

# CALCULUS I NOTES



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# Calculus I Notes

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# TABLE OF CONTENTS

## Licensing

## Preface

## Introduction

### 1: Limits and Continuity

- 1.1: The Limit of a Function
- 1.2: The Limit Laws
- 1.3: Continuity

### 2: Derivatives

- 2.1: The Definition of the Derivative
- 2.2: The Derivative as a Function
- 2.3: Differentiation Rules
- 2.4: Derivatives as Rates of Change
- 2.5: Derivatives of Trigonometric Functions
- 2.6: The Chain Rule
- 2.7: Derivatives of Inverse Functions
- 2.8: Implicit Differentiation
- 2.9: Derivatives of Exponential and Logarithmic Functions

### 3: Applications of Derivatives

- 3.1: Related Rates
- 3.2: Linear Approximation
- 3.3: The Mean Value Theorem
- 3.4: Maxima and Minima
- 3.5: Relationship between Derivatives and the Shape of a Graph
- 3.6: Limits at Infinity, Horizontal Asymptotes, and How to Use Calculus to Sketch the Graph of a Function
- 3.7: Applied Optimization Problems
- 3.8: L'Hôpital's Rule
- 3.9: Newton's Method
- 3.10: Antiderivatives

### 4: Introduction to Integrals

- 4.1: Riemann Sums
- 4.2: The Definite Integral
- 4.3: The Fundamental Theorem of Calculus
- 4.4: The Substitution Rule

**Index**

**Glossary**

**Detailed Licensing**

**Detailed Licensing**

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# Preface

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## About this Open Educational Resource (OER)

This book and its ancillary materials were created by Kevin Palencia at Northern Illinois University for MATH 229 Calculus I Notes and was made possible through an Illinois Secretary of State/Illinois State Library (ISL) OER Grant awarded to Northern Illinois University Libraries' project: *Huskies Unbound: Developing OER for Undergraduate Success in High-Impact Courses*. Funded by a one-time \$3 million allocation to the Illinois Secretary of State's Budget in FY24, and facilitated by the Consortium of Academic and Research Libraries in Illinois (CARLI), the ISL OER Grant Program was created to support the development of textbooks and ancillaries as open educational resources to reduce students' costs, encourage student success, and allow faculty to adapt materials to meet local needs.

I wish to express my gratitude to Illinois Governor JB Pritzker, Illinois Secretary of State Alexander Giannoulas, and Greg McCormick, Director of the Illinois State Library, for their commitment to Open Education.

## Acknowledgements

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The Peer Reviewer process was completed by the following reviewer(s):

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# Introduction

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This text was developed to support the teaching and learning of Calculus I in an accessible, flexible, and instructor-friendly format. It is designed as a set of organized notes rather than a traditional textbook, allowing instructors to shape lessons, discussions, and classroom activities around clearly presented concepts and examples.

The material highlights key ideas, foundational theorems, and representative problems that illustrate how calculus is used to describe change and model real-world phenomena. Rather than offering fully solved exercises, examples are intentionally left without final answers so that instructors can guide solution processes during class meetings, and students can actively engage in problem solving. This structure promotes dialogue, collaborative work, and deeper understanding through participation.

These notes are intended to be adapted. Instructors may supplement them with additional exercises, technology activities, or contextual applications that match their course goals and student needs. Students may use them for organization, review, and conceptual reinforcement. Furthermore, these notes reflect the philosophy that learning calculus is not only about arriving at answers, but about discovering patterns, interpreting relationships, and building confidence in reasoning. I hope that these notes serve as a useful tool for educators and learners as they explore the beauty, power, and relevance of Calculus I.

# 1: LIMITS AND CONTINUITY

1.1: The Limit of a Function

1.2: The Limit Laws

1.3: Continuity

## Overview

Calculus is built on two fundamental ideas:

- The **derivative**, which measures the *instantaneous rate of change* of a function.
- The **integral**, which measures the *accumulated total* of a quantity over an interval.

Both rely on a single underlying concept: the **limit**. Understanding limits enables us to describe how a function behaves near a point, even when the function is not explicitly defined at that point. Limits provide the foundation for derivatives and integrals, making them one of the most important ideas in calculus.

In this chapter, we focus on two closely related topics: **limits** and **continuity**. Together, they give us a precise language for describing when a function behaves smoothly, and when it does not.

---

## Topics in this Chapter

### 1. The Limit of a Function

- We introduce the definition of a limit, both intuitively and formally.
- Limits describe the value a function *approaches*, not necessarily the value it *takes*.
- This allows us to handle situations with jumps in a graph or asymptotes.

### 2. Limit Laws

- Once we know what limits are, we need tools to compute them.
- The limit laws give us algebraic rules for combining limits, which make evaluating limits of complex functions much easier.

### 3. Continuity

- Continuity captures the idea of a function with no “jumps,” “holes,” or “breaks.”
  - Informally, a continuous function is one you can draw without lifting your pencil.
  - Formally, continuity connects directly to the definition of limits.
- 

## Why this Matters

- Limits and continuity provide the rigorous foundation for both derivatives and integrals.
- They describe how functions behave near difficult or undefined points.
- They allow us to work with infinite processes and approximations.

In short: this chapter develops the **basic language of calculus**. Once you are comfortable with limits and continuity, you will be ready to explore differentiation and integration in the chapters that follow.

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# 1.1: The Limit of a Function

## Section 1.1 The Limit of a Function

### Learning Objectives

- Use a table of values and graphs to estimate and/or evaluate limits and identify when limits do not exist.
- Evaluate and construct examples illustrating one-sided limits.
- Explain that a two-sided limit exists if and only if the left-hand and right-hand limits exist and are equal.
- Describe infinite limits and vertical asymptotes, identifying them from graphs and equations.

The foundation of calculus is *limits*. It is a method to describe the behavior of a function. Let's analyze the graphs below.

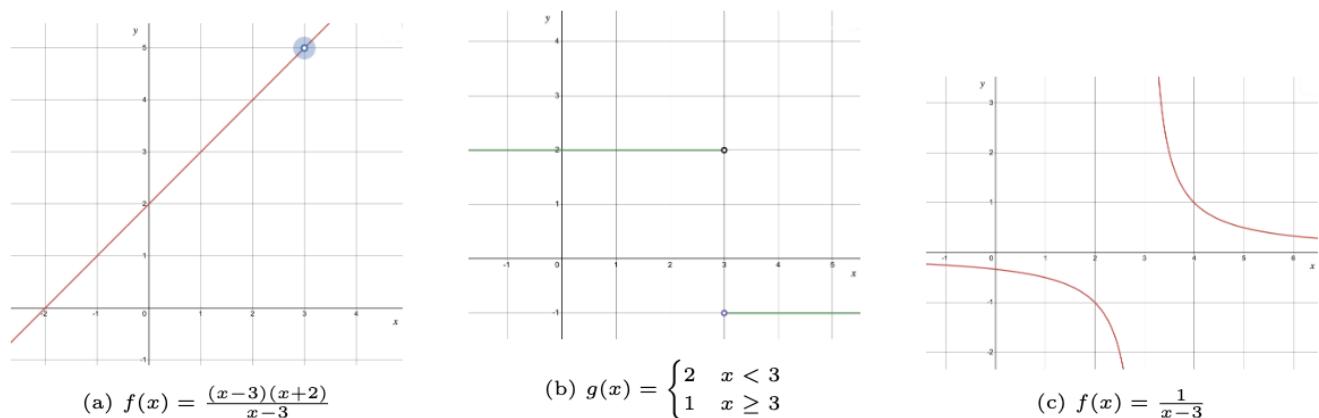


Figure 1.1.1: Three graphs with different behaviors around  $x = 3$

1. What do they have in common?
2. How do the graphs differ from one another?

## Intuitive Definition of a Limit

### Definition 1.1.1: Limits

Suppose that  $f(x)$  is a function defined at all values in an open interval containing  $a$ , except possibly at  $a$ , and suppose that  $L$  is a real number. If  $f(x)$  is getting closer to  $L$  as  $x$  gets closer to  $a$ , then the limit of  $f(x)$  as  $x$  approaches  $a$  is the real number  $L$ . We use the following notation:

$$\lim_{x \rightarrow a} f(x) = L. \quad (1.1.1)$$

If we cannot make  $f(x)$  as close to a single value as we would like as  $x$  approaches  $a$ , then we say that  $f$  **does not have a limit as  $x$  approaches  $a$** .

## Estimating Limits

1. Numerical Approach: Construct a table of values
2. Graphical Approach: Draw the graph (by hand or with the aid of technology)

### Example 1.1.1

Suppose that  $f(x)$  has the following values:

$$\text{[Redacted Table]} \quad (1.1.2)$$

Calculate

$$\lim_{x \rightarrow 8} f(x) = \quad (1.1.3)$$

### Example 1.1.2

Estimate the limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (1.1.4)$$

by using a table of values.

#### Solution

Note that

$$\text{[Redacted Table]} \quad (1.1.5)$$

What do we conclude about  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ ?

### Example 1.1.3

The graph of the function  $g(x)$  is given below. Evaluate  $\lim_{x \rightarrow 2} g(x)$

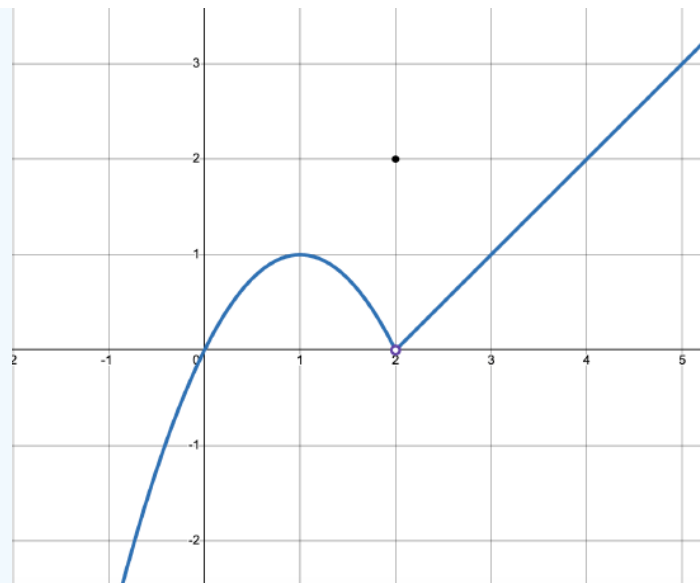


Figure 1.1.2: Graph of  $g(x)$

#### Example 1.1.4

The graph of the function  $f(x) = \sin\left(\frac{1}{x}\right)$  is given below. Evaluate the limit  $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

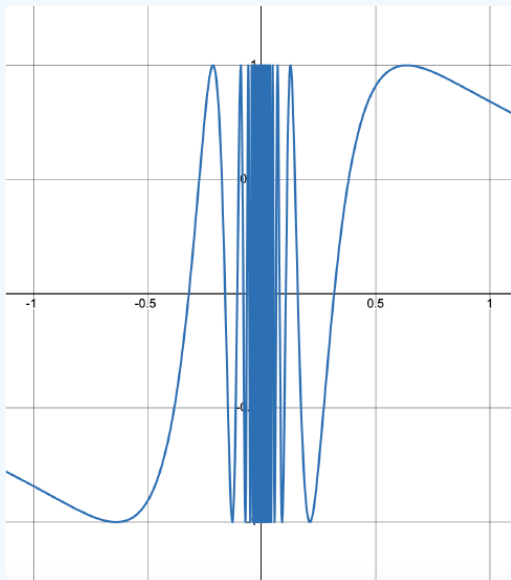


Figure 1.1.3: Graph of  $f(x) = \sin\left(\frac{1}{x}\right)$

#### Example 1.1.5

Evaluate the limit  $\lim_{x \rightarrow 2} f(x)$ , where  $f(x) = \begin{cases} -x^2 + 2x + 1, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$

**Solution**

We can use a table or the graph of  $f(x)$  to evaluate the limit



(1.1.6)

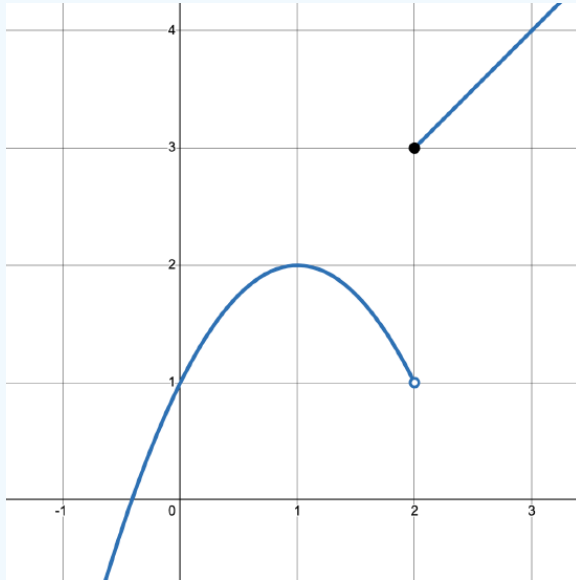


Figure 1.1.4: Graph of  $f(x)$

1. As the  $x$ -values get closer and closer to 2 (but are a little less than 2), what pattern do you notice in the outputs of the function  $f(x) = -x^2 + 2x + 1$ ?
2. Now, look at the values of  $f(x) = x + 1$  when  $x$  is very close to 2 (but a little more than 2). What pattern do you notice in those values?
3. What does this tell you about how the function behaves near 2?
4. What is  $\lim_{x \rightarrow 2} f(x) = ?$

## One-Sided Limits

### Definition 1.1.2: One-Sided Limits

We say that a function  $f$  has a limit  $L_1$  as  $x$  approaches  $a$  from the left, and we write

$$\lim_{x \rightarrow a^-} f(x) = L_1 \quad (1.1.7)$$

if we can make the value of  $f(x)$  as close as we like to  $L_1$  by choosing  $x$  values that are very close to  $a$ , but always with  $x < a$ . We call  $L_1$  the left-hand limit of  $f$  as  $x$  approaches  $a$ . Similarly, we say that  $f$  has a limit  $L_2$  as  $x$  approaches  $a$  from the right, and we write

$$\lim_{x \rightarrow a^+} f(x) = L_2 \quad (1.1.8)$$

if we can make the value of  $f(x)$  as close as we like to  $L_2$  by choosing  $x$  values that are very close to  $a$ , but always with  $x > a$ . We call  $L_2$  the right-hand limit of  $f$  as  $x$  approaches  $a$ .

Consider the function from Example 1.5. Calculate:

$$\lim_{x \rightarrow 2^-} f(x) \text{ and } \lim_{x \rightarrow 2^+} f(x) \quad (1.1.9)$$

## Limits

### Theorem 1.1.1

Let  $f(x)$  be a function defined at all values in an open interval containing  $a$ , except possibly at  $a$ , and let  $L$  be a real number.

Then

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x) \quad (1.1.10)$$

### Example 1.1.6

Use the graph of  $f(x)$  given below to evaluate the limits.

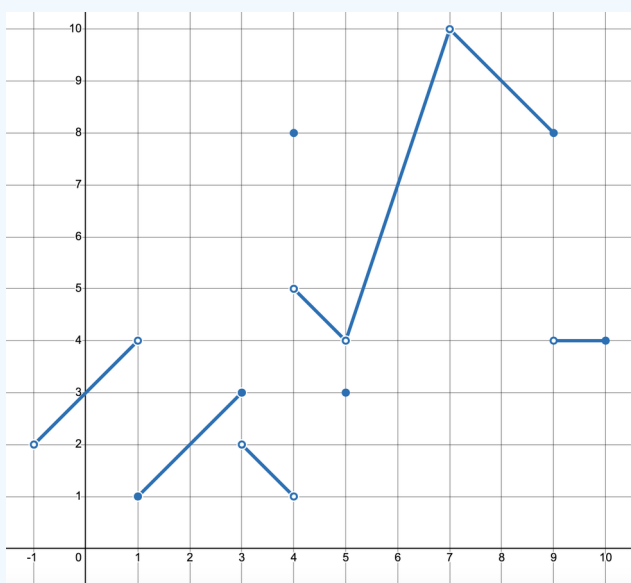


Figure 1.1.5: Graph of  $f(x)$ .

|                                   |                                   |
|-----------------------------------|-----------------------------------|
| $\lim_{x \rightarrow 5} f(x) =$   | $\lim_{x \rightarrow 4^-} f(x) =$ |
| $\lim_{x \rightarrow 3^-} f(x) =$ | $\lim_{x \rightarrow 4^+} f(x) =$ |
| $\lim_{x \rightarrow 3^+} f(x) =$ | $\lim_{x \rightarrow 4} f(x) =$   |
| $\lim_{x \rightarrow 3} f(x) =$   | $\lim_{x \rightarrow 7^-} f(x) =$ |
| $\lim_{x \rightarrow 1^-} f(x) =$ | $\lim_{x \rightarrow 7^+} f(x) =$ |
| $\lim_{x \rightarrow 1^+} f(x) =$ | $\lim_{x \rightarrow 7} f(x) =$   |
| $\lim_{x \rightarrow 1} f(x) =$   | $\lim_{x \rightarrow 9} f(x) =$   |

## Infinite Limits

### Definition 1.1.6: Infinite Limits

Suppose that  $f$  is defined on both sides of  $a$ , except possibly at  $a$ . If  $f(x)$  increases without bound as  $x$  approaches  $a$ , we say that the limit of  $f(x)$ , as  $x$  approaches  $a$ , is infinity. In symbols,

$$\lim_{x \rightarrow a} f(x) = \infty \quad (1.1.11)$$

Similarly, we can define  $\lim_{x \rightarrow a} f(x) = -\infty$ ,  $\lim_{x \rightarrow a^-} f(x) = \infty$ ,  $\lim_{x \rightarrow a^+} f(x) = \infty$ ,  $\lim_{x \rightarrow a^-} f(x) = -\infty$ , and  $\lim_{x \rightarrow a^+} f(x) = -\infty$

### Note

Infinite limits describe the behavior of the function near  $x = a$ ; they do not mean that the limit exists in the usual sense.

For infinite limits, we do not use "DNE" (does not exist).

### Example 1.1.7

Use the graph of the function  $f(x)$  to evaluate the following limits:

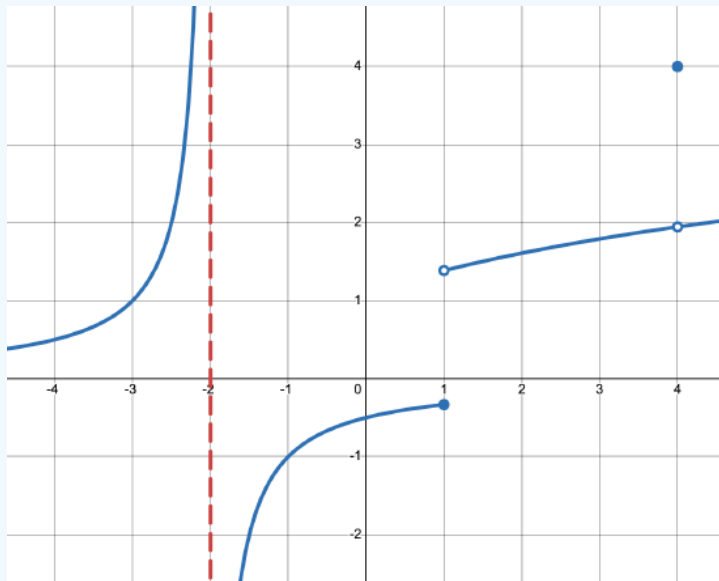


Figure 1.1.6: Graph of  $f(x)$ .

1.  $\lim_{x \rightarrow -2^-} f(x)$
2.  $\lim_{x \rightarrow -2^+} f(x)$
3.  $\lim_{x \rightarrow 1} f(x)$
4.  $\lim_{x \rightarrow 4} f(x)$

### Example 1.1.8

Evaluate each of the following limits, if possible. Use a table of functional values and graph  $f(x) = 1/x^2$  to confirm your conclusion.

- a.  $\lim_{x \rightarrow 0^-} \frac{1}{x}$
- b.  $\lim_{x \rightarrow 0^+} \frac{1}{x}$
- c.  $\lim_{x \rightarrow 0} \frac{1}{x}$
- d.  $\lim_{x \rightarrow 0^-} \frac{1}{x^2}$
- e.  $\lim_{x \rightarrow 0^+} \frac{1}{x^2}$
- f.  $\lim_{x \rightarrow 0} \frac{1}{x^2}$

Analyze the behaviors of the functions  $\frac{1}{x-2}$  and  $\frac{1}{(x-2)^2}$ . Do you notice any patterns?

### Theorem 1.1.2

If  $n$  is a positive even integer, then

$$\lim_{x \rightarrow a} \frac{1}{(x-a)^n} = +\infty \quad (1.1.12)$$

If  $n$  is a positive odd integer, then

$$\lim_{x \rightarrow a^+} \frac{1}{(x-a)^n} = +\infty \quad (1.1.13)$$

and

$$\lim_{x \rightarrow a^-} \frac{1}{(x-a)^n} = -\infty \quad (1.1.14)$$

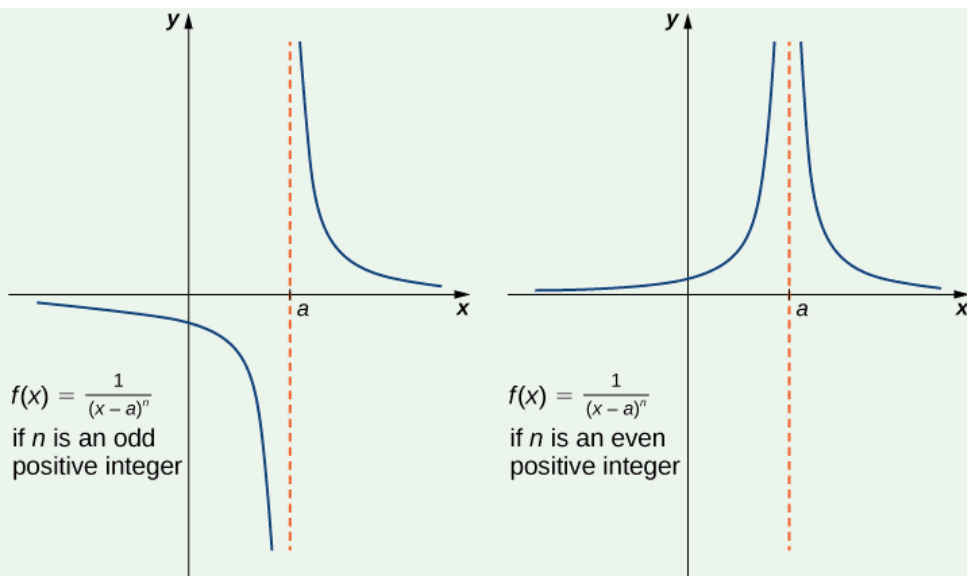


Figure 1.1.7: The function  $f(x) = 1/(x - a)^n$  has infinite limits at  $a$ . From OpenStax Calculus Volume 1, licensed under CC BY.

### Definition: Vertical Asymptotes

Let  $f(x)$  be a function. If any of the following conditions hold, then the line  $x = a$  is a vertical asymptote of  $f(x)$ .

$$\lim_{x \rightarrow a^-} f(x) = +\infty \text{ or } -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = +\infty \text{ or } -\infty$$

$$\lim_{x \rightarrow a} f(x) = +\infty \text{ or } -\infty$$

### Example 1.1.9

- Find the vertical asymptote from the graph of the function in Example 1.1.7.
- Find the vertical asymptotes of the following functions, if any.

- $\frac{1}{x-2}$
- $\frac{1}{(x-3)^4}$
- $\frac{1}{(x-1)^7}$

### Example 1.1.10

Sketch a graph of a function  $f(x)$  that satisfies all of the following conditions:

$\lim_{x \rightarrow 1^-} f(x) = 3$ ,  $\lim_{x \rightarrow 1^+} f(x) = 1$ ,  $f(1) = 2$ ,  $\lim_{x \rightarrow 3} f(x) = 0$ ,  $f(3) = 5$ ,  $\lim_{x \rightarrow 5^-} f(x) = \infty$ ,  $\lim_{x \rightarrow 5^+} f(x) = -\infty$ , and a vertical asymptote at  $x = -3$ .

Be sure to:

- Clearly show your graph approaching the indicated values from the correct direction.
- Plot any given function values as solid points.

- If the function approaches infinity or negative infinity, show this behavior clearly.
- Label important  $x$ -values on the horizontal axis.

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# 1.2: The Limit Laws

## Section 1.2 The Limit Laws

### Learning Objectives

- Evaluate the limit of a function using the limit laws.
- Evaluate the limit of a function at a specific point by factoring the expression or using conjugates.
- Apply the squeeze theorem to evaluate the limit of a function at a point where direct substitution leads to an indeterminate form.

Suppose that  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$  exist and that  $c$  is a constant.

- (a)  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$
- (b)  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M$
- (c)  $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x) = cL$
- (d)  $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M$
- (e)  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$ , provided  $\lim_{x \rightarrow a} g(x) \neq 0$
- (f)  $\lim_{x \rightarrow a} [f(x)]^n = \left( \lim_{x \rightarrow a} f(x) \right)^n = L^n$ , for any positive integer  $n$ .
- (g)  $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L}$ , for all  $L$  if  $n$  is odd, and for  $L \geq 0$  if  $n$  is even and  $f(x) \geq 0$ .
- (h)  $\lim_{x \rightarrow a} c = c$ ,  $\lim_{x \rightarrow a} x = a$ ,  $\lim_{x \rightarrow a} x^n = a^n$ ,  $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$  (when defined).

### Example 1.2.11

Use the limit laws to evaluate the following limits:

$$\lim_{x \rightarrow -1} (2x + 1), \quad \lim_{x \rightarrow 2} \frac{x^2 - 3x + 4}{x + 2} \quad (1.2.1)$$

### Theorem 1.2.3

Suppose that  $f$  is a polynomial or rational function. If  $a$  is in the domain of  $f$ , then

$$\lim_{x \rightarrow a} f(x) = f(a). \quad (1.2.2)$$

### Example 1.2.12

Use Theorem 1.2.3 to evaluate the following limits:

$$\lim_{x \rightarrow -1} (2x + 1), \lim_{x \rightarrow 2} \frac{x^2 - 3x + 4}{x + 2} \quad (1.2.3)$$

Evaluate the following limit:  $\lim_{x \rightarrow 1} \frac{x - 1}{x^2 + 3x - 4}$ .

- What happens if you try to factor the numerator? How does this help simplify the expression?
- After factoring, what does the new simplified expression suggest about the limit as  $x$  approaches 1?

## Techniques to Evaluate Limits

Calculating a limit when  $f(x)/g(x)$  has the indeterminate form  $0/0$

1. Check if the limit can be calculated using the limit laws.

2. Task: Find a function equal to  $h(x) = \frac{f(x)}{g(x)}$  for all  $x \neq a$  over some interval containing  $a$ .

(a) If  $f(x)$  and  $g(x)$  are polynomials, factor each function and cancel out any common factors.

(b) If the numerator or denominator contains a difference involving a square root, try multiplying the numerator and denominator by the conjugate of the expression involving the square root.

### Example 1.2.13

Evaluate the following limits:

(a)  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

(b)  $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

(c)  $\lim_{x \rightarrow 1} \frac{\sqrt{(x+3)} - 2}{x - 1}$

(d)  $\lim_{x \rightarrow 0} \frac{\sqrt{(x^2 + 1)} - 1}{x^2}$

(e)  $\lim_{x \rightarrow 2} \frac{\frac{1}{x-1} - 1}{x - 2}$

(f)  $\lim_{x \rightarrow 1} \left( \frac{1}{x-1} - \frac{2}{x^2 - 1} \right)$

(g)  $\lim_{x \rightarrow 4} f(x)$ , where  $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x < 4 \\ 5 & \text{if } x = 4 \\ x + 2 & \text{if } x > 4 \end{cases}$

## The Squeeze Theorem

### Theorem 1.2.4

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be defined for all  $x \neq a$  over an open interval containing  $a$ , and let  $L$  be a real number. If  $f(x) \leq g(x) \leq h(x)$  for all  $x \neq a$  in an open interval containing  $a$ , and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x), \quad (1.2.4)$$

then

$$\lim_{x \rightarrow a} g(x) = L. \quad (1.2.5)$$

#### Example 1.2.14

Use the limit  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  to evaluate the limit

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \quad (1.2.6)$$

#### Example 1.2.15

(a) Let  $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ . Use the Squeeze Theorem to evaluate:  $\lim_{x \rightarrow 0} f(x)$

(b) Let  $g(x) = \begin{cases} x \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ . Use the Squeeze Theorem to evaluate:  $\lim_{x \rightarrow 0} g(x)$

## 1.3: Continuity

### 1.3 Continuity

#### Learning Objectives

- Apply the three conditions for continuity to determine whether a function is continuous or discontinuous at a given point.
- Identify and describe the three types of continuity in functions.
- Use the Intermediate Value Theorem to demonstrate the existence of a solution within a given interval for continuous functions.

#### Definition: Continuity at a Point

A function  $f$  is continuous at  $a$  if the following conditions are satisfied:

1.  $f(a)$  is defined
2.  $\lim_{x \rightarrow a} f(x)$  exists
3.  $\lim_{x \rightarrow a} f(x) = f(a)$

When can we say that  $f$  is not continuous (or that it is discontinuous) at  $a$ ?

#### Example 1.3.16

Determine whether the following functions are continuous at the given value:

1.  $f(x) = x^2 + 5, a = 1$

2.  $f(x) = \frac{x^2 - 9}{x - 3}, a = 3$

3.

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x < 4 \\ 5 & \text{if } x = 4 \\ x + 2 & \text{if } x > 4 \end{cases} \quad a = 4$$

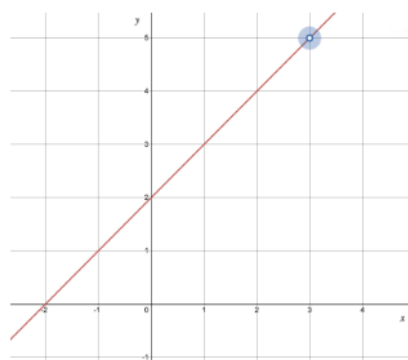
4.  $f(x) = \begin{cases} \frac{x^2 + 2x}{x}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases} \quad a = 0$

5.

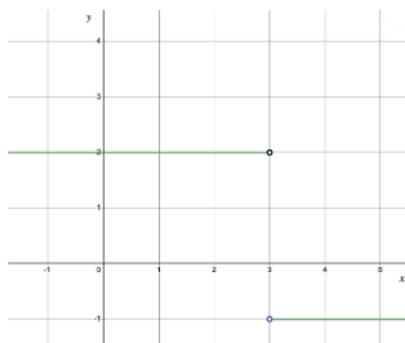
$$f(x) = \begin{cases} \frac{1 - \cos x}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad a = 0$$

6.  $f(x) = \frac{1}{(x - 2)^2}, a = 2$

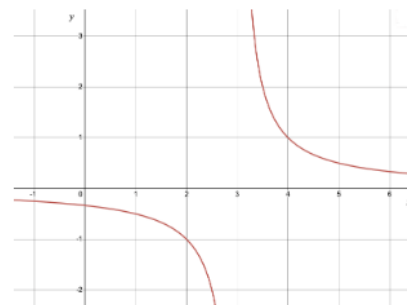
## Types of Discontinuities



(a) removable discontinuity



(b) jump discontinuity



(c) infinite discontinuity

Figure 1.3.8: Types of discontinuities (a) removable, (b) jump, or (c) infinite

Note that if  $f$  has a discontinuity at  $a$ , then:

1. The discontinuity at  $a$  is *removable* if  $\lim_{x \rightarrow a} f(x)$  exists.
2. The discontinuity at  $a$  is a *jump discontinuity* if  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  both exist, but they are not equal.
3. The discontinuity at  $a$  is an *infinite discontinuity* if  $\lim_{x \rightarrow a^-} f(x) = \pm\infty$  and/or  $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ .

### Example 1.3.17

The graph of the function  $f(x) = \frac{x^2-1}{x-1}$  is given below. Determine whether the function is continuous at  $x = 1$ . If it is not continuous, what type of discontinuity occurs at that point?

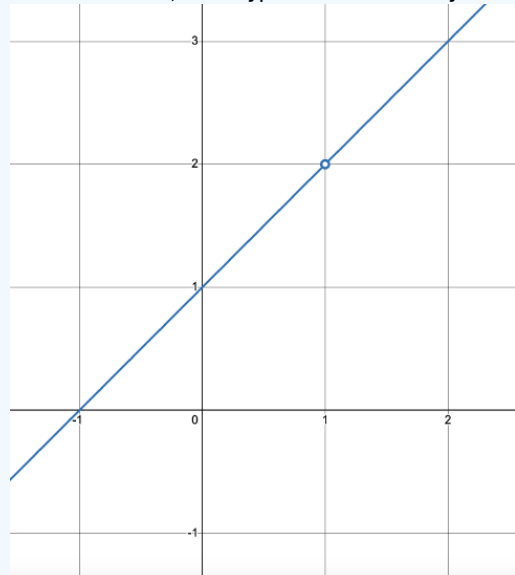


Figure 1.3.9: Graph of  $f(x)$  (from Desmos)

### Example 1.3.18

The graph of the function  $f(x) = \begin{cases} -x^2 + 2x + 1, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$  is given below. Determine whether the function is continuous at  $x = 2$ . If it is not continuous, what type of discontinuity occurs at that point?

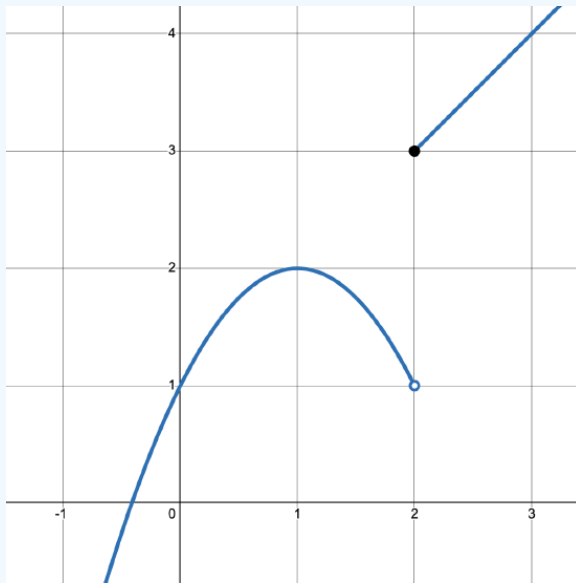


Figure 1.3.10: Graph of  $f(x)$ (from Desmos)

### Example 1.3.19

The graph of the function  $f(x)$  is given below. Determine whether the function is continuous at  $x = 1$ . If it is not continuous, what type of discontinuity occurs at that point?

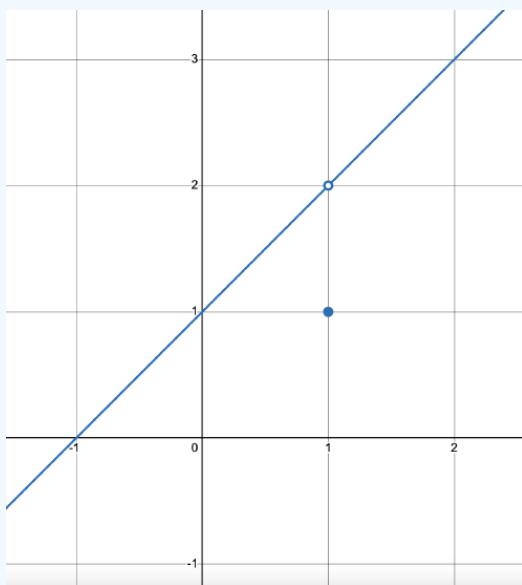


Figure 1.3.11: Graph of  $f(x)$ (from Desmos)

### Example 1.3.20

The graph of the function  $f(x)$  is given below. Determine whether the function is continuous at  $x = -2$ . If it is not continuous, what type of discontinuity occurs at that point?

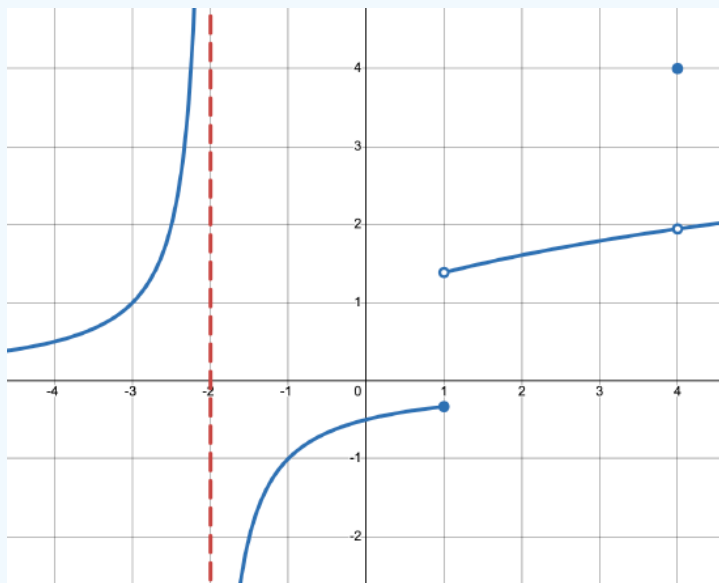


Figure 1.3.12: Graph of  $f(x)$ (from Desmos)

### Example 1.3.21

Classify the discontinuities (if any) of the following functions:

1.  $f(x) = x^2 + 5, a = 1$
2.  $f(x) = \frac{x^2 - 9}{x - 3}, a = 3$
3. 
$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x < 4 \\ 5 & \text{if } x = 4 \\ x + 2 & \text{if } x > 4 \end{cases} \quad a = 4$$
4.  $f(x) = \begin{cases} \frac{x^2 + 2x}{x}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases} \quad a = 0$
5.  $f(x) = \begin{cases} \frac{1 - \cos x}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad a = 0$
6.  $f(x) = \frac{1}{(x - 2)^2}, a = 2$

### Definition: Continuity on an Interval

A function  $f$  is continuous on an interval if it is continuous at every number in the interval. Note that at an endpoint, you must check one-sided continuity — left-hand continuity at a right endpoint and right-hand continuity at a left endpoint.

### Theorem 1.3.4

Polynomial, rational, root, and trigonometric functions are continuous at every number in their domains.

### Example 1.3.22

State the interval(s) over which the following functions are continuous:

1.  $y = 3x^3 + 6x^2 + 2$
2.  $y = \frac{2x + 7}{x^2 - 3x + 2}$
3.  $y = \sqrt{x + 3}$
4.  $y = \tan x$

### Theorem 1.3.5: Intermediate Value Theorem

Suppose that  $f$  is continuous on the interval  $[a, b]$ , and let  $z$  be any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ . Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = z$ .

### Example 1.3.1

Use the Intermediate Value Theorem to show that:

1.  $f(x) = \ln(x) - x + 2$  has at least one zero.
2.  $f(x) = x^3 - 4x + 1$  has a zero over the interval  $[1, 2]$ .

## 2: DERIVATIVES

### Overview

The derivative is one of the central ideas in calculus. It provides a precise way to describe how a quantity changes at an instant. If  $y = f(x)$ , then the derivative  $f'(x)$  measures the *instantaneous rate of change* of  $f$  with respect to  $x$ .

Geometrically,  $f'(x)$  is the *slope of the tangent line* to the graph of  $y = f(x)$  at  $x$ . Physically, derivatives connect to velocity, acceleration, growth rates, and many other real-world rates of change.

The derivative is defined via a limit:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (2.1)$$

In this chapter, we move from the definition to practical rules and applications that make differentiation both powerful and broadly useful.

### Topics in this Chapter

1. The Definition of the Derivative
  - The formal limit definition of  $f'(x)$ .
  - Interpreted as the instantaneous rate of change and tangent line slope.
2. The Derivative as a Function
  - Understanding the properties of the derivative function.
  - Studying how  $f'(x)$  and  $f(x)$  are related.
3. Differentiation Rules
  - Rules for computing derivatives: constant rule, power rule, product rule, and quotient rule.
4. Derivatives as Rates of Change
  - Interpreting  $f'(x)$  in applied contexts such as velocity, acceleration, and growth.
5. Derivatives of Trigonometric Functions
  - Compute and apply in different contexts the derivative of functions involving trigonometric functions.
6. The Chain Rule
  - Find the derivative of composite functions.
7. Derivatives of Inverse Functions
  - Compute the derivative of functions involving inverse trigonometric functions.
8. Implicit Differentiation
  - Compute derivatives of functions given implicitly, and apply them in different contexts.
9. Derivatives of Exponential and Logarithmic Functions
  - Compute the derivative of functions involving exponential or logarithmic functions, and use them to solve real-life problems.

### Why this matters

- Derivatives quantify instantaneous change across science, engineering, and economics.
- They help analyze graphs through slopes, monotonicity, and concavity.
- They are the foundation for optimization, curve sketching, and modeling real-world processes.

#### 2.1: The Definition of the Derivative

[2.2: The Derivative as a Function](#)

[2.3: Differentiation Rules](#)

[2.4: Derivatives as Rates of Change](#)

[2.5: Derivatives of Trigonometric Functions](#)

[2.6: The Chain Rule](#)

[2.7: Derivatives of Inverse Functions](#)

[2.8: Implicit Differentiation](#)

[2.9: Derivatives of Exponential and Logarithmic Functions](#)

## 2.1: The Definition of the Derivative

### 2.1 The Definition of the Derivative

#### Learning Objectives

- Given a curve and a specific point, calculate the slope of the tangent line using the limit of the difference quotient, with correct application of the formula.
- Identify and apply the derivative as the limit of a difference quotient to find instantaneous rates of change.

### Tangent Lines

#### Definition 2.1.1

Let  $f(x)$  be a function defined on an open interval containing two distinct points  $a$  and  $x$ , where  $x \neq a$ . The secant line to the curve  $y = f(x)$  through the points  $P(a, f(a))$  and  $Q(x, f(x))$  is the line passing through both points with slope

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a} \quad (2.1.1)$$

We can also write this slope as

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{h}, \quad (2.1.2)$$

where  $h \neq 0$  and  $x = a + h$ .

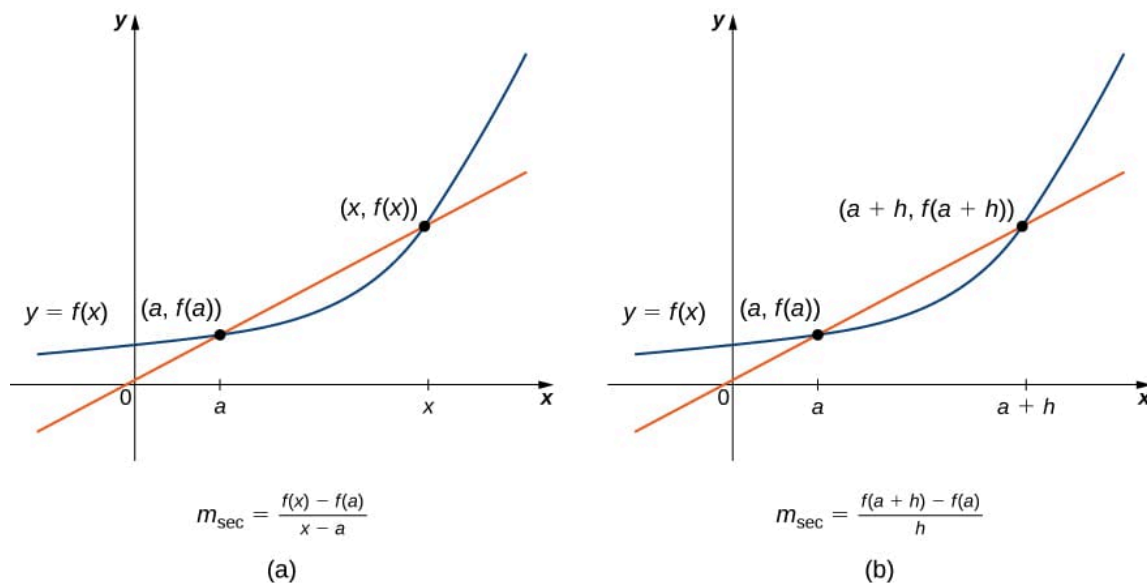


Figure 2.1.1: The slope of a secant line. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

### Example 2.1.1

Find the equation of the secant line to the curve  $y = x^2$  that passes through the points  $(3, 9)$  and  $(5, 25)$ .

### Definition 2.1.2

Let  $f(x)$  be a function defined on an open interval containing  $a$ . The tangent line to the curve  $y = f(x)$  at the point  $P(a, f(a))$  is the line passing through  $P$  with slope

$$m_{\text{tan}} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (2.1.3)$$

provided that the limit exists.

We can also express the above limit as

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}, \quad (2.1.4)$$

where  $h \neq 0$  and  $x = a + h$ .

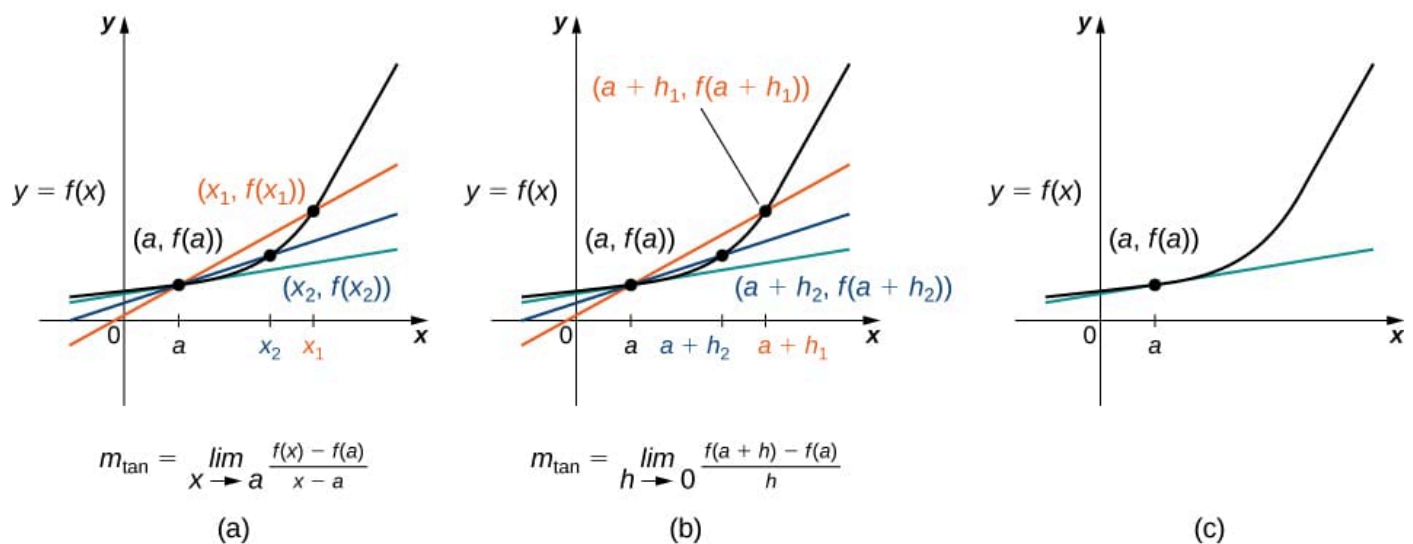


Figure 2.1.2: The slope of tangent line. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

### Example 2.1.2

Find the equation of the tangent line to the curve  $f(x) = x^2$  at  $x = 3$ .

### Example 2.1.3

The graph of  $f(x)$  is shown below. Estimate and list all values of  $x$  where  $f(x)$  appears to have a horizontal tangent line.

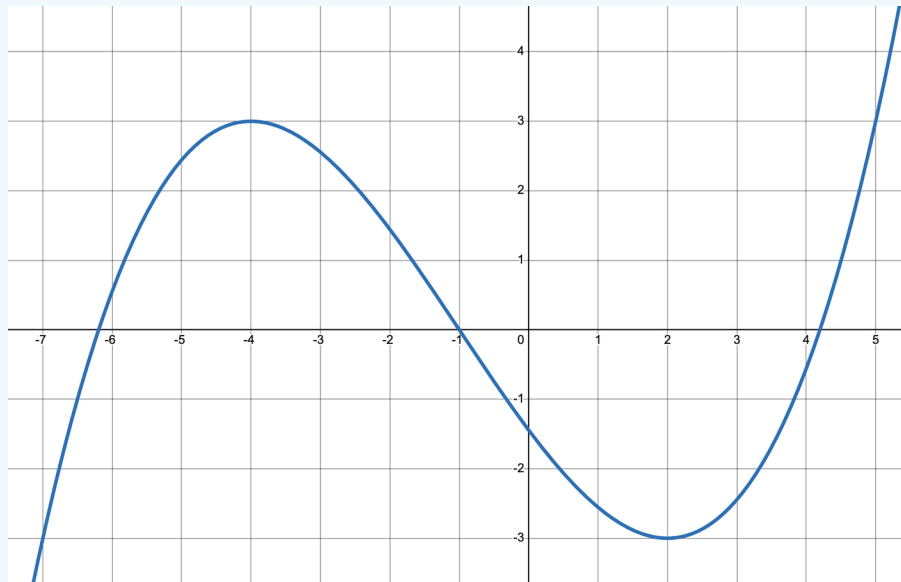


Figure 2.1.3: Graph of  $f(x)$

## The Derivative of a Function at a Point

### Definition 2.1.3

Let  $f(x)$  be a function defined in an open interval containing  $a$ . The derivative of the function  $f(x)$  at  $a$ , denoted by  $f'(a)$ , is defined by

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (2.1.5)$$

provided that the limit exists.

We can also write the above limit as

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}, \quad (2.1.6)$$

where  $h \neq 0$  and  $x = a + h$ .

### Example 2.1.4

Calculate the following derivatives using the definition of the derivative:

(a)  $f'(3)$ , where  $f(x) = x^2$

(b)  $f'(2)$ , where  $f(x) = 3x^2 - 4x + 1$

## 2.2: The Derivative as a Function

### 2.2 The Derivative as a Function

#### Learning Objectives

- Apply the definition of the derivative (difference quotient) to find the derivative of basic functions (e.g., polynomials, rational functions) at specific points.
- Given the graph of a function, accurately sketch the graph of its derivative by identifying the slopes of the tangent lines at key points within a specified interval.
- Identify and describe the three conditions (sharp corners, vertical tangents, and discontinuities) under which a function does not have a derivative at a specific point.
- Describe the meaning and interpretation of higher order derivatives.

#### Definition 2.2.4

Let  $f$  be a function. The derivative function, denoted by  $f'$ , is the function whose domain consists of those values  $x$  such that the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (2.2.1)$$

exists.

We can also use other notations for the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f(x)) = D_x f(x) \quad (2.2.2)$$

A function  $f$  is said to be differentiable at  $x = a$  if  $f'(a)$  exists. We may also use the notation:

$$\left. \frac{dy}{dx} \right|_{x=a} \quad (2.2.3)$$

#### Example 2.2.5

Calculate the following derivatives using the Definition 2.2.4

(a)  $y = x^2$

(b)  $y = \sqrt{x}$

Analyze the both the graph of the function and the graph of the derivative function:

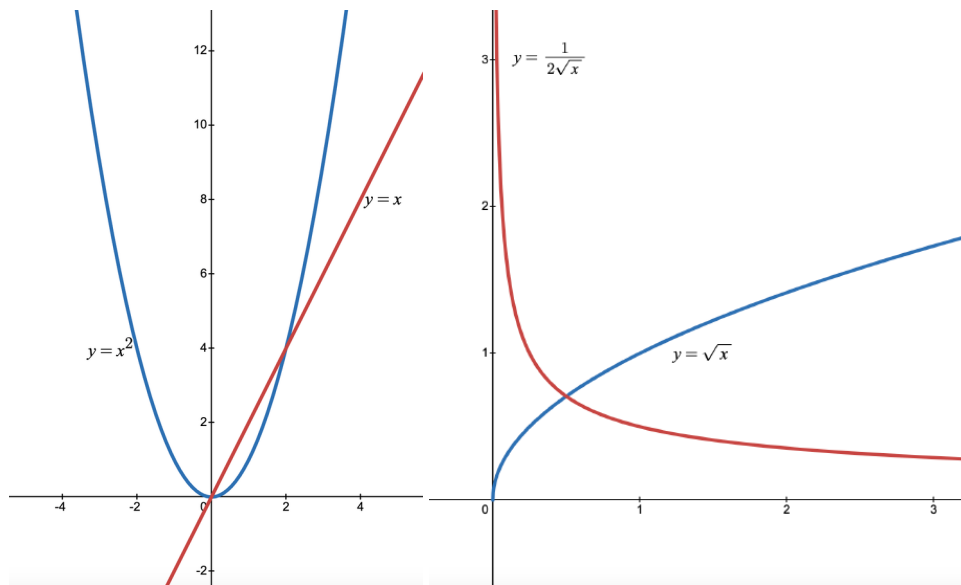


Figure 2.2.4: Graph of functions and their derivative functions

#### Example 2.2.6

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#### Example 2.2.7

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#### Example 2.2.8

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#### Example 2.2.9

The graph of the function  $f(x)$  is given below. Sketch the graph of the derivative function  $f'(x)$ .

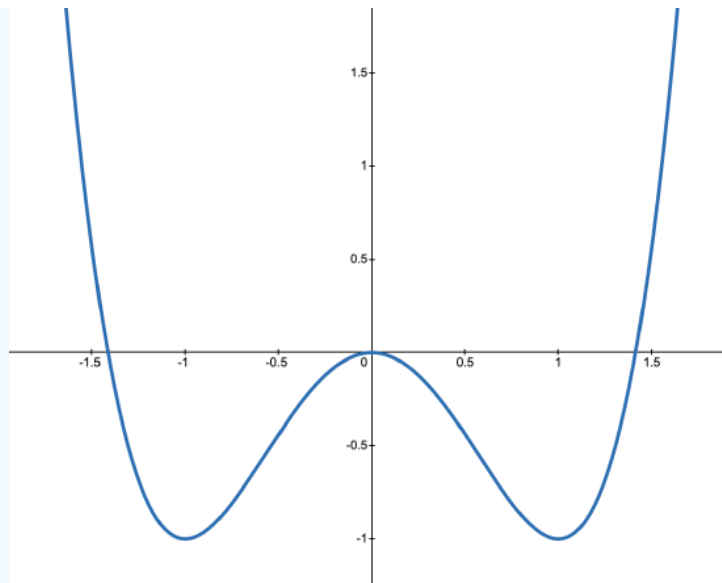


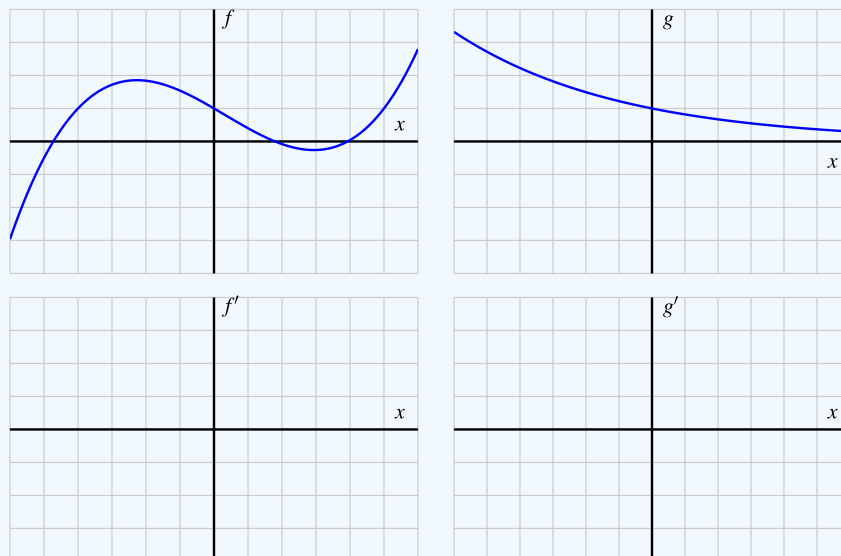
Figure 2.2.5: Graph of  $f(x)$  to sketch its derivative function

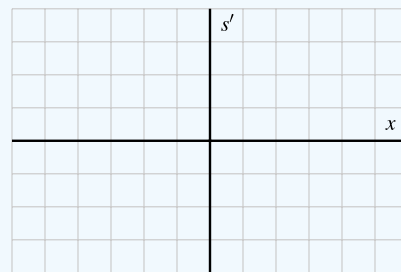
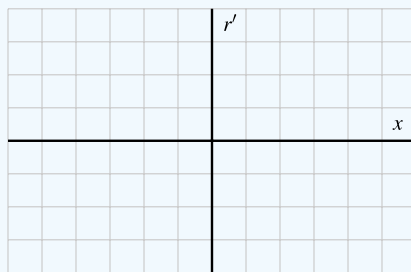
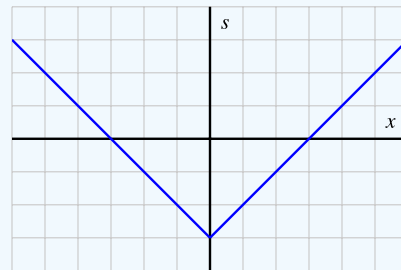
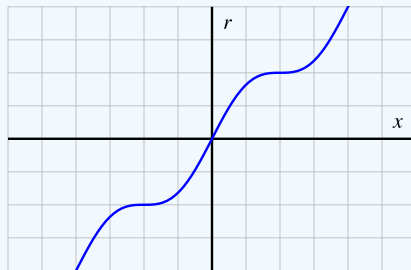
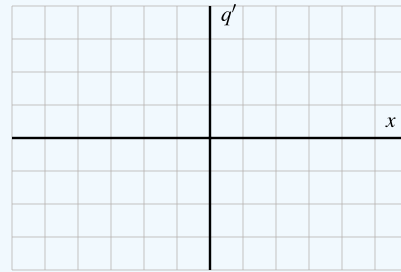
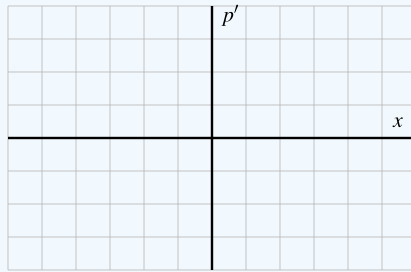
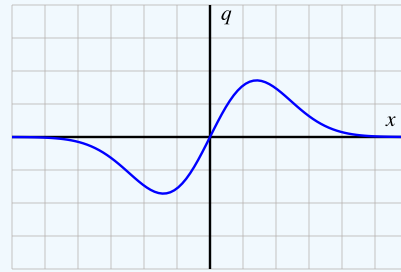
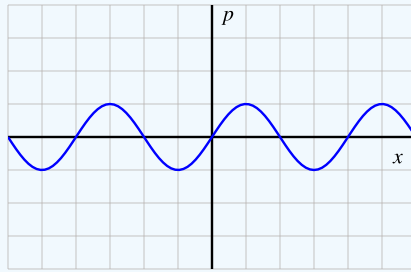
### Example 2.2.9 – 1

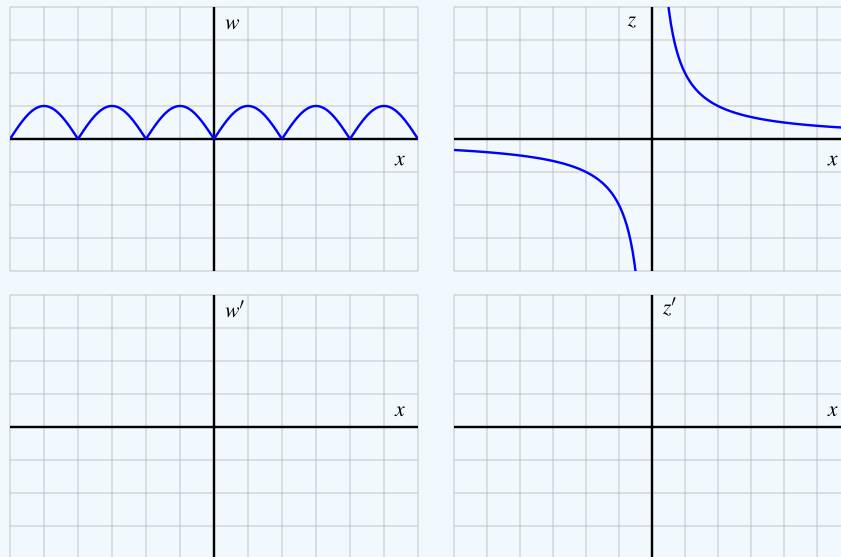
#### Activity 2.2.2

For each given graph of  $y = f(x)$ , sketch an approximate graph of its derivative function,  $y = f'(x)$ , on the axes immediately below. The scale of the grid for the graph of  $f$  is  $1 \times 1$ ; assume the horizontal scale of the grid for the graph of  $f'$  is identical to that for  $f$ . If necessary, adjust and label the vertical scale on the axes for  $f'$ .

When you are finished with all 8 graphs, write several sentences that describe your overall process for sketching the graph of the derivative function, given the graph the original function. What are the values of the derivative function that you tend to identify first? What do you do thereafter? How do key traits of the graph of the derivative function exemplify properties of the graph of the original function?







## Derivatives and Continuity

### Theorem 2.2.1

Suppose that  $f$  is differentiable at  $a$ . Then  $f$  is continuous at  $a$ .

1. Consider the function  $f(x) = |x|$ . Is  $f(x)$  continuous at  $x = 0$ ? What happens if you try to compute the limit of  $f(x)$  as  $x \rightarrow 0$ ? Does the function "jump" or "break" at  $x = 0$ ?
2. Now, examine the derivative of  $f(x) = |x|$  at  $x = 0$ . Try to compute the left-hand derivative and the right-hand derivative at  $x = 0$ . Are they the same? What does this tell you about the differentiability of  $f(x)$  at  $x = 0$ ?
3. If a function is continuous at a point, can we automatically conclude that it is differentiable at that point? Based on your observations with  $f(x) = |x|$ , why is it that continuity alone does not imply differentiability?

### Example 2.2.10

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## Cases Where a Function Fails to Be Differentiable

1. If a function  $f$  is not continuous at  $x = a$ , then the function is not differentiable at  $x = a$ .
2. The function  $f$  is differentiable at  $x = a$  if the function is smooth at  $x = a$ . That is, the graph of  $f$  at the point  $(a, f(a))$  does not have abrupt bends, corners, or breaks.
3. A function  $f$  fails to be differentiable at  $x = a$  if there is a vertical tangent at  $(a, f(a))$ .
4. More complicated behavior can also cause failure of differentiability. For example:

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad (2.2.4)$$

### Example 2.2.11

The function  $y = \sqrt[3]{x-1}$  is not differentiable at  $x = 1$ .

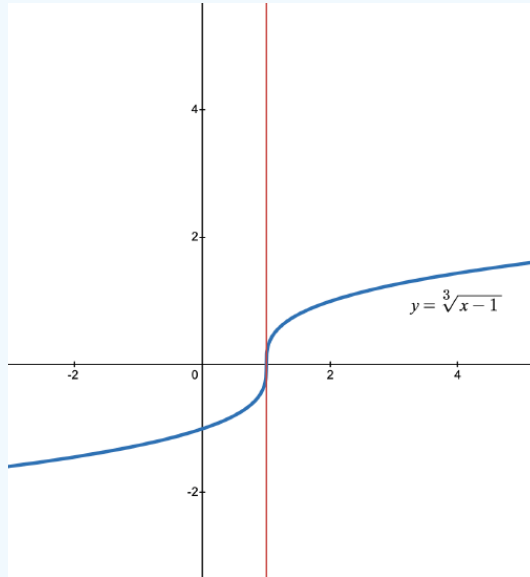


Figure 2.2.6: Graph of  $y = \sqrt[3]{x-1}$  to sketch its derivative function

### Example 2.2.12

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### Example 2.2.13

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## Higher Order Derivatives

### Definition 2.2.5

Let  $y = f(x)$  be a differentiable function. We denote successive derivatives of  $f(x)$  using the following standard notations:

$$\begin{aligned}\text{First derivative: } & f'(x) \\ \text{Second derivative: } & f''(x) = y'' = \frac{d^2y}{dx^2} \\ \text{Third derivative: } & f'''(x) = y''' = \frac{d^3y}{dx^3} \\ \text{Fourth derivative: } & f^{(4)}(x) = y^{(4)} = \frac{d^4y}{dx^4} \\ & \vdots \\ n\text{-th derivative: } & f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n} \\ & \vdots\end{aligned}\tag{2.2.5}$$

### Example 2.2.14

Let  $f(x) = x^2 + 3x + 5$ . Find  $f''(x)$ .

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## 2.3: Differentiation Rules

### 2.3 Differentiation Rules

#### Learning Objectives

- Apply the constant, power, sum and difference, product, and quotient rules to calculate the derivatives of a variety of functions, including polynomials, rational functions, and products of functions.
- Use the extended power rule to find the derivatives of functions with negative exponents, demonstrating the correct application to both positive and negative integer exponents.
- Combine the constant, power, sum and difference, product, and quotient rules to calculate the derivatives of polynomial and rational functions.

Differentiation Formulas:

1. Derivative of a constant function:

$$\frac{d}{dx}(c) = 0 \quad (2.3.1)$$

2. Power rule (for any real number  $n$ ):

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad (2.3.2)$$

3. Constant multiple rule:

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \quad (2.3.3)$$

4. Sum rule:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \quad (2.3.4)$$

5. Difference rule:

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) \quad (2.3.5)$$

6. Product rule:

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)] \quad (2.3.6)$$

7. Quotient rule:

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2} \quad (2.3.7)$$

#### Example 2.3.15

Find the derivative of each function below.

1.  $y = 7$
2.  $y = x^5$
3.  $y = 4x^9 - x^2 + 1$

4.  $y = x^\pi$
5.  $y = x^2 - \frac{3}{x}$
6.  $y = x^5(3x^2 + 2x + 1)$
7.  $y = x^4\left(\frac{10}{x^4} + \frac{1}{x^2}\right)$
8.  $y = \frac{8x^6}{3x^2 - 1}$
9.  $y = \frac{\sqrt{x} + 1}{x^3 - 4x^2 + 5}$

#### Example 2.3.16

Find the equation of the tangent line  $T(x)$  to the graph of the given function at the specified point.

- (1)  $f(x) = x^2 - \frac{3}{x}$  at  $(3, 8)$
- (2)  $f(x) = x^4\left(\frac{10}{x^4} + \frac{1}{x^2}\right)$  at  $(2, 14)$
- (3)  $f(x) = \frac{\sqrt{x} + 1}{x^3 - 4x^2 + 5}$  at  $(1, 1)$

#### Example 2.3.17

Determine all points on the graph of

$$f(x) = x^3 - 6x^2 + 9x \quad (2.3.8)$$

for which:

- (a) the tangent line is horizontal
- (b) the tangent line has a slope of  $-3$

## 2.4: Derivatives as Rates of Change

### 2.4 Derivatives as Rates of Change

#### Learning Objectives

- Calculate the average rate of change of a function over a specified interval and explain how it differs from the instantaneous rate of change.
- Use the concept of rate of change to calculate displacement, velocity, and acceleration for an object moving in a straight line, applying the appropriate derivatives to describe motion.
- Interpret the derivative as the instantaneous rate of change of the output variable with respect to the input variable, applying it to various real-world contexts such as motion and population growth.

#### Definition 2.4.6

The **rate of change** of a function  $y = f(x)$  describes how the dependent variable,  $y$ , is changing with respect to the independent variable,  $x$ .

#### Example 2.4.18

Consider the function  $y = f(x)$ , where  $y$  represents the population of a certain city, and  $x$  is time in years. The *rate of change* of the city's population is 100 people per year. That is, the city's population is increasing at a rate of 100 people per year (100/year).

We would like to be more specific and study the rate of change over a particular period of time.

#### Definition 2.4.7

The **average** rate of change of a function  $y = f(x)$  over  $[a, b]$  describes how the dependent variable is changing with respect to the independent variable on the interval  $[a, b]$ .

#### Example 2.4.19

Following Example 2.18, we could say that the *average rate of change* of the city's population between 2000 and 2010 is 100 people per year. That is, between 2000 and 2010, the city's population increased at a rate of 100 people per year (100/year).

#### Definition 2.4.8

The average rate of change of the function  $f(x)$  over the interval  $[a, a + h]$  is given by

$$\frac{f(a + h) - f(a)}{h} \quad (2.4.1)$$

Suppose that  $f$  is a function whose graph is given below. Answer the following questions.

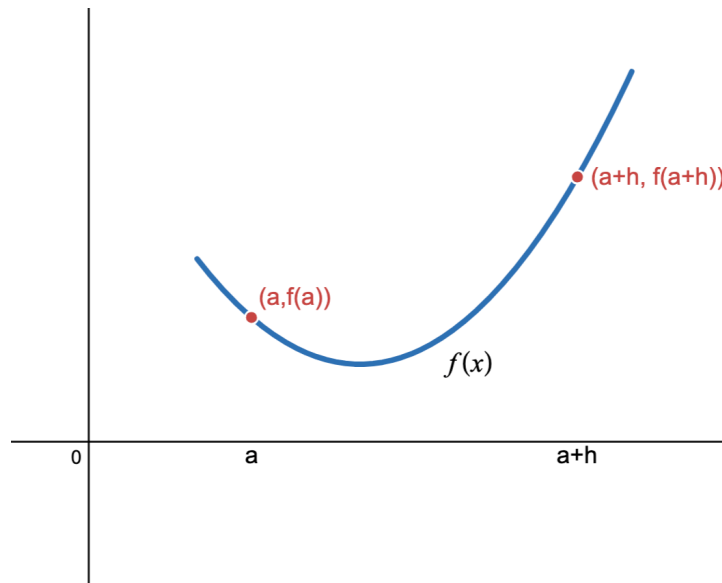


Figure 2.4.7: Graph of  $f(x)$  to explore rates of change

1. Construct a right triangle whose hypotenuse is the line segment from  $(a, f(a))$  to  $(a + h, f(a + h))$ . What are the lengths of the respective legs of this triangle?
2. What is the slope of the line that joins the points  $(a, f(a))$  and  $(a + h, f(a + h))$ ?
3. Write a meaningful sentence that explains how the average rate of change of the function on a given interval and the slope of a related line are connected.

#### Definition 2.4.9

The instantaneous rate of change of a function  $f(x)$  at a value  $a$  is given by

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (2.4.2)$$

How is the derivative  $f'(a)$  related to the instantaneous rate of change of  $f(x)$  at  $x = a$ ?

Note that we can approximate  $f(a + h) \approx f(a) + f'(a)h$ .

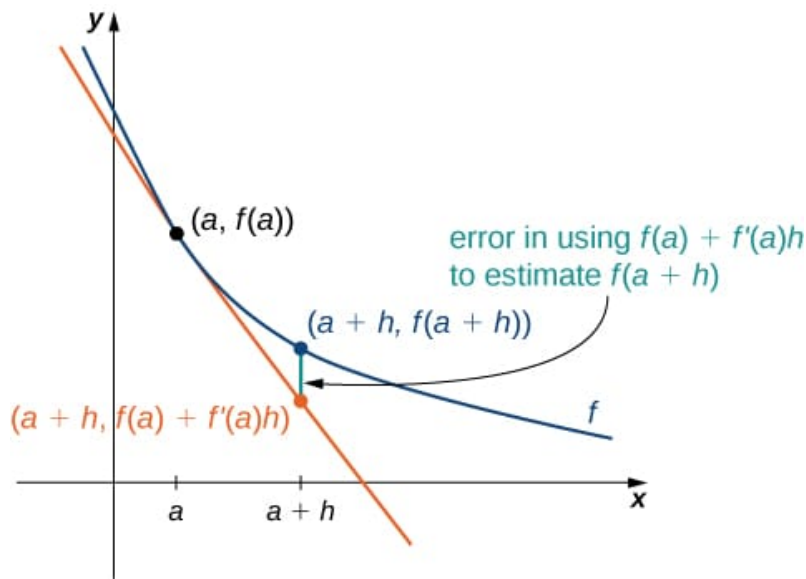


Figure 2.4.8: Approximating  $f(a+h) \approx f(a) + f'(a)h$ . From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

For example, if  $f(x)$  is a function such that  $f(3) = 2$  and  $f'(3) = 5$ , we can approximate  $f(3.2)$  as:

$$f(3.2) = f(3 + 0.2) \approx f(3) + 0.2 \cdot f'(3) = 2 + 0.2(5) = 3 \quad (2.4.3)$$

## Velocities and Rates of Change

### Definition 2.4.10

Let  $s(t)$  be the position of an object moving along a coordinate axis at time  $t$ .

The **average velocity** of the object over a time interval  $[a, t]$ , where  $a < t$  (or  $[t, a]$  if  $t < a$ ), is given by:

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a} \quad (2.4.4)$$

The instantaneous velocity at  $a$ , denoted  $v(a)$ , is given by:

$$v(a) = s'(a) = \lim_{t \rightarrow a} \frac{s(t) - s(a)}{t - a} \quad (2.4.5)$$

### Example 2.4.20

The position in feet of a race car along a straight track after  $t$  seconds is modeled by the function:

$$s(t) = 8t^2 - \frac{1}{16}t^3 \quad (2.4.6)$$

1. Find the average velocity of the vehicle over the following time intervals (round your answers to four decimal places):

a.  $[4, 4.1]$

- b.  $[4, 4.01]$
- c.  $[4, 4.001]$
- d.  $[4, 4.0001]$

2. Use your answers from part (1) to draw a conclusion about the instantaneous velocity of the vehicle at  $t = 4$  seconds.

### Definition 2.4.11

Let  $s(t)$  be a function giving the position of an object at time  $t$ .

- The velocity of the object at time  $t$  is given by  $v(t) = s'(t)$ .
- The speed of the object at time  $t$  is given by  $|v(t)|$ .
- The acceleration of the object at time  $t$  is given by  $a(t) = v'(t) = s''(t)$ .

### Example 2.4.21

A ball is dropped from a height of 64 feet. Its height above ground (in feet)  $t$  seconds later is given by:

$$s(t) = -16t^2 + 64 \quad (2.4.7)$$

1. What is the instantaneous velocity of the ball when it hits the ground?
2. What is the average velocity during its fall?

### Example 2.4.22

*This interactive element is not available in this format. Visit the link or scan the QR code to access it online.*

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## Population Change

### Example 2.4.23

The population of a city (in thousands) is modeled by the function:

$$P(t) = -0.2t^2 + 3t + 120 \quad (2.4.8)$$

where  $t$  is the number of years since 2020.

1. Find the rate of population change function  $P'(t)$ .
2. Calculate and interpret  $P'(5)$  and  $P'(10)$ . What do these values tell you about the population growth or decline at years 2025 and 2030?

### Example 2.4.24

A population of bacteria in a petri dish is modeled by the function

$$P(t) = \frac{500t}{t+5} \quad (2.4.9)$$

where  $P(t)$  represents the number of bacteria (in thousands) at time  $t$  hours after the experiment begins.

1. Find the rate of change function  $P'(t)$ . What are its units?
2. Calculate  $P'(2)$ . What does this tell you about the bacteria population at that time?
3. Use  $P'(2)$  to estimate the number of bacteria at time  $t = 2.1$ .

## Cost

### Example 2.4.25

The total cost (in dollars) to produce  $x$  units of a product is modeled by the function:

$$C(x) = 0.05x^3 - 2x^2 + 50x + 1000 \quad (2.4.10)$$

1. Find the marginal cost function  $C'(x)$ , which represents the rate of change of cost with respect to production.
2. Calculate and interpret  $C'(10)$  and  $C'(20)$ . What do these values tell you about the cost of producing the 11th and 21st units?

### Example 2.4.26

A company manufactures rope, and the total cost of producing  $r$  feet of rope is  $C(r)$  dollars.

1. What does it mean to say that  $C(3000) = 120$ ?
2. What are the units of  $C'(r)$ ?
3. Suppose that  $C(3000) = 120$  and  $C'(3000) = 0.45$ . Estimate  $C(3005)$ .
4. What is the meaning of  $C'(7000) = -0.3$ ? Is this possible? Why or why not?

## 2.5: Derivatives of Trigonometric Functions

### 2.5 Derivatives of Trigonometric Functions

#### Learning Objectives

- Calculate the derivatives of standard trigonometric functions (sine, cosine, tangent, etc.) at specific points using standard differentiation rules.
- Calculate the second and higher-order derivatives of the sine and cosine functions, demonstrating the pattern of derivatives for these functions.

Known Formulas:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0 \quad (2.5.1)$$

Using the above formulas, we can prove that:

$$\frac{d}{dx}(\sin x) = \cos x \quad (2.5.2)$$

Similarly, we can prove that:

$$\frac{d}{dx}(\cos x) = -\sin x \quad (2.5.3)$$

Additionally, we can use the quotient rule to calculate the derivative of  $\tan x = \frac{\sin x}{\cos x}$ .

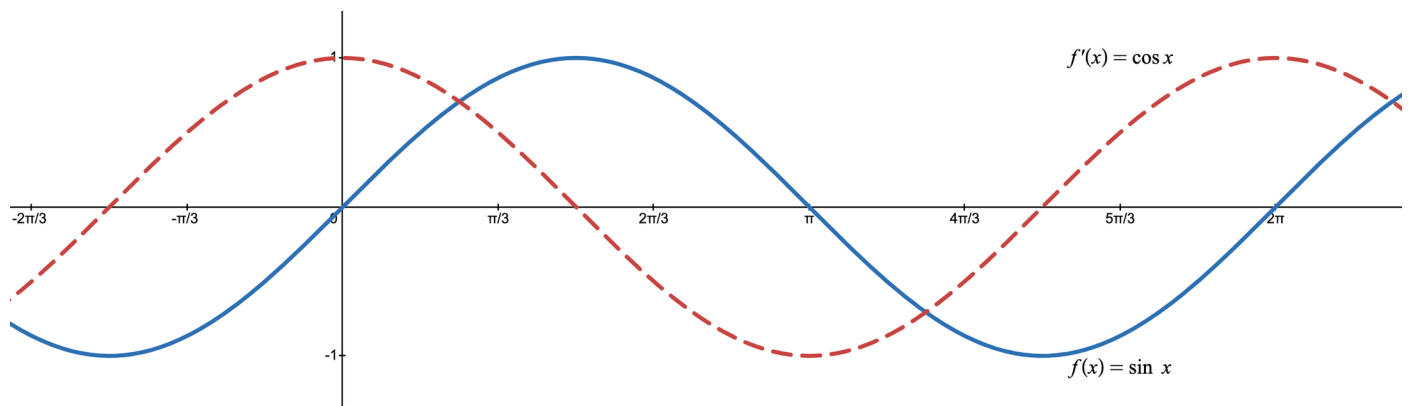


Figure 2.5.9: The sine and cosine functions

### Derivatives of Trigonometric Functions

1.  $\frac{d}{dx}(\sin x) = \cos x$
2.  $\frac{d}{dx}(\cos x) = -\sin x$
3.  $\frac{d}{dx}(\tan x) = \sec^2 x$
4.  $\frac{d}{dx}(\csc x) = -\csc x \cot x$

5.  $\frac{d}{dx}(\sec x) = \sec x \tan x$
6.  $\frac{d}{dx}(\cot x) = -\csc^2 x$

### Example 2.5.27

Calculate the following derivatives:

1.  $f(x) = 10\sqrt{x} \sin x$
2.  $f(x) = 4 \cos x \tan x$
3.  $f(x) = \frac{\sin x}{5x^3 + 2}$
4.  $f(x) = \frac{x^3}{\tan x}$
5.  $f(x) = \sec x + \sqrt{x} \cos x$
6.  $f(x) = 3 \sin x - \csc x$

### Example 2.5.28

Find the equation of the tangent line to the graph of

$$f(x) = 10\sqrt{x} \sin x$$

at  $x = \pi$ .

### Example 2.5.29

A particle moves along a coordinate axis in such a way that its position at time  $t$  is given by:

$$s(t) = 2 - \sin t$$

Find  $v\left(\frac{\pi}{3}\right)$  and  $a\left(\frac{\pi}{3}\right)$ . Decide whether the particle is speeding up or slowing down at that time.

## Higher-Order Derivatives of $\sin x$ and $\cos x$

$$\begin{aligned}
 \frac{d^4}{dx^4}(\sin x) &= \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \cdots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x \\
 \frac{d^5}{dx^5}(\sin x) &= \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \cdots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x \\
 \frac{d^6}{dx^6}(\sin x) &= \frac{d^{10}}{dx^{10}}(\sin x) = \frac{d^{14}}{dx^{14}}(\sin x) = \cdots = \frac{d^{4n+2}}{dx^{4n+2}}(\sin x) = -\sin x \\
 \frac{d^7}{dx^7}(\sin x) &= \frac{d^{11}}{dx^{11}}(\sin x) = \frac{d^{15}}{dx^{15}}(\sin x) = \cdots = \frac{d^{4n+3}}{dx^{4n+3}}(\sin x) = -\cos x
 \end{aligned}
 \tag{2.5.4}$$

### Example 2.5.30

Calculate the following derivatives:

1.  $\frac{d^4}{dx^4}(\cos x)$ . What pattern do you observe for repeated derivatives of  $\cos x$ ?
2.  $\frac{d^{74}}{dx^{74}}(\sin x)$

3.  $\frac{d^{59}}{dx^{59}}(\sin x)$

## 2.6: The Chain Rule

### 2.6 The Chain Rule

#### Learning Objectives

- Given a composite function, identify the inner and outer functions, and describe how they relate to the overall structure of the function.
- Use the chain rule in conjunction with the power, product, and quotient rules to calculate the derivative of composite functions, applying the appropriate rules in the correct order.

#### Definition 2.6.11

Let  $h(x)$  be the composite function  $h(x) = f(g(x))$ . The function  $g$  is applied first to the input  $x$ , and is referred to as the *inner* function.

The function  $f$  is applied to the result  $g(x)$ , and is referred to as the *outer* function.

#### Example 2.6.31

For each composite function below, identify the inner and outer functions.

- $h(x) = (x^2 + 5)^{10}$
- $h(x) = \sqrt{x^3 + 4x + 1}$
- $h(x) = \sin(x^7)$
- $h(x) = \cos^3(x)$

#### The Chain Rule

The Chain Rule is a method to calculate the derivative of composite functions:

Suppose that  $g$  is differentiable at  $a$  and  $f$  is differentiable at  $g(a)$ .

Then the composite function  $h = f \circ g$ , defined by  $h(x) = f(g(x))$ , is differentiable at  $a$ , and

$$h'(a) = f'(g(a)) \cdot g'(a) \quad (2.6.1)$$

**Problem-solving strategy to find the derivative of  $h(x) = f(g(x))$ :**

- Identify  $f(x)$  and  $g(x)$  in the composition  $h(x) = f(g(x))$
- Find  $f'(x)$ , then evaluate it at  $g(x)$  to obtain  $f'(g(x))$
- Find  $g'(x)$
- Write the derivative:  $h'(x) = f'(g(x)) \cdot g'(x)$

#### Example 2.6.32

Calculate the derivative of each function:

- $h(x) = (x^2 + 5)^{10}$
- $h(x) = \sqrt{x^3 + 4x + 1}$
- $h(x) = \cos^3 x$

4.  $h(x) = \sin(x^7 + 4x^3)$
5.  $h(x) = \frac{1}{\sin^3 x}$
6.  $h(x) = (3x + 8)^4(x^2 - 3x)^2$
7.  $h(x) = \frac{3x + \tan x}{(6x^2 + 1)^5}$
8.  $h(x) = \cos^{31}(3x^5 + 2x)$

We can also use the Leibniz notation to find the derivative of  $h(x) = f(g(x))$ .

If  $y = f(u)$  and  $u = g(x)$  are both differentiable, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (2.6.2)$$

#### Example 2.6.33

Use the Leibniz notation to find the derivative of:

1.  $y = (x^2 + 5)^{10}$
2.  $y = \sin(x^7 + 4x^3)$

#### Example 2.6.34

Find an equation of the tangent line to the graph of  $f(x) = \cos(\pi x)$  at the point  $(\frac{1}{2}, 0)$ .

#### Example 2.6.35

The position function of a train is given by

$$s(t) = \frac{200}{(1+t)^3} \quad (2.6.3)$$

where  $s$  is in meters and  $t$  is in seconds.

1. Find the velocity and acceleration of the train at  $t = 6$ .
2. Is the train speeding up or slowing down at that moment?

## 2.7: Derivatives of Inverse Functions

### 2.7 Derivatives of Inverse Functions

#### Learning Objectives

- Apply the formula for the derivative of an inverse function to calculate the derivative of a given inverse function.
- Identify the derivatives of standard inverse trigonometric functions (e.g., arcsin, arccos, arctan) and apply these derivatives to compute the derivatives of functions that involve inverse trigonometric expressions.

#### Inverse functions

Recall that function  $f: A \rightarrow B$  is a rule that associates each element in the set  $A$  to one and only one element in the set  $B$ . We call  $A$  the *domain* of  $f$  and  $B$  the *codomain* of  $f$ . If there exists a function  $g: B \rightarrow A$  such that  $g(f(a)) = a$  for every possible choice of  $a$  in the set  $A$  and  $f(g(b)) = b$  for every  $b$  in the set  $B$ , then we say that  $g$  is the *inverse* of  $f$ .

We often use the notation  $f^{-1}$  (read “ $f$ -inverse”) to denote the inverse of  $f$ . The inverse function undoes the work of  $f$ . Indeed, if  $y = f(x)$ , then

$$f^{-1}(y) = f^{-1}(f(x)) = x \quad (2.7.1)$$

Thus, the equations  $y = f(x)$  and  $x = f^{-1}(y)$  say the same thing. The only difference between the two equations is one of perspective — one is solved for  $x$ , while the other is solved for  $y$ .

Here we briefly remind ourselves of some key facts about inverse functions.

#### Note

For a function  $f: A \rightarrow B$ ,

- $f$  has an inverse if and only if  $f$  is one-to-one<sup>1</sup> and onto<sup>2</sup>;
- provided  $f^{-1}$  exists, the domain of  $f^{-1}$  is the codomain of  $f$ , and the codomain of  $f^{-1}$  is the domain of  $f$ ;
- $f^{-1}(f(x)) = x$  for every  $x$  in the domain of  $f$  and  $f(f^{-1}(y)) = y$  for every  $y$  in the codomain of  $f$ ;
- $y = f(x)$  if and only if  $x = f^{-1}(y)$ .

A function  $f$  is *one-to-one* provided that no two distinct inputs lead to the same output.

A function  $f$  is *onto* provided that every possible element of the codomain can be realized as an output of the function for some choice of input from the domain.

The last fact reveals a special relationship between the graphs of  $f$  and  $f^{-1}$ . If a point  $(x, y)$  that lies on the graph of  $y = f(x)$ , then it is also true that  $x = f^{-1}(y)$ , which means that the point  $(y, x)$  lies on the graph of  $f^{-1}$ . This shows us that the graphs of  $f$  and  $f^{-1}$  are the reflections of each other across the line  $y = x$ , because this reflection is precisely the geometric action that swaps the coordinates in an ordered pair. In Figure 2.6.2, we see this illustrated by the function  $y = f(x) = 2^x$  and its inverse, with the points  $(-1, \frac{1}{2})$  and  $(\frac{1}{2}, -1)$  highlighting the reflection of the curves across  $y = x$ .

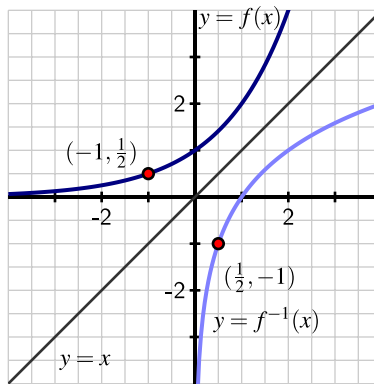


Figure 2.7.10. A graph of a function  $y = f(x)$  along with its inverse,  $y = f^{-1}(x)$ . From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

### Theorem 2.7.2

Let  $f(x)$  be a function that is both invertible and differentiable. Let  $y = f^{-1}(x)$  be the inverse of  $f(x)$ . Then:

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad (2.7.2)$$

This is true for all  $x$  such that  $f'(f^{-1}(x)) \neq 0$ .

### Example 2.7.36

Let  $f(x) = x^2$ . Calculate  $(f^{-1})'(4)$ .

### Example 2.7.37

Let  $f(x) = \frac{1}{2}x^3 + x - 1$ . Calculate  $(f^{-1})'(5)$ .

## Inverse Trigonometric Functions

The function  $f(x) = \sin x$ , defined on the interval  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , is one-to-one.

Thus, the inverse exists and is denoted by  $\sin^{-1}$  or  $\arcsin$ . This function is called the *inverse sine function* or the *arcsine function*:

$$\sin^{-1} x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad (2.7.3)$$

The function  $f(x) = \cos x$ , defined on the interval  $0 \leq x \leq \pi$ , is also one-to-one. The *inverse cosine function*, denoted by  $\cos^{-1}$  or the *arccosine function*, is given by:

$$\cos^{-1} x = y \iff \cos y = x \quad \text{and} \quad 0 \leq y \leq \pi \quad (2.7.4)$$

The tangent function is one-to-one on the interval  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ . The *inverse tangent function*, denoted by  $\tan^{-1}$  or *arctangent function*, is defined by:

$$\tan^{-1} x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \quad (2.7.5)$$

The graph of  $\tan^{-1}(x)$  has horizontal asymptotes at  $y = -\frac{\pi}{2}$  and  $y = \frac{\pi}{2}$

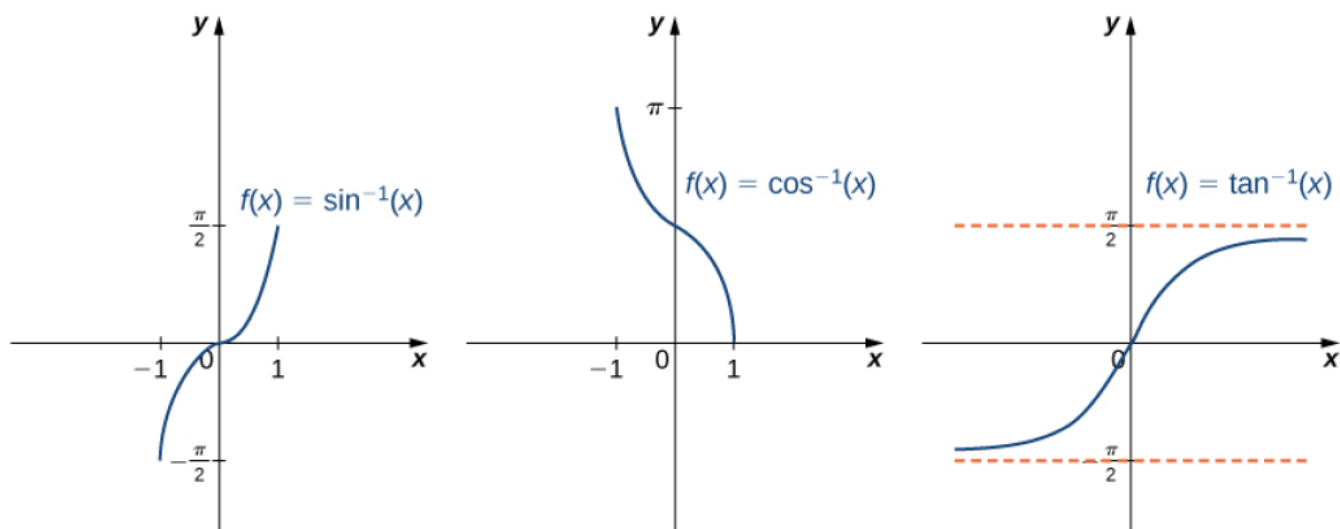


Figure 2.7.11. Graphs of the arcsine, arccosine, and arctangent functions. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

List of the derivatives of the main inverse trigonometric functions:

1.  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2.  $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
3.  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

#### Example 2.7.38

Compute the derivative of each function:

1.  $y = \tan^{-1}(x^3 + 6)$
2.  $y = \sin^{-1}(7x)$
3.  $y = x^3 \sin^{-1}(7x)$
4.  $y = \cos^{-1}(x^2 + 5x)$
5.  $y = \tan^{-1}(\sqrt{x})$
6.  $y = (1 + \cos^{-1} x)^2$

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## 2.8: Implicit Differentiation

### 2.8 Implicit Differentiation

#### Learning Objectives

- Apply implicit differentiation to find the derivative of an implicit equation.
- Use implicit differentiation to calculate the derivative of an implicit function at a given point and apply it to determine the equation of the tangent line at a given point.

Usually, we define  $y$  as an explicit function of  $x$ . For example:  $y = \sqrt{4 - x^2}$

However, the equation

$$x^2 + y^2 = 4 \quad (2.8.1)$$

defines  $y$  implicitly as a function of  $x$ .

The graph of the equation can be broken into two pieces when we solve for  $y$ :

$$y = \pm \sqrt{4 - x^2} \quad (2.8.2)$$

What about the derivative of a function given implicitly? First, let's analyze the following example:

Suppose that the explicit formula of  $y = f(x)$  is unknown and that  $f$  is differentiable.

Calculate:

$$\frac{d}{dx} [x + f(x)^2] = \frac{d}{dx} [x + y^2] \quad (2.8.3)$$

We differentiate both sides with respect to  $x$ :

$$\frac{d}{dx}(x) + \frac{d}{dx}(y^2) = 1 + 2y \cdot \frac{dy}{dx} \quad (2.8.4)$$

So the result is:

$$\frac{d}{dx} [x + y^2] = 1 + 2y \cdot \frac{dy}{dx} \quad (2.8.5)$$

#### Problem-Solving Strategy:

Suppose that we define a function  $y$  implicitly in terms of  $x$ . To calculate the derivative of  $y$  using implicit differentiation, follow these steps:

1. Take the derivative of both sides of the equation with respect to  $x$ . For example:

$$\frac{d}{dx}(\cos x) = -\sin x \quad \text{but} \quad \frac{d}{dx}(\cos y) = -\sin y \cdot \frac{dy}{dx} \quad (2.8.6)$$

2. Rewrite the resulting equation so that all terms involving  $\frac{dy}{dx}$  are on the left-hand side, and all other terms are on the right-hand side.
3. Factor out  $\frac{dy}{dx}$  on the left-hand side.
4. Solve the equation for  $\frac{dy}{dx}$ .

### Example 2.8.39

Compute the derivative  $\frac{dy}{dx}$  of the following equations (assume that  $y$  is a function of  $x$ ):

1.  $x^2 + y^2 = 4$
2.  $x^3 + y^2 - 3xy = 3$
3.  $x^3 \sin y = x^4$
4.  $\arctan(x^3 y) = xy^3$
5.  $\sqrt{x+y} = 1 + x^2 y^2$

### Example 2.8.40

Find the equations of the tangent lines to the curve  $x^2 + y^2 = 4$  at the following points:

- (a)  $(0, 2)$
- (b)  $(-2, 0)$
- (c)  $(1, \sqrt{3})$

### Example 2.8.41

Find the equation of the tangent line to the curve  $\frac{x^2}{9} + \frac{y^2}{36} = 1$  at the point  $(-2, -2\sqrt{5})$ .

### Example 2.8.42

Find the equation of the tangent line to the curve  $x^3 + y^2 - 3xy = 3$  at the point  $(-1, 1)$ .

## 2.9: Derivatives of Exponential and Logarithmic Functions

### 2.9 Derivatives of Exponential and Logarithmic Functions

#### Learning Objectives

- Apply differentiation rules to calculate the derivatives of functions involving exponential and logarithmic functions.
- Use derivatives to compare the rate of growth or decay of a function at specific points in time, interpreting the derivative as the instantaneous rate of change.

#### Derivative rules for exponential functions:

Let  $b > 0$ ,  $b \neq 1$ , and let  $g(x)$  be a differentiable function.

1. Let  $y = e^x$  be the natural exponential function. Then:

$$\frac{dy}{dx} = e^x \quad (2.9.1)$$

In general:

$$\frac{d}{dx} (e^{g(x)}) = e^{g(x)} \cdot g'(x) \quad (2.9.2)$$

2. Consider the general exponential function  $y = b^x$ . Then:

$$\frac{dy}{dx} = \ln b \cdot b^x \quad (2.9.3)$$

In general:

$$\frac{d}{dx} (b^{g(x)}) = \ln b \cdot b^{g(x)} \cdot g'(x) \quad (2.9.4)$$

#### Example 2.9.43

Compute the derivative of each of the following functions:

1.  $y = e^{x^4}$
2.  $y = 5xe^{x^4}$
3.  $y = \frac{e^{x^4}}{5x}$
4.  $y = 2^{\tan x}$
5.  $e^{x^3}y = 2x - 7y$  Here  $y$  is an implicit function of  $x$

Let  $b > 0$ ,  $b \neq 1$ , and let  $g(x)$  be a differentiable function.

1. Let  $y = \ln x$  be the natural logarithmic function. Then:

$$y' = \frac{1}{x} \quad (2.9.5)$$

In general, if  $y = \ln(g(x))$ , then:

$$\frac{d}{dx} [\ln(g(x))] = \frac{1}{g(x)} \cdot g'(x) \quad (2.9.6)$$

for all values of  $x$  such that  $g(x) > 0$ .

2. Consider the general logarithmic function  $y = \log_b x$ . Then:

$$y' = \frac{1}{\ln b \cdot x} \quad (2.9.7)$$

In general, if  $y = \log_b(g(x))$ , then:

$$\frac{d}{dx} [\log_b(g(x))] = \frac{1}{\ln b \cdot g(x)} \cdot g'(x) \quad (2.9.8)$$

for all values of  $x$  such that  $g(x) > 0$ .

Recall that you can use logarithmic properties to simplify expressions before differentiating. For example:

$$y = \ln((x^2 + 1)^{10}) = 10 \ln(x^2 + 1) \quad (2.9.9)$$

#### Example 2.9.44

Compute the derivative of each of the following functions:

1.  $y = \ln((x^3 + 5x + 2)^6)$
2.  $y = \sin x \cdot \ln(x^3 - 1)$
3.  $y = \ln\left(\frac{\sin x}{x^3 - 1}\right)$
4.  $y = \log_3(x^7)$
5.  $y = \ln(2x^2 + 6y^2)$  \ \ Here  $y$  is an implicit function of  $x$

#### Example 2.9.45

Find the equation of the tangent line to the graph of  $y = 10 + 12x - 13e^x$  at the point  $(0, -3)$ .

#### Example 2.9.46

The temperature (in degrees Fahrenheit) of a heated metal object is modeled by the function

$$T(t) = 70 + 120e^{-0.2t^2} \quad (2.9.10)$$

where  $t$  is the time in minutes after the object is removed from the heat source.

1. Find  $T'(t)$ , the rate at which the temperature is changing at time  $t$ .
2. Evaluate  $T'(3)$  and interpret its meaning in the context of the cooling process.
3. Is the object cooling down faster at  $t = 1$  minute or at  $t = 3$  minutes? Explain using the derivative.

#### Example 2.9.47

The intensity of sound (in decibels) received by a microphone is modeled by the function:

$$I(d) = 10 \ln\left(\frac{500}{d^2 + 1}\right) \quad (2.9.11)$$

where  $d$  is the distance (in meters) from the sound source.

1. Find  $I'(d)$ , the rate at which the intensity changes with respect to distance.

2. Evaluate  $I'(2)$ . Interpret this value in the context of how the sound intensity is changing at 2 meters from the source.
3. Is the intensity decreasing more rapidly when the microphone is closer to or farther from the source? Support your answer using the derivative.

## 3: APPLICATIONS OF DERIVATIVES

### Overview

Derivatives provide powerful tools to analyze and understand the behavior of functions beyond instantaneous change. They allow us to approximate functions, determine growth and decay, optimize real-world scenarios, and study the shape of curves. In this chapter, we apply differentiation techniques to solve a wide variety of problems in mathematics, science, and engineering.

Applications of derivatives include analyzing related rates of change, approximating functions with tangent lines, and studying function behavior through the mean value theorem and extrema. Derivatives also connect to graphing by describing monotonicity, concavity, and asymptotes. Further, we apply derivatives to optimization, limits, and numerical methods for solving equations. Finally, we revisit antiderivatives as the natural inverse process to differentiation.

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### Topics in this Chapter

1. **Related Rates**
    - Solve problems involving two or more related quantities that change with respect to time.
    - Translate real-world motion and geometry into equations involving derivatives.
  2. **Linear Approximation**
    - Use tangent lines to approximate function values near a point.
    - Understand differentials and estimation errors.
  3. **The Mean Value Theorem**
    - State and apply the Mean Value Theorem (MVT).
    - Connect average rate of change to instantaneous rate of change.
    - Explore consequences of MVT in calculus and applied contexts.
  4. **Maxima and Minima**
    - Identify critical points using the derivative.
    - Use the first and second derivative tests to classify local extrema.
    - Apply these ideas to real-world optimization problems.
  5. **Relationship Between Derivatives and the Shape of a Graph**
    - Use the first derivative to determine intervals of increase and decrease.
    - Use the second derivative to determine concavity and inflection points.
    - Build complete graph sketches using derivative information.
  6. **Limits at Infinity and Horizontal Asymptotes**
    - Evaluate limits of functions as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ .
    - Identify horizontal asymptotes from limits.
  7. **Applied Optimization Problems**
    - Formulate optimization models in physics, economics, geometry, and engineering.
    - Use derivatives to find and justify maximum or minimum values.
  8. **L'Hôpital's Rule**
    - Evaluate indeterminate forms such as  $\frac{0}{0}$  and  $\frac{\infty}{\infty}$ .
    - Apply L'Hôpital's Rule to compute limits in applied contexts.
  9. **Newton's Method**
    - Use iterative procedures to approximate roots of equations.
    - Analyze convergence and limitations of Newton's Method.
  10. **Antiderivatives**
    - Introduce the reverse process of differentiation.
    - Compute basic antiderivatives and use them to solve simple differential equations.
-

## Why This Matters

Applications of derivatives extend far beyond computation. They allow us to:

- Model dynamic change in physics, biology, and economics.
- Approximate and predict values in complex systems.
- Optimize resources in engineering and business contexts.
- Understand the geometry and shape of graphs.
- Provide numerical methods when exact algebraic solutions are not possible.

Derivatives not only describe instantaneous change, but also open the door to deeper mathematical insights and practical problem-solving tools.

[3.1: Related Rates](#)

[3.2: Linear Approximation](#)

[3.3: The Mean Value Theorem](#)

[3.4: Maxima and Minima](#)

[3.5: Relationship between Derivatives and the Shape of a Graph](#)

[3.6: Limits at Infinity, Horizontal Asymptotes, and How to Use Calculus to Sketch the Graph of a Function](#)

[3.7: Applied Optimization Problems](#)

[3.8: L'Hôpital's Rule](#)

[3.9: Newton's Method](#)

[3.10: Antiderivatives](#)

## 3.1: Related Rates

### 3.1 Related Rates

#### Learning Objectives

- Analyze real-world problems to identify the relationships between changes in quantities, and express these relationships mathematically using equations or formulas.
- Set up and solve real-world problems involving related rates of change by differentiating relevant equations with respect to time and applying the chain rule to find rates of change for different quantities.

**Task:** A soccer ball is being inflated at a constant rate of  $4 \text{ cm}^3/\text{s}$ . Assume the ball is a perfect sphere with a diameter of 22 cm. You are asked to determine how the radius and the surface area of the ball are changing over time as it is being inflated.

Answer the following questions:

1. Draw several spheres with different radii. What quantities change as the volume changes?
2. Recall that the volume of a sphere of radius  $r$  is given by  $V = \frac{4}{3}\pi r^3$ , where  $V$  and  $r$  could be seen as implicit functions of time  $t$ . Find an equation that relates the rates of change of  $V$  and  $r$  with respect to time.
3. At what rate is the radius of the soccer ball changing when the radius is 5 cm?
4. At what rate is the surface area of the ball changing at that moment? Recall that the surface area of the sphere is given by  $S = 4\pi r^2$ .

#### Problem-Solving Strategy (Related Rates)

1. Draw a figure representing the situation and identify all relevant variables.
2. Identify the rate(s) given in the problem and the rate to be determined.
3. Use known mathematical formulas from the context of the problem to write an equation that relates the variables.
4. Differentiate both sides of the equation with respect to the independent variable (usually time  $t$ ) to obtain an equation that relates the rates of change (derivatives). You may use implicit differentiation if necessary.
5. Substitute the known values into the differentiated equation and solve for the unknown rate of change.

#### Example 3.1.20

An airplane is flying overhead at a constant elevation of 5,000 ft. A person is viewing the plane from a position 4,000 ft from the base of a radio tower. The airplane is flying horizontally away from the person. If the plane is flying at the rate of 800 ft/sec, at what rate is the distance between the person and the plane increasing when the plane passes over the radio tower?

#### Example 3.1.21

A rocket is launched so that it rises vertically. A camera is positioned 5,000 ft from the launch pad. When the rocket is 1,000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

#### Example 3.1.22

A 15-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 3 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 9 ft from the wall?

### Example 3.1.23

A water tank has the shape of an inverted circular cone with a base radius of 6 feet and a depth of 8 feet. Suppose water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute. At what rate is the water level rising when the water depth is 4 feet?

### Example 3.1.24

Two airplanes are flying at the same altitude:

- Airplane A is flying east at 450 mi/h and is currently 80 miles west of an airport.
- Airplane B is flying north at 500 mi/h and is currently 100 miles south of the same airport.

At what rate is the distance between the two airplanes changing at this moment?

### Example 3.1.25

A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?

## 3.2: Linear Approximation

### 3.2 Linear Approximation

#### Learning Objectives

- Explain the concept of linear approximation (tangent line approximation) to a function at a given point and use it to estimate values of the function near that point.

#### Definition 3.2.1

Let  $f$  be a differentiable function at  $x = a$ . The equation of the tangent line to  $f$  at  $x = a$  is:

$$y - f(a) = f'(a)(x - a) \quad (3.2.1)$$

We can use this linear function to approximate  $f(x)$  for values of  $x$  near  $a$ . This gives the approximation:

$$f(x) \approx f(a) + f'(a)(x - a) \quad (3.2.2)$$

The linear approximation, or tangent line approximation, of  $f$  at  $x = a$  is the function:

$$L(x) = f(a) + f'(a)(x - a) \quad (3.2.3)$$

The function  $L$  is also called the linearization of  $f$  at  $x = a$ .

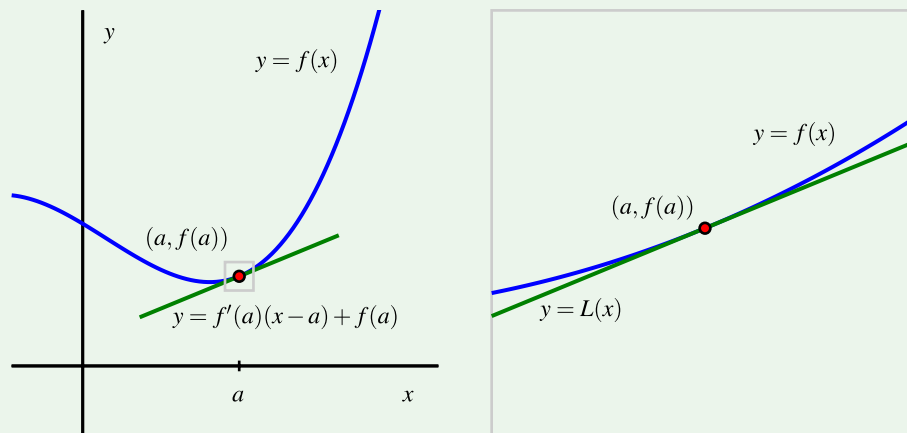


Figure 3.2.1. A function  $y = f(x)$  and its tangent line at the point  $(a, f(a))$ : From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

#### Example 3.2.1

Use the linear approximation of  $f(x) = \sqrt{x}$  at  $x = 4$  to approximate  $\sqrt{4.01}$ .

**Example 3.2.2**

Use the linear approximation of  $f(x) = \cos x$  at  $x = \frac{\pi}{6}$  (which is  $30^\circ$ ) to approximate  $\cos(31^\circ)$ .

**Example 3.2.3**

Use the linear approximation of  $f(x) = (1 + x)^n$  at  $x = 1$  to approximate  $(2.001)^5$ .

## 3.3: The Mean Value Theorem

### 3.3 The Mean Value Theorem

#### Learning Objectives

- Use the Mean Value Theorem to solve real-world problems by applying the theorem to find the average rate of change of a function over an interval and relating it to the instantaneous rate of change at a specific point.

#### Theorem 3.3.1. Rolle's Theorem

Suppose that  $f$  is a function that satisfies the following conditions:

- $f$  is continuous on the closed interval  $[a, b]$
- $f$  is differentiable on the open interval  $(a, b)$
- $f(a) = f(b)$

Then there exists a number  $c$  in  $(a, b)$  such that:  $f'(c) = 0$

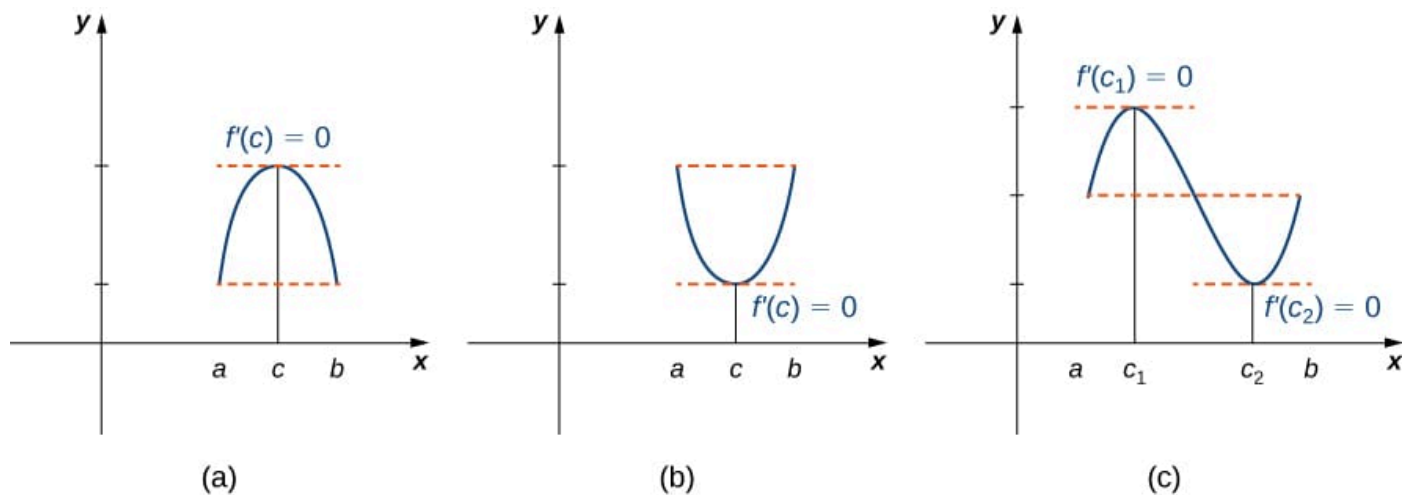


Figure 3.3.2. Different functions satisfying Rolle's Theorem. From [OpenStax Calculus Volume 1](#), licensed under CC-BY-NC-SA.

#### Example 3.3.4

Let  $f(x) = x^2 - 2x - 3$ , defined on the interval  $[-1, 3]$ . Verify that  $f$  satisfies the hypotheses of Rolle's Theorem and find all values  $c$  in  $(-1, 3)$  such that  $f'(c) = 0$ .

#### Theorem 3.3.2. The Mean Value Theorem

Suppose that  $f$  is a function that satisfies the following conditions:

- $f$  is continuous on the closed interval  $[a, b]$
- $f$  is differentiable on the open interval  $(a, b)$

Then there exists a number  $c^{\text{int}}(a, b)$  such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a) \quad (3.3.1)$$

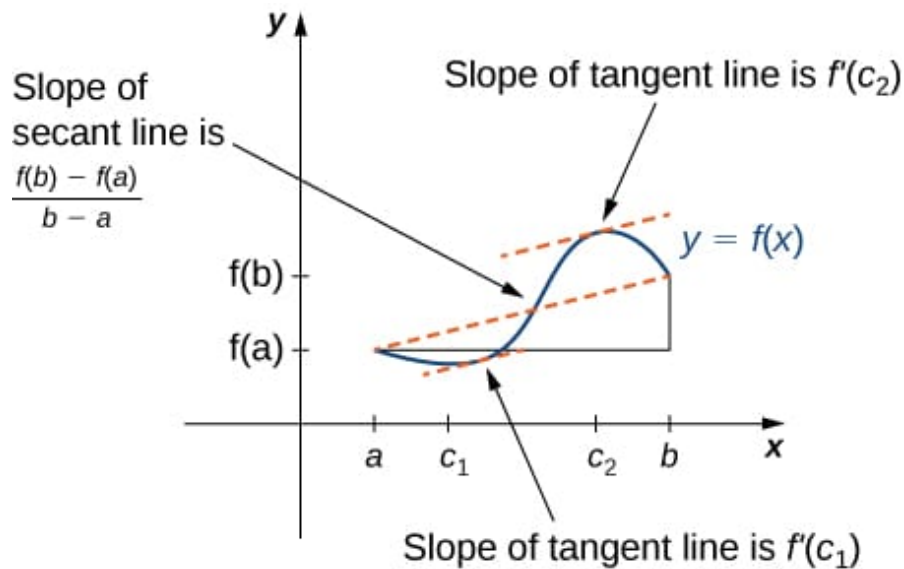


Figure 3.3.3. A function illustrating the Mean Value Theorem. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

Task: Describe the Mean Value Theorem with your own words.

### Example 3.3.5

If a rock is dropped from a height of 144 ft, its position  $s(t)$  in feet after  $t$  seconds is given by:  $s(t) = -16t^2 + 144$

1. Determine how long it takes before the rock hits the ground.
2. Find the average velocity  $v_{\text{avg}}$  of the rock from the moment it is released until it hits the ground.
3. Find the time  $t$  guaranteed by the Mean Value Theorem when the instantaneous velocity equals  $v_{\text{avg}}$ .

## 3.4: Maxima and Minima

### 3.4 Maxima and Minima

#### Learning Objectives

- Explain the difference between absolute and local extrema, identifying how they relate to the global behavior and local behavior of a function.
- Determine the critical points of a function over a closed interval and use them, along with the endpoints of the interval, to identify absolute and local extreme values.

#### Definition 3.4.3

Let  $c$  be a number in the domain  $D$  of a function  $f$ . Then:

- $f(c)$  is the absolute maximum value of  $f$  on  $D$  if  $f(c) \geq f(x)$  for all  $x \in D$
- $f(c)$  is the absolute minimum value of  $f$  on  $D$  if  $f(c) \leq f(x)$  for all  $x \in D$

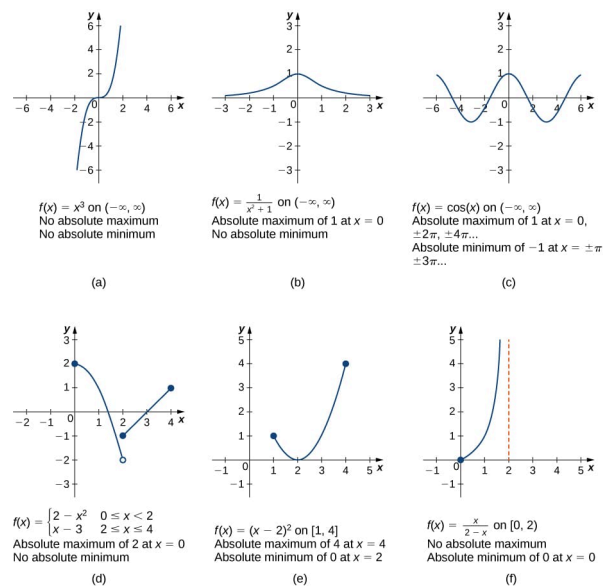


Figure 3.4.4: Graphs showing several possibilities for absolute extrema for functions with a domain that is a bounded interval. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

#### Example 3.4.6

Use the graph of  $y = f(x)$  below to find the absolute minimum and maximum values (if any).

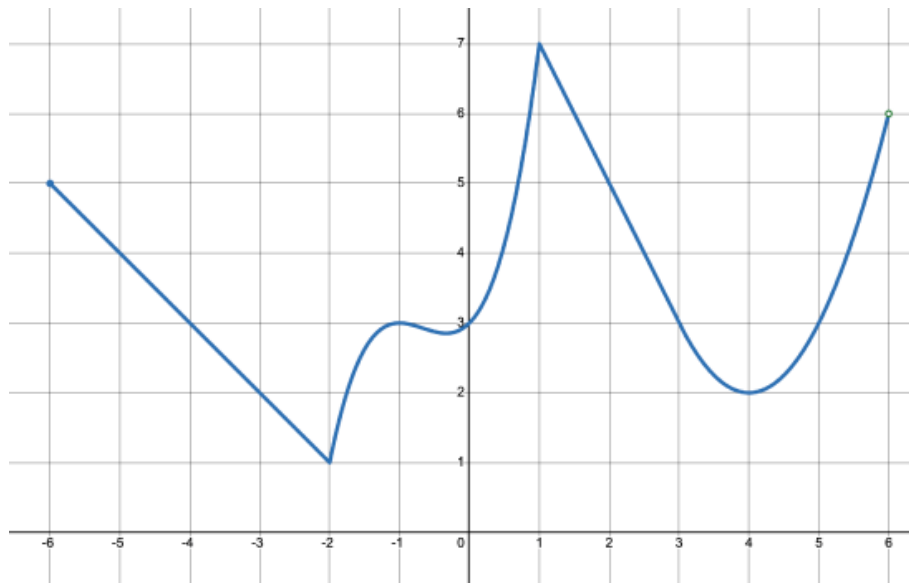


Figure 3.4.5. A function to find absolute maximum and minimum values

### Theorem 3.4.3 The Extreme Value Theorem

Suppose that  $f$  is continuous on the closed interval  $[a, b]$ . Then  $f$  attains both an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c$  and  $d$  in  $[a, b]$ .

### Definition 3.4.4

Let  $c$  be a number in the domain  $D$  of a function  $f$ . Then:

- $f(c)$  is a local maximum value of  $f$  if  $f(c) \geq f(x)$  for all  $x$  near  $c$
- $f(c)$  is a local minimum value of  $f$  if  $f(c) \leq f(x)$  for all  $x$  near  $c$

The maximum and minimum values of  $f$  are called **local extreme values** of  $f$ .

### Example 3.4.7

Find local extreme values from the graph above (Figure 3.5)

### Definition 3.4.5

A critical number of a function  $f$  is a number  $c$  in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist. The point  $(c, f(c))$  is called a **critical point** of  $f$ .

### Theorem 3.4.4 Fermat's Theorem

Suppose that  $f$  has a local maximum or minimum at  $c$ . If  $f'(c)$  exists, then  $f'(c) = 0$

## Note

If  $f$  has a local maximum or minimum at  $c$ , then  $c$  is a critical number of  $f$ .

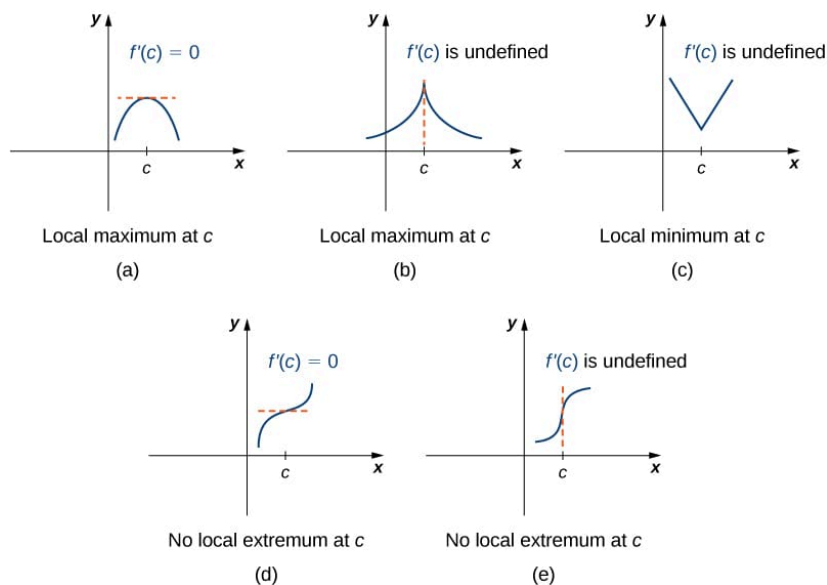


Figure 3.4.6: Functions illustrating possible local extreme values. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

## Example 3.4.8

For the functions below, find all critical numbers.

1.  $y = x^3 - 6x^2 + 9x + 5$
2.  $y = 2(x^2 - 4)^3$
3.  $y = (4x - 4)e^{-2x}$
4.  $y = \frac{x + 5}{x^2 - 9}$

## The Closed Interval Method:

If we want to find the absolute maximum and minimum values of a continuous function  $f$  on a closed interval  $[a, b]$ , we can follow these steps:

1. Find the critical numbers of  $f$  in the open interval  $(a, b)$ .
2. Evaluate  $f$  at each critical number found in Step 1.
3. Evaluate  $f$  at the endpoints of the interval:  $x = a$  and  $x = b$ .
4. The largest of the values from Steps 2 and 3 is the absolute maximum value. The smallest of these values is the absolute minimum value.

## Example 3.4.9

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval. State the values and where they occur.

1.  $y = x^3 - 6x^2 + 9x + 5$  over the interval  $[-1, 5]$

2.  $y = 2x - 4 \sin x$  over the interval  $[0, \pi]$
3.  $y = 13 \ln x - 14\sqrt{x} + 10$  over the interval  $[2, 9]$

## 3.5: Relationship between Derivatives and the Shape of a Graph

### 3.5 Relationship between Derivatives and the Shape of a Graph

#### Learning Objectives

- Use the first derivative of a function to determine the intervals where the function is increasing or decreasing, based on the sign of the derivative.
- Apply the first derivative test at critical points to determine whether a function has local extreme values (maximum or minimum) and identify their locations.
- Use the second derivative of a function to determine the intervals where the function is concave up or concave down, based on the sign of the second derivative.
- Apply the second derivative test at critical points to determine whether the function has local extrema and classify them as maxima, minima, or points of inflection.

#### Increasing–Decreasing Test:

- If  $f'(x) > 0$  on an interval, then  $f$  is increasing on that interval.
- If  $f'(x) < 0$  on an interval, then  $f$  is decreasing on that interval.

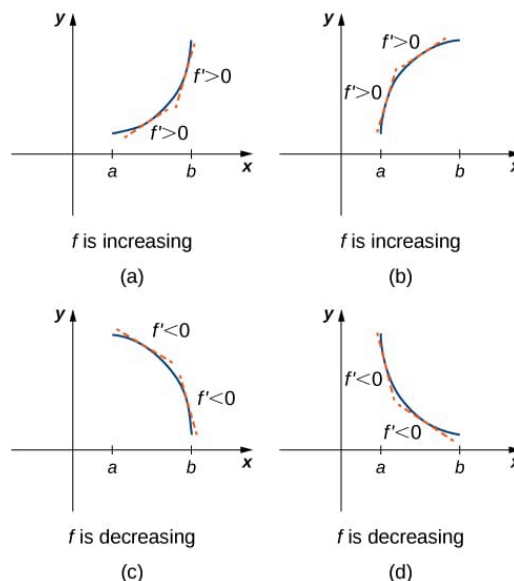


Figure 3.5.7: Functions describing when a function is decreasing or increasing. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

#### The First Derivative Test:

Let  $f$  be a continuous function on an interval  $I$  containing a critical point  $c$  of  $f$ . If  $f$  is differentiable on  $I$ , except possibly at  $c$ , then:

1. If  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local maximum at  $c$ . That is,  $f(c)$  is a local maximum value of  $f$ .
2. If  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local minimum at  $c$ . That is,  $f(c)$  is a local minimum value of  $f$ .

3. If  $f'$  does not change sign at  $c$  (i.e., remains positive or negative on both sides), then  $f$  has no local maximum or minimum at  $c$ .

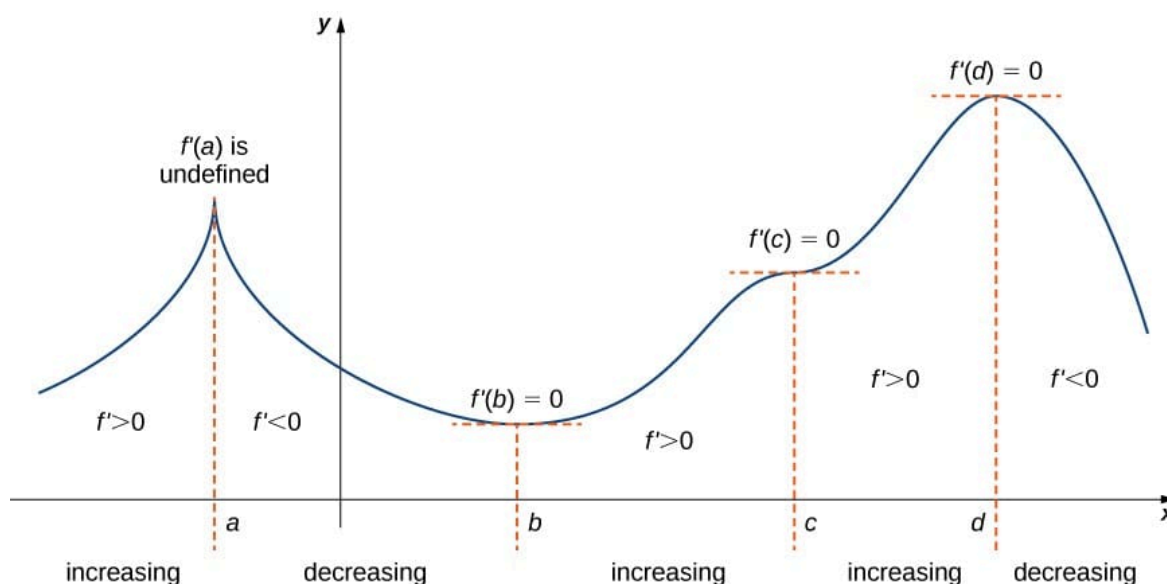


Figure 3.5.8: Several applications of the First Derivative Test to determine if the critical numbers of the function are local extrema. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

**Problem-Solving Strategy:** Use the First Derivative Test to find local extreme values.

Suppose that  $f$  is continuous over an interval  $I$ . Follow these steps to find local extrema:

1. Find all critical points of  $f$ , and divide the interval  $I$  into smaller subintervals using the critical points as endpoints.
2. Analyze the sign of  $f'$  in each subinterval. Usually,  $f'$  is continuous over each subinterval, so its sign does not change within that subinterval. You can determine the sign of  $f'$  by choosing a test value  $x$  in the subinterval and evaluating  $f'(x)$ .
3. Use the sign of  $f'$  to determine whether  $f$  is increasing or decreasing over each subinterval.
4. Apply the First Derivative Test using the results from Step 3 to determine whether  $f$  has a local maximum, a local minimum, or neither at each critical point.

### Example 3.5.10

Use the First Derivative Test to find the local extrema of the following functions, if any.

1.  $y = x^3 - 6x^2 + 9x + 5$
2.  $y = 2x^4 + 8x^3 + 9$
3.  $y = 2(x^2 - 4)^3$
4.  $y = 3x^2e^x$
5.  $y = \frac{x^2 - 2x + 1}{x + 1}$

### Definition 3.5.6

Let  $f$  be a function that is differentiable over an open interval  $I$ .

- If  $f'$  is increasing over  $I$ , then  $f$  is said to be concave up on  $I$ .
- If  $f'$  is decreasing over  $I$ , then  $f$  is said to be concave down on  $I$ .

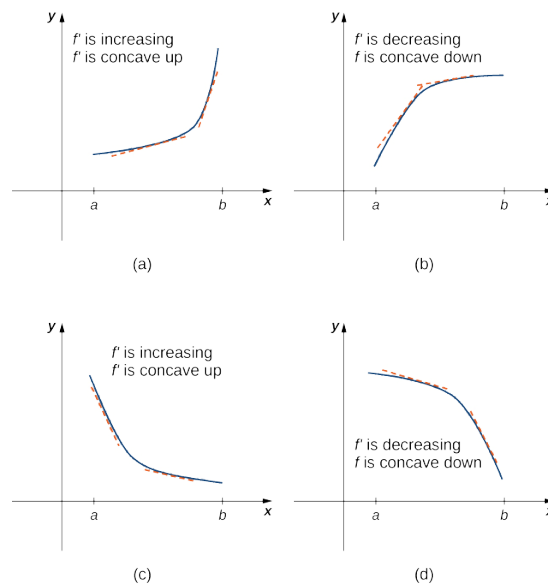


Figure 3.5.9: Functions describing when a function is concave up or concave down. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

## Concavity Test:

- If  $f''(x) > 0$  for all  $x$  in an interval  $I$ , then the graph of  $f$  is concave up on  $I$ .
- If  $f''(x) < 0$  for all  $x$  in an interval  $I$ , then the graph of  $f$  is concave down on  $I$ .

### Definition 3.5.7

Let  $f$  be a function continuous at  $x = a$ . If  $f$  changes concavity at  $a$ , then the point  $(a, f(a))$  on the graph of  $f$  is called an inflection point. This means that  $(a, f(a))$  is an inflection point of  $f$  if  $f''(a) = 0$  or  $f''(a)$  is undefined, and the concavity of  $f$  changes at that point.

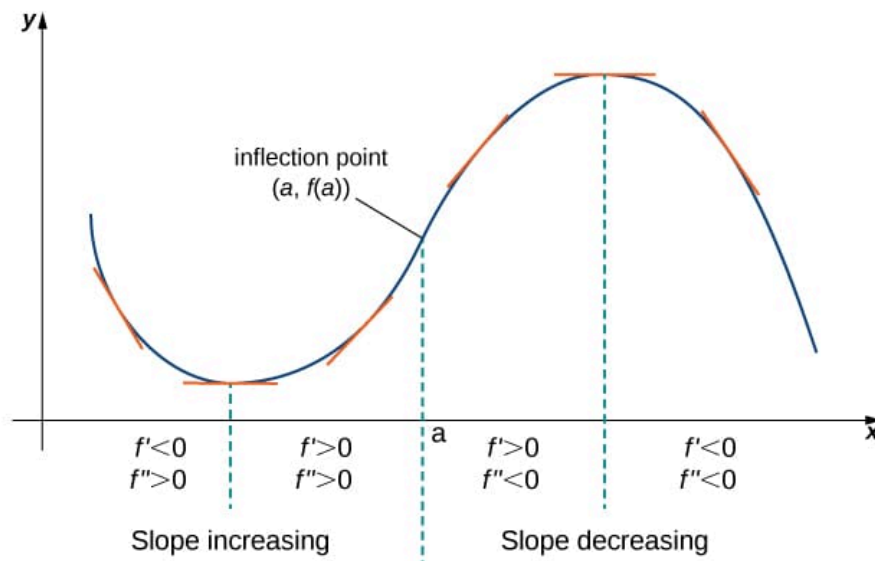


Figure 3.5.10: A function illustrating an inflection point. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

### Example 3.5.11

For each of the following functions, determine:

- The intervals where the function is concave up.
- The intervals where the function is concave down.
- All inflection points of the function.

1.  $y = x^4 - 4x^3 + 5$
2.  $y = \frac{3}{20}x^5 - \frac{3}{2}x^4 + 4x^3 + 3$
3.  $y = \frac{2x+8}{3x+3}$

## The Second Derivative Test:

Suppose that  $f''$  is continuous near  $c$ .

- If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $c$ .
- If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $c$ .
- If  $f'(c) = 0$  and  $f''(c) = 0$ , then the test is inconclusive.

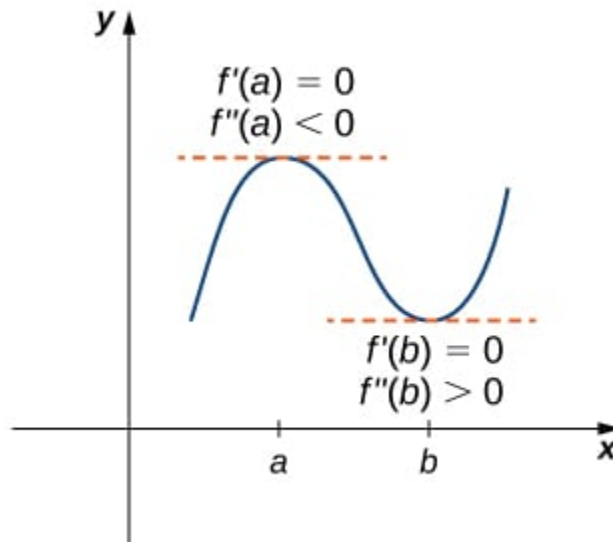


Figure 3.5.11: A function illustrating the Second Derivative Test. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

### Example 3.5.12

Use the second derivative test to find the local extrema of the functions from Example 3.5.10.

## 3.6: Limits at Infinity, Horizontal Asymptotes, and How to Use Calculus to Sketch the Graph of a Function

### 3.5 Limits at Infinity, Horizontal Asymptotes, and How to Use Calculus to Sketch the Graph of a Function

#### Learning Objectives

- Identify horizontal asymptotes from the graph of a function by observing the behavior of the function as  $x$  approaches positive or negative infinity.
- Calculate the limits of a function as  $x$  approaches infinity or negative infinity to determine the horizontal asymptotes of the function.
- Use the first and second derivatives of a function to analyze its increasing/decreasing behavior, concavity, and identify key features (such as critical points and inflection points), and use this information to sketch its graph accurately.

#### Definition 3.6.8

Let  $y = f(x)$  be a function.

The line  $y = L$  is called a horizontal asymptote of the graph of  $f(x)$  if either:

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L \quad (3.6.1)$$

These limits are called limits at infinity and are used to describe the end behavior of the function  $f(x)$ .

For example, the line  $y = 1$  is a horizontal asymptote of the graph of

$$f(x) = 1 + \frac{1}{x} \quad (3.6.2)$$

because:

$$\lim_{x \rightarrow \infty} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 1 \quad (3.6.3)$$

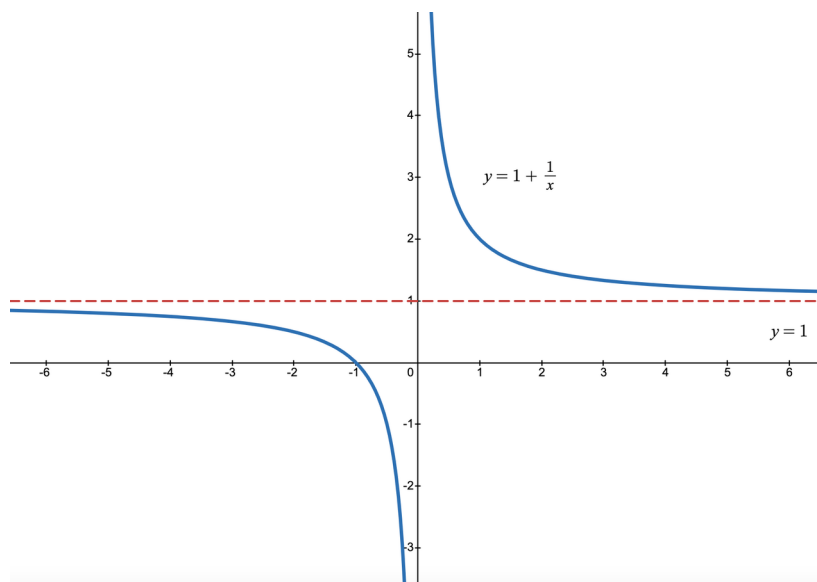


Figure 3.6.12: The graph of  $y = 1 + \frac{1}{x}$  and its horizontal asymptote  $y = 1$

### Theorem 3.6.5

Let  $r > 0$  be a rational number. If  $x^r$  is defined for all  $x$ , then:

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0 \quad (3.6.4)$$

### Theorem 3.6.6

Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  be a polynomial. Then:

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} a_n x^n \quad (3.6.5)$$

That is, the term of highest degree dominates the behavior of the polynomial as  $x \rightarrow \pm\infty$ .

### Example 3.6.13

For each of the following functions:

- Evaluate  $\lim_{x \rightarrow \infty} f(x)$
- Evaluate  $\lim_{x \rightarrow -\infty} f(x)$
- Determine the horizontal asymptotes, if any.

1.  $y = 2x^7$
2.  $y = -4x^5 + 3x^2 + 2$
3.  $y = 7 + \frac{4}{x^3}$
4.  $y = 1 - \frac{x^2}{4x+2}$
5.  $y = \frac{3x-7}{x^2-4x}$
6.  $y = \frac{x^2-4x}{6x^3-2x^2+9}$

$$7. \quad y = \frac{2x^5 + 3x^2 + 2}{3x^3 + 6}$$

$$8. \quad y = \frac{\sqrt{9x^2 + 3}}{5x + 2}$$

## Problem-Solving Strategy: Sketching the Graph of a Function

Let  $f(x)$  be a function. Follow the steps below to sketch the graph of  $f(x)$  by hand:

1. Determine the domain and range of the function.
2. Intercepts: Find the  $y$ -intercept and  $x$ -intercepts (if possible or easy to compute).
3. Symmetry: Determine whether  $f(x)$  is even, odd, or neither.
4. Asymptotes: Find any horizontal and vertical asymptotes.
5. Intervals of increase or decrease: Find critical numbers. Determine the intervals where  $f'(x) > 0$  (increasing) and where  $f'(x) < 0$  (decreasing).
6. Local maximum and minimum values: Use the first or second derivative test to identify local extrema.
7. Concavity and inflection points: Use the concavity test by determining where  $f''(x) > 0$  (concave up) and  $f''(x) < 0$  (concave down). Inflection points occur where the concavity changes.
8. Sketch the graph:
  - Sketch asymptotes as dashed lines.
  - Plot the intercepts, local extrema, and inflection points.
  - Draw the curve passing through these key points.

### Example 3.6.14

Use the above problem solving strategy to sketch the graph of each of the following functions:

1.  $y = -2x^3 + 6x^2 + 1$
2.  $y = x^4 - 2x^2 + 4$
3.  $y = \frac{x^2 + 4x}{x^2 - 16}$

## 3.7: Applied Optimization Problems

### 3.7 Applied Optimization Problems

#### Learning Objectives

- Formulate and solve applied optimization problems by identifying the objective function, determining the constraints, and applying differentiation techniques to find the maximum or minimum values of the function.

**Task:** Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

Answer the following questions:

1. What is the formula for the perimeter of a rectangle? Write it in terms of the length  $l$  and width  $w$ .
2. Given that the perimeter is 100 meters, how can you express  $l$  in terms of  $w$  (or vice versa)?
3. What is the formula for the area of a rectangle in terms of  $l$  and  $w$ ?
4. Substitute the expression for  $l$  (from Question 2) into the formula for the area. What is the area  $A$  as a function of  $w$ ?
5. To maximize the area, what should you do with the function  $A(w)$ ? What method can you use to find the maximum area? Hint: Consider the derivative of  $A(w)$ .
6. What is the critical point of  $A(w)$ ? How can you determine if this critical point corresponds to a maximum?
7. Once you have the value of  $w$ , how can you find the corresponding value of  $l$ ? What are the dimensions of the rectangle that maximize the area?
8. What is the maximum area of the rectangle with a perimeter of 100 meters?

Based on the guiding questions above, let's try to solve the following problem.

#### Example 3.7.15

A rancher is planning to build a rectangular fence around his property, with one side of the property being next to a cliff. The property is 5,000 square yards, and only the three sides not adjacent to the cliff will be fenced off. What is the least amount of fencing required for this job?

### Problem-Solving Strategy (Optimization Problems)

1. Identify what the question is asking. Determine the known quantities in the problem and introduce variables to represent them.
2. Draw a diagram (if appropriate) and label all relevant variables.
3. Determine the quantity to be optimized and identify the possible range of values for that quantity, if known.
4. Write a formula for the quantity to be optimized in terms of the variables involved.
5. Write any necessary equations that relate the variables. Use the equations to rewrite the expression from step 4 as a function of a single variable.
6. Determine the domain on which to consider the function to be optimized.
7. Use methods from Section 3.5 (e.g., finding critical points and using tests for extrema) to find the maximum or minimum value of the function.

#### Example 3.7.16

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 200 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

#### Example 3.7.17

An open-top box is to be made from a 20 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

#### Example 3.7.18

A rectangular box with a square base, an open top, and a volume of  $216 \text{ in}^3$  is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

#### Example 3.7.19

A microwaveable cup-of-soup package needs to be constructed in the shape of cylinder to hold 550 cubic centimeters of soup. The sides and bottom of the container will be made of Styrofoam costing 0.04 cents per square centimeter. The top will be made of glued paper, costing 0.06 cents per square centimeter. Find the dimensions for the package that will minimize production cost.

## 3.8: L'Hôpital's Rule

### 3.8 L'Hôpital's Rule

#### Learning Objectives

- Identify indeterminate forms such as  $0/0$  or  $\infty/\infty$  in limits, and apply L'Hôpital's rule to evaluate these limits by differentiating the numerator and denominator until a determinate form is obtained.

Consider the following limit:

$$\lim_{x \rightarrow -3} \frac{x^2 + 4x + 3}{x^2 - 9} \quad (3.8.1)$$

(a) How do we evaluate the limit above?

(b) Once you identify the way to evaluate it, then evaluate the limit.

(c) Let  $f(x) = x^2 + 4x + 3$  and  $g(x) = x^2 - 9$ . Find the linearization of  $f(x)$  at  $a = -3$ , denoted  $L_f(x)$ , and the linearization of  $g(x)$  at  $a = -3$ , denoted  $L_g(x)$ .

(d) Evaluate the limit:  $\lim_{x \rightarrow -3} \frac{L_f(x)}{L_g(x)}$

(e) Compare the results from parts (b) and (d).

Now, let's suppose that  $f$  and  $g$  are differentiable functions at  $x = a$  such that:

$$\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x), \quad (3.8.2)$$

and both  $f'(x)$  and  $g'(x)$  are continuous at  $a$ , with  $g'(a) \neq 0$ . If  $x$  is near  $a$ , we can use linear approximation:

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{and} \quad g(x) \approx g(a) + g'(a)(x - a) \quad (3.8.3)$$

Therefore, the quotient can be approximated as:

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} \quad (3.8.4)$$

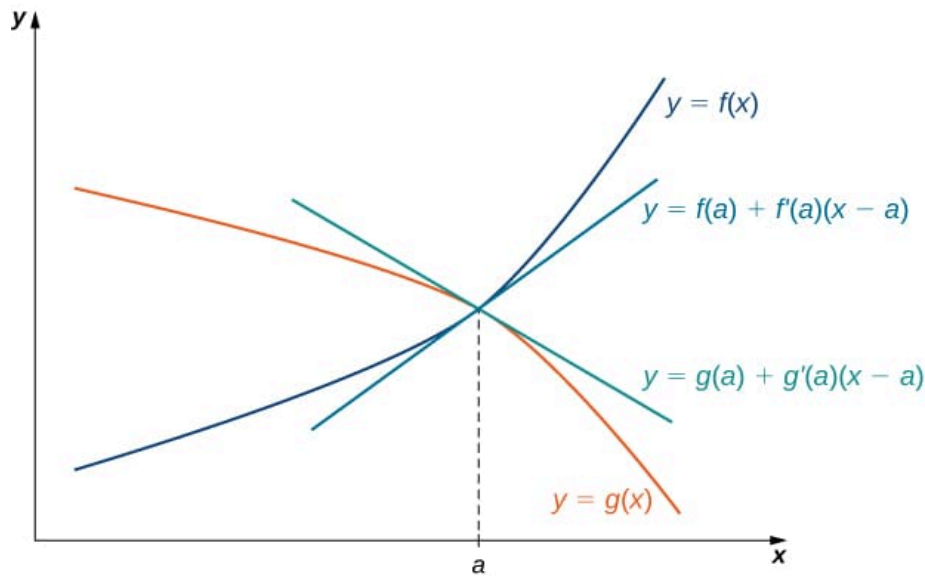


Figure 3.8.13: The graph of two functions and their linear approximations. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

Both  $f$  and  $g$  are continuous at  $a$  because they are differentiable at  $a$ , and differentiability implies continuity.

Therefore:

$$f(a) = \lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad g(a) = \lim_{x \rightarrow a} g(x) = 0 \quad (3.8.5)$$

If  $g'(a) \neq 0$ , then we can apply linear approximations:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} = \lim_{x \rightarrow a} \frac{f'(a)(x - a)}{g'(a)(x - a)} = \lim_{x \rightarrow a} \frac{f'(a)}{g'(a)} \quad (3.8.6)$$

So,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \quad (3.8.7)$$

#### Note

Observe that we are assuming that  $f'$  and  $g'$  are continuous.

#### Theorem 3.8.7 L'Hôpital Rule- Indeterminate Form $\frac{0}{0}$ .

Suppose that  $f$  and  $g$  are differentiable on an open interval containing  $a$ , except possibly at  $a$ , and that:

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0, \quad (3.8.8)$$

with  $g'(x) \neq 0$  near  $a$ . Then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \quad (3.8.9)$$

provided the limit on the right exists.

The same result applies for one-sided limits or limits as  $x \rightarrow \pm\infty$ .

### Example 3.8.26

Evaluate the following limits:

1.  $\lim_{x \rightarrow 1} \frac{x^3 - 4x^2 + 3}{x^2 - 1}$
2.  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$
3.  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$
4.  $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin(\pi x)}$
5.  $\lim_{x \rightarrow 0} \frac{\tan x}{4x}$
6.  $\lim_{x \rightarrow \infty} \frac{e^{5/x} - 1}{1/x^2}$

### Theorem 3.8.8 L'Hôpital Rule - Indeterminate Form $\frac{\infty}{\infty}$

Suppose that  $f$  and  $g$  are differentiable on an open interval containing  $a$ , except possibly at  $a$ . If

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{or} \quad -\infty, \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \infty \quad \text{or} \quad -\infty, \quad (3.8.10)$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \quad (3.8.11)$$

provided the limit on the right exists.

This result also holds for one-sided limits and limits as  $x \rightarrow \pm\infty$ .

### Example 3.8.27

Evaluate the following limits:

1.  $\lim_{x \rightarrow \infty} \frac{x^2 + 8x}{3x + 1}$
2.  $\lim_{x \rightarrow \infty} x e^{-x^2}$
3.  $\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}}$

## Other Indeterminate Forms

### Example 3.8.28

Evaluate the following limits.

1.  $\lim_{x \rightarrow 0^+} x \ln x$ . Indeterminate form:  $0 \cdot \infty$

2.  $\lim_{x \rightarrow \pi/2^+} (\sec x - \tan x)$ . Indeterminate form:  $\infty - \infty$
3.  $\lim_{x \rightarrow \infty} x^{1/\sqrt{x}}$ . Indeterminate form:  $\infty^0$
4.  $\lim_{x \rightarrow 0^+} x^{3/x}$ . Indeterminate form:  $0^\infty$
5.  $\lim_{x \rightarrow 0^+} (1 + \sin 6x)^{1/x}$ . Indeterminate form:  $1^\infty$
6.  $\lim_{x \rightarrow 0^+} x^x$ . Indeterminate form:  $0^0$

## 3.9: Newton's Method

### 3.9 Newton's Method

#### Learning Objectives

- Apply Newton's method to approximate solutions of equations by using iterative processes.

**Task:** Find the solutions of the equation  $f(x) = 0$ .

Note that the solutions of the above equation corresponds to the  $x$ -intercepts of the graph of  $f$ . We use the following iterative process to obtain the solutions.

#### Newton's Method

By sketching a graph of  $f$ , we can estimate a root of  $f(x) = 0$ . Let's call this estimate  $x_0$ . We then draw the tangent line to  $f$  at  $x_0$ . If  $f'(x_0) \neq 0$ , this tangent line intersects the  $x$ -axis at some point  $(x_1, 0)$ . Now let  $x_1$  be the next approximation to the actual root. Typically,  $x_1$  is closer than  $x_0$  to an actual root. Next we draw the tangent line to  $f$  at  $x_1$ . If  $f'(x_1) \neq 0$ , this tangent line also intersects the  $x$ -axis, producing another approximation,  $x_2$ . We continue in this way, deriving a list of approximations:  $x_0, x_1, x_2, \dots$ . Typically, the numbers  $x_0, x_1, x_2, \dots$  quickly approach an actual root  $x^*$ , as shown in the following figure.

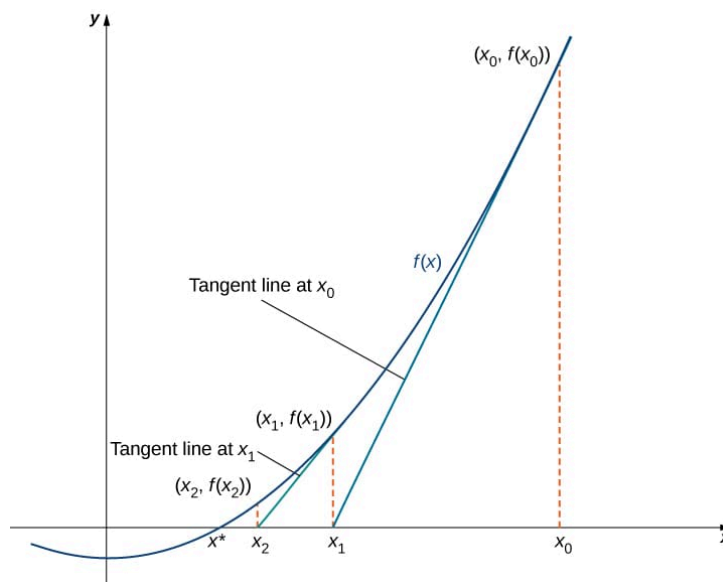


Figure 3.9.14: A graph illustrating the iterative process of Newton's method. From [OpenStax Calculus Volume 1](#), licensed under [CC-BY-NC-SA](#).

Now, let's calculate the approximations  $x_0, x_1, x_2, \dots$ . Suppose that  $x_0$  is our first approximation. Usually, this approximation is obtained by guessing, or from a sketch of the graph, or from a graphing device or app. Then, the approximation  $x_1$  is defined by letting  $(x_1, 0)$  be the  $x$ -intercept of the tangent line to  $f$  at  $x_0$ . The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore,  $(x_1, 0)$  is on the line and must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

If  $f'(x_0) \neq 0$ , we can solve this equation for  $x_1$ , so

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point  $(x_2, 0)$  is the  $x$ -intercept of the tangent line to  $f$  at  $x_1$ . Therefore,  $x_2$  satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for  $n > 0$ ,  $x_n$  satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \quad (3.9.1)$$

#### Example 3.9.29

Use Newton's Method to approximate a root of the equation  $2x^3 + 3x + 1 = 0$ . Let  $x_0 = -1$  be the initial approximation.

#### Example 3.9.30

Use Newton's Method to find  $\sqrt[5]{2}$  corrected to six decimal places.

#### Example 3.9.31

Use Newton's Method to find, correct to eight decimal places, a solution to the equation  $\sin x = x$ . Let  $x_0 = 1$  be the initial approximation.

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## 3.10: Antiderivatives

### 3.10 Antiderivatives

#### Learning Objectives

- Find the general antiderivative of a given function by applying integration techniques, including basic rules for integrals.
- Apply basic rules for integrals (e.g., power rule, sum rule) to compute antiderivatives of functions.
- Solve initial-value problems by finding the antiderivative of a function and applying the given initial conditions to determine the constant of integration.

#### Definition 3.10.9

A function  $F$  is called an antiderivative of  $f$  on an interval  $I$  if  $F'(x) = f(x)$  for all  $x$  in  $I$ .

For example, if  $f(x) = x^2$ , then  $F(x) = \frac{x^3}{3}$  is an antiderivative of  $f$ .

#### Theorem 3.10.9

Suppose that  $F$  is an antiderivative of  $f$  on an interval  $I$ . Then the most general antiderivative of  $f$  on  $I$  is:

$$F(x) + C \quad (3.10.1)$$

where  $C$  is a constant. Furthermore, if  $G$  is an antiderivative of  $f$  over  $I$ , then there exists a constant  $C$  such that:

$$G(x) = F(x) + C \quad \text{on } I \quad (3.10.2)$$

#### Example 3.10.32

Find the general antiderivative of the following functions:

1.  $f(x) = e^x$
2.  $f(x) = \sin x$
3.  $f(x) = \cos x$
4.  $f(x) = \frac{1}{x}$

#### Definition 3.10.10

Let  $f$  be a function. The indefinite integral of  $f$ , denoted

$$\int f(x) dx \quad (3.10.3)$$

is defined as the most general antiderivative of  $f$ . That is, if  $F$  is an antiderivative of  $f$ , then

$$\int f(x) dx = F(x) + C \quad (3.10.4)$$

where  $C$  is an arbitrary constant.

| Derivative Formula                                    | Indefinite Integral                                                |
|-------------------------------------------------------|--------------------------------------------------------------------|
| $\frac{d}{dx}[k] = 0$                                 | $\int k \, dx = kx + C$                                            |
| $\frac{d}{dx}[x^n] = nx^{n-1}$                        | $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \text{ for } n \neq -1$ |
| $\frac{d}{dx}[\ln x ] = \frac{1}{x}$                  | $\int \frac{1}{x} \, dx = \ln x  + C$                              |
| $\frac{d}{dx}[e^x] = e^x$                             | $\int e^x \, dx = e^x + C$                                         |
| $\frac{d}{dx}[\sin x] = \cos x$                       | $\int \cos x \, dx = \sin x + C$                                   |
| $\frac{d}{dx}[\cos x] = -\sin x$                      | $\int \sin x \, dx = -\cos x + C$                                  |
| $\frac{d}{dx}[\tan x] = \sec^2 x$                     | $\int \sec^2 x \, dx = \tan x + C$                                 |
| $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$  | $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$              |
| $\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$ | $\int -\frac{1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x + C$             |
| $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$         | $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$                     |

### Theorem 3.10.10

#### Properties of Indefinite Integrals:

Let  $F$  and  $G$  be antiderivatives of  $f$  and  $g$ , respectively. Let  $k$  be a constant.

- $\int (f(x) \pm g(x)) \, dx = F(x) \pm G(x) + C$
- $\int kf(x) \, dx = kF(x) + C$

### Example 3.10.33

Evaluate the following indefinite integrals:

- $\int (8x^9 + 2x^6 - 4x^2 - 5) \, dx$
- $\int (4x^5 + e^x) \, dx$
- $\int \frac{3x^5 + 10x^3 + 4x}{x^5} \, dx$
- $\int \frac{3x^5 + 7\sqrt[5]{x} + 2}{x} \, dx$
- $\int \frac{2}{\sqrt{1-x^2}} \, dx$
- $\int (\cos x - 4\sec^2 x) \, dx$

## Initial-value problem

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \quad (3.10.5)$$

is a simple example of a differential equation. Solving this equation means finding a function  $y$  with a derivative  $f$ . Therefore, the solutions of Equation 3.10.5 are the antiderivatives of  $f$ . If  $F$  is one antiderivative of  $f$ , every function of the form  $y = F(x) + C$  is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 4x^3$$

are given by

$$y = \int 4x^3 dx = x^4 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point  $(x_0, y_0)$  —that is,  $y(x_0) = y_0$ . The problem of finding a function  $y$  that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an initial-value problem. The condition  $y(x_0) = y_0$  is known as an initial condition. For example, looking for a function  $y$  that satisfies the differential equation

$$\frac{dy}{dx} = 4x^3$$

and the initial condition

$$y(2) = 6$$

is an example of an initial-value problem. Since the solutions of the differential equation are  $y = x^4 + C$ , to find a function  $y$  that also satisfies the initial condition, we need to find  $C$  such that  $y(2) = (2)^4 + C = 6$ . From this equation, we see that  $C = -10$ , and we conclude that  $y = x^4 - 10$  is the solution of this initial-value problem.

#### Example 3.10.34

Solve the following initial value problems:

1.  $\frac{dy}{dx} = 4 + \sqrt{x}, \quad y(4) = 20$
2.  $\frac{dy}{dx} = 4x^5 + e^x, \quad y(1) = e$
3.  $\frac{dy}{dx} = \cos x - 4 \sec^2 x, \quad y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

#### Example 3.10.35

A particle in motion has acceleration given by:

$$a(t) = 12t^2 - 6 \text{ and } s(2) = 5. \quad (3.10.6)$$

Find the position function  $s(t)$  of the particle.

### Example 3.10.36

A car is traveling at a speed of 66 ft/sec (45 mph) when the brakes are applied. The car begins decelerating at a constant rate of  $10 \text{ ft/sec}^2$ .

1. How long does it take for the car to stop?
2. What distance does the car cover before stopping?

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## 4: INTRODUCTION TO INTEGRALS

### Overview

While derivatives measure instantaneous rates of change, integrals allow us to measure accumulation. Integrals provide a mathematical framework to compute areas, total change, and accumulated quantities.

The concept begins with approximating area using rectangles, formalized through Riemann sums. From there, we define the definite integral and explore its properties. The **Fundamental Theorem of Calculus** reveals the deep connection between differentiation and integration, providing a powerful tool to evaluate integrals. Finally, we introduce the **Substitution Rule**, a method that simplifies integrals by reversing the Chain Rule.

Integrals are essential in mathematics, physics, engineering, and economics, where they represent accumulated quantities such as distance traveled, total growth, and area under a curve.

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### Topics in this Chapter

1. **Riemann Sums**
  - Approximate area under a curve using sums of rectangles.
  - Understand left, right, and midpoint Riemann sums.
  - Connect the idea of approximation to the exact value of an integral.
2. **The Definite Integral**
  - Define the definite integral as the limit of Riemann sums.
  - Interpret the definite integral as the net area under a curve.
  - Learn the properties of the definite integral (linearity, additivity, etc.).
3. **The Fundamental Theorem of Calculus**
  - State and apply both parts of the Fundamental Theorem of Calculus.
  - Connect differentiation and integration as inverse processes.
  - Evaluate definite integrals efficiently using antiderivatives.
4. **The Substitution Rule**
  - Apply substitution to compute integrals, reversing the Chain Rule.
  - Solve integrals involving composite functions.
  - Practice substitution in both indefinite and definite integrals.

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### Why This Matters

Integration is a central idea of calculus, balancing the concept of differentiation. It allows us to:

- Compute exact areas, accumulated quantities, and total change.
- Solve real-world problems in physics, engineering, biology, and economics.
- Establish the foundation for advanced applications such as volumes, work, probability, and differential equations.

Integrals transform local rates of change into global accumulation, providing one of the most powerful tools in mathematics.

[4.1: Riemann Sums](#)

[4.2: The Definite Integral](#)

[4.3: The Fundamental Theorem of Calculus](#)

[4.4: The Substitution Rule](#)



## 4.1: Riemann Sums

### 4.1 Riemann Sums

#### Learning Objectives

- Approximate the area under a curve by calculating the sum of rectangular regions, using an appropriate method such as left, right, or midpoint rectangles.
- Calculate the area under a curve using Riemann sums by partitioning the interval into subintervals and applying the left, right, or midpoint approximation methods.

### Sigma Notation

We anticipate dealing with sums of a large number of terms. To do so, we introduce *sigma notation*, named for the Greek letter  $\Sigma$ , which is the capital letter *S* in the Greek alphabet.

For example, say we are interested in the sum of the first 100 natural numbers:  $1 + 2 + 3 + \cdots + 100$ .

In sigma notation we write

$$\sum_{k=1}^{100} k = 1 + 2 + 3 + \cdots + 100.$$

We read the symbol  $\sum_{k=1}^{100} k$  as “the sum from  $k$  equals 1 to 100 of  $k$ .” The variable  $k$  is called the index of summation, and any letter can be used for this variable. The pattern in the terms of the sum is denoted by a function of the index; for example,

$$\sum_{k=1}^{10} (k^2 + 2k) = (1^2 + 2 \cdot 1) + (2^2 + 2 \cdot 2) + (3^2 + 2 \cdot 3) + \cdots + (10^2 + 2 \cdot 10),$$

and more generally,

$$\sum_{k=1}^n f(k) = f(1) + f(2) + \cdots + f(n).$$

Sigma notation allows us to easily vary the function being used to describe the terms in the sum, and to adjust the number of terms in the sum simply by changing the value of  $n$ . We will test our understanding of this new notation in the following activity.

### Riemann Sums

Consider an arbitrary positive function  $f$  on  $[a, b]$  with the interval subdivided as shown in the figure below.

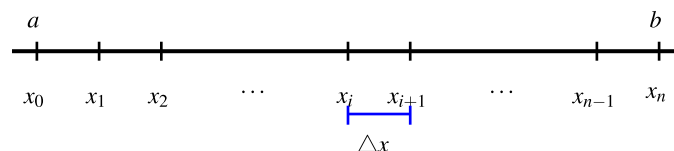


Figure 4.1.1. Subdividing the interval  $[a, b]$  into  $n$  subintervals of equal length  $\Delta x$ . From Active Calculus by Matthew, Boelkins, licensed under CC BY.

Let's use left endpoints. Then on each interval  $[x_i, x_{i+1}]$ , the area of the rectangle formed is given by

$$A_{i+1} = f(x_i) \cdot \Delta x,$$

as seen in the figure below.

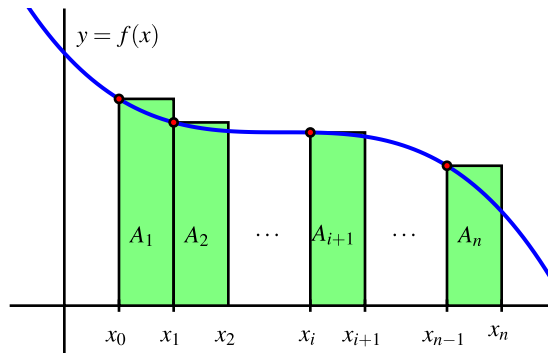


Figure 4.1.2. Subdividing the interval  $[a, b]$  into  $n$  subintervals of equal length  $\Delta x$  and

approximating the area under  $y = f(x)$  over  $[a, b]$  using left rectangles. From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

If we let  $L_n$  denote the sum of the areas of these rectangles, we see that

$$\begin{aligned} L_n &= A_1 + A_2 + \cdots + A_{i+1} + \cdots + A_n \\ &= f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + \cdots + f(x_i) \cdot \Delta x + \cdots + f(x_{n-1}) \cdot \Delta x. \end{aligned}$$

In the more compact sigma notation, we have

$$L_n = \sum_{i=0}^{n-1} f(x_i) \Delta x.$$

Note that since the index of summation begins at 0 and ends at  $n - 1$ , there are indeed  $n$  terms in this sum. We call  $L_n$  the *left Riemann sum* for the function  $f$  on the interval  $[a, b]$ .

To see how the Riemann sums for right endpoints and midpoints are constructed, consider Figure 4.1.3.

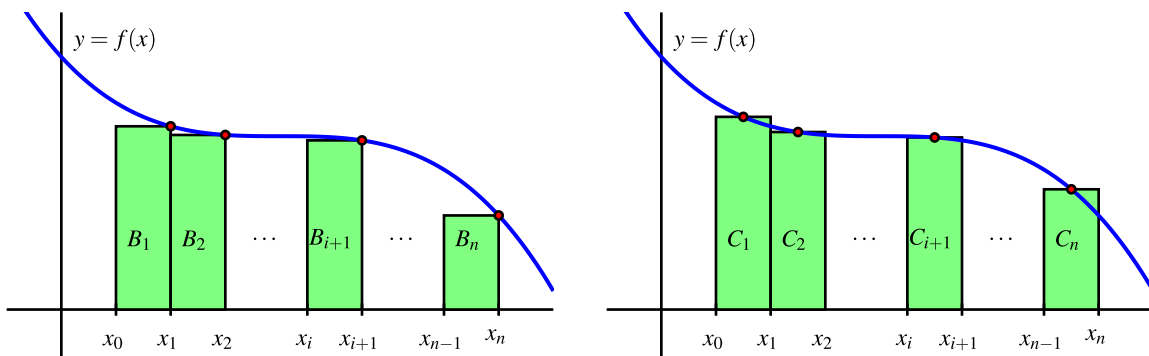


Figure 4.1.3. Riemann sums using right endpoints and midpoints. From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

For the sum with right endpoints, we see that the area of the rectangle on an arbitrary interval  $[x_i, x_{i+1}]$  is given by  $B_{i+1} = f(x_{i+1}) \cdot \Delta x$ , and that the sum of all such areas of rectangles is given by

$$\begin{aligned} R_n &= B_1 + B_2 + \cdots + B_{i+1} + \cdots + B_n \\ &= f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_{i+1}) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x \\ &= \sum_{i=1}^n f(x_i) \Delta x. \end{aligned}$$

We call  $R_n$  the *right Riemann sum* for the function  $f$  on the interval  $[a, b]$ .

For the sum that uses midpoints, we introduce the notation

$$\bar{x}_{i+1} = \frac{x_i + x_{i+1}}{2}$$

so that  $\bar{x}_{i+1}$  is the midpoint of the interval  $[x_i, x_{i+1}]$ . For instance, for the rectangle with area  $C_1$  in Figure 4.1.3, we now have

$$C_1 = f(\bar{x}_1) \cdot \Delta x.$$

Hence, the sum of all the areas of rectangles that use midpoints is

$$\begin{aligned} M_n &= C_1 + C_2 + \cdots + C_{i+1} + \cdots + C_n \\ &= f(\bar{x}_1) \cdot \Delta x + f(\bar{x}_2) \cdot \Delta x + \cdots + f(\bar{x}_{i+1}) \cdot \Delta x + \cdots + f(\bar{x}_n) \cdot \Delta x \\ &= \sum_{i=1}^n f(\bar{x}_i) \Delta x, \end{aligned}$$

and we say that  $M_n$  is the *middle Riemann sum* for  $f$  on  $[a, b]$ .

#### Query4.1.1

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## 4.2: The Definite Integral

### 4.2 The Definite Integral

#### Learning Objectives

- Identify the integrand, limits of integration, and variable of integration in a given integral.
- Evaluate definite integrals using geometric interpretations (e.g., area under a curve) and properties of definite integrals.
- Interpret definite integrals as signed areas under a curve, understanding how the area can be positive or negative depending on the direction of the curve relative to the x-axis.
- Apply definite integrals to calculate the change in position of an object over a given interval or the total distance traveled, considering the speed or velocity function.

As the number of rectangles got larger and larger, the values of  $L_n$ ,  $M_n$ , and  $R_n$  all grew closer and closer to the same value. It turns out that this occurs for any continuous function on an interval  $[a, b]$ , and also for a Riemann sum using any point  $x_{i+1}^*$  in the interval  $[x_i, x_{i+1}]$ . Thus, as we let  $n \rightarrow \infty$ , it doesn't really matter where we choose to evaluate the function within a given subinterval, because

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} M_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

That these limits always exist (and share the same value) when  $f$  is continuous allows us to make the following definition.

It turns out that a function need not be continuous in order to have a definite integral. For our purposes, we assume that the functions we consider are continuous on the interval(s) of interest. It is straightforward to see that any function that is piecewise continuous on an interval of interest will also have a well-defined definite integral.

#### Definition 4.2.1

The **definite integral** of a continuous function  $f$  on the interval  $[a, b]$ , denoted  $\int_a^b f(x) \, dx$ , is the real number given by

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x,$$

where  $\Delta x = \frac{b-a}{n}$ ,  $x_i = a + i\Delta x$  (for  $i = 0, \dots, n$ ), and  $x_i^*$  satisfies  $x_{i-1} \leq x_i^* \leq x_i$  (for  $i = 1, \dots, n$ ).

We call the symbol  $\int$  the *integral sign*, the values  $a$  and  $b$  the *limits of integration*, and the function  $f$  the *integrand*. The process of determining the real number  $\int_a^b f(x) \, dx$  is called *evaluating the definite integral*. While there are several different interpretations of the definite integral, for now the most important is that  $\int_a^b f(x) \, dx$  measures the net signed area bounded by  $y = f(x)$  and the  $x$ -axis on the interval  $[a, b]$ .

### Definite Integrals and Areas

When  $f(x) \geq 0$  on  $[a, b]$ , the definite integral  $\int_a^b f(x) \, dx$  represents the area under the curve  $y = f(x)$  over the interval  $[a, b]$ .

For example, if  $f$  is the function pictured in Figure 4.2.4, and  $A_1$ ,  $A_2$ , and  $A_3$  are the exact areas bounded by  $f$  and the  $x$ -axis on the respective intervals  $[a, b]$ ,  $[b, c]$ , and  $[c, d]$ , then

$$\int_a^b f(x) dx = A_1, \quad \int_b^c f(x) dx = -A_2,$$

$$\int_c^d f(x) dx = A_3,$$

and  $\int_a^d f(x) dx = A_1 - A_2 + A_3.$

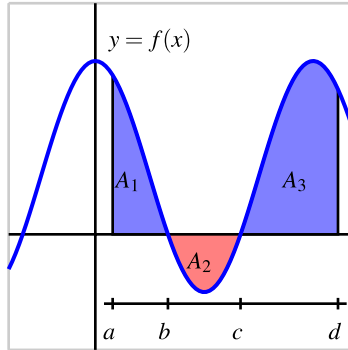


Figure 4.2.4. A continuous function  $f$  on the interval  $[a, d]$ . From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

### Definition 4.2.2

Let  $f(x)$  be an integrable function defined on an interval  $[a, b]$ . Let  $A_1$  represent the area between  $f(x)$  and the  $x$ -axis that lies above the axis, and let  $A_2$  represent the area between  $f(x)$  and the  $x$ -axis that lies below the axis.

Then, the **net signed area** between  $f(x)$  and the  $x$ -axis is:

$$\int_a^b f(x) dx = A_1 - A_2 \quad (4.2.1)$$

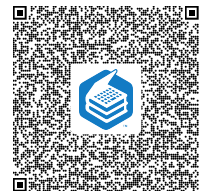
The **total area** between  $f(x)$  and the  $x$ -axis is:

$$\int_a^b |f(x)| dx = A_1 + A_2 \quad (4.2.2)$$

### Query 4.2.2

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### Example 4.2.1

Let  $f(x) = x - 3$ .

1. Find the net signed area of  $f(x)$  over the interval  $[0, 7]$ .
2. Find the total area between  $f(x) = x - 3$  and the  $x$ -axis over the interval  $[0, 7]$ .

For now, our goal is to understand the meaning and properties of the definite integral, rather than to compute its value. To do this,

we will rely on the net signed area interpretation of the definite integral. So we will use as examples curves that produce regions whose areas we can compute exactly through area formulas. We can thus compute the exact value of the corresponding integral.

For instance, if we wish to evaluate the definite integral  $\int_1^4 (2x + 1) dx$ , we observe that the region bounded by this function and the  $x$ -axis is the trapezoid shown in Figure 4.2.5. By the formula for the area of a trapezoid,  $A = \frac{1}{2}(3 + 9) \cdot 3 = 18$ , so

$$\int_1^4 (2x + 1) dx = 18.$$

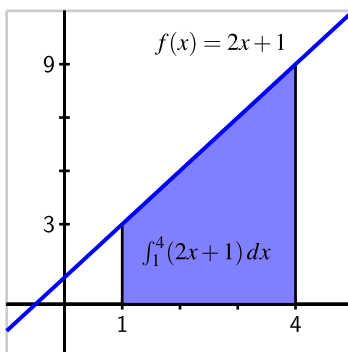


Figure 4.2.5. The area bounded by  $f(x) = 2x + 1$  and the  $x$ -axis on the interval  $[1, 4]$ . From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

### Example 4.2.2

Use known geometric formulas and the net signed area interpretation of the definite integral to evaluate each of the definite integrals below.

1.  $\int_0^1 3x dx$
2.  $\int_{-1}^4 (2 - 2x) dx$
3.  $\int_{-1}^1 \sqrt{1 - x^2} dx$
4.  $\int_{-3}^4 g(x) dx$ , where  $g$  is the function pictured in Figure 4.2.6. Assume that each portion of  $g$  is either part of a line or part of a circle.

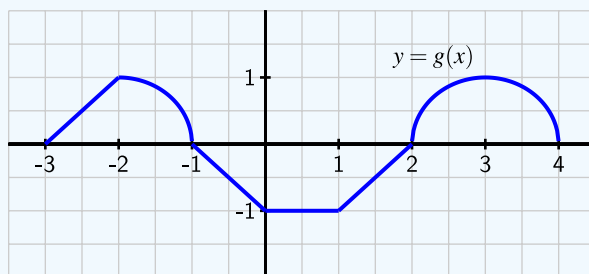


Figure 4.2.6. The function  $g$  for part (d). Note that  $g$  is piecewise defined, and each piece of the function is part of a circle or part of a line. From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

### Example 4.2.3

Use geometry to evaluate the following definite integrals:

1.  $\int_{-2}^2 \sqrt{4-x^2} \, dx$
2.  $\int_3^6 \sqrt{16-(x-4)^2} \, dx$
3.  $\int_0^4 (x+2) \, dx$

## Definite Integrals and Change in Position

If an object moves along a straight line with position  $s(t)$ , then its velocity is  $v(t) = s'(t)$ . The net change of position (or displacement) of the object during the time interval  $[t_1, t_2]$  is:

$$\int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1) \quad (4.2.3)$$

If we want to calculate the total distance traveled by the object during the interval  $[t_1, t_2]$ , we evaluate:

$$\int_{t_1}^{t_2} |v(t)| \, dt = \text{total distance traveled} \quad (4.2.4)$$

### Query 4.2.3

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## Definite Integrals and the Function's Average Value

One of the most valuable applications of the definite integral is that it provides a way to discuss the average value of a function, even for a function that takes on infinitely many values. Recall that if we wish to take the average of  $n$  numbers  $y_1, y_2, \dots, y_n$ , we compute

$$AVG = \frac{y_1 + y_2 + \dots + y_n}{n}.$$

Since integrals arise from Riemann sums in which we add  $n$  values of a function, it should not be surprising that evaluating an integral is similar to averaging the output values of a function. Consider, for instance, the right Riemann sum  $R_n$  of a function  $f$ , which is given by

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = (f(x_1) + f(x_2) + \dots + f(x_n))\Delta x.$$

Since  $\Delta x = \frac{b-a}{n}$ , we can thus write

$$\begin{aligned} R_n &= (f(x_1) + f(x_2) + \dots + f(x_n)) \cdot \frac{b-a}{n} \\ &= (b-a) \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}. \end{aligned} \quad (4.2.11)$$

We see that the right Riemann sum with  $n$  subintervals is just the length of the interval  $(b-a)$  times the average of the  $n$  function values found at the right endpoints. And just as with our efforts to compute area, the larger the value of  $n$  we use, the more accurate our average will be. Indeed, we will define the average value of  $f$  on  $[a, b]$  to be

$$f_{\text{AVG}[a,b]} = \lim_{n \rightarrow \infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

But we also know that for any continuous function  $f$  on  $[a, b]$ , taking the limit of a Riemann sum leads precisely to the definite integral. That is,  $\lim_{n \rightarrow \infty} R_n = \int_a^b f(x) \, dx$ , and thus taking the limit as  $n \rightarrow \infty$  in Equation (4.2.11), we have that

$$\int_a^b f(x) \, dx = (b - a) \cdot f_{\text{AVG}[a,b]}. \quad (4.2.12)$$

Solving Equation 4.2.12 for  $f_{\text{AVG}[a,b]}$ , we have the following general principle.

### Average value of a function

If  $f$  is a continuous function on  $[a, b]$ , then its **average value** on  $[a, b]$  is given by the formula

$$f_{\text{AVG}[a,b]} = \frac{1}{b - a} \cdot \int_a^b f(x) \, dx.$$

### Example 4.2.4

Suppose that  $v(t) = \sqrt{4 - (t - 2)^2}$  tells us the instantaneous velocity of a moving object on the interval  $0 \leq t \leq 4$ , where  $t$  is measured in minutes and  $v$  is measured in meters per minute.

1. Sketch an accurate graph of  $y = v(t)$ . What kind of curve is  $y = \sqrt{4 - (t - 2)^2}$ ?
2. Evaluate  $\int_0^4 v(t) \, dt$  exactly.
3. In terms of the physical problem of the moving object with velocity  $v(t)$ , what is the meaning of  $\int_0^4 v(t) \, dt$ ? Include units on your answer.
4. Determine the exact average value of  $v(t)$  on  $[0, 4]$ . Include units on your answer.
5. Sketch a rectangle whose base is the line segment from  $t = 0$  to  $t = 4$  on the  $t$ -axis such that the rectangle's area is equal to the value of  $\int_0^4 v(t) \, dt$ . What is the rectangle's exact height?
6. How can you use the average value you found in (d) to compute the total distance traveled by the moving object over  $[0, 4]$ ?

## Properties of the Definite Integral

Let  $f(x)$  and  $g(x)$  be continuous functions and let  $a, b, c, m, M$  be constants.

1.  $\int_a^a f(x) \, dx = - \int_a^b f(x) \, dx$
2.  $\int_a^a f(x) \, dx = 0$
3.  $\int_a^b c \, dx = c(b - a)$
4.  $\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$
5.  $\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$
6.  $\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$
7.  $\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$
8. If  $f(x) \geq 0$  for  $a \leq x \leq b$ , then  $\int_a^b f(x) \, dx \geq 0$
9. If  $f(x) \geq g(x)$  for  $a \leq x \leq b$ , then  $\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$

10. If  $m \leq f(x) \leq M$  for  $a \leq x \leq b$ , then  $m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a)$

### Example 4.2.5

Suppose that the following information is known about the functions  $f$ ,  $g$ ,  $x^2$ , and  $x^3$ :

- $\int_0^2 f(x) \, dx = -3$ ;  $\int_2^5 f(x) \, dx = 2$
- $\int_0^2 g(x) \, dx = 4$ ;  $\int_2^5 g(x) \, dx = -1$
- $\int_0^2 x^2 \, dx = \frac{8}{3}$ ;  $\int_2^5 x^2 \, dx = \frac{117}{3}$
- $\int_0^2 x^3 \, dx = 4$ ;  $\int_2^5 x^3 \, dx = \frac{609}{4}$

Use the provided information and the rules discussed in the preceding section to evaluate each of the following definite integrals.

1.  $\int_5^2 f(x) \, dx$
2.  $\int_0^5 g(x) \, dx$
3.  $\int_0^5 (f(x) + g(x)) \, dx$
4.  $\int_5^2 (3x^2 - 4x^3) \, dx$
5.  $\int_5^2 (2x^3 - 7g(x)) \, dx$

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## 4.3: The Fundamental Theorem of Calculus

### 4.3 The Fundamental Theorem of Calculus

#### Learning Objectives

- Apply the FTC, Part 1, to evaluate the derivative of an integral with respect to its upper limit of integration, recognizing how it simplifies the computation.
- Use the FTC, Part 2, to evaluate definite integrals by finding the antiderivative and applying the limits of integration.

#### Note

Recall that a function  $F$  is called an antiderivative of  $f$  on the interval  $I$  if  $F'(x) = f(x)$  for all  $x \in I$ . The general antiderivative has the form  $F(x) + C$ , where  $C$  is a constant.

For example, if  $f(x) = x^n$ , where  $n \geq 0$ , the general antiderivative is:

$$F(x) = \frac{x^{n+1}}{n+1} + C \quad (4.3.1)$$

#### Theorem 4.3.1 FTC Part 1

Suppose that  $f$  is continuous on  $[a, b]$ . Then the function

$$F(x) = \int_a^x f(t) dt \quad \text{for } a \leq x \leq b \quad (4.3.2)$$

is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $F'(x) = f(x)$

#### Example 4.3.6

Find the derivative of the following functions:

1.  $y = \int_0^x \sqrt{4+t^2} dt$
2.  $y = \int_1^{\sqrt{x}} \sin t dt$
3.  $y = \int_2^{1/x} (t^2 - 1)^{10} dt$

#### Query 4.3.4

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### Example 4.3.7

Suppose that pollutants are leaking out of an underground storage tank at a rate of  $r(t)$  gallons/day, where  $t$  is measured in days. It is conjectured that  $r(t)$  is given by the formula  $r(t) = 0.0069t^3 - 0.125t^2 + 11.079$  over a certain 12-day period. The graph of  $y = r(t)$  is given in Figure 4.3.7. What is the meaning of  $\int_4^{10} r(t) dt$  and what is its value? What is the average rate at which pollutants are leaving the tank on the time interval  $4 \leq t \leq 10$ ?

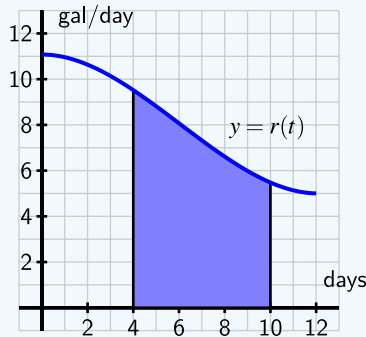


Figure 4.3.7. The rate  $r(t)$  of pollution leaking from a tank, measured in gallons per day. From [Active Calculus](#) by Matthew, Boelkins, licensed under [CC BY SA](#).

### Query 4.3.5

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Recall the integration formulas from Chapter 3:

| Derivative Formula                                   | Indefinite Integral                                             |
|------------------------------------------------------|-----------------------------------------------------------------|
| $\frac{d}{dx}[k] = 0$                                | $\int k dx = kx + C$                                            |
| $\frac{d}{dx}[x^n] = nx^{n-1}$                       | $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \text{ for } n \neq -1$ |
| $\frac{d}{dx}[\ln x ] = \frac{1}{x}$                 | $\int \frac{1}{x} dx = \ln x  + C$                              |
| $\frac{d}{dx}[e^x] = e^x$                            | $\int e^x dx = e^x + C$                                         |
| $\frac{d}{dx}[\sin x] = \cos x$                      | $\int \cos x dx = \sin x + C$                                   |
| $\frac{d}{dx}[\cos x] = -\sin x$                     | $\int \sin x dx = -\cos x + C$                                  |
| $\frac{d}{dx}[\tan x] = \sec^2 x$                    | $\int \sec^2 x dx = \tan x + C$                                 |
| $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$ | $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$              |

|                                                        |                                                     |
|--------------------------------------------------------|-----------------------------------------------------|
| $\frac{d}{dx} [\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$ | $\int -\frac{1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$ |
| $\frac{d}{dx} [\tan^{-1} x] = \frac{1}{1+x^2}$         | $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$         |

### Theorem 4.3.2 FTC Part 2

Suppose that  $f$  is continuous on  $[a, b]$ . Then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (4.3.3)$$

where  $F$  is any antiderivative of  $f$ .

### Example 4.3.8

Evaluate the following definite integrals:

- $\int_{-\frac{2}{4}}^2 (x^2 - 4) dx$
- $\int_{\frac{1}{4}}^9 3\sqrt{x} dx$
- $\int_1^9 \frac{x-1}{\sqrt{x}} dx$
- $\int_0^{\pi/4} \sec^2 x dx$
- $\int_{\frac{1}{4}}^{\pi} \sqrt{x}(1+x) dx$
- $\int_0^1 (\sin x - \cos x) dx$

## Integrals as Accumulated Change

If a function  $R(t)$  represents the rate at which a quantity changes, then its definite integral over an interval measures the net change in the quantity itself.

The Fundamental Theorem of Calculus states that

$$\int_a^b R(t) dt = F(b) - F(a), \quad (4.3.4)$$

where  $F(t)$  is any antiderivative of  $R(t)$ .

Thus:

- If  $R(t)$  is a growth rate, the integral gives change in population.
- If  $R(t)$  is velocity, the integral gives displacement (change in position).
- If  $R(t)$  is acceleration, the integral gives change in velocity.

In short, integrating a rate function accumulates how much the underlying quantity changes over the interval.

### Example 4.3.9

A lake's fish population grows at a rate modeled by

$$g(t) = 30 - 4t \quad (4.3.5)$$

fish per month, where  $t$  is measured in months.  
Find and interpret

$$\int_0^5 g(t) dt. \quad (4.3.6)$$

**Solution**

$$\int_0^5 (30 - 4t) dt = (30t - 2t^2) \Big|_0^5 = (150 - 50) - 0 = 100. \quad (4.3.7)$$

This model predicts that over the first 5 months, the population increases by **100 fish**. Even though the growth rate decreases over time, the definite integral measures the accumulated change.

### Example 4.3.10

A particle moves on a line with velocity and acceleration

$$v(t) = 8t - 6, \quad a(t) = 3 \text{ m/s}^2 \quad (4.3.8)$$

meters per second, where  $t$  is in seconds.  
Compute and interpret

$$\int_1^4 v(t) dt \quad \text{and} \quad \int_0^5 a(t) dt. \quad (4.3.9)$$

**Solution**

$$\int_1^4 (8t - 6) dt = (4t^2 - 6t) \Big|_1^4 = (64 - 24) - (4 - 6) = 40 - (-2) = 42. \quad (4.3.10)$$

The object's position changes by **42 meters** between  $t = 1$  and  $t = 4$ . The integral gives displacement, not *total distance* traveled.

Now consider acceleration. If the acceleration is constant

$$a(t) = 3 \text{ m/s}^2, \quad (4.3.11)$$

then

$\int_0^5 3 dt = 15$  tells us that during the first 5 seconds, the object's speed increases by **15 m/s**.

## Displacement and Total Distance

If  $v(t)$  represents velocity, then integrating it over an interval gives displacement:

$$\int_a^b v(t) dt = s(b) - s(a), \quad (4.3.12)$$

where  $s(t)$  is position. Displacement measures the net change in location and can be positive, negative, or zero depending on direction of motion.

Total distance traveled is different. Distance cannot be negative, so we must account for times when the velocity is negative (motion in the opposite direction). Total distance is computed by

$$\int_a^b |v(t)| dt. \quad (4.3.13)$$

Thus:

1.  $\int_a^b v(t) dt$  gives displacement.
2.  $\int_a^b |v(t)| dt$  gives total distance traveled.

Whenever velocity changes sign, the definite integral of  $v(t)$  underestimates actual distance because movement in opposite directions cancels in displacement but counts positively in distance.

### Example 4.3.11

A particle moves along a straight line according to the velocity function

$$v(t) = 4t - 6, \quad (4.3.14)$$

where  $t$  is measured in seconds and  $0 \leq t \leq 4$ .

(a) Find the displacement of the particle from  $t = 0$  to  $t = 4$ .

$$\int_0^4 (4t - 6) dt = (2t^2 - 6t) \Big|_0^4 = (32 - 24) - 0 = 8. \quad (4.3.15)$$

The net change in position over the time interval is **8 meters**.

(b) Find the total distance traveled over the same time interval.

First determine when velocity changes sign:

$$4t - 6 = 0 \Rightarrow t = \frac{3}{2}. \quad (4.3.16)$$

Since velocity is negative on  $0 \leq t < \frac{3}{2}$  and positive on  $\frac{3}{2} \leq t \leq 4$ , compute the distance in two pieces.

\medskip

On  $0 \leq t \leq \frac{3}{2}$ :

$$\int_0^{3/2} |4t - 6| dt = \int_0^{3/2} (6 - 4t) dt = (6t - 2t^2) \Big|_0^{3/2} = 9 - \frac{9}{2} = \frac{9}{2}. \quad (4.3.17)$$

On  $\frac{3}{2} \leq t \leq 4$ :

$$\int_{3/2}^4 |4t - 6| dt = \int_{3/2}^4 (4t - 6) dt = (2t^2 - 6t) \Big|_{3/2}^4 = 8 - \left(-\frac{9}{2}\right) = \frac{25}{2}. \quad (4.3.18)$$

Adding the pieces:

$$\frac{9}{2} + \frac{25}{2} = \frac{34}{2} = 17. \quad (4.3.19)$$

Thus, the total distance traveled is **17 meters**.

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## 4.4: The Substitution Rule

### 4.4 The Substitution Rule

#### Learning Objectives

- Apply the substitution rule to evaluate both definite and indefinite integrals by making an appropriate substitution to simplify the integrand and compute the integral.

### Reversing the Chain Rule: $u$ -substitution

How can we find the antiderivative of the function

$$g(x) = xe^{x^2} \text{ and } h(x) = e^{x^2}?$$

It is important to remember that differentiation and antidifferentiation are almost inverse processes (that they are not is due to the  $+C$  that arises when antidifferentiating). This almost-inverse relationship enables us to take any known derivative rule and rewrite it as a corresponding rule for an indefinite integral. For example, since

$$\frac{d}{dx} [x^5] = 5x^4,$$

we can equivalently write

$$\int 5x^4 dx = x^5 + C.$$

Recall that the Chain Rule states that

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x).$$

Restating this relationship in terms of an indefinite integral,

$$\int f'(g(x))g'(x) dx = f(g(x)) + C. \quad (4.4.1)$$

Equation 4.4.1 tells us that if we can view a given function as  $f'(g(x))g'(x)$  for some appropriate choices of  $f$  and  $g$ , then we can antidifferentiate the function by reversing the Chain Rule. Note that both  $g(x)$  and  $g'(x)$  appear in the form of  $f'(g(x))g'(x)$ ; we will sometimes say that we seek to *identify a function-derivative pair* ( $g(x)$  and  $g'(x)$ ) when trying to apply the rule in Equation 4.4.1.

If we can identify a function-derivative pair, we will introduce a new variable  $u$  to represent the function  $g(x)$ . With  $u = g(x)$ , it follows in Leibniz notation that  $\frac{du}{dx} = g'(x)$ , so that in terms of differentials,  $du = g'(x) dx$ . Now converting the indefinite integral to a new one in terms of  $u$ , we have

$$\int f'(g(x))g'(x) dx = \int f'(u) du.$$

If we recall from the definition of the derivative that  $\frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$  and use the fact that  $\frac{du}{dx} = g'(x)$ , then we see that  $g'(x) \approx \frac{\Delta u}{\Delta x}$ . Solving for  $\Delta u$ ,  $\Delta u \approx g'(x)\Delta x$ . It is this last relationship that, when expressed in “differential” notation enables us to write  $du = g'(x) dx$  in the change of variable formula.

Provided that  $f'$  is an elementary function whose antiderivative is known, we can easily evaluate the indefinite integral in  $u$ , and then go on to determine the desired overall antiderivative of  $f'(g(x))g'(x)$ . We call this process  *$u$ -substitution*, and summarize the rule as follows:

Note

With the substitution  $u = g(x)$ ,

$$\int f'(g(x))g'(x) dx = \int f'(u) du = f(u) + C = f(g(x)) + C.$$

To see  $u$ -substitution at work, we consider the following example.

Example 4.4.9

Evaluate the indefinite integral

$$\int 5x^4 \cdot \sin(x^5 + 1) dx$$

and check the result by differentiating.

## Evaluating Definite Integrals via $u$ -substitution

We have introduced  $u$ -substitution as a means to evaluate indefinite integrals of functions that can be written, up to a constant multiple, in the form  $f(g(x))g'(x)$ . This same technique can be used to evaluate definite integrals involving such functions, though we need to be careful with the corresponding limits of integration. Consider, for instance, the definite integral

$$\int_2^5 xe^{x^2} dx.$$

Whenever we write a definite integral, it is implicit that the limits of integration correspond to the variable of integration. To be more explicit, observe that

$$\int_2^5 xe^{x^2} dx = \int_{x=2}^{x=5} xe^{x^2} dx.$$

When we execute a  $u$ -substitution, we change the *variable* of integration; it is essential to note that this also changes the *limits* of integration. For instance, with the substitution  $u = x^2$  and  $du = 2x dx$ , it also follows that when  $x = 2$ ,  $u = 2^2 = 4$ , and when  $x = 5$ ,  $u = 5^2 = 25$ . Thus, under the change of variables of  $u$ -substitution, we now have

$$\begin{aligned} \int_{x=2}^{x=5} xe^{x^2} dx &= \int_{u=4}^{u=25} e^u \cdot \frac{1}{2} du \\ &= \frac{1}{2} e^u \Big|_{u=4}^{u=25} \\ &= \frac{1}{2} e^{25} - \frac{1}{2} e^4. \end{aligned}$$

Alternatively, we could consider the related indefinite integral  $\int xe^{x^2} dx$ , find the antiderivative  $\frac{1}{2}e^{x^2}$  through  $u$ -substitution, and then evaluate the original definite integral. With that method, we'd have

$$\begin{aligned} \int_2^5 xe^{x^2} dx &= \frac{1}{2} e^{x^2} \Big|_2^5 \\ &= \frac{1}{2} e^{25} - \frac{1}{2} e^4, \end{aligned}$$

which is, of course, the same result.

### Example 4.4.10

Evaluate the following integrals:

1.  $\int 6x(3x^2 + 4)^4 dx$

2.  $\int x\sqrt{x^2 - 5} dx$

3.  $\int \frac{\sin t}{\cos^3 x} dx$

4.  $\int \frac{x}{\sqrt{x-1}} dx$

5.  $\int_0^1 x^2(1 + 2x^3)^5 dx$

6.  $\int_0^1 xe^{4x^2+3} dx$

7.  $\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right) dx$

8.  $\int_0^{\pi/2} \cos^2 x dx$

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## INDEX

No index terms found.

## GLOSSARY

### S

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**Sample Word 1** | Sample Definition 1

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