2. If possible, compress the solution set to a single equation (this is most often the case with equations involving the tangent or the cotangent functions).
5. If u is a function of another variable, solve the equations in Step 4 (or Step 1, if the angle is a quadrantal angle) for the "true" variable.
6. Check for extraneous solutions!

Therefore, our ultimate goal when solving *any* trigonometric equation is to get it into the form $\text{trig}(u) = c$.

Example 10.4.1

Solve the following equations and list the solutions which lie in the interval $[0, 2\pi)$.

a. $3\cos(x)\sin^2(x) = \sin^2(x)$
b. $\sec^2(x) = \tan(x) + 3$
c. $\cos^2(A) - \sin^2(A) - \sin(A) = 1$

Solutions

a. We resist the temptation to divide both sides of $3\cos(x)\sin^2(x) = \sin^2(x)$ by $\sin^2(x)$.[1] Instead, we look at our desired destination - $\text{trig}(u) = c$. One thing is certain about that desired form: We need all trigonometric functions on one side. Let's start there.

$$\begin{aligned} & & 3\cos(x)\sin^2(x) &= \sin^2(x) & \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & 3\cos(x)\sin^2(x) - \sin^2(x) &= 0 & \text{(move all variable expressions to one side)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & \sin^2(x)[3\cos(x) - 1] &= 0 & \text{(factor out the GCF)} \end{aligned}$$

We now use the Zero Factor Property from Algebra to break the equation into two different (and simpler) equations.

$$\begin{aligned} & \sin^2(x) = 0 \quad \text{or} \quad 3\cos(x) - 1 = 0 \\ \Rightarrow \quad & \sin(x) = 0 \quad \text{or} \quad \cos(x) = \frac{1}{3} \end{aligned}$$

These equations match the form $\text{trig}(u) = c$, so solving them will mirror what we did in the previous section.

For $\sin(x) = 0$
The right side of $\sin(x) = 0$ implies a quadrantal angle.

Figure 10.4.1

From the graph of $y = \sin(x)$, we know the solution to this equation is $x = \pi k$, with $x = 0$ and $x = \pi$ being the two solutions which lie in $[0, 2\pi)$.

For $\cos(x) = \frac{1}{3}$

Since the cosine is positive, its argument is an angle that terminates in **QI** or **QIV**. Additionally, the ratio, $\frac{1}{3}$, is not a special ratio, so we will eventually need the use of technology. For now, we note that the reference angle is $\hat{x} = \cos^{-1}\left(\left|\frac{1}{3}\right|\right) = \cos^{-1}\left(\frac{1}{3}\right)$. Hence, the two families of solutions are

$$\text{Solutions associated with QI}: \quad x = \cos^{-1}\left(\frac{1}{3}\right) + 2\pi k$$

$$\text{Solutions associated with QIV}: \quad x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right) + 2\pi k$$

where $k \in \mathbb{Z}$. We find the two solutions which lie in $[0, 2\pi)$ to be

$$x = \cos^{-1}\left(\frac{1}{3}\right) \quad \text{and} \quad x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right).$$

Thus, the solutions to the original equation on the interval $[0, 2\pi)$ are

$$x = 0,\ x = \cos^{-1}\left(\frac{1}{3}\right),\ x = \pi, \text{ and } x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right).$$

Grabbing a calculator (and rounding to four decimal places), we get

$$x = 0,\ x = \pi,\ x \approx 1.2310, \text{ and } x \approx 5.0522.$$

b. Again, our ultimate goal is to get an equation (or equations) equivalent to $\text{trig}(u) = c$. At the very least, this means we need to move all the trigonometric functions to one side of the equation.

$$\begin{aligned} \sec^2(x) &= \tan(x) + 3 \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} \quad \sec^2(x) - \tan(x) &= 3 \qquad \text{(move all variable expressions to one side)} \end{aligned}$$

Again, we have a mixture of different trigonometric functions on the left side, but this time, Algebra (specifically, factoring) is not going to help (do you see why?). Instead, we look for any trigonometric identities that might be of use. The $\sec^2(x)$ in an equation with $\tan(x)$ screams "Pythagorean Identity!"

$$\begin{aligned} \sec^2(x) - \tan(x) &= 3 \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry}} \quad \tan^2(x) + 1 - \tan(x) &= 3 \qquad \text{(Pythagorean Identities)} \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} \quad \tan^2(x) - \tan(x) - 2 &= 0 \qquad \text{(subtract 3 from both sides)} \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} \quad u^2 - u - 2 &= 0 \qquad \text{(substitute } u = \tan(x)\text{)} \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} \quad (u+1)(u-2) &= 0 \qquad \text{(factor)} \end{aligned}$$

This gives $u = -1$ or $u = 2$. Since $u = \tan(x)$, we have $\tan(x) = -1$ or $\tan(x) = 2$.

For $\tan(x) = -1$

Since the tangent is negative, we know the argument terminates in either **QII** or **QIV**. Moreover, the reference angle is $\frac{\pi}{4}$. Hence,

$$\text{Solutions associated with QII and IV}: \quad x = \frac{3\pi}{4} + \pi k$$

where $k \in \mathbb{Z}$. Notice that we used the fact that the period of the tangent is π to compress our *two* solution sets into *one*. Letting $k = 0$ and then $k = 1$, we get the solutions $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, respectively. Any further, and our values for x will go beyond the requested interval.

For $\tan(x) = 2$

The tangent is positive in **QI** and **QIII**. We should recognize that the right side is *not* a special ratio for the tangent function. Therefore, we set the reference angle to $\hat{x} = \tan^{-1}(|2|) = \tan^{-1}(2)$. Thus,

$$\text{Solutions associated with QI and III}: \quad x = \tan^{-1}(2) + \pi k$$

where $k \in \mathbb{Z}$. On $[0, 2\pi)$, $x = \tan^{-1}(2)$ and $x = \pi + \tan^{-1}(2)$ are the solutions.

Hence, the solution set to the original equation on the interval $[0, 2\pi)$ is

$$x \in \left\{\tan^{-1}(2),\ \frac{3\pi}{4},\ \pi + \tan^{-1}(2),\ \frac{7\pi}{4}\right\},$$

where $\tan^{-1}(2) \approx 1.1071$ and $\pi + \tan^{-1}(2) \approx 4.2487$.

c. Since this equation involves both the sine and cosine functions, we will need to either perform some Algebra or look for trigonometric identities to get to a point where we have a simple equation (or equations) like $\text{trig}(u) = c$. Factoring out the GCF is out of the question because the three terms on the left side of

$$\cos^2(A) - \sin^2(A) - \sin(A) = 1$$

do not share a common factor. Therefore, we immediately look for obvious trigonometric identities. We know by this point that any time we encounter the square of a trigonometric function, we can rewrite it using a Pythagorean Identity. In this case, we have *two* terms containing squares of trigonometric functions. We will rewrite $\cos^2(A)$ in terms of $\sin^2(A)$ so our equation will only have terms involving the sine function.

$$\begin{aligned}
& & \cos^2(A) - \sin^2(A) - \sin(A) &= 1 & \\
\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry} \quad &\Longrightarrow & 1 - \sin^2(A) - \sin^2(A) - \sin(A) &= 1 & \text{(Pythagorean Identity)} \\
\text{Arithmetic} \to \text{Algebra} \quad &\Longrightarrow & 1 - 2\sin^2(A) - \sin(A) &= 1 & \text{(combine like terms)} \\
\text{Arithmetic} \to \text{Algebra} \quad &\Longrightarrow & -2\sin^2(A) - \sin(A) &= 0 & \text{(subtract 1 from both sides)} \\
\text{Arithmetic} \to \text{Algebra} \quad &\Longrightarrow & 2\sin^2(A) + \sin(A) &= 0 & \text{(multiply both sides by } -1\text{)} \\
\text{Arithmetic} \to \text{Algebra} \quad &\Longrightarrow & \sin(A)[2\sin(A) + 1] &= 0 & \text{(factor out the GCF)}
\end{aligned}$$

Thus, by the Zero Factor Property,

$$\sin(A) = 0 \quad \text{or} \quad \sin(A) = -\frac{1}{2}.$$

For $\sin(A) = 0$
From part (a), we know the solutions on $[0, 2\pi)$ are $x = 0$ and $x = \pi$.

For $\sin(A) = -\frac{1}{2}$
The sine is negative in QIII and QIV, and the reference angle is $\frac{\pi}{6}$. Therefore, the solutions on $[0, 2\pi)$ are $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$.

Thus, the solution set to this equation on the interval $[0, 2\pi)$ is

$$x \in \left\{0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}.$$

Advice: Squares on Trigonometric Functions Are Clues

In Examples **10.4.1*b*** and **10.4.1*c***, we used a Pythagorean Identity to transform the square of a trigonometric function into a different trigonometric function. This should convince you that squares on trigonometric functions are often desirable because they allow us to rewrite an expression in an equivalent form using a different trigonometric function.

? Checkpoint 10.4.1

Solve each equation on the indicated interval.

a. $1 - \sin^2\theta + \cos\theta = 0$ for $0° \le \theta < 360°$
b. $2\tan(\theta)\sin(\theta) + \tan(\theta) = 1 + 2\sin(\theta)$ for $\theta \in [0, 2\pi)$

Answers

a. $90°, 180°, 270°$
b. $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Solving Equations Involving Trigonometric Functions Having Different Angular Frequencies

Suppose an equation contains more than one trigonometric function (whether the same or not) but with different angular frequencies. In that case, we will need to use the Double-Angle Identities, Half-Angle Identities, or, in rare cases, the Sum-to-Product or Product-to-Sum Identities to help simplify the equation into the form $\text{trig}(u) = c$.

✓ Example 10.4.2

Solve the following equations and list the solutions in the interval $[0, 2\pi)$.

a. $\cos(2x) = 3\cos(x) - 2$
b. $\cos(3x) = \cos(5x)$

c. $\sin(2x) = \sqrt{3}\cos(x)$
d. $\sin(x)\cos\left(\frac{\pi}{2}\right) + \cos(x)\sin\left(\frac{\pi}{2}\right) = 1$
e. $\sin\left(\frac{\pi}{2}\right) - \cos(x) = 0$

Solutions

a. In the equation $\cos(2x) = 3\cos(x) - 2$, we have the same circular function, namely cosine, on both sides but the arguments differ. Using the Double-Angle Identity

$$\cos(2x) = 2\cos^2(x) - 1,$$

we obtain a "quadratic in disguise" and proceed as we have done in the past.

$$\begin{aligned}
& & \cos(2x) &= 3\cos(x) - 2 & \\
\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry} \implies & & 2\cos^2(x) - 1 &= 3\cos(x) - 2 & \text{(Double-Angle Identity)} \\
\text{Arithmetic} \to \text{Algebra} \implies & & 2\cos^2(x) - 3\cos(x) + 1 &= 0 & \text{(gathering terms on the left side)} \\
\text{Arithmetic} \to \text{Algebra} \implies & & 2u^2 - 3u + 1 &= 0 & (u - \text{Substitution}) \\
\text{Arithmetic} \to \text{Algebra} \implies & & (2u-1)(u-1) &= 0 & \text{(factor)}
\end{aligned}$$

This gives $u = \frac{1}{2}$ or $u = 1$. Since $u = \cos(x)$, we get $\cos(x) = \frac{1}{2}$ or $\cos(x) = 1$.

For $\cos(x) = \frac{1}{2}$
Cosine is positive in **QI** and **QIV** and the reference angle is $\hat{x} = \frac{\pi}{3}$. Therefore, we have the following:

$$\begin{aligned}
\textbf{Solutions associated with QI:} \quad x &= \tfrac{\pi}{3} + 2\pi k \\
\textbf{Solutions associated with QIV:} \quad x &= \tfrac{5\pi}{3} + 2\pi k,
\end{aligned}$$

where $k \in \mathbb{Z}$.

For $\cos(x) = 1$
This implies x is a quadrantal angle. Quickly sketching the cosine curve,

Figure 10.4.2

we see that $\cos(x) = 1$ when $x = 2\pi k$.

Thus, the answers which lie in $[0, 2\pi)$ are $x = 0$, $x = \frac{\pi}{3}$, and $x = \frac{5\pi}{3}$.

b. The given equation, $\cos(3x) = \cos(5x)$, is troublesome for a few reasons. First, while the equation involves a single trigonometric function (which happens to occur twice), the angular frequency of each function is different. The second issue is that, if we use our previous tactic of moving all trigonometric functions to one side, we get

$$\cos(3x) - \cos(5x) = 0,$$

and there is nothing from Algebra that can help us (e.g., there isn't a common factor). Finally, we cannot easily use the Double-Angle Identities for the cosine because the angular frequency of each function is not even.[2] What do we do?

If you have exhausted all hope of rewriting your equation using algebraic manipulations and fundamental trigonometric identities, then you likely need to reach into your "abstract bag of tricks." Recall that I told you not to memorize the Sum-to-Product and Product-to-Sum

Identities; however, I did state that you needed to know they exist. This is because, every once in a while, you will run into a need for them. Specifically, in this case, we need the following Sum-to-Product Identity:

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

Therefore, our solution process goes like this:

$$\begin{aligned}
& & \cos(3x) &= \cos(5x) & \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & \cos(3x) - \cos(5x) &= 0 & \text{(variables expressions to one side)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & -2\sin\left(\frac{3x+5x}{2}\right)\sin\left(\frac{3x-5x}{2}\right) &= 0 & \text{(Sum-To-Product Identities)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & -2\sin(4x)\sin(-x) &= 0 & \text{(simplify)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & 2\sin(4x)\sin(x) &= 0 & \text{(Symmetry Identities)}
\end{aligned}$$

Hence, the equation $\cos(3x) = \cos(5x)$ is equivalent to $2\sin(4x)\sin(x) = 0$. From this, we get $\sin(4x) = 0$ or $\sin(x) = 0$.

For $\sin(4x) = 0$
From the base graph for the sine,

Figure 10.4.3

we know the sine is zero when its argument is an integer multiple of π. That is,

$$4x = \pi k,$$

where $k \in \mathbb{Z}$. This means

$$x = \frac{\pi}{4}k.$$

On the interval $[0, 2\pi)$, this leads to the solutions $x = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{4}$.

For $\sin(x) = 0$
We know the solutions on $[0, 2\pi)$ are $x = 0$ and π; however, these are already represented in the list of solutions for $\sin(4x) = 0$.

Thus, the solutions to the original equation on the interval $[0, 2\pi)$ are

$$x = 0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4},\ \frac{3\pi}{2},\ \text{and}\ \frac{7\pi}{4}.$$

c. Here is where I shorten the discussion and rely on your ability to follow the logic we have been using up to this point.

$$\begin{aligned}
& & \sin(2x) &= \sqrt{3}\cos(x) & \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & \sin(2x) - \sqrt{3}\cos(x) &= 0 & \text{(move variable terms to one side)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & 2\sin(x)\cos(x) - \sqrt{3}\cos(x) &= 0 & \text{(Double-Angle Identity)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & \cos(x)\left(2\sin(x) - \sqrt{3}\right) &= 0 & \text{(factor out the GCF)}
\end{aligned}$$

From this, we get $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$.

For $\cos(x) = 0$
Since 0 is a quadrantal ratio for the cosine, we quickly sketch the cosine function.

Figure 10.4.4

We see the solutions are $x = \frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$. On this interval $[0, 2\pi)$, this gives the restricted solutions $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

For $\sin(x) = \frac{\sqrt{3}}{2}$
The sine is positive in **QI** and **QII**, and the reference angle for sine with the ratio $\frac{\sqrt{3}}{2}$ is $\hat{x} = \frac{\pi}{3}$. Thus, we get the solutions $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for $k \in \mathbb{Z}$.[3]

The answers which lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

d. There seems to be no quick way to get the trigonometric functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$; however, if we stare at it long enough, we realize that the left side is the expanded form of the Sum of Angles Identity for the sine.

$$\begin{aligned} & & \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) &= 1 & \\ \text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry} \quad & \Longrightarrow & \sin\left(x + \frac{x}{2}\right) &= 1 & \text{(Sum of Angles Identity)} \\ \text{Arithmetic} \to \text{Algebra} \quad & \Longrightarrow & \sin\left(\frac{3}{2}x\right) &= 1 & \text{(combine like terms)} \end{aligned}$$

Since 1 is a quadrantal ratio for the sine, we can quickly sketch $y = \sin(\theta)$

Figure 10.4.5

to see that $\sin(\theta) = 1$ when $\theta = \frac{\pi}{2} + 2\pi k$, where $k \in \mathbb{Z}$. Therefore,

$$\frac{3}{2}x = \frac{\pi}{2} + 2\pi k \implies x = \frac{\pi}{3} + \frac{4\pi}{3}k.$$

Two of these solutions lie in $[0, 2\pi)$: $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

e. As in part (c), we have two different trigonometric functions with different angular frequencies. Moreover, we immediately identify that we are locked from doing any impactful Arithmetic or Algebra; however, in this case, there is a fundamental identity we can use - the Half-Angle Identity for sine.

$$\begin{aligned}
& & \sin\left(\frac{x}{2}\right)-\cos(x) &= 0 & \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & \pm\sqrt{\frac{1-\cos(x)}{2}}-\cos(x) &= 0 & \text{(Half-Angle Identity)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \pm\sqrt{\frac{1-\cos(x)}{2}} &= \cos(x) & \text{(isolating the radical)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \frac{1-\cos(x)}{2} &= \cos^2(x) & \text{(squaring both sides} \\
& & & & \Longrightarrow \textbf{CHECK SOLUTIONS!)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 1-\cos(x) &= 2\cos^2(x) & \text{(multiplying both sides by 2)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 0 &= 2\cos^2(x)+\cos(x)-1 & \text{(recognizing the equation is quadratic-in-form)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 0 &= (2\cos(x)-1)(\cos(x)+1) & \text{(factoring)}
\end{aligned}$$

Hence, either $\cos(x)=\frac{1}{2}$ or $\cos(x)=-1$.

For $\cos(x)=\frac{1}{2}$
The cosine function is positive in **QI** and **QIV**, and the reference angle is $\hat{x}=\frac{\pi}{3}$. Therefore, we have the following:

$$\begin{aligned}
\textbf{Solutions associated with QI:} \quad x &= \tfrac{\pi}{3}+2\pi k \\
\textbf{Solutions associated with QIV:} \quad x &= \tfrac{5\pi}{3}+2\pi k,
\end{aligned}$$

where $k\in\mathbb{Z}$. On the interval $[0,2\pi)$, this equates to $x=\frac{\pi}{3}$ and $x=\frac{5\pi}{3}$; *however*, since we squared both sides of the equation while solving (see the step with "**CHECK SOLUTIONS**"), we need to check to see if these solutions work with the *original* equation.

Letting $x=\frac{\pi}{3}$ in the original equation, we get

$$\begin{aligned}
\sin\left(\frac{\pi/3}{2}\right)-\cos\left(\frac{\pi}{3}\right) &= \sin\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{3}\right) \\
&= \frac{1}{2}-\frac{1}{2} \\
&= 0 \\
&= \textbf{RHS}
\end{aligned}$$

Thus, $x=\frac{\pi}{3}$ is a solution.

Trying $x=\frac{5\pi}{3}$, we get

$$\begin{aligned}
\sin\left(\frac{5\pi/3}{2}\right)-\cos\left(\frac{5\pi}{3}\right) &= \sin\left(\frac{5\pi}{6}\right)-\cos\left(\frac{5\pi}{3}\right) \\
&= \frac{1}{2}-\frac{1}{2} \\
&= 0 \\
&= \textbf{RHS}
\end{aligned}$$

Thus, $x=\frac{5\pi}{3}$ is also a solution.

For $\cos(x)=-1$
From the graph of the cosine,

Figure 10.4.6

we see that the solutions to this equation occur at $x = \pi + 2\pi k$, where $k \in \mathbb{Z}$. On the interval $[0, 2\pi)$, this gives $x = \pi$. Again, we must check this solution.

Letting $x = \pi$ in the original equation, we get

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) - \cos(\pi) &= 1 - (-1) \\ &= 2 \\ &\neq \textbf{RHS} \end{aligned}$$

Thus, $x = \pi$ is an **extraneous solution**, and we throw it out.

In the end, our solutions on the interval $[0, 2\pi)$ to the original equation are $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Instructors will tell you to "check your solutions" after solving an equation. While this is always great advice, let's be realistic - most people don't check their solutions. With that sense of realism in mind, I want to clarify something from Algebra and tell you when you *must* check the validity of your solutions for the risk of them being wrong. There are three general cases when you *must* check solutions after solving an equation:

1. when the equation contains a function with a restricted domain,
2. when you multiply both sides of an equation by a variable expression, and
3. when you apply an even power (e.g., a square) to both sides of an equation.

When solving trigonometric equations, you can run into any of these issues. In Example **10.4.2e**, we ended up squaring both sides of the equation in the process of finding the solution (see the step with "**CHECK SOLUTIONS!**"). When doing so, you risk introducing "false" solutions - called **extraneous solutions**. Indeed, that is precisely what happened!

? Checkpoint 10.4.2

Find the solutions to each equation on requested the interval.

a. $\tan(\theta) + \cot(\theta) = 4\sin(2\theta)$ on $[0, 2\pi)$
b. $\tan\left(\frac{\theta}{2}\right) - \sin(\theta) = 0$ for $x \in [0, 2\pi)$

Answers

a. $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$
b. $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$

Solving Other Trigonometric Equations

At any given step while solving an equation, the tactic I have continually modeled throughout this material is to do the Arithmetic first (i.e., performing simple operations on constants). Once this is done, you move on to see if there is any Algebra to be done (e.g., combining like terms, factoring, etc.). Once you have exhausted all possibilities to use Arithmetic and Algebra, then (and only then) you move on to looking for identities or concepts from Trigonometry.

But what happens when you exhaust all three and you're stuck?

In truth, this means one of two things - either the equation you were given involves a combination of trigonometric functions and non-trigonometric functions (in which case you have no choice but to use technology), or you missed some not-so-obvious Algebra or Trigonometry (Example **10.4.2*b***, which required the abstract Sum-to-Product Identities, is an excellent example of this). There is no way to cover all possible "oddball" cases. Still, the following example covers a case that comes up often enough that we should investigate it.

Example 10.4.3

Solve the equation and list the solutions that lie in the interval $[0, 2\pi)$.

$$\cos(x) = \sin(x)$$

Solutions

The angular frequencies are the same, but the circular functions are different. This has been fine up to this point because all the previous examples involving two different circular functions had at least one of these functions being squared. Squares of trigonometric functions are blessings in disguise, as we can always try to change them using the Pythagorean Identities. In this case, there are no squares. So what do we do? Since we would like squares, why don't we square both sides?[4]

$$\begin{aligned}
& & \cos(x) &= \sin(x) & & \\
\text{Arithmetic} \to \text{Algebra} \quad & \Longrightarrow & \cos^2(x) &= \sin^2(x) & & \text{(square both sides} \\
& & & & & \textbf{CHECK SOLUTIONS!)} \\
\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry} \quad & \Longrightarrow & 1 - \sin^2(x) &= \sin^2(x) & & \text{(Pythagorean Identity)} \\
\text{Arithmetic} \to \text{Algebra} \quad & \Longrightarrow & 1 &= 2\sin^2(x) & & \text{(move all variable terms to one side)} \\
\text{Arithmetic} \to \text{Algebra} \quad & \Longrightarrow & \frac{1}{2} &= \sin^2(x) & & \text{(divide both sides by 2)} \\
\text{Arithmetic} \to \text{Algebra} \quad & \Longrightarrow & \pm\frac{1}{\sqrt{2}} &= \sin(x) & & \text{(Extraction of Roots)}
\end{aligned}$$

Therefore, $\hat{x} = \frac{\pi}{4}$ and we need to check *all* quadrants.

$$\begin{aligned}
\textbf{Solutions associated with QI:} \quad & x = \hat{x} + 2\pi k = \frac{\pi}{4} + 2\pi k \\
\textbf{Solutions associated with QII:} \quad & x = \pi - \hat{x} + 2\pi k = \pi - \frac{\pi}{4} + 2\pi k = \frac{3\pi}{4} + 2\pi k \\
\textbf{Solutions associated with QIII:} \quad & x = \pi + \hat{x} + 2\pi k = \pi + \frac{\pi}{4} + 2\pi k = \frac{5\pi}{4} + 2\pi k \\
\textbf{Solutions associated with QIV:} \quad & x = 2\pi - \hat{x} + 2\pi k = 2\pi - \frac{\pi}{4} + 2\pi k = \frac{7\pi}{4} + 2\pi k
\end{aligned}$$

where $k \in \mathbb{Z}$. This is a case where, rather than compressing these solutions into two families (which we *could* do), it's easier to check the four possibilities separately. In all four cases, the reference angles are $\hat{x} = \frac{\pi}{4}$. Therefore, the sine and cosine functions become $\pm\frac{1}{\sqrt{2}}$. We just require the signs to be the same.

In **QI**, $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$. In **QIII**, $\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} = \sin\left(\frac{5\pi}{4}\right)$; however, we can throw out the other two possibilities because the signs of the functions are opposite in both **QII** and **QIV**. Thus, the solution set on the interval $[0, 2\pi)$ is $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Another option with Example **10.4.3** is to start by dividing both sides by $\cos(x)$. Doing so, we get

$$1 = \tan(x).$$

This is *much* easier to solve, however, by dividing by the cosine, we have introduced a possible issue - what if $\cos(x) = 0$?

A little thought goes a long way here. If $\cos(x) = 0$, then $x = \frac{\pi}{2} + \pi k$. If we go back to the original equation and plug in $\frac{\pi}{2}$, we would get

$$\begin{aligned}
\sin\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) \\
\Longrightarrow \qquad 1 &= 0
\end{aligned}$$

Since this is obviously not true, $x = \frac{\pi}{2}$ is not a solution to the equation. It is a simple task to verify that $x = \frac{3\pi}{2}$ is also not a solution. Therefore, we don't have to worry about $\cos(x) = 0$, and we can go back to solving

$$\tan(x) = 1.$$

Tangent is positive in **QI** and **QIII**, and $\hat{x} = \frac{\pi}{4}$. Hence, the solutions are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Combining Waves Having Equal Angular Frequencies

Recall that a sinusoidal function of the form $A\sin(Bx+\phi)$ can be rewritten using the Sum of Angles Identity. That is,

$$\begin{aligned} A\sin(Bx+\phi) &= A[\sin(Bx)\cos(\phi)+\cos(Bx)\sin(\phi)] && \text{(Sum of Angles Identity)} \\ &= A[\cos(\phi)\sin(Bx)+\sin(\phi)\cos(Bx)] && \text{(rearrange terms)} \\ &= A\cos(\phi)\sin(Bx)+A\sin(\phi)\cos(Bx) && \text{(distribute)} \\ &= m\sin(Bx)+n\cos(Bx) && \text{(substitute } m = A\cos(\phi) \text{ and } n = A\sin(\phi)) \end{aligned} \tag{10.4.1}$$

Notice that the result of 10.4.1, $m\sin(Bx)+n\cos(Bx)$, is a vertical stretch (or compression) of the sine added to a different vertical stretch (or compression) of the cosine, but both have the same angular frequency.

Looking at the work in 10.4.1, but in reverse order, we reach a fascinating conclusion - if you have a function that is the sum (or difference) of a sine and cosine function having the same angular frequencies, the result is a single sine function with the same angular frequency, but having a phase shift. This is true even if the original sine and cosine functions have different amplitudes.

The question that we should have is if given a function of the form

$$f(x) = m\sin(Bx)+n\cos(Bx),$$

how do we find A and ϕ to write this as

$$f(x) = A\sin(Bx+\phi)?$$

The clue to answering this question is in the last line of 10.4.1. In that step, we let $m = A\cos(Bx)$ and $n = A\sin(Bx)$. Therefore, we can find the value of A by using the following ingenious move.

$$\begin{aligned} m^2+n^2 &= A^2\cos^2(Bx)+n^2\sin^2(Bx) && \text{(substitute)} \\ &= A^2[\cos^2(Bx)+\sin^2(Bx)] && \text{(factor)} \\ &= A^2\cdot[1] && \text{(Pythagorean Identity)} \\ &= A^2 \end{aligned}$$

Thus, we will choose

$$A = \sqrt{m^2+n^2}.$$

Note that we will deliberately forgo the possibility of A being negative.

Now that we have a value for A, we still need to find the value of ϕ so that we can properly convert $f(x) = m\sin(Bx)+n\cos(Bx)$ into $f(x) = A\sin(Bx+\phi)$. Again, the clue to finding ϕ is in 10.4.1. In the last line, we made the substitutions

$$m = A\cos(\phi) \text{ and } n = A\sin(\phi).$$

This means

$$\cos(\phi) = \frac{m}{A} \text{ and } \sin(\phi) = \frac{n}{A}.$$

Since $A = \sqrt{m^2+n^2}$ is positive, we can determine the quadrant in which ϕ terminates using the signs of m and n and how those affect the quadrant choices for $\cos(\phi)$ and $\sin(\phi)$, respectively. Once we have the proper quadrant for ϕ, we find the reference angle, $\widehat{\phi}$, using an inverse trigonometric function (arcsine, arccosine, or arctangent all work fine).

The following theorem summarizes these results.

Theorem: Same-Frequency Sinusoid Summation

$$m\sin(Bx)+n\cos(Bx) = A\sin(Bx+\phi),$$

where

$$A = \sqrt{m^2+n^2}$$

and

$$\cos(\phi) = \frac{m}{A} \text{ and } \sin(\phi) = \frac{n}{A}.$$

It's time we showcase the method using a concrete example.

Example 10.4.4

Rewrite $4\sqrt{3}\sin(2x) - 4\cos(2x)$ as a single sinusoidal function.

Solution

Let $m = 4\sqrt{3}$ and $n = -4$. Therefore,

$$A = \sqrt{\left(4\sqrt{3}\right)^2 + (-4)^2} = \sqrt{16 \cdot 3 + 16} = \sqrt{64} = 8.$$

We now turn to finding ϕ.We know

$$\cos(\phi) = \frac{m}{A} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(\phi) = \frac{n}{A} = \frac{-4}{8} = -\frac{1}{2}.$$

Cosine is positive in **QI** and **QIV**, and sine is negative in **QIII** and **QIV**. The intersection of these choices is **QIV**. Thus, $\phi \in$ **QIV**.

We can use either relation to find the reference angle, but I will use the cosine. The reference angle for cosine that returns $\frac{\sqrt{3}}{2}$ is $\widehat{\phi} = \frac{\pi}{6}$. Therefore, $\phi = \frac{11\pi}{6}$.

Combining these results gives us the expression

$$8\sin\left(2x + \frac{11\pi}{6}\right).$$

? Checkpoint 10.4.4

Rewrite $-3\sqrt{2}\sin(5x) + 3\sqrt{2}\cos(5x)$ as a single sinusoidal function.

Answer

$$6\sin\left(5x + \frac{3\pi}{4}\right)$$

Rewriting a combination of sine and cosine of equal periods as a single sinusoidal function provides an alternative approach for solving some equations.

Example 10.4.5

Find all solutions to the equation $3\sin(2x) + 4\cos(2x) = 1$ on the interval $[0, 2\pi)$.

Solution

There are two ways to approach this problem: recognizing that you could isolate one of the trigonometric functions and square both sides or using what we just learned (combining sine and cosine into a single sinusoidal function). I encourage you to try the first method on your own (it slightly mimics Example **10.4.3**, but it is more challenging). For now, we will pretend that we cannot see the possibility of using Algebra in the initial equation.

Not having any Arithmetic or Algebra to do, we then recall what we just learned. We let

$$A = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5,$$

where

$$\cos(\phi) = \frac{3}{5} \quad \text{and} \quad \sin(\phi) = \frac{4}{5}.$$

The sine and cosine are both positive in **QI**. Therefore, $\phi \in$ **QI**. Finally,

$$\widehat{\phi} = \cos^{-1}\left(\left|\frac{3}{5}\right|\right) = \cos^{-1}\left(\frac{3}{5}\right).$$

Hence,

$$\phi = \cos^{-1}\left(\frac{3}{5}\right).$$

We now can rewrite the left side of the original equation as $A\sin(2x + \phi)$ and get

$$5\sin\left(2x+\cos^{-1}\left(\frac{3}{5}\right)\right)=1.$$

This is a much easier equation to solve.

$$\begin{aligned} 5\sin\left(2x+\cos^{-1}\left(\frac{3}{5}\right)\right) &= 1 \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \sin\left(2x+\cos^{-1}\left(\frac{3}{5}\right)\right) &= \frac{1}{5} \qquad \text{(divide both sides by 5)} \end{aligned}$$

Sine is positive in QI and QII and the reference angle is $\hat{\theta}=\sin^{-1}\left(\frac{1}{5}\right)$, where $\theta=2x+\cos^{-1}\left(\frac{3}{5}\right)$. Therefore, we get:

$$\text{Solutions associated with QI:} \quad 2x+\cos^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{1}{5}\right)+2\pi k$$

$$\text{Solutions associated with QII:} \quad 2x+\cos^{-1}\left(\frac{3}{5}\right)=\pi-\sin^{-1}\left(\frac{1}{5}\right)+2\pi k$$

for $k\in\mathbb{Z}$. Solving each equation for x, we get:

$$\text{Solutions associated with QI:} \quad x=\frac{1}{2}\sin^{-1}\left(\frac{1}{5}\right)+\pi k-\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\approx -0.362968649+\pi k$$

$$\text{Solutions associated with QII:} \quad 2x+\cos^{-1}\left(\frac{3}{5}\right)=\frac{\pi}{2}-\frac{1}{2}\sin^{-1}\left(\frac{1}{5}\right)+\pi k-\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\approx 1.006469757+\pi k$$

We only want solutions on $[0,2\pi)$, so we build a quick table.

k	$-0.362968649+\pi k$	$1.006469757+\pi k$
0	−0.362968649	1.006469757
1	2.778624005	4.148062411
2	5.920216658	7.289655065
3	9.061809312	⋮

Of these results, the solutions on the requested interval are $x\approx 1.0065$, $x\approx 2.7786$, $x\approx 4.1481$, and $x\approx 5.9202$.

? Checkpoint 10.4.5

Find all solutions of the given equations on the specified intervals.

a. $\cos(2t)-\sqrt{3}\sin(2t)=1$ for $t\in[0,2\pi)$
b. $\frac{1}{2}\sin(\theta)+\frac{\sqrt{3}}{2}\cos(\theta)=1$ for $0\le\theta<2\pi$
c. $\cos(\theta)+1=\sin(\theta)$ on $[0,2\pi)$

Answers

a. $t=0$, $t=\frac{2\pi}{3}$, $t=\pi$, and $t=\frac{5\pi}{3}$
b. $\theta=\frac{\pi}{6}$
c. $\theta=\frac{\pi}{2}$ and $\theta=\pi$

Footnotes

[1] Recall from Algebra that when you divide both sides of an equation by a variable expression, you are assuming the expression you are dividing by is nonzero. Therefore, in the given equation, unless you can guarantee that $\sin^2(x)\neq 0$, you cannot divide both sides of the equation by $\sin^2(x)$.

[2] Technically, we can use the Double-Angle Identities, however, they would not be very helpful. This is because, for example,

$$\cos(3x)=\cos\left(2\left(\frac{3}{2}x\right)\right)=2\cos^2\left(\frac{3}{2}x\right)-1,$$

and this looks *much* worse than the original expression.

[3] Notice that I am speeding up descriptions as we move through. That is, the amount of detail I describe once a topic is introduced becomes less and less. This is purposeful. You need to be fluent in this material so it becomes second nature.

[4] Remember, when you square both sides of an equation, you run the risk of introducing extraneous solutions. Therefore, you must *always* check your solutions when you perform this action.

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10.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Solving Trigonometric Equations
  - Solving Trigonometric Equations - Equations Requiring the Use of Identities
  - Solving Trigonometric Equations - Multiple-Angle Equations
  - Solving Trigonometric Equations - Equations Requiring the Use of the Sum to Product Formulas
  - Solving Trigonometric Equations - Equations Requiring Technology and a Warning

KEY CONCEPTS

THEOREMS

- **Same-Frequency Sinusoid Summation:** An expression of the form $m\sin(Bx) + n\cos(Bx)$ can be rewritten as a single sinusoidal function $A\sin(Bx + \phi)$, where $A = \sqrt{m^2 + n^2}$, $\cos(\phi) = \frac{m}{A}$, and $\sin(\phi) = \frac{n}{A}$.

COMMON MISTAKES

- **Dividing by a Variable Expression:** Do not divide both sides of an equation by an expression that could be equal to zero, such as $\sin^2(x)$. This can lead to losing solutions. Instead, move all terms to one side and factor.
- **Forgetting to Check for Extraneous Solutions:** When you square both sides of an equation, you risk introducing extraneous solutions. You must check all potential solutions in the original equation to ensure they are valid.
- **Ignoring Simpler Paths:** It is easy to get focused on one solution method. Always look for the simplest approach. For example, after finding $\sin(2\theta)$ and $\cos(2\theta)$, it's often easier to find $\tan(2\theta)$ using the Ratio Identity rather than a more complex identity.

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10.4.2: HOMEWORK

READING QUESTIONS

1. What is the ultimate goal when solving a trigonometric equation, according to the text?
2. When an equation contains a squared trigonometric function, what type of identity is often useful for simplifying it?
3. When solving an equation like $3\cos(x)\sin^2(x) = \sin^2(x)$, what is the recommended first step instead of dividing by $\sin^2(x)$?
4. What algebraic principle allows you to solve a factored trigonometric equation like $\sin(A)[2\sin(A)+1]=0$?
5. If an equation contains trigonometric functions with different arguments, like $\cos(2x)$ and $\cos(x)$, what is the general strategy?
6. If an equation contains a sum or difference of trigonometric functions that cannot be easily factored, like $\cos(3x) - \cos(5x) = 0$, what type of identity might be required?
7. Under what circumstances must you check your solutions for extraneous solutions?
8. An expression of the form $m\sin(Bx) + n\cos(Bx)$ can be rewritten as a single sinusoidal function. What is the amplitude of this new function?

HOMEWORK

CONCEPT CHECK

1. Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.
2. When solving a trigonometric equation involving more than one trigonometric function, do we always want to try to rewrite it so it is expressed in one trigonometric function? Why or why not?

BASIC SKILLS

For the following exercises, solve the equation, giving exact solutions on the interval $[0, 2\pi)$.

3. $\sin(2\theta) + \sqrt{2}\cos(\theta) = 0$
4. $\sin(2\alpha)\sin(\alpha) = \cos(\alpha)$
5. $\cos(2t) - 5\cos(t) + 3 = 0$
6. $\cos(2x) + 3\sin(x) = 2$
7. $\tan(2\beta) + 2\sin(\beta) = 0$
8. $\tan(2z) - 2\cos(z) = 0$
9. $3\cos(\phi) - \sin\left(\frac{\pi}{2} - \phi\right) = \sqrt{3}$
10. $\sin(w) + \cos\left(\frac{\pi}{2} - w\right) = 1$
11. $\sin(2\phi)\cos(\phi) + \cos(2\phi)\sin(\phi) = 1$
12. $\cos(\theta)\cos(3\theta) + \sin(\theta)\sin(3\theta) = \frac{\sqrt{2}}{2}$
13. $\sin(x) = \cos(x)$
14. $\sin(2x) = \sin(x)$
15. $\sin(2x) = \cos(x)$
16. $2\tan^2(t) = 3\sec(t)$
17. $\cos(2x) = \sin(x)$
18. $\cos(2x) = \cos(x)$
19. $\cos(2x) = 2 - 5\cos(x)$
20. $3\cos(2x) + \cos(x) + 2 = 0$
21. $\cos(2x) = 5\sin(x) - 2$
22. $3\cos(2x) = \sin(x) + 2$
23. $2\sec^2(x) = 3 - \tan(x)$
24. $\tan^2(x) = 1 - \sec(x)$
25. $\cot^2(x) = 3\csc(x) - 3$
26. $\sec(x) = 2\csc(x)$
27. $\cos(x)\csc(x)\cot(x) = 6 - \cot^2(x)$
28. $\sin(2x) = \tan(x)$
29. $\cot^4(x) = 4\csc^2(x) - 7$
30. $\cos(2x) + \csc^2(x) = 0$
31. $\tan^3(x) = 3\tan(x)$
32. $\cos(6x) - \cos(3x) = 0$
33. $\sin^2(x) - \cos^2(x) - \sin(x) = 0$
34. $\tan^2(x) = \frac{3}{2}\sec(x)$
35. $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$
36. $\cos^3(x) = -\cos(x)$
37. $\tan(2x) - 2\cos(x) = 0$
38. $\csc^3(x) + \csc^2(x) = 4\csc(x) + 4$
39. $2\tan(x) = -1 - \tan^2(x)$
40. $\tan(x) = \sec(x)$
41. $\sin(6x)\cos(x) = -\cos(6x)\sin(x)$
42. $\sin(3x)\cos(x) = \cos(3x)\sin(x)$
43. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$
44. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \frac{\sqrt{3}}{2}$
45. $\sin(x) + \cos(x) = 1$
46. $\sin(x) + \sqrt{3}\cos(x) = 1$
47. $\sqrt{2}\cos(x) - \sqrt{2}\sin(x) = 1$
48. $\sqrt{3}\sin(2x) + \cos(2x) = 1$
49. $\cos(2x) - \sqrt{3}\sin(2x) = \sqrt{2}$
50. $9\cos(2\theta) = 9\cos^2(\theta) - 4$
51. $3\sqrt{3}\sin(3x) - 3\cos(3x) = 3\sqrt{3}$
52. $\cos(3x) = \cos(5x)$
53. $\cos(4x) = \cos(2x)$
54. $\sin(5x) = \sin(3x)$
55. $\cos(5x) = -\cos(2x)$
56. $\sin(6x) + \sin(x) = 0$
57. $\tan(x) = \cos(x)$

For the following exercises, solve the equation for $0° \le \theta \le 360°$. Round angles to three decimal places if necessary.

58. $\cos(\theta) - \sin^2(\theta) + 1 = 0$
59. $4\sin(\theta) + 2\cos^2(\theta) - 3 = -1$
60. $1 - \sin(\theta) - 2\cos^2(\theta) = 0$

61. $3\cos^2(\theta) - \sin^2(\theta) = 2$

62. $2\cos(\theta)\tan(\theta) + 1 = 0$

63. $\cos(\theta) - \sin(\theta) = 0$

64. $\frac{1}{3}\cos(\theta) = \sin(\theta)$

65. $5\sin(C) = 2\cos(C)$

For the following exercises, find all radian solutions to the equation.

66. $\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

67. $\sin(6x)\cos(11x) - \cos(6x)\sin(11x) = -0.1$

68. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

69. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \frac{\sqrt{3}}{2}$

70. $6\sin(2t) + 9\sin(t) = 0$

For the following exercises, find all radian solutions to the equation.

71. $\cos(5x) = -\cos(2x)$

72. $\sin(5x) = \sin(3x)$

73. $\cos(6\theta) - \cos(2\theta) = \sin(4\theta)$

74. $\cos(8\theta) - \cos(2\theta) = \sin(5\theta)$

For the following exercises, rewrite the expression as a single function of the form $A\sin(Bx + \phi)$.

75. $4\sin(x) - 6\cos(x)$

76. $-\sin(x) - 5\cos(x)$

77. $5\sin(3x) + 2\cos(3x)$

78. $-3\sin(5x) + 4\cos(5x)$

For the following exercises, find the first two positive (radian) solutions.

79. $-5\sin(x) + 3\cos(x) = 1$

80. $3\sin(x) + \cos(x) = 2$

81. $3\sin(2x) - 5\cos(2x) = 3$

82. $-3\sin(4x) - 2\cos(4x) = 1$

For the following exercises, solve the equation for $0° \le x < 360°$.

83. $9\sin^2(\theta) - 6\sin(\theta) = 1$

84. $4\cos^2(\theta) + 4\cos(\theta) = 1$

85. $\sec^2(\alpha) - 2\sec(\alpha) - 3 = 0$

86. $\csc^2(\beta) + 4\csc(\beta) - 10 = 0$

87. $\csc^2(x) + 4\csc(x) - 7 = 0$

88. $3\cot^2(x) - 3\cot(x) - 1 = 0$

89. $2\sin^2(x) = 1 - \cos(x)$

90. $\cos^2(\alpha) + 4 = 2\sin(\alpha) - 3$

91. $\cos^2(\beta) - 3\sin(\beta) + 2\sin^2(\beta) = 0$

92. $\sin^2(\theta) = 2\cos(\theta) + 3\cos^2(\theta)$

93. $\sec^2(x) = 2\tan(x) + 4$

94. $3\tan^2(x) = \sec(x) + 2$

95. $\cos(\alpha) + 1 = 2\cos(2\alpha)$

96. $\cos(2x) - 3\sin(x) - 2 = 0$

97. $\csc^2(\theta) = \cot(\theta) + 5$

98. $\csc(\theta) + 5 = 2\cot^2(\theta) + 2$

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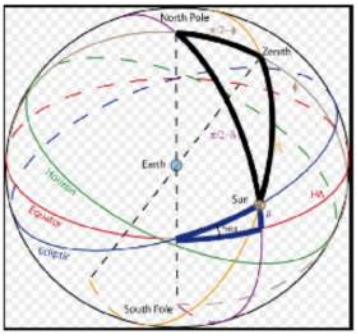
CHAPTER OVERVIEW

Chapter 11: Non-Right Triangle Trigonometry

Figure 1



The first science developed by humans is probably Astronomy. Before the invention of clocks and calendars, early people looked to the night sky to help them keep track of time. What is the best time to plant crops, and when will they ripen? On what day exactly do important religious festivals fall?

By tracking the motions of the stars, early astronomers could identify the summer and winter solstices and the equinoxes. The rising and setting of certain stars marked the hours of the night.

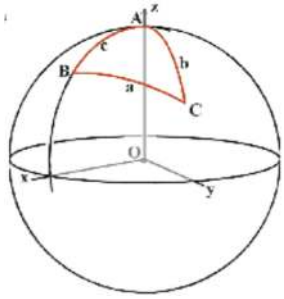
If we think of the stars as traveling on a dome above the Earth, we create the **celestial sphere**. Actually, of course, the Earth itself rotates among the stars, but for calculating the motions of heavenly objects, this model works very well.

Figure 2

Babylonian astronomers kept detailed records on the motion of the planets, and were able to predict solar and lunar eclipses. All of this required familiarity with angular distances measured on the celestial sphere.

To find angles and distances on this imaginary sphere, astronomers invented techniques that are now part of **spherical trigonometry**. The laws of sines and cosines were first stated in this context, in a slightly different form than the laws for plane trigonometry.

Figure 3



On a sphere, a **great-circle** lies in a plane passing through the sphere's center. It gives the shortest distance between any two points on a sphere, and is the analogue of a straight line on a plane. A spherical angle is formed where two such arcs intersect, and a spherical triangle is made up of three arcs of great circles.

The spherical law of sines was first introduced in Europe in 1464 by Johann Muller, also known as Regiomontus, who wrote:

> *"You, who wish to study great and wondrous things, who wonder about the movement of the stars, must read these theorems about triangles. ... For no one can bypass the science of triangles and reach a satisfying knowledge of the stars."*

11.1: The Law of Sines
- 11.1.1: Resources and Key Concepts
- 11.1.2: Homework

11.2: The Law of Cosines
- 11.2.1: Resources and Key Concepts
- 11.2.2: Homework

11.3: Areas
- 11.3.1: Resources and Key Concepts
- 11.3.2: Homework

11.4: Vectors - A Geometric Approach
- 11.4.1: Resources and Key Concepts
- 11.4.2: Homework

11.5: Vectors - An Algebraic Approach
- 11.5.1: Resources and Key Concepts
- 11.5.2: Homework

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11.1: The Law of Sines

Note to the Instructor - All of the Law of Sines

This is a very thorough section of all cases for the Law of Sines and their applications. Please pay close attention to my tabular method.

Learning Objectives

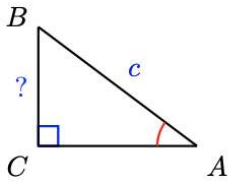
- Use the Law of Sines to find a missing side in an oblique triangle.
- Solve an application using the Law of Sines.
- Use the Law of Sines to find all possible solutions for the ambiguous case.
- Solve an application involving the ambiguous case.

Hawk A.I. Section-Specific Tutor

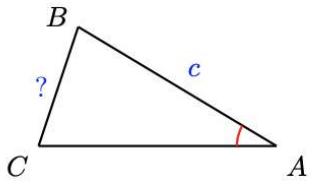
Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

We have learned to use trigonometric ratios to solve right triangles. However, the ratios are only valid for the sides of right triangles. Can we find unknown sides or angles in an oblique triangle?

Figure 11.1.1



We can find a if we know A and c (and $C = 90°$).

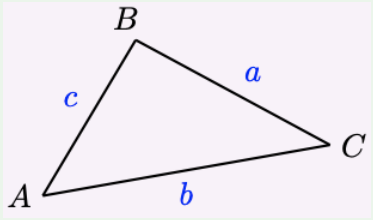


Can find a if we know A and c and C?

In this section and the next, we find relationships between the sides and angles of oblique triangles. These relationships are called the Law of Sines and the Law of Cosines. We use what we already know about right triangles to derive these new rules.

The Law of Sines

Theorem: Law of Sines



If the angles of a triangle are A, B, and C, and the opposite sides are a, b, and c, respectively, then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},$$

or equivalently,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

Proof

Consider the oblique triangle in the figure below.

By drawing in the altitude h of the triangle, we create two right triangles, $\triangle BCD$ and $\triangle ABD$. We can write expressions in terms of h for $\sin(A)$ and for $\sin(C)$.

Looking at $\triangle BCD$, we see that

$$\frac{h}{a} = \sin(C).$$

Looking at $\triangle ABD$, we see that

$$\frac{h}{c} = \sin(A).$$

Now we solve each of these equations for h:

$$\frac{h}{a} = \sin(C) \implies h = a\sin(C)$$

$$\text{and} \quad \frac{h}{c} = \sin(A) \implies h = c\sin(A)$$

Therefore,

$$a\sin(C) = c\sin(A)$$

$$\implies \frac{\sin(C)}{c} = \frac{\sin(A)}{a} \qquad \text{(divide both sides by } ac\text{)}$$

We have derived a relationship between the angles A and C and their opposite sides, a and c. If we know any three of these quantities, we can find the fourth.

In a similar way, by drawing in the altitude from the vertex C, we can show that

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}.$$

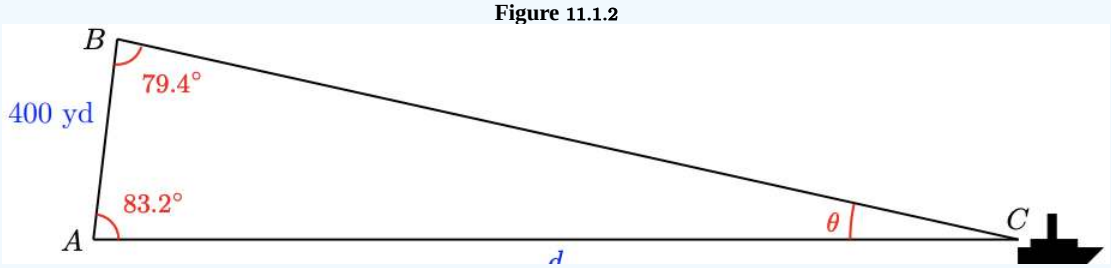
Putting both results together, we have proved the Law of Sines.

The Law of Sines is true for any triangle, whether acute, right, or obtuse. In the following example, we use the Law of Sines to find a distance.

Example 11.1.1

Two observers on the shore sight a ship at an unknown distance from the shoreline. The observers are 400 yards apart at points A and B, and they each measure the angle from the shoreline to the ship, as shown below. How far is the ship from the observer at A?

Figure 11.1.2



Solution

First, note that $\triangle ABC$ is not a right triangle, so we cannot use the trigonometric ratios directly to find the sides of this triangle.

Let d be the unknown distance opposite $\angle B = 79.4°$. To use the Law of Sines, we must know another angle and the side opposite that angle. Luckily, we know that side $c = 400$, and we can compute the angle at the ship, $\angle C$.

$$\angle C = 180° - (79.4° + 83.2°) = 17.4°.$$

Now we apply the Law of Sines, using angles B and C.

$$\begin{aligned} & \frac{b}{\sin(B)} = \frac{c}{\sin(C)} && \text{(Law of Sines)} \\ \Longrightarrow\ & \frac{d}{\sin(79.4°)} = \frac{400}{\sin(17.4°)} && \text{(substitute)} \\ \Longrightarrow\ & d = \frac{400\sin(79.4°)}{\sin(17.4°)} && \text{(multiply both sides by } \sin(79.4°)\text{)} \\ \Longrightarrow\ & d \approx 1315 \end{aligned}$$

The ship is about 1315 yards from the observer at A.

In Example 11.1.1, we saved all calculator work for the last step. That is, we did not approximate the values of $\sin(79.4°)$ and $\sin(17.4°)$, and then use those approximations to get a further approximation - doing so would propagate errors.

Advice: Save Calculator Work for the Last Step

As much as possible, only reach for technology in any mathematical work at the very last moment.

Another piece of advice is to collect information into tables when working through an application. This habit is especially helpful with the current material (and the material in the next section).

In Example 11.1.1, we could have collected the information into a table where the columns are the angles and their opposite side lengths. When missing information, we place a temporary question mark. Therefore, we would have started with the table

Angle	Opposite Side
$\angle A = 83.2°$	$a = ?$
$\angle B = 79.4°$	$b = ?$
$\angle C = ?$	$c = 400$

Looking at that first column, we realize these are the interior angles of a triangle and, as such, should sum to $180°$. Hence,

$$\angle C = 180° - 83.2° - 79.4° = 17.4°.$$

Thus, our table becomes

Angle	Opposite Side
$\angle A = 83.2°$	$a = ?$
$\angle B = 79.4°$	$b = ?$
$\angle C = 17.4°$	$c = 400\checkmark$

That checkmark (✓) is not an accident. Recall, the Law of Sines tells us

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

If we have the sine of an angle and the length of its opposite side, then we know the value of these ratios.

That is, once we have a complete row in our table, we can try to use the Law of Sines.[1] Since our goal is to find the side length opposite $\angle B$ (which, for the sake of building the table, I call b), we work with the equation

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

We substitute the values from the table to get

$$\frac{b}{\sin(79.4°)} = \frac{400}{\sin(17.4°)}$$

and finish as we did in Example **11.1.1**.

? Checkpoint 11.1.1

Delbert and Francine are 40 feet apart on one side of a river. They make angle measurements to a pine tree on the opposite shore, as shown below. What is the distance from Francine to the pine tree?

Figure 11.1.3

Answer

About 114.8 feet

Solving Triangles with the Law of Sines

To apply the Law of Sines to find a side, we must know one angle of the triangle and its opposite side (either a and A, or b and B, or c and C), and one other angle. Then, we can find the side opposite that angle.

✓ Example 11.1.2

In the triangle shown below, $A = 37°$, $B = 54°$, and $a = 11$.

a. Find b.
b. Solve the triangle.

Figure 11.1.4

Solution

We begin by building our table of given information (using question marks where we aren't explicitly given information).

Angle	Opposite Side
$A = 37°$	$a = 11\checkmark$
$B = 54°$	$b = ?$
$C = ?$	$c = ?$

We already have a complete row! This means we can immediately try using the Law of Sines; however, I cannot resist finding the value of C when it is so easy to compute.

$$C = 180° - 54° - 37° = 89°.$$

Hence, our table becomes

Angle	Opposite Side
$A = 37°$	$a = 11$✓
$B = 54°$	$b =$?
$C = 89°$	$c =$?

a. We use the Law of Sines with a and angle A to find b.

$$\begin{aligned} & \frac{a}{\sin(A)} = \frac{b}{\sin(B)} && \text{(Law of Sines)} \\ \Longrightarrow \quad & \frac{11}{\sin(37°)} = \frac{b}{\sin(54°)} && \text{(substitute)} \\ \Longrightarrow \quad & \frac{11\sin(54°)}{\sin(37°)} = b && \text{(multiply both sides by } \sin(54°)\text{)} \end{aligned}$$

Evaluating this expression with a calculator, we find that $b \approx 14.79$. We fill this in our table for display purposes only (we will not use the rounded value for computing).

Angle	Opposite Side
$A = 37°$	$a = 11$✓
$B = 54°$	$b \approx 14.79$
$C = 89°$	$c =$?

b. At this point, we are very close to having the triangle solved. We only need the length of side c. Using the given side, a, is safer than using the value we calculated for b.

$$\begin{aligned} & \frac{a}{\sin(A)} = \frac{c}{\sin(C)} && \text{(Law of Sines)} \\ \Longrightarrow \quad & \frac{11}{\sin(37°)} = \frac{c}{\sin(89°)} && \text{(substitute)} \\ \Longrightarrow \quad & \frac{11\sin(89°)}{\sin(37°)} = c && \text{(multiply both sides by } \sin(89°)\text{)} \end{aligned}$$

So $c \approx 18.28$. Now our table is complete, and we have solved the triangle.

Angle	Opposite Side
$A = 37°$	$a = 11$✓
$B = 54°$	$b \approx 14.79$
$C = 89°$	$c \approx 18.28$

? Checkpoint 11.1.2

In the triangle below, $A = 65°$, $C = 42°$, and $c = 16$. Solve the triangle.

Figure 11.1.5

Answer

$B = 73°, b \approx 22.87, a \approx 21.68$

Finding an Angle

We can also use the Law of Sines to find an unknown angle of a triangle. We must know two sides of the triangle and the angle opposite one of them.

Example 11.1.3

In the triangle below, $B = 55°$, $a = 5$, and $b = 11$. Solve the triangle.

Figure 11.1.6

Solution

Again, let's start by filling out a table.

Angle	Opposite Side
$A = ?$	$a = 5$
$B = 55°$	$b = 11 \checkmark$
$C = ?$	$c = ?$

This is the first time where we cannot simply find the remaining angles using the fact that the sum of angles in a triangle is $180°$. Luckily, we still have a complete row so we can use the Law of Sines, and we will do so to find $\sin(A)$.

$$\begin{aligned} & \frac{\sin(A)}{a} = \frac{\sin(B)}{b} && \text{(Law of Sines)} \\ \Longrightarrow\ & \frac{\sin(A)}{5} = \frac{\sin(55°)}{11} && \text{(substitute)} \\ \Longrightarrow\ & \sin(A) = \frac{5\sin(55°)}{11} && \text{(multiply both sides by 5)} \\ \Longrightarrow\ & \widehat{A} = \sin^{-1}\left(\frac{5\sin(55°)}{11}\right) && \text{(finding the reference angle -} \\ & && \text{see } \textbf{Unpacking Law of Sines Subtleties}\text{)} \end{aligned}$$

We have a *reference* angle, and we know that each angle in a triangle must be positive but less than $180°$. Therefore, A is either in QI or QII.[2] Here's where a little extra logic comes into play. Since the length of side b is greater than the length of side a, it must be the case that $B > A$. Since $B = 55°$, $0 < A < 55°$. So

$$A = \sin^{-1}\left(\frac{5\sin(55°)}{11}\right) \approx 21.9°.$$

Let's update our table.

Angle	Opposite Side
$A \approx 21.9°$	$a = 5$
$B = 55°$	$b = 11 \checkmark$
$C = ?$	$c = ?$

Now that we know two angles, we can find the third.

$$C = 180° - B - A \approx 103.1°.$$

Again, we update the table.

Angle	Opposite Side
$A \approx 21.9°$	$a = 5$
$B = 55°$	$b = 11$✓
$C \approx 103.1°$	$c =$?

Finally, we use the Law of Sines to find side c.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad \text{(Law of Sines)}$$

$$\implies \frac{b\sin(C)}{\sin(B)} = c \qquad \text{(multiplying both sides by } \sin(C)\text{)}$$

Substituting in the values of b, B, and C (stored in the calculator) and evaluating this expression with a calculator gives $c \approx 13.1$. Thus, our final table is the following:

Angle	Opposite Side
$A \approx 21.9°$	$a = 5$
$B = 55°$	$b = 11$✓
$C \approx 103.1°$	$c \approx 13.1$

Caution: Do Not Use Rounded Values to Calculate

Example **11.1.3** has a lot going on; however, the simplest thing to cover at this point is proper etiquette when it comes to calculations.

While I am *displaying* the approximations in the tables within Example **11.1.3**, any computations relying on these are done using the exact or stored values in the calculator. For example, when I do any computation relying upon A, I either use $\sin^{-1}\left(\frac{5\sin(55°)}{11}\right)$ or the unrounded approximation stored in my calculator. This is done to avoid the propagation of rounding errors. All modern calculators can store results, and I suggest you learn how to use that feature. For example, on the calculator I use (the TI-30XS), there is a button labeled sto → and another button labeled $\boxed{X^{yzt}_{abc}}$. To store the result of $\sin^{-1}\left(\frac{5\sin(55°)}{11}\right)$, I use the following button sequence:

sto → $\boxed{X^{yzt}_{abc}}$ (tapping this a few times until a is displayed) enter

When I need to access that stored value, I tap $\boxed{X^{yzt}_{abc}}$ until I get the variable I stored it as, and then hit enter.

Checkpoint 11.1.3

Sketch a triangle with $C = 93°$, $a = 7$, and $c = 11$.

a. Use the Law of Sines to find another angle of the triangle.
b. Solve the triangle, and label your sketch with the results.

Answers

a. $A \approx 39.5°$
b. $b \approx 8.13, B \approx 47.5°$

In Examples **11.1.1** and **11.1.2**, we used the Law of Sines to find missing side lengths, but we didn't need the Law of Sines to find angles. In Example **11.1.3** (and Checkpoint **11.1.3**), on the other hand, we had no choice but to use the Law of Sines to find a missing angle - this is where we need to be careful when using the Law of Sines.

Note: Unpacking Law of Sines Subtleties

It's imperative to recognize that all angles within a triangle must be positive but less than $180°$. Thus, the sine of *any* angle within a triangle will always be positive. Since the side lengths are also positive, the arcsine we use to find the missing angle will always return an acute angle (i.e., an angle between $0°$ and $90°$).

"Why do we need to be aware of this?"

When using the Law of Sines to find a missing angle in a triangle, there could be two such triangles satisfying the given conditions - one containing the acute returned by the arcsine, and the other containing its supplement.

"How do we know when we have two triangles for a given set of information?"

When using the Law of Sines to find a missing angle, we consider the angle returned by the arcsine to be a *reference angle*. To help with this discussion, let's call this reference angle $\hat{\theta}$.

$\hat{\theta}$ gives two possibilities for the true value of the angle we seek. The first is $\theta = \hat{\theta}$. The second is its supplement, $\theta' = 180° - \hat{\theta}$ (the "prime" notation is traditionally used for the supplementary angle). While θ is the solution that gives one triangle, θ' is a candidate for a second value (and, therefore, a second triangle). This is because the sine is positive and, therefore, its argument could also be between $90°$ and $180°$.

With our candidate angles, θ and θ', in hand, we then check whether the other given angle works with each of these computed values. Sometimes, the provided information works with *both* candidate values; however, at the very least, the acute angle always works.

To illustrate this concept, let's look back at Example 11.1.3. We found $\widehat{A} = \sin^{-1}\left(\frac{5\sin(55°)}{11}\right) \approx 21.9°$. Thus, we immediately knew $A \approx 21.9°$ worked and would yield a triangle; however, what about the supplement of this angle?

In QII, our alternate option for A is $A' = 180° - \widehat{A} \approx 158.1°$; however, if we tried to fit this obtuse angle into a triangle, we would compute the final missing angle to be

$$C = 180° - A - B \approx 180° - 158.1° - 55° = -33.1°.$$

This is impossible because angles within a triangle are never negative. Thus, we only have one triangle (given by the acute angle $A \approx 21.9°$).

Again, sometimes both angles produce (different) triangles, and sometimes only the acute angle will work. You should always check whether both angles provide solutions.

Because there can be more than one solution for a triangle when using the Law of Sines to find a missing angle, we call this situation the **ambiguous case** for the Law of Sines.

✔ Example 11.1.4

Solve the triangle in which $B = 14.4°$, $a = 8$, and $b = 3$. Sketch the resulting triangle.

Solution

We start, as usual, by building a table.

Angle	Opposite Side
$A = ?$	$a = 8$
$B = 14.4°$	$b = 3\checkmark$
$C = ?$	$c = ?$

Other than the numbers, this looks like it's going to mimic what we did in Example 11.1.3.

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(B)}{b} && \text{(Law of Sines)} \\ \Longrightarrow \quad \sin(A) &= \frac{a\sin(B)}{b} && \text{(multiplying both sides by } a\text{)} \\ \Longrightarrow \quad \widehat{A} &= \sin^{-1}\left(\frac{a\sin(B)}{b}\right) && \text{(finding the reference angle)} \end{aligned}$$

Grabbing a calculator, substituting in the known values of B, a, and b, we get $\widehat{A} \approx 41.5°$ (again, the exact value is stored in the calculator to be used for other calculations). As stated previously, the acute angle will always work because if $A \approx 41.5°$, then $C = 180° - A - B \approx 124.1°$; however, before throwing all of this into a table, let's check to see if the supplementary (obtuse) angle works as well. Let

$$A' = 180° - A \approx 138.5°.$$

Then

$$C' = 180° - A' - B \approx 27.1°.$$

Since this is not a negative angle, this creates another workable triangle!

Thus, if $A \approx 41.5°$, there is room in the triangle for the final, third angle $C \approx 124.1°$, and if $A' \approx 138.5°$, there is room in another triangle for the final, third angle $C' \approx 27.1°$. In these ambiguous cases, we build *two* tables of information, one for the acute angle and one for the obtuse angle.

Angle	Opposite Side
$A \approx 41.5°$	$a = 8$
$B = 14.4°$	$b = 3$✓
$C \approx 124.1°$	$c =$?

Angle	Opposite Side
$A' \approx 138.5°$	$a = 8$
$B = 14.4°$	$b = 3$✓
$C' \approx 27.1°$	$c' =$?

In either case, we need to compute the final missing sides, c and c', using the Law of Sines. Remember, when using the Law of Sines, we want to use as much of the given information as possible because it's cleaner than our approximated information. Moreover, we use the non-rounded values and save rounding for the end.

$$\begin{aligned} \frac{c}{\sin(C)} &= \frac{b}{\sin(B)} && \text{(Law of Sines)} \\ \Longrightarrow \quad c &= \frac{b\sin(C)}{\sin(B)} && \text{(multiply both sides by } \sin(C)\text{)} \end{aligned}$$

This formula is the same for c' (just replace C with C'). Hence, we grab a calculator, substitute the necessary values, and get our approximations (listed in the tables below).

Angle	Opposite Side
$A \approx 41.5°$	$a = 8$
$B = 14.4°$	$b = 3$✓
$C \approx 124.1°$	$c \approx 9.99$

Angle	Opposite Side
$A' \approx 138.5°$	$a = 8$
$B = 14.4°$	$b = 3$✓
$C' \approx 27.1°$	$c' \approx 5.5$

The following figure shows both triangles.

Figure 11.1.7

? Checkpoint 11.1.4

Suppose that $C = 29.7°$, $b = 8$, and $c = 5$.

a. Find two possible values for angle B.

b. Solve the triangle for both values of B, and sketch both solutions.

Answer

a. $B = 52.4°$ or $B = 127.6°$

b. $A = 97.9°, a = 10$

Figure 11.1.8

or $A = 22.7°, a = 3.9$

Figure 11.1.9

Example 11.1.5

Solve the triangle in which $C = 73°$, $a = 15$, and $c = 2$. Sketch your result.

Solution

Building a table, we get the following:

Angle	Opposite Side
A =?	$a = 15$
B =?	b =?
$C = 73°$	$c = 2$✓

Seeing that we have a complete row, we use that to help us find the missing angle from the first row (angle A).

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c} \quad \text{(Law of Sines)}$$

$$\Longrightarrow \quad \sin(A) = \frac{a\sin(C)}{c} \quad \text{(multiplying both sides by } a\text{)}$$

$$\Longrightarrow \quad \widehat{A} = \sin^{-1}\left(\frac{a\sin(C)}{c}\right) \quad \text{(finding the reference angle)}$$

Grabbing a calculator, substituting in the known values of C, a, and c, we find the calculator returns an error!

What happened?

Consider the following value:

$$\frac{a\sin(C)}{c} = \frac{15\sin(73°)}{2} \approx 7.2$$

This is what $\sin(A)$ is supposed to equal; however, we know that the sine function is always between -1 and 1.

What does this mean?

This means there is no such triangle where $C = 73°$, $a = 15$, and $c = 2$. We can see this from a sketch.

Figure 11.1.10

The angle C is large, but the length of its opposite side is relatively small (when compared to side length a). If we imagine side c being able to swing freely, we can see it will never intersect side b. Therefore, forming a triangle with the given information is impossible.

Example **11.1.5** illustrates the final possibility when using the Law of Sines to find a missing angle. In summary, when using the Law of Sines to find a missing angle within a triangle, one of three possibilities occurs:

1. There is a single triangle satisfying the given information. In this case, the angle will directly result from the arcsine when using the Law of Sines.
2. Two triangles satisfy the given information. In this case, one triangle will be the direct result of the arcsine, and the second triangle will contain the supplement of that angle.
3. There is no such triangle satisfying the given information. In this case, your calculator will return a domain error when computing the arcsine.

? Checkpoint 11.1.5

You are told that, in a triangle, $B = 30°$, $b = 7$, and $c = 15$. Without a calculator, explain how you know no such triangle satisfies the given information.

Answer

While using the Law of Sines to find a missing angle in a triangle, you must compute

$$sin^{-1}\left(\frac{15\sin(30°)}{7}\right) = \sin^{-1}\left(\frac{15}{14}\right).$$

This means that $\sin(C) = \frac{15}{14}$, which is not possible.

Visualizing the Law of Sines Cases

Up to this point, we have computationally justified that the Law of Sines results in one, two, or no triangles for a given set of information; however, the visual of why this is true is often more impactful.

If we know two sides a and b of a triangle and the acute angle α opposite one of them, there may be one solution, two solutions, or no solution, depending on the size of a in relation to b and α, as shown below.

The Law of Sines - Ambiguous Case

In each of the following cases, $h = b\sin(\alpha)$ is the altitude of the triangle. Therefore, $b > h$.

The "No Solution" Case

Condition: $a < h$

Reason: a is shorter than the altitude of the (non-existent) triangle. Therefore, it is impossible for side a to intersect the base of the "triangle."

The "One Solution" Cases

Condition #1: $a = h$

Reason: a is exactly the right length to make a right triangle.

Condition #2: $a > b > h$

Reason: If $a > b$, then one triangle is created as side a swings downward to intersect the base; however, as that side continues to swing, it is too long to strike the base again. Hence, there can be only one triangle.

The "Two Solutions" Case

Condition: $b > a > h$

Reason: Since $a > h$, side a will strike the base a it swings downward, but before becoming vertical. Moreover, because $a < b$, side a will strike the base once more as it continues to swing clockwise.

Applications Requiring the Law of Sines

The Law of Sines is used quite a bit in applications within Trigonometry. While I could not cover all possible applications, I have chosen two that commonly occur.

Example 11.1.6

A vertical communications tower is located at the top of a steep hill, as shown below. The angle of inclination of the hill is $56°$. A guy wire is to be attached to the top of the tower and the ground 175 yards downhill from the base of the tower. The angle formed by the guy wire is $26°$. Find the length of the cable required for the guy wire. Round your answer to the nearest yard.

Figure 11.1.11

Solution

While it's nice that we have been given a sketch of the situation, I find it's best to redraw the sketch without the "fluff," labeling vertices in uppercase and side lengths in lowercase (as we have been doing this entire time).

Figure 11.1.12

We now build our table.

Angle	Opposite Side
$A = 26^\circ$	$a = ?$
$B = ?$	$b = ?$
$C = ?$	$c = 175$

We do not have a complete row in this table, so we might think the Law of Sines is not going to be useable; however, looking at the provided sketch, we see one additional piece of information that we have not used.

The green line (side c) makes an angle of $56°$ with the horizontal. If we draw a horizontal line at vertex B, we can label the angle between the horizontal and the extended green line as $56°$ (see Figure 11.1.13 below). Since the communications tower (side a) is vertical, the angle between the extended green line and the tower is the complement of $56°$ (again, see Figure 11.1.13 below). This is $34°$.

Figure 11.1.13

Finally, we can see that the angle within the triangle at vertex B is the supplement of this $34°$ angle. Hence, $B = 180° - 34° = 146°$. Let's go ahead and place this information in our table.

Angle	Opposite Side
$A = 26°$	$a = ?$
$B = 146°$	$b = ?$
$C = ?$	$c = 175$

Again, we do not have a complete row; however, now that we have two angles, we can easily find the third.

$$C = 180° - A - B = 8°$$

Filling in the table with this information, we get the following:

Angle	Opposite Side
$A = 26°$	$a = ?$
$B = 146°$	$b = ?$
$C = 8°$	$c = 175$ ✓

Finally, we have a complete row in the table, which tells us that the Law of Sines is now available. We are being asked to find side length b. Therefore,

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad \text{(Law of Sines)}$$

$$\Longrightarrow \quad b = \frac{c\sin(B)}{\sin(C)} \qquad \text{(multiplying both sides by } \sin(B)\text{)}$$

https://math.libretexts.org/@go/page/197622

Substituting in the values of B, C, and c, we get

$$b = \frac{175 \sin(146°)}{\sin(8°)} \approx 703.$$

Hence, the length of the guy wire is approximately 703 yards.

Checkpoint 11.1.6

Lap and Piyali are on the north side of a river that runs east to west. Lap is standing 225 meters due west of Piyali. They spot a treasure chest on the south shore of the river. For Lap, the angle between the chest and Piyali is $81°$, and for Piyali, the angle between the chest and Lap is $58°$ (see Figure 11.1.14 below). How far is Lap from the chest? Round your answer to the nearest meter.

Figure 11.1.14

Answer

291 meters.

The following example (and the subsequent Checkpoint) illustrates that the Law of Sines can be used when you might otherwise use Right Triangle Trigonometry.

Example 11.1.7

Ron wants to measure the height of a castle controlled by hostile forces. When he is as close as he can get to the castle, the angle of elevation to the top of the wall is $18.5°$. He then retreats 20 yards and measures the angle of elevation again; this time, it is $15.9°$. How tall is the castle?

Figure 11.1.15

Solution

Notice that h is one side of the right triangle $\triangle ADC$. If we find its hypotenuse, labeled r in Figure 11.1.15, we can use the sine ratio to find h. To find r, we consider a second triangle, $\triangle ABC$, as shown in Figure 11.1.16.

Figure 11.1.16

In this triangle, we know side $BC = 20$ and would like to find side $AC = r$. We can use the Law of Sines to find r, but first, we must calculate the other angles of the triangle.

Now, the angle opposite r, $\angle ABC$, is the complement of $18.5°$, so

$$\angle ABC = 180° - 18.5° = 161.5°$$

The angle opposite the 20-yard side, $\angle BAC$, is

$$\angle BAC = 180° - (161.5° + 15.9°) = 2.6°$$

Now we can apply the Law of Sines to find r. We have

$$\frac{r}{\sin(161.5°)} = \frac{20}{\sin(2.6°)} \quad \text{(Law of Sines)}$$

$$\implies r = \frac{20\sin(161.5°)}{\sin(2.6°)} \quad \text{(multiplying both sides by } \sin(161.5°)\text{)}$$

Finally, using the right triangle $\triangle ADC$, we can write

$$\frac{h}{r} = \sin(15.9°) \quad \text{(right triangle definition of the sine)}$$

$$\implies h = r\sin(15.9°) \quad \text{(multiplying both sides by } r\text{)}$$

$$= \frac{20\sin(161.5°)}{\sin(2.6°)} \cdot \sin(15.9°)$$

$$\approx 38.33$$

The castle is about 38.33 yards tall.

Checkpoint 11.1.7

Solve the problem in the previous example again, but instead of finding r, find the length AB, and then use $\triangle ABD$ to find h.

Answer

$AB = 120.79$, the castle is about 38.33 yards tall.

If you look at a nearby object and alternately close your left and right eyes, the object seems to jump in position. This apparent change occurs because your eyes view the object from two positions spaced several centimeters apart. If the object at point O is straight ahead of one eye, it appears to be at some angle p away from the line of sight of the other eye. The angle p is called the **parallax** of the object.

Use Figure $11.1.17$ to see that p is also the angle between the directions to your two eyes when viewed from point O. (What fact from Geometry justifies this statement?)

Figure 11.1.17

Astronomers use parallax to determine the distance from Earth to stars and other celestial objects. Two observers on Earth at a known distance apart both measure the direction to the star. The difference in angle between those two directions is the parallax.

Example 11.1.8

Astronomers 1000 kilometers apart observe an asteroid with a parallax of $0.001°$. How far is the asteroid from Earth?

Solution

We let x represent the distance to the asteroid. The asteroid and the two observers make an isosceles triangle with a base of approximately 1000 km and equal sides of length x, as shown below.

Figure 11.1.18

The base angles of the triangle are both $\frac{180° - 0.001°}{2} = 89.999°$. Thus,

$$\frac{x}{\sin(89.999°)} = \frac{1000}{\sin(0.001°)} \qquad \text{(Law of Sines)}$$

$$\implies x = \frac{1000\sin(89.999°)}{\sin(0.001°)} \qquad \text{(multiplying both sides by } \sin(89.999°)\text{)}$$

$$\implies x \approx 57,000,000$$

The asteroid is about 57 million kilometers from Earth (roughly one-third of the distance to the Sun).

Checkpoint 11.1.8

Two observers 800 kilometers apart observe an object with a parallax of $0.0005°$. How far is the object from Earth?

Answer

About $91,673,247$ km

Footnotes

[1] I say "try" because sometimes solving a triangle using the Law of Sines will not work. We will see those cases momentarily.

[2] In all honesty, since A is an angle within a triangle and *not* an angle in standard position within a prescribed coordinate system, it is incorrect to say that $A \in \text{QI}$ or $A \in \text{QII}$; however, I am okay with the "looseness" of this language.

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11.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Law of Sines
  - Non-Right Triangles - Law of Sines: Theorem and Proof
  - Non-Right Triangles - Law of Sines: A Conceptual Investigation of the Ambiguous Case
  - Non-Right Triangles - Law of Sines: Using the Law of Sines to Solve Triangles
  - Non-Right Triangles - Law of Sines: Applications of Non-Right Triangle Trigonometry (Part 1)
  - The Law of Sines - Review of the Law of Sines
  - The Law of Sines - Review of Applications of Non-Right Triangle Trigonometry (Part 1)

KEY CONCEPTS

DEFINITIONS

- **Ambiguous Case:** A situation that arises when using the Law of Sines to find a missing angle, where the given information (typically two sides and a non-included angle, or SSA) could result in one triangle, two different triangles, or no triangle at all.

THEOREMS

- **Law of Sines:** If the angles of a triangle are A, B, and C, and the opposite sides are a, b, and c, respectively, then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad (11.1.1.1)$$

  or equivalently,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad (11.1.1.2)$$

- **The Law of Sines - Ambiguous Case (SSA):** Given two sides a and b and an acute angle α opposite side a, let $h = b\sin(\alpha)$ be the altitude of the triangle.
  - **No Solution:** If $a < h$, no triangle can be formed.
  - **One Solution:** If $a = h$ (a right triangle) or if $a > b$, one triangle can be formed.
  - **Two Solutions:** If $h < a < b$, two distinct triangles can be formed.

COMMON MISTAKES

- **Do Not Use Rounded Values to Calculate:** When solving a triangle, use the exact or stored values from your calculator for all intermediate steps. Using rounded approximations can propagate errors and lead to an inaccurate final answer.
- **Forgetting the Ambiguous Case:** When using the Law of Sines to solve for a missing angle (the SSA case), remember that there might be two possible triangles. The arcsine function on a calculator will only return an acute angle; you must also check its supplement as a potential second solution for the angle.
- **Misidentifying the Number of Solutions:** In the ambiguous case, failing to compare the side opposite the given angle to the altitude and the other given side can lead to incorrectly concluding there is one triangle when there are two or none, or vice versa.

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11.1.2: HOMEWORK

READING QUESTIONS

1. What is an oblique triangle?
2. State the Law of Sines.
3. To use the Law of Sines to solve for a missing side or angle, what information must you already have about the triangle?
4. What does it mean to "solve a triangle"?
5. When using the Law of Sines to find an unknown angle, why must you be careful?
6. What is the "ambiguous case" for the Law of Sines? What given information (e.g., SSS, ASA, SSA) leads to this case?
7. If you use the Law of Sines to find that a missing angle θ could be 40°, what is the other possible value for θ?
8. How do you determine if both possible angles in the ambiguous case lead to valid triangles?
9. If your calculator returns an error when you try to use the arcsin function to find a missing angle, what does this imply about the triangle?
10. Why is it better to use the given values in calculations rather than values you have already calculated and rounded?

HOMEWORK

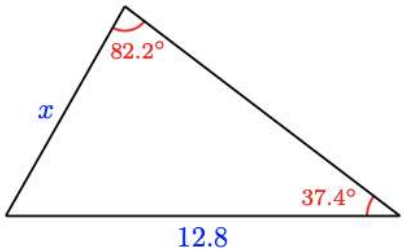
CONCEPT CHECK

1. Can we use the Law of Sines to solve a right triangle?
2. Explain why we cannot use the Law of Sines to solve the triangle with $a = 8$, $b = c$, and $C = 35°$.
3. Linda says, "I'm thinking of an angle whose sine is 0.3420 (rounded to four decimal places)." Matt says, "The angle must be $20°$ (rounded to the nearest degree)." Is he correct? Why or why not?
4. Sketch two possible triangles with $A = 25°$, $b = 18$, and $a = 10$.
5. Try to sketch a triangle with $A = 65°$, $b = 18$, and $a = 10$. What went wrong?

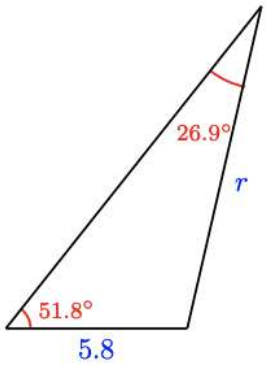
BASIC SKILLS

For the following exercises, find the indicated side. Round to two decimal places.

6.



7.



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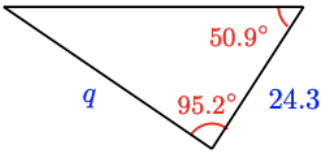
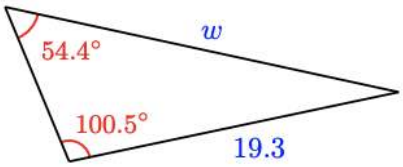
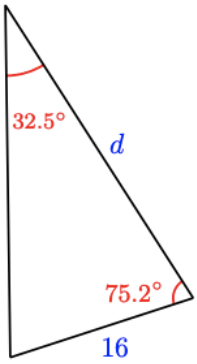


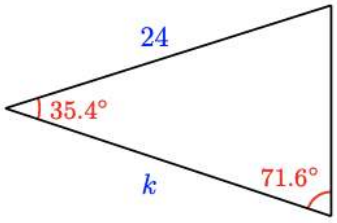
9.



10.

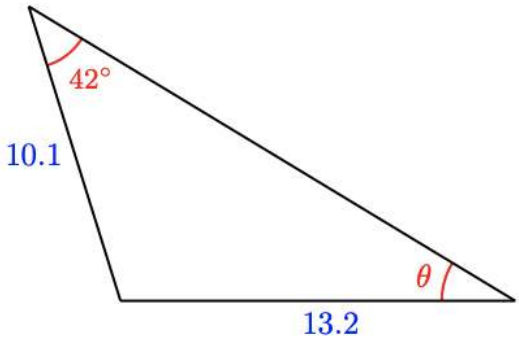


11.



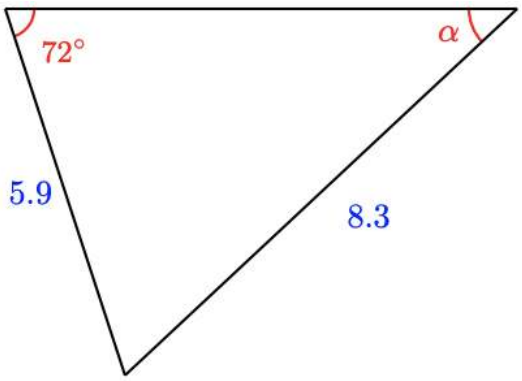
For the following exercises, find the indicated angle. Round to two decimal places.

12.



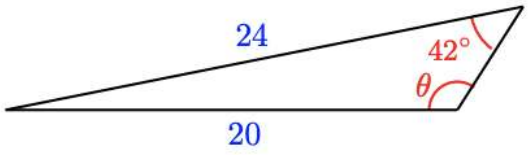
(θ is acute.)

13.



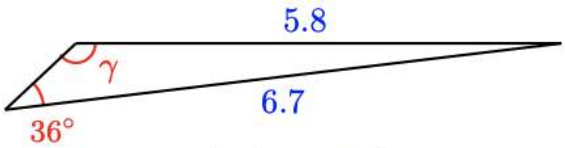
(α is acute.)

14.



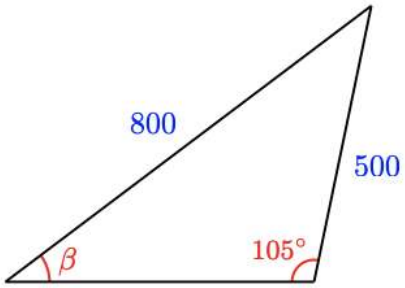
(θ is acute.)

15.

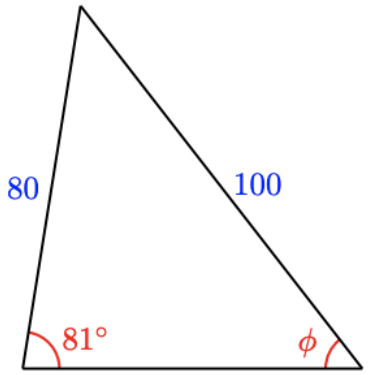


(γ is acute.)

16.



17.



For the following exercises, sketch the triangle and solve. Round answers to two decimal places.

18. $b = 7$, $A = 23°$, $B = 42°$
19. $c = 34$, $A = 53°$, $C = 26°$
20. $a = 1.8$, $c = 2.1$, $C = 44°$
21. $b = 8.5$, $c = 6.8$, $B = 23°$
22. $c = 75$, $A = 35°$, $B = 46°$
23. $a = 94$, $B = 29°$, $C = 84°$
24. In $\triangle ABC$, $\angle A = 30°$ and $c = 12$. How many triangles are possible for each of the following lengths for side a? Sketch the solutions in each case.
    a. $a = 6$
    b. $a = 4$
    c. $a = 9$
    d. $a = 15$

For the following exercises, find the remaining angles of the triangle (or triangles, if this is the ambiguous case of the Law of Sines). Round answers to two decimal places.

25. $a = 66$, $c = 43$, $C = 25°$
26. $b = 10$, $c = 14$, $B = 20°$
27. $b = 100$, $c = 80$, $B = 49°$
28. $b = 4.7$, $c = 6.3$, $C = 54°$
29.
    a. Sketch a triangle with $A = 25°$, $B = 35°$, and $b = 16$.
    b. Use the Law of Sines to find a.
    c. Use the Law of Sines to find c.
    d. Find c without using the Law of Sines. (Hint: Sketch the altitude, h, from C to make two right triangles. Find h, then use h to find c.)

SYNTHESIS QUESTIONS

30. In this problem, we show that there are two different triangles $\triangle ABC$ with $A = 30°$, $a = 2$, and $c = 3$.
    a. Use a protractor to draw an angle $A = 30°$. Mark point B on one side of the angle so that $\overline{AB}$ is 3 inches long.

b. Locate two distinct points on the other side of the angle that are each 2 inches from point B. These points are both possible locations for point C.

c. Use the Law of Sines to find two distinct possible measures for $\angle C$.

31. Consider the triangle $\triangle ABC$ shown below.

a. Express the length of the altitude in terms of $\angle A$ and c.

b. Suppose we keep $\angle A$ and side c fixed but allow a to vary in length. What is the smallest value a can have and still be long enough to make a triangle?

c. What are the largest and smallest values that a can have in order to produce two distinct triangles $\triangle ABC$ (without changing $\angle A$ and side c)

32. For the triangle in the previous problem, suppose $A = 40°$ and $c = 8$.

a. Sketch and solve the triangle if $a = 12$.

b. Sketch and solve the triangle if $a = 6$.

c. Sketch and solve the triangle if $a = 4$.

d. For what value of a is c the hypotenuse of a right triangle?

33. For the triangle in Problem 31, suppose $A = 70°$ and $c = 20$.

a. For what value of a is the triangle a right triangle?

b. For what values of a are there two solutions for the triangle?

c. For what values of a is there one obtuse solution for the triangle?

d. For what value of a is there no solution?

34.

a. Sketch a triangle with $A = 75°$, $a = 15$, and $b = 6$.

b. Use the Law of Sines to find c.

c. Find c without using the Law of Sines.

35. Here is a method for solving certain oblique triangles by dividing them into two right triangles. In the triangle shown, we know two angles, A and B, and the side opposite one of them, say a. We would like to find side b.

a. Draw the altitude h from angle C.

b. Write an expression for b in terms of h and angle A.

c. Write an expression for h in terms of angle B.

d. Substitute your expression for h into your expression for b.

e. Which of the following is equivalent to the formula you wrote in part (d)?

i. $a\sin(A) = b\sin(B)$

ii. $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$

iii. $\frac{a}{\sin(B)} = \frac{b}{\sin(A)}$

PROOFS

For the following exercises, you will prove the Law of Sines using the trigonometric formula for the area of a triangle. We will derive this formul later. For now, you can use the fact that the area of triangle $\triangle ABC$ is $\frac{1}{2}ab\sin(\theta)$, where θ is the angle between sides a and b.

36. Sketch a triangle with angles A, B, and C, and opposite sides of lengths a, b, and c, respectively.

a. Write the area of the triangle in terms of a, b, and angle C.

b. Write the area of the triangle in terms of a, c, and angle B.

c. Write the area of the triangle in terms of b, c, and angle A.

37. Equate the three different expressions from Problem 36 for the area of the triangle. Multiply through by $\frac{2}{abc}$ and simplify to deduce the Law of Sines.

APPLICATIONS

For the following exercises,
(a) Sketch and label a triangle to illustrate the problem.
(b) Solve the problem.

38. Phuong wants to know the height of a cliff on the other side of a ravine. The angle of elevation from the edge of the ravine to the cliff top is $84.6°$. When she moves 30 feet back from the ravine, the angle of elevation is $82.5°$. How tall is the cliff?

39. Nam wants to know the height of a tree in the median strip of a highway. The angle of elevation from the highway shoulder to the treetop is $43.5°$. When he moves 10 feet farther away from the tree, the angle of elevation is $37.2°$. How tall is the tree?

40. Mike and Loi are 10 kilometers apart, observing a satellite that passes directly over their heads. At a moment when the satellite is between them, Loi measures its angle of elevation as $84.6°$, and Mike measures an angle of $87°$. How far is the satellite from Mike?

41. Paige rows her kayak due east. When she began, she spotted a lighthouse 2000 meters in the distance at an angle of $14°$ south of east. After traveling for an hour, the lighthouse was at an angle of $83°$ south of east. How far did Paige travel, and what was her average speed?

42. Ivan is hiking along a straight path but needs to detour around a large pond. He turns $23°$ from his path until clear of the pond, then walks back to his original path, intercepting it at an angle of $29°$ and at a distance of 2 miles from where he had left the path. How far did Ivan walk in each of the two segments of his detour, and how much farther did his detour require compared with a straight line through the pond?

43. Wyatt is flying to Monterey but must change course to avoid a storm. He flies $19°$ off from his original direction until he clears the storm, then turns again to return to his original flight path, intercepting it at an angle of $54.9°$ and at a distance of 50 miles from where he had left it. How much farther did his detour require compared with his original course?

44. Geologists find an outcropping from an underground rock formation that usually indicates the presence of oil. The outcropping is on a hillside, and the formation dips another $17°$ from the surface. If an oil well is placed 1000 meters downhill

from the outcropping, how far will they drill before reaching the formation?

45. A proposed ski lift will rise from a point near the base of the slope with an angle of $27°$. At a distance of 400 meters further from the hill, the angle of elevation to the top of the ski lift is $19°$. How long is the ski lift?

46. Nhat wants to measure the height of a hill. She first plants a 50-foot-tall antenna at the hill's peak. Then, she descends the hill and finds a point where she can see the antenna's top and bottom. The angle of elevation to the bottom of the antenna is $23°$, and the angle of elevation to the top of the antenna is $24°$.

    a. Find $\angle ACB$.

    b. Find $\angle CAB$ at the top of the antenna.

    c. How long is BC, the distance from the bottom of the antenna to C?

    d. How tall is the hill?

47. Jorge and Camille are 1000 yards apart. The angle Jorge sees between Camille and a particular tree is $38°$. If the tree is 800 yards from Camille, how far is it from Jorge? (There are two possible answers.)

48. From the lookout point on Fabrick Rock, Sang can not only see the famous "Crooked Spire" in Chesterfield, which is 8 miles away but also the red phone box in the village of Alton. Chesterfield and Alton are 7 miles apart. Fabrick Rock has a plaque that shows directions to famous sites, and from the plaque, Sang determines that the angle between the lines to the spire and the phone box measures $19°$. How far is Fabrick Rock from the phone box? (There are two possible answers.)

49. A billboard of California's gubernatorial candidate, Min, is located on the roof of a building. At a distance of 180 feet from the building, the angles of elevation to the bottom and top of the billboard are respectively $39.8°$ and $47.3°$. How tall is the billboard?

For the following exercises, compute the distances in Astronomical Units. Then convert to kilometers, using the fact that 1 AU $\approx 1.5 \times 108$ km.

50. When observed from opposite sides of Earth's orbit, the star Alpha Centauri has a parallax of $0.76''$. How far from the Sun is Alpha Centauri?

51. How far from the Sun is Barnard's star, which has a parallax of $1.1''$ when observed at opposite ends of Earth's orbit?

52. How far from the Sun is Tau Ceti, which has a parallax of $0.55''$ when observed from opposite ends of Earth's orbit?

53. How far from the Sun is Sirius, which has a parallax of $0.75''$ when observed from opposite ends of Earth's orbit?

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11.2: The Law of Cosines

Note to the Instructor - Not Much to Say

This section is a very traditional treatment of the Law of Cosines and applications involving the Law of Cosines. You will likely spend a good chunk of time on navigation problems, as these are often difficult for students to master.

Learning Objectives

- Use the Law of Cosines to find a missing side or a missing angle in an oblique triangle.
- Use the Law of Sines to find the remaining angle once the Law of Cosines has been used to find one angle in an oblique triangle.
- Use the Law of Cosines to solve an application involving heading or bearing.
- Use the Law of Cosines to solve an application.

EST. 1970 COSUMNES RIVER COLLEGE

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The Law of Cosines

When learning the Law of Sines, we used a table to help organize information and determine which unknown to find first. For example, suppose the information in the following table was what we were given.

Angle	Opposite Side
$\angle A = ?$	$a = ?$
$\angle B = 79.4^\circ$	$b = ?$
$\angle C = 17.4^\circ$	$c = 400 \checkmark$

I mentioned in the previous section that a complete row in this table meant we could use the Law of Sines. Therefore, we can see that, given two angles and a side opposite one of those angles, we can use the Law of Sines to solve the triangle.

Suppose, instead, the given information looked as follows:

Angle	Opposite Side
$\angle A = ?$	$a = ?$
$\angle B = 79.4^\circ$	$b = 1600 \checkmark$
$\angle C = ?$	$c = 400$

Again, we have a complete row, so we can use the Law of Sines. Therefore, given two sides and an angle opposite one of those sides, we could use the Law of Sines to solve the triangle (if one exists[1]).

In contrast, given any other combination of sides or angles, the Law of Sines is not helpful. If you consider what is required for this to happen, we just need a table with no complete rows. The two majors examples are the following:

Angle	Opposite Side
$\angle A = ?$	$a = 7$
$\angle B = 115^\circ$	$b = ?$
$\angle C = ?$	$c = 5$

or

Angle	Opposite Side
$\angle A = ?$	$a = 7$
$\angle B = ?$	$b = 9$
$\angle C = ?$	$c = 5$

Notice that there is not a complete row within either of these tables and, as of now, we have no simple way of completing a row. Not having an angle-side pairing presents an issue that the Law of Sines cannot surmount (unless you were given two angles, in which

case you can find the third and proceed to use the Law of Sines).

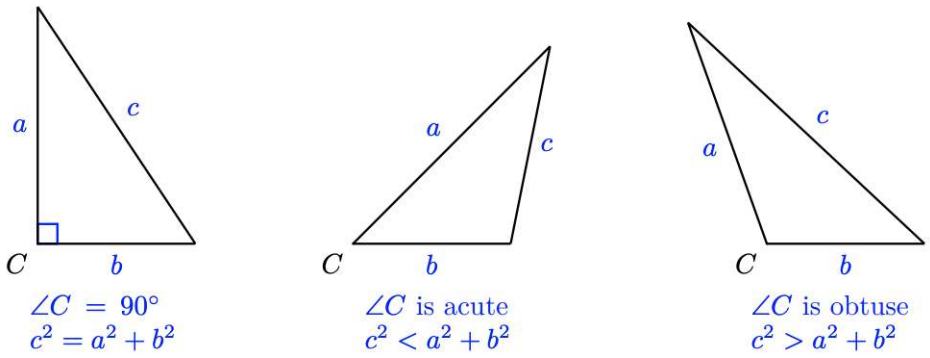
To solve a triangle when we know two sides and the included angle, we will need a generalization of the Pythagorean Theorem.

In a right triangle, with $C = 90°$, the Pythagorean Theorem tells us that

$$c^2 = a^2 + b^2.$$

If we allow angle C to vary (rather than being locked in at a right angle) but keep a and b the same length, the side c will grow or shrink, depending on whether we increase or decrease the angle C, as shown below.

Figure 11.2.1



The Pythagorean Theorem is a special case of a more general law that applies to *all* triangles, no matter the size of angle C. This law is known as the **Law of Cosines**.

Theorem: Law of Cosines

If the angles of a triangle are A, B, and C, and the opposite sides are respectively a, b, and c, then

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) \\ b^2 &= a^2 + c^2 - 2ac\cos(B) \\ c^2 &= a^2 + b^2 - 2ab\cos(C) \end{aligned}$$

Proof

Start with general triangles as seen in the figures below.

Two general triangles, one acute (left) and one obtuse (right), with altitudes labeled as h

We will perform the proof for the acute triangle (the proof associated with the obtuse triangle is similar).

The acute triangle is comprised of two right triangles - $\triangle ABD$ and $\triangle BCD$. Let x be the length of the line segment $\overline{AD}$. Then the length of the segment $\overline{CD}$ is $b - x$. We show these facts in the following figure, splitting the general right triangle into its two right triangle pieces.

"Splitting" the acute triangle into two right triangles

With the leftmost triangle, we have the following:

$$\begin{aligned} (b-x)^2+h^2 &= a^2 && \text{(Pythagorean Theorem)} \\ \implies \quad b^2-2bx+x^2+h^2 &= a^2 && \text{(distributing)} \end{aligned}$$

With the rightmost triangle, we have

$$x^2+h^2=c^2 \quad \text{(Pythagorean Theorem)}$$

Therefore,

$$b^2-2bx+x^2+h^2=a^2 \quad \text{and} \quad x^2+h^2=c^2.$$

Replace x^2+h^2 in the first equation with c^2. Doing so yields the following:

$$b^2-2bx+c^2=a^2 \implies b^2+c^2-2bx=a^2.$$

From the rightmost triangle in the figure above, we see that $\cos(A)=\frac{x}{c}$, and so $c\ \cos(A)=x$. Hence, we have developed the following relationship between the sides of *any* triangle.

$$\begin{aligned} b^2+c^2-2bx &= a^2 \\ \implies \quad b^2+c^2-2b\,(c\,\cos(A)) &= a^2 && \text{(substitution)} \\ \implies \quad b^2+c^2-2bc\,\cos(A) &= a^2 \end{aligned}$$

When A is a right angle, $\cos(A)=\cos(90°)=0$, so the equation reduces to the Pythagorean Theorem.

We can write similar equations involving the angles B or C. In all cases, the angle within the cosine is opposite the side playing the role of the "hypotenuse" (really, a **quasi-hypotenuse**).

The Law of Cosines is also known as the **Generalized Pythagorean Theorem**.

Rather than committing all three forms to memory, I find it much easier to think of these as modifications of the Pythagorean Theorem.[2] We have a sum of squares of sides *almost* equal to the square of a quasi-hypotenuse. The adjustment we make to the sum of squares side is to subtract twice the product of the non-quasi-hypotenuse sides and the cosine of the angle opposite the quasi-hypotenuse.

Now that I have written that out, I can see that it might be confusing to most people, so let me say it symbolically. If s_1 and s_2 are playing the roles of the non-quasi-hypotenuse sides, and h is playing the role of the quasi-hypotenuse, then

$$s_1^2+s_2^2-2s_1s_2\cos(H)=h^2,$$

where H is the angle opposite side h.

Finding a Side

Example 11.2.1

The town of Avery lies 48 miles due east of Baker, and Clio is 34 miles from Baker, in the direction $35°$ west of north. How far is it from Avery to Clio?

Solution

We start with a simple sketch.

Figure 11.2.2

The angle $\angle ABC=35°+90°=125°$. Thus, in $\triangle ABC$ we have the following:

Angle	Opposite Side
$\angle A=?$	$a=34$
$\angle B=125°$	$b=?$
$\angle C=?$	$c=48$

Since we do not have a complete row, we must use the Law of Cosines. We are interested in knowing the length of side b, so that will be our quasi-hypotenuse.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(B) && \text{(Law of Cosines)} \\ \Longrightarrow\ b^2 &= 34^2 + 48^2 - 2(34)(48)\cos(125°) && \text{(substitute)} \\ \Longrightarrow\ b^2 &= 3460 - 3264\cos(125°) \\ \Longrightarrow\ b^2 &\approx 5332.153 \\ \Longrightarrow\ b &\approx 73.02 \end{aligned}$$

Therefore, Avery is about 73 miles from Clio.

Caution: Follow the Order of Operations

When simplifying the Law of Cosines, follow the Order of Operations carefully. In Example **11.2.1**, the right side of the equation

$$b^2 = 34^2 + 48^2 - 2(34)(48)\cos(125°)$$

has three terms, and simplifies to

$$\begin{aligned} b^2 &= 1156 + 2304 - 3264\cos(125°) \\ \Longrightarrow\ b^2 &\approx 3460 - 3264(-0.573567364\ldots) \end{aligned}$$

Note that 3264 is the coefficient of $\cos(125°)$, so it would be incorrect to subtract 3264 from 3460. Using a graphing calculator, you can enter the right side of the equation exactly as it is written.

Checkpoint 11.2.1

In $\triangle ABC$, $a = 11$, $c = 23$, and $B = 87°$. Find b, and round your answer to two decimal places.

Answer

24.97

Finding an Angle

We can also use the Law of Cosines to find an angle when we know all three sides of a triangle. In the following example, pay close attention to the algebraic steps used to solve the equation.

Example 11.2.2

In the triangle below, $a = 6$, $b = 7$, and $c = 11$. Find angle C.

Figure 11.2.3

Solution

Our table looks like

Angle	Opposite Side
$\angle A = ?$	$a = 6$
$\angle B = ?$	$b = 7$
$\angle C = ?$	$c = 11$

Since there is no way to get a complete row, we must use the Law of Cosines. Moreover, because we are asked to find C, we choose the version of the Law of Cosines that uses angle C.

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos(C) && \text{(Law of Cosines)}\\
\implies 11^2 &= 6^2 + 7^2 - 2(6)(7)\cos(C) && \text{(substitute)}\\
\implies 121 &= 36 + 49 - 84\cos(C) && \text{(apply powers and compute the product)}\\
\implies 36 &= -84\cos(C) && \text{(subtract 85 from both sides)}\\
\implies -\frac{3}{7} &= \cos(C) && \text{(divide both sides by } -84\text{)}\\
\implies \widehat{C} &= \cos^{-1}\left(\frac{3}{7}\right) && \text{(finding the reference angle)}\\
\implies C &\approx 115.4^\circ && \text{(arccosine is either in QI or QII.}\\
& && \text{Since the cosine is negative, } C \in \text{QII.}\\
& && \text{Thus, } C = 180^\circ - \widehat{C}\text{)}
\end{aligned}
$$

Angle C is roughly 115.4°.

? Checkpoint 11.2.2

In $\triangle ABC$, $a = 5.3$, $b = 4.7$, and $c = 6.1$. Find angle B, and round your answer to two decimal places.

Answer

48.07°

Once we have calculated one of the angles in a triangle, we can use either the Law of Sines or the Law of Cosines to find a second angle. Here is how we would use the Law of Sines to find angle A in Example 11.2.2.

$$
\begin{aligned}
\frac{\sin(A)}{a} &= \frac{\sin(C)}{c} && \text{(Law of Sines)}\\
\implies \sin(A) &= \frac{a\sin(C)}{c} && \text{(multiplying both sides by } a\text{)}\\
\implies \widehat{A} &= \sin^{-1}\left(\frac{a\sin(C)}{c}\right) && \text{(finding the reference angle)}
\end{aligned}
$$

Substituting in the given values, we get $\widehat{A} \approx 29.5^\circ$. Since A is the shortest side of the triangle, A must be acute. Therefore, $A \approx 29.5^\circ$. Finally,

$$B = 180^\circ - A - C \approx 35.1^\circ.$$

Alternatively, we can use the Law of Cosines to find angle A.

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos(A) && \text{(Law of Cosines)}\\
\implies a^2 - b^2 - c^2 &= -2bc\cos(A) && \text{(subtracting } b^2 \text{ and } c^2 \text{ from both sides)}\\
\implies \frac{a^2 - b^2 - c^2}{-2bc} &= \cos(A) && \text{(dividing both sides by } -2bc\text{)}\\
\implies \cos^{-1}\left(\left|\frac{a^2 - b^2 - c^2}{-2bc}\right|\right) &= \widehat{A} && \text{(finding the reference angle)}
\end{aligned}
$$

Substituting the given values into this result, we get the reference angle of $\widehat{A} \approx 29.5^\circ$. If we had computed the value of $\frac{a^2-b^2-c^2}{-2bc}$, we would have found that this fraction is positive. Since the arccosine returns in QI or QII, and, of those two quadrants, the cosine is positive in QI, our actual angle is $A \approx 29.5^\circ$.

From this conversation about finding a second angle once you have used the Law of Cosines to find an initial angle, I hope you agree that the amount of computational work is minimized when using the Law of Sines. You can choose either method to find a

second angle, but I will always choose the Law of Sines over the Law of Cosines.

Advice: Choose Your Angle Wisely!

When you have used the Law of Cosines to find the first missing angle within a triangle, you can use the Law of Sines or the Law of Cosines to find the next unknown angle. If using the Law of Sines, however, *always* choose to find the angle opposite the smallest remaining side. This is because the angle opposite the smallest remaining side is *guaranteed* to be acute. Hence, the reference angle returned when using the Law of Sines will be the actual angle!

Applications Involving the Law of Cosines

The applications of the Law of Cosines are countless, more so than the Law of Sines; however, they all somewhat follow the structure of the following example.

Example 11.2.3

A researcher wishes to determine the width of a vernal pond, as drawn below. From a point P, he finds the distance to the eastern-most point of the pond to be 950 feet, while the distance to the western-most point of the pond from P is 1000 feet (obviously, the perspective of the provided image is such that due north is roughly facing the 4 o'clock position). If the angle between the two lines of sight is $60°$, find the width of the pond.

Figure 11.2.4

Solution

Let p be the side length opposite angle P, $a = 1000$, and $b = 950$. Then our table looks like the following:

Angle	Opposite Side
$\angle A = ?$	$a = 1000$
$\angle B = ?$	$b = 950$
$\angle P = 60°$	$p = ?$

No complete rows (and no hope of getting a complete row using simple arithmetic) means we are using the Law of Cosines. We want to find length p.

$$\begin{aligned} p^2 &= a^2 + b^2 - 2ab\cos(P) && \text{(Law of Cosines)} \\ \Longrightarrow \quad p &= \sqrt{a^2 + b^2 - 2ab\cos(P)} && \text{(Extraction of Roots)} \end{aligned}$$

Substituting in the given values, we get

$$p = \sqrt{950^2 + 1000^2 - 2(950)(1000)\cos(60°)} = \sqrt{952{,}500} \approx 976.$$

Thus, the pond is roughly 976 feet wide.

Checkpoint 11.2.3

The hour hand on Kim's antique clock is 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.

Answer

8.26 inches.

Revisiting Navigation Problems

In a previous section, we were introduced to navigation problems (heading and bearing problems). At that time, however, we were only able to work with these styles of problems when they resulted in right triangles. We now have enough Trigonometry "under our belts" to approach these problems in *any* situation.

Example 11.2.4

The sailing club leaves the marina on a heading of $15°$ and sails for 18 miles. They then change course, and after traveling for 12 miles on a heading of $35°$, they experience engine trouble and radio for help. The marina sends a speed boat to rescue them. How far should the speed boat go, and on what heading?

Solution

As we did when first introduced to heading problems, we sketch the situation, placing small axes at each pivot point.

Figure 11.2.5

This results in the following figure.

Figure 11.2.6

We want to find the distance d and the angle $\angle BAC$ shown in the figure. We use alternate interior angles to get further information about the angles within $\triangle ABC$.

Figure 11.2.7

This results in the final, clean sketch.

Figure 11.2.8

In $\triangle ABC$, we calculated the angle at point B where the sailing club changed course as follows:

$$B = 180^\circ - 35^\circ + 15^\circ = 160^\circ.$$

We now build our table.use the Law of Cosines to find b and $\angle A$.

Angle	Opposite Side
$\angle A = ?$	$a = 12$
$\angle B = 160^\circ$	$b = ?$
$\angle C = ?$	$c = 18$

Since there is not a complete row, and there is no way to get a complete row without Trigonometry, we use the Law of Cosines to find the length of side b (accidentally labeled as side d in the figure above).

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(B) && \text{(Law of Cosines)} \\ \Longrightarrow \quad b &= \sqrt{a^2 + c^2 - 2ac\cos(B)} && \text{(Extraction of Roots)} \end{aligned}$$

Substituting in the given values, we get $b \approx 29.56$ miles. Taking the time to add this to our table, we get the following:

Angle	Opposite Side
$\angle A = ?$	$a = 12$
$\angle B = 160^\circ$	$b \approx 29.56 \checkmark$
$\angle C = ?$	$c = 18$

Now that we have a complete row, we can use the Law of Sines to find one of the remaining angles. The advice given previously was to find the angle opposite the smallest remaining side. This would be A (which is, in fact, the angle we need to find).

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(B)}{b} && \text{(Law of Sines)} \\ \Longrightarrow \quad \sin(A) &= \frac{a\sin(B)}{b} && \text{(multiplying both sides by } a\text{)} \\ \Longrightarrow \quad \widehat{A} &= \sin^{-1}\left(\frac{a\sin(B)}{b}\right) && \text{(finding the reference angle)} \end{aligned}$$

Substituting in the nonrounded values for a, b, and B, we get $\widehat{A} \approx 8.0°$. Since A is opposite the smallest remaining side, A must be acute. Therefore, $A \approx 8.0°$.

We finish this problem with a bit of extra logic. We know the speed boat must travel $b \approx 29.56$ miles; however, the heading of the speed boat is not $8.0°$. This is because they must rotate through $15°$ and then an additional $8.0°$ to face the sailing club's boat. Thus, their heading is $23.0°$.

? Checkpoint 11.2.4

Roy wants to fly from Anchorage to Nome, Alaska, a distance of 540 miles on a heading of $303°$. After flying for some time, he discovers that his heading is in error and is actually flying at a heading of $313°$. Roy corrects his flight plan and changes course when he is 200 miles from Anchorage. What is his new heading, and how far is he from Nome?

Answer

$297.2°$, 344.8 miles

Footnotes

[1] Recall that we might find one, two, or no triangles when we need to use the Law of Sines to find a missing angle.

[2] In truth, the Pythagorean Theorem is a modification of the Law of Cosines.

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11.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Law of Cosines
  - Non-Right Triangles - Law of Cosines: Theorem and Proof
  - Non-Right Triangles - Law of Cosines: Using the Law of Cosines to Solve Triangles
  - Non-Right Triangles - Law of Cosines: Applications of Non-Right Triangle Trigonometry (Part 2)
  - The Law of Cosines - Review of the Law of Cosines
  - The Law of Cosines - Applications of Non-Right Triangle Trigonometry (Part 2)
  - The Law of Cosines - Applications of Non-Right Triangle Trigonometry (Part 3): Navigation Problems

KEY CONCEPTS

THEOREMS

- **Law of Cosines:** If the angles of a triangle are A, B, and C, and the opposite sides are respectively a, b, and c, then:
  - $a^2 = b^2 + c^2 - 2bc\cos(A)$
  - $b^2 = a^2 + c^2 - 2ac\cos(B)$
  - $c^2 = a^2 + b^2 - 2ab\cos(C)$

COMMON MISTAKES

- **Order of Operations:** When simplifying the Law of Cosines, follow the order of operations carefully. In an expression like $b^2 + c^2 - 2bc\cos(A)$, the term $2bc\cos(A)$ must be calculated before subtracting from the sum of squares. Do not subtract $2bc$ from $b^2 + c^2$ first.
- **Incorrectly Applying Law of Sines After Law of Cosines:** After using the Law of Cosines to find an angle, you can use the Law of Sines to find a second angle. To avoid the ambiguous case, you should always choose to find the angle opposite the smallest remaining side, as this angle is guaranteed to be acute.

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11.2.2: HOMEWORK

READING QUESTIONS

1. The Law of Cosines is a generalization of what famous theorem?
2. Under what two conditions (e.g., SSS, ASA, etc.) is the Law of Cosines useful for solving a triangle when the Law of Sines is not?
3. State one of the three versions of the Law of Cosines.
4. When using the Law of Cosines to find a missing side, what information must you have?
5. When using the Law of Cosines to find a missing angle, what information must you have?
6. What is a common mistake regarding the order of operations when simplifying the right side of the Law of Cosines formula?
7. After using the Law of Cosines to find one angle in a triangle, what is the recommended method for finding a second angle?
8. When using the Law of Sines after using the Law of Cosines, which angle should you choose to find next to avoid the ambiguous case? Why?

HOMEWORK

CONCEPT CHECK

1. The Law of Cosines is a generalization of what theorem?
2. If you know all three sides of a triangle and one angle, what might be the advantage of using the Law of Cosines to find another angle, instead of the Law of Sines?
3. Piyali is solving a triangle in which $a = 20$, $b = 16$, and $A = 26°$. She finds that $\sin(B) \approx 0.3507$. How does she know that $B \approx 20.5°$, and not $159.5°$?

BASIC SKILLS

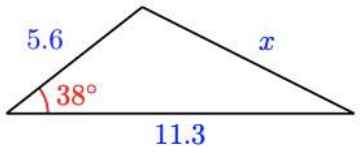
4.
   a. Simplify $5^2 + 7^2 - 2(5)(7)\cos(\theta)$.
   b. Evaluate the expression in part (a) for $\theta = 29°$.
   c. Evaluate the expression in part (a) for $\theta = 151°$.

5.
   a. Simplify $26.1^2 + 32.5^2 - 2(26.1)(32.5)\cos(\phi)$.
   b. Evaluate the expression in part (a) for $\phi = 64°$.
   c. Evaluate the expression in part (a) for $\phi = 116°$.

6.
   a. Solve $b^2 = a^2 + c^2 - 2ac\cos(\beta)$ for $\cos(\beta)$.
   b. For the equation in part (a), find $\cos(\beta)$ if $a = 5$, $b = 11$, and $c = 8$.

7.
   a. Solve $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ for $\cos(\alpha)$.
   b. For the equation in part (a), find $\cos(\alpha)$ if $a = 4.6$, $b = 7.2$, and $c = 9.4$.

For the following exercises, find the indicated side. Round to two decimal places.

8.


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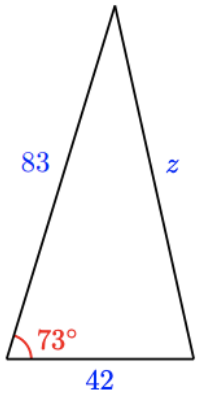

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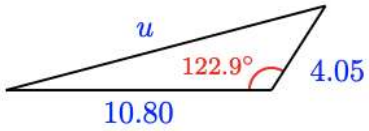

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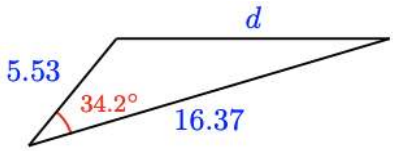

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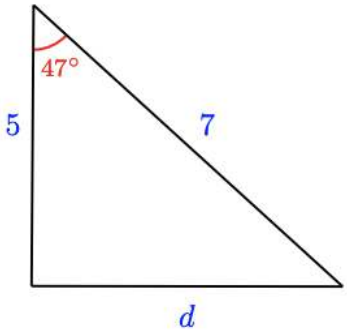

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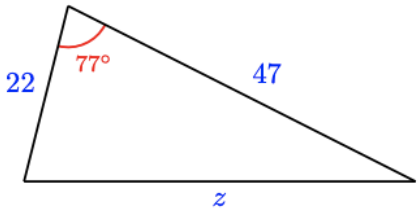
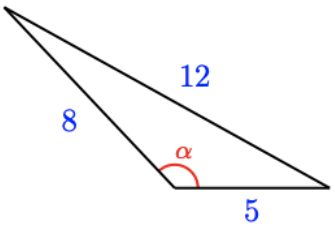


For the following exercises, find the indicated angle. Round to two decimal places.

14.



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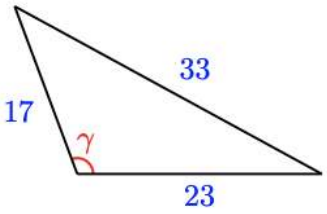
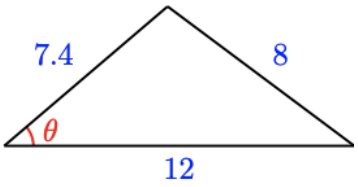
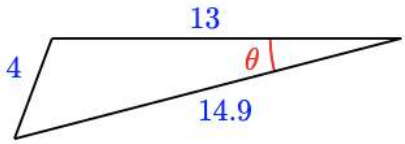


16.



17.



For the following exercises, find the angles of the triangle. Round answers to two decimal places.

18. $a = 23, b = 14, c = 18$

19. $a = 18, b = 25, c = 19$

20. $a = 16.3, b = 28.1, c = 19.4$

21. $a = 82.3, b = 22.5, c = 66.8$

For the following exercises, find the unknown side. Round your answers to two decimal places.

22.

23.

24.

25.

26.

27.

For the following exercises, which law should you use to find the labeled unknown value, the Law of Sines or the Law of Cosines? Write an equation you can solve to find the unknown value. For Problems 32 - 35, you may need two steps to find the unknown value.

28.

29.

30.

31.

32.

33.

34.

35.

For the following exercises,
(a) Sketch and label the triangle.
(b) Solve the triangle. Round answers to two decimal places.

36. $B = 47°, a = 23, c = 17$

37. $C = 32°, a = 14, b = 18$

38. $a = 8, b = 7, c = 9$

39. $a = 23, b = 34, c = 45$

40. $b = 72, c = 98, B = 38°$

41. $a = 28, c = 41, A = 27°$

42. $c = 5.7, A = 59°, B = 82°$

43. $b = 82, A = 11°, C = 42°$

SYNTHESIS QUESTIONS

44. The sides of a parallelogram are 10 inches and 8 inches, and form an angle of $130°$. Find the lengths of the diagonals of the parallelogram.

For the following exercises, find x, the distance from one vertex to the foot of the altitude.

45.

46.

47.

48.

PROOFS

49.

a. Copy the three figures above showing the three possibilities for an angle C in a triangle: C is acute, obtuse, or a right

angle. For each figure, explain why it is true that $c^2 = (b-x)^2 + y^2$, then rewrite the right side to get $c^2 = (x^2+y^2) + b^2 - 2bx$.

b. For each figure, explain why it is true that $x^2 + y^2 = a^2$.

c. For all three figures, a is the distance from the origin to the point (x, y). Use the definition of cosine to write $\cos(C)$ in terms of a and x, then solve your equation for x.

d. Start with the last equation from (a), and substitute expressions from (b) and (c) to conclude one case of the Law of Cosines.

50. Demonstrate the other two cases of the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

and

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

(Hint: See the previous problem and switch the roles of a and c, etc.)

51. Use the Law of Cosines to prove the projection laws:

$$a = b\cos(C) + c\cos(B)$$
$$b = c\cos(A) + a\cos(C)$$
$$c = a\cos(B) + b\cos(A)$$

Illustrate with a sketch. (Hint: Add together two of the versions of the Law of Cosines.)

52. If $\triangle ABC$ is isosceles with $a = b$, show that $c^2 = 2a^2(1 - \cos(C))$.

53. Use the Law of Cosines to prove:

$$1 + \cos(A) = \frac{(a+b+c)(-a+b+c)}{2bc}$$
$$1 - \cos(A) = \frac{(a-b+c)(-a+b+c)}{2bc}$$

54. Prove that

$$\frac{\cos(A)}{a} + \frac{\cos(B)}{b} + \frac{\cos(C)}{c} = \frac{a^2+b^2+c^2}{2abc}$$

APPLICATIONS

55. A surveyor would like to know the distance, PQ, across a small lake, as shown in the figure. She stands at point O and measures the angle between the lines of sight to points P and Q at $76°$. She also finds $OP = 1400$ meters and $OQ = 600$ meters. Calculate the distance PQ.

56. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is $117°$, what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.

57. Highway engineers plan to drill a tunnel through Boney Mountain from G to H, as shown in the figure. The angle at point F is $41°$, and the distances to G and H are 900 yards and 2500 yards, respectively. How long will the tunnel be?

58. From the Pedimaxus International Airport a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of N 8.2° E for 192 miles and it can fly to Bigfoot Falls by following a bearing of S 68.5° E for 207 miles. Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.

59. Cliffs of Insanity Point and Bigfoot Falls from Problem 58 above both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.

60. Two pilots leave an airport at the same time. One pilot flies at a heading of $3°$ and at a speed of 320 miles per hour, the other flies in the heading $157°$ at a speed of 406 miles per hour. How far apart are the two pilots after 3 hours? What is the heading from the first plane to the second plane at that time?

61. Two boats leave port at the same time. One boat sails due west at a speed of 17 miles per hour, the other moves in the heading $42°$ at a speed of 23 miles per hour. How far apart are the two boats after 2 hours? What is the heading from the first boat to the second boat at that time?

62. Linda wants to fly directly south from Indianapolis to Cancun, Mexico, a distance of 1290 miles. However, to avoid bad weather, she flies for 400 miles on a heading of $162°$. What is the heading to Cancun from that location, and how far is it?

63. Lap sails 8 miles from Key West, Florida on a heading of $140°$. He then changes course and sails for 10 miles due east. What is the heading back to Key West from that point, and how far is it?

64. A naturalist sets off on a hike from a lodge on a bearing of S 80° W. After 1.5 miles, she changes her bearing to S 17° W and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.

65. The HMS Sasquatch leaves port on a bearing of N 23° E and travels for 5 miles. It then changes course and follows a bearing of S 41° E for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.

66. The SS Bigfoot leaves a harbor bound for Nessie Island which is 300 miles away at a bearing of N 32° E. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now S 70° W, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.

67. The phone company wants to erect a cell tower on a steep hill inclined $26°$ to the horizontal. The installation crew plans to run a guy wire from a point on the ground 20 feet uphill from the base

of the tower and attach it to the tower at a height of 100 feet. How long should the guy wire be?

68. Sandstone Peak rises 3500 above the desert. The Park Service plans to run an aerial tramway up the north face, which is inclined at an angle of $68°$ to the horizontal. The base station will be located 500 feet from the foot of Sandstone Peak. Ignoring any slack in the cable, how long should it be?

69. From a point 300 feet above level ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression made by the line of sight from the ranger to the first fire is $2.5°$ and the angle of depression made by line of sight from the ranger to the second fire is $1.3°$. The angle formed by the two lines of sight is $117°$. Find the distance between the two fires. Round your answer to the nearest foot. (Hint: In order to use the $117°$ angle between the lines of sight, you will first need to use right angle Trigonometry to find the lengths of the lines of sight. This will allow you to apply the Law of Cosines.)

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11.3: Areas

Note to the Instructor - Why Did We Wait So Long?

I deliberately chose to not cover the included angle formula for the area of a triangle until this section. It was not needed previously, and Heron's formula, by itself, is too short of a topic for its own. Thus, I waited until we had enough mathematical background to do both!

Learning Objectives

- Find the area of a triangle given two sides and the included angle.
- Find the area of a triangle using Heron's Formula.

COSUMNES RIVER COLLEGE

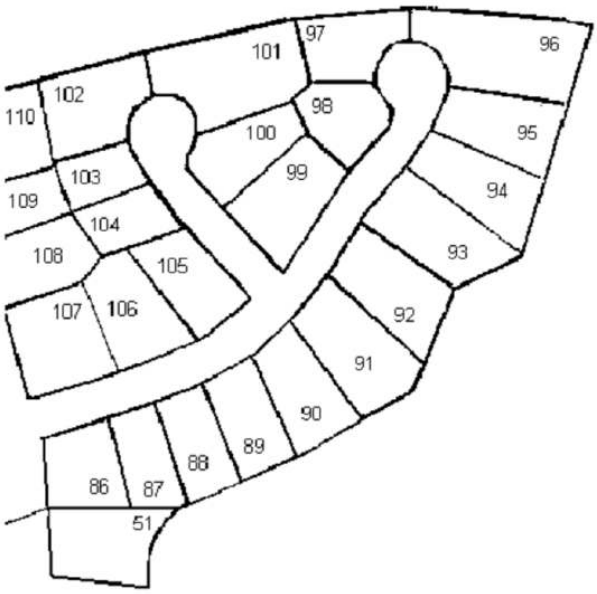
Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The Area of a Triangle

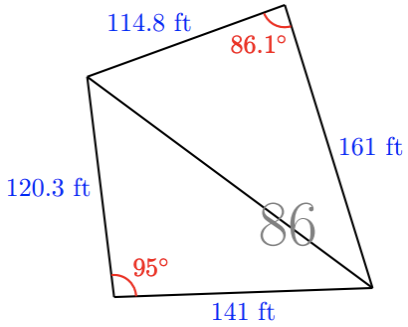
Figure **11.3.1** shows part of the map for a new housing development, Pacific Shores. You are interested in the corner lot, number 86, and you would like to know the area of the lot in square feet. The sales representative for Pacific Shores provides you with the dimensions of the lot. Still, you need to know a formula for the area of an irregularly shaped quadrilateral.

Figure 11.3.1: *Map of Pacific Shores development*



It occurs to you that you can divide the quadrilateral into two triangles and find the area of each.

Figure 11.3.2: *Map of Lot 86*



You know a formula for the area of a triangle in terms of its base and height, namely, $A = \frac{1}{2}bh$, but unfortunately, you don't know the height of either triangle. You can, however, quickly measure the angles at the corners of the lot using the plot map and a protractor (see Figure **11.3.2**). With this information, we can find the area of each triangle via the following theorem.

Theorem: Area of a Triangle

If a triangle has sides of length a and b, and the angle between those two sides is θ, then the area of the triangle is given by

$$A = \frac{1}{2}ab\sin(\theta).$$

Proof

The figure below shows three possibilities for triangles, depending on whether the angle θ is acute, obtuse, or $90°$.

In each case, b is the base of the triangle, and the altitude is h. In all three triangles

$$\sin(\theta) = \frac{h}{a}.$$

(In the third triangle, if $h = a$, then $\frac{h}{a} = 1$. Since the sine of $90°$ is 1, $\sin(\theta) = 1 = \frac{h}{a}$)

Solving for h yields $h = a\sin(\theta)$. Substituting this expression for h into our triangle area formula from Geometry, we get

$$A = \frac{1}{2}bh = \frac{1}{2}ba\sin(\theta) = \frac{1}{2}ab\sin(\theta).$$

Example 11.3.1

Find the area of lot 86.

Solution

For the triangle in the lower portion of lot 86, $a = 120.3$, $b = 141$, and $\theta = 95°$. The area of that portion is

$$\begin{aligned} \text{First Area} &= \frac{1}{2}ab\sin(\theta) \\ &= \frac{1}{2}(120.3)(141)\sin(95^\circ) \\ &\approx 8448.88 \end{aligned}$$

For the triangle in the upper portion of the lot, $a = 161$, $b = 114.8$, and $\theta = 86.1^\circ$. The area of that portion is

$$\begin{aligned} \text{Second Area} &= \frac{1}{2}ab\sin(\theta) \\ &= \frac{1}{2}(161)(114.8)\sin(86.1^\circ) \\ &\approx 9220.00 \end{aligned}$$

The total area of the lot is the sum of the areas of the triangles. Therefore,

$$\text{Total area} = \text{First Area} + \text{Second Area} \approx 17668.88.$$

Lot 86 has an area of approximately 17,669 square feet.

Caution: θ is between the given sides

The formula, $A = \frac{1}{2}ab\sin(\theta)$, does not mean that we always use the sides labeled a and b to find the area of a triangle. The variables a and b represent the lengths of the sides that *include* the known angle. For example, the area of the triangle below is given by $A = \frac{1}{2}(5c)\sin(\phi)$.

Figure 11.3.3

Checkpoint 11.3.1

A triangle has sides of length 6 and 7, and the angle between those sides is 150°. Find the area of the triangle.

Answer

$$\frac{21}{2}$$

Area of a Triangle - Heron's Formula

To use the previous theorem when finding the area of a triangle, we must be given two sides and an included angle; however, is it possible to find the area of a triangle given all three sides but no angles? The answer to this question is yes, and the following theorem, Heron's Formula, explains how to find such an area.

Theorem: Heron's Formula

Suppose a, b, and c denote the lengths of the three sides of a triangle. Let s be the **semiperimeter** of the triangle. That is, let

$$s = \frac{1}{2}(a+b+c).$$

Then the area, A, enclosed by the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof

Using the convention that the angle C is opposite the side c, and borrowing the area theorem from earlier in this section, we have

$$A = \frac{1}{2}ab\sin(C).$$

To simplify computations, we start by manipulating the expression for A^2.

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https://math.libretexts.org/@go/page/197628

$$
\begin{aligned}
A^2 &= \left(\tfrac{1}{2}ab\sin(C)\right)^2 && \text{(squaring both sides of the area formula)}\\
\Longrightarrow\ A^2 &= \frac{1}{4}a^2b^2\sin^2(C) && \text{(Laws of Exponents)}\\
\Longrightarrow\ A^2 &= \frac{a^2b^2}{4}\left(1-\cos^2(C)\right) && \text{(Pythagorean Identity)}\\
\Longrightarrow\ A^2 &= \frac{a^2b^2}{4}\left[1-(\cos(C))^2\right] &&\\
\Longrightarrow\ A^2 &= \frac{a^2b^2}{4}\left[1-\left(\frac{c^2-a^2-b^2}{-2ab}\right)^2\right] && \text{(Law of Cosines)}\\
\Longrightarrow\ A^2 &= \frac{a^2b^2}{4}\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] && \text{(multiplying numerator and denominator by } -1)\\
\Longrightarrow\ A^2 &= \frac{a^2b^2}{4}\left[1-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] && \text{(Laws of Exponents)}\\
\Longrightarrow\ A^2 &= \frac{a^2b^2}{4}\left[\frac{4a^2b^2}{4a^2b^2}-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] && \text{(making equivalent fractions)}\\
\Longrightarrow\ A^2 &= \frac{a^2b^2}{4}\left[\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] && \text{(combining like fractions)}\\
\Longrightarrow\ A^2 &= \frac{\cancel{a^{2^1}}\cancel{b^{2^1}}}{4}\left[\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4\cancel{a^{2^1}}\cancel{b^{2^1}}}\right] && \text{(cancel like factors)}\\
\Longrightarrow\ A^2 &= \frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{16} && \text{(arithmetic)}\\
\Longrightarrow\ A^2 &= \frac{\left(2ab-\left(a^2+b^2-c^2\right)\right)\left(2ab+\left(a^2+b^2-c^2\right)\right)}{16} && \text{(factoring via Difference of Squares)}\\
\Longrightarrow\ A^2 &= \frac{\left(2ab-a^2-b^2+c^2\right)\left(2ab+a^2+b^2-c^2\right)}{16} && \text{(distribution)}\\
\Longrightarrow\ A^2 &= \frac{\left(c^2-a^2+2ab-b^2\right)\left(a^2+2ab+b^2-c^2\right)}{16} && \text{(Commutative Property of Addition)}\\
\Longrightarrow\ A^2 &= \frac{\left[c^2-\left(a^2-2ab+b^2\right)\right]\left[\left(a^2+2ab+b^2\right)-c^2\right]}{16} && \text{(factoring out a negative and grouping)}\\
\Longrightarrow\ A^2 &= \frac{\left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right]}{16} && \text{(factoring trinomials)}\\
\Longrightarrow\ A^2 &= \frac{[c-(a-b)]\,[c+(a-b)]\,[(a+b)-c]\,[(a+b)+c]}{16} && \text{(factoring via Difference of Squares)}\\
\Longrightarrow\ A^2 &= \frac{(c-a+b)\,(c+a-b)\,(a+b-c)\,(a+b+c)}{16} && \text{(distribution)}\\
\Longrightarrow\ A^2 &= \frac{c-a+b}{2}\cdot\frac{c+a-b}{2}\cdot\frac{a+b-c}{2}\cdot\frac{a+b+c}{2} && \text{(arithmetic)}\\
\Longrightarrow\ A^2 &= \frac{c-a+b}{2}\cdot\frac{c+a-b}{2}\cdot\frac{a+b-c}{2}\cdot s && \left(\text{let } s=\frac{a+b+c}{2}\right)\\
\Longrightarrow\ A^2 &= (s-a)\cdot\frac{c+a-b}{2}\cdot\frac{a+b-c}{2}\cdot s && \left(\text{then } s-a=\frac{a+b+c}{2}-a=\frac{b+c-a}{2}\right)\\
\Longrightarrow\ A^2 &= (s-a)\cdot(s-b)\cdot\frac{a+b-c}{2}\cdot s && \left(\text{and } s-b=\frac{a+b+c}{2}-b=\frac{a+c-b}{2}\right)\\
\Longrightarrow\ A^2 &= (s-a)\cdot(s-b)\cdot(s-c)\cdot s && \left(\text{and } s-c=\frac{a+b+c}{2}-c=\frac{a+b-c}{2}\right)\\
\Longrightarrow\ A &= \sqrt{s(s-a)(s-b)(s-c)} && \text{(Extraction of Roots)}
\end{aligned}
$$

$\Longrightarrow \quad A = \sqrt{s(s-a)(s-b)(s-c)}$ (Extraction of roots)

Example 11.3.3

Find the area of the given triangle.

Figure 11.3.4

Solution

Let $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(10+15+7) = \frac{1}{2}(32) = 16$. Then, by Heron's Formula,

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16(16-10)(16-15)(16-7)} = \sqrt{16(6)(1)(9)} = 12\sqrt{6} \approx 29.4 \text{ square units.}$$

Checkpoint 11.3.3

Find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Answer

≈ 552.3

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11.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Geometry**
  - Common Areas: The section builds upon the basic formula for the area of a triangle, $A = \frac{1}{2}bh$, to derive the trigonometric formula for the area. This basic formula is also needed for the "Skills Refresher" exercises.

VIDEOS

- Area of a Triangle
  - Trigonometric Functions of Angles - Areas of Triangles
  - Non-Right Triangles - Area of a Triangle (Part 1)
- Heron's Formula
  - The Law of Cosines - Heron's Formula: Theorem, Proof, and Examples

KEY CONCEPTS

DEFINITIONS

- **Semiperimeter:** Half the perimeter of a triangle. For a triangle with side lengths a, b, and c, the semiperimeter s is calculated as $s = \frac{1}{2}(a+b+c)$.

THEOREMS

- **Area of a Triangle:** If a triangle has two sides of length a and b, and the angle between those two sides is θ, then the area of the triangle is given by $A = \frac{1}{2}ab\sin(\theta)$.
- **Heron's Formula:** If a triangle has sides of length a, b, and c, and the semiperimeter is s, then the area of the triangle is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$.

COMMON MISTAKES

- **Using a Non-Included Angle:** The formula $A = \frac{1}{2}ab\sin(\theta)$ is only valid if the angle θ is the angle included between the sides of length a and b. Using an angle that is not between the two given sides will result in an incorrect area.
- **Confusing Perimeter and Semiperimeter:** Heron's formula requires the semiperimeter (s), which is half the perimeter. Using the full perimeter in the formula will produce an incorrect result.

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11.3.2: HOMEWORK

READING QUESTIONS

1. What three pieces of information about a triangle are required to use the area formula $A = \frac{1}{2}ab\sin(\theta)$?
2. In the formula $A = \frac{1}{2}ab\sin(\theta)$, what is the specific relationship between the angle θ and the sides a and b?
3. What information about a triangle is required to use Heron's Formula?
4. What does the variable s represent in Heron's Formula, and how is it calculated?
5. How can you find the area of an irregular quadrilateral using the methods from this section?
6. The proof of Heron's Formula provided in the text relies on which two other major triangle formulas?

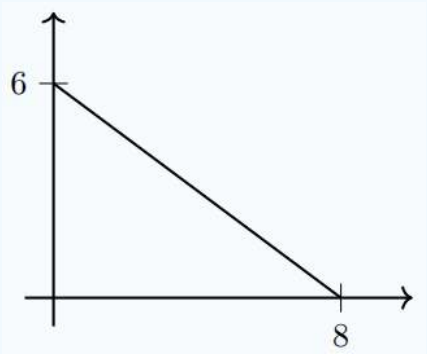
SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, find the area of the triangle.

1.



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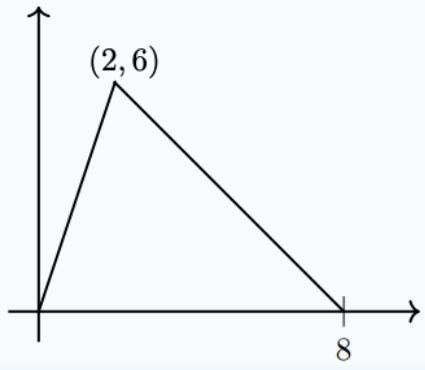

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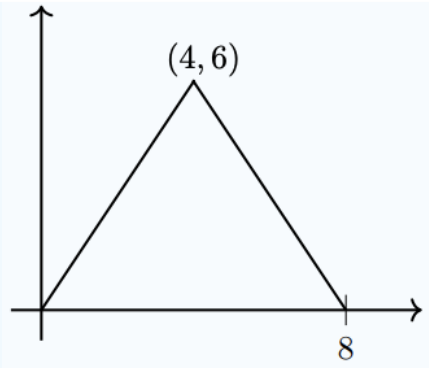

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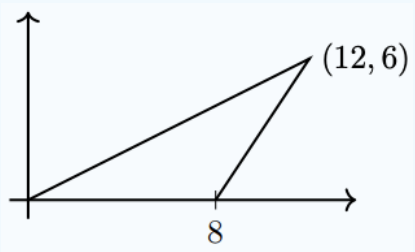


Answers

1. 24
2. 24
3. 24
4. 24

HOMEWORK

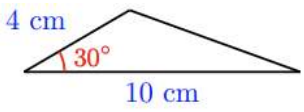
CONCEPT CHECK

1. Explain what s represents in Heron's formula.

BASIC SKILLS

For the following exercises, find an exact value for the area of the triangle.

2.



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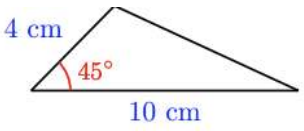

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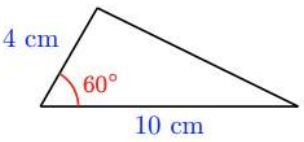
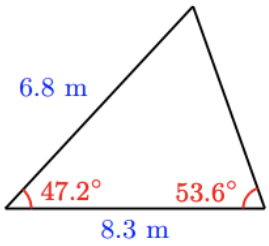
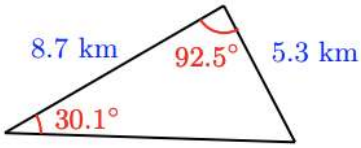


For the following exercises, find the area of the triangle. Round your answer to the nearest tenth.

5.



6.



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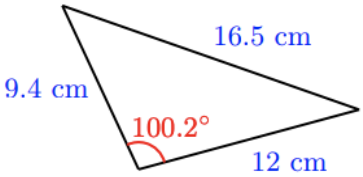
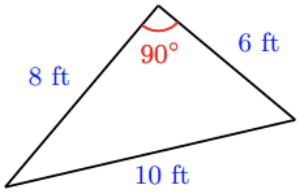
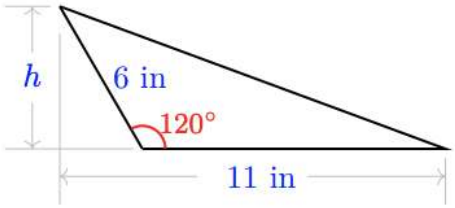


8.

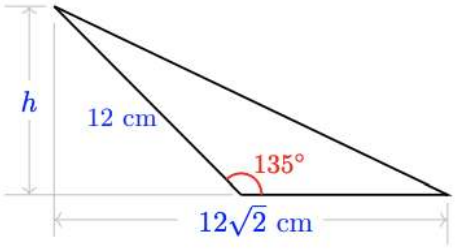


For the following exercises,
(a) Find exact values for the base and height of the triangle.
(b) Compute an exact value for the area of the triangle.

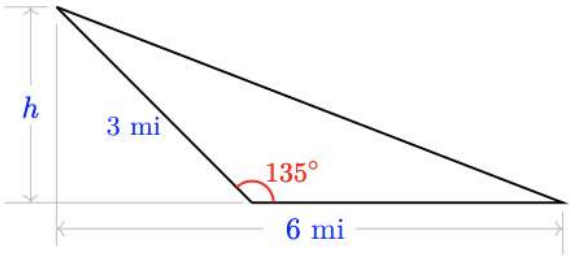
9.



10.



11.



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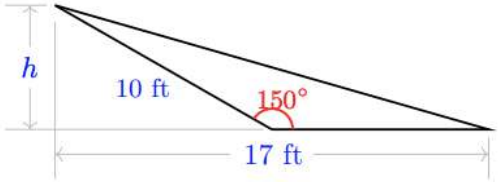

For the following exercises, find the area of the triangle with the given properties. Round to the nearest tenth.

13. (MyOpenMath) one side has length 37 feet, another side has length 11 feet, and the included angle measures $51°$.
14. $b = 2.5$ in, $c = 7.6$ in, $A = 138°$
15. $a = 0.8$ m, $c = 0.15$ m, $B = 15°$
16. the sides of the triangle are 27 cm, 15 cm, and 20 cm.
17. the side lengths are 18 in, 21 in, and 32 in.
18. the triangle has sides of length 20 cm, 26 cm, and 37 cm.
19. (MyOpenMath) An isosceles triangle has an area of 168 square feet, and the angle between the two sides of equal length is $37°$. Find the length of one of the two equal sides. Round your answer to one decimal place.
20. (MyOpenMath) A triangle has an area of 352 square meters, and two sides of the triangle have lengths 42 meters and 19 meters. Find the angle included by these two sides. Find the length of one of the two equal sides. Round your answer to one decimal place.
21. Write an expression for the area of the triangle.

SYNTHESIS QUESTIONS

19. A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.
20. The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is $117°$. What is the area of this quadrilateral?
21. The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is $106°$. What is the area of this quadrilateral?
22. Find the area of the regular pentagon shown below. (Hint: The pentagon can be divided into five congruent triangles.)
23.

a. Find the perimeter of a regular hexagon if the apothegm is 8 cm long. (The apothegm is the segment from the center of the hexagon and perpendicular to one of its sides.)

b. Find the area of the hexagon.

24. Find the area of the regular hexagon shown below. (Hint: The hexagon can be divided into six congruent triangles.)

APPLICATIONS

25. Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

26. Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

For the following exercises, lots from a housing development have been subdivided into triangles. Find the total area of each lot by computing and adding the areas of each triangle.

27.

28.

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11.4: Vectors - A Geometric Approach

Note to the Instructor - What Goes Where?

It was this specific section (and the next one as well) that prompted me to write these "Note to the Instructor" messages. As I was building the online homework for the sections, I had a hard time remembering which topics go where when it came to vectors.

This section introduces vectors from a geometric perspective. **It is important to realize that all vectors in this section are represented either geometrically or through verbal description (in the case of applications). Components of a vector, $\vec{i}$-$\vec{j}$ notation, and the like are held off for the next section.** The topics include:

- geometrically defining...
  - vectors
  - operations on vectors
  - vector magnitude
  - resultant vectors
- interpreting...
  - true heading and velocity as associated with the resultant vector
- computing...
  - the angle a vector makes with the positive x-axis (not given in component form)
  - the magnitude of a vector (not given in component form)

Learning Objectives

- Sketch vectors and sums of vectors.
- Find the magnitude of the horizontal and vertical vector components for a vector.
- Find a vector's magnitude and its angle with the positive x-axis.
- Solve an applied problem using vectors.

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Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Introduction

One way to specify a location is to give a direction and a distance from a fixed landmark. For example, the airport is 8 miles northeast of the town hall, or a ship has been sighted 20 miles from the lighthouse in the direction $10°$ west of north. It's not enough to give just the distance to the object or just the direction; we need both to describe the object's location. Thus, we have the following definition.

Definition: Vector

In Mathematics, a quantity defined by both a magnitude and a direction is called a **vector**.

Vectors are used to analyze motion and velocity in multiple dimensions, study forces such as gravity and electric fields, and design computer graphics and animation. They are also used throughout Engineering, Mathematics, and Physics.

The types of vectors we will concern ourselves with in this section are position vectors.

Definition: Position Vector

A **position vector** is a vector used to designate a location relative to a fixed landmark.

We often illustrate a vector with an arrow. The length of the arrow represents the magnitude (size) of the vector, and the head of the arrow indicates its direction.

Example 11.4.1

Draw an arrow to represent each vector.

a. The campsite is 3 miles away, in a direction $30°$ north of east.
b. The wind is blowing due west at 50 kilometers per hour.

Solutions

a. We draw an arrow making an angle of $30°$ from east. The length of the arrow is 3 units, representing 3 miles. See Figure **11.4.1***a*.
b. We draw an arrow pointing due west. That is, making an angle of $180°$ from east. The arrow's length represents the wind's speed, 50 kilometers per hour. See Figure **11.4.1***b*.

Figure 11.4.1

In both images in Figure **11.4.1**, we intuitively understand that the direction of the vector is where the arrow is pointing; however, to be able to discuss vectors better, it's best to laboriously define the geometric parts of the graphical representation of a vector.

Definition: Head and Tail of a Vector

Given the graph of a vector, which is a finite-length ray, we define the **head** as the location of the terminal end of the ray (the arrowhead) and the **tail** as the initial point of the ray (the non-arrowhead end).

In Figure **11.4.1***a*, the head of the vector is the campsite, and the tail is the origin.

Checkpoint 11.4.1

Draw an arrow representing the velocity of an airplane traveling southeast at 300 miles per hour.

Answer

Figure 11.4.2

Notation for Vectors

Some texts use boldface characters such as $\mathbf{u}$ and $\mathbf{v}$ to represent vectors; however, this is difficult to replicate by hand. Therefore, we choose the alternative notation, $\vec{u}$ and $\vec{v}$, which uses arrows above a variable to indicate that it is a vector.

It is vital to distinguish vector quantities from ones with magnitude only, such as length or temperature.

Definition: Scalar

A quantity having only magnitude (and no direction) is called a **scalar**.

Therefore, scalars are the constants and variables you are used to dealing with, usually denoted by italic letters such as x or k. They have a magnitude (size), which we typically consider their absolute value; however, we refine the definition of magnitude when speaking of vectors.

Definition: Magnitude of a Vector (geometric)

The length of a vector $\vec{v}$ is called its **magnitude** and is denoted by $\|\vec{v}\|$.

Caution: Magnitude is a Scalar Quantity

Note that the magnitude of a vector is a scalar quantity. Thus, $\vec{v}$ denotes a vector (a distance and direction), but $\|\vec{v}\|$ represents a scalar (the length of the vector).

Figure **11.4.3** shows two vectors, $\vec{u}$ and $\vec{v}$, that have the same magnitude but different directions. For this example, $\|\vec{u}\| = \|\vec{v}\|$ but $\vec{u} \neq \vec{v}$. It makes sense that two vectors may have the same length but point in different directions.

Figure 11.4.3

This begs the question: what does it mean for two vectors to be equal? Believe it or not, we have to define what this means!

Definition: Vector Equality (geometric)

Two vectors are **equal** if and only if they have the same length and direction.

Thus, we could have two equal vectors (same magnitude and direction) but with their tails (and, therefore, heads) at different locations. The fact that we can move vectors from one location to another is valuable.

Example 11.4.2

Which of the vectors are equal?

Figure 11.4.4

Solution

Only vectors $\vec{c}$ and $\vec{g}$ are equal. Vectors $\vec{b}$, $\vec{c}$, and $\vec{f}$ are all the same length but have different directions. Vectors $\vec{a}$ and $\vec{d}$ also have equal lengths but different directions. Vectors $\vec{d}$ and $\vec{e}$ have the same direction but different lengths.

Checkpoint 11.4.2

Sketch a vector $\vec{w}$ that is equal to the vector $\vec{v}$, but with tail at the point $(-4, 2)$.

Figure 11.4.5

Answer

Figure 11.4.6

Scalar Multiplication of Vectors

We can define multiplication of a vector by a scalar.

Definition: Scalar Multiplication (geometric)

The notation $k\vec{v}$, where k is a scalar, is called the **scalar multiple** (or **scalar product**) of the vector $\vec{v}$ with the scalar k and it is defined to be the vector having same direction as $\vec{v}$, but magnitude k times the magnitude of $\vec{v}$.

Figure **11.4.7** shows the vector $\vec{w}$ and the vector $\vec{v} = 3\vec{w}$. Multiplying by a positive scalar changes a vector's length (magnitude) but not its direction.

Figure 11.4.7

Thus, the vector $\vec{v}$ is 3 times as long as the vector $\vec{w}$. If we multiply a vector by a negative scalar, we alter its length and reverse its direction; that is, we change the direction by $180°$. For example, the vector $\vec{u} = -\frac{1}{2}\vec{w}$. It is half the length of $\vec{w}$ and points in the opposite direction.

In general, if k is a real number, then $k\vec{v}$ represents the vector with magnitude k times the magnitude of $\vec{v}$. It points in the same direction as $\vec{v}$ when $k > 0$ and in the opposite direction from $\vec{v}$ when $k < 0$. Real numbers are called scalars because they "scale" vectors in this way.

Example 11.4.3

The figure below shows the vector $\vec{v}$ and two scalar multiples of $\vec{v}$.

Figure 11.4.8

The vector $\frac{2}{3}\vec{v}$ points in the same direction as $\vec{v}$, but is only two-thirds as long as the vector $\vec{v}$. The vector $-\sqrt{3}\vec{v}$ points in the direction opposite to $\vec{v}$ and is $\sqrt{3}$ or approximately 1.7 times as long as $\vec{v}$.

Note that because $\vec{v}$ has a slope of $\frac{-2}{5}$, so does any nonzero multiple of $\vec{v}$.

Checkpoint 11.4.3

For the vector $\vec{w}$ shown below, draw the vectors $-0.6\vec{w}$ and $\sqrt{2}\vec{w}$.

Figure 11.4.9

Answer

Figure 11.4.10

Addition of Vectors

Definition: Displacement Vector

A **displacement vector** represents the change in position from one point to another.

For example, you leave home and travel 6 miles east and 8 miles north. These two displacements are represented by the vectors $\vec{u}$ and $\vec{v}$, in Figure 11.4.11. When we follow one displacement vector with a second one, the net displacement is a new vector, $\vec{w}$, starting at the tail of the first vector and ending at the head of the second vector.

Figure 11.4.11

In this example, notice that $\vec{w}$ forms the hypotenuse of a right triangle, so we can calculate its magnitude and direction.

$$\|\vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 = 6^2 + 8^2 = 100.$$

Therefore,

$$\|\vec{w}\| = 10.$$

To find the direction of the vector $\vec{w}$, we rely on the methods we have established up to this point in Trigonometry.

$$\begin{aligned} & \tan(\theta) = \frac{8}{6} \\ \Longrightarrow \quad & \tan(\theta) = \frac{4}{3} \\ \Longrightarrow \quad & \hat{\theta} = \tan^{-1}\left(\frac{4}{3}\right) \qquad \text{(finding the reference angle)} \end{aligned}$$

Since $0° < \theta < 90°$, $\theta = \hat{\theta} \approx 53.1°$.

The net displacement gives your current position relative to home: 10 miles in the direction $53.1°$ north of east.

When we follow one vector by a second vector as described above, we are adding the two vectors. As Figure 11.4.11 illustrates, the sum of vectors results in a new vector. This new vector gets a special name.

Definition: Resultant Vector

The vector resulting from a sum of vectors is the **resultant vector**.

Example 11.4.4

For each pair of vectors, draw the resultant vector $\vec{w} = \vec{u} + \vec{v}$.

Figure 11.4.12

Solutions

We draw a copy of the vector $\vec{v}$ so that its tail (or starting point) is placed on the head (or ending point) of $\vec{u}$, and then we draw vector $\vec{w}$ from the tail of $\vec{u}$ to the head of $\vec{v}$.

Figure 11.4.13

Checkpoint 11.4.4

Draw the resultant vector $\vec{w} = \vec{u} + \vec{v}$

Figure 11.4.14

Answer

Figure 11.4.15

When the resultant vector forms the third side of a triangle, we can calculate its length and direction using the Law of Sines and the Law of Cosines.

Example 11.4.5

You are camping in a state park. In the morning, you start from your campground at point C and hike 4 miles bearing $\mathbf{S\,45^\circ\,W}$ to point B. After taking a break, you continue hiking, and this time you cover 3 miles in the bearing $\mathbf{N\,30^\circ\,E}$, and arrive at point A, as shown below. What is the position of the campground relative to your position?

Figure 11.4.16

Solution

The resultant vector, $\vec{w}$, forms the side opposite a 15° angle.

$$\begin{aligned} \|\vec{w}\|^2 &= a^2 + c^2 - 2ac\cos(\angle ABC) && \text{(Law of Cosines)} \\ \Longrightarrow \quad \|\vec{w}\| &= \sqrt{a^2 + c^2 - 2ac\cos(\angle ABC)} && \text{(Extraction of Roots)} \end{aligned}$$

In this case, $a = 4$, $c = 3$, and $\angle ABC = 15^\circ$. Substituting these values into our result, we get $\|\vec{w}\| \approx 1.348$, so you are approximately 1.348 miles from the campground. To find the direction back to the campground, we must calculate the angle $\angle ACB$. Following the advice in the section covering the Law of Cosines, we find the angle opposite the smallest remaining side. Luckily, that angle is $\angle ACB$.

$$\frac{\sin(\angle ACB)}{c} = \frac{\sin(\angle ABC)}{b} \quad \text{(Law of Sines)}$$

$$\implies \sin(\angle ACB) = \frac{c\sin(\angle ABC)}{b} \quad \text{(multiplying both sides by } c\text{)}$$

$$\implies \widehat{\angle ACB} = \sin^{-1}\left(\frac{c\sin(\angle ABC)}{b}\right) \quad \text{(finding the reference angle)}$$

Since $\angle ACB < 90°$, $\angle ACB = \widehat{\angle ACB} \approx 35.2°$. It's best to get a visual of our angles, so we redraw the figure with the newly found angles.

Figure 11.4.17

Since $\angle ABC = 15°$ and $\angle ACB \approx 35.2°$, $\angle BAC \approx 129.8°$. However, this means

$$30° + 90° + (90° - \theta) \approx 129.8° \implies \theta \approx 80.2°.$$

Thus, the campground is 1.348 miles away at a bearing of N 80.2° E.

? Checkpoint 11.4.5

Ron has gone sailing with friends. After leaving the marina, they sail 5 miles on a heading of 160° and stop at Gull Island for lunch. After lunch, they have sailed for 3 miles on a heading of 25° when they get a phone call to return home. What heading is the most direct route back to the marina, and how far is it?

Answer

Heading of 303.6° for 3.58 miles

It doesn't matter in which order we choose to add two vectors. As with the ordinary addition of scalars, the addition of vectors is commutative, so that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. To see this, first draw $\vec{u}$ and $\vec{v}$ starting at the same point.

Figure 11.4.18

- To represent $\vec{u} + \vec{v}$, we place the tail of $\vec{v}$ at the head of $\vec{u}$, and draw the resultant vector $\vec{u} + \vec{v}$ as shown in Figure 11.4.18.
- To represent $\vec{v} + \vec{u}$, we place the tail of $\vec{u}$ at the head of $\vec{v}$, and the resultant vector $\vec{v} + \vec{u}$ is the same as the vector $\vec{u} + \vec{v}$.

Because the vector sum forms the diagonal of a parallelogram in this picture, the rule for adding vectors is sometimes called the **parallelogram rule**.

We summarize the operations on vectors as follows.

Summary: Operations on Vectors

- We can multiply a vector, $\vec{v}$, by a scalar, k.
  - If $k > 0$, the magnitude of $k\vec{v}$ is k times the magnitude of $\vec{v}$. The direction of $k\vec{v}$ is the same as the direction of $\vec{v}$.
  - If $k < 0$, the direction of $k\vec{v}$ is opposite the direction of $\vec{v}$.
- We can add two vectors $\vec{v}$ and $\vec{w}$ with the parallelogram rule.

Caution: Lengths of vectors don't add (in general)

Unless $\vec{u}$ and $\vec{v}$ are parallel vectors, it is not true that the length of $\vec{u} + \vec{v}$ is just the sum of the lengths of $\vec{u}$ and $\vec{v}$. Vector addition is a geometric operation; the length of $\vec{u} + \vec{v}$ depends on the lengths of $\vec{u}$ and $\vec{v}$ and on the angle between them. Be careful to distinguish between regular addition of scalars, such as $\|\vec{u}\| + \|\vec{v}\|$, and vector addition, $\vec{u} + \vec{v}$, which requires the parallelogram rule.

Velocity

Physicists and mathematicians use the word velocity to mean not simply speed but the combination of speed and direction of motion. The interesting thing about velocities is that they add like vectors; if an object's motion consists of two simultaneous components, the resulting displacement is the same as if the motions had occurred one after the other.

For example, imagine a beetle walking across a moving conveyor belt, as shown in the leftmost image in Figure **11.4.19**. Suppose the beetle starts at the lower left corner. The conveyor belt is moving at a speed of 4 inches per second, and the beetle walks at right angles to the motion of the belt at 2 inches per second. After 1 second, the beetle has traveled upwards from its starting point, a total distance of 2 inches. Simultaneously, the conveyor belt has moved the beetle to the right a distance of 4 inches. Thus, the beetle has moved a distance of $\sqrt{2^2+4^2}=\sqrt{20}$, or about 4.47 inches. His actual velocity relative to the ground is approximately 4.47 inches per second at an angle of $\theta=\tan^{-1}\left(\frac{2}{4}\right)=26.6°$ from the direction of the conveyor belt.

Figure 11.4.19

Notice that the resulting displacement would be the same if the two motions were performed in succession instead of simultaneously, as shown in the rightmost image in Figure **11.4.19**. In other words, if the beetle had walked across the belt for 1 second before the belt started moving and then ridden the belt for 1 second without walking, he would still end up at point Q.

Thus, velocity is a vector quantity, and we can calculate the result of two simultaneous motions using the parallelogram rule. We treat the two motions as if they had occurred one after the other by starting one vector at the endpoint of the other. (Remember, we can move a vector from one location to another if preserve its length and direction.)

Example 11.4.6

A ship travels 15 miles per hour relative to the water on a heading of $280°$. The water current flows 6 miles per hour on a heading of $160°$. What is the actual speed and direction of the ship?

Solution

We represent the ship's velocity by a vector $\vec{v}$ and the velocity of the water current by vector $\vec{w}$, as shown in Figure **11.4.20***a*. Note that the angle $\vec{v}$ makes with the horizontal is $10°$, and angle $\vec{w}$ makes with the vertical is $20°$ - these will be important in a few moments.

Figure 11.4.20

The actual motion of the ship is the sum of these two vectors. We can calculate the sum as if the two motions had occurred separately, one after the other, as shown in Figure **11.4.20***b*. So we will add $\vec{w}$ to $\vec{v}$ by placing the tail of $\vec{w}$ at the head of $\vec{v}$. The resultant vector, $\vec{u}$, represents the actual motion of the ship. Let's zoom in to Figure **11.4.20***b*.

Figure 11.4.21

The angle α in Figure **11.4.21** is alternate interior to the $10°$ angle $\vec{v}$ makes with the horizontal, so $\alpha=10°$. Moreover, we already mentioned that $\vec{w}$ makes a $20°$ with the vertical. Therefore, if we let β be the angle between the head of $\vec{v}$ and the tail of $\vec{w}$ (as seen in Figure **11.4.21**), we have $20°+\beta+10°=90°$, so $\beta=60°$. Hence,

$$\begin{aligned} \|\vec{u}\|^2 &= \|\vec{v}\|^2+\|\vec{w}\|^2-2\|\vec{v}\|\cdot\|\vec{w}\|\cos(\beta) && \text{(Law of Cosines)} \\ \Longrightarrow \quad \|\vec{u}\| &= \sqrt{\|\vec{v}\|^2+\|\vec{w}\|^2-2\|\vec{v}\|\cdot\|\vec{w}\|\cos(\beta)} && \text{(Extraction of Roots)} \end{aligned}$$

Substitution in $\|\vec{v}\|=15$, $\|\vec{w}\|=6$, and $\beta=60°$, we find $\|\vec{u}\|=\sqrt{171}\approx 13.1$. Thus, the ship's speed is approximately 13.1 miles per hour.

Next, we calculate the ship's heading (also known as its **true heading**). This means we need to calculate another angle within the triangle shown in Figure 11.4.21. We already know β. Of the remaining angles, we want to find the measure of γ (mostly because it is opposite the smallest remaining side, but also because this is the angle that will help us calculate the ship's true ehading).

$$\begin{aligned} \frac{\sin(\gamma)}{\|\vec{w}\|} &= \frac{\sin(\beta)}{\|\vec{u}\|} && \text{(Law of Sines)} \\ \Longrightarrow \quad \sin(\gamma) &= \frac{\|\vec{w}\|\sin(\beta)}{\|\vec{u}\|} && \text{(multiplying both sides by } \|\vec{w}\|) \\ \Longrightarrow \quad \hat{\gamma} &= \sin^{-1}\left(\frac{\|\vec{w}\|\sin(\beta)}{\|\vec{u}\|}\right) && \text{(finding the reference angle)} \end{aligned}$$

Since γ is opposite the smallest remaining side, we can guarantee $0° < \gamma < 90°$. Hence, $\gamma = \hat{\gamma} \approx 23.5°$.

Looking back at Figure 11.4.21, we see the angle that $\vec{u}$ makes with the horizontal is $\gamma - 10° = 13.4°$. Hence, the ship is heading $270° - 13.4° = 256.6°$ at 13.1 miles per hour.

? Checkpoint 11.4.6

A plane heads due north at an airspeed of 120 miles per hour. A 45 mph wind blows at a heading of $95°$. What are the plane's actual speed and direction relative to the ground?

Answer

Heading of $21.1°$ moving at 124.4 mph.

In some situations, we would like to find a vector to produce a particular sum instead of calculating a vector. That is, we know vectors $\vec{u}$ and $\vec{w}$, and we want to find a vector $\vec{v}$ so that $\vec{u} + \vec{v} = \vec{w}$.

✔ Example 11.4.7

Phuong intends to travel west to an island at 15 miles per hour. However, she must compensate for a current running at a bearing of $\mathbf{N\,45°\,E}$ at a speed of 3 miles per hour. In what direction and at what speed should Phuong move her boat?

Solution

We draw a triangle using vectors to represent the desired velocity of Phuong's boat, $\vec{w}$, and the velocity of the current, $\vec{u}$, as shown in Figure 11.4.22. We want to find a vector $\vec{v}$ that represents the speed and bearing Phuong should take to compensate for the current.

Figure 11.4.22

We first use the Law of Cosines to calculate the length or magnitude of the vector $\vec{v}$. The angle at C is $135°$. (Do you see why?). Thus,

$$\begin{aligned} \|\vec{v}\|^2 &= 3^2 + 15^2 - 2(3)(15)\cos(135°) \\ &\approx 297.64 \\ \Longrightarrow \quad \|\vec{v}\| &\approx 17.25 \end{aligned}$$

So, Phuong should travel at about 17.25 miles per hour. To find her bearing, we use the Law of Sines to calculate the angle θ.

$$\begin{aligned} \frac{\sin(\theta)}{3} &= \frac{\sin(45^\circ)}{15} \\ \implies \sin(\theta) &= \frac{3/\sqrt{2}}{15} \\ \implies \theta &= \sin^{-1}\left(\frac{1}{5\sqrt{2}}\right) \\ &\approx 8.1^\circ \end{aligned}$$

This means that the angle $\vec{v}$ makes with the vertical is approximately 81.9°. Therefore, Phuong should move her boat in the bearing of $\mathrm{S}\,81.9^\circ\,\mathrm{W}$.

? Checkpoint 11.4.7

Matt would like to sail at 20 kilometers per hour due west towards a whale reported at that position. However, a steady ocean current is moving 52° east of north at 8 kilometers per hour. At what speed and heading should Matt sail?

Answer

$26.76\mathrm{kph}$ and $10.6^\circ\mathrm{S}$ of W

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11.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

KEYWORDS

- Vector
  - Vectors - A Geometric Approach

KEY CONCEPTS

DEFINITIONS

- **Vector:** A mathematical quantity defined by both a magnitude and a direction.
- **Position Vector:** A vector used to specify a location relative to a fixed landmark.
- **Head and Tail of a Vector:** For a vector represented by an arrow, the head is the terminal end (arrowhead), and the tail is the initial point.
- **Scalar:** A quantity that has only magnitude and no direction.
- **Magnitude of a Vector (geometric):** The length of a vector $\vec{v}$, denoted by $||\vec{v}||$.
- **Vector Equality (geometric):** Two vectors are equal if and only if they have the same magnitude and direction.
- **Scalar Multiplication (geometric):** The product of a scalar k and a vector $\vec{v}$, denoted $k\vec{v}$, is a vector with the same direction as $\vec{v}$ if $k > 0$ and the opposite direction if $k < 0$. Its magnitude is $|k|$ times the magnitude of $\vec{v}$.
- **Displacement Vector:** A vector that represents the change in position from one point to another.
- **Resultant Vector:** The vector that results from the sum of two or more vectors.

COMMON MISTAKES

- **Confusing Vectors and Scalars:** A vector has both magnitude and direction, while a scalar has only magnitude. The magnitude of a vector, $||\vec{v}||$, is a scalar.
- **Adding Magnitudes Instead of Vectors:** The magnitude of a sum of vectors, $||\vec{u}+\vec{v}||$, is generally not equal to the sum of their magnitudes, $||\vec{u}|| + ||\vec{v}||$. Vector addition must be performed geometrically using the head-to-tail or parallelogram rule.

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11.4.2: HOMEWORK

READING QUESTIONS

1. What two properties define a vector?
2. What is the difference between a vector and a scalar?
3. In a graphical representation of a vector, what does the length of the arrow represent? What does the arrowhead represent?
4. When are two vectors considered equal?
5. What is the term for the length of a vector?
6. Describe what happens to a vector when it is multiplied by a positive scalar $k > 1$.
7. Describe what happens to a vector when it is multiplied by a negative scalar, such as -1.
8. What is a resultant vector?
9. Explain the "head-to-tail" method for adding two vectors.
10. What is the "parallelogram rule" for vector addition?
11. Is the magnitude of the sum of two vectors, $||\vec{u} + \vec{v}||$, typically equal to the sum of their individual magnitudes, $||\vec{u}|| + ||\vec{v}||$? Why or why not?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

1. State two versions of the Law of Sines.
2. State two versions of the Law of Cosines.

For the following exercises, find the unknown part of the triangle. Round to two decimal places.

3. $A = 15°$, $B = 125°$, and $b = 12$ cm. Find a.
4. $C = 87°$, $b = 11$ inches, and $c = 13$ inches. Find B.
5. $A = 37°$, $b = 6$, and $c = 14$. Find a.
6. $a = 9$, $b = 4$, and $c = 7$. Find B.

Answers

1. $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}, \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
2. $a^2 = b^2 + c^2 - 2bc\cos(A), b^2 = a^2 + c^2 - 2ac\cos(B)$
3. 3.79
4. $57.67°$
5. 9.89
6. $25.21°$

HOMEWORK

VOCABULARY CHECK

1. A quantity defined by both a magnitude (such as a distance) and a direction is called a ___.
2. Two vectors are equal if they have the same ___ and ___; it does not matter where the vector starts.
3. A quantity having only magnitude (and no direction) is called a ___.
4. The length of a vector $\vec{v}$ is called its ___, and is denoted by $\|\vec{v}\|$.
5. The sum of two vectors $\vec{u}$ and $\vec{v}$ is a new vector, $\vec{w}$, starting at the ___ of the first vector and ending at the ___ of the second vector.
6. The sum of two vectors is called the ___ vector.
7. The rule for adding vectors is sometimes called the ___ rule.

CONCEPT CHECK

8. What is the difference between a scalar and a vector?
9. If velocity is represented by a vector, what is its magnitude called?
10. Does $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$?
11. Does $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$?
12. What is the parallelogram rule?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

13. If $k < 0$, the magnitude of $k\vec{v}$ is k times the magnitude of $\vec{v}$.
14. The direction of $k\vec{v}$ is the same as the direction of $\vec{v}$.

BASIC SKILLS

For the following exercises, sketch a vector to represent the quantity.

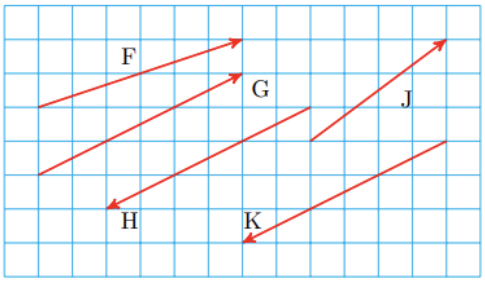
15. The waterfall is 3 km away, at a bearing of $\mathbf{S\,75°\,W}$.
16. The cave entrance is 450 meters away, at a bearing of $\mathbf{N\,45°\,E}$.
17. The current is moving 6 feet per second at a heading of $60°$.
18. The bird is flying due south at 45 miles per hour.
19. The projectile was launched at a speed of 40 meters per second, at an angle of $30°$ above horizontal.
20. The baseball was hit straight up at a speed of 60 miles per hour.

For the following exercises, which vectors are equal?

21.

22.

23.



24.

For the following exercises, sketch a vector equal to $\vec{v}$, but starting at the given point.

25.

26.

27.

28.

For the following exercises, draw the scalar multiples of the given vectors.

29.

$-2v$ and $1.5v$

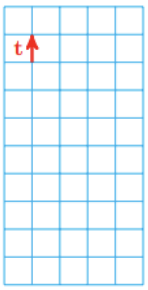
30.

$-\frac{1}{2}w$ and $3w$

31.

$-2.5u$ and $\sqrt{2}u$

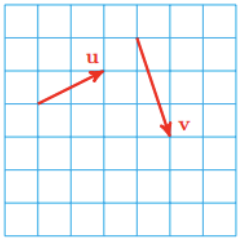
32.



$-\sqrt{6}t$ and $5.4t$

For the following exercises,
(a) draw the resultant vector,
(b) calculate the length and direction of the resultant vector.

33. $\vec{A} = \vec{u} + \vec{v}$

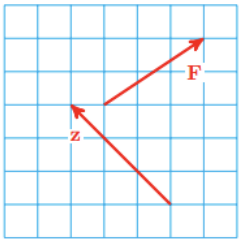


34. $\vec{B} = \vec{z} + \vec{u}$

35. $\vec{C} = \vec{w} + \vec{u}$

36. $\vec{D} = \vec{G} + \vec{z}$

37. $\vec{E} = \vec{z} + \vec{F}$



38. $\vec{F} = \vec{w} + \vec{v}$

39. $\vec{G} = \vec{w} + \vec{w}$

40. $\vec{H} = \vec{G} + \vec{G}$

For the following exercises, find the magnitude and direction of the vector.

41. $v_x = 5, v_y = -12$

42. $v_x = -8, v_y = 15$

43. $v_x = -6, v_y = -7$

44. $v_x = 1, v_y = -3$

For the following exercises, sketch the vectors, then calculate the resultant.

45. Add the vector $\vec{v}$ of length 45 pointing $26°$ east of north to the vector $\vec{w}$ of length 32 pointing $17°$ south of west.

46. Add the vector $\vec{v}$ of length 105 pointing $41°$ west of south to the vector w of length 77 pointing $8°$ west of north.

47. Let $\vec{v}$ have length 8 and point in the heading $10°$. Let $\vec{w}$ have length 13 and point in the heading $250°$. Find $\vec{v} + \vec{w}$.

48. Let a have length 43 and point in the heading $343°$. Let $\vec{b}$ have length 19 and point in the heading $141°$. Find $\vec{a} + \vec{b}$.

SYNTHESIS QUESTIONS

Multiplying a vector $\vec{v}$ by -1 gives a vector $-\vec{v}$ that has the same magnitude as $\vec{v}$ but points in the opposite direction. We define subtraction of two vectors the same way we define subtraction of integers:

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}).$$

That is, to subtract a vector $\vec{v}$, we add its opposite.

For the following exercises, draw the resultant vector.

49. $\vec{A} = \vec{u} - \vec{v}$

50. $\vec{B} = \vec{F} - \vec{z}$

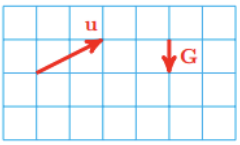
51. $\vec{C} = \vec{v} - \vec{u}$

52. $\vec{D} = \vec{z} - \vec{G}$

53. $\vec{P} = \vec{w} - \vec{F}$

54. $\vec{Q} = \vec{u} - \vec{w}$

55. $\vec{R} = \vec{G} - \vec{u}$



56. $\vec{S} = \vec{v} - \vec{F}$

APPLICATIONS

For the following exercises, sketch the vectors, then calculate the resultant.

57. Loi swam 3.6 miles at a bearing of $\mathbf{N\ 20^\circ\ E}$. However, the water current displaced her by 0.9 miles at a bearing of $\mathbf{N\ 37^\circ\ E}$. How far is Loi from her starting point, and at what bearing?

58. Paige paddles her canoe 4.5 miles at a bearing of $\mathbf{N\ 12^\circ\ W}$. The water current pushes her 0.3 miles off course at a bearing of $\mathbf{N\ 5^\circ\ E}$. How far is Paige from her starting point, and at what bearing?

59. Nhat wants to fly to an airport that is 103 miles due west in 1 hour. The prevailing winds blow at a heading of $\mathbf{112^\circ}$ at 28 miles per hour, so Nhat will head her plane somewhat north of due west to compensate. What airspeed and heading should Nhat take?

60. Nam wants to cross a 300 meter wide river, but the river is running due south at 80 meters per minute. There are rocks upstream and rapids downstream, so he wants to paddle straight across from east to west. At what heading should he point his kayak, and how fast should his water speed be in order to cross the river in 2 minutes? (Hint: The current will move him 160 meters due south compared with where his speed and heading would take him if the current stopped. Compute the distance he would have traveled, then divide by 2 minutes to get the speed.)

61. A ship maintains a heading of $\mathbf{30^\circ}$ and a speed of 20 miles per hour. There is a current in the water running at a heading of $\mathbf{135^\circ}$ and at a speed of 10 miles per hour. What is the actual heading and speed of the ship?

62. A plane is heading due south, with an airspeed of 180 kilometers per hour. The wind is blowing at 50 kilometers per hour at a heading of $\mathbf{225^\circ}$. What is the actual heading and speed of the plane?

63. The campground is 3.6 kilometers from the trail head at a bearing of $\mathbf{N\ 20^\circ\ W}$. A ranger station is located 2.3 kilometers from the campsite at a bearing of $\mathbf{S\ 8^\circ\ W}$. What is the distance and bearing from the trail head to the ranger station?

64. There is treasure buried 40 paces due east from a dead tree. From the buried treasure, a hidden mine shaft is 100 paces at a bearing of $\mathbf{N\ 58^\circ\ W}$. What is the distance and bearing from the dead tree to the mine shaft?

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11.5: Vectors - An Algebraic Approach

Note to the Instructor - The Rest of Vectors

Now that students have (hopefully) a decent grasp of vectors from a geometric perspective, this section introduces vectors from an algebraic perspective. Topics include:

- the component form of a vector
- unit vectors
- formulas for computing vector components and vector magnitude
- normalizing a vector
- static equilibrium

In all, students might feel like this section is a repeat of the previous section; however, it will feel a bit "crunchier" because of the computations involved.

Learning Objectives

- Express a vector in coordinate form.
- Convert a vector between geometric and coordinate forms.
- Find the unit vector in the same direction as a given vector.
- Scale a vector.
- Perform arithmetic on vectors.
- Use vectors in force and static equilibrium problems.

Hawk A.I. Section-Specific Tutor

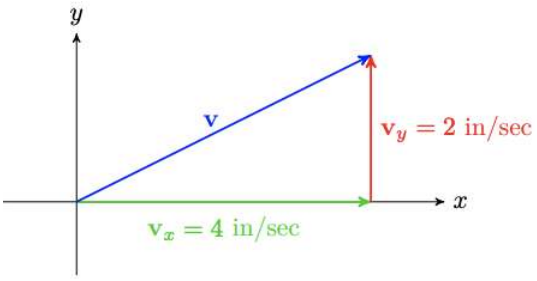
Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Components of a Vector

We have seen that we can add two vectors to get a third, or resultant, vector. We can also break down a vector into two or more component vectors. Breaking a vector into horizontal and vertical components is helpful for many applications.

For example, the beetle on the conveyor belt moved at 4.47 inches per second in the direction $26.6°$. If we set up coordinate axes, as shown in Figure 11.5.1, the vector $\vec{v}$ representing its velocity is the sum of the beetle's motion in the x-direction, $\vec{v}_x$, at 4 inches per second, and its motion in the y-direction, $\vec{v}_y$, at 2 inches per second.

Figure 11.5.1



We can break down any vector into its x- and y-components and the sum of those components is equal to the original vector. In other words,

$$\vec{v} = \vec{v}_x + \vec{v}_y.$$

Definition: Vector Components

A (two-dimensional) vector $\vec{v}$ has two **vector components** - a horizontal vector component and a vertical vector component. These vector components are themselves vectors. The horizontal vector component is denoted $\vec{v}_x$ and the vertical vector component is denoted $\vec{v}_y$. These are the only two vectors (of horizontal and vertical direction) whose sum is the original vector $\vec{v}$.

To find the component vectors, we use the following theorem.

Theorem: Components

If θ is the angle measured counter-clockwise from the positive x-axis to the vector $\vec{v}$, where the tail of $\vec{v}$ is at the origin, then the scalar quantities given by

$$v_x = \|\vec{v}\| \cos(\theta)$$

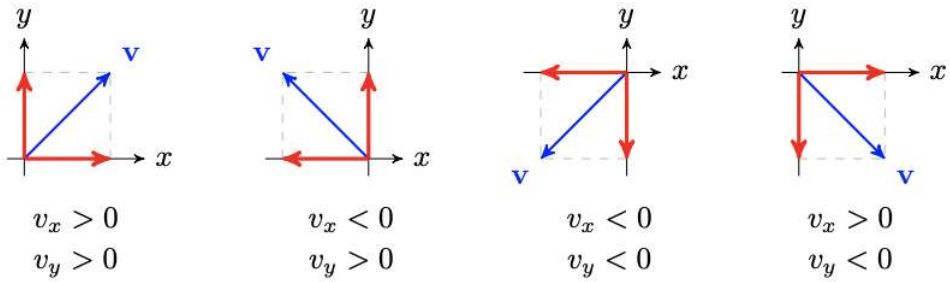
and

$$v_y = \|\vec{v}\| \sin(\theta)$$

are called simply the **components** of the vector $\vec{v}$.

Note that the components v_x and v_y of a vector are scalars; they are not vectors. They can be either positive or negative (or zero), as shown in Figure 11.5.2.

Figure 11.5.2



Again, for clarification, $\vec{v}_x$ is the red horizontal *vector* in each of the images in Figure 11.5.2. At the same time, v_x is a *scalar* representing the signed length of $\vec{v}_x$. A similar statement holds for the vertical *vector* $\vec{v}_y$.

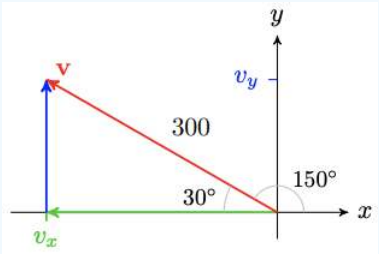
Example 11.5.1

A plane flies 300 miles per hour at a heading of 300°. Find the x- and y-components of its velocity.

Solution

We draw a triangle showing the plane's velocity, $\vec{v}$, as the sum of components in the x - and y-directions.

Figure 11.5.3



Since the *heading* is 300°, $\vec{v}$ makes an angle of 30° with the horizontal. The angle θ, on the other hand, is 150° because θ is *always* measured from the positive x-axis. Therefore,

$$v_x = 300\cos(150°) \approx -259.81$$

and

$$v_y = 300\sin(150°) = 150$$

Checkpoint 11.5.1

The wind blows 50 kilometers per hour at a heading of $100°$. Find the x- and y-components of its velocity.

Answer

$v_x \approx 49.2$ kph and $v_y \approx 8.7$ kph

We now know how to resolve a vector into horizontal and vertical vector components. Thus, every vector can be expressed as the sum of a horizontal vector and a vertical vector.

Example 11.5.2

A vector $\vec{w}$ has magnitude 4 and direction $\theta = 29°$, where θ is measured counter-clockwise from the positive x-axis. Express $\vec{w}$ as the sum of a horizontal vector, $\vec{w}_x$, and a vertical vector, $\vec{w}_y$.

Solution

The components of $\vec{w}$ are given by

$$\begin{aligned} w_x &= \|\vec{w}\|\cos(\theta) \\ &= 4\cos(29°) \\ &\approx 3.50 \end{aligned}$$

and

$$\begin{aligned} w_y &= \|\vec{w}\|\sin(\theta) \\ &= 4\sin(29°) \\ &\approx 1.94 \end{aligned}$$

Figure 11.5.4

Then $\vec{w} = \vec{w}_x + \vec{w}_y$, where $\vec{w}_x$ is a horizontal vector of magnitude 3.50, and $\vec{w}_y$ is a vertical vector of magnitude 1.94. See the figure above.

Checkpoint 11.5.2

A vector $\vec{u}$ has magnitude 2 and direction $\theta = 116°$, where θ is in standard position. Express $\vec{u}$ as the sum of a horizontal vector, $\vec{u}_x$, and a vertical vector, $\vec{u}_y$.

Answer

$\vec{u} = \vec{u}_x + \vec{u}_y$, where $\|\vec{u}_x\| = -1.943, \|\vec{u}_y\| = 0.473$

Using Components

We can describe a vector completely using either magnitude and direction, or components. Many calculations with vectors are more straightforward when we use components. For example, to add two vectors using components, we don't need the Law of Sines and the Law of Cosines. We resolve each vector into its horizontal and vertical components, add the corresponding components, and then compute the magnitude and direction of the resultant vector.

Before we get there, however, let's introduce a couple of theorems that should make sense to you.

Theorem: Vector Component Formula

$$\vec{v}_x = \|\vec{v}\| \cos(\theta) \quad \text{and} \quad \vec{v}_y = \|\vec{v}\| \sin(\theta).$$

This naturally leads to the next theorem.

Theorem: Vector Magnitude Formula

$$\|\vec{v}\| = \sqrt{(v_x)^2 + (v_y)^2} \quad \text{and} \quad \tan(\theta) = \frac{v_y}{v_x}.$$

To calculate magnitude and direction from the components, we need only Right Triangle Trigonometry.

Example 11.5.3

After flying for some time, the airplane in the previous example encounters a steady wind blowing at 40 miles per hour at a heading of $100°$. What are the actual speed and heading of the airplane relative to the ground?

Solution

We want to add the vectors $\vec{v}$, representing the plane's intended velocity, and $\vec{u}$, representing the velocity of the wind. We first resolve $\vec{u}$ into its components.

$$\begin{aligned} u_x &= 40\cos(-10°) \approx 39.39 \\ &\text{and} \\ u_y &= 40\sin(-10°) \approx -6.95 \end{aligned}$$

We find the components of the resultant, $\vec{w}$, by adding the components of $\vec{u}$ and $\vec{v}$, as shown below.

Figure 11.5.5

That is,

$$\begin{aligned} w_x &= u_x + v_x \approx 39.39 - 259.81 = -220.42 \\ &\text{and} \\ w_y &= u_y + v_y \approx -6.95 + 150 = 143.05 \end{aligned}$$

Finally, we compute the magnitude and direction of the resultant vector.

$$\|\vec{w}\|^2 \approx (-220.49)^2 + (143.05)^2 \approx 69{,}079.14$$

Therefore, $\|\vec{w}\| \approx 262.83$. So, the airplane's ground speed is approximately 262.83 miles per hour.

To find the plane's true *heading* we compute

$$\tan(\theta) \approx \frac{143.05}{-220.42} \implies \hat{\theta} \approx \tan^{-1}\left(\frac{143.05}{220.42}\right).$$

We don't need to compute the value of θ because we are going to state the true *heading* of the plane. We only need to know that θ is a second-quadrant angle. We will use the reference angle, $\hat{\theta}$, to get the heading.

Thus, the true heading of the plane is $270° + \hat{\theta} \approx 302.98°$.

Caution: Use Reference Angles and Quadrants

In Example 11.5.3, the airplane's heading is not $\tan^{-1}\left(-\frac{143.02}{220.42}\right) \approx -32.98°$. Remember that there are always two angles with a given tangent. This is why we only consider reference angles when computing the values of inverse trigonometric functions. We can refer to a sketch of the vector or to the signs of its components to decide which of those two angles is appropriate.

Checkpoint 11.5.3

A plane is flying in a wind blowing 50 kilometers per hour in a direction $10°$ south of due west. The plane has an airspeed of 200kph and heading $142°$. What is the ground speed and actual direction of the plane?

Answer

groundspeed $182\text{kph}, 66°\text{S}$ of E

Unit Vectors

It is often helpful to describe a vector by giving its horizontal and vertical vector components instead of its magnitude and direction. To make the notation easier, we give names to the vectors of length 1 that point in the x- and y-directions.

Definition: Unit Vector

A vector of magnitude 1 is called a **unit vector**.

We can have unit vectors in any direction, but the unit vector in the x-direction is traditionally denoted by $\vec{i}$, and the unit vector in the y-direction is represented by $\vec{j}$, as shown in Figure 11.5.6.

Figure 11.5.6

By taking scalar multiples of $\vec{i}$ and $\vec{j}$, we can describe any vector that lies in the directions of the coordinate axes. For example, $4\vec{i}$ represents the vector of magnitude 4 pointing in the x-direction, and $3\vec{j}$ represents the vector of magnitude 3 pointing in the y-direction. By adding multiples of $\vec{i}$ and $\vec{j}$, we can represent any vector we like. The vector $\vec{v} = 4\vec{i} + 3\vec{j}$ is shown in Figure 11.5.6.

Because the components of the vector are chosen to align with the coordinate system, we call this the vector's **coordinate form**.

Definition: Coordinate Form (of a vector)

The vector $\vec{v}$ is said to be wirtten in **coordinate form** when it is written as

$$\vec{v} = v_x\vec{i} + v_y\vec{j},$$

where v_x and v_y are the horizontal and vertical components of $\vec{v}$, respectively.

The coordinate form of the vector $\vec{w}$ in the previous example is $\vec{w} = 3.50\vec{i} + 1.94\vec{j}$.

Example 11.5.4

State the coordinate form of the vector shown below.

Figure 11.5.7

Solution

From its base to its tip, the vector extends 4 units in the negative $\vec{i}$ direction and 6 units in the $\vec{j}$ direction. Thus, $\vec{v} = -4\vec{i} + 6\vec{j}$.

Checkpoint 11.5.4

State the coordinate form of the vector shown below.

Figure 11.5.8

Answer

$\vec{w} = 3\vec{i} - 5\vec{j}$

Converting Between Geometric and Coordinate Form

Finding the magnitude and direction of a vector given in coordinate form is a simple matter. The vector $\vec{v} = 4\vec{i} + 3\vec{j}$ has magnitude

$$\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

and direction $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9°$. Thus, we can readily convert vectors from geometric form to coordinate form or vice versa.

Theorem: Conversion Between Geometric and Coordinate Forms

Given the vector $\vec{v}$ with direction θ (in standard position) and coordinate form $\vec{v} = v_x\vec{i} + v_y\vec{j}$, then

$$v_x = \|\vec{v}\|\cos(\theta)$$

and

$$v_y = \|\vec{v}\|\sin(\theta)$$

where $\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$ and $\tan(\theta) = \frac{v_y}{v_x}$.

Example 11.5.5

a. Find the geometric form of the vector $\vec{w} = -3\vec{i} - 2\vec{j}$.
b. Find the coordinate form of the vector whose magnitude is 5 and whose direction is the angle $\theta = 135°$.

Solutions

a. The magnitude of $\vec{w}$ is $\|\vec{w}\| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$. The direction satisfies the equation $\tan(\theta) = \frac{-2}{-3} = \frac{2}{3}$. Because the point $(-3, -2)$ lies in the third quadrant, we know that θ must be a third quadrant angle. Thus, $\theta = \arctan\left(\frac{2}{3}\right) + 180° \approx 213.69°$.

b. The coordinates of the vector are given by

$$a = 5\cos(135°) = -\frac{5}{\sqrt{2}}, \quad b = 5\sin(135°) = \frac{5}{\sqrt{2}},$$

so the coordinate form is $\vec{w} = -\frac{5}{\sqrt{2}}\vec{i} + \frac{5}{\sqrt{2}}\vec{j}$.

Checkpoint 11.5.5

Find the geometric form of the vector $\vec{F} = -6\vec{i} + 8\vec{j}$.

Answer

magnitude 10, direction $127°$

Scalar Multiples of Vectors in Coordinate Form

Scalar multiplication is easy to compute in coordinate form. This is because the sides of similar triangles are proportional.

Figure 11.5.9 shows the vectors $\vec{u} = 3\vec{i} + 2\vec{j}$ and $\vec{v} = 2\vec{u}$. The vector $\vec{v}$ is twice as long as $\vec{u}$ and points in the same direction as $\vec{u}$. You can see that each component of $\vec{v}$ is twice the corresponding component of $\vec{u}$, so that $\vec{v} = 6\vec{i} + 4\vec{j}$.

Figure 11.5.9

Theorem: Scalar Multiplication (coordinate form)

If $\vec{v} = v_x\,\vec{i} + v_y\,\vec{j}$ and k is a scalar, then

$$k\vec{v} = kv_x\,\vec{i} + kv_y\,\vec{j}.$$

In other words, to find a scalar multiple of a vector in coordinate form, we multiply each component by the scalar.

Example 11.5.6

The vector $\vec{v}$ has coordinate form $\vec{v} = 4\vec{i} - 7\vec{j}$. Find the coordinate form for the vector $\vec{w} = -\frac{1}{2}\vec{v}$.

Solution

We multiply each component of $\vec{v}$ by $-\frac{1}{2}$ to get $\vec{w} = -2\vec{i} + \frac{7}{2}\vec{j}$.

Checkpoint 11.5.6

Find the coordinate form for the vector $\vec{u} = 3\vec{v}$, where $\vec{v} = -12\vec{i} + 15\vec{j}$.

Answer

$\vec{u} = -36\vec{i} + 45\vec{j}$

If we divide a non-zero vector $\vec{v}$ by its length, we create a unit vector that points in the same direction as $\vec{v}$.

Definition: Normalizing (a vector)

The process of dividing a vector by its magnitude is called **normalizing**.

Normalizing vectors becomes incredibly important in Linear Algebra. Dividing a vector by a scalar k is the same as multiplying the vector by $\frac{1}{k}$.

Example 11.5.7

Find a unit vector pointing in the same direction as $\vec{v} = 2\vec{i} - 4\vec{j}$.

Solution

We first compute the magnitude of $\vec{v}$.

$$\|\vec{v}\| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}.$$

Next we multiply $\vec{v}$ by the reciprocal of its length. The unit vector in the same direction as $\vec{v}$ is then

$$\begin{aligned}\vec{u} &= \frac{1}{\|\vec{v}\|}\vec{v} \\ &= \frac{1}{2\sqrt{5}}(2\vec{i}-4\vec{j}) \\ &= \frac{1}{2\sqrt{5}}\cdot 2\vec{i}+\frac{1}{2\sqrt{5}}\cdot(-4\vec{j}) \\ &= \frac{1}{\sqrt{5}}\vec{i}+\frac{-2}{\sqrt{5}}\vec{j}\end{aligned}$$

By computing its length, you can check that the vector $\vec{u}$ found in the Example **11.5.7** is a unit vector.

$$\|\vec{u}\| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2+\left(\frac{-2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5}+\frac{4}{5}} = \sqrt{11}.$$

Once we have a unit vector, $\vec{u}$, that points in a given direction, we can create a vector of any length in that direction, simply by scaling $\vec{u}$ by the length we desire. For example, the vector of length 10 pointing in the same direction as $\vec{v}$ in the previous example is

$$\begin{aligned}10\vec{u} &= 10\cdot\left(\frac{1}{\sqrt{5}}\vec{i}+\frac{-2}{\sqrt{5}}\vec{j}\right) \\ &= 10\cdot\frac{1}{\sqrt{5}}\vec{i}+10\cdot\frac{-2}{\sqrt{5}}\vec{j} \\ &= 2\sqrt{5}\vec{i}-4\sqrt{5}\vec{i}\end{aligned}$$

Theorem: Unit Vector Scaling

The unit vector $\vec{u}$ in the direction of $\vec{v}$ is given by $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$.

A vector $\vec{w}$ of length k in the direction of $\vec{v}$ is given by $\vec{w} = \frac{k}{\|\vec{v}\|}\vec{v}$.

Checkpoint 11.5.7

Find a unit vector $\vec{u}$ and a vector $\vec{w}$ of length 10 pointing in the same direction as $\vec{v} = -\vec{i}+3\vec{j}$.

Answer

$$\vec{u} = -\frac{1}{\sqrt{10}}\vec{i}+\frac{3}{\sqrt{10}}\vec{j}, \quad \vec{w} = -\sqrt{10}\vec{i}+3\sqrt{10}\vec{j}$$

Adding Vectors in Coordinate Form

It is also easy to add vectors in coordinate form. Figure **11.5.10** shows the sum of $\vec{u} = \vec{i}+2\vec{j}$ and $\vec{v} = 5\vec{i}+3\vec{j}$. Remember that we add two vectors by following the first vector by the second.

Figure 11.5.10

The vector sum describes the following path: 1 unit in the x-direction, then 2 units in the y-direction, then 5 units in the x-direction, then 3 units in the y-direction.

However, we arrive at the same endpoint by traveling 1 + 5 = 6 units in the x-direction, then 2 + 3 = 5 units in the y-direction. The resultant vector has components that are just the sums of the components of the vectors $\vec{u}$ and $\vec{v}$. To add two vectors in coordinate form, we add the corresponding components.

Theorem: Vector Addition (coordinate form)

If $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$, then

$$\vec{u} + \vec{v} = (a + c)\vec{i} + (b + d)\vec{j}.$$

Example 11.5.8

Mike flew 15 miles southwest, followed by 20 miles on a heading of $260°$. What is his current position relative to his starting point?

Solution

A sketch of Mike's journey is shown below. We begin by converting the vector for each leg of his journey into coordinate form.

$$\begin{aligned} v_x &= 15\cos(225°) \approx -10.607 \\ v_y &= 15\sin(225°) \approx -10.607 \\ w_x &= 20\cos(190°) \approx -19.696 \\ w_y &= 20\sin(190°) \approx -3.473 \end{aligned}$$

Figure 11.5.11

Thus,

$$\vec{v} = -10.607\vec{i} - 10.607\vec{j} \quad \text{and} \quad \vec{w} = -19.696\vec{i} - 3.473\vec{j}.$$

We find the resultant vector, $\vec{r}$, by adding $\vec{v}$ and $\vec{w}$.

$$\begin{aligned} \vec{r} &\approx (-10.607 - 19.696)\vec{i} + (-10.607 - 3.473)\vec{j} \\ &= -30.303\vec{i} - 14.08\vec{j} \end{aligned}$$

Mike's position vector is $\vec{r} = 30.303\vec{i} - 14.08\vec{j}$, or about 30.3 miles west and 14.1 miles south of his starting position.

Checkpoint 11.5.8

Min's yacht moves at a heading of $350°$ at 20 kilometers per hour relative to the water. However, the water moves at a heading of $70°$ at 4 kilometers per hour. What is Min's velocity relative to land?

Answer

22.7 kph due north

The coordinate form is especially efficient if we want to add more than two vectors.

Example 11.5.9

While exploring an abandoned cottage in the woods, you discover an old map showing the location of a buried treasure. According to the map, from the cottage, go 10 km bearing $\mathrm{N}\,20°\,\mathrm{E}$. From that point, go 12.8 km bearing $\mathrm{S}\,78°\,\mathrm{E}$. Finally, go 9.5 km bearing $\mathrm{N}\,39°\,\mathrm{W}$.

You wonder if the directions will lead you safely to the treasure. You draw vectors to illustrate the path described by the directions; however, instead of following the map directions, you plan to fly by helicopter from the abandoned cottage directly to the treasure. You will need to know the correct bearing and distance to fly. Draw the vector for the net displacement from the cottage to the treasure and give its length and direction.

Solution

The helicopter's flight plan is the vector sum of the three vectors $\vec{v}_1$ (for the first part of the journey), $\vec{v}_2$ (for the second part), and $\vec{v}_3$ (for the third part). We resolve each of the three vectors into its components to find their sum. We start by finding the angles each vector makes with the positive x-axis. From the beginning to the end of the journey, these angles are $\theta_1 = 70°$, $\theta_2 = -12°$, and $\theta_3 = 129°$ (You should verify this.). In coordinate form, the three displacement vectors are

$$\begin{aligned} \vec{v}_1 &= (10\cos(70°))\,\vec{i} + (10\sin(70°))\,\vec{j} &&\approx 3.42\vec{i} + 9.40\vec{j} \\ \vec{v}_2 &= (12.8\cos(-12°))\,\vec{i} + (12.8\sin(-12°))\,\vec{j} &&\approx 12.52\vec{i} - 2.66\vec{j} \\ \vec{v}_3 &= (9.5\cos(129°))\,\vec{i} + (9.5\sin(129°))\,\vec{j} &&\approx -5.98\vec{i} + 7.38\vec{j} \end{aligned}$$

The net displacement is given by the resultant vector, $\vec{r}$. We add the corresponding components of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$\begin{aligned} \vec{r} &\approx (3.42\vec{i} + 9.40\vec{j}) + (12.52\vec{i} - 2.66\vec{j}) + (-5.98\vec{i} + 7.38\vec{j}) \\ &= (3.42 + 12.52 - 5.98)\vec{i} + (9.40 - 2.66 + 7.38)\vec{j} \\ &= 10\vec{i} + 14.12\vec{j} \end{aligned}$$

The treasure is at the point 10 km east and 14.12 km north of the cottage. To find the flight plan for the helicopter, we compute the magnitude and direction of the vector $\vec{r}$.

$$\begin{aligned} \|\vec{r}\| &= \sqrt{10^2 + 14.12^2} = 17.3 \\ &\text{and} \\ \tan(\theta) &= \frac{14.12}{10} \\ \Longrightarrow \quad \hat{\theta} &= \tan^{-1}\left(\frac{14.12}{10}\right) \\ &\approx 54.69° \end{aligned}$$

However, since we know the treasure is to the east and north, we know $0° < \theta < 90°$. Hence, $\theta \approx 54.7°$. This menas the treasure is 17.3 km from the cottage at a bearing of $\mathrm{N}\,35.3°\,\mathrm{E}$.

? Checkpoint 11.5.9

From your campsite you hike 1.6 km heading $175°$, then 0.8 km at the heading of $65°$, and finally 1.2 km in the heading $350°$.

a. Draw a diagram for your hike, using vectors to represent each of the three segments. Use a grid in which 1 square represents 0.1 km.
b. Resolve each vector into components, and determine your location after your hike.

Answers

a.

Figure 11.5.12

b. $-1.594\vec{i} + 0.139\vec{j}, 0.338\vec{i} + 0.725\vec{j}, 1.182\vec{i} - 0.208\vec{j}, \quad -0.074\vec{i} + 0.656\vec{j}$

Force

When you push or pull on something, you exert a **force** on the object. For example, an object's weight is actually a force, the result of gravity pulling the object towards the earth. A force has magnitude (measured in pounds) and direction, so it is a vector quantity. A force applied to an object causes the object to accelerate in the direction of the force.

When two or more forces act simultaneously on an object, the object moves as if the sum of the individual force vectors acted on it. The sum of all the forces acting on an object is the **resultant force**.

Example 11.5.10

As shown below, Ivan and Wyatt are trying to move a refrigerator by pulling on it with forces of 150 pounds and 120 pounds, respectively. $\vec{A}$ represents the force with which Ivan is pulling, and $\vec{B}$ represents the force with which Wyatt is pulling. What is the resulting force on the refrigerator, and in what direction will it move?

Figure 11.5.13

Solution

We write each force in its coordinate form.

$$\begin{aligned} \vec{A} &= 150\cos(20°)\,\vec{i} + 150\sin(20°)\,\vec{j} &\approx& \ 140.95\vec{i} + 51.30\vec{j} \\ \vec{B} &= 120\cos(80°)\,\vec{i} + 120\sin(80°)\,\vec{j} &\approx& \ 20.84\vec{i} + 118.18\vec{j} \end{aligned}$$

Next, we add the forces applied by Ivan and Wyatt to find the resultant force, $\vec{r} = \vec{A} + \vec{B}$.

$$\begin{aligned} \vec{r} &\approx (140.95\vec{i} + 51.30\vec{j}) + (20.84\vec{i} + 118.18\vec{j}) \\ &= 161.75\vec{i} + 169.48\vec{j} \end{aligned}$$

The magnitude of the resultant force is $\|\vec{r}\| \approx \sqrt{161.75^2 + 169.48^2} \approx 234.28$ and the direction of the force is given by

$$\tan^{-1}\left(\frac{169.48}{161.75}\right) = 46.33°.$$

Figure 11.5.14

Therefore, the force exerted on the refrigerator is about 234.3 pounds, and it will move at an angle of approximately $46.33°$ from the horizontal.

Checkpoint 11.5.10

Two tugboats pull on a barge in the river with forces $\vec{u} = 20\vec{i} + 8\vec{j}$ and $\vec{v} = 28\vec{i} - 6\vec{j}$, measured in thousands of pounds. In what direction will the barge move, and what is the magnitude of the force propelling it?

Answer

$2.4°$, 48.04 thousand lbs

If vectors $\vec{u}$ and $\vec{v}$ have the same magnitude but opposite directions, then $\vec{u} + \vec{v}$ has zero magnitude and is called the **zero vector**.

Definition: Zero Vector

A **zero vector**, also known as a **null vector**, is a vector with a magnitude of zero and an undefined direction. It is denoted $\vec{0}$.

A zero displacement vector means that the object started and ended in the same place; a zero velocity vector means that the object is not moving.

The vector that has the same magnitude as $\vec{v}$ but the opposite direction is called the opposite of $\vec{v}$ and denoted by $-\vec{v}$. Then

$$\vec{v} + (-\vec{v}) = \vec{0}.$$

We now have enough language to tackle some of the most interesting (and important) applications of vectors - those involving **static equilibrium**.

Definition: Static Equilibrium

When the sum of the forces (as vectors) acting on a particle or object is $\vec{0}$, we say the object is in **static equilibrium** (or simply **equilibrium**).

Example 11.5.11

Recall the refrigerator in the previous example. Jorge does not want Ivan and Wyatt to move it. How hard must Jorge pull so that it remains motionless?

Solution

Let $\vec{C}$ be the force that Jorge applies. It must be equal in magnitude to the sum of the forces applied by Ivan and Wyatt but point in the opposite direction. That is,

$$\vec{C} = -\vec{r} = -(\vec{A} + \vec{B}).$$

So, Jorge must pull with 234.3 pounds in the direction $180° + 46.33° = 226.33°$.

Checkpoint 11.5.11

Suppose that $\vec{u} + \vec{v} + \vec{w} = \vec{0}$, and that $\vec{u} = -8\vec{i} + 32\vec{j}$ and $\vec{v} = 14\vec{i} - 26\vec{j}$. What is the coordinate form of vector $\vec{w}$?

Answer

$\vec{w} = -6\vec{i} - 6\vec{j}$

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11.5.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Algebraic Vectors
  - Vectors - An Algebraic Approach
- Vector Arithmetic
  - Vectors - Vector Arithmetic and Properties of Vectors
- Component Form and Unit Vectors
  - Vectors - Unit Vectors and Component Form
- Applications
  - Vectors - Applications Involving Vectors

KEY CONCEPTS

DEFINITIONS

- **Vector Components:** A vector $\vec{v}$ can be broken down into a horizontal vector component, $\vec{v}_x$, and a vertical vector component, $\vec{v}_y$, such that $\vec{v} = \vec{v}_x + \vec{v}_y$. The scalar values v_x and v_y are called the components of the vector.
- **Unit Vector:** A vector with a magnitude of 1. The unit vectors in the horizontal and vertical directions are denoted by $\vec{i}$ and $\vec{j}$, respectively.
- **Coordinate Form (of a vector):** A vector $\vec{v}$ is in coordinate form when it is written as the sum of its scalar components multiplied by the standard unit vectors: $\vec{v} = v_x\vec{i} + v_y\vec{j}$.
- **Normalizing (a vector):** The process of creating a unit vector in the same direction as a given non-zero vector by dividing the vector by its magnitude.
- **Zero Vector:** A vector with a magnitude of zero and an undefined direction, denoted $\vec{0}$. It is also known as a null vector.
- **Static Equilibrium:** A state in which an object or particle is motionless because the sum of all forces acting on it is the zero vector.

THEOREMS

- **Components:** If θ is the angle measured counter-clockwise from the positive x-axis to the vector $\vec{v}$, then the scalar components of the vector are given by $v_x = ||\vec{v}||\cos(\theta)$ and $v_y = ||\vec{v}||\sin(\theta)$.
- **Vector Component Formula:** The vector components of $\vec{v}$ are $\vec{v}_x = ||\vec{v}||\cos(\theta)\vec{i}$ and $\vec{v}_y = ||\vec{v}||\sin(\theta)\vec{j}$.
- **Vector Magnitude Formula:** For a vector $\vec{v} = v_x\vec{i} + v_y\vec{j}$, the magnitude is $||\vec{v}|| = \sqrt{v_x^2 + v_y^2}$ and the direction angle θ satisfies $\tan(\theta) = \frac{v_y}{v_x}$.
- **Conversion Between Geometric and Coordinate Forms:** The formulas for components and magnitude/direction allow for conversion between the geometric representation (magnitude and direction) and the coordinate form of a vector.
- **Scalar Multiplication (coordinate form):** If $\vec{v} = v_x\vec{i} + v_y\vec{j}$ and k is a scalar, then $k\vec{v} = (kv_x)\vec{i} + (kv_y)\vec{j}$.
- **Unit Vector Scaling:** The unit vector $\vec{u}$ in the direction of $\vec{v}$ is given by $\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$. A vector $\vec{w}$ of length k in the direction of $\vec{v}$ is given by $\vec{w} = k\vec{u}$.
- **Vector Addition (coordinate form):** If $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$, then $\vec{u} + \vec{v} = (a + c)\vec{i} + (b + d)\vec{j}$.

COMMON MISTAKES

- **Confusing Vector Components and Components:** The vector components $(\vec{v}_x, \vec{v}_y)$ are vectors, while the components (v_x, v_y) are scalars that represent the signed lengths of the vector components.
- **Using Heading as the Standard Angle θ:** When finding components, the angle θ must be the angle in standard position (measured counter-clockwise from the positive x-axis), not necessarily the heading or bearing angle given in a navigation problem.
- **Incorrectly Determining the Quadrant for Direction:** When finding the direction angle θ from components using $\arctan(\frac{v_y}{v_x})$, you must use the signs of v_x and v_y to determine the correct quadrant for θ, as the arctangent function only returns angles in Quadrants I or IV.

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11.5.2: HOMEWORK

READING QUESTIONS

1. What are the two vector components of a two-dimensional vector $\vec{v}$?
2. If a vector $\vec{v}$ has magnitude $||\vec{v}||$ and direction angle θ, what are the formulas for its scalar components v_x and v_y?
3. What is a unit vector?
4. What is the process of "normalizing" a vector?
5. How do you write a vector $\vec{v}$ in coordinate form?
6. If you have a vector in coordinate form, $\vec{v} = v_x\vec{i} + v_y\vec{j}$, how do you find its magnitude?
7. How do you add two vectors when they are in coordinate form?
8. How do you perform scalar multiplication on a vector that is in coordinate form?
9. What does it mean for an object to be in static equilibrium?
10. What is the zero vector?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

1. State the sum of angles formula for sine.
2. State the sum of angles formula for cosine.
3. State the sum of angles formula for tangent.
4. Explain how the difference of angles formulas differ from the sum of angles formulas.

For the following exercises, let $\cos(\alpha) = -\frac{2}{\sqrt{5}}$, where $90° < \alpha < 180°$.

5. Find an exact value for $\cos(\alpha + 30°)$.
6. Find an exact value for $\tan(\alpha - 30°)$.

Answers

1. $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
2. $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
3. $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1\tan(\alpha)\tan(\beta)}$
4. Change subtraction signs to addition and vice versa.
5. $\frac{-2\sqrt{3}-1}{2\sqrt{5}}$
6. $\frac{-\sqrt{3}-2}{1+2\sqrt{3}}$

HOMEWORK

VOCABULARY CHECK

1. A vector having a magnitude of 1 is called a ___ vector.
2. The ___ form of a vector expresses the vector in terms of $\vec{i}$ and $\vec{j}$.
3. Then ___ vector has magnitude 0.
4. $\vec{v}_x$ and $\vec{v}_y$ are called the ___ of the vector $\vec{v}$, while v_x and v_y are simply called the ___.

CONCEPT CHECK

5. Is it easier to add vectors in geometric form or coordinate form? Why?
6. To find a unit vector in the direction of $\vec{v}$, multiply the coordinates of $\vec{v}$ by ___ .
7. To find a vector of length k in the direction of $\vec{v}$, multiply the coordinates of $\vec{v}$ by ___.
8. Name two physical quantities that are represented by vectors.
9. What are the components of a vector?
10. Does $\|\vec{v}\| = \|\vec{v}_x\| + \|\vec{v}_y\|$?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

11. Any vector can be written as the sum of its horizontal and vertical vector components, $\vec{v}_x$ and $\vec{v}_y$.
12. The component vectors of a vector $\vec{v}$ whose direction is given by the angle θ in standard position are the scalar quantities

$$\begin{aligned} \vec{v}_x &= \|\vec{v}\|\sin(\theta) \\ \vec{v}_y &= \|\vec{v}\|\cos(\theta) \end{aligned}$$

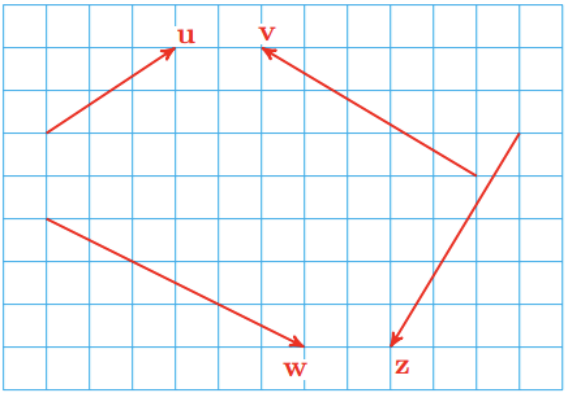
13. The magnitude and direction of a vector with components $\|\vec{v}_x\|$ and $\|\vec{v}_x\|$ are given by

$$\|\vec{v}\| = \sqrt{(\|\vec{v}_x\|)^2 + (\|\vec{v}_y\|)^2} \quad \text{and} \quad \tan(\theta) = \frac{\|\vec{v}_y\|}{\|\vec{v}_x\|},$$

respectively.

BASIC SKILLS

For the following exercises, use the coordinate form of each vector shown in the figure to answer the question.



14.

a. $\|\vec{u}\|$

b. $2\vec{u}$

c. $\|2\vec{u}\|$

15.

a. $\|\vec{v}\|$

b. $\frac{1}{2}\vec{v}$

c. $\|\frac{1}{2}\vec{v}\|$

16.

a. $\|\vec{w}\|$

b. $-\vec{w}$

c. $\|-\vec{w}\|$

17.

a. $\|\vec{z}\|$

b. $-3z$

c. $\|-3\vec{z}\|$

18.

a. Calculate $\vec{u}+\vec{v}$ and $\|\vec{u}+\vec{v}\|$.

b. Which of the following statements is true?

i. $\|\vec{u}\|+\|\vec{v}\| \leq \|\vec{u}+\vec{v}\|$

ii. $\|\vec{u}\|+\|\vec{v}\| = \|\vec{u}+\vec{v}\|$

iii. $\|\vec{u}\|+\|\vec{v}\| \geq \|\vec{u}+\vec{v}\|$

19.

a. Calculate $\vec{w}+\vec{z}$ and $\|\vec{w}+\vec{z}\|$.

b. Which of the following statements is true?

i. $\|\vec{w}\|+\|\vec{z}\| \leq \|\vec{w}+\vec{z}\|$

ii. $\|\vec{w}\|+\|\vec{z}\| = \|\vec{w}+\vec{z}\|$

iii. $\|\vec{w}\|+\|\vec{z}\| \geq \|\vec{w}+\vec{z}\|$

For the following exercises,
(a) Sketch the vector and give its coordinate form.
(b) Find the magnitude and direction of the vector.

20. The displacement vector from (1, −2) to (−4, 6).

21. The displacement vector from (−5, 2) to (4, 7).

22. The displacement vector from (−2, 9) to (−4, 8).

23. The displacement vector from (−6, 2) to (3, 0).

24. Camille is 12 meters east and 3 meters north of Roy. Lap is 6 meters east and 9 meters north of Camille.

a. Calculate the displacement vector from Roy to Lap in coordinate form. Let $\vec{i}$ point east and $\vec{j}$ point north.

b. Find the magnitude and direction of the displacement vector.

25. Ron and Sang are climbing a rock wall. Ron is 8 feet to the right and and 23 feet above their starting point. Sang is 2 feet to the right and 7 feet above Ron.

a. Calculate the displacement vector from the starting point to Sang in coordinate form. Let $\vec{i}$ point right and $\vec{j}$ point up.

b. Find the magnitude and direction of the displacement vector.

For the following exercises, find the magnitude and direction of the vector.

26. $\vec{v} = -6\vec{i} + 6\vec{j}$

27. $\vec{p} = -12\vec{i} - 5\vec{j}$

28. $\vec{w} = 7\sqrt{3}\vec{i} - 7\vec{j}$

29. $z = -6\sqrt{2}\vec{i} + 6\sqrt{6}\vec{j}$

30. $\vec{q} = 52\vec{i} + 96\vec{j}$

31. $\vec{s} = 3.2\vec{i} - 1.8\vec{j}$

For the following exercises, find the coordinate form of the vector.

32. $\|\vec{v}\| = 6, \quad \theta = -45°$

33. $\|\vec{v}\| = 200, \quad \theta = 240°$

34. $\|\vec{v}\| = 8.3, \quad \theta = 37°$

35. $\|\vec{v}\| = 23, \quad \theta = 200°$

For the following exercises, sketch each vector and its components. Use the coordinate form to find the resultant vector $\vec{u}+\vec{v}$, and sketch it.

36. $\vec{u} = -3\vec{i} + 2\vec{j}, \quad \vec{v} = 4\vec{i} - 4\vec{j}$

37. $\vec{u} = 5\vec{i} + \vec{j}, \quad \vec{v} = 2\vec{i} - 3\vec{j}$

38. $\vec{u} = -5\vec{i} - 2\vec{j}, \quad \vec{v} = \vec{i} + 6\vec{j}$

39. $\vec{u} = 8\vec{i} - 3\vec{j}, \quad \vec{v} = -4\vec{i} - 2\vec{j}$

For the following exercises, find the sum $\vec{u}+\vec{v}$ of the given vectors.

40. $\vec{u} = 13\vec{i} - 8\vec{j}, \quad \vec{v} = -1\vec{i} + 11\vec{j}$

41. $\vec{u} = 3.7\vec{i} + 2.6\vec{j}, \quad \vec{v} = -1.3\vec{i} - 5.7\vec{j}$

42. $\vec{u} = -3\vec{i} + 9\vec{j}, \quad \vec{v} = 5.8\vec{i} - 7.1\vec{j}$

43. $\vec{u} = 6\vec{i} - 8\vec{j}, \quad \vec{v} = 23\vec{i} + 42\vec{j}$

For the following exercises, find the coordinate form of the vector, where

$$\vec{u} = 2\vec{i} + 3\vec{j}, \quad \vec{v} = -5\vec{i} + 4\vec{j}, \quad \vec{w} = -2\vec{i} - 5\vec{j}, \quad \vec{z} = 8\vec{i} - 3\vec{j}$$

44. $\vec{u}+\vec{v}$

45. $\vec{w}-\vec{z}$

46. $4\vec{w}$

47. $-3\vec{v}$

48. $2\vec{z}-\vec{u}$

49. $-\vec{w}+5\vec{u}$

50. $3\vec{v}-\vec{w}+2\vec{u}$

51. $\vec{z}-2(\vec{v}+\vec{w})$

For the following exercises, find a unit vector $\vec{u}$ in the same direction as the given vector.

52. $\vec{r} = -12\vec{i} + 5\vec{j}$

53. $\vec{s} = 7\vec{i} - 24\vec{j}$

54. $\vec{t} = \vec{i} - \vec{j}$

55. $\vec{w} = -2\vec{i} - 3\vec{j}$

For the following exercises, find a vector $\vec{v}$ in the same direction as $\vec{w}$, but with the given length.

56. $\vec{w} = 8\vec{i} + 15\vec{j}, \quad \|\vec{v}\| = 51$

57. $\vec{w} = -20\vec{i} - 21\vec{j}, \quad \|\vec{v}\| = 58$

58. $\vec{w} = -3\vec{i} + \vec{j}, \quad \|\vec{v}\| = 4$

59. $\vec{w} = \vec{i} - 2\vec{j}, \quad \|\vec{v}\| = 7$

For the following exercises,

(a) Draw a diagram using arrows to represent the vectors.

(b) Convert each vector to coordinate form.

(c) Use the coordinate form to add or subtract the vectors.

60. Find $\vec{u} + \vec{v}$, where $\vec{u}$ has magnitude 2.6 and direction $\theta = 23°$, $\vec{v}$ has magnitude 5.8 and direction $\theta = 223°$.

61. Find $\vec{u} + \vec{v}$, where $\vec{u}$ has magnitude 50 and direction $\theta = 173°$, $\vec{v}$ has magnitude 70 and direction $\theta = 308°$.

62. Find $\vec{u} - \vec{v}$, where $\vec{u}$ has magnitude 35 and direction $\theta = 110°$, $\vec{v}$ has magnitude 60 and direction $\theta = 165°$.

63. Find $\vec{u} - \vec{v}$, where $\vec{u}$ has magnitude 12.4 and direction $\theta = 250°$, $\vec{v}$ has magnitude 8.8 and direction $\theta = 315°$.

For the following exercises,

(a) Find the resultant force.

(b) Find the additional force needed for the system to be in equilibrium.

64. $\vec{F}_1 = -3\vec{i} + \vec{j}, \quad \vec{F}_2 = 5\vec{i} - 2\vec{j}, \quad \vec{F}_3 = -6\vec{i} - 4\vec{j}$

65. $\vec{F}_1 = 10\vec{i} + 4\vec{j}, \quad \vec{F}_2 = -12\vec{i} - 9\vec{j}, \quad \vec{F}_3 = -3\vec{i} + 5\vec{j}$

66.

67.

68.

a. Find the magnitude of the vector $\vec{v} = 6\vec{i} - 8\vec{j}$, and the magnitude of the vector $2\vec{v} = 12\vec{i} - 16\vec{j}$, and verify that $\|2\vec{v}\| = 2\|\vec{v}\|$.

b. Let $\vec{v} = a\vec{i} + b\vec{j}$, and verify that $\|k\vec{v}\| = k\|\vec{v}\|$, for $k > 0$.

69.

a. Find the magnitude of the vector $\vec{v} = 3\vec{i} + 5\vec{j}$, and verify that the vector $\frac{\vec{v}}{\|\vec{v}\|}$ has magnitude 1.

b. Let $\vec{v} = a\vec{i} + b\vec{j}$, where a and b are not both 0, and verify that $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector.

SYNTHESIS QUESTIONS

70. Find the horizontal and vertical components of $\vec{u}, \vec{v}$, and $\vec{A}$ from Problem 55 . What do you notice when you compare the horizontal components of two vectors with the horizontal component of the difference?

71. Find the horizontal and vertical components of $\vec{z}, \vec{y}$, and $\vec{B}$ from Problem 56 . What do you notice when you compare the horizontal components of two vectors with the horizontal component of the difference?

APPLICATIONS

For the following exercises,

(a) Make a sketch using vectors to illustrate the problem.

(b) Use the coordinate form of the vectors to solve problem.

72. The tornado displaced the trash bin to a spot 500 meters north and 800 meters east of its original position, and the flood later displaced the bin 2000 meters due south from there. How far and in what direction was the trash bin moved from it original position?

73. A radio-controlled model plane pointed due west with an airspeed of 15 miles per hour, but there was a crosswind from the north at a speed of 8 miles per hour. How fast and in what direction is the plane moving relative to the ground?

74. Matt flies 10 km in a direction 175° north from east, then turns and flies an additional 12 km due west. How far and in what direction is Matt's final position relative to his starting point?

75. Kim sails 500 yards due south, then turns and sails 350 yards in the direction 300° from east. How far and in what direction is Kim's final position relative to her starting point?

76. After leaving the airport, Piyali flew 30 miles at a heading 30° east of north, then 50 miles 70° east of north, and finally 12 miles 20° south of east. What is her current position relative to the airport?

77. On a whale-watching trip, the SS Dolphin sailed 15 miles from port on a heading of 40°, then 8 miles on a heading of 320°, and then 4 miles on a heading of 250°. What is the current position of the SS Dolphin relative to port?

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CHAPTER OVERVIEW

Chapter 12: Polar Coordinates and Complex Numbers

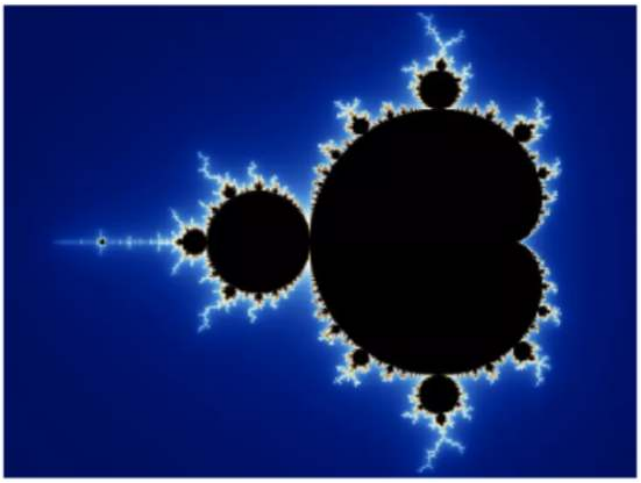
The picture on the left below is a photo of a fern leaf. The picture on the right is a computer-generated image called the Barnsley fern. It is named after the British mathematician Michael Barnsley, who developed it in the 1980's.

Figure 1

The Barnsley fern is an example of a **fractal**. A fractal is an infinitely complex pattern created by iteration: repeating a process over and over in an ongoing feedback loop. You can watch a Barnsley fern being created at https://www.youtube.com/watch?v=iGMGVpLMtMs.

One of the most famous and intriguing fractals is the Mandelbrot set, named after the French mathematician Benoît Mandelbrot, who lived from 1924 to 2010. The Mandelbrot set looks like a cardioid, or heart-shaped figure, studded with circles all around its boundary.

Figure 2



Like the Barnsley fern, we can create the Mandelbrot set by repeatedly evaluating a simple equation over and over. This equation uses complex numbers.

The plane of complex numbers is similar to a two dimensional coordinate system. Every point on the plane is represented by a complex number $c = x + iy$, and we perform the following steps for every point:

Starting with $z_1 = 0$, we create an infinite sequence of complex numbers z_n according to the rule

$$z_{n+1} = (z_n)^2 + c$$

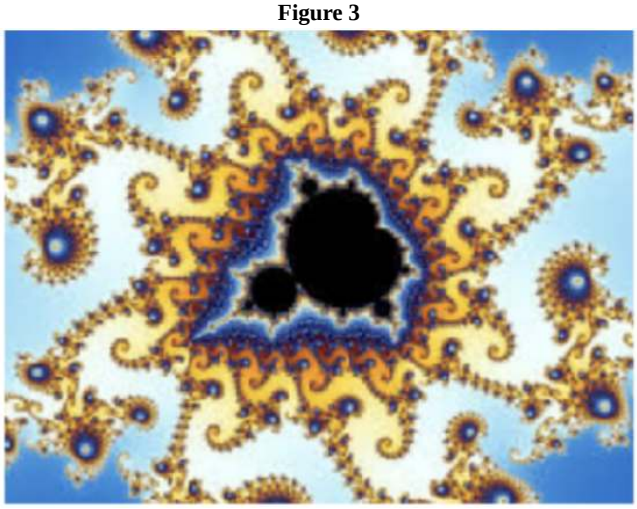
If this sequence increases to infinity (it diverges), we color the point c white. If the sequence does not increase beyond a certain limit (it is bounded), we color the point black.

We create a sequence for every point c in the plane. The collection of all the black points is the Mandelbrot set. It sits in the portion of the plane where x is between -1 and 2, and y is between -1.5 and 1.5. Outside the set, all the values of c cause the sequence to go to infinity, and the color of the point is proportional to the speed at which the sequence diverges.

Because the calculations must be performed thousands or millions of times, we need computers to study them. The higher the number of iterations, the better the quality of the image produced. Not coincidentally, the Mandelbrot set was discovered in 1980, shortly after the invention of the personal computer.

One of the intriguing features of fractal images is **self-similarity**. If we zoom in on a fractal, we see the same pattern repeated again and again, sometimes with interesting variations. Here is a portion of the boundary of the Mandelbrot set, blown up to many times its original size. You can see the copy of whole set embedded in the image.

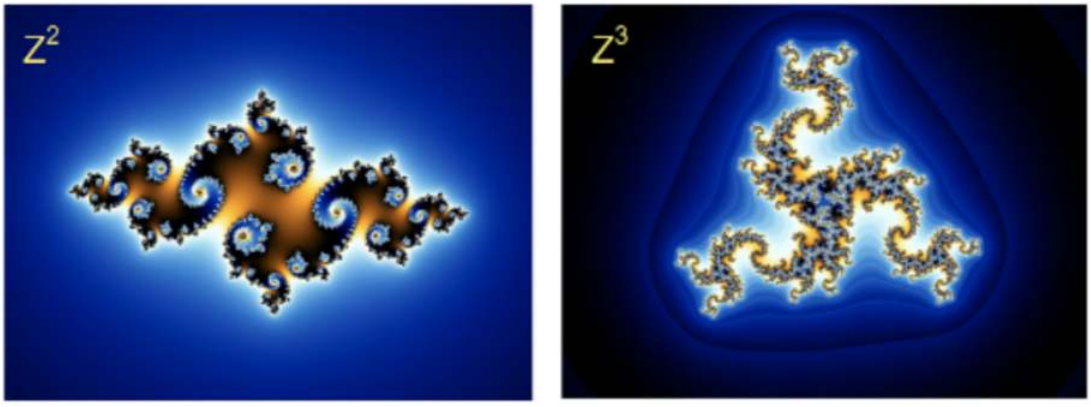
Figure 3



You can find a ”Mandelbrot Zoomer” at https://mathigon.org/course/fractals/introduction.

Here are some more fractals, called Julia sets:

Figure 4



Fractal patterns seem familiar because the laws that govern the creation of fractals are found throughout the natural world. Tree branches, rivers, ice crystals, and seashells all form in fractal shapes. You can see some examples at https://www.treehugger.com/earth-matters/wilderness-resources/blogs.

12.1: Polar Coordinates
12.1.1: Resources and Key Concepts
12.1.2: Homework

12.2: Polar Graphs
12.2.1: Resources and Key Concepts
12.2.2: Homework
12.3: Complex Numbers
12.3.1: Resources and Key Concepts
12.3.2: Homework
12.4: Polar Form of Complex Numbers
12.4.1: Resources and Key Concepts
12.4.2: Homework

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12.1: Polar Coordinates

Note to the Instructor - Polars Part One: The Crunch

The material in this section is the traditional introduction to polar coordinates. You will cover all the basics *except* graphing polar equations (which is done in the next section).

Learning Objectives

- Graph an ordered pair given in polar coordinates.
- Give multiple representations for a point in polar coordinates.
- Sketch a region in the polar coordinate system.
- Convert an ordered pair from polar to rectangular coordinates or vice-versa.
- Express a polar equation in rectangular coordinates.
- Express a rectangular equation in polar coordinates.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Location by Coordinates

One of the most valuable applications of trigonometric ratios allows us to find distances or locations specified by angles. Starting with the definitions of sine and cosine,

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

we can solve for x and y, the coordinates of points on the terminal side of the angle, and obtain the following results.

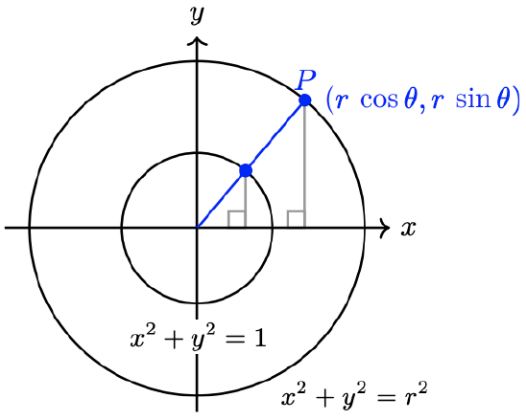
Theorem: Cartesian Coordinates in Terms of Cosine and Sine

If point P is located at a distance r from the origin in the direction specified by angle θ in standard position, then the coordinates of P are

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

These formulas make sense when we think of the unit circle. On a unit circle, the coordinates of a point designated by an angle θ are $(\cos(\theta), \sin(\theta))$, as shown in Figure 12.1.1. On a circle of radius r, the angle θ forms a similar triangle whose dimensions are scaled up by a factor of r. In particular, the legs of the new triangle are r times larger than the original triangle.

Figure 12.1.1



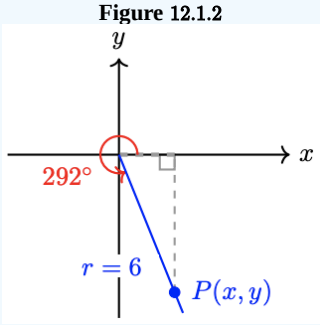
Example 12.1.1

Point P is located 6 centimeters from the origin in the direction of $292°$. Find the coordinates of P, rounded to the nearest hundredths.

Solution

The location of point P is shown below. We see that $r = 6$, and we can use a calculator to evaluate $\cos(292°)$ and $\sin(292°)$.

Figure 12.1.2



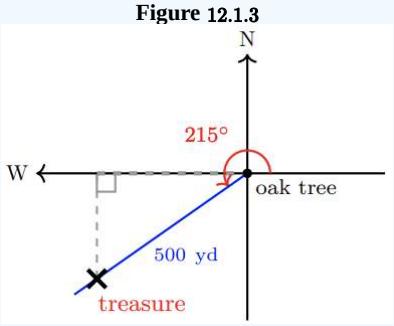
$$\begin{aligned} x &= r\cos(292°) & \text{and} \quad y &= r\sin(292°) \\ &\approx 6(0.3746) & &\approx 6(-0.9272) \\ &\approx 2.2476 & &\approx -5.5632 \end{aligned}$$

The coordinates of P are approximately $(2.25, -5.56)$.

Checkpoint 12.1.1

You find an old map that shows a buried treasure located 500 yards from the big oak tree in the direction $215°$, as shown below. You don't have anything with you to measure angles, but you have your calculator.

Figure 12.1.3



a. Find the cosine of $215°$. How far west should you walk from the big oak to be directly north of the treasure?
b. Find the tangent of $215°$. How far south should you walk from your present location before you begin digging?

Answer

a. 409.58 yds
b. 286.79 yds

The Cartesian coordinate system is also called the rectangular coordinate system because it describes a location in the plane as the vertex of a rectangle. To construct a rectangular coordinate system, we begin with two perpendicular axes intersecting at the origin. The x- and y-coordinates of a point indicate the length and width of a rectangle with one vertex at the origin. The point $P(x, y)$ sits at the opposite vertex of the rectangle, as shown in Figure 12.1.4.

Figure 12.1.4

Previously, we used vectors to specify a location by giving a distance and a direction. For example, we might say that the airport is located 5 miles southwest of the city center. This method of designating locations is so helpful that we will construct a new coordinate system using the same tools: distance and direction.

Definition: Polar Coordinate System

The **polar coordinate system** is a two-dimensional system for specifying points in a plane using a distance from a central point (called the **pole**) and an angle from a reference direction (called the **polar axis**). Typically, the pole is aligned with what we would consider the origin in the rectangular coordinate system, and the polar axis is aligned with the positive x-axis.

We describe the location of a point P in the plane by measuring the distance, $|r|$, from P to the pole, and the angle, θ, that $\vec{OP}$ makes with the polar axis (measured counterclockwise). The components of the ordered pair, (r, θ), are called the **polar coordinates** of the point P.

Definition: Polar Coordinates

The **polar coordinates** of a point P in the plane are (r, θ), where $|r|$ is the distance from P to the pole, and θ is the signed angle measured counterclockwise from the polar axis to the ray through P from the pole.

In our work with polar coordinates, we *always* use radians for the angle θ. For example, the point $P\left(2, \frac{\pi}{2}\right)$ is located 2 units from the pole at an angle of $\frac{\pi}{2}$ radians, and $Q(3, 4)$ is located 3 units from the pole at an angle of 4 radians. The graphs of P and Q are shown in Figure 12.1.5.[1]

Figure 12.1.5

A piece of graph paper in rectangular coordinates consists of a grid of horizontal and vertical lines. These are the lines $x = k$ and $y = k$, for evenly spaced values of k. Each vertical grid line consists of points with the same x-coordinate, and each horizontal grid line is made up of points with the same y-coordinate. The grid lines are benchmarks to help us locate points with specific rectangular coordinates.

A piece of graph paper in polar coordinates consists of a grid of concentric circles and radial lines, as shown in Figure 12.1.6. Each circle consists of points with the same r-coordinate, all the same distance from the pole. All points with the same θ-coordinate lie on the same radial line. This grid helps us locate points with specific polar coordinates.

Figure 12.1.6

Plotting Points in the Polar Coordinate System

You are used to thinking in rectangular coordinates: to locate a point, we move so many units left or right and so many units up or down. When working in polar coordinates, we want to "think radially" - how far to move from the pole and in which direction.

Example 12.1.2

Plot the points whose polar coordinates are given: $A\left(1, \frac{\pi}{2}\right)$ and $B\left(2, \frac{7\pi}{4}\right)$.

Solution

To plot point A, we move 1 unit away from the pole in the direction $\frac{\pi}{2}$, as shown below. To plot point B, we move 2 units away from the pole in the direction $\frac{7\pi}{4}$.

Figure 12.1.7

Caution: Polar Backwardness

We are used to "plugging in" a value for x and using the given equation or relation to find the corresponding value of y. This is why most of us think of x as the default independent variable and y as depending on x (and, hence, is called the dependent variable). We get the ordered pair, (x, y), which is listed in the order $(\textbf{independent variable}, \textbf{dependent variable})$. This convention, however, is a bit backward with polar coordinates. Even though the independent, or input, variable in polar coordinates is almost always θ, it is customary to list the dependent or output variable first: (r, θ).

Checkpoint 12.1.2

Give the polar coordinates of the points S and T shown below, with $0 \leq \theta \leq 2\pi$.

Figure 12.1.8

Answer

$$S(2, \pi), T\left(3, \frac{2\pi}{3}\right)$$

Each point has infinitely many polar coordinates because we can add multiples of 2π to the value of θ. For example, the point with polar coordinates $\left(1, \frac{\pi}{2}\right)$ also has polar coordinates $\left(1, \frac{5\pi}{2}\right)$ and $\left(1, \frac{-3\pi}{2}\right)$.

If we allow negative values of r, there are even more ways to write the coordinates of a point. Each angle designates a line through the pole, and each line has both a positive and a negative direction.

On the line at $\theta = \frac{\pi}{4}$, for instance, the positive direction lies in the first quadrant, so that the point $\left(2, \frac{\pi}{4}\right)$ is designated by P in the figure below. To plot the point $\left(-2, \frac{\pi}{4}\right)$, we move in the opposite direction from the pole, arriving at point Q. Note that the point Q can also be designated by the coordinates $\left(2, \frac{5\pi}{4}\right)$.

Figure 12.1.9

Theorem: Non-Uniqueness of Polar Coordinates

Any point with polar coordinates (r, θ) also has coordinates $(r, \theta + 2k\pi)$, where $k \in \mathbb{Z}$. The point can also be designated by $(-r, \theta + \pi)$. Finally, the pole has coordinates $(0, \theta)$, for any value of θ.

Example 12.1.3

Give polar coordinates with negative r-values for the points in the previous exercise.

Solution

The point $S(2, \pi)$ is also $(-2, 2\pi)$ or $(-2, 0)$. The point $T\left(3, \frac{2\pi}{3}\right)$ can also be designated by $\left(-3, \frac{2\pi}{3} + \pi\right)$, or $\left(-3, \frac{5\pi}{3}\right)$.

Checkpoint 12.1.3

Give polar coordinates with positive r-values and $0 \le \theta \le 2\pi$ for each point given in polar coordinates.

a. $\left(-2.5, \frac{\pi}{6}\right)$
b. $(-0.7, 5)$

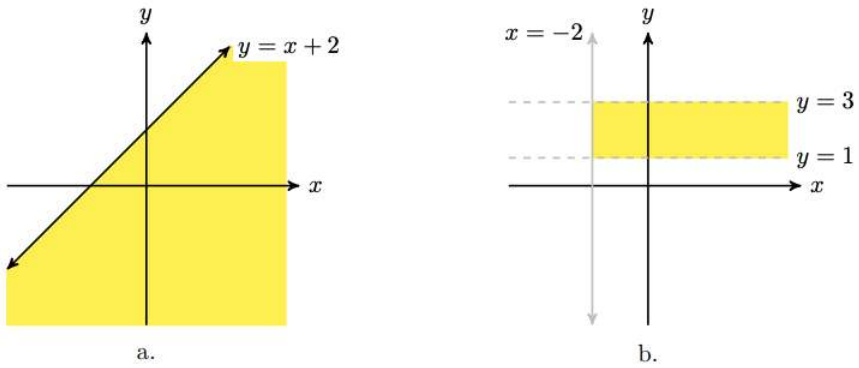
Answer

a. $\left(2.5, \frac{7\pi}{6}\right)$
b. $(0.7, 5-\pi)$

Regions in the Plane

In Cartesian coordinates, we use equations and inequalities to describe regions in the plane. For example, the shaded region in Figure 12.1.10*a* is the graph of $y \le x+2$. The pair of inequalities $x \ge -2$ and $1 < y < 3$ describes the region in Figure 12.1.10*b*. This region is particularly simple in that its boundaries are portions of the coordinate grid lines $x = k$ and $y = k$, where k is a constant.

Figure 12.1.10



In the polar plane, the coordinate grid lines are circles centered at the pole, with equations $r = k$, and lines through the pole, with equations $\theta = k$.

Example 12.1.4

Sketch the region designated by each set of inequalities.

a. $1 \le r < 3$
b. $0 \le r \le 2, \frac{\pi}{6} < \theta < \frac{\pi}{3}$

Solutions

a. The region consists of all points between 1 (inclusive) and 3 units from the pole. No restriction is given on θ, so the region is the annular ring shown in Figure 12.1.6*a*.

Figure 12.1.11

b. The region includes all points up to 2 units from the pole, but only between the lines $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$. These curves bound the pie-shaped wedge (circular sector) shown in Figure 12.1.6*b*.

Checkpoint 12.1.4

Write inequalities to describe each region shown below.

Figure 12.1.12

Answers

a. $r \ge 2, \frac{3\pi}{4} < \theta \le \frac{5\pi}{4}$

b. $2 < r < 3, 0 < \theta < \pi$

Converting Between Polar and Cartesian Coordinates

Using Trigonometry, converting from polar coordinates to Cartesian is easy, or vice versa. Figure 12.1.13 shows how the two coordinates are related.

Figure 12.1.13

If we know the polar coordinates (r, θ), we can compute the Cartesian coordinates (x, y) as the legs of a right triangle.

$$\begin{aligned} \frac{x}{r} &= \cos(\theta) \implies x = r\cos(\theta) \\ \frac{y}{r} &= \sin(\theta) \implies y = r\sin(\theta) \end{aligned}$$

Example 12.1.5

Find the Cartesian coordinates of the point with polar coordinates $\left(\sqrt{2}, \frac{3\pi}{4}\right)$.

Solution

The polar coordinates tell us that $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$, as shown below. The x-coordinate of the point is

$$\begin{aligned} x &= r\cos(\theta) \\ &= \sqrt{2}\cos\left(\frac{3\pi}{4}\right) \\ &= \sqrt{2}\cdot\frac{-1}{\sqrt{2}} \\ &= -1 \end{aligned}$$

Figure 12.1.14

and its y-coordinate is

$$\begin{aligned} y &= r\sin(\theta) \\ &= \sqrt{2}\sin\left(\frac{3\pi}{4}\right) \\ &= \sqrt{2}\cdot\frac{1}{\sqrt{2}} \\ &= 1 \end{aligned}$$

Thus, the Cartesian coordinates are $(-1, 1)$.

Checkpoint 12.1.5

Find the Cartesian coordinates of the point with polar coordinates $\left(4, \frac{5\pi}{3}\right)$.

Answer

$(2, -2\sqrt{3})$

If we know the Cartesian coordinates of a point, we can compute the polar coordinates as follows.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ \implies r &= \sqrt{x^2 + y^2} \\ \tan(\theta) &= \frac{y}{x} \end{aligned}$$

As we are used to at this point, to find the angle θ, we first compute its reference angle, $\hat{\theta} = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$. We use this reference angle to help find the value of θ by considering the given point's quadrant.

Example 12.1.6

Find the polar coordinates of the point with Cartesian coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Solution

We first compute the value of r.

$$r^2 = x^2 + y^2 = \left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$$

Thus, $r = \sqrt{1} = 1$. To find θ, we compute

$$\tan(\theta) = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}.$$

Hence, $\hat{\theta} = \frac{\pi}{3}$. There are infinitely many values of θ that satisfy this equation, but our point lies in the third quadrant, so we take $\theta = \frac{4\pi}{3}$.

Figure 12.1.15

Caution: Reference Angles and Quadrants!

In Example 12.1.6, the angle θ is not given by $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$, because the given point lies in the third quadrant. Remember that there are always two angles between 0 and 2π with a given tangent.

Checkpoint 12.1.6

Find polar coordinates for the points with Cartesian coordinates (2, −5). Give decimal approximations rounded to hundredths.

Answer

(5.39, 5.09)

Equations in Polar Coordinates

Polar coordinates are useful for studying objects or phenomena with radial symmetry, such as circles, spheres, and cylinders, or the central forces (those that act equally in all directions), such as gravity and electric charge. The equations describing these objects are often simpler in polar coordinates than in Cartesian coordinates.

It is not hard to convert an equation from Cartesian to polar coordinates: we replace each x with $r\cos(\theta)$ and each y with $r\sin(\theta)$.

Example 12.1.7

Convert the equation $2x + 3y = 6$ to polar coordinates.

Solution

Replacing by x by $r\cos(\theta)$ and y by $r\sin(\theta)$, we get

$$2r\cos(\theta) + 3r\sin(\theta) = 6.$$

If we solve for r in terms of θ, we find

$$r\left(2\cos\left(\theta\right)+3\sin\left(\theta\right)\right) = 6$$

$$\Longrightarrow \qquad r = \frac{6}{2\cos\left(\theta\right)+3\sin\left(\theta\right)}$$

Neither of these equations is simpler than the Cartesian version; however, that should not surprise us because the original equation describes a line. Cartesian coordinates are better suited to rectilinear graphs.

? Checkpoint 12.1.7

Convert the equation $x^2+y^2=4$ to polar coordinates.

Answer

$r=2$

You may recognize the equation in Checkpoint **12.1.7** as a circle. Note also that its polar equation is more straightforward than the familiar Cartesian version.

Converting equations from polar to Cartesian form is more complex than the other direction (and is sometimes impossible). Here are some strategies to try:

- Expressions of the form $r\cos\left(\theta\right)$ or $r\sin\left(\theta\right)$ can be replaced by x and y, respectively.
- r^2 can be replaced by x^2+y^2.
- $\tan\left(\theta\right)$ can be replaced by $\frac{y}{x}$.

✓ Example 12.1.8

Convert the equation $r=3\cos\left(\theta\right)$ to Cartesian coordinates.

Solution

We multiply both sides of the equation by r to obtain

$$r^2=3r\cos\left(\theta\right).$$

Now we can replace the left side by x^2+y^2 and the right side by $3x$ to obtain

$$x^2+y^2=3x,$$

This is the equation of a circle. By completing the square in each variable, we can write the equation in standard form,

$$\left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}.$$

The center of the circle is $\left(\frac{3}{2},0\right)$, and its radius is $\frac{3}{2}$.

? Checkpoint 12.1.8

Convert the equation $r=\tan\left(\theta\right)$ to Cartesian coordinates.

Answer

$y^2=\frac{x^4}{1-x^2}$

Footnotes

[1] Sadly, it was not until I finished editing this page that I realized this example makes it look like we are graphing the three-dimensional Cartesian coordinate system. Sorry.

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12.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Quadratic Functions**
  - Vertex Form and Completing the Square: This skill is required to convert some polar equations into the standard form of a circle in rectangular coordinates, as seen in Example 8.
- **Graphing Inequalities**
  - Graphing Linear Inequalities in Two Variables: The concept of graphing regions in the plane defined by inequalities is extended from the Cartesian system to the polar system in this section (Example 4).

VIDEOS

- Polar Coordinates
  - Polar Coordinates - The Polar Coordinate System
- Converting Coordinates
  - Polar Coordinates - Converting Ordered Pairs from Polar to Rectangular Coordinates
  - Polar Coordinates - Converting Ordered Pairs from Rectangular to Polar Coordinates
- Polar Equations
  - Polar Coordinates - Converting Equations from Polar to Rectangular Form
  - Polar Coordinates - Converting Equations from Rectangular to Polar Form

KEY CONCEPTS

DEFINITIONS

- **Polar Coordinate System:** A two-dimensional system for specifying points in a plane using a distance from a central point (the pole) and an angle from a reference direction (the polar axis).
- **Pole:** The central point in the polar coordinate system, equivalent to the origin in the rectangular coordinate system.
- **Polar Axis:** The reference direction from the pole, equivalent to the positive x-axis in the rectangular coordinate system.
- **Polar Coordinates:** The ordered pair (r, θ) that describes the location of a point, where $|r|$ is the distance from the point to the pole and θ is the signed angle from the polar axis to the ray passing through the point.

THEOREMS

- **Cartesian Coordinates in Terms of Cosine and Sine:** If a point P is located at a distance r from the origin in the direction specified by an angle θ in standard position, then the Cartesian coordinates of P are $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- **Non-Uniqueness of Polar Coordinates:** Any point with polar coordinates (r, θ) can also be represented by $(r, \theta + 2k\pi)$ for any integer k, or by $(-r, \theta + \pi)$. The pole has coordinates $(0, \theta)$ for any angle θ.

COMMON MISTAKES

- **Polar Backwardness:** It is customary to write polar coordinates as (r, θ), which lists the dependent variable r before the independent variable θ. This is the reverse of the standard (x, y) convention for rectangular coordinates.
- **Reference Angles and Quadrants:** When converting from rectangular coordinates (x, y) to polar coordinates (r, θ), the angle θ is not simply $\arctan(\frac{y}{x})$. The arctangent function only returns angles in Quadrants I or IV. You must use the quadrant of the point (x, y) to find the correct angle θ.
- **Plotting Negative r-values:** A negative value for r in polar coordinates, such as $(-2, \frac{\pi}{4})$, means you move in the direction opposite to the angle. In this case, you would face the direction $\frac{\pi}{4}$ but move 2 units in the opposite direction, ending up on the ray for $\frac{5\pi}{4}$.

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12.1.2: HOMEWORK

READING QUESTIONS

1. What are the two components of an ordered pair in the polar coordinate system?
2. What is the "pole" and what is the "polar axis"?
3. If a point has polar coordinates (r, θ), what are the formulas to convert it to rectangular coordinates (x, y)?
4. If a point has rectangular coordinates (x, y), how do you find its distance from the origin, r?
5. Explain why a single point can be represented by infinitely many different polar coordinates.
6. If a point has polar coordinates $(3, \frac{\pi}{4})$, give another representation for the same point where r is negative.
7. What does the graph of the polar equation $r = 3$ look like?
8. What does the graph of the polar equation $\theta = \frac{\pi}{3}$ look like?
9. What is the general strategy for converting a rectangular equation into a polar equation?
10. What are some algebraic strategies you can use to convert a polar equation into a rectangular equation?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

Recall the standard form for the equation of a circle of radius r centered at (h, k):

$$(x-h)^2 + (y-k)^2 = r^2.$$

For the following exercises, use completing the square to write each equation in standard form.

1. $x^2 + y^2 - 4x + 2y - 4 = 0$
2. $x^2 + y^2 = 88 + 18y$
3. $x^2 + y^2 + 25 = 9 - 10x - 9y$
4. $5y^2 = 85 - 5x^2 - 10x$

Answers

1. $(x-2)^2 + (y+1)^2 = 3^2$
2. $x^2 + (y-9)^2 = 13^2$
3. $(x-5)^2 + (y-3)^2 = (3\sqrt{2})^2$
4. $(x-1)^2 + y^2 = 4^2$

HOMEWORK

VOCABULARY CHECK

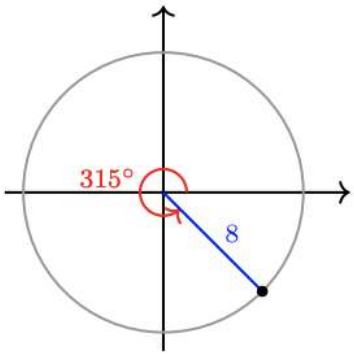
1. The origin in the polar coordinate system is called the ___.
2. The positive x-axis in the polar coordinate system is called the ___.

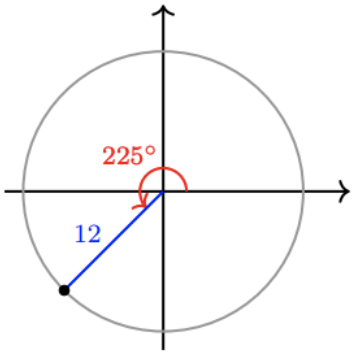
CONCEPT CHECK

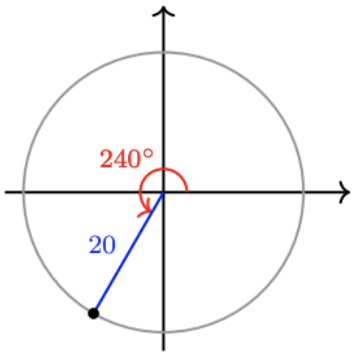
3. Why is the Cartesian coordinate system sometimes called the rectangular system?
4. What do the graphs of the polar equations $r = k$ and $\theta = k$, where k is a constant, look like?
5. Fill in the blank: the point with polar coordinates $(-6, \theta)$ also has coordinates $(6, ?)$.
6. What does the region described by $a \leq r \leq b$ look like? What does the region described by $a \leq \theta \leq b$ look like?
7. Sketch a figure that explains how to convert between polar and Cartesian coordinates.

BASIC SKILLS

For the following exercises, find exact values for the coordinates of the point.

8. 

9. 

10. 

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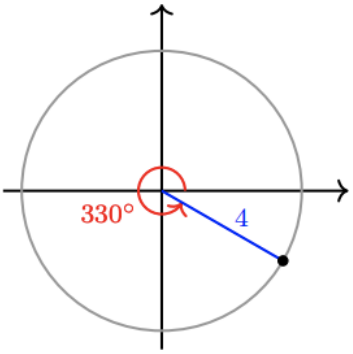

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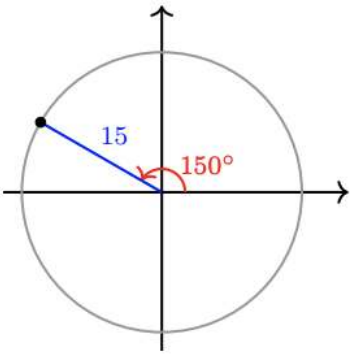

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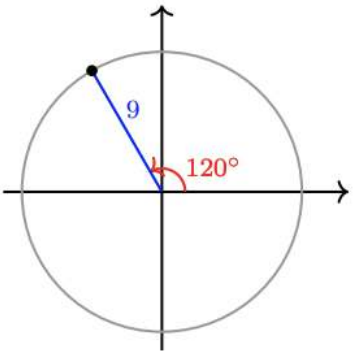
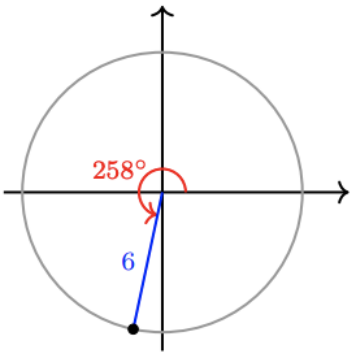


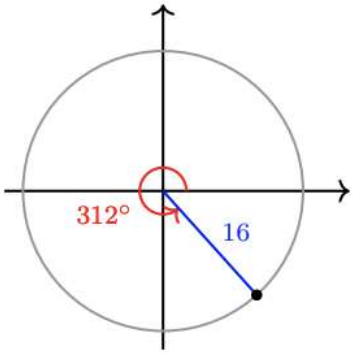
For the following exercises, find the coordinates of the point, rounded to hundredths.

14.



15.

16.



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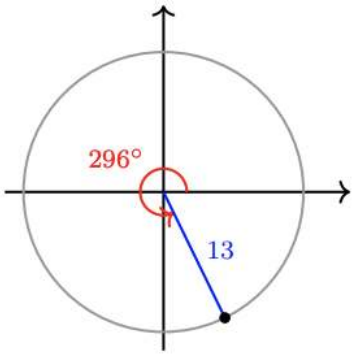

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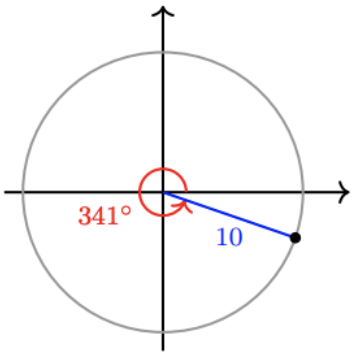

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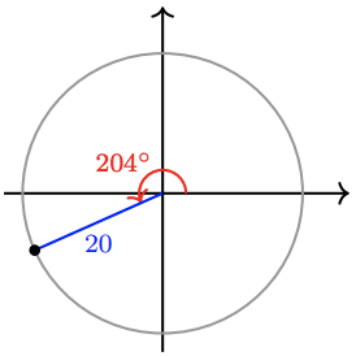
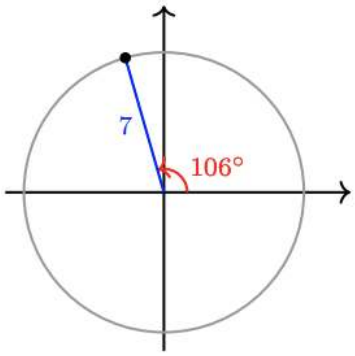
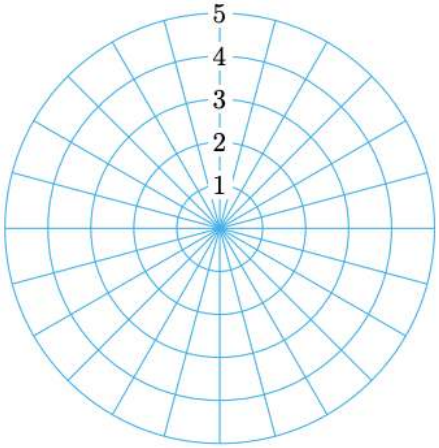




For the following exercises, use the grid below to plot the points whose polar coordinates are given.



20. $\left(2, \frac{2\pi}{3}\right)$
21. $\left(3, \frac{5\pi}{4}\right)$
22. $\left(3, \frac{3\pi}{2}\right)$
23. $(4, \pi)$
24. $\left(-4, \frac{\pi}{6}\right)$
25. $\left(-1, \frac{\pi}{2}\right)$
26. $\left(-1, \frac{7\pi}{4}\right)$
27. $\left(-2, \frac{5\pi}{3}\right)$

For the following exercises, give polar coordinates for each point shown below, with $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

28. A
29. B
30. C
31. D
32. E
33. F
34. G
35. H

For the following exercises, convert the polar coordinates to Cartesian coordinates.

36. $\left(6, \frac{2\pi}{3}\right)$
37. $\left(5, \frac{7\pi}{6}\right)$
38. $\left(-3, \frac{3\pi}{4}\right)$
39. $\left(-4, \frac{5\pi}{3}\right)$
40. $(2.4, 3.6)$
41. $(1.7, 5.2)$
42. $(-2, 1.5)$
43. $(-3, 4.5)$

For the following exercises, convert the Cartesian coordinates to polar coordinates with $r \geq 0$ and $0 \leq \theta \leq 2\pi$. Give exact values for r and θ.

44. $(7, 7)$
45. $(-7, -7)$
46. $(\sqrt{6}, -\sqrt{2})$
47. $(-\sqrt{3}, 3)$
48. $(-3, -2)$
49. $(1, 4)$
50. $(-2, 0)$
51. $(0, -5)$

For the following exercises, polar coordinates are given.
(a) Convert to polar coordinates with $r \leq 0, 0 \leq \theta < 2\pi$.
(b) Convert to polar coordinates with $r \geq 0, \quad -2\pi < \theta \leq 0$.

52. $\left(2, \frac{5\pi}{6}\right)$
53. $\left(1, \frac{4\pi}{3}\right)$
54. $(3, \pi)$
55. $(4, 0)$
56. $(2.3, 5.2)$
57. $(1.2, 1.2)$

For the following exercises, sketch the region described by the inequalities.

58. $r \leq 3$
59. $2 \leq r \leq 3$
60. $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$
61. $0 \leq \theta \leq \frac{\pi}{2}$
62. $r \geq 4, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$
63. $2 \leq r \leq 3, \frac{5\pi}{3} \leq \theta \leq \frac{11\pi}{6}$

For the following exercises, write inequalities to describe the region.

64.
65.
66.
67.

68.

69.

For the following exercises, convert the equation into Cartesian coordinates.

70. $r^2 = 2$

71. $r = 2$

72. $r = 4\cos(\theta)$

73. $r = \sin(\theta)$

74. $r = \csc(\theta)$

75. $r = 2\sec(\theta)$

76. $\tan(\theta) = 2$

77. $r^2 = \tan(\theta)$

78. $r\sec(\theta) = 3$

79. $\theta = \frac{\pi}{3}$

80. $r = \frac{2}{1+\sin(\theta)}$

81. $r = \frac{4}{4-\cos(\theta)}$

82. $2r\cos(\theta) + r\sin(\theta) = 1$

83. $\tan(\theta) = r\cos(\theta) - 2$

For the following exercises, convert the equation into polar coordinates.

84. $x = 2$

85. $y^2 = 2x - x^2$

86. $2xy = 1$

87. $y = -x$

88. $y^2 = 4x$

89. $x^2 - y^2 = 1$

90. $x^2 + y^2 = 4\sqrt{x^2 + y^2}$

91. $(x^2 + y^2)^3 = 9x^2y^2$

SYNTHESIS QUESTIONS

92. Show that the graph of $r = a\cos(\theta) + b\sin(\theta)$ is a circle. Find its center and radius.

PROOFS

93. Use the Law of Cosines to prove the Distance Formula in polar coordinates:

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

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12.2: Polar Graphs

Note to the Instructor - Polars Part Two: The Graphs

There's not much to say here other than that you will be covering the standard polar curves. I don't focus on symmetry properties of polar curves for the simple reason that students are likely exhausted at this point, you are likely low on time, and the symmetry properties are not worth the time spent if either of those first two conditions are true.

Learning Objectives

- Graph an equation in polar coordinates by plotting points.
- Graph an equation in polar coordinates using a graphing calculator.
- Identify the graph of a polar equation.
- Determine the intersection points of polar curves.

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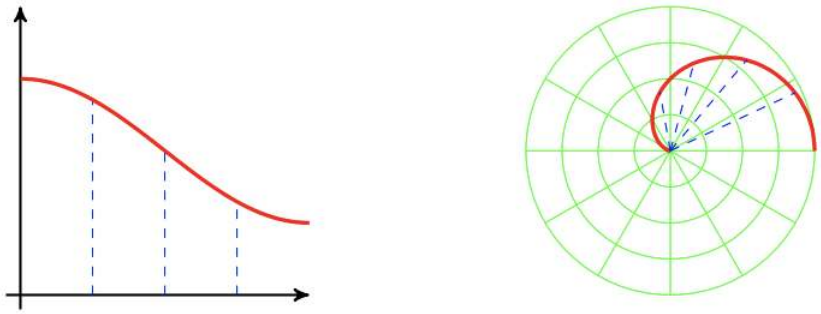
Graphing in Polar Coordinates

When we plot points in Cartesian coordinates, we start at the origin and move a distance right or left given by the x-coordinate of the point, then move up or down according to the y-coordinate. When we sketch the graph of an equation or function, we think of drawing the graph from left to right, with the "height" of the graph at each x-value given by the function, as shown in Figure 12.2.1a.

Figure 12.2.1



a. b.

In polar coordinates, however, the dependent variable, r, gives not a height but a distance from the pole in direction θ, as shown in Figure 12.2.1b. When graphing an equation in polar coordinates, we think of sweeping around the pole in the counterclockwise direction. At each angle θ, the r-value tells us how far the graph is from the pole.

Example 12.2.1

Graph the polar equation $r = 2\sin(\theta)$.

Solution

We make a table of values, choosing the special values for θ. For each value of θ, we evaluate $r = 2\sin(\theta)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

First, we'll plot the points in the first quadrant. Observe that as θ increases from 0 to $\frac{\pi}{2}$, r increases from 0 to 2. Starting at the pole, we connect the points in order of increasing θ. Imagine a radial line sweeping around the graph through the first quadrant: as the angle increases, the length of the segment increases so that its tip traces out the graph shown in Figure 12.2.2*a*.

Figure 12.2.2

Now continue plotting the points in the table as θ increases from $\frac{\pi}{2}$ to π. In the second quadrant, r decreases as θ increases, as shown in Figure 12.2.2*b*. The graph we obtain is, in fact, a circle, which we will prove algebraically shortly. However, we have not yet plotted points for θ between 0 and 2π. Because $\sin(\theta)$ is negative in the third and fourth quadrants, all the r-values for these angles are negative.

θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	π
r	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2	$-\sqrt{3}$	$-\sqrt{2}$	-1	0

Plotting the points in this table shows that the original graph is traced out again. For example, the point $\left(\frac{7\pi}{6}, -1\right)$ is the same as the point $\left(\frac{\pi}{6}, 1\right)$, the point $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is the same as the point $\left(\frac{\pi}{4}, \sqrt{2}\right)$, and so on, around the circle. Thus, the graph of $r = 2\sin(\theta)$ is a circle, traced twice for $0 \le \theta \le 2\pi$.

To prove that the graph in Example 12.2.1 is a circle, we convert the equation $r = 2\sin(\theta)$ to Cartesian form. First, multiply both sides by r to obtain

$$r^2 = 2r\sin(\theta).$$

Next, replace r^2 by $x^2 + y^2$ and $r\sin(\theta)$ by y, to get

$$x^2 + y^2 = 2y.$$

This equation is quadratic in two variables. Hence, its graph is a conic section. We put the equation in standard form by completing the square in y.

$$\begin{aligned} & x^2 + y^2 = 2y \\ \Longrightarrow \quad & x^2 + y^2 - 2y = 0 && \text{(subtract } 2y \text{ from both sides)} \\ \Longrightarrow \quad & x^2 + (y^2 - 2y + 1) = 1 && \text{(complete the square)} \\ \Longrightarrow \quad & (x-0)^2 + (y-1)^2 = 1 && \text{(factor to write in standard form)} \end{aligned}$$

We have the equation of a circle with center $(0, 1)$ and radius 1.

? Checkpoint 12.2.1

Graph the polar equation $r = 4\cos(\theta)$.

Answer

Figure 12.2.3

Using a Graphing Calculator

You are familiar with the graphs of many equations in Cartesian coordinates, including lines, parabolas, and other conic sections, as well as the graphs of basic functions. You should now become familiar with some standard graphs in polar coordinates. These include circles, **roses**, **cardioids**, **limaçons**, **lemniscates**, and **spirals**.

At the end of this section, you will find a Catalog of Polar Curves which contains the basic polar graphs and their properties. You can use your calculator, set in Polar mode, to experiment with these graphs.[1]

Example 12.2.2

Graph the Archimedean spiral $r = 2\theta$, for $0 \le \theta \le 6\pi$.

Solution

After setting the calculator in Polar mode, we enter the equation $r = 2\theta$ and enter the window settings

$$\theta\,\text{min} = 0 \quad \theta\,\text{max} = 6\pi \quad \theta\ \text{step} = 0.1$$
$$\text{Xmin} = -25 \quad \text{Xmax} = 25 \quad \text{X step} = 5$$
$$\text{Ymin} = -25 \quad \text{Ymax} = 25 \quad \text{Yscl} = 5$$

We then press Zoom 5 to set a square window.

The graph is shown below.

Figure 12.2.4

Studying a table of values can help us understand the shape of the graph.

θ	0.25	0.50	0.75	1.00	1.25
r	0.50	1.00	1.50	2.00	2.50

As θ increases, r also increases at a constant rate. We wind our way around the pole, steadily increasing our distance from the pole as we go. We spiral outwards, tracing the graph shown above.

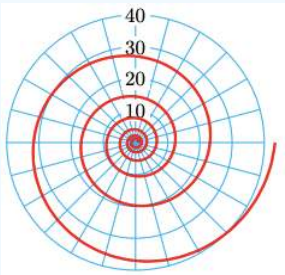
Checkpoint 12.2.2

Use a calculator to graph the polar equation $r = e^{0.1\theta}$. Use the window settings

$$\theta\,\text{min} = 0 \quad \theta\,\text{max} = 12\pi \quad \theta\ \text{step} = 0.5$$
$$\text{Xmin} = -45 \quad \text{Xmax} = 45 \quad \text{Xstep} = 10$$
$$\text{Ymin} = -35 \quad \text{Ymax} = 35 \quad \text{Yscl} = 10$$

Answer

Figure 12.2.5



It is essential to connect the points on the graph in order of increasing θ.

Example 12.2.3

Graph the polar equation $r = 2\sin(3\theta)$.

Solution

We'll graph the equation in stages to see how the graph is traced out. Begin with the window settings

$$\theta\,\text{min} = 0 \quad \theta\,\text{max} = \tfrac{\pi}{3} \quad \theta\ \text{step} = 0.5$$
$$\text{Xmin} = -4.5 \quad \text{Xmax} = 4.5 \quad \text{Xstep} = 1$$
$$\text{Ymin} = -3 \quad \text{Ymax} = 3 \quad \text{Yscl}_{=1}$$

Watch as your calculator produces the graph shown in Figure 12.2.6*a*.

Figure 12.2.6

Observe that, as θ increases from 0 to $\frac{\pi}{3}$, r first increases, reaching its maximum value of 2 at $\theta = \frac{\pi}{6}$ and then decreases back to 0. You can verify the values in the table below, which shows the points at multiples of $\frac{\pi}{18}$. These points create the first loop of the graph.

θ	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$
3θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	1	$\sqrt{3}$	2	$\sqrt{3}$	1	0

Next, change $\theta\,\text{max}$ to $\frac{2\pi}{3}$, and graph again. The calculator traces out two loops this time, as shown in Figure 12.2.6*b*. For θ between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, r is negative, so the second loop lies in the third and fourth quadrants.

θ	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$	$\frac{5\pi}{9}$	$\frac{11\pi}{18}$	$\frac{2\pi}{3}$
3θ	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	0	-1	$-\sqrt{3}$	-2	$-\sqrt{3}$	-1	0

Finally, change $\theta\,\text{max}$ to π, and graph again. For θ between $\frac{2\pi}{3}$ and π, the graph traces out a third loop, as shown in Figure 12.2.6*c*. For θ between π and 2π, the entire graph is traced a second time. The finished graph is a **rose** with 3 petals of length 2.

Checkpoint 12.2.3

a. Use your calculator to graph the polar equation $r = 2\cos(3\theta)$.

b. Complete the table of values for the function.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r							

c. Sketch the graph by hand on the grid below.

Figure 12.2.7

Answer

Figure 12.2.8

Sketching Familiar Equations

You should also be able to sketch the standard polar graphs by hand. Once you recognize an equation as a particular type of graph, say a rose or a limaçon, you can sketch the graph quickly by finding just a few well-chosen guide points. The following example demonstrates a technique for sketching a rose.

Example 12.2.4

Graph the polar equation $r = 3\sin(2\theta)$.

Solution

Referring to the Catalog of Polar Curves, we see that the graph of this equation is a rose, with petal length $a = 3$ and four petals, because $2n = 4$. If we can locate the tips of the petals, we can use them as guide points to sketch the graph.

The points at the tips of the petals have $r = 3$, so we substitute $r = 3$ into the equation of the rose to find the values of θ at those points.

$$\begin{aligned} & 3 = 3\sin(2\theta) && \text{(letting } r = 3\text{)} \\ \Longrightarrow\ & 1 = \sin(2\theta) && \text{(dividing both sides by 3)} \\ \Longrightarrow\ & \frac{\pi}{2} = 2\theta && \text{(using the arcsine to find } 2\theta\text{)} \\ \Longrightarrow\ & \frac{\pi}{4} = \theta && \text{(dividing both sides by 2)} \end{aligned}$$

Thus, one of the petal tips is located at $\theta = \frac{\pi}{4}$. Because the 4 petals are evenly spaced around the pole, the angle between the petals is $\frac{2\pi}{4} = \frac{\pi}{2}$, and the other petal tips occur at

$$\begin{aligned} \theta &= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \\ \theta &= \frac{\pi}{4} + 2\left(\frac{\pi}{2}\right) = \frac{5\pi}{4} \\ \theta &= \frac{\pi}{4} + 3\left(\frac{\pi}{2}\right) = \frac{7\pi}{4} \end{aligned}$$

We plot the tips of the petals as guide points at $\left(3, \frac{\pi}{4}\right)$, $\left(3, \frac{3\pi}{4}\right)$, $\left(3, \frac{5\pi}{4}\right)$ and $\left(3, \frac{7\pi}{4}\right)$. We can sketch a rose with 4 petals of length 3, as shown below.

Figure 12.2.9

Checkpoint 12.2.4

Graph the polar equation $r = 4\cos(5\theta)$.

Answer

Figure 12.2.10

The **limaçons**, $r = a \pm \sin(\theta)$ and $r = a \pm \cos(\theta)$, are another family of polar graphs. In particular, the **cardioid** is a special case of a limaçon with $a = b$.

Example 12.2.5

Graph the polar equation $r = 3 + 2\cos(\theta)$.

Solution

This is the equation for a limaçon, with $a = 3$ and $b = 2$. Because $a > b$, the limaçon will have a dent, like a lima bean, rather than a loop. As guide points, we locate the points at the four quadrantal angles.

θ	$r = 3 + 2\cos(\theta)$
0	5
$\frac{\pi}{2}$	3
π	1
$\frac{3\pi}{2}$	3

We plot the guide points and connect them in order with a smooth curve, as shown below. Note that this limaçon involving cosine is symmetric about the x-axis; limaçons involving sine are symmetric about the y-axis.

Figure 12.2.11

Checkpoint 12.2.5

Graph the polar equation $r = 1 - 3\sin(\theta)$.

Answer

Figure 12.2.12

You should also be able to identify a polar graph and write its equation.

Example 12.2.6

Give a polar equation for each of the graphs below.

Figure 12.2.13

Solutions

a. The graph is a circle centered at (-2,0) and with radius 2, so we choose the equation $r = 2a\cos(\theta)$, with $a = -2$. Thus, $r = -4\cos(\theta)$.

b. The graph is a cardioid with its axis of symmetry on the x-axis, and the "bottom" of the heart points in the positive x direction, so its equation has the form $r = a + a\cos(\theta)$. At $\theta = 0, r = 2$, so we can solve for a:

$$\begin{aligned} 2 &= a + a\cos(0) \\ \Longrightarrow \quad 2 &= a + a(1) \\ \Longrightarrow \quad 2 &= 2a \end{aligned}$$

Thus, $a = 1$ and we choose the equation $r = 1 + \cos(\theta)$.

Checkpoint 12.2.6

Give a polar equation for each of the graphs below.

Figure 12.2.14

Answer

a. $r^2 = 4\sin(2\theta)$

b. $r = 1 - \sin(\theta)$

Finding Intersection Points

To find the intersection points of two graphs, we solve the system made up of their equations. If the equations are $y = f(x)$ and $y = g(x)$, we simply solve the equation $f(x) = g(x)$. For example, we find the intersection points of $y = x^2$ and $y = x + 2$ by solving the equation $x^2 - x - 2 = 0$ to get

$$\begin{aligned} x^2 - x - 2 &= 0 \\ \Rightarrow \quad (x-2)(x+1) &= 0 \end{aligned}$$

Therefore, $x - 2 = 0$ or $x + 1 = 0$, which leads us to $x = 2$ or $x = -1$. These are the x-coordinates of the intersection points, and we can find the y-coordinates by substituting these values into either equation.

- For $x = 2$, we find $y = 2^2 = 4$.
- For $x = -1$, we find $y = (-1)^2 = 1$.

Thus, the intersection points are $(2, 4)$ and $(-1, 1)$, as shown in Figure 12.2.15.

Figure 12.2.15

To find the intersection points of the polar graphs $r = f(\theta)$ and $r = g(\theta)$ we solve the equation $f(\theta) = g(\theta)$.

Example 12.2.7

Find all intersection points of the graphs of $r = 2 + 2\sin(\theta)$ and $r = 2 + 2\cos(\theta)$.

Solution

We equate the two expressions for r and solve for θ.

$$\begin{aligned} & 2 + 2\sin(\theta) = 2 + 2\cos(\theta) && \text{(equating the two equations)} \\ \Rightarrow \quad & \sin(\theta) = \cos(\theta) && \text{(subtracting 2 from both sides and then dividing both sides by 2)} \\ \Rightarrow \quad & \tan(\theta) = 1 && \text{(dividing both sides by } \cos(\theta)\text{)} \\ & && \textbf{WARNING: check solutions)} \\ \Rightarrow \quad & \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} && \text{(solutions on } [0, 2\pi)\text{)} \end{aligned}$$

We evaluate either expression for θ to find the other coordinate of the intersection point.

$$\text{when} \quad \theta = \frac{\pi}{4}, \quad r = 2 + 2\sin\left(\frac{\pi}{4}\right) = 2 + \left(\frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2}$$

and

$$\text{when} \quad \theta = \frac{5\pi}{4}, \quad r = 2 + 2\sin\left(\frac{5\pi}{4}\right) = 2 + \left(\frac{-1}{\sqrt{2}}\right) = 2 - \sqrt{2}.$$

Thus, two of the intersection points are $\left(2 + \sqrt{2}, \frac{\pi}{4}\right)$ and $\left(2 - \sqrt{2}, \frac{5\pi}{4}\right)$, as shown in Figure 12.2.16.

Figure 12.2.16

However, you can see in the figure that the graphs also appear to intersect at the pole. To verify that the pole indeed lies on both graphs, we can solve for θ in each equation when $r = 0$.

$$\begin{aligned} 0 &= 2 + 2\sin(\theta) & 0 &= 2 + 2\cos(\theta) \\ \Rightarrow \quad -1 &= \sin(\theta) & -1 &= \cos(\theta) \\ \Rightarrow \quad \theta &= \frac{3\pi}{2} & \theta &= \pi \end{aligned}$$

Both points, $\left(0, \frac{3\pi}{2}\right)$ and $(0, \pi)$, represent the pole. Thus, the graphs intersect at three points: $\left(2 + \sqrt{2}, \frac{\pi}{4}\right)$, $\left(2 - \sqrt{2}, \frac{5\pi}{4}\right)$, and the pole.

Caution: Check the Pole Separately!

As we saw in Example 12.2.7, solving a system of equations $r = f(\theta)$ and $r = g(\theta)$ will not always reveal an intersection at the pole, because r may be equal to zero for different values of θ in the two equations. We should always check separately whether the pole is a point on both graphs.

Checkpoint 12.2.7

Find all intersection points of the graphs of $r = 1$ and $r = 2\cos(\theta)$.

Answer

$\left(1, \frac{\pi}{3}\right), \left(1, \frac{5\pi}{3}\right)$

Footnotes

[1] If you are using Desmos, there is no need to worry about Polar mode. Desmos is coded in such a way that it will graph a polar curve if you choose to use the variables r and θ.

Catalog of Polar Curves

Catalog of Polar Curves

Lines

Line (through the pole)
$\theta = k$, *where* k *a constant*
k *gives the angle of inclination of the line (in radians)*

Circles

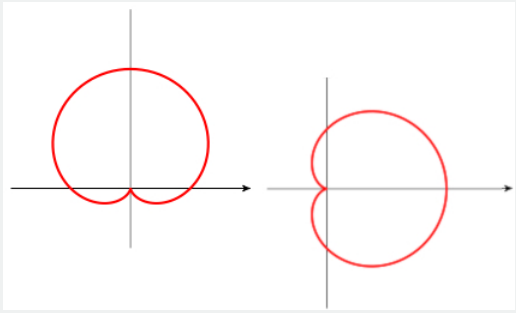
Circles
Left (Center at Pole): $r = k$, *where* k *a constant;* k *is the radius of the circle*
Middle (Centered on the vertical axis): $r = 2a\sin(\theta)$; *Center (0,* a*) on the vertical axis and radius* $|a|$
Right (centered on the horizontal axis): $r = 2a\cos(\theta)$; *Center (*a*, 0) on the horizontal axis and radius* $|a|$

Roses

Roses
Left: $r = a\sin(n\theta)$; *petal length* $|a|$; n *petals if* n *is odd;* $2n$ *petals if* n *is even*
Right: $r = a\cos(n\theta)$; *petal length* $|a|$; n *petals if* n *is odd;* $2n$ *petals if* n *is even*

Cardioids

Cardioids
Left: $r = a \pm b\sin(\theta)$, *where* $|a| = |b|$; *touches pole; "heavy" on positive vertical axis if* $b > 0$; *"heavy" on negative vertical axis if* $b < 0$
Right: $r = a \pm b\cos(\theta)$, *where* $|a| = |b|$; *touches pole; "heavy" on positive horizontal axis if* $b > 0$; *"heavy" on negative horizontal axis if* $b < 0$



Limaçons

Limaçons

Left: $r = a \pm b\sin(\theta)$, *where* $|a| > |b|$; *doesn't touch pole; "heavy" on positive vertical axis if* $b > 0$; *"heavy" on negative vertical axis if* $b < 0$

Right: $r = a \pm b\cos(\theta)$, *where* $|a| > |b|$; *doesn't touch pole; "heavy" on positive horizontal axis if* $b > 0$; *"heavy" on negative horizontal axis if* $b < 0$

Limaçons with Inner Loops

Limaçons with Inner Loops

Left: $r = a \pm b\sin(\theta)$, *where* $|a| < |b|$; *goes through pole; "heavy" on positive vertical axis if* $b > 0$; *"heavy" on negative vertical axis if* $b < 0$

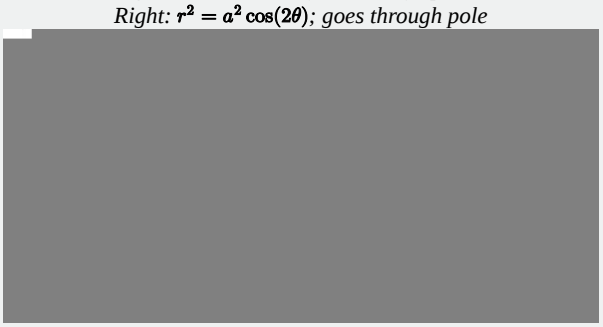
Right: $r = a \pm b\cos(\theta)$, *where* $|a| < |b|$; *goes through pole; "heavy" on positive horizontal axis if* $b > 0$; *"heavy" on negative horizontal axis if* $b < 0$

Lemniscates

Lemniscates

Left: $r^2 = a^2\sin(2\theta)$; *goes through pole*

Right: $r^2 = a^2\cos(2\theta)$; *goes through pole*



Spirals

Spirals

*Left (**Archimedean Spiral**):* $r = a\theta$

*Right (**Logarithmic Spiral**):* $r = ae^{b\theta}$

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12.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Systems**
  - Solving Systems of Nonlinear Equations: Finding the intersection points of two polar curves is equivalent to solving a system of nonlinear equations.

VIDEOS

- Polar Graphs
  - Graphs of Polar Equations - The General Idea Behind Graphing Polar Curves
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = A\sin(\theta)$ or $r(\theta) = A\cos(\theta)$
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = A \pm B\sin(\theta)$ or $r(\theta) = A \pm B\cos(\theta)$
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = A\sin(B\theta)$ or $r(\theta) = A\cos(B\theta)$
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = c$ or $\theta = c$
  - Graphs of Polar Equations - Naming the Polar Curves

KEY CONCEPTS

DEFINITIONS

- This section focuses on graphing polar equations and does not introduce any new vocabulary terms. The names of various polar curves (circles, roses, cardioids, etc.) are introduced as categories of graphs.

COMMON MISTAKES

- **Incorrectly Tracing the Graph:** When plotting points for a polar graph, it is essential to connect them in order of increasing θ. Failing to do so can result in an incorrect shape, especially for curves like limaçons and roses that may trace over themselves or have loops.
- **Missing Intersection Points at the Pole:** When finding the intersection of two polar graphs $r = f(\theta)$ and $r = g(\theta)$, solving $f(\theta) = g(\theta)$ may not reveal an intersection at the pole. This is because the pole can be represented by $r = 0$ for many different values of θ. You must always check separately if the pole ($r = 0$) is a point on both graphs.
- **Assuming a Single Representation:** A single polar graph can often be represented by multiple different equations. For example, a circle centered on the y-axis is typically written with sine, but a phase-shifted cosine function could produce the same graph.

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12.2.2: HOMEWORK

READING QUESTIONS

1. When graphing a polar equation, what does the variable r represent at each angle θ?
2. Explain why the graph of $r = 2\sin(\theta)$ for $0 \le \theta \le 2\pi$ traces the circle twice.
3. What is the general shape of the polar graph for an equation of the form $r = a\sin(n\theta)$ when n is an integer?
4. How do you determine the number of petals for a rose curve $r = a\cos(n\theta)$?
5. What is the difference between the graph of a cardioid and a limaçon with a dent?
6. What is the general strategy for finding the points of intersection of two polar graphs, $r = f(\theta)$ and $r = g(\theta)$?
7. Why must you check for an intersection at the pole separately from solving the equation $f(\theta) = g(\theta)$?
8. What is the shape of the graph for the polar equation $r = k$, where k is a constant?
9. What is the shape of the graph for the polar equation $\theta = k$, where k is a constant?
10. In the equation for a limaçon, $r = a + b\cos(\theta)$, what condition on a and b results in a cardioid? What condition results in a limaçon with an inner loop?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

Solve each equation. Give two solutions, $0 \le \theta \le 2\pi$.

1. $\cos(\theta) = \dfrac{\sqrt{3}}{2}$
2. $\sin(\theta) = \dfrac{1}{\sqrt{2}}$
3. $\tan(\theta) = \sqrt{3}$
4. $\tan(\theta) = \dfrac{1}{\sqrt{3}}$
5. $\sin(\theta) = \dfrac{11}{\sqrt{2}}$
6. $\cos(\theta) = \dfrac{-1}{2}$
7. $\tan(\theta) = -1$
8. $\cos(\theta) = 0$

Answers

1. $\dfrac{\pi}{6}, \dfrac{11\pi}{6}$
2. $\dfrac{\pi}{4}, \dfrac{3\pi}{4}$
3. $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$
4. $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$
5. $\dfrac{5\pi}{4}, \dfrac{7\pi}{4}$
6. $\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$
7. $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$
8. $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

HOMEWORK

CONCEPT CHECK

1. Mike says that the graph of $r = 2\sin(\theta)$ in the Example **12.2.2.1** cannot be correct because there are no points on the graph for angles between π and 2π. How do you respond?
2. Is it possible to have a rose with only two petals? What would its equation be?
3. Linda says that a circle of the form $r = 2a\cos(\theta)$ is a special case of a limaçon. Support or refute her statement.
4. There are no points on the graph of $r^2 = 4\cos(2\theta)$ for angles between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ Why is that?

BASIC SKILLS

For the following exercises, use your calculator to graph the equation.

5.
   a. Graph $r = k$, for $k = 1, 2, 3$. How do these graphs change for different values of k?
   b. Write a Cartesian equation for each graph in part (a).
6.
   a. Graph $r = k$, for $k = -1, -2, -3$. How do these graphs compare to the graphs in Problem 5?
   b. Write a Cartesian equation for each graph in part (a).
7.
   a. Graph $\theta = k$, for $k = \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{5\pi}{6}$. How do the graphs change for different values of k?
   b. Write a Cartesian equation for each graph in part (a).
8.
   a. Graph $\theta = k$, for $k = \dfrac{7\pi}{6}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{11\pi}{6}$. How do the graphs change for different values of k?
   b. Write a Cartesian equation for each graph in part (a).
9. Complete the table of values for each equation. Plot the points in order of increasing θ. What is different about the two graphs?
   a. $r = 2$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r = 2$								

   b. $r = -2$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r=-2$								

10. Graph each line and label the points with their coordinates. How are the points on the two lines related?

a. $\theta=\frac{\pi}{4}$

$\theta=\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
r	-2	-1	0	1	2

b. $\theta=\frac{5\pi}{4}$

$\theta=\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$
r	-2	-1	0	1	2

11.

a. Graph the circle $r=4\cos(\theta)$. Label the points corresponding to $\theta=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4}$, and π.

b. Complete the table of values. What happens to the graph as θ increases from π to 2π?

θ	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r					

c. Find the center and radius of the circle.

d. Give the Cartesian equation of the circle.

12.

a. Graph the circle $r=4\sin(\theta)$. Label the points corresponding to $\theta=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4}$, and π.

b. Complete the table of values. What happens to the graph as θ increases from π to 2π?

θ	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r					

c. Find the center and radius of the circle.

d. Give the Cartesian equation of the circle.

13.

a. Graph $r=2a\sin(\theta)$ for $a=-2,-1,1,2$.

b. How do the graphs change for different values of a?

14.

a. Graph $r=2a\cos(\theta)$ for $a=-2,-1,1,2$.

b. How do the graphs change for different values of a?

15. Complete the table of values for each cardioid and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

a. $r=1+\sin(\theta)$

b. $r=-1+\sin(\theta)$

c. $r=1-\sin(\theta)$

d. $r=-1-\sin(\theta)$

16. Complete the table of values for each cardioid and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

a. $r=1+\cos(\theta)$

b. $r=-1+\cos(\theta)$

c. c $r=1-\cos(\theta)$

d. $r=-1-\cos(\theta)$

17. Complete the table of values for each limaçon and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

a. $r=2+\cos(\theta)$

b. $r=2-\cos(\theta)$

c. $r=1+2\cos(\theta)$

d. $r=1-2\cos(\theta)$

18. Complete the table of values for each limaçon and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

a. $r=2+\sin(\theta)$

b. $r=2-\sin(\theta)$

c. $r=1+2\sin(\theta)$

d. $r=1-2\sin(\theta)$

19.

a. Graph the following roses and compare. How is the number of petals related to the value of n in the equation $r=a\sin n\theta$?

$$r=\sin(2\theta),\quad r=\sin(3\theta),\quad r=\sin(4\theta),\quad r=\sin(5\theta)\backslash)$$

b. For each graph above, list the values of θ where the tips of the petals occur.

c. Graph $r=a\sin(3\theta)$ for $a=1$, 2, and 3. How does the value of a affect the graph?

20.

a. Graph the following roses and compare. How is the number of petals related to the value of n in the equation $r=a\cos n\theta$?

$$r=\cos(2\theta),\quad r=\cos(3\theta),\quad r=\cos(4\theta),\quad r=\cos(5\theta)$$

b. For each graph above, list the values of θ where the tips of the petals occur.

c. Graph $r=a\cos(3\theta)$ for $a=1,2$, and 3 . How does the value of a affect the graph?

21.

a. Solve $r^2 = 9\cos(2\theta)$ for r. (You should get two equations for r.)

b. Graph both equations together. Change the θ step to 0.02 to see the whole graph.

c. How does the value of a affect the graph of $r^2 = a^2\cos(2\theta)$?

22.

a. Solve $r^2 = 9\cos(2\theta)$ for r. (You should get two equations for r.)

b. Graph both equations together. Change the θ step to 0.02 to see the whole graph.

c. How does this graph differ from the graph in Problem 21?

23. Graph the Archimedean spiral $r = \theta$. Set your window to

$$\theta\min = 0 \quad \theta\max = 8\pi$$
$$\text{Xmin} = -20 \quad \text{X max} = 20$$
$$\text{Y min} = -20 \quad \text{Y max} = 20$$

Then graph by pressing Zoom 5.

24. Graph the logarithmic spiral $r = e^{0.2\theta}$. Set your window to

$$\theta\min = 0 \quad \theta\max = 8\pi$$
$$\text{Xmin} = -100 \quad \text{X max} = 100$$
$$\text{Ymin} = -100 \quad \text{Y max} = 100$$

Then graph by pressing Zoom 5.

25.

a. Complete the table and graph the equation $y = \sin(3\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ									
3θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = \sin(3\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ									
3θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

26.

a. Complete the table and graph the equation $y = \cos(2\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ									
2θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = \cos(2\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ									
2θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

27.

a. Complete the table and graph the equation $y = 2 + 2\cos(\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = 2 + 2\cos(\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

28.

a. Complete the table and graph the equation $y = 1 - \sin(\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = 1 - \sin(\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

For the following exercises, use the catalog of polar graphs to help you identify and sketch the curve. Check your work by graphing with a calculator.

29. $r = 3\cos(\theta)$
30. $r = 2\sin(\theta)$
31. $\theta = \frac{\pi}{4}$
32. $\theta = \frac{4\pi}{3}$
33. $r = 4$
34. $r = 2$
35. $r = 2 + 2\sin(\theta)$
36. $r = 3 + 3\cos(\theta)$
37. $r = 2 - \cos(\theta)$
38. $r = 1 - 3\sin(\theta)$
39. $r = 3\sin(2\theta)$
40. $r = 2\cos(3\theta)$
41. $r = 2\cos(5\theta)$
42. $r = 4\sin(4\theta)$
43. $r = 2 + 3\sin(\theta)$
44. $r = 3 + 2\sin(\theta)$
45. $r^2 = \cos(2\theta)$

46. $r^2 = 4\sin(2\theta)$

For the following exercises, identify the curve and graph it.

47. $r\csc\theta = 2$

48. $r = 2\sec\theta$

49. $r^2 = 4, 0 \le \theta \le \frac{3\pi}{4}$

50. $\theta = \frac{\pi}{4}, |r| < 2$

51. $r = \sin(\theta), \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$

52. $r = \cos(\theta), 0 \le \theta \le \frac{\pi}{2}$

53. $r = 2\sin(2\theta)\cos(2\theta)$

54. $r = \cos^2\theta - \sin^2\theta$

55. $r(1 - \cos(\theta)) = \sin^2\theta$

56. $r\sec\theta = \sec\theta - \tan(\theta)$

For the following exercises, graph the polar curve. Do you recognize it?

57. $r = \frac{2}{1 - \cos(\theta)}$

58. $r = \frac{6}{2 + \sin(\theta)}$

59. $r = \frac{2}{2 - \cos(\theta)}$

60. $r = \frac{1}{1 + \sin(\theta)}$

61. $r = \frac{1}{1 + 2\sin(\theta)}$

62. $r = \frac{3}{2 - 3\cos(\theta)}$

For the following exercises, write a polar equation for the graph.

63.

64.

65.

66.

67.

68.

69.

70.

For the following exercises, find the coordinates of the intersection points of the two curves analytically. Then, graph the curves to verify your answers.

71. $r = \cos(\theta), r = 1 - \cos(\theta)$

72. $r = \sin(\theta), r = \cos(\theta)$

73. $r = 3\sin(\theta), r = 3\cos(\theta)$

74. $r = \sin(2\theta), r = \cos(2\theta)$

75. $r = 1, r = 1 - \cos(\theta)$

76. $r = 3\cos(\theta), r = 1 + \cos(\theta)$

77. $r = 2 + \sin(\theta), r = 2 - \cos(\theta)$

78. $r = \sin(\theta), r = \sin(2\theta)$

For the following exercises, graph the polar curve.

79. $r^2 = \tan(\theta)$

80. $r^2 = \cot\theta$

81. $r = \csc\theta - 2$ (conchoid)

82. $r = \tan(\theta)$ (kappa curve)

83. $r = \cos(2\theta)\sec\theta$ (strophoid)

84. $r = \sin(\theta)\tan(\theta)$ (cissoid)

85. $r = \frac{1}{\sqrt{\theta}}$

86. $r = \cos\frac{\theta}{2}, 0 \le \theta \le 4\pi$

87. Graph the polar curves $r = 1 - 2\sin n\theta$ for $n = 2, 3, 4, 5, 6$. Explain how the value of the parameter n affects the curve.

88. Graph the polar curves $r = 1 - 3\cos n\theta$ for $n = 2, 3, 4, 5, 6$. Explain how the value of the parameter n affects the curve.

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12.3: Complex Numbers

Note to the Instructor - Can You Make It?

You have arrived at a decision point. While this material is technically *not* optional, you need to weigh whether or not squeezing it into a short period of time is worth it.

If you have roughly 2 or more hours of instructional time remaining, you can easily cover this and the next (and final) section. It's worth it because of the beauty of the material; however, it is not encountered in Mathematics (at least for most students) anywhere after this course.

If you have less than 2 hours of instructional time remaining, review the course. Squeezing this material into a repid fire session is not going to be worth it.

Learning Objectives

- Express the square root of a negative number as an imaginary number.
- Add and subtract complex numbers.
- Simplify powers of i.
- Multiply and divide complex numbers.
- Graph complex numbers.
- Use the Fundamental Theorem of Algebra to find all complex-valued zeros of a polynomial.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

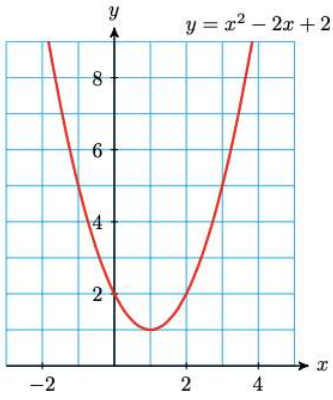
So far, most of your work in Mathematics has been done using the set of real numbers. We often represent the real numbers by a number line because they can be matched up one-for-one with the points on the line. Every real number is either rational or irrational. It can be expressed as a decimal number, although irrational numbers are non-repeating non-terminating decimals. However, the real numbers are a subset of a more extensive set of numbers called the **complex numbers**.

You may have first encountered complex numbers as solutions to certain quadratic equations. For example, the graph of

$$f(x) = x^2 - 2x + 2$$

has no x-intercepts (as shown in Figure 12.3.1), because the equation $x^2 - 2x + 2 = 0$ has no real-valued solutions.

Figure 12.3.1



Applying the Quadratic Formula, we find

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2}.$$

The solutions of the equation are $\frac{2+\sqrt{-4}}{2}$ and $\frac{2-\sqrt{-4}}{2}$, but they are not real numbers. Because $\sqrt{-4}$ is not a real number, the equation $x^2-2x+2-0$ has no real solutions.

Imaginary Numbers

Although square roots of negative numbers such as $\sqrt{-4}$ are not real numbers, they occur often in Mathematics and its applications. Mathematicians began working with square roots of negative numbers in the sixteenth century in their attempts to solve quadratic and cubic equations. René Descartes gave them the name **imaginary numbers**, which reflected the mistrust with which mathematicians regarded them at the time. Today, however, such numbers are well understood and used routinely by scientists and engineers.

We begin by defining a new number, called i, whose square is -1.

Definition: Imaginary Unit

The **unit imaginary number**, denoted as i, is defined as the square root of negative 1. That is,

$$\sqrt{-1}=i.$$

The letter i used in this way is *not* a variable; it is the name of a specific number and hence is a constant. We have seen this sort of deisgnation before - π represents the approximation $3.14159\ldots$ and e represents the approximation $2.71828\ldots$.

The square root of any negative number can be written as the product of a real number and i. For example,

$$\begin{aligned}\sqrt{-4} &= \sqrt{-1\cdot 4}\\ &= \sqrt{-1}\sqrt{4}\\ &= i\cdot 2\\ &= 2i\end{aligned}$$

Any number that is the product of i and a real number is called an **imaginary number**.

Definition: Imaginary Number

Any number of the form bi, where $b\neq 0$, is called a **pure imaginary number** (or simply an **imaginary number**).

Examples of imaginary numbers are $3i$, $\frac{7}{8}i$, $-38i$, and $i\sqrt{5}$.

Example 12.3.1

Write each radical as an imaginary number.

a. $\sqrt{-25}$
b. $2\sqrt{-3}$

Solutions

a. $\sqrt{-25}=\sqrt{-1}\sqrt{25}=i\sqrt{25}=5i$
b. $2\sqrt{-3}=\sqrt{-1}\sqrt{3}=2i\sqrt{3}$

Checkpoint 12.3.1

Write each radical as an imaginary number.

a. $\sqrt{-18}$
b. $-6\sqrt{-5}$

Answers

a. $3i\sqrt{2}$
b. $-6i\sqrt{5}$

Just as each positive number has two real-valued square roots, every negative number has two imaginary square roots. For example, the two square roots of -9 are $3i$ and $-3i$.

Complex Numbers

Consider the quadratic equation

$$x^2 - 2x + 5 = 0.$$

Using the Quadratic Formula to solve the equation, we find

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2}.$$

If we now replace $\sqrt{-16}$ by $4i$, we have

$$x = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

The two solutions are $1 + 2i$ and $1 - 2i$. These are examples of **complex numbers**.

Definition: Complex Number

A **complex number** is a number that can be written in the form $a + bi$ where

- a is called the **real part**, and
- b is called the **imaginary part** of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and b is not equal to 0, the complex number is called an **imaginary number**. The form, $a + bi$ is called the **standard form** of the complex number.

Examples of complex numbers are

$$3 - 5i, \quad 2 + \sqrt{7}i, \quad \frac{4 - i}{3}, \quad 6i, \text{ and } \quad -9.$$

The real and imaginary parts of a complex number cannot be combined. Thus, two complex numbers z_1 and z_2 are equal if and only if their real parts are equal and their imaginary parts are equal.

Theorem: Equality of Complex Numbers

Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are said to be **equal** if and only if

$$a = c \text{ and } c = d.$$

Example 12.3.2

The solutions of the equation $x^2 - 2x + 2 = 0$ are $\frac{2 \pm \sqrt{-4}}{2}$. Write these solutions as complex numbers and simplify.

Solution

Because $\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$, we have $\frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$, or $1 \pm i$. The solutions are $1 + i$ and $1 - i$.

Checkpoint 12.3.2

Use Extraction of Roots to solve $(2x + 1)^2 + 9 = 0$. Write your answers as complex numbers.

Answer

$$x = \frac{-1}{2} \pm \frac{3}{2}i$$

Arithmetic of Complex Numbers

We add and subtract complex numbers by combining their real and imaginary parts separately. For example,

$$\begin{aligned}(4+5i)+(2-3i) &= (4+2)+(5-3)i \\ &= 6+2i\end{aligned}$$

The algebraic form of this rule can be stated as follows.

Definition: Addition and Subtraction of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define addition and subtraction as follows:

Addition:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

Subtraction:

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

Example 12.3.3

Subtract: $(8 - 6i) - (5 + 2i)$.

Solution

Combine the real and imaginary parts.

$$\begin{aligned}(8-6i)-(5+2i) &= (8-5)+(-6-2)i \\ &= 3+(-8)i = 3-8i\end{aligned} \tag{12.3.1}$$

Checkpoint 12.3.3

Subtract: $(-3 + 2i) - (-3 - 2i)$

Answer

$4i$

Products of Complex Numbers

We use the following theorem to help us find the product of two imaginary numbers.

Theorem: The Square of the Imaginary Unit

$$i^2 = -1$$

Proof

$$i^2 = (i)^2 = \left(\sqrt{-1}\right)^2 = -1$$

For example,

$$\begin{aligned}(3i)(4i) &= 3 \cdot 4i^2 \\ &= 12(-1) \\ &= -12\end{aligned}$$

To find the product of two complex numbers, we use the Distributive Law as if the numbers were binomials.

Definition: Multiplication of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define multiplication as follows:

$$\begin{aligned}z_1 \cdot z_2 &= (z_1)(z_2) \\ &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + (ad + bc)i + bc(-1) \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Example 12.3.4

Multiply $(2 + 3i)(3 - 5i)$.

Solution

We multiply each term of the first complex number by each term of the second one.

$$(2 + 3i)(3 - 5i) = 6 - 10i + 9i - 15i^2.$$

Because $i^2 = -1$, the last term, $-15i^2$, can be replaced by $-15(-1)$, or 15, to obtain

$$6 - 10i + 9i + 15.$$

Finally, we combine the real parts and imaginary parts to obtain

$$(6 + 15) + (-10i + 9i) = 21 - i.$$

Checkpoint 12.3.4

Multiply $(-3 + 2i)(-3 - 2i)$.

Answer

13

Caution: A Property that Doesn't Carry Over from Arithmetic of Real Numbers

One property of real numbers that is *not* true of complex numbers is $\sqrt{ab} = \sqrt{a}\sqrt{b}$. This identity fails when a and b are negative. For example, if $a = b = -2$, we have

$$\sqrt{ab} = \sqrt{(-2)(-2)} = \sqrt{4} = 2,$$

but

$$\begin{aligned}\sqrt{a}\cdot\sqrt{b} &= \sqrt{(-2)}\cdot\sqrt{(-2)}\\ &= \sqrt{(-1\cdot 2)}\cdot\sqrt{(-1\cdot 2)}\\ &= i\sqrt{2}\cdot i\sqrt{2}\\ &= i^2(\sqrt{2})^2\\ &= -2\end{aligned}$$

so $\sqrt{ab}\neq\sqrt{a}\sqrt{b}$. We can avoid possible errors by writing the square roots of negative numbers as imaginary numbers before attempting any calculations.

Quotients of Complex Numbers

To find the quotient of two complex numbers, we use the technique of rationalizing the denominator. First, consider division by a pure imaginary number.

Example 12.3.5

Divide $\frac{3+4i}{2i}$.

Solution

Because i is a radical (remember that $i=\sqrt{-1}$), we multiply the numerator and denominator of the quotient by i to obtain

$$\begin{aligned}\frac{(3+4i)\cdot i}{2i\cdot i} &= \frac{3i+4i^2}{2i^2} && \text{(multiplying numerator and denominator by } i)\\ &= \frac{3i-4}{-2} && \text{(distributing (in the numerator) and recalling that } i^2=-1)\end{aligned}$$

To write the quotient in the form $a+bi$, we divide -2 into each term of the numerator.

$$\frac{3i}{-2}-\frac{4}{-2}=\frac{-3}{2}i+2.$$

The quotient is $2-\frac{3}{2}i$.

Checkpoint 12.3.5

Divide $\frac{8+9i}{3i}$.

Answer

$3-\frac{8}{3}i$

Perhaps you recall that to rationalize a binomial denominator, we multiply by its conjugate. For example, to rationalize the denominator of $\frac{5}{2+\sqrt{3}}$, we multiply numerator and denominator by $2-\sqrt{3}$. A similar technique works for dividing complex numbers. We first define the **conjugate** of a complex number.

Definition: Complex Conjugate

The **complex conjugate** of $z=a+bi$ is defined to be $\bar{z}=a-bi$. Likewise, the complex conjugate of $z=a-bi$ is defined to be $\bar{z}=a+bi$.

The conjugate of a complex number has several useful properties. In particular, the product of a nonzero complex number and its conjugate is always a positive real number.

Theorem: Products of Complex Conjugates

Let $z = a + bi$, so that its complex conjugate is $\bar{z} = a - bi$. Then

$$z \cdot \bar{z} = a^2 + b^2.$$

Proof

Let $z = a + bi$. Then

$$\begin{aligned} z \cdot \bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 - \cancel{abi} + \cancel{abi} - b^2(-1) \\ &= a^2 + b^2 \end{aligned}$$

Now that we have an idea of what a complex conjugate is, and what the product of a complex number and its conjugate become, we are ready to define division of complex numbers.

Definition: Division of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define division as follows:

$$\begin{aligned} z_1 \div z_2 &= \frac{z_1}{z_2} \\ &= \frac{a+bi}{c+di} \\ &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 + d^2} \\ &= \frac{ac + (-ad+bc)i - bd(-1)}{c^2 + d^2} \\ &= \frac{(ac+bd) + (bc-ad)i}{c^2 + d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i \\ &= r + si \end{aligned}$$

where r and s are real numbers.

To illustrate, we calculate the quotient $\frac{5}{2+3i}$. We multiply the numerator and denominator by the complex conjugate of the denominator, $2 - 3i$, to obtain

$$\frac{5}{2+3i} \cdot \frac{2-3i}{2-3i}.$$

The denominator is then a real number, because

$$(2+3i)(2-3i) = 4 - 6i + 6i - 9i^2 = 4 + 9 = 13$$

and the quotient is

$$\frac{5}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{10-15i}{13} = \frac{10}{13} - \frac{15}{13}i.$$

Example 12.3.6

Divide $\frac{2+3i}{4-2i}$.

Solution

We multiply the numerator and denominator by $4+2i$, the complex conjugate of the denominator.

$$\begin{aligned}
\frac{2+3i}{4-2i} &= \frac{(2+3i)(4+2i)}{(4-2i)(4+2i)} && \text{(multiplying both numerator and denominator by the complex conjugate } 4+2i\text{)} \\
&= \frac{8+4i+12i+6i^2}{16+8i-8i-4i^2} && \text{(distributing)} \\
&= \frac{8+16i-6}{16-(-4)} && \text{(combining like terms in the numerator and using } i^2=-1 \text{ in the denominator)} \\
&= \frac{2+16i}{20} && \text{(combining like terms)} \\
&= \frac{2}{20}+\frac{16i}{20} && \text{(splitting the fraction)} \\
&= \frac{1}{10}+\frac{4}{5}i && \text{(simplifying fractions)}
\end{aligned}$$

Checkpoint 12.3.6

Write the quotient $\frac{4-2i}{1+i}$ in the form $a+bi$.

Answer

$1-3i$

Graphing Complex Numbers

Real numbers can be plotted on a number line, but we use the **complex plane** to graph a complex number.

Definition: Complex Plane

The **complex plane** is a coordinate system in which the horizontal axis represents the real part of a complex number and the vertical axis represents the imaginary part. In this sense, we define the horizontal axis to be the **real axis** and the vertical axis to be the **complex axis**. Complex numbers are the points on the plane, expressed as ordered pairs (a,b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

To plot a complex number $a+bi$, we move a units from the origin in the horizontal direction and b units in the vertical direction.

Example 12.3.7

Plot the numbers $z=2+3i$ and $\bar{z}=2-3i$ in the complex plane.

Solutions

To plot $z=2+3i$, we move from the origin 2 units to the right and 3 units up. To plot $\bar{z}=2-3i$, we move from the origin 2 units to the right and 3 units down. The numbers $2+3i$ and $2-3i$ are plotted in the figure below.

Figure 12.3.2

Checkpoint 12.3.7

Plot $z = -3 + 5i$ and its conjugate in the complex plane.

Answer

Figure 12.3.3

Caution: Points in the Complex Plane Represent Numbers

You are used to plotting ordered pairs (x, y) in the Cartesian plane so that one point represents two real numbers given by its coordinates. However, in the complex plane, one point represents a *single* complex number $z = a + bi$, and the coordinates of the point are the real and imaginary parts of z. For example, the point $(2, -6)$ in the real plane represents the pair of real numbers $x = 2$ and $y = -6$, but the point $(2, -6)$ in the complex plane represents the complex number $z = 2 - 6i$.

The **modulus**, or length, of a complex number is its distance from the origin in the complex plane. The modulus of a complex number is analogous to the absolute value of a real number. It is similarly denoted by $|z|$. If $z = a + bi$, we can use the Pythagorean Theorem to compute its modulus.

Definition: Modulus (of a complex number)

The **modulus** of a complex number $z = a + bi$ is given by

$$|z| = \sqrt{a^2 + b^2}.$$

For example, the modulus of $z = 2 + 3i$ is

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

Example 12.3.8

Plot all points in the complex plane that satisfy the equation $|z| = 4$.

Solution

The points that satisfy $|z| = 4$ all lie 4 units from the origin. In other words, they lie on a circle of radius 4 centered at the origin, as shown below.

Figure 12.3.4

? Checkpoint 12.3.8

Write an equation for the circle of radius 1 centered at the origin in the complex plane.

Answer

$|z| = 1$

We can think of the graph of a complex number $z = a + bi$ as the vector that starts at the origin and ends at the point (a, b) in the complex plane, as shown in Figure 12.3.5*a*.

Figure 12.3.5

Then, the sum of two complex numbers corresponds to the sum of the two vectors representing them. Figure 12.3.5*b* illustrates the sum of two complex numbers, z_1 and z_2, by vector addition using the parallelogram law: we form a parallelogram with the vectors z_1 and z_2 and as adjacent sides. Their sum is the vector that forms the diagonal of the parallelogram, starting at the origin.

✓ Example 12.3.9

Illustrate the sum of $z_1 = 5 - i$ and $z_2 = -7 + 4i$ by vector addition.

Solution

Draw a vector starting at the origin and ending at the point (5,-1) in the complex plane to represent z_1, and another vector ending at (-7,4) to represent z_2. Now sketch the opposite sides of a parallelogram and draw the diagonal starting at the origin, as shown below.

Figure 12.3.6

This vector represents the sum $z_1 + z_2$. You can check that the vector ends at the point

$$z_1 + z_2 = (5 - 7) + (-1 + 4)i = -2 + 3i.$$

To visualize subtraction, we add the opposite of the second vector because

$$z_1 - z_2 = z_1 + (-z_2).$$

Recall that the opposite of a vector has the same length, but it points in the opposite direction.

? Checkpoint 12.3.9

Illustrate $z_1 - z_2$, for $z_1 = 3 - 2i$ and $z_2 = -4 + 5i$.

Answer

Figure 12.3.7

Zeros of Polynomials

A polynomial with real-number coefficients may or may not have real-valued zeros. For example, the polynomial $x^4 + 4$ has no real-valued zeros. But a polynomial always has a zero if we allow complex numbers as inputs. Because we can add, subtract, and multiply any two complex numbers, we can evaluate a polynomial function at a complex number. Thus, we can extend the domain of any polynomial to include all complex numbers.

✓ Example 12.3.10

Evaluate the polynomial $f(x) = x^2 - 2x + 2$ for $x = 1 + i$, and simplify.

Solution

We substitute $x = 1 + i$, to find

$$\begin{aligned} f(1+i) &= (1+i)^2 - 2(1+i) + 2 \\ &= 1^2 + 2i + i^2 - 2 - 2i + 2 \\ &= 1 + 2i + (-1) - 2 - 2i + 2 \end{aligned}$$

Thus, $f(1+i) = 0$, so $1+i$ is a zero of the polynomial.

? Checkpoint 12.3.10

If $f(x) = x^2 - 6x + 13$, evaluate $f(3+2i)$.

Answer

0

The zeros of a quadratic polynomial $ax^2 + bx + c$, of course, are the solutions of the quadratic equation $ax^2 + bx + c = 0$, and those solutions are given by the Quadratic Formula,

$$x = \frac{-b}{2a} + \frac{b^2 - 4ac}{2a} \text{ and } x = \frac{-b}{2a} - \frac{b^2 - 4ac}{2a}.$$

If the discriminant $D = b^2 - 4ac$ is negative, the two solutions are complex conjugates. For example, the solutions of the equation $x^2 - 4x + 5 = 0$ are

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2}$$

or $z = 2 + i$ and $\bar{z} = 2 - i$. Thus, if we know that $z = a + bi$ is one complex solution of a quadratic equation, we know that $\bar{z} = a - bi$ is the other solution.

We can now write a quadratic polynomial, with real coefficients, having any complex number as one of its zeros. The factored form of the quadratic polynomial with zeros z and $\bar{z}$ is

$$p(x) = (x - z)(x - \bar{z}).$$

Expanding the right side, we find

$$p(x) = x^2 - (z + \bar{z})x + z\bar{z}.$$

Because $(z + \bar{z})$ and $z\bar{z}$ are both real numbers, this polynomial has real-valued coefficients.

Theorem: Complex Zeros Theorem

If the polynomial function f has *real* coefficients and a complex zero of the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero of f.

Proof

The proof requires several theorems from complex numbers. These are mentioned within the proof. If $z = a + bi$ is a zero of a polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, then

$$
\begin{aligned}
f(\overline{z}) &= a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1\overline{z} + a_0 && \text{(substituting)} \\
&= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \cdots + a_1\overline{z} + a_0 && \left(\overline{z^n} = (\overline{z})^n\right) \\
&= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} && (\text{if } a_i \in \mathbb{R}, \text{ then } a_i = \overline{a_i}) \\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && (\overline{z_1}\cdot\overline{z_2} = \overline{z_1\cdot z_2}) \\
&= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} && \left(\overline{z_1}+\overline{z_2}+\cdots+\overline{z_k} = \overline{z_1+z_2+\cdots+z_k}\right) \\
&= \overline{f(z)} \\
&= \overline{0} && (z \text{ is a zero of } f(x)) \\
&= 0 && \left(\overline{k} = k \text{ for real numbers } k\right)
\end{aligned}
$$

Hence, $\overline{z}$ (the conjugate) is also a zero of $f(x)$.

Example 12.3.11

a. Let $z = 7 - 5i$. Compute $(z + \bar{z})$ and $z\bar{z}$.
b. Find a quadratic polynomial with one zero being $z = 7 - 5i$.

Solutions

a. The conjugate of $z = 7 - 5i$ is $\bar{z} = 7 + 5i$, so

$$(z + \bar{z}) = (7 - 5i) + (7 + 5i) = 14$$

and

$$
\begin{aligned}
z\bar{z} &= (7 - 5i)(7 + 5i) \\
&= 49 - 25i^2 \\
&= 49 + 25 \\
&= 74
\end{aligned}
$$

b. The polynomial is

$$
\begin{aligned}
p(x) &= x^2 + (z + \bar{z})x + z\bar{z} \\
&= x^2 - 14x + 74
\end{aligned}
$$

Checkpoint 12.3.11

a. Let $z = -3 + 4i$. Compute $(z + \bar{z})$ and $z\bar{z}$.
b. Find a quadratic polynomial with one zero being $z = -3 + 4i$.

Answer

a. $z + \bar{z} = -6, \quad z\bar{z} = 25$
b. $x^2 + 6x + 25 = 0$

One of the most important results in mathematics is the **Fundamental Theorem of Algebra**.

Theorem: Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued coefficients, then $f(x)$ has at least one complex-valued zero.

Proof

The proof of this theorem is beyond the scope of lower-division mathematics.

Despite the name, however, the Fundamental Theorem of Algebra needs one linked theorem (called a corollary) to be useful for our current conversation.

Corollary: Complete Factorization Theorem

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued coefficients, then $f(x)$ has exactly n linear factors over the complex number system. That is,

$$f(x) = a(x - z_1)(x - z_2)...(x - z_n),$$

where z_1, z_2, ..., z_n are complex numbers and $a \in \mathbb{R}$.

Therefore, $f(x)$ has n complex-valued zeros if we allow for multiplicities.

Proof

The proof of this theorem is done using mathematical induction and the Fundamental Theorem of Algebra. Since this topic is usually covered *prior* to learning proofs by mathematical induction, the proof is omitted.

Before we continue, let's define the **multiplicity** of a zero of a polynomial.

Definition: Multiplicity of a Zero

If a polynomial contains a factor of the form $(x - h)^p$, and if this is the greatest such factor of $(x - p)$, we say that $x = h$ is a **zero of multiplicity** p.

For example, the polynomial $p(x) = (x - 5)^2$ has a zero of multiplicity 2 at $x = 5$.

As a result of the Fundamental Theorem of Algebra and the Complete Factorization Theorem, every polynomial can be factored as the product of linear terms.

For example, although the graph of $y = x^4 + 4$ shown below has no x-intercepts, the Fundamental Theorem of Algebra tells us that there are four complex solutions to $x^4 + 4 = 0$, and that $x^4 + 4$ can be factored. You can check that the four solutions to $x^4 + 4 = 0$ are

$$1 + i,\ -1 + i,\ -1 - i, \text{ and } 1 - i.$$

Figure 12.3.8

For example, if $x = 1 + i$, then

$$x^2 = (1 + i)^2 = 1 + 2i + i^2 = 2i$$

and

$$x^4 = (x^2)^2 = (2i)^2 = -4.$$

Because each zero corresponds to a factor of the polynomial, the factored form of $x^4 + 4$ is

$$x^4 + 4 = [x - (1 + i)][x - (-1 + i)][x - (-1 - i)][x - (1 - i)].$$

The four solutions to $x^4 + 4 = 0$ form two complex conjugate pairs, namely $1 \pm i$ and $-1 \pm i$. We see again that, for every polynomial with real coefficients, the nonreal zeros always occur in complex conjugate pairs.

Example 12.3.12

Find a fourth degree polynomial with real coefficients, two of whose zeros are $3i$ and $2 + i$.

Solution

The other two zeros are $-3i$ and $2 - i$. The factored form of the polynomial is

$$(x-3i)(x+3i)[x-(2+i)][x-(2-i)].$$

We multiply together the factors to find the polynomial. The product of $(x-3i)(x+3i)$ is x^2+9, and

$$\begin{aligned}[x-(2+i)][x-(2-i)] &= x^2-(2+i+2-i)x+(2+i)(2-i)\\ &= x^2-4x+5\end{aligned}$$

Finally, we multiply these two partial products to find the polynomial we seek,

$$\left(x^2+9\right)\left(x^2-4x+5\right)=x^4-4x^3+14x^2-36x+45.$$

? Checkpoint 12.3.11

a. Find the zeros of the polynomial x^4+15x^2-16.
b. Write the polynomial in factored form.

Answer

a. $\pm 1, \pm 4i$
b. $(x-1)(x+1)(x-4i)(x+4i)$

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12.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Arithmetic**
  - Mathematical Operations - Powers and Square Roots: This section introduces the imaginary unit by considering the square root of negative numbers.
- **Polynomial Functions**
  - The Basics of Polynomial Functions: This section concludes with a discussion of the zeros of polynomials, requiring a basic understanding of what a polynomial is.

VIDEOS

- Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Definitions, Theorem, and Example
- Complex Plane
  - Arithmetic - Sets - Complex Numbers - The Complex Plane and Plotting Complex Numbers
- Arithmetic with Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Adding and Subtracting Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Multiplying Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Dividing Complex Numbers
- Power of the Imaginary Unit
  - Arithmetic - Sets - Complex Numbers - Simplifying Powers of i
- Fundamental Theorem of Algebra
  - Complex Zeros and the Fundamental Theorem of Algebra - Intro to the Fundamental Theorem of Algebra
- Complete Factorization Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Complete Factorization Theorem
- Conjugate Zeros Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Zeros and Conjugate Zeros Theorems
  - Complex Zeros and the Fundamental Theorem of Algebra - Using the Fundamental Theorem of Algebra

KEY CONCEPTS

DEFINITIONS

- **Imaginary Unit:** The number i, defined as $i = \sqrt{-1}$.
- **Imaginary Number:** Any number of the form bi, where b is a non-zero real number.
- **Complex Number:** A number that can be written in the standard form $a + bi$, where a is the real part and b is the imaginary part.
- **Addition and Subtraction of Complex Numbers:** The process of adding or subtracting two complex numbers by combining their real and imaginary parts separately.
- **Multiplication of Complex Numbers:** The process of multiplying two complex numbers using the Distributive Law, as if they were binomials.
- **Complex Conjugate:** The complex conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$.
- **Division of Complex Numbers:** The process of finding the quotient of two complex numbers, typically by multiplying the numerator and denominator by the complex conjugate of the denominator.
- **Complex Plane:** A coordinate system used to graph complex numbers, where the horizontal axis represents the real part and the vertical axis represents the imaginary part.
- **Modulus (of a complex number):** The distance of a complex number from the origin in the complex plane, given by $|z| = \sqrt{a^2 + b^2}$ for $z = a + bi$.
- **Multiplicity of a Zero:** If $(x - h)^p$ is the greatest power of the factor $(x - h)$ in a polynomial, then h is a zero of multiplicity p.

THEOREMS

- **Equality of Complex Numbers:** Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are equal if and only if their real parts are equal ($a = c$) and their imaginary parts are equal ($b = d$).
- **The Square of the Imaginary Unit:** $i^2 = -1$.
- **Products of Complex Conjugates:** For a complex number $z = a + bi$, the product of the number and its conjugate is a real number: $z\bar{z} = a^2 + b^2$.
- **Complex Zeros Theorem:** If a polynomial function f has real coefficients and a complex zero of the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero of f.
- **Fundamental Theorem of Algebra:** If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued coefficients, then $f(x)$ has at least one complex-valued zero.
- **Corollary: Complete Factorization Theorem:** If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued

coefficients, then $f(x)$ has exactly n linear factors over the complex number system (counting multiplicities).

COMMON MISTAKES

- **A Property that Doesn't Carry Over from Arithmetic of Real Numbers:** The property $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is not true for complex numbers when both a and b are negative. Always write the square roots of negative numbers as imaginary numbers first before performing calculations.
- **Points in the Complex Plane Represent Numbers:** A single point (a, b) in the complex plane represents one complex number, $z = a + bi$, not two separate real numbers as in the Cartesian plane.

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12.3.2: HOMEWORK

READING QUESTIONS

1. What is the imaginary unit, i, defined to be?
2. What are the real and imaginary parts of the complex number $3-5i$?
3. How do you add two complex numbers?
4. What is the complex conjugate of $z=a+bi$?
5. What is the result of multiplying a complex number by its conjugate?
6. When dividing by a complex number, what is the general technique used to eliminate the imaginary part from the denominator?
7. What do the horizontal and vertical axes represent in the complex plane?
8. What is the modulus of a complex number, and what does it represent geometrically?
9. What does the Fundamental Theorem of Algebra guarantee for any polynomial of degree 1 or higher?
10. If a polynomial with real coefficients has a zero of $4-i$, what is another zero that it must have?
11. What is the multiplicity of the zero at $x=2$ for the polynomial $p(x)=(x-2)^3(x+1)$?

SKILLS REFRESHER

Review the following skills you will need for this section.

? SKILLS REFRESHER

For the following exercises, solve.

1 $3x^2=6x$

2 $16x^2=9$

3 $x^2+4x=12$

4 $2(x-3)^2=8$

5 $3x^2-8=10x$

6 $x^2-3x+1=0$

Answers

1 $0, 2$

2 $\frac{\pm 3}{4}$

3 $-6, 2$

4 $1, 5$

5 $\frac{-2}{3}, 4$

6 $\frac{3\pm\sqrt{5}}{2}$

HOMEWORK

CONCEPT CHECK

1. What are imaginary numbers, and why were they invented?
2. Simplify the following powers of i:

   $$i^2, i^3, i^4, i^5, i^6, i^7, i^8.$$

   What do you notice?
3. Explain how the complex conjugate is used in dividing complex numbers.
4. If one solution of a quadratic equation is $3+i\sqrt{2}$, what is the other solution?
5. If $P(x)$ is a polynomial of degree 7, how many zeros does $P(x)$ have? How many x-intercepts could its graph have? How many complex zeros could $P(x)$ have?

BASIC SKILLS

For the following exercises, write the complex number in the form $a+bi$, where a and b are real numbers.

6.
   a. $\sqrt{-25}-4$
   b. $\frac{-8+\sqrt{-4}}{2}$
   c. $\frac{-5-\sqrt{-2}}{6}$
7.
   a. $\sqrt{-9}+3$
   b. $\frac{6-\sqrt{-36}}{2}$
   c. $\frac{7+\sqrt{-3}}{4}$

For the following exercises, find the zeros of the quadratic polynomial. Write each zero in the form $a+bi$, where a and b are real numbers.

8. $x^2+6x+13$
9. $x^2-2x+10$
10. $3x^2-x+1$
11. $5x^2+2x+2$

For the following exercises, add or subtract.

12. $(11-4i)-(-2-8i)$
13. $(7i-2)+(6-4i)$
14. $(2.1+5.6i)+(-1.8i-2.9)$
15. $\left(\frac{1}{5}i-\frac{2}{5}\right)-\left(\frac{4}{5}-\frac{3}{5}i\right)$

For the following exercises, multiply.

16. $5i(2-4i)$
17. $-7i(-1+4i)$
18. $(4-i)(-6+7i)$
19. $(2-3i)(2-3i)$
20. $(7+i\sqrt{3})^2$

21. $(5-i\sqrt{2})^2$

22. $(7+i\sqrt{3})(7-i\sqrt{3})$

23. $(5-i\sqrt{2})(5+i\sqrt{2})$

24. $(1-i)^3$

25. $(2+i)^3$

For the following exercises, divide.

26. $\frac{12+3i}{-3i}$

27. $\frac{12+4i}{8i}$

28. $\frac{10+15i}{2+i}$

29. $\frac{4-6i}{1-i}$

30. $\frac{5i}{2-5i}$

31. $\frac{-2i}{7+2i}$

32. $\frac{\sqrt{3}}{\sqrt{3}+i}$

33. $\frac{2\sqrt{2}}{1-i\sqrt{2}}$

34. $\frac{1+i\sqrt{5}}{1-i\sqrt{5}}$

35. $\frac{\sqrt{2}-i}{\sqrt{2}+i}$

36. $\frac{3+2i}{2-3i}$

37. $\frac{4-6i}{-3-2i}$

38. Simplify.

    a. i^6

    b. i^{12}

    c. i^{15}

    d. i^{102}

39. Express with a positive exponent and simplify.

    a. i^{-1}

    b. i^{-2}

    c. i^{-3}

    d. i^{-6}

For the following exercises, evaluate the polynomial for the given values of the variable.

40. z^2+9

    a. $z=3i$

    b. $z=-3i$

41. $2y^2-y-2$

    a. $y=2-i$

    b. $y=-2-i$

42. x^2-2x+2

    a. $x=1-i$

    b. $x=1+i$

43. $3w^2+5$

    a. $w=2i$

    b. $w=-2i$

44. $q^2+4q+13$

    a. $q=-2+3i$

    b. $q=-2-3i$

45. v^2+2v+3

    a. $v=1+i$

    b. $v=-1+i$

For the following exercises, expand the product of polynomials.

46. $(2z+7i)(2z-7i)$

47. $(5w+3i)(5w-3i)$

48. $[x+(3+i)][x+(3-i)]$

49. $[s-(1+2i)][s-(1-2i)]$

50. $[v-(4+i)][v-(4-i)]$

51. $[Z+(2+i)][Z+(2-i)]$

For the following exercises, plot the number and its complex conjugate in the complex plane. What is the geometric relationship between complex conjugates?

52. $z=-3+2i$

53. $z=4-3i$

54. $z=\frac{\sqrt{3}}{2}-\frac{1}{2}i$

55. $z=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

For the following exercises, sketch the set of points in the complex plane.

56. $|z|\leq 2$

57. $1<|z|<3$

58. The set of all $z=a+bi$ for which $a\geq b$

59. The set of all $z=a+bi$ for which $a+b<2$

For the following exercises, illustrate addition using the parallelogram rule in the complex plane.

60. $(1-4i)+(-3+2i)$

61. $(-4+2i)+(-2+i)$

62. $(2+6i)-(3+3i)$

63. $(-5-2i)-(-3+2)$

For the following exercises,
(a) Given one solution of a quadratic equation with rational coefficients, find the other solution.
(b) Write a quadratic equation that has those solutions.

64. $2+\sqrt{5}$

65. $3-\sqrt{2}$

66. $4-3i$

67. $5+i$

For the following exercises, find a fourth degree polynomial with real coefficients that has the given complex numbers as two of its zeros.

68. $1-3i, 2-i$

69. $5-4i, -i$

70. $\frac{1}{2}-\frac{\sqrt{3}}{2}i, 3+2i$

71. $-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, 4-i$

SYNTHESIS QUESTIONS

72. Suppose t, w, and z are complex numbers. If $w=t+z$, is it necessarily true that $|w|=|t|+|z|$? Provide examples to support your conclusion.

PROOFS

73. Prove that the product of two complex numbers $z_1=a+bi$ and $z_2=c+di$ is

$$z_1z_2=(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$$

74. Prove that the quotient of two complex numbers $z_1=a+bi$ and $z_2=c+di$ is

$$\frac{z_1}{z_2}=\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i.$$

75. Prove the commutative laws for addition and multiplication of complex numbers:

$$z_1+z_2 = z_2+z_1$$

and

$$z_1z_2 = z_2z_1$$

76. Prove the Distributive Law for complex numbers:

$$z_1(z_2+z_3)=z_1z_2+z_1z_3.$$

77.

a. Prove that $z+\bar{z}$ is a real number, and $z-\bar{z}$ is an imaginary number.

b. Prove that $z\bar{z}=|z|^2$

78. Prove that $\overline{zw}=\bar{z}\cdot\bar{w}$

79. Prove the triangle inequality for complex numbers:

$$|w+z|\leq|w|+|z|.$$

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12.4: Polar Form of Complex Numbers

Note to the Instructor - Bragging Rights

If you made it to (and through) this section, congratulate yourself and your students. You have performed a substantial accomplishment.

Learning Objectives

- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

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> *"God made the integers; all else is the work of man."*

This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

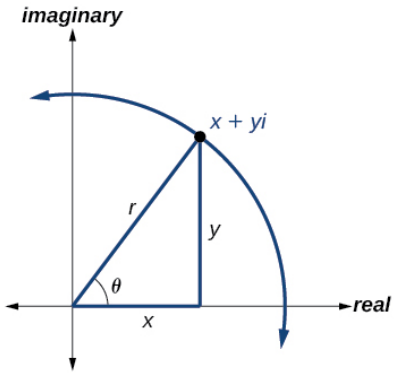
Writing Complex Numbers in Polar Form

Given a complex number expressed as $z = x + y\,i$, we can use the material from the previous section to visual z as the point (x, y) on the Cartesian plane.[1] We then observe that we can rewrite both x and y in trigonometric form:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

where θ is the angle in standard position with terminal side being the line segment connecting the origin to the point (x, y). We review these relationships in Figure **12.4.1**.

Figure 12.4.1



Thus, we can rewrite the complex number, $z = x + y\,i$, in the form

$$z = r\cos(\theta) + (r\sin(\theta))\,i = r\,(\cos(\theta) + i\,\sin(\theta))\,.$$

This form is known as the **polar form** of the complex number.

Definition: Polar Form of a Complex Number

The complex number, $z = x + yi$, can be written in the form

$$z = r(\cos(\theta) + i\sin(\theta)) \equiv r\,\text{cis}(\theta),$$

where

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

r is the modulus of z, and θ is the argument. This form is called the **polar form** of the complex number.

The notation $z = r\,\text{cis}(\theta)$ is read as "cosine plus i sine of theta."

As we did in the previous section, we use the term modulus to represent the "size" of a complex number, or the distance from the origin to the point (x, y) on the complex plane. The modulus, therefore, is the same as r, the radius in polar form.

Theorem: Modulus in Polar Form

The modulus of the complex number z, written in polar form $z = r(\cos(\theta) + i\sin(\theta))$, is r.

Example 12.4.1

Express the complex number $4i$ in polar form.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. To write this in polar form, we must first calculate r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{0^2 + 4^2} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

Next, we look at x. If $x = r\cos(\theta)$ and $x = 0$, then $\theta = \frac{\pi}{2}$. Therefore, in polar form, the complex number $z = 0 + 4i$ can be written as

$$z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 4\,\text{cis}\left(\frac{\pi}{2}\right).$$

See Figure 12.4.2.

Figure 12.4.2

Checkpoint 12.4.1

Express $z = 3i$ as $r\,\text{cis}(\theta)$ in polar form.

Answer

$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ or $z = 3\,\text{cis}\left(\frac{\pi}{2}\right)$

Example 12.4.2

Find the polar form of $-4 + 4i$.

Solution

We begin by finding the value of r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(-4)^2+4^2} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

We now find the angle θ.

$$\begin{aligned} \cos(\theta) &= \frac{x}{r} \\ \Longrightarrow \quad \cos(\theta) &= -\frac{4}{4\sqrt{2}} \\ \Longrightarrow \quad \cos(\theta) &= -\frac{1}{\sqrt{2}} \\ \Longrightarrow \quad \hat{\theta} &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ \Longrightarrow \quad \hat{\theta} &= \frac{\pi}{4} \end{aligned}$$

Using the fact that $z=-4+4i$ is in the second quadrant, we get $\theta = \frac{3\pi}{4}$. Thus, the polar form is

$$z = 4\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right).$$

It's very tempting to evaluate the trigonometric functions at $\frac{3\pi}{4}$ in Example 12.4.2. I would not blame you for wanting to do that because that's what we have been doing this entire course; however, in doing so, you would be reverting back to the standard (recangular) form of the complex number (thereby undoing all of your work to get the number in polar form).

? Checkpoint 12.4.2

Write $z=\sqrt{3}+i$ in polar form.

Answer

$z = 2\left(\cos\left(\frac{\pi}{6}\right)+i\,\sin\left(\frac{\pi}{6}\right)\right) = 2\text{cis}\left(\frac{\pi}{6}\right)$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given

$$z = r\left(\cos(\theta)+i\,\sin(\theta)\right),$$

we first evaluate the trigonometric functions $\cos(\theta)$ and $\sin(\theta)$, then multiply through by r.

✓ Example 12.4.3

Convert the polar form of the given complex number to rectangular form:

$$z = 12\left(\cos\left(\frac{\pi}{6}\right)+i\,\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

After substitution, the complex number is

$$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 6\sqrt{3} + 6i$$

Hence, the rectangular form of the given point in polar form is $6\sqrt{3} + 6i$.

? Checkpoint 12.4.3

Convert the complex number to rectangular form:

$$z = 4\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$$

Answer

$z = 2\sqrt{3} - 2i$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham de Moivre (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

Theorem: Products of Complex Numbers Written in Polar Form

If $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, then the product of these numbers is given as

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2).$$

Notice that the product calls for multiplying the moduli and adding the angles. The proof of this theorem is left as a homework exercise.

✓ Example 12.4.4

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80°) + i\sin(80°))$ and $z_2 = 2\operatorname{cis}(145°)$.

Solution

Follow the formula

$$\begin{aligned} z_1 z_2 &= 4 \cdot 2\left[\cos(80° + 145°) + i\sin(80° + 145°)\right] \\ &= 8\left[\cos(225°) + i\sin(225°)\right] \\ &= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ &= -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

? Checkpoint 12.4.4

Let $z_1 = -\sqrt{3} + i$ and $z_2 = 3 + 3i$.

a. Find the product of z_1 and z_2 in rectangular form (without converting to polar form).
b. Without converting to polar form, determine the angle $z_1 z_2$ makes with the positive x-axis.
c. Rewrite z_1 and z_2 in polar form.
d. Find the product of z_1 and z_2 in polar form.
e. Using your answer from part (d), determine the angle $z_1 z_2$ makes with the positive x-axis.

Answers

a. $z_1 z_2 = (-3 - 3\sqrt{3}) + (3 - 3\sqrt{3})i$
b. $195°$
c. $z_1 = 2(\cos(150°) + i\sin(150°))$ and $z_2 = 3(\cos(45°) + i\sin(45°))$

d. $z_1 z_2 = 6\left(\cos\left(195^\circ\right) + i\,\sin\left(195^\circ\right)\right)$
e. 195°

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form follows the same patterns set for products.

Theorem: Quotients of Complex Numbers Written in Polar Form

If $z_1 = r_1\left(\cos\left(\theta_1\right) + i\,\sin\left(\theta_1\right)\right)$ and $z_2 = r_2\left(\cos\left(\theta_2\right) + i\,\sin\left(\theta_2\right)\right)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\,\sin\left(\theta_1 - \theta_2\right)\right] = \frac{r_1}{r_2}\,\text{cis}\left(\theta_1 - \theta_2\right),$$

where $z_2 \neq 0$.

The proof of this theorem is left as a homework exercise.

Example 12.4.5

Find the quotient of $z_1 = 2\left(\cos(213^\circ) + i\,\sin(213^\circ)\right)$ and $z_2 = 4\left(\cos(33^\circ) + i\,\sin(33^\circ)\right)$.

Solution

Using the formula, we have

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{2}{4}\left[\cos(213^\circ - 33^\circ) + i\,\sin(213^\circ - 33^\circ)\right] \\
&= \frac{1}{2}\left[\cos(180^\circ) + i\,\sin(180^\circ)\right] \\
&= \frac{1}{2}[-1 + 0i] \\
&= -\frac{1}{2} + 0i \\
&= -\frac{1}{2}
\end{aligned}$$

Checkpoint 12.4.5

Find the product and the quotient of $z_1 = 2\sqrt{3}\left(\cos(150^\circ) + i\,\sin(150^\circ)\right)$ and $z_2 = 2\left(\cos(30^\circ) + i\,\sin(30^\circ)\right)$.

Answer

$z_1 z_2 = -4\sqrt{3}$; $\frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is *greatly* simplified when they are written in polar form. For example, let $z = 1 + i$. This is a fairly innocuous complex number; however, let's try to compute $z^5 = (1 + i)^5$.

$$\begin{aligned} z^5 &= (1+i)^5 \\ &= (1+i)^2 (1+i)^2 (1+i) \\ &= (1+2i+i^2)(1+2i+i^2)(1+i) \\ &= (1+2i-1)(1+2i-1)(1+i) \\ &= (2i)(2i)(1+i) \\ &= 4i^2(1+i) \\ &= -4(1+i) \\ &= -4-4i \end{aligned}$$

This is only the fifth power of a somewhat simple complex number. Imagine working with the tenth power of something like $z = -\sqrt{3} + i$. If our complex number is written in polar form, however, something fantastic happens.

(Note: The following derivation of *why* De Moivre's Theorem works requires a proof method called Mathematical Induction. If you have already taken College Algebra, then you have worked with, and performed, proofs using Mathematical Induction;[2] however, since College Algebra is not a prerequisite for this course, you might not have seen it. As such, I provide the proof only for those who are interested.)

Theorem: De Moivre's Theorem

If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number and $n \in \mathbb{N}$, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n \operatorname{cis}(n\theta).$$

Proof

Let $z = r(\cos(\theta) + i\sin(\theta))$ and $n \in \mathbb{N}$.

CLAIM: $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$.

For $n = 1$,

$$\begin{aligned} z^1 &= (r(\cos(\theta) + i\sin(\theta)))^1 \\ &= r^1(\cos(\theta) + i\sin(\theta))^1 && \text{(Laws of Exponents)} \\ &= r^1(\cos(1\cdot\theta) + i\sin(1\cdot\theta)) && \text{(Multiplicative Identity)} \end{aligned}$$

Thus the claim is true for $n = 1$.

Induction Hypothesis: Suppose the claim is true for $n = k$. That is, suppose $z^k = r^k[\cos(k\theta) + i\sin(k\theta)]$.

Assuming this Induction Hypothesis,

$$\begin{aligned} z^{k+1} &= z^k \cdot z^1 && \text{(Laws of Exponents)} \\ &= r^k[\cos(k\theta) + i\sin(k\theta)] \cdot z^1 && \text{(Induction Hypothesis)} \\ &= r^k[\cos(k\theta) + i\sin(k\theta)] \cdot r[\cos(\theta) + i\sin(\theta)] && \text{(substituting } z = r(\cos(\theta) + i\sin(\theta))) \\ &= r^{k+1}[\cos(k\theta) + i\sin(k\theta)][\cos(\theta) + i\sin(\theta)] && \text{(Commutative Property of Multiplication)} \\ &= r^{k+1}[\cos(k\theta)\cos(\theta) + i\cos(k\theta)\sin(\theta) + i\sin(k\theta)\cos(\theta) + i^2\sin(k\theta)\sin(\theta)] && \text{(distributing)} \\ &= r^{k+1}[\cos(k\theta)\cos(\theta) + i\cos(k\theta)\sin(\theta) + i\sin(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)] && (i^2 = -1) \\ &= r^{k+1}[\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) + i\cos(k\theta)\sin(\theta) + i\sin(k\theta)\cos(\theta)] && \text{(Commutative Property of Addition)} \\ &= r^{k+1}[(\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) + i(\cos(k\theta)\sin(\theta) + \sin(k\theta)\cos(\theta))] && \text{(grouping and factoring)} \\ &= r^{k+1}[\cos((k+1)\theta) + i\sin((k+1)\theta)] && \text{(Sum of Angles Identities)} \end{aligned}$$

Thus the claim is true for $n = k+1$.

Since the claim holds for $n = 1$, and subsequently for $n = k+1$ when it is assumed it works for $n = k$, the claim must be true for all $n \in \mathbb{N}$.

De Moivre's Theorem states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern Mathematics.

Example 12.4.6

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(1)^2+(1)^2} \\ &= \sqrt{2} \end{aligned}$$

Then we find θ. Using the formula $\tan\theta = \dfrac{y}{x}$ gives

$$\begin{aligned} \tan(\theta) &= \frac{1}{1} \\ \Longrightarrow \quad \tan(\theta) &= 1 \\ \Longrightarrow \quad \hat{\theta} &= \frac{\pi}{4} \end{aligned}$$

Since $(1+i)$ is in the first quadrant, this means $\theta = \frac{\pi}{4}$.

Now we use De Moivre's Theorem to evaluate the expression.

$$\begin{aligned} (1+i)^5 &= (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right)+i\sin\left(5\cdot\frac{\pi}{4}\right)\right] \\ &= 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right] \\ &= 4\sqrt{2}\left[-\frac{\sqrt{2}}{2}+i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ &= -4-4i \end{aligned}$$

Checkpoint 12.4.6

Compute $\left(-\sqrt{3}+i\right)^{10}$.

Answer

$$2^{10}\operatorname{cis}\left(\frac{25\pi}{3}\right) = 1024\operatorname{cis}\left(\frac{\pi}{3}\right) = 1024\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 512+512i\sqrt{3}$$

Finding Roots of Complex Numbers in Polar Form

DeMoivre's Theorem also works for fractional values of n, so we can compute roots of complex numbers. For example, by applying the theorem with $n=\frac{1}{2}$, we see that one of the square roots of $z = r(\cos(\theta)+i\sin(\theta))$ is

$$z^{1/2} = r^{1/2}\left(\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right).$$

Now, every number, whether real or complex, has two square roots. To find the other root, remember that we can add a multiple of 2π to the argument of z, that is, we can also write the polar form of z as\[z=r(\cos (\theta+2 \pi)+i \, \sin (\theta+2 \pi))\].

The second square root of z is thus

$$r^{1/2}\left(\cos\left(\frac{\theta}{2}+\pi\right)+i\sin\left(\frac{\theta}{2}+\pi\right)\right).$$

As an example, consider $w = -8 + 8i\sqrt{3}$, whose polar form is $16\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$, or by adding 2π to the argument, $16\left(\cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right)\right)$. The two square roots of w are

$$\begin{aligned} z_1 &= 16^{1/2}\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) & z_2 &= 16^{1/2}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) \\ &= 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) & &= 4\left(\frac{-1}{2} + i\frac{-\sqrt{3}}{2}\right) \\ &= 2 + 2i\sqrt{3} & &= -2 - 2i\sqrt{3} \end{aligned}$$

You can verify that both of these numbers are square roots of $w = -8 + 8i\sqrt{3}$, for instance,

$$\begin{aligned} (2 + 2i\sqrt{3})^2 &= 4 + 2(2)(2i\sqrt{3}) + 12i^2 \\ &= -8 + 8i\sqrt{3} \end{aligned}$$

The graphs of w and its two square roots are shown below.

Figure 12.4.3

It is not hard to show that every number has three complex cube roots, four complex fourth roots, and so on.

In general, to find the nth root of a complex number in polar form, we use the nth Root Theorem, or we can use De Moivre's Theorem as we did in our work above. The following theorem is presented without proof.

Theorem: n^{th} Root Theorem

If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number, the nth root of z is

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right].$$

where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Example 12.4.7

Find the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution

We have

$$\begin{aligned} z^{\frac{1}{3}} &= 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right)\right] \\ &= 2\left[\cos\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)\right] \end{aligned}$$

There will be three roots - one for each of $k = 0$, 1, and 2.

When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)\right).$$

When $k = 1$, we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right)\right] \qquad \left(\text{adding } \frac{2(1)\pi}{3} \text{ to each angle}\right) \\ &= 2\left(\cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)\right) \end{aligned}$$

When $k = 2$, we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right)\right] \qquad \left(\text{adding } \frac{2(2)\pi}{3} \text{ to each angle}\right) \\ &= 2\left(\cos\left(\frac{14\pi}{9}\right) + i\sin\left(\frac{14\pi}{9}\right)\right) \end{aligned}$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\begin{aligned} \frac{\frac{2\pi}{3}}{3} + \frac{2(1)\pi}{3} &= \frac{2\pi}{3}\left(\frac{1}{3}\right) + \frac{2(1)\pi}{3}\left(\frac{3}{3}\right) \\ &= \frac{2\pi}{9} + \frac{6\pi}{9} \\ &= \frac{8\pi}{9} \end{aligned}$$

? Exercise 12.4.7

Find the four fourth roots of $16(\cos(120°) + i\,\sin(120°))$.

Answer

$z_0 = 2(\cos(30°) + i\,\sin(30°))$

$z_1 = 2(\cos(120°) + i\,\sin(120°))$

$z_2 = 2(\cos(210°) + i\,\sin(210°))$

$z_3 = 2(\cos(300°) + i\,\sin(300°))$

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12.4.1: RESOURCES AND KEY CONCEPTS

KEY CONCEPTS

DEFINITIONS

- **Polar Form of a Complex Number:** A way of writing a complex number $z = x + yi$ using its modulus r and argument θ. The form is $z = r(\cos(\theta) + i\sin(\theta))$, which is often abbreviated as $z = r\text{cis}(\theta)$.
- **Modulus:** The distance of the point (x, y) from the origin in the complex plane, given by $r = \sqrt{x^2 + y^2}$. It is the same as the radius r in polar form.
- **Argument:** The angle θ in standard position whose terminal side passes through the point (x, y) in the complex plane.

THEOREMS

- **Cartesian Coordinates in Terms of Cosine and Sine:** If a point P is located at a distance r from the origin in the direction specified by an angle θ in standard position, then the coordinates of P are $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- **Modulus in Polar Form:** The modulus of the complex number z, written in polar form $z = r(\cos(\theta) + i\sin(\theta))$, is r.
- **Products of Complex Numbers Written in Polar Form:** If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then their product is $z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$.
- **Quotients of Complex Numbers Written in Polar Form:** If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then their quotient is $\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$.
- **De Moivre's Theorem:** If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number and n is a positive integer, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$, or $z^n = r^n\text{cis}(n\theta)$.
- **nth Root Theorem:** If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number, then its n distinct nth roots are given by $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$, where $k = 0, 1, 2, \ldots, n-1$.

COMMON MISTAKES

- **Incorrectly Determining the Argument:** When converting a complex number from rectangular form to polar form, the argument θ is not simply $\arctan(\frac{y}{x})$. You must consider the quadrant of the point (x, y) to find the correct angle.
- **Evaluating Trigonometric Functions Prematurely:** When converting a complex number *to* polar form, do not evaluate the trigonometric functions. The goal is to leave the number in the form $r(\cos\theta + i\sin\theta)$. Evaluating them would convert the number back to rectangular form.
- **Forgetting to Find All Roots:** A complex number has n distinct nth roots. A common mistake is to only find the principal root (for $k = 0$) and forget to find the other roots by adding multiples of $\frac{2\pi}{n}$ to the angle.

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12.4.2: HOMEWORK

READING QUESTIONS

1. What is the polar form of a complex number $z = x + yi$?
2. In the polar form $z = r(\cos\theta + i\sin\theta)$, what is another name for r? What is another name for θ?
3. What does the notation $r\text{cis}(\theta)$ represent?
4. When multiplying two complex numbers in polar form, what do you do with their moduli and what do you do with their arguments?
5. When dividing two complex numbers in polar form, what do you do with their moduli and what do you do with their arguments?
6. State De Moivre's Theorem for finding the power of a complex number.
7. How many distinct cube roots does a complex number have?
8. According to the nth Root Theorem, how do you find all the nth roots of a complex number?
9. When converting $z = -4 + 4i$ to polar form, how do you determine that the argument θ is $\frac{3\pi}{4}$ and not its reference angle $\frac{\pi}{4}$?

HOMEWORK

BASIC SKILLS

For the following exercises, simplify and plot each complex number as a point on the complex plane.

1. $1, i, i^2, i^3$ and i^4
2. $-1, -i, -i^2, -i^3$ and $-i^4$
3. $1 + 2i$ and $i(1 + 2i)$
4. $3 - 4i$ and $i(3 - 4i)$

For the following exercises, write the complex numbers in standard form. Give exact values for your answers.

5. $6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
6. $4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$
7. $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
8. $\frac{3}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

For the following exercises, write the complex numbers in standard form. Round your answers to hundredths.

9. $5(\cos 5.2 + i\sin 5.2)$
10. $3(\cos 3.5 + i\sin 3.5)$
11. $12\,(\cos 115^\circ + i\sin 115^\circ)$
12. $20\,(\cos 250^\circ + i\sin 250^\circ)$

For the following exercises, write the complex numbers in polar form. Give exact values for your answers.

13. $3i$ and $-3i$
14. $2 + 2i$ and $2 - 2i$
15. $-3 - \sqrt{3}i$ and $3 - \sqrt{3}i$
16. $2\sqrt{3} + 2i$ and $-2\sqrt{3} + 2i$

For the following exercises, write the complex numbers in polar form. Round your answers to hundredths.

17. $-4 + 2i$ and $4 - 2i$
18. $-3 - 8i$ and $3 + 8i$
19. $9 - 5i$ and $9 + 5i$
20. $2 + 6i$ and $2 - 6i$
21. $3 + 4i, 3 - 4i, -3 + 4i$, and $-3 - 4i$
22. $1 + 3i, 1 - 3i, -1 + 3i$, and $-1 - 3i$
23. What can you conclude about the polar forms of z and $\bar{z}$?
24. What can you conclude about the polar forms of z and $-z$?

For the following exercises, find the product $z_1 z_2$ and the quotient $\frac{z_1}{z_2}$.

25. $z_1 = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ and $z_2 = \frac{1}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
26. $z_1 = 6\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$ and $z_2 = \frac{3}{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$
27. $z_1 = 3\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$ and $z_2 = 2\left(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}\right)$
28. $z_1 = 4\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$ and $z_2 = 6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

For the following exercises, convert the complex number to polar form, then find the product $z_1 z_2$ and the quotient $\frac{z_1}{z_2}$.

29. $z_1 = 2i, z_2 = 4i$
30. $z_1 = -2i, z_2 = 3i$
31. $z_1 = 2\sqrt{3} - 2i, \quad z_2 = -1 + i$
32. $z_1 = \sqrt{3} + i, \quad z_2 = -1 + \sqrt{3}$

For the following exercises, find the power.

33. $(2 + 2i)^5$
34. $(\sqrt{2} - \sqrt{2}i)^6$
35. $(-1 + \sqrt{3}i)^8$
36. $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{12}$
37. $(\sqrt{3} - i)^{10}$
38. $(1 - i)^{20}$

For the following exercises, use De Moivre's theorem to find the reciprocal.

39. $2 - 2i$
40. $3 + \sqrt{3}i$
41. $-\sqrt{2} + \sqrt{6}i$
42. $-1 - i$

For the following exercises,
(a) Find the roots and plot them in the complex plane.
(b) Write the roots in standard form.

43. The square roots of $9i$.

44. The fourth roots of -81.

45. The fifth roots of 32.

46. The cube roots of i.

47. The cube roots of $4\sqrt{3}+4i$.

48. The square roots of $-2+2\sqrt{3}i$.

49. Show that any complex number of the form $z=\cos\theta+i\sin\theta$ lies on the unit circle in the complex plane.

50. Show that if $|z|=1$, then $\frac{1}{z}=\bar{z}$.

51.

a. Find three distinct cube roots of 1.

b. Find four distinct fourth roots of 1.

c. Find five distinct fifth roots of 1.

d. Find six distinct sixth roots of 1.

52.

a. Find the sum of the three distinct cube roots of 1. (Hint: Plot the roots.)

b. Find the sum of the four distinct fourth roots of 1.

c. Find the sum of the five distinct fifth roots of 1.

d. Find the sum of the six distinct sixth roots of 1.

53. If n is a positive integer, define

$$\omega_k=\cos\frac{2\pi k}{n}+i\sin\frac{2\pi k}{n}$$

for $k=0,1,2,\cdots,n-1$. Show that $(\omega_k)^n=1$. (We call ω_k an nth **root of unity**.)

54. Let $\omega=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, where n is a positive integer. Show that the n distinct n^{th} roots of unity are $\omega,\omega^2,\omega^3,\cdots,\omega^{n-1}$.

For the following exercises, solve the equation.

55. $z^4+4z^2+8=0$

56. $z^6+4z^3+8=0$

57. $z^6-8=0$

58. $z^4-9i=0$

59. $z^4+2z^2+4=0$

60. $z^4-2z^2+4=0$

61.

a. Let $z=\cos\theta+i\sin\theta$. Compute z^2 by expanding the product.

b. Use DeMoivre's theorem to compute z^2.

c. Compare your answers to (a) and (b) to write identities for $\sin 2\theta$ and $\cos 2\theta$.

62.

a. Let $z=\cos\theta+i\sin\theta$. Compute z^3 by expanding the product.

b. Use DeMoivre's theorem to compute z^3.

c. Compare your answers to (a) and (b) to write identities for $\sin 3\theta$ and $\cos 3\theta$.

For the following exercises, show that multiplication by i results in a rotation of 90°.

63. Suppose that $z=a+bi$ and that the real numbers a and b are both nonzero.

a. What is the slope of the segment in the complex plane joining the origin to z?

b. What is the slope of the segment in the complex plane joining the origin to zi?

c. What is the product of the slopes of the two segments from parts (a) and (b)? What can you conclude about the angle between the two segments?

64. Suppose that $z=a+bi$ and that a and b are both real numbers.

a. If $a\neq 0$ and $b=0$, then what is the slope of the segment in the complex plane joining the origin to z? What is the slope of the segment joining the origin to zi?

b. If $a=0$ and $b\neq 0$, then what is the slope of the segment in the complex plane joining the origin to z? What is the slope of the segment joining the origin to zi?

c. What can you conclude about the angle between the two segments from parts (a) and (b)?

PROOFS

65. Prove the product rule by following the steps.

a. Suppose $z_1=a+bi$ and $z_2=c+di$. Compute z_1z_2.

b. Now suppose that $z_1=r(\cos\alpha+i\sin\alpha)$ and $z_2=R(\cos\beta+i\sin\beta)$. Write a,b,c and d in terms of r,R,α and β.

c. Substitute your expressions for a,b,c and d into your formula for z_1z_2.

d. Use the laws of sines and cosines to simplify your answer to part (c).

66. Let $z_1=r(\cos\alpha+i\sin\alpha)$ and $z_2=R(\cos\beta+i\sin\beta)$. Prove the quotient rule as follows: Set $w=\frac{r}{R}(\cos(\alpha-\beta)+i\sin(\alpha-\beta))$, and show that $z_1=wz_2$.

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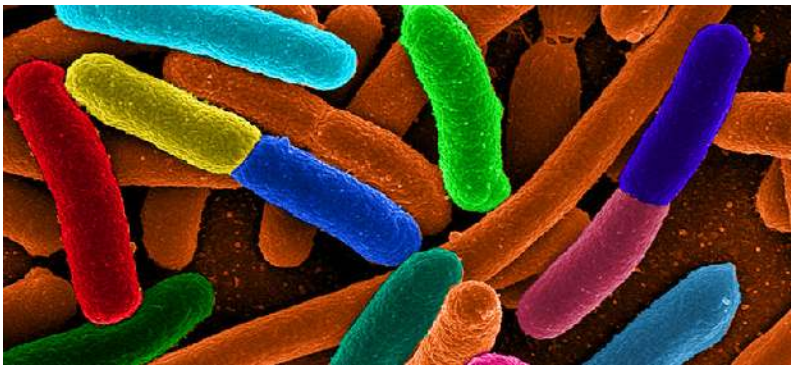
CHAPTER OVERVIEW

Chapter 13: Exponential and Logarithmic Functions

Electron micrograph of E.Coli bacteria (credit: "Mattosaurus," Wikimedia Commons)



Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. The table below shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

13.1: Exponential Functions

13.1.1: Resources and Key Concepts

13.1.2: Homework

13.2: Graphs of Exponential Functions

13.2.1: Resources and Key Concepts

13.2.2: Homework

13.3: Logarithmic Functions

13.3.1: Resources and Key Concepts

13.3.2: Homework

13.4: Graphs of Logarithmic Functions

13.4.1: Resources and Key Concepts

13.4.2: Homework

13.5: Laws of Logarithms
13.5.1: Resources and Key Concepts
13.5.2: Homework
13.6: Exponential and Logarithmic Equations
13.6.1: Resources and Key Concepts
13.6.2: Homework
13.7: Exponential and Logarithmic Models
13.7.1: Resources and Key Concepts
13.7.2: Homework

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13.1: Exponential Functions

Learning Objectives

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

Hawk A.I. Section-Specific Tutor

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When the original iteration of this textbook was written (2013), India was the second most populous country in the world with a population of about 1.25 billion people. At that time, the population was growing at a rate of about 1.2% each year. If that rate continued, the population of India would have exceeded China's population by the year 2031. Interestingly, the rate of growth for India's population actually sped up! In 2025, India's population was 1.42 billion while China's was 1.41 billion. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at **exponential functions**, which model this kind of rapid growth.

Identifying Exponential Functions

When exploring linear growth back in Algebra, we observed a constant rate of change - a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by 3 each time the input increases by 1. We call this the **Slope Addition Property** and it is unique to linear functions. If you never knew of this property from your old Algebra class, then it's best we codify it here.

Theorem: Slope Addition Property

For a linear function with slope m, every increase of 1 in input results in adding m to the function value. This is called the **slope addition property**.

The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products - no meat, poultry, fish, dairy, or eggs. If this rate continued, vegans would have made up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the

corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See Table $13.1.1$.

Table 13.1.1

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12
7	128	14
8	256	16
9	512	18
10	1024	20

From Table $13.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** means the original value from the range increases by the *same percentage* over equal increments in the domain.
- **Linear growth** means the original value from the range increases by the *same amount* over equal increments in the domain (again, this is called the Slope Addition Property).

Apparently, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. If we agree to call this multiplier the **base**, then this aspect that is unique to exponential functions is called the **Base Multiplier Property**.

Theorem: Base Multiplier Property

For an exponential function with base b, every increase of 1 in input results in multiplying the previous function value by b. This is called the **base multiplier property**.

So, how do we define exponential functions?

Definition: Exponential Function

For any real number x, an **exponential function** is a function with the form

$$f(x) = ab^x,$$

where a is a non-zero real number called the **initial value** and b is any positive real number, called the **base**, such that $b \neq 1$. This function is also referred to as the **exponential function base b**.

Let's look at the function $f(x) = 2^x$ from our example. We will create a table (Table $13.1.2$) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table 13.1.2

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$2^{-3}=\frac{1}{8}$	$2^{-2}=\frac{1}{4}$	$2^{-1}=\frac{1}{2}$	$2^0=1$	$2^1=2$	$2^2=4$	$2^3=8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in Figure 13.1.1, and then make a few observations.

Figure 13.1.1

From the graph, it appears that the following are some key characteristics of $f(x)=2^x$:

- the domain is $(-\infty,\infty)$,
- the range is $(0,\infty)$,
- as $x\to\infty$, $f(x)\to\infty$,
- as $x\to-\infty$, $f(x)\to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the x-axis because base two raised to any exponent never has the result of zero,
- $y=0$ appears to be the horizontal asymptote, and
- the y-intercept is 1.

In fact, these characteristics are not unique to the exponential function base 2.

Theorem: Properties of the Exponential Function

Let $f(x)=ab^x$, with $a>0$:

- The domain of f is all real numbers.
- The range of f is all positive real numbers.
- The y-intercept is $(0,a)$.
- The graph of f includes the point $(1,ab)$.
- The horizontal asymptote is $y=0$.
- If $b>1$,
  - f is increasing,
  - as $x\to-\infty$, $f(x)\to 0$, and
  - as $x\to\infty$, $f(x)\to\infty$.
- If $0<b<1$,
  - $f(x)$ is decreasing,
  - as $x\to-\infty$, $f(x)\to\infty$, and
  - as $x\to\infty$, $f(x)\to 0$.

We state this theorem assuming $a > 0$ for sake of simplicity. If $a < 0$, then we can visualize the graphical change in f as a reflection about the x-axis. Moreover, we require $a \neq 0$. Otherwise, $f(x) = 0 \cdot b^x = 0$, and this is the horizontal line along the x-axis. Hence, it is not exponentially increasing nor decreasing.

The domain of an exponential function is all real numbers because we can evaluate b^{power} for any power we desire. The range is all positive real numbers because, as long as $b > 0$, $b^{\text{power}} > 0$.

If $f(x) = ab^x$, where $b \neq 0$, then $f(0) = a \cdot b^0 = a \cdot 1 = a$ (remember, we cannot determine the value of 0^0, so we cannot let $b = 0$). Furthermore, $f(1) = a \cdot b^1 = ab$. Thus, the points $(0, 1)$ and $(1, ab)$ are on the graph of the exponential function.

Example 13.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Checkpoint 13.1.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values?

Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{1/2} = \sqrt{-9}$, which is not a real number. This should convince us that allowing b to be negative would cause too much of a headache. Therefore, it is convention to restrict the base of exponential functions to positive values only.

Why do we limit the base to positive values other than 1?

Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x. Since this is a constant, it is neither exponentially increasing nor exponentially decreasing. As such, we further restrict the base of exponential functions to not include 1.

To evaluate an exponential function with the form $f(x) = b^x$, we simply substitute x with the given value, and calculate the resulting power. For example, let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} & f(x) = 2^x \\ \Longrightarrow \quad & f(3) = 2^3 && \text{(substituting)} \\ & = 8 && \text{(simplifying)} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the Order of Operations. For example, let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} & f(x) = 30(2)^x & \\ \Longrightarrow \quad & f(3) = 30(2)^3 & \text{(substituting)} \\ & = 30(8) & \text{(exponents before multiplication)} \\ & = 240 & \text{(multiplication)} \end{aligned}$$

Note that if the Order of Operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216,000.$$

Example 13.1.2: Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution

Follow the Order of Operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} & f(x) = 5(3)^{x+1} & \\ \Longrightarrow \quad & f(2) = 5(3)^{2+1} & \text{(substituting)} \\ & = 5(3)^3 & \text{(simplifying)} \\ & = 5(27) & \text{(simplifying)} \\ & = 135 & \text{(simplifying)} \end{aligned}$$

Checkpoint 13.1.2

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Definition: Exponential Growth and Decay

A function that models **exponential growth** or **exponential decay** grows or decays by a rate proportional to the amount present. For any real number t and any positive real numbers P_0 and b such that $b \neq 1$, an exponential growth or decay function has the form

$$P(t) = P_0\, b^t,$$

where

- P_0 is the **initial** or **starting value** of the function, and
- b is the **growth** or **decay factor**
  - if $b > 1$, it is a growth factor or **growth multiplier** per unit t
  - if $0 < b < 1$, it is a decay factor or **decay multiplier** per unit t.

Note that this definition mirrors the definition of an exponential function, but with f replaced by P, a replaced by P_0, and x replaced by t. The traditional notation introduced with the original definition (using $f(x)$ and a) only differs from this new notation in letters. Most applications involving exponential functions will use time (t) as the independent variable and are modeling

populations (thus, the choice of function name, $P(t)$). Moreover, P_0 (read as, "P naught") stands for the initial population size (or the time-zero population size).

To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table **13.1.3**.

Table 13.1.3

Year, x	Stores, Company A	Stores, Company B
0	$100 + 50(0) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50(1) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50(2) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50(3) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A(x) = 100 + 50x$	$B(x) = 100(1 + 0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in Figure **13.1.2**. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

Figure 13.1.2

The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the *realistic* domain for both functions is $[0, \infty)$, and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the **growth rate**, and $1 + 0.5 = 1.5$ represents the

growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$, where 100 is the initial value, 1.5 is the base, and x is the exponent.

Example 13.1.3: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, and using this model, what will the population of India be in 2031?

Solution

To estimate the population in 2031, we evaluate the model for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549.$$

There will be about 1.549 billion people in India in the year 2031.

Checkpoint 13.1.3

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example $13.1.3$?

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

How to... find the equation of an exponential function given two points

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 13.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an exponential function $N(t)$ representing the population (N) of deer over time t.

Solution

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: $(0, 80)$ and $(6, 180)$. Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b:

$$
\begin{aligned}
N(t) &= 80b^t \\
\implies 180 &= 80b^6 && \text{(substituting)} \\
\implies \frac{9}{4} &= b^6 && \text{(dividing both sides by 80)} \\
\implies \left(\frac{9}{4}\right)^{1/6} &= b && \left(\text{raising both sides to the } \tfrac{1}{6} \text{ power}\right) \\
\implies b &\approx 1.1447 && \text{(rounding to four decimal places)}
\end{aligned}
$$

NOTES: When we took the sixth root of both sides, we did not bother writing $b = \pm\left(\frac{9}{4}\right)^{1/6}$. This is because the base of an exponential function is never negative. Moreover, *do not round any intermediate calculations. Doing so leads to error propagation.*

The exponential model for the population of deer is $N(t) \approx 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure 13.1.3 passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$, and the range for the function is $[80, \infty)$.

Figure 13.1.3

Graph showing the population of deer over time, $N(t) \approx= 80(1.1447)^t$, t years after 2006

? Checkpoint 13.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

✓ Example 13.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = a\,b^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = a\,b^{-2}$
- Substituting $(2, 1)$ gives $1 = a\,b^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} & 6 = a\,b^{-2} \\ \Rightarrow \quad & 6b^2 = a \qquad \text{(multiplying both sides by } b^2\text{)} \end{aligned}$$

Substitute a in the second equation, and solve for b:

$$\begin{aligned} & 1 = a\,b^2 \\ \Rightarrow \quad & 1 = 6b^2b^2 \qquad \text{(substituting } a = 6b^2\text{)} \\ \Rightarrow \quad & 1 = 6b^4 \qquad \text{(Laws of Exponents)} \\ \Rightarrow \quad & \frac{1}{6} = b^4 \qquad \text{(dividing both sides by 6)} \\ \Rightarrow \quad & \left(\frac{1}{6}\right)^{1/4} = b \qquad \text{(raise both sides to the } \tfrac{1}{4} \text{ power)} \end{aligned}$$

Now that we have b, we go back to the first equation to find a:

$$a = 6b^2 = 6\left(\left(\frac{1}{6}\right)^{1/4}\right)^2 = 6\left(\frac{1}{6}\right)^{2/4} = 6\left(\frac{1}{6}\right)^{1/2} = \frac{6}{6^{1/2}} = 6^{1-1/2} = 6^{1/2} = \sqrt{6}.$$

Thus, the equation is

$$f(x) = \sqrt{6}\left(\left(\frac{1}{6}\right)^{1/4}\right)^x = \sqrt{6}\left(\frac{1}{6}\right)^{x/4}.$$

We can graph our model to check our work. Notice that the graph in Figure **13.1.4** passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

Figure 13.1.4

The graph of $f(x) = \sqrt{6}\left(\frac{1}{6}\right)^{x/4}$ models exponential decay.

Checkpoint 13.1.5

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Q&A

Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x, which in many real world cases involves time.

Example 13.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 13.1.5.

Figure 13.1.5

Solution

We can choose the y-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$
\begin{aligned}
& y = ab^x && \text{(writing the general form)} \\
\Longrightarrow\ & y = 3b^x && \text{(substituting } a = 3\text{)} \\
\Longrightarrow\ & 12 = 3b^2 && \text{(substituting } x = 2 \text{ and } y = 12\text{)} \\
\Longrightarrow\ & 4 = b^2 && \text{(dividing both sides by 3)} \\
\Longrightarrow\ & \pm 2 = b && \text{(Extraction of Roots)} \\
\Longrightarrow\ & 2 = b && \text{(exponential bases are positive)}
\end{aligned}
$$

Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Checkpoint 13.1.6

Find an equation for the exponential function graphed in Figure 13.1.6.

Figure 13.1.6

Compound Interest

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, earn interest that is compounded.

Definition: Compounding

Compounding refers to the process of earning interest not only on the initial investment (**principal**) but also on the accumulated interest itself.

To convey to the customer how much their money is earning annually (per year), investors cite the **annual percentage rate** (**APR**).

Definition: Annual Percentage Rate

The **annual percentage rate** (**APR**) of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account.

The term nominal is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the **effective interest rate** ends up being *greater* than the nominal rate! This is a powerful tool for investing.

Definition: Effective Rate

The **effective interest rate** (**EIR**), **effective annual interest rate**, **annual equivalent rate** (**AER**) or simply **effective rate** is the percentage of interest on a loan or financial product if compound interest accumulates in periods different than a year. Another, very popular, name for the effective rate is the **annual percentage yield** (**APY**).

Typical compounding periods (other than once per year) are:

- semiannual (twice per year, at six-month intervals)
- quarterly (once per three months)
- monthly
- daily

Other compounding periods exist (e.g., weekly), but they are not as common.

Developing a formula to predict the effects of compound interest on an investment takes a little bit of patience and insight. Let's assume we have $1,000 and invest it for one year at a bank that pays 10% interest.

If the bank is only paying **simple interest** (which means it is compounding annually), then at the end of the year we would have

$$1000 + 10\% \text{ of } 1000 = 1000 + 0.10 \times 1000 = 1000 + 100 = 1100.$$

Our money has grown from $1,000 to $1,100, earning us $100 in interest!

What would happen if the bank was paying 10% APR, but compounding semiannually? Since they are compounding twice per year, at the end of six months they pay 5% on what we invested. Then, at the end of the second six months, they pay another 5% *of what is now in our account*! That is, the second compounding period earns interest not only on what we invested, but also on the interest we received during the first six months.

$$\begin{aligned} \textbf{End of First Six Months:} \quad & 1000 + 5\% \text{ of } 1000 = 1000 + 0.05 \times 1000 = 1000 + 50 = 1050 \\ \textbf{End of Second Six Months:} \quad & 1050 + 5\% \text{ of } 1050 = 1050 + 0.05 \times 1050 \\ & = 1050 + 52.50 = 1102.50 \end{aligned}$$

We end the year with $1,102.50. We earned more interest than before!

Let's show only more case before stating the observation. In this case, let's assume that the bank compounds interest quarterly. Moreover, instead of simplifying our computations, we will use some Algebra to showcase a pattern.

End of First Quarter:	1000 $+\frac{10\%}{4}$ of 1000	$=$	1000 $+\frac{0.10}{4}\times 1000$	
		$=$	$1000\left(1+\frac{0.10}{4}\right)$	(factoring out 1000)
		$=$	P_1	(labeling this amount as P_1)
End of Second Quarter:	P_1 $+\frac{10\%}{4}$ of P_1	$=$	$P_1+\frac{0.10}{4}\times P_1$	(earning interest on the previous quarter's value, P_1)
		$=$	$P_1\left(1+\frac{0.10}{4}\right)$	(factoring out P_1)
		$=$	$1000\left(1+\frac{0.10}{4}\right)\left(1+\frac{0.10}{4}\right)$	(substituting)
		$=$	$1000\left(1+\frac{0.10}{4}\right)^2$	(simplifying)
		$=$	P_2	(labeling this amount as P_2)
End of Third Quarter:	P_2 $+\frac{10\%}{4}$ of P_2	$=$	$P_2+\frac{0.10}{4}\times P_2$	(earning interest on the previous quarter's value, P_2)
		$=$	$P_2\left(1+\frac{0.10}{4}\right)$	(factoring out P_2)
		$=$	$1000\left(1+\frac{0.10}{4}\right)^2\left(1+\frac{0.10}{4}\right)$	(substituting)
		$=$	$1000\left(1+\frac{0.10}{4}\right)^3$	(simplifying)
		$=$	P_3	(labeling this amount as P_3)
End of Fourth Quarter:	P_3 $+\frac{10\%}{4}$ of P_3	$=$	$P_3+\frac{0.10}{4}\times P_3$	(earning interest on the previous quarter's value, P_3)
		$=$	$P_3\left(1+\frac{0.10}{4}\right)$	(factoring out P_3)
		$=$	$1000\left(1+\frac{0.10}{4}\right)^3\left(1+\frac{0.10}{4}\right)$	(substituting)
		$=$	$1000\left(1+\frac{0.10}{4}\right)^4$	(simplifying)

We can grab a calculator to compute this final value to be approximately $1,103.81. This is even more than our previous amount!

Now that we have taken our time, hopefully we can see a pattern. If we replace the interest rate (10%) with r, the number of compounding periods (4) with n, and the principal invested ($1000) with P_0, we have the amount of money at the end of a single year as

$$A = P_0\left(1+\frac{r}{n}\right)^n.$$

Allowing our money to sit over t years gives us the **compound interest formula**.

Theorem: Compound Interest Formula

The amount an investment of P_0 dollars is worth after t years earning an annual percentage rate of $r \times 100\%$ compounding n times per year is given by

$$A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}.$$

P_0 is called the **principal**, or, more generally, the **present value**.

For example, observe Table 13.1.4, which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 13.1.4

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Example 13.1.7: Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} \\ \Rightarrow \quad A(10) &= 3000\left(1 + \frac{0.03}{4}\right)^{4\cdot 10} && \text{(substituting known values)} \\ &\approx \$4,045.05 && \text{(rounding to two decimal places)} \end{aligned}$$

The account will be worth about $4,045.05 in 10 years.

Checkpoint 13.1.7

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Example 13.1.8: Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth \$40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{n}\right)^{nt} \\ \implies 40,000 &= P\left(1+\frac{0.06}{2}\right)^{2\cdot 18} && \text{(substituting known values)} \\ \implies 40,000 &= P(1.03)^{36} && \text{(simplifying)} \\ \implies \frac{40,000}{1.03^{36}} &= P && \text{(isolating } P\text{)} \\ \implies P &\approx \$13,801 && \text{(rounding to the nearest dollar)} \end{aligned}$$

Lily will need to invest \$13,801 to have \$40,000 in 18 years.

? Checkpoint 13.1.8

Refer to Example 13.1.8. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Evaluating Functions with Base e

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 13.1.5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of \$1 invested at 100% interest for 1 year, compounded at various frequencies, listed in Table 13.1.5.

Table 13.1.5

Frequency	$A(n) = \left(1+\frac{1}{n}\right)^n$	Value
Annually	$\left(1+\frac{1}{1}\right)^1$	\$2
Semiannually	$\left(1+\frac{1}{2}\right)^2$	\$2.25
Quarterly	$\left(1+\frac{1}{4}\right)^4$	\$2.441406
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	\$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	\$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	\$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	\$2.718279
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	\$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in Mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating.

Definition: e

The letter e represents the irrational number

$$\left(1+\frac{1}{n}\right)^n, \text{ as } n \text{ increases without bound.}$$

It is approximately **2.718281828459045**. This is also known as **Euler's number**, **Euler's constant**, and the **natural exponential base**.

The letter e is used as a base for many real-world exponential models. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example 13.1.9: Using a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution

On a calculator, press the button labeled [e^x]. The window shows $e^($. Type **3.14** and then close parenthesis, **)**. Press [ENTER]. Rounding to **5** decimal places, $e^{3.14} \approx 23.10387$.

Caution: Many scientific calculators have an **Exp** button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Checkpoint 13.1.9

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called **continuous growth or decay models**. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Definition: Continuous Growth/Decay

For all real numbers t, and all positive numbers a and r, **continuous exponential growth** (or, simply **continuous growth**) or **continuous exponential decay** (**continuous decay**) is represented by the formula

$$P(t) = P_0 e^{rt},$$

where r is called the **continuous growth** (or **decay**) **rate** per unit time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

Example 13.1.10: Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} \\ &= 1000e^{0.1} && \text{(substituting known values)} \\ &\approx 1105.17 && \text{(rounding to the nearest hundredth)} \end{aligned}$$

The account is worth $1,105.17 after one year.

? Checkpoint 13.1.10

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

✓ Example 13.1.11: Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} \\ &= 100e^{-0.173(3)} && \text{(substituting known values)} \\ &\approx 59.5115 && \text{(rounding to four decimal places)} \end{aligned}$$

So approximately 59.5115 mg of radon-222 will remain.

? Checkpoint 13.1.11

Using the data in Example 13.1.11, how much radon-222 will remain after one year?

13.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Arithmetic**
  - Percents: Understanding how to convert percentages to decimals (e.g., 9% to 0.09) is crucial for working with growth/decay rates in exponential functions and compound interest formulas.
- **Systems**
  - Solving 2x2 Systems of Linear Equations: This skill is needed to find equations of exponential functions going through two given points.

VIDEOS

- Exponential functions
  - Intermediate Algebra - Exponential Functions: The Concepts
  - Exponential Functions - A Brief Review of Exponential Functions
- Exponential growth and decay
  - Exponential Functions - Types of Exponential Growth and Decay
- Finding equation of exponential function
  - Exponential Functions - Finding Equations of Exponential Functions
- Compound interest formula
  - Exponential Functions - Applying the Compound Interest Formula
- Natural exponential base e
  - Exponential Functions - Investigating Continuous Growth
  - Exponential Functions - Calculus and the Natural Base
- Applications of exponential functions
  - Exponential Functions - Applications Involving Exponential Functions
  - Intermediate Algebra - Exponential Functions: Creating from Applications

KEY CONCEPTS

DEFINITIONS

- **Percent Change**: A change based on a percent of the original amount.
- **Exponential Growth**: An increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.
- **Exponential Decay**: A decrease based on a constant multiplicative rate of change over equal increments of time, that is, a percent decrease of the original amount over time.
- **Exponential Function**: A function with the form $f(x) = ab^x$, where a is a non-zero real number called the initial value and b is any positive real number, called the base, such that $b \neq 1$.
- **Initial Value (of an exponential function)**: In the form $f(x) = ab^x$, a is the initial value (the value of the function when $x = 0$).
- **Base (of an exponential function)**: In the form $f(x) = ab^x$, b is the base. It must be a positive real number not equal to 1.
- **Exponential Growth/Decay Function (alternative form)**: A function that models exponential growth or decay can be written as $P(t) = P_0 b^t$, where P_0 is the initial value, and b is the growth/decay factor. If $b > 1$, it's a growth factor. If $0 < b < 1$, it's a decay factor.
- **Compounding**: The process of earning interest not only on the initial investment (principal) but also on the accumulated interest itself.
- **Annual Percentage Rate (APR) / Nominal Rate**: The yearly interest rate earned by an investment account before considering the effect of compounding frequency.
- **Effective Rate / Annual Percentage Yield (APY)**: The percentage of interest on a loan or financial product if compound interest accumulates in periods different than a year; it reflects the actual annual rate earned after compounding.
- **The number e (Euler's number / natural exponential base)**: An irrational number approximately equal to 2.71828, which arises as the limit of $(1 + \frac{1}{n})^n$ as n increases without bound. It is used as the base for continuous growth/decay models.
- **Continuous Growth/Decay**: Exponential growth or decay represented by the formula $P(t) = P_0 e^{rt}$, where P_0 is the initial value, r is the continuous growth rate (if $r > 0$) or continuous decay rate (if $r < 0$) per unit time, and t is time.
- **Continuous Growth/Decay Rate**: The parameter r in the continuous growth/decay formula $P(t) = P_0 e^{rt}$.

THEOREMS

- **Slope Addition Property**: For a linear function with slope m, every increase of 1 in input results in adding m to the function value.
- **Base Multiplier Property**: For an exponential function with base b, every increase of 1 in input results in multiplying the previous function value by b.
- **Properties of the Exponential Function $f(x) = ab^x$, $a > 0$**:
  - Domain: All real numbers.
  - Range: All positive real numbers $(0, \infty)$.
  - y-intercept: $(0, a)$.
  - Includes point $(1, ab)$.
  - Horizontal asymptote: $y = 0$.

- If $b > 1$, f is increasing; as $x \to -\infty, f(x) \to 0$; as $x \to \infty, f(x) \to \infty$.
- If $0 < b < 1$, f is decreasing; as $x \to -\infty, f(x) \to \infty$; as $x \to \infty, f(x) \to 0$.

- **Compound Interest Formula**: The amount $A(t)$ an investment of P_0 dollars is worth after t years earning an annual percentage rate of $r \times 100\%$ compounding n times per year is given by $A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$.

COMMON MISTAKES

- **Confusing Exponential Growth with Linear Growth**: Mistaking a situation with a constant additive change for exponential growth, or vice-versa. Exponential growth involves a constant multiplicative factor (percent change).
- **Incorrect Base for Exponential Functions**: Using a base $b \leq 0$ or $b = 1$ for an exponential function $f(x) = ab^x$. The base must be positive and not equal to 1.
- **Order of Operations Errors in Evaluation**: When evaluating exponential functions like $f(x) = ab^{x+c} + d$, performing operations in the wrong order (e.g., multiplying by a before evaluating the exponent).
- **Incorrectly Using the Continuous Compounding Formula**: Applying $A = P_0e^{rt}$ when interest is compounded discretely (e.g., monthly, quarterly), or using the discrete compounding formula for continuous compounding.
- **Sign Errors with Decay Rates**: For continuous decay $P(t) = P_0e^{rt}$, forgetting that the rate r must be negative.

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13.1.2: HOMEWORK

READING QUESTIONS

1. Explain the difference between linear growth and exponential growth.
2. What is the general form of an exponential function? Define the "initial value" and the "base," including any restrictions on the base.
3. Describe the key characteristics of the graph of $f(x) = ab^x$ when $a > 0$ and $b > 1$ (domain, range, intercepts, asymptotes, increasing/decreasing behavior).
4. How does the graph of $f(x) = ab^x$ change if $0 < b < 1$ compared to when $b > 1$?
5. If you are given two points, how can you find the formula for an exponential function $f(x) = ab^x$ that passes through them? What if one of the points is the y-intercept $(0, a)$?
6. What is the compound interest formula? Define each variable in the formula.
7. What is the number e? Why is it significant in the context of exponential functions?
8. What is the formula for continuous exponential growth or decay? Define each variable.
9. If a population is decaying continuously at a rate of 5% per year, what would be the value of r in the formula $P(t) = P_0e^{rt}$?
10. What is the difference between the annual percentage rate (APR) and the annual percentage yield (APY)?

HOMEWORK

VERBAL

11. Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.
12. Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.
13. The Oxford Dictionary defines the word **nominal** as a value that is "stated or expressed but not necessarily corresponding exactly to the real value." Develop a reasonable argument for why the term **nominal rate** is used to describe the annual percentage rate of an investment account that compounds interest.

ALGEBRAIC

For the following exercises, identify whether the statement represents an exponential function. Explain.

14. The average annual population increase of a pack of wolves is 25.
15. A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.
16. The value of a coin collection has increased by 3.25% annually over the last 20 years.
17. For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.
18. The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

19. Which forest's population is growing at a faster rate?
20. Which forest had a greater number of trees initially? By how many?
21. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?
22. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?
23. Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

24. $y = 300(1 - t)^5$
25. $y = 220(1.06)^x$
26. $y = 16.5(1.025)^{1/x}$
27. $y = 11,701(0.97)^t$

For the following exercises, find the formula for an exponential function that passes through the two points given.

28. $(0, 6)$ and $(3, 750)$
29. $(0, 2000)$ and $(2, 20)$
30. $(-1, \frac{3}{2})$ and $(3, 24)$
31. $(-2, 6)$ and $(3, 1)$
32. $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

33.

x	1	2	3	4
$f(x)$	70	40	10	-20

34.

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

35.

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

36.

x	1	2	3	4
$f(x)$	10	20	40	80

37.

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

For the following exercises, use the compound interest formula, $A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$.

38. After a certain number of years, the value of an investment account is represented by the equation $A = 10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?
39. What was the initial deposit made to the account in the previous exercise?
40. How many years had the account from the previous exercise been accumulating interest?
41. An account is opened with an initial deposit of $6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?
42. How much more would the account in the previous exercise have been worth if the interest were compounding weekly?
43. Solve the compound interest formula for the principal, P_0.
44. Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)
45. How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?
46. Use properties of rational exponents to solve the compound interest formula for the interest rate, r.
47. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.
48. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

49. $y = 3742e^{0.75t}$
50. $y = 150e^{3.25/t}$
51. $y = 2.25e^{-2t}$
52. Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?
53. How much less would the account from the previous exercise be worth after 30 years if it were compounded monthly instead?

NUMERIC

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

54. $f(x) = 2(5)^x$, for $f(-3)$
55. $f(x) = -4^{2x+3}$, for $f(-1)$
56. $f(x) = e^x$, for $f(3)$
57. $f(x) = -2e^{x-1}$, for $f(-1)$
58. $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$
59. $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$
60. $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f(2)$

TECHNOLOGY

For the following exercises, use graphing technology to find the equation of an exponential function given the points on the curve.

61. $(0, 3)$ and $(3, 375)$
62. $(3, 222.62)$ and $(10, 77.456)$
63. $(20, 29.495)$ and $(150, 730.89)$
64. $(5, 2.909)$ and $(13, 0.005)$
65. $(11, 310.035)$ and $(25, 356365.2)$

EXTENSIONS

66. The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1 + \frac{r}{12}\right)^{12} - 1$.
67. Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.
68. Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.
69. In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = ae^{-nx}$ for some positive number n.
70. The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = P_0e^{rt}$, where P_0 is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage

of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$.

REAL-WORLD APPLICATIONS

71. The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?
72. A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?
73. In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?
74. A car was valued at $38,000 in the year 2007. By 2013, the value had depreciated to $11,000. If the car's value continues to drop by the same percentage, what will it be worth by 2017?
75. Jamal wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?
76. Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)
77. Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?
78. An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000. Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

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13.2: Graphs of Exponential Functions

Learning Objectives

- Graph exponential functions.
- Graph exponential functions using transformations.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table **13.2.1** change as the input increases by **1**.

Table 13.2.1

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the **constant ratio**. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a. If this sounds familiar, it's because we gave this property a name in the previous section - the Base Multiplier Property.

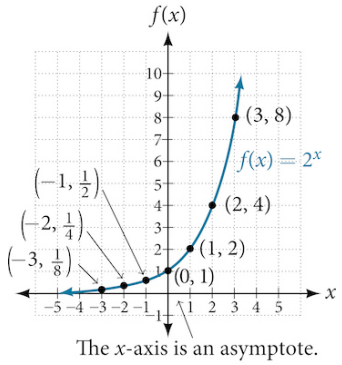
Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure **13.2.1** shows the exponential growth function $f(x) = 2^x$.

Figure 13.2.1

Notice that the graph gets close to the x-axis, but never touches it.



The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 13.2.2 change as the input increases by 1.

Table 13.2.2

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

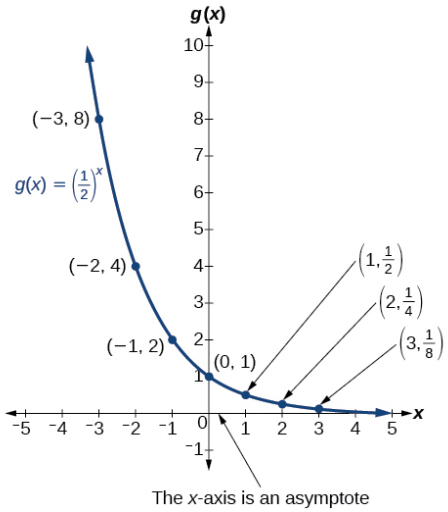
Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure 13.2.2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

Figure 13.2.2



The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

Theorem: Properties of the Graphs of Exponential Functions

An exponential function with the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$
- Secondary key point: $(1, b)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$

Example 13.2.1: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and create a table of points for the graph.

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table 13.2.3.

Table 13.2.3

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the y-intercept, $(0, 1)$, along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure 13.2.3.

Figure 13.2.3

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

? Checkpoint 13.2.1

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other toolkit functions, we can apply the four types of transformations - shifts, reflections, stretches, and compressions - to the function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 13.2.4.

Figure 13.2.4

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.

- The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 13.2.5.

Figure 13.2.5

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$, so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept becomes $\left(0, \frac{1}{8}\right)$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right)2^x$, so the initial value of the function is $\frac{1}{8}$.

Example 13.2.2: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

- Draw the horizontal asymptote $y = d$, so draw $y = -3$.
- Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.
- Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.

Figure 13.2.6

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

? Checkpoint 13.2.2

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

✓ Example 13.2.3: Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ using graphing technology. Round to the nearest thousandth.

Solution

If you use Desmos, then you would go to the graphing calculator and type in $f(x) = 1.2(5)^x + 2.8$. On the next line, type $y = 42$. For a window, use the values –3 to 3 for x and –5 to 55 for y. If you click where the two graphs intersect, Desmos will display the point $(2.1662, 42)$. Therefore, the solution, which is the value of x where these graphs intersect, is approximately 2.166.

? Checkpoint 13.2.3

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph the stretch, using $a = 3$, to get $g(x) = 3(2)^x$ as shown on the left in Figure 13.2.7, and the compression, using $a = \frac{1}{3}$, to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 13.2.7.

Figure 13.2.7

(a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3.
(b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

✓ Example 13.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.

- Create a table of points as shown in Table 13.2.4.

Table 13.2.4

x	-3	-2	-1	0	1	2	3
$f(x)=4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0,4)$, along with two other points. We can use $(-1,8)$ and $(1,2)$.

Draw a smooth curve connecting the points, as shown in Figure 13.2.8.

Figure 13.2.8

The domain is $(-\infty,\infty)$; the range is $(0,\infty)$; the horizontal asymptote is $y=0$.

? Checkpoint 13.2.4

Sketch the graph of $f(x)=\frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the x-axis or the y-axis. When we multiply the function $f(x)=b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection about the y-axis. For example, if we begin by graphing the function $f(x)=2^x$, we can then graph the two reflections alongside it. The reflection about the x-axis, $g(x)=-2^x$, is shown on the left side of Figure 13.2.9, and the reflection about the y-axis $h(x)=2^{-x}$, is shown on the right side of Figure 13.2.9.

Figure 13.2.9

(a) $g(x)=-2^x$ reflects the graph of $f(x)=2^x$ about the x-axis.
(b) $g(x)=2^{-x}$ reflects the graph of $f(x)=2^x$ about the y-axis.

Example 13.2.5: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis. State its domain, range, and asymptote.

Solution

Since we want to reflect the parent function $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$. Next we create a table of points as in Table 13.2.5.

Table 13.2.5

x	-3	-2	-1	0	1	2	3
$g(x) = -\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the y-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

Figure 13.2.10

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Checkpoint 13.2.5

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in Table 13.2.6 to arrive at the general equation for translating exponential functions.

Table 13.2.6

Transformations of the Parent Function $f(x) = b^x$

Transformation	Form
Shift • Horizontally c units to the left (if $c > 0$) • Vertically d units up (if $d > 0$)	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\|a\| > 1$ • Compression if $0 < \|a\| < 1$	$f(x) = ab^x$
Reflect about the x-axis	$f(x) = -b^x$
Reflect about the y-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General equation for all transformations	$f(x) = ab^{x+c} + d$

Example 13.2.6: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2 , reflected across the y-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2 , so $a = 2$.
- The function is reflected about the y-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$f(x) = ab^{x+c} + d = 2e^{-x+0} + 4 = 2e^{-x} + 4.$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

Checkpoint 13.2.6

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x-axis and then shifted down 2 units.

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13.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Graphs of exponential functions
  - Graphs of Exponential Functions - Graphing Exponential Functions
  - Exponential Functions - Advanced Topics in Exponential Functions
  - Intermediate Algebra - Exponential Functions: Graphs of Exponential Functions

KEY CONCEPTS

DEFINITIONS

- **Constant Ratio (for exponential functions)**: For an exponential function $f(x) = ab^x$, the base b is the constant ratio by which the output is multiplied when the input increases by 1.

THEOREMS

- **Properties of the Graphs of Exponential Functions $f(x) = b^x$ (where $b > 0, b \neq 1$)**:
  - One-to-one function.
  - Horizontal asymptote: $y = 0$.
  - Domain: $(-\infty, \infty)$.
  - Range: $(0, \infty)$.
  - x-intercept: none.
  - y-intercept: $(0, 1)$.
  - Increasing if $b > 1$.
  - Decreasing if $0 < b < 1$.

COMMON MISTAKES

- **Incorrect Order of Transformations**: Applying shifts, reflections, and stretches/compressions in the wrong order, which can lead to an incorrect final graph. For example, applying a vertical shift before a vertical stretch will result in the shift also being stretched.
- **Errors in Horizontal Shifts**: Confusing the direction of horizontal shifts (e.g., $f(x + c)$ shifts left, $f(x - c)$ shifts right for $c > 0$).
- **Misinterpreting Stretches vs. Shifts**: Confusing the effect of multiplying the function/input (stretch/compression) with adding to the function/input (shift).
- **Incorrectly Identifying the Asymptote after Transformations**: Forgetting that vertical shifts change the horizontal asymptote, while horizontal shifts, reflections, and vertical stretches/compressions (by a non-zero factor) do not change the location of the horizontal asymptote $y = 0$ unless there's also a vertical shift.
- **Sign Errors with Reflections**: Incorrectly applying reflections; $-f(x)$ reflects across the x-axis, and $f(-x)$ reflects across the y-axis.
- **Errors in Graphing Key Points after Transformation**: Incorrectly transforming key points like the y-intercept or the point $(1, b)$ from the parent function.

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13.2.2: HOMEWORK

READING QUESTIONS

1. What is the "constant ratio" for an exponential function of the form $f(x) = ab^x$, and what does it signify about the function's values as the input increases by 1?
2. Describe the domain, range, and horizontal asymptote of the basic exponential function $f(x) = b^x$ where $b > 0$ and $b \neq 1$.
3. How does a vertical shift by a constant d (i.e., $g(x) = b^x + d$) affect the domain, range, and horizontal asymptote of $f(x) = b^x$?
4. How does a horizontal shift by a constant c (i.e., $g(x) = b^{x+c}$) affect the domain, range, and horizontal asymptote of $f(x) = b^x$?
5. Explain how the graph of $g(x) = a \cdot b^x$ is related to the graph of $f(x) = b^x$ if $|a| > 1$. What if $0 < |a| < 1$?
6. How is the graph of $g(x) = -b^x$ related to the graph of $f(x) = b^x$?
7. How is the graph of $g(x) = b^{-x}$ related to the graph of $f(x) = b^x$?
8. When graphing $f(x) = ab^{x+c} + d$, what is a general order in which transformations can be applied to the parent graph $y = b^x$?
9. If the graph of $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units, what is the equation of the new function?
10. What is the y-intercept of the function $f(x) = 4(\frac{1}{2})^x$?
11. What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?
12. What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

HOMEWORK

ALGEBRAIC

13. The graph of $f(x) = 3^x$ is reflected about the y-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
14. The graph of $f(x) = (\frac{1}{2})^{-x}$ is reflected about the y-axis and compressed vertically by a factor of $\frac{1}{5}$. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
15. The graph of $f(x) = 10^x$ is reflected about the x-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
16. The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its y-intercept (to the nearest thousandth), domain, and range.
17. The graph of $f(x) = -\frac{1}{2}(\frac{1}{4})^{x-2} + 4$ is shifted downward 4 units, and then shifted left 2 units, stretched vertically by a factor of 4, and reflected about the x-axis. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

GRAPHICAL

For the following exercises, graph the function and its reflection about the y-axis on the same axes, and give the y-intercept.

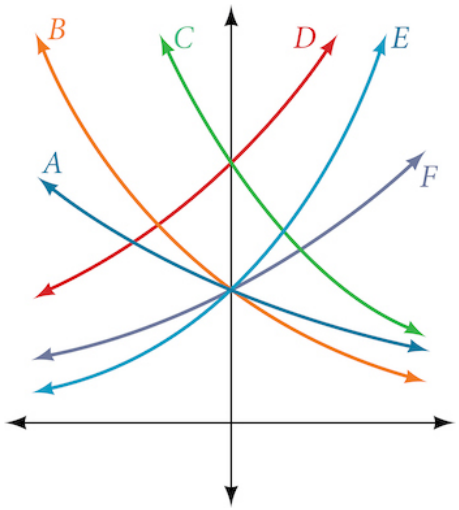
18. $f(x) = 3(\frac{1}{2})^x$
19. $g(x) = -2(0.25)^x$
20. $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

21. $f(x) = 3(\frac{1}{4})^x$, $g(x) = 3(2)^x$, and $h(x) = 3(4)^x$
22. $f(x) = \frac{1}{4}(3)^x$, $g(x) = 2(3)^x$, and $h(x) = 4(3)^x$

For the following exercises, match each function with one of the graphs in Figure 13.2.2.1.

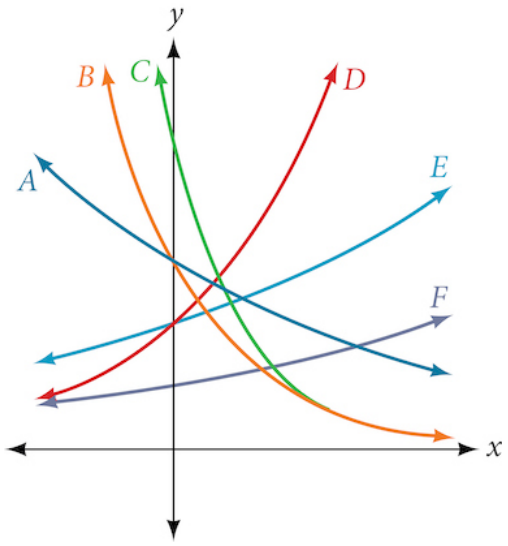
Figure 13.2.2.1



23. $f(x) = 2(0.69)^x$
24. $f(x) = 2(1.28)^x$
25. $f(x) = 2(0.81)^x$
26. $f(x) = 4(1.28)^x$
27. $f(x) = 2(1.59)^x$
28. $f(x) = 4(0.69)^x$

For the following exercises, use the graphs shown in Figure 13.2.2.2. All have the form $f(x) = ab^x$.

Figure 13.2.2.2



29. Which graph has the largest value for b?
30. Which graph has the smallest value for b?
31. Which graph has the largest value for a?
32. Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the x-axis on the same axes.

33. $f(x) = \frac{1}{2}(4)^x$
34. $f(x) = 3(0.75)^x - 1$
35. $f(x) = -4(2)^x + 2$

For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.

36. $f(x) = 2^{-x}$
37. $h(x) = 2^x + 3$
38. $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

39. $f(x) = -5(4)^x - 1$
40. $f(x) = 3\left(\frac{1}{2}\right)^x - 2$
41. $f(x) = 3(4)^{-x} + 2$

For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.

42. Shift $f(x)$ 4 units upward
43. Shift $f(x)$ 3 units downward
44. Shift $f(x)$ 2 units left
45. Shift $f(x)$ 5 units right
46. Reflect $f(x)$ about the x-axis
47. Reflect $f(x)$ about the y-axis

For the following exercises, each graph is a transformation of $y = 2^x$. Write an equation describing the transformation.

48.

49.

50.

For the following exercises, find an exponential equation for the graph.

51.

52.

NUMERIC

For the following exercises, evaluate the exponential functions for the indicated value of x.

53. $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.
54. $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.
55. $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^x + 6$ for $h(-7)$.

TECHNOLOGY

For the following exercises, use graphing technology to approximate the solutions of the equation. Round to the nearest thousandth.

56. $-50 = -\left(\frac{1}{2}\right)^{-x}$
57. $116 = \frac{1}{4}\left(\frac{1}{8}\right)^x$
58. $12 = 2(3)^x + 1$
59. $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$
60. $-30 = -4(2)^{x+2} + 2$

EXTENSIONS

61. Explore and discuss the graphs of $F(x) = b^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.
62. Prove the conjecture made in the previous exercise.
63. Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b^n}\right)b^x$ for any real number n and real number $b > 0$.
64. Prove the conjecture made in the previous exercise.

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13.3: Logarithmic Functions

Learning Objectives

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 13.3.1

Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure **13.3.1**. Even though both caused substantial damage, the earthquake in 2011 was **100** times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the **Richter Scale**. The Haitian earthquake registered a **7.0** on the Richter Scale whereas the Japanese earthquake registered a **9.0**.

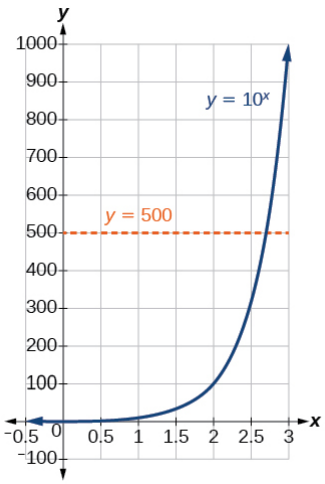
The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude **8** is not twice as great as an earthquake of magnitude **4**. It is $10^{8-4} = 10^4 = 10,000$ times as great! In this section, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were **500** times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between **2** and **3**, since $y = 10^x$ is increasing. We can examine a graph, as in Figure **13.3.2**, to better estimate the solution.

Figure 13.3.2



Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 13.3.2 passes the Horizontal Line Test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a **logarithmic function** of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

Definition: Logarithmic Function

The **base b logarithmic function** is defined as follows:

$$y = \log_b(x) \text{ is equivalent to } b^y = x,$$

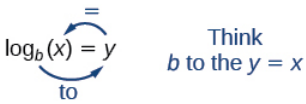
where, $x > 0$, $b > 0$, and $b \neq 1$. We read $\log_b(x)$ as, "the logarithm with base b of x," or the "log base b of x."

It is critical to recognize that logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2(32) = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \iff b^y = x, b > 0, b \neq 1.$$

The symbol, $\iff$, means "**if and only if**." Since b is the base of an exponential function, b is always positive.



Note: Lax Notation

Because the logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. While this "lax" notation has been adopted as acceptable by most mathematicians and scientists, we will stick with parentheses as often as possible to reinforce the concept that the logarithm is a function.

Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions. This leads us to quickly develop some fundamental theory about logarithmic functions.

Theorem: Domain and Range of the Logarithmic Function

The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Proof

Let $y = \log_b(x)$. Since the logarithm base b is the inverse of the exponential function base b, the domain of the logarithm is the range of the exponential function. This is precisely $(0, \infty)$. Moreover, the range of the logarithm is the domain of the exponential function, which is $(-\infty, \infty)$.

In fact, it is due to the inverse nature of the logarithmic and exponential functions that $x > 0$, $b > 0$, and $b \neq 1$ are restrictions for the logarithmic function. Since $y = \log_b(x)$ is the same as $b^y = x$, and we know the base of an exponential function must be positive, $b > 0$. Moreover, when defining the exponential function, we took great care to state that the base could not be 1. Thus, $b \neq 1$. Finally, exponential functions only return positive values. Hence, $x > 0$.

Q&A

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Example 13.3.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution

First, identify the values of b, y, and x. Then, write the equation in the form $b^y = x$.

a. Here, $b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{1/2} = \sqrt{6}.$$

b. Here, $b = 3$, $y = 2$, and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

Checkpoint 13.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

Example 13.3.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10,000}$

Solution

First, identify the values of b, y, and x. Then, write the equation in the form $x = \log_b(y)$.

a. Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.
b. Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.
c. Here, $b = 10$, $x = -4$, and $y = \frac{1}{10,000}$. Therefore, the equation $10^{-4} = \frac{1}{10,000}$ is equivalent to $\log_{10}\left(\frac{1}{10,000}\right) = -4$.

Checkpoint 13.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2(8)$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2(8) = 3$.

Now consider simplifying $\log_7(49)$ and $\log_3(27)$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7(49) = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3(27) = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{2/3}\left(\frac{4}{9}\right)$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{2/3}\left(\frac{4}{9}\right) = 2$.

Example 13.3.3: Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64.$$

Therefore,

$$\log_4(64) = 3.$$

Checkpoint 13.3.3

Solve $y = \log_{121}(11)$ without using a calculator.

Example 13.3.4: Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$? "

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}.$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Checkpoint 13.3.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**.

Definition: Common Logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$.

The common logarithm otherwise follows the standard definition of a logarithmic function: For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x.$$

Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars, decibel levels of sounds, and the pH of acids and bases also use common logarithms.

Example 13.3.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000.$$

Therefore, $\log(1000) = 3$.

Checkpoint 13.3.5

Evaluate $y = \log(1,000,000)$.

Example 13.3.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by **)**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

We can perform a quick "logic check" on our answer to Example 13.3.6. Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100) = 2$ and $\log(1000) = 3$.

Checkpoint 13.3.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Example 13.3.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$\begin{aligned} & 10^x = 500 \\ \Longrightarrow \quad & \log(500) = x && \text{(definition of the common logarithm)} \\ \Longrightarrow \quad & x \approx 2.699 && \text{(using a calculator)} \end{aligned}$$

The difference in Richter Scale magnitudes was about 2.699.

Checkpoint 13.3.7

The amount of energy released from one earthquake was 8,500 times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in Calculus and some scientific applications; they are called **natural logarithms**.

Definition: Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$.

The natural logarithm otherwise follows the standard definition of a logarithmic function: For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x.$$

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln(1) = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Example 13.3.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution

- Press **[LN]**.
- Enter 500, followed by **)**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Checkpoint 13.3.8

Evaluate $\ln(-500)$.

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13.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Logarithmic functions
  - Intermediate Algebra - Logarithmic Functions: Concepts and Basic Use
  - Intermediate Algebra - Logarithms: Logarithmic Functions
  - Logarithmic Functions - A Focus on Logarithmic Functions
- Converting between logarithmic to exponential forms
  - Logarithmic Functions - Converting Between Logarithmic and Exponential Forms
- Evaluating logarithms
  - Intermediate Algebra - Logarithmic Functions: Evaluating Logarithms

KEY CONCEPTS

DEFINITIONS

- **Logarithmic Function (base b)**: The function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. It is defined as: $y = \log_b(x)$ is equivalent to $b^y = x$, where $x > 0$, $b > 0$, and $b \neq 1$. The base b logarithm of a number x is the exponent by which we must raise b to get x.
- **Common Logarithm**: A logarithm with base 10. It is written as $\log(x)$, which is shorthand for $\log_{10}(x)$.
- **Natural Logarithm**: A logarithm with base e (Euler's number). It is written as $\ln(x)$, which is shorthand for $\log_e(x)$.

THEOREMS

- **Domain and Range of the Logarithmic Function**: The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

COMMON MISTAKES

- **Incorrect Conversion Between Logarithmic and Exponential Form**: Misplacing the base, argument, or exponent when converting between $y = \log_b(x)$ and $b^y = x$.
- **Taking the Logarithm of a Non-Positive Number**: Attempting to evaluate the logarithm of zero or a negative number (e.g., $\ln(-500)$ in Checkpoint 4.3.8). The argument of a logarithm must be positive.
- **Errors in Evaluating Basic Logarithms**: Difficulty in mentally evaluating logarithms by not recognizing them as exponents (e.g., for $\log_2(8)$, thinking "2 to what power is 8?").
- **Calculator Errors**: Incorrectly inputting expressions, especially with parentheses, when using a calculator.
- **Assuming $\log_b(0) = 1$ or similar misconceptions**: The logarithm of 0 is undefined. $\log_b(1) = 0$ because $b^0 = 1$.

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13.3.2: HOMEWORK

READING QUESTIONS

1. What is a base b logarithm? Explain its relationship to an exponential function.
2. How do you convert the logarithmic equation $\log_b(x) = y$ into exponential form?
3. How do you convert the exponential equation $b^y = x$ into logarithmic form?
4. What is the domain and range of the logarithmic function $f(x) = \log_b(x)$?
5. Can you take the logarithm of a negative number or zero? Explain why or why not based on the properties of exponential functions.
6. What is a "common logarithm"? What is its base, and how is it typically notated?
7. What is a "natural logarithm"? What is its base, and how is it typically notated?
8. How would you mentally evaluate $\log_5(25)$?
9. If $10^x = 500$, how can you rewrite this equation using a common logarithm to solve for x?
10. What is the value of $\ln(1)$? What is the value of $\log(1)$? Why?
11. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b(x) = y$ for $b > 0$, $b \neq 1$.
12. How is the logarithmic function $f(x) = \log_b(x)$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?
13. How can the logarithmic equation $\log_b(x) = y$ be solved for x using the properties of exponents?
14. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?
15. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

HOMEWORK

ALGEBRAIC

For the following exercises, rewrite each equation in exponential form.

16. $\log_4(q) = m$
17. $\log_a(b) = c$
18. $\log_{16}(y) = x$
19. $\log_x(64) = y$
20. $\log_y(x) = -11$
21. $\log_{15}(a) = b$
22. $\log_y(137) = x$
23. $\log_{13}(142) = a$
24. $\log(v) = t$
25. $\ln(w) = n$

For the following exercises, rewrite each equation in logarithmic form.

26. $4^x = y$
27. $c^d = k$
28. $m^{-7} = n$
29. $19^x = y$
30. $x^{-10/13} = y$
31. $n^4 = 103$
32. $\left(\frac{7}{5}\right)^m = n$
33. $y^x = \frac{39}{100}$
34. $10^a = b$
35. $e^k = h$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

36. $\log_3(x) = 2$
37. $\log_2(x) = -3$
38. $\log_5(x) = 2$
39. $\log_3(x) = 3$
40. $\log_2(x) = 6$
41. $\log_9(x) = \frac{1}{2}$
42. $\log_{18}(x) = 2$
43. $\log_6(x) = -3$
44. $\log(x) = 3$
45. $\ln(x) = 2$

For the following exercises, use the definition of common and natural logarithms to simplify.

46. $\log(100^8)$
47. $10^{\log(32)}$
48. $2\log(0.0001)$
49. $e^{\ln(1.06)}$
50. $\ln(e^{-5.03})$
51. $e^{\ln(10.125)} + 4$

NUMERIC

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

52. $\log_3\left(\frac{1}{27}\right)$
53. $\log_6(\sqrt{6})$
54. $\log_2\left(\frac{1}{8}\right) + 4$
55. $6\log_8(4)$

For the following exercises, evaluate the common logarithmic expression without using a calculator.

56. $\log(10,000)$
57. $\log(0.001)$
58. $\log(1) + 7$
59. $2\log(100^{-3})$

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

60. $\ln(e^{1/3})$

61. $\ln(1)$
62. $\ln(e^{-0.225}) - 3$
63. $25\ln(e^{2/5})$

TECHNOLOGY

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

64. $\log(0.04)$
65. $\ln(15)$
66. $\ln\left(\frac{4}{5}\right)$
67. $\log(\sqrt{2})$
68. $\ln(\sqrt{2})$

EXTENSIONS

69. Is $x = 0$ in the domain of the function $f(x) = \log(x)$? If so, what is the value of the function when $x = 0$? Verify the result.
70. Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.
71. Is there a number x such that $\ln(x) = 2$? If so, what is that number? Verify the result.
72. Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.
73. Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$? Verify the result.

REAL-WORLD APPLICATIONS

74. The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?
75. Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?
76. The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula

$$\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2,$$

where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

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13.4: Graphs of Logarithmic Functions

Learning Objectives

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

Hawk A.I. Section-Specific Tutor

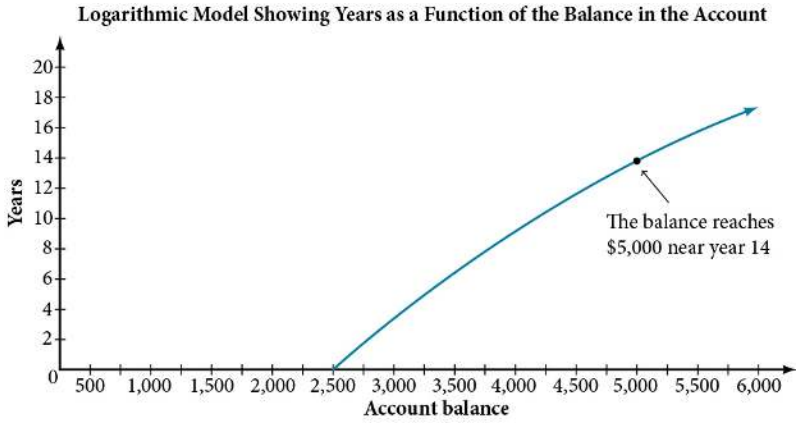
Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

When we investigated graphs of exponential functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $2500 in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure 13.4.1 shows this point on the logarithmic graph.

Figure 13.4.1



In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where the domain of y is $(-\infty, \infty)$ and the range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.

- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Theorem: Domain and Range of the Logarithmic Function

The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Proof

Let $y = \log_b(x)$. Since the logarithm base b is the inverse of the exponential function base b, the domain of the logarithm is the range of the exponential function. This is precisely $(0, \infty)$. Moreover, the range of the logarithm is the domain of the exponential function, which is $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations — shifts, stretches, compressions, and reflections — to the parent function without loss of shape.

We have seen that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be strictly greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} & 2x - 3 > 0 & \\ \implies \quad & 2x > 3 & \text{(adding 3 to both sides)} \\ \implies \quad & x > 1.5 & \text{(dividing both sides by 2)} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

Example 13.4.1: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$x + 3 > 0 \implies x > -3.$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

Checkpoint 13.4.1

What is the domain of $f(x) = \log_5(x - 2) + 1$?

Example 13.4.2: Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5 - 2x)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $5 - 2x > 0$. Solving this inequality,

$$\begin{aligned} 5-2x &> 0 \\ \Longrightarrow \quad -2x &> -5 \qquad \text{(subtracting 5 from both sides)} \\ \Longrightarrow \quad x &< \frac{5}{2} \qquad \text{(dividing both sides by } -2 \text{ and switching the inequality)} \end{aligned}$$

The domain of $f(x) = \log(5-2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

? Checkpoint 13.4.2

What is the domain of $f(x) = \log(x-5) + 2$?

Of course, there is no need to restrict the argument of a logarithm to linear expressions. We can rely on the graphing skills we built in previous chapters to find the domain of the following example.

✓ Example 13.4.3: Identifying the Domain of a Logarithm Composed with a Nonlinear Function

State the domain of the function.

$$g(x) = -2\log_5\left(\frac{2x^3-3x^2-3x+2}{3x^3+375}\right)$$

Solution

Since the argument of a logarithm must be positive, answering this question requires us to solve

$$\frac{2x^3-3x^2-3x+2}{3x^3+375} > 0.$$

From our previous work, we have two ways to solve this rational inequality - sign charts or a quick graph. As you probably will guess, I am in favor of a quick sketch. In either case, we will need to factor the numerator and denominator of the rational expression.

Factoring the denominator is the easier task, so we start there.

$$\begin{aligned} 3x^3+375 &= 3\left(x^3+125\right) && \text{(factoring out the GCF)} \\ &= 3(x+5)\left(x^2-5x+25\right) && \text{(factoring the sum of cubes)} \end{aligned}$$

Hence, the rational inequality is

$$\frac{2x^3-3x^2-3x+2}{3(x+5)\left(x^2-5x+25\right)} > 0,$$

where the trinomial in the denominator is prime (i.e., it does not have real zeros).

Turning our attention to the numerator, we are forced to use the Rational Zeros Theorem. The candidate roots are ± 1, ± 2, and $\pm\frac{1}{2}$. Starting with 1, we get

$$\begin{array}{r|rrrr} \underline{1} & 2 & -3 & -3 & 2 \\ & & 2 & -1 & -4 \\ \hline & 2 & -1 & -4 & -2 \end{array}$$

Since the remainder is nonzero, $(x-1)$ is not a factor. Let's move to -1.

$$\begin{array}{r|rrrr} \underline{-1} & 2 & -3 & -3 & 2 \\ & & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

The remainder is zero. Therefore, $(x-(-1)) = (x+1)$ is a factor. Moreover, the remaining factor is $2x^2-5x+2 = (2x-1)(x-2)$. Thus, the rational inequality we need to solve can be rewritten as

$$\frac{(x+1)(2x-1)(x-2)}{3(x+5)(x^2-5x+25)} > 0.$$

The intercepts of the function on the left side are $(-1,0)$, $(\frac{1}{2},0)$, $(2,0)$, and $(0,\frac{2}{375})$. The vertical asymptote is $x=-5$ and the horizontal asymptote is $y=\frac{2}{3}$. Noting that the multiplicities of the x-intercepts are all 1 and the multiplicity of the vertical asymptote is odd, we quickly sketch the following graph.

We can see this graph is above the x-axis on the interval $(-\infty,-5)\cup\left(-1,\frac{1}{2}\right)\cup(2,\infty)$. Hence, this is the domain of the logarithmic function.

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y=\log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y=\log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y=b^x$, their graphs will be reflections of each other across the line $y=x$. To illustrate this, we can observe the relationship between the input and output values of $y=2^x$ and its equivalent $x=\log_2(y)$ in Table 13.4.1.

Table 13.4.1

x	-3	-2	-1	0	1	2	3
$2^x=y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y)=x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table 13.4.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x)=2^x$ and $g(x)=\log_2(x)$. See Table 13.4.2.

Table 13.4.2

$f(x)=2^x$	$\left(-3,\frac{1}{8}\right)$	$\left(-2,\frac{1}{4}\right)$	$\left(-1,\frac{1}{2}\right)$	$(0,1)$	$(1,2)$	$(2,4)$	$(3,8)$
$g(x)=\log_2(x)$	$\left(\frac{1}{8},-3\right)$	$\left(\frac{1}{4},-2\right)$	$\left(\frac{1}{2},-1\right)$	$(1,0)$	$(2,1)$	$(4,2)$	$(8,3)$

As we'd expect, the x- and y-coordinates are reversed for the inverse functions. Figure 13.4.2 shows the graph of f and g.

Figure 13.4.2

Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

This leads to the following theorem.

Theorem: Properties of the Graphs of Logarithmic Functions

A logarithmic function with the form $f(x) = \log_b(x)$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$
- Secondary key point: $(b, 1)$
- y-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

Figure 13.4.3 shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e \approx 2.718$).

Figure 13.4.3

The graphs of three logarithmic functions with different bases, all greater than 1.

Example 13.4.4: Graphing a Logarithmic Function with the Form $f(x) = \log_b(x)$.

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The x-intercept is $(-1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure 13.4.4).

Figure 13.4.4

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Checkpoint 13.4.4

Graph $f(x) = \log_{1/5}(x)$. State the domain, range, and asymptote.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = \log_b(x)$

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 13.4.5.

Figure 13.4.5

Example 13.4.5: Graphing a Horizontal Shift of the Parent Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x$–2.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$, and $(5, 1)$.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

Figure 13.4.6

Checkpoint 13.4.5

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Graphing a Vertical Shift of $y = \log_b(x)$

When a constant d is added to the parent function $f(x) = \log_b(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$. See Figure 13.4.7.

Figure 13.4.7

Example 13.4.6: Graphing a Vertical Shift of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x) - 2$, we will notice $d = -2$. Thus $d < 0$. This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the y coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Figure 13.4.8

? Checkpoint 13.4.6

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$. See Figure 13.4.9.

Figure 13.4.9

Example 13.4.7: Graphing a Stretch or Compression of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = 2\log_4(x)$, we will notice $a = 2$. This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$, $(-1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$, $(-1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure 13.4.10.

Figure 13.4.10

? Checkpoint 13.4.7

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

✓ Example 13.4.8: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x+2)$. State the domain, range, and asymptote.

Solution

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure 13.4.11. The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

Figure 13.4.11

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2$.

? Checkpoint 13.4.8

Sketch a graph of the function $f(x) = 3\log(x-2) + 1$. State the domain, range, and asymptote.

Graphing Reflections of $f(x) = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1, the result is a reflection about the x-axis. When the *input* is multiplied by -1, the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.

Figure 13.4.12

Example 13.4.9: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, f is a reflection of the parent graph about the y-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The x-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

Figure 13.4.13

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Checkpoint 13.4.9

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Example 13.4.10: Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x - 1)$ using graphing technology. Round to the nearest thousandth.

Solution

In the graphing calculator portion of Desmos, we enter $y = 4\ln(x) + 1$ on one line and $y = -2\ln(x - 1)$ on the next line. For a window, use the values 0 to 5 for x and -10 to 10 for y. The graphs should intersect somewhere a little to right of $x = 1$. Clicking on their intersection point, we see the x-coordinate of the point of intersection is displayed as 1.3385. So, to the nearest thousandth, $x \approx 1.339$.

Checkpoint 13.4.10

Solve $5\log(x + 2) = 4 - \log(x)$ using graphing technology. Round to the nearest thousandth.

Summarizing Transformations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 13.4.3 to arrive at the general equation for translating exponential functions.

Table 13.4.3

Transformations of the Parent Function $y = \log_b(x)$

Transformation	Form
Shift • Horizontally c units to the left (if $c > 0$) • Vertically d units up (if $d > 0$)	$y = \log_b(x + c) + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $\lvert a \rvert < 1$	$y = a \log_b(x)$
Reflect about the x-axis	$y = -\log_b(x)$
Reflect about the y-axis	$y = \log_b(-x)$
General equation for all transformations	$y = a \log_b(x + c) + d$

Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a \log_b(x + c) + d$$

where the parent function, $y = \log_b(x)$, $b > 1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $\lvert a \rvert$ if $\lvert a \rvert > 0$.
- compressed vertically by a factor of $\lvert a \rvert$ if $0 < \lvert a \rvert < 1$.
- reflected about the x-axis when $a < 0$.

For $f(x) = \log_b(-x)$, the graph of the parent function is reflected about the y-axis.

Example 13.4.11: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2 \log_3(x + 4) + 5$?

Solution

The vertical asymptote is at $x = -4$.

In Example 13.4.11, the coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

Checkpoint 13.4.11

What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$?

Example 13.4.12: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 13.4.14.

Figure 13.4.14

Solution

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x+2) + k$$

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

$$\begin{aligned} & 1 = -a\log(-1+2) + k && \text{(substituting)} \\ \Rightarrow\ & 1 = -a\log(1) + k && \text{(simplifying)} \\ \Rightarrow\ & 1 = -a(0) + k && (\log(1) = 0) \\ \Rightarrow\ & 1 = k && (\log(1) = 0) \end{aligned}$$

Next, substituting in $(2, -1)$,

$$\begin{aligned} & -1 = -a\log(2+2) + 1 && \text{(substituting)} \\ \Rightarrow\ & -1 = -a\log(4) + 1 && \text{(simplifying)} \\ \Rightarrow\ & -2 = -a\log(4) && \text{(subtracting 1 from both sides)} \\ \Rightarrow\ & -\frac{2}{\log(4)} = a && \text{(dividing both sides by } -\log(4)) \end{aligned}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2) + 1$.

We can verify this answer by comparing the function values in Table **13.4.4** with the points on the graph in Figure **13.4.14**.

Table 13.4.4

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

? Checkpoint 13.4.12

Give the equation of the natural logarithm graphed in Figure **13.4.15**.

Figure 13.4.15

Q&A

Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure **13.4.15**. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $x \mid x > -3$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

More Complex Transformations

Just as we encountered early in this course, logarithmic functions can also have horizontal stretches or compressions. Since these more robust transformations occur in Calculus, we spend a few moments reminding ourselves how to deal with graphing such types of functions.

✔ Example **13.4.13**: Graphing a More Complex Logarithmic Function

Sketch a graph of the function

$$h(x) = 2\ln\left(-\frac{1}{4}x + 1\right) - 1.$$

Solution

As we have done previously, we begin by factoring the argument of the function.

$$h(x) = 2\ln\left(-\frac{1}{4}(x - 4)\right) - 1.$$

Starting with the parent graph (which contains the anchor points $(1, 0)$ and $(e, 1)$), we read our function from left to right:

- The parent graph, $y = \ln(x)$, has been **stretched vertically** by a factor of 2. Thus, we multiply the y-coordinates of our anchor points by 2 to get $(1, 0)$ and $(e, 2)$ to get the graph of $y = 2\ln(x)$.

- The graph of $y = 2\ln(x)$ has been **reflected horizontally** about the y-axis and **stretched horizontally** by a factor of 4. Therefore, we multiply the x-coordinates of our anchor points by -4 to get $(-4, 0)$ and $(-4e, 2)$ to get the graph of $y = 2\ln\left(-\frac{1}{4}x\right)$.

- The graph of $y = 2\ln\left(-\frac{1}{4}x\right)$ gets **shifted horizontally** 4 units to the right. In this horizontal shift, the vertical asymptote also shifts. Hence, we add 4 to all the x-coordinates of our anchor points and to the vertical asymptote. Thus, the graph of $y = 2\ln\left(-\frac{1}{4}(x-4)\right)$ contains the points $(0, 0)$ and $(-4e + 4, 2)$, and the vertical asymptote is at $x = 4$.

- The graph of $y = 2\ln\left(-\frac{1}{4}(x-4)\right)$ gets **shifted vertically 1** unit down to arrive at our final graph of the function $h(x) = 2\ln\left(-\frac{1}{4}(x-4)\right) - 1$. The anchor points get moved to $(0,-1)$ and $(-4e+4,1)$. The final graph is shown below.

Revisiting Graphs of Absolute Value Functions

Finally, while you might not know it yet, absolute values and natural logarithms will appear together quite often in Calculus. It's best that we spend an example reminding ourselves of the effect of the absolute value on a function.

Example 13.4.14

Graph

$$k(x) = \left|2\ln\left(-\frac{1}{4}x+1\right)-1\right|.$$

Solution

If this function looks familiar, it is because it is the absolute value of the function $h(x)$ from Example **13.4.13**. The easiest way to graph the absolute value of a function is to recall that the absolute value keeps non-negative values the same, but forces negative values to become positive. Graphically, this means that any coordinate on the graph of $h(x)$ with negative y-values will be reflected about the x-axis to their positive values. All other coordinates remain unchanged.

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13.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Graphs of logarithmic functions
  - Graphs of Logarithmic Functions - Graphing Logarithmic Functions
  - Logarithmic Functions - Graphs of Logarithmic Functions
  - Intermediate Algebra - Logarithms: Graphing Logarithmic Functions
  - Intermediate Algebra - Logarithms: Graphing Logarithmic Functions Using Transformations
- Domain of logarithmic function
  - Graphs of Logarithmic Functions - Finding the Domain of a Logarithmic Function
- Logarithms and You
  - Logarithmic Functions - Logarithms and You

KEY CONCEPTS

THEOREMS

- **Domain and Range of the Logarithmic Function $f(x) = \log_b(x)$**: The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
- **Properties of the Graphs of Logarithmic Functions $f(x) = \log_b(x)$ (where $b > 0, b \neq 1$)**:
  - One-to-one function.
  - Vertical asymptote: $x = 0$.
  - Domain: $(0, \infty)$.
  - Range: $(-\infty, \infty)$.
  - x-intercept: $(1, 0)$.
  - Secondary key point: $(b, 1)$.
  - y-intercept: none.
  - Increasing if $b > 1$.
  - Decreasing if $0 < b < 1$.

COMMON MISTAKES

- **Incorrectly Identifying the Domain of Transformed Logarithmic Functions**: The argument of the logarithm must always be positive. For $f(x) = \log_b(expression)$, one must solve $expression > 0$. Errors occur if this inequality is set up or solved incorrectly, especially with reflections or multiple transformations.
- **Errors in Horizontal Shifts and Vertical Asymptote Location**: Confusing the direction of horizontal shifts (e.g., $\log_b(x + c)$ shifts left, $\log_b(x - c)$ shifts right for $c > 0$) and consequently misplacing the vertical asymptote. The vertical asymptote shifts with the horizontal shift.
- **Order of Transformations**: Applying shifts, reflections, and stretches/compressions in the wrong order, particularly when combining horizontal transformations (e.g., in $h(x) = 2\ln(-\frac{1}{4}x + 1) - 1$, factoring the argument is key: $-\frac{1}{4}(x - 4)$).
- **Misinterpreting Stretches vs. Shifts**: Confusing the effect of multiplying the function/input (stretch/compression) with adding to the function/input (shift).
- **Incorrectly Identifying the Range of Logarithmic Functions**: The range of any untransformed or vertically stretched/compressed/shifted logarithmic function $y = a\log_b(expression) + d$ is always $(-\infty, \infty)$. Horizontal transformations do not affect the range.
- **Sign Errors with Reflections**: Incorrectly applying reflections; $-f(x)$ reflects across the x-axis, and $f(-x)$ reflects across the y-axis. This also impacts the domain for $f(-x)$.
- **Graphing Absolute Value of Logarithmic Function**: Incorrectly reflecting parts of the graph. Only the portions of the original logarithmic graph that are *below* the x-axis should be reflected *above* the x-axis. Portions already above remain unchanged.

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13.4.2: HOMEWORK

READING QUESTIONS

1. What is the domain of the basic logarithmic function $f(x) = \log_b(x)$? What is its range?
2. What is the equation of the vertical asymptote for the basic logarithmic function $f(x) = \log_b(x)$?
3. How does a horizontal shift by a constant c (i.e., $g(x) = \log_b(x + c)$) affect the domain and vertical asymptote of $f(x) = \log_b(x)$?
4. How does a vertical shift by a constant d (i.e., $g(x) = \log_b(x) + d$) affect the domain, range, and vertical asymptote of $f(x) = \log_b(x)$?
5. Explain how the graph of $g(x) = a\log_b(x)$ is related to the graph of $f(x) = \log_b(x)$ if $|a| > 1$. What if $0 < |a| < 1$?
6. How is the graph of $g(x) = -\log_b(x)$ related to the graph of $f(x) = \log_b(x)$?
7. How is the graph of $g(x) = \log_b(-x)$ related to the graph of $f(x) = \log_b(x)$? How does this affect the domain?
8. When graphing $f(x) = a\log_b(B(x - h)) + k$, what is a general order in which transformations can be applied to the parent graph $y = \log_b(x)$? Why is factoring the argument $B(x - h)$ important?
9. What are the two "anchor" points typically used when graphing transformations of $y = \log_b(x)$?
10. How do you graph the function $k(x) = |h(x)|$ if you know the graph of $h(x) = 2\ln(-\frac{1}{4}x + 1) - 1$?
11. The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?
12. What type(s) of transformation(s), if any, affect the range of a logarithmic function?
13. What type(s) of transformation(s), if any, affect the domain of a logarithmic function?
14. Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't x be zero?
15. Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

HOMEWORK

ALGEBRAIC

For the following exercises, state the domain and range of the function.

16. $f(x) = \log_3(x + 4)$
17. $h(x) = \ln\left(\frac{1}{2} - x\right)$
18. $g(x) = \log_5(2x + 9) - 2$
19. $h(x) = \ln(4x + 17) - 5$
20. $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

21. $f(x) = \log_b(x - 5)$
22. $g(x) = \ln(3 - x)$
23. $f(x) = \log(3x + 1)$
24. $f(x) = 3\log(-x) + 2$
25. $g(x) = -\ln(3x + 9) - 7$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

26. $f(x) = \ln(2 - x)$
27. $f(x) = \log\left(x - \frac{3}{7}\right)$
28. $h(x) = -\log(3x - 4) + 3$
29. $g(x) = \ln(2x + 6) - 5$
30. $f(x) = \log_3(15 - 5x) + 6$

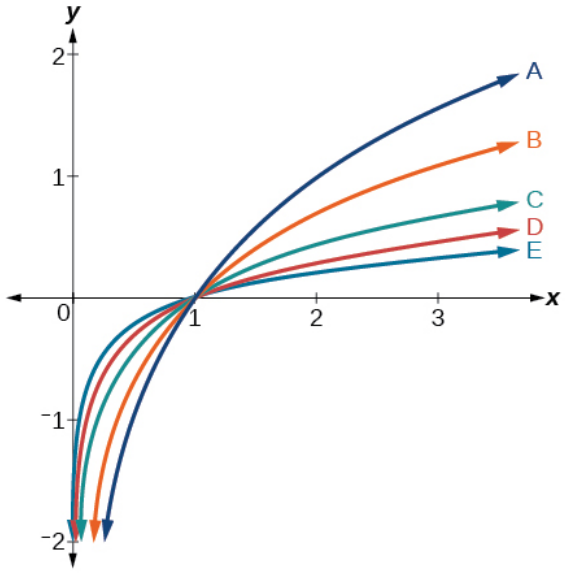
For the following exercises, state the domain, range, and x- and y -intercepts, if they exist. If they do not exist, write DNE.

31. $h(x) = \log_4(x - 1) + 1$
32. $f(x) = \log(5x + 10) + 3$
33. $g(x) = \ln(-x) - 2$
34. $f(x) = \log_2(x + 2) - 5$
35. $h(x) = 3\ln(x) - 9$

GRAPHICAL

For the following exercises, match each function in Figure 13.4.2.1 with the letter corresponding to its graph.

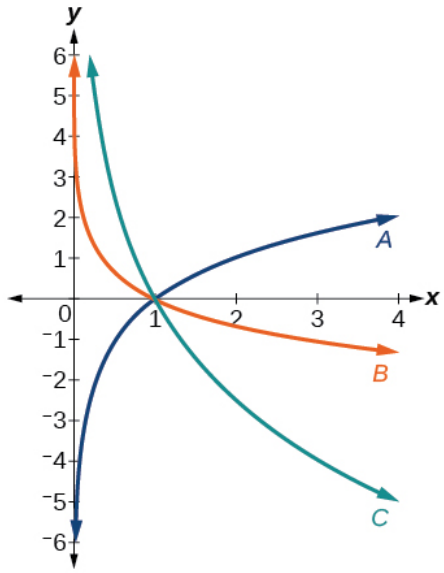
Figure 13.4.2.1



36. $d(x) = \log(x)$
37. $f(x) = \ln(x)$
38. $g(x) = \log_2(x)$
39. $h(x) = \log_5(x)$
40. $j(x) = \log_{25}(x)$

For the following exercises, match each function in Figure 13.4.2.2 with the letter corresponding to its graph.

Figure 13.4.2.2



41. $f(x) = \log_{1/3}(x)$
42. $g(x) = \log_2(x)$
43. $h(x) = \log_{3/4}(x)$

For the following exercises, sketch the graphs of each pair of functions on the same axis.

44. $f(x) = \log(x)$ and $g(x) = 10^x$
45. $f(x) = \log(x)$ and $g(x) = \log_{1/2}(x)$
46. $f(x) = \log_4(x)$ and $g(x) = \ln(x)$
47. $f(x) = e^x$ and $g(x) = \ln(x)$

For the following exercises, match each function in Figure 13.4.2.3 with the letter corresponding to its graph.

Figure 13.4.2.3

48. $f(x) = \log_4(-x + 2)$
49. $g(x) = -\log_4(x + 2)$
50. $h(x) = \log_4(x + 2)$

For the following exercises, sketch the graph of the indicated function.

51. $f(x) = \log_2(x + 2)$
52. $f(x) = 2\log(x)$
53. $f(x) = \ln(-x)$
54. $g(x) = \log(4x + 16) + 4$
55. $g(x) = \log(6 - 3x) + 1$
56. $h(x) = -\frac{1}{2}\ln(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

57. Use $y = \log_2(x)$ as the parent function.
58. Use $f(x) = \log_3(x)$ as the parent function.
59. Use $f(x) = \log_4(x)$ as the parent function.
60. Use $f(x) = \log_5(x)$ as the parent function.

TECHNOLOGY

For the following exercises, use graphing technology to find approximate solutions to each equation.

61. $\log(x - 1) + 2 = \ln(x - 1) + 2$
62. $\log(2x - 3) + 2 = -\log(2x - 3) + 5$
63. $\ln(x - 2) = -\ln(x + 1)$
64. $2\ln(5x + 1) = \frac{1}{2}\ln(-5x) + 1$
65. $\frac{1}{3}\log(1 - x) = \log(x + 1) + \frac{1}{3}$

EXTENSIONS

66. Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.
67. Explore and discuss the graphs of $f(x) = \log_{1/2}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.
68. Prove the conjecture made in the previous exercise.
69. What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.
70. Use properties of exponents to find the x-intercepts of the function $f(x) = \log(x^2 + 4x + 4)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

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13.5: Laws of Logarithms

Learning Objectives

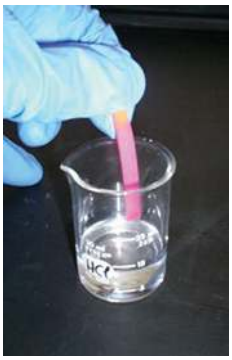
- Use the Product Law for logarithms.
- Use the Quotient Law for logarithms.
- Use the Power Law for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the Change-of-Base Formula for logarithms.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 13.5.1

The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)



In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where H^+, is the concentration of hydrogen ion in the solution

$$\text{pH} = -\log([H^+]) = \log\left(\frac{1}{[H^+]}\right).$$

The equivalence of $-\log([H^+])$ and $\log\left(\frac{1}{[H^+]}\right)$, is one of the Laws of Logarithms we will examine in this section.

Using the Product Law for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar laws to exponents. Some important laws of logarithms are given here. First, the following "Base Laws" are easy to prove.

Theorem: Laws of Logarithms - Base Laws

$$\log_b(1) = 0 \quad \text{and} \quad \log_b(b) = 1.$$

Proofs

By definition of a logarithm, $\log_b(1)$ is the exponent, p, for which $b^p = 1$. Therefore, $p = 0$. As such,

$$\log_b(1) = 0.$$

Similarly, $\log_b(b)$ is the exponent, q, such that $b^q = b$. Since $b = b^1$, this means that $b^q = b^1$. In other words, $q = 1$. Hence,

$$\log_b(b) = 1.$$

For example, $\log_5 1 = 0$, since $5^0 = 1$, and $\log_5 5 = 1$, since $5^1 = 5$.

Next, we have the Inverse Laws.

Theorem: Laws of Logarithms - Inverse Laws

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b(x)} = x, \text{ where } x > 0.$$

Proofs

Consider $\log_b(b^x)$. By the definition of a logarithm, this is the exponent, p, such that $b^p = b^x$. That is, $x = p$. Thus, $\log_b(b^x) = x$.

Now consider $b^{\log_b(x)}$, where $x > 0$. Recall that $\log_b(x)$ is the exponent, q, such that $b^q = x$. That is,

$$b^{\log_b(x)} = b^q = x.$$

To illustrate these laws, we imagine evaluating $\log(100)$. We can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the Inverse Law $\log_b(b^x) = x$, to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the Inverse Law $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the One-to-One Law.

Theorem: Laws of Logarithms - One-to-One Law

$$\log_b(M) = \log_b(N) \text{ if and only if } M = N.$$

Proof

From the definition of a logarithm, if we let $\log_b(M) = m$ and $\log_b(N) = n$, then $b^m = M$ and $b^n = N$.

($\Longrightarrow$): If $\log_b(M) = \log_b(N)$, then $m = n$. However, if $m = n$, then $b^m = b^n$. Therefore, $M = N$.

($\Longleftarrow$): If $M = N$, then $b^m = b^n$. However, by laws of exponents, this means $m = n$. Hence, $\log_b(M) = \log_b(N)$.

We can use the One-to-One Law to solve the equation $\log_3(3x) = \log_3(2x + 5)$, for x. Since the bases are the same, we can apply the One-to-One Law by setting the arguments equal and solving for x:

$$\begin{aligned} 3x &= 2x + 5 && \text{(setting the arguments equal to each other)} \\ \Longrightarrow \quad x &= 5 && \text{(subtracting } 2x \text{ from both sides)} \end{aligned}$$

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The One-to-One Law does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *Product Law of Exponents* to combine the product of powers by adding exponents: $x^a x^b = x^{a+b}$. We have a similar law for logarithms, called the **Product Law for Logarithms**.

Theorem: Laws of Logarithms - Product Law

Given any real number x, and positive real numbers M and N,

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Proof

Let $m = \log_b(M)$, and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{(substituting)} \\ &= \log_b(b^{m+n}) && \text{(Laws of Exponents: Product Rule)} \\ &= m+n && \text{(Laws of Logarithms: Inverse Law)} \\ &= \log_b(M) + \log_b(N) && \text{(substituting)} \end{aligned}$$

Note that repeated applications of the Product Law for Logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the Product Law for Logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b(w) + \log_b(x) + \log_b(y) + \log_b(z).$$

Example 13.5.1: Using the Product Law for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution

We begin by recognizing the argument as a product of three factors.

$$\log_3(30x(3x+4)) = \log_3(30 \cdot x \cdot (3x+4)).$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30 \cdot x \cdot (3x+4)) = \log_3(30) + \log_3(x) + \log_3(3x+4).$$

Finally, we need to be aware of the base and the possibiliy that the 30 can be factored into a power of the base and some other constant. Specifically, $30 = 3 \cdot 10$. Therefore,

$$\begin{aligned} \log_3(30x(3x+4)) &= \log_3(3 \cdot 10) + \log_3(x) + \log_3(3x+4) \\ &= \log_3(3) + \log_3(10) + \log_3(x) + \log_3(3x+4) && \text{(Laws of Logorithms: Product Law)} \\ &= 1 + \log_3(10) + \log_3(x) + \log_3(3x+4) && \text{(Laws of Logorithms: } \log_b b = 1\text{)} \end{aligned}$$

Checkpoint 13.5.1

Expand $\log_b(8k)$.

Using the Quotient Law for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *Quotient Law of Exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b} = x^{a-b}$. The **Quotient Law for Logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the Product Law, we can use the Inverse Law to derive the Quotient Law.

Theorem: Laws of Logarithms - Quotient Law

Given any real number x, and positive real numbers M and N,

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Proof

Let $m = \log_b(M)$, and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$, and $b^n = N$. It follows that

$$\begin{aligned} \log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{(substituting)} \\ &= \log_b\left(b^{m-n}\right) && \text{(Laws of Exponents: Quotient Law)} \\ &= m-n && \text{(Laws of Exponents: Inverse Law)} \\ &= \log_b(M)-\log_b(N) && \text{(substituting)} \end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned} \log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{(factoring out the GCF from numerator and denominator)} \\ &= \log\left(\frac{2x\ \cancel{(x+3)}}{3\ \cancel{(x+3)}}\right) && \text{(canceling like factors)} \\ &= \log\left(\frac{2x}{3}\right) && \text{(simplifying)} \end{aligned}$$

Next we apply the Quotient Law by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the Product Law.

$$\log\left(\frac{2x}{3}\right) = \log(2x) - \log(3) = \log(2) + \log(x) - \log(3).$$

Caution: Assumptions Abound

When using the Product and Quotient Laws of Logarithms, we are *assuming* (often falsely) that the component logarithms are well-defined. That is, when expanding $\log\left(\frac{2x^2+6x}{3x+9}\right)$ to arrive at $\log(2)+\log(x)-\log(3)$, we are assuming

1. $x > 0$, and
2. $x+3 \neq 0$ (remember, this is the factor we canceled in the process of simplifying the logarithm).

These assumptions are likely *not correct*! However, when *practicing* the Laws of Logarithms, we allow ourselves the "training wheels" of assuming everything is going to work out. When we start to solve equations involving logarithms, on the other hand, we remove those "training wheels" and become concerned with the possibility that the component logarithms are not well-defined.

Example 13.5.2: Using the Quotient Law for Logarithms

Expand

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right).$$

Solution

First we note that the quotient is factored and in lowest terms, so we apply the Quotient Law.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x)).$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the Product Law, noting that 15 is not divisible by a power of the base (2).

$$\begin{aligned} \log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= \log_2(15)+\log_2(x)+\log_2(x-1)-[\log_2(3x+4)+\log_2(2-x)] && \text{(Product Law)} \\ &= \log_2(15)+\log_2(x)+\log_2(x-1)-\log_2(3x+4)-\log_2(2-x) && \text{(distributing)} \end{aligned}$$

Let's restate our "Caution" from above. There are exceptions to consider in Example 13.5.2. First, because denominators must never be zero, the expression in Example 13.5.2 is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive,

we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

? Checkpoint 13.5.2

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Using the Power Law for Logarithms

We've explored the Product Law and the Quotient Law, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned} \log_b(x^2) &= \log_b(x \cdot x) && \text{(Laws of Exponents)} \\ &= \log_b x + \log_b x && \text{(Laws of Logarithms: Product Law)} \\ &= 2\log_b x && \text{(combining like terms)} \end{aligned}$$

Notice that we used the Product Law for Logarithms to find a solution for the example above. By doing so, we have derived the **Power Law for Logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{1/2} \quad \frac{1}{e} = e^{-1}.$$

Laws of Logarithms: Power Law

$$\log_b(M^p) = p\log_b(M).$$

Proof

Let $\log_b(M) = m$.

$$\begin{aligned} \implies b^m &= M && \text{(definition of the logarithm)} \\ \implies (b^m)^p &= M^p && \text{(Laws of Exponents)} \\ \implies b^{mp} &= M^p && \text{(Laws of Exponents)} \\ \implies \log_b\left(b^{mp}\right) &= \log_b\left(M^p\right) && \text{(One-to-One Law of Logarithms)} \\ \implies \log_b\left(b^m \cdot b^p\right) &= \log_b\left(M^p\right) && \text{(Laws of Exponents)} \\ \implies \log_b\left(b^m\right) \cdot \log_b\left(b^p\right) &= \log_b\left(M^p\right) && \text{(Product Law of Logarithms)} \\ \implies m \cdot p &= \log_b\left(M^p\right) && \text{(Base Laws of Logarithms)} \\ \implies \log_b(M) \cdot p &= \log_b\left(M^p\right) && \text{(our original substitution } \log_b(M) = m\text{)} \\ \implies p\log_b(M) &= \log_b\left(M^p\right) && \text{(Commutative Property of Multiplication)} \end{aligned}$$

✓ Example 13.5.3: Expanding a Logarithm with Powers

Expand $\log_2(x^5)$.

Solution

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

? Checkpoint 13.5.3

Expand $\ln(x^2)$.

✓ Example 13.5.4: Rewriting an Expression as a Power before Using the Power Law

Expand $\log_3(25)$, using the Power Law for Logs.

Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(5^2) = 2\log_3(5)$$

? Checkpoint 13.5.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

✓ Example 13.5.5: Using the Power Law in Reverse

Rewrite $4\ln(x)$, using the Power Law for logs to a single logarithm with a leading coefficient of 1.

Solution

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

? Checkpoint 13.5.5

Rewrite $2\log_3 4$, using the Power Law for logs to a single logarithm with a leading coefficient of 1.

Expanding Logarithmic Expressions

Taken together, the laws introduced in this section so far are collectively called the "Laws of Logarithms." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the Power Law to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the Quotient Law for Logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the Product Law to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots — never with addition or subtraction inside the argument of the logarithm.

Example 13.5.6: Expanding Logarithms Using Product, Quotient, and Power Laws

Rewrite $\ln\left(\frac{x^4y}{7}\right)$, as a sum or difference of logs.

Solution

First, because we have a quotient of two expressions, we can use the Quotient Law:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7).$$

Then seeing the product in the first term, we use the Product Law:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7).$$

Finally, we use the Power Law on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7).$$

Checkpoint 13.5.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Example 13.5.7: Using the Power Law for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(\sqrt{x})$.

Solution

$$\log(\sqrt{x}) = \log x^{1/2} = \frac{1}{2}\log x$$

Checkpoint 13.5.7

Expand $\ln(\sqrt[3]{x^2})$.

Q&A

Can we expand $\ln(x^2+y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example 13.5.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{2x-1}\right)$.

Solution

We can expand by applying the Product and Quotient Laws.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{2x-1}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1)\\ &= \log_6 64 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1)\end{aligned}$$

Note that we *could* have rewritten $\log_6 64$ as $\log_6 2^6 = 6\log_6 2$, however, this is not any better than $\log_6 64$. When it comes to logarithms of constants, we are only concerned with the possibility that a power of the base is a factor of the constant.

? Checkpoint 13.5.8

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{x^2-9}\right)$.

Condensing Logarithmic Expressions

We can use the Laws of Logarithms to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that *the logarithms must have the same base to be combined*. We will learn later how to change the base of any logarithm before condensing. Because students sometimes make mistakes on the order with which they apply the Laws of Logarithms, we provide the following "How To" as a beginner's reference guide.

How To

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the Power Law first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the Product Law. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the Quotient Law last. Rewrite differences of logarithms as the logarithm of a quotient.

✓ Example 13.5.9: Using the Product and Quotient Laws to Combine Logarithms

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution

Using the product and Quotient Laws

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40).$$

This reduces our original expression to

$$\log_3(40)-\log_3(2).$$

Then, using the Quotient Law

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20).$$

? Checkpoint 13.5.9

Condense $\log 3-\log 4+\log 5-\log 6$.

✓ Example 13.5.10: Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)$.

Solution

We apply the Power Law first:

$$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)=\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6).$$

Next we apply the Product Law to the sum:

$$\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)=\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6).$$

Finally, we apply the Quotient Law to the difference:

$$\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)=\log_2\left(\frac{x^2\sqrt{x-1}}{(x+3)^6}\right).$$

? Checkpoint 13.5.10

Rewrite $\log(5)+0.5\log(x)-\log(7x-1)+3\log(x-1)$ as a single logarithm.

✓ Example 13.5.11: Rewriting as a Single Logarithm

Rewrite $2\log x-4\log(x+5)+\frac{1}{x}\log(3x+5)$ as a single logarithm.

Solution

We apply the Power Law first:

$$2\log x-4\log(x+5)+\frac{1}{x}\log(3x+5)=\log(x^2)-\log(x+5)^4+log\left((3x+5)^{x^{-1}}\right).$$

Next, we rearrange and apply the Product Law to the sum:

$$\begin{aligned}\log(x^2)-\log(x+5)^4+\log\left((3x+5)^{x^{-1}}\right) &= \log(x^2)+\log\left((3x+5)^{x^{-1}}\right)-\log(x+5)^4\\ &= \log\left(x^2(3x+5)^{x^{-1}}\right)-\log(x+5)^4\end{aligned}$$

Finally, we apply the Quotient Law to the difference:

$$=\log\left(x^2(3x+5)^{x^{-1}}\right)-\log(x+5)^4=\log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4}.$$

? Checkpoint 13.5.11

Condense $4(3\log(x)+\log(x+5)-\log(2x+3))$.

✓ Example 13.5.12: Applying of the Laws of Logs

Recall that, in chemistry, $\text{pH}=-\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C, is the original concentration of hydrogen ions, and P, is the original pH of the liquid. Then $P=-\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$\text{pH}=-\log(2C).$$

Using the Product Law of logs

$$\text{pH}=-\log(2C)=-(\log(2)+\log(C))=-\log(2)-\log(C).$$

Since $P=-\log(C)$, the new pH is

$$\text{pH}=P-\log(2)\approx P-0.301.$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

? Checkpoint 13.5.12

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **Change-of-Base Formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

Theorem: Change-of-Base Formula

Given any positive real numbers M, b, and n, where $n \neq 1$, and $b \neq 1$,

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)}.$$

Proof

Let $y = \log_b(M)$. By the definition of a logarithm, $b^y = M$. It follows that

$$\begin{aligned} & \log_n(b^y) = \log_n(M) && \text{(One-to-One Law of Logarithms)} \\ \Longrightarrow\ & y\log_n(b) = \log_n(M) && \text{(Power Law of Logarithms)} \\ \Longrightarrow\ & y = \frac{\log_n(M)}{\log_n(b)} && \text{(dividing both sides by } \log_n(b)\text{)} \\ \Longrightarrow\ & \log_b(M) = \frac{\log_n(M)}{\log_n(b)} && \text{(substituting)} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\log_5 36 = \frac{\log(36)}{\log(5)} \approx 2.2266.$$

Example 13.5.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5(3)$ to a quotient of natural logarithms.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the Change-of-Base Formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_5 3 = \frac{\ln 3}{\ln 5}.$$

Checkpoint 13.5.13

Change $\log_{0.5}(8)$ to a quotient of natural logarithms.

Q&A

Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$, means $\log_{10} 9$. So, $\log 9 = \frac{\ln 9}{\ln 10}$.

Example 13.5.14: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$, using the Change-of-Base Formula with a calculator.

Solution

According to the Change-of-Base Formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2 10 = \frac{\ln 10}{\ln 2} \approx 3.3219.$$

Checkpoint 13.5.14

Evaluate $\log_5(100)$, using the Change-of-Base Formula.

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13.5.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Laws of Logarithms
  - Intermediate Algebra - Logarithms: Properties of Logarithms
  - Logarithmic Properties - A Brief Review of the Properties of Logarithms
  - Logarithmic Functions - A Brief Review of Properties of Logarithms
  - Logarithmic Properties - Using Properties of Logarithms
- Expanding and condensing logarithmic expressions
  - Logarithmic Functions - Expanding and Combining Logarithms

KEY CONCEPTS

THEOREMS

- **Laws of Logarithms - Base Laws**:
  - $\log_b(1) = 0$
  - $\log_b(b) = 1$
- **Laws of Logarithms - Inverse Laws**:
  - $\log_b(b^x) = x$
  - $b^{\log_b(x)} = x$, where $x > 0$
- **Laws of Logarithms - One-to-One Law**: $\log_b(M) = \log_b(N)$ if and only if $M = N$.
- **Laws of Logarithms - Product Law**: Given any real number x, and positive real numbers M and N, $\log_b(MN) = \log_b(M) + \log_b(N)$.
- **Laws of Logarithms - Quotient Law**: Given any real number x, and positive real numbers M and N, $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$.
- **Laws of Logarithms - Power Law**: For any real number p and positive real number M, $\log_b(M^p) = p\log_b(M)$.
- **Change-of-Base Formula**: Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, $\log_b(M) = \frac{\log_n(M)}{\log_n(b)}$.

COMMON MISTAKES

- **Incorrectly Applying Logarithm Laws**:
  - $\log_b(M + N) \neq \log_b(M) + \log_b(N)$ (Log of a sum is not sum of logs).
  - $\log_b(M - N) \neq \log_b(M) - \log_b(N)$ (Log of a difference is not difference of logs).
  - $\frac{\log_b(M)}{\log_b(N)} \neq \log_b(M) - \log_b(N)$ (Quotient of logs is not log of a quotient).
  - $(\log_b(M))^p \neq p\log_b(M)$ (Power of a log is not product of exponent and log, unless the power is on the argument).
- **Errors in Expanding Logarithms**: Not fully expanding an expression, or incorrectly applying laws (e.g., distributing a log over a sum/difference inside the argument).
- **Errors in Condensing Logarithms**:
  - Incorrectly combining terms, especially with subtraction (order matters for division).
  - Not applying the power law first to make coefficients into exponents before combining with product/quotient laws.
  - Combining logarithms with different bases.
- **Ignoring Domain Restrictions (Conceptual vs. Practical Application)**: While the "Caution: Assumptions Abound" notes that for practice with laws, well-definedness is assumed, in solving equations or other applications, the domain of each logarithmic expression must be considered.
- **Errors with Change-of-Base Formula**: Incorrectly setting up the ratio, e.g., $\frac{\log_n(b)}{\log_n(M)}$ instead of $\frac{\log_n(M)}{\log_n(b)}$.
- **Calculator Errors with Change-of-Base**: Using the wrong base on the calculator or inputting the fraction incorrectly.

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13.5.2: HOMEWORK

READING QUESTIONS

1. What is the Product Law for Logarithms? State the formula.
2. What is the Quotient Law for Logarithms? State the formula.
3. What is the Power Law for Logarithms? State the formula.
4. How does the Power Law for Logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?
5. When expanding a complex logarithmic expression, what is the general order in which you might apply the Product, Quotient, and Power Laws?
6. When condensing multiple logarithmic terms into a single logarithm, what must be true about the bases of the logarithms?
7. When condensing an expression like $2\log(x)+3\log(y)-\log(z)$, which law should generally be applied first to the terms $2\log(x)$ and $3\log(y)$?
8. What is the Change-of-Base Formula for logarithms? Why is it particularly useful when using a calculator?
9. Rewrite $\log_5(36)$ as a quotient of common logarithms (base 10).
10. Rewrite $\log_5(3)$ as a quotient of natural logarithms (base e).
11. Can $\log_b(A+C)$ be expanded using the Product Law for Logarithms? Why or why not?
12. How does the Power Law for Logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?
13. What does the change-of-base formula do? Why is it useful when using a calculator?

HOMEWORK

ALGEBRAIC

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

14. $\log_b(7x\cdot 2y)$
15. $\ln(3ab\cdot 5c)$
16. $\log_b\left(\frac{13}{17}\right)$
17. $\log_4\left(\frac{x/z}{w}\right)$
18. $\ln\left(\frac{1}{4^k}\right)$
19. $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

20. $\ln(7)+\ln(x)+\ln(y)$
21. $\log_3(2)+\log_3(a)+\log_3(11)+\log_3(b)$
22. $\log_b(28)-\log_b(7)$
23. $\ln(a)-\ln(d)-\ln(c)$
24. $-\log_b\left(\frac{1}{7}\right)$
25. $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

26. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$
27. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$
28. $\log(\sqrt{x^3y^{-4}})$
29. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$
30. $\log(x^2y^3\sqrt[3]{x^2y^5})$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

31. $\log(2x^4)+\log(3x^5)$
32. $\ln(6x^9)-\ln(3x^2)$
33. $2\log(x)+3\log(x+1)$
34. $\log(x)-\frac{1}{2}\log(y)+3\log(z)$
35. $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

36. $\log_7(15)$ to base e
37. $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6)=a$ and $\log_5(11)=b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

38. $\log_{11}(5)$
39. $\log_6(55)$
40. $\log_{11}\left(\frac{6}{11}\right)$

NUMERIC

For the following exercises, use properties of logarithms to evaluate without using a calculator.

41. $\log_3\left(\frac{1}{9}\right)-3\log_3(3)$
42. $6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$
43. $2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

44. $\log_3(22)$
45. $\log_8(65)$
46. $\log_6(5.38)$
47. $\log_4\left(\frac{15}{2}\right)$
48. $\log_{1/2}(4.7)$

EXTENSIONS

49. Use the Product Law of Logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$. Show the steps for solving.
50. Use the Quotient Law of Logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$. Show the steps for solving.
51. Can the Power Law of Logarithms be derived from the Power Law of Exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.
52. Prove that $\log_b(n)=\frac{1}{\log_n(b)}$ for any positive integers $b>1$ and $n>1$.
53. Does $\log_{81}(2401)=\log_3(7)$? Verify the claim algebraically.

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13.6: Exponential and Logarithmic Equations

Learning Objectives

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the One-to-One Property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 13.6.1

Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall the following theorem from Algebra (the proof of which requires recalling that logarithms "undo" exponential functions).

Theorem: One-to-One Property of Exponents

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T.$$

Proof

Let S and T be algebraic expressions, and b be a positive real number not equal to 1.

$$
\begin{aligned}
& & b^S &= b^T & \\
\iff & \ln(b^S) &= \ln(b^T) & \text{(taking the natural log of both sides)} \\
\iff & S\ln(b) &= T\ln(b) & \text{(Laws of Logarithms)} \\
\iff & S &= T & \text{(dividing both sides by } \ln(b)\text{)}
\end{aligned}
$$

In other words, when an exponential equation has the same base on each side, the exponents must be equal. Therefore, we can solve many exponential equations by using the Laws of Exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \frac{3^{2x}}{3}$. To solve for x, we use the Quotient Rule for Exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the One-to-One Property of exponents by setting the exponents equal to one another and solving for x:

$$
\begin{aligned}
& 3^{4x-7} &= \frac{3^{2x}}{3} & \\
\implies & 3^{4x-7} &= \frac{3^{2x}}{3^1} & (3 = 3^1) \\
\implies & 3^{4x-7} &= 3^{2x-1} & \text{(Laws of Exponents: Quotient Rule)} \\
\implies & 4x-7 &= 2x-1 & \text{(One-to-One Property of Exponents)} \\
\implies & 2x &= 6 & \text{(subtracting } 2x \text{ and adding 7 to both sides)} \\
\implies & x &= 3 & \text{(dividing both sides by 2)}
\end{aligned}
$$

Example 13.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution

$$
\begin{aligned}
& 2^{x-1} &= 2^{2x-4} & \\
\implies & x-1 &= 2x-4 & \text{(One-to-One Property)} \\
\implies & 3 &= x & \text{(subtracting } x \text{ and adding 4 to both sides)}
\end{aligned}
$$

Checkpoint 13.6.1

Solve $5^{2x} = 5^{3x+2}$.

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the One-to-One Property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the Laws of Exponents, along with the One-to-One Property, to solve for x:

$$
\begin{aligned}
& 256 &= 4^{x-5} & \\
\implies & 4^4 &= 4^{x-5} & \text{(rewriting both sides in the same base)} \\
\implies & 4 &= x-5 & \text{(One-to-One Property)} \\
\implies & 9 &= x & \text{(adding 5 to both sides)}
\end{aligned}
$$

Example 13.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Solution

$$
\begin{aligned}
8^{x+2} &= 16^{x+1} \\
\Rightarrow \quad (2^3)^{x+2} &= (2^4)^{x+1} && \text{(rewriting both sides with the same base)} \\
\Rightarrow \quad 2^{3(x+2)} &= 2^{4(x+1)} && \text{(Laws of Exponents)} \\
\Rightarrow \quad 2^{3x+6} &= 2^{4x+4} && \text{(distributing)} \\
\Rightarrow \quad 3x+6 &= 4x+4 && \text{(One-to-One Property)} \\
\Rightarrow \quad 2 &= x && \text{(subtracting } 3x \text{ and 4 from both sides)}
\end{aligned}
$$

Checkpoint 13.6.2

Solve $5^{2x} = 25^{3x+2}$.

Example 13.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Solution

$$
\begin{aligned}
2^{5x} &= \sqrt{2} \\
\Rightarrow \quad 2^{5x} &= 2^{1/2} && \text{(rewriting the radical as a rational exponent)} \\
\Rightarrow \quad 5x &= \frac{1}{2} && \text{(One-to-One Property)} \\
\Rightarrow \quad x &= \frac{1}{10} && \text{(dividing both sides by 5)}
\end{aligned}
$$

Checkpoint 13.6.3

Solve $5^x = \sqrt{5}$.

Q&A

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Example 13.6.4: Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

Solution

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Figure 13.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

Figure 13.6.2

? Checkpoint 13.6.4

Solve $2^x = -100$.

Solving Exponential Equations Using Logarithms

Most of the time the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

✓ Example 13.6.5: Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Solution

$$
\begin{aligned}
& 5^{x+2} = 4^x && \text{(notice there is no common base between 5 and 4)} \\
\Longrightarrow\ & \ln\left(5^{x+2}\right) = \ln\left(4^x\right) && \text{(take the natural log of both sides)} \\
\Longrightarrow\ & (x+2)\ln(5) = x\ln(4) && \text{(Laws of Logarithms)} \\
\Longrightarrow\ & x\ln(5) + 2\ln(5) = x\ln(4) && \text{(distributing)} \\
\Longrightarrow\ & x\ln(5) - x\ln(4) = -2\ln(5) && \text{(moving all terms containing } x \text{ to one side)} \\
\Longrightarrow\ & x(\ln(5) - \ln(4)) = -2\ln(5) && \text{(factoring out } x\text{)} \\
\Longrightarrow\ & x = \frac{-2\ln(5)}{\ln(5) - \ln(4)} && \text{(dividing both sides by } \ln(5) - \ln(4)\text{)}
\end{aligned}
$$

Some could correctly argue that we could further simplify the final expression using the Laws of Logarithms.

$$x = \frac{-2\ln(5)}{\ln(5) - \ln(4)} = \frac{\ln\left(5^{-2}\right)}{\ln\left(\frac{5}{4}\right)}.$$

However, this is as correct as $\frac{-2\ln(5)}{\ln(5)-\ln(4)}$. In fact, you could go even further in your simplification to

$$x = \frac{\ln\left(5^{-2}\right)}{\ln\left(\frac{5}{4}\right)} = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}.$$

Be sure to check with your instructor as to their preference, but my guess is that any of these is acceptable as equivalent.

Checkpoint 13.6.5

Solve $2^x = 3^{x+1}$.

Q&A

Is there any way to solve $2^x = 3^x$ without logarithms?

Yes. The solution is 0.

Equations Containing e

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in Mathematics, in Science, in Engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

Example 13.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution

$$\begin{aligned} & 100 = 20e^{2t} & \\ \Longrightarrow \quad & 5 = e^{2t} & \text{(dividing both sides by 20)} \\ \Longrightarrow \quad & \ln(5) = \ln\left(e^{2t}\right) & \text{(taking the natural log of both sides)} \\ \Longrightarrow \quad & \ln(5) = 2t & \text{(Laws of Logarithms)} \\ \Longrightarrow \quad & \frac{\ln(5)}{2} = t & \text{(dividing both sides by 2)} \end{aligned}$$

Using Laws of Logarithms, we can also write this answer in the form $t = \ln\sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

Checkpoint 13.6.6

Solve $3e^{0.5t} = 11$.

Q&A

Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$, and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

Example 13.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution

$$
\begin{aligned}
4e^{2x}+5 &= 12 && \\
\Longrightarrow \quad 4e^{2x} &= 7 && \text{(subtracting 5 from both sides)} \\
\Longrightarrow \quad e^{2x} &= \frac{7}{4} && \text{(dividing both sides by 4)} \\
\Longrightarrow \quad \ln\left(e^{2x}\right) &= \ln\left(\frac{7}{4}\right) && \text{(taking the natural log of both sides)} \\
\Longrightarrow \quad 2x &= \ln\left(\frac{7}{4}\right) && \text{(Laws of Logarithms)} \\
\Longrightarrow \quad x &= \frac{\ln(7/4)}{2} && \text{(dividing both sides by2)}
\end{aligned}
$$

? Checkpoint 13.6.7

Solve $3+e^{2t}=7e^{2t}$.

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an extraneous solution, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

✓ Example 13.6.8: Solving Exponential Functions in Quadratic Form

Solve $e^{2x}-e^{x}=56$.

Solution

$$
\begin{aligned}
e^{2x}-e^{x} &= 56 && \\
\Longrightarrow \quad e^{2x}-e^{x}-56 &= 0 && \text{(subtracting 56 from both sides)} \\
\Longrightarrow \quad (e^{x})^{2}-e^{x}-56 &= 0 && \text{(Laws of Exponents)} \\
\Longrightarrow \quad u^{2}-u-56 &= 0 && \text{(substituting } u=e^{x}\text{)} \\
\Longrightarrow \quad (u-8)(u+7) &= 0 && \text{(factoring the trinomial)}
\end{aligned}
$$

Therefore, $u=8$ or $u=-7$. Resubstituting $u=e^{x}$, we get

$$e^{x}=8 \quad \text{or} \quad e^{x}=-7.$$

Since $e^{x}>0$, we throw out the possibility that $e^{x}=-7$. To solve $e^{x}=8$, we take the natural log of both sides to get $x=\ln(8)$.

When we plan to use factoring to solve a problem, as we did in Example 13.6.8, we always get zero on one side of the equation. This is because zero has the unique property that when a product is zero, one or both of the factors must be zero. We rejected the equation $e^{x}=-7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

? Checkpoint 13.6.8

Solve $e^{2x}=e^{x}+2$.

Q&A

Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the Laws of Logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use Laws of Logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned} & \log_2(2) + \log_2(3x-5) = 3 & \\ \Longrightarrow \quad & \log_2(2(3x-5)) = 3 & \text{(Laws of Logarithms)} \\ \Longrightarrow \quad & \log_2(6x-10) = 3 & \text{(distributing)} \\ \Longrightarrow \quad & 6x - 10 = 2^3 & \text{(definition of a logarithm)} \\ \Longrightarrow \quad & 6x - 10 = 8 & \text{(simplifying)} \\ \Longrightarrow \quad & 6x = 18 & \text{(adding 10 to both sides)} \\ \Longrightarrow \quad & x = 3 & \text{(dividing both sides by 6)} \end{aligned}$$

As you can see, when solving a logarithmic equation, we rely on the following definition of logarithms:

For any algebraic expression S and real numbers b and c, where $b > 0$ and $b \neq 1$,
$\log_b(S) = c$ if and only if $b^c = S$.

Example 13.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln(x) + 3 = 7$.

Solution

$$\begin{aligned} & 2\ln(x) + 3 = 7 & \\ \Longrightarrow \quad & 2\ln(x) = 4 & \text{(subtracting 3 from both sides)} \\ \Longrightarrow \quad & \ln(x) = 2 & \text{(dividing both sides by 2)} \\ \Longrightarrow \quad & x = e^2 & \text{(rewriting in exponential form)} \end{aligned}$$

Checkpoint 13.6.9

Solve $6 + \ln x = 10$.

Example 13.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

Solution

$$\begin{aligned} & 2\ln(6x) = 7 & \\ \Longrightarrow \quad & \ln(6x) = \frac{7}{2} & \text{(dividing both sides by 2)} \\ \Longrightarrow \quad & 6x = e^{7/2} & \text{(rewriting in exponential form)} \\ \Longrightarrow \quad & x = \frac{1}{6}e^{7/2} & \text{(dividing both sides by 6)} \end{aligned}$$

? Checkpoint 13.6.10

Solve $2\ln(x+1)=10$.

✓ Example 13.6.11: Showcasing the Need to Check Solutions

Solve $\log_7(1-2x)=1-\log_7(3-x)$.

Solution

$$\begin{aligned}
&\log_7(1-2x) &&= 1-\log_7(3-x) && \\
\Longrightarrow\ &\log_7(1-2x)+\log_7(3-x) &&= 1 && \text{(adding } \log_7(3-x) \text{ to both sides)}\\
\Longrightarrow\ &\log_7\left((1-2x)(3-x)\right) &&= 1 && \text{(Laws of Logarithms)}\\
\Longrightarrow\ &(1-2x)(3-x) &&= 7^1 && \text{(rewriting in exponential form)}\\
\Longrightarrow\ &2x^2-7x+3 &&= 7 && \text{(distributing)}\\
\Longrightarrow\ &2x^2-7x-4 &&= 0 && \text{(subtracting 7 from both sides)}\\
\Longrightarrow\ &(2x+1)(x-4) &&= 0 && \text{(factoring the trinomial)}
\end{aligned}$$

Therefore, either $x=-\frac{1}{2}$ or $x=4$. We now check our candidate solutions (as we always must do when solving logarithmic equations).

Substituting $x=-\frac{1}{2}$ into the original equation, we get

$$\begin{aligned}
&\log_7\left(1-2\left(-\frac{1}{2}\right)\right) &&\stackrel{?}{=} 1-\log_7\left(3-\left(-\frac{1}{2}\right)\right) && \text{(substituting)}\\
\Longrightarrow\ &\log_7(1+1) &&\stackrel{?}{=} 1-\log_7\left(\frac{7}{2}\right) && \text{(simplifying)}\\
\Longrightarrow\ &\log_7(2) &&\stackrel{?}{=} 1-(\log_7(7)-\log_7(2)) && \text{(simplifying and Laws of Logarithms)}\\
\Longrightarrow\ &\log_7(2) &&\stackrel{?}{=} 1-\log_7(7)+\log_7(2) && \text{(distributing)}\\
\Longrightarrow\ &\log_7(2) &&\stackrel{?}{=} 1-1+\log_7(2) && \text{(Laws of Logarithms)}\\
\Longrightarrow\ &\log_7(2) &&\stackrel{\checkmark}{=} \log_7(2) && \text{(simplifying)}
\end{aligned}$$

Thus, $x=-\frac{1}{2}$ is a solution.

Substituting $x=4$ into the original equation, we get the following.

$$\begin{aligned}
&\log_7(1-2(4)) &&\stackrel{?}{=} 1-\log_7(3-4)\\
\Longrightarrow\ &\log_7(-7) &&\stackrel{?}{=} \log_7(-1)
\end{aligned}$$

We can stop at this point and confidently state that $x=4$ is an extraneous solution.

WARNING: Our determination that $x=4$ is extraneous is *not* because $-7\neq -1$. Instead, it is due to the fact that the argument of a logarithm can never be negative!

Hence, the only solution to the equation is $-\frac{1}{2}$.

✓ Example 13.6.12: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x=3$ and use a graph to interpret the solution.

Solution

$$\ln x = 3 \implies x = e^3.$$

Figure 13.6.3 represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

Figure 13.6.3

The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.

? Checkpoint 13.6.12

Use graphing technology to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the One-to-One Property to solve logarithmic equations. The One-to-One Property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x, and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use the Laws of Logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the One-to-One Property to solve for x:

$$\begin{aligned}
& \log(3x-2) - \log(2) = \log(x+4) && \\
\implies & \log\left(\frac{3x-2}{2}\right) = \log(x+4) && \text{(Laws of Logarithms)} \\
\implies & \frac{3x-2}{2} = x+4 && \text{(One-to-One Property)} \\
\implies & 3x-2 = 2x+8 && \text{(multiplying both sides by 2)} \\
\implies & x = 10 && \text{(subtracting } 2x \text{ and adding 2 to both sides)}
\end{aligned}$$

To check the result, substitute $x = 10$ into the original equation.

$$\begin{aligned} & \log(3(10)-2)-\log(2) \stackrel{?}{=} \log(10+4) \\ \Longrightarrow \quad & \log(28)-\log(2) \stackrel{?}{=} \log(14) \\ \Longrightarrow \quad & \log\left(\frac{28}{2}\right) \stackrel{?}{=} \log(14) \\ \Longrightarrow \quad & \log(14) \stackrel{\checkmark}{=} \log(14) \end{aligned}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

Example 13.6.13: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

Solution

$$\begin{aligned} & \ln(x^2) = \ln(2x+3) & \\ \Longrightarrow \quad & x^2 = 2x+3 & \text{(One-to-One Property)} \\ \Longrightarrow \quad & x^2-2x-3 = 0 & \text{(subtracting } 2x \text{ and 3 from both sides)} \\ \Longrightarrow \quad & (x-3)(x+1) = 0 & \text{(factoring the trinomial)} \end{aligned}$$

Therefore, $x = 3$ or $x = -1$

We leave it to the reader to check the solutions; however, it is important to note that, while the solution -1 is negative, it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Checkpoint 13.6.13

Solve $\ln(x^2) = \ln 1$.

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13.6.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Solving exponential equations
  - Exponential and Logarithmic Equations - Solving Exponential Equations and Inequalities
- Like bases property exponents
  - Intermediate Algebra - Exponential Functions: Solving Exponential Equations (no logs)
- Using logarithms to solve exponential equations
  - Exponential and Logarithmic Equations - Solving Exponential Equations Using Logarithms
  - Intermediate Algebra - Logarithms: Solving Any Exponential Equation
  - Intermediate Algebra - Exponential and Logarithmic Functions: Solving Equations
- Solving logarithmic equations
  - Exponential and Logarithmic Equations - Solving Logarithmic Equations
  - Exponential and Logarithmic Equations - Solving Logarithmic Equations and Inequalities

KEY CONCEPTS

THEOREMS

- **One-to-One Property of Exponents**: For any algebraic expressions S and T, and any positive real number $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

COMMON MISTAKES

- **Incorrectly Applying the One-to-One Property of Exponents**: Attempting to equate exponents when the bases are not the same, without first rewriting them to have a common base.
- **Errors in Using Logarithms to Solve Exponential Equations**: Incorrectly applying logarithm laws, especially the power rule (e.g., $\ln(5^{x+2}) = (x+2)\ln(5)$, not $x\ln(5)+2$).
- **Forgetting to Check for Extraneous Solutions in Logarithmic Equations**: Failing to verify that all solutions obtained make the arguments of the original logarithms positive.
- **Errors in Using the One-to-One Property of Logarithms**: Not first condensing logarithmic expressions on each side of the equation to a single logarithm before equating arguments.
- **Treating $\ln(A-B)$ as $\ln(A)-\ln(B)$ or $\frac{\ln(A)}{\ln(B)}$**: Incorrectly splitting logarithms of differences or quotients of logarithms. (Common algebraic errors with log laws)
- **Solving Equations Quadratic-in-Form Incorrectly**: Forgetting to substitute back after solving for u in an equation like $e^{2x} - e^x - 56 = 0$, or incorrectly solving the resulting exponential/logarithmic equations after substitution.

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13.6.2: HOMEWORK

READING QUESTIONS

1. What is the One-to-One Property of Exponents, and how is it used to solve exponential equations?
2. If an exponential equation cannot be easily rewritten with like bases, what is the general strategy for solving it using logarithms?
3. When solving an exponential equation by taking the logarithm of both sides, does it matter which base of logarithm you use (e.g., common log vs. natural log)?
4. What is an extraneous solution in the context of solving exponential or logarithmic equations? Why do they sometimes occur?
5. How is the definition of a logarithm used to solve logarithmic equations of the form $\log_b(S) = c$?
6. What is the One-to-One Property of Logarithms, and how can it be used to solve logarithmic equations? What must be true about the bases of the logarithms?
7. Before applying the One-to-One Property of Logarithms to an equation like $\log(3x-2) - \log(2) = \log(x+4)$, what must you do first?
8. Why is it essential to check solutions when solving logarithmic equations?
9. Describe how to solve an equation like $e^{2x} - e^x = 56$ which is quadratic in form.
10. If you solve a logarithmic equation and obtain a candidate solution that makes an argument of a logarithm in the original equation negative or zero, what do you conclude about that candidate solution?
11. How can an exponential equation be solved?
12. When does an extraneous solution occur? How can an extraneous solution be recognized?
13. When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

HOMEWORK

ALGEBRAIC

For the following exercises, use like bases to solve the exponential equation.

14. $4^{-3v-2} = 4^{-v}$
15. $64 \cdot 4^{3x} = 16$
16. $3^{2x+1} \cdot 3^x = 243$
17. $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$
18. $625 \cdot 5^{3x+3} = 125$
19. $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$
20. $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^6$

For the following exercises, use logarithms to solve.

21. $9^{x-10} = 1$
22. $2e^{6x} = 13$
23. $e^{r+10} - 10 = -42$
24. $2 \cdot 10^{9a} = 29$
25. $-8 \cdot 10^{p+7} - 7 = -24$
26. $7e^{3n-5} + 5 = -89$
27. $e^{-3k} + 6 = 44$
28. $-5e^{9x-8} - 8 = -62$
29. $-6e^{9x+8} + 2 = -74$
30. $2^{x+1} = 5^{2x-1}$
31. $e^{2x} - e^x - 132 = 0$
32. $7e^{8x+8} - 5 = -95$
33. $10e^{8x+3} + 2 = 8$
34. $4e^{3x+3} - 7 = 53$
35. $8e^{-5x-2} - 4 = -90$
36. $3^{2x+1} = 7^{x-2}$
37. $e^{2x} - e^x - 6 = 0$
38. $3e^{3-3x} + 6 = -31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

39. $\log\left(\frac{1}{100}\right) = -2$
40. $\log_{324}(18) = \frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

41. $5\log_7(n) = 10$
42. $-8\log_9(x) = 16$
43. $4 + \log_2(9k) = 2$
44. $2\log(8n+4) + 6 = 10$
45. $10 - 4\ln(9-8x) = 6$

For the following exercises, use the one-to-one property of logarithms to solve.

46. $\ln(10-3x) = \ln(-4x)$
47. $\log_{13}(5n-2) = \log_{13}(8-5n)$
48. $\log(x+3) - \log(x) = \log(74)$
49. $\ln(-3x) = \ln(x^2 - 6x)$
50. $\log_4(6-m) = \log_4(3m)$
51. $\ln(x-2) - \ln(x) = \ln(54)$
52. $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$
53. $\ln(x^2 - 10) + \ln(9) = \ln(10)$

For the following exercises, solve each equation for x.

54. $\log(x+12) = \log(x) + \log(12)$
55. $\ln(x) + \ln(x-3) = \ln(7x)$
56. $\log_2(7x+6) = 3$
57. $\ln(7) + \ln(2-4x^2) = \ln(14)$
58. $\log_8(x+6) - \log_8(x) = \log_8(58)$
59. $\ln(3) - \ln(3-3x) = \ln(4)$
60. $\log_3(3x) - \log_3(6) = \log_3(77)$

GRAPHICAL

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

61. $\log_9(x) - 5 = -4$
62. $\log_3(x) + 3 = 2$
63. $\ln(3x) = 2$
64. $\ln(x - 5) = 1$
65. $\log(4) + \log(-5x) = 2$
66. $-7 + \log_3(4 - x) = -6$
67. $\ln(4x - 10) - 6 = -5$
68. $\log(4 - 2x) = \log(-4x)$
69. $\log_{11}(-2x^2 - 7x) = \log_{11}(x - 2)$
70. $\ln(2x + 9) = \ln(-5x)$
71. $\log_9(3 - x) = \log_9(4x - 8)$
72. $\log(x^2 + 13) = \log(7x + 3)$
73. $\frac{3}{\log_2(10)} - \log(x - 9) = \log(44)$
74. $\ln(x) - \ln(x + 3) = \ln(6)$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

75. An account with an initial deposit of $6,500 earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?
76. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8.3 \cdot 10^2$ watts per square meter?
77. The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach 20,000?

TECHNOLOGY

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

78. $1000(1.03)^t = 5000$ using the common log
79. $e^{5x} = 17$ using the natural log
80. $3(1.04)^{3t} = 8$ using the common log
81. $3^{4x-5} = 38$ using the common log
82. $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

83. $7e^{3x-5} + 7.9 = 47$
84. $\ln(3) + \ln(4.4x + 6.8) = 2$
85. $\log(-0.7x - 9) = 1 + 5\log(5)$
86. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (*Hint*: there are 5,280 feet in a mile)
87. The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules and $E_0 = 10^{4.4}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

EXTENSIONS

88. Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b(x)} = x$.
89. Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.
90. Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.
91. Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

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13.7: Exponential and Logarithmic Models

Learning Objectives

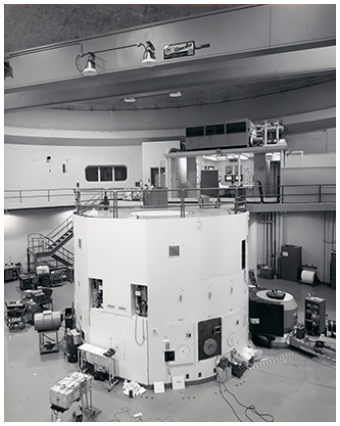
- Model exponential growth and decay.
- Use Newton's Law of Cooling.
- Use logistic-growth models.
- Express an exponential model in base e.
- Use logarithmic models.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 13.7.1

A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)



We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and **Newton's Law of Cooling**.

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function $P(t) = P_0e^{kt}$, where P_0 is equal to the value at time zero, e is Euler's constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double.

Definition: Doubling Time

The time it takes for a quantity undergoing exponential growth to double is called the quantity's **doubling time**.

Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the exponential decay model. Again, we have the form $P(t) = P_0e^{kt}$ where P_0 is the starting value, and e is Euler's constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating **half-life**, or

the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

Definition: Half-Life

The time it takes for a quantity undergoing exponential decay to decay by half is called the quantity's **half-life**.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure 13.7.2 and Figure 13.7.3. It is important to remember that, although parts of each of the two graphs seem to lie on the x-axis, they are really a tiny distance above the x-axis.

Figure 13.7.2

A graph showing exponential growth. The equation is $y = 2e^{3x}$.

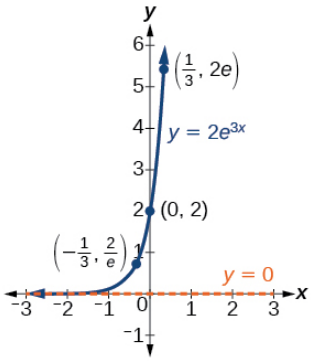


Figure 13.7.3

A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude.

Definition: Order of Magnitude

The **order of magnitude** of a number is the power of ten of the number when written in scientific notation.

For example, the distance to the nearest star, Proxima Centauri, measured in kilometers, is $40,113,497,200,000$ kilometers. Expressed in scientific notation, this is $4.01134972 \times 10^{13}$. So, we could describe this number as having order of magnitude 13.

Example 13.7.1: Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

Solution

When an amount grows at a fixed percent per unit time, the growth is exponential. To find P_0 we use the fact that P_0 is the amount at time zero, so $P_0 = 10$. To find k, use the fact that after one hour ($t = 1$) the population doubles from 10 to 20. The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k\cdot 1} && \text{(substituting)} \\ \Longrightarrow \quad 2 &= e^{k} && \text{(dividing both sides by 10)} \\ \Longrightarrow \quad \ln 2 &= k && \text{(applying the natural logarithm to both sides)} \end{aligned}$$

So $k = \ln(2)$. Thus, the equation we want to graph is

$$y = 10e^{(\ln 2)t} = 10\left(e^{\ln 2}\right)^{t} = 10 \cdot 2^{t}.$$

The graph is shown in Figure 13.7.4.

Figure 13.7.4

The graph of $P(t) = 10e^{(\ln 2)t}$

The population of bacteria after ten hours is $10,240$. We could describe this amount as being an order of magnitude 4 because it can be written as 1.024×10^4. The population of bacteria after twenty hours is $10,485,760 \approx 1.049 \times 10^7$, which is an order of magnitude of 7, so we could say that the population has increased by three orders of magnitude in ten hours.

Half-Life

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. Table 13.7.1 lists the half-life for several of the more common radioactive substances.

Table 13.7.1

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

To find the half-life of a function describing exponential decay, we need to know how long it takes for the initial population, P_0, to decay by half to $\frac{1}{2}P_0$. This requires us to solve the following equation:

$$\frac{1}{2}P_0 = P_0 e^{kt}.$$

We find that the half-life depends only on the constant k and not on the starting quantity P_0.

The formula is derived as follows:

$$\begin{aligned}
\frac{1}{2}P_0 &= P_0 e^{kt} \\
\implies \frac{1}{2} &= e^{kt} && \text{(dividing both sides by } P_0\text{)} \\
\implies \ln\left(\frac{1}{2}\right) &= kt && \text{(taking the natural logarithm of both sides)} \\
\implies \frac{\ln(1/2)}{k} &= t && \text{(dividing both sides by } k\text{)} \\
\implies \frac{\ln(2^{-1})}{k} &= t && \text{(Laws of Exponents)} \\
\implies -\frac{\ln(2)}{k} &= t && \text{(Laws of Logarithms)}
\end{aligned}$$

Since t, the time, is positive, k must, as expected, be negative. This gives us the half-life formula.

Theorem: Half-Life Formula

If an exponentially decaying population is modeled by the function $P(t) = P_0 e^{kt}$, then the time it takes for the population to decay by half is

$$t = -\frac{\ln(2)}{k}.$$

While most students tend to memorize this formula, your are expected to understand it and derive it, if needed.

Example 13.7.2: Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, t.

Solution

This formula is derived as follows:

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ \Longrightarrow \quad \frac{1}{2}P_0 &= P_0 e^{k\cdot 5730} && \text{(substituting)} \\ \Longrightarrow \quad \frac{1}{2} &= e^{k\cdot 5730} && \text{(dividing both sides by } P_0\text{)} \\ \Longrightarrow \quad \ln\left(\frac{1}{2}\right) &= 5730k && \text{(taking the natural logarithm of both sides)} \\ \Longrightarrow \quad \frac{\ln(1/2)}{5730} &= k && \text{(dividing both sides by 5730)} \end{aligned}$$

Substituting this back into the original function, we get

$$P(t) = P_0 e^{\left(\frac{\ln(1/2)}{5730}\right)t}.$$

We observe that the coefficient of t, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$, is negative, as expected in the case of exponential decay.

? Checkpoint 13.7.2

The half-life of plutonium-244 is 80,000,000 years. Find a function that gives the amount of plutonium-244 remaining as a function of time, measured in years.

Radiocarbon Dating

The formula for exponential decay is important in radiocarbon dating, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of 5,730 years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last 60,000 years, using tree rings and other organic samples of known dates - although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, we derive a formula for the amount present at any time t as follows:

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ \Longrightarrow \quad \frac{1}{2}P_0 &= P_0 e^{k\cdot 5730} && \text{(after 5,730 years, there is half the initial amount remains)} \\ \Longrightarrow \quad \frac{1}{2} &= e^{5730k} && \text{(dividing both sides by } P_0\text{)} \\ \Longrightarrow \quad \ln\left(\tfrac{1}{2}\right) &= 5730k && \text{(taking the natural logarithm of both sides)} \\ \Longrightarrow \quad \frac{\ln(0.5)}{5730} &= k && \text{(dividing both sides by 5730)} \end{aligned}$$

Thus, the formula for the amount of carbon-14 remaining after t years is

$$P(t) \approx P_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t},$$

where

- $P(t)$ is the amount of carbon-14 remaining at time t, and
- P_0 is the amount of carbon-14 when the plant or animal began decaying.

Again, I *highly* discourage you from memorizing this formula. It's fast to derive, and the half-life changes based on substance given, that it is much preferred to derive the formula each time.

Example 13.7.3: Finding the Age of a Bone

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Solution

We are told that 20% of an initial sample remains. That is, the current amount of carbon-14 in the sample is $0.20P_0$. Therefore, we want to solve $0.2P_0 = P_0e^{kt}$. This gives

$$0.2 = e^{kt} \implies \ln 0.2 = kt \implies t = \frac{\ln 0.2}{k}.$$

Since we already have computed k for carbon-14 eariler, we substitute that value into this formula:

$$t = \frac{\ln 0.2}{k} = \frac{\ln 0.2}{\frac{\ln 0.5}{5730}} = \frac{5730 \ln(0.2)}{\ln 0.5} \approx 13301$$

The bone fragment is about 13,301 years old.

In reality, the instruments that measure the percentage of carbon-14 are extremely sensitive and a scientist would need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so the age of the sample from Example **13.7.3** should be given as

$$13,301 \text{ years} \pm 113,301 \text{ years} = 13,301 \pm 133 \text{ years}.$$

Checkpoint 13.7.3

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the doubling time.

Given the basic exponential growth equation $P(t) = P_0e^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2P_0 = P_0e^{kt}$. Rather than attracting you to memorize a formula, let's show how to derive it while in an example.

Example 13.7.4: Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Solution

The formula is derived as follows:

$$\begin{aligned} & P(t) = P_0 e^{kt} & \\ \implies & 2P_0 = P_0 e^{k\cdot 2} & \text{(substituting } t=2 \text{ and } P(2)=2P_0\text{)} \\ \implies & 2 = e^{2k} & \text{(dividing both sides by } P_0\text{)} \\ \implies & \ln 2 = 2k & \text{(taking the natural log of both sides)} \\ \implies & \frac{\ln 2}{2} = k & \text{(dividing both sides by 2)} \end{aligned}$$

Thus, the function is

$$P(t) = P_0 e^{(\ln(2)/2)t}.$$

? Checkpoint 13.7.4

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature.

Theorem: Newton's Law of Cooling

The temperature of an object, T, in a surrounding medium having temperature T_s will behave according to the formula

$$T(t) = T_s + (T_0 - T_s)e^{kt},$$

where t is the time the object has been immersed in the medium, T_0 is the temperature of the object at time 0 (usually before being immersed in the medium), and k is a constant called the **continuous rate of cooling** of the object.

Example 13.7.5: Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of $165°\text{F}$ and is placed into a $35°\text{F}$ refrigerator. After 10 minutes, the cheesecake has cooled to $150°\text{F}$. If we must wait until the cheesecake has cooled to $70°\text{F}$ before we eat it, how long will we have to wait?

Solution

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = 35 + (T_0 - 35)e^{kt}.$$

We know the initial temperature was 165, so $T(0) = T_0 = 165$. Therefore,

$$T(t) = 35 + (165 - 35)e^{k\cdot t} = 35 + 130e^{kt}.$$

We were given another data point, $T(10) = 150$, which we can use to solve for k.

$$
\begin{aligned}
150 &= 35 + 130e^{k\cdot 10} && \text{(subsituting the point } (10, 150)\text{)} \\
\implies 115 &= 130e^{10k} && \text{(subtracting 35 from both sides)} \\
\implies \frac{115}{130} &= e^{10k} && \text{(dividing both sides by 130)} \\
\implies \ln\left(\tfrac{115}{130}\right) &= 10k && \text{(taking the natural log of both sides)} \\
\implies \frac{\ln\left(\frac{115}{130}\right)}{10} &= k && \text{(dividing both sides by 10)}
\end{aligned}
$$

Hence, $k \approx -0.0123$. This gives us the equation for the cooling of the cheesecake:

$$T(t) \approx 35 + 130e^{-0.0123t}.$$

Now we can solve for the time it will take for the temperature to cool to 70 degrees. To keep numbers as exact as possible, we wait until the end to replace k with our found value.

$$
\begin{aligned}
70 &= 35 + 130e^{kt} && \text{(substituting 70 for } T(t)\text{)} \\
\implies 35 &= 130e^{kt} && \text{(subtracting 35 from both sides)} \\
\implies \frac{35}{130} &= e^{kt} && \text{(dividing both sides by 130)} \\
\implies \ln\left(\tfrac{35}{130}\right) &= kt && \text{(taking the natural log of both sides)} \\
\implies \frac{\ln(35/130)}{k} &= t && \text{(dividing both sides by } k\text{)} \\
\implies \frac{\ln(35/130)}{\frac{\ln(115/130)}{10}} &= t && \left(\text{substituting } k = \frac{\ln\left(\frac{115}{130}\right)}{10}\right)
\end{aligned}
$$

Computing this value, we get $t \approx 106.68$. Thus, it will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to 70°F.

? Checkpoint 13.7.5

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the exponential growth model is still useful over a short term, before approaching the limiting value.

Definition: Logistic Growth Model

A **logistic growth model** is any model that can be written in the form

$$P(t) = \frac{c}{1 + ae^{-bt}},$$

where a, b, and c are constants. c is called the **carrying capacity** (also known as the **limiting value**) of the population, b is a constant determined by the rate of growth, and the ratio $\frac{c}{1+a}$ is the initial value.

The logistic growth model is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, or carrying capacity. The graph in Figure **13.7.5** shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

Figure 13.7.5

Example 13.7.6: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution

We substitute the given data into the logistic growth model

$$f(t) = \frac{c}{1 + ae^{-bt}}.$$

Because at most 1,000 people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find a, we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$. This model predicts that, after ten days, the number of people who have had the flu is $f(t) = \frac{1000}{1+999e^{-0.6030t}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round *down* to 293. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure **13.7.6** gives a good picture of how this model fits the data.

Figure 13.7.6

The graph of $f(t) = \frac{1000}{1+999e^{-0.6030t}}$.

? Checkpoint 13.7.6

Using the model in Example **13.7.6**, estimate the number of cases of flu on day 15.

Expressing an Exponential Model in Base e

While powers and logarithms of any base can be used in modeling, the two most common bases are **10** and e. In Science and Mathematics, the base e is often preferred. We can use Laws of Exponents and Laws of Logarithms to change any base to base e.

Given a model with the form $y = ab^x$, we change it to the form $y = A_0e^{kx}$ using the following steps.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the Power Rule of Logarithms to rewrite y as $y = ae^{x\ln(b)} = ae^{\ln(b)x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

✓ Example 13.7.7: Changing to base e

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0e^{kx}$.

Solution

The formula is derived as follows:

$$\begin{aligned} y &= 2.5(3.1)^x \\ &= 2.5e^{\ln(3.1^x)} && \text{(Inverse Property of Exponentials and Logarithms)} \\ &= 2.5e^{x\ln(3.1)} && \text{(Laws of Logarithms)} \\ &= 2.5e^{\ln(3.1)x} && \text{(Commutative Property of Multiplication)} \end{aligned}$$

? Checkpoint 13.7.7

Change the function $y = 3(0.5)^x$ to one having e as the base.

Logarithmic Scales

For quantities that vary greatly in magnitude, a standard scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

Table 13.7.2

Planet	Distance (millions of km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale – one with equally spaced values – these values get bunched up (see Figure 13.7.7).

Figure 13.7.7

Linear scale of astronomical distances in our solar system.

However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent, as seen in Figure 13.7.8 below Table 13.7.3.(It is interesting to note the large gap between Mars and Jupiter on the log number line. The asteroid belt is located there, which scientists believe is a planet that never formed because of the effects of the gravity of Jupiter.)

Table 13.7.3

Planet	Distance (millions of km)	log(distance)
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65

Figure 13.7.8

Logarithmic scale of astronomical distances in our solar system.

Sometimes, as shown above in Figure **13.7.8**, the scale on a logarithmic number line will show the log values, but more commonly the original values are listed as powers of 10, as shown below in Figure **13.7.9**.

Figure 13.7.9

Example 13.7.8

Estimate the value of point P on the log scale above

Solution

The point P appears to be half way between -2 and -1 in log value. So if V is the value of this point,

$$\log(V) \approx -1.5.$$

Rewriting in exponential form,

$$V \approx 10^{-1.5} \approx 0.0316.$$

Example 13.7.9

Place the number 6000 on a logarithmic scale.

Solution

Since $\log(6000) \approx 3.8$, this point would belong on the log scale about here:

Checkpoint 13.7.9

Plot the data in the table below on a logarithmic scale (From http://www.epd.gov.hk/epd/noise_educ...1/intro_5.html, retrieved Oct 2, 2010).

Table 13.7.4

Source of Sound/Noise	Approximate Sound Pressure in μ Pa (micro Pascals)
Launching of the Space Shuttle	2000,000,000
Full Symphony Orchestra	2000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio	200
Softest Sound a human can hear	20

Answer

Notice that on the log scale above Figure **13.7.10**, the visual distance on the scale between points A and B and between C and D is the same. When looking at the values these points correspond to, notice B is ten times the value of A, and D is ten times the value of C. A visual (linear) difference between points corresponds to a relative (ratio) change between the corresponding values.

Logarithms are useful for showing these relative changes. For example, comparing \$1,000,000 to \$10,000, the first is 100 times larger than the second.

$$\frac{1,000,000}{10,000} = 100 = 10^2.$$

Likewise, comparing \$1000 to \$10, the first is 100 times larger than the second.

$$\frac{1,000}{10} = 100 = 10^2.$$

When one quantity is roughly ten times larger than another, its exponent when written in scientific notation increases by 1. Thus, we say it is one order of magnitude larger. In both cases described above, the first number was two orders of magnitude larger than the second.

Notice that the order of magnitude can be found as the common logarithm of the ratio of the quantities. On the log scale above, B is one order of magnitude larger than A, and D is one order of magnitude larger than C.

Theorem: Determining the Order of Magnitude Between Two Numbers

Given two values A and B, the number of orders of magnitude A is greater than B is

$$\log\left(\frac{A}{B}\right).$$

Example 13.7.10

On the log scale in Figure **13.7.10**, how many orders of magnitude larger is C than B?

Solution

The value B corresponds to $10^2 = 100$ and the value C corresponds to $10^5 = 100,000$. The relative change is $\frac{100,000}{100} = 1000 = \frac{10^5}{10^2} = 10^3$. The log of this value is 3. Therefore, C is three orders of magnitude greater than B, which can be seen on the log scale by the visual difference between the points on the scale.

? Checkpoint 13.7.10

Using the table from Checkpoint **13.7.9**, what is the difference of order of magnitude between the softest sound a human can hear and the launching of the space shuttle?

Answer

$\frac{2\times10^9}{2\times10^1} = 10^8$. The sound pressure in μ Pa created by launching the space shuttle is 8 orders of magnitude greater than the sound pressure in μ Pa created by the softest sound a human ear can hear.

Earthquakes

An example of a logarithmic scale is the **Richter Scale** used for earthquakes. While not the only scale for measuring earthquakes, it is the most commonly known. The Richter scale compares the amplitude, A, of the earthquake's seismographic trace with the amplitude, A_0, of the smallest detectable earthquake.

Definition: Richter Scale

The **Richter scale** is a logarithmic scale used to express the magnitude of an earthquake, specifically the energy released during it. For an earthquake with **seismic trace amplitude** A (a measurement from the baseline of the seismograph), the **Richter value**, or **magnitude** of the earthquake, is defined to be

$$M = \log\left(\frac{A}{A_0}\right),$$

where A_0 is a baseline measure for the seismograph.

Example 13.7.11

If one earthquake has a Richter value of 6.0, and another has a magnitude of 8.0, how much more powerful (in terms of earth movement) is the second earthquake?

Solution

Since the first earthquake has Richter value of 6.0, we can find the amount of earth movement for that quake, which we'll denote A_1. The value of A_0 is not particularity relevant, so we will not replace it with its value.

$$\begin{aligned} 6.0 &= \log\left(\frac{A_1}{A_0}\right) \\ \Longrightarrow \quad 10^{6.0} &= \frac{A_1}{A_0} && \text{(rewriting the logarithm in exponential form)} \\ \Longrightarrow \quad 10^{6.0}A_0 &= A_1 && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

This tells us the first earthquake has about 10^6 times more earth movement than the baseline measure.

Doing the same with the second earthquake, A_2, with a magnitude of 8.0,

$$\begin{aligned} 8.0 &= \log\left(\frac{A_2}{A_0}\right) \\ \Longrightarrow \quad 10^{8.0} &= \frac{A_2}{A_0} && \text{(rewriting the logarithm in exponential form)} \\ \Longrightarrow \quad 10^{8.0}A_0 &= A_2 && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

Comparing the earth movement of the second earthquake to the first,

$$\frac{A_2}{A_1} = \frac{10^8 A_0}{10^6 A_0} = 10^2 = 100.$$

The second value's earth movement is 100 times as large as the first earthquake.

Example 13.7.12

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Solution

Since the first quake has magnitude 3.0,

$$3.0 = \log\left(\frac{A}{A_0}\right).$$

Solving for A,

$$\begin{aligned} 3.0 &= \log\left(\frac{A}{A_0}\right) \\ \implies 10^{3.0} &= \frac{A}{A_0} && \text{(rewriting in exponential form)} \\ \implies 10^{3.0} A_0 &= A && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

Since the second earthquake has twice as much earth movement, for the second quake we get

$$S = 2 \cdot 10^{3.0} S_0.$$

Finding the magnitude,

$$\begin{aligned} M &= \log\left(\frac{2 \cdot 10^{3.0} A_0}{A_0}\right) \\ &= \log\left(2 \cdot 10^{3.0}\right) \\ &\approx 3.3010 \end{aligned}$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.3.

In fact, using the Laws of Logarithms, we could show that whenever the earth movement doubles, the Richter magnitude will increase by about 0.301:

$$\begin{aligned} M &= \log\left(\frac{2A}{A_0}\right) \\ &= \log\left(2 \cdot \frac{A}{A_0}\right) \\ &= \log(2) + \log\left(\frac{A}{A_0}\right) \\ &\approx 0.3010 + \log\left(\frac{A}{A_0}\right) \end{aligned}$$

This illustrates the most important feature of a log scale: that multiplying the quantity being considered will add to the scale value, and vice versa.

Logarithms are useful in measuring the intensities of many quantities based on senses. Earthquakes (the Richter scale), sound (decibels) and acids and bases (pH) are common examples. We now present yet a different use of the basic logarithm function,

password strength.

Example 13.7.13

The information entropy H, in bits, of a randomly generated password consisting of L characters is given by

$$H = L\log_2(N),$$

where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

a. If a 7 character case-sensitive password is comprised of letters and numbers only, find the associated information entropy.
b. How many possible symbol options per character are required to produce a 7 character password with an information entropy of 50 bits?

Solutions

a. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.
b. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.

Chemical systems known as buffer solutions can adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications, from maintaining a healthy fish tank to regulating the pH levels in blood.

Example 13.7.14

Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream, it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base, to partially neutralize the acid. The equation which models blood pH in this situation is

$$\text{pH} = 6.1 + \log\left(\frac{800}{x}\right),$$

where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution

We set $\text{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr.

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13.7.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Sets and Numbers**
  - Scientific Notation: Scientific notation is often used when working with exponential models.

VIDEOS

- Exponential growth and decay models
  - Exponential Functions - Applications Involving Exponential Functions
- Half-life and doubling time models
  - Exponential and Logarithmic Models - Doubling Time and Half-Life
  - Modeling with Exponential and Logarithmic Equations - Applications Involving Half Life
- Newton's Law of Cooling
  - Modeling with Exponential and Logarithmic Equations - Applications Involving Newton's Law of Cooling
- Logarithmic models
  - Modeling with Exponential and Logarithmic Equations - Logarithmic Applications

KEY CONCEPTS

DEFINITIONS

- **Doubling Time**: The time it takes for a quantity undergoing exponential growth to double its initial amount.
- **Half-Life**: The time it takes for a quantity undergoing exponential decay to decay to half of its original quantity.
- **Order of Magnitude**: The power of ten of a number when written in scientific notation. It describes how large or small a number is relative to powers of ten.
- **Radiocarbon Dating**: A method used to calculate the approximate date a plant or animal died by comparing the ratio of carbon-14 to carbon-12 in a sample to the known ratio in the atmosphere.
- **Continuous Rate of Cooling**: The constant k in Newton's Law of Cooling formula.
- **Logistic Growth Model**: A model for growth that is approximately exponential at first but has a reduced rate of growth as the output approaches the model's upper bound (carrying capacity). The formula is $P(t) = \frac{c}{1+ae^{-kt}}$.
- **Carrying Capacity (Limiting Value)**: The constant c in the logistic growth model, representing the upper bound or maximum sustainable population/value.
- **Richter Scale**: A logarithmic scale used to express the magnitude of an earthquake, based on the seismic trace amplitude. Defined as $M = \log\left(\frac{A}{A_0}\right)$.

THEOREMS

- **Half-Life Formula**: If an exponentially decaying population is modeled by $P(t) = P_0e^{kt}$, then the time it takes for the population to decay by half (half-life) is $t = -\frac{\ln(2)}{k}$ (or $t = \frac{\ln(1/2)}{k}$).
- **Newton's Law of Cooling**: The temperature of an object, T, in a surrounding medium having temperature T_s will behave according to the formula $T(t) = T_s + (T_0 - T_s)e^{kt}$, where t is time, T_0 is the initial temperature, and k is the continuous rate of cooling.
- **Determining the Order of Magnitude Between Two Numbers**: Given two values A and B, the number of orders of magnitude A is greater than B is $\log\left(\frac{A}{B}\right)$.

COMMON MISTAKES

- **Incorrect Sign for Rate Constant k**: Using a positive k for decay models (like half-life or Newton's cooling) or a negative k for growth models (like doubling time) in the $P(t) = P_0e^{kt}$ formula.
- **Confusing Half-Life/Doubling Time with Rate Constant**: Mistaking the half-life or doubling time itself for the rate constant k, or vice-versa, instead of using the derived formulas (e.g., $k = -\frac{\ln(2)}{T_{half}}$) to relate them.
- **Errors in Setting Up Newton's Law of Cooling**: Incorrectly identifying T_s (ambient temperature), T_0 (initial temperature), or the temperature at a given time $T(t)$.
- **Algebraic Errors When Solving for t or k**: Making mistakes in isolating the exponential term, taking logarithms, or using logarithm properties when solving for time or the rate constant in various models.
- **Misinterpreting Logistic Growth Parameters**: Incorrectly identifying the carrying capacity c or the initial value $\frac{c}{1+a}$ from the logistic model formula.
- **Using Linear Models for Exponential Situations**: Attempting to model situations like population growth or radioactive decay with linear functions when an exponential model is appropriate.
- **Rounding Too Early**: Rounding intermediate calculations, which can lead to significant errors in the final answer, especially in exponential models.
- **Incorrectly Applying Logarithmic Scales**: Misunderstanding that a linear difference on a logarithmic scale corresponds to a multiplicative (ratio) difference in actual values (e.g., Richter scale).

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Simpson.

13.7.2: HOMEWORK

READING QUESTIONS

1. What is the difference between "doubling time" and "half-life" in the context of exponential models? Which applies to growth and which to decay?
2. What is the general formula for exponential growth/decay using base e? Define each variable.
3. How is the half-life T related to the decay constant k in the formula $P(t) = P_0e^{kt}$?
4. Describe Newton's Law of Cooling. What does each variable in the formula $T(t) = T_s + (T_0 - T_s)e^{kt}$ represent?
5. What is a logistic growth model, and how does its behavior differ from a simple exponential growth model, especially in the long term?
6. What is the "carrying capacity" in a logistic growth model?
7. If an exponential function is given as $y = ab^x$, how can you convert it to the form $y = A_0e^{kx}$? What are A_0 and k in terms of a and b?
8. Explain what an "order of magnitude" means. If one quantity is 3 orders of magnitude larger than another, approximately how many times larger is it?
9. What is the Richter scale used to measure, and is it a linear or logarithmic scale?
10. In the context of radiocarbon dating, what does the half-life of Carbon-14 (5,730 years) signify?
11. With what kind of exponential model would half-life be associated? What role does half-life play in these models?
12. What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.
13. With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?
14. Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.
15. What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

HOMEWORK

NUMERIC

16. The temperature of an object in degrees Fahrenheit after t minutes is represented by the equation $T(t) = 72 + 68e^{-0.0174t}$. To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f(x) = \frac{150}{1+8e^{-2x}}$.

17. Find and interpret $f(0)$. Round to the nearest tenth.
18. Find and interpret $f(4)$. Round to the nearest tenth.
19. Find the carrying capacity.
20. Graph the model.
21. Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074
5	2.488

22. Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base e to five decimal places.

TECHNOLOGY

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

23.

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

24.

x	$f(x)$
1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

25.

x	$f(x)$
4	9.429
5	9.972
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

26.

x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P(t) = \frac{1000}{1+9e^{-0.8t}}$.

27. Graph the function.
28. What is the initial population of fish?
29. To the nearest tenth, what is the doubling time for the fish population?
30. To the nearest whole number, what will the fish population be after 2 years?
31. To the nearest tenth, how long will it take for the population to reach 900?
32. What is the carrying capacity for the fish population? Justify your answer using the graph of P.

EXTENSIONS

33. A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?
34. The formula for an increasing population is given by $P(t) = P_0e^{rt}$ where P_0 is the initial population and $r > 0$. Derive a general formula for the time t it takes for the population to increase by a factor of M.
35. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment S.
36. What is the y-intercept of the logistic growth model $y = \frac{c}{1+ae^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?
37. Prove that $b^x = e^{x\ln(b)}$ for positive $b \neq 1$.

REAL-WORLD APPLICATIONS

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

38. To the nearest hour, what is the half-life of the drug?
39. Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.
40. Using the model found in the previous exercise, find $f(10)$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15 per day.

41. To the nearest day, how long will it take for half of the Iodine-125 to decay?
42. Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.
43. A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?
44. The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.
45. The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.
46. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)
47. A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

48. To the nearest whole number, what was the initial population in the culture?
49. Rounding to six decimal places, write an exponential equation representing this situation. To the nearest minute, how long did it

take the population to double?

For the following exercises, use this scenario: A pot of warm soup with an internal temperature of 100° Fahrenheit was taken off the stove to cool in a 69° F room. After fifteen minutes, the internal temperature of the soup was 95° F.

50. Use Newton's Law of Cooling to write a formula that models this situation.
51. To the nearest minute, how long will it take the soup to cool to 80° F?
52. To the nearest degree, what will the temperature be after 2 and a half hours?

For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165° F and is allowed to cool in a 75° F room. After half an hour, the internal temperature of the turkey is 145° F.

53. Write a formula that models this situation.
54. To the nearest degree, what will the temperature be after 50 minutes?
55. To the nearest minute, how long will it take the turkey to cool to 110° F?

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.

56.

57.

58. Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$, Jet: $10^{2}\frac{W}{m^2}$.
59. Recall the formula for calculating the magnitude of an earthquake, $M=\frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

For the following exercises, use this scenario: The equation $N(t)=\frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after t days.

60. How many people started the rumor?
61. To the nearest whole number, how many people will have heard the rumor after 3 days?
62. As t increases without bound, what value does $N(t)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

63. A doctor injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?
    a. $f(t)=13(0.0805)^t$
    b. $f(t)=13e^{0.9195t}$
    c. $f(t)=13e^{(}-0.0839t)$
    d. $f(t)=\frac{4.75}{1+13e^{-0.83925t}}$

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CHAPTER OVERVIEW

Chapter 14: Sequences, Summations, and Logic

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A lottery winner has some big decisions to make regarding what to do with the winnings. Buy a villa in Saint Barthélemy? A luxury convertible? A cruise around the world?

The likelihood of winning the lottery is slim, but we all love to fantasize about what we could buy with the winnings. One of the first things a lottery winner has to decide is whether to take the winnings in the form of a lump sum or as a series of regular payments, called an annuity, over the next 30 years or so.

This decision is often based on many factors, such as tax implications, interest rates, and investment strategies. There are also personal reasons to consider when making the choice, and one can make many arguments for either decision. However, most lottery winners opt for the lump sum.

In this chapter, we will explore the mathematics behind situations such as these. We will take an in-depth look at annuities. We will also look at the branch of mathematics that would allow us to calculate the number of ways to choose lottery numbers and the probability of winning.

14.1: Sequences and Their Notations

14.1.1: Resources and Key Concepts

14.1.2: Homework

14.2: Arithmetic Sequences

14.2.1: Resources and Key Concepts

14.2.2: Homework

14.3: Geometric Sequences

14.3.1: Resources and Key Concepts

14.3.2: Homework

14.4: Series and Their Notations

14.4.1: Resources and Key Concepts

14.4.2: Homework

14.5: Mathematical Induction

14.5.1: Resources and Key Concepts

14.5.2: Homework

14.6: Binomial Theorem

14.6.1: Resources and Key Concepts

14.6.2: Homework

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14.1: Sequences and Their Notations

Learning Objectives

- Write the terms of a sequence defined by an explicit formula.
- Write the terms of a sequence defined by a recursive formula.
- Use factorial notation.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

A video game company launches an exciting new advertising campaign. They predict the number of online visits to their website, or hits, will double each day. The model they are using shows 2 hits the first day, 4 hits the second day, 8 hits the third day, and so on. See Table 14.1.1.

Table 14.1.1

Day	1	2	3	4	5	...
Hits	2	4	8	16	32	...

If their model continues, how many hits will there be at the end of the month? To answer this question, we'll first need to know how to determine a list of numbers written in a specific order. In this section, we will explore these kinds of ordered lists.

Writing the Terms of a Sequence Defined by an Explicit Formula

One way to describe an ordered list of numbers is as a **sequence**.

Definition: Sequence

A **sequence** is a function whose domain is the set of natural numbers. The numbers in a sequence are called **terms**.

For example, the sequence established by the number of hits on the website from Table 14.1.1 is

$$\{2, 4, 8, 16, 32, ...\}.$$

The first five terms of this sequence are 2, 4, 8, 16, and 32. The ellipsis (...) indicates that the sequence continues indefinitely.

Definition: Infinite Sequence

A sequence that continues indefinitely is an **infinite sequence**.

Therefore, the sequence of website hits is considered an infinite sequence.

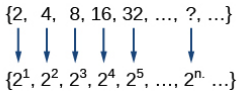
Listing all of the terms for a sequence can be cumbersome. For example, finding the number of hits on the website at the end of the month would require listing out as many as 31 terms. A more efficient way to determine a specific term is by writing a formula to define the sequence.

One type of formula is an **explicit formula**, which defines the terms of a sequence using their position in the sequence.

Definition: Explicit Formula

Given the sequence $\{a_1, a_2, a_3, ...\}$, where a_1 is the first term of the sequence, a_2 is the second term of the sequence, a_3 the third term of the sequence, and so on, an **explicit formula** defines a_n, the nth term of the sequence, in relation to the position of the term, n. That is, if a function formula, $f(n)$, can be found to represent the give sequence, the explicit formula would be $a_n = f(n)$.

Explicit formulas are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the nth term of the sequence, where n is any positive integer. In our example, each number in the sequence is double the previous number, so we can use powers of 2 to write a formula for the nth term.



The first term of the sequence is $2^1 = 2$, the second term is $2^2 = 4$, the third term is $2^3 = 8$, and so on. The nth term of the sequence can be found by raising 2 to the nth power. An explicit formula for a sequence is named by a lower case letter a, b, c, ... with the subscript n. The explicit formula for this sequence is

$$a_n = 2^n.$$

Now that we have a formula for the nth term of the sequence, we can answer the question posed at the beginning of this section. We were asked to find the number of hits at the end of the month, which we will take to be 31 days. To find the number of hits on the last day of the month, we need to find the 31st term of the sequence. We will substitute 31 for n in the formula.

$$a_{31} = 2^{31} = 2,147,483,648.$$

If the doubling trend continues, the company will get $2,147,483,648$ hits on the last day of the month. That is over 2.1 billion hits! The huge number is probably a little unrealistic because it does not take consumer interest and competition into account. It does, however, give the company a starting point from which to consider business decisions.

Another way to represent the sequence is by using a table. The first five terms of the sequence and the nth term of the sequence are shown in Table $14.1.2$.

Table 14.1.2

n	1	2	3	4	5	n
a_n	2	4	8	16	32	2^n

Graphing provides a visual representation of the sequence as a set of distinct points. We can see from the graph in Figure $14.1.1$ that the number of hits is rising at an exponential rate. This particular sequence forms an exponential function.

Figure 14.1.1

Lastly, we can write this particular sequence as

$$\{2, 4, 8, 16, 32, \ldots, 2n, \ldots\}.$$

If we consider only the first 10 terms of the sequence, we could write

$$\{2, 4, 8, 16, 32, \ldots, 2n, \ldots, 1024\}.$$

This sequence is called a **finite sequence** because it does not continue indefinitely.

Definition: Finite Sequence

A **finite sequence** is a sequence whose domain consists of only the first n natural numbers.

Q&A

Does a sequence always have to begin with a_1?

No. In certain problems, it may be useful to define the initial term as a_0 instead of a_1. In these problems, the domain of the function includes 0.

Example 14.1.1: Writing the Terms of a Sequence Defined by an Explicit Formula

Write the first five terms of the sequence defined by the explicit formula $a_n = -3n + 8$.

Solution

Substitute $n = 1$ into the formula. Repeat with values 2 through 5 for n.

$$\begin{aligned} n &= 1 \qquad a_1 = -3(1) + 8 = 5 \\ n &= 2 \qquad a_2 = -3(2) + 8 = 2 \\ n &= 3 \qquad a_3 = -3(3) + 8 = -1 \\ n &= 4 \qquad a_4 = -3(4) + 8 = -4 \\ n &= 5 \qquad a_5 = -3(5) + 8 = -7 \end{aligned}$$

The first five terms are $\{5, 2, -1, -4, -7\}$.

The sequence values in Example **14.1.1** can be listed in a table. A table, such as Table **14.1.3**, is a convenient way to input the function into graphing technology like Desmos.

Table 14.1.3

n	1	2	3	4	5
a_n	5	2	–1	–4	–7

A graph can be made from this table of values. From the graph in Figure **14.1.2**, we can see that this sequence represents a linear function, but notice the graph is not continuous because the domain is over the natural numbers only.

Figure 14.1.2

? Checkpoint 14.1.1

Write the first five terms of the sequence defined by the explicit formula $t_n = 5n - 4$.

Investigating Alternating Sequences

Sometimes sequences have terms that alternate in sign.

Definition: Alternating Sequence

A sequence in which consecutive terms alternate in sign, or which can be simplified to a sequence where consecutive terms alternate in sign, is called an **alternating sequence**.

The steps to finding terms of such sequences are the same as if the signs did not alternate. However, the resulting terms will not show increasing or decreasing behvior as n increases. Let's take a look at the following sequence.

$$\{2, -4, 6, -8\}$$

Notice the first term is greater than the second term, the second term is less than the third term, and the third term is greater than the fourth term. This trend continues forever. *Do not* rearrange the terms in numerical order to interpret the sequence.

Example 14.1.2: Writing the Terms of an Alternating Sequence Defined by an Explicit Formula

Write the first five terms of the sequence.

$$a_n = \frac{(-1)^n n^2}{n+1}$$

Solution

Substitute $n = 1$, $n = 2$, and so on into the formula.

$$n = 1 \qquad a_1 = dfrac(-1)^1 1^2 1 + 1 = -\frac{1}{2}$$

$$n = 2 \qquad a_2 = dfrac(-1)^2 2^2 2 + 1 = \frac{4}{3}$$

$$n = 3 \qquad a_3 = dfrac(-1)^3 3^2 3 + 1 = -\frac{9}{4}$$

$$n = 4 \qquad a_4 = dfrac(-1)^4 4^2 4 + 1 = \frac{16}{5}$$

$$n = 5 \qquad a_5 = dfrac(-1)^5 5^2 5 + 1 = -\frac{25}{6}$$

The first five terms are $\left\{-\frac{1}{2}, \frac{4}{3}, -\frac{9}{4}, \frac{16}{5}, -\frac{25}{6}\right\}$.

The graph of this function, shown in Figure **14.1.3**, looks different from the ones we have seen previously in this section because the terms of the sequence alternate between positive and negative values.

Figure 14.1.3

Q&A

In Example 14.1.2, does the (–1) to the power of n account for the oscillations of signs?

Yes, the power might be n, $n+1$, $n-1$, and so on, but any odd powers will result in a negative term, and any even power will result in a positive term.

Checkpoint **14.1.2**

Write the first five terms of the sequence.

$$a_n = \frac{4n}{(-2)^n}$$

Investigating Piecewise Explicit Formulas

We've learned that sequences are functions whose domain is over the natural numbers. This is true for other types of functions, including some piecewise functions. Recall that a piecewise function is a function defined by multiple subsections. A different formula might represent each individual subsection.

Example **14.1.3**: Writing the Terms of a Sequence Defined by a Piecewise Explicit Formula

Write the first six terms of the sequence.

$$a_n = \begin{cases} n^2, & \text{if } n \text{ is not divisible by } 3 \\ \frac{n}{3}, & \text{if } n \text{ is divisible by } 3 \end{cases}$$

Solution

Substitute $n = 1$, $n = 2$, and so on into the appropriate formula. Use n^2 when n is not a multiple of 3. Use $\frac{n}{3}$ when n is a multiple of 3.

$$
\begin{array}{lll}
n=1 & a_1=1^2=1 & (1 \text{ is not a multiple of } 3 \implies \text{Use } n^2) \\
n=2 & a_2=2^2=4 & (2 \text{ is not a multiple of } 3 \implies \text{Use } n^2) \\
n=3 & a_3=\dfrac{3}{3}=1 & \left(3 \text{ is a multiple of } 3 \implies \text{Use } \dfrac{n}{3}\right) \\
n=4 & a_4=4^2=16 & (4 \text{ is not a multiple of } 3 \implies \text{Use } n^2) \\
n=5 & a_5=5^2=25 & (5 \text{ is not a multiple of } 3 \implies \text{Use } n^2) \\
n=6 & a_6=\dfrac{6}{3}=2 & \left(6 \text{ is a multiple of } 3 \implies \text{Use } \dfrac{n}{3}\right)
\end{array}
$$

The first six terms are $\{1, 4, 1, 16, 25, 2\}$.

Every third point on the graph shown in Figure 14.1.4 stands out from the two nearby points. This occurs because the sequence was defined by a piecewise function.

Figure 14.1.4

? Checkpoint 14.1.3

Write the first six terms of the sequence.

$$a_n = \begin{cases} 2n^3, & \text{if } n \text{ is odd} \\ \frac{5n}{2}, & \text{if } n \text{ is even} \end{cases}$$

Finding an Explicit Formula

Thus far, we have been given the explicit formula and asked to find a number of terms of the sequence. Sometimes, the explicit formula for the nth term of a sequence is not given. Instead, we are given several terms from the sequence. When this happens, we can work in reverse to find an explicit formula from the first few terms of a sequence. The key to finding an explicit formula is to look for a pattern in the terms. Keep in mind that the pattern may involve alternating terms, formulas for numerators, formulas for denominators, exponents, or bases.

✓ Example 14.1.4: Writing an Explicit Formula for the nth Term of a Sequence

Write an explicit formula for the nth term of each sequence.

a. $\left\{-\frac{2}{11}, \frac{3}{13}, -\frac{4}{15}, \frac{5}{17}, -\frac{6}{19}, \ldots\right\}$
b. $\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3{,}125}, -\frac{2}{15{,}625}, \ldots\right\}$
c. $\{e^4, e^5, e^6, e^7, e^8, \ldots\}$

Solutions

Look for the pattern in each sequence.

a. The terms alternate between positive and negative. We can use $(-1)^n$ to make the terms alternate. The numerator can be represented by $n+1$. The denominator can be represented by $2n+9$. Therefore,

$$a_n = \frac{(-1)^n(n+1)}{2n+9}$$

b. The terms are all negative.

$$\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3,125}, -\frac{2}{15,625}, \ldots\right\}$$

The numerator is always 2.

$$\left\{-\frac{2}{5^2}, -\frac{2}{5^3}, -\frac{2}{5^4}, -\frac{2}{5^5}, -\frac{2}{5^6}, \ldots\right\}$$

The denominators are increasing powers of 5, starting with the second power when $n=1$. So we know that the denominator can be represented by 5^{n+1}. Therefore,

$$a_n = -\frac{2}{5^{n+1}}.$$

c. The terms are powers of e. For $n=1$, we get e^4. Therefore, the exponent must be $n+3$. That is,

$$a_n = e^{n+3}.$$

? Checkpoint 14.1.4

Write an explicit formula for the nth term of each sequence.

a. $\{9, -81, 729, -6561, 59049, \ldots\}$
b. $\left\{-\frac{3}{4}, -\frac{9}{8}, -\frac{27}{12}, -\frac{81}{16}, -\frac{243}{20}, \ldots\right\}$
c. $\left\{\frac{1}{e^2}, \frac{1}{e}, 1, e, e^2, \ldots\right\}$

Writing the Terms of a Sequence Defined by a Recursive Formula

Sequences occur naturally in the growth patterns of nautilus shells, pinecones, tree branches, and many other natural structures. We may see the sequence in the leaf or branch arrangement, the number of petals of a flower, or the pattern of the chambers in a nautilus shell. Their growth follows the **Fibonacci sequence**, a famous sequence in which each term can be found by adding the preceding two terms. The numbers in the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Other examples from the natural world that exhibit the Fibonacci sequence are the Calla Lily, which has just one petal, the Black-Eyed Susan with 13 petals, and different varieties of daisies that may have 21 or 34 petals.

Each term of the Fibonacci sequence depends on the terms that come before it. The Fibonacci sequence cannot easily be written using an explicit formula. Instead, we describe the sequence using a **recursive formula**.

Definition: Recursive Formula

A **recursive formula** is a formula that defines each term of a sequence using a preceding term (or multiple preceding terms). Recursive formulas must always state the **initial term**, or terms, of the sequence.

For example, suppose we know the following:

$$\begin{aligned} a_1 &= 3 \\ a_n &= 2a_{n-1} - 1 \qquad \text{for } n \geq 2 \end{aligned}$$

We can find the subsequent terms of the sequence using the first term.

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 2a_1 - 1 = 2(3) - 1 = 5 \\ a_3 &= 2a_2 - 1 = 2(5) - 1 = 9 \\ a_4 &= 2a_3 - 1 = 2(9) - 1 = 17 \end{aligned}$$

So the first four terms of the sequence are $\{3, 5, 9, 17\}$.

The recursive formula for the Fibonacci sequence states the first two terms and defines each successive term as the sum of the preceding two terms.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \end{aligned}$$

To find the tenth term of the sequence, for example, we would need to add the eighth and ninth terms. We were told previously that the eighth and ninth terms are **21** and **34**, so

$$a_{10} = a_9 + a_8 = 34 + 21 = 55.$$

Q&A

Must the first two terms always be given in a recursive formula?

No. The Fibonacci sequence defines each term using the two preceding terms, but many recursive formulas define each term using only one preceding term. These sequences need only the first term to be defined.

Example 14.1.5: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first five terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 9 \\ a_n &= 3a_{n-1} - 20 \qquad \text{for } n \geq 2 \end{aligned}$$

Solution

The first term is given in the formula. For each subsequent term, we replace a_{n-1} with the value of the preceding term.

$$\begin{aligned} n = 1 \qquad & a_1 = 9 \\ n = 2 \qquad & a_2 = 3a_1 - 20 = 3(9) - 20 = 27 - 20 = 7 \\ n = 3 \qquad & a_3 = 3a_2 - 20 = 3(7) - 20 = 21 - 20 = 1 \\ n = 4 \qquad & a_4 = 3a_3 - 20 = 3(1) - 20 = 3 - 20 = -17 \\ n = 5 \qquad & a_5 = 3a_4 - 20 = 3(-17) - 20 = -51 - 20 = -71 \end{aligned}$$

The first five terms are $\{9, 7, 1, -17, -71\}$. See Figure 14.1.5

Figure 14.1.5

Checkpoint 14.1.5

Write the first five terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 2 \\ a_n &= 2a_{n-1} + 1 \qquad \text{for } n \geq 2 \end{aligned}$$

Example 14.1.6: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first six terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_n &= 3a_{n-1} + 4a_{n-2} \qquad \text{for } n \geq 3 \end{aligned}$$

Solution

The first two terms are given. For each subsequent term, we replace a_{n-1} and a_{n-2} with the values of the two preceding terms.

$$\begin{aligned} n = 3 \qquad & a_3 = 3a_2 + 4a_1 = 3(2) + 4(1) = 10 \\ n = 4 \qquad & a_4 = 3a_3 + 4a_2 = 3(10) + 4(2) = 38 \\ n = 5 \qquad & a_5 = 3a_4 + 4a_3 = 3(38) + 4(10) = 154 \\ n = 6 \qquad & a_6 = 3a_5 + 4a_4 = 3(154) + 4(38) = 614 \end{aligned}$$

The first six terms are $\{1, 2, 10, 38, 154, 614\}$. See Figure 14.1.6.

Figure 14.1.6

? Checkpoint 14.1.7

Write the first 8 terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= 1 \\ a_n &= \frac{a_{n-1}}{a_{n-2}} + a_{n-3} \qquad \text{for } n \geq 4 \end{aligned}$$

Using Factorial Notation

The formulas for some sequences include products of consecutive natural numbers - these are known as **factorials** and there are two (somewhat equivalent) definitions. Since we just learned about recursive formulas, let's start with the recursive formula definition of a factorial.

Definition: Factorial (Recursive Definition)

The **factorial** of a natural number n, denoted $n!$ and reads as "n **factorial**," is defined by the recursive formula

$$\begin{aligned} a_0 &= 0! = 1 \\ a_n &= n! = n \cdot a_{n-1} \qquad \text{for } n \geq 1 \end{aligned}$$

Therefore,

$$\begin{aligned} 0! &= a_0 = 1 \\ 1! &= a_1 = 1 \cdot a_0 = 1 \cdot 1 = 1 \\ 2! &= a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2 \\ 3! &= a_3 = 3 \cdot a_2 = 3 \cdot 2 = 6 \\ 4! &= a_4 = 4 \cdot a_3 = 4 \cdot 6 = 24 \\ 5! &= a_5 = 5 \cdot a_4 = 5 \cdot 24 = 120 \end{aligned}$$

We could continue forever, but there is a pattern to notice here. A habit we need to introduce at this point is to *not* simplify when trying to find patterns. Ignoring $0!$ for the moment,

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \cdot 1 \\ 3! &= 3 \cdot 2 \cdot 1 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

In general, $n!$ is the product of the natural numbers from 1 to n. This gives us our second (somewhat equivalent) definition.

Definition: Factorial (Product of Natural Numbers Definition)

The **factorial** of a natural number n, denoted $n!$ and reads as "n **factorial**," is defined to be the product of all natural numbers less than or equal to n. That is,

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

For example,

$$\begin{aligned} 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

An example of formula containing a factorial is $a_n = (n+1)!$. The sixth term of the sequence can be found by substituting 6 for n.

$$a_6 = (6+1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

The factorial of any natural number n is $n(n-1)!$ We can therefore also think of $5!$ as $5 \cdot 4!$.

I made sure to carefully state that these definitions are *somewhat* equivalent. In truth, the recursive definition is the stronger of the two because it not only includes the definition of $0!$ (which is 1), but when you reach Calculus, it will be expanded to being useable on rational numbers.

Q&A

Can factorials always be found using a calculator?

No. Factorials get large very quickly - faster than exponential functions! When the output gets too large for the calculator, it will not be able to calculate the factorial.

Example 14.1.7: Writing the Terms of a Sequence Using Factorials

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{5n}{(n+2)!}$.

Solution

Substitute $n = 1$, $n = 2$, and so on in the formula.

$$
\begin{aligned}
n=1 \qquad a_1 &= \frac{5(1)}{(1+2)!} = \frac{5}{3!} = \frac{5}{3\cdot 2\cdot 1} = \frac{5}{6} \\
n=2 \qquad a_2 &= \frac{5(2)}{(2+2)!} = \frac{10}{4!} = \frac{10}{4\cdot 3\cdot 2\cdot 1} = \frac{5}{12} \\
n=3 \qquad a_3 &= \frac{5(3)}{(3+2)!} = \frac{15}{5!} = \frac{15}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{8} \\
n=4 \qquad a_4 &= \frac{5(4)}{(4+2)!} = \frac{20}{6!} = \frac{20}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{36} \\
n=5 \qquad a_5 &= \frac{5(5)}{(5+2)!} = \frac{25}{7!} = \frac{25}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{5}{1008}
\end{aligned}
$$

The first five terms are $\left\{\frac{5}{6}, \frac{5}{12}, \frac{1}{8}, \frac{1}{36}, \frac{5}{1008}\right\}$.

Figure **14.1.7** shows the graph of the sequence. Notice that, since factorials grow very quickly, the presence of the factorial term in the denominator results in the denominator becoming much larger than the numerator as nn increases. This means the quotient gets smaller and, as the plot of the terms shows, the terms are decreasing and nearing zero.

Figure 14.1.7

? Checkpoint 14.1.7

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{(n+1)!}{2n}$.

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14.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Sequences
  - Sequences and Summation Notation - Sequences
- Recursive formula for sequence
  - Sequences and Summation Notation - Recursively-Defined Sequences

KEY CONCEPTS

DEFINITIONS

- **Sequence**: A function whose domain is the set of natural numbers.
- **Terms (of a sequence)**: The numbers in a sequence; the output values of the sequence function.
- **Infinite Sequence**: A sequence that continues indefinitely.
- **Finite Sequence**: A sequence whose domain consists of only the first n natural numbers.
- **Explicit Formula (for a sequence)**: A formula that defines the nth term of a sequence, a_n, in relation to its position n.
- **Alternating Sequence**: A sequence whose terms alternate in sign (positive, negative, positive, negative, ... or vice-versa).
- **Recursive Formula (for a sequence)**: A formula that defines each term of a sequence using a preceding term (or multiple preceding terms). Recursive formulas must always state the initial term(s).
- **Initial Term(s) (of a recursive sequence)**: The first term (or first few terms) of a sequence that are given directly in a recursive definition, before the recursive part of the formula applies.
- **Factorial (Recursive Definition)**: The factorial of a natural number n, denoted $n!$, is defined by $0! = 1$ and $n! = n \cdot (n-1)!$ for $n \geq 1$.
- **Factorial (Product of Natural Numbers Definition)**: The factorial of a natural number n, denoted $n!$, is defined as the product of all natural numbers less than or equal to n. That is, $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$.

COMMON MISTAKES

- **Incorrectly Identifying Patterns for Explicit Formulas**: When given terms and asked to find an explicit formula, misinterpreting the pattern of numerators, denominators, signs, or exponents.
- **Errors in Evaluating Recursive Formulas**: Not starting with the given initial term(s).
- **Misinterpreting Alternating Sign Notation**: Incorrectly using or interpreting powers of (-1) (e.g., $(-1)^n$, $(-1)^{n+1}$) to generate alternating signs.
- **Errors in Factorial Calculations**: Incorrectly expanding or simplifying expressions involving factorials (e.g., $\frac{n!}{(n-2)!} \neq \frac{1}{-2!}$).
- **Domain Issues with Piecewise Explicit Formulas**: Applying the wrong piece of the formula for a given term number n.
- **Assuming a Sequence is Continuous**: Graphing a sequence as a continuous curve rather than a set of distinct points, as the domain is the natural numbers.

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14.1.2: HOMEWORK

READING QUESTIONS

1. What is a sequence? What are the numbers in a sequence called?
2. What is the difference between an infinite sequence and a finite sequence?
3. What is an explicit formula for a sequence? How does it define the terms?
4. If a sequence alternates in sign, what common factor is often used in its explicit formula to achieve this alternation?
5. What is a recursive formula for a sequence? What must always be stated along with a recursive formula?
6. Give an example of a sequence that is easily defined by a recursive formula but might be more difficult to define with an explicit formula.
7. What is factorial notation (e.g., $n!$)? Provide both the recursive and product definitions.
8. What is the value of $0!$?
9. How would you find the 5th term of the sequence defined by $a_1 = 2$ and $a_n = 3a_{n-1} + 1$ for $n \geq 2$?
10. If the terms of a sequence are $\{2, 4, 8, 16, ...\}$, what is a possible explicit formula for a_n?
11. When graphing a sequence, why are the points distinct and not connected by a continuous curve?
12. Discuss the meaning of a sequence. If a finite sequence is defined by a formula, what is its domain? What about an infinite sequence?
13. Describe three ways that a sequence can be defined.
14. Is the ordered set of even numbers an infinite sequence? What about the ordered set of odd numbers? Explain why or why not.
15. What happens to the terms, a_n, of a sequence when there is a negative factor in the formula that is raised to a power that includes n? What is the term used to describe this phenomenon?
16. What is a factorial, and how is it denoted? Use an example to illustrate how factorial notation can be beneficial.

HOMEWORK

ALGEBRAIC

For the following exercises, write the first four terms of the sequence.

17. $a_n = 2^n - 2$
18. $a_n = -\frac{16}{n+1}$
19. $a_n = -(-5)^{n-1}$
20. $a_n = \frac{2^n}{n^3}$
21. $a_n = \frac{2n+1}{n^3}$
22. $a_n = 1.25 \cdot (-4)^{n-1}$
23. $a_n = -4 \cdot (-6)^{n-1}$
24. $a_n = \frac{n^2}{2n+1}$
25. $a_n = (-10)^n + 1$
26. $a_n = -\left(\frac{4 \cdot (-5)^{n-1}}{5}\right)$

For the following exercises, write the first eight terms of the piecewise sequence.

27. $a_n = \begin{cases} (-2)^n - 2, & \text{if } n \text{ is even} \\ (3)^{n-1}, & \text{if } n \text{ is odd} \end{cases}$
28. $a_n = \begin{cases} \frac{n^2}{2n+1}, & \text{if } n \leq 5 \\ n^2 - 5, & \text{if } n > 5 \end{cases}$
29. $a_n = \begin{cases} (2n+1)^2, & \text{if } n \text{ is divisible by 4} \\ \frac{2}{n}, & \text{if } n \text{ is not dividible by 4} \end{cases}$
30. $a_n = \begin{cases} -0.6 \cdot 5^{n-1}, & \text{if } n \text{ is prime or 1} \\ 2.5 \cdot (-2)^{n-1}, & \text{if } n \text{ is composite} \end{cases}$
31. $a_n = \begin{cases} 4(n^2 - 2), & \text{if } n \leq 3 \text{ or } n > 6 \\ \frac{n^2-2}{4}, & \text{if } 3 < n \leq 6 \end{cases}$

For the following exercises, write an explicit formula for each sequence.

32. $4, 7, 12, 19, 28, ...$
33. $-4, 2, -10, 14, -34, ...$
34. $1, 1, \frac{4}{3}, 2, \frac{16}{5}, ...$
35. $0, \frac{1-e^1}{1+e^2}, \frac{1-e^2}{1+e^3}, \frac{1-e^3}{1+e^4}, \frac{1-e^4}{1+e^5}, ...$
36. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, ...$

For the following exercises, write the first five terms of the sequence.

37. $a_1 = 9, a_n = a_{n-1} + n$
38. $a_1 = 3, a_n = (-3)a_{n-1}$
39. $a_1 = -4, a_n = \frac{a_{n-1}+2n}{a_{n-1}-1}$
40. $a_1 = -1, a_n = \frac{(-3)^{n-1}}{a_{n-1}-2}$
41. $a_1 = -30, a_n = (2 + a_{n-1})\left(\frac{1}{2}\right)^n$

For the following exercises, write the first eight terms of the sequence.

42. $a_1 = \frac{1}{24}, a_2 = 1, a_n = (2a_{n-2})(3a_{n-1})$
43. $a_1 = -1, a_2 = 5, a_n = a_{n-2}(3 - a_{n-1})$
44. $a_1 = 2, a_2 = 10, a_n = \frac{2(a_{n-1}+2)}{a_{n-2}}$

For the following exercises, write a recursive formula for each sequence.

45. $-2.5, -5, -10, -20, -40, ...$
46. $-8, -6, -3, 1, 6, ...$
47. $2, 4, 12, 48, 240, ...$
48. $35, 38, 41, 44, 47, ...$
49. $15, 3, \frac{3}{5}, \frac{3}{25}, \frac{3}{125}, ...$

For the following exercises, evaluate the factorial.

50. $6!$
51. $\left(\frac{12}{6}\right)!$
52. $\frac{12!}{6!}$
53. $\frac{100!}{99!}$

For the following exercises, write the first four terms of the sequence.

54. $a_n = \frac{n!}{n^2}$
55. $a_n = \frac{3 \cdot n!}{4 \cdot n!}$

56. $a_n = \frac{n!}{n^2-n-1}$
57. $a_n = \frac{100n}{n(n-1)!}$

GRAPHICAL

For the following exercises, graph the first five terms of the indicated sequence

58. $a_n = \frac{(-1)^n}{n} + n$
59. $a_n = \begin{cases} \frac{4+n}{2n}, & \text{if } n \text{ is even} \\ 3+n, & \text{if } n \text{ is odd} \end{cases}$
60. $a_1 = 2,\ a_n = (-a_{n-1}+1)^2$
61. $a_1 = 1,\ a_n = a_{n-1} + 8$
62. $a_n = \frac{(n+1)!}{(n-1)!}$

For the following exercises, write an explicit formula for the sequence using the first five points shown on the graph.

63.

64.

65.

For the following exercises, write a recursive formula for the sequence using the first five points shown on the graph.

66.

67.

TECHNOLOGY

Follow these steps to evaluate a sequence defined recursively using a graphing calculator:

- On the home screen, key in the value for the initial term a_1 and press [ENTER].
- Enter the recursive formula by keying in all numerical values given in the formula, along with the key strokes [2ND]ANS for the previous term a_{n-1}. Press [ENTER].
- Continue pressing [ENTER] to calculate the values for each successive term.

For the following exercises, use the steps above to find the indicated term or terms for the sequence.

68. Find the first five terms of the sequence $a_1 = \frac{87}{111},\ a_n = \frac{4}{3}a_{n-1} + \frac{12}{37}$. Use the >Frac feature to give fractional results.
69. Find the 15th term of the sequence $a_1 = 625,\ a_n = 0.8a_{n-1} + 18$.
70. Find the first five terms of the sequence $a_1 = 2,\ a_n = 2^{(a_{n-1}-1)} + 1$.
71. Find the first ten terms of the sequence $a_1 = 8,\ a_n = \frac{(a_{n-1}+1)!}{a_{n-1}!}$.
72. Find the tenth term of the sequence $a_1 = 2,\ a_n = na_{n-1}$

EXTENSIONS

73. Consider the sequence defined by $a_n = -6 - 8n$. Is $a_n = -421$ a term in the sequence? Verify the result.
74. What term in the sequence $a_n = \frac{n^2+4n+4}{2(n+2)}$ has the value 41? Verify the result.
75. Find a recursive formula for the sequence $1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots$. (Hint: find a pattern for a_n based on the first two terms.)
76. Calculate the first eight terms of the sequences $a_n = \frac{(n+2)!}{(n-1)!}$ and $b_n = n^3 + 3n^2 + 2n$, and then make a conjecture about the relationship between these two sequences.
77. Prove the conjecture made in the preceding exercise.

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14.2: Arithmetic Sequences

Learning Objectives

- Find the common difference for an arithmetic sequence.
- Write terms of an arithmetic sequence.
- Use a recursive formula for an arithmetic sequence.
- Use an explicit formula for an arithmetic sequence.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Companies often make large purchases, such as computers and vehicles, for business use. The book-value of these supplies decreases each year for tax purposes. This decrease in value is called **depreciation**. One method of calculating depreciation is **straight-line depreciation**, in which the value of the asset decreases by the same amount each year.

As an example, consider a woman who starts a small contracting business. She purchases a new truck for $25,000. After five years, she estimates that she will be able to sell the truck for $8,000. The loss in value of the truck will therefore be $17,000, which is $3,400 per year for five years. The truck will be worth $21,600 after the first year; $18,200 after two years; $14,800 after three years; $11,400 after four years; and $8,000 at the end of five years. In this section, we will consider specific kinds of sequences that will allow us to calculate depreciation, such as the truck's value.

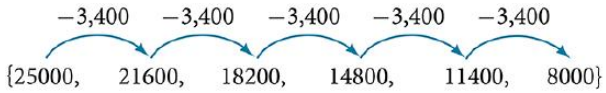
An Introduction to Arithmetic Sequences

The values of the truck in the example are said to form an **arithmetic sequence** because they change by a constant amount each year. Each term increases or decreases by the same constant value called the **common difference** of the sequence.

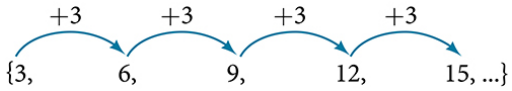
Definition: Arithmetic Sequence

An **arithmetic sequence** is a sequence having the property that the difference between any two consecutive terms is constant. This constant is called the **common difference**. Therefore, if a_1 is the first term of an arithmetic sequence and d is the common difference, the sequence will be:$$\{a_n\} = \{ a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots \}.$$

For this sequence representing the value of the truck, the common difference is -3400.



The sequence below is another example of an arithmetic sequence. In this case, the constant difference is 3. You can choose any term of the sequence, and add 3 to find the subsequent term.



Example $\PageIndex{1}$: Finding Common Differences

Is each sequence arithmetic? If so, find the common difference.

a. $\{a_n\} = \{1,2,4,8,16,\ldots\}$
b. $\{b_n\} = \{-3,1,5,9,13,\ldots\}$

Solution

Subtract each term from the subsequent term to determine whether a common difference exists.

a. The sequence is not arithmetic because there is no common difference.

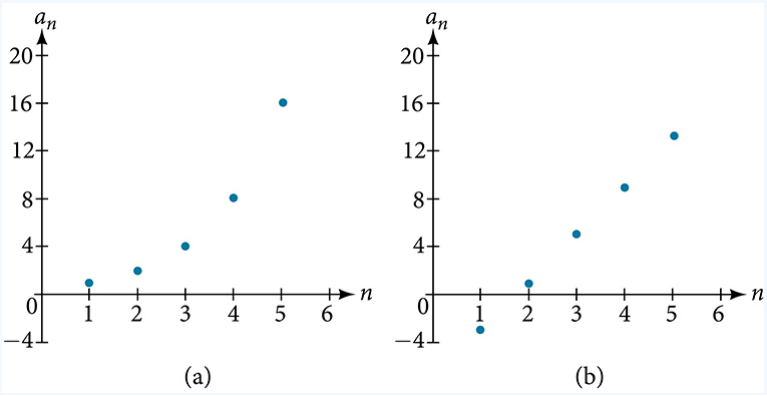
$$2 - 1 = 1 \qquad 4 - 2 = 2 \qquad 8 - 4 = 4 \qquad 16 - 8 = 8$$

b. The sequence is arithmetic because there is a common difference. The common difference is 4.

$$1 - (-3) = 4 \qquad 5 - 1 = 4 \qquad 9 - 5 = 4 \qquad 13 - 9 = 4$$

The graph of each of these sequences is shown in Figure \(\PageIndex{ 1 } \). We can see from the graphs that, although both sequences show growth, \(\{a_n\}\) is not linear whereas \(\{b_n\}\) is linear. Arithmetic sequences have a constant rate of change so their graphs will always be points on a line.

Figure \(\PageIndex{ 1 } \)



Q&A

If we are told that a sequence is arithmetic, do we have to subtract every term from the following term to find the common difference?

No. If we know that the sequence is arithmetic, we can choose any one term in the sequence, and subtract it from the subsequent term to find the common difference.

Checkpoint \(\PageIndex{ 1 } \)

Is the given sequence arithmetic? If so, find the common difference.

a. \(\{c_n\} = \{18,16,14,12,10, \ldots \}\)
b. \(\{ d_n \} = \{1,3,6,10,15, \ldots \}\)

The Recursive Formula for Arithmetic Sequences

Now that we can recognize an arithmetic sequence, we will find the terms using a recursive formula. For example, if the common difference is 5, then each term is the previous term plus 5. As with any recursive formula, the first term must be given.\[a_n = a_{n−1} + d, \quad n \geq 2 \nonumber \]

Theorem: Recursive Formula of an Arithmetic Sequence

The recursive formula for the arithmetic sequence with first term \(a_1 \) and common difference \(d \) is\[a_n = a_{n - 1} + d. \nonumber \]

Proof

This is a direct result of the definition of an arithmetic sequence.

Example \(\PageIndex{ 2 } \): Writing a Recursive Formula for an Arithmetic Sequence

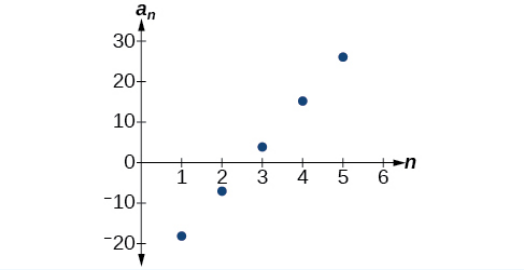
Write a recursive formula for the arithmetic sequence.\[\{−18, −7, 4, 15, 26, \ldots \} \nonumber \]

Solution

The first term is given as \(−18\) . The common difference can be found by subtracting the first term from the second term.\[d = −7 − (−18) = 11 \nonumber \]Substitute the initial term and the common difference into the recursive formula for arithmetic sequences.\[\begin{array}{rcl} a_1 & = & 18 \\[6pt] a_n & = & a_{n - 1} + 11, \text{ for }n \geq 2 \\[6pt] \end{array} \nonumber \]

We see that the common difference is the slope of the line formed when we graph the terms of the sequence, as shown in Figure \(\PageIndex{ 2 } \). The growth pattern of the sequence shows the constant difference of 11 units.

Figure \(\PageIndex{ 2 } \)



Q&A

Do we have to subtract the first term from the second term to find the common difference?

No. We can subtract any term in the sequence from the subsequent term. It is, however, most common to subtract the first term from the second term because it is often the easiest method of finding the common difference.

Checkpoint \(\PageIndex{ 2 } \)

Write a recursive formula for the arithmetic sequence.\[\{ 25, 37, 49, 61, \ldots \} \nonumber \]

The Explicit Formula for Arithmetic Sequences

We now find the terms if we are given the first term and the common difference. The terms can be found by beginning with the first term and adding the common difference repeatedly. Doing so results in the explicit formula for an arithmetic sequence.

Theorem: Explicit Formula of an Arithmetic Sequence

The explicit formula for the arithmetic sequence with first term \(a_1 \) and common difference \(d \) is\[a_n = a_1 + (n - 1)d. \nonumber \]

Proof (by Mathematical Induction)

NOTE: The method of proof for this theorem is Mathematical Induction, which is a topic generally covered *after* learning about sequences and series. If you do not know about proofs by Mathematical Induction, feel free to come back to this proof later.

Let \(a_n \) be the \(n^{\text{th}} \) term of an arithmetic sequence with common difference \(d \). Our claim is that \(a_n = a_1 + (n - 1)d \).

Base Case (\(n=1 \)): For \(n = 1 \), we get \(a_1 = a_1 + (1 - 1)d = a_1 + 0 = a_1 \), which is correct. Therefore, the claim is true for \(n = 1 \).

Induction Hypothesis: Assume the claim is true for $n = k$. That is, assume $a_k = a_1 + (k - 1)d$.

Inductive Step ($n = k + 1$): For $n = k + 1$, we would like to show that $a_{k+1} = a_1 + ((k + 1) - 1)d$.

$$\begin{array}{rclcl}
a_{k+1} & = & a_k + d & \quad & \left(\text{since the sequence is arithmetic, the next term is } a_k + d \right) \\
& = & a_1 + (k - 1)d + d & \quad & \left(\text{from the Induction Hypothesis} \right) \\
& = & a_1 + (k - 1 + 1)d & \quad & \left(\text{factoring out } d \right) \\
& = & a_1 + (k + 1 - 1)d & \quad & \left(\text{Commutative Property of Addition} \right) \\
& = & a_1 + ((k + 1) - 1)d & \quad & \left(\text{Associative Property of Addition} \right)
\end{array}$$

Thus, $a_{k+1} = a_1 + ((k + 1) - 1)\, d$, and so the claim must be true for $n = k + 1$.

Hence, the claim is true for all natural numbers by Mathematical Induction.

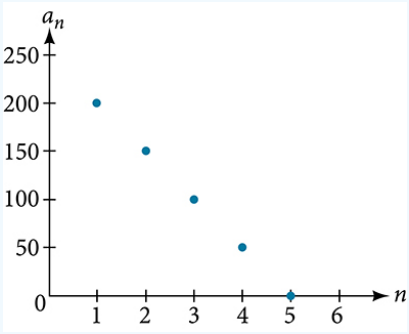
Example $\PageIndex{3}$: Writing Terms of Arithmetic Sequences

Write the first five terms of the arithmetic sequence with $a_1 = 17$ and $d = -3$.

Solution

Adding -3 is the same as subtracting 3. Beginning with the first term, subtract 3 from each term to find the next term. The first five terms are $\{17,14,11,8,5\}$. As expected, the graph of the sequence consists of points on a line as shown in Figure $\PageIndex{3}$.

Figure $\PageIndex{3}$



Checkpoint $\PageIndex{3}$

List the first five terms of the arithmetic sequence with $a_1 = 1$ and $d = 5$.

Example $\PageIndex{4}$: Writing Terms of Arithmetic Sequences

Given $a_1 = 8$ and $a_4 = 14$, find a_5.

Solution

The sequence can be written in terms of the initial term, 8, and the common difference d. $\{8, 8+d, 8+2d, 8+3d\}$. We know the fourth term equals 14; we know the fourth term has the form $a_1 + 3d = 8+3d$.

We can find the common difference d.

$$\begin{array}{rclcl}
a_n & = & a_1 + (n - 1)d & & \\
\implies 14 & = & 8 + 3d & \quad & \left(\text{substituting } n = 4,\ a_1 = 8, \text{ and } a_4 = 14 \right) \\
\implies 6 & = & 3d & \quad & \left(\text{subtracting } 8 \text{ from both sides} \right) \\
\implies 2 & = & d & \quad & \left(\text{dividing both sides by } 3 \right)
\end{array}$$

Find the fifth term by adding the common difference to the fourth term. $a_5 = a_4 + 2 = 16$.

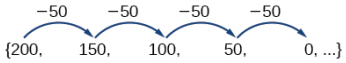
Notice that the common difference is added to the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by adding the common difference to the first term nine times or by using the equation $a_n = a_1 + (n-1)d$.

Checkpoint $\PageIndex{4}$

Given $a_3 = 7$ and $a_5 = 17$, find a_2.

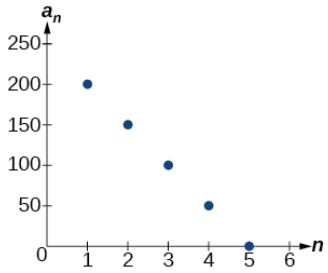
Using Explicit Formulas for Arithmetic Sequences

We can think of an arithmetic sequence as a "linear-like" function on the domain of the natural numbers - it is a "linear-like" function because it has a constant rate of change. The common difference is the constant rate of change, or the slope of the function. We can construct the linear function if we know the slope and the vertical intercept.$$a_n = a_1 + d(n-1)$$To find the y-intercept of the function, we can subtract the common difference from the first term of the sequence. Consider the following sequence.



The common difference is -50, so the sequence represents a linear function with a slope of -50. To find the y-intercept, we subtract -50 from 200:$$200 - (-50) = 200 + 50 = 250.$$The graph is shown in Figure $\PageIndex{4}$.

Figure $\PageIndex{4}$



Recall the slope-intercept form of a line is $y = mx+b$. When dealing with sequences, we use a_n in place of y and n in place of x. If we know the slope and vertical intercept of the function, we can substitute them for m and b in the slope-intercept form of a line. Substituting -50 for the slope and 250 for the vertical intercept, we get the following equation:$$a_n = -50n + 250.$$We do not need to find the vertical intercept to write an explicit formula for an arithmetic sequence. Another explicit formula for this sequence is $a_n = 200 - 50(n-1)$, which simplifies to $a_n = -50n + 250$.

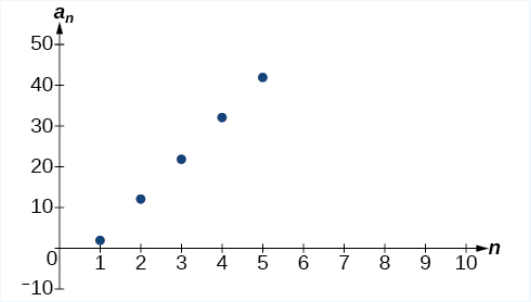
Example $\PageIndex{5}$: Writing the nth Term Explicit Formula for an Arithmetic Sequence

Write an explicit formula for the arithmetic sequence.$$\{2, 12, 22, 32, 42, \ldots\}$$

Solution

The common difference can be found by subtracting the first term from the second term.$$\begin{array}{rcl} d &=& a_2 - a_1 \\ &=& 12 - 2 \\ &=& 10 \end{array}$$The common difference is 10. Substitute the common difference and the first term of the sequence into the formula and simplify.$$\begin{array}{rcl} a_n &=& 2 + 10(n - 1) \\ \implies a_n &=& 10n - 8 \end{array}$$The graph of this sequence, represented in Figure $\PageIndex{5}$, shows a slope of 10 and a vertical intercept of -8 .

Figure \(\PageIndex{ 5 } \)



? Checkpoint \(\PageIndex{ 5 } \)

Write an explicit formula for the following arithmetic sequence.\[\{50,47,44,41, \ldots \} \nonumber \]

Finding the Number of Terms in a Finite Arithmetic Sequence

Explicit formulas can be used to determine the number of terms in a finite arithmetic sequence. We need to find the common difference, and then determine how many times the common difference must be added to the first term to obtain the final term of the sequence.

✔ Example \(\PageIndex{ 6 } \): Finding the Number of Terms in a Finite Arithmetic Sequence

Find the number of terms in the finite arithmetic sequence.\[\{ 8, 1, –6, \ldots, –41 \} \nonumber \]

Solution

The common difference can be found by subtracting the first term from the second term.\[1−8=−7 \nonumber \]The common difference is \(−7\) . Substitute the common difference and the initial term of the sequence into the \(n^{\text{th}}\) term formula and simplify.\[\begin{array}{rrclcl} & a_n & = & a_1 + d(n - 1) & & \\[6pt] \implies & a_n & = & 8 + (-7)(n - 1) & \quad & \left(\text{substituting} \right) \\[6pt] \implies & a_n & = & 15 - 7n & \quad & \left(\text{simplifying} \right) \\[6pt] \end{array} \nonumber \]Substitute \(−41\) for \(a_n \) and solve for \(n\).\[\begin{array}{rrclcl} & -41 & = & 15 - 7n & & \\[6pt] \implies & -56 & = & -7n & \quad & \left(\text{subtracting }15\text{ from both sides} \right) \\[6pt] \implies & 8 & = & n & \quad & \left(\text{dividing both sides by }-7 \right) \\[6pt] \end{array} \nonumber \]There are eight terms in the sequence.

? Checkpoint \(\PageIndex{ 6 } \)

Find the number of terms in the finite arithmetic sequence.\[\{6, 11, 16, \ldots, 56 \} \nonumber \]

Solving Application Problems with Arithmetic Sequences

In many application problems, it often makes sense to use an initial term of \(a_0 \) instead of \(a_1 \). In these problems, we alter the explicit formula slightly to account for the difference in initial terms. We use the following formula:\[a_n = a_0 + d n. \nonumber \]

✔ Example \(\PageIndex{ 7 } \): Solving Application Problems with Arithmetic Sequences

A five-year old child receives an allowance of $1 each week. His parents promise him an annual increase of $2 per week.

a. Write a formula for the child's weekly allowance in a given year.
b. What will the child's allowance be when he is 16 years old?

Solution

a. The situation can be modeled by an arithmetic sequence with an initial term of 1 and a common difference of 2. Let \(A\) be the amount of the allowance and \(n\) be the number of years after age 5. Using the altered explicit formula for an arithmetic sequence we get:\[A_n = 1 + 2 n. \nonumber \]

b. We can find the number of years since age 5 by subtracting.\[16 − 5 = 11 \nonumber \]We are looking for the child's allowance after 11 years. Substitute 11 into the formula to find the child's allowance at age 16.\[A_{11} = 1 + 2 (11) = 23 \nonumber \]The child's allowance at age 16 will be $23 per week.

? Checkpoint \(\PageIndex{ 7 } \)

A woman decides to go for a 10-minute run every day this week and plans to increase the time of her daily run by 4 minutes each week. Write a formula for the time of her run after n weeks. How long will her daily run be 8 weeks from today?

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14.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Linear Functions**
  - Equations of Lines: The explicit formula for an arithmetic sequence, $a_n = a_1 + (n-1)d$, resembles the point-slope form of a linear equation.

VIDEOS

- Arithmetic sequence
  - Arithmetic Sequences - Arithmetic Sequences

KEY CONCEPTS

DEFINITIONS

- **Arithmetic Sequence**: A sequence that has the property that the difference between any two consecutive terms is a constant.
- **Common Difference (d)**: The constant difference between any two consecutive terms in an arithmetic sequence.

THEOREMS

- **Recursive Formula of an Arithmetic Sequence**: The recursive formula for the arithmetic sequence with first term a_1 and common difference d is $a_n = a_{n-1} + d$.
- **Explicit Formula of an Arithmetic Sequence**: The explicit formula for the arithmetic sequence with first term a_1 and common difference d is $a_n = a_1 + (n-1)d$.

COMMON MISTAKES

- **Incorrectly Calculating the Common Difference**: Subtracting terms in the wrong order (e.g., $a_n - a_{n+1}$ instead of $a_{n+1} - a_n$) or assuming a sequence is arithmetic without checking if the difference is constant between all consecutive terms.
- **Errors in Applying the Recursive Formula**: Using the wrong previous term or an incorrect common difference when generating terms recursively.
- **Errors in Applying the Explicit Formula**: Using n instead of $(n-1)$ in the term $(n-1)d$, i.e., writing $a_n = a_1 + nd$.
- **Confusing Arithmetic Sequences with Other Types of Sequences**: Assuming a sequence is arithmetic when it might be geometric or neither, especially if only looking at the first few terms.
- **Off-by-One Errors with a_0 vs. a_1**: When using an explicit formula based on an initial term a_0 (like $a_n = a_0 + dn$), confusing it with the formula based on a_1 ($a_n = a_1 + (n-1)d$), or misinterpreting which term is considered the "starting" term in an application.
- **Misinterpreting "Number of Terms" in a Finite Sequence**: Errors in setting up or solving the explicit formula to find the total number of terms in a finite arithmetic sequence.

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14.2.2: HOMEWORK

READING QUESTIONS

1. What defines an arithmetic sequence?
2. How is the common difference of an arithmetic sequence found?
3. If you are told a sequence is arithmetic, do you need to check the difference between every pair of consecutive terms to find the common difference? Explain.
4. What is the recursive formula for an arithmetic sequence with first term a_1 and common difference d?
5. What is the explicit formula for an arithmetic sequence with first term a_1 and common difference d?
6. How can an arithmetic sequence be thought of as a "linear-like" function? What does the common difference correspond to in this analogy?
7. If the first term of an arithmetic sequence is $a_1 = 8$ and the fourth term is $a_4 = 14$, how can you find the common difference d?
8. How can you use an explicit formula to find the number of terms in a finite arithmetic sequence if you know the first term, the last term, and the common difference?
9. What is the difference between the explicit formula $a_n = a_1 + (n-1)d$ and $a_n = a_0 + dn$? When might the second formula be more convenient?
10. When graphing an arithmetic sequence, why are the points collinear (lie on a straight line)?
11. What is an arithmetic sequence?
12. How is the common difference of an arithmetic sequence found?
13. How do we determine whether a sequence is arithmetic?
14. What are the main differences between using a recursive formula and using an explicit formula to describe an arithmetic sequence?
15. Describe how linear functions and arithmetic sequences are similar. How are they different?

HOMEWORK

ALGEBRAIC

For the following exercises, find the common difference for the arithmetic sequence provided.

16. $\{5, 11, 17, 23, 29, ...\}$
17. $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\}$

For the following exercises, determine whether the sequence is arithmetic. If so find the common difference.

18. $\{11.4, 9.3, 7.2, 5.1, 3, ...\}$
19. $\{4, 16, 64, 256, 1024, ...\}$

For the following exercises, write the first five terms of the arithmetic sequence given the first term and common difference.

20. $a_1 = -25$, $d = -9$
21. $a_1 = 0$, $d = \frac{2}{3}$

For the following exercises, write the first five terms of the arithmetic series given two terms.

22. $a_1 = 17$, $a_7 = -31$
23. $a_{13} = -60$, $a_{33} = -160$

For the following exercises, find the specified term for the arithmetic sequence given the first term and common difference.

24. First term is 3, common difference is 4, find the 5th term.
25. First term is 4, common difference is 5, find the 4th term.
26. First term is 5, common difference is 6, find the 8th term.
27. First term is 6, common difference is 7, find the 6th term.
28. First term is 7, common difference is 8, find the 7th term.

For the following exercises, find the first term given two terms from an arithmetic sequence.

29. Find the first term or a_1 of an arithmetic sequence if $a_6 = 12$ and $a_{14} = 28$.
30. Find the first term or a_1 of an arithmetic sequence if $a_7 = 21$ and $a_{15} = 42$.
31. Find the first term or a_1 of an arithmetic sequence if $a_8 = 40$ and $a_{23} = 115$.
32. Find the first term or a_1 of an arithmetic sequence if $a_9 = 54$ and $a_{17} = 102$.
33. Find the first term or a_1 of an arithmetic sequence if $a_{11} = 11$ and $a_{21} = 16$.

For the following exercises, find the specified term given two terms from an arithmetic sequence.

34. $a_1 = 33$ and $a_7 = -15$. Find a_4.
35. $a_3 = -17.1$ and $a_{10} = -15.7$. Find a_{21}.

For the following exercises, use the recursive formula to write the first five terms of the arithmetic sequence.

36. $a_1 = 39$; $a_n = a_{n-1} - 3$
37. $a_1 = -19$; $a_n = a_{n-1} - 1.4$

For the following exercises, write a recursive formula for each arithmetic sequence.

38. $a = \{40, 60, 80, ...\}$
39. $a = \{17, 26, 35, ...\}$
40. $a = \{-1, 2, 5, ...\}$
41. $a = \{12, 17, 22, ...\}$
42. $a = \{-15, -7, 1, ...\}$
43. $a = \{8.9, 10.3, 11.7, ...\}$
44. $a = \{-0.52, -1.02, -1.52, ...\}$
45. $a = \{\frac{1}{5}, \frac{9}{20}, \frac{7}{10}, ...\}$
46. $a = \{-\frac{1}{2}, -\frac{5}{4}, -2, ...\}$
47. $a = \{\frac{1}{6}, -\frac{11}{12}, -2, ...\}$

For the following exercises, write a recursive formula for the given arithmetic sequence, and then find the specified term.

48. $a = \{7, 4, 1, ...\}$; Find the 17th term.
49. $a = \{4, 11, 18, ...\}$; Find the 14th term.
50. $a = \{2, 6, 10, ...\}$; Find the 12th term.

For the following exercises, use the explicit formula to write the first five terms of the arithmetic sequence.

51. $a_n = 24 - 4n$
52. $a_n = \frac{1}{2}n - \frac{1}{2}$

For the following exercises, write an explicit formula for each arithmetic sequence.

53. $a = \{3, 5, 7, \ldots\}$
54. $a = \{32, 24, 16, \ldots\}$
55. $a = \{-5, 95, 195, \ldots\}$
56. $a = \{-17, -217, -417, \ldots\}$
57. $a = \{1.8, 3.6, 5.4, \ldots\}$
58. $a = \{-18.1, -16.2, -14.3, \ldots\}$
59. $a = \{15.8, 18.5, 21.2, \ldots\}$
60. $a = \left\{\frac{1}{3}, -\frac{4}{3}, -3, \ldots\right\}$
61. $a = \left\{0, \frac{1}{3}, \frac{2}{3}, \ldots\right\}$
62. $a = \left\{-5, -\frac{10}{3}, -\frac{5}{3}, \ldots\right\}$

For the following exercises, find the number of terms in the given finite arithmetic sequence.

63. $a = \{3, -4, -11, \ldots, -60\}$
64. $a = \{1.2, 1.4, 1.6, \ldots, 3.8\}$
65. $a = \left\{\frac{1}{2}, 2, \frac{7}{2}, \ldots, 8\right\}$

GRAPHICAL

For the following exercises, determine whether the graph shown represents an arithmetic sequence.

66.

67.

For the following exercises, use the information provided to graph the first 5 terms of the arithmetic sequence.

68. $a_1 = 0$, $d = 4$
69. $a_1 = 9$; $a_n = a_{n-1} - 10$
70. $a_n = -12 + 5n$

EXTENSIONS

71. Give two examples of arithmetic sequences whose 4th terms are 9.
72. Give two examples of arithmetic sequences whose 10th terms are 206.
73. Find the 5th term of the arithmetic sequence $\{9b, 5b, b, \ldots\}$.
74. Find the 11th term of the arithmetic sequence $\{3a - 2b, a + 2b, -a + 6b \ldots\}$.
75. At which term does the sequence $\{5.4, 14.5, 23.6, \ldots\}$ exceed 151?
76. At which term does the sequence $\left\{\frac{17}{3}, \frac{31}{6}, \frac{14}{3}, \ldots\right\}$ begin to have negative values?
77. For which terms does the finite arithmetic sequence $\left\{\frac{5}{2}, \frac{19}{8}, \frac{9}{4}, \ldots, \frac{1}{8}\right\}$ have integer values?
78. Write an arithmetic sequence using a recursive formula. Show the first 4 terms, and then find the 31st term.
79. Write an arithmetic sequence using an explicit formula. Show the first 4 terms, and then find the 28th term.

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14.3: Geometric Sequences

Learning Objectives

- Find the common ratio for a geometric sequence.
- List the terms of a geometric sequence.
- Use a recursive formula for a geometric sequence.
- Use an explicit formula for a geometric sequence.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Many jobs offer an annual cost-of-living increase to keep salaries consistent with inflation. Suppose, for example, a recent college graduate finds a position as a sales manager earning an annual salary of $26,000. He is promised a 2% cost of living increase each year. His annual salary in any given year can be found by multiplying his salary from the previous year by 102%. His salary will be $26,520 after one year; $27,050.40 after two years; $27,591.41 after three years; and so on. When a salary increases by a constant rate each year, the salary grows by a constant factor. In this section, we will review sequences that grow in this way.

An Introduction to Geometric Sequences

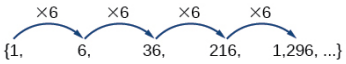
The yearly salary values described form a **geometric sequence** because they change by a constant factor each year. Each term of a geometric sequence increases or decreases by a constant factor called the **common ratio**.

Definition: Geometric Sequence

A **geometric sequence** is a sequence having the property that the ratio between any two consecutive terms is constant. This constant is called the **common ratio**. Therefore, if a_1 is the first term of a geometric sequence and r is the common ratio, the sequence will be:

$$\{a_n\} = \{a_1, a_1 r, a_1 r^2, a_1 r^3, \ldots\}.$$

The sequence below is an example of a geometric sequence because each term increases by a constant factor of 6. Multiplying any term of the sequence by the common ratio 6 generates the subsequent term.



Example 14.3.1: Finding Common Ratios

Is the sequence geometric? If so, find the common ratio.

a. $1, 2, 4, 8, 16, \ldots$
b. $48, 12, 4, 2, \ldots$

Solution

Divide each term by the previous term to determine whether a common ratio exists.

a. The sequence is geometric because there is a common ratio. The common ratio is 2.
b. The sequence is not geometric because there is not a common ratio.

The graph of each sequence is shown in Figure 14.3.1. It seems from the graphs that both (a) and (b) appear have the form of the graph of an exponential function in this viewing window. However, we know that (a) is geometric and so this interpretation holds, but (b) is not.

Figure 14.3.1

Q&A

If you are told that a sequence is geometric, do you have to divide every term by the previous term to find the common ratio?

No. If you know that the sequence is geometric, you can choose any one term in the sequence and divide it by the previous term to find the common ratio.

Checkpoint 14.3.1

Is the sequence geometric? If so, find the common ratio.

a. $5, 10, 15, 20, \ldots$
b. $100, 20, 4, \frac{4}{5}, \ldots$

The Recursive Formula for Geometric Sequences

We begin finding a formula for a geometric sequence as we did in the previous section - by developing a recursive formula. This allows us to find any term of a geometric sequence by using the previous term. Each term is the product of the common ratio and the previous term. For example, suppose the common ratio is 9. Then each term is nine times the previous term. As with any recursive formula, the initial term must be given.

Theorem: Recursive Formula of a Geometric Sequence

The recursive formula for the geometric sequence with first term a_1 and common ratio r is

$$a_n = a_{n-1}\, r.$$

Proof

This is a direct result of the definition of a geometric sequence.

Example 14.3.2: Using Recursive Formulas for Geometric Sequences

Write a recursive formula for the following geometric sequence.

$$\{6, 9, 13.5, 20.25, \ldots\}$$

Solution

The first term is given as 6. The common ratio can be found by dividing the second term by the first term.

$$r = \frac{9}{6} = 1.5$$

Substitute the common ratio into the recursive formula for geometric sequences and define a_1.

$$\begin{aligned} a_n &= 1.5a_{n-1}, \text{ for } n \geq 2 \\ a_1 &= 6 \end{aligned}$$

The sequence of data points follows an exponential pattern. The common ratio is also the base of an exponential function as shown in Figure **14.3.2**.

Figure 14.3.2

Q&A

Do we have to divide the second term by the first term to find the common ratio?

No. We can divide any term in the sequence by the previous term. It is, however, most common to divide the second term by the first term because it is often the easiest method of finding the common ratio.

Checkpoint **14.3.2**

Write a recursive formula for the following geometric sequence.

$$\left\{2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots\right\}$$

The Explicit Formula for Geometric Sequences

Now that we can identify a geometric sequence, we will learn how to find the terms of a geometric sequence if we are given the first term and the common ratio. The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. Do so results in the following theorem.

Theorem: Explicit Formula of a Geometric Sequence

The explicit formula for the geometric sequence with first term a_1 and common ratio r is

$$a_n = a_1 r^{n-1}.$$

Proof (by Mathematical Induction)

NOTE: The method of proof for this theorem is Mathematical Induction, which is a topic generally covered *after* learning about sequences and series. If you do not know about proofs by Mathematical Induction, feel free to come back to this proof later.

Let a_n be the nth term of a geometric sequence with common ratio r. Our claim is that $a_n = a_1 r^{n-1}$.

Base Case ($n = 1$): For $n = 1$, we get $a_1 = a_1 r^{1-1} = a_1 r^0 = a_1 \cdot 1 = a_1$, which is correct. Therefore, the claim is true for n

$= 1$.

Induction Hypothesis: Assume the claim is true for $n = k$. That is, assume $a_k = a_1 r^{k-1}$.

Inductive Step ($n = k+1$): For $n = k+1$, we would like to show that $a_{k+1} = a_1 r^{(k+1)-1}$.

$$\begin{aligned} a_{k+1} &= a_k r && \text{(since the sequence is geometric, the next term is } a_k r) \\ &= a_1 r^{k-1} \cdot r && \text{(from the Induction Hypothesis)} \\ &= a_1 r^{k-1+1} && \text{(Laws of Exponents)} \\ &= a_1 r^{k+1-1} && \text{(Commutative Property of Addition)} \\ &= a_1 r^{(k+1)-1} && \text{(Associative Property of Addition)} \end{aligned}$$

Thus, $a_{k+1} = a_1 r^{(k+1)-1}$, and so the claim must be true for $n = k+1$.

Hence, the claim is true for all natural numbers by Mathematical Induction.

Example 14.3.3: Writing the Terms of a Geometric Sequence

List the first four terms of the geometric sequence with $a_1 = 5$ and $r = -2$.

Solution

Multiply a_1 by -2 to find a_2. Repeat the process, using a_2 to find a_3, and so on.

$$\begin{aligned} a_1 &= 5 \\ a_2 &= -2a_1 = (-2) \cdot 5 = -10 \\ a_3 &= -2a_2 = (-2) \cdot (-10) = 20 \\ a_4 &= -2a_3 = (-2) \cdot 20 = -40 \end{aligned}$$

The first four terms are $\{5, -10, 20, -40\}$.

Checkpoint 14.3.3

List the first five terms of the geometric sequence with $a_1 = 18$ and $r = \frac{1}{3}$.

Using Explicit Formulas for Geometric Sequences

Let's take a look at the sequence $\{18, 36, 72, 144, 288, ...\}$. This is a geometric sequence with a common ratio of 2 and an "exponential-like" function with a base of 2. An explicit formula for this sequence is

$$a_n = 18 \cdot 2^{n-1}.$$

The graph of the sequence is shown in Figure 14.3.3.

Figure 14.3.3

Example 14.3.4: Writing Terms of Geometric Sequences Using the Explicit Formula

Given a geometric sequence with $a_1 = 3$ and $a_4 = 24$, find a_2.

Solution

The sequence can be written in terms of the initial term and the common ratio r.

$$3, 3r, 3r^2, 3r^3, \ldots$$

Find the common ratio using the given fourth term.

$$\begin{aligned} & a_n = a_1 r^{n-1} & \\ \Longrightarrow \quad & a_4 = 3r^3 & \text{(substituting } n = 4\text{)} \\ \Longrightarrow \quad & 24 = 3r^3 & \text{(substituting } a_4 = 24\text{)} \\ \Longrightarrow \quad & 8 = r^3 & \text{(dividing both sides by 3)} \\ \Longrightarrow \quad & 2 = r & \text{(taking the cube root of both sides)} \end{aligned}$$

Find the second term by multiplying the first term by the common ratio.

$$a_2 = 2a_1 = 2(3) = 6$$

Note that the common ratio is multiplied by the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by multiplying the first term by the common ratio nine times or by multiplying by the common ratio raised to the ninth power.

Checkpoint 14.3.4

Given a geometric sequence with $a_2 = 4$ and $a_3 = 32$, find a_6.

Example 14.3.5: Writing an Explicit Formula for the nth Term of a Geometric Sequence

Write an explicit formula for the nth term of the following geometric sequence.

$$\{2, 10, 50, 250, \ldots\}$$

Solution

The first term is 2. The common ratio can be found by dividing the second term by the first term.

$$\frac{10}{2} = 5$$

The common ratio is 5. Substitute the common ratio and the first term of the sequence into the formula.

$$a_n = 2 \cdot 5^{n-1}$$

The graph of this sequence in Figure **14.3.4** shows an exponential pattern.

Figure 14.3.4

? Checkpoint 14.3.5

Write an explicit formula for the following geometric sequence.

$$\{-1, 3, -9, 27, \ldots\}$$

Solving Application Problems with Geometric Sequences

In real-world scenarios involving geometric sequences, we may need to use an initial term of a_0 instead of a_1. In these problems, we can alter the explicit formula slightly by using the following formula:

$$a_n = a_0 r^n.$$

✓ Example 14.3.6: Solving Application Problems with Geometric Sequences

In 2013, the number of students in a small school is 284. It is estimated that the student population will increase by 4% each year.

a. Write a formula for the student population.
b. Estimate the student population in 2030.

Solution

a. The situation can be modeled by a geometric sequence with an initial term of 284. The student population will be 104% of the prior year, so the common ratio is 1.04.

   Let P be the student population and n be the number of years after 2013. Using the explicit formula for a geometric sequence we get

$$P_n = 284 \cdot 1.04^n.$$

b. We can find the number of years since 2013 by subtracting.

$$2030 - 2013 = 17$$

   We are looking for the population after 17 years. We can substitute 17 for n to estimate the population in 2030.

$$P_{17} = 284 \cdot 1.04^{17} \approx 553.$$

The student population will be about 553 in 2030.

? Checkpoint 14.3.6

A business starts a_new website. Initially the number of hits is 293 due to the curiosity factor. The business estimates the number of hits will increase by 2.6% per week.

a. Write a formula for the number of hits.
b. Estimate the number of hits in 5 weeks.

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14.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Geometric sequence
  - Geometric Sequences - Geometric Sequences

KEY CONCEPTS

DEFINITIONS

- **Geometric Sequence**: A sequence having the property that the ratio between any two consecutive terms is constant.
- **Common Ratio (r)**: The constant ratio between any two consecutive terms in a geometric sequence. It is found by dividing any term by its preceding term ($r = \frac{a_n}{a_{n-1}}$).

THEOREMS

- **Recursive Formula of a Geometric Sequence**: The recursive formula for the geometric sequence with first term a_1 and common ratio r is $a_n = a_{n-1} \cdot r$.
- **Explicit Formula of a Geometric Sequence**: The explicit formula for the geometric sequence with first term a_1 and common ratio r is $a_n = a_1 r^{n-1}$.

COMMON MISTAKES

- **Incorrectly Calculating the Common Ratio**: Dividing terms in the wrong order (e.g., a_n/a_{n+1} instead of a_{n+1}/a_n) or assuming a sequence is geometric without checking if the ratio is constant between all consecutive terms.
- **Errors in Applying the Recursive Formula**: Using the wrong previous term or an incorrect common ratio when generating terms recursively.
- **Errors in Applying the Explicit Formula**: Using r^n instead of r^{n-1} in the formula $a_n = a_1 r^{n-1}$.
- **Confusing Geometric Sequences with Arithmetic Sequences**: Applying concepts of common difference to geometric sequences or common ratio to arithmetic sequences.
- **Off-by-One Errors with a_0 vs. a_1**: When using an explicit formula based on an initial term a_0 (like $a_n = a_0 r^n$), confusing it with the formula based on a_1 ($a_n = a_1 r^{n-1}$), or misinterpreting which term is considered the "starting" term in an application.
- **Sign Errors with Negative Common Ratios**: Forgetting that terms will alternate in sign if the common ratio r is negative.

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14.3.2: HOMEWORK

READING QUESTIONS

1. What defines a geometric sequence?
2. How is the common ratio of a geometric sequence found?
3. If you are told a sequence is geometric, do you need to divide every term by its preceding term to find the common ratio? Explain.
4. What is the recursive formula for a geometric sequence with first term a_1 and common ratio r?
5. What is the explicit formula for a geometric sequence with first term a_1 and common ratio r?
6. How are geometric sequences similar to exponential functions? What corresponds to the base of the exponential function?
7. If the first term of a geometric sequence is $a_1 = 3$ and the fourth term is $a_4 = 24$, how can you find the common ratio r?
8. What is the difference between the explicit formula $a_n = a_1 r^{n-1}$ and $a_n = a_0 r^n$? When might the second formula be more convenient?
9. If the common ratio r of a geometric sequence is negative, what happens to the signs of the terms?
10. When graphing a geometric sequence, what kind of pattern do the points typically form if all terms are positive and $r > 1$?
11. What is a geometric sequence?
12. How is the common ratio of a geometric sequence found?
13. What is the procedure for determining whether a sequence is geometric?
14. What is the difference between an arithmetic sequence and a geometric sequence?
15. Describe how exponential functions and geometric sequences are similar. How are they different?

HOMEWORK

ALGEBRAIC

For the following exercises, find the common ratio for the geometric sequence.

16. $1, 3, 9, 27, 81, \ldots$
17. $-0.125, 0.25, -0.5, 1, -2, \ldots$
18. $-2, -\frac{1}{2}, -\frac{1}{8}, -\frac{1}{32}, -\frac{1}{128}, \ldots$

For the following exercises, determine whether the sequence is geometric. If so, find the common ratio.

19. $-6, -12, -24, -48, -96, \ldots$
20. $5, 5.2, 5.4, 5.6, 5.8, \ldots$
21. $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$
22. $6, 8, 11, 15, 20, \ldots$
23. $0.8, 4, 20, 100, 500, \ldots$

For the following exercises, write the first five terms of the geometric sequence, given the first term and common ratio.

24. $a_1 = 8$, $r = 0.3$
25. $a_1 = 5$, $r = \frac{1}{5}$

For the following exercises, write the first five terms of the geometric sequence, given any two terms.

26. $a_7 = 64$, $a_{10} = 512$
27. $a_6 = 25$, $a_8 = 6.25$

For the following exercises, find the specified term for the geometric sequence, given the first term and common ratio.

28. The first term is 2, and the common ratio is 3. Find the 5th term.
29. The first term is 16 and the common ratio is $-\frac{1}{3}$. Find the 4th term.

For the following exercises, find the specified term for the geometric sequence, given the first four terms.

30. $a_n = \{-1, 2, -4, 8, \ldots\}$. Find a_{12}.
31. $a_n = \left\{-2, \frac{2}{3}, -\frac{2}{9}, \frac{2}{27}, \ldots\right\}$. Find a_7.

For the following exercises, write the first five terms of the geometric sequence.

32. $a_1 = -486$, $a_n = -\frac{1}{3}a_{n-1}$
33. $a_1 = 7$, $a_n = 0.2a_{n-1}$

For the following exercises, write a recursive formula for each geometric sequence.

34. $a_n = \{-1, 5, -25, 125, \ldots\}$
35. $a_n = \{-32, -16, -8, -4, \ldots\}$
36. $a_n = \{14, 56, 224, 896, \ldots\}$
37. $a_n = \{10, -3, 0.9, -0.27, \ldots\}$
38. $a_n = \{0.61, 1.83, 5.49, 16.47, \ldots\}$
39. $a_n = \left\{\frac{3}{5}, \frac{1}{10}, \frac{1}{60}, \frac{1}{360}, \ldots\right\}$
40. $a_n = \left\{-2, \frac{4}{3}, -\frac{8}{9}, \frac{16}{27}, \ldots\right\}$
41. $a_n = \left\{\frac{1}{512}, -\frac{1}{128}, \frac{1}{32}, -\frac{1}{8}, \ldots\right\}$

For the following exercises, write the first five terms of the geometric sequence.

42. $a_n = -4 \cdot 5^{n-1}$
43. $a_n = 12 \cdot \left(-\frac{1}{2}\right)^{n-1}$

For the following exercises, write an explicit formula for each geometric sequence.

44. $a_n = \{-2, -4, -8, -16, \ldots\}$
45. $a_n = \{1, 3, 9, 27, \ldots\}$
46. $a_n = \{-4, -12, -36, -108, \ldots\}$
47. $a_n = \{0.8, -4, 20, -100, \ldots\}$
48. $a_n = \{-1.25, -5, -20, -80, \ldots\}$
49. $a_n = \left\{-1, -\frac{4}{5}, -\frac{16}{25}, -\frac{64}{125}, \ldots\right\}$
50. $a_n = \left\{2, \frac{1}{3}, \frac{1}{18}, \frac{1}{108}, \ldots\right\}$
51. $a_n = \left\{3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots\right\}$

For the following exercises, find the specified term for the geometric sequence given.

52. Let $a_1 = 4$, $a_n = -3a_{n-1}$. Find a_8.
53. Let $a_n = -\left(-\frac{1}{3}\right)^{n-1}$. Find a_{12}.

For the following exercises, find the number of terms in the given finite geometric sequence.

54. $a_n = \{-1, 3, -9, \ldots, 2187\}$
55. $a_n = \left\{2, 1, \frac{1}{2}, \ldots, \frac{1}{1024}\right\}$

GRAPHICAL

For the following exercises, determine whether the graph shown represents a geometric sequence.

56.

57.

For the following exercises, use the information provided to graph the first five terms of the geometric sequence.

58. $a_1 = 1$, $r = \frac{1}{2}$
59. $a_1 = 3$, $a_n = 2a_{n-1}$
60. $a_n = 27 \cdot 0.3^{n-1}$

EXTENSIONS

61. Use recursive formulas to give two examples of geometric sequences whose 3rd terms are 200.
62. Use explicit formulas to give two examples of geometric sequences whose 7th terms are 1024.
63. Find the 5th term of the geometric sequence $\{b, 4b, 16b, \ldots\}$.
64. Find the 7th term of the geometric sequence $\{64a(-b), 32a(-3b), 16a(-9b), \ldots\}$.
65. At which term does the sequence $\{10, 12, 14.4, 17.28, \ldots\}$ exceed 100?
66. At which term does the sequence $\left\{\frac{1}{2187}, \frac{1}{729}, \frac{1}{243}, \frac{1}{81}, \ldots\right\}$ begin to have integer values?
67. For which term does the geometric sequence $a_n = -36\left(\frac{2}{3}\right)^{n-1}$ first have a non-integer value?
68. Use the recursive formula to write a geometric sequence whose common ratio is an integer. Show the first four terms, and then find the 10th term.
69. Use the explicit formula to write a geometric sequence whose common ratio is a decimal number between 0 and 1. Show the first 4 terms, and then find the 8th term.
70. Is it possible for a sequence to be both arithmetic and geometric? If so, give an example.

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14.4: Series and Their Notations

Learning Objectives

- Use summation notation.
- Use the formula for the sum of the first n terms of an arithmetic series.
- Use the formula for the sum of the first n terms of a geometric series.
- Use the formula for the sum of an infinite geometric series.
- Solve annuity problems.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

A couple decides to start a college fund for their daughter. They plan to invest $50 in the fund each month. The fund pays 6% annual interest, compounded monthly. How much money will they have saved when their daughter is ready to start college in 6 years? In this section, we will learn how to answer this question. To do so, we need to consider the amount of money invested and the amount of interest earned.

Using Summation Notation

To find the total amount of money in the college fund and the sum of the amounts deposited, we need to add the amounts deposited each month and the amounts earned in interest monthly. The sum of the terms of a sequence is called a **series**.

Definition: Series

The sum of the terms of a (finite or infinite) sequence is called a **series**.

Consider, for example, the following sequence.

$$\{a_n\} = \{3, 7, 11, 15, 19, ...\}$$

The series for this sequence is the infinite summation

$$3 + 7 + 11 + 15 + 19 + ...$$

Obviously, adding terms of a sequence forever is impossible, so we begin our excursion into summations of sequences by asking ourselves what would happen if we only added up, say, the first n terms of the sequence. This is called the nth **partial sum**.

Definition: nth Partial Sum of a Series

The nth **partial sum** of a series is the sum of the first n terms of that series, and it is denoted S_n.

For example, with the series above, the first few partial sums are as follows:

$$\begin{aligned} S_1 &= 3 \\ S_2 &= 3 + 7 = 10 \\ S_3 &= 3 + 7 + 11 = 21 \\ S_4 &= 3 + 7 + 11 + 15 = 36 \end{aligned}$$

Since partial sums can get pretty bulky to write, we introduce a new notation. **Summation notation** is used to represent series.

Definition: Summation Notation

Summation notation, also known as **Sigma notation**, is a concise way to express the sum of a series of terms. Given a function f, we write

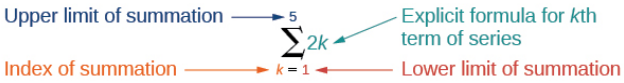
$$\sum_{k=a}^{b} f(k)$$

to represent the operation

$$f(a) + f(a+1) + f(a+2) + \cdots + f(n).$$

Summation notation uses the Greek capital letter Sigma (Σ) to indicate operation of summation, along with an **index variable** (k in the notation above, but this choice is arbitrary), a **lower limit** of summation (a in the notation above), and an **upper limit** of summation (n in the notation above). We read the notation $\sum_{k=a}^{b} f(k)$ as "the sum from $k = a$ to $k = b$ of $f(k)$."

The index variable is also called the **index of summation**.



If we interpret the given notation, we see that it asks us to find the sum of the terms in the series $a_k = 2k$ for $k = 1$ through $k = 5$. We can begin by substituting the terms for k and listing out the terms of this series.

$$\begin{aligned} a_1 &= 2(1) = 2 \\ a_2 &= 2(2) = 4 \\ a_3 &= 2(3) = 6 \\ a_4 &= 2(4) = 8 \\ a_5 &= 2(5) = 10 \end{aligned}$$

We can find the sum of the series by adding the terms:

$$\sum_{k=1}^{5} 2k = 2 + 4 + 6 + 8 + 10 = 30.$$

Q&A

Does the lower limit of summation have to be 1?

No (and this is why our definition didn't start with the lower limit set as 1). The lower limit of summation can be any number, but 1 is frequently used. We will look at examples with lower limits of summation other than 1.

Example 14.4.1: Using Summation Notation

Evaluate $\sum_{k=3}^{7} k^2$.

Solution

According to the notation, the lower limit of summation is 3 and the upper limit is 7. So we need to find the sum of k^2 from $k = 3$ to $k = 7$. We find the terms of the series by substituting $k = 3, 4, 5, 6,$ and 7 into the function k^2. We add the terms to find the sum.

$$\begin{aligned} \sum_{k=1}^{7} k^2 &= 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \\ &= 9 + 16 + 25 + 36 + 49 \\ &= 135 \end{aligned}$$

Checkpoint 14.4.1

Evaluate $\sum_{k=2}^{5} (3k - 1)$.

Arithmetic Series

Just as we studied special types of sequences, we will look at special types of series. Recall that an arithmetic sequence is a sequence in which the difference between any two consecutive terms is the common difference, d. That is, if d is the common difference and a_1 is the first term for the arithmetic sequence, then $a_2 - a_1 = d$. Moreover, in general, $a_n - a_{n-1} = d$. Since this difference is constant between any two terms in the arithmetic sequence, we could write the sequence terms as follows:

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_1 + 2d \\ a_4 &= a_1 + 3d \\ \vdots &= \vdots \\ a_n &= a_1 + (n-1)d \end{aligned}$$

Definition: Arithmetic Series

The sum of the terms of a (finite or infinite) arithmetic sequence is called an **arithmetic series**.

Arithmetic series are special in that we have a closed formula for their n^{th} partial sum.

Theorem: nth Partial Sum of an Arithmetic Series

Given the arithmetic sequence $a_n = a_1 + (n-1)d$, the sum of the first n terms of the corresponding arithmetic series is

$$S_n = n \cdot \frac{a_1 + a_n}{2}.$$

Proof

We can write the sum of the first n terms of the arithmetic series as:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-3)d) + (a_1 + (n-2)d) + (a_1 + (n-1)d).$$

We can also reverse the order of the terms and write the sum as

$$S_n = (a_1 + (n–1)d) + (a_1 + (n-2)d) + (a_1 + (n-3)d) + \ldots + (a_1 + 2d) + (a_1 + d) + a_1.$$

If we add these two expressions for the sum of the first n terms of the arithmetic series, we can derive a formula for the sum of the first n terms of any arithmetic series.

$$\begin{array}{rcccccccccccccc} S_n & = & a_1 & + & a_1+d & + & a_1+2d & + & \cdots & + & a_1+(n-3)d & + & a_1+(n-2)d & + & a_1+(n-1)d \\ + \; S_n & = & a_1+(n-1)d & + & a_1+(n-2)d & + & a_1+(n-3)d & + & \cdots & + & a_1+2d & + & a_1+d & + & a_1 \\ \hline 2S_n & = & 2a_1+(n-1)d & + & 2a_1+(n-1)d & + & 2a_1+(n-1)d & + & \cdots & + & 2a_1+(n-1)d & + & 2a_1+(n-1)d & + & 2a_1+(n-1)d \end{array}$$

Because there are n terms in the series, we can simplify this sum to

$$2S_n = n(2a_1 + (n-1)d).$$

However, we know that $a_1 + (n-1)d = a_n$. Therefore, we can rewrite the relation as

$$2S_n = n(a_1 + a_1 + (n-1)d) = n(a_1 + a_2).$$

We divide by 2 to find the formula for the sum of the first n terms of an arithmetic series.

$$S_n = n \cdot \frac{a_1 + a_n}{2}.$$

That is, the sum of the first n terms of an arithmetic sequence is the product of n and the average of the first and n^{th} terms of the sequence.

Example 14.4.2 Finding the First n Terms of an Arithmetic Series

Find the sum of each arithmetic series.

a. $5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32$

b. $20 + 15 + 10 + \ldots + (-50)$

c. $\sum_{k=1}^{12} 3k - 8$

Solutions

a. We are given $a_1 = 5$ and $a_n = 32$. A quick count shows the number of terms to be 10. Thus, $n = 10$.

Substitute values for a_1, a_n, and n into the formula and simplify.

$$S_{10} = 10 \cdot \frac{5 + 32}{2} = 5 \cdot (37) = 185.$$

b. We are given $a_1 = 20$ and $a_n = -50$. The sequence given suggests that $d = -5$. Use the formula for the general term of an arithmetic sequence to find n.

$$\begin{array}{rcll} a_n & = & a_1 + (n-1)d & \\ \Rightarrow \quad -50 & = & 20 + (n-1)(-5) & \text{(substituting)} \\ \Rightarrow \quad -70 & = & (n-1)(-5) & \text{(subtracting 20 from both sides)} \\ \Rightarrow \quad 14 & = & n-1 & \text{(dividing both sides by } -5\text{)} \\ \Rightarrow \quad 15 & = & n & \text{(adding 1 to both sides)} \end{array}$$

Substitute values for a_1, a_n, and n into the formula and simplify.

$$S_{15} = 15 \cdot \frac{20 - 50}{2} = 15 \cdot (-15) = -225.$$

c. To find a_1, substitute $k = 1$ into the given explicit formula.

$$\begin{aligned} a_k &= 3k - 8 \\ \Rightarrow \quad a_1 &= 3(1) - 8 = -5 \end{aligned}$$

We are given that $n = 12$. To find a_{12}, substitute $k = 12$ into the given explicit formula.

$$\begin{aligned} a_k &= 3k - 8 \\ \Rightarrow \quad a_{12} &= 3(12) - 8 = 28 \end{aligned}$$

Substitute values for a_1, a_n, and n into the formula and simplify.

$$\begin{aligned} S_n &= n \cdot \frac{a_1 + a_n}{2} \\ \Rightarrow \quad S_{12} &= 12 \cdot \frac{-5 + 28}{2} = 6 \cdot 23 = 138 \end{aligned}$$

? Checkpoint 14.4.2

Find the sum of each arithmetic series.

a. $1.4 + 1.6 + 1.8 + 2.0 + 2.2 + 2.4 + 2.6 + 2.8 + 3.0 + 3.2 + 3.4$

b. $13 + 21 + 29 + \ldots + 69$

c. $\sum_{k=1}^{10} 5 - 6k$

✓ Example 14.4.3: Solving Application Problems with Arithmetic Series

On the Sunday after a minor surgery, a woman is able to walk a half-mile. Each Sunday, she walks an additional quarter-mile. After 8 weeks, what will be the total number of miles she has walked?

Solution

This problem can be modeled by an arithmetic series with $a_1 = \frac{1}{2}$ and $d = \frac{1}{4}$. We are looking for the total number of miles walked after 8 weeks, so we know that $n = 8$, and we are looking for S_8. To find a_8, we can use the explicit formula for an arithmetic sequence.

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ \Rightarrow \quad a_8 &= \frac{1}{2} + \frac{1}{4}(8-1) = \frac{9}{4} \end{aligned}$$

We can now use the formula for arithmetic series.

$$\begin{aligned} S_n &= n \cdot \frac{a_1 + a_n}{2} \\ \Rightarrow \quad S_8 &= 8 \cdot \frac{\frac{1}{2} + \frac{9}{4}}{2} = 4 \cdot \left(\frac{11}{4}\right) = 11 \end{aligned}$$

She will have walked a total of 11 miles.

? Checkpoint 14.4.3

A man earns $100 in the first week of June. Each week, he earns $12.50 more than the previous week. After 12 weeks, how much has he earned?

Geometric Series

Just as the sum of the terms of an arithmetic sequence is called an arithmetic series, the sum of the terms in a geometric sequence is called a **geometric series**.

Definition: Geometric Series

The sum of the terms of a (finite or infinite) geometric sequence is called an **geometric series**.

Likewise, we can derive a closed formula for the nth partial sum of a geometric series.

Theorem: nth Partial Sum of a Geometric Series

Given the geometric sequence $a_n = a_1 r^{n-1}$, the sum of the first n terms of the corresponding geometric series is

$$S_n = \frac{a_1(1-r^n)}{1-r}.$$

Proof

We can write the sum of the first n terms of a geometric series as

$$S_n = a_1 + ra_1 + r^2a_1 + \ldots + r^{n-1}a_1.$$

Multiplying both sides of the equation by r, we get the following:

$$rS_n = ra_1 + r^2a_1 + r^3a_1 + \ldots + r^na_1.$$

Next, we subtract this equation from the original equation.

$$\begin{array}{rrcccccccccccc} & S_n & = & a_1 & + & ra_1 & + & r^2a_1 & + \cdots + & r^{n-3}a_1 & + & r^{n-2}a_1 & + & r^{n-1}a_1 \\ - & rS_n & = - (& & & ra_1 & + & r^2a_1 & + \cdots + & r^{n-3}a_1 & + & r^{n-2}a_1 & + & r^{n-1}a_1 & + & r^na_1) \\ \hline & (1-r)S_n & = & a_1 & + & 0 & + & 0 & + \cdots + & 0 & + & 0 & + & 0 & - & r^na_1 \end{array}$$

Notice that when we subtract, all but the first term of the top equation and the last term of the bottom equation cancel out. To obtain a formula for S_n, divide both sides by $1-r$.

$$S_n = \frac{a_1(1-r^n)}{1-r}, \text{ where } r \neq 1.$$

Notice that this formula does not exist when $r = 1$. This is not an issue because geometric sequences (which geometric series are based upon) are required to have common ratios not equal to 1.

Example 14.4.4: Finding the First n Terms of a Geometric Series

Use the formula to find the indicated partial sum of each geometric series.

a. S_{11} for the series $8 + (-4) + 2 + \ldots$

b. $\sum_{k=1}^{6} 3 \cdot 2^k$

Solutions

a. $a_1 = 8$, and we are given that $n = 11$. We can find r by dividing the second term of the series by the first.

$$r = \frac{-4}{8} = -\frac{1}{2}.$$

Substitute values for a_1, r, and n into the formula and simplify.

$$\begin{aligned} S_n &= \frac{a_1(1-r^n)}{1-r} \\ \Rightarrow \quad S_{11} &= \frac{8\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} \approx 5.336 \end{aligned}$$

b. Find a_1 by substituting $k = 1$ into the given explicit formula.

$$a_1 = 3 \cdot 2^1 = 6.$$

We can see from the given explicit formula that $r = 2$. The upper limit of summation is 6, so $n = 6$.

Substitute values for a_1, r, and n into the formula, and simplify.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$\Longrightarrow \quad S_6 = \frac{6\left(1-(2)^6\right)}{1-2} = 378$$

? Checkpoint 14.4.4

Use the formula to find the indicated partial sum of each geometric series.

a. S_{20} for the series $1,000 + 500 + 250 + \ldots$

b. $\sum_{k=1}^{8} 3^k$

✔ Example 14.4.5: Solving an Application Problem with a Geometric Series

At a new job, an employee's starting salary is $26,750. He receives a 1.6% annual raise. Find his total earnings at the end of 5 years.

Solution

The problem can be represented by a geometric series with $a_1 = 26,750$, $n = 5$, and $r = 1.016$. Substitute values for a_1, r, and n into the formula and simplify to find the total amount earned at the end of 5 years.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$\Longrightarrow \quad S_5 = \frac{26,750\left(1-1.016^5\right)}{1-1.016} \approx 138,099.03$$

He will have earned a total of $138,099.03 by the end of 5 years.

? Checkpoint 14.4.5

At a new job, an employee's starting salary is $32,100. She receives a 2% annual raise. How much will she have earned by the end of 8 years?

Infinite Series

Thus far, we have only considered partial sums of series. These are also known as **finite series**. Sometimes, however, we are interested in the sum of the terms of an infinite sequence rather than the sum of only the first n terms.

Definition: Infinite Series

An **infinite series** is the sum of the terms of an infinite sequence.

An example of an infinite series is $2 + 4 + 6 + 8 + \ldots$.

This series can also be written in summation notation as $\sum_{k=1}^{\infty} 2k$, where the upper limit of summation is infinity. Because the terms are not tending to zero, the sum of the series increases without bound as we add more terms. Therefore, the sum of this infinite series is not defined; however, some infinite series tend to a finite value. A major focus of Calculus is determining which infinite series tend to finite values. As such, we need to introduce new language.

Definition: Convergent/Divergent Infinite Series

When the sum of an infinite series is not a real number, we say the series **diverges** and we call the series a **divergent series**. On the other hand, if the sum of an infinite series becomes finite, we say the series **converges** and we call the series a **convergent series**.

Caution: Convergence is Not Intuitive (at first)

The study of Calculus will present you with many concepts that seem counterintuitive. For example, it seems reasonable to state that an infinite series where the terms are tending to zero must become finite.

For example, consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The terms are tending to zero as we keep writing out the series. In fact, by the time we get to the thousandth term, we are only adding $\frac{1}{1000}$. Therefore, it *seems* logical to state that this summation would approach a finite value. Afterall, we are essentially adding zero as time goes on.

Unfortunately, this is *not true*!

In general, if the terms of a series tend to zero, we cannot guarantee that the sum of those terms will become finite. The series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

have terms that tend to zero as n tends to get large; however, the first of these series is divergent, while the second is convergent (in fact, eerily, it converges to $\ln(2)$).

In general, finding the sum of a convergent (infinite) series is a task left for Calculus.

Infinite Geometric Series

Among infinite series, infinite *geometric* series claim a special place. This is because, when an infinite geometric series converges, we can calculate its sum - this is often not true for other type of convergent infinite series. The formula for the sum of an infinite geometric series is related to the formula for the sum of the first n terms of the geometric series:

$$S_n = \frac{a_1(1-r^n)}{1-r}.$$

We will examine an infinite series with $r = \frac{1}{2}$. What happens to r^n as n increases?

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$
$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

The value of r^n decreases rapidly. What happens for greater values of n?

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{1,024}$$
$$\left(\frac{1}{2}\right)^{20} = \frac{1}{1,048,576}$$
$$\left(\frac{1}{2}\right)^{30} = \frac{1}{1,073,741,824}$$

As n gets very large, r^n gets very small. We say that, as n increases without bound, r^n approaches 0. Borrowing notation from our exploration of rational functions, as $n \to \infty$, $r^n \to 0$. As r^n approaches 0, $1 - r^n$ approaches 1. When this happens, the numerator approaches a_1. This give us a formula for the sum of a convergent infinite geometric series.

Theorem: Sum of a Convergent Infinite Geometric Series

The infinite geometric series

$$\sum_{n=1}^{\infty} a_1 r^{n-1}$$

converges to $\frac{a_1}{1-r}$ as long as $-1 < r < 1$. That is,

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1-r}, \quad \text{if } |r| < 1.$$

The infinite geometric series diverges otherwise.

Proof

A true proof of this theorem cannot be done without Calculus; however, your instructor can *justify* this theorem in class using a little bit of "hand-waving."

Example 14.4.6: Determining Whether the Sum of an Infinite Series is Defined

Determine whether the sum of each infinite series is defined.

a. $\frac{3}{4}+\frac{1}{2}+\frac{1}{3}+\ldots$

b. $\sum_{k=1}^{\infty} 27 \cdot \left(\frac{1}{3}\right)^k$

Solutions

a. The ratio of the second term to the first is the same as the ratio of the third term to the second. The series is geometric with a common ratio of $\frac{2}{3}$. Since $-1 < \frac{2}{3} < 1$, this series must converge.

b. The given formula is exponential with a base of $\frac{1}{3}$. Therefore, the series is geometric with a common ratio of $\frac{1}{3}$, and so the the infinite series is convergent.

Checkpoint 14.4.6

Determine whether the sum of the infinite series is defined.

a. $24+(-12)+6+(-3)+\ldots$

b. $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\frac{9}{8}+\ldots$

c. $\sum_{k=1}^{\infty} 15 \cdot (-0.3)^k$

Example 14.4.7: Finding the Sum of an Infinite Geometric Series

Find the sum, if it exists, for the following:

a. $10+9+8+7+\ldots$

b. $248.6+99.44+39.776+\ldots$

c. $\sum_{k=1}^{\infty} 4,374 \cdot \left(-\frac{1}{3}\right)^{k-1}$

d. $\sum_{k=1}^{\infty} \frac{1}{9} \cdot \left(\frac{4}{3}\right)^k$

Solutions

a. Since there is not a constant ratio between consecutive terms of the series, the series is not geometric. As such, even if it did converge (which it doesn't), we don't have a theorem to help us find its value.

b. There is a constant ratio; the series is geometric. $a_1 = 248.6$ and $r = \frac{99.44}{248.6} = 0.4$, which is between -1 and 1. Therefore, the geometric series must converge. Substitute $a_1 = 248.6$ and $r = 0.4$ into the formula and simplify to find the sum:

$$S = \frac{a_1}{1-r}$$

$$\Rightarrow \quad S = \frac{248.6}{1-0.4} = 414.\overline{3}$$

c. The formula is exponential, so the series is geometric with $r = -\frac{1}{3}$. Find a_1 by substituting $k = 1$ into the given explicit formula:

$$a_1 = 4,374 \cdot \left(-\frac{1}{3}\right)^{1-1} = 4,374$$

Substitute $a_1 = 4,374$ and $r = -\frac{1}{3}$ into the formula, and simplify to find the sum:

$$S = \frac{a_1}{1-r}$$

$$\Rightarrow \quad S = \frac{4,374}{1-\left(-\frac{1}{3}\right)} = 3,280.5$$

d. The formula is exponential, so the series is geometric, but $r \geq 1$. The series must diverge.

Checkpoint 14.4.7

Find the sum, if it exists.

a. $2+\frac{2}{3}+\frac{2}{9}+\ldots$

b. $\sum_{k=1}^{\infty} 0.76k + 1$

c. $\sum_{k=1}^{\infty} \left(-\frac{3}{8}\right)^k$

A common "classroom" use of convergent infinite geometric series is to find fractional equivalents of repeating decimal numbers.

Example 14.4.8: Finding an Equivalent Fraction for a Repeating Decimal

Find an equivalent fraction for the repeating decimal $0.\overline{3}$

Solution

We notice the repeating decimal $0.\overline{3} = 0.333...$, so we can rewrite the repeating decimal as a sum of terms.

$$0.\overline{3} = 0.3 + 0.03 + 0.003 + ...$$

Looking for a pattern, we rewrite the sum, noticing that $0.3 = 0.3 \cdot 1 = 0.3 \cdot (0.1)^0$, $0.03 = 0.3 \cdot 0.1 = 0.3 \cdot (0.1)^1$, $0.003 = 0.3 \cdot 0.01 = 0.33 \cdot (0.1)^2$, and so on.

$$0.\overline{3} = 0.3 + 3(0.1)^1 + 3(0.1)^2 + \cdots$$

Notice the pattern; we multiply each consecutive term by a common ratio of 0.1 starting with the first term of 0.3. So, substituting into our formula for an infinite geometric sum, we have

$$S_n = \frac{a_1}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}.$$

Solving Annuity Problems

At the beginning of the section, we looked at a problem in which a couple invested a set amount of money each month into a college fund for six years. An annuity is an investment in which the purchaser makes a sequence of periodic, equal payments. To find the amount of an annuity, we need to find the sum of all the payments and the interest earned. In the example, the couple invests $50 each month. This is the value of the initial deposit. The account paid 6% annual interest, compounded monthly. To find the interest rate per payment period, we need to divide the 6% annual percentage rate (APR) by 12. So the monthly interest rate is 0.5%. We can multiply the amount in the account each month by 100.5% to find the value of the account after interest has been added.

We can find the value of the annuity right after the last deposit by using a geometric series with $a_1 = 50$ and $r = 100.5$. After the first deposit, the value of the annuity will be $50. Let us see if we can determine the amount in the college fund and the interest earned.

We can find the value of the annuity after n deposits using the formula for the sum of the first n terms of a geometric series. In 6 years, there are 72 months, so $n = 72$. We can substitute $a_1 = 50$, $r = 1.005$, and $n = 72$ into the formula, and simplify to find the value of the annuity after 6 years.

$$S_{72} = \frac{50(1-1.005^{72})}{1-1.005} \approx 4,320.44.$$

After the last deposit, the couple will have a total of $4,320.44 in the account. Notice, the couple made 72 payments of $50 each for a total of $72(50) = \$3,600$. This means that because of the annuity, the couple earned $720.44 interest in their college fund.

Example 14.4.9: Solving an Annuity Problem

A deposit of $100 is placed into a college fund at the beginning of every month for 10 years. The fund earns 9% annual interest, compounded monthly, and paid at the end of the month. How much is in the account right after the last deposit?

Solution

The value of the initial deposit is $100, so $a_1 = 100$. A total of 120 monthly deposits are made in the 10 years, so $n = 120$. To find r, divide the annual interest rate by 12 to find the monthly interest rate and add 1 to represent the new monthly deposit.

$$r = 1 + \frac{0.09}{12} = 1.0075.$$

Substitute $a_1 = 100$, $r = 1.0075$, and $n = 120$ into the formula for the sum of the first n terms of a geometric series, and simplify to find the value of the annuity.

$$S_{120} = \frac{100(1-1.0075^{120})}{1-1.0075} \approx 19,351.43$$

So the account has $19,351.43 after the last deposit is made.

? Checkpoint 14.4.9

At the beginning of each month, $200 is deposited into a retirement fund. The fund earns 6% annual interest, compounded monthly, and paid into the account at the end of the month. How much is in the account if deposits are made for 10 years?

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14.4.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Partial sums
  - Sequences and Summation Notation - Partial Sums of a Sequence
  - Arithmetic Sequences - Partial Sums of Arithmetic Sequences
  - Geometric Sequences - Partial Sums of Geometric Sequences
- Summation notation
  - Sequences and Summation Notation - Using Sigma Notation
- Infinite series
  - Geometric Series - Infinite Series
- Infinite geometric series
  - Geometric Series - Infinite Geometric Series
  - Geometric Series - Applications of Infinite Geometric Series

KEY CONCEPTS

DEFINITIONS

- **Series**: The sum of the terms of a (finite or infinite) sequence.
- **n^{th} Partial Sum (S_n)**: The sum of the first n terms of a sequence.
- **Summation Notation (Sigma Notation)**: A concise way to express the sum of a series of terms using the Greek capital letter Sigma (Σ). It includes an index variable (index of summation), a lower limit of summation, an upper limit of summation, and an explicit formula for the kth term. For example, $\sum_{k=a}^{b} f(k)$.
- **Index Variable (Index of Summation)**: The variable used in summation notation (e.g., k in $\sum_{k=a}^{b} f(k)$) that takes on integer values from the lower limit to the upper limit.
- **Lower Limit of Summation**: The starting value for the index variable in summation notation.
- **Upper Limit of Summation**: The ending value for the index variable in summation notation.
- **Arithmetic Series**: The sum of the terms of a (finite or infinite) arithmetic sequence.
- **Geometric Series**: The sum of the terms of a (finite or infinite) geometric sequence.
- **Infinite Series**: The sum of the terms of an infinite sequence.
- **Convergent Series**: An infinite series whose sum approaches a finite real number.
- **Divergent Series**: An infinite series whose sum is not a real number (e.g., tends to infinity or oscillates without approaching a limit).
- **Annuity**: An investment in which the purchaser makes a sequence of periodic, equal payments.

THEOREMS

- **Formula for the nth Partial Sum of an Arithmetic Series**: Given the arithmetic sequence $a_n = a_1 + (n-1)d$, the sum of the first n terms is $S_n = n\frac{a_1+a_n}{2}$ or $S_n = \frac{n}{2}(2a_1 + (n-1)d)$. (The first form is explicitly derived and used in the text, the second is a common alternative.)
- **Formula for the nth Partial Sum of a Geometric Series**: Given the geometric sequence $a_n = a_1 r^{n-1}$, the sum of the first n terms is $S_n = \frac{a_1(1-r^n)}{1-r}$, for $r \neq 1$.
- **Formula for the Sum of a Convergent Infinite Geometric Series**: The infinite geometric series $\sum_{n=1}^{\infty} a_1 r^{n-1}$ converges to the sum $S = \frac{a_1}{1-r}$ if and only if $|r| < 1$. If $|r| \geq 1$, the series diverges.

COMMON MISTAKES

- **Confusing Sequences with Series**: A sequence is a list of terms, while a series is the sum of those terms.
- **Errors in Summation Notation**: Incorrectly identifying the lower or upper limits of summation.
- **Using the Wrong Sum Formula**: Applying the arithmetic series sum formula to a geometric series, or vice-versa. (Implicit from having distinct formulas)
- **Errors in Arithmetic Series Sum Formula**: Incorrectly identifying a_1, a_n, or n.
- **Errors in Geometric Series Sum Formula (S_n)**: Incorrectly identifying a_1, r, or n.
- **Errors with Infinite Geometric Series**: Attempting to find the sum of a divergent infinite geometric series (i.e., when $|r| \geq 1$).
- **Assuming All Series with Terms Tending to Zero Converge**: Believing that if the terms of an infinite series approach zero, the series must converge. This is not true (e.g., the harmonic series $\sum \frac{1}{n}$ diverges).

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14.4.2: HOMEWORK

READING QUESTIONS

1. What is an nth partial sum of a series?
2. What is the difference between an arithmetic sequence and an arithmetic series?
3. What is a geometric series?
4. How is finding the sum of an infinite geometric series different from finding the nth partial sum of a geometric series? What condition must be met for an infinite geometric series to have a sum?
5. What is an annuity in the context of finance?
6. Explain what each part of the summation notation $\sum_{k=1}^{N} a_k$ represents.
7. What is the formula for the sum of the first n terms of an arithmetic series?
8. What is the formula for the sum of the first n terms of a geometric series?
9. Under what condition does an infinite geometric series converge (have a finite sum)? What is the formula for this sum?
10. If the terms of an infinite series tend to zero, does the series necessarily converge? Explain.
11. What is an nth partial sum?
12. What is the difference between an arithmetic sequence and an arithmetic series?
13. What is a geometric series?
14. How is finding the sum of an infinite geometric series different from finding the nth partial sum?
15. What is an annuity?

HOMEWORK

ALGEBRAIC

For the following exercises, express each description of a sum using summation notation.

16. The sum of terms m^2+3m from $m=1$ to $m=5$
17. The sum from of $n=0$ to $n=4$ of $5n$
18. The sum of $6k-5$ from $k=-2$ to $k=1$
19. The sum that results from adding the number 4 five times

For the following exercises, express each arithmetic sum using summation notation.

20. $5+10+15+20+25+30+35+40+45+50$
21. $10+18+26+\ldots+162$
22. $\frac{1}{2}+1+\frac{3}{2}+2+\ldots+4$

For the following exercises, use the formula for the sum of the first n terms of each arithmetic sequence.

23. $\frac{3}{2}+2+\frac{5}{2}+3+\frac{7}{2}$
24. $19+25+31+\ldots+73$
25. $3.2+3.4+3.6+\ldots+5.6$

For the following exercises, express each geometric sum using summation notation.

26. $1+3+9+27+81+243+729+2187$
27. $8+4+2+\ldots+0.125$
28. $-\frac{1}{6}+\frac{1}{12}-\frac{1}{24}+\ldots+\frac{1}{768}$

For the following exercises, use the formula for the sum of the first n terms of each geometric sequence, and then state the indicated sum.

29. $9+3+1+\frac{1}{3}+\frac{1}{9}$
30. $\sum_{n=1}^{9} 5\cdot 2^{n-1}$
31. $\sum_{a=1}^{11} 64\cdot 0.2^{a-1}$

For the following exercises, determine whether the infinite series has a sum. If so, write the formula for the sum. If not, state the reason.

32. $12+18+24+30+\ldots$
33. $2+1.6+1.28+1.024+\ldots$
34. $\sum_{m=1}^{\infty} 4^{m-1}$
35. $\sum_{k=1}^{\infty} -\left(-\frac{1}{2}\right)^{k-1}$

GRAPHICAL

For the following exercises, use the following scenario. Javier makes monthly deposits into a savings account. He opened the account with an initial deposit of $50. Each month thereafter he increased the previous deposit amount by $20.

36. Graph the arithmetic sequence showing one year of Javier's deposits.
37. Graph the arithmetic series showing the monthly sums of one year of Javier's deposits.

For the following exercises, use the geometric series $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$.

38. Graph the first 7 partial sums of the series.
39. What number does S_n seem to be approaching in the graph? Find the sum to explain why this makes sense.

NUMERIC

For the following exercises, find the indicated sum.

40. $\sum_{a=1}^{14} a$
41. $\sum_{n=1}^{6} n(n-2)$
42. $\sum_{k=1}^{17} k^2$
43. $\sum_{k=1}^{7} 2^k$

For the following exercises, use the formula for the sum of the first n terms of an arithmetic series to find the sum.

44. $-1.7+-0.4+0.9+2.2+3.5+4.8$
45. $6+\frac{15}{2}+9+\frac{21}{2}+12+\frac{27}{2}+15$
46. $-1+3+7+\ldots+31$

47. $\sum_{k=1}^{11}\left(\frac{k}{2}-\frac{1}{2}\right)$

For the following exercises, use the formula for the sum of the first n terms of a geometric series to find the partial sum.

48. S_6 for the series $-2-10-50-250-\ldots$
49. S_7 for the series $0.4-2+10-50+\ldots$
50. $\sum_{k=1}^{9} 2^{k-1}$
51. $\sum_{n=1}^{10} -2\cdot\left(\frac{1}{2}\right)^{n-1}$

For the following exercises, find the sum of the infinite geometric series.

52. $4+2+1+\frac{1}{2}+\ldots$
53. $-1-\frac{1}{4}-\frac{1}{16}-\frac{1}{64}-\ldots$
54. $\sum_{k=1}^{\infty} 3\cdot\left(\frac{1}{4}\right)^{k-1}$
55. $\sum_{n=1}^{\infty} 4.6\cdot 0.5^{n-1}$

For the following exercises, determine the value of the annuity for the indicated monthly deposit amount, the number of deposits, and the interest rate.

56. Deposit amount: \$50; total deposits: 60; interest rate: 5%, compounded monthly
57. Deposit amount: \$150; total deposits: 24; interest rate: 3%, compounded monthly
58. Deposit amount: \$450; total deposits: 60; interest rate: 4.5%, compounded quarterly
59. Deposit amount: \$100; total deposits: 120; interest rate: 10%, compounded semi-annually

EXTENSIONS

60. The sum of terms $50-k^2$ from $k=x$ through 7 is 115. What is x?
61. Write an explicit formula for a_k such that $\sum_{k=0}^{6} a_k = 189$. Assume this is an arithmetic series.
62. Find the smallest value of n such that $\sum_{k=1}^{n}(3k-5) > 100$.
63. How many terms must be added before the series $-1-3-5-7-\ldots$ has a sum less than -75?
64. Write $0.\overline{65}$ as an infinite geometric series using summation notation. Then use the formula for finding the sum of an infinite geometric series to convert $0.\overline{65}$ to a fraction.
65. The sum of an infinite geometric series is five times the value of the first term. What is the common ratio of the series?
66. To get the best loan rates available, the Riches want to save enough money to place 20% down on a \$160,000 home. They plan to make monthly deposits of \$125 in an investment account that offers 8.5% annual interest compounded semi-annually. Will the Riches have enough for a 20% down payment after five years of saving? How much money will they have saved?
67. Karl has two years to save \$10,000 to buy a used car when he graduates. To the nearest dollar, what would his monthly deposits need to be if he invests in an account offering a 4.2% annual interest rate that compounds monthly?

REAL-WORLD APPLICATIONS

68. Keisha devised a week-long study plan to prepare for finals. On the first day, she plans to study for 1 hour, and each successive day she will increase her study time by 30 minutes. How many hours will Keisha have studied after one week?
69. A boulder rolled down a mountain, traveling 6 feet in the first second. Each successive second, its distance increased by 8 feet. How far did the boulder travel after 10 seconds?
70. A scientist places 50 cells in a petri dish. Every hour, the population increases by 1.5%. What will the cell count be after 1 day?
71. A pendulum travels a distance of 3 feet on its first swing. On each successive swing, it travels $\frac{3}{4}$ the distance of the previous swing. What is the total distance traveled by the pendulum when it stops swinging?
72. Rachael deposits \$1,500 into a retirement fund each year. The fund earns 8.2% annual interest, compounded monthly. If she opened her account when she was 19 years old, how much will she have by the time she is 55? How much of that amount will be interest earned?

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14.5: Mathematical Induction

Learning Objectives

- Use the Principle of Mathematical Induction to prove algebraic identities.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The Chinese philosopher Confucius is credited with the saying, "A journey of a thousand miles begins with a single step." In many ways, this is the central theme of this section. Here, we introduce a method of proof, **Mathematical Induction**, which allows us to prove many of the formulas we have merely motivated in previous sections. A good example is the formula for arithmetic sequences.

As mentioned previously, arithmetic sequences can be defined recursively, starting with $a_1 = a$ and then $a_{n+1} = a_n + d$ for $n \geq 1$. This tells us that we start the sequence with a and go from one term to the next by successively adding d. In symbols,

$$a, a+d, a+2d, a+3d, a+4d, \ldots$$

The pattern suggested here is that to reach the nth term, we start with a and add d to it exactly $n-1$ times, which lead us to our formula $a_n = a + (n-1)d$ for $n \geq 1$. But how do we prove this to be the case?

The Principle of Mathematical Induction (PMI)

Suppose $P(n)$ is a sentence involving the natural number n.

IF

$P(1)$ is true and whenever $P(k)$ is true, it follows that $P(k+1)$ is also true,

THEN

the sentence $P(n)$ is true for all natural numbers n.

The **Principle of Mathematical Induction**, or **PMI** for short, is exactly that - a principle (also known as an **axiom**). It is a property of the natural numbers we either choose to accept or reject. In English, it says that if we want to prove a formula works for all natural numbers n, we start by showing it is true for $n = 1$ (the "**base case**") and then show that if it is true for an arbitrary natural number k, it must be true for the next natural number, $k+1$ (the "**inductive step**").

The notation $P(n)$ acts just like function notation. For example, if $P(n)$ is the claim that "$n^2 + 1 = 3$," then $P(1)$ would be "$1^2 + 1 = 3$," which is false. The construction $P(k+1)$ would be "$(k+1)^2 + 1 = 3$." As usual, this new concept is best illustrated with an example.

Returning to our quest to prove the formula for an arithmetic sequence, we first identify $P(n)$ as the claim $a_n = a + (n-1)d$. We need to do two things to prove this claim is true for all natural numbers n. First, we need to establish that $P(1)$ is true. In other words, is it true that $a_1 = a + (1-1)d$? The answer is yes since this simplifies to $a_1 = a$, which is part of the definition of the arithmetic sequence. The second thing we need to show is that whenever $P(k)$ is true, it follows that $P(k+1)$ is true. In other words, we *assume* $P(k)$ is true (this is called the "**induction hypothesis**") and *deduce* (through logical arguments) that $P(k+1)$ is also true.

Assuming $P(k)$ to be true seems to invite disaster - after all, isn't this essentially what we're trying to prove in the first place? To help explain this step better, we show how this works for specific values of n. We've already established that $P(1)$ is true, and we

now want to show that $P(2)$ is true. Thus, we need to show that $a_2 = a + (2-1)d$. Since $P(1)$ is true, we have $a_1 = a$, and by the definition of an arithmetic sequence, $a_2 = a_1 + d = a + d = a + (2-1)d$. So $P(2)$ is true.

We now use the fact that $P(2)$ is true to show that $P(3)$ is true. Using the fact that $a_2 = a + (2-1)d$, we show $a_3 = a + (3-1)d$. Since $a_3 = a_2 + d$, we get $a_3 = (a + (2-1)d) + d = a + 2d = a + (3-1)d$, so we have shown $P(3)$ is true. Similarly, we can use the fact that $P(3)$ is true to show that $P(4)$ is true, and so forth.

In general, if $P(k)$ is true (i.e., $a_k = a + (k-1)d$) we set out to show that $P(k+1)$ is true (i.e., $a_{k+1} = a + ((k+1)-1)d$). Assuming $a_k = a + (k-1)d$, we have by the definition of an arithmetic sequence that $a_{k+1} = a_k + d$ so we get $a_{k+1} = (a + (k-1)d) + d = a + kd = a + ((k+1)-1)d$. Hence, $P(k+1)$ is true.

In essence, by showing that $P(k+1)$ must always be true when $P(k)$ is true, we are showing that the formula $P(1)$ can be used to get the formula $P(2)$, which in turn can be used to derive the formula $P(3)$, which in turn can be used to establish the formula $P(4)$, and so on. Thus, as long as $P(k)$ is true for some natural number k, $P(n)$ is true for all of the natural numbers n which follow k. Coupling this with the fact that $P(1)$ is true, we have established $P(k)$ is true for all natural numbers that follow $n = 1$; in other words, all natural numbers n. One might liken Mathematical Induction to a repetitive process like climbing stairs. If you are sure that (1) you can get on the stairs (the base case) and (2) you can climb from any one step to the next step (the inductive step), then presumably you can climb the entire staircase (or falling dominoes, which is the most widely used metaphor in mainstream College Algebra books). We get some more practice with induction in the following example.

Example 14.5.1: Proving a Summation Formula

Prove the following, where $n \in \mathbb{N}$.

$$\sum_{j=1}^{n}(a + (j-1)d) = \frac{n}{2}(2a + (n-1)d)$$

Solution

We set $P(n)$ to be the equation we are asked to prove. For $n = 1$, we compare both sides of the equation given in $P(n)$:

$$\begin{aligned} \sum_{j=1}^{1}(a + (j-1)d) &\stackrel{?}{=} \frac{1}{2}(2a + (1-1)d) \\ a + (1-1)d &\stackrel{?}{=} \frac{1}{2}(2a) \\ a &= a \checkmark \end{aligned}$$

This shows the base case $P(1)$ is true. Next we assume $P(k)$ is true, that is, we assume

$$\sum_{j=1}^{k}(a + (j-1)d) = \frac{k}{2}(2a + (k-1)d)$$

and attempt to use this to show $P(k+1)$ is true. Namely, we must show

$$\sum_{j=1}^{k+1}(a + (j-1)d) = \frac{k+1}{2}(2a + (k+1-1)d).$$

To see how we can use $P(k)$ in this case to prove $P(k+1)$, we note that the sum in $P(k+1)$ is the sum of the first $k+1$ terms of the sequence $a_k = a + (k-1)d$ for $k \geq 1$ while the sum in $P(k)$ is the sum of the first k terms. We compare both sides of the equation in $P(k+1)$.

$$\underbrace{\sum_{j=1}^{k+1}(a+(j-1)d)}_{\text{summing the first } k+1 \text{ terms}} \stackrel{?}{=} \frac{k+1}{2}(2a+(k+1-1)d)$$

$$\underbrace{\sum_{j=1}^{k}(a+(j-1)d)}_{\text{summing the first } k \text{ terms}} + \underbrace{(a+(k+1-1)d)}_{\text{adding the } (k+1)\text{st term}} \stackrel{?}{=} \frac{k+1}{2}(2a+kd)$$

$$\underbrace{\frac{k}{2}(2a+(k-1)d)}_{\text{Using } P(k)} + (a+kd) \stackrel{?}{=} \frac{(k+1)(2a+kd)}{2}$$

$$\frac{k(2a+(k-1)d)+2(a+kd)}{2} \stackrel{?}{=} \frac{2ka+k^2d+2a+kd}{2}$$

$$\frac{2ka+2a+k^2d+kd}{2} = \frac{2ka+2a+k^2d+kd}{2} \checkmark$$

Since all of our steps on both sides of the string of equations are reversible, we conclude that the two sides of the equation are equivalent, and hence, $P(k+1)$ is true. By the Principle of Mathematical Induction, we have that $P(n)$ is true for all natural numbers n.

Warning!

Pay close attention to the fact that we used the symbol $\stackrel{?}{=}$ when comparing both sides of the equations in Example 14.5.1 for $P(1)$ and $P(k+1)$. This symbol means that we are not certain if those two sides are actually equal. As such, we cannot treat it like an equality. That is, we are not allowed to add, subtract, multiply, or divide between sides (as you can only do that if you have an equality or inequality). In fact, we cannot perform *any* operations *between* sides!

To be clear, it would be mathematically *illegal* to do the following when comparing both sides of $P(n)$ for $n=1$:

$$\sum_{j=1}^{1}(a+(j-1)d) \stackrel{?}{=} \frac{1}{2}(2a+(1-1)d)$$

$$a+(1-1)d \stackrel{?}{=} \frac{1}{2}(2a)$$

$$a \stackrel{?}{=} \frac{1}{2}(2a)$$

$$0 \stackrel{?}{=} \frac{1}{2}(2a)-a \quad \text{(subtracting } a \text{ from both sides)}$$

$$0 \stackrel{?}{=} a-a$$

$$0 = 0 \checkmark$$

Once you have performed an operation between the two sides of the $\stackrel{?}{=}$ symbol, you have *assumed* that the equation is already true.

Example 14.5.2: Proving an Algebraic Claim

Prove that, for a complex number z, $(\overline{z})^n = \overline{z^n}$, where $n \geq 1$ and $n \in \mathbb{N}$.

Solution

Let $P(n)$ be the formula $(\overline{z})^n = \overline{z^n}$.

For $n=1$, we have $(\overline{z})^1 = \overline{z} = \overline{z^1}$. Thus, $P(1)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $(\overline{z})^k = \overline{z^k}$.

To show $P(k+1)$ is true, we need to show that $(\overline{z})^{k+1} = \overline{z^{k+1}}$.

$$\begin{aligned}(\overline{z})^{k+1} &= (\overline{z})^k\,\overline{z}\\ &= \overline{z^k}\,\overline{z} && \text{(Induction Hypothesis)}\\ &= \overline{z^k z} && \text{(Product Rule for Conjugates)}\\ &= \overline{z^{k+1}}\end{aligned}$$

This establishes $(\overline{z})^{k+1} = \overline{z^{k+1}}$, so that $P(k+1)$ is true. Hence, by the Principle of Mathematical Induction, $(\overline{z})^n = \overline{z^n}$ for all $n \geq 1$.

Example 14.5.3: Proving a Summation Formula

Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$, where $n \in \mathbb{N}$.

Solution

Let $P(n)$ be the formula $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

For $n = 1$, we have $1^3 = \frac{1^2(1+1)^2}{4}$. Thus, $P(1)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$.

To show $P(k+1)$ is true, we need to show that $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$.

$$\begin{aligned}1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 &= (1^3 + 2^3 + 3^3 + \ldots + k^3) + (k+1)^3\\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 && \text{(Induction Hypothesis)}\\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}\\ &= \frac{(k+1)^2\left(k^2 + 4(k+1)\right)}{4}\\ &= \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4}\\ &= \frac{(k+1)^2(k+2)^2}{4}\end{aligned}$$

Therefore, $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$, and so $P(k+1)$ is true. Hence, by the Principle of Mathematical Induction, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers n.

Advice

Notice in the Induction Hypothesis step of Example 14.5.3 that we clearly stated, "Assume $P(k)$ is true. That is, assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$." This statement is intentional. It tells us that, from this point forward, whenever we see $1^3 + 2^3 + 3^3 + \cdots + k^3$, we can replace it with $\frac{k^2(k+1)^2}{4}$. This replacement always happens in the next step of the proof.

Additionally, when starting our work on $P(k+1)$, we clearing stated that, "we need to show that $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$." Clearly stating what we need to show, and simplifying that down to $\frac{(k+1)^2(k+2)^2}{4}$, provides us with a goal - we need to manipulate $P(k+1)$ (using the Induction Hypothesis) to eventually become $\frac{(k+1)^2(k+2)^2}{4}$.

Example 14.5.4: Proving an Inequality Claim

Prove that $3^n > 100n$ for $n > 5$, where $n \in \mathbb{N}$.

Solution

The first wrinkle we encounter in this problem is that we are asked to prove this formula for $n > 5$ instead of $n \geq 1$. Since n is a natural number, our base step occurs at $n = 6$. We can still use the PMI in this case, but we will conclude that the formula is valid for all $n \geq 6$.

Let $P(n)$ be the inequality $3^n > 100n$.

For $n = 6$, we have $3^6 = 729 > 600 = 100(6)$. Thus, $P(6)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $3^k > 100k$.

To show $P(k+1)$ is true, we need to show $3^{k+1} > 100(k+1) = 100k + 100$.

$$\begin{aligned} 3^{k+1} &= 3^k \cdot 3 \\ &> 100k \cdot 3 && \text{(Induction Hypothesis)} \\ &= 100k + 200k && \text{(our goal is to get to } 100k+100\text{, so isolating } 100k \text{ is helpful)} \\ &> 100k + 100 && \left(200k > 100 \text{ when } k > \tfrac{1}{2}\text{. Since } k \geq 6,\ 200k > 100\right) \end{aligned}$$

Hence $P(k+1)$ is true. By induction, $3^n > 100n$ for all $n \geq 6$.

Example 14.5.4 showcases a challenge when working with inequalities and Mathematical Induction - you need to be very creative and truly understand your prerequisite mathematics to do well with these types of induction proofs. Read through Example 14.5.4 a few times to truly understand what we did.

Example 14.5.5: Proving a Divisibility Claim

Prove $n^2 + n$ is divisible by 2, where $n \in \mathbb{N}$.

Solution

Let $(P(n)$ be the statement that $n^2 + n$ is divisible by 2.

For $n = 1$, we have $1^1 + 1 = 2$. This is definitely divisible by 2 (since $\frac{2}{2} = 1$), so $P(1)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $k^2 + k$ is divisible by 2.

To show $P(k+1)$ is true, we need to show that $(k+1)^2 + (k+1)$ is divisible by 2.

$$\begin{aligned} (k+1)^2 + (k+1) &= k^2 + 2k + 1 + k + 1 \\ &= (k^2 + k) + 2k + 1 + 1 && \text{(Associative and Commutative Properties of Addition)} \\ &= (k^2 + k) + 2k + 2 \\ &= (k^2 + k) + 2(k+1) \end{aligned}$$

We know that $k^2 + k$ is divisible by 2, by the Induction Hypothesis. We also know that $2(k+1)$ is divisible by 2. The sum of numbers divisible by 2 is also divisible by 2, so $(k+1)^2 + (k+1)$ must be divisible by 2. Hence, $P(k+1)$ is true. Thus, the claim is true for all natural numbers n by Mathematical Induction.

While we have used the Principle of Mathematical Induction to prove some of the formulas we have merely motivated in the text, our primary use of this result is to prove the celebrated Binomial Theorem. The ardent Mathematics student will undoubtedly see the PMI in many courses that have yet to be taught. Sometimes, it is explicitly stated, and sometimes, it remains hidden in the background. If ever you see a property expressed as being true "for all natural numbers n," it's a solid bet that the formal proof requires the Principle of Mathematical Induction.

14.5.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Principle of Mathematical Induction (PMI)
  - Mathematical Induction - What is Mathematical Induction, and How Does it Work?
- Proving summation formulas
  - Mathematical Induction - Proving Summation Conjectures Using Mathematical Induction
- Proving divisibility and inequalities by induction
  - Mathematical Induction - Proving Non-Summation Conjectures Using Mathematical Induction

KEY CONCEPTS

DEFINITIONS

- **Mathematical Induction**: A method of proof used to establish that a given statement is true for all natural numbers (or for all natural numbers from a certain point onwards).
- **Base Case (in Mathematical Induction)**: The first step in an inductive proof, where the statement $P(n)$ is shown to be true for the initial value (usually $n = 1$, or the starting value specified in the claim).
- **Induction Hypothesis (Inductive Hypothesis)**: The second step in an inductive proof, where it is assumed that the statement $P(k)$ is true for an arbitrary natural number k (or k greater than or equal to the base case value).
- **Inductive Step (in Mathematical Induction)**: The part of an inductive proof where, using the induction hypothesis (assuming $P(k)$ is true), it is shown that $P(k+1)$ must also be true.

THEOREMS

- **The Principle of Mathematical Induction (PMI)**: Suppose $P(n)$ is a claim involving the natural number n. If 1. $P(1)$ is true (base case) and 2. assuming $P(k)$ is true (Induction Hypothesis), it follows that $P(k+1)$ is also true (inductive step), then the claim $P(n)$ is true for all natural numbers n.

COMMON MISTAKES

- **Skipping or Incorrectly Proving the Base Case**: The base case is a crucial first step. Failing to establish its truth, or choosing the wrong starting value for n, invalidates the entire proof.
- **Incorrectly Stating or Using the Inductive Hypothesis**: The inductive hypothesis must be clearly stated as an assumption ($P(k)$ is true). In the inductive step, this assumption must be explicitly used to prove $P(k+1)$.
- **Proving $P(k)$ instead of $P(k+1)$ in the Inductive Step**: The goal of the inductive step is to show that *if* $P(k)$ is true, *then* $P(k+1)$ is true. Simply restating or manipulating $P(k)$ is not sufficient.
- **Assuming $P(k+1)$ is True**: Circular reasoning where the statement to be proven ($P(k+1)$) is assumed as part of its own proof.
- **Algebraic Errors in the Inductive Step**: Making mistakes in algebraic manipulation when trying to transform the expression for $P(k+1)$ to show it follows from $P(k)$. This is common in summation, inequality, and divisibility proofs.
- **Not Clearly Showing How $P(k)$ Leads to $P(k+1)$**: The logical connection between the assumption of $P(k)$ and the conclusion of $P(k+1)$ must be explicit and clear.
- **Incorrectly Handling Inequalities in Inductive Step**: When proving inequalities, not carefully maintaining the direction of the inequality or making invalid steps (e.g., multiplying by a negative without reversing the sign).

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14.5.2: HOMEWORK

READING QUESTIONS

1. What is the Principle of Mathematical Induction (PMI)? Describe its two main parts.
2. What is the "base case" in a proof by mathematical induction, and why is it important?
3. What is the "inductive hypothesis" in a proof by mathematical induction?
4. What is the "inductive step" in a proof by mathematical induction? What are you trying to show in this step?
5. Explain the "domino effect" analogy for mathematical induction.
6. When proving a summation formula like $\sum_{j=1}^{n}(a+(j-1)d) = \frac{n}{2}(2a+(n-1)d)$ using induction, what does $P(k)$ represent, and what does $P(k+1)$ represent?
7. In the inductive step of proving a summation formula, how is the sum for $P(k+1)$ typically related to the sum for $P(k)$?
8. If you are asked to prove a statement for $n > 5$, what would be your base case?
9. When proving a divisibility claim like "n^2+n is divisible by 2," what does it mean to assume $P(k)$ is true (i.e., k^2+k is divisible by 2)? How might you express this algebraically?
10. Why is it important not to perform operations *between* the two sides of an equation when you are in the process of verifying if $P(k+1)$ is true (using the $\stackrel{?}{=}$ notation)?

HOMEWORK

For the following exercises, prove the claim using the Principle of Mathematical Induction.

11. $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
12. $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$
13. $2^n > 500n$ for $n > 12$
14. $3^n \geq n^3$ for $n \geq 4$
15. Use the Product Rule for Absolute Value to show $|x^n| = |x|^n$ for all real numbers x and all natural numbers $n \geq 1$
16. Use the Product Rule for Logarithms to show $\log(x^n) = n\log(x)$ for all real numbers $x > 0$ and all natural numbers $n \geq 1$.
17. For the sequence $a_1 = a$, $a_{n+1} = ra_n$, for $n \geq 1$, prove $a_n = ar^{n-1}$.
18. Discuss the classic 'paradox' All Horses are the Same Color problem with your classmates.

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14.6: Binomial Theorem

Learning Objectives

- Simplify expressions involving factorials.
- Understand and use binomial coefficients and the Binomial Theorem to expand the powers of binomials quickly.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Recall that a polynomial with two terms is called a binomial. We have already learned how to multiply binomials and to raise binomials to powers, but raising a binomial to a high power can be tedious and time-consuming. In this section, we will discuss a shortcut that will allow us to find $(x+y)^n$ without multiplying the binomial by itself n times.

In this section, we aim to prove the celebrated **Binomial Theorem**. Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers n. In Elementary and Intermediate Algebra, you should have seen specific instances of the formula, namely

$$\begin{aligned}(a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

If we wanted the expansion for $(a+b)^4$, we would write $(a+b)^4=(a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get

$$(a+b)^4=(a+b)\left(a^3+3a^2b+3ab^2+b^3\right)=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

Generalizing this a bit, we see that if we have a formula for $(a+b)^k$, we can obtain a formula for $(a+b)^{k+1}$ by rewriting the latter as

$$(a+b)^{k+1}=(a+b)(a+b)^k.$$

This means Mathematical Induction plays a significant role in the proof of the Binomial Theorem.

Factorials

Before we can state the theorem, we need to revisit the sequence of factorials that were introduced earlier in this chapter.

Definition: Factorial (Recursive Definition)

The **factorial** of a natural number n, denoted $n!$ and reads as "n **factorial**," is defined by the recursive formula

$$\begin{aligned}a_0 &= 0! = 1\\ a_n &= n! = n\cdot a_{n-1} \qquad \text{for } n\geq 1\end{aligned}$$

Recall this means $0!=1$ and $n!=n(n-1)!$ for $n\geq 1$. Using the recursive definition, we get: $1!=1\cdot 0!=1\cdot 1=1$, $2!=2\cdot 1!=2\cdot 1=2$, $3!=3\cdot 2!=3\cdot 2\cdot 1=6$, and $4!=4\cdot 3!=4\cdot 3\cdot 2\cdot 1=24$. Informally, $n!=n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$ with $0!=1$ as our "base case." Our first example refamiliarizes us with some of the basic computations involving factorials.

Example 14.6.1: Computing Factorials

a. Simplify the following expressions.

i. $\frac{3!\,2!}{0!}$

ii. $\frac{7!}{5!}$

iii. $\frac{1000!}{998!\,2!}$

iv. $\frac{(k+2)!}{(k-1)!}$, $k \geq 1$

b. Prove $n! > 3^n$ for all $n \geq 7$.

Solutions

a. We keep in mind the mantra, "When in doubt, write it out!" as we simplify the following.

i. We have been programmed to react with alarm to the presence of a 0 in the denominator, but in this case, $0! = 1$, so the fraction is defined after all. As for the numerator, $3! = 3 \cdot 2 \cdot 1 = 6$ and $2! = 2 \cdot 1 = 2$, so we have $\frac{3!\,2!}{0!} = \frac{(6)(2)}{1} = 12$.

ii. We have $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ while $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Dividing, we get $\frac{7!}{5!} = \frac{5040}{120} = 42$. While this is correct, we note that we could have saved ourselves some time had we proceeded as follows

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42.$$

In fact, should we want to exploit the recursive nature of the factorial fully, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42.$$

iii. Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

iv. This problem continues the theme we have seen in the previous two problems. We first note that since $k+2$ is larger than $k-1$, $(k+2)!$ contains all of the factors of $(k-1)!$ and as a result we can get the $(k-1)!$ to cancel from the denominator. To see this, we begin by writing out $(k+2)!$ starting with $(k+2)$ and multiplying it by the numbers which precede it until we reach $(k-1)$: $(k+2)! = (k+2)(k+1)(k)(k-1)!$. As a result, we have

$$\frac{(k+2)!}{(k-1)!} = \frac{(k+2)(k+1)(k)(k-1)!}{(k-1)!} = \frac{(k+2)(k+1)(k)\,\cancel{(k-1)!}}{\cancel{(k-1)!}} = k(k+1)(k+2)$$

The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

b. This will require a proof by Mathematical Induction. In this case, the claim should only hold for $n \geq 7$. Let $P(n)$ be the statement $n! > 3^n$.

For $n = 7$, we have $7! = 5040 > 2187 = 3^7$. Thus, $P(7)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $k! > 3^k$.

We want to show that $P(k+1)$ is true. That is, we want to show $(k+1)! > 3^{k+1} = 3 \cdot 3^k$.

$$\begin{aligned}
(k+1)! &= (k+1)k! && \text{(Definition of a factorial)} \\
&> (k+1)3^k && \text{(Induction Hypothesis)} \\
&> (3)3^k && (k+1 > 3 \text{ when } k > 2. \text{ Luckily, we know that } k \geq 7, \text{ so } k+1 > 3) \\
&= 3^{k+1}
\end{aligned}$$

Hence, we have shown $(k+1)! > 3^{k+1}$. Thus, $P(k+1)$ is true. By the Principle of Mathematical Induction, we have $n! > 3^n$ for all $n \geq 7$.

Of all the mathematical animals we discussed in the text, factorials grow most quickly. In Example 14.6.1*b*, we proved that $n!$ overtakes 3^n at $n = 7$. "Overtakes" may be too polite a word since $n!$ thoroughly trounces 3^n for $n \geq 7$, as any reasonable set of data will show. It can be shown that for any real number $x > 0$, not only does $n!$ eventually overtake x^n, but the ratio $\frac{x^n}{n!} \to 0$ as $n \to \infty$. This fact is far more important than you might realize right now, but wait until Calculus and you *might* begin to understand the significance of this statement.

Binomial Coefficients

Applications of factorials in the wild often involve counting arrangements. For example, suppose you have fifty songs on your phone and wish to arrange these songs in a playlist in which the order of the songs matters. There are $50!$ different possible playlists in that case. That equates to $30,414,093,201,713,378,043,612,608,166,064,768,844,377,641,568,960,512,000,000,000,000$ possible playlists. If you wish to select only ten of the songs to create a playlist, then there are $\frac{50!}{40!} = 37,276,043,023,296,000$ such playlists. If, on the other hand, you want to select ten song files out of the fifty to play in *any* order, there are $\frac{50!}{40!10!} = 10,272,278,170$ ways to achieve this. The authors encourage you to take courses such as Finite Mathematics, Discrete Mathematics, and Statistics if you are interested in concepts like this. We introduce these concepts here because this is how the factorials make their way into the Binomial Theorem, as the following definition indicates.

Definition: Binomial Coefficient

Given two whole numbers n and j with $n \geq j$, the **binomial coefficient** $\binom{n}{j}$ (read, "n choose j") is the whole number given by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

The name "binomial coefficient" will be justified shortly. For now, we can physically interpret $\binom{n}{j}$ as the number of ways to select j items from n items where the order of the items selected is unimportant. For example, suppose you win two free tickets to a special screening of the latest Hollywood blockbuster and have five friends, each of whom would love to accompany you to the movies. There are $\binom{5}{2}$ ways to choose who goes with you. Applying the definition of the binomial coefficient, we get

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10.$$

So there are 10 different ways to distribute those two tickets among five friends. (Some will see it as 10 ways to decide which three friends must stay home.) The reader is encouraged to verify this by listing all the possibilities.

Example 14.6.2: Finding the Number of Combinations Using the Formula

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.

a. How many ways can you select your side dishes?
b. How many ways can you select 3 side dishes?

Solutions

a. We want to choose 2 side dishes from 5 options.

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \frac{20}{2} = 10.$$

There are ten ways to choose 2 side dishes from 5 options.

b. We want to choose 3 side dishes from 5 options.

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = \frac{20}{2} = 10.$$

Therefore, there are also ten ways to choose 3 side dishes from 5 options.

The equivalence of the answers to Example **14.6.2***a* and **14.6.2***b* is not a coincidence. Asking how many ways to choose 2 items from 5 should be the same as asking how to *exclude* 3 items from 5.

? Checkpoint 14.6.2

An ice cream shop offers 10 flavors of ice cream. How many ways are there to choose 3 flavors for a banana split?

Q&A

Is a binomial coefficient always a whole number?

Yes. Just as the number of combinations must always be a whole number, a binomial coefficient will always be a whole number.

✔ Example 14.6.3: Finding Binomial Coefficients

Find each binomial coefficient.

a. $\binom{5}{3}$
b. $\binom{9}{2}$
c. $\binom{9}{7}$

Solutions

Use the formula to calculate each binomial coefficient:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

a. $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = \frac{20}{2} = 10$
b. $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9 \cdot 8 \cdot \cancel{7!}}{2! \cdot \cancel{7!}} = \frac{72}{2} = 36$
c. $\binom{9}{7} = \frac{9!}{7!(9-7)!} = \frac{9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!} \cdot 2!} = \frac{72}{2} = 36$

? Checkpoint 14.6.3

Find each binomial coefficient.

a. $\binom{7}{3}$
b. $\binom{11}{4}$

The Binomial Theorem

We now state and prove a theorem, named after the celebrated French mathematician Blaise Pascal, which is crucial to the proof of the Binomial Theorem.

Theorem: Pascal's Rule

For natural numbers n and j with $n \geq j$,

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}.$$

Proof

$$\begin{aligned}
\binom{n}{j-1}+\binom{n}{j} &= \frac{n!}{(j-1)!(n-(j-1))!}+\frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)!}+\frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)(n-j)!}+\frac{n!}{j(j-1)!(n-j)!} \\
&= \frac{n!\,j}{j(j-1)!(n-j+1)(n-j)!}+\frac{n!(n-j+1)}{j(j-1)!(n-j+1)(n-j)!} \\
&= \frac{n!\,j}{j!(n-j+1)!}+\frac{n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,j+n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,(j+(n-j+1))}{j!(n-j+1)!} \\
&= \frac{(n+1)n!}{j!(n+1-j))!} \\
&= \frac{(n+1)!}{j!((n+1)-j))!} \\
&= \binom{n+1}{j}\checkmark
\end{aligned}$$

We can now state and prove the Binomial Theorem. This theorem states that binomial coefficients are just that - coefficients in the binomial expansion.

Theorem: Binomial Theorem

For nonzero real numbers a and b,

$$(a+b)^n=\sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j$$

for all natural numbers n.

Proof

To prove the Binomial Theorem, we let $P(n)$ be the expansion formula given in the statement of the theorem, and we note that $P(1)$ is true since

$$\begin{aligned}
(a+b)^1 &\stackrel{?}{=} \sum_{j=0}^{1}\binom{1}{j}a^{1-j}b^j \\
a+b &\stackrel{?}{=} \binom{1}{0}a^{1-0}b^0+\binom{1}{1}a^{1-1}b^1 \\
a+b &= a+b\checkmark
\end{aligned}$$

Now we assume that $P(k)$ is true. That is, we assume that we can expand $(a+b)^k$ using the formula given in the Binomial Theorem and attempt to show that $P(k+1)$ is true.

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k \\ &= (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\ &= a\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j + b\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\ &= \sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1}\end{aligned}$$

We aim to combine as many terms as possible within the two summations. As the counter j in the first summation runs from 0 through k, we get terms involving a^{k+1}, $a^k b$, $a^{k-1}b^2$, \ldots , ab^k. In the second summation, we get terms involving $a^k b$, $a^{k-1}b^2$, \ldots , ab^k, b^{k+1}. In other words, they have common terms apart from the first term in the first summation and the last term in the second summation. Our next move is to "kick out" the terms that we cannot combine and rewrite the summations so that we can combine them. To that end, we note

$$\sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j$$

and

$$\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1} = \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}$$

so that

$$(a+b)^{k+1} = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}.$$

We now wish to write

$$\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}$$

as a single summation. The wrinkle is that the first summation starts with $j=1$, while the second starts with $j=0$. Even though the sums produce terms with the same powers of a and b, they do so for different values of j. To resolve this, we need to shift the index on the second summation so that the index j starts at $j=1$ instead of $j=0$.

$$\begin{aligned}\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=0+1}^{k-1+1}\binom{k}{j-1}a^{k-(j-1)}b^{(j-1)+1} \\ &= \sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^j\end{aligned}$$

We can now combine our two sums and simplify.

$$\begin{aligned}\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^j \\ &= \sum_{j=1}^{k}\left[\binom{k}{j} + \binom{k}{j-1}\right]a^{k+1-j}b^j \\ &= \sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^j\end{aligned}$$

Using this and the fact that $\binom{k+1}{0} = 1$ and $\binom{k+1}{k+1} = 1$, we get

$$\begin{aligned}(a+b)^{k+1} &= a^{k+1}+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^j+b^{k+1}\\ &= \binom{k+1}{0}a^{k+1}b^0+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^j+\binom{k+1}{k+1}a^0b^{k+1}\\ &= \sum_{j=0}^{k+1}\binom{k+1}{j}a^{(k+1)-j}b^j\end{aligned}$$

which shows that $P(k+1)$ is true. Hence, by induction, we have established that the Binomial Theorem holds for all natural numbers n.

To get a feel of what this theorem is saying and how it is easier to remember than it may first appear, let's consider the specific case of $n=4$. According to the theorem, we have

$$\begin{aligned}(a+b)^4 &= \sum_{j=0}^{4}\binom{4}{j}a^{4-j}b^j\\ &= \binom{4}{0}a^{4-0}b^0+\binom{4}{1}a^{4-1}b^1+\binom{4}{2}a^{4-2}b^2+\binom{4}{3}a^{4-3}b^3+\binom{4}{4}a^{4-4}b^4\\ &= \binom{4}{0}a^4+\binom{4}{1}a^3b+\binom{4}{2}a^2b^2+\binom{4}{3}ab^3+\binom{4}{4}b^4\end{aligned}$$

We forgo the simplification of the coefficients to note the pattern in the expansion. First, note that in each term, the total of the exponents is 4, which matches the exponent of the binomial $(a+b)^4$. The exponent on a begins at 4 and decreases by one as we move from one term to the next, while the exponent on b starts at 0 and increases by one each time. Also, note that the binomial coefficients themselves have a pattern. The upper number, 4, matches the exponent on the binomial $(a+b)^4$, whereas the lower number changes from term to term and matches the exponent of b in that term. This is no coincidence and corresponds to the kind of counting we discussed earlier. If we think of obtaining $(a+b)^4$ by multiplying $(a+b)^2(a+b)^2$, our answer is the sum of all possible products with exactly four factors - some a, some b. If we wish to count, for instance, the number of ways we obtain 1 factor of b out of a total of 4 possible factors, thereby forcing the remaining 3 factors to be a, the answer is $\binom{4}{1}$. Hence, the term $\binom{4}{1}a^3b$ is in the expansion. The other terms which appear cover the remaining cases.

Example 14.6.4: Evaluating Binomials Using the Binomial Theorem

Use the Binomial Theorem to find the following.

a. $(x-2)^4$
b. 2.1^3
c. The term containing x^3 in the expansion $(2x+y)^5$

Solutions

a. Since $(x-2)^4=(x+(-2))^4$, we identify $a=x$, $b=-2$ and $n=4$ and obtain

$$\begin{aligned}(x-2)^4 &= \sum_{j=0}^{4}\binom{4}{j}x^{4-j}(-2)^j\\ &= \binom{4}{0}x^{4-0}(-2)^0+\binom{4}{1}x^{4-1}(-2)^1+\binom{4}{2}x^{4-2}(-2)^2+\binom{4}{3}x^{4-3}(-2)^3+\binom{4}{4}x^{4-4}(-2)^4\\ &= x^4-8x^3+24x^2-32x+16\end{aligned}$$

b. At first, this problem seems misplaced, but we can write $2.1^3=(2+0.1)^3$. Identifying $a=2$, $b=0.1=\frac{1}{10}$ and $n=3$, we get

$$\begin{aligned}\left(2+\frac{1}{10}\right)^3 &= \sum_{j=0}^{3}\binom{3}{j}2^{3-j}\left(\frac{1}{10}\right)^j \\ &= \binom{3}{0}2^{3-0}\left(\frac{1}{10}\right)^0+\binom{3}{1}2^{3-1}\left(\frac{1}{10}\right)^1+\binom{3}{2}2^{3-2}\left(\frac{1}{10}\right)^2+\binom{3}{3}2^{3-3}\left(\frac{1}{10}\right)^3 \\ &= 8+\frac{12}{10}+\frac{6}{100}+\frac{1}{1000} \\ &= 8+1.2+0.06+0.001 \\ &= 9.261\end{aligned}$$

c. Identifying $a=2x$, $b=y$ and $n=5$, the Binomial Theorem gives

$$(2x+y)^5=\sum_{j=0}^{5}\binom{5}{j}(2x)^{5-j}y^j$$

Since we are concerned with only the term containing x^3, there is no need to expand the entire sum. The exponents on each term must add to 5 and if the exponent on x is 3, the exponent on y must be 2. Plucking out the term $j=2$, we get

$$\binom{5}{2}(2x)^{5-2}y^2=10(2x)^3y^2=80x^3y^2$$

Checkpoint 14.6.4

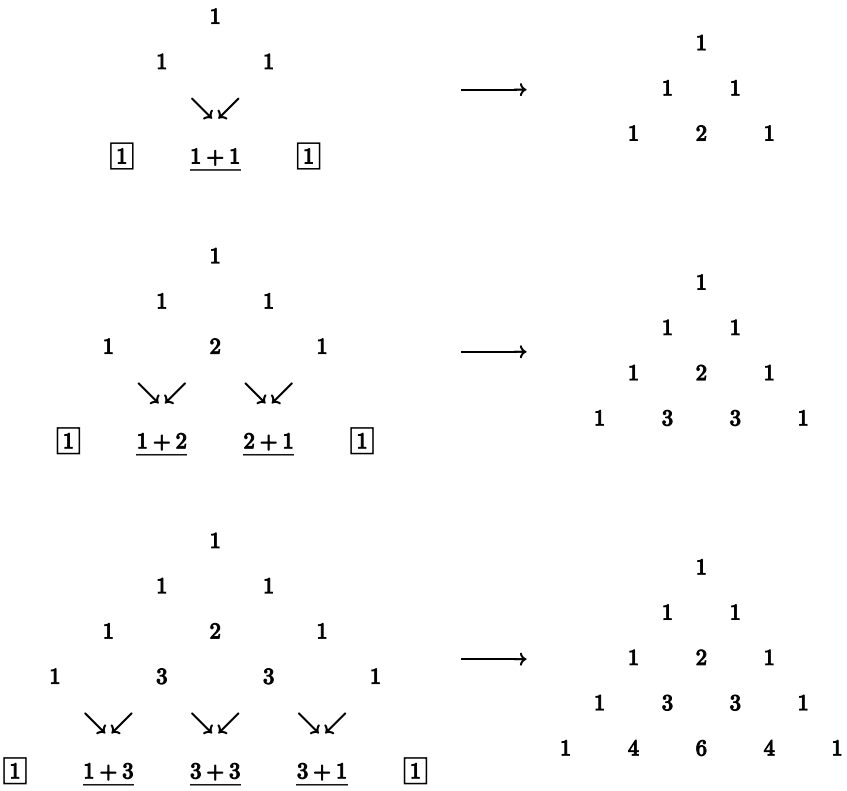
Find the sixth term of $(3x-y)^9$ without fully expanding the binomial.

Pascal's Triangle

We close this section with Pascal's Triangle. Pascal's Triangle is obtained by arranging the binomial coefficients in the triangular fashion below.

$$\begin{array}{ccccccccccc}
 & & & & \binom{0}{0} & & & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \searrow\swarrow & & & & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \searrow\swarrow & & \searrow\swarrow & & & & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & & & & \vdots & & & & & &
\end{array}$$

Since $\binom{n}{0}=1$ and $\binom{n}{n}=1$ for all whole numbers n, we get that each row of Pascal's Triangle begins and ends with 1. We use the additive relationship expressed in our first theorem in this section to generate the numbers in the middle of the rows (from the third row onwards). For instance, $\binom{1}{0}+\binom{1}{1}=\binom{2}{1}$, $\binom{2}{0}+\binom{2}{1}=\binom{3}{1}$, and so forth. The arrows in the array above indicate this relationship. With these two facts, we can quickly generate Pascal's Triangle. We start with the first two rows, 1 and $1\quad 1$. From that point on, each successive row begins and ends with 1, and the middle numbers are generated using the first theorem in this section. Below we attempt to demonstrate this building process to generate the first five rows of Pascal's Triangle.



To see how we can use Pascal's Triangle to expedite the Binomial Theorem, suppose we wish to expand $(3x-y)^4$. The coefficients we need are $\binom{4}{j}$ for $j=0,1,2,3,4$ and are the numbers that form the fifth row of Pascal's Triangle. Since we know that the exponent of $3x$ in the first term is 4 and then decreases by one as we go from left to right while the exponent of $-y$ starts at 0 in the first term and then increases by one as we move from left to right, we quickly obtain

$$\begin{aligned}(3x-y)^4 &= (1)(3x)^4+(4)(3x)^3(-y)+(6)(3x)^2(-y)^2+4(3x)(-y)^3+1(-y)^4\\ &= 81x^4-108x^3y+54x^2y^2-12xy^3+y^4\end{aligned}$$

We would like to stress that Pascal's Triangle is a rapid method to expand an *entire* binomial. If only a term (or two or three) is required, then the Binomial Theorem is the way to go.

? Checkpoint 14.6.5

Write in expanded form.

a. $(x-y)^5$
b. $(2x+5y)^3$

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14.6.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Binomial Theorem
  - The Binomial Theorem - The Binomial Theorem
- Binomial coefficients
  - The Binomial Theorem - Factorials and Combinations
- Pascal's Triangle
  - The Binomial Theorem - Pascal's Triangle and Powers of Binomials

KEY CONCEPTS

DEFINITIONS

- **Binomial Coefficient**: Given two whole numbers n and j with $n \geq j$, the binomial coefficient $\binom{n}{j}$ (read "n choose j") is the whole number given by $\binom{n}{j} = \frac{n!}{j!(n-j)!}$. It represents the number of ways to select j items from n items where order is unimportant.
- **Pascal's Triangle**: A triangular arrangement of binomial coefficients, where each number is the sum of the two numbers directly above it. The kth number (starting $k = 0$) in row n (starting $n = 0$) is $\binom{n}{k}$.

THEOREMS

- **Pascal's Rule**: For natural numbers n and j with $n \geq j$, $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$.
- **Binomial Theorem**: For nonzero real numbers a and b, and any natural number n, $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$. This can be expanded as: $(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$.

COMMON MISTAKES

- **Miscalculating Binomial Coefficients**: Errors in applying the formula $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.
- **Errors in Applying the Binomial Theorem**:
  - Using incorrect binomial coefficients for terms.
  - Errors in the exponents of a or b for a given term (they should sum to n; the exponent of b matches j, the lower index of the coefficient).
  - Forgetting to apply exponents to the entire term if a or b are themselves products or have coefficients (e.g., in $(2x+y)^5$, a term involves $(2x)^k$, not just $2x^k$).
  - Sign errors when one of the terms in the binomial is negative (e.g., $(x-2)^4$).
- **Incorrectly Identifying Terms in Pascal's Triangle**: Using the wrong row or miscounting positions when using Pascal's Triangle for binomial coefficients. Remember row n gives coefficients for $(a+b)^n$, and rows/positions start counting from 0.
- **Using Pascal's Triangle for Single Term vs. Full Expansion**: Pascal's Triangle is efficient for full expansion. For finding a single term, the Binomial Theorem formula with binomial coefficients is generally better.
- **Assuming $\binom{n}{j} = \binom{n}{n-j}$ is Pascal's Rule**: While $\binom{n}{j} = \binom{n}{n-j}$ is a true property (symmetry of binomial coefficients), Pascal's Rule is the additive relationship $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$.

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14.6.2: HOMEWORK

READING QUESTIONS

1. What is a factorial, and how is $n!$ defined recursively? What is the value of $0!$?
2. What is a binomial coefficient $\binom{n}{j}$, and how is it calculated using factorials? What does it represent in terms of combinations?
3. State the Binomial Theorem. What does it provide a formula for?
4. In the expansion of $(a+b)^n$, what is the sum of the exponents of a and b in each term?
5. In the term $\binom{n}{j}a^{n-j}b^j$ from the Binomial Theorem, how does the value of j relate to the exponent of b?
6. Describe how Pascal's Triangle is constructed. How do its entries relate to binomial coefficients?
7. When is it more advantageous to use Pascal's Triangle for binomial expansion compared to using the Binomial Theorem formula directly?
8. What is Pascal's Rule for binomial coefficients?
9. How many terms are there in the expansion of $(a+b)^n$?
10. What are the first three terms in the expansion of $(x+y)^5$ using the Binomial Theorem?
11. What is a binomial coefficient, and how it is calculated?
12. What role do binomial coefficients play in a binomial expansion? Are they restricted to any type of number?
13. What is the Binomial Theorem and what is its use?
14. When is it an advantage to use the Binomial Theorem? Explain.

HOMEWORK

ALGEBRAIC

For the following exercises, evaluate the binomial coefficient.

15. $\binom{6}{2}$
16. $\binom{5}{3}$
17. $\binom{7}{4}$
18. $\binom{9}{7}$
19. $\binom{10}{9}$
20. $\binom{25}{11}$
21. $\binom{17}{6}$
22. $\binom{200}{199}$

For the following exercises, use the Binomial Theorem to expand each binomial.

23. $(4a-b)^3$
24. $(5a+2)^3$
25. $(3a+2b)^3$
26. $(2x+3y)^4$
27. $(4x+2y)^5$
28. $(3x-2y)^4$
29. $(4x-3y)^5$
30. $\left(\frac{1}{x}+3y\right)^5$
31. $(x^{-1}+2y^{-1})^4$
32. $(\sqrt{x}-\sqrt{y})^5$

For the following exercises, use the Binomial Theorem to write the first three terms of each binomial.

33. $(a+b)^{17}$
34. $(x-1)^{18}$
35. $(a-2b)^{15}$
36. $(x-2y)^8$
37. $(3a+b)^{20}$
38. $(2a+4b)^7$
39. $(x^3-\sqrt{y})^8$

For the following exercises, find the indicated term of each binomial without fully expanding the binomial.

40. The fourth term of $(2x-3y)^4$
41. The fourth term of $(3x-2y)^5$
42. The third term of $(6x-3y)^7$
43. The eighth term of $(7+5y)^{14}$
44. The seventh term of $(a+b)^{11}$
45. The fifth term of $(x-y)^7$
46. The tenth term of $(x-1)^{12}$
47. The ninth term of $(a-3b^2)^{11}$
48. The fourth term of $\left(x^3-\frac{1}{2}\right)^{10}$
49. The eighth term of $\left(\frac{y}{2}+\frac{2}{x}\right)^9$

GRAPHICAL

For the following exercises, use the Binomial Theorem to expand the binomial $f(x)=(x+3)^4$. Then find and graph each indicated sum on one set of axes.

50. Find and graph $f_1(x)$, such that $f_1(x)$ is the first term of the expansion.
51. Find and graph $f_2(x)$, such that $f_2(x)$ is the sum of the first two terms of the expansion.
52. Find and graph $f_3(x)$, such that $f_3(x)$ is the sum of the first three terms of the expansion.
53. Find and graph $f_4(x)$, such that $f_4(x)$ is the sum of the first four terms of the expansion.
54. Find and graph $f_5(x)$, such that $f_5(x)$ is the sum of the first five terms of the expansion.

EXTENSIONS

55. In the expansion of $(5x+3y)^n$, each term has the form $\binom{n}{k}a^{n-k}b^k$, where k successively takes on the value 0, 1, 2, ..., n. If $\binom{n}{k}=\binom{7}{2}$, what is the corresponding term?
56. In the expansion of $(a+b)^n$, the coefficient of $a^{n-k}b^k$ is the same as the coefficient of which other term?
57. Consider the expansion of $(x+b)^{40}$. What is the exponent of b in the kth term?
58. Find $\binom{n}{k-1}+\binom{n}{k}$ and write the answer as a binomial coefficient in the form $\binom{n}{k}$. Prove it. *Hint:* Use the fact that, for any integer p, such that $p\geq 1$, $p!=p(p-1)!$.
59. Which expression cannot be expanded using the Binomial Theorem? Explain.
    a. (x^2-2x+1)

b. $(\sqrt{a}+4\sqrt{a}-5)^8$
c. $(x^3+2y^2-z)^5$
d. $(3x^2-\sqrt{2y^3})^{12}$

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CHAPTER OVERVIEW

Chapter 15: Analytic Geometry

(a) Greek philosopher Aristotle (384–322 BCE) (b) German mathematician and astronomer Johannes Kepler (1571–1630)

Videos

- A Visual Introduction to Conic Sections

The Greek mathematician Menaechmus (c. 380 - c. 320 BCE) is generally credited with discovering the shapes formed by the intersection of a plane and a right circular cone. Depending on how he tilted the plane when it intersected the cone, he formed different shapes at the intersection–beautiful shapes with near-perfect symmetry.

It was also said that Aristotle may have had an intuitive understanding of these shapes, as he observed the orbit of the planet to be circular. He presumed that the planets moved in circular orbits around Earth, and for nearly 2000 years this was the commonly held belief.

It was not until the Renaissance movement that Johannes Kepler noticed that the orbits of the planet were not circular in nature. His published law of planetary motion in the 1600s changed our view of the solar system forever. He claimed that the sun was at one end of the orbits, and the planets revolved around the sun in an oval-shaped path.

In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems.

15.1: The Ellipse

15.1.1: Resources and Key Concepts

15.1.2: Homework

15.2: The Hyperbola

15.2.1: Resources and Key Concepts

15.2.2: Homework

15.3: The Parabola

15.3.1: Resources and Key Concepts

15.3.2: Homework

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15.1: The Ellipse

Learning Objectives

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 15.1.1

The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure **15.1.1**, is such a room. It is an oval-shaped room called a **whispering chamber** because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

Ellipses as Conic Sections

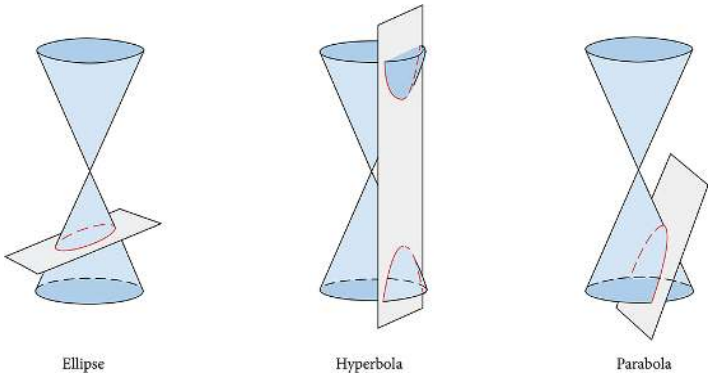
We begin this material with the general concept of a **conic section**.

Definition: Conic Section

A **conic section**, or **conic**, is a shape resulting from intersecting a right circular cone with a plane.

The angle at which the plane intersects the cone determines the shape, as shown in Figure **15.1.2**.

Figure 15.1.2



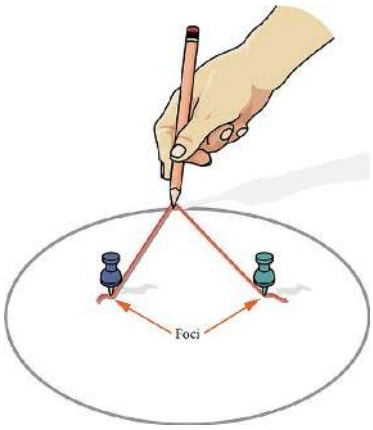
Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs within the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the **ellipse**.

Definition: Ellipse

An **ellipse** is the set of all points in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a **focus** (plural: **foci**).

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure **15.1.3**.

Figure 15.1.3



Every ellipse has two axes of symmetry and several key points.

Definition: Major and Minor Axes, Vertex and Co-Vertex, and Center of an Ellipse

The longer axis of an ellipse is called the **major axis**, and the shorter axis is called the **minor axis**. Each endpoint of the major axis is called a **vertex** of the ellipse (plural: **vertices**), and each endpoint of the minor axis is called a **co-vertex** of the ellipse. The **center** of the ellipse is the midpoint of both the major and minor axes.

The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci. See Figure **15.1.4**.

Figure 15.1.4

In this course, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the x- and y-axes.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First, we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

Standard Form of an Ellipse Centered at the Origin

Since there are two ways we are going to orient our ellipses (horizontally and vertically), there are two equations for the standard form of an ellipse.

Theorem: Standard Form of the Equation of an Ellipse Centered at the Origin

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the x-axis is

$$\frac{x^2}{v_x^2}+\frac{y^2}{v_y^2}=1,$$

where

- v_x and v_y are positive
- $v_x > v_y$
- the length of the major axis is $2v_x$
- the coordinates of the vertices are $(\pm v_x, 0)$
- the length of the minor axis is $2v_y$
- the coordinates of the co-vertices are $(0, \pm v_y)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = |v_x^2 - v_y^2|$.

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the y-axis is

$$\frac{x^2}{v_x^2}+\frac{y^2}{v_y^2}=1,$$

where

- v_x and v_y are positive
- $v_y > v_x$
- the length of the major axis is $2v_y$
- the coordinates of the vertices are $(0, \pm v_y)$
- the length of the minor axis is $2v_x$
- the coordinates of the co-vertices are $(\pm v_x, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = |v_x^2 - v_y^2|$.

Proof

To derive the equation of an ellipse centered at the origin, we begin with the foci $(-c,0)$ and $(c,0)$. The ellipse is the set of all points (x,y) such that the sum of the distances from (x,y) to the foci is constant, as shown in the figure below.

If $(v_x,0)$ is a vertex of the ellipse (on the major axis), the distance from $(-c,0)$ to $(v_x,0)$ is $v_x-(-c)=v_x+c$. The distance from $(c,0)$ to $(v_x,0)$ is v_x-c. The sum of the distances from the foci to the vertex is

$$(v_x+c)+(v_x-c)=2v_x$$

If (x,y) is a point on the ellipse, then we can define the following variables:

$$\begin{aligned} d_1 &= \text{the distance from } (-c,0) \text{ to } (x,y) \\ d_2 &= \text{the distance from } (c,0) \text{ to } (x,y) \end{aligned}$$

By the definition of an ellipse, d_1+d_2 is constant for any point (x,y) on the ellipse. We know that the sum of these distances is $2v_x$ for the vertex $(v_x,0)$. It follows that $d_1+d_2=2v_x$ for any point on the ellipse. We will begin the derivation by applying the Distance Formula. The rest of the derivation is algebraic.

$$
\begin{aligned}
& & d_1+d_2 &= 2v_x & \\
\Longrightarrow & & \sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2} &= 2v_x & \text{(substituting)} \\
\Longrightarrow & & \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2v_x & \text{(simplifying)} \\
\Longrightarrow & & \sqrt{(x+c)^2+y^2} &= 2v_x-\sqrt{(x-c)^2+y^2} & \text{(isolating a radical)} \\
\Longrightarrow & & (x+c)^2+y^2 &= \left[2v_x-\sqrt{(x-c)^2+y^2}\right]^2 & \text{(squaring both sides)} \\
\Longrightarrow & & (x+c)^2+y^2 &= 4v_x^2-4v_x\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 & \text{(distributing)} \\
\Longrightarrow & & x^2+2cx+c^2+y^2 &= 4v_x^2-4v_x\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 & \text{(distributing)} \\
\Longrightarrow & & 2cx &= 4v_x^2-4v_x\sqrt{(x-c)^2+y^2}-2cx & \text{(subtracting } x^2,\ c^2,\ \text{and } y^2 \\
& & & & \text{from both sides)} \\
\Longrightarrow & & 4cx-4v_x^2 &= -4v_x\sqrt{(x-c)^2+y^2} & \text{(isolating the radical)} \\
\Longrightarrow & & cx-v_x^2 &= -v_x\sqrt{(x-c)^2+y^2} & \text{(dividing both sides by 4)} \\
\Longrightarrow & & (cx-v_x^2)^2 &= v_x^2\left((x-c)^2+y^2\right) & \text{(squaring both sides)} \\
\Longrightarrow & & c^2x^2-2v_x^2cx+v_x^4 &= v_x^2\left(x^2-2cx+c^2+y^2\right) & \text{(distributing)} \\
\Longrightarrow & & c^2x^2-2v_x^2cx+v_x^4 &= v_x^2x^2-2v_x^2cx+v_x^2c^2+v_x^2y^2 & \text{(distributing)} \\
\Longrightarrow & & c^2x^2+v_x^4 &= v_x^2x^2+v_x^2c^2+v_x^2y^2 & \text{(adding } 2v_x^2cx \text{ to both sides)} \\
\Longrightarrow & & v_x^4-v_x^2c^2 &= v_x^2x^2-c^2x^2+v_x^2y^2 & \text{(subtracting } v_x^2c^2 \text{ and } c^2x^2 \\
& & & & \text{from both sides)} \\
\Longrightarrow & & v_x^2(v_x^2-c^2) &= x^2(v_x^2-c^2)+v_x^2y^2 & \text{(factor)} \\
\Longrightarrow & & v_x^2b^2 &= x^2b^2+v_x^2y^2 & \text{(let } b^2=v_x^2-c^2\text{)} \\
\Longrightarrow & & \frac{v_x^2b^2}{v_x^2b^2} &= \frac{x^2b^2}{v_x^2b^2}+\frac{v_x^2y^2}{v_x^2b^2} & \text{(divide both sides by } v_x^2b^2\text{)} \\
\Longrightarrow & & 1 &= \frac{x^2}{v_x^2}+\frac{y^2}{b^2} & \text{(simplify)}
\end{aligned}
$$

If we let $x=0$ in our final result, we get $y^2=b^2$, which implies $y=\pm b$. Therefore, the point $(0,b)$ must be the co-vertex of the ellipse. As such, we relabel $b=v_y$ to arrive at the standard equation of an ellipse:

$$\frac{x^2}{v_x^2}+\frac{y^2}{v_y^2}=1.$$

This equation defines an ellipse centered at the origin. If $v_x>v_y$, the ellipse is stretched further in the horizontal direction, and if $v_y>v_x$, the ellipse is stretched further in the vertical direction.

You should spend time reading through the proof of this theorem (or, better yet, you should ask your instructor to go over this theorem in class!).

This theorem can be shortened quite a bit by saying that the larger of v_x and v_y dictates the major axis (and the smaller dictates the minor axis). In any case, the vertices in the horizontal direction are located v_x units from the center (at $(\pm v_x,0)$) and the vertices in the vertical direction are v_y units from the center (at $(0,\pm v_y)$). The foci are located c units from the center along the major axis, where $c^2=|v_x^2-v_y^2|$. The absolute values in this final equation force positivity of the right side.

Example 15.1.1: Writing the Equation of an Ellipse Centered at the Origin in Standard Form

What is the standard form equation of the ellipse that has vertices $(\pm 8,0)$ and foci $(\pm 5,0)$?

Solution

The foci are on the x-axis, so the major axis is the x-axis. Thus, the equation will have the form

$$\frac{x^2}{v_x^2}+\frac{y^2}{v_y^2}=1$$

The vertices are $(\pm 8, 0)$, so $v_x = 8$ and $v_x^2 = 64$.

The foci are $(\pm 5, 0)$, so $c = 5$ and $c^2 = 25$.

We know that the vertices and foci are related by the equation $c^2 = |v_x^2 - v_y^2|$. Solving for v_y^2, we have:

$$\begin{aligned} c^2 &= |v_x^2 - v_y^2| \\ \implies 25 &= |64 - v_y^2| && \text{(substituting)} \\ \implies 25 &= 64 - v_y^2 && \text{(since the major axis is horizontal, } v_x^2 > v_y^2\text{)} \\ \implies v_y^2 &= 39 && \text{(solving for } v_y^2\text{)} \end{aligned}$$

Now we need only substitute $v_x^2 = 64$ and $v_y^2 = 39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

? Checkpoint 15.1.1

What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $(0, \pm\sqrt{15})$?

Q&A

Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?

Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm v_x, 0)$ or $(0, \pm v_y)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use v_x (or v_y) and c from the given points, along with the equation $c^2 = |v_x^2 - v_y^2|$, to find the missing denominator.

Standard Form of an Ellipse Centered Off the Origin

Like the graphs of other equations, the graph of an ellipse can be transformed - specifically, in this conversation, shifted. If an ellipse is shifted h units horizontally and k units vertically, the center of the ellipse will be (h, k). This transformation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

Theorem: The Standard Form of the Equation of an Ellipse with Center (h, k)

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- $v_x^2 > v_y^2$
- the length of the major axis is $2v_x$
- the coordinates of the vertices are $(h \pm v_x, k)$
- the length of the minor axis is $2v_y$
- the coordinates of the co-vertices are $(h, k \pm v_y)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = |v_x^2 - v_y^2|$.

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the y-axis is

$$\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- $v_y^2 > v_x^2$
- the length of the major axis is $2v_y$
- the coordinates of the vertices are $(h, k \pm v_y)$
- the length of the minor axis is $2v_x$
- the coordinates of the co-vertices are $(h \pm v_x, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = |v_x^2 - v_y^2|$.

Again, this theorem can be shortened to saying that the larger of v_x and v_y dictates the major axis. In any case, the vertices in the horizontal direction are located v_x units from the center (at $(h \pm v_x, k)$) and the vertices in the vertical direction are v_y units from the center (at $(h, k \pm v_y)$). The foci are located c

units from the center along the major axis, where $c^2 = |v_x^2 - v_y^2|$.

Example 15.1.2: Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

Solution

The x-coordinates of the vertices and foci are the same, so the major axis is parallel to the y-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$$

First, we identify the center, (h,k). The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the Midpoint Formula, we have:

$$\begin{aligned}(h,k) &= \left(\frac{-2+(-2)}{2}, \frac{-8+2}{2}\right) \\ &= (-2,-3)\end{aligned}$$

Next, we find v_y^2. The length of the major axis, $2v_y$, is bounded by the vertices. We solve for v_y by finding the distance between the y-coordinates of the vertices.

$$\begin{aligned} 2v_y &= 2-(-8) \\ \implies 2v_y &= 10 \\ \implies v_y &= 5\end{aligned}$$

So $v_y^2 = 25$.

Now we find c^2. The foci are given by $(h, k \pm c)$. So, $(h, k-c) = (-2,-7)$ and $(h, k+c) = (-2,1)$. We substitute $k = -3$ using either of these points to solve for c.

$$\begin{aligned} k+c &= 1 \\ \implies -3+c &= 1 \\ \implies c &= 4\end{aligned}$$

So $c^2 = 16$.

Next, we solve for v_x^2 using the equation $c^2 = |v_x^2 - v_y^2|$.

$$\begin{aligned} c^2 &= |v_x^2 - v_y^2| && \\ \implies 16 &= |v_x^2 - 25| && \text{(substituting)} \\ \implies 16 &= 25 - v_x^2 && \text{(since } v_x^2 > v_x^2\text{)} \\ \implies v_x^2 &= 9 && \text{(solving for } v_x^2\text{)}\end{aligned}$$

Finally, we substitute the values found for h, k, v_x^2, and v_y^2 into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

Checkpoint 15.1.2

What is the standard form equation of the ellipse that has vertices $(-3,3)$ and $(5,3)$ and foci $(1-2\sqrt{3},3)$ and $(1+2\sqrt{3},3)$?

Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given information about its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form $\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1, a > b$.

Example 15.1.3: Graphing an Ellipse Given an Equation

Graph the ellipse given by the equation, $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Identify and label the center, vertices, covertices, and foci.

Solution

From the equation, $v_x^2 = 9$ and $v_y^2 = 25$. Moreover, $h = 0$ and $k = 0$. It follows that:

- the major axis is vertical ($v_y^2 > v_x^2$)
- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(0, \pm v_y) = (0, \pm\sqrt{25}) = (0, \pm 5)$
- the coordinates of the co-vertices are $(\pm v_x, 0) = (\pm\sqrt{9}, 0) = (\pm 3, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = |v_x^2 - v_y^2|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |9 - 25| \\ &= |-16| \\ &= 16 \\ \implies \quad c &= 4 \end{aligned}$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. See Figure 15.1.5

Figure 15.1.5

? Checkpoint 15.1.3

Graph the ellipse given by the equation $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Identify and label the center, vertices, covertices, and foci.

✓ Exercise 15.1.4: Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2 + 25y^2 = 100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Solution

First, use algebra to rewrite the equation in standard form.

$$\begin{aligned} 4x^2 + 25y^2 &= 100 \\ \implies \quad \frac{4x^2}{100} + \frac{25y^2}{100} &= \frac{100}{100} \\ \implies \quad \frac{x^2}{25} + \frac{y^2}{4} &= 1 \end{aligned}$$

From the equation, $v_x^2 = 25$ and $v_y^2 = 4$. Moreover, $h = 0$ and $k = 0$. It follows that:

- the major axis is horizontal ($v_x^2 > v_y^2$)
- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(\pm v_x, 0) = (\pm\sqrt{25}, 0) = (\pm 5, 0)$
- the coordinates of the co-vertices are $(0, \pm v_y) = (0, \pm\sqrt{4}) = (0, \pm 2)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = |v_x^2 - v_y^2|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |25-4| \\ &= |21| \\ &= 21 \\ \Rightarrow \quad c &= \sqrt{21} \end{aligned}$$

Therefore the coordinates of the foci are $(\pm\sqrt{21}, 0)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

Figure 15.1.6

? Checkpoint 15.1.4

Graph the ellipse given by the equation $49x^2 + 16y^2 = 784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Graphing Ellipses Not Centered at the Origin

When an ellipse is not centered at the origin, we can still use the standard form to find the key features of the graph. When the ellipse is centered at some point, (h, k), we use the standard form $\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$. From this standard equation, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

✓ Exercise 15.1.5: Graphing an Ellipse Centered at (h, k)

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4} + \frac{(y-5)^2}{9} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Solution

From the equation, $v_x^2 = 4$ and $v_y^2 = 9$. Moreover, $h = -2$ and $k = 5$. It follows that:

- the major axis is vertical $(v_y^2 > v_x^2)$
- the center of the ellipse is $(-2, 5)$
- the coordinates of the vertices are $(-2, 5 \pm v_y) = (-2, 5 \pm \sqrt{9}) = (-2, 5 \pm 3)$. That is, the vertices are at $(-2, 8)$ and $(-2, 2)$.
- the coordinates of the co-vertices are $(-2 \pm v_x, 5) = (-2 \pm \sqrt{4}, 5) = (-2 \pm 2, 5)$. Hence, they are at $(-4, 5)$ and $(0, 5)$.
- the coordinates of the foci are $(-2, 5 \pm c)$, where $c^2 = |v_x^2 - v_y^2|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |4-9| \\ &= |-5| \\ &= 5 \\ \Rightarrow \quad c &= \sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(-2, 5-\sqrt{5})$ and $(-2, 5+\sqrt{5})$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

Figure 15.1.7

? Checkpoint 15.1.5

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

Ellipses in General Form

We are sometimes given the equation of a conic in **general form**.

Definition: General Form of the Equation of a Conic Section

The **general form** of a conic is the unfactored form of the conic's equation, which is generally written in the form

$$ax^2+by^2+cx+dy+e=0,$$

where a and b are not both zero.

To get equations like these into the celebrated standard form, we use our old friend - Completing the Square.

Example 15.1.6: Graphing an Ellipse Centered at (h,k) by First Writing It in Standard Form

Graph the ellipse given by the equation $4x^2+9y^2-40x+36y+100=0$. Identify and label the center, vertices, co-vertices, and foci.

Solution

We must begin by rewriting the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$\left(4x^2-40x\right)+\left(9y^2+36y\right)=-100$$

Factor out the coefficients of the squared terms.

$$4\left(x^2-10x\right)+9\left(y^2+4y\right)=-100$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4\left(x^2-10x+25\right)+9\left(y^2+4y+4\right)=-100+100+36$$

Rewrite as perfect squares.

$$4(x-5)^2+9(y+2)^2=36$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-5)^2}{9}+\frac{(y+2)^2}{4}=1$$

Now that the equation is in standard form, we can determine the position of the major axis. From the equation, $v_x^2=9$ and $v_y^2=4$. Moreover, $h=5$ and $k=-2$. It follows that:

- the major axis is horizontal ($v_x^2>v_y^2$)
- the center of the ellipse is $(5,-2)$
- the coordinates of the vertices are $(5\pm v_x,-2)=(5\pm\sqrt{9},-2)=(5\pm3,-2)$. That is, $(2,-2)$ and $(8,-2)$
- the coordinates of the co-vertices are $(5,-2\pm v_y)=(5,-2\pm\sqrt{4})=(5,-2\pm2)$. That is, $(5,-4)$ and $(5,0)$
- the coordinates of the foci are $(5\pm c,-2)$, where $c^2=\left|v_x^2-v_y^2\right|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |9-4| \\ &= |5| \\ &= 5 \\ \Rightarrow \quad c &= \sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(5-\sqrt{5},-2)$ and $(5+\sqrt{5},-2)$.

Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 15.1.8.

Figure 15.1.8

? Checkpoint 15.1.6

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus. See Figure 15.1.9. In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci — about 43 feet apart — can hear each other whisper.

Figure 15.1.9

Sound waves are reflected between foci in an elliptical room, called a whispering chamber.

✓ Example 15.1.7: Locating the Foci of a Whispering Chamber

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are 46 feet wide by 96 feet long as shown in Figure 15.1.10.

a. What is the standard form of the equation of the ellipse representing the outline of the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0,0)$.
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.

Figure 15.1.10

Solution

a. We are assuming a horizontal ellipse with center $(0,0)$, so we need to find an equation of the form $\frac{x^2}{v_x^2}+\frac{y^2}{v_y^2}=1$, where $v_x^2>v_y^2$. We know that the length of the major axis, $2v_x$, is longer than the length of the minor axis, $2v_y$. So the length of the room, 96 , is represented by the major axis, and the width of the room, 46 , is represented by the minor axis.

Solving for v_x, we have $2v_x=96$, so $v_x=48$, and $v_x^2=2304$. Additionally, solving for v_y, we have $2v_y=46$, so $v_y=23$, and $v_y^2=529$.

Therefore, the equation of the ellipse is

$$\frac{x^2}{2304}+\frac{y^2}{529}=1.$$

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c,0)$, where $c^2=|a^2-b^2|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |2304-529| \\ &= |1775| \\ &= 1775 \\ \Longrightarrow \quad c &= \sqrt{1775} \\ &\approx 42 \end{aligned}$$

? Checkpoint 15.1.7

Suppose a whispering chamber is 480 feet long and 320 feet wide.

a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0,0)$.
b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot

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15.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: The distance formula is used to derive the standard equation of an ellipse from its definition. The midpoint formula is used to find the center of an ellipse given its vertices or foci. (LibreTexts 6.1.3, Example 6.1.2)

VIDEOS

- Ellipse
  - The Ellipse - Definitions and Concept
- Equation of an ellipse
  - The Ellipse - Building Equations for Ellipses
- Graphing ellipses
  - The Ellipse - Graphing Ellipses

KEY CONCEPTS

DEFINITIONS

- **Conic Section (Conic)**: A curve formed by the intersection of a plane and a double-napped right circular cone.
- **Ellipse**: The set of all points in a plane such that the sum of their distances from two fixed points is a constant.
- **Focus (plural: foci)**: One of the two fixed points used to define an ellipse. The sum of the distances from any point on the ellipse to the two foci is constant.
- **Major Axis**: The longer axis of an ellipse; it passes through the center and the vertices.
- **Minor Axis**: The shorter axis of an ellipse; it passes through the center and the co-vertices and is perpendicular to the major axis.
- **Vertex (of an ellipse)**: An endpoint of the major axis.
- **Co-vertex (of an ellipse)**: An endpoint of the minor axis.
- **Center (of an ellipse)**: The midpoint of both the major and minor axes.
- **General Form of the Equation of a Conic Section**: An equation of the form $ax^2 + by^2 + cx + dy + e = 0$, where a and b are not both zero.

THEOREMS

- **Standard Form of the Equation of an Ellipse with Center at the Origin**:
  - The standard form for an ellipse with its major axis on the x-axis is $\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1$, where $v_x > v_y$. The vertices are $(\pm v_x, 0)$ and the foci are $(\pm c, 0)$ where $c^2 = v_x^2 - v_y^2$.
  - The standard form for an ellipse with its major axis on the y-axis is $\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1$, where $v_y > v_x$. The vertices are $(0, \pm v_y)$ and the foci are $(0, \pm c)$ where $c^2 = v_y^2 - v_x^2$.
- **Standard Form of the Equation of an Ellipse with Center at (h, k)**:
  - The standard form for an ellipse with its major axis parallel to the x-axis is $\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$, where $v_x > v_y$. The vertices are $(h \pm v_x, k)$ and the foci are $(h \pm c, k)$ where $c^2 = v_x^2 - v_y^2$.
  - The standard form for an ellipse with its major axis parallel to the y-axis is $\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$, where $v_y > v_x$. The vertices are $(h, k \pm v_y)$ and the foci are $(h, k \pm c)$ where $c^2 = v_y^2 - v_x^2$.

COMMON MISTAKES

- **Confusing Major and Minor Axes**: Incorrectly identifying the major axis. The major axis corresponds to the larger denominator in the standard form equation.
- **Mixing up Vertices and Co-vertices**: Vertices are the endpoints of the major axis, while co-vertices are the endpoints of the minor axis.
- **Errors in Finding Foci**: Using the wrong formula for c^2. For an ellipse, c^2 is the difference of the squared semi-axis lengths ($c^2 = |v_x^2 - v_y^2|$), not the sum.
- **Forgetting to Take Square Roots**: The denominators in the standard form are v_x^2 and v_y^2. Forgetting to take the square root to find the values of v_x and v_y, which represent the distances from the center to the vertices and co-vertices.
- **Errors in Completing the Square**: When converting from general form to standard form, making algebraic mistakes during the completing the square process, especially forgetting to properly balance the equation when a coefficient has been factored out (e.g., adding $4(25)$ to the right side, not just 25).
- **Sign Errors with Center Coordinates**: When identifying the center (h, k) from the standard form $\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$, incorrectly determining the signs of h and k. For example, in $(x+2)^2$, $h = -2$.

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15.1.2: HOMEWORK

READING QUESTIONS

1. Define an ellipse in terms of its foci.
2. What is the difference between the major axis and the minor axis of an ellipse?
3. What are the vertices and co-vertices of an ellipse?
4. What is the standard form of the equation for an ellipse centered at the origin with its major axis on the x-axis?
5. How can you determine if the major axis of an ellipse is horizontal or vertical by looking at its standard form equation?
6. What is the relationship between the values v_x, v_y, and c (the distance from the center to a focus) for an ellipse?
7. How does the standard form equation of an ellipse change if its center is at (h, k) instead of the origin?
8. What algebraic process is used to convert the general form of an ellipse's equation into standard form?
9. What is a whispering chamber and what property of an ellipse does it utilize?
10. If an ellipse has vertices at $(\pm 5, 0)$ and co-vertices at $(0, \pm 3)$, what are the values of v_x and v_y?
11. Define an ellipse in terms of its foci.
12. Where must the foci of an ellipse lie?
13. What special case of the ellipse do we have when the major and minor axis are of the same length?
14. For the special case mentioned in the previous question, what would be true about the foci of that ellipse?
15. What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the y-axis?

HOMEWORK

ALGEBRAIC

For the following exercises, determine whether the given equations represent ellipses. If yes, write in standard form.

16. $2x^2 + y = 4$
17. $4x^2 + 9y^2 = 36$
18. $4x^2 - y^2 = 4$
19. $4x^2 + 9y^2 = 1$
20. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

For the following exercises, write the equation of an ellipse in standard form, and identify the end points of the major and minor axes as well as the foci.

21. $\frac{x^2}{4} + \frac{y^2}{49} = 1$
22. $\frac{x^2}{100} + \frac{y^2}{64} = 1$
23. $x^2 + 9y^2 = 1$
24. $4x^2 + 16y^2 = 1$
25. $\frac{(x-2)^2}{49} + \frac{(y-4)^2}{25} = 1$
26. $\frac{(x-2)^2}{81} + \frac{(y+1)^2}{16} = 1$
27. $\frac{(x+5)^2}{4} + \frac{(y-7)^2}{9} = 1$
28. $\frac{(x-7)^2}{49} + \frac{(y-7)^2}{49} = 1$
29. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$
30. $9x^2 - 54x + 9y^2 - 54y + 81 = 0$
31. $4x^2 - 24x + 36y^2 - 360y + 864 = 0$
32. $4x^2 + 24x + 16y^2 - 128y + 228 = 0$
33. $4x^2 + 40x + 25y^2 - 100y + 100 = 0$
34. $x^2 + 2x + 100y^2 - 1000y + 2401 = 0$
35. $4x^2 + 24x + 25y^2 + 200y + 336 = 0$
36. $9x^2 + 72x + 16y^2 + 16y + 4 = 0$

For the following exercises, find the foci for the given ellipses.

37. $\frac{(x+3)^2}{25} + \frac{(y+1)^2}{36} = 1$
38. $\frac{(x+1)^2}{100} + \frac{(y-2)^2}{4} = 1$
39. $x^2 + y^2 = 1$
40. $x^2 + 4y^2 + 4x + 8y = 1$
41. $10x^2 + y^2 + 200x = 0$

GRAPHICAL

For the following exercises, graph the given ellipses, noting center, vertices, and foci.

42. $\frac{x^2}{25} + \frac{y^2}{36} = 1$
43. $\frac{x^2}{16} + \frac{y^2}{9} = 1$
44. $4x^2 + 9y^2 = 1$
45. $81x^2 + 49y^2 = 1$
46. $\frac{(x-2)^2}{64} + \frac{(y-4)^2}{16} = 1$
47. $\frac{(x+3)^2}{9} + \frac{(y-3)^2}{9} = 1$
48. $\frac{x^2}{2} + \frac{(y+1)^2}{5} = 1$
49. $4x^2 - 8x + 16y^2 - 32y - 44 = 0$
50. $x^2 - 8x + 25y^2 - 100y + 91 = 0$
51. $x^2 + 8x + 4y^2 - 40y + 112 = 0$
52. $64x^2 + 128x + 9y^2 - 72y - 368 = 0$
53. $16x^2 + 64x + 4y^2 - 8y + 4 = 0$
54. $100x^2 + 1000x + y^2 - 10y + 2425 = 0$
55. $4x^2 + 16x + 4y^2 + 16y + 16 = 0$

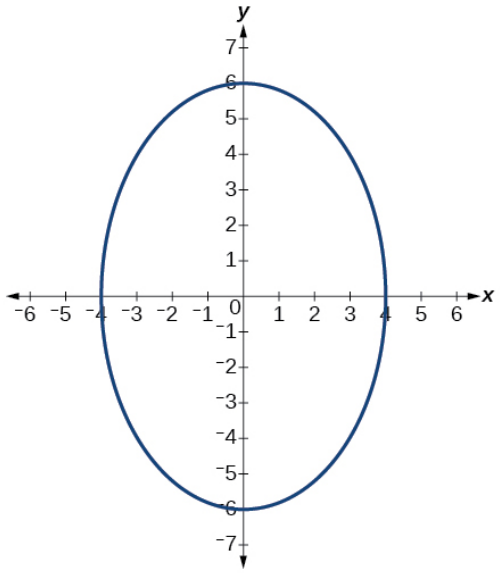
For the following exercises, use the given information about the graph of each ellipse to determine its equation.

56. Center at the origin, symmetric with respect to the x- and y-axes, focus at $(4, 0)$, and point on graph $(0, 3)$.
57. Center at the origin, symmetric with respect to the x- and y-axes, focus at $(0, -2)$, and point on graph $(5, 0)$.
58. Center at the origin, symmetric with respect to the x- and y-axes, focus at $(3, 0)$, and major axis is twice as long as minor axis.
59. Center $(4, 2)$; vertex $(9, 2)$; one focus: $(4 + 2\sqrt{6}, 2)$.
60. Center $(3, 5)$; vertex $(3, 11)$; one focus: $(3, 5 + 4\sqrt{2})$
61. Center $(-3, 4)$; vertex $(1, 4)$; one focus: $(-3 + 2\sqrt{3}, 4)$

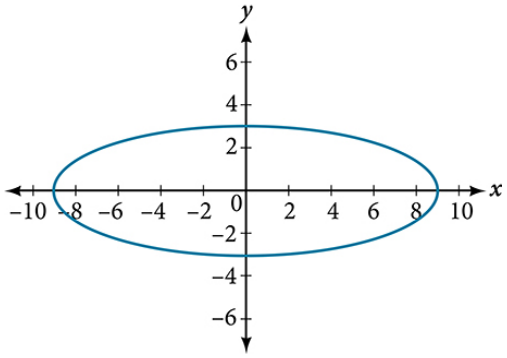
For the following exercises, given the graph of the ellipse, determine its equation.

62.

63.



64.



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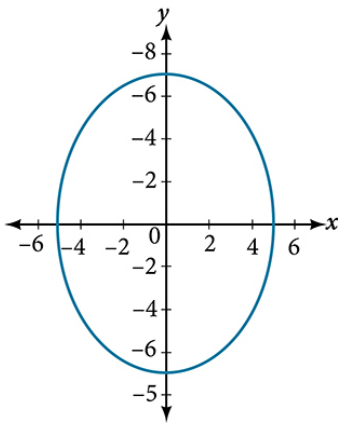

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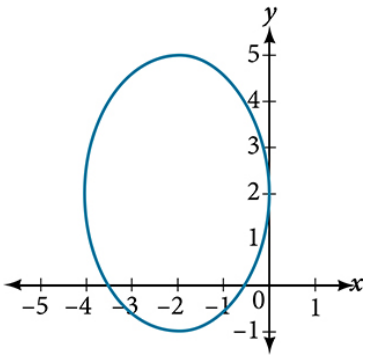

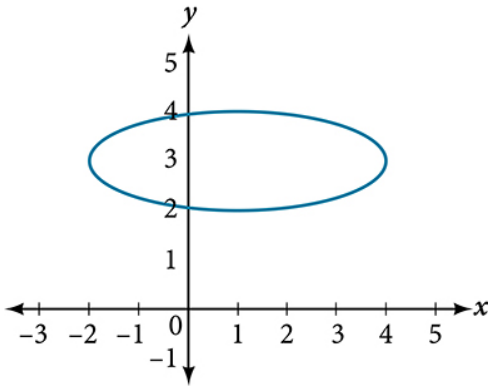

EXTENSIONS

For the following exercises, find the area of the ellipse. The area of an ellipse is given by the formula Area $= v_x \cdot v_y \cdot \pi$.

67. $\frac{(x-3)^2}{9} + \frac{(y-3)^2}{16} = 1$
68. $\frac{(x+6)^2}{16} + \frac{(y-6)^2}{36} = 1$
69. $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{5} = 1$
70. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$
71. $9x^2 - 54x + 9y^2 - 54y + 81 = 0$

REAL-WORLD APPLICATIONS

72. Find the equation of the ellipse that will just fit inside a box that is 8 units wide and 4 units high.
73. Find the equation of the ellipse that will just fit inside a box that is four times as wide as it is high. Express in terms of h, the height.
74. An arch has the shape of a semi-ellipse (the top half of an ellipse). The arch has a height of 8 feet and a span of 20 feet. Find an equation for the ellipse, and use that to find the height to the nearest 0.01 foot of the arch at a distance of 4 feet from the center.
75. An arch has the shape of a semi-ellipse. The arch has a height of 12 feet and a span of 40 feet. Find an equation for the ellipse, and use that to find the distance from the center to a point at which the height is 6 feet. Round to the nearest hundredth.
76. A bridge is to be built in the shape of a semi-elliptical arch and is to have a span of 120 feet. The height of the arch at a distance of 40 feet from the center is to be 8 feet. Find the height of the arch at its center.
77. A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the

other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet, and the foci are located 30 feet from the center, find the height of the ceiling at the center.

78. A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is 80 feet away, what is the length and height at the center of the gallery?

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15.2: The Hyperbola

Learning Objectives

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

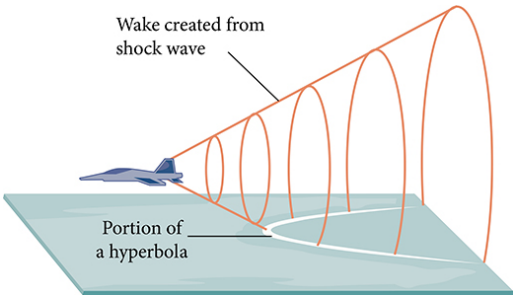
Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of conic. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom. See Figure **15.2.1**.

Figure 15.2.1

A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.



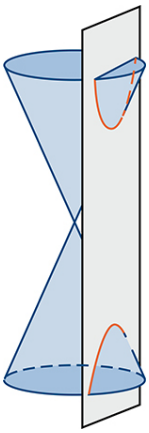
Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

Hyperbolas as Conic Sections

In analytic Geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other. See Figure **15.2.2**.

Figure 15.2.2

A hyperbola



Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane.

Definition: Hyperbola

A **hyperbola** is the set of all points in a plane such that the difference of the distances between the point and two fixed points is a positive constant. Each fixed point is called a **focus** (plural: **foci**).

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two axes of symmetry and several key points.

Definition: Transverse and Conjugate Axes, Vertex and Co-Vertex, and Center of a Hyperbola

The line segment that passes through the center of the hyperbola and has **vertices** as its endpoints is called **transverse axis**. The **conjugate axis** is perpendicular to the transverse axis and has the **co-vertices** as its endpoints. The **center** of the hyperbola is the midpoint of both the transverse and conjugate axes.

Students often think of the transverse axis of a hyperbola as analogous to the major axis for an ellipse. Likewise, they consider conjugate axis of a hyperbola as analogous to the minor axis.

Every hyperbola also has two asymptotes that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the center of the hyperbola with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle. See Figure **15.2.3**.

Figure 15.2.3

Key features of the hyperbola

As we did with ellipses, we limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the x- and y-axes. We will consider two cases: those that are centered at the origin, and those that are centered at a point other than the origin.

Standard Form of a Hyperbola Centered at the Origin

Theorem: Standard Form of the Equation of a Hyperbola Centered at the Origin

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the x-axis is

$$\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_x$
- the coordinates of the vertices are $(\pm v_x, 0)$
- the length of the conjugate axis is $2v_y$
- the coordinates of the co-vertices are $(0, \pm v_y)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y = \pm \frac{v_y}{v_x} x$.

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the y-axis is

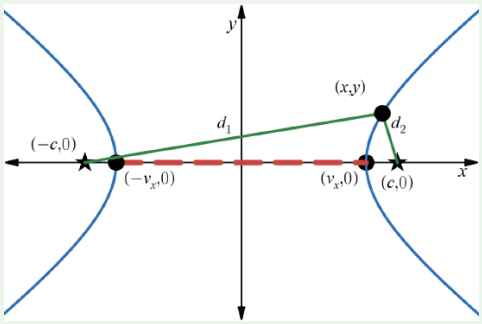
$$\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_y$
- the coordinates of the vertices are $(0, \pm v_y)$
- the length of the conjugate axis is $2v_x$
- the coordinates of the co-vertices are $(\pm v_x, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y = \pm \frac{v_y}{v_x} x$.

Proof

To derive the equation of a hyperbola centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The hyperbola is the set of all points (x, y) such that the difference of the distances from (x, y) to the foci is constant, as shown in the figure below.



If $(v_x, 0)$ is a vertex of the hyperbola (on the transverse axis), the distance from $(-c, 0)$ to $(v_x, 0)$ is $v_x - (-c) = v_x + c$. The distance from $(c, 0)$ to $(v_x, 0)$ is $c - v_x$ (this is where the difference in the proof between the ellipse and the hyperbola begin to differ - notice from the figure above that $c > v_x$). The difference of the distances from the foci to the vertex is

$$(v_x + c) - (c - v_x) = 2v_x.$$

If (x, y) is a point on the hyperbola, then we can define the following variables:

$$\begin{aligned} d_1 &= \text{the distance from } (-c, 0) \text{ to } (x, y) \\ d_2 &= \text{the distance from } (c, 0) \text{ to } (x, y) \end{aligned}$$

By the definition of a hyperbola, $d_1 - d_2$ is constant for any point (x, y) on the hyperbola. We know that the difference of these distances is $2v_x$ for the vertex $(v_x, 0)$. It follows that $d_1 - d_2 = 2v_x$ for any point on the hyperbola. We will begin the derivation by applying the Distance Formula. The rest of the derivation is algebraic.

$$
\begin{aligned}
& & d_1 - d_2 &= 2v_x & & \\
\implies & \sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2v_x & & \text{(substituting)} \\
\implies & \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2v_x & & \text{(simplifying)} \\
\implies & \sqrt{(x+c)^2+y^2} &= 2v_x + \sqrt{(x-c)^2+y^2} & & \text{(isolating a radical)} \\
\implies & (x+c)^2+y^2 &= \left[2v_x + \sqrt{(x-c)^2+y^2}\right]^2 & & \text{(squaring both sides)} \\
\implies & (x+c)^2+y^2 &= 4v_x^2 + 4v_x\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 & & \text{(distributing)} \\
\implies & x^2+2cx+c^2+y^2 &= 4v_x^2 + 4v_x\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2 & & \text{(distributing)} \\
\implies & 2cx &= 4v_x^2 + 4v_x\sqrt{(x-c)^2+y^2} - 2cx & & \text{(subtracting } x^2, c^2, \text{ and } y^2 \\
& & & & \text{ from both sides)} \\
\implies & 4cx - 4v_x^2 &= 4v_x\sqrt{(x-c)^2+y^2} & & \text{(isolating the radical)} \\
\implies & cx - v_x^2 &= v_x\sqrt{(x-c)^2+y^2} & & \text{(dividing both sides by 4)} \\
\implies & (cx-v_x^2)^2 &= v_x^2\left((x-c)^2+y^2\right) & & \text{(squaring both sides)} \\
\implies & c^2x^2 - 2v_x^2cx + v_x^4 &= v_x^2\left(x^2 - 2cx + c^2 + y^2\right) & & \text{(distributing)} \\
\implies & c^2x^2 - 2v_x^2cx + v_x^4 &= v_x^2x^2 - 2v_x^2cx + v_x^2c^2 + v_x^2y^2 & & \text{(distributing)} \\
\implies & c^2x^2 + v_x^4 &= v_x^2x^2 + v_x^2c^2 + v_x^2y^2 & & \text{(adding } 2v_x^2cx \text{ to both sides)} \\
\implies & v_x^4 - v_x^2c^2 &= v_x^2x^2 - c^2x^2 + v_x^2y^2 & & \text{(subtracting } v_x^2c^2 \text{ and } c^2x^2 \\
& & & & \text{ from both sides)} \\
\implies & v_x^2(v_x^2-c^2) &= x^2(v_x^2-c^2) + v_x^2y^2 & & \text{(factor)} \\
\implies & -v_x^2(v_x^2-c^2) &= -x^2(v_x^2-c^2) - v_x^2y^2 & & \text{(multiplying both sides by } -1) \\
\implies & v_x^2(c^2-v_x^2) &= x^2(c^2-v_x^2) - v_x^2y^2 & & \text{(distributing)} \\
\implies & v_x^2b^2 &= x^2b^2 - v_x^2y^2 & & \text{(let } b^2 = c^2 - v_x^2) \\
\implies & \frac{v_x^2b^2}{v_x^2b^2} &= \frac{x^2b^2}{v_x^2b^2} - \frac{v_x^2y^2}{v_x^2b^2} & & \text{(divide both sides by } v_x^2b^2) \\
\implies & 1 &= \frac{x^2}{v_x^2} - \frac{y^2}{b^2} & & \text{(simplify)}
\end{aligned}
$$

At this point in the proof of the equation for the *ellipse*, we were able to let $x = 0$ to determine the meaning of b; however, the hyperbola opening horizontally in the figure above does not exist at $x = 0$. Therefore, such a simple maneuver is not possible here. Instead, let's consider our result,

$$\frac{x^2}{v_x^2} - \frac{y^2}{b^2} = 1,$$

and solve for y.

$$\begin{aligned}
&\frac{x^2}{v_x^2}-\frac{y^2}{b^2} = 1 \\
\Rightarrow \quad &\frac{x^2}{v_x^2} = 1+\frac{y^2}{b^2} && \left(\text{adding } \tfrac{y^2}{b^2} \text{ to both sides}\right) \\
\Rightarrow \quad &\frac{x^2}{v_x^2}-1 = \frac{y^2}{b^2} && \text{(subtracting 1 from both sides)} \\
\Rightarrow \quad &\pm\sqrt{\frac{x^2}{v_x^2}-1} = \frac{y}{b} && \text{(taking the square root of both sides)} \\
\Rightarrow \quad &\pm b\sqrt{\frac{x^2}{v_x^2}-1} = y && \text{(multiplying both sides by } b\text{)} \\
\Rightarrow \quad &\pm b\sqrt{\frac{x^2}{v_x^2}\left(1-\frac{v_x^2}{x^2}\right)} = y && \left(\text{factoring } \tfrac{x^2}{v_x^2} \text{ within the radicand}\right) \\
\Rightarrow \quad &\pm b\left|\frac{x}{v_x}\right|\sqrt{1-\frac{v_x^2}{x^2}} = y && \left(\text{simplifying using } \sqrt{\blacksquare^2}=|\blacksquare|\right)
\end{aligned}$$

Since we are requiring $v_x > 0$, $|v_x| = v_x$. Moreover, for simplicity (and without loss of generality), let's only consider points on the right branch of the hyperbola. That is, let's assume $x > 0$. Then our final result becomes

$$y = \pm\frac{b}{v_x}x\sqrt{1-\frac{v_x^2}{x^2}}.$$

Using the same logic we developed when discussing slant asymptotes (see the section on graphing rational functions), we consider what happens as $x \to \infty$. To truly grasp this concept, you must keep in mind that v_x is a constant.

$$\begin{aligned}
&\text{As } x \to \infty, && \frac{v_x^2}{x^2} \to 0 \\
&\text{As } \frac{v_x^2}{x^2} \to 0, && 1-\frac{v_x^2}{x^2} \to 1 \\
&\text{As } 1-\frac{v_x^2}{x^2} \to 1, && \sqrt{1-\frac{v_x^2}{x^2}} \to 1
\end{aligned}$$

Thus, as $x \to \infty$, the y-values for the points on our hyperbola approach

$$y = \pm\frac{b}{v_x}x\sqrt{1-\frac{v_x^2}{x^2}} \to \pm\frac{b}{v_x}x \cdot 1 = \pm\frac{b}{v_x}x.$$

That is, the lines $y = \frac{b}{v_x}x$ and $y = -\frac{b}{v_x}x$ are slant asymptotes (guidelines for graphing) as $x \to \infty$. Both of these are linear equations where the slopes are $\pm\frac{b}{v_x}$. Since slope is "rise over run," or "increase in y over increase in x," we can interpret the numerator of these slopes as the corresponding change in y when x-values change by v_x. Making the notational change, $b = v_y$, we arrive at our desired result: the equation of the hyperbola centered at the origin and opening horizontally is

$$\frac{x^2}{v_x^2}-\frac{y^2}{v_y^2} = 1,$$

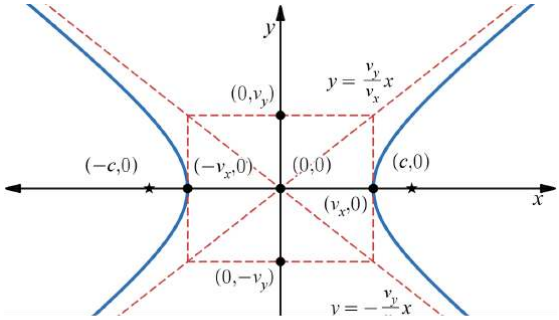
where the slant asymptotes are $y = \pm\frac{v_y}{v_x}x$.

The proof for the equation of the hyperbola opening vertically is similar.

Figure 15.2.4 shows the graphs of the horizontal and vertical hyperbolas centered at the origin.

Figure 15.2.4

Top: $\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$
Bottom: $\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$



One of the major differences between the ellipse and the hyperbola is the fact that one of the variable terms involving either x^2 or y^2 is negative in the equation of the hyperbola. This means that the order of the terms matters. That is,

$$\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1 \text{ is not the same as } \frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1.$$

The former results in a graph with branches opening left and right (this is what we derived above). The latter results in a graph with branches opening up and down. In either case, $c^2 = |v_x^2 + v_y^2|$.

This theorem can be shortened quite a bit by saying that the variable associated with the positive term dictates the transverse axis (and the variable associated with the negative terms dictates the conjugate axis). In any case, $c^2 = |v_x^2 + v_y^2|$. While there is no need for the absolute values in this equation, it doesn't hurt to have them and it aligns nicely with our method for ellipses.

✔ Example 15.2.1: Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

Solution

The equation has the form $\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$, so the transverse axis lies on the y-axis. The hyperbola is centered at the origin, so the vertices serve as the y-intercepts of the graph. To find the vertices, set $x = 0$ and solve for y

$$\begin{aligned} & 1 = \frac{y^2}{49} - \frac{x^2}{32} \\ \Rightarrow \quad & 1 = \frac{y^2}{49} - \frac{0^2}{32} && \text{(substituting)} \\ \Rightarrow \quad & 1 = \frac{y^2}{49} && \text{(simplifying)} \\ \Rightarrow \quad & 49 = y^2 && \text{(multiplying both sides by 49)} \\ \Rightarrow \quad & \pm 7 = y && \text{(Extraction of Roots)} \end{aligned}$$

The foci are located at $(0, \pm c)$. Solving for c,

$$\begin{aligned} c^2 &= |v_x^2 + v_y^2| \\ \Longrightarrow \quad c^2 &= |49 + 32| && \text{(substituting)} \\ \Longrightarrow \quad c^2 &= 81 && \text{(simplifying)} \\ \Longrightarrow \quad c &= 9 && \text{(Extraction of Roots)} \end{aligned}$$

Note that we stick with the convention that v_x, v_y, and c are all positive. Thus, we have no need to worry about negative radicals when performing Extraction of Roots.

We have found that the vertices are located at $(0, \pm 7)$, and the foci are at $(0, \pm 9)$.

? Checkpoint 15.2.1

Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

Reviewing the standard forms given for hyperbolas centered at $(0, 0)$, we see that the vertices, co-vertices, and foci are related by the equation $c^2 = |a^2 + b^2|$. Note that this equation can also be rewritten as $c^2 = a^2 + b^2$, which can be rewritten as $b^2 = c^2 - a^2$. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

✓ Example 15.2.2: Finding the Equation of a Hyperbola Centered at (0,0) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $\left(\pm 2\sqrt{10}, 0\right)$?

Solution

The vertices and foci are on the x-axis. Thus, the equation for the hyperbola will have the form $\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$.

The vertices are $(\pm 6, 0)$, so $v_x = 6$ and $v_x^2 = 36$.

The foci are $\left(\pm 2\sqrt{10}, 0\right)$, so $c = 2\sqrt{10}$ and $c^2 = 40$.

Solving for v_y^2, we have

$$\begin{aligned} c^2 &= |v_x^2 + v_y^2| \\ \Longrightarrow \quad 40 &= |36 + v_y^2| && \text{(substituting)} \\ \Longrightarrow \quad 40 &= 36 + v_y^2 && \text{(removing the unnecessary absolute values)} \\ \Longrightarrow \quad 4 &= v_y^2 && \text{(subtracting 36 from both sides)} \end{aligned}$$

Therefore, the equation of the hyperbola is

$$\frac{x^2}{36} - \frac{y^2}{4} = 1,$$

as shown in Figure 15.2.5.

Figure 15.2.5

? Checkpoint 15.2.2

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $\left(0, \pm 2\sqrt{5}\right)$?

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form

$$\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$$

for horizontal hyperbolas and the standard form

$$\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$$

for vertical hyperbolas.

✔ Example 15.2.3: Graphing a Hyperbola Centered at $(0, 0)$ Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64} - \frac{x^2}{36} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Solution

The standard form that applies to the given equation is $\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$. Thus, the transverse axis is on the y-axis.

The coordinates of the vertices are $(0, \pm v_y) = (0, \pm\sqrt{64}) = (0, \pm 8)$.

The coordinates of the co-vertices are $(\pm v_x, 0) = (\pm\sqrt{36}, 0) = (\pm 6, 0)$.

The coordinates of the foci are $(0, \pm c)$, where $c = \sqrt{v_x^2 + v_y^2}$. Solving for c, we have

$$c = \sqrt{64 + 36} = \sqrt{100} = 10.$$

Therefore, the coordinates of the foci are $(0, \pm 10)$.

The equations of the asymptotes are $y = \pm\frac{v_y}{v_x}x = \pm\frac{8}{6}x = \pm\frac{4}{3}x$.

Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola, as shown in Figure 15.2.6.

Figure 15.2.6

Checkpoint 15.2.3

Graph the hyperbola given by the equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Standard Form of a Hyperbola Centered Off the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated h units horizontally and k units vertically, the center of the hyperbola will be (h, k). This translation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

Theorem: Standard Form of the Equation of a Hyperbola Centered Off-Origin

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the x-axis is

$$\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_x$
- the coordinates of the vertices are $(h \pm v_x, k)$
- the length of the conjugate axis is $2v_y$
- the coordinates of the co-vertices are $(h, k \pm v_y)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y - k = \pm\frac{v_y}{v_x}(x - h)$.

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the y-axis is

$$\frac{(y-k)^2}{v_y^2} - \frac{(x-h)^2}{v_x^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_y$
- the coordinates of the vertices are $(h, k \pm v_y)$
- the length of the conjugate axis is $2v_x$
- the coordinates of the co-vertices are $(h \pm v_x, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y - k = \pm\frac{v_y}{v_x}(x - h)$.

Figure 15.2.7 shows the graphs of the horizontal and vertical hyperbolas centered at the origin.

Figure 15.2.7

Top: $\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1$
Bottom: $\frac{(y-k)^2}{v_y^2} - \frac{(x-h)^2}{v_x^2} = 1$

Like hyperbolas centered at the origin, hyperbolas centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2 = |v_x^2 + v_y^2|$. Again, there is no need for the absolute values; however, including them while introducing the concepts allows a firmer link between the equations for ellipses and equations for hyperbolas. We can use this relationship along with the Midpoint and Distance Formulas to find the standard equation of a hyperbola when the vertices and foci are given.

✔ Example 15.2.4: Finding the Equation of a Hyperbola Centered at (h, k) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

Solution

The y-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the x-axis. Thus, the equation of the hyperbola will have the form

$$\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1.$$

First, we identify the center, (h, k). The center is halfway between the vertices $(0, -2)$ and $(6, -2)$. Applying the Midpoint Formula, we have

$$(h, k) = \left(\frac{0+6}{2}, \frac{-2+(-2)}{2}\right) = (3, -2).$$

Next, we find v_x^2. The length of the transverse axis, $2v_x$, is bounded by the vertices. So, we can find v_x^2 by finding the distance between the x-coordinates of the vertices.

$$\begin{aligned} & 2v_x = |0-6| \\ \Rightarrow\ & 2v_x = 6 && \text{(simplifying)} \\ \Rightarrow\ & v_x = 3 && \text{(dividing both sides by 2)} \\ \Rightarrow\ & v_x^2 = 9 && \text{(squaring both sides)} \end{aligned}$$

Now we need to find c^2. The coordinates of the foci are $(h \pm c, k)$. So $(h-c, k) = (-2, -2)$ and $(h+c, k) = (8, -2)$. We can use the x-coordinate from either of these points to solve for c. Using the point $(8, -2)$ and substituting $h = 3$, we get

$$\begin{aligned} h+c &= 8 \\ \Rightarrow \quad 3+c &= 8 && \text{(substituting)} \\ \Rightarrow \quad c &= 5 && \text{(subtracting 3 from both sides)} \\ \Rightarrow \quad c^2 &= 25 && \text{(squaring both sides)} \end{aligned}$$

Next, solve for v_y^2 using the equation $c^2 = |v_x^2 + v_y^2| = v_x^2 + v_y^2$.

$$\begin{aligned} c^2 &= v_x^2 + v_y^2 \\ \Rightarrow \quad 25 &= 9 + v_y^2 && \text{(substituting)} \\ \Rightarrow \quad 16 &= v_y^2 && \text{(subtracting 9 from both sides)} \end{aligned}$$

Finally, substitute the values found for h, k, v_x^2, and v_y^2 into the standard form of the equation.

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$$

? Checkpoint 15.2.4

What is the standard form equation of the hyperbola that has vertices $(1,-2)$ and $(1,8)$ and foci $(1,-10)$ and $(1,16)$?

Graphing hyperbolas centered at a point (h,k) other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms

$$\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1$$

for horizontal hyperbolas, and

$$\frac{(y-k)^2}{v_y^2} - \frac{(x-h)^2}{v_x^2} = 1$$

for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

✔ Example 15.2.5: Graphing a Hyperbola Centered at (h,k) Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solution

Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(9x^2 - 36x) - (4y^2 + 40y) = 388$$

Factor the leading coefficient of each expression.

$$9(x^2 - 4x) - 4(y^2 + 10y) = 388$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$9(x^2 - 4x + 4) - 4(y^2 + 10y + 25) = 388 + 36 - 100$$

Rewrite as perfect squares.

$$9(x-2)^2 - 4(y+5)^2 = 324$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-2)^2}{36} - \frac{(y+5)^2}{81} = 1$$

The standard form that applies to the given equation is $\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1$, where $v_x^2 = 36$ and $v_y^2 = 81$, or $v_x = 6$ and $v_y = 9$. Thus, the transverse axis is parallel to the x-axis. It follows that:

- the center of the ellipse is $(h,k) = (2,-5)$
- the coordinates of the vertices are $(h \pm v_x, k) = (2 \pm 6, -5)$, or $(-4,-5)$ and $(8,-5)$

- the coordinates of the co-vertices are $(h, k \pm v_y) = (2, -5 \pm 9)$, or $(2, -14)$ and $(2, 4)$
- the coordinates of the foci are $(h \pm c, k)$, where $c = \sqrt{v_x^2 + v_y^2}$. Solving for c, we have

$$c = \sqrt{36 + 81} = \sqrt{117} = 3\sqrt{13}$$

  Therefore, the coordinates of the foci are $\left(2 - 3\sqrt{13}, -5\right)$ and $\left(2 + 3\sqrt{13}, -5\right)$.
- The equations of the asymptotes are

$$y - k = \pm\frac{v_y}{v_x}(x - h) \implies y - (-5) = \pm\frac{9}{6}(x - 2) \implies y = \pm\frac{3}{2}(x - 2) - 5.$$

Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola, as shown in Figure **15.2.8**.

Figure 15.2.8

? Checkpoint 15.2.5

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100} - \frac{(x-3)^2}{64} = 1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as Astronomy, Physics, Engineering, and Architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength. See Figure **15.2.9**. For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!

Figure 15.2.9

Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example **15.2.6** we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

Example 15.2.6: Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 15.2.10. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

Figure 15.2.10

Project design for a natural draft cooling tower

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola - indicated by the intersection of dashed perpendicular lines in the figure - is the origin of the coordinate plane. Round final values to four decimal places.

Solution

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of v_x^2 and v_y^2 to complete the model.

First, we find v_x^2. Recall that the length of the transverse axis of a hyperbola is $2v_x$. This length is represented by the distance where the sides are closest, which is given as 60 meters. So, $2v_x = 60$. Therefore, $v_x = 30$ and $v_x^2 = 900$.

To solve for v_y^2, we need to substitute for x and y in our equation using a known point. To do this, we can use the dimensions of the tower to find some point (x, y) that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the y-axis bisects the tower, our x-value can be represented by the radius of the top, or 36 meters. The y-value is represented by the distance from the origin to the top, which is given as 79.6 meters. Therefore,

$$\begin{aligned}
& \frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1 \\
\Rightarrow \quad & \frac{36^2}{900} - \frac{79.6^2}{v_y^2} = 1 && \text{(substituting)} \\
\Rightarrow \quad & \frac{36^2}{900} - 1 = \frac{79.6^2}{v_y^2} && \left(\text{subtracting 1 and adding } \tfrac{79.6^2}{v_y^2} \text{ to both sides}\right) \\
\Rightarrow \quad & \frac{1}{\frac{36^2}{900} - 1} = \frac{v_y^2}{79.6^2} && \text{(taking the reciprocal of both sides)} \\
\Rightarrow \quad & \frac{79.6^2}{\frac{36^2}{900} - 1} = v_y^2 && \left(\text{multiplying both sides by } 79.6^2\right) \\
\Rightarrow \quad & v_y^2 \approx 14400.3636 && \text{(Extraction of Roots)}
\end{aligned}$$

The sides of the tower can be modeled by the hyperbolic equation

$$\frac{x^2}{900} - \frac{y^2}{14400.3636} = 1.$$

Checkpoint 15.2.6

A design for a cooling tower project is shown in Figure 15.2.11. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

Figure 15.2.11

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15.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Hyperbola
  - The Hyperbola - Definitions and Concept
- Equation of a hyperbola
  - The Hyperbola - Building Equations for Hyperbolas
- Graphing hyperbolas
  - The Hyperbola - Graphing Hyperbolas

KEY CONCEPTS

DEFINITIONS

- **Hyperbola**: The set of all points in a plane such that the absolute value of the difference of the distances between the point and two fixed points is a positive constant.
- **Focus (plural: foci)**: One of the two fixed points used to define a hyperbola.
- **Transverse Axis**: The line segment that passes through the center of the hyperbola and has the vertices as its endpoints.
- **Conjugate Axis**: The line segment that is perpendicular to the transverse axis, passes through the center, and has the co-vertices as its endpoints.
- **Center (of a hyperbola)**: The midpoint of both the transverse and conjugate axes.
- **Vertices (of a hyperbola)**: The endpoints of the transverse axis.
- **Co-vertices (of a hyperbola)**: The endpoints of the conjugate axis.
- **Central Rectangle**: A rectangle centered at the center of the hyperbola with sides that pass through each vertex and co-vertex. The diagonals of this rectangle form the asymptotes of the hyperbola.

THEOREMS

- **Standard Form of the Equation of a Hyperbola with Center at the Origin**:
  - The standard form for a hyperbola with its transverse axis on the x-axis is $\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$. The vertices are $(\pm v_x, 0)$, the foci are $(\pm c, 0)$ where $c^2 = v_x^2 + v_y^2$, and the asymptotes are $y = \pm \frac{v_y}{v_x} x$.
  - The standard form for a hyperbola with its transverse axis on the y-axis is $\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$. The vertices are $(0, \pm v_y)$, the foci are $(0, \pm c)$ where $c^2 = v_x^2 + v_y^2$, and the asymptotes are $y = \pm \frac{v_y}{v_x} x$.
- **Standard Form of the Equation of a Hyperbola with Center at (h, k)**:
  - The standard form for a hyperbola with its transverse axis parallel to the x-axis is $\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1$. The equations of the asymptotes are $y - k = \pm \frac{v_y}{v_x}(x - h)$.
  - The standard form for a hyperbola with its transverse axis parallel to the y-axis is $\frac{(y-k)^2}{v_y^2} - \frac{(x-h)^2}{v_x^2} = 1$. The equations of the asymptotes are $y - k = \pm \frac{v_y}{v_x}(x - h)$.

COMMON MISTAKES

- **Confusing the Foci Equation with that of an Ellipse**: For a hyperbola, the relationship is $c^2 = v_x^2 + v_y^2$. This is different from the ellipse, which uses subtraction.
- **Incorrectly Identifying the Transverse Axis**: For a hyperbola, the transverse axis corresponds to the positive term in the standard form equation, not necessarily the term with the larger denominator.
- **Errors in Asymptote Equations**: Mixing up the numerator and denominator in the slope of the asymptotes. The slope is always a variation of $\pm \frac{\text{rise}}{\text{run}}$ or $\pm \frac{v_y}{v_x}$.
- **Errors in Completing the Square**: When converting from general to standard form, failing to correctly account for factored coefficients when balancing the equation. For example, when changing $9x^2 - 36x$ to $9(x^2 - 4x + 4)$, you have added $9 \times 4 = 36$ to that side of the equation.
- **Sign Errors with Center Coordinates**: Incorrectly identifying the center (h, k) from the standard form. For example, in $\frac{(x-2)^2}{36}$, $h = 2$, and in $\frac{(y+5)^2}{81}$, $k = -5$.

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15.2.2: HOMEWORK

READING QUESTIONS

1. Define a hyperbola in terms of its foci. How does this definition differ from the definition of an ellipse?
2. What is the transverse axis of a hyperbola? How can you identify which axis is the transverse axis from the standard form equation?
3. What is the role of the "central rectangle" in graphing a hyperbola?
4. What is the equation relating the distances from the center to the vertices (v_x, v_y) and the distance from the center to the foci (c)? How does this differ from the corresponding equation for an ellipse?
5. What is the standard form of the equation for a hyperbola with a horizontal transverse axis, centered at (h, k)?
6. What is the formula for the slopes of the asymptotes of a hyperbola centered at (h, k)?
7. If the asymptotes of a hyperbola intersect at the origin, what can you conclude about the hyperbola?
8. Do the foci of a hyperbola lie on the transverse axis or the conjugate axis?
9. If the transverse axis of a hyperbola is vertical, what does this tell you about its graph and its standard form equation?
10. Can the graph of a hyperbola ever cross its asymptotes? Can it cross its transverse axis? Can it cross its conjugate axis?
11. Define a hyperbola in terms of its foci.
12. What can we conclude about a hyperbola if its asymptotes intersect at the origin?
13. What must be true of the foci of a hyperbola?
14. If the transverse axis of a hyperbola is vertical, what do we know about the graph?
15. Where must the center of hyperbola be relative to its foci?

HOMEWORK

ALGEBRAIC

For the following exercises, determine whether the following equations represent hyperbolas. If so, write in standard form.

16. $3y^2 + 2x = 6$
17. $\frac{x^2}{36} - \frac{y^2}{9} = 1$
18. $5y^2 + 4x^2 = 6x$
19. $25x^2 - 16y^2 = 400$
20. $-9x^2 + 18x + y^2 + 4y - 14 = 0$

For the following exercises, write the equation for the hyperbola in standard form if it is not already, and identify the vertices and foci, and write equations of asymptotes.

21. $\frac{x^2}{25} - \frac{y^2}{36} = 1$
22. $\frac{x^2}{100} - \frac{y^2}{9} = 1$
23. $\frac{y^2}{4} - \frac{x^2}{81} = 1$
24. $9y^2 - 4x^2 = 1$
25. $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$
26. $\frac{(y-6)^2}{36} - \frac{(x+1)^2}{16} = 1$
27. $\frac{(x-2)^2}{49} - \frac{(y+7)^2}{49} = 1$
28. $4x^2 - 8x - 9y^2 - 72y + 112 = 0$
29. $-9x^2 - 54x + 9y^2 - 54y + 81 = 0$
30. $4x^2 - 24x - 36y^2 - 360y + 864 = 0$
31. $-4x^2 + 24x + 16y^2 - 128y + 156 = 0$
32. $-4x^2 + 40x + 25y^2 - 100y + 100 = 0$
33. $x^2 + 2x - 100y^2 - 1000y + 2401 = 0$
34. $-9x^2 + 72x + 16y^2 + 16y + 4 = 0$
35. $4x^2 + 24x - 25y^2 + 200y - 464 = 0$

For the following exercises, find the equations of the asymptotes for each hyperbola.

36. $\frac{y^2}{3^2} - \frac{x^2}{3^2} = 1$
37. $\frac{(x-3)^2}{5^2} - \frac{(y+4)^2}{2^2} = 1$
38. $\frac{(y-3)^2}{3^2} - \frac{(x+5)^2}{6^2} = 1$
39. $9x^2 - 18x - 16y^2 + 32y - 151 = 0$
40. $16y^2 + 96y - 4x^2 + 16x + 112 = 0$

GRAPHICAL

For the following exercises, sketch a graph of the hyperbola, labeling vertices and foci.

41. $\frac{x^2}{49} - \frac{y^2}{16} = 1$
42. $\frac{x^2}{64} - \frac{y^2}{4} = 1$
43. $\frac{y^2}{9} - \frac{x^2}{25} = 1$
44. $81x^2 - 9y^2 = 1$
45. $\frac{(y+5)^2}{9} - \frac{(x-4)^2}{25} = 1$
46. $\frac{(x-2)^2}{8} - \frac{(y+3)^2}{27} = 1$
47. $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$
48. $-4x^2 - 8x + 16y^2 - 32y - 52 = 0$
49. $x^2 - 8x - 25y^2 - 100y - 109 = 0$
50. $-x^2 + 8x + 4y^2 - 40y + 88 = 0$
51. $64x^2 + 128x - 9y^2 - 72y - 656 = 0$
52. $16x^2 + 64x - 4y^2 - 8y - 4 = 0$
53. $-100x^2 + 1000x + y^2 - 10y - 2575 = 0$
54. $4x^2 + 16x - 4y^2 + 16y + 16 = 0$

For the following exercises, given information about the graph of the hyperbola, find its equation.

55. Vertices at $(3, 0)$ and $(-3, 0)$ and one focus at $(5, 0)$.
56. Vertices at $(0, 6)$ and $(0, -6)$ and one focus at $(0, -8)$.
57. Vertices at $(1, 1)$ and $(11, 1)$ and one focus at $(12, 1)$.
58. Center: $(0, 0)$; vertex: $(0, -13)$; one focus: $(0, \sqrt{313})$.
59. Center: $(4, 2)$; vertex: $(9, 2)$; one focus: $(4 + \sqrt{26}, 2)$.
60. Center: $(3, 5)$; vertex: $(3, 11)$; one focus: $(3, 5 + 2\sqrt{10})$.

For the following exercises, given the graph of the hyperbola, find its equation.

61.

62.

63.

64.

65.

EXTENSIONS

For the following exercises, express the equation for the hyperbola as two functions, with y as a function of x. Express as simply as possible. Use graphing technology to sketch the graph of the two functions on the same axes.

66. $\frac{x^2}{4} - \frac{y^2}{9} = 1$
67. $\frac{y^2}{9} - x^2 = 1$
68. $\frac{(x-2)^2}{16} - \frac{(y+3)^2}{25} = 1$
69. $-4x^2 - 16x + y^2 - 2y - 19 = 0$
70. $4x^2 - 24x - y^2 - 4y + 16 = 0$

REAL-WORLD APPLICATIONS

For the following exercises, a hedge is to be constructed in the shape of a hyperbola near a fountain at the center of the yard. Find the equation of the hyperbola and sketch the graph.

71. The hedge will follow the asymptotes $y = x$ and $y = -x$, and its closest distance to the center fountain is 5 yards.
72. The hedge will follow the asymptotes $y = 2x$ and $y = -2x$, and its closest distance to the center fountain is 6 yards.
73. The hedge will follow the asymptotes $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$, and its closest distance to the center fountain is 10 yards.
74. The hedge will follow the asymptotes $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$, and its closest distance to the center fountain is 12 yards.
75. The hedge will follow the asymptotes $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$, and its closest distance to the center fountain is 20 yards.

For the following exercises, assume an object enters our solar system and we want to graph its path on a coordinate system with the sun at the origin and the x-axis as the axis of symmetry for the object's path. Give the equation of the flight path of each object using the given information.

76. The object enters along a path approximated by the line $y = x - 2$ and passes within 1 au (astronomical unit) of the sun at its closest approach, so that the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -x + 2$.
77. The object enters along a path approximated by the line $y = 2x - 2$ and passes within 0.5 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -2x + 2$.
78. The object enters along a path approximated by the line $y = 0.5x + 2$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -0.5x - 2$.
79. The object enters along a path approximated by the line $y = \frac{1}{3}x - 1$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -\frac{1}{3}x + 1$.
80. The object enters along a path approximated by the line $y = 3x - 9$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -3x + 9$.

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15.3: The Parabola

Learning Objectives

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 15.3.1

The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror (see Figure **15.3.1**), which focuses light rays from the sun to ignite the flame.

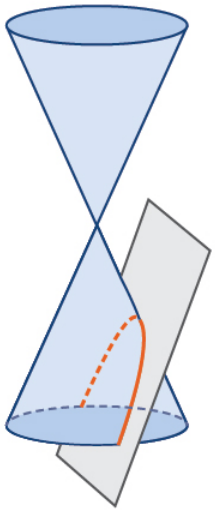
Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

Graphing Parabolas with Vertices at the Origin

We previously saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a **parabola**. See Figure **15.3.2**.

Figure 15.3.2

Parabola



Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane.

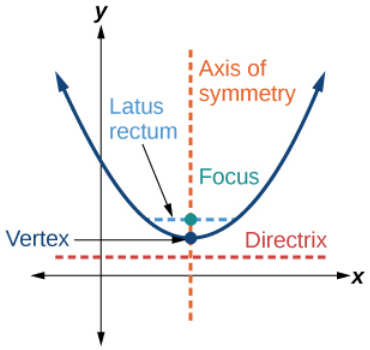
Definition: Parabola

A **parabola** is the set of all points in a plane that are the same distance from a fixed line, called the **directrix**, and a fixed point called the **focus** (where the focus is not on the directrix).

When studying quadratic functions, we learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola. See Figure **15.3.3**. Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus.

Figure 15.3.3

Key features of the parabola



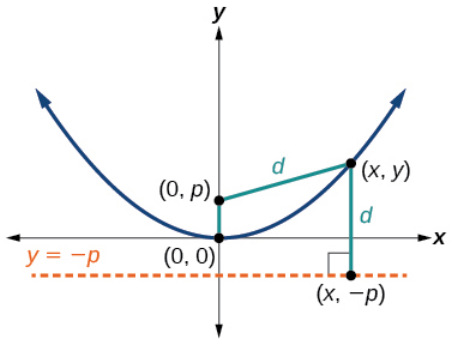
Definition: Latus Rectum

The line segment that passes through the focus and is parallel to the directrix is called the **latus rectum**.

The endpoints of the latus rectum lie on the curve. By definition, the distance d from the focus to any point P on the parabola is equal to the distance from P to the directrix.

To work with parabolas in the coordinate plane, we consider two cases: those with a vertex at the origin and those with a vertex at a point other than the origin. We begin with the former.

Figure 15.3.4



Let (x, y) be a point on the parabola with vertex $(0, 0)$, focus $(0, p)$, and directrix $y = -p$ as shown in Figure 15.3.4. The distance d from point (x, y) to point $(x, -p)$ on the directrix is the difference of the y-values: $d = y + p$. The distance from the focus $(0, p)$ to the point (x, y) is also equal to d and can be expressed using the Distance Formula.

$$d = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}.$$

Set the two expressions for d equal to each other and solve for y to derive the equation of the parabola. We do this because the distance from (x, y) to $(0, p)$ equals the distance from (x, y) to $(x, -p$.

$$\sqrt{x^2 + (y-p)^2} = y + p$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$\begin{aligned} & x^2 + (y-p)^2 = (y+p)^2 && \text{(squaring both sides)} \\ \Longrightarrow\ & x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 && \text{(distributing)} \\ \Longrightarrow\ & x^2 - 2py = 2py && \text{(subtracting } y^2 \text{ and } p^2 \text{ from both sides)} \\ \Longrightarrow\ & x^2 = 4py && \text{(adding } 2py \text{ to both sides)} \end{aligned}$$

The equations of parabolas with vertex $(0, 0)$ are $y^2 = 4px$ when the x-axis is the axis of symmetry and $x^2 = 4py$ when the y-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

Theorem: Standard Form of the Equation of a Parabola with Vertex at the Origin

The table below and the following figure summarize the standard features of parabolas with a vertex at the origin.

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
x-axis	$y^2 = 4px$	$(p, 0)$	$x = -p$	$(p, \pm 2p)$
y-axis	$x^2 = 4py$	$(0, p)$	$y = -p$	$(\pm 2p, p)$

(a) When $p > 0$ and the axis of symmetry is the x-axis, the parabola opens right.
(b) When $p < 0$ and the axis of symmetry is the x-axis, the parabola opens left.
(c) When $p > 0$ and the axis of symmetry is the y-axis, the parabola opens up.
(d) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and latus rectum. When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola.

A line is said to be **tangent** to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the latus rectum, these lines intersect on the axis of symmetry, as shown in Figure 15.3.5.

Figure 15.3.5

Example 15.3.1: Graphing a Parabola with Vertex $(0, 0)$ and the x-axis as the Axis of Symmetry

Graph $y^2 = 24x$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $y^2 = 4px$. Thus, the axis of symmetry is the x-axis. It follows that:

- $24 = 4p$, so $p = 6$. Since $p > 0$, the parabola opens right
- the coordinates of the focus are $(p, 0) = (6, 0)$
- the equation of the directrix is $x = -p = -6$
- the endpoints of the latus rectum have the same x-coordinate at the focus. To find the endpoints, substitute $x = 6$ into the original equation: $(6, \pm 12)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

Figure 15.3.6

? Checkpoint 15.3.1

Graph $y^2 = -16x$. Identify and label the focus, directrix, and endpoints of the latus rectum.

✓ Example 15.3.2: Graphing a Parabola with Vertex $(0,0)$ and the y-axis as the Axis of Symmetry

Graph $x^2 = -6y$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $x^2 = 4py$. Thus, the axis of symmetry is the y-axis. It follows that:

- $-6 = 4p$, so $p = -\frac{3}{2}$. Since $p < 0$, the parabola opens down
- the coordinates of the focus are $(0, p) = \left(0, -\frac{3}{2}\right)$
- the equation of the directrix is $y = -p = \frac{3}{2}$
- the endpoints of the latus rectum can be found by substituting $y = \frac{3}{2}$ into the original equation: $\left(\pm 3, -\frac{3}{2}\right)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

Figure 15.3.7

? Checkpoint 15.3.2

Graph $x^2 = 8y$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

✓ Example 15.3.3: Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $x = \frac{1}{2}$?

Solution

The focus has the form $(p, 0)$, so the equation will have the form $y^2 = 4px$. Multiplying $4p$, we have $4p = 4\left(-\frac{1}{2}\right) = -2$. Substituting for $4p$, we have $y^2 = 4px = -2x$. Therefore, the equation for the parabola is $y^2 = -2x$.

? Exercise 15.3.3

What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y = -\frac{7}{2}$?

Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k). This translation results in the standard form of the equation we saw previously with x replaced by $(x - h)$ and y replaced by $(y - k)$.

To graph parabolas with a vertex (h, k) other than the origin, we use the standard form $(y - k)^2 = 4p(x - h)$ for parabolas that have an axis of symmetry parallel to the x-axis, and $(x - h)^2 = 4p(y - k)$ for parabolas that have an axis of symmetry parallel to the y-axis. These standard forms are given below, along with their general graphs and key features.

Theorem: Standard Form of the Equation of a Parabola with Vertex Off-Origin

The table below and the following figure summarize the standard features of parabolas with a vertex at the origin.

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
$y=k$	$(y-k)^2=4p(x-h)$	$(h+p,k)$	$x=h-p$	$(h+p,k\pm 2p)$
$x=h$	$(x-h)^2=4p(y-k)$	$(h,k+p)$	$y=k-p$	$(h\pm 2p,k+p)$

(a) When $p>0$, the parabola opens right.
(b) When $p<0$, the parabola opens left.
(c) When $p>0$, the parabola opens up.
(d) When $p<0$, the parabola opens down.

Example 15.3.4: Graphing a Parabola with Vertex (h,k) and Axis of Symmetry Parallel to the x-axis

Graph $(y-1)^2=-16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $(y-k)^2=4p(x-h)$. Thus, the axis of symmetry is parallel to the x-axis. It follows that:

- the vertex is $(h,k)=(-3,1)$
- the axis of symmetry is $y=k=1$
- $-16=4p$, so $p=-4$. Since $p<0$, the parabola opens left.
- the coordinates of the focus are $(h+p,k)=(-3+(-4),1)=(-7,1)$
- the equation of the directrix is $x=h-p=-3-(-4)=1$
- the endpoints of the latus rectum are $(h+p,k\pm 2p)=(-3+(-4),1\pm 2(-4))$, or $(-7,-7)$ and $(-7,9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola. See Figure 15.3.8.

Figure 15.3.8

? Checkpoint 15.3.4

Graph $(y+1)^2 = 4(x-8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

✓ Example 15.3.5: Graphing a Parabola from an Equation Given in General Form

Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution

Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x-h)^2 = 4p(y-k)$. Thus, the axis of symmetry is parallel to the y-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable x in order to complete the square.

$$\begin{aligned} x^2 - 8x - 28y - 208 &= 0 \\ \Longrightarrow \quad x^2 - 8x &= 28y + 208 \\ \Longrightarrow \quad x2 - 8x + 16 &= 28y + 208 + 16 \\ \Longrightarrow \quad (x-4)^2 &= 28y + 224 \\ \Longrightarrow \quad (x-4)^2 &= 28(y+8) \\ \Longrightarrow \quad (x-4)^2 &= 4 \cdot 7(y+8) \end{aligned}$$

It follows that:

- the vertex is $(h,k) = (4,-8)$
- the axis of symmetry is $x = h = 4$
- since $p = 7$, $p > 0$ and so the parabola opens up
- the coordinates of the focus are $(h, k+p) = (4, -8+7) = (4,-1)$
- the equation of the directrix is $y = k - p = -8 - 7 = -15$
- the endpoints of the latus rectum are $(h \pm 2p, k+p) = (4 \pm 2(7), -8+7)$, or $(-10,-1)$ and $(18,-1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola. See Figure 15.3.9.

Figure 15.3.9

? Checkpoint 15.3.5

Graph $(x+2)^2 = -20(y-3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus. See Figure 15.3.12. This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.

Figure 15.3.10

Reflecting property of parabolas

Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

✓ Example 15.3.6: Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 15.3.11. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

a. Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
b. Use the equation found in part (a) to find the depth of the fire starter.

Figure 15.3.11

Cross-section of a travel-sized solar fire starter

Solution

a. The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^2 = 4py$, where $p > 0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p = 1.7$.

$$\begin{aligned} & x^2 = 4py && \text{(Standard form of upward-facing parabola with vertex } (0,0)) \\ \Longrightarrow\ & x^2 = 4(1.7)y && \text{(substituting)} \\ \Longrightarrow\ & x^2 = 6.8y && \text{(multiplying)} \end{aligned}$$

b. The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute 2.25 for x in the equation from part (a) to find the depth of the dish.

$$\begin{aligned} & x^2 = 6.8y \\ \Longrightarrow\ & (2.25)^2 = 6.8y && \text{(substituting)} \\ \Longrightarrow\ & 0.74 \approx y && \text{(solving for } y) \end{aligned}$$

The dish is about 0.74 inches deep.

? Checkpoint 15.3.6

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed 320 mm from the base.

a. Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the x-axis as its axis of symmetry).
b. Use the equation found in part (a) to find the depth of the cooker.

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15.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- Parabola
  - Algebra Mini-Lecture: Conic Sections - The Parabola

KEY CONCEPTS

DEFINITIONS

- **Parabola**: The set of all points in a plane that are the same distance from a fixed line, called the directrix, and a fixed point called the focus (where the focus is not on the directrix).
- **Directrix**: A fixed line used to define a parabola. Every point on the parabola is equidistant from the directrix and the focus.
- **Focus (of a parabola)**: A fixed point used to define a parabola. Every point on the parabola is equidistant from the focus and the directrix.
- **Axis of Symmetry (of a parabola)**: A line that passes through the focus and vertex and is perpendicular to the directrix.
- **Vertex (of a parabola)**: The point where the axis of symmetry intersects the parabola; it is the midpoint between the focus and the directrix.
- **Latus Rectum**: The line segment that passes through the focus and is parallel to the directrix; its endpoints lie on the parabola.
- **Tangent (to a curve)**: A line that intersects a curve at exactly one point.

THEOREMS

- **Standard Form of the Equation of a Parabola with Vertex at the Origin**:
  - If the axis of symmetry is the y-axis, the standard form is $x^2 = 4py$. The focus is $(0, p)$, and the directrix is the line $y = -p$.
  - If the axis of symmetry is the x-axis, the standard form is $y^2 = 4px$. The focus is $(p, 0)$, and the directrix is the line $x = -p$.
- **Standard Form of the Equation of a Parabola with Vertex at (h, k)**:
  - If the axis of symmetry is parallel to the y-axis, the standard form is $(x - h)^2 = 4p(y - k)$.
  - If the axis of symmetry is parallel to the x-axis, the standard form is $(y - k)^2 = 4p(x - h)$.

COMMON MISTAKES

- **Confusing the Parabola's Orientation**: Mistaking the standard form $y^2 = 4px$ (opens left/right) for $x^2 = 4py$ (opens up/down). The variable that is squared determines the axis of symmetry.
- **Incorrect Sign or Value for p**: The standard form has a $4p$ term. A common error is to mistake the coefficient for p instead of $4p$ (e.g., in $y^2 = 24x$, $p = 6$, not 24). Also, the sign of p determines the direction the parabola opens.
- **Incorrect Formulas for Focus and Directrix**: Applying the focus/directrix formula for a vertical parabola to a horizontal one, or vice-versa. For example, for $y^2 = 4px$, the focus is $(p, 0)$ and directrix is $x = -p$.
- **Errors in Completing the Square**: When converting from general form to standard form, making algebraic mistakes, especially when factoring out a leading coefficient before completing the square and forgetting to balance the other side of the equation accordingly.
- **Sign Errors with Center Coordinates (h, k)**: Incorrectly identifying the center (h, k) from the standard form. For example, in $(y - 1)^2 = -16(x + 3)$, the vertex is $(-3, 1)$.

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15.3.2: HOMEWORK

READING QUESTIONS

1. Define a parabola in terms of its focus and directrix.
2. If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about its graph? (Which way does it open?)
3. If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about its graph? (Which way does it open?)
4. What is the effect on the graph of a parabola if its equation in standard form has increasing values of $|p|$?
5. What are the two standard forms for the equation of a parabola with its vertex at the origin? What determines which way the parabola opens?
6. How are the focus and directrix related to the value of p?
7. How does the standard form equation of a parabola change if its vertex is at (h,k) instead of the origin?
8. What algebraic process is used to convert the general form of a parabola's equation into standard form?
9. What is the reflecting property of a parabola and how is it used in real-world objects like car headlights or satellite dishes?
10. What is the latus rectum of a parabola?
11. Define a parabola in terms of its focus and directrix.
12. If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about its graph?
13. If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about its graph?
14. What is the effect on the graph of a parabola if its equation in standard form has increasing values of p?
15. As the graph of a parabola becomes wider, what will happen to the distance between the focus and directrix?

HOMEWORK

ALGEBRAIC

For the following exercises, determine whether the given equation is a parabola. If so, rewrite the equation in standard form.

16. $y^2=4-x^2$
17. $y=4x^2$
18. $3x^2-6y^2=12$
19. $(y-3)^2=8(x-2)$
20. $y^2+12x-6y-51=0$

For the following exercises, rewrite the given equation in standard form, and then determine the vertex (V), focus (F), and directrix (d) of the parabola.

21. $x=8y^2$
22. $y=\frac{1}{4}x^2$
23. $y=-4x^2$
24. $x=\frac{1}{8}y^2$
25. $x=36y^2$
26. $x=\frac{1}{36}y^2$
27. $(x-1)^2=4(y-1)$
28. $(y-2)^2=\frac{4}{5}(x+4)$
29. $(y-4)^2=2(x+3)$
30. $(x+1)^2=2(y+4)$
31. $(x+4)^2=24(y+1)$
32. $(y+4)^2=16(x+4)$
33. $y^2+12x-6y+21=0$
34. $x^2-4x-24y+28=0$
35. $5x^2-50x-4y+113=0$
36. $y^2-24x+4y-68=0$
37. $x^2-4x+2y-6=0$
38. $y^2-6y+12x-3=0$
39. $3y^2-4x-6y+23=0$
40. $x^2+4x+8y-4=0$

GRAPHICAL

For the following exercises, graph the parabola, labeling the focus and the directrix.

41. $x=\frac{1}{8}18y^2$
42. $y=36x^2$
43. $y=\frac{1}{36}x^2$
44. $y=-9x^2$
45. $(y-2)^2=-\frac{4}{3}(x+2)$
46. $-5(x+5)^2=4(y+5)$
47. $-6(y+5)^2=4(x-4)$
48. $y^2-6y-8x+1=0$
49. $x^2+8x+4y+20=0$
50. $3x^2+30x-4y+95=0$
51. $y^2-8x+10y+9=0$
52. $x^2+4x+2y+2=0$
53. $y^2+2y-12x+61=0$
54. $-2x^2+8x-4y-24=0$

For the following exercises, find the equation of the parabola given information about its graph.

55. Vertex is $(0,0)$; directrix is $y=4$, focus is $(0,-4)$.
56. Vertex is $(0,0)$; directrix is $x=4$, focus is $(-4,0)$.
57. Vertex is $(2,2)$; directrix is $x=2-\sqrt{2}$, focus is $(2+\sqrt{2},2)$.
58. Vertex is $(-2,3)$; directrix is $x=-\frac{7}{2}$, focus is $\left(-\frac{1}{2},3\right)$.
59. Vertex is $(\sqrt{2},-\sqrt{3})$; directrix is $x=2\sqrt{2}$, focus is $(0,-\sqrt{3})$.
60. Vertex is $(1,2)$; directrix is $y=\frac{11}{3}$, focus is $\left(1,\frac{1}{3}\right)$.

For the following exercises, determine the equation for the parabola from its graph.

61.

62.

63.

64.

65.

EXTENSIONS

For the following exercises, the vertex and endpoints of the latus rectum of a parabola are given. Find the equation.

66. $V(0,0)$, Endpoints $(2,1)$, $(-2,1)$
67. $V(0,0)$, Endpoints $(-2,4)$, $(-2,-4)$
68. $V(1,2)$, Endpoints $(-5,5)$, $(7,5)$
69. $V(-3,-1)$, Endpoints $(0,5)$, $(0,-7)$
70. $V(4,-3)$, Endpoints $(5,-\frac{7}{2})$, $(3,-\frac{7}{2})$

REAL-WORLD APPLICATIONS

71. The mirror in an automobile headlight has a parabolic cross-section with the light bulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the light bulb?
72. If we want to construct the mirror from the previous exercise such that the focus is located at $(0, 0.25)$, what should the equation of the parabola be?
73. A satellite dish is shaped like a paraboloid of revolution. This means that it can be formed by rotating a parabola around its axis of symmetry. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?
74. Consider the satellite dish from the previous exercise. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?
75. The reflector in a searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.
76. If the reflector in the searchlight from the previous exercise has the light source located 6 inches from the base along the axis of symmetry and the opening is 4 feet, find the depth.
77. An arch is in the shape of a parabola. It has a span of 100 feet and a maximum height of 20 feet. Find the equation of the parabola, and determine the height of the arch 40 feet from the center.
78. If the arch from the previous exercise has a span of 160 feet and a maximum height of 40 feet, find the equation of the parabola, and determine the distance from the center at which the height is 20 feet.
79. An object is projected so as to follow a parabolic path given by $y = -x^2 + 96x$, where x is the horizontal distance traveled in feet and y is the height. Determine the maximum height the object reaches.
80. For the object from the previous exercise, assume the path followed is given by $y = -0.5x^2 + 80x$. Determine how far along the horizontal the object traveled to reach maximum height.

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CHAPTER OVERVIEW

Chapter 16: Systems of Equations and Inequalities

Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

At the start of the Second World War, British military and intelligence officers recognized that defeating Nazi Germany would require the Allies to know what the enemy was planning. This task was complicated by the fact that the German military transmitted all of its communications through a presumably uncrackable code created by a machine called Enigma. The Germans had been encoding their messages with this machine since the early 1930s. Not long after the war started, the British recruited a team of brilliant codebreakers to crack the Enigma code. The codebreakers used what they knew about the Enigma machine to build a mechanical computer that could crack the code. The Germans were so confident that the code could not be cracked, that they felt comfortable transmitting all manner of battlefield intelligence encoded with the machine. But the Allies had cracked it. And that knowledge of what the Germans were planning proved to be a key part of the ultimate Allied victory of Nazi Germany in 1945.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

16.1: Systems of Linear Equations - Two Variables

16.1.1: Resources and Key Concepts

16.1.2: Homework

16.2: Systems of Linear Equations - Three Variables

16.2.1: Resources and Key Concepts

16.2.2: Homework

16.3: Systems of Nonlinear Equations and Inequalities - Two Variables

16.3.1: Resources and Key Concepts

16.3.2: Homework

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16.1: Systems of Linear Equations - Two Variables

Learning Objectives

- Solve systems of equations by graphing.
- Solve systems of equations by Substitution.
- Solve systems of equations by Elimination.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 16.1.1

(credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. This is where **systems of equations** come into play.

Definition: System of Equations

A **system of equations**, sometimes simply called a **system**, consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.

It would not be an overstatement to say that systems of equations are one of the most important mathematical structures in modern society. From cell phone operation to web search to generative artificial intelligence, the ability to work with and solve systems of equations is a fundamental necessity in technology.

The simplest of systems involves only linear equations.

Definition: System of Linear Equations

A **system of linear equations**, sometimes simply called a **linear system**, consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously.

For the remainder of this section (and the next), we will focus on systems of linear equations.

Since a system involves two or more variables, we can think of the values of the variables as dials on a safe. We are looking for the very special combination of those dials that makes all equations within the system true simultaneously.

Definition: Solution to a System

A combination of values that causes each equation in a system to become a true statement is called a **solution** to the system.

For example, we might have a system of three equations, each containing the variables x, y, and z. If the combination $x = 0$, $y = -1$, and $z = \frac{1}{2}$ makes all three equations in the system true, we say that the solution to the system is $x = 0$, $y = -1$, and $z = \frac{1}{2}$; however, we have a much more succinct way of writing a solution to a system.

Definition: n-tuple

An **n-tuple** is an ordered collection of n elements of the form

$$(x_1, x_2, x_3 \ldots, x_n),$$

where n is a non-negative integer. It's a generalization of ordered pairs ($n = 2$), triples ($n = 3$), etc.

Therefore, in our case mentioned above, we could say the solution to the system is $\left(0, -1, \frac{1}{2}\right)$ (where we assume the entries in the n-tuple are written in alphabetical order).

Caution: Ambiguous Notation

When dealing with ordered pairs, an ambiguity appears in our notation. For example, without context, $(1, 5)$ could mean the ordered pair (the point on the plane) where $x = 1$ and $y = 5$, or it could mean the interval $1 < x < 5$.

You need to use the context of the given situation to determine which interpretation should be used.

As we shall see shortly, some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{cases} 2x + y = 15 \\ 3x - y = 5 \end{cases}$$

The ordered pair $(4, 7)$ is the solution to this system (for now, don't worry how we arrived at that). We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. We will investigate methods of finding such a solution, if it exists.

$$\begin{cases} 2(4) + (7) = 15 & \text{True} \\ 3(4) - (7) = 5 & \text{True} \end{cases}$$

In addition to considering the number of equations and variables, we can categorize systems (not just linear systems, but systems in general) by the number of solutions.

Systems can be defined in terms of whether or not they have solutions, and they can be further categorized based upon the number of solutions (finite or infinite).

Definition: Types of Systems

A system having one or more solutions is called a **consistent system**. A system having no solutions is called an **inconsistent system**.

Consistent systems are further categorized into **independent systems**, which are consistent systems with a finite number of solutions, and **dependent systems**, which are consistent systems with an infinite number of solutions.

Table 16.1.1 summarizes these possibilities.

Table 16.1.1

Types of Systems

Consistent Systems		Inconsistent Sys…	Inconsistent Systems
Independent Systems	**Dependent Systems**		
Finite Solutions	Infinite Solutions		No Solutions

Consider a system of linear equations involving two equations with two unknowns. Recall that the graph of an equation is a visual representation of the solution set for that specific equation. Thus, when we graph a system of two linear equations, we are seeing the solution sets to each component equation separately, and we are trying to find out if these solution sets ever overlap. Figure 16.1.2 shows us the possibilities when considering a system of two linear equations.

Figure 16.1.2

The independent and dependent systems are consistent because they have at least one solution. For the independent system, the two lines have different slopes and intersect at one point in the plane. For the dependent system, the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions. Finally, the inconsistent system is one in which the equations represent two *different* parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

This should convince us of the following theorem.

Theorem: Number of Solutions of Linear Systems

A consistent linear system has either exactly one solution or an infinite number of solutions.

Example 16.1.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5,1)$ is a solution to the given system of equations.

$$\begin{cases} x + 3y = 8 \\ 2x - 9 = y \end{cases}$$

Solution

Substitute the ordered pair $(5,1)$ into both equations.

$$
\begin{aligned}
(5)+3(1) &= 8 \\
\Longrightarrow \quad 8 &= 8 \qquad \text{True} \\
2(5)-9 &= (1) \\
\Longrightarrow \quad 1 &= 1 \qquad \text{True}
\end{aligned}
$$

The ordered pair $(5,1)$ satisfies both equations, so it is the solution to the system.

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure **16.1.3**.

Figure 16.1.3

? Checkpoint 16.1.1

Determine whether the ordered pair $(8,5)$ is a solution to the following system.

$$
\begin{cases} 5x - 4y = 20 \\ 2x + 1 = 3y \end{cases}
$$

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

✓ Example 16.1.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$
\begin{cases} 2x + y = -8 \\ x - y = -1 \end{cases}
$$

Solution

Solve the first equation for y.

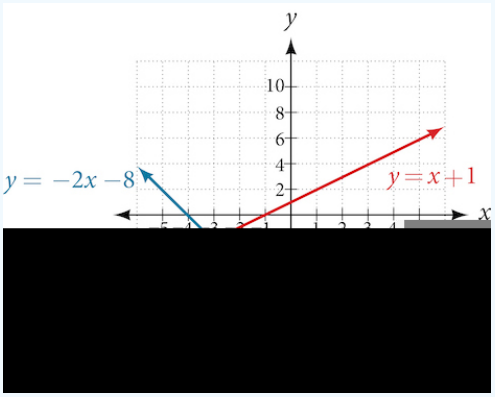
$$
\begin{aligned}
2x+y &= -8 \\
\Longrightarrow \quad y &= -2x-8
\end{aligned}
$$

Solve the second equation for y.

$$
\begin{aligned}
x-y &= -1 \\
\Longrightarrow \quad y &= x+1
\end{aligned}
$$

Graph both equations on the same set of axes as in Figure **16.1.4**.

Figure 16.1.4



The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} & 2(-3)+(-2) = -8 \\ \Longrightarrow \quad & -8 = -8 \qquad \text{True} \\ & (-3)-(-2) = -1 \\ \Longrightarrow \quad & -1 = -1 \qquad \text{True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$, so the system is independent.

Checkpoint 16.1.2

Solve the following system of equations by graphing.

$$\begin{cases} 2x - 5y = -25 \\ -4x + 5y = 35 \end{cases}$$

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **Substitution Method**.

Definition: Substitution Method

The **Substitution Method** is a heuristic process in which we solve one equation in a system for one variable and then substitute that result into another equation in the system.

The word "heuristic" used within this definition is purposeful. A process is heuristic if it requires the user to investigate using trial and error. In this regard, solving systems using the Substitution Method is not ideal - especially for larger systems (systems involving more than two equations and two variables). However, we can learn much from the Substitution Method, and it's use cannot be avoided.

How To... Solve a 2x2 System Using Substitution

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Before we continue, I should clarify what I mean by "2x2." A system of m equations involving n unknowns is called an $m \times n$ **system**. Therefore, a system of 2 equations with 2 unknowns is also called a $\mathbf{2 \times 2}$ **system**.

Example 16.1.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{cases} -x + y = -5 \\ 2x - 5y = 1 \end{cases}$$

Solution

We solve the first equation for y.

$$\begin{aligned} -x + y &= -5 \\ \Longrightarrow \quad y &= x - 5 \end{aligned}$$

Now we can substitute the expression $x - 5$ for y in the second equation.

$$\begin{aligned} 2x - 5y &= 1 \\ \Longrightarrow \quad 2x - 5(x-5) &= 1 && \text{(substituting)} \\ \Longrightarrow \quad 2x - 5x + 25 &= 1 && \text{(distributing)} \\ \Longrightarrow \quad -3x + 25 &= 1 && \text{(combining like terms)} \\ \Longrightarrow \quad -3x &= -24 && \text{(subtracting 25 from both sides)} \\ \Longrightarrow \quad x &= 8 && \text{(dividing both sides by } -3\text{)} \end{aligned}$$

Now, we substitute $x = 8$ into the first equation and solve for y.

$$\begin{aligned} -x + y &= -5 \\ \Longrightarrow \quad -(8) + y &= -5 && \text{(substituting)} \\ \Longrightarrow \quad y &= 3 && \text{(adding 8 to both sides)} \end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.

$$\begin{aligned} -x + y &= -5 \\ \Longrightarrow \quad -(8) + (3) &= -5 && \text{True} \\ 2x - 5y &= 1 \\ \Longrightarrow \quad 2(8) - 5(3) &= -1 && \text{True} \end{aligned}$$

Checkpoint 16.1.3

Solve the following system of equations by substitution.

$$\begin{cases} x &= y &+ 3 \\ 4 &= 3x &- 2y \end{cases}$$

Q&A

Can the Substitution Method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of **1** or **–1** so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Elimination Method

A third method of solving systems of linear equations is the **Elimination Method**.

Definition: Elimination Method (of solving a system)

The **Elimination Method** is an algorithmic process in which we add two equations within a system that have the opposite coefficients on the same variable so that the sum of the terms involving that variable is zero. This *eliminates* that variable from the system and forms a new, smaller system.

As with the Substitution Method, our choice of wording here is purposeful. Specifically, the use of the word "algorithm." An algorithm is a prescribed set of rules to achieve a specific goal. As such, the Elimination Method tends to be the better method to solve systems in the long run.

Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How To... Solve a 2x2 System Using Elimination

1. Write both equations with x- and y-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 16.1.4: Solving a System by the Elimination Method

Solve the given system of equations using the Elimination Method.

$$\begin{cases} x &+ 2y &= -1 \\ -x &+ y &= 3 \end{cases}$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{array}{rrrrr} x &+ 2y &= -1 \\ + -x &+ y &= 3 \\ \hline & 3y &= 2 \end{array}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$3y = 2 \implies y = \frac{2}{3}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} & -x + y = 3 \\ \implies & -x + \left(\frac{2}{3}\right) = 3 && \text{(substituting)} \\ \implies & -x = \frac{7}{3} && \left(\text{subtracting } \tfrac{7}{3} \text{ from both sides}\right) \\ \implies & x = -\frac{7}{3} && \text{(multiplying both sides by } -1) \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} & x + 2y = -1 \\ \implies & \left(-\frac{7}{3}\right) + 2\left(\frac{2}{3}\right) = -1 && \text{(substituting)} \\ \implies & -7 + 2(2) = -3 && \text{(multiplying both sides by 3)} \\ \implies & -7 + 4 = -3 && \text{(simplifying)} \\ \implies & -3 = -3 && \text{(combining like terms)} \end{aligned}$$

Thus, the solution works in the first equation. We leave it to the reader to verify it satisfies the second equation.

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 16.1.5 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

Figure 16.1.5

Example 16.1.5: Using the Elimination Method When Multiplication of One Equation Is Required

Solve the given system of equations by the Elimination Method.

$$\begin{cases} 3x + 5y = -11 \\ x - 2y = 11 \end{cases}$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3, the x-terms will add to zero.

$$\begin{aligned} x &- 2y = 11 \\ \Longrightarrow \quad -3(x &- 2y) = -3(11) \\ \Longrightarrow \quad -3x &+ 6y = -33 \end{aligned}$$

Now, let's add them.

$$\begin{array}{rrrrrr} & 3x & + & 5y & = & -11 \\ + & -3x & + & 6y & = & -33 \\ \hline & & & 11y & = & -44 \\ \Longrightarrow & & & y & = & -4 \end{array}$$

Substituting this into the second equation (in the original system), we get the following:

$$\begin{aligned} x - 2y &= 11 \\ \Longrightarrow \quad x - 2(-4) &= 11 \qquad \text{(substituting)} \\ \Longrightarrow \quad x + 8 &= 11 \qquad \text{(simplifying)} \\ \Longrightarrow \quad x &= 3 \qquad \text{(subtracting 8 from both sides)} \end{aligned}$$

Our solution is the ordered pair $(3, -4)$ (we leave it to the reader to algebraically check the solution). See Figure 16.1.6.

Figure 16.1.6

? Checkpoint 16.1.5

Solve the system of equations using the Elimination Method.

$$\begin{cases} 2x - 7y = 2 \\ 3x + y = -20 \end{cases}$$

✔ Example 16.1.6: Using the Elimination Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables using the Elimination Method.

$$\begin{cases} 2x + 3y = -16 \\ 5x - 10y = 30 \end{cases}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} & 2x + 3y = -16 \\ \Rightarrow\ & -5(2x + 3y) = -5(-16) \\ \Rightarrow\ & -10x - 15y = 80 \\ & 5x - 10y = 30 \\ \Rightarrow\ & 2(5x - 10y) = 2(30) \\ \Rightarrow\ & 10x - 20y = 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{array}{rrrrr} & -10x & - 15y & = & 80 \\ + & 10x & - 20y & = & 60 \\ \hline & & - 35y & = & 140 \\ \Rightarrow & & y & = & -4 \end{array}$$

Substitute $y = -4$ into the original first equation.

$$\begin{aligned} & 2x + 3y = -16 & & \\ \Rightarrow\ & 2x + 3(-4) = -16 & & \text{(substituting)} \\ \Rightarrow\ & 2x - 12 = -16 & & \text{(simplifying)} \\ \Rightarrow\ & 2x = -4 & & \text{(adding 12 to both sides)} \\ \Rightarrow\ & x = -2 & & \text{(dividing both sides by 2)} \end{aligned}$$

The solution is $(-2, -4)$ (again, we leave it to the reader to check this). See Figure 16.1.7.

Figure 16.1.7

Example 16.1.7: Using the Elimination Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables using the Elimination Method.

$$\begin{cases} \frac{x}{3} + \frac{y}{6} = 3 \\ \frac{x}{2} - \frac{y}{4} = 1 \end{cases}$$

Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$\begin{aligned} \frac{x}{3} + \frac{y}{6} &= 3 \\ \Longrightarrow \quad 6\left(\frac{x}{3} + \frac{y}{6}\right) &= 6(3) \\ \Longrightarrow \quad 2x + y &= 18 \\ \frac{x}{2} - \frac{y}{4} &= 1 \\ \Longrightarrow \quad 4\left(\frac{x}{2} - \frac{y}{4}\right) &= 4(1) \\ \Longrightarrow \quad 2x - y &= 4 \end{aligned}$$

Add the two equations to eliminate the y-variable and solve the resulting equation.

$$\begin{array}{rrcrcl} & 2x & + & y & = & 18 \\ + & 2x & - & y & = & 4 \\ \hline & 4x & & & = & 22 \\ \Longrightarrow & x & & & = & \frac{11}{2} \end{array}$$

Substitute $x = \frac{11}{2}$ into the first equation.

$$\begin{aligned} 2x + y &= 18 \\ \Longrightarrow \quad 2\left(\frac{11}{2}\right) + y &= 18 \qquad \text{(substituting)} \\ \Longrightarrow \quad 11 + y &= 18 \qquad \text{(simplifying)} \\ \Longrightarrow \quad y &= 7 \qquad \text{(subtracting 11 from both sides)} \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. We leave it to the reader to check it in the other equation.

? Checkpoint 16.1.7

Solve the system of equations using the Elimination Method.

$$\begin{cases} 2x + 3y = 8 \\ 3x + 5y = 10 \end{cases}$$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

✔ Example 16.1.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{cases} x = 9 - 2y \\ x + 2y = 13 \end{cases}$$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x+2y &= 13 \\ \Rightarrow \quad (9-2y)+2y &= 13 \qquad \text{(substituting)} \\ \Rightarrow \quad 9 &= 13 \qquad \text{(simplifying)} \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9-2y \\ \Rightarrow \quad 2y &= -x+9 \qquad \text{(subtracting } x \text{ from and adding } 2y \text{ to both sides)} \\ \Rightarrow \quad y &= -\frac{1}{2}x+\frac{9}{2} \qquad \text{(dividing both sides by 2)} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x+2y &= 13 \\ \Rightarrow \quad 2y &= -x+13 \qquad \text{(subtracting } 2y \text{ from both sides)} \\ \Rightarrow \quad y &= -\frac{1}{2}x+\frac{13}{2} \qquad \text{(dividing both sides by 2)} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines (with different y-intercepts) will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 16.1.8.

Figure 16.1.8

? Checkpoint 16.1.8

Solve the following system of equations in two variables.

$$\begin{cases} 2y - 2x = 2 \\ 2y - 2x = 6 \end{cases}$$

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using Substitution or Elimination, the resulting equation will be an identity, such as $0 = 0$.

Example 16.1.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the Elimination Method.

$$\begin{cases} x + 3y = 2 \\ 3x + 9y = 6 \end{cases}$$

Solution

With the Elimination Method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$\begin{aligned} & x + 3y = 2 \\ \Longrightarrow \quad & (-3)(x + 3y) = (-3)(2) \\ \Longrightarrow \quad & -3x - 9y = -6 \end{aligned}$$

Now add the equations.

$$\begin{array}{rrcl} & -3x - 9y & = & -6 \\ & 3x + 9y & = & 6 \\ \hline + & 0 & = & 0 \end{array}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x + 3y &= 2 \\ \Longrightarrow \quad 3y &= -x + 2 \\ \Longrightarrow \quad y &= -\frac{1}{3}x + \frac{2}{3} \\ 3x + 9y &= 6 \\ \Longrightarrow \quad 9y &= -3x + 6 \\ \Longrightarrow \quad y &= -\frac{1}{3}x + \frac{2}{3} \end{aligned}$$

See Figure 16.1.9. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x + \frac{2}{3}\right)$.

Figure 16.1.9

? Checkpoint 16.1.9

Solve the following system of equations in two variables.

$$\begin{cases} y - 2x = 5 \\ -3y + 6x = -15 \end{cases}$$

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's **revenue function** is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x = \text{quantity}$ and $p = \text{price}$. The revenue function is shown in orange in Figure 16.1.10.

The **cost function** is the function used to calculate the costs of doing business. It includes **fixed costs**, such as rent and salaries, and **variable costs**, such as utilities. The cost function is shown in blue in Figure 16.1.10. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

Figure 16.1.10

The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

✓ Example 16.1.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution

Write the system of equations using y to replace function notation.

$$\begin{cases} y = 0.85x + 35000 \\ y = 1.55x \end{cases}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35000 &= 1.55x \\ \Longrightarrow \quad 35000 &= 0.7x \qquad \text{(subtracting } 0.85x \text{ from both sides)} \\ \Longrightarrow \quad 50000 &= x \qquad \text{(dividing both sides by } 0.7\text{)} \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50000) = 77500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) \quad R(x) - C(x) & \\ = \quad & 1.55x - (0.85x + 35000) \\ = \quad & 0.7x - 35000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See Figure **16.1.11**.

Figure 16.1.11

We see from the graph in Figure **16.1.12** that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

Figure 16.1.12

Example 16.1.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $25.00 for children and $50.00 for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is $70,000. How many children and how many adults bought tickets?

Solution

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2,000$$

The revenue from all children can be found by multiplying $25.00 by the number of children, $25c$. The revenue from all adults can be found by multiplying $50.00 by the number of adults, $50a$. The total revenue is $70,000. We can use this to write an equation for the revenue.

$$25c + 50a = 70,000$$

We now have a system of linear equations in two variables.

$$\begin{cases} c + a = 2,000 \\ 25c + 50a = 70,000 \end{cases}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$c + a = 2,000 \implies a = 2,000 - c.$$

Substitute the expression $2,000 - c$ in the second equation for a and solve for c.

$$\begin{aligned} & 25c + 50a = 70,000 \\ \implies \quad & 25c + 50(2,000 - c) = 70,000 && \text{(substituting)} \\ \implies \quad & 25c + 100,000 - 50c = 70,000 && \text{(distributing)} \\ \implies \quad & -25c + 100,000 = 70,000 && \text{(combining like terms)} \\ \implies \quad & -25c = -30,000 && \text{(subtracting 100,000 from both sides)} \\ \implies \quad & c = 1,200 && \text{(dividing both sides by } -25) \end{aligned}$$

Substitute $c = 1,200$ into the first equation to solve for a.

$$\begin{aligned} & c + a = 2,000 \\ \implies \quad & 1,200 + a = 2,000 && \text{(substituting)} \\ \implies \quad & a = 800 && \text{(subtracting 1,200 from both sides)} \end{aligned}$$

We find that 1,200 children and 800 adults bought tickets to the circus that day.

Checkpoint 16.1.11

Meal tickets at the circus cost $4.00 for children and $12.00 for adults. If 1,650 meal tickets were bought for a total of $14,200, how many children and how many adults bought meal tickets?

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16.1.1: RESOURCES AND KEY CONCEPTS

RESOURCES

PREREQUISITE TOPICS

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Linear Functions**
  - Graphing Linear Functions: A direct prerequisite for solving systems of linear equations by graphing.
  - Linear Models: Required for working with applications of systems of linear equations.
  - Parallel and Perpendicular Lines: Understanding that parallel lines have the same slope is key to identifying inconsistent systems graphically.

VIDEOS

- Systems of linear equations
  - Intermediate Algebra - An Introduction to Systems of Equations (Part A)
  - Intermediate Algebra - An Introduction to Systems of Equations (Part B)
  - Intermediate Algebra - An Introduction to Systems of Equations (Part C)
- Solving systems by substitution
  - Intermediate Algebra - Solving Small Systems of Linear Equations Algebraically (Part A)
- Solving systems by elimination
  - Intermediate Algebra - Solving Small Systems of Linear Equations Algebraically (Part B)

KEY CONCEPTS

DEFINITIONS

- **System of Equations**: Two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- **System of Linear Equations (Linear System)**: A system consisting of two or more linear equations.
- **Solution to a System**: A combination of values for the variables that causes each equation in the system to become a true statement. For a 2x2 system, this is an ordered pair (x, y).
- **n-tuple**: An ordered collection of n elements, used to represent solutions to systems with n variables.
- **Consistent System**: A system of equations that has one or more solutions.
- **Inconsistent System**: A system of equations that has no solutions. Graphically, this corresponds to parallel lines.
- **Independent System**: A consistent system with a finite number of solutions (for a 2x2 linear system, this means exactly one solution). Graphically, this corresponds to intersecting lines.
- **Dependent System**: A consistent system with an infinite number of solutions. Graphically, this corresponds to two identical lines (coincident lines).
- **Substitution Method**: A heuristic process for solving a system where one equation is solved for one variable, and that expression is then substituted into another equation in the system.
- **Elimination Method**: An algorithmic process for solving a system where two equations are added together to eliminate a variable. This often requires first multiplying one or both equations by a constant to ensure one variable has opposite coefficients.

THEOREMS

- **Number of Solutions of Linear Systems**: A consistent linear system has either exactly one solution or an infinite number of solutions. (This implies a linear system cannot have exactly two, three, etc., solutions).

COMMON MISTAKES

- **Solving for Only One Variable**: After finding the value of the first variable (e.g., x), forgetting to substitute it back into an original equation to solve for the second variable.
- **Errors in Substitution**: When substituting an expression into another equation, making algebraic errors, especially with distribution if the expression is being multiplied by a coefficient.
- **Errors in Elimination**:
  - Incorrectly multiplying an entire equation by a constant (e.g., forgetting to multiply the constant term on the other side of the equals sign).
  - Making addition errors after multiplying, especially with negative numbers.
  - Multiplying by a constant that does not result in opposite coefficients for one of the variables.
- **Assuming a Solution Exists**: Concluding there is one solution when the system is actually inconsistent (parallel lines) or dependent (same line). Arriving at a false statement (e.g., $9 = 13$) indicates no solution. Arriving at a true identity (e.g., $0 = 0$) indicates infinite solutions.
- **Incorrectly Stating the Solution for a Dependent System**: For a dependent system with infinite solutions, simply writing "infinite solutions" is often insufficient. The solution should be expressed as an ordered pair in terms of one of the variables (e.g., $(x, -\frac{1}{3}x + \frac{2}{3})$).
- **Errors in Setting Up Systems from Word Problems**: Misinterpreting the problem and creating incorrect linear equations to represent the given information.

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16.1.2: HOMEWORK

READING QUESTIONS

1. What is a system of linear equations?
2. Can a system of two linear equations in two variables have exactly two solutions? Why or why not?
3. Describe the three types of systems based on their number of solutions (independent, dependent, and inconsistent) and what their graphs look like.
4. Outline the steps of the Substitution Method for solving a 2x2 system.
5. Outline the steps of the Elimination Method for solving a 2x2 system.
6. If you are solving a system and arrive at a false statement like $9 = 13$, what can you conclude about the system?
7. If you are solving a system and arrive at an identity like $0 = 0$, what can you conclude about the system, and how should you express the solution?
8. In a business context, what is the "break-even point"? How is it found using a system of equations?
9. What is the profit function, $P(x)$, in relation to the cost function, $C(x)$, and the revenue function, $R(x)$?
10. List the three methods for solving a system of linear equations in two variables discussed in this section.
11. Can a system of linear equations have exactly two solutions? Explain why or why not.
12. If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.
13. If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?
14. If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?
15. Given a system of equations, explain at least two different methods of solving that system.

HOMEWORK

ALGEBRAIC

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

16. $\begin{cases} 5x - y = 4 \\ x + 6y = 2 \end{cases}$ and $(4, 0)$
17. $\begin{cases} -3x - 5y = 13 \\ -x + 4y = 10 \end{cases}$ and $(-6, 1)$
18. $\begin{cases} 3x + 7y = 1 \\ 2x + 4y = 0 \end{cases}$ and $(2, 3)$
19. $\begin{cases} -2x + 5y = 7 \\ 2x + 9y = 7 \end{cases}$ and $(-1, 1)$
20. $\begin{cases} x + 8y = 43 \\ 3x - 2y = -1 \end{cases}$ and $(3, 5)$

For the following exercises, solve each system by Substitution.

21. $\begin{cases} x + 3y = 5 \\ 2x + 3y = 4 \end{cases}$
22. $\begin{cases} 3x - 2y = 18 \\ 5x + 10y = -10 \end{cases}$
23. $\begin{cases} 4x + 2y = -10 \\ 3x + 9y = 0 \end{cases}$
24. $\begin{cases} 2x + 4y = -3.8 \\ 9x - 5y = 1.3 \end{cases}$
25. $\begin{cases} -2x + 3y = 1.2 \\ -3x - 6y = 1.8 \end{cases}$
26. $\begin{cases} x - 0.2y = 1 \\ -10x + 2y = 5 \end{cases}$
27. $\begin{cases} 3x + 5y = 9 \\ 30x + 50y = -90 \end{cases}$
28. $\begin{cases} -3x + y = 2 \\ 12x - 4y = -8 \end{cases}$
29. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 16 \\ \frac{1}{6}x + \frac{1}{4}y = 9 \end{cases}$
30. $\begin{cases} -\frac{1}{4}x + \frac{3}{2}y = 11 \\ -\frac{1}{8}x + \frac{1}{3}y = 3 \end{cases}$

For the following exercises, solve each system by Elimination.

31. $\begin{cases} -2x + 5y = -42 \\ 7x + 2y = 30 \end{cases}$
32. $\begin{cases} 6x - 5y = -34 \\ 2x + 6y = 4 \end{cases}$
33. $\begin{cases} 5x - y = -2.6 \\ -4x - 6y = 1.4 \end{cases}$
34. $\begin{cases} 7x - 2y = 3 \\ 4x + 5y = 3.25 \end{cases}$
35. $\begin{cases} -x + 2y = -1 \\ 5x - 10y = 6 \end{cases}$
36. $\begin{cases} 7x + 6y = 2 \\ -28x - 24y = -8 \end{cases}$
37. $\begin{cases} \frac{5}{6}x + \frac{1}{4}y = 0 \\ \frac{1}{8}x - \frac{1}{2}y = -\frac{43}{120} \end{cases}$
38. $\begin{cases} \frac{1}{3}x + \frac{1}{9}y = \frac{2}{9} \\ -\frac{1}{2}x + \frac{4}{5}y = -\frac{1}{3} \end{cases}$
39. $\begin{cases} -0.2x + 0.4y = 0.6 \\ x - 2y = -3 \end{cases}$
40. $\begin{cases} -0.1x + 0.2y = 0.6 \\ 5x - 10y = 1 \end{cases}$

For the following exercises, solve each system by any method.

41. $\begin{cases} 5x + 9y = 16 \\ x + 2y = 4 \end{cases}$
42. $\begin{cases} 6x - 8y = -0.6 \\ 3x + 2y = 0.9 \end{cases}$
43. $\begin{cases} 5x - 2y = 2.25 \\ 7x - 4y = 3 \end{cases}$
44. $\begin{cases} x - \frac{5}{12}y = -\frac{55}{12} \\ -6x + \frac{5}{2}y = \frac{55}{2} \end{cases}$
45. $\begin{cases} 7x - 4y = \frac{7}{6} \\ 2x + 4y = \frac{1}{3} \end{cases}$
46. $\begin{cases} 3x + 6y = 11 \\ 2x + 4y = 9 \end{cases}$
47. $\begin{cases} \frac{7}{3}x - \frac{1}{6}y = 2 \\ -\frac{21}{6}x + \frac{3}{12}y = -3 \end{cases}$

48. $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=\frac{1}{3} \\ \frac{3}{2}x+\frac{1}{4}y=-\frac{1}{8} \end{cases}$
49. $\begin{cases} 2.2x+1.3y=-0.1 \\ 4.2x+4.2y=2.1 \end{cases}$
50. $\begin{cases} 0.1x+0.2y=2 \\ 0.35x-0.3y=0 \end{cases}$

GRAPHICAL

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

51. $\begin{cases} 3x-y=0.6 \\ x-2y=1.3 \end{cases}$
52. $\begin{cases} -x+2y=4 \\ 2x-4y=1 \end{cases}$
53. $\begin{cases} x+2y=7 \\ 2x+6y=12 \end{cases}$
54. $\begin{cases} 3x-5y=7 \\ x-2y=3 \end{cases}$
55. $\begin{cases} 3x-2y=5 \\ -9x+6y=-15 \end{cases}$

TECHNOLOGY

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

56. $\begin{cases} 0.1x+0.2y=0.3 \\ -0.3x+0.5y=1 \end{cases}$
57. $\begin{cases} -0.01x+0.12y=0.62 \\ 0.15x+0.20y=0.52 \end{cases}$
58. $\begin{cases} 0.5x+0.3y=4 \\ 0.25x-0.9y=0.46 \end{cases}$
59. $\begin{cases} 0.15x+0.27y=0.39 \\ -0.34x+0.56y=1.8 \end{cases}$
60. $\begin{cases} -0.71x+0.92y=0.13 \\ 0.83x+0.05y=2.1 \end{cases}$

EXTENSIONS

For the following exercises, solve each system in terms of A, B, C, D, E, and F where A–F are nonzero numbers. Note that $A \neq B$ and $A \cdot E \neq B \cdot D$.

61. $\begin{cases} x+y=A \\ x-y=B \end{cases}$
62. $\begin{cases} x+Ay=1 \\ x+By=1 \end{cases}$
63. $\begin{cases} Ax+y=0 \\ Bx+y=1 \end{cases}$
64. $\begin{cases} Ax+By=C \\ x+y=1 \end{cases}$
65. $\begin{cases} Ax+By=C \\ Dx+Ey=F \end{cases}$

REAL-WORLD APPLICATIONS

For the following exercises, solve for the desired quantity.

66. A stuffed animal business has a total cost of production $C=12x+30$ and a revenue function $R=20x$. Find the break-even point.
67. A fast-food restaurant has a cost of production $C(x)=11x+120$ and a revenue function $R(x)=5x$. When does the company start to turn a profit?
68. A cell phone factory has a cost of production $C(x)=150x+10,000$ and a revenue function $R(x)=200x$. What is the break-even point?
69. A musician charges $C(x)=64x+20,000$ where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?
70. A guitar factory has a cost of production $C(x)=75x+50,000$. If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

71. Find two numbers whose sum is 28 and difference is 13.
72. A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.
73. The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?
74. A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?
75. A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?
76. 276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.
77. There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?
78. A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.
79. If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?
80. An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?
81. An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

82. If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}$% interest, and the investor earns $737.50 annual interest, how much was invested in each account?
83. If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?
84. CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?
85. A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?
86. A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?
87. Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

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16.2: Systems of Linear Equations - Three Variables

Learning Objectives

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Figure 16.2.1

(credit: "Elembis," Wikimedia Commons)

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as the one above makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as **three-by-three systems**, the primary tool we will be using is called **Gaussian Elimination**, named after the prolific German mathematician Carl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve **upper triangular form**, the ideal form for a three-by-three system because it allows for straightforward **back-substitution** to find a solution (x, y, z), which we call an ordered triple.

Definition: Upper Triangular Form

A 3×3 system is said to be in **upper triangular form** if it is written as follows:

$$\begin{cases} Ax + By + Cz = D \\ \quad\quad Ey + Fz = G \\ \quad\quad\quad\quad Hz = K \end{cases}$$

The upper triangular form for an $n \times n$ systems is likewise defined to be a system in the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = c_1 \\ \quad\quad a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = c_2 \\ \quad\quad\quad\quad a_{33}x_3 + \cdots + a_{3n}x_n = c_3 \\ \quad\quad\quad\quad\quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ \quad\quad\quad\quad\quad\quad\quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad a_{nn}x_n = c_n \end{cases}$$

There are entire branches of Mathematics dedicated to developing faster methods for solving large systems; however, in this section, we limit ourselves to 3×3 systems. Therefore, let's focus on the upper triangular form for a 3×3 system.

The upper triangular form is useful because, in the 3×3 case, the third equation can be solved for z, and then we back-substitute to find y and x. Therefore, our task is to find a method to write a given system in upper triangular form. To achieve our goal, we introduce the following theorem.

Theorem: Invariant Solutions Under Row Operations

The solution to a system of linear equations in invariant (remains unchanged) under the following operations:

1. **Row Exchange:** Interchange the order of any two equations.
2. **Row Scaling:** Multiply both sides of an equation by a nonzero constant.
3. **Row Replacement:** Add a nonzero multiple of one equation to another equation.

As mentioned in the previous section, the solution set to a three-by-three system is an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Theorem: Number of Solutions to 3x3 Linear Systems

A system of three linear equations involving three unknowns has either one, none, or infinitely many solutions.

Figures **16.2.2** and **16.2.3** illustrate possible solution scenarios for three-by-three systems.

Figure 16.2.2

(a): Three planes intersect at a single point, representing a three-by-three system with a single solution.
(b): Three planes intersect in a line, representing a three-by-three system with infinite solutions.

Figure 16.2.3

All three figures represent three-by-three systems with no solution.
(a): The three planes intersect with each other, but not at a common point.
(b): Two of the planes are parallel and intersect with the third plane, but not with each other.
(c): All three planes are parallel, so there is no point of intersection.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

Example 16.2.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{cases} x + y + z = 2 \\ 6x - 4y + 5z = 31 \\ 5x + 2y + 2z = 13 \end{cases}$$

Solution

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$$\begin{aligned} x + y + z &= 2 \\ \implies (3) + (-2) + (1) &= 2 \\ \implies 2 &= 2\checkmark \end{aligned}$$

$$\begin{aligned} 6x - 4y + 5z &= 31 \\ \implies 6(3) - 4(-2) + 5(1) &= 31 \\ \implies 18 + 8 + 5 &= 31 \\ \implies 31 &= 31\checkmark \end{aligned}$$

$$\begin{aligned} 5x + 2y + 2z &= 13 \\ \implies 5(3) + 2(-2) + 2(1) &= 13 \\ \implies 15 - 4 + 2 &= 13 \\ \implies 13 &= 13\checkmark \end{aligned}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

The process of solving a linear system of three equations involving three unknowns can be cumbersome at times - especially if you do not have a systematic approach. We will begin by taking the following approach, which employs the Substitution Method:

1. Pick a variable and solve one of the equations for that variable.
2. Substitute this solution into the remaining two equations.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

It is important to note that the method we have written is not ideal, but it links well with previously-learned methods. We will introduce a more algorithmic approach in a moment.

Example 16.2.2: Solving a System of Three Equations in Three Variables by Substitution

Find a solution to the following system:

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y - z = -6 \\ 2x - 5y + 5z = 17 \end{cases}$$

Solution

We begin by choosing a variable to solve for. While we could choose any of the three variables, x is easy to solve for in the first equation, so that will be our choice.

$$x - 2y + 3z = 9 \implies x = 9 + 2y - 3z.$$

Substituting this for x in the remaining two equations, we get the following:

$$\begin{aligned} & -x + 3y - z = -6 && && 2x - 5y + 5z = 17 \\ \implies & -(9 + 2y - 3z) + 3y - z = -6 && && \implies 2(9 + 2y - 3z) - 5y + 5z = 17 \\ \implies & -9 - 2y + 3z + 3y - z = -6 && \text{and} && \implies 18 + 4y - 6z - 5y + 5z = 17 \\ \implies & -9 + y + 2z = -6 && && \implies 18 - y - z = 17 \\ \implies & y + 2z = 3 && && \implies -y - z = -1 \end{aligned}$$

At this point, we have reduced the system of three equations in three unknowns down to the following system of two linear equations in two unknowns.

$$\begin{cases} y + 2z = 3 \\ -y - z = -1 \end{cases}$$

We can solve this 2×2 system using the Elimination Method. Adding the two equations, we get

$$\begin{array}{rrrrr} & y & + & 2z & = & 3 \\ + & -y & - & z & = & -1 \\ \hline & & & z & = & 2 \end{array}$$

Substituting $z = 2$ into $y + 2z = 3$, we get

$$y + 2(2) = 3 \implies y = -1.$$

Finally, going back to our original equation solved for x, we let $z = 2$ and $y = -1$:

$$\begin{aligned} & x = 9 + 2y - 3z && \\ \implies & x = 9 + 2(-1) - 3(2) && \text{(substituting)} \\ \implies & x = 9 - 2 - 6 && \text{(simplifying)} \\ \implies & x = 1 && \text{(simplifying)} \end{aligned}$$

The solution is the ordered triple $(1, -1, 2)$. See Figure 16.2.4.

Figure 16.2.4

Solving linear systems using the method applied in Example **16.2.2** can get us into trouble when dealing with larger systems. Therefore, it is beneficial to showcase a method based on the Elimination Method we learned previously. The process is as follows:

1. Pick any pair of equations and use the technique discussed in the previous section to eliminate one variable.
2. Pick another pair of equations and eliminate the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 16.2.3: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{cases} x - 2y + 3z = 9 & (E1) \\ -x + 3y - z = -6 & (E2) \\ 2x - 5y + 5z = 17 & (E3) \end{cases}$$

Solution

When learning the Elimination Method, it's best to label the equations (in this case, we used $(E1)$, $(E2)$, and $(E3)$). We begin by choosing a variable to eliminate. The ideal looks to be x. We will add $(E1)$ to $(E2)$ to eliminate the x from those equations.

$$\begin{array}{rrcrcrcr} & (E1): & x & - & 2y & + & 3z & = & 9 \\ + & (E2): & -x & + & 3y & - & z & = & -6 \\ \hline & & & & y & + & 2z & = & 3 \end{array}$$

We now and $2 \times (E2)$ to $(E3)$ to eliminate the x from those.

$$\begin{array}{rrcrcrcr} & 2 \times (E2): & -2x & + & 6y & - & 2z & = & -12 \\ + & (E3): & 2x & - & 5y & + & 5z & = & 17 \\ \hline & & & & y & + & 3z & = & 5 \end{array}$$

At this point, we have reduced the system of three equations in three unknowns down to the following system of two linear equations in two unknowns.

$$\begin{cases} y + 2z = 3 & (E4) \\ y + 3z = 5 & (E5) \end{cases}$$

We choose to continue with the Elimination Method by adding the opposite of $(E4)$ to $(E5)$.

$$\begin{array}{rrcrcrcr} & -(E4): & -y & - & 2z & = & -3 \\ + & (E5): & y & + & 3z & = & 5 \\ \hline & & & & z & = & 2 \end{array}$$

Substituting $z = 2$ into $y + 2z = 3$, we get

$$y + 2(2) = 3 \implies y = -1.$$

Finally, we can use any of the equations involving x and we let $z = 2$ and $y = -1$. In this case, let's just use $(E1)$:

$$\begin{aligned} & x - 2y + 3z &&= 9 && \\ \implies & x - 2(-1) + 3(2) &&= 9 && \text{(substituting)} \\ \implies & x + 2 + 6 &&= 9 && \text{(simplifying)} \\ \implies & x + 8 &&= 9 && \text{(simplifying)} \\ \implies & x &&= 1 && \text{(subtracting 8 from both sides)} \end{aligned}$$

The solution is the ordered triple $(1, -1, 2)$.

? Checkpoint 16.2.3

Solve the system of equations in three variables.

$$\begin{cases} 2x + y - 2z = -1 \\ 3x - 3y - z = 5 \\ x - 2y + 3z = 6 \end{cases}$$

✓ Example 16.2.4: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $670. How much did he invest in each type of fund?

Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$\begin{aligned} x &= \text{amount invested in money-market fund} \\ y &= \text{amount invested in municipal bonds} \\ z &= \text{amount invested in mutual funds} \end{aligned}$$

The first equation indicates that the sum of the three principal amounts is $12,000.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $4,000 more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $670.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{cases} x + y + z = 12,000 \\ -y + z = 4,000 \\ 0.03x + 0.04y + 0.07z = 670 \end{cases}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{cases} x + y + z = 12,000 & (E1) \\ -y + z = 4,000 & (E2) \\ 3x + 4y + 7z = 67,000 & (E3) \end{cases}$$

Since one of the equations is already missing a term involving x, let's eliminate the x in the other two equations. This requires us to add $-3 \times (E1)$ to $(E3)$.

$$\begin{array}{rrcrcrcr} -3(E1): & -3x & - & 3y & - & 3z & = & -36,000 \\ + \quad (E3): & 3x & + & 4y & + & 7z & = & 67,000 \\ \hline & & & y & + & 4z & = & 31,000 \end{array}$$

We can take this result and place it in a new system involving $(E2)$ - this is because $(E2)$ is already missing a term with x.

$$\begin{cases} -y + z = 4,000 & (E2) \\ y + 4z = 31,000 & (E4) \end{cases}$$

Adding these two equations together immediately eliminates y to get

$$5z = 35,000 \implies z = 7,000.$$

We now substitute this into $(E2)$ to get

$$-y + 7,000 = 4,000 \implies -y = -3,000 \implies y = 3,000.$$

Finally, we substitute these values into any equation involving x, y, and z. Let's choose $(E1)$:

$$x + 3,000 + 7,000 = 12,000 \implies x = 2,000.$$

John invested \$2,000 in a money-market fund, \$3,000 in municipal bonds, and \$7,000 in mutual funds.

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The Elimination Method will result in a false statement, such as $3 = 7$ or some other contradiction.

Example 16.2.5: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$\begin{cases} x - 3y + z = 4 & (E1) \\ -x + 2y - 5z = 3 & (E2) \\ 5x - 13y + 13z = 8 & (E3) \end{cases}$$

Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding equation $(E1)$ to equation $(E2)$.

$$\begin{array}{rrcrcrcr} (E1): & x & - & 3y & + & z & = & 4 \\ + \quad (E2): & -x & + & 2y & - & 5z & = & 3 \\ \hline & & & -y & - & 4z & = & 7 \end{array}$$

Next, we multiply equation $(E1)$ by -5 and add it to equation $(E3)$.

$$\begin{array}{rrcrcrcr} -5(E1): & -5x & + & 15y & - & 5z & = & -20 \\ + \quad (E3): & 5x & - & 13y & + & 13z & = & 8 \\ \hline & & & 2y & + & 8z & = & -12 \end{array}$$

We now have the simpler system

$$\begin{cases} -y - 4z = 7 & (E4) \\ 2y + 8z = -12 & (E5) \end{cases}$$

We now multiply equation $(E4)$ by 2 and add it to equation $(E5)$.

$$\begin{array}{rrrrrrr} 2(E4): & -2y & - & 8z & = & 14 \\ + \quad (E5): & 2y & + & 8z & = & -12 \\ \hline & 0 & & & = & 2 \end{array}$$

The final equation, $0 = 2$, is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

? Checkpoint 16.2.5

Solve the system of three equations in three variables.

$$\begin{cases} x + y + z = 2 \\ \quad y - 3z = 1 \\ 2x + y + 5z = 0 \end{cases}$$

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

✓ Example 16.2.6: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{cases} 2x + y - 3z = 0 & (E1) \\ 4x + 2y - 6z = 0 & (E2) \\ x - y + z = 0 & (E3) \end{cases}$$

Solution

First, we can multiply equation $(E1)$ by -2 and add it to equation $(E2)$.

$$\begin{array}{rrrrrrr} -2(E1): & -4x & - & 2y & + & 6z & = & 0 \\ + \quad (E2): & 4x & + & 2y & - & 6z & = & 0 \\ \hline & & & & & 0 & = & 0 \end{array}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation $(E3)$ by -2, and adding it to equation $(E1)$. We then perform the same steps as above and find the same result, $0 = 0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations $(E1)$ and $(E3)$, we have

$$\begin{array}{rrrrrrr} (E1): & 2x & + & y & - & 3z & = & 0 \\ + \quad (E3): & x & - & y & + & z & = & 0 \\ \hline & 3x & & & - & 2z & = & 0 \end{array}$$

We then solve the resulting equation for z.

$$3x - 2z = 0 \implies z = \frac{3}{2}x$$

We back-substitute the expression for z into one of the equations and solve for y.

$$
\begin{aligned}
& 2x + y - 3z = 0 \\
\Rightarrow\ & 2x + y - 3\left(\frac{3}{2}x\right) = 0 && \text{(substituting)} \\
\Rightarrow\ & 2x + y - \frac{9}{2}x = 0 && \text{(simplifying)} \\
\Rightarrow\ & -\frac{5}{2}x + y = 0 && \text{(combining like terms)} \\
\Rightarrow\ & y = \frac{5}{2}x && \left(\text{adding } \tfrac{5}{2}x \text{ to both sides}\right)
\end{aligned}
$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

As shown in Figure **16.2.5**, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

Figure 16.2.5

Q&A

Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x or, if needed, x and y.

? Checkpoint 16.2.6

Solve the following system.

$$
\begin{cases} x + y + z = 7 \\ 3x - 2y - z = 4 \\ x + 6y + 5z = 24 \end{cases}
$$

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16.2.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- 3x3 systems
  - Systems of Linear Equations: Three Unknowns - Review of Solving Systems of 3 Equations in 3 Unknowns
  - Systems of Linear Equations: Three Unknowns - Creating Your Own Systems
  - Intermediate Algebra - Solving Medium Systems of Linear Equations (Part A)
- Gaussian Elimination
  - CIA: Solving 3x3 Systems Algebraically
- Dependent systems 3x3
  - Systems of Linear Equations: Three Unknowns - Parameterized Solutions to Linear Systems
- Applications of 3x3 systems
  - Intermediate Algebra - Solving Medium Systems of Linear Equations (Part B)
  - CIA: Solving Applications Using Systems
  - Systems of Linear Equations: Three Unknowns - Review of Applications Requiring Systems of Equations

KEY CONCEPTS

DEFINITIONS

- **Upper Triangular Form**: A system of linear equations is in upper triangular form if the variables are eliminated from the equations in a way that creates a "triangle" shape. For a 3x3 system, the first equation may contain x, y, z; the second contains only y, z; and the third contains only z.
- **Gaussian Elimination**: An algorithmic process used to solve systems of linear equations by applying a series of row operations to transform the system into upper triangular form.
- **Ordered Triple**: An ordered collection of three elements, (x, y, z), that represents the solution to a system of three equations in three variables. Graphically, it is the point of intersection of three planes in space.

THEOREMS

- **Invariant Solutions Under Row Operations**: The solution to a system of linear equations remains unchanged (is invariant) under the following operations:
  1. **Row Exchange**: Interchange the order of any two equations.
  2. **Row Scaling**: Multiply both sides of an equation by a nonzero constant.
  3. **Row Replacement**: Add a nonzero multiple of one equation to another equation.
- **Number of Solutions to 3x3 Linear Systems**: A system of three linear equations involving three unknowns has either one solution, no solution, or infinitely many solutions.

COMMON MISTAKES

- **Not Eliminating the Same Variable**: When using elimination, a common error is to eliminate one variable from a first pair of equations (e.g., eliminate x from E1 and E2) but a different variable from a second pair (e.g., eliminate y from E1 and E3). This does not successfully reduce the system to a 2x2 system.
- **Arithmetic Errors in Row Operations**: Making simple addition or multiplication errors when performing row scaling or row replacement. For example, when multiplying an entire equation by a constant, forgetting to multiply the term on the other side of the equals sign.
- **Sign Errors During Elimination**: Forgetting to distribute a negative sign correctly when adding a multiple of one equation to another, leading to incorrect cancellation of terms.
- **Incorrectly Expressing Dependent Solutions**: When a system is found to be dependent (infinite solutions), stating "infinite solutions" is not enough. A common mistake is not expressing the general solution as an ordered triple in terms of one of the variables (e.g., $(x, \frac{5}{2}x, \frac{3}{2}x)$).
- **Misinterpreting Contradictions and Identities**:
  - Arriving at a contradiction (e.g., $0 = 2$) and not concluding that the system is inconsistent and has no solution.
  - Arriving at an identity (e.g., $0 = 0$) and not concluding that the system is dependent and has infinite solutions.
- **Errors in Back-Substitution**: After finding the value of the first variable (e.g., z) from the upper triangular form, making arithmetic errors when substituting it back into the other equations to find the remaining variables.

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16.2.2: HOMEWORK

READING QUESTIONS

1. Can a linear system of three equations have exactly two solutions? Explain why or why not.
2. What is Gaussian Elimination and what is its goal when solving a 3x3 system?
3. Describe what "upper triangular form" means for a system of three linear equations. Why is this form useful?
4. List the three "row operations" that can be performed on a system of equations without changing its solution set.
5. From a geometric perspective, what does the solution to a 3x3 system of linear equations represent?
6. If you use elimination to solve a 3x3 system and arrive at the statement $0 = 2$, what does this tell you about the system and its solution?
7. If you use elimination to solve a 3x3 system and arrive at the statement $0 = 0$, what does this tell you about the system? How should you express the solution?
8. When using the elimination method, after you have combined one pair of equations to eliminate a variable, what must you do next?
9. What is "back-substitution"?
10. If a given ordered triple does not solve a system of equations, does that mean there is no solution to the system? Explain.
11. Can a linear system of three equations have exactly two solutions? Explain why or why not?
12. If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.
13. If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.
14. Using the method of addition, is there only one way to solve the system?
15. Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

HOMEWORK

ALGEBRAIC

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

16. $\begin{cases} 2x-6y+6z &= -12 \\ x+4y+5z &= -1 \\ -x+2y+3z &= -1 \end{cases}$ and $(0,1,-1)$

17. $\begin{cases} 6x-y+3z &= 6 \\ 3x+5y+2z &= 0 \\ x+y &= 0 \end{cases}$ and $(3,-3,-5)$

18. $\begin{cases} 6x-7y+z &= 2 \\ -x-y+3z &= 4 \\ 2x+y-z &= 1 \end{cases}$ and $(4,2,-6)$

19. $\begin{cases} x-y &= 0 \\ x-z &= 5 \\ x-y+z &= -1 \end{cases}$ and $(4,4,-1)$

20. $\begin{cases} -x-y+2z &= 3 \\ 5x+8y-3z &= 4 \\ -x+3y-5z &= -5 \end{cases}$ and $(4,1,-7)$

For the following exercises, solve each system by Elimination.

21. $\begin{cases} 3x-4y+2z &= -15 \\ 2x+4y+z &= 16 \\ 2x+3y+5z &= 20 \end{cases}$

22. $\begin{cases} 5x-2y+3z &= 20 \\ 2x-4y-3z &= -9 \\ x+6y-8z &= 21 \end{cases}$

23. $\begin{cases} 5x+2y+4z &= 9 \\ -3x+2y+z &= 10 \\ 4x-3y+5z &= -3 \end{cases}$

24. $\begin{cases} 4x-3y+5z &= 31 \\ -x+2y+4z &= 20 \\ x+5y-2z &= -29 \end{cases}$

25. $\begin{cases} 5x-2y+3z &= 4 \\ -4x+6y-7z &= -1 \\ 3x+2y-z &= 4 \end{cases}$

26. $\begin{cases} 4x+6y+9z &= 0 \\ -5x+2y-6z &= 3 \\ 7x-4y+3z &= -3 \end{cases}$

For the following exercises, solve each system by Gaussian Elimination.

27. $\begin{cases} 2x-y+3z &= 17 \\ -5x+4y-2z &= -46 \\ 2y+5z &= -7 \end{cases}$

28. $\begin{cases} 5x-6y+3z &= 50 \\ -x+4y &= 10 \\ 2x-z &= 10 \end{cases}$

29. $\begin{cases} 2x+3y-6z &= 1 \\ -4x-6y+12z &= -2 \\ x+2y+5z &= 10 \end{cases}$

30. $\begin{cases} 4x+6y-2z &= 8 \\ 6x+9y-3z &= 12 \\ -2x-3y+z &= -4 \end{cases}$

31. $\begin{cases} 2x+3y-4z &= 5 \\ -3x+2y+z &= 11 \\ -x+5y+3z &= 4 \end{cases}$

32. $\begin{cases} 10x+2y-14z &= 8 \\ -x-2y-4z &= -1 \\ -12x-6y+6z &= -12 \end{cases}$

33. $\begin{cases} x+y+z &= 14 \\ 2y+3z &= -14 \\ -16y-24z &= -112 \end{cases}$

34. $\begin{cases} 5x-3y+4z &= -1 \\ -4x+2y-3z &= 0 \\ -x+5y+7z &= -11 \end{cases}$

35. $\begin{cases} x+y+z &= 0 \\ 2x-y+3z &= 0 \\ x-z &= 0 \end{cases}$

36. $\begin{cases} 3x+2y-5z &= 6 \\ 5x-4y+3z &= -12 \\ 4x+5y-2z &= 15 \end{cases}$

37. $\begin{cases} x+y+z &= 0 \\ 2x-y+3z &= 0 \\ x-z &= 1 \end{cases}$

38. $\begin{cases} 3x-\frac{1}{2}y-z &= -\frac{1}{2} \\ 4x+z &= 3 \\ -x+\frac{3}{2}y &= \frac{5}{2} \end{cases}$

39. $\begin{cases} 6x-5y+6z &= 38 \\ \frac{1}{5}x-\frac{1}{2}y+\frac{3}{5}z &= 1 \\ -4x-\frac{3}{2}y-z &= -74 \end{cases}$

40. $\begin{cases} \frac{1}{2}x-\frac{1}{5}y+\frac{2}{5}z &= -\frac{13}{10} \\ \frac{1}{4}x-\frac{2}{5}y-\frac{1}{5}z &= -\frac{7}{20} \\ -\frac{1}{2}x-\frac{3}{4}y-\frac{1}{2}z &= -\frac{5}{4} \end{cases}$

41. $\begin{cases} -\frac{1}{3}x-\frac{1}{2}y-\frac{1}{4}z &= \frac{3}{4} \\ -\frac{1}{2}x-\frac{1}{4}y-\frac{1}{2}z &= 2 \\ -\frac{1}{4}x-\frac{3}{4}y-\frac{1}{2}z &= -\frac{1}{2} \end{cases}$

42. $\begin{cases} \frac{1}{2}x-\frac{1}{4}y+\frac{3}{4}z &= 0 \\ \frac{1}{4}x-\frac{1}{10}y+\frac{2}{5}z &= -2 \\ \frac{1}{8}x+\frac{1}{5}y-\frac{1}{8}z &= 2 \end{cases}$

43. $\begin{cases} \frac{4}{5}x-\frac{7}{8}y+\frac{1}{2}z &= 1 \\ -\frac{4}{5}x-\frac{3}{4}y+\frac{1}{3}z &= -8 \\ -\frac{2}{5}x-\frac{7}{8}y+\frac{1}{2}z &= -5 \end{cases}$

44. $\begin{cases} -\frac{1}{3}x-\frac{1}{8}y+\frac{1}{6}z &= -\frac{4}{3} \\ -\frac{2}{3}x-\frac{7}{8}y+\frac{1}{3}z &= -\frac{23}{3} \\ -\frac{1}{3}x-\frac{5}{8}y+\frac{5}{6}z &= 0 \end{cases}$

45. $\begin{cases} -\frac{1}{4}x-\frac{5}{4}y+\frac{5}{2}z &= -5 \\ -\frac{1}{2}x-\frac{5}{3}y+\frac{5}{4}z &= \frac{55}{12} \\ -\frac{1}{3}x-\frac{1}{3}y+\frac{1}{3}z &= \frac{5}{3} \end{cases}$

46. $\begin{cases} \frac{1}{40}x+\frac{1}{60}y+\frac{1}{80}z &= \frac{1}{100} \\ -\frac{1}{2}x-\frac{1}{3}y-\frac{1}{4}z &= -\frac{1}{5} \\ \frac{3}{8}x+\frac{3}{12}y+\frac{3}{16}z &= \frac{3}{20} \end{cases}$

47. $\begin{cases} 0.1x-0.2y+0.3z &= 2 \\ 0.5x-0.1y+0.4z &= 8 \\ 0.7x-0.2y+0.3z &= 8 \end{cases}$

48. $\begin{cases} 0.2x+0.1y-0.3z &= 0.2 \\ 0.8x+0.4y-1.2z &= 0.1 \\ 1.6x+0.8y-2.4z &= 0.2 \end{cases}$

49. $\begin{cases} 1.1x+0.7y-3.1z &= -1.79 \\ 2.1x+0.5y-1.6z &= -0.13 \\ 0.5x+0.4y-0.5z &= -0.07 \end{cases}$

50. $\begin{cases} 0.5x-0.5y+0.5z &= 10 \\ 0.2x-0.2y+0.2z &= 4 \\ 0.1x-0.1y+0.1z &= 2 \end{cases}$

51. $\begin{cases} 0.1x+0.2y+0.3z &= 0.37 \\ 0.1x-0.2y-0.3z &= -0.27 \\ 0.5x-0.1y-0.3z &= -0.03 \end{cases}$

52. $\begin{cases} 0.5x-0.5y-0.3z &= 0.13 \\ 0.4x-0.1y-0.3z &= 0.11 \\ 0.2x-0.8y-0.9z &= -0.32 \end{cases}$

53. $\begin{cases} 0.5x+0.2y-0.3z &= 1 \\ 0.4x-0.6y+0.7z &= 0.8 \\ 0.3x-0.1y-0.9z &= 0.6 \end{cases}$

54. $\begin{cases} 0.3x+0.3y+0.5z &= 0.6 \\ 0.4x+0.4y+0.4z &= 1.8 \\ 0.4x+0.2y+0.1z &= 1.6 \end{cases}$

55. $\begin{cases} 0.8x+0.8y+0.8z &= 2.4 \\ 0.3x-0.5y+0.2z &= 0 \\ 0.1x+0.2y+0.3z &= 0.6 \end{cases}$

EXTENSIONS

For the following exercises, solve the system for x, y, and z.

56. $\begin{cases} x+y+z &= 3 \\ \frac{x-1}{2}+\frac{y-3}{2}+\frac{z+1}{2} &= 0 \\ \frac{x-2}{3}+\frac{y+4}{3}+\frac{z-3}{3} &= \frac{2}{3} \end{cases}$

57. $\begin{cases} 5x-3y-\frac{z+1}{2} &= \frac{1}{2} \\ 6x+\frac{y-9}{2}+2z &= -3 \\ \frac{x+8}{2}-4y+z &= 4 \end{cases}$

58. $\begin{cases} \frac{x+4}{7}-\frac{y-1}{6}+\frac{z+2}{3} &= 1 \\ \frac{x-2}{4}+\frac{y+1}{8}-\frac{z+8}{12} &= 0 \\ \frac{x+6}{3}-\frac{y+2}{3}+\frac{z+4}{2} &= 3 \end{cases}$

59. $\begin{cases} \frac{x-3}{6}+\frac{y+2}{2}-\frac{z-3}{3} &= 2 \\ \frac{x+2}{4}+\frac{y-5}{2}+\frac{z+4}{2} &= 1 \\ \frac{x+6}{2}-\frac{y-3}{2}+z+1 &= 9 \end{cases}$

60. $\begin{cases} \frac{x-1}{3}+\frac{y+3}{4}+\frac{z+2}{6} &= 1 \\ 4x+3y-2z &= 11 \\ 0.02x+0.015y-0.01z &= 0.065 \end{cases}$

REAL-WORLD APPLICATIONS

61. Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?
62. Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?
63. At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

64. An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?
65. Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate's groceries were $2.10 more than your groceries. How much was each of your share of the groceries?
66. Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.
67. Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?
68. At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?
69. A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?
70. In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?
71. Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?
72. A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $3\frac{1}{8}$% simple interest, and one that paid $2\frac{1}{2}$% simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?
73. You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?
74. You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?
75. The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?
76. The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?
77. The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?
78. The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?
79. At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

80. Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?

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16.3: Systems of Nonlinear Equations and Inequalities - Two Variables

Learning Objectives

- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Halley's Comet (Figure **16.3.1**) orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.

Figure 16.3.1

Halley's Comet (credit: "NASA Blueshift"/Flickr)

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving **systems of nonlinear equations** are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

Definition: System of Nonlinear Equations

A **system of nonlinear equations** is a system of two or more equations in two or more variables containing at least one equation that is not linear.

Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form is **nonlinear**. The Substitution Method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

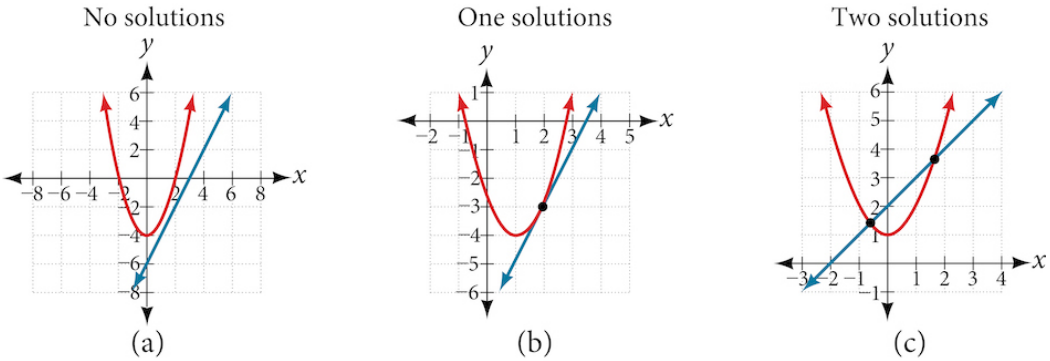
Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line. Figure **16.3.2** illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution. The line will never intersect the parabola.

- One solution. The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions. The line crosses on the inside of the parabola and intersects the parabola at two points.

Figure 16.3.2



Example 16.3.1: Solving a System of Nonlinear Equations Representing a Parabola and a Line

Solve the system of equations.

$$\begin{cases} x - y = -1 \\ y = x^2 + 1 \end{cases}$$

Solution

Solve the first equation for x.

$$\begin{aligned} x - y &= -1 \\ \Rightarrow \quad x &= y - 1 \qquad \text{(adding } y \text{ to both sides)} \end{aligned}$$

We now substitute the resulting expression into the second equation.

$$\begin{aligned} y &= x^2 + 1 \\ \Rightarrow \quad y &= (y-1)^2 + 1 && \text{(substituting)} \\ \Rightarrow \quad y &= y^2 - 2y + 1 + 1 && \text{(distributing)} \\ \Rightarrow \quad y &= y^2 - 2y + 2 && \text{(combining like terms)} \\ \Rightarrow \quad 0 &= y^2 - 3y + 2 && \text{(subtracting } y \text{ from both sides)} \\ \Rightarrow \quad 0 &= (y-2)(y-1) && \text{(factoring)} \end{aligned}$$

Solving for y gives $y = 2$ and $y = 1$. Next, substitute each value for y into the first equation to solve for x. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} x - y &= -1 \\ \Rightarrow \quad x - (2) &= -1 && \text{(substituting)} \\ \Rightarrow \quad x &= 1 && \text{(adding 2 to both sides)} \end{aligned}$$

$$\begin{aligned} x - y &= -1 \\ \Rightarrow \quad x - (1) &= -1 && \text{(substituting)} \\ \Rightarrow \quad x &= 0 && \text{(adding 1 to both sides)} \end{aligned}$$

The solutions are $(1, 2)$ and $(0, 1)$, which can be verified by substituting these (x, y) values into both of the original equations. See Figure 16.3.3.

Figure 16.3.3

Q&A

Could we have substituted values for y into the second equation to solve for x in Example 16.3.1?

Yes, but because x is squared in the second equation this could give us extraneous solutions for x.

For $y = 1$

$$\begin{aligned} & y = x^2 + 1 & \\ \Rightarrow \quad & 1 = x^2 + 1 & \text{(substituting)} \\ \Rightarrow \quad & 0 = x^2 & \text{(subtracting 1 from both sides)} \\ \Rightarrow \quad & \pm\sqrt{0} = x & \text{(Extraction of Roots)} \\ \Rightarrow \quad & 0 = x & \text{(simplifying)} \end{aligned}$$

This gives us the same value as in the solution.

For $y = 2$

$$\begin{aligned} & y = x^2 + 1 & \\ \Rightarrow \quad & 2 = x^2 + 1 & \text{(substituting)} \\ \Rightarrow \quad & 1 = x^2 & \text{(subtracting 1 from both sides)} \\ \Rightarrow \quad & \pm\sqrt{1} = x & \text{(Extraction of Roots)} \\ \Rightarrow \quad & \pm 1 = x & \text{(simplifying)} \end{aligned}$$

Notice that -1 is an extraneous solution.

? Checkpoint 16.3.1

Solve the given system of equations by substitution.

$$\begin{cases} 3x - y = -2 \\ 2x^2 - y = 0 \end{cases}$$

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

- No solution. The line does not intersect the circle.
- One solution. The line is tangent to the circle and intersects the circle at exactly one point.

- Two solutions. The line crosses the circle and intersects it at two points.

Figure 16.3.4

Example 16.3.2: Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

$$\begin{cases} x^2 + y^2 = 5 \\ y = 3x - 5 \end{cases}$$

Solution

One of the equations has already been solved for y. We will substitute $y = 3x - 5$ into the equation for the circle.

$$\begin{aligned} & x^2 + y^2 = 5 \\ \Longrightarrow \quad & x^2 + (3x-5)^2 = 5 && \text{(substituting)} \\ \Longrightarrow \quad & x^2 + 9x^2 - 30x + 25 = 5 && \text{(distributing)} \\ \Longrightarrow \quad & 10x^2 - 30x + 25 = 5 && \text{(combining like terms)} \\ \Longrightarrow \quad & 10x^2 - 30x + 20 = 0 && \text{(subtracting 5 from both sides)} \\ \Longrightarrow \quad & x^2 - 3x + 2 = 0 && \text{(dividing both sides by 10)} \\ \Longrightarrow \quad & (x-2)(x-1) = 0 && \text{(factoring)} \end{aligned}$$

Substitute the two x-values into the original linear equation to solve for y.

$$\begin{aligned} y &= 3(2) - 5 = 1 \\ y &= 3(1) - 5 = -2 \end{aligned}$$

The line intersects the circle at $(2, 1)$ and $(1, -2)$, which can be verified by substituting these (x, y) values into both of the original equations. See Figure 16.3.5.

Figure 16.3.5

Checkpoint 16.3.2

Solve the system of nonlinear equations.

$$\begin{cases} x^2+y^2 &= 10 \\ x-3y &= -10 \end{cases}$$

Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

Possible Types of Solutions for the Points of Intersection of a Circle and an Ellipse

Figure **16.3.6** illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution. The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution. The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions. The circle and the ellipse intersect at two points.
- Three solutions. The circle and the ellipse intersect at three points.
- Four solutions. The circle and the ellipse intersect at four points.

Figure 16.3.6

Example 16.3.3: Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

$$\begin{cases} x^2+y^2 &= 26 \quad (1) \\ 3x^2+25y^2 &= 100 \quad (2) \end{cases}$$

Solution

Let's begin by multiplying equation (1) by -3. This gives $-3x^2-3y^2=-78$. We now add this to equation (2).

$$\begin{array}{rcrcr} -3x^2 & - & 3y^2 & = & -78 \\ 3x^2 & + & 25y^2 & = & 100 \\ \hline & & 22y^2 & = & 22 \end{array}$$

After we add the two equations together, we solve for y.

$$y^2=1 \implies y=\pm\sqrt{1}=\pm 1.$$

Substitute $y=\pm 1$ into one of the equations and solve for x.

$$\begin{aligned} x^2+(1)^2 &= 26 \\ \Rightarrow \quad x^2+1 &= 26 \\ \Rightarrow \quad x^2 &= 25 \\ \Rightarrow \quad x &= \pm 5 \end{aligned}$$

and

$$\begin{aligned} x^2+(-1)^2 &= 26 \\ \Rightarrow \quad x^2+1 &= 26 \\ \Rightarrow \quad x^2 &= 25 \\ \Rightarrow \quad x &= \pm 5 \end{aligned}$$

There are four solutions: $(5,1)$, $(-5,1)$, $(5,-1)$, and $(-5,-1)$. See Figure **16.3.7**.

Figure 16.3.7

? Checkpoint 16.3.3

Find the solution set for the given system of nonlinear equations.

$$\begin{cases} 4x^2+y^2 &= 13 \\ x^2+y^2 &= 10 \end{cases}$$

Graphing a Nonlinear Inequality

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. Now, we will follow similar steps to graph a nonlinear inequality so that we can learn to solve systems of nonlinear inequalities. A nonlinear inequality is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

Recall that when the inequality is greater than, $y > a$, or less than, $y < a$, the graph is drawn with a dashed line. When the inequality is greater than or equal to, $y \geq a$, or less than or equal to, $y \leq a$, the graph is drawn with a solid line. The graphs will create regions in the plane, and we will test each region for a solution. If one point in the region works, the whole region works. That is the region we shade. See Figure **16.3.8**.

Figure 16.3.8

(a) an example of $y > a$;
(b) an example of $y \geq a$;
(c) an example of $y < a$;
(d) an example of $y \leq a$

How To

Given an inequality bounded by a parabola, sketch a graph.

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is $\leq$ or $\geq$), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is < or >), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5. Shade the region representing the solution set.

Example **16.3.4**: Graphing an Inequality for a Parabola

Graph the inequality $y > x^2 + 1$.

Solution

First, graph the corresponding equation $y = x^2 + 1$. Since $y > x^2 + 1$ has a greater than symbol, we draw the graph with a dashed line. Then we choose points to test both inside and outside the parabola. Let's test the points $(0, 2)$ and $(2, 0)$. One point is clearly inside the parabola and the other point is clearly outside.

$$\begin{aligned} & y > x^2 + 1 & \\ \Longrightarrow \quad & 2 > (0)2 + 1 & \textbf{(substituting)} \\ \Longrightarrow \quad & 2 > 1 & \textbf{(True)} \end{aligned}$$

and

$$\begin{aligned} & y > x^2 + 1 & \\ \Longrightarrow \quad & 0 > (2)2 + 1 & \textbf{(substituting)} \\ \Longrightarrow \quad & 0 > 5 & \textbf{(False)} \end{aligned}$$

The graph is shown in Figure **16.3.9**. We can see that the solution set consists of all points inside the parabola, but not on the graph itself.

Figure 16.3.9

Graphing a System of Nonlinear Inequalities

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A system of nonlinear inequalities is a system of two or more inequalities in two or more variables containing at least one inequality that is not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the feasible region.

How To... Solve a System of Nonlinear Inequalities

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

Example 16.3.5: Graphing a System of Inequalities

Graph the given system of inequalities.

$$\begin{cases} x^2 - y & \leq & 0 \\ 2x^2 + y & \leq & 12 \end{cases}$$

Solution

These two equations are clearly parabolas. We can find the points of intersection by the elimination process: Add both equations and the variable y will be eliminated. Then we solve for x.

$$\begin{array}{rrcl} x^2 & - \; y & = & 0 \\ 2x^2 & + \; y & = & 12 \\ \hline 3x^2 & & = & 12 \\ \Longrightarrow \quad x^2 & & = & 4 \\ \Longrightarrow \quad x & & = & \pm 2 \end{array}$$

Substitute the x-values into one of the equations and solve for y.

$$\begin{array}{rrcl} & x^2 - y & = & 0 \\ \Longrightarrow & (2)^2 - y & = & 0 \\ \Longrightarrow & 4 - y & = & 0 \\ \Longrightarrow & y & = & 4 \end{array}$$

and

$$\begin{aligned} x^2 - y &= 0 \\ \Rightarrow \quad (-2)^2 - y &= 0 \\ \Rightarrow \quad 4 - y &= 0 \\ \Rightarrow \quad y &= 4 \end{aligned}$$

The two points of intersection are $(2, 4)$ and $(-2, 4)$. Notice that the equations can be rewritten as follows.

$$\begin{aligned} x^2 - y &\leq 0 \\ \Rightarrow \quad x^2 &\leq y \\ \Rightarrow \quad y &\geq x^2 \end{aligned}$$

and

$$\begin{aligned} 2x^2 + y &\leq 12 \\ \Rightarrow \quad y &\leq -2x^2 + 12 \end{aligned}$$

Graph each inequality. See Figure 16.3.10. The feasible region is the region between the two equations bounded by $2x^2 + y \leq 12$ on the top and $x^2 - y \leq 0$ on the bottom.

Figure 16.3.10

? Checkpoint 16.3.4

Graph the given system of inequalities.

$$\begin{cases} y & \geq x^2 - 1 \\ x - y & \geq -1 \end{cases}$$

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16.3.1: RESOURCES AND KEY CONCEPTS

RESOURCES

VIDEOS

- System of nonlinear equations
  - Intermediate Algebra - Solving Non-Linear Systems of Equations (Part A)
- System of nonlinear inequalities
  - Intermediate Algebra - Graphing Systems of Linear Inequalities
  - CIA: Solving a System of Linear Inequalities

KEY CONCEPTS

DEFINITIONS

- **System of Nonlinear Equations**: A system of two or more equations in two or more variables containing at least one equation that is not linear.
- **Substitution Method (for nonlinear systems)**: A method for solving systems where one equation is solved for one variable and the result is substituted into the other equation.
- **Elimination Method (for nonlinear systems)**: A method for solving systems where two equations are added or subtracted to eliminate a variable, often used when both equations have like variables of the second degree.
- **Nonlinear Inequality**: An inequality containing a nonlinear expression.
- **System of Nonlinear Inequalities**: A system of two or more inequalities in two or more variables containing at least one inequality that is not linear.
- **Feasible Region**: The region of a graph where the shaded regions of each inequality in a system overlap; this region represents the set of all solutions to the system of inequalities.

COMMON MISTAKES

- **Errors in Substitution**: When substituting one equation into another, especially when squaring a binomial, algebraic errors are common. For example, incorrectly expanding $(y-1)^2$ as y^2-1.
- **Forgetting All Solutions**: When solving an equation like $x^2=4$, only finding one solution ($x=2$) instead of both ($x=\pm 2$). This can lead to missing intersection points in the system.
- **Introducing Extraneous Solutions**: When solving a system, substituting a variable into the more complex (higher-degree) equation can sometimes introduce extraneous solutions that must be checked in the simpler (often linear) equation.
- **Errors in Elimination**: Making sign errors when adding or subtracting entire equations, or incorrectly multiplying an equation by a constant before adding.
- **Graphing Inequality Boundaries Incorrectly**: Using a solid line for a strict inequality ($<$ or $>$) or a dashed line for an inclusive inequality ($\leq$ or $\geq$).
- **Shading the Wrong Region for an Inequality**: After graphing the boundary curve, choosing an incorrect test point or misinterpreting the result, leading to the wrong region being shaded as the solution set.
- **Incorrectly Identifying the Feasible Region**: When graphing a system of inequalities, incorrectly identifying the final solution set, which should only be the region where *all* individual shaded areas overlap.

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16.3.2: HOMEWORK

READING QUESTIONS

1. Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three?
2. When graphing a nonlinear inequality, why do we only need to test one point in each region created by the boundary curve?
3. Will a system of nonlinear inequalities always have a feasible region (a solution set)? Explain why or why not.
4. What are the three possible solution outcomes for a system consisting of a parabola and a line?
5. What are the five possible solution outcomes for a system consisting of a circle and an ellipse?
6. Which method, substitution or elimination, is generally preferred for solving a system with one linear and one nonlinear equation?
7. Which method, substitution or elimination, is often easier for solving a system where both equations have like variables of the second degree (e.g., x^2 and y^2)?
8. When graphing a nonlinear inequality, how do you determine whether the boundary curve should be solid or dashed?
9. What is a "feasible region" in the context of a system of inequalities?
10. Why is it important to check for extraneous solutions when solving nonlinear systems?
11. Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three? If not, explain why not. If so, give an example of such a system, in graph form, and explain why your choice gives two or three answers.
12. When graphing an inequality, explain why we only need to test one point to determine whether an entire region is the solution?
13. When you graph a system of inequalities, will there always be a feasible region? If so, explain why. If not, give an example of a graph of inequalities that does not have a feasible region. Why does it not have a feasible region?
14. If you graph a revenue and cost function, explain how to determine in what regions there is profit.
15. If you perform your break-even analysis and there is more than one solution, explain how you would determine which x-values are profit and which are not.

HOMEWORK

ALGEBRAIC

For the following exercises, solve the system of nonlinear equations using Substitution.

16. $\begin{cases} x+y &= 4 \\ x^2+y^2 &= 9 \end{cases}$

17. $\begin{cases} y &= x-3 \\ x^2+y^2 &= 9 \end{cases}$

18. $\begin{cases} y &= x \\ x^2+y^2 &= 9 \end{cases}$

19. $\begin{cases} y &= -x \\ x^2+y^2 &= 9 \end{cases}$

20. $\begin{cases} x &= 2 \\ x^2-y^2 &= 9 \end{cases}$

For the following exercises, solve the system of nonlinear equations using Elimination.

21. $\begin{cases} 4x^2-9y^2 &= 36 \\ 4x^2+9y^2 &= 36 \end{cases}$

22. $\begin{cases} x^2+y^2 &= 25 \\ x^2-y^2 &= 1 \end{cases}$

23. $\begin{cases} 2x^2+4y^2 &= 4 \\ 2x^2-4y^2 &= 25x-10 \end{cases}$

24. $\begin{cases} y^2-x^2 &= 9 \\ 3x^2+2y^2 &= 8 \end{cases}$

25. $\begin{cases} x^2+y^2+\frac{1}{16} &= 2500 \\ y &= 2x^2 \end{cases}$

For the following exercises, use any method to solve the system of nonlinear equations.

26. $\begin{cases} -2x^2+y &= -5 \\ 6x-y &= 9 \end{cases}$

27. $\begin{cases} -x^2+y &= 2 \\ -x+y &= 2 \end{cases}$

28. $\begin{cases} x^2+y^2 &= 1 \\ y &= 20x^2-1 \end{cases}$

29. $\begin{cases} x^2+y^2 &= 1 \\ y &= -x^2 \end{cases}$

30. $\begin{cases} 2x^3-x^2 &= y \\ y &= \frac{1}{2}-x \end{cases}$

31. $\begin{cases} 9x^2+25y^2 &= 225 \\ (x-6)^2+y^2 &= 1 \end{cases}$

32. $\begin{cases} x^4-x^2 &= y \\ x^2+y &= 0 \end{cases}$

33. $\begin{cases} 2x^3-x^2 &= y \\ x^2+y &= 0 \end{cases}$

For the following exercises, use any method to solve the nonlinear system.

34. $\begin{cases} x^2+y^2 &= 9 \\ y &= 3-x^2 \end{cases}$

35. $\begin{cases} x^2-y^2 &= 9 \\ x &= 3 \end{cases}$

36. $\begin{cases} x^2-y^2 &= 9 \\ y &= 3 \end{cases}$

37. $\begin{cases} x^2-y^2 &= 9 \\ x-y &= 0 \end{cases}$

38. $\begin{cases} -x^2+y &= 2 \\ -4x+y &= -1 \end{cases}$

39. $\begin{cases} -x^2+y &= 2 \\ 2y &= -x \end{cases}$

40. $\begin{cases} x^2+y^2 &= 25 \\ x^2-y^2 &= 36 \end{cases}$

41. $\begin{cases} x^2+y^2 &= 1 \\ y^2 &= x^2 \end{cases}$

42. $\begin{cases} 16x^2-9y^2+144 &= 0 \\ y^2+x^2 &= 16 \end{cases}$

43. $\begin{cases} 3x^2-y^2 &= 12 \\ (x-1)^2+y^2 &= 1 \end{cases}$

44. $\begin{cases} 3x^2-y^2 &= 12 \\ (x-1)^2+y^2 &= 4 \end{cases}$

45. $\begin{cases} 3x^2-y^2 &= 12 \\ x^2+y^2 &= 16 \end{cases}$

46. $\begin{cases} x^2-y^2-6x-4y-11 &= 0 \\ -x^2+y^2 &= 5 \end{cases}$

47. $\begin{cases} x^2+y^2-6y &= 7 \\ x^2+y &= 1 \end{cases}$

48. $\begin{cases} x^2+y^2 &= 6 \\ xy &= 1 \end{cases}$

GRAPHICAL

For the following exercises, graph the inequality.

49. $x^2+y<9$
50. $x^2+y^2<4$

For the following exercises, graph the system of inequalities. Label all points of intersection.

51. $\begin{cases} x^2+y &< 1 \\ y &> 2x \end{cases}$

52. $\begin{cases} x^2+y &< -5 \\ y &> 5x+10 \end{cases}$

53. $\begin{cases} x^2+y^2 &< 25 \\ 3x^2-y^2 &> 12 \end{cases}$

54. $\begin{cases} x^2-y^2 &> -4 \\ x^2+y^2 &< 12 \end{cases}$

55. $\begin{cases} x^2+3y^2 &> 16 \\ 3x^2-y^2 &< 1 \end{cases}$

EXTENSIONS

For the following exercises, graph the inequality.

56. $\begin{cases} y &\geq e^x \\ y &\leq \ln(x)+5 \end{cases}$

57. $\begin{cases} y &\leq -\log(x) \\ y &\leq e^x \end{cases}$

For the following exercises, find the solutions to the nonlinear equations with two variables.

58. $\begin{cases} \frac{4}{x^2}+\frac{1}{y^2} &= 24 \\ \frac{5}{x^2}-\frac{2}{y^2}+4 &= 0 \end{cases}$

59. $\begin{cases} \frac{6}{x^2}-\frac{1}{y^2} &= 8 \\ \frac{1}{x^2}-\frac{6}{y^2} &= \frac{1}{8} \end{cases}$

60. $\begin{cases} x^2-xy+y^2-2 &= 0 \\ x+3y &= 4 \end{cases}$

61. $\begin{cases} x^2-xy-2y^2-6 &= 0 \\ x^2+y^2 &= 1 \end{cases}$

62. $\begin{cases} x^2+4xy-2y^2-6 &= 0 \\ x &= y+2 \end{cases}$

TECHNOLOGY

For the following exercises, solve the system of inequalities. Use a calculator to graph the system to confirm the answer.

63. $\begin{cases} xy &< 1 \\ y &> \sqrt{x} \end{cases}$

64. $\begin{cases} x^2+y &< 3 \\ y &> 2x \end{cases}$

REAL-WORLD APPLICATIONS

For the following exercises, construct a system of nonlinear equations to describe the given behavior, then solve for the requested solutions.

65. Two numbers add up to 300. One number is twice the square of the other number. What are the numbers?
66. The squares of two numbers add to 360. The second number is half the value of the first number squared. What are the numbers?

67. A laptop company has discovered their cost and revenue functions for each day: $C(x) = x^2 - 10x + 200$ and $R(x) = -2x^2 + 100x + 50$. If they want to make a profit, what is the range of laptops per day that they should produce? Round to the nearest number which would generate profit.
68. A cell phone company has the following cost and revenue functions: $C(x) = 8x^2 - 600x + 21,500$ and $R(x) = -3x^2 + 480x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

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